A Truthmaker Semantics Approach to Modal Logic

MSc Thesis *(Afstudeerscriptie)*

written by

Giuliano Rosella
(born 23 September 1996, in Benevento, Italy)

under the supervision of **Dr. Maria Aloni** and **Dr. Thomas Schindler**, and submitted to the Board of Examiners in partial fulfillment of the requirements for the degree of

MSc Logic

at the *Universiteit van Amsterdam*.

Date of the public defense:
September 30, 2019

Members of the Thesis Committee:
Dr. Maria Aloni (supervisor)
Dr. Paul Dekker (chair)
Dr. Peter Hawke
Dr. Thomas Schindler (supervisor)
Dr. Julian Schlöder
Dr. Katrin Schulz

INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Abstract

The aim of this work is to find an answer to the following questions: is it possible to develop a truthmaker semantics for modal statements? And how? The answer to the first question is assumed to be positive and we will focus on seeking the answer to the second one. We believe that the truthmaker semantic account originally developed by Johannes Korbmacher in some unpublished work constitutes a satisfactory answer to the second question. So, we will prove some results on the connections between Korbmacher's account and already existing logics and we will also extend the framework and the related results to the first-order case. Moreover, we will engage in a philosophical discussion about the nature of truthmakers of modal statements and the way we should conceive of it in the light of this novel modal truthmaker approach. In the end, we will discuss the details of a possible application of the new semantic framework analysed in the thesis.

Contents

Introduction

Truthmaker semantics is a novel formal semantic framework which has been recently developed in a series of publications ((Fine, forthcoming), (Fine, 2017), (Fine, 2016)) by Kit Fine starting from the work done by Van Fraassen's in (Van Fraassen, 1969); it is based upon the notion of truthmaking, that is, as Fine's points out,

> "The idea of something on the side of the world - a fact, perhaps, a state of affairs- verifying, or making true, something on the side of the language or thought - a statement, perhaps, or a proposition" (see (Fine, 2017)).

That *something* on the side of the world which is responsible for the truth of a certain proposition A is called a *truthmaker* of A. Intuitively, there are different ways a proposition can be made true by a fact in the world; for instance, assume that now that it is raining and windy in the city of Amsterdam. We would say that the proposition "it is raining in Amsterdam" (B, for short) is indeed true. Furthermore, according to the above idea of truthmaking, we would say that B is made true by the *fact that* it is raining in Amsterdam as well as it is verified by the more *complex fact* that it is raining and windy in Amsterdam. The way the former fact makes B true is different from the way the latter fact makes B true: the fact that it is raining and windy in Amsterdam *contains* something, namely the sub-fact that it is windy in Amsterdam, which is not relevant for the truth of B.

From this intuitive observations, we can distinguish between (at least) two ways of truthmaking, which we will indicate respectively "exact" and "inexact" one. Fine's truthmaker semantics aims to formally describe the former way of truthmaking, namely, it is concerned with providing the conditions for a fact (something on the side of the world) to be an *exact* truthmaker of a sentence. Let us refer to these conditions as *(exact) truthmaker conditions* and let us call the conditions for a fact to be an exact truthmaker of A "exact truthmaker conditions for A". Now, we can provide more explicitly the definition of an exact truthmaker: an exact truthmaker of a sentence A is defined as that fact in the world which is responsible and *wholly* relevant for the truth of A (see (Fine, 2017)). The key to understanding the primitive idea of exact truthmaking is in the *whole relevance*: the fact that it is raining in Amsterdam is an exact truthmaker of sentence B : "it is raining in Amsterdam" as it contains nothing irrelevant for the truth of B. In terms of exact truthmaking, it is possible to define the notion of inexact truthmaker: a fact is an inexact truthmaker of a sentence A if and only if it contains, among its parts, an exact truthmaker of A. In the light of this definition, we can see how the fact that it is raining and windy in Amsterdam is an *inexact* truthmaker for the sentence B.

All these ideas have been formally worked out by Kit Fine to build the framework of truthmaker semantics. However, a modal truthmaker semantics to uniformly account for the truthmaker conditions of statements like "necessarily A" or "possibly A", is still lacking in Fine's framework.

The objective of this thesis is to provide a satisfactory answer to the following two questions: *is it possible to develop a truthmaker semantics to account for the exact truthmaker conditions for modal statements? And how?*

We assume the answer to the first question to be positive: *yes*, it is possible to account for the (exact) truthmaker conditions for modal statements. Hence, the focus of this work will be mostly on *how* to construct such semantic framework.

The structure of the present work is the following:

- In the first chapter, I will introduce a formal modal truthmaker semantics based on the ideas of Van Fraassen and some unpublished work of Johannes Korbmacher; and I will extend it to the first-order case.
- The second chapter is focused on developing some leading ideas for a philosophical account of truthmakers of modal truths which is compatible with the intuitions behind the semantics introduced in the first chapter. The general question I will try to address is: *what* is a(n) (exact) truthmaker of a modal statement (like "necessarily p" or "possibly p")?
- In the third and last chapter I will try to analyze a possible application of the semantic framework developed in the thesis.

1 Formal Framework

1.1 Background

1.1.1 Fine's Framework

In some recent papers ((Fine, forthcoming), (Fine, 2017), (Fine, 2016)), Kit Fine has developed, starting from the work done by Van Fraassen in (Van Fraassen, 1969), a new formal semantic framework based on the idea of truthmaking introduced above.

In the following presentation of Fine's work, we use letters A, B, C... to denote formulas and we stick to a language consisting of propositional variables p, q, r..., logical constants "$\neg, \wedge, \vee$" and auxiliary symbols "(,)"; a well-formed formula in this language is defined as:

$$A := p \mid \neg B \mid B \vee C \mid B \wedge C$$

Moreover, we assume some familiarity of the reader with orders and basic definitions such as those of *greatest lower bound* and *least upper bound*.

A *state model* is a tuple $\mathcal{M} = \langle S, \sqsubseteq, |.|^+, |.|^- \rangle$ with:

- $\langle S, \sqsubseteq \rangle$ a *state space* where
  - S non-empty set of states/facts;
  - $\sqsubseteq$ (*parthood relation*) over S such that:
    - for any $s \in S$, $s \sqsubseteq s$ (reflexivity);
    - for any $s, t, u \in S$, if $s \sqsubseteq t$ and $t \sqsubseteq u$, then $s \sqsubseteq u$ (transitivity);
    - for any $s, t \in S$, if $s \sqsubseteq t$ and $t \sqsubseteq s$ then $t = s$ (anti-symmetry);
    - S is complete, namely every $T \subseteq S$ has a least upper bound $\bigsqcup T \in S$ ($s \sqcup t$ denotes the fusion $\bigsqcup\{s, t\}$ of s and t);
    - $\bigsqcup \emptyset = 0$ and $\bigsqcup \emptyset \in S$ is the *null* element such that $0 \sqsubseteq s$ for any $s \in S$;

- $|.|^+, |.|^- : \mathcal{L}_{prop} \to \mathcal{P}(S)$ are valuation functions such that
  - $|p|^+ \subseteq S$ is the set of exact truthmakers of p;
  - $|p|^- \subseteq S$ is the set of exact falsemakers of p.

We define a relation of *overlapping* between states in a model $\mathcal{M}$: for all states $s, t \in S$, we say that s overlaps with t (Ost) if and only if there is a non-null state u in $\mathcal{M}$ such that $u \sqsubseteq s$ and $u \sqsubseteq t$.

The set S has to be understood as the *facts* on the side of the world among which we seek for truthmakers for propositions. The relation of composition among facts ($\sqsubseteq$) amount to the relation of parthood: for instance the fact that it is raining in Amsterdam (s) will be part of the fact that it is raining and windy in Amsterdam; and this latter fact will be conceived as the fusion ($s \sqcup t$) of the fact that it is raining in Amsterdam (s) and it is windy in Amsterdam (t).

The null fact "0" is understood as the fact which is part of every fact; intuitively we can understand "0" in analogy with the empty set: 0 is the fact that doesn't require anything in order to obtain; it is the fact which stands for *nothing*, in a suggestive slogan: *it is the fact of no fact*.

Given a *state model* $\mathcal{M} = \langle S, \sqsubseteq, |.|^+, |.|^- \rangle$ we recursively define the conditions for a formula to be *exactly verified* ($\Vdash$) or *exactly falsified* ($\dashv\vert$) by a state $s \in S$:

$$
\begin{array}{lcl}
s \Vdash p & \Leftrightarrow & s \in |p|^+ \\
s \dashv\vert\, p & \Leftrightarrow & s \in |p|^- \\
s \Vdash \neg A & \Leftrightarrow & s \dashv\vert\, A \\
s \dashv\vert\, \neg A & \Leftrightarrow & s \Vdash A \\
s \Vdash A \wedge B & \Leftrightarrow & \text{for some } t, u \ (t \Vdash A, u \Vdash B \text{ and } s = t \sqcup u) \\
s \dashv\vert\, A \wedge B & \Leftrightarrow & s \dashv\vert\, A \text{ or } s \dashv\vert\, B \\
s \Vdash A \vee B & \Leftrightarrow & s \Vdash A \text{ or } s \Vdash B \\
s \dashv\vert\, A \vee B & \Leftrightarrow & \text{for some } t, u \ (t \dashv\vert\, A,\ u \dashv\vert\, B \text{ and } s = t \sqcup u)
\end{array}
$$

where $s \Vdash A$ stands for "s is an exact truthmaker of A". More informally, a state is a(n) (exact) truthmaker of a disjunction $A \vee B$ if and only if it is a(n) (exact) truthmaker of one of its disjuncts; and a state is a(n) (exact) truthmaker of a conjunction $A \wedge B$ if and only if it is the fusion of (exact) truthmakers of both its conjuncts.

We are now ready to provide the following definitions:

Definition 1 Exact Consequence*: for any formula A, B, A exactly entails B ($A \Vdash_e B$) if and only if for any state model $\mathcal{M}$ and any s in $\mathcal{M}$, $\mathcal{M}, s \Vdash A$ implies $\mathcal{M}, s \Vdash B$.*

Definition 2 Inexact Verification*: Given a state model $\mathcal{M} = \langle S, \sqsubseteq, |.|^+, |.|^- \rangle$, for any $s \in S$, we say that s* inexactly verifies *a formula A if s contains and exact verifier of A; more formally $s \Vvdash A$ iff for some $t \sqsubseteq s$, $t \Vdash A$.*

Definition 3 Inexact Consequence*: for any formula A, B, A inexactly entails B ($A \Vvdash_i B$) if and only if for any state model $\mathcal{M}$ and any s in $\mathcal{M}$, $\mathcal{M}, s \Vvdash A$ implies $\mathcal{M}, s \Vvdash B$.*

We refer to the above semantic framework as *TS*.

At this point a question arises: what kind of entailment are the above notions modeling? How should we understand *exact* and *inexact* consequences?

The logic of exact consequence has been the subject of investigation of a recent paper by Kit Fine and Mark Jago (Fine & Jago, 2017) but its applications still have to be fully explored. On the other hand, the notion of inexact consequence can be understood in terms of already known notions of logical consequence. In particular, it has been shown, originally by Van Fraassen in (Van Fraassen, 1969) and more recently by Fine in (Fine, 2016), that the notion of inexact consequence can be characterized via *first-degree entailment*. Before making explicit this characterization we need to briefly introduce the non-classical logic of first-degree entailment (*FDE*).

The language of *FDE* consists of propositional variables $p, q, r...$; logical connectives "$\neg, \wedge, \vee$"; a well-formed formula in the language is defined as

$$A := p \mid \neg B \mid B \vee C \mid B \wedge C$$

and the classical implication and bi-conditional are standardly defined: $A \rightarrow B := \neg A \vee B$; $A \leftrightarrow B := (A \rightarrow B) \wedge (B \rightarrow A)$.

The logic of *FDE* can be presented from a syntactical perspective as it is originally done in (Anderson & Belnap, 1962) as the logic of a certain class of entailments, called *first-degree* entailments. In (Anderson & Belnap, 1962) a an entailment $A \Rightarrow B$ is said to be a valid *first-degree* entailment if and only if $A \Rightarrow B$ is a *tautological* entailment; and an entailment $A \Rightarrow B$ is tautological if and only if it can be put in a normal form $A_1 \vee A_2 \vee ... \vee A_m \Rightarrow B_1 \wedge B_2 \wedge ... \wedge B_n$ where each A_j is in conjunction of atoms and each B_i is a conjunction of atoms and for all $A_j \Rightarrow B_i$, A_j and B_i share an atom (in this context an atom is a propositional variable p or its negated version $\neg p$).

More intuitively, the logic *FDE* can be characterized, from a semantic perspective, as it is done in (Belnap, 1977) and (Priest, 2008), as a four-valued logic in which the notion of logical consequence amounts preservation of truth under four-valued semantics.

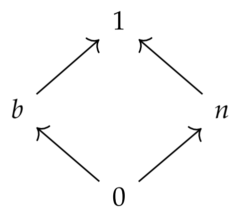
The aim of *FDE* logic is to account for a non-classical notion of entailment which does not validate the so-called "paradoxes" of strict implication, such as $(A \wedge \neg A) \rightarrow B$ and $A \rightarrow (B \vee \neg B)$; indeed $(A \wedge \neg A) \Rightarrow B$ and $A \Rightarrow (B \vee \neg B)$ are not valid first-degree entailments.

In the following, we will provide a more detailed and systematic presentation of a four-valued semantics of *FDE*.

An *FDE* four-valued model is a tuple $M = \langle S_L, v \rangle$ where:

- $S_L = \{V, D, f_\neg, f_\vee, f_\wedge\}$, generally we indicate $f_c \in \{f_\neg, f_\vee, f_\wedge\}$, with
  - $V = \{1, b, n, 0\}$ is the set of the four values where 1 stands for *only true*, b for *both true and false*, n for *neither true nor false* and 0 for *only false*;
  - $D = \{1, b\} \subseteq V$;
  - $f_\neg : V \rightarrow V$;
  - $f_\wedge : V \times V \rightarrow V$;
  - $f_\vee : V \times V \rightarrow V$;
  - $v : \mathcal{L}_{prop} \rightarrow V$, namely it is an assignment mapping any propositional letter to a truth value $v(p) \in V$.

The set of truth values V comes with a partial ordering, hence we would have $\mathcal{V} = \langle V, \leq \rangle$ such that 0 is the bottom element, namely $1 = LubV$, 1 is the top element, namely $0 = GlbV$ (where $LubX$ and $GlbX$ stand rispectively for the *least upper bound* and the *greatest lower bound* of a set X), b and n are incomparable to each other and $0 \leq b \leq 1$ and $0 \leq n \leq 1$. In a picture, $\mathcal{V}$ is the lattice:



where the arrow stands for the relation $\leq$.

We can now make explicit the role played by f_c: $f_\wedge, f_\vee$ are respectively the *meet* and the *join* operation on the lattice $\mathcal{V}$, namely $f_\wedge(x, y)$ and $f_\vee(x, y)$ are respectively the greatest lower bound (*Glb*) and the least upper bound (*Lub*) of x and y, and $f_\neg$ maps 1 to 0, 0 to 1 and each of b and n to itself.
For any formula A, its truth value ($v(A) \in V$)[1] is recursively defined in the following way:

- $v(\neg A) = f_\neg(v(A))$;
- $v(A \wedge B) = f_\wedge(v(A), v(B))$;
- $v(A \vee B) = f_\vee(v(A), v(B))$;

We can now define the notion of logical consequence (or *FDE* entailment) under this semantics:

Definition 4 $\Gamma \models^4_{K_{FDE}} B$ *if and only if for every four-valued model* $\langle S_L, v \rangle$, *if* $v(A) \in D$ *for any* $A \in \Gamma$, *then* $v(B) \in D$.

Now, we can make explicit the characterization of inexact consequence in terms of *FDE* entailment; we refer to the following theorem as Van Fraassen's theorem:

Theorem 1 $A \Vdash_i B$ *if and only if* $A \models_{FDE} B$

Proof: see (Van Fraassen, 1969) or (Fine, 2016).

[1] $v(A)$ is an abuse meant to simplify the notation.

1.1.2 Van Fraassen-Korbmacher's work

The original idea of a truthmaker semantics for modal statements can be found in (Van Fraassen, 1969); at the end of his paper form 1969, Van Fraassen mentions some intuitions to expand its semantics of facts to more complex modal statements. He claims:

> "The facts that make *Necessarily A* true would then be the conjunctions of facts that make A true in the various possible worlds; for *Necessarily A* is true if and only if A is true in world α_1, *and* in world α_2 and so forth"

In the light of this, it seems that the truthmaker conditions of statement of the form "necessarily A" ($\Box A$, for short) must, in some sense, resemble those of a conjunction: the conjunctive fact that it is raining in Amsterdam *and* it is windy in Amsterdam is an exact truthmaker of the sentence "it is raining and windy in Amsterdam", analogously the conjunctive fact making A true in world α_1 *and* A true in world α_2 *and*... should be an exact truthmaker of "necessarily A". At this point some questions arise: how do we formally account for modalities? How should we understand the notion of making a sentence *true in a world*?

The answer to the first question is straightforward: modalities should be understood as quantification over possible accessible worlds and this presumably is also what Van Fraassen had in mind.

The answer to the second question comes from an idea of Johanness Korbmacher (see (Korbmacher, 2016)): the truthmaking relation must be relativized to possible worlds. This move seems rather natural: just as we evaluate in classical modal logic the truth of a sentence with respect to a world, we can relativize the notion of (exact) truthmaking to possible worlds. A sentence A, then, would be made true (or false) with respect to a possible world. For instance, consider the actual world $w_@$ where it is raining in Amsterdam and a world w in which it is not raining in Amsterdam: clearly, the sentence "it is raining in Amsterdam" is made true at $w_@$ but not in w. What is left now it to establish the truthmaker conditions for any sentence in the light of this new notion.

For non-modal statements, it doesn't seem very problematic, we just take Fine's truthmaker conditions and relativize them to possible worlds: for instance, an exact truthmaker of $A \wedge B$ at a world w would be the fusion of an exact truthmaker of A at w and an exact truthmaker of B at w. For a modal statement like $\Box A$, Korbmacher, in (Korbmacher, 2016), provides the following truthmaker conditions in the light of what Van Fraassen claims in his 1969 paper: take w_1, w_2,... as the worlds accessible from w,

> an exact truthmaker s of $\Box A$ at w is the fusion of an exact truthmaker s_1 of A at w_1 and an exact truthmaker s_2 of A at w_2 and...

namely $s = s_1 \sqcup s_2 \sqcup ....$ The similarity of these truthmaker conditions with Fine's ones for conjunctions is evident. A statement like "possibly A" ($\Diamond A$) can be understood, instead, as a long disjunction, in the sense that a fact that makes "possibly A" true would then be the fact that makes A true at world w_1 *or* A true at world w_2 *or*... The truthmaker conditions for $\Diamond A$ would, then, intuitively resemble Fine's ones for disjunctions; thus Korbmacher provides the following truthmaker conditions for $\Diamond A$: take w_1, w_2,... as the worlds accessible from w,

> an exact truthmaker s of $\Diamond A$ at w is an exact truthmaker of A at w_1 or an exact truthmaker of A at w_2 or...

In the next section we will present more systematically Korbmacher's formal framework arising from these ideas.

1.2 Modal Truthmaker Semantics for Exact Verification

We will now formally introduce Korbmacher's semantic framework. We start with the language consisting of: propositional variables p, q, r...; logical constants "$\neg, \vee, \wedge$"; modal operators $\Box$, $\Diamond$; and auxiliary symbols "(,)". We use letters A, B, C... to denote formulas; a (well-formed) formula in this language is defined as

$$A ::= p \mid \neg B \mid B \wedge C \mid B \vee C \mid \Diamond B \mid \Box B$$

An E-Kripke model is a tuple $\mathcal{E} = \langle \mathcal{G}, v^+, v^- \rangle$ where:

- $\mathcal{G} = \langle \mathcal{F}, \mathcal{S} \rangle$ with
  - $\mathcal{F} = \langle W, R \rangle$ is a Kripke frame;
  - $\mathcal{S} = \langle S, \sqsubseteq \rangle$ is a state space;
- $v^+, v^- : W \times \mathcal{L}_{prop} \to \mathcal{P}(S)$ are assignments such that
  - $v^+_w(p) \subseteq S$ is the set of states making p true at w;
  - $v^-_w(p) \subseteq S$ is the set of states making p false at w.

Given an E-Kripke model $\mathcal{E} = \langle W, R, S, \sqsubseteq, v^+, v^- \rangle$, we recursively define the conditions for a formula to be *verified* or *falsified* at a world w by a state s:

$$
\begin{array}{lll}
s \Vdash_w p & \Leftrightarrow & s \in v^+_w(p) \\
s \dashv\!\vert_w p & \Leftrightarrow & s \in v^-_w(p) \\
s \Vdash_w \neg A & \Leftrightarrow & s \dashv\!\vert_w A \\
s \dashv\!\vert_w \neg A & \Leftrightarrow & s \Vdash_w A \\
s \Vdash_w A \wedge B & \Leftrightarrow & \text{for some } t, u\ (t \Vdash_w A, u \Vdash_w B \text{ and } s = t \sqcup u) \\
s \dashv\!\vert_w A \wedge B & \Leftrightarrow & s \dashv\!\vert_w A \text{ or } s \dashv\!\vert_w B \\
s \Vdash_w A \vee B & \Leftrightarrow & s \Vdash_w A \text{ or } s \Vdash_w B \\
s \dashv\!\vert_w A \vee B & \Leftrightarrow & \text{for some } t, u\ (t \dashv\!\vert_w A, u \dashv\!\vert_w B \text{ and } s = t \sqcup u) \\
s \Vdash_w \Box A & \Leftrightarrow & \text{there is a function } f : W \to S \text{ and for any } v \text{ such that } wRv,\ f(v) \Vdash_v A \\
 & & \text{and } s = \bigsqcup(\bigcup_{wRv}\{f(v)\}) \\
s \dashv\!\vert_w \Box A & \Leftrightarrow & \text{for some } v \text{ such that } wRv, s \dashv\!\vert_v A \\
s \Vdash_w \Diamond A & \Leftrightarrow & \text{for some } v \text{ such that } wRv, s \Vdash_v A \\
s \dashv\!\vert_w \Diamond A & \Leftrightarrow & \text{there is a function } f : W \to S \text{ and for any } v \text{ such that } wRv,\ f(v) \dashv\!\vert_v A \\
 & & \text{and } s = \bigsqcup(\bigcup_{wRv}\{f(v)\})
\end{array}
$$

We can now see more formally that s is an exact truthmaker of $\Box A$ at w if and only if it is the fusion of an exact truthmaker of A at w_1 and an exact truthmaker of A at w_2 and so forth for all the accessible worlds w_1 and w_1 and... form w; dually, s is an exact truthmaker of $\Diamond A$ at w if and only if s is an exact truthmaker of A at w_1 or an exact truthmaker of A at w_2 and so forth.

For any formula A, we define with respect to an $\mathcal{E}$ its the *positive meaning* at a world w, $[A]^+_w$, and its *negative meaning* at a world w $[A]^-_w$, as:

$$[A]^+_w = \{s \in S : s \Vdash_w A\}$$

$$[A]^-_w = \{s \in S : s \dashv\!\vert_w A\}$$

We can now provide the conditions for a formula to be made true or false at a world $w \in W$:

$$w \models A \quad \Leftrightarrow \quad [A]^+_w \neq \emptyset$$
$$w \dashv A \quad \Leftrightarrow \quad [A]^-_w \neq \emptyset$$

namely, w makes A true (false) if there is some possible state that makes A true (false) with respect to w.

Definition 5 Modal Exact Verification*: Given an E-Kripke Model $\mathcal{E} = \langle W, R, S, \sqsubseteq, v^+, v^- \rangle$, for any $s \in S$ and any $w \in W$, we say that s* exactly verifies *a formula A at w if $s \Vdash_w A$.*

Definition 6 Modal Exact Consequence*: for any formula A, B, $A \Vdash_{K_i} B$ iff for any E-Kripke model $\mathcal{E}$ and any s and w in $\mathcal{E}$, $s \Vdash_w A$, the implies $\mathcal{M}, s \Vdash_w B$.*

Definition 7 Modal Inexact Verification*: Given an E-Kripke Model $\mathcal{E} = \langle W, R, S, \sqsubseteq, v^+, v^- \rangle$, for any $s \in S$ and any $w \in W$, we say that s* inexactly verifies *($\Vvdash$) a formula A at w if s contains and exact verifier of A at w; more formally $s \Vvdash_w A$ iff for some $t \sqsubseteq s$, $t \Vdash_w A$.*

Definition 8 Modal Inexact Consequence*: for any formula A, B, $A \Vvdash_{K_i} B$ iff for any E-Kripke model $\mathcal{E}$ and any s and w in $\mathcal{E}$, $s \Vvdash_w A$, the implies $\mathcal{M}, s \Vvdash_w B$.*

We refer to the above semantic framework as $TS_\Box$.

1.3 Modal First-Degree and Modal Truthmaker Semantics

In this section, we will explore the connection between a modal extension of *FDE* (denoted by K_{FDE}) developed by Priest in (Priest, 2008), and Korbmacher's $TS_\Box$.

The language of the K_{FDE} consists of the language of *FDE* plus modal operators $\Box$, $\Diamond$. As before, we use $A, B, C..$ to refer to formulas. A (well-formed) formula in the language of K_{FDE} is defined as

$$A ::= p \mid \neg B \mid B \wedge C \mid B \vee C \mid \Diamond B \mid \Box B$$

At first, we will introduce a possible worlds semantics for K_{FDE} sketched in (Omori, 2017). A K_{FDE}-Kripke model is a tuple $\mathcal{M} = \langle W, R, a^+, a^- \rangle$ such that

- W is a non-empty set of worlds;
- $R : W \times W$ is an accessibility relation;
- $a^+, a^- : \mathcal{L}_{prop} \to \mathcal{P}(W)$ are valuation functions such that
  - $a^+(p) \subseteq W$ is the set of world where p is true;
  - $a^-(p) \subseteq W$ is the set of world where p is false.

Given a K_{FDE}-Kripke model $\mathcal{M} = \langle W, R, a^+, a^- \rangle$ we recursively define the conditions for a formula to be made true at a world w ($\models$) and false at a world w ($\dashv$):

$$
\begin{array}{lll}
w \models p & \Leftrightarrow & w \in a^+(p) \\
w \dashv p & \Leftrightarrow & w \in a^-(p) \\
w \models \neg A & \Leftrightarrow & w \dashv A \\
w \dashv \neg A & \Leftrightarrow & w \models A \\
w \models A \wedge B & \Leftrightarrow & w \models A \text{ and } w \models B \\
w \dashv A \wedge B & \Leftrightarrow & w \dashv A \text{ or } w \dashv B \\
w \models A \vee B & \Leftrightarrow & w \models A \text{ or } w \models B \\
w \dashv A \vee B & \Leftrightarrow & w \dashv A \text{ and } w \dashv B \\
w \models \Box A & \Leftrightarrow & \text{for any } v \text{ such that } wRv\ v \models A \\
w \dashv \Box A & \Leftrightarrow & \text{for some } v \text{ such that } wRv\ v \dashv A \\
w \models \Diamond A & \Leftrightarrow & \text{for some } v \text{ such that } wRv\ v \models A \\
w \dashv \Diamond A & \Leftrightarrow & \text{for any } v \text{ such that } wRv\ v \dashv A
\end{array}
$$

The following definition naturally follows:

Definition 9 K_{FDE} Consequence*: for any formula B and any set of formula Γ, $\Gamma \models_{K_{FDE}} B$ iff for any K_{FDE}-Kripke model $\mathcal{M}$ and any w in $\mathcal{M}$, $\mathcal{M}, w \models \bigwedge \Gamma$ implies $\mathcal{M}, w \models B$.*

For the sake of completeness, we will introduce Priest's four-valued semantics for K_{FDE} (see (Priest, 2008)) and show that the above semantics for K_{FDE} is equivalent to Priest's one.

A K_{FDE} four-valued model is a tuple $F = \langle W, R, S_L, v \rangle$ where:

- W is a non-empty set of worlds;
- $R \subseteq W \times X$, is an accessibility relation on W;
- S_L is defined as in the non-modal case
- $v_w : \mathcal{L}_{prop} \times W \to V$, namely it is an assignment mapping any propositional letter and a world to a truth value $v_w(p) \in V$.

For any formula A, its truth value at a world w ($v_w(A) \in V$) is recursively defined in the following way:

- $v_w(\neg A) = f_\neg(v_w(A))$;

- $v_w(A \wedge B) = f_\wedge(v_w(A), v_w(B))$;
- $v_w(A \vee B) = f_\vee(v_w(A), v_w(B))$;
- $v_w(\Box A) = Glb\{v_t(A) : wRt\}$;
- $v_w(\Diamond A) = Lub\{v_t(A) : wRt\}$

Notice that by definition of Glb and Lub, $Glb\emptyset = 1$ and $Lub\emptyset = 0$ since the set of lower bounds and upper bound of $\emptyset$ is the whole V. This guarantees that when there is no accessible worlds from w, $v_w(\Box A) = 1$ and $v_w(\Diamond A) = 0$.

We can now define K_{FDE} entailment under this semantics:

Definition 10 *$\Gamma \models^4_{K_{FDE}} B$ if and only if for every four-valued model $\langle W, R, S_L, v\rangle$ and any $w \in W$, if $v_w(B) \in D$ for any $B \in \Gamma$, then $v_w(B) \in D$.*

Now, it is possible to prove that

Theorem 2 *$\Gamma \models^4_{K_{FDE}} B$ if and only if $\Gamma \models_{K_{FDE}} B$.*

Proof: see appendix **A.1**

1.3.1 Characterizations

In this section, we will go through Korbmacher's proof of a modal extension of Van Fraassen's Theorem.

Johannes Korbmacher has defined new operations to transform each E-Kripke model into an ordinary K_{FDE} model and vice versa by preserving truth with respect to possible worlds; in the following we will introduce such definitions and explore their properties.

Definition 11 *Given an E-Kripke model $\mathcal{E} = \langle W, R, S, \sqsubseteq, v^+, v^-\rangle$ we define its* ordinarification *as $\mathbf{O}(\mathcal{E}) = \langle W, R, a^+, a^-\rangle$ where W and R are the same as in $\mathcal{E}$ and*

- $a^+, a^- : \mathcal{L}_{prop} \to \mathcal{P}(W)$
  - $a^+(p) = \{w \in W \ : \ v^+_w(p) \neq \emptyset\}$
  - $a^-(p) = \{w \in W \ : \ v^-_w(p) \neq \emptyset\}$

Notice that, by construction, $\mathbf{O}(\mathcal{E})$ is an ordinary possible-worlds model of K_{FDE} and that truth at a world w of a formula is preserved under ordinarification:

Lemma 1 *For any E-Kripke model $\mathcal{E} = \langle W, R, S, \sqsubseteq, v^+, v^-\rangle$, given its ordinarification $\mathbf{O}(\mathcal{E})$, for any formula A and any $w \in W$,*

$[\mathcal{E}, w \models A$ if and only if $\boldsymbol{O}(\mathcal{E}), w \models A]$ and $[\mathcal{E}, w \dashv A$ if and only if $\boldsymbol{O}(\mathcal{E}), w \dashv A]$

Proof: see appendix **A.2**.

