A COALGEBRAIC APPROACH TO DUALITIES FOR NEIGHBORHOOD FRAMES

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ABSTRACT. We develop a uniform coalgebraic approach to Thomason and Jónsson-Tarski type dualities for various classes of neighborhood frames and neighborhood algebras. In the first part of the paper we construct an endofunctor on the category of complete and atomic Boolean algebras that is dual to the double powerset functor on Set. This allows us to show that Thomason duality for neighborhood frames can be viewed as an algebra-coalgebra duality. We generalize this approach to any class of algebras for an endofunctor presented by one-step axioms in the language of infinitary modal logic. As a consequence, we obtain a uniform approach to dualities for various classes of neighborhood frames, including monotone neighborhood frames, pretopological spaces, and topological spaces.

In the second part of the paper we develop a coalgebraic approach to Jónsson-Tarski duality for neighborhood algebras and descriptive neighborhood frames. We introduce an analogue of the Vietoris endofunctor on the category of Stone spaces and show that descriptive neighborhood frames are isomorphic to coalgebras for this endofunctor. This allows us to obtain a coalgebraic proof of the duality between descriptive neighborhood frames and neighborhood algebras. Using one-step axioms in the language of finitary modal logic, we restrict this duality to other classes of neighborhood algebras studied in the literature, including monotone modal algebras and contingency algebras.

We conclude the paper by connecting the two types of dualities via canonical extensions, and discuss when these extensions are functorial.

CONTENTS

1. Introduction

Dualities between algebraic and relational semantics are an important tool in the study of modal logics. Among these, Jónsson-Tarski and Thomason dualities play a prominent rôle. Jónsson-Tarski duality establishes that the category of modal algebras is dually equivalent to the category of descriptive Kripke frames, while Thomason duality establishes a dual equivalence between the category of Kripke frames and the category of complete and atomic modal algebras $(A, \Box)$ such that $\Box$ is completely multiplicative. Jónsson-Tarski duality generalizes Stone duality between Boolean algebras and Stone spaces (compact Hausdorff zero-dimensional spaces), and Thomason duality generalizes Tarski duality between complete and atomic Boolean algebras and sets.

A coalgebraic proof of Jónsson-Tarski duality is obtained by lifting Stone duality to an appropriate duality of functors [1, 38]. More precisely, let Stone be the category of Stone spaces and $\mathcal{V} :$ Stone $\to$ Stone the Vietoris endofunctor on Stone. Then the category DKF of descriptive Kripke frames is isomorphic to the category Coalg$(\mathcal{V})$ of coalgebras for $\mathcal{V}$. In [38] an endofunctor $\mathcal{K} :$ BA $\to$ BA on the category BA of Boolean algebras is constructed that is dual to $\mathcal{V}$. Consequently, the category Alg$(\mathcal{K})$ of algebras for $\mathcal{K}$ is dually equivalent to Coalg$(\mathcal{V})$. Since Alg$(\mathcal{K})$ is isomorphic to the category MA of modal algebras, putting the pieces together yields Jónsson-Tarski duality, stating that MA is dually equivalent to DKF.

Recently a similar approach was undertaken to prove Thomason duality. It is well known that the category KF of Kripke frames is isomorphic to the category Coalg$(\mathcal{P})$ of coalgebras for the covariant powerset endofunctor $\mathcal{P}$ on the category Set of sets. In [10] an endofunctor $\mathcal{H}$ is constructed on the category CABA of complete and atomic Boolean algebras that is dual to $\mathcal{P}$. Let CAMA be the category of complete and atomic modal algebras $(A, \Box)$ such that $\Box$ is completely multiplicative. Then CAMA is isomorphic to the category Alg$(\mathcal{H})$ of algebras for $\mathcal{H}$, and we arrive at Thomason