The dual operation to build an E-Kripke model out of a Kripke K_{FDE} model is:

Definition 12 *Given a Kripke model $\mathcal{M} = \langle W, R, a^+, a^- \rangle$ we define its* exactification *as $\boldsymbol{E}(\mathcal{M}) = \langle W, R, S, \sqsubseteq, v^+, v^- \rangle$ where W and R are the same as in $\mathcal{M}$*

- *$S = \mathcal{P}(W \times (\mathcal{L}_{prop} \cup \overline{\mathcal{L}_{prop}}))$ where $\overline{\mathcal{L}_{prop}} = \{\neg p \ : \ p \in \mathcal{L}_{prop}\}$*
- *$\sqsubseteq$ is the relation of set inclusion ($\subseteq$) over S and, consequently, $\bigsqcup$ amounts the operation of union $\bigcup$ over S;*
- *$v^+, v^- : \mathcal{L}_{prop} \times W \rightarrow \mathcal{P}(S)$*
  - $v^+_w(p) = \{\{(w, p)\} : w \in a^+(p)\}$
  - $v^-_w(p) = \{\{(w, \neg p)\} : w \in a^-(p)\}$

It is easy to show that $\mathbf{E}(\mathcal{M})$ is indeed an E-Kripke model. It is evident by the fact the relation of set inclusion is reflexive, transitive and anti-symmetric, hence it amounts to a parthood relation on S; moreover, for any $X \subseteq S$, $\bigsqcup X$ amounts to $\bigcup X$, as $\bigcup \emptyset = \emptyset$ and $\emptyset$ is such that $\emptyset \sqsubseteq Y$ for any $Y \in S$; moreover $\bigcup X \in S$, in fact for any $x \in X$, $x \subseteq (W \times (\mathcal{L}_{prop} \cup \overline{\mathcal{L}_{prop}}))$ and, clearly, union of subsets of $W \times (\mathcal{L}_{prop} \cup \overline{\mathcal{L}_{prop}})$ returns a subset of $W \times (\mathcal{L}_{prop} \cup \overline{\mathcal{L}_{prop}})$; furthermore $\bigcup X$ is the least upper bound of the elements in X, in fact consider $Z \in S$ such that for all $x \in X$, $x \sqsubseteq Z$; since every element of X is a subset of Z, then $\bigcup X \subseteq Z$ and the union among sets is unique.

Analogously to the case of ordinarification, truth at a world w of a formula is preserved under exactification:

Lemma 2 *For any Kripke model $\mathcal{M} = \langle W, R, a^+, a^- \rangle$, given its exactification $\boldsymbol{E}(\mathcal{M})$ it is the case that for any formula A, and any $w \in W$,*

$[\mathcal{M}, w \models A$ if and only if $\boldsymbol{E}(\mathcal{M}), w \models A]$ and $[\mathcal{M}, w \dashv A$ if and only if $\boldsymbol{E}(\mathcal{M}), w \dashv A]$

Proof: see appendix **A.3**.

We will introduce a small and useful result.

Consider an arbitrary E-Kripke model $\langle S, \sqsubseteq, W, R, v^+, v^- \rangle$; take an arbitrary state $s \in S$; we define the restriction of $\mathcal{E}$ with respect to s that E-Kripke model *downward generated* by s, namely $\mathcal{E}^* = \langle S^*, \sqsubseteq^*, W^*, R^*, v^+_*, v^-_* \rangle$ where

- $S^* = \{t \in S : t \sqsubseteq s\}$
- $\sqsubseteq^* \subseteq (S^* \times S^*)$ namely $\sqsubseteq$ is the restriction of the parthood relation on S^*
- $W^* = W$

- $R^* = R$
- $v^+_*, v^-_* : \mathcal{L}_{prop} \times W \rightarrow S^*$, namely v^+_* and v^-_* are the restriction of v^+ and v^- on S:
  - $v^+_{*,w}(p) = \{s \in (v^+_w \cap S^*)\}$;
  - $v^-_{*,w}(p) = \{s \in (v^-_w \cap S^*)\}$.

The following lemma is readily provable by induction:

Lemma 3 *For any E-Kripke model $\mathcal{E} = \langle S, \sqsubseteq, W, R, v^+, v^- \rangle$; given the restriction $\mathcal{E}^* = \langle S^*, \sqsubseteq^*, W^*, R^*, v^+_*, v^-_* \rangle$ of $\mathcal{E}$ with respect to s, it is the case that for any $t \in S^*$, any $w \in W^*$ and any formula A, $\mathcal{E}, t \Vdash_w A$ if and only if $\mathcal{E}^*, t \Vdash_w A$.*

We are now ready to introduce Korbmacher's modal extension of Van Fraassen's theorem

Theorem 3 $A \Vvdash_{K_i} B \Leftrightarrow A \models_{K_{FDE}} B$

Proof:

($\Rightarrow$) By contrapositio; assume $A \not\models_{K_{FDE}} B$, so there is a K_{FDE}-Kripke models $\mathcal{M}$ and some w in $\mathcal{M}$ such that $\mathcal{M}, w \models A$ and $\mathcal{M}, w \not\models B$. Now, let's consider $\mathbf{E}(\mathcal{M})$. By **Lemma 2** we have that since $\mathcal{M}, w \models A$ and $\mathcal{M}, w \not\models B$ it is also the case that $\mathbf{E}(\mathcal{M}), w \models A$ and $\mathbf{E}(\mathcal{M}), w \not\models B$. This means that $[A]^+_w \neq \emptyset$ and $[B]^+_w = \emptyset$, so there is a $s \in S$ such that $\mathbf{E}(\mathcal{M}), s \Vdash_w A$ and no $t \in S$ such that $\mathbf{E}(\mathcal{M}), t \Vdash_w B$. Now, it is the case that s inexactly verifies A at w ($\mathbf{E}(\mathcal{M}), s \Vvdash_w A$), since there is $s \sqsubseteq s$ and $\mathbf{E}(\mathcal{M}), s \Vdash_w A$. Now, consider an arbitrary $z \sqsubseteq s$; since $[B]^+_w = \emptyset$, then it is cannot be the case that $z \Vdash_w B$. Since z was taken arbitrarily among the parts of s, it is the case that for any $z \sqsubseteq s$, $s \nVdash_w B$, namely s does not inexactly verify B. And so, it is not the case that $A \Vvdash_{K_i} B$

($\Leftarrow$) By contrapositio; assume that it is not the case that $A \Vvdash_{K_i} B$, so there is some model $\mathcal{E}$ and some state u and some world w such that $\mathcal{E}, u \Vvdash_w A$ and it is not the case that $\mathcal{E}, u \Vvdash_w B$, namely there is some $s \sqsubseteq u$ such that $\mathcal{E}, s \Vdash_w A$ and for any $z \sqsubseteq u$, $\mathcal{E}, z \nVdash_w B$. Now consider the restriction $\mathcal{E}^* = \langle W, S^*, R^*, \sqsubseteq, v^*_+, v^*_- \rangle$ of $\mathcal{E}$ with respect to u. By the **Lemma 3**, it follows that $\mathcal{E}^*, u \Vvdash_w A$ and it is not the case that $\mathcal{E}^*, u \Vvdash_w B$. Moreover, it is the case that $\mathcal{E}^*, w \models A$, in fact, since by construction all the states in S^* makes A true, $[A]^*_w \neq \emptyset$. By construction, since all the states t in S^* are such that $t \nVdash_w B$, it is the case that $[B]^+_w = \emptyset$, namley $\mathcal{E}^*, w \dashv B$.

Let's consider the ordinarification of $\mathcal{E}^*$, namely $\mathbf{O}(\mathcal{E})$. Since, $\mathcal{E}^*, w \models A$ and $\mathcal{E}^*, w \not\models B$, by **Lemma 1**, it is the case that $\mathbf{O}(\mathcal{E}^*), w \models A$ and $\mathbf{O}(\mathcal{E}^*), w \not\models B$. So, we found a countermodel to $A \models_{K_{FDE}} B$

With the above theorem by Korbmacher, we have found a characterization of the notion of modal inexact consequences in terms of preservation of truth under four-valued semantics.

1.4 Classical Modal Logic and Modal Truthmaker Semantics

In this chapter, we will investigate the relation between $TS_{\Box}$ and classical Modal logic (K). In particular, our aim is to discuss whether it possible to characterize classical modal logic consequence via modal inexact consequence, just as in the case of K_{FDE}.

So far, we have also admitted *impossible* worlds in E-Kripke models; *impossible* in the sense of *logically* impossible (an analogous characterization of impossible worlds can be found in (Priest, 1997)). This means, more explicitly, that the worlds in an E-Kripke model do not necessarily obey the laws of classical logic; in particular, they do not respect the principle of non-contradiction and excluded middle, namely that no formula can be both made true and false and that any formula is made either true or false. Indeed, consider the model $\mathcal{E} = \langle S, \sqsubseteq, W, R, v^+, v^- \rangle$ with

$$v^+_w(p) = \{s\}$$

$$v^-_w(p) = \{t\}$$

$$v^+_w(q) = v^-_w(q) = \emptyset$$

Consider the state $s \sqcup t$, we know that $s \sqcup t$ must exist by the completeness of S; by semantic conditions, it is the case that $s \sqcup t \Vdash_w p \wedge \neg p$, hence $w \models p \wedge \neg p$, namely $w \dashv \neg(p \wedge \neg p)$. At the same time, by construction, there is no truthmaker or falsemaker of q at w, hence $w \nvDash p \vee \neg q$, namely $w \nvDash q$ and $w \nvDash \neg q$.

So, it would be reasonable to restrict the models so that they include only logically possible worlds, namely worlds that make no formula both true and false and any formula either true or false. In order to meet these classical constraints, it would be plausible to impose some conditions on the valuations of an E-Kripke model $\mathcal{E} = \langle S, \sqsubseteq, W, R, v^+, v^- \rangle$: for any propositional letter p and any world w

(C) exactly one between

(i) $v^+_w(p) \neq \emptyset$ and

(ii) $v^-_w(p) \neq \emptyset$

holds.

We can now introduce the following definition:

Definition 13 *Given an E-Kripke Model $\mathcal{E} = \langle S, \sqsubseteq, W, R, v^+, v^- \rangle$ we say that v^+ and v^- are classical if and only if they meet (C); $\mathcal{E}$ is called classical if and only if v^+ and v^- are classical.*

Now, it is readily provable by induction that

Lemma 4 *For any classical E-Kripke model $\mathcal{E} = \langle S, \sqsubseteq, W, R, v^+, v^- \rangle$, any formula A and any world $w \in W$, $w \nvDash A \wedge \neg A$ and it is the case that either $w \models A$ or $w \models \neg A$.*

and that falsity of a formula at a world amounts to non-truth of that formula at that world, more explicitly by an easy induction it is possible to prove:

Lemma 5 *For any classical E-Kripke model* $\mathcal{E} = \langle S, \sqsubseteq, W, R, v^+, v^- \rangle$*, and any world* $w \in W$ *and any formula* A*,* $w \not\models A$ *if and only if* $w \dashv A$*.*

Now, in order to analyze the interaction between classical modal logic and the modal truthmaker semantic account, it will be useful to look at the connection between K_{FDE} and K. Again, the worlds in a K_{FDE} can be (logically) impossible, in the sense that they do not necessarily obey the principle of non-contradiction and the excluded middle; indeed consider a K_{FDE} Kripke model $\mathcal{M} = \langle W, R, a^+, a^- \rangle$ with

$$a^+_w(p) = \{w\}$$

$$a^-_w(p) = \{w\}$$

$$a^+_w(q) = a^+_w(q) = \emptyset$$

The world w is such that it makes p both true and false, $w \models p \wedge \neg p$ and q neither true nor false, namely $w \not\models q$ and $w \not\models \neg q$. Again, in order to meet the classical principles, it would be reasonable to impose some constraints on the valuations of a K_{FDE} Kripke model $\mathcal{M} = \langle W, R, a^+, a^- \rangle$: for any world $w \in W$ and any propositional letter p

(NC) $a^+_w(p) \cap a^-_w(p) = \emptyset$

(EM) $a^+_w(p) \cup a^-_w(p) = W$

(NC) corresponds to the conditions that no formula can be made both true and false; (EM) corresponds to the constraint that every formula is made either true or false. It is convenient to introduce the following definition:

Definition 14 *Given a* K_{FDE} *Kripke model* $\mathcal{M} = \langle W, R, a^+, a^- \rangle$ *we say that* a^+ *and* a^- *are classical if and only if they meet (EM) and (NC); and* $\mathcal{M}$ *is called classical if and only if* a^+ *and* a^- *are classical.*

And by an easy induction we can prove

Lemma 6 *For any classical* K_{FDE} *Kripke model* $\mathcal{M} = \langle W, R, a^+, a^- \rangle$*, any formula* A *and any world* $w \in W$*,* $w \not\models A \wedge \neg A$ *and it is the case that either* $w \models A$ *or* $w \models \neg A$*.*

So, possible worlds semantics for K can be regarded as a restriction of the possible worlds semantics for K_{FDE}, in particular, we can show that every classical K_{FDE} model can be transformed into a standard Kripke model for K. Before illustrating this transformation, notice that in every classical K_{FDE} model, the falsity of a formula at a world amounts to non-truth of that formula at that world, more explicitly the following result holds and can easily be proven by induction:

Lemma 7 *For any classical K_{FDE} Kripke model $\mathcal{M} = \langle W, R, a^+, a^- \rangle$, and any world $w \in W$ and any formula A, $w \nvDash A$ if and only if $w \dashv A$.*

Now, we are able to define a new operation of *restriction* over a classical K_{FDE} Kripke model in order to obtain a classical Kripke model:

Definition 15 *Given a classical K_{FDE} Kripke model $\mathcal{M} = \langle W, R, a^+, a^- \rangle$, we define its restriction $\boldsymbol{R}(\mathcal{M}) = \langle W^*, R^*, a \rangle$ where*

- $W^* = W$
- $R^* = R$
- $a = a^+$.

It is clear that $a : \mathcal{L}_{prop} \rightarrow W$ is a classical valuation mapping every propositional letter p to the set of worlds where p is true, hence $\mathbf{R}(\mathcal{M}) = \langle W^*, R^*, a \rangle$ is a classical Kripke model for K. Now, by an easy induction we can establish:

Lemma 8 *For any classical K_{FDE} Kripke model $\mathcal{M}$ and any formula A, $\mathcal{M}, w \models A$ if and only if $\boldsymbol{R}(\mathcal{M}), w \models A$.*

As we can expect, also the opposite holds, namely any classical Kripke model for K can be transformed into a classical model for K_{FDE}:

Definition 16 *Given a classical Kripke model for K, $\mathcal{M} = \langle W, R, a \rangle$, we define its inflation $\boldsymbol{I}(\mathcal{M}) = \langle W^*, R^*, a^+, a^- \rangle$ where*

- $W^* = W$
- $R^* = R$
- $a^+ = a$
- $a^- : \mathcal{L}_{prop} \rightarrow W$ *such that for any propositional letter p*
  - $a^-(p) = W / a^+(p)$

By definition, for any p, $a^+(p) \cup a^-(p) = W$ and $a^+(p) \cap a^-(p) = \emptyset$, namely a^+ and a^- are classical and so, $\mathbf{I}(\mathcal{M})$ is classical. Now, by easy induction, we can prove that

Lemma 9 *For any classical Kripke model for K $\mathcal{M}$ and any formula A, $\mathcal{M}, w \models A$ if and only if $\boldsymbol{I}(\mathcal{M}), w \models A$.*

Now, one could ask whether it is possible to recover a classical K_{FDE} Kripke model from a classical E-Kripke model and vice versa. The answer is positive and it relies on the operation of the operations of exactification and ordinarification, indeed we can easily prove the following two lemmas:

Lemma 10 *For any classical E-Kripke model $\mathcal{E} = \langle S, \sqsubseteq, W, R, v^+, v^- \rangle$ its ordinarification $\mathbf{O}(\mathcal{E})$ is classical.*

Proof:

Consider an arbitrary classical E-Kripke model $\mathcal{E} = \langle S, \sqsubseteq, W, R, v^+, v^- \rangle$ and its ordinarifiaction $\mathbf{O}(\mathcal{E}) = \langle W, R, a^+, a^- \rangle$. Now, take an arbitrary propositional letter p and an arbitrary world w; since v^+ and v^- are classical we have two cases to consider:

(i) $v_w^+(p) = \emptyset$ and $v_w^-(p) \neq \emptyset$. This means, by definition of $\mathbf{O}(\mathcal{E})$, that $w \in a^-(p)$ but $w \notin a^+(p)$;

(ii) $v_w^-(p) = \emptyset$ and $v_w^+(p) \neq \emptyset$. Analogously to (i), we obtain $w \in a^+(p)$ but $w \notin a^-(p)$.

In both (i) or (ii), we have that $w \notin a^+(p) \cap a^-(p)$. Since w was taken arbitrarily, we have that for all $w \in W$, $w \notin a^+(p) \cap a^-(p)$, namely $a^+(p) \cap a^-(p) = \emptyset$. So, since p was taken arbitrarily, a^+ and a^- meet (NC).

Moreover, notice that in both (i) and (ii), w is such that $w \in a^+(p) \cup a^-(p)$. Since w was taken arbitrarily, we have that for all $w \in W$, $w \in a^+(p) \cup a^-(p)$, namely $a^+(p) \cup a^-(p) = W$. So, since p was taken arbitrarily, a^+ and a^- meet (EM).

Hence, it is the case that $\mathbf{O}(\mathcal{E})$ is classical.

Lemma 11 *For any classical K_{FDE} Kripke model $\mathcal{M} = \langle W, R, a^+, a^- \rangle$ its exactification $\mathbf{E}(\mathcal{M}) = \langle S, \sqsubseteq, W, v^+, v^- \rangle$ is classical.*

Proof:

Consider an arbitrary classical K_{FDE} Kripke model $\mathcal{M} = \langle W, R, a^+, a^- \rangle$ and its exactification $\mathbf{E}(\mathcal{M}) = \langle S, \sqsubseteq, W, R, v^+, v^- \rangle$. Now, take an arbitrary propositional letter p and an arbitrary world w; since a^+ and a^- are classical we have two cases to consider:

(i) $w \in a^+(p)$ and $w \notin a^-(p)$ (by (EM) and (NC));

(ii) $w \in a^-(p)$ and $w \notin a^+(p)$ (by (EM) and (NC)).

If (i) holds, then, by definition of $\mathbf{E}(\mathcal{M})$, $\{(p, w)\} \in v_w^+(p)$ but $\{(\neg p, w)\} \notin v_w^-(p)$. This means that $v_w^+(p) \neq \emptyset$ and $v_w^-(p) = \emptyset$. Analogously if (ii) holds, we have that $v_w^-(p) \neq \emptyset$ and $v_w^+(p) = \emptyset$. Hence, in both cases, only one between $v_w^+(p) \neq \emptyset$ and $v_w^-(p) \neq \emptyset$ holds.

So, since p and w were taken arbitrarily, we have that v^+ and v^- are classical. Hence, $\mathbf{E}(\mathcal{M})$ is classical.

It is predictable that the truth (and falsity) of any formula with respect to a world is preserved under ordinarification and exactification of classical models, in particular, the following two results hold:

Lemma 12 *For any classical E-Kripke model* $\mathcal{E} = \langle S, \sqsubseteq, W, R, v^+, v^- \rangle$*, any world* $w \in W$ *and any formula* A*,* $\mathcal{E}, w \models A$ *if and only if* $\mathbf{O}(\mathcal{E}), w \models A$*.*

Proof: analogously to **Lemma 1**.

Lemma 13 *For any classical* K_{FDE} *Kripke model* $\mathcal{M} = \langle W, R, a^+, a^- \rangle$*, any world* $w \in W$ *and any formula* A*,* $\mathcal{M}, w \models A$ *if and only if* $\mathbf{E}(\mathcal{M}), w \models A$*.*

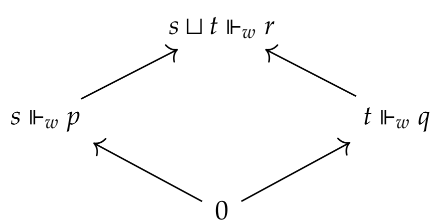
Proof: analogously to **Lemma 2**.

By combining the results shown above, we obtain an effective procedure to transform every classical E-Kripke model into a classical Kripke model for K (by applying ordinarification and restriction) and vice versa (by applying inflation and exactification). So, at this point, one could ask whether classical modal logic consequence ($\models_K$) can be characterized in terms of exact or inexact consequence for classical E-Kripke models. It turns out, however, that K consequence cannot be characterized in this way. Here some counterexample:

$p \vee \neg p \models r \vee \neg r$ is a clear classical K-consequence, however, consider the E-Kripke model $\mathcal{E} = \langle S, \sqsubseteq, W, R, v^+, v^- \rangle$ with

- $S = \{0, s, t, s \sqcup t\}$
- v^+ and v^- classical such that $v^+_w(p) = \{s\}$, $v^-_w(p) = \emptyset$, $v^+_w(q) = \{t\}$, $v^-_w(q) = \emptyset$, $v^+_w(r) = \{s \sqcup t\}$, $v^-_w(r) = \emptyset$

in a picture



Clearly, $s \Vdash_w p \vee \neg p$ since $s \Vdash_w p$, however, it is not the case that $s \Vdash_w r \vee \neg r$ since $s \notin v^+_w(r)$ and $s \notin v^-_w(r)$. This means that it is not the case that $p \vee \neg p \Vdash r \vee \neg r$. Notice also that since $0 \notin v^+_w(r)$ and $0 \notin v^-_w(r)$, it also holds that $s \Vvdash_w p \vee \neg p$ but it is not the case that $s \Vvdash_w r \vee \neg r$, so $p \vee \neg p \Vvdash_{K_i} r \vee \neg r$ doesn't hold as well. The same goes for simply validities: clearly $r \vee \neg r$ is a validity in K, however, it is not the case that $r \vee \neg r$ is inexactly or exactly valid in all classical E-Kripke model. Indeed, consider $\mathcal{E}$ above and s: it is not the case that $s \Vdash_w r \vee \neg r$ nor $s \Vvdash_w r \vee \neg r$. Hence, classical K validities are not expressible as validities in a classical E-Kripke models and classical K logical consequences are not expressible as exact or inexact consequences under classical E-Kripke models. Notice, however, that classical modal logical consequences in standard Kripke semantics is expressible as preservation of truth at worlds in classical E-Kripke models, namely:

Theorem 4 *$\Gamma \models_K B$ if and only if for any classical E-Kripke model $\mathcal{E}$ and any world w in $\mathcal{E}$, $\mathcal{E}, w \models \bigwedge \Gamma$ implies $\mathcal{E}, w \models B$.*

Proof:

(⇐) Straightforward by inflation and exactification.

(⇒) Straightforward by ordinarification and restriction.

At this point, we have seen how models with impossible and possible worlds behave separately. One might want to generalize the structure and have models including both possible and impossible worlds. This can be done by introducing a restriction of possible states over E-Kripke models. More formally, we can define a new structure, call it *general* E-Kripke model:

Definition 17 *A general E-Kripke model is a tuple $\mathcal{E} = \langle S, F, \sqsubseteq, W, N, R, v^+, v^- \rangle$ where*

- *$\langle S, \sqsubseteq \rangle$ is a state space;*
- *$\langle W, R \rangle$ is a Kripke frame;*
- *$F \subseteq S$ is the set of* possible *states, such that for any $X \subseteq S$, $X \subseteq F \Leftrightarrow \bigsqcup X \in F$;*
- *$N \subseteq W$ is the set of possible worlds and W/N is set the of impossible worlds*
- *$v^+, v^- : W \times \mathcal{L}_{prop} \to \mathcal{P}(S)$ are the evaluation function such that*
  - *for any $w \in N$ and any propositional letter p, only one between the following holds:*
    - *(i)* $v^+_w(p) \cap F \neq \emptyset$
    - *(ii)* $v^-_w(p) \cap F \neq \emptyset$

where F behaves as the set of logically possible states.

A general E-Kripke model resembles very much a Fine's modalized state model (see (Fine, fortcoming)) which is a tuple $\mathcal{S} = \langle S, \sqsubseteq, P, v^+, v^- \rangle$ in which

- $\langle S, \sqsubseteq, v^+, v^- \rangle$ is a state model
- $v^+, v^- : \mathcal{L}_{prop} \to \mathcal{P}(S)$ such that

(i) for any propositional letter p, for any state $s \in v^+(P)$ and $t \in v^-(p)$, their fusion $s \sqcup t$ is not in P ($s \sqcup t \notin P$);

(ii) for any propositional letter p, for any state $s \in P$, there is a $t \in v^+(p)$ such that $s \sqcup t \in P$ or there is a $u \in v^-(p)$ such that $s \sqcup u \in P$

in which (i) is analogous of condition (NC) and (ii) analogous of condition (EM).

All the results we have obtained so far can be extended, *mutatis mutandis*, to any general E-Kripke model; hence, this structure could serve as the unique general tool to analyze the connection between modal truthmaker semantics, K_{FDE} and classical modal logic.

1.5 First-Order Modal Truthmaker Semantics

In this section we will propose a first-order extension of Korbmacher's $TS_{\Box}$; our point of departure is the work done by Fine (2017). Fine's idea for developing a semantics for quantified formulas is to reduce them to truth-functional statements; hence, intuitively, an exact truth-maker of a universally quantified formula $\forall xFx$ corresponds to a truthmaker of the conjunction $Fa_1 \wedge Fa_2$... for all the objects a$_1$, a$_2$,... (denoted respectively by a_1, a_2...) in the domain. Conversely, a truthmaker of an existential statement $\exists xFx$ intuitively corresponds to a truthmaker of the disjunction $Fa_1 \vee Fa_2$... for all the objects a$_1$, a$_2$,... (denoted respectively by a_1, a_2...) in the domain. Hence, intuitively, exact truthmaker conditions for $\forall$ and $\exists$ should respectively resemble the ones for $\wedge$ and $\vee$.

The language we use for the following presentation consists of individual variables $x, y, ...$; logical constants $\neg, \vee, \wedge$; n-ary predicate variables $P^n, Q^n, R^n...$; quantifiers $\forall$, $\exists$; modal operators $\Box$, $\Diamond$; and auxiliary symbols (,). Here we use Greek letters φ, ψ... to refer to (well-formed) formulas in the language. A (well-formed) formula is defined, as :

$$A := F^n x_1, ..., x_n \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \Diamond\varphi \mid \Box\varphi$$

plus if φ is a formula and x has some free occurrences in φ, then $\forall x\varphi$ is a formula and analogously for $\exists$.

A first-order E-Kripke model is a tuple $\mathcal{E} = \langle \mathcal{G}, D, v^+, v^- \rangle$ where $\mathcal{G}$ is standardly defined and:

- D is a non-empty domain of individuals;
- $v^+, v^- : W \times (\mathcal{L}_{Pred^n} \times D^n) \to \mathcal{P}(S)$ are assignments such that
  - $v^+_w((F^n, (d_1, ..., d_n))) \subseteq S$ is the set of states verifying F^n of $d_1, ..., d_n$ at w;
  - $v^-_w((F^n, (d_1, ..., d_n))) \subseteq S$ is the set of states falsifying F^n of $d_1, ..., d_n$ at w.

For any first-order E-Kripke model $\mathcal{E} = \langle \mathcal{G}, D, I, v^+, v^- \rangle$ we define an interpretation I of the language such that $I : \mathcal{L}_{Var} \rightarrow D$ is an assignment mapping each variable in the language to an individual in the domain D. Notice that I is not relativized to worlds, hence variables behave as rigid designators, namely the interpretation of every variable is fixed across the possible worlds.

We define the *x-variant* of an interpretation I:

Definition 18 *for any variable x in the language, an x-variant assignment I^* of I is that assignment which differs, if at all, from I only in its assignment to x.*

Given a first-order E-Kripke model $\mathcal{E} = \langle W, R, S, \sqsubseteq, D, v^+, v^- \rangle$, we recursively define in the following the conditions for a formula to be *verified* or *falsified* at a world w by a state s with respect to an assignment I; for the Boolean and modal operators the truth conditions are analogous to the propositional case, just relativized to I:

$s \Vdash^I_w F^n x_1, ..., x_n \quad \Leftrightarrow \quad s \in v^+_w((F^n, (I(a_1), ..., I(a_n))))$

$s \dashv\Vdash^I_w F^n x_1, ..., x_n \quad \Leftrightarrow \quad s \in v^-_w((F^n, (I(a_1), ..., I(a_n))))$

$s \Vdash^I_w \forall x\varphi \quad \Leftrightarrow$ there is a function $f : D \rightarrow S$ such that for all x-variant I^* of I there is a $f(d)$ (with $d \in D$) such that $f(d) \Vdash^{I^*}_w \varphi$ and $s = \bigsqcup(\bigcup_{(d \in D)}\{f(d)\})$

$s \dashv\Vdash^I_w \forall x\varphi \quad \Leftrightarrow$ there is an x-variant I^* of I such that $s \dashv\Vdash^{I^*}_w \varphi$

$s \Vdash^I_w \exists x\varphi \quad \Leftrightarrow$ there is an x-variant I^* of I such that $s \Vdash^{I^*}_w \varphi$

$s \dashv\Vdash^I_w \exists x\varphi \quad \Leftrightarrow$ there is a function $f : D \rightarrow S$ such that for all x-variant I^* of I there is a $f(d)$ (with $d \in D$) such that $f(d) \dashv\Vdash^{I^*}_w \varphi$ and $s = \bigsqcup(\bigcup_{(d \in D)}\{f(d)\})$

Now, for any formula φ, we define, with respect to an E-Kripke model $\mathcal{E}$, its *positive meaning* at a world w with respect to an assignment I, $[A]^+_{(I,w)}$, and its *negative meaning* at a world w with respect to an assignment I, $[A]^-_{(I,w)}$, as:

- $[A]^+_{(I,w)} = \{s \in S : s \Vdash_w A\}$
- $[A]^-_{(I,w)} = \{s \in S : s \dashv\Vdash_w A\}$

We define the conditions for a formula to be made true or false at a world $w \in W$ with respect to an assignment I in the standard way:

$w \models^I A \quad \Leftrightarrow \quad [A]^+_{(I,w)} \neq \emptyset$

$w \dashv^I A \quad \Leftrightarrow \quad [A]^-_{(I,w)} \neq \emptyset$

The notion of inexact verification and inexact consequence within first-order modal truthmaker semantics are standardly defined with respect to an assignment I:

Definition 19 First-order Modal Inexact Verification*: Given a first-order E-Kripke Model $\mathcal{E} = \langle W, R, S, D, \sqsubseteq, v^+, v^- \rangle$, for any $s \in S$ and any $w \in W$, we say that s* inexactly verifies *($\Vdash$) a formula φ at w with respect to I, if s contains and exact verifier of φ at w; more formally $s \Vdash^I_w A$ iff for some $t \sqsubseteq s$, $t \Vdash^I_w A$.*

Definition 20 First-order Modal Inexact Consequence*: for any formula φ, ψ, $\varphi \Vdash_{FOK_i} \psi$ iff for any E-Kripke model $\mathcal{E}$, any s, any w in $\mathcal{E}$ and any I, $\mathcal{E}, s \Vdash^I_w A$ implies $\mathcal{M}, s \Vdash^I_w B$.*

Analogously to before we will now explore the connection between this framework and first-order K_{FDE} trying to extend Van Fraassen's Theorem.

1.6 Quantified Modal First-Degree and First-Order Modal Truthmaker Semantics

In this section, we will show that the relations between modal truthmaker semantics and K_{FDE} carry over to the first-order case. In particular, I will try to extend Korbmacher's framework and related results to the modal case. At first, we will present a natural first-order extension of the possible worlds semantics for K_{FDE} introduced in the previous sections. The language of first-order K_{FDE} (FOK_{FDE}) is made of individual variables $x, y, ...$; logical constants $\neg, \vee, \wedge$; n-ary predicate variables $P, Q, R...$; quantifiers $\forall$, $\exists$; modal operators $\Box$, $\Diamond$; and auxiliary symbols (,). As before, we use Greek letters $\varphi, \psi...$ to refer to (well-formed) formulas in the language; a well-formed formula is defined as in the case of the language for first-order $TS_{\Box}$.

A FOK_{FDE} Kripke model is a tuple $\mathcal{M} = \langle W, R, D, a^+, a^- \rangle$ where W and R are standardly defined, the interpretation of the language I is defined as above and:

- D is a non-empty domain of quantification;
- $a^+, a^- : W \times \mathcal{L}_{Pred^n} \to \mathcal{P}(D^n)$ such that
  - $a^+_w(F^n) \subseteq D^n$ is the positive extension of F^n, namely the n-tuple of objects of which F^n is true;
  - $a^-_w(F^n) \subseteq D^n$ is the negative extension of F^n, namely the n-tuple of objects of which F^n is false.

Given a model $\mathcal{M} = \langle W, R, D, a^+, a^- \rangle$ and an assignment I, we are now ready to define the conditions for a formula to be true at a world w in $\mathcal{M}$ with respect to I; for the Boolean and modal operators the truth conditions are analogous to the propositional case, just relativized to I:

$$\begin{array}{lll}
w \models^{I} F^n x_1, ..., x^n & \Leftrightarrow & \langle I(a_1), ..., I(a_n)\rangle, \in a^{+}_{w}(F^n) \\
w \dashv^{I} F^n x_1, ..., x^n & \Leftrightarrow & \langle I(a_1), ..., I(a_n)\rangle, \in a^{-}_{w}(F^n) \\
w \models^{I} \forall x\varphi & \Leftrightarrow & \text{for any } x\text{-variant } I^{*}\ w \models^{I^{*}} \varphi \\
w \dashv^{I} \forall x\varphi & \Leftrightarrow & \text{there is some } x\text{-variant } I^{*} \text{ such that } w \dashv^{I^{*}} \varphi \\
w \models^{I} \exists x\varphi & \Leftrightarrow & \text{there is some } x\text{-variant } I^{*} \text{ such that } w \models^{I^{*}} \varphi \\
w \dashv^{I} \exists x\varphi & \Leftrightarrow & \text{for any } x\text{-variant } I^{*}\ w \dashv^{I^{*}} \varphi
\end{array}$$

The next definition naturally follows:

Definition 21 FOK_{FDE} Consequence*: for any formula B and any set of formula* Γ*,* $\Gamma \models_{FOK_{FDE}} B$ *iff for any ordinary* FOK_{FDE}*-Kripke model* $\mathcal{M}$*, any* w *in* $\mathcal{M}$ *and any* I*,* $\mathcal{M}, w \models^{I} \bigwedge \Gamma$ *implies* $\mathcal{M}, w \models^{I} B$*.*

As before, we will show that the above semantics is equivalent to a natural modal extension of the four-valued semantics for first-order *FDE* developed in (Priest, 2008).

A FOK_{FDE} four-valued model is a tuple $\mathcal{F} = \langle W, R, S_L, D, v^{\mathcal{E}}, v^{\mathcal{A}}\rangle$ where, W and R are standardly defined, I is the interpretation standardly defined and:

- $S_L = \{V, A, f_{\neg}, f_{\vee}, f_{\wedge}, f_{\forall}, f_{\exists}\}$, generally we indicate $f_c \in \{f_{\neg}, f_{\vee}, f_{\wedge}\}$ and $f_q \in \{f_{\forall}, f_{\exists}\}$, with
  - $V = \{1, b, n, 0\}$ is the set of the four values;
  - $A = \{1, b\} \subseteq V$;
  - $f_{\neg}, f_{\wedge}, f_{\vee}$ are defined as in the propositional case;
  - $f_{\forall}(X) = Glb(X)$
  - $f_{\exists}(X) = Lub(X)$
  - $v^{\mathcal{E}}, v^{\mathcal{A}} : W \times \mathcal{L}_{Pred^n} \rightarrow D^n$, namely
    - * $v^{\mathcal{E}}_{w}(F^n) \subseteq D^n$ is the positive extension of F^n;
    - * $v^{\mathcal{E}}_{w}(F^n) \subseteq D^n$ is the negative extension of F^n.

Given a four-valued assignment $\mu : W \times \{I : I \text{ is an assignment of the language of } FOK_{FDE}\} \times For \rightarrow V$ (where For is the set of formulas of FOK_{FDE}) we are ready to define for any formula φ, its truth value at a world w with respect to an assignment I ($\mu^{I}_{w}(\varphi) \in V$); for the Boolean and modal operators, the assignment is defined as in the propositional case, just relativized to I:

- $\mu^{I}_{w}(\forall x\varphi) = Glb\{\mu^{I^{*}}_{w}(\varphi) : I^{*} \text{is an } x\text{-variant of } I\}$;
- $\mu^{I}_{w}(\exists x\varphi) = Lub\{\mu^{I^{*}}_{w}(\varphi) : I^{*} \text{is an } x\text{-variant of } I\}$.

We can now define FOK_{FDE} entailment under this semantics:

Definition 22 $\Gamma \models^{4}_{K_{FDE}} \psi$ *if and only if for every four-valued model* $\langle W, R, S_L, D, I, v^{\mathcal{E}}, v^{\mathcal{A}}\rangle$*, any* $w \in W$ *and any* I*, if* $\mu^{I}_{w}(\varphi) \in A$ *for any* $\varphi \in \Gamma$*, then* $\mu^{I}_{w}(\psi) \in A$*.*

Like in the propositional case, we define operations to translate each four-valued FOK_{FDE} into a Kripke FOK_{FDE} model and vice versa and prove that four-valued FOK_{FDE} consequence is equivalent to FOK_{FDE} consequence under possible worlds semantics.

Theorem 5 *$A \models^4_{FOK_{FDE}} B$ if and only if $A \models_{FOK_{FDE}} B$*

Proof: see appendix **A.4**.

1.6.1 First-order Characterizations

Like in the propositional case, in this section we will try to extend Van Fraassen's Theorem to the first-order modal case by following the same strategy.

We will define new operations to transform each first-order E-Kripke model into an ordinary FOK_{FDE} model and vice versa:

Definition 23 *Given a first-order E-Kripke model $\mathcal{E} = \langle W, R, S, D, I, \sqsubseteq, v^+, v^- \rangle$ we define its* ordinarification *as $\boldsymbol{O}(\mathcal{E}) = \langle W, R, D, I, a^+, a^- \rangle$ where W and R are the same as in $\mathcal{E}$ and*

- $a^+, a^- : \mathcal{L}_{prop} \rightarrow \mathcal{P}(W)$
  - $a^+_w(F^n) = \{\langle d_1, d_2, ..., d_n \rangle \in D^n : v^+_w((F^n, (d_1, d_2, ..., d_n)) \neq \emptyset\}$
  - $a^-_w(F^n) = \{\langle d_1, d_2, ..., d_n \rangle \in D^n : v^+_w((F^n, (d_1, d_2, ..., d_n)) \neq \emptyset\}$

We will now prove the following useful lemma:

Lemma 14 *for any first-order E-Kripke model $\mathcal{E} = \langle W, R, S, D, I, \sqsubseteq, v^+, v^- \rangle$, given its ordinarification $\boldsymbol{O}(\mathcal{E})$, for any formula φ and any $w \in W$,*

[$\mathcal{E}, w \models^I \varphi$ if and only if $\boldsymbol{O}(\mathcal{E}), w \models^I \varphi$] and [$\mathcal{E}, w \mathrel{=\!\!|}^I \varphi$ if and only if $\boldsymbol{O}(\mathcal{E}), w \mathrel{=\!\!|}^I \varphi$]

Proof: see appendix **A.5**.

And analogously we define an operation of exactification:

Definition 24 *Given a Kripke model $\mathcal{M} = \langle W, R, D, I, a^+, a^- \rangle$ we define its* exactification *as $\boldsymbol{E}(\mathcal{M}) = \langle W, R, S, D, I, \sqsubseteq, v^+, v^- \rangle$ where W and R are the same as in $\mathcal{M}$*

- $S = \mathcal{P}(W \times ((\mathcal{L}_{Pred^n} \cup \overline{\mathcal{L}_{Pred^n}}) \times D^n))$ *where* $\overline{\mathcal{L}_{Pred^n}} = \{\neg F^n \ : \ F^n \in \mathcal{L}_{Pred}\}$
- $\sqsubseteq$ *is the relation of set inclusion ($\subseteq$) over S and, consequently, $\bigsqcup$ amounts the operation of union $\bigcup$ over S;*
- $v^+, v^- : W \times (\mathcal{L}_{Pred^n} \times D^n) \rightarrow \mathcal{P}(S)$ *such that*
  - $v^+_w((F^n, (d_1, ..., d_n))) = \{\{(w, (F^n, (d_1, ..., d_n)))\} : \langle d_1, ..., d_n \rangle \in a^+_w(F^n)\}$

- $v^-_w((F^n, (d_1, ..., d_n))) = \{\{(w, (\neg F^n, (d_1, ..., d_n)))\} : \langle d_1, ..., d_n\rangle \in a^-_w(F^n)\}$

As expected the following lemma holds:

Lemma 15 *For any first-order ordinary Kripke model $\mathcal{M} = \langle W, R, D, I, a^+, a^-\rangle$, given its exactification $\mathbf{E}(\mathcal{M})$ it is the case that for any formula φ, and any $w \in W$,*

$$[\mathcal{M}, w \models^I \varphi \text{ if and only if } \mathbf{E}(\mathcal{M}), w \models^I \varphi] \text{ and } [\mathcal{M}, w \dashv^I \varphi \text{ if and only if } \mathbf{E}(\mathcal{M}), w \dashv^I \varphi]$$

Proof: see appendix **A.6**.

Now, we are ready to prove the following theorem

Theorem 6 $\varphi \Vdash_{FOK_i} \psi \Leftrightarrow \varphi \models_{FOK_{FDE}} \psi$

Proof: the proof proceeds analogously to the proof of **Theorem 3**.

By proving the above result, we have been able to extend Van Fraassen's original result to the first-order modal case: the characterization of modal inexact consequence in terms of preservation of truth under four-valued semantics is preserved in the first-order extension of $TS_\Box$.

1.6.2 First-Order Modal Truthmaker Semantics and Classical First-Order Modal Logic

In this section, we will outline the connection between our first-order modal truthmaker semantics and classical first-order modal logic (*FOK*). It is predictable that by imposing on the valuations constraints analogous to the propositional case, we can preserve the same results as the propositional cases.

Given a FOK_{FDE} Kripke model $\langle W, R, D, a^+, a-\rangle$, we say that a^+ and a^- are classical if and only if they meet the following constraint:

(NC_q) for any world $w \in W$, any predicate F^n, $a^+_w(F^n) \cap a^-_w(F^n) = \emptyset$

(EM_q) for any world $w \in W$, any predicate F^n $a^+_w(F^n) \cup a^-_w(F^n) = D$

Hence we obtain:

Definition 25 *A FOK_{FDE} Kripke model $\langle W, R, D, a^+, a-\rangle$ is said to be classical if and only if a^+ and a^- are classical*

Analogously, given a first-order E-Kripke model $\langle S, \sqsubseteq, W, R, D, v^+, v^-\rangle$ we say that v^+ and v^- are classical if and only if they meet the following constraint:

(C_q) for any world w, any predicate F^n and tuple of individuals $\langle d_1, ...d_n \rangle$, exactly one between the following holds:

(i) $v^+_w((F^n, (d_1, ...d_n)) \cap S \neq \emptyset$

(ii) $v^-_w((F^n, (d_1, ...d_n)) \cap S \neq \emptyset$

So, we have:

Definition 26 *A first-order E-Kripke model $\langle S, \sqsubseteq, W, R, D, v^+, v^- \rangle$ is said to be classical if and only if v^+ and v^- are classical.*

As one would expect, the results we proved for the propositional case are extendable to the first-order case following the same strategy without any problem; from those, we will obtain that classical first-order modal logical consequence cannot be characterized via classical first-order modal inexact (or exact) consequence, while truth at a world is preserved from classical first-order E-Kripke models to classical first-order modal logic and vice versa.

1.6.3 Identity

In this section we will try to briefly analyze the behavior of the identity predicate in our semantics.

Within the framework outlined above, identity is intuitively treated as a dyadic predicate and, in principle, it could be the case that there are truthmakers for the identity of two distinct objects, as well as there could be falsemakers for the identity of the same object with itself. It is not our intention to engage with a metaphysical discussion about the plausibility of these situations: our aim is to investigate which semantic constraints make sense to impose on our framework in order to validate, in the object language, standard commonly accepted laws about identity, namely *reflexivity* (R), *indiscernibility of identicals* (II) and *necessity of identity* (NI).

(R) (R) stands for the principle that for any object d in the domain of individuals, is identical with itself. For (R) to be validated we mean that the identity of every individual with itself must be true at every world, namely, it must have a truthmaker at any worlds.

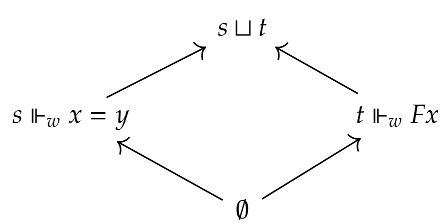
(II) We take (II) to stand for the principle that *if* two individuals are identical *then* they share all the properties. First of all, we need to clarify the meaning of the conditional "*if..then...*" in the formulation of *II*. What do we want to express when we say that if two individuals are identical they share the same properties? Do we mean that every (exact) truthmaker of the identity of x and y is also a(n) (exact) truthmaker of the proposition that they share the same properties? (exact consequence) Or, do we mean that every (inexact) truthmaker of their identity contains a(n) (exact) truthmaker of their sharing the same properties? (inexact consequences). It seems that the interpretation of the meaning of a conditional sentence within truthmaker semantic is very controversial. It is not our aim to provide truthmaker conditions for conditional sentences, however we need to stick to some interpretation of the "*if...then*" involved in the formulation of (II). Let's try to be the most general as possible in interpreting (II): we should want that if $x = y$ is true at a world w, then it must also be true at w that x and y share all the properties. More explicitly, if there is a truthmaker for the identity of x and y at w, then, for any property F, Fx having a truthmaker at w implies Fy having a truthmaker at w and vice versa (Fy having a truthmaker at w implies Fx having a truthmaker at w). This means, more formally, that (II) amounts to say that if $|x = y|^+_{(w,I)}$ is non-empty, then, for any property F, $|Fx|^+_{(w,I)}$ is non-empty if and only if $|Fy|^+_{(w,I)}$ is non-empty.

(NI) We take (NI) to be the principle that *if* two individuals are identical, *then* they are necessarily identical. Again, the same problem arises: how do we interpret the "*if...then...*" involved in the formulation of (NI)? Analogously to the case of (II), we stick to the interpretation of (NI) as the principle that if $x = y$ has a(n) (exact) truthmaker at a world w, then $\Box x = y$ also has a(n) (exact) truthmaker at w, more formally, for any world w, if $|x = y|^+_{(w,I)}$ is non-empty, $|\Box x = y|^+_{(w,I)}$ is non-empty as well.

The validity of (R), (NI) and (II), so formulated, fails in the semantic account presented above; for instance, consider a first-order E-Kripke model $\mathcal{E} = \langle S, \sqsubseteq, W, R, D, v^+, v^- \rangle$ with

- $S = \{0, s, t, s \sqcup t\}$
- $W = \{w, v\}$
- $R = \{(w, v)\}$
- $v^+_w((=, (I(x), I(y)))) = \{s\}$,
- $v^+_w((F, I(y))) = \emptyset$
- $v^+_w((F, I(x))) = \{t\}$
- $v^+_v((=, (I(x), I(x)))) = \emptyset$

in a picture



Clearly, it is the case that $|x = y|^+_{(w,I)} \neq \emptyset$ since $s \Vdash^I_w x = y$, that $|Fx|^+_{(w,I)}$ since $t \Vdash^I_w Fx$, but $|Fy|^+_{(w,I)} = \emptyset$; hence (II) fails in $\mathcal{E}$; moreover $|x = y|^+_{(v,I)} = \emptyset$, hence $|\Box x = y|^+_{(w,I)} = \emptyset$, namely (NI), (R) and (II) fail. So, what constraint should we introduce in our semantic account in order to validate (R), (II) and (NI)? In the following we will consider some options:

(i) The first straightforward option to validate all the three principles would be to introduce = in the language as a logical constant and the exact truthmaker conditions of $x = y$ would be:

$s \Vdash^I_w x = y \quad \Leftrightarrow \quad I(x) = I(y)$

However, this solution implies an over-generation of exact truthmakers of identity statements: given that we fix $I(x) = I(y)$, then we would have that every state in every model, every fact is an exact truthmaker of $x = y$ with respect to any world w. We would lose the *relevance* and the *responsibility* constraints that an exact truthmaker should meet. How can the fact, say, that it is raining in Amsterdam *be responsible* and relevant for the truth of *Hesperus is equal to Phosphorus*?

(ii) Another solution for the validity of all (R), (NI), (II) is to impose that the (positive) valuation of every tuple of predicate and individuals in the domain must be non empty, namely for any F^n and $\langle d_1, ...d_n \rangle$, and any world w, $v^+_w((F^n, \langle d_1, ...d_n \rangle)) \neq \emptyset$. However, this constraint is counterintuitive: every statement would have a truthmaker at every world.

(iii) There are at least two ways (R) could be validated: (1) for any $d \in D$, we impose on every E-Kripke model that for any world w, $v^+_w((=, (d, d))) = \{0\}$; (2) for any $d \in D$, we impose on every E-Kripke model that for any world w, $v^+_w((=, (d, d))) \neq \emptyset$. (1) corresponds to the intuition that (R) is an a priori principle, namely, nothing is required for its truth, and so its only truthmaker would intuitively be the null fact: (R) is made true by nothing substantially. (2) on the other hand corresponds to the intuition that the identity of every object with itself is indeed a necessary and universally valid principle, and so it would be anyway made true by something in any world. Establishing what this *something* is depends on one's favourite view about truthmakers of identity (for instance, the identity of d with itself could be made true by the existence of d itself). Notice that both the constraints in (1) and (2) would imply that the identity of an individual with itself is also necessary: under (1) $|\Box x = x|^+_w = \{0\}$ for any world w, hence $\Box x = x$ is made vacuously true; and under (2) $|\Box x = x|^+_w \neq \emptyset$ for any world w.

(iv) Notice that the failure of (II) depends upon the possibility of having truthmakers of statements like $x = y$ where $I(x)$ and $I(x)$ are different. Indeed, we can have a model with $v^+_{(w,I)}((=, (d_1, d_2)))$ non-empty where d_1 and d_2 are two distinct individuals. Indeed, being d_1 and d_2 different allows us to play with the valuation and make $v^+_w((F, d_1))$ empty and

$v^+_w((F, d_2))$ non-empty for some property F. If d_1 and d_2 were *the same* individuals, the consequent of (II) would never fail. It seems, then, that a plausible condition to impose is that the valuation of the identity predicate having as argument a pair of distinct individuals must be empty. Indeed, what can make the identity between two distinct individuals d_1 and d_2 true at w? Under this condition (II) is safe since we are ruling out the possibility of truthmakers for the identity of distinct individuals. This constraint, from a philosophical point of view, seems to amount to the move of banning some *metaphysically impossible* worlds from our model, namely those worlds where two distinct individuals can be identical.

(v) The failure of (NI) depends, instead, on the fact that some valuations for the tuple $((=, (d, d))$ could be non-empty at some world w while being empty at some other world v accessible from w; the emptiness of $v^-_v((=, (d, d)))$ implies that we cannot find a truthmaker at w for $\Box x = y$ where $I(x) = d = I(y)$. The easiest and most plausible way to solve this issue is to impose the non-emptiness of the truthmaker sets of identity statements over accessible worlds, namely, for any world w and v, if wRv then for any individual d in the domain, $v^+_v((=, (d, d,))) \neq \emptyset$. But now, think of a model in which w is not accessible to itself but access to some other possible world v different from itself with an evaluation $v^+_w((=, (d, d))) = \emptyset$ and $v^+_v((=, (d, d))) \neq \emptyset$. Clearly (NI) holds in this model but a question arises: what reason do we have to impose the truth of identity statement at v while not at w? This seems a very arbitrary and not justified assumption.

(vi) Maybe one wants to impose that every identity statement must have a truthmaker at any world, and this would validate (NI), but this move would need further philosophical justification. Or one can directly impose (NI) as a constraint on our semantics by saying that for any world w and v, and any individual d, if wRv and $v^+_w((=, (d, d))) \neq \emptyset$ then $v^+_v((=, (d, d))) \neq \emptyset$.

It was not our aim to choose one of the constraints we mentioned above, however, it is important to highlight that some constraints are more metaphysically heavier than others: (iv) seems to me a quite innocent constraint to impose on our semantics in order to have (II) valid. (iv) does not commit us to any more (things) in particular, unlike (i) and (ii), actually, it just rules out metaphysically impossible worlds. On the other hand, the constraints aimed at validating (NI) seem metaphysically much more loaded: (v), in addition to being implausible, commits us to a big realm of truthmakers; as well as (vi) seems to hide a philosophical assumption behind the identity of individuals which commits us to truthmakers of identity statements in every possible worlds. Hence, the philosophical intuitions behind the plausibility of the constraints discussed above require further metaphysical discussion, which is beyond the scope of this thesis.

However, in the next chapter, we will outline a philosophical idea of truthmakers for modal truths which could serve as a background theory to develop a more general and systematic conception of truthmakers.

2 Philosophical Foundations

So far, the account presented above seems to require no extra foundation or explanation: if Van Fraassen's aim was to show that "facts can be represented within the framework of standard metalogic" and that they can "provide us with a semantic explanation of tautological entailment" (see (Van Fraassen, 1969, p. 477)) then, Korbmacher's account can be simply be conceived as accomplishing this task for the modal case in the spirit of Van Fraassen, indeed the framework also preserves the nice characterization of (modal) inexact consequence in terms of (modal) first-degree entailment in the first-order case. However, one may still wonder about the metaphysical implications of this semantic account. I believe that no matter how hard one tries to escape the metaphysical discourse related to truthmaker semantics, he is still forced at some point to answer the metaphysical question of *what* a(n) (exact) truthmaker of a certain proposition is. Indeed, when we posit truthmakers of propositions, we are actually speaking of facts which belong to the objective world and which the truth of certain propositions depends on. Hence, this talk about *facts* of reality is, then, fundamental in the philosophy of *truthmaking*. It is not possible, I believe, to be ontologically or metaphysically neutral in doing truthmaker semantics: Van Fraassen's himself has to stick to some leading metaphysical principles in order to build his semantics account, namely, he accepts to be committed to negative fact and conjunctive facts of the forms $e.e'$ that obtains if and only if its components e and e' obtain (see (Van Fraassen, 1969)).

We could, of course, ignore the metaphysical talk behind truthmaker semantics, as Kit Fine would do: building a systematic philosophical theory supporting the semantic account is, actually, beyond the scope of this thesis; however, I believe, the interaction in the framework of modal truthmaker semantics between possible worlds, individuals and facts requires, at least, some metaphysical leading principles that could inspire and promote further philosophical discussion on the topic.

In light of that, in this section, we will try to answer to the philosophical question of *what* is a(n) (exact) truthmaker of modal sentences like $\Box A$ and $\Diamond A$ and *how* it must be conceived within the framework. The answer we will try to seek must be compatible with the philosophical intuitions behind the formal framework.

In our analysis, we will rely upon some assumptions: (i) the principle that a truth (even a modal truth) is made true by something on the side of the world; (ii) the relativization of the (exact) truthmaking relation to worlds, namely an exact truthmaker of something is always a truthmaker *at* a certain world; (iii) the reduction of modalities to quantification over worlds, in the sense that we understand necessity and possibility as quantification over possible worlds. Hence a truthmaker for a modal truth must be an *exact* truthmaker at a certain world; so, very presumably, possible worlds must play a key role in this characterization.

Notice that principle (ii) and (iii) are the original ideas which the semantic framework developed in chapter **1** relies on. Principle (i) seems to be a novel assumption for our work: (i) is usually identified with the position of *truthmaker maximalism*, namely the view that every truth is made true by something in the world (a supporter of this view is, for instance, Armstrong (see Armstrong, 2004)). It is not the aim of this manuscript to argue explicitly for the validity of this assumption, however, for the sake of completeness, it is worth mention that there exist alternative views, incompatible with our assumptions, and according to those there is no such a thing as a truthmaker for necessary truths. In the following, we will precisely show where this incompatibility comes from.

The supporters of this view (see (Beebee & Dodd, 2005)) maintain that necessary truths do not need truthmakers: if they are necessarily true, namely, true *however* the world is, it cannot be the case that *there is something* in the world that *makes* them true. In replying to this view, we follow Merrick's argument (see (Merricks, 2007)). First of all, it is important to distinguish between two readings of the above claim: (i) a more moderate reading according to which necessary true are not made true by anything *substantial* in the sense that they are trivially made true without requiring anything in particular, they are made true by anything; (ii) a radical reading according to which *there isn't* anything in the world making necessary truths true, in the sense that all necessary truth are trivially true and *nothing* makes them true.

- The first reading is incompatible with our assumption about exactness: take, say, the necessary truth "I am identical with myself" (A, for short). Consider the actual world w in which it is raining in Amsterdam. If A is trivially made true by anything in any world, then A should be made (exactly) true in the actual world by the fact that it is raining in Amsterdam. But, clearly, the rain in Amsterdam is not responsible and wholly relevant for the truth of $\Box A$.

- The incompatibility of the second radical reading depends on the interpretation of "nothing". If *nothing* is understood as the trivial null fact, in the sense that no substantial truthmaker is required for a necessary truth, then the opponent view can be, in principle, compatible with our assumptions: for instance, consider an exact truthmaker of a necessary truth at w, with w having no successor. Any necessary truth of he form of $\Box A$ requires nothing in order to be made true at w, as it would be *vacuously* true in virtue of w having no successor. $TS_\Box$ expresses the *triviality* of a truth of the form $\Box A$ at w by the fact that the only truthmaker of $\Box A$ at w is indeed the null state, the fact which stands for *nothing/no fact*, as *nothing* is required for $\Box A$ to be true at w. However, not all the necessary truths are made true by the null fact. For example, as Merricks (2007) and Jago (2018) argue, consider truths of the form φ :"x exists". φ is made true by the existence of x. If x is some necessary existent, like God or platonic objects, then "necessarily x exists" is true and must depend upon x's existence as well. And x's existence cannot be a null fact.

  Instead, under a more radical reading of "nothing", we would have that there is *no* truthmaker of necessary truths. But this is clearly incompatible with our assumption that a truth (even a modal one) is made true by something on the side of the world.

So, after having clarified our philosophical assumptions, in the following, we will first try to establish what an exact truthmaker for a modal truth is *not*, coherently with the intuitions behind the framework; then we will provide some leading ideas that could shed some new light on the notion of a modal truthmaker.

2.1 Truthmakers for Modal Truths

In our philosophical investigation, we start with the assumption that there are such things as *exact truthmakers* for modal truths.

We assume that an exact truthmaker of a proposition is a *fact*, something on the side of the world, which is directly responsible and wholly relevant for the truth of that proposition (see (Fine, 2017)). What is the fact that makes "it is necessary that A" ("it is possible that A") true? We will assume that the truthmaker of the sentence "Socrates is a philosopher" (Ok) would simply be the fact that Socrates has the property of being a philosopher) and the truthmaker of the sentence "Socrates is a philosopher and Eva is not a philosopher" ($Ok \wedge \neg Oe$) is the complex fact that Socrates has the property of being a philosopher *and* Eva doesn't.

Now, when does the modality come in? What is the (exact) truthmaker of, for instance, the sentence that *it is possible that Eva is a philosopher* ($\Diamond Oe$)? Let's take $[Oe]$ to stand for the fact that Eva is a philosopher; at first, one could conceive of modalities as primitive notions: properties of facts (*de dicto*) or relations between objects and properties (*de re*); so what that makes $\Diamond Oe$ true could be *the fact* that Eva being a philosopher *is possible* (or, in a different formulation, that the fact that Eva is a philosopher has the property of being possible) or that Eva is possibly a philosopher, and analogously for necessitated statements. However, following Linsky's argument (see (Linsky, 1994)), this predicate approach to modalities and truthmaking is problematic. At first, a dilemma concerning the reading of the modality arises: how can we distinguish between *de re* and *de dicto* modalities? Is *that e being O* is possible equivalent to the fact that *e is possibly O*? In the former case we are attributing a property to a fact, namely $[Oe]$, while in the latter we are taking necessity/possibility to be a two-places relation between an object and its property, namely *e is O* possibly. However, no matter the status of the *de re* and *de dicto* distinctions, I believe, the troubles of taking modalities as primitive notions still occur, indeed, the following argument can be applied to both the de dicto and de re reading. Consider a(n) (exact) truthmaker s of the statement $\Box(Ok \vee \neg Oe)$; what is a(n) (exact) truthmaker of $\Box(Ok \vee Oe)$? It cannot be any (exact) truthmaker of $Ok \vee Oe$ which is necessary (in a de dicto or de re reading), otherwise we would have strange consequences. Indeed, if we accept the intuitive principle that a truthmaker of a sentence of the form $A \vee B$ is a truthmaker of A or a truthmaker of A, then $\Box(Ok \vee Oe)$ and $\Box Ok \vee \Box Oe$ have the same truthmakers, while it is intuitive to accept that not all the truthmakers of $\Box(Ok \vee Oe)$ are also truthmakers of $\Box Ok \vee \Box Oe$. Indeed, any state would be a truthmaker of $\Box(Ok \vee Oe)$ if and only if it is a truthmaker of *Ok* or *Oe and* it is necessary; and so, it would also make true $\Box Ok$ or $\Box Oe$, namely $\Box Ok \vee \Box Oe$. Then, it seems that in order to avoid this weird consequences, we should accept a *primitive* truthmaker of $Ok \vee Oe$, namely the disjunctive facts $[Ok \vee Oe]$ which, if having the property of being necessary, would count as a truthmaker of $\Box(Ok \vee Ye)$ without being a truthmaker of $\Box Ok \vee \Box Oe$. But accepting disjunctive facts seems incompatible with the intuitive principle underlying the truthmaker conditions of disjunctions: a fact is a truthmaker of a disjunction $A \vee B$ if and only if it is an exact truthmaker of one of its disjuncts, indeed it is not clear how the fact $[Ok \vee Oe]$ can be responsible and wholly relevant for the truth of *Ok* or *Oe*. Hence, from a philosophical point of view, taking modalities as primitives seems inconsistent with our assumption. Furthermore, from a more technical perspective, we understand modalities in terms of quantification over possible worlds: our treatment of modalities is in principle incompatible with a predicate approach to them.

Then, how should we make sense of truthmakers of modal proposition? At first, I believe, a closer look at the relation among facts and possible worlds is needed. A truthmaker, by assumption, is a fact; the (exact) truthmaking relation is relativized to possible worlds, namely every formula A is made true by a state s with respect to a world w. The most intuitive interpretation of s being a(n) (exact) truthmaker of A at w is to take s as the fact *at w* which is responsible for the truth of A. But, what does this "at" stand for? Following Lewis (see (Lewis, 1986)), we could regard the "at" as a mereological relation: s being a truth maker of A at w means that s is a *part* of w which is responsible for the truth of A. Then, according to this view, worlds would count as *maximal* mereological sum of facts/individuals; similar views connecting the idea of world to that of maximality can be also found in Plantinga (Plantinga, 1978) who regards worlds as *maximal state of affairs*. This kind of *bottom-up* view about possible worlds can be also found in (Restall, 1996): in the version of truthmaker semantics that Restall develops, worlds are maximal sets of states/facts and truthmakers are just the elements of those sets. The leading idea of Restall's framework is that

> "Instead of taking possible worlds as atomic, we look inside possible worlds to see their fine structure of truthmakers".

However, this kind of view holding that regarding truthmakers of a proposition *at* a world are something *inside* that world seems incompatible with the account of modal truthmaker semantics when it comes to the truthmakers of modal statements. In the following, I will consider some examples of this view and discuss how, in my opinion, they happen to be inconsistent with the account presented in the previous chapters.

2.1.1 Mereological sums

I take David Lewis's stance as the example of the view that a possible world is the *maximal* mereological sum of its parts and truthmakers of modal propositions are to be found among these parts. More specifically, according to Lewis, a world is

> **M1**: "the mereological sum of all the possible individuals that are parts of it. [...] It is a maximal sum: anything that is a worldmate of any part of it it is itself a part" and "every part of a world bears some such relation to every other part, but no part of one world ever bears any such relation to any part of another" (see (Lewis, 1986, ch. 1.6))

where two individuals are said to be worldmates whenever they are both parts of the same world.

Now, consider a(n) (exact) truthmaker s of $\Box A$ at w ($s \Vdash_w \Box A$); intuitively, such s must stand for something on the side of the world w which makes $\Box A$ true. According to Lewis's view about truthmakers, I believe, we are legitimated to regard s as in individual in w which *makes* $\Box A$ true, where $\Box A$ is a predication about the individual s. For instance, in a paper from 2003 (Lewis, 2003), Lewis holds that the truthmaker of the proposition "the cat Long is black" is *Long qua black*, where "Long qua black is none other than Long himself" (see (Lewis, 2003, p.31))[2]. According to **M1**, s, being an individual at w, must be a part (in the mereological sense) of w; hence, we are advocating a mereological understanding of the *at* relation between s and w. However, by semantic conditions, s being a truthmaker of $\Box A$ at w corresponds to the fusion (in the broader sense) of all the truthmaker of A at the accessible worlds from w, namely s contains some *parts* of other worlds responsible for the truth of A; in Lewis terminology, then, s will count as a *trans-world individual*, namely and *impossible* individual (see (Lewis, 1986, ch. 4.3)). Now, it seems that we are committed to inconsistency. By assumption, s would count as a part of w (something on the side of w) and s is made of parts of other worlds; this means that w share some parts with all its accessible worlds. But this is in contradiction in general with Lewis's account of possible worlds and, specifically, with **M1**, which implies that worlds are disjoint and cannot overlap (see also (Lewis, 2003, ch. 4.2)). A similar inconsistency arises from truthmaker of possible statements. For instance, consider $\Diamond A$: a(n) (exact) truthmaker s of $\Diamond A$ at w would be *the same* truthmaker s of A at some world w_1 accessible from w ($s \Vdash_w \Diamond A \Leftrightarrow$ for some v such that $wRv, s \Vdash_v A$). Under a mereological understanding of the *at* relation, we would have that such truthmaker s of $\Diamond A$ at w is a part of w, and, by semantic conditions, s is also the truthmaker of A at w_1, namely s is a part of both w and w_1 and so w and w_1 would overlap; hence, this situation would generate the same inconsistency as before. However, the

[2]Likewise, for all the other predications, Lewis claims "the truthmaker of a true predication is identical with the subject of that predication", (see (Lewis, 2003, p. 35))

overlapping among worlds is not the only problem with possibility statements like $\Diamond A$. Indeed, given $S5$ as the intended system for modalities (where every world is accessible from each other and the necessity of A and possibility of A respectively amount to A holding in all and some possible worlds), every fact in every world would be possible with respect to any world; hence, every world would collapse to (be part of) one world and maximality and the concept of different possible worlds itself wouldn't make sense. For instance, without loss of generality, consider two worlds w and w_1 accessible to each other. Consider A true in w; this means that, by the accessibility of w from w_1, there is a truthmaker t of $\Diamond A$ in w_1 such that t is a part of w_1; but t, by semantic conditions, must also be a truthmaker of A at w, namely it is a part of w which is responsible for the truth of A at w. Analogously from w_1 in w. But then, any part of w is also a part of w_1 and vice versa, so w and w_1 would amount to the same world. This implies that, in $S5$, all the worlds will collapse to just one possible world (accessible to itself) and so, every truth (in such world), even a contingent one, is also a necessary truth.

Of course, one could overcome these problems by employing, in a very Lewisian spirit, a counterpart relations among parts of different worlds; hence a certain part s of w would be a truthmaker of $\Diamond A$ whenever s has a counterpart s_1 in some w_1 (namely s_1 is a part of w) accessible from w such that s_1 is a truthmaker of A at w_1. This strategy could also be used for necessitated statements like $\Box A$: a truthmaker of $\Box A$ at w is such if and only if its counterparts $s_1, s_2, ...$ in all the worlds $w_1, w_2 ...$ accessible form w are truthmakers of A at each $w_1, w_2 ...$ However, this move doesn't seem compatible with the philosophical intuition behind the semantic machinery: indeed notice that a truthmaker t of $\Box A$ at w is taken to be the *fusion* of other truthmakers of A in all the accessible worlds from w. But this idea of fusion doesn't seem to play any role in the counterparts-based view mentioned above.

The argument above, I believe, is not oriented at showing specifically the inconsistency of Lewisian account of possible worlds with modal truthmaker semantics; more in general, I think, it can serve as an argument against those views according to which worlds are *maximal and disjoint sums of* facts/individuals and truthmaker *at* worlds have to be understood as *parts* of those worlds; and we take Lewis's account as an example of this kind of view. Indeed, notice that the argument discussed for truthmakers of possibility statements of the form $\Diamond A$ can be easily extended to any view based on the principle that truthmakers which make propositions true *at* a world are parts of that world.

2.1.2 States of affairs

The relation between truthmakers/facts and worlds could be understood as a non-mereological relation: a fact s *at* w could, for instance, be understood as a fact *included* in w and w as a maximal inclusive entity. The champion of this view can be found in Plantinga; he claims that

M2: "A possible world is simply a possible state of affairs that is maximal"

where *possible* is understood in a broadly logical sense as *consistency* (see (Plantinga, 1978, p.45); for instance, the conjunctive state of affairs that *John is tall and John is not tall* is an impossible state of affairs) and *maximality* is understood as a relation among states of affairs: a state of affair w is maximal if and only if for any state of affair S, w includes S of S precludes S (see *ibidem*), where w is said to includes S if it is not possible that w obtains (or is actual), and S fails to obtain and w is said to precludes S if it is not possible that both w and S obtain. In this sense, we understand a fact *s at w* as s being included in w.

This view, I believe, can handle without big difficulties the truthmakers of necessity statements: s is a truthmaker of $\Box A$ at w whenever s is a state of affair (or fact) which includes all the truthmakers of $\Box A$ at the different possible worlds accessible from w. For this view, *overlapping* among worlds is not problematic: a state of affair can be included in different possible worlds. However, **M2** is subject to the same modal collapse when it comes to possibility statements of the form $\Diamond A$; as in the case of **M1**, take, without loss of generality, two worlds w_1 and w_2 accessible from each other. Consider A true in w_2, this means that there is a truthmaker t of $\Diamond A$ included in w_1; but, by semantic conditions, such t is also included in w_1. But then, every state of affair included in w_2 is included in w_1; analogously we can reason from w_2 to w_1 and show that w_1 and w_2 include the same states of affairs. Hence w_1 and w_2 amount to the same possible world; this means that in a $S5$ system, all the worlds collapse in just one world w and every truth, even contingent in w, becomes a necessary truth.

The same counterargument could be repeated for a combinatorialist account of possible worlds too; according to the combinatorialist Armstrong:

M3: "[...]possible atomic states of affairs may then be combined in all ways to yield possible molecular states of affairs. If such a possible molecular state of affairs is thought of as the totality of being, then it is a possible world." (see (Armstrong, 1986, p. 579)

where possible states of affairs are obtained from *actual* objects and properties. Again, if we regard truthmaker as states of affairs and possible worlds as maximal re-combinations of states of affairs, then we would be committed to a modal collapse.

2.2 Truthmakers and Possible Worlds

Given the failure of some of the traditional accounts of possible worlds, a different idea is needed. I think that the philosophical key is to consider facts relativized to possible worlds as truthmakers as a whole; we can find a similar idea in (Linsky, 1994): a truthmaker of a true proposition A at w would be the relational fact that $[A]$ *holds at* w, namely the fact $[At([A], w)]$. Analogously, a truthmaker of $\Diamond A$ at w would be the fact $[At([\Diamond A], w)]$; more explicitly, according to Lisnky, we can reduce the modal operator to quantification over possible worlds and hence obtain that the truthmaker of $\Diamond A$ at w amounts to the more complex relational fact $[At([At(A, w_1)], w)]$ where w_1 is an accessible worlds from w. In this way, possibility and necessity become a matter of quantification over worlds and iteration of relational properties between facts and worlds. Indeed, within Lisnky's account, the key for making possibilities true is the iteration of the relation At: "possibly A" at w is made true by the fact that there occurs an At relation between $[At([A], w_1])$ (with w_1 accessible from w) and w itself. However, this iteration is not clearly expressible within the semantic account: a truthmaker of $\Diamond p$ at w is such if and only if it is also the truthmaker of p at w_1 accessible from w. Actually, I believe, we have no clear need of iterating the At relation in order to construct truthmakers for modal proposition: we could simply take, according to the semantic clauses, the truthmaker of $\Diamond p$ at w to be the relational fact $[At([p], w_1)]$ with w_1 accessible from w, namely the truthmaker of p at w_1 is identical with the truthmaker of $\Diamond p$ at w (with w_1 accessible from w). However, our idea still shares the conceptual core with Lisnky's account, namely that

> "Being possible would not be a primitive property of non actual-facts, but rather a relational property of belonging to a certain maximal fact." (see (Lisnky, 1984, p. 203)).

Following the semantic clauses we would have that the truthmaker of an atomic sentence p at w is simply the fact $At([p], w)$ and the truthmaker of a conjunction $p \wedge q$ is simply the fusion $[At([p], w)] \sqcup [At([q], w)]$. It seems, at this point, acceptable to impose some constraints on the $\sqcup$ operation among At facts, for instance it is plausible to impose the constraint that $[At([p], w)] \sqcup [At([q], w)]$ returns the fact $[At([p] \sqcup [q], w)]$.

So, the semantic account seems more compatible with such a view according to which a truthmaker amounts to a relation between facts and worlds, without requiring any iteration of such relations. Notice, however, that we have no problem in accepting in our ontology facts of the form $[At([At(A, w_1)], w)]$: they are simply more complex facts encoding a relation between two other facts.

At this point, it remains unclear what the At relation stands for and what conception of possible worlds is more compatible with the semantic account. At could be taken as a primitive unique relation as Linsky seems to suggests: his "account relies upon a primitive modal notion of truth at a world for which we need an account." (see (Linsky, 1994, p. 203)). What should be the features of such a relation? In the following we will analyze some possible answers:

- At, for instance, could be taken intuitively as the relation of parthood between facts and worlds: for instance, we would have that $At([p], w)$ stands for the fact that $[p]$ is a part of w and, by iterating At, $At([At([p], w)], w)$

means that $[At([p], w)]$ is a part of w. Under this conception, it would be legitimated to ask what's the nature of the possible worlds and, the most intuitive answer, would be that a world is the mereological sum of all the facts that are parts of it. But this view cannot be reduced to Lewis's notion of possible worlds. Consider, as before, two possible worlds w_1 and w_2 which access to themselves and to each other and consider a fact $[p]$ such that $[p]Atw_1$ (where $[p]Atw_1 := At([p], w_1)$). By accessibility relations, we have that $[[p]Atw_1]$ is a part of both w_1 and w_2, as $[[p]Atw_1]$ is the truthmaker of $\Diamond p$ at w_1 as well as it is the truthmaker of $\Diamond p$ at w_2. Hence, we are committed to w_1 and w_2 sharing the part $[[p]Atw_1]$, which is inconsistent with the non-overlapping constraint of Lewis's account. Hence, understanding *At* as a mereological relation needs a new notion of possible world, different from the Lewisian idea of worlds as mereological sums of their parts.

- More appealing is an understanding of *At* as an inclusion relation *à la* Plantinga: *At* holds between a fact $[p]$ and a world w if and only if it is not possible that w obtains without $[p]$ obtaining. However, this idea is not consistent with Plantinga's notion of possible worlds as *maximal* states of affairs, more specifically it seems that under this understanding of *At*, possible worlds turn out not to be *maximal*. Indeed, consider two inaccessible worlds w_1 and w_2 and the fact that $[p]$ included in w_2, namely $[[p]Atw_2]$. By non-accessibility, it's not necessary that $\Diamond p$ is made true at w_1, hence it's not necessary that w_1 obtains without $[[p]Atw_2]$ obtaining; on the other hand, there is no clear reason why it shouldn't be possible that both w_1 and $[[p]Atw_2]$ obtain, indeed, why should the fact that $[p]$ is included in w_1 *logically* incompatible with w_1? Under this understanding of *At*, the possible worlds would lose their status of *maximal* states of affairs, hence, a new notion of possible worlds different from the Plantinga's would be needed.

- Alternatively, we can simply regard *At* as a primitive relation between facts and world; however this idea would be inconsistent with the classical combinatorialist notion of possible worlds: by accepting a truthmaker in the actual world of some possible fact $[p]$, namely the fact that $[p]Atw_1$, where w_1 is a possible world accessible from the actual one, we would be committed to the actuality of the possible world w_1, namely to the actuality of all the combinations of object and properties in w_1. But, some of those combinations could be incompatible with some other actual combinations, for instance the weather in Amsterdam may be raining in the actual world and not raining in w_1. This would exclude the possibility of w_1 being actual.

- Otherwise, one could go for a very peculiar view of possible worlds as *atomic* entities which are not maximal nor molecular, nor inclusive, nor total. In this peculiar view, facts and worlds would be connected by a primitive relation *At* which cannot be understood in terms of parthood nor inclusion. Besides the implausibility of this account of possible worlds, such view stresses the facts that possibility and necessity are merely relational properties and truthmakers of modal statements are just *facts* instantiating these relations.

But now, some problems arises with merely possible facts and individuals: given our commitment to truthmakers of the form $[p]Atw$ what kind of commitment do we have to merely possible facts and individuals? For instance, consider the merely possible fact that Wittgenstein might have had a child in the actual world $w_@$ (with $\Diamond Ftc$ being the sentence expressing that Wittgenstein (t) is the father of c). Presumably, the truthmaker at $w_@$ of $\Diamond Ftc$ would be the fact $[[Ftc]Atw_1]$ with w_1 accessible from w. At a first glance, being $\Diamond Ftc$ true and made true by the fact $[[Ftc]Atw_1]$ simply commits us to the fact that a certain relation holds between the fact that Wittgenstein had a child and a possible world, without forcing us to accept the obtaining of $[Ftc]$ or the existence of c. However, whether $[[Ftc]Atw_1]$ carries over the commitment to the obtaining of $[Ftc]$, requires the answer to the more general question of whether facts (complex or atomic) carry over the commitment to their constituents (such answer is, I think, independent of $[[Ftc]Atw_1]$ being conceived as atomic or complex, the commitment to Ftc or c would be in any case problematic). All this implies a deeper discussion on the metaphysics of facts which is behind the scope of this thesis.

2.3 Conclusions

In the previous sections, I tried to argue for the incompatibility of certain views of truthmakers for modal truths with the traditional conceptions of possible world. However, developing a precise metaphysical view on possible worlds and *At* relation falls outside the scope of this thesis. In principle, the semantic account would be neutral with respect to the different understandings of *At* relation: we just confined ourselves with showing that *some* understandings of the *At* relation are incompatible with *some* conceptions of possible worlds. What is essential for our aim is to have shown that the formal treatment of modal truthmaker semantics requires the conceptual move of relativizing truthmaking at worlds as well as truthmakers at worlds. Formally, we have a semantic account of *how* a state makes *true* a sentence with respect to a world, for instance a state s is a truthmaker of $\Diamond p$ at w if and only if s is the truthmaker of p at some accessible world w_1. *What* is such truthmaker at w is a matter of philosophical discussion. We argued that the key idea is to relativize facts at worlds, for instance s truthmaker of $\Diamond p$ at w would be the *fact* (wider scope) that the fact (narrow scope) $[p]$ is in a certain relation *At* with w. What such *At* amounts to is a matter of philosophical discussion as well.

This view, I believe, open the doors to an understanding of the possibility as a merely relational property, within the truthmaker framework, in the sense that a truthmaker of a possible facts with respect to w is simply the relation (no matter how we understand it) which occurs between that fact and the corresponding possible world. For instance, the truthmaker of "Wittgenstein might have had a child" in the actual world $w_@$ is the fact that [Wittgensteing had a child] is part of/included in/obtaining at a possible world w_1 accessible from $w_@$. Notice, however, that the notion of accessibility relation is still fundamental to understand how possibility works: a world is *possible* relative to another (see (Kripke, 1963)), and accessibility relations among worlds allow us to model this concept of being possible *relative to*. One could think of promoting an understanding of the accessibility relations in terms of the iteration of the *At* relation, for instance, a worlds w_2 is said to be accessible from w_1 if and only if every facts of the form $[x]Atw_2$ is also a fact *at* w_1, namely the *At* relation holds between $[x]Atw_2$ and w_1 ($At([x]Atw_2, w_1)$). This conceptual move would be of a very Lewisian flavour: it would simplify in *ideology* our account: a relation of a certain kind (accessibility relation) would be understood in terms of a relation of another kind (*At* relation) already present in our account.

2.4 Comparisons

In the literature about truthmaker semantics, it seems that a uniform formal treatment of modalities, especially alethic modalities (*necessities and possibilities*) is still lacking. Kit Fine in one of his paper (Fine, 2016) mentions some ideas concerning how modalities are supposed to be treated within the framework of truthmaker semantics; given a state space $\langle S, \sqsubseteq \rangle$, he distinguishes a subset P of S of possible states such that P is downward closed, namely for any state $s \in P$ and for any other state $t \in S$, if $t \sqsubseteq s$ then $t \in P$. This corresponds to the intuition that if a fact is possible, then also its parts must be possible. Moreover, two states re said to be compatible if their fusion is a possible state. Notice that *compatibility* implies *possibility*, in facts, if two states s, t are compatible, then their fusion must be possible, namely $s \sqcup t \in P$, and, by downward closure of P, it is also the case that s and t are in P, namely they are possible. Kit Fine, in the same paper, also provides us with a definition of a necessary state: a state s is said to be necessary if it is compatible with all the possible states. Finally, a state w is said to be a possible world when it is maximal, in the sense that for any state t is S, if t is compatible with w, then $t \sqsubseteq w$. These ideas, however, lack a precise and faithful formal treatment in the sense that no (exact) truthmaker semantics for modal statements has been developed starting from them.

In the following we will try to compare Korbmacher's framework with that of Fine; we will try to argue for the fact that Korbmacher's semantic framework has several formal and philosophical advantages over Fine's semantics.

- *Modalities.* At first, it is not clear what Fine precisely mean by *possible* states. One could think that a possible state is defined in terms of compatibility, namely s is said to be possible if and only if it is compatible with itself, indeed $s \in P$ if and only if $s \sqcup s \in P$ (notice that the fusion of every state with itself returns the same state). But this definition is circular: compatibility can be defined in terms of possibility and vice versa, namely x is compatible with y if and only if $x \sqcup y$ is possible. So, what is prior to what? It doesn't matter anyway: what is crucial is that both *compatibility* and *possibility* are modal notions and if one takes one of them as a primitive, she lacks explanatory power. Notice, instead, that in our account we can define possible states in purely extensional terms according to our favourite view of possible worlds, for instance, we can define a possible state as a part of a certain world and two compatible states as part of the same world. So, it seems that where Fine postulates primitives to account for some phenomena, we have a familiar explanation of them. In this sense his account lacks explanatory power: he has to postulate primitives modal notions while we can provide a definition of them.

Taking modal concept as primitives also generates other problems concerning the formulation of exact truthmaking conditions for modal statement: what is a truthmaker of a modal sentence like $\Box A$ or $\Diamond A$? More likely, a truthmaker of $\Box A$ must be a state which is *necessary* and makes A true. We have already encountered this problematic situation: defining truthmaking conditions in this way generates a problematic commitment to disjunctive facts (see chapter **2**). So, if one wants to maintain modalities as primitives, then he should also define suitable exact truthmaking conditions for modal statements; but such a semantic account is still lacking in Fine's framework.

- *Exact Truthmaking Conditions.* It is also possible, in principle, to define possibility within Fine's framework in terms of logical compatibility, in the sense that two states $x \sqcup y$ are said to be compatible if and only if for any formula A it is not the case that $x \Vdash A$ and $y \Vdash \neg A$. Notice, actually that Fine imposes an exclusivity constraint over the possible states in his models, namely for any formula A, no *possible* truthmaker of A is compatible with a *possible* falsemaker of A (and this follows by imposing an exclusivity constraint on the valuation, namely, for any propositional letter p, no *possible* truthmakers of p is compatible with a possible falsemaker of p). Consequently, Fine could go further and take possible worlds to be maximal sum of compatible states and this definition would resemble very much Plantinga's view on possible worlds. This would solve the problem of having modalities as primitives but leaves open the problem of *how* to define truthmaking conditions for modal statements. For instance, what would be a truthmaker for $\Diamond p$? More likely it would be a possible truthmaker of p, and so a state compatible with itself which makes p true. But what happens if p is a *mere* possible fact, namely something which is not *actual* but it might have been such? For instance consider the statement "Wittgenstein might have had a child" ($\Diamond Ftc$). Clearly, it is not the case that Wittgenstein had a child, but it might have been so. Hence, there must be a(n) (exact) truthmaker s for the sentence $\Diamond Ftc$, namely a state which is possible and makes Ftc true. But now, what prevents s to make also true the simple sentence *Ftc*? In which sense *s* is a(n) (exact) truthmaker of *Ftc*? This shouldn't be ambiguous: *s* make also *Ftc* true, which is clearly counterintuitive, indeed it is not true that Wittgenstein had a child and so, there can be nothing which makes it true.

It seems that, in principle, Korbmacher's framework is more expressive than Fine's one since we have been able to provide an exact truthmakers semantics for modal statements while this is lacking in Fine's account. For all these reasons, the framework presented and developed in the previous chapter seems to be a more satisfactory answer to the question "how to construct a truthmaker semantics for modal statements?".

3 Applications

In this section, we will focus on one possible application of the semantic framework developed in the previous sections. We will try to analyze how and whether it can furnish a good semantic characterization of the modal extension of the logic of *analytic containment*. Kit Fine, in (Fine, 2016) has shown that the propositional version of the logic of analytic containment (AC), originally presented in (Angell, 1989), is sound and complete with respect to (an inclusive version of) truthmaker semantics; our goal is to expand the work done in (Fine, 2016) to the modal case by employing $TS_\Box$. More specifically we will proceed by:

(i) introducing the propositional logic of analytic containment and presenting a natural modal extension of it (we will denote this extension by “$AC_\Box$”);

(ii) developing a modal truthmaker semantics for $AC_\Box$ by expanding the semantics presented in (Fine, 2016) with $TS_\Box$;

(iii) proving soundness of $AC_\Box$ and extending some results demonstrated in (Fine, 2016) to $AC_\Box$.

3.1 Analytic Containment

Richard B. Angell in 1989 introduced the first original formulation of the propositional logic of analytic containment (AC). Angell's aim was to formalize a notion of entailment understood in terms of containment of meanings:

> “The concept of entailment [...] has also been connected to the concept of containment in Kant's sense of analytic containment: A entails B only if the meaning of B is contained in the meaning of A” (Angell, 1989, p.1).

Angell claims that this concept of analytic entailment is also connected to the concept of synonymity:

> “S_1 is synonymous with S_2 if and only if S_1 entails S_2 and S_2 entails S_1” (*ibidem*)

In the original formalization and axiomatization of AC (see (Angell, 1989)), Angell takes synonymity as the a primitive symbol in the language “$<>$”, so $A <> B$ stands for “A is analytically equivalent to B”, i.e. “A is synonymous with B” . By means of $<>$, the concept of analytic entailment, i.e. containment of meaning, ($>$) can be defined as $A > B := A \wedge B <> A$, where $A > B$ stands for “A analytically entails B”; alternatively, taking $>$ as primitive, one can define $<>$ as $A <> B := (A > B) \wedge B > A$. For instance, an alternative axiomatization of AC in terms of $>$ as the primitive in the language can be found in (Correia, 2004).

A well-formed formulas in the language of AC has the form of $A <> B$ where A and B are classical well-formed formulas; here we stick to the following definition of well-formed formulas:

$$A ::= p \mid \neg B \mid B \wedge C \mid B \vee C$$

where implication and equivalence are defined in the standard way: $A \to B := \neg A \vee B$ and $A \leftrightarrow B := (A \to B) \wedge (B \to C)$. Notice that, by the definition, $<>$ cannot be iterated, namely $<>$ cannot be flanked by formulas containing $<>$: on both sides of $<>$ a classical formula must occur which, by definition, cannot contain $<>$.

Angell syntactically defines theoremhood in his system by introducing some intuitive constraints on the relation of analytic containment/equivalence; we will mention some among the most relevant:

- if $A <> B$ is a theorem, then they share the same propositional letters. This constraint is meant to rule out the "paradoxes of strict implication", like (i) $A > B \vee \neg B$ (ii) and $A \wedge \neg A > B$ from the theorems of AC. Indeed, consider some instances of (i) and (ii), namely $p > (q \vee \neg q)$ and $(p \wedge \neg p) > q$: intuitively, it is not clear at all how the meaning of $q \vee \neg q$ can be contained in p or the meaning of q can be contained in $p \wedge \neg p$. For example, take p to stand for "it is raining in Amsterdam" and q for "Socrates if philosopher": intuitively the proposition that it is raining in Amsterdam has nothing to do with Socrates being a philosopher, hence "it is raining in Amsterdam" cannot contain analytically entail that either Socrates is a philosopher or not (and analogously for $(p \wedge \neg p) > q$. Moreover, this constraint, rules out the law of addition (iii) $A > A \vee B$ and absorption (iv) $A <> A \wedge (A \vee B)$; form the same reason as before, (iii) and (iv) shouldn't count intuitively as theorems of AC.

- if $A <> B$ is a theorem then every variable occurs positively (negatively) in A if and only if it occurs positively (negatively) in B. This rules out from the theorems of AC formulas like $(A \wedge \neg A) \wedge B <> A \wedge (B \wedge \neg B)$

Angell ends up with the following axiomatization

(Ax.1) $A <> \neg\neg A$

(Ax.2) $A <> (A \wedge A)$

(Ax.3) $(A \wedge B) <> (B \wedge A)$

(Ax.4) $A \wedge (B \wedge C) <> (A \wedge B) \wedge C$

(Ax.5) $A \vee (B \wedge C) <> (A \vee B) \wedge (A \vee C)$

(R.1) From $\vdash A <> B$ and $\vdash X$ infer $\vdash X(A/B)$

where $X(A/B)$ is the formula obtained by replacing come occurrences of A in X with B. Notice that any theorem of AC has the form of $A <> C$[3]. Correia in (Correia, 2016) takes a specific fragment of AC to stand for a faithful formalization of his notion of *factual equivalence* which relies upon the intuition that two formulas are factually equivalent "when they describe the same facts or situations understood as worldly items, i.e. as bits of reality rather than as representations of reality" (see (Correia, 2016)).

We will proceed in our analysis of the modal extension of AC trying to be as neutral as possible about the philosophical interpretations of AC: our aim is not to provide a formalization of the modal version of the notion of *factual equivalence* or *partial content*, we are rather concerned with investigating whether, when we expand the language of AC with modal operators, $TS_{\Box}$ can provide the obtained system with a good semantic structure.

3.1.1 Modal Analytic Containment

In the following, we will try to develop a natural modal extension of AC ($AC_{\Box}$).

The most intuitive expansion is to augment the language of AC by adding modal operators: we would now define classical modal formulas as

$$A := p \mid \neg B \mid B \wedge C \mid B \vee C \mid \Box B$$

where the possibility operator $\Diamond$ is defined in the standard way as $\Diamond A := \neg\Box\neg A$. A formula in the language of $AC_{\Box}$ ($\mathcal{L}_{AC_{\Box}}$) would be recursively defined as:

> if A and B are classical modal formulas, then $A <> B$ is a well-formed formula of $\mathcal{L}_{AC_{\Box}}$.

Notice that by definition, in $\mathcal{L}_{AC_{\Box}}$, $<>$ cannot fall into the range of a modal operator and it cannot be iterated, as for the non-modal case.

We want to preserve the intuitive demand that theoremhood (i.e. validity) in $AC_{\Box}$ amounts to modal analytic containment, namely that $A <> B$ is a theorem (i. e. valid) in $AC_{\Box}$ if and only if the meaning of B is contained in the meaning of A and vice versa. Now, our attempt now will be to determine how the relationship of modal analytic containment between two formula A and B holds or fails in virtue of semantic properties of A and B. Notice that this is analogous but, in some sense, reverse with respect to Angell's attempt: he originally defined the relationship of analytic containment between two formulas A and B by appealing to the syntactic properties of A and B. The points of departure of our plan are Fine's semantic for AC developed in (Fine, 2016) and $TS_{\Box}$. We will now introduce some new definitions which corresponds to a modal extension of the notions we find in (Fine, 2016):

Definition 27 *An* inclusive *E-Kripke model is a tuple $\mathcal{E} = \langle S, \sqsubseteq, W, R, v^+, v^- \rangle$ where $\langle S, \sqsubseteq, W, R \rangle$ is an E-Kripke frame and $v^+, v^- : \mathcal{L}^{AC_{\Box}}_{prop} \rightarrow \mathcal{P}(S)$, where $\mathcal{L}^{AC_{\Box}}_{prop}$ consists of*

[3]Fine proposes an alternative axiomatization of his system which is equivalent to the original AC.

the propositional letters in the language of $AC_{\Box}$, such that for any propositional letter p and any $w \in W$:

$$X \subseteq v^+_w(p) \quad \Rightarrow \quad \bigsqcup X \in v^+_w(p)$$
$$X \subseteq v^-_w(p) \quad \Rightarrow \quad \bigsqcup X \in v^-_w(p)$$

and

$$v^+_w(p) \neq \emptyset$$
$$v^-_w(p) \neq \emptyset$$

namely the set of exact truthmakers (falsemakers) of an atomic sentence is non-empty and closed under $\bigsqcup$ operation. This constraints correspond to the intuition that the fusion of two exact truthmakers s and t of the same proposition p should still be *wholly relevant* for the truth of p. Indeed, if both p and t doesn't contain anything irrelevant for the truth of p, then their fusion also should not have any irrelevant parts for the truth of p. This conditions would be essential for proving **Lemma 16** and **Lemma 19**.

In the following, we will define different notions of *content* of a sentence which has been originally formulated in (Fine, 2016). The main and only difference with respect to Fine's system is that our notions of content will be relativized to worlds. The long definition which will follow are essential in order to construct a semantics for the modal extension of the system AC.

Definition 28 *Given an inclusive E-Kripke model $\mathcal{E} = \langle S, \sqsubseteq, W, R, v^+, v^- \rangle$, the* positive (negative) exact content *of a formula A at $\mathcal{E}$, with respect to a world $w \in W$, $|A|^+_w$ ($|A|^-_w$), is the set of its exact truthmakers (falsemakers), namely $|A|^+_w = \{s : s \Vdash_w A\}$ ($|A|^-_w = \{s : s \dashv\!\!\vert_w A\}$).*

Definition 29 *Given an inclusive E-Kripke model $\mathcal{E} = \langle S, \sqsubseteq, W, R, v^+, v^- \rangle$, the* positive (negative) complete content *of a formula A at $\mathcal{E}$, with respect to a world $w \in W$, $\overline{|A|}^+_w$ ($\overline{|A|}^-_w$), is the $\bigsqcup$-complete closure set of its positive (negative) exact content at $\mathcal{E}$, namely $\overline{|A|}^+_w = \{s : s = \bigsqcup T$, with $T \neq \emptyset$ and $T \subseteq |A|^+_w\}$ ($\overline{|A|}^-_w = \{s : s = \bigsqcup T$, with $T \neq \emptyset$ and$T \subseteq |A|^-_w\}$).*

Definition 30 *Given an inclusive E-Kripke model $\mathcal{E} = \langle S, \sqsubseteq, W, R, v^+, v^- \rangle$, the* positive (negative) replete content *of a formula A at $\mathcal{E}$, with respect to a world $w \in W$, $[A]^+_w$ ($[A]^-_w$), is the convex-closure of its positive (negative) complete content at $\mathcal{E}$, namely $[A]^+_w = \{s : s_1 \sqsubseteq s \sqsubseteq s_2$, for some $s_1, s_2 \in \overline{|A|}^+_w\}$ ($[A]^-_w = \{s : s_1 \sqsubseteq s \sqsubseteq s_2$, for some $s_1, s_2 \in \overline{|A|}^-_w\}$).*

We will now define three relations between sets of states:

Definition 31 *Given an inclusive E-Kripke model $\mathcal{E} = \langle S, \sqsubseteq, W, R, v^+, v^- \rangle$, for any two subset T, Z of S, we say that T* subsumes *Z at $\mathcal{E}$, $T \sqsupseteq Z$, if and only if for any $t \in T$ there is a $z \in Z$ such that $t \sqsupseteq z$.*

Definition 32 *Given an inclusive E-Kripke model $\mathcal{E} = \langle S, \sqsubseteq, W, R, v^+, v^- \rangle$, for any two subset T, Z of S, we say that T* subserves *Z at $\mathcal{E}$, $T \sqsubseteq Z$, if and only if for any $t \in T$ there is a $z \in Z$ such that $t \sqsubseteq z$.*

Definition 33 *Given an inclusive E-Kripke model $\mathcal{E} = \langle S, \sqsubseteq, W, R, v^+, v^- \rangle$, for any two subset T, Z of S, we say that T* analytically contains *Z, at $\mathcal{E}$, $T > Z$, if and only if $T \sqsupseteq Z$ and $Z \sqsubseteq T$*[4].

We have now all the ingredients to introduce a semantics for $AC_{\Box}$ which is a coherent modal expansion of the one in (Fine, 2016): given an inclusive E-Kripke model $\mathcal{E}$, we say that a formula $A > B$ is positively true at $\mathcal{E}$ with respect to a world w in $\mathcal{E}$ if and only if the positive replete content of A at w analytically contains the positive replete content of B at w. More formally

$$\mathcal{E}, w \models A > B \quad \Leftrightarrow \quad [A]^+_w > [B]^+_w$$

We say that $A > B$ is valid in an inclusive E-Kripke models $\mathcal{E}$ if and only if $A > B$ is true at all the world w in $\mathcal{E}$. More formally

$$\mathcal{E} \models A > B \quad \Leftrightarrow \quad \text{for any } w \in W,\ [A]^+_w > [B]^+_w$$

Now, we will introduce new truthmaker conditions which should correspond to an *inclusive* intuition about the relation of truthmaking. We have seen above that the inclusive constraint on E-Kripke models was intended to model the principle that the fusion of two exact truthmakers of a sentence A is also an exact truthmaker of A. The same argument can be applied to the truthmakers of all formulas. An interesting case for this argument is the one of disjunctions: assume s and t are truthmakers of a disjunction $A \vee B$, then they both cannot contain anything irrelevant for the truth of $A \vee B$. This seems to intuitively implies that also their fusion shouldn't contain anything irrelevant for the truth of A. Notice that from this principle, it follows that it can be the case that if $s \Vdash_w A$ and $t \Vdash_w B$ then $s \sqcup t \Vdash_w A \vee B$ since both s and t are exact truthmakers of $A \vee B$. Namely, truthmaker of a conjunction $A \wedge B$ are also truthmakers of the disjunction $A \vee B$ (the same reasoning, with same consequence, can be applied to the truthmaker conditions of $\Diamond A$, and falsemaker condition of $\Box A$ and $A \wedge B$). Hence, we will introduce new truth conditions for $\wedge$, $\vee$, $\Box$ and $\Diamond$ to simplify the semantics for $AC_{\Box}$; let us call the resulting semantics *inclusive semantics* for $AC_{\Box}$.

Given an inclusive E-Kripke model $\mathcal{E}$, we inductively define the *inclusive* semantic conditions for a classical formula A to be verified or falsified by a state

[4] Notice that $T > Z$ is an abuse meant to simplify the notation as the relation $>$ by definition holds between two formulas.

s with respect to a world w:

$$
\begin{array}{lll}
s \dashv\!|_w A \wedge B & \Leftrightarrow & s \dashv\!|_w A \text{ or } s \dashv\!|_w B \text{ or } s \dashv\!|_w A \vee B \\
s \Vdash_w A \vee B & \Leftrightarrow & s \Vdash_w A \text{ or } s \Vdash_w B \text{ or } s \Vdash_w A \wedge B \\
s \dashv\!|_w \Box A & \Leftrightarrow & s \dashv\!|_v A \text{ for some } v : wRv \text{ or} \\
 & & \text{there is a function } f : W \to S \text{ such that for a non-empty } Y \subseteq R[w], \\
 & & \text{for any } v \in R[w],\ f(v) \dashv\!|_v A \text{ and } s = \bigsqcup(\bigcup_{v \in R[w]}\{f(v)\}) \\
s \Vdash_w \Diamond A & \Leftrightarrow & s \Vdash_v A \text{ for some } v : wRv \text{ or} \\
 & & \text{there is a function } f : W \to S \text{ such that for a non-empty } Y \subseteq R[w], \\
 & & \text{for any } v \in R[w],\ f(v) \Vdash_v A \text{ and } s = \bigsqcup(\bigcup_{v \in R[w]}\{f(v)\})
\end{array}
$$

where $R[w]$ is the set of worlds accessible from w. The remaining case are analogous to the ones introduced for $TS_{\Box}$. Notice that the non-emptiness constraint is justified by the intuition that the null-state cannot make $\Diamond A$ true at w when w has no successor. Indeed, consider an E-Kripke model $\mathcal{E}$ in which a world w has no successor; clearly $\Box A$ would be vacuously true at w and the *vacuous* true of $\Box A$ at w is also expressed by the fact that, by semantic conditions, the only truthmaker of $\Box A$ at w is the null state. Indeed, *nothing* (no positive fact) is required for the truth of $\Box A$ at w. Now, consider $\Diamond A$ at w; intuitively, the possibility of A requires substantial conditions in order for it to obtain: there must be *some* world in which A is true (analogously we can reason for the falsity of necessity). If we had formulated the inclusive semantic conditions dropping the non-emptiness conditions of $Y \subseteq R[w]$ it would have happened that in the model $\mathcal{E}$ we have considered, $\Diamond A$ would have been made true at w by the null-state (since $\bigcup_{v:wRv}\{f(v)\}$ would have been empty and $\bigsqcup \emptyset = 0$) although w had no successor. From now on, we will stick to this inclusive version of the semantics: whenever we use $\Vdash_w$ and $\dashv\!|_w$ we will refer to the above semantic conditions.

Analogously to (Fine, 2016), we can the following three results are provable:

Lemma 16 *For any formula A, any inclusive E-Kripke model $\mathcal{E}$, any world w in $\mathcal{E}$ and any state s, t in $\mathcal{E}$,*

[if $\mathcal{E}, s \Vdash_w A$ and $\mathcal{E}, t \Vdash_w A$, then $\mathcal{E}, s \sqcup t \Vdash_w A$] and [if $\mathcal{E}, s \dashv\!|_w A$ and $\mathcal{E}, t \dashv\!|_w A$, then $\mathcal{E}, s \sqcup t \dashv\!|_w A$]

Proof: see appendix **A.6**.

The inclusive condition we imposed on any E-Kripke model was essential to prove the above lemma which is essential to prove soundness of *AC*: it will allow us, as we will see further, to prove the validity of the principle that any proposition A is synonymous with $A \wedge A$, which is Ax.2 in Angell's *AC*. Moreover, **Lemma 16** correspond also to to the inclusive intuition on the notion of truthmaking.

Lemma 17 *Given an inclusive E-Kripke model $\mathcal{E}$, for any classical formula A, the positive complete content of A at $\mathcal{E}$ is equal to the positive exact content of A at $\mathcal{E}$ under* *the inclusive semantics, more formally*

$$\overline{|A|}^{+}_{w} = \{s : s \text{ verifies } A \text{ under the inclusive semantics } \}$$

Proof: analogously to (Fine, 2016).

Lemma 18 *Given an inclusive E-Kripke model $\mathcal{E} = \langle S, \sqsubseteq, W, R, v^+, v^- \rangle$, for any two subsets T, Z, of S, their convex closure, T_*, Z_*. does not affect analytic containment, more formally*

$$T > Z \text{ if and only if } T_* > Z_*$$

Proof: analogously to (Fine, 2016).

Now, from the above two lemmas, we can easily prove a simpler characterization of the truth conditions of a formula $A > B$ with respect to a world w in an inclusive E-Kripke model $\mathcal{E}$. By **Lemma 18**, $[A]^+_w > [B]^+_w$ if and only if $\overline{|A|}^{+}_{w} > \overline{|B|}^{+}_{w}$, in fact, by definition, $[A]^+_w$ and $[B]^+_w$ are respectively the convex closures of $\overline{|A|}^{+}_{w}$ and $\overline{|B|}^{+}_{w}$. Moreover, by **Lemma 17**, we have that $\overline{|A|}^{+}_{w}$ and $\overline{|B|}^{+}_{w}$ amount respectively to the positive exact content of A and B under the inclusive semantics, hence, we have that

$$\mathcal{E}, w \models A > B \quad \Leftrightarrow \quad \text{(i) } \overline{|A|}^{+}_{w} \sqsupseteq \overline{|B|}^{+}_{w} \text{ and (ii) } \overline{|B|}^{+}_{w} \sqsubseteq \overline{|B|}^{+}_{w}$$

Inclusive semantics allows us to give a semantics for $AC_{\Box}$ in a more natural way, namely, by **Lemma 17**,

$A > B$ is true with respect to a world w if and only if (i) every verifier of A at w under the inclusive semantics contains a verifier of B at w under the inclusive semantics and (ii) every verifier of B at w under the inclusive semantics is contained in a verifier of B at w under the inclusive semantics.

In a very natural way, we can now provide semantic conditions for *analytic equivalence* (<>):

$$\mathcal{E}, w \models A <> B \quad \Leftrightarrow \quad \text{(i) } \mathcal{E}, w \models A > B \text{ and (ii) } \mathcal{E}, w \models B > A$$

3.1.2 Axiomatization

Johannes Korbmacher in (Johannes, 2016) has proposed an axiomatization of $AC_{\Box}$ which consists of all the axioms and rules of AC plus the following axioms:

(Dis) $\Box(A \wedge B) <> \Box A \wedge \Box B$

(Dual) $\neg\Box\neg A <> \Diamond B$

He also conjectured that this system is sound and complete with respect to the semantics for $AC_\Box$ we presented above. However, in the following, we will show that his conjectured is not true. First, we will start with some considerations on the semantics for $AC_\Box$.

Angell, in his paper from 1989, points out that the principle of *simplification*, namely $(A \wedge B) > B$ is left as "a sort of paradigm" (see (Angell, 1989, p. 121)) of analytic entailment. Indeed, it is quite natural to think that the meaning of B is contained in the meaning of $A \wedge B$; notice, in fact, that this principle is very similar to the intuition behind Fine's notion of partial content. So, all the formulas of the form of $(A \wedge B) > B$ should be validated within the semantics for $AC_\Box$, as it is in the case of AC. However, consider the formula $(\Diamond p \wedge q) > q$ and an inclusive E-Kripke model $\mathcal{E} = \langle S, \sqsubseteq, W, R, v^+, v^- \rangle$ in which a world $w \in W$ has no successor. It is evident that $|\Diamond p|^+_w$ in $\mathcal{E}$ is empty since w has no successor, namely, by semantic conditions, there is no state in $\mathcal{E}$ making $\Diamond p$ true. This clearly implies that also $\overline{|\Diamond p \wedge q|}^+_w$ is empty; moreover notice that, by definition of the valuation in the inclusive models, $\overline{|q|}^+_w$ is non-empty. Hence, it turns out that it is not the case that $\mathcal{E}, w \models (\Diamond p \wedge q) > q$ since $\overline{|q|}^+_w$ does not subserve $\overline{|\Diamond p \wedge q|}^+_w$ (see **Definition 28**). Consequently, $(\Diamond p \wedge q) > q$ cannot be valid in $AC_\Box$. This is an undesired result: being the principle of simplification the *paradigm* of analytic entailment, which should be preserved for $AC_\Box$, how can it happen that the meaning of q is not contained in the meaning of $\Diamond p \wedge q$?

One way to overcome this problem would be to impose a seriality constraint over the inclusive E-Kripke models of $\mathcal{E}$, namely, every world in every model must have (at least) one successor. The results also implies that Korbmacher's axiomatization is not complete, indeed, under the seriality constraint there must be, at least, another validity (which will show later in the paper) which should be encountered among the axioms, namely $\Box A > \Diamond A$, which amounts to $\Box A \wedge \Diamond A <> \Box A$.

We now propose a new conjecture: the system $AC_{\mathbf{D}}$ made of all the axioms and rules of AC plus

(*Dis*) $\Box(A \wedge B) <> \Box A \wedge \Box B$

(*Dual*) $\neg\Box\neg A <> \Diamond B$

(**D**) $(\Box A \wedge \Diamond B) <> \Box A$

is sound and complete with respect to the inclusive and serial version of the semantic for $AC_\Box$.

Now, given the seriality constraint, it is possible to prove the following result, which is crucial also in order to validate the simplification paradigm:

Lemma 19 *Given a inclusive and serial E-Kripke model* $\mathcal{E} = \langle S, \sqsubseteq, W, R, v^+, v^- \rangle$*, for any formula* A *and any* $w \in W$*,* $|A|^+_w \neq \emptyset$ *and* $|A|^-_w \neq \emptyset$*.*

Proof: see appendix **A.7**.

Notice that, by definition, for any formula A and any inclusive and serial E-Kripke model $\mathcal{E}$ and any world w in A, it is the case that $|A|^+_w \subseteq \overline{|A|}^+_w \subseteq [A]^+_w$ and $|A|^-_w \subseteq \overline{|A|}^-_w \subseteq [A]^-_w$. Hence, by **Lemma 19** the following corollary is readily provable:

Corollary 1 $\overline{|A|}^+_w$*,* $[A]^+_w$*,* $\overline{|A|}^-_w$ *and* $[A]^-_w$ *are non-empty for any formula* A *and any inclusive E-Kripke model.*

Given the results above, we can prove the semantic characterization of analytic containment ($>$) in terms of analytic equivalence (i. e. synonymity, $<>$), in particular:

Lemma 20 *For any inclusive E-Kripke model* $\mathcal{E}$ *and any world* w *in* $\mathcal{E}$*,* $\mathcal{E}, w \models A > B$ *if and only if* $\mathcal{E}, w \models (A \wedge B) <> A$*.*

Proof:

($\Rightarrow$) Consider an arbitrary inclusive and serial E-Kripke model $\mathcal{E}$ and an arbitrary w in $\mathcal{E}$ and assume $\mathcal{E}, w \models A > B$.

Since by **Lemma 19** $\overline{|A \wedge B|}^+_w \neq \emptyset$, consider an arbitrary $s \in \overline{|A \wedge B|}^+_w$; since $s \Vdash_w A \wedge B$, it is the case that $s = u \sqcup t$ such that $u \Vdash_w A$ and $t \Vdash B$, namely $u \in \overline{|A|}^+_w$ Clearly, since by definition $u \sqsubseteq s$ and s was taken arbitrarily in $\overline{|A \wedge B|}^+_w$, it holds that $\overline{|A \wedge B|}^+_w \sqsupseteq \overline{|A|}^+_w$.

Since by **Lemma 19** $\overline{|A|}^+_w \neq \emptyset$, consider an arbitrary $s \in \overline{|A|}^+_w$, it is the case that $s \Vdash_w A$; now consider the formula B. By **Corollary 1**, it is the case that $\overline{|B|}^+_w \neq \emptyset$; hence, take a $t \in \overline{|B|}^+_w$, it is such that $t \Vdash_w B$. By completeness of S, consider $u = s \sqcup t$; since $t \Vdash_w B$ and $s \Vdash_w A$, we have that $u \Vdash_w A \wedge B$, namely $u \in \overline{|A \wedge B|}^+_w$ and, by definition, $s \sqsubseteq u$. Since s was taken arbitrarily in $\overline{|A|}^+_w$, we have that $\overline{|A|}^+_w \sqsubseteq \overline{|A \wedge B|}^+_w$.

Since by **Lemma 19** $\overline{|A|}^+_w \neq \emptyset$, consider an arbitrary $s \in \overline{|A|}^+_w$; it is such that $s \Vdash_w A$. Since by assumption $\mathcal{E}, w \models A > B$, we have that there is a $t \in \overline{|B|}^+_w$ such that $s \sqsupseteq t$. Given that $t \Vdash_w B$, $s \Vdash_w A$ and $t \sqsubseteq s$ we have that $s = t \sqcup s$ and so $s \Vdash_w A \wedge B$, namely $s \in \overline{|A \wedge B|}^+_w$. Since $s \sqsubseteq s$ and s was taken arbitrarily in $\overline{|A|}^+_w$, we have that $\overline{|A|}^+_w \sqsupseteq \overline{|A \wedge B|}^+_w$.

Since by **Lemma 19** $\overline{|A \wedge B|}^+_w \neq \emptyset$, consider an arbitrary $s \in \overline{|A \wedge B|}^+_w$; it is such that $s = t \sqcup u$ for some $t \Vdash_w A$ and $u \Vdash_w B$. Since by assumption $\mathcal{E}, w \models A > B$, we have that there is a $z \in \overline{|A|}^+_w$, namely $z \Vdash_w A$, such that

$u \sqsubseteq z$. Now, by completeness of S, consider $t \sqcup z$; by **Lemma 16** it is the case that $t \sqcup z \Vdash_w A$, namely $t \sqcup z \in \overline{|A|}^{+}_{w}$. Notice that $t \sqsubseteq t \sqcup u$, $u \sqsubseteq z$ and $u \sqsubseteq t \sqcup z$; hence, by definition of least upper bound, it must be the case that $t \sqcup u \sqsubseteq t \sqcup z$, namely $s \sqsubseteq t \sqcup z$. So, since s was taken arbitrarily in $\overline{|A \wedge B|}^{+}_{w}$, we have that $\overline{|A \wedge B|}^{+}_{w} \sqsubseteq \overline{|A|}^{+}_{w}$.

($\Leftarrow$) Consider an arbitrary inclusive E-Kripke model $\mathcal{E}$ and an arbitrary w in $\mathcal{E}$ and assume $\mathcal{E}, w \models (A \wedge B) <> A$.

By assumption, $\mathcal{E}, w \models A > (A \wedge B)$, namely $\overline{|A|}^{+}_{w} \sqsupseteq \overline{|A \wedge B|}^{+}_{w}$; hence, since $\overline{|A|}^{+}_{w} \neq \emptyset$ by **Lemma 19**, consider an arbitrary $s \in \overline{|A|}^{+}_{w}$. It is the case that there is a $t \in \overline{|A \wedge B|}^{+}_{w}$, namely $t \Vdash_w A \wedge B$, such that $s \sqsupseteq t$. Since $t \Vdash_w A \wedge B$, it is the case that $t = u \sqcup z$ for some $u \Vdash_w A$ and $z \Vdash_w B$, namely $z \in \overline{|B|}^{+}_{w}$. Being the case that $z \sqsubseteq t$, then, since s was taken arbitrarily in $\overline{|A|}^{+}_{w}$, we have that $\overline{|A|}^{+}_{w} \sqsupseteq \overline{|B|}^{+}_{w}$.

Now, since $\overline{|B|}^{+}_{w} \neq \emptyset$ by **Lemma 19**, consider an arbitrary $s \in \overline{|B|}^{+}_{w}$, namely $s \Vdash_w B$; since $\overline{|A|}^{+}_{w} \neq \emptyset$ for **Lemma 19**, take an arbitrary $t \in \overline{|A|}^{+}_{w}$, namely $t \Vdash_w A$. Consider $s \sqcup t$, it is such that $s \sqcup t \Vdash_w A \wedge B$, namely $s \sqcup t \in \overline{|A \wedge B|}^{+}_{w}$. By assumption it is the case that $\mathcal{E}, w \models A > (A \wedge B)$, namely $\overline{|A \wedge B|}^{+}_{w} \sqsubseteq \overline{|A|}^{+}_{w}$. Hence there is a $u \in \overline{|A|}^{+}_{w}$ such that $s \sqcup t \sqsubseteq u$, but then, it also holds that $s \sqsubseteq u$. Since s was taken arbitrarily in $\overline{|B|}^{+}_{w}$, we have that $\overline{|B|}^{+}_{w} \sqsubseteq \overline{|A|}^{+}_{w}$.

So far have provided a modal extension of the logic analytic containment from a semantic point of view; we have shown how to conveniently characterize this semantics in terms of inclusive modal truthmaker semantics; we have shown the importance of imposing the seriality constraint over inclusive E-Kripke models and we have proposed a new axiomatization $AC_{\mathbf{D}}$ of the system of modal analytic containment. In the following, we will discuss the properties of such system.

3.1.3 Properties of Modal Analytic Containment

Johannes Korbmacher in (Korbmacher, 2016) has conjectured that his system $AC_{\Box}$ corresponds to a fragment of the logic of K_{FDE}, more specifically, that the validities of $AC_{\Box}$ correspond to a certain class of valid entailments of K_{FDE}. In order to understand the formulation of his conjecture, we first have to introduce some notions.

For every formula A, B we recursively define the *positive*, and *negative*, occurrences of atomic sentences in A: for any p

- p occurs positively in p;
- if p occurs positively (negatively) in A then it occurs negatively (positively) in $\neg A$;

- if p occurs positively (negatively) in A then it occurs positively (negatively) in $A \wedge B$;
- if p occurs positively (negatively) in A then it occurs positively (negatively) in $A \vee B$;

For any formula A, we define the set of its *positive literals* as

- $t^+(A) = \{p : p \text{ occurs positively in } A\}$,

and the set of its *negative literals* as

- $t^-(A) = \{\neg p : p \text{ occurs negatively in } A\}$

For convenience, we write $t(A) \subseteq t(B)$ for $(t^+(A) \subseteq t^+(B)) \wedge (t^-(A) \subseteq t^-(B))$.

Definition 34 *We say that A preserves the valence of B if and only if* $t(B) \subseteq t(A)$.

Preservation of valence has a direct and more intuitive semantic characterization: Fine has shown that for any formula A, B, A preserves the valence of B if and only if A preserves the *partial truth* of B (see (Fine, 2016)).

Partial truth has been first introduced by Humberstone in (Humberstone, 2003) and further developed in (Fine, forthcoming). The intuition behind the notion of partial truth is that a proposition, although not true, can be *partially true* when some parts of it are true, namely it is true with respect to some of its parts. For instance, assume that it is raining and foggy in Amsterdam. The sentence A :"it is raining but not foggy in Amsterdam" is, of course, false, but can be regarded as true with respect to some of its part. In particular, it is true that the weather outside Amsterdam is raining, hence A can be intuitively said to be partially true. Kit Fine goes further in analyzing this notion and claims that

> "One reasonable way to explain the concept [...] is to say that a proposition is partially true if the actual worlds goes some way towards making it true." (Fine, forthcoming)

Taking this as his leading idea, Fine has developed a truthmaker semantic characterization of the notion of partial truth. The question he tries to answer precisely is: what is a partial-truth-maker of a proposition A? Intuitively, following the leading principle mentioned above, it would be something, in the actual world, which partially contributes to the truth of A. Translated into the truthmaker framework, a partial-truth-maker would be something which contributes a non-null part to an exact truthmaker of A. More precisely, this intuition can be formulated as: a partial-truth-maker s of A should have a part s_1 shared with an exact truthmaker of A. Hence, that part would be responsible for the truth of *a part A* and s would indeed be responsible *in part* for the truth of A.

This intuition can be easily formalized within the truthmaker framework and, in fact, Fine ended up with the following characterization of partial-truth-maker: for any formula A, a state s in a model makes A partially true if and only if s overlaps with an exact truthmaker of A.

Analogously we could reason for the *partial falsity* of a formula and end up with the following characterization:

a state s in a model makes A partially false if and only if s overlaps with an exact falsemaker of A.

(alternatively we say that A is partially true (false) at (with respect to) s if and only if s overlaps with an exact truthmaker (falsemaker) of A). By imposing an *overlapping* constraint on the state models, Kit Fine succeeded in providing a recursive characterization of the partial-truthmaker conditions for any formula. The overlapping constraint on a state model $\mathcal{S} = \langle S, \sqsubseteq, v^+, v^- \rangle$ amounts to:

- for any state $s \in S$, is s overlaps with $\bigsqcup T$ ($Os \bigsqcup T$) for some $t \subseteq S$, then s overlaps with come member of T (recall the definition of overlapping in section **1.1.1**)

For any state model $\mathcal{S} = \langle S, \sqsubseteq, v^+, v^- \rangle$ meeting this constraint, for any state $s \in S$, Fine in (Fine, 2016) has shown that.

- $\neg A$ is partially true (false) at s if and only if A is partially false (true) at s;
- $A \wedge B$ is partially true (false) at s if and only if A is partially true (false) at s or B is partially true (false) at s.
- $A \vee B$ is partially true (false) at s if and only if A is partially true (false) at s or B is partially true (false) at s.

From this, the following definition naturally follows:

Definition 35 *A preserves the partial truth of B if and only if for any state s in any state model, if B is partially true at s, then A is partially true at s.*

Kit Fine, in (Fine, 2016), has shown that this semantic notion of preservation of partial truth can be characterized in terms of preservation of valence, namely

Theorem 7 *A preserves the valence of B if and only if A preserves the partial truth of B*

Form this result, in the same paper, Fine has been able to characterize the notion of analytic entailment in terms of partial truth and K_{FDE} entailment:

Theorem 8 $\models_{AC} A > B \Leftrightarrow$ *[$A \models_{FDE} B$ and A preserves the partial truth of B]* where $\models_{AC} A > B$ indicates that $A > B$ is a validity in $AC_{\Box}$.

The need of finding an alternative semantic characterization of analytic entailment comes from the fact that one might want to better understand the novel semantic notion of analytic containment, maybe by appealing to already existing notions of entailment. And, indeed, Fine has been able to show that an analytic entailment amounts to preservation of truth in *FDE* and preservation of partial truth, more specifically: A analytically entails B if and only if B preserves the truth of A under four-valued semantics and A preserves the partial truth of B.

The syntactic characterization of partial truth, has been employed by Rohan French in (French, 2017) to show that the system *AC* corresponds to a specific fragment of the system of *FDE*. Johannes Korbmacher has conjecture in (Korbmacher, 2016) that this characterization can be easily extended to his system $AC_{\Box}$, namely he holds that the following is true

Conjecture 1 $\models_{AC_{\Box}} A > C \Leftrightarrow [A \models_{K_{FDE}}$ *and A preserves the partial truth of B*$]$

or alternatively, by **Conjecture 1**:

Conjecture 2 $\models_{AC_{\Box}} A > C \Leftrightarrow [A \models_{K_{FDE}}$ *and* $t(B) \subseteq t(A)]$

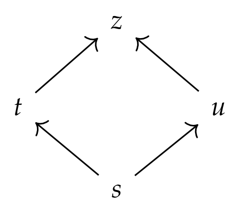
where $t(B) \subseteq t(A)$ is preservation of valence standardly defined (without taking into consideration the occurrences of atomic letter under modal operators).

This conjecture (if true) would tell us, along the line of French's result for non-modal *AC*, that the system $AC_{\Box}$ corresponds to a certain fragment of K_{FDE}, more specifically that the analytic entailments correspond to some class of K_{FDE} entailments, namely those K_{FDE} preserving the valence of the formulas.

However,is possible to find a *counterexample* to the above conjecture: consider the formula $(p \wedge q) > (\Box p \vee q)$. It is the case that $(p \wedge q) \models_{K_{FDE}} (\Box p \vee q)$; indeed, take a K_{FDE} model $\mathcal{M}$ and a world w in $\mathcal{M}$ and assume $w \models (p \wedge q)$. By semantic conditions it is the case that $w \models p$ and $w \models q$, hence $w \models \Box p \vee q$, namely $(p \wedge q) \models_{K_{FDE}} (\Box p \vee q)$. Moreover, by definition, it also holds that $t(\Box p \vee q) \subseteq t(p \wedge q)$. But now consider an inclusive E-Kripke model $\mathcal{E} = \langle S, \sqsubseteq, W, R, v^+, v^- \rangle$ for $AC_{\Box}$ such that:

- $W = \{w, v\}$
- $R = \{(w, v), (w, w)\}$
- $S = \{s, t, u, z\}$
- $v_v^+(p) = \{t\}$
- $v_w^+(q) = \{u\}$
- $v_v^+(q) = \{t\}$
- $v_w^+(p) = \{u\}$

with



Notice that $z \Vdash_w \Box p$ as $u \sqcup t = z$ and $u \Vdash_w p$ and $t \Vdash_v p$, but, clearly, it is not the case that z is part of any truthmaker of $p \wedge q$ at w. Hence, it doesn't hold that $(p \wedge q) > (\Box p \vee q)$ is valid in $AC_\Box$.

So, modal analytic containment cannot be characterize via K_{FDE} entailment and preservation of valence.

However, it seems evident that the failure of the conjecture arises from the characterization of the notion of preservation of valence for the modal case. So, in the following we will try to overcome this problem by finding a suitable modal expansion of the notion of preservation of valence to the modal case, we will do this via a modal extension of the notion of partial truth.

3.1.4 Modal Partial Truth

Our goal now is to find a good characterization of the notion partial truth for the modal case. Given the inclusive and serial $TS_\Box$, it seems natural to extend the semantic characterization of partial truth of a formula by relativizing it to a possible world; we would then have

> a state s in a model makes A partially true (false) with respect to a world w if and only if s overlaps with an exact truthmaker (falsemaker) of A at w.

By imposing the overlapping constraint on the E-Kripke models, it is possible to show that for any E-Kripke model $\mathcal{E} = \langle S, \sqsubseteq, W, R, v^+, v^- \rangle$, any states $s \in S$, any world $w \in W$, we will obtain the following recursive characterization of the partial-truthmaker conditions for any formula:

- $\neg A$ is partially true (false) at s w.r.t. w if and only if A is partially false (true) at s w.r.t. w;
- $A \wedge B$ is partially true (false) at s w.r.t. w if and only if A is partially true (false) at s w.r.t. w or B is partially true (false) at s w.r.t. w.
- $A \vee B$ is partially true (false) at s w.r.t. w if and only if A is partially true (false) at s w.r.t. w or B is partially true (false) at s w.r.t. w.

- $\Box A$ is partially true (false) at s w.r.t. w if and only if A is partially true (false) at s w.r.t. some v such that wRv;
- $\Diamond A$ is partially true (false) at s w.r.t. w if and only if A is partially true (false) at s w.r.t. some v such that wRv.

Proof: by induction

Base Case. Straightforward by definition of modal partial truth.

Inductive Case. $\neg$ (Analogously to the proof in (Fine, 2016).

Inductive Case. $\vee$ (Analogously to the proof in (Fine, 2016).

Inductive Case. $\wedge$ (Analogously to the proof in (Fine, 2016).

Inductive Case. $\Box$

($\models$) Consider an arbitrary inclusive and serial E-Kripke model $\mathcal{E} = \langle W, R, S, \sqsubseteq, v^+, v^- \rangle$ and an arbitrary world w. ($\Rightarrow$) Assume $\Box A$ is partially true at a state s with respect to w. Then s overlaps with an exact verifier t of $\Box A$ at w, namely $t \Vdash_w \Box A$. Since, by semantic conditions, $t = \bigsqcup \bigcup_{(v:wRv)}(\{f(v)\})$ such that $f(v) \Vdash_v A$, for some function f, we have that, by the overlapping condition of $\mathcal{E}$, s also overlaps some $f(v) \in \bigcup_{(v:wRv)}(\{f(v)\})$. Hence, we have that s overlaps some exact verifier of A at v for some $v : wRv$. This means that A is partially true at s w.r.t. v for some $v : wRv$. ($\Leftarrow$) Assume Ost for some t such that $t \Vdash_v A$ for some $v : wRv$. Now, take a function g such that for any $u \in R[w]/\{v\}$, $g(u) \Vdash_u A$ ((we know that such verifier $g(u)$ of A exists by **Lemma 19**) and consider the set $T = \bigcup_{(u \in R[w]/\{v\})}(\{g(u)\})$. Now, consider the set $T \cup \{f(v)\}$ and its least upper bound $z = \bigsqcup(T \cup \{f(v)\})$; consider the function h which is just like g with the only difference that $h(v) = f(v)$; notice that $T \cup \{f(v)\} = \bigcup_{v:wRv}\{h(v)\}$, hence, by semantic conditions, we have that $z \Vdash_w \Box$. Since $t \sqsubseteq z$ and Ost, it also holds that Osz, namely $\Box A$ is partially true at s w.r.t. w.

($\dashv$) Consider an arbitrary inclusive and serial E-Kripke model $\mathcal{E} = \langle W, R, S, \sqsubseteq, v^+, v^- \rangle$ and a world w ($\Rightarrow$) Assume $\Box A$ is partially false at a state s with respect to w. Then s overlaps with an exact falsifier t of $\Box A$ at w, namely $t \dashv\!\vert_w \Box A$. Now we have two cases to consider: (i) there is some v such that wRv and $t \dashv\!\vert_v A$; (ii) for a non-empty $Y \subseteq R[w]$, there is a function f such that for any $v \in Y$, $f(v) \dashv\!\vert_v A$ and $s = \bigsqcup \bigcup_{(v \in Y)}(\{f(v)\})$. If (i) holds, then, since Ost, it is also the case that s partially verifies A w.r.t. some v such that wRv, since $t \dashv\!\vert_v A$ for some $v : wRv$. If (ii) holds, then we can reason analogously to the ($\models$) case, and show that s overlaps with some exact falsifier of A w.r.t. some $v : wRv$. ($\Leftarrow$) Assume Ost for some t such that $t \dashv\!\vert_v A$ for some $v : wRv$. We can reason analogously to the ($\models$) case.

Inductive Case. $\Diamond$

($\models$) We can reason analogously to the $\dashv$ case of $\Box$.

($\dashv$) We can reason analogously to the $\models$ case of $\Box$.

We can now define a relation of preservation of *modal* partial truth:

Definition 36 *For all formulas A, B we say that B preserves the partial truth of A if and only if [for any inclusive and serial E-Kripke model meeting the overlap constraint $\mathcal{E}$, any state s and any world w in $\mathcal{E}$, if A is partially true at s w.r.t. w, then B is partially true at s w.r.t. w.]*

It is interesting to notice that for the $\Box$ case we have a characterization of partial truth in terms of *existential* quantification over accessible worlds. This is not surprising. We could think of $\Box A$ within the truthmaker framework as a sort of long conjunction, indeed, recall the original intuition by van Fraassen's: "Necessarily A is true if and only if A is true in world α_1 and in world α_2 and so forth." Not surprisingly, the truthmaker conditions of $\Box A$ resembles the one for conjunctions. From this perspective, given that a conjunction is partially true only if one of its conjunct is partially true, it seems natural to think that *some* worlds suffice for making $\Box A$ partial true, namely some *parts* of the conjunction that $\Box A$ is intended to represent are true

However, given this characterization of the partial truth of $\Box A$, some counterintutive situations arise. Suppose we have an E-Kripke model $\mathcal{E}$ (meeting the overlapping constraint) in which a world w has no successor. Clearly, by semantic conditions, a formula $\Box A$ would be true at the null state 0 with respect to w, but there is no state $s \neq 0$ in $\mathcal{E}$ such that $\Box A$ is true at s with respect to w. Moreover notice that 0 technically doesn't overlap with any state since the definition of the overlap relation requires the sharing of a non-null part (see section **1.1.1**). Hence, since no state overlaps with 0, $\Box A$ is not partially true at any state with respect to w. Namely, $\Box A$, although being true (at 0), is not partially true. This seems a counterintuitive result.

At first glance, it seems intuitively to accept the principle that every formula which is true is also partially true (let us denote this principle as "*truth implies partial truth*"). In the light of this, the example given above would count as an undesired consequence of our characterization of the notion of partial truth. Instead, in the following, will argue for the fact that $\Box A$ being not partially true although true shouldn't be worrying as the intuitive principle that *truth implies partial truth* is not universally acceptable. In order to show this, we will follow the argument exposed by Kite in (Fine, forthcoming): holding the principle that every true formula is also partially true commits us to the weird consequence that every formula is partially true. In fact:

(1) assume the principle of *truth implies partial truth*;

(2) consider a world w with no successor and in which it is not the case that it is raining in Amsterdam;

(3) consider a sentence A which is identical (has the same truthmakers as) to $\varphi \wedge \top$;

(4) since $\top$ is vacuously true in any world, then, by (1) $\top$ would be partially true at w.

(5) by (4) and definition of modal partial truth $A \wedge \top$ will be partially true in w as well;

(6) by (3) A is partially true.

Now, repeat the argument above by considering the case in which A is the proposition "it is raining in Amsterdam". It follows that A is partially true at w although in w. However, in w it is not the case that it is raining in Amsterdam: how can A be partially true at w? It seems there is no way w goes to make A true, hence no way that A can be partially true at w.

Along the same line as Fine, we believe that the argument above shows the universal validity of the intuitive principle that *truth implies partial truth* commits us to weird consequence. In order to avoid this kind of consequences, we should reject the validity of the principle so that the argument above is not triggered anymore (in particular, step (4) would not follow).

We believe that the the principle is not valid for a certain class of truths, more specifically, for the class of truths of the form of $\top$. Truths like $\top$ has the feature of being *vacuously* true, in the sense that *nothing* is required to make them true: there is no specific way the world goes towards to make them true, since they do not require such way. The example of $\Box A$ being true at w, with w having no successor, falls into the same class: $\Box A$ is *vacuously* true at w as there is no accessible world from w. The *vacuous* feature is also reflected in the fact that the only truthmaker of $\Box A$ at w is the null state 0 which requires nothing in order for it to obtain. Indeed, nothing substantially is required for the truth of $\Box A$ at w. There is no particular way the world w goes that is responsible for the truth of $\Box A$.

On the other hand, it seems that the notion of partial truth requires something substantial: there must be *some* way the world goes towards making a proposition partially true. But vacuous truths like $\top$ or $\Box A$ at w require no such way. This implies that trivially verified truths cannot be partially true. Hence, under this substantial (and correct) understanding of the notion of partial truth, it seems that the principle of *truth implies partial truth* is not valid and so, cases like $\Box A$ being true but not partially true at w do not represent anymore an undesired consequence of our semantic characterization of partial truth.

3.1.5 Characterizations

In this section we will discuss whether it is possible to extend **Theorem 8** to $AC_{\mathbf{D}}$ and our new notion of modal partial truth. In order to do this some new characterization result is needed; in particular, we will try to prove van Fraassen's theorem for the inclusive and serial version of the semantics. Notice, in fact that it is not straightforward to extend the proof strategy of **Theorem 3** to the new inclusive and serial version of the semantics; indeed the problems will arise from the fact that the operation of exactification (see **Definition 11**) over ordinary K_{FDE} Kripke model does not return necessarily an inclusive and serial Kripke model meeting the overlapping constraint. For instance consider an ordinary K_{FDE} Kripke model $\mathcal{M} = \langle W, R, a^+, a^- \rangle$ in which a certain world $w \in W$ is such that for some propositional letter p, $w \notin a^+(p)$ and $w \notin a^-(p)$. This would imply that $\mathbf{E}(\mathcal{M}) = \langle S, \sqsubseteq, W, R, v^+, v^- \rangle$ is not inclusive since $v^+_w(p) = \emptyset = v^-_w(p)$ (recall **Definition 11**).

However, in **A.9** we outline a proof strategy used by Korbmacher to show that:

Theorem 9 *$A \Vdash_{\mathbf{D}_i} B$ if and only if $A \models_{\mathbf{D}_{FDE}} B$*

where $A \Vdash_{\mathbf{D}_i} B$ means that for any inclusive and serial E-Kripke model meeting the overlapping constraint $\mathcal{E}$ and any state s ad world w in $\mathcal{E}$, $\mathcal{E}, s \Vdash_w A$ implies $\mathcal{E}, \mathcal{E}, s \Vdash_w B$; and $A \models_{\mathbf{D}_{FDE}}$ means that for any serial K_{FDE} Kripke model $\mathcal{M}$ and any world w in $\mathcal{M}$, $\mathcal{M}, w \models A$ implies $\mathcal{M}, w \models B$.

Proof: see appendix **A.9**.

We will now prove the following result, which correspond to a modal extension of **Theorem 8**:

Theorem 10 *$\models_{AC_{\mathbf{D}}} A > B$ if and only if [$A \models_{\mathbf{D}_{FDE}} B$ and A preserves modal partial truth of B]*

Proof:

(⇒) Assume $\models_{AC_{\Box}} A > B$, then we have for any inclusive and serial E-Kripke model $\mathcal{E}$ meeting the overlap constraint and any s, w in the model: (i) $\overline{|A|}^+_w \sqsupseteq \overline{|B|}^+_w$ and (ii) $\overline{|B|}^+_w \sqsubseteq \overline{|A|}^+_w$. Consider an arbitrary inclusive and serial model $\mathcal{E}$ meeting the overlap constraint and arbitrary w in $\mathcal{E}$, by assumption, it is the case that:

(i) $\overline{|A|}^+_w \sqsupseteq \overline{|B|}^+_w$. Consider an arbitrary s such that s inexactly verifies A, $s \Vdash_w A$ (under the inclusive semantics); then there is a t such that $t \sqsubseteq s$ and $t \Vdash_w A$, namely $t \in \overline{|A|}^+_w$. Hence, since $\overline{|A|}^+_w \sqsupseteq \overline{|B|}^+_w$ by assumption, it must be the case that there is a u such that $u \Vdash_w B$ and $u \sqsubseteq t$; by transitivity it also holds that $u \sqsubseteq s$, namely, s inexactly verifies B. Since $\mathcal{E}$, s and w were taken arbitrarily, we have that $A \Vdash_{\mathbf{D}_i} B$. Hence, by **Theorem 9**, we have that $A \models_{\mathbf{D}_{FDE}} B$.

(ii) $\overline{|B|}^+_w \sqsubseteq \overline{|A|}^+_w$. Consider an arbitrary s such that s partially verifies B, namely there is a t such that $t \Vdash_w B$ and Ost. This implies that $t \in \overline{|B|}^+_w$; hence, since by assumption $\overline{|B|}^+_w \sqsubseteq \overline{|A|}^+_w$, there is a u such that $u \Vdash_w A$

and $t \sqsubseteq u$. Since Ost, and $t \sqsubseteq u$, it is also the case that Osu, namely A is partially true at s w.r.t. w. Since $\mathcal{E}, s$ and w were taken arbitrarily, we have that A preserves the partial truth of B.

($\Leftarrow$) Assume (a) $A \models_{D_{FDE}} B$ and that (b) A preserves the modal partial truth of A. Since (a) then, by **Theorem 9**, $A \Vdash_D B$. By contraposition, assume that there is an inclusive and serial E-Kripke model $\mathcal{E} = \langle S, \sqsubseteq, W, R, v^+, v^- \rangle$ satisfying the overlapping constraint and a s and a w in $\mathcal{E}$ such that $s \Vdash_w B$ but for no t such that $s \sqsubseteq t$, $t \Vdash_w A$. Now consider the model $\mathcal{E}^* = \langle S^*, \sqsubseteq^*, W, R, v^+, v^- \rangle$ where W, R, v^+ and v^- are the same as in $\mathcal{E}$ and

- $S^* = S \cup \{z\}$ (where for any $s \in S$, $z \neq s$)
- $\sqsubseteq^* = \sqsubseteq \cup \{(0, z)\} \cup \{(z, t) : s \sqsubseteq t\}$ ($\sqsubseteq^*$ is a partial order).

where 0 is the null state. We can show that for $s \in S$ and $w \in W$, $\mathcal{E}, s \Vdash_w B$ iff $\mathcal{E}^*, s \Vdash_w B$. Now, notice that in $\mathcal{E}^*$, z overlaps only with states having s among their parts and by assumption there is no state t such that $s \sqsubseteq t$ and $t \Vdash_w A$. Hence, z doesn't overlap with any truthmaker of A, namely A is not partially true at z w.r.t. w. Moreover notice that $s \Vdash_w B$ by assumption, and so B is partially true at z w.r.t. w. So, A doesn't preserve the modal partial truth of B.

3.1.6 Soundness

In the following, we will prove soundness of modal $AC_{\mathbf{D}}$ with respect to the semantics presented above, namely we will prove the validity of the following axioms:

(Dis) $\Box(A \wedge B) <> (\Box A \wedge \Box B)$

($Dual$) $\neg\Diamond\neg A <> \Box A$

(***D***) $(\Box A \wedge \Diamond A) <> \Box A$ (or equivalently, by **Lemma 20**, $(\Box A > \Diamond A)$)

The rest of the proof is analogous to the soundness proof in (Fine, 2016).

(Dis) For any E-Kripke model $\mathcal{E}$, $\mathcal{E} \models \Box(A \wedge B) <> (\Box A \wedge \Box B)$

Proof:

Consider an arbitrary inclusive and serial E-Kripke model (meeting the overlapping constraint) $\mathcal{E} = \langle S, \sqsubseteq, W, R, v+, v- \rangle$, consider a world $w \in W$

(>) (i) take an $s \in S$ such that $s \in \overline{|\Box(A \wedge B)|}^+_w$. It must be the case that $s \Vdash_w \Box(A \wedge B)$; by inclusive semantic conditions $s = \bigsqcup(\bigcup_{wRv}\{f(v)\})$ with, for any v such that wRv, $f(v) = t \sqcup u$ and $t \Vdash_v A$ and $u \Vdash_v B$; 8Consider the set $\bigcup_{wRv}\{f(v)\}$ and call it X; now, consider the function

$g : X \to S$ which maps every element $f(v)$ in X with its component u such that $u \Vdash_v A$. Take the set $\bigcup_{f(v)\in X}\{g(f(v))\}$; notice that $f \circ g$ is a function from W to S and, by construction, for any v such that wRv, $g(f(v)) \Vdash_v A$. Hence, $\bigsqcup(\bigcup_{f(v)\in X}\{g(f(v))\}) \Vdash_w \Box A$; for convenience, call Y the set $\bigcup_{f(v)\in X}\{g(f(v))\}$. We can reason analogously with B and construct a function h analogous to g such that $\bigsqcup(\bigcup_{f(v)\in X}\{h(f(v))\}) \Vdash_w \Box B$; for convenience, call Z the set $\bigsqcup(\bigcup_{f(v)\in X}\{h(f(v))\})$. Notice that, by semantic conditions, $\bigsqcup Y \sqcup \bigsqcup Z \Vdash_w \Box A \wedge \Box B$, namely $\bigsqcup Y \sqcup \bigsqcup Z \in \overline{|\Box A \wedge \Box B|_w^+}$. Moreover, notice that for any v such that wRv, $g(f(v)) \sqsubseteq f(v)$ and $h(f(v)) \sqsubseteq f(v)$, hence $\bigsqcup Y \sqsubseteq s$ and $\bigsqcup Z \sqsubseteq s$. This means that $(\bigsqcup Y \sqcup \bigsqcup Z) \sqsubseteq s$. Since s was taken arbitrarily, we have that it holds in $\mathcal{E}$ that $\overline{|\Box(A \wedge B)|_w^+} \sqsupseteq \overline{|(\Box A \wedge \Box B)|_w^+}$.

(ii) Consider a $s \in S$ such that $s \in \overline{|\Box A \wedge \Box B)|_w^+}$. It must be the case that $s \Vdash_w \Box A \wedge \Box B$, namely, by semantic conditions, $s = t \sqcup u$ with $t \Vdash_w \Box A$ and $u \Vdash_w \Box B$. By semantic conditions, $t = \bigsqcup(\bigcup_{wRv}\{f(v)\})$ with, for any v such that wRv, $f(v) \Vdash_v A$ and $u = \bigsqcup(\bigcup_{wRv}\{g(v)\})$ with, for any v such that wRv, $g(v) = t \Vdash_v B$. Consider the sets $\bigcup_{wRv}\{f(v)\}$ and $\bigcup_{wRv}\{g(v)\}$ and call them respectively X and Y. Now, consider the set $Z = \{g(v) \sqcup f(v) : g(v) \in Y \text{ and } f(v) \in X \text{ and } wRv\}$; notice that, by semantic conditions, for any v such that wRv, $g(v) \sqcup f(v) \Vdash_v A \wedge B$. Consider the function $h : W \to S$ such that $h(v) = g(v) \sqcup f(v)$. Notice that it is the case that for any wRv, $h(v) \Vdash_v A \wedge B$. Take the set $\bigcup_{wRv}\{h(v)\}$; notice that by semantic conditions $\bigsqcup(\bigcup_{wRv}\{h(v)\}) \Vdash_w \Box(A \wedge B)$, namely $\bigsqcup(\bigcup_{wRv}\{h(v)\}) \in \overline{|\Box(A \wedge B)|_w^+}$. Moreover, since for any v such that wRv, $g(v) \sqsubseteq g(v) \sqcup f(v)$ and $f(v) \sqsubseteq g(v) \sqcup f(v)$, hence $t \sqsubseteq \bigsqcup(\bigcup_{wRv}\{h(v)\})$ and $u \sqsubseteq \bigsqcup(\bigcup_{wRv}\{h(v)\})$. This means that $s \sqsubseteq \bigsqcup(\bigcup_{wRv}\{h(v)\})$. Since s was taken arbitrarily, we have that it holds in $\mathcal{E}$ that $\overline{|(\Box A \wedge \Box B)|_w^+} \sqsubseteq \overline{|\Box(A \wedge B)|_w^+}$.

Since w was taken arbitrarily, we have that $\overline{|(\Box A \wedge \Box B)|_w^+} \sqsubseteq \overline{|\Box(A \wedge B)|_w^+}$ and $\overline{|\Box(A \wedge B)|_w^+} \sqsupseteq \overline{|(\Box A \wedge \Box B)|_w^+}$ for any w.

(<) The proof proceeds analogously to the > case. Just notice that in case (ii) for any v such that wRv, $g(v) \sqcup f(v) \sqsubseteq s$ since $f(v) \sqsubseteq s$ and $g(v) \sqsubseteq s$ by definition of s; and so $s \sqsupseteq \bigsqcup(\bigcup_{wRv}\{h(v)\})$.
And in case
(i) for any v such that wRv, $f(v) \sqsubseteq \bigsqcup Y \sqcup \bigsqcup Z$, since $f(v) = g(f(v)) \sqcup h(f(v))$ and $g(f(v)) \sqsubseteq Y$ and $h(f(v)) \sqsubseteq Z$. And so $s \sqsubseteq \bigsqcup Y \sqcup \bigsqcup Z$.

Since $\mathcal{E}$ was taken arbitrarily, we have that *Dis* is a validity of $AC_{\mathbf{D}}$.

(*Dual*) For any E-Kripke model $\mathcal{E}$, $\mathcal{E} \models \neg\Diamond\neg A <> \Box A$

Consider an arbitrary inclusive and serial E-Kripke model (meeting the overlapping constraint) $\mathcal{E} = \langle S, \sqsubseteq, W, R, v+, v-\rangle$, consider a world $w \in W$

(>) take an $s \in S$ such that $s \in \overline{|\neg\Diamond\neg A|}^{+}_{w}$. It must be the case that $s \Vdash_w \neg\Diamond\neg A$, namely, by semantic conditions, $s \Dashv_w \Diamond A$ that means $s = \bigsqcup(\bigcup_{wRv}\{f(v)\})$ with, for any v such that wRv, $f(v) \Dashv_w \neg A$, namely $f(v) \Vdash_w A$. Hence, by semantic conditions, it is also the case that $s \Vdash_w \Box A$ and since by reflexivity $s \sqsubseteq s$, we have that $\overline{|\neg\Diamond\neg A|}^{+}_{w} \sqsupseteq \overline{|\Box A|}^{+}_{w}$.

Now, consider an $s \in S$ such that $s \in \overline{|\Box A|}^{+}_{w}$; by semantic conditions it is the case that $s = \bigsqcup(\bigcup_{wRv}\{f(v)\})$ with, for any v such that wRv, $f(v) \Vdash_w A$, namely $f(v) \Dashv_w \neg A$. Hence, by semantic conditions, it is also the case that $s \Vdash_w \neg\Diamond\neg A$ and, since by reflexivity $s \sqsubseteq s$, we have that $\overline{|\Box A|}^{+}_{w} \sqsubseteq \overline{|\neg\Diamond\neg A|}^{+}_{w}$.

(<) This direction proceeds analogously to >.

Since w was taken arbitrarily, we have that $\overline{|\Box A|}^{+}_{w} \sqsubseteq \overline{|\neg\Diamond\neg A|}^{+}_{w}$ and $\overline{|\neg\Diamond\neg A|}^{+}_{w} \sqsupseteq \overline{|\Box A|}^{+}_{w}$ for any w.

Since $\mathcal{E}$ was taken arbitrarily, we have that *Dual* is valid in $AC_{\mathbf{D}}$

(**D**) For any E-Kripke model $\mathcal{E}$, $\mathcal{E} \models \Box A > \Diamond A$

Consider an arbitrary inclusive E-Kripke model $\mathcal{E} = \langle S, \sqsubseteq, W, R, v+, v-\rangle$, consider a world $w \in W$

(>) take an $s \in S$ such that $s \in \overline{|\Box A|}^{+}_{w}$. It must be the case that $s \Vdash_w \Box A$ under inclusive semantics, that means $s = \bigsqcup(\bigcup_{wRv}\{f(v)\})$ with, for any v such that wRv, $f(v) \Vdash_v A$. Consider an arbitrary v such that wRv; we know that such v exists by the seriality of R. Hence by semantic conditions, $f(v) \Vdash_v A$ under inclusive semantics; this implies that the state $f(v)$ is such that $f(v) \Vdash_w \Diamond A$ (it exists by **Lemma 19**) under inclusive semantics, since wRv. Notice that $s = \bigsqcup(\bigcup_{wRv}\{f(v)\})$, and so $f(v) \sqsubseteq s$. Namely $\overline{|\Box A|}^{+}_{w} \sqsupseteq \overline{|\Diamond A|}^{+}_{w}$.

Now, consider an $s \in S$ such that $s \in \overline{|\Diamond A|}^{+}_{w}$; by semantic conditions, it is the case that $s \Vdash_w \Diamond A$ under inclusive semantics, namely (i) $s \Vdash_v A$ or (ii) for a non-empty $Y \subseteq R[w]$, for any $v \in R[w]$, $f(v) \Vdash_v A$ and $s = \bigsqcup(\bigcup_{v \in R[w]}\{f(v)\})$. Assume (i) holds; by **Lemma 19**, we can infer that for any u such that $u \neq v$ and wRu, it is the case that $|A|^{+}_{u} \neq \emptyset$. So, consider a function from the set of worlds to S such that $f(v) = s$ and for any $u \neq v$ such that wRv, $f(u) \Vdash_u A$ (by **Lemma 19** such $f(u)$ must exists). Consider the set $(\{f(u) : u \neq v \wedge wRu\}) \cup \{s\}$. Clearly $t = \bigsqcup((\{f(u) : u \neq v \wedge wRu\}) \cup \{s\})$ exists and by semantic conditions $t \Vdash_w \Box A$. Notice also that by construction $s \sqsubseteq t$. Assume (ii) holds, we can reason analogously to (i) construct f in such a way that $f(v) \Vdash_v A$ for all $v \in Y$ and obtain that $t = \bigsqcup(((\bigcup\{f(u) : (\forall v \in Y(u \neq v \wedge wRu)\}) \cup \{s\})$; so $s \sqsubseteq t$. Since s was taken arbitrarily, we have that $\overline{|\Diamond A|}^{+}_{w} \sqsubseteq \overline{|\Box A|}^{+}_{w}$.

Since w was taken arbitrarily we have that $\overline{|\Box A|}^{+}_{w} \sqsupseteq \overline{|\Diamond A|}^{+}_{w}$ and $\overline{|\Diamond A|}^{+}_{w} \sqsubseteq \overline{|\Box A|}^{+}_{w}$ for any w.

Since $\mathcal{E}$ was taken arbitrarily, we have that *Dual* is valid in $AC_{\mathbf{D}}$.

4 Conclusions and Further Work

We believe that this thesis could serve as a programmatic work for the development of a modal truthmaker semantics. We think that Johannes Korbmacher has provided us with a very intuitive framework to analyse the truthmaker conditions of modal statements; moreover, it is also coherent with the original Van Fraasen's idea of a truthmaker of a sentence of the form "necessarily A". Indeed, we have proved how some nice results about Korbmacher's semantics, such as the characterization of inexact consequence in terms of first-degree entailment, are preserved under a natural first-order extension of the semantics. Moreover, we hope that the philosophical intuitions we worked out in the second section could serve as leading ideas to develop a more uniform and systematic conception of truthmakers for modal truths: it is of a great importance to go deeper in the analysis of the *at* relation in order to have a clearer picture on the nature of truthmakers of modal truths. We believe that the relation that a fact entertains with the world in which it obtains (in a very broadly sense) plays a key role in the truthmaking relation and must be taken into consideration in the definition of a truthmaker for a modal sentence. Finally, we hope that the extension of the logic of analytic containment by means of modal truthmaker semantics can find a more systematic treatment; we have shown some key results in this direction, like the importance of a seriality constraint on the models and the characterization of the notion of modal partial truth, and we have discussed some characterization results about the logic of modal analytic containment; we also have proposed an axiomatization of such system and proved that it is sound with respect to the semantics we developed.

Hence, points for further work in this direction could be: (i) proving, if possible, completeness of $AC_{\mathbf{D}}$; (ii) developing a modal extension of the logic of exact entailment in (Fine & Jago, 2017).

A Formal Appendix

A.1 Theorem 2

In the following, we will define new operations to transform each four-valued model into an ordinary K_{FDE} model and viceversa:

Definition 37 *Given a four-valued model $\mathcal{F} = \langle W, R, S_L, v\rangle$, we define its* standardification $\boldsymbol{S}(\mathcal{F}) = \langle W', R', a^+, a^-\rangle$ *where*

- $W' = W$
- $R' = R$
- $a^+(p) = \{w : v_w(p) \in D\}$
- $a^-(p) = \{w : v_w(p) \in \{b, 0\}\}$.

Where D is a subset of the set of the four values $\{1, b, n, 0\}$, $D = \{1, b\}$.

Definition 38 *Given an ordinary K_{FDE} model, $\mathcal{M} = \langle W, R, a^+, a^-\rangle$, we define its* generalization $\boldsymbol{G}(\mathcal{M}) = \langle W', R', S_L, v\rangle$ *where*

- $W' = W$;
- $R' = R$;
- *S_L is defined in the standard way (see section **1.3**);*
- $v_w(p) = \begin{cases} 1 & \text{if } [w \in a^+(p) \text{ and } w \notin a^-(p)] \\ b & \text{if } [w \in a^+(p) \text{ and } w \in a^-(p)] \\ n & \text{if } [w \notin a^+(p) \text{ and } w \notin a^-(p)] \\ 0 & \text{if } [w \notin a^+(p) \text{ and } w \in a^-(p)] \end{cases}$

We will now prove the following lemmas:

Lemma 21 *Given a four-valued model $\mathcal{F} = \langle W, R, S_L, v\rangle$, for any formula A and any world w in $\mathcal{F}$,*

$$[\mathcal{F}, v_w(A) \in D \text{ if and only if } \boldsymbol{S}(\mathcal{F}), w \models A] \text{ and } [\mathcal{F}, v_w(A) \in \{b, 0\} \text{ if and only if } \boldsymbol{S}(\mathcal{F}), w \dashv A\ ].$$

Proof: by induction

Base Case.

($\models$) Consider an arbitrary four-valued model $\mathcal{F}$ and its standardification $\mathbf{S}(\mathcal{F})$. For an arbitrary w ($\Rightarrow$) assume $\mathcal{F}, v_w(p) \in D$; by definition of $\mathbf{S}(\mathcal{F}) = \langle W, R, a^+, a^- \rangle$, since $v_w(p) \in D$, then $w \in a^+(p)$, hence $\mathbf{S}(\mathcal{F}), w \models p$. ($\Leftarrow$) Assume $\mathbf{S}(\mathcal{F}), w \models p$, then by semantic conditions $w \in a^+(p)$. By definition of $\mathbf{S}(\mathcal{F}) = \langle W, R, a^+, a^- \rangle$, since $w \in a^+(p)$, then $\mathcal{F}, v_w(p) \in D$.

($\dashv$) Analogously we reason for $\mathcal{F}, v_w(p) \in \{b, 0\}$ and $w \dashv p$.

Inductive Case $\neg$.

($\models$) Consider an arbitrary four-valued model $\mathcal{F}$ and its standardification $\mathbf{S}(\mathcal{F})$. For an abitrary w, ($\Rightarrow$) assume $\mathcal{F}, v_w(\neg A) \in D$ for an arbitrary w; by definition of v_w and $f_\neg$, since $v_w(\neg A) \in D$, then $v_w(A) \in \{0, b\}$. Then by IH, it is the case that $\mathbf{S}(\mathcal{F}), w \dashv A$, and so, by semantic conditions, $\mathbf{S}(\mathcal{F}), w \models \neg A$ ($\Leftarrow$) assume $\mathbf{S}(\mathcal{F}), w \models \neg A$, then by semantic conditions $w \dashv A$. So, by IH $v_w(A) \in \{b, 0\}$ and by definition of $v_w(\neg A)$ and $f_\neg$, it is the case that $v_w(\neg A) \in D$.

($\dashv$) Analogously we reason for $\mathcal{F}, v_w(\neg A) \in \{b, 0\}$ and $w \dashv \neg A$.

Inductive Case $\wedge$.

($\models$) Consider an arbitrary four-valued model $\mathcal{F}$ and its standardification $\mathbf{S}(\mathcal{F})$. For an arbitrary w ($\Rightarrow$) Assume $\mathcal{F}, v_w(A \wedge B) \in D$ for an arbitrary w; by definition of v_w and $f_\wedge$, since $v_w(A \wedge B) \in D$, then it must be the case that $v_w(A) \in D$ and $v_w(A) \in D$, otherwise if one between A or B is such that its value at w is n or 0, then the $Glb(\{v_w(A), v_w(B)\}$, namely $v_w(A \wedge B)$ is not in D. So, by IH, $\mathbf{S}(\mathcal{F}), w \models A$, and $\mathbf{S}(\mathcal{F}), w \models B$ and by semantic conditions, $\mathbf{S}(\mathcal{F}), w \models A \wedge B$. ($\Leftarrow$) Assume $\mathbf{S}(\mathcal{F}), w \models A \wedge B$, then by semantic conditions $w \models A$ and $w \models B$. So, by IH $v_w(A) \in D$ and $v_w(B) \in D$. By definition of $v_w(A \wedge B)$ and $f_\wedge$, it is the case that $v_w(A \wedge B) \in D$.

($\dashv$) Consider an arbitrary four-valued model $\mathcal{F}$ and its standardification $\mathbf{S}(\mathcal{F})$. For an arbitrary w ($\Rightarrow$) assume $\mathcal{F}, v_w(A \wedge B) \in \{b, 0\}$ for an arbitrary w; by definition of v_w and $f_\wedge$, since $v_w(A \wedge B) \in \{b, 0\}$, then (i) $v_w(A) \in \{b, 0\}$ or (ii) $v_w(A) \in \{b, 0\}$, otherwise if this is not the case, then the $Glb(\{v_w(A), v_w(B)\})$, namely $v_w(A \wedge B)$, is not in $\{b, 0\}$. If (i) holds then by IH $\mathbf{S}(\mathcal{F}), w \dashv A$, hence $w \dashv A \wedge B$; analogously if (ii) holds. ($\Leftarrow$) Assume $\mathbf{S}(\mathcal{F}), w \dashv A \wedge B$, then by semantic conditions, $w \dashv A$ or $w \dashv B$. If the former holds, then by IH $v_w(A) \in \{b, 0\}$; hence, the Glb between $v_w(A)$ and any other value must be in $\{b, 0\}$ and so $v_w(A \wedge B) \in \{b, 0\}$; analogously if the latter holds.

Inductive Case $\vee$.

($\models$) For $\models$ and D, we reason analogously to the case of $\dashv$ and $\{b, 0\}$ for $\wedge$.

($\dashv$) For $\dashv$ and $\{b, 0\}$ we reason analogously to the case of $\models$ and D of $\wedge$

Inductive Case $\Box$.

($\models$) Consider an arbitrary four-valued model $\mathcal{F}$ and its standardification $\mathbf{S}(\mathcal{F})$. For an arbitrary w ($\Rightarrow$) assume $\mathcal{F}, v_w(\Box A) \in D$; by definition of $v_w(\Box A)$ and *Glb*, since $v_w(\Box A) \in D$, then it must be the case that $v_t(A) \in D$ for any t such that wRt. Take, if any, an arbitrary t such that wRt; by IH it must be the case that $t \models A$, hence, by semantic conditions, since t was taken arbitrarily, we have that $w \models \Box A$. ($\Leftarrow$) Assume $\mathbf{S}(\mathcal{F}), w \models \Box A$, then by semantic conditions, for any t such that wRt, $t \models A$. Take, if any, an arbitrary world t such that wRt; then by IH $v_t(A) \in D$; hence, since t was taken arbitrarily, we have that $v_t(A) \in D$ for any t such that wRt; hence, by definition of *Glb* it must be the case that $v_w(\Box A) \in D$.

($=\!|$) Consider an arbitrary four-valued model $\mathcal{F}$ and its standardification $\mathbf{S}(\mathcal{F})$. For an arbitrary w ($\Rightarrow$) Assume $\mathcal{F}, v_w(\Box A) \in \{b, 0\}$; by definition of $v_w(\Box A)$ and *Glb*, since $v_w(\Box A) \in \{b, 0\}$, then it must be the case that there is a t such that wRt and $v_t(A) \in \{b, 0\}$. Take such t: by IH it must be the case that $t =\!| A$, hence, by semantic conditions, since wRt, we have that $w =\!| \Box A$. ($\Leftarrow$) Assume $\mathbf{S}(\mathcal{F}), w =\!| \Box A$, then by semantic conditions, there is a t such that wRt and $t =\!| A$. Take such t: by IH $v_t(A) \in \{b, 0\}$; hence, since $t : wRt$ and $v_t(A) \in \{b, 0\}$, by definition of *Glb*, it is the case that $v_w(\Box A) \in \{b, 0\}$.

Inductive Case $\Diamond$.

($\models$) For $\models$ and D we reason analogously to the case of $=\!|$ and $\{b, 0\}$ for $\Box$.

($=\!|$) For $=\!|$ and $\{b, 0\}$ we reason analogously to the case of $\models$ and D for $\Box$.

Lemma 22 *Given an ordinary K_{FDE} model $\mathcal{M} = \langle W, R, a^+, a^- \rangle$, for any formula A and any world w in $\mathcal{M}$,*

$$[\mathcal{M}, w \models A \text{ if and only if } \mathbf{G}(\mathcal{M}), v_w(A) \in D] \text{ and } [\mathcal{M}, w =\!| A \text{ if and only if } \mathbf{G}(\mathcal{M}), v_w(A) \in \{b, 0\}]$$

Proof: by induction

Base Case.

($\models$) Consider an arbitrary ordinary K_{FDE} model $\mathcal{M}$ and its generalization $\mathbf{G}(\mathcal{M})$. For an arbitrary w ($\Rightarrow$) Assume $\mathcal{M}, w \models p$; by definition of $\mathbf{G}(\mathcal{M}) = \langle W, R, a^+, a^- \rangle$, since $w \models p$, then it is the case that $v_w(p) = 1$ or $v_w(p) = b$, hence $\mathbf{G}(\mathcal{M}), v_w(p) \in D$. ($\Leftarrow$) Assume $\mathbf{G}(\mathcal{M}), v_w(p) \in D$; hence either $v_w(p) = 1$ or $v_w(p) = b$. Then by definition of $\mathbf{G}(\mathcal{M})$, in both cases it holds that $\mathcal{M}, w \models p$.

($=\!|$) Analogously for $w =\!| A$ and $v_w(A) \in \{b, 0\}$.

The rest of the proof proceeds analogously to the proof of the previous lemma.

We are now ready to prove **Theorem 2**:

$\Gamma \models^4_{K_{FDE}} B$ if and only if $\Gamma \models_{K_{FDE}} B$.

Proof:

($\Rightarrow$) By contrapositio: assume it is not the case that $\Gamma \models_{K_{FDE}} \Delta$, hence there must be some some ordinary K_{FDE} model $\mathcal{M} = \langle W, R, a^+, a^- \rangle$ and some $w \in W$ such that $\mathcal{M}, w \models \bigwedge \Gamma$ and $\mathcal{M}, w \not\models B$. So, for any $A \in \Gamma$, $w \models A$. Now, let's consider $\mathbf{G}(\mathcal{M}) = \langle W, R, S_L, v \rangle$. By **Lemma 24**, since $\mathcal{M}, w \models A$ for any $A \in \Gamma$ and $\mathcal{M}, w \not\models B$ for any $B \in \Delta$, it follows that $\mathbf{G}(\mathcal{M}), v_w(A) \in D$ for any $A \in \Gamma$ and $\mathbf{G}(\mathcal{M}), v_w(B) \notin D$. So it is not the case that $\Gamma \models^4_{K_{FDE}} \Delta$.

($\Leftarrow$) By contrapositio: assume it is not the case that $\Gamma \models^4_{K_{FDE}} \Delta$, hence there must be some some four-valued model $\mathcal{F} = \langle W, R, S_L, v \rangle$ and some $w \in W$ such that for all $A \in \Gamma$, $\mathcal{F}, v_w(A) \in D$ and $\mathcal{F}, v_w(B) \notin D$. Now, let's consider $\mathbf{S}(\mathcal{F}) = \langle W, R, a^+, a^- \rangle$. By **Lemma 23**, since $\mathcal{F}, v_w(A) \in D$ for any $A \in \Gamma$ and $\mathcal{F}, v_w(B) \notin D$, it follows that $\mathbf{S}(\mathcal{F}), w \models A$ for any $A \in \Gamma$ and $\mathbf{S}(\mathcal{F}), w \not\models B$ for any $B \in \Delta$. So we found a model such that $\mathbf{S}(\mathcal{F}), w \models \bigwedge \Gamma$ but $\mathbf{S}(\mathcal{F}), w \not\models B$, namely it is not the case that $\Gamma \models^4_{K_{FDE}} B$.

A.2 Lemma 1

Proof by induction:

Base Case.

($\models$) Given an E-Kripke model $\mathcal{E} = \langle W, R, S, \sqsubseteq, v^+, v^- \rangle$ and its ordinarification $\mathbf{O}(\mathcal{E}) = \langle W, R, a^+, a^- \rangle$, consider an arbitrary $w \in W$. ($\Rightarrow$) Assume $\mathcal{E}, w \models p$, then $[p]^+_w \neq \emptyset$, namely there is a $s \in S$ such that $s \Vdash_w p$, so, $s \in v^+_w(p)$; this means that $v^+_w(p) \neq \emptyset$; it follows that $w \in a^+(p)$ and, by semantic conditions, $\mathbf{O}(\mathcal{E}), w \models p$. ($\Leftarrow$) Assume $\mathbf{O}(\mathcal{E}), w \models p$, then $w \in a^+(p)$, namely, by definition of $a^+_w(p)$, $v^+_w(p) \neq \emptyset$. This means that there is a $s \in S$ such that $s \Vdash_w p$; so, $s \in [p]^+_w$. It follows that $[p]^+_w \neq \emptyset$ and, by semantic condition $\mathcal{E}, w \models p$.

($=\!\mid$) Analogously we can reason for $=\!\mid$.

Inductive Case $\neg$.

($\models$) Given an E-Kripke model $\mathcal{E} = \langle W, R, S, \sqsubseteq, v^+, v^- \rangle$ and its ordinarification $\mathbf{O}(\mathcal{E}) = \langle W, R, a^+, a^- \rangle$, consider an arbitrary $w \in W$. ($\Rightarrow$) Assume $\mathcal{E}, w \models \neg A$, then $[\neg A]^+_w \neq \emptyset$. Consider a $s \in [\neg A]^+_w$, it is the case that

$s \Vdash_w \neg A$, namely $s \dashv\!\mid_w A$. This means that $s \in [A]^-_w$ and so $\mathcal{E}, w \dashv A$. By IH, it follows that $\mathbf{O}(\mathcal{E}), w \dashv A$, namely $\mathbf{O}(\mathcal{E}), w \models \neg A$. ($\Leftarrow$) Assume $\mathbf{O}(\mathcal{E}), w \models \neg A$, then by semantic conditions, $\mathbf{O}(\mathcal{E}), w \dashv A$. By IH, it follows that $\mathcal{E}, w \dashv A$, namely $[A]^-_w \neq \emptyset$. Consider a $s \in [A]^-_w$: it is the case that $s \dashv\!\mid_w A$, and so, by semantic conditions, $s \Vdash_w \neg A$. This means that $s \in [\neg A]^+_w \neq \emptyset$, namely $\mathcal{E}, w \models \neg A$.

($\dashv$) Analogously we can reason for $\dashv$.

Inductive Case $\wedge$.

($\models$) Given an E-Kripke model $\mathcal{E} = \langle W, R, S, \sqsubseteq, v^+, v^- \rangle$ and its ordinarification $\mathbf{O}(\mathcal{E}) = \langle W, R, a^+, a^- \rangle$, consider an arbitrary $w \in W$. ($\Rightarrow$) Assume $\mathcal{E}, w \models A \wedge B$, then, by semantic conditions, $[A \wedge B]^+_w \neq \emptyset$. Consider a $s \in [A \wedge B]^+_w$; this means that $s \Vdash_w A \wedge B$, namely $s = u \sqcup t$ for $t \Vdash_w A$ and $u \Vdash_w B$. But then, $t \in [A]^+_w$ and $u \in [B]^+_w$, hence $[A]^+_w \neq \emptyset$ and $[B]^+_w \neq \emptyset$. It follows that $\mathcal{E}, w \models A$ and $\mathcal{E}, w \models B$; so, by IH, $\mathbf{O}(\mathcal{E}), w \models A$ and $\mathbf{O}(\mathcal{E}), w \models B$, namely $\mathbf{O}(\mathcal{E}), w \models A \wedge B$ ($\Leftarrow$) Assume $\mathbf{O}(\mathcal{E}), w \models A \wedge B$, then, by semantic conditions, $\mathbf{O}(\mathcal{E}), w \models A$ and $\mathbf{O}(\mathcal{E}), w \models B$. By IH it follows that $\mathcal{E}, w \models A$ and $\mathcal{E}, w \models B$. This means that $[A]^+_w \neq \emptyset$ and $[A]^+_w \neq \emptyset$. Now, consider a $t \in [A]^+_w$ and $u \in [B]^+_w$; by completeness of S, $s = t \sqcup u$ exists and $s \Vdash_w A \wedge B$ since $t \Vdash_w A$ and $u \Vdash_w B$, namely $s \in [A \wedge B]^+_w$. Hence, $[A \wedge B]^+_w \neq \emptyset$. So, $\mathcal{E}, w \models A \wedge B$.

($\dashv$) Given an E-Kripke model $\mathcal{E} = \langle W, R, S, \sqsubseteq, v^+, v^- \rangle$ and its ordinarification $\mathbf{O}(\mathcal{E}) = \langle W, R, a^+, a^- \rangle$, consider an arbitrary $w \in W$. ($\Rightarrow$) Assume $\mathcal{E}, w \dashv A \wedge B$, then, by semantic conditions, $[A \wedge B]^-_w \neq \emptyset$. Consider a $s \in [A \wedge B]^-_w$; it is such that $s \dashv\!\mid_w A \wedge B$, namely $s \dashv\!\mid_w A$ or $s \dashv\!\mid_w B$. If the former holds, then $s \in [A]^-_w$ and so $[A]^-_w \neq \emptyset$; this means that $\mathcal{E}, w \dashv A$ and by IH $\mathbf{O}(\mathcal{E}), w \dashv A$, namely $\mathbf{O}(\mathcal{E}), w \dashv A \wedge B$. Analogously if the latter holds. ($\Leftarrow$) Assume $\mathbf{O}(\mathcal{E}), w \dashv A \wedge B$, then, by semantic conditions, (i) $\mathbf{O}(\mathcal{E}), w \dashv A$ or (ii) $\mathbf{O}(\mathcal{E}), w \dashv B$. If (i) holds, then by IH it follows that $\mathcal{E}, w \dashv A$, namely $[A]^-_w \neq \emptyset$; consider a $s \in [A]^-_w$, it is the case that $s \dashv\!\mid_w A$ and, by semantic conditions, $s \dashv\!\mid_w A \wedge B$. Hence $s \in [A \wedge B]^-_w$; this means that $[A \wedge B]^-_w \neq \emptyset$ and so $\mathcal{E}, w \dashv A \wedge B$. Analogously if (ii) holds.

Inductive Case $\vee$.

($\models$) For $\models$, we reason analogously to the case of $\dashv$ for $\wedge$.

($\dashv$) For $\dashv$ we reason analogously to the case of $\models$ for $\wedge$.

Inductive case $\Box$.

($\models$) Given an E-Kripke model $\mathcal{E} = \langle W, R, S, \sqsubseteq, v^+, v^- \rangle$ and its ordinarification $\mathbf{O}(\mathcal{E}) = \langle W, R, a^+, a^- \rangle$, consider an arbitrary $w \in W$. ($\Rightarrow$) Assume $\mathcal{E}, w \models \Box A$; we have two cases to consider: (i) there is no v such that wRv and (ii) there is some v such that wRv. If (i) holds then, by semantic conditions, $[\Box A]^+_w \neq \emptyset$ as $0 \in [\Box A]^+_w$. Now, vacuously, it is also the case that $\mathbf{O}(\mathcal{E}), w \models \Box A$. If (ii) holds, by semantic conditions, it is the case that $[\Box A]^+_w \neq \emptyset$; consider a $s \in [\Box A]^+_w$; it is such that $s \Vdash_w \Box A$, namely there is a f such that for any v such that wRv, $f(v) \Vdash_w A$ and $s = \bigsqcup(\bigcup_{(v:wRv)}\{f(v)\})$. Consider an arbitrary $f(v) \in \{f(v) : f(v) \Vdash_w A \wedge wRv\}$; since $f(v) \Vdash_w A$, $[A]^+_v \neq \emptyset$, namely $\mathcal{E}, v \models A$ and by IH $\mathbf{O}(\mathcal{E}), v \models A$. Since v was taken arbitrarily among the successors of w, we have that for all v such that wRv, $\mathbf{O}(\mathcal{E}), v \models A$, and so $\mathbf{O}(\mathcal{E}), w \models \Box A$. ($\Leftarrow$) Assume $\mathbf{O}(\mathcal{E}), w \models \Box A$, then we have two cases to consider: (i) there is no v such that wRv and (ii) there is some v such that wRv. If (i) holds, then vacuously $\mathcal{E}, w \models \Box A$ since $[A]^+_w = \{0\}$; if (ii) holds then consider an arbitrary v such that wRv. It is the case that $\mathbf{O}(\mathcal{E}), v \models A$ and by IH $\mathcal{E}, v \models A$. This means that $[A]^+_v \neq \emptyset$. Since v was taken arbitrarily, we have that for all v such that wRv, it is the case that $[A]^+_v \neq \emptyset$, namely there is a f such that for any v such that wRv, $f(v) \in [A]^+_v$; by completeness of S, there exists a s such that $s = \bigsqcup(\bigcup_{(v:wRv)}\{f(v)\})$, namely $s \Vdash_w \Box A$. Hence $[\Box A]^+_w \neq \emptyset$, and so $\mathcal{E}, w \models \Box A$.

($\dashv$) Given an E-Kripke model $\mathcal{E} = \langle W, R, S, \sqsubseteq, v^+, v^- \rangle$ and its ordinarification $\mathbf{O}(\mathcal{E}) = \langle W, R, a^+, a^- \rangle$, consider an arbitrary $w \in W$. ($\Rightarrow$) Assume $\mathcal{E}, w \dashv \Box A$; then by semantic conditions $[\Box A]^-_w \neq \emptyset$. Now, consider a $s \in [\Box A]^-_w$; since $s \dashv\!\mid_w \Box A$, there is a v such that wRv and $s \dashv\!\mid_v A$. Hence, $[A]^-_v \neq \emptyset$, namely $\mathcal{E}, v \dashv A$. By IH, $\mathbf{O}(\mathcal{E}), v \dashv A$ and since wRv, it is the case that $\mathbf{O}(\mathcal{E}), w \dashv \Box A$. ($\Leftarrow$) Assume $\mathbf{O}(\mathcal{E}), w \dashv \Box A$, then there is some v such that wRv and $\mathbf{O}(\mathcal{E}), v \dashv A$. By IH, $\mathcal{E}, v \dashv A$, hence $[A]^-_v \neq \emptyset$. Consider a $s \in [A]^-_v$: it is the case that $s \dashv\!\mid_v A$ and so, since wRv, $s \dashv\!\mid_w \Box$. Hence $[\Box A]^-_w \neq \emptyset$, namely $\mathcal{E}, w \dashv \Box A$

Inductive case $\Diamond$.

($\models$) For $\models$ we reason analogously to $\dashv$ for $\Box$.

($\dashv$) For $\dashv$ we reason analogously to $\models$ for $\Box$.

A.3 Lemma 2

Proof by induction :

Base case.

($\models$) Given a Kripke model $\mathcal{M} = \langle W, R, a^+, a^- \rangle$ and its exactification $\mathbf{E}(\mathcal{M}) = \langle W, R, S, \sqsubseteq, v^+, v^- \rangle$, consider an arbitrary $w \in W$. ($\Rightarrow$) Assume $\mathcal{M}, w \models p$, then $w \in a^+(p)$. Hence, by definition of v^+_w, $\{(w, p)\} \in v^+_w(p)$ and clearly $\{(w, p)\} \Vdash_w p$, namely $\{(w, p)\} \in [p]^+_w$. So, it is the case that $[p]^+_w \neq \emptyset$. Then, by semantic conditions, $\mathbf{E}(\mathcal{M}), w \models p$. ($\Leftarrow$) Assume $\mathbf{E}(\mathcal{M}), w \models p$, then $[p]^+_w \neq \emptyset$. This means that there is some s such that $s \Vdash_w p$; so $s \in v^+_w(p)$, namely $s = \{(w, p)\}$ with $w \in a^+(p)$. Then, it follows that $\mathcal{M}, w \models p$ since $w \in a^+(p)$.

($\dashv$) analogously we can reason for $\dashv$.

The rest of the the proof proceeds analogously to the proof of **Lemma 3.**.

A.4 Theorem 5

Definition 39 *Given a four-valued model* $\mathcal{F} = \langle W, R, S_L, D, v\rangle$*, we define its* standardification $\boldsymbol{S}(\mathcal{F}) = \langle W', R', D, a^+, a^-\rangle$ *where*

- $W' = W$
- $R' = R$
- $a^+_w(F^n) = v^{\mathcal{E}}_w(F^n)$
- $a^-_w(F^n) = v^{\mathcal{A}}_w(F^n)$.

Definition 40 *Given an ordinary* K_{FDE} *model,* $\mathcal{M} = \langle W, R, D, a^+, a^-\rangle$*, we define its* generalization $\boldsymbol{G}(\mathcal{M}) = \langle W', R', D, S_L, v\rangle$ *where*

- $W' = W$;
- $R' = R$;
- S_L *is defined in the standard way:*
- $v^{\mathcal{E}}_w(F^n) = a^+_w(F^n)$
- $v^{\mathcal{A}}_w(F^n) = a^-_w(F^n)$

We are now ready to prove the following results :

Lemma 23 *Given a four-valued model* $\mathcal{F} = \langle W, R, D, S_L, v\rangle$*, for any formula* φ *and any world* w *in* $\mathcal{F}$*,*

$$[\mathcal{F}, \mu^I_w(\varphi) \in A \text{ if and only if } \boldsymbol{S}(\mathcal{F}), w \models^I \varphi] \text{ and } [\mathcal{F}, \mu^I_w(\varphi) \in \{b, 0\} \text{ if and only if } \boldsymbol{S}(\mathcal{F}), w \dashv^I \varphi\ ].$$

Proof: by induction on φ

Base Case. Straightforward by definition of $v^{\mathcal{E}}_w$ and $v^{\mathcal{A}}_w$ in $\mathbf{G}(\mathcal{M})$.

Inductive case $\forall$.

($\models$) Consider an arbitrary four-valued FOK_{FDE} model $\mathcal{F}$ and its standardification $\mathbf{S}(\mathcal{F})$. For an arbitrary w ($\Rightarrow$) Assume $\mathcal{F}, \mu^I_w(\forall x\varphi) \in A$; by definition of $\mu^I_w(\forall x\varphi)$ and Glb, since $\mu^I_w(\forall x\varphi) \in A$, then it must be the case that $\mu^{I^*}_w(\varphi) \in A$ for any x-variant I^* of I. Take an arbitrary x-variant I^* of I, by IH it must be the case that $w \models^{I^*} \varphi$, hence, by semantic conditions, since I^* was taken arbitrarily, we have that $w \models^{I^*} \varphi$

for any x-variant I^* of I, so $w \models^I \forall x\varphi$. ($\Leftarrow$) Assume $\mathbf{S}(\mathcal{F}), w \models^I \forall x\varphi$, then by semantic conditions, for any x-variant I^* of I, $w \models^{I^*} \varphi$. Take, an arbitrary x-variant I^* of I; then by IH $\mu_w^{I^*}(\varphi) \in A$; hence, since I^* was taken arbitrarily, we have that $\mu_w^{I^*}(\varphi) \in A$ for any x-variant I^* of I; and by definition of *Glb* it must be the case that $\mu_w^I(\forall x\varphi) \in A$.

($\dashv$) Consider an arbitrary four-valued FOK_{FDE} model $\mathcal{F}$ and its standardification $\mathbf{S}(\mathcal{F})$. For an arbitrary w ($\Rightarrow$) Assume $\mathcal{F}, \mu_w^I(\forall x\varphi) \in \{b, 0\}$; by definition of $\mu_w^I(\forall x\varphi)$ and *Glb*, since $\mu_w^I(\forall x\varphi) \in \{b, 0\}$, then it must be the case that $\mu_w^{I^*}(\varphi) \in \{b, 0\}$ for some x-variant I^* of I. By IH it must be the case that $w \dashv^{I^*} \varphi$, hence, by semantic conditions, we have that $w \models^{I^*} \varphi$ for some x-variant I^* of I, so $w \dashv^I \forall x\varphi$. ($\Leftarrow$) Assume $\mathbf{S}(\mathcal{F}), w \dashv^I \forall x\varphi$, then by semantic conditions, for some x-variant I^* of I, $w \models^{I^*} \varphi$. By IH $\mu_w^{I^*}(\varphi) \in \{b, 0\}$; hence we have that $\mu_w^{I^*}(\varphi) \in A$ for some x-variant I^* of I; and by definition of *Glb* it must be the case that $\mu_w^I(\forall x\varphi) \in \{b, 0\}$.

Inductive case $\exists$.

($\models$) Analogously to the ($\dashv$) case of $\forall$.

($\dashv$) Analogously to the ($\models$) case of $\forall$.

The rest of the proof is analogous to the propositional case.

Lemma 24 *Given an ordinary* FOK_{FDE} *model* $\mathcal{M} = \langle W, R, D, a^+, a^- \rangle$*, for any formula* φ *and any world* w *in* $\mathcal{M}$*,*

$$[\mathcal{M}, w \models^I \varphi \text{ if and only if } \mathbf{G}(\mathcal{M}), \mu_w^I(\varphi) \in A] \text{ and } [\mathcal{M}, w \dashv^I \varphi \text{ if and only if } \mathbf{G}(\mathcal{M}), \mu_w^I(\varphi) \in \{b, 0\}]$$

Proof: by induction

Base Case. Straightforward by definition of $v_w^{\mathcal{E}}$ and $v_w^{\mathcal{A}}$ in $\mathbf{G}(\mathcal{M})$.

The rest of the proof proceeds analogously to the one of the previous lemma.

A.5 Lemma 13

Proof: by induction

Base Case.

($\models$) Given a first-order E-Kripke model $\mathcal{E} = \langle W, R, S, D, \sqsubseteq, v^+, v^- \rangle$ and its ordinarification $\mathbf{O}(\mathcal{E}) = \langle W, R, D, a^+, a^- \rangle$, consider an arbitrary $w \in W$. ($\Rightarrow$) Assume $\mathcal{E}, w \models^I F^n x_1, ..., x_n$, then $[F^n x_1, ..., x_n]^I_{(w,+)} \neq \emptyset$, namely there is a $s \in S$ such that $s \Vdash^I_w F^n x_1, ..., x_n$, so, $s \in v^+_w(F^n, I(x_1), ..., I(x_n))$; this means that $v^+_w(F^n I(x_1), ..., I(x_n)) \neq \emptyset$. It follows, by definition of a^+_w, that $\langle I(x_1), ..., I(x_n) \rangle \in a^+_w(F^n)$ and, by semantic conditions, $\mathbf{O}(\mathcal{E}), w \models^I F^n x_1, ..., x_n$. ($\Leftarrow$) Assume $\mathbf{O}(\mathcal{E}), w \models^I F^n x_1, ..., x_n$, then $\langle x_1, ..., x_n \rangle \in a^+_w(F^n)$, namely, by definition of $a^+_w(F^n)$, $v^+_w((F^n, (I(x_1), .., I(x_n))) \neq \emptyset$. This means that there is a $s \in S$ such that $s \Vdash^I_w F^n x_1, ..., x_n$; so, $s \in [F^n x_1, ..., x_n]^I_{(w,+)}$. It follows that $[F^n x_1, ..., x_n]^+_w \neq \emptyset$ and by semantic conditions $\mathcal{E}, w \models^I F^n x_1, ..., x_n$.

($\Dashv$) Analogously to $\models$ case

Inductive Case $\forall$.

($\models$) Given a first-order E-Kripke model $\mathcal{E} = \langle W, R, S, D, \sqsubseteq, v^+, v^- \rangle$ and its ordinarification $\mathbf{O}(\mathcal{E}) = \langle W, R, D, a^+, a^- \rangle$, consider an arbitrary $w \in W$. ($\Rightarrow$) Assume $\mathcal{E}, w \models^I \forall x \varphi$; by semantic conditions, it is the case that $[\forall x \varphi]^I_{(w,+)} \neq \emptyset$; consider a $s \in [\forall x \varphi]^I_{(w,+)}$; it is such that $s \Vdash^I_w \forall x \varphi$, namely for any x-variant I^* of I, $s \Vdash^{I^*}_w \varphi$. Consider an arbitrary x-variant I^* of I; it is the case that $s \Vdash^{I^*}_w \varphi$ and by IH $\mathbf{O}(\mathcal{E}), w \models^{I^*} \varphi$. Since I^* was taken arbitrarily we have that for all the x-variant I^* of I, $\mathbf{O}(\mathcal{E}), w \models^{I^*} \varphi$, and so $\mathbf{O}(\mathcal{E}), w \models^I \forall x \varphi$. ($\Leftarrow$) Analogously to the ($\Rightarrow$) case.

($\Dashv$) Given a first-order E-Kripke model $\mathcal{E} = \langle W, R, S, D, \sqsubseteq, v^+, v^- \rangle$ and its ordinarification $\mathbf{O}(\mathcal{E}) = \langle W, R, D, a^+, a^- \rangle$, consider an arbitrary $w \in W$. ($\Rightarrow$) Assume $\mathcal{E}, w \Dashv^I \forall x \varphi$; by semantic conditions, it is the case that $[\forall x \varphi]^I_{(w,)} \neq \emptyset$; consider a $s \in [\forall x \varphi]^I_{(w,-)}$; it is such that $s \dashVv^I_w \forall x \varphi$, namely for some x-variant I^* of I, $s \dashVv^{I^*}_w \varphi$. Consider such I^*; it is the case that $s \dashVv^{I^*}_w \varphi$ and by IH $\mathbf{O}(\mathcal{E}), w \Dashv^{I^*} \varphi$. Hence we have that for some x-variant I^* of I, $\mathbf{O}(\mathcal{E}), w \Dashv^{I^*} \varphi$, and so $\mathbf{O}(\mathcal{E}), w \Dashv^I \forall x \varphi$. ($\Leftarrow$) Analogously to the ($\Rightarrow$) case.

Inductive Case $\exists$.

($\models$) Analogously to the $\Dashv$ case of $\forall$.

($\Dashv$) Analogously to the $\models$ case of $\forall$.

The rest of the proof proceeds analogously to the propositional case.

A.6 Lemma 14

Proof: by induction

Base Case.

(⊨) Given an ordinary first-order Kripke model $\mathcal{M} = \langle W, R, D, a^+, a^- \rangle$ and its exactification $\mathbf{E}(\mathcal{M}) = \langle W, R, S, \sqsubseteq, D, v^+, v^- \rangle$, consider an arbitrary $w \in W$. (⇒) Assume $\mathcal{M}, w \models^I F^n x_1, ..., x_n$, then $\langle I(x_1), ..., I(x_n) \rangle \in a^+_w(F^n)$; this means that, by definition of v^+, $\{(w, (F^n, (I(x_1), ..., I(x_n)))\} \in v^+_w((F^n, (I(x_1), ..., I(x_n)))$. Call s the state $\{(w, (F^n, (I(x_1), ..., I(x_n))\}$. Since $s \in v^+_w((F^n, (I(x_1), ..., I(x_n)))$, it is the case that $s \Vdash^I_w F^n x_1, ..., x_n$; it follows that $[F^n x_1, ..., x_n]^{I+}_w \neq \emptyset$, hence $\mathbf{E}(\mathcal{M}), w \models^I F^n x_1, ..., x_n$. (⇐) Assume $\mathbf{E}(\mathcal{M}), w \models^I F^n x_1, ..., x_n$, then $[F^n x_1, ..., x_n] \neq \emptyset$, hence there is some $s \in S$ such that $s \Vdash^I_w F^n x_1, ..., x_n$, and so $s \in v^+_w(F^n x_1, ..., x_n)$. This means that $s = \{w, (F^n, (I(x_1), ..., I(x_n))\}$ with $\langle I(x_1), ..., I(x_n) \rangle \in a^+_w(F^n)$. Hence, by semantic conditions, $\mathcal{M}, w \models^I F^n x_1, ..., x_n$.

(⫤) Analogously to ⊨ case

The rest of the proof proceeds analogously to the one of the previous lemma.

A.7 Lemma 19

Proof: by induction

Base Case.

(⊨) Consider an arbitrary inclusive E-Kripke model $\mathcal{E}$ and arbitrary world w and states s, t in $\mathcal{E}$. Assume $\mathcal{E}, s \Vdash_w p$ and $\mathcal{E}, t \Vdash_w p$; it is the case that $s \in v^+_w(p)$ and $t \in v^+_p(p)$. Hence, by definition of $v^+_w(p)$, it is also the case that $s \sqcup t \in v^+_w(p)$.

(⫤) Analogously to the (⊨) case

Inductive Case ¬

(⊨) Consider an arbitrary inclusive E-Kripke model $\mathcal{E}$ and arbitrary world w and states s, t in $\mathcal{E}$. Assume $\mathcal{E}, s \Vdash_w \neg A$ and $\mathcal{E}, t \Vdash_w \neg A$; by semantic conditions it is the case that $s \dashv\!\mid_w A$ and $t \dashv\!\mid_w A$. By IH, it holds that $s \sqcup t \dashv\!\mid_w A$, namely, by semantic conditions, $s \sqcup t \Vdash_w \neg A$.

(⫤) Analogously to the (⊨) case

Inductive Case ∧

(⊨) Analogously to the proof of lemma 3.1 for ∧ in (Fine & Jago, 2016)

(⫤) Analogously to the proof of lemma 3.1 for ∨ in (Fine & Jago, 2016)

Inductive Case ∨

(⊨) Analogously to the proof of lemma 3.1 for ∨ in (Fine & Jago, 2016)

(⫤) Analogously to the proof of lemma 3.1 for ∧ in (Fine & Jago, 2016)

Inductive Case $\Box$

($\models$) Consider an arbitrary inclusive E-Kripke model $\mathcal{E}$ and arbitrary world w and states s, t in $\mathcal{E}$. Assume $\mathcal{E}, s \Vdash_w \Box A$ and $\mathcal{E}, t \Vdash_w \Box A$; by semantic conditions, it is the case that there is a f such that for any v such that wRv there is a state $f(v)$ such that $f(v) \Vdash_v A$ and $s = \bigsqcup(\bigcup_{(v:wRv)}\{f(v)\}$, call X such $\bigcup_{(v:wRv)}\{f(v)\}$; and analogously for $t = \bigsqcup(\bigcup_{(v:wRv)}\{g(v)\}$, call Y such $\bigcup_{(v:wRv)}\{g(v)\}$. Consider a successor v of w for an arbitrary v, consider$f(v) \in \bigcup_{(v:wRv)}\{f(v)\}$ and $g(v) \in \bigcup_{(v:wRv)}\{g(v)\}$. Now consider the set $Z = \{f(v) \sqcup g(v) : f(v) \in \bigcup_{(v:wRv)}\{f(v)\} \wedge g(v) \in \bigcup_{(v:wRv)}\{g(v)\}\}$ and its least upper bound $\bigsqcup Z$. By IH, we have that $f(v) \sqcup g(v) \Vdash_v A$, since $f(v) \Vdash_v A$ and $g(v) \Vdash_v A$. Consider a function h from the set of worlds to S such that for any v such that wRv, $h(v) = f(v) \sqcup g(v)$. Notice that $\bigcup_{v:wRv}\{h(v)\} = Z$ and since for any v, $f(v) \sqcup (v) \Vdash_v A$, by semantic conditions, it is the case that $\bigsqcup Z \Vdash_w \Box A$. Moreover, it is readily provable that $\bigsqcup X \sqcup \bigsqcup Y = \bigsqcup Z$ and since $s = \bigsqcup X$ and $t = \bigsqcup Y$, we have that $s \sqcup t = \bigsqcup Z$ and so $s \sqcup t \Vdash_w \Box A$.

($\dashv$) Consider an arbitrary inclusive E-Kripke model $\mathcal{E}$ and arbitrary world w and states s, t in $\mathcal{E}$. Assume $\mathcal{E}, s \dashv\!\mid_w \Box A$ and $\mathcal{E}, t \dashv\!\mid_w \Box A$; by semantic conditions we have two case to consider for each t and s: (i) there is some v such that wRv and $s \dashv\!\mid_v A$; (ii) there is a non-empty set of successors of w, Y, such that there is an f such that for any $v \in Y$, $f(v) \dashv\!\mid_v A$ and $s = \bigsqcup \bigcup_{(v \in Y)}\{f(v)\}$, call X such $\bigcup_{(v \in Y)}\{f(v)\}$; (iii) there is some v such that wRv and $t \dashv\!\mid_v A$; (iv) there is a non-empty set of successors of w, Y, there is a g such that for any $v \in Y$ there is a $g(v)$ such that $g(v) \dashv\!\mid_v A$ and $t = \bigsqcup \bigcup_{(v \in Y)}\{g(v)\}$, call Z such $\bigcup_{(v \in Y)}\{g(v)\}$.

If (i) and (iii) hold, then consider the set $\{s, t\}$; assumptions $s \dashv\!\mid_v A$, for some v such that wRv and $t \dashv\!\mid_u A$, for some u such that wRu. Consider a function h from the set of worlds to S such that $h(v) = s$ and $h(u) = t$. By inclusive semantic conditions, we have that $s \sqcup t \dashv\!\mid_w \Box A$.

If (i) and (iv) hold, then $t = \bigsqcup Z$ and $s \dashv\!\mid_v A$ for some v such that wRv. Consider two scenarios: (1) for such v there is a $g(v) \in Z$; (2) it is not the case that (1). If (1) holds, then take the set $T = Z/\{g(v)\} \cup \{s \sqcup g(v)\}$. Now, consider a function h from the set of worlds to S just like g with the only difference that $h(v) = s \sqcup g(v)$ notice that since $g(v) \dashv\!\mid_v A$ and $s \dashv\!\mid_v A$ we have that, by IH, $s \sqcup g(v) \dashv\!\mid_v A$. Hence any $u \in T$ is such that there is some v such that wRv, $h(v) = u$ and $u \dashv\!\mid_v A$. Moreover, it is readily provable that $s \sqcup t = \bigsqcup T$, hence $s \sqcup t \dashv\!\mid_w \Box A$. If (2) holds, take the set $U = Z \cup \{s\}$; then analogously $s \sqcup t = \bigsqcup U$ and so $s \sqcup t \dashv\!\mid_w \Box A$.

If (ii) and (iii) hold, then we can reason analogously to the case of (i) & (iv).

If (ii) and (iv) hold, then $s = \bigsqcup X$ and $t = \bigsqcup Z$. Consider two scenarios: (1) there is a non-empty set Y^* of successors of w such that there are functions f, g such hat for any $v \in Y^*$ there are some

$f(v) \in X$ and $g(v) \in Z$ such that $f(v) \dashv\Vdash_v A$ and $g(v) \dashv\Vdash_v A$ and take $V = \bigcup_{(v \in Y^*)}(\{f(v) \sqcup g(v)\})$; (2) it is not the case that (1). If (1) holds, then take the set $U = (X \,/\, \bigcup_{(v \in Y^*)}\{f(v)\}) \cup (Z \,/\, \bigcup_{(v \in Y^*)}\{g(v)\}) \cup V$; notice that, by assumption, for any $v \in Y^*$, $f(v) \dashv\Vdash_v A$ and $g(v) \dashv\Vdash_v A$. Hence, by IH, for any $u \in V$, it is the case that $u \dashv\Vdash_v A$, for any $v \in Y^*$. Consider the function h from the set of worlds to S such that for any $v \in Y^*$, $h(v) = f(v) \sqcup g(v)$ and for any other successor z of w, if $f(z) \in X$ then $h(z) = f(v)$ and if $g(z) \in Z$, then $h(z) = g(z)$. This means that $U = \bigcup_{u:wRv}\{g(u)\}$ and any state in the set U falsifies A at v for some successor v of W; so, $\bigsqcup U \dashv\Vdash_w \Box A$ by inclusive semantic conditions; moreover, it is readily provable that $s \sqcup t = \bigsqcup U$, and so, $s \sqcup t \dashv\Vdash_w \Box A$. If (2) holds, then we reason analogously to (1) without consider the restriction V but just the union of X and Z.

Inductive Case $\Diamond$

($\models$) Analogously to the ($\Dashv$) case of $\Box$.

($\Dashv$) Analogously to the $\models$ case of $\Box$.

A.8 Lemma

Proof: by induction

Base Case.

($\models$ Consider an atomic letter p and an arbitrary world $w \in W$; by definition of inclusive v^+ and v^- it is the case that $v^+_w(p) \neq \emptyset$ Hence there are some s such that $s \Vdash_w p$. Hence, $|p|^+ \neq \emptyset$.

($\Dashv$) Analogously to the ($\models$) case.

Inductive Case $\neg$.

($\models$) Consider $|\neg A|^+_w$ for an arbitrary $w \in W$. By IH, it is the case that $|A|^-_w \neq \emptyset$; take a $s \in |A|^-_w$, it is such that $s \dashv\Vdash_w A$ and, by semantic conditions, $s \Vdash_w \neg A$.

($\Dashv$) Analogously to ($\models$).

Inductive Case $\wedge$.

($\models$) Consider $|A \wedge B|^+_w$ for an arbitrary $w \in W$; by IH it is the case that $|A|^+_w \neq \emptyset$ and $|B|^+_w \neq \emptyset$. Now, take a $t \in |A|^+_w$ and a $u \in |B|^+_w$, they are such that $t \Vdash_w A$ and $u \Vdash_w B$. By completeness of S, $s = t \sqcup u$ exists and, by semantic conditions, $s \Vdash_w A \wedge B$. This means that $s \in |A \wedge B|^+_w$, namely $|A \wedge B|^+_w \neq \emptyset$.

($\Dashv$) Consider $|A \wedge B|^-_w$ for an arbitrary $w \in W$; by IH it is the case that $|A|^-_w \neq \emptyset$ and $|B|^-_w \neq \emptyset$. Now, take a $t \in |A|^-_w$, it is such that $t \dashv\Vdash_w A$ and, by semantic conditions, $t \dashv\Vdash_w A \wedge B$. Analogously if we take a $u \in |B|^-_w$.

Inductive Case $\vee$.

($\models$) Analogously to the ($\dashv$) case of $\wedge$.

($\dashv$) Analogously to the $\models$ case of $\wedge$

Inductive case $\Box$.

($\models$) Consider $|\Box A|^+_w$ for an arbitrary $w \in W$. By IH, it is the case that $|A|^+_v \neq \emptyset$ for a $v \in W$. Consider the set $\{t : t \Vdash_v A \text{and} wRv\}$; clearly this set is non-empty since w has at least one successor, by seriality, and by IH $|A|^+_v \neq \emptyset$ for any $v \in W$. Now, by completeness of S and non-emptiness of $\{t : t \Vdash_v A \text{and} wRv\}$, $u = \bigsqcup\{t : t \Vdash_v A \text{and} wRv\}$ exists and it is non-null. Consider the function f such that for any v such that wRv, $f(v) \Vdash_v A$. Notice that $\{t : t \Vdash_v A \text{and} wRv\} = \bigcup_{v:wRV}\{f(v)\}$. Hence, $u \Vdash_w \Box A$, namely $|\Box A|^+_w \neq \emptyset$.

($\dashv$) Consider $|\Box A|^-_w$ for an arbitrary w. By IH, it is the case that $|A|^-_v \neq \emptyset$ for an $v \in W$. Since w has at least one successor v , by seriality, consider the exact content of A with respect to such v. By IH, $|A|^-_v \neq \emptyset$; now, take a $s \in |A|^-_v$, it is the case that $s \dashv\!|_v A$ and, since wRv, it also holds that $s \dashv\!|_w \Box A$, namely $s \in |\Box A|^-_w$ and so $|\Box A|^-_w \neq \emptyset$.

Inductive Case $\Diamond$.

($\models$) Analogously to the ($\dashv$) case of $\Box$

($\dashv$) Analogously to the ($\models$) case of $\Box$

A.9 Modal Van Fraassen's Theorem for inclusive E-Kripke models

Consider any inclusive and serial E-Kripke model $\mathcal{E} = \langle S, F, \sqsubseteq, W, R, v^+, v^- \rangle$ putting in evidence a new element $F \subseteq S$. F it is such that it closed under parts and $\bigsqcup$ operation, namely

$$X \subseteq F \Leftrightarrow \bigsqcup X \in F$$

Moreover, we define new semantic conditions for a formula A to be true or false at a world w in an inclusive and serial E-Kripke model:

$$w \models A \quad \Leftrightarrow \quad [A]^+_w \cap F \neq \emptyset$$
$$w \dashv A \quad \Leftrightarrow \quad [A]^-_w \cap F \neq \emptyset$$

Now, given an inclusive and serial E-Kripke model, we define the ordinarification $\mathbf{O}(\mathcal{E})$ in the standard way with the only difference that:

- $a^+, a^- : \mathcal{L}_{prop} \rightarrow \mathcal{P}(W)$
  - $a^+(p) = \{w \in W \ : \ v^+_w(p) \cap F \neq \emptyset\}$

- $a^{-}(p) = \{w \in W \;:\; v^{-}_{w}(p) \cap F \neq \emptyset\}$

Now, it is possible to prove the following result:

Lemma 25 *For any E-Kripke model $\mathcal{E} = \langle W, R, S, \sqsubseteq, F, v^{+}, v^{-}\rangle$, given its ordinarification $\mathbf{O}(\mathcal{E})$, for any formula A and any $w \in W$,*

$[\mathcal{E}, w \models A$ if and only if $\mathbf{O}(\mathcal{E}), w \models A]$ and $[\mathcal{E}, w \dashv A$ if and only if $\mathbf{O}(\mathcal{E}), w \dashv A]$

Proof: by induction :

Base Case.

($\models$) Given an E-Kripke model $\mathcal{E} = \langle W, R, S, \sqsubseteq, F, v^{+}, v^{-}\rangle$ and its ordinarification $\mathbf{O}(\mathcal{E}) = \langle W, R, a^{+}, a^{-}\rangle$, consider an arbitrary $w \in W$. ($\Rightarrow$) Assume $\mathcal{E}, w \models p$, then $[p]^{+}_{w} \neq \emptyset$, namely there is a $s \in F$ such that $s \Vdash_{w} p$, so, $s \in v^{+}_{w}(p)$; this means that $v^{+}_{w}(p) \cap F \neq \emptyset$; it follows that $w \in a^{+}(p)$ and, by semantic conditions, $\mathbf{O}(\mathcal{E}), w \models p$. ($\Leftarrow$) Assume $\mathbf{O}(\mathcal{E}), w \models p$, then $w \in a^{+}(p)$, namely, by definition of $a^{+}_{w}(p)$, $v^{+}_{w}(p) \cap F \neq \emptyset$. This means that there is a $s \in S$ such that $s \Vdash_{w} p$; so, $s \in [p]^{+}_{w}$. It follows that $[p]^{+}_{w} \cap F \neq \emptyset$ and, by semantic condition $\mathcal{E}, w \models p$.

($\dashv$) Analogously we can reason for $\dashv$.

The rest of the the proof proceeds analogously to the proof of **Lemma 3.** by employing the fact that F is closed under parts and $\bigsqcup$ operation.

Given an ordinary D_{FDE} model $\mathcal{M}$ (namely a model of K_{FDE} whose accessibility relation is serial), we define its exactification $\mathbf{E}(\mathcal{M})$ in the standard way with the only difference that:

- $F = \mathcal{P}(\{(w, p) \;:\; w \in a^{+}(p)\} \cup \{(w, \neg p) \;:\; w \in a^{-}(p)\})$
- $v^{+}, v^{-} : \mathcal{L}_{prop} \times W \to \mathcal{P}(S)$
  - $v^{+}_{w}(p) = \{\{(w, p)\}\}$
  - $v^{-}_{w}(p) = \{\{(w, \neg p)\}\}$

It is easily provable that $\mathbf{E}(\mathcal{M})$ is indeed inclusive, since, by definition $v^{+}_{w}(p)$ and $v^{-}_{w}(p)$ are non-empty and closed under $\bigsqcup$ (as they contain only one element) and serial, as the starting $\mathcal{M}$ was serial. Moreover, by definition of S in $\mathbf{E}(\mathcal{M})$, it is easy to prove that $\mathbf{E}(\mathcal{M})$ conforms to the overlap constraint.

Now, it is possible to prove the following lemma for D_{FDE} models:

Lemma 26 *For any D_{FDE} model $\mathcal{M} = \langle W, R, a^+, a^- \rangle$, given its exactification $\mathbf{E}(\mathcal{M})$ it is the case that for any formula A, and any $w \in W$,*

$[\mathcal{M}, w \models A$ if and only if $\mathbf{E}(\mathcal{M}), w \models A]$ and $[\mathcal{M}, w \Dashv A$ if and only if $\mathbf{E}(\mathcal{M}), w \Dashv A]$

Proof by induction :

Base case.

($\models$) Given a D_{FDE} model $\mathcal{M} = \langle W, R, a^+, a^- \rangle$ and its exactification $\mathbf{E}(\mathcal{M}) = \langle W, R, S, \sqsubseteq, F, v^+, v^- \rangle$, consider an arbitrary $w \in W$. ($\Rightarrow$) Assume $\mathcal{M}, w \models p$, then $w \in a^+(p)$. Hence, $\{(w, p)\} \in F$ and by definition of v_w^+, $\{(w, p)\} \in v_w^+(p)$; so, clearly, $\{(w, p)\} \Vdash_w p$, namely $\{(w, p)\} \in [p]_w^+$. So, it is the case that $[p]_w^+ \cap F \neq \emptyset$. Then, by semantic conditions, $\mathbf{E}(\mathcal{M}), w \models p$. ($\Leftarrow$) Assume $\mathbf{E}(\mathcal{M}), w \models p$, then $[p]_w^+ \cap F \neq \emptyset$. This means that there is some $s \in F$ such that $s \Vdash_w p$; so $s \in v_w^+(p)$, namely $s = \{(w, p)\}$ with $w \in a^+(p)$. Then, it follows that $\mathcal{M}, w \models p$ since $w \in a^+(p)$.

($\Dashv$) analogously we can reason for $\Dashv$.

The rest of the the proof proceeds analogously to the proof in **A.3** by employing the fact that F is closed under parts and $\bigsqcup$ operation.

Now, it is possible to prove an analogous of Van Fraassen's theorem for inclusive semantics under inclusive serial E-Kripke models, namely

Theorem 11 **($A \Vvdash_{\mathbf{D}_i} B$)** *if and only if $[A \models_{\mathbf{D}_{FDE}} B]$ (where $\mathbf{D}_{FDE}$ is the serial extension of K_{FDE}).*

Proof:

($\Rightarrow$) By contrapositio; assume $A \not\models_{D_{FDE}} B$, so there is a D_{FDE}-Kripke model $\mathcal{M}$ and some w in $\mathcal{M}$ such that $\mathcal{M}, w \models A$ and $\mathcal{M}, w \not\models B$. Now, let's consider $\mathbf{E}(\mathcal{M})$. By **Lemma 28.** we have that since $\mathcal{M}, w \models A$ and $\mathcal{M}, w \not\models B$ it is also the case that $\mathbf{E}(\mathcal{M}), w \models A$ and $\mathbf{E}(\mathcal{M}), w \not\models B$. This means that $[A]_w^+ \cap F \neq \emptyset$ and $[B]_w^+ \cap F = \emptyset$, so there is a $s \in F$ such that $\mathbf{E}(\mathcal{M}), s \Vdash_w A$ and no $t \in F$ (indeed by **Lemma 19** $[B]_w^+ \neq \emptyset$) such that $\mathbf{E}(\mathcal{M}), t \Vdash_w B$. Now, it is the case that s inexactly verifies A at w since there is $s \sqsubseteq s$ and $\mathbf{E}(\mathcal{M}), s \Vdash_w A$. Now, consider an arbitrary $z \sqsubseteq s$; since F is closed under parts, $z \in F$, but, by assumption there is no $t \in F$ such that $\mathbf{E}(\mathcal{M}), t \Vdash_w B$. Then, it cannot be the case that $z \Vdash_w B$. Since z was taken arbitrarily among the parts of s, it is the case that for any $z \sqsubseteq s$, $s \nVdash_w B$, namely s does not inexactly verify B. And so, it is not the case that $A \Vvdash_{K_i} B$

($\Leftarrow$) By contrapositio; assume that it is not the case that $A \Vvdash_{K_i} B$, so there is some inclusive and serial E-Kripke model $\mathcal{E}$ and some state u and some world w such that $\mathcal{E}, u \Vvdash_w A$ and it is not the case that $\mathcal{E}, u \Vvdash_w B$, namely there is some $s \sqsubseteq u$ such that $\mathcal{E}, s \Vdash_w A$ and for any $z \sqsubseteq u$, $\mathcal{E}, z \nVdash_w B$. Now consider a modification of $\mathcal{E}$ in which F is the down-set of such u, namely $\mathcal{E}^* = \langle W, S, R, \sqsubseteq, F^*, v^*_+, v^*_- \rangle$ where F^* is the set of all the states t such that $t \sqsubseteq u$ so that u is the top element of F^*; it is readily provable that F^* is closed under parts and $\bigsqcup$. By construction, since all the states t in F^* are such that $t \nVdash_w B$, it is the case that $[B]^+_w \cap F = \emptyset$. and $[A]^+_w \cap F \neq \emptyset$ since $\mathcal{E}, u \Vdash_w A$ and F is closed under parts.

Let's consider the ordinarification of $\mathcal{E}^*$ with, namely $\mathbf{O}(\mathcal{E}^*)$. Since, $\mathcal{E}^*, w \models A$ and $\mathcal{E}^*, w \nvDash B$, by **Lemma 27**, it is the case that $\mathbf{O}(\mathcal{E}^*), w \models A$ and $\mathbf{O}(\mathcal{E}^*), w \nvDash B$. So, we found a countermodel for $A \models_{D_{FDE}} B$.

References

Anderson A., R. and Belnap N. D., (1962), "Tautological entailments", *Philosophical Studies* 13 (1-2):9 - 24.

Angell, R., B., (1989), "Deducibility, Entailment and Analytic Containment", in *Directions in Relevant Logic*, Jean Norman & Richard Sylvan (eds.), Kluwer Academic, 119-143.

Anglberger, A. J. J. and Korbmacher, J. and Faroldi, F., L., G., (2016), "An Exact Truthmaker Semantics for Permission and Obligation", In Olivier Roy, Allard Tamminga & Malte Willer (eds.), *Deontic Logic and Normative Systems*, College Publications. pp. 16-31.

Armstrong, D., (1986), "The Nature of Possibility", *Canadian Journal of Philosophy*, 16(4): 575-594.

Armstrong, D., (1993), "A World of States of Affairs", *Philosophical Perspectives*, 7 (3):429-440.

Armstrong, D., (2004), *Truth and Truthmakers*, Cambridge University Press.

Beebee, H. and J. Dodd. (2005), "Introduction", *Truthmakers: The Contemporary Debate*, (eds.) H. Beebee and J. Dodd., Oxford: Oxford UP, 1–16.

Belnap, N., D., (1977), "A useful four-valued logic", in J. M. Dunn and G. Epstein (eds.), *Modern Uses of Multiple-Valued Logic*, D. Reidel Publishing Company, Dordrecht, pp. 8-37.

Berto, F., and Jago, M., (2019), *Impossible worlds*, Oxford: Oxford University Press.

Blackburn, P. and De Rijke, M., and Venema, Y., (2002), *Modal Logic*, Cambridge University Press.

Cameron, R., P., (2008), "Truthmakers and Modality", *Synthese*, 164 (2):261 - 280.

Correia F., (2004), "Semantics for analytic containment", *Studia Logica*, 77 (1):87-104.

Correia, F., (2016), "On the logic of factual equivalence", *Review of Symbolic Logic*, 9 (1):103-122.

Davey, B., A., and Priestly, H., A., (2002), *Introduction to Lattices and Order*, Cambridge University Press.

Fine, K., (2013), "A note on partial content",*Analysis*, 73 (3):413-419.

Fine, K., (2016) "Angellic content", *Journal of Philosophical Logic*, 45(2):199–226.

Fine, K., (2017), "Truthmaker Semantics", in *A companion to the Philosophy of Language*, : B.Hale -C. Wright-A. Miller (eds.), Blackwell.

Fine, K., (2017a), "A Theory of Truthmaker Content I: Conjunction, Disjunction and Negation", *Journal of Philosophical Logic*, 46(6):625–674.

Fine, K., (2017b) "A Theory of Truthmaker Content II: Subject Matter, Common Content, Remainder and Ground", *Journal of Philosophical Logic*, 46(6):675–702.

Fine, K. and Jago, M., (2018), "Logic for Exact Entailment", *Review of Symbolic Logic*, 1-21.

Fine, K., (forthcoming*a*) "A Theory of Partial Truth".

Fine, K., (forthcoming*b*) "Constructing the Impossible", In Walters and Hawthorne, (eds.), *Conditionals, Probability, and Paradox: Themes from the Philosohy of Dorothy Edgington*, Oxford University Press.

Fitting, M. and Mendelsohn, R., L., (1998), *Firt-Order Modal Logic*, Kluwer Academic Publishers.

French, R., (2017), "A Simple Sequent Calculus for Angell's Logic of Analytic Containment", *Studia Logica*, 105 (5):971-994.

Humberstone, L., (1981), "From worlds to possibilities", *Journal of Philosophical Logic*, 10 (3):313 - 339 (1981).

Humberstone, L., (2003), "False though partly true – an experiment in logic", *Journal of Philosophical Logic*, 32 (6):613-665.

Jago, M., (2018), *What truth is*, Oxford: Oxford University Press.

Korbmacher, J., (2016), "Exact Truthmaker Semantic for Propositional Modal Logic(s)", *Unpublished presentation.* Groningen, 2016.

Kripke, S., (1963), "Semantical Considerations on Modal Logic", *Acta Philosophica Fennica*, 16 (1963):83-94 (1963).

Linsky, B., (1994), "Truth Makers for Modal Propositions", *The Monist*, 77 (2):192-206 (1994).

Lewis, D., (1968), "Counterpart Theory and Quantified Modal Logic", *The Journal of Philosophy*, 65 (5):113-126.

Lewis, D., (1986), "On the Plurality of Worlds", Oxford: Basil Blackwell.

Lewis, D., (2003), "Things Qua Truthmakers", *Real Metaphysics: Essays in honor of D. H. Mellor*, in Hallvard Lillehammer & Gonzalo Rodriguez-Pereyra (eds.), Routledge.

MacBride, F., (2019), "Truthmakers", The Stanford Encyclopedia of Philosophy (Spring 2019 Edition), Edward N. Zalta (ed.),
URL = https://plato.stanford.edu/archives/spr2019/entries/truthmakers/.

Menzel, C., (2016), "Possible Worlds", *The Stanford Encyclopedia of Philosophy* (Winter 2017 Edition), Edward N. Zalta (ed.),
URL = https://plato.stanford.edu/archives/win2017/entries/possible-worlds/.

Merricks, T., (2007), *Truth and Ontology*, Oxford University Press.

Mulligan, K. and Correia, F., (2017), "Facts", The Stanford Encyclopedia of Philosophy (Winter 2017 Edition), Edward N. Zalta (ed.),
URL = https://plato.stanford.edu/archives/win2017/entries/facts/.

Omori, H. and Wansing H., (2017), "40 years of FDE: An Introductory Overview", *Studia Logica* 105 (6):1021-1049.

Plantinga, A., (1978), *The Nature of Necessity*, Oxford: Oxford University Press.

Priest, G. (1997). "Editor's introduction", *Notre Dame Journal of Formal Logic*, 38: 481–7.

Priest, G., (2008), *An introduction to non-classical logic*, Cambridge University Press.

Restall, G., (1996), "Truthmakers, entailment and necessity", *Australasian Journal of Philosophy*, 74 (2):331 – 340 (1996).

Rodriguez-Pereyra, G. (2006), "Truthmakers", *Philosophy Compass* 1 (2):186–200.

Shramko, Y. and Zaitsev D. and Belikov A., (2017), "First-Degree Entailment and its Relatives", *Studia Logica* 105 (6):1291-1317.

Textor, M., (2016), "States of Affairs", The Stanford Encyclopedia of Philosophy (Winter 2016 Edition), Edward N. Zalta (ed.),
URL = https://plato.stanford.edu/archives/win2016/entries/states-of-affairs/.

Van Fraassen, B., C., (1969), "Facts and Tautological Entailment", *Journal of Philosophy*, 66 (15):477-487.

Williamson, T. (1999), "Truthmakers and the Converse Barcan Formula" *Dialectica* (1999): 253–70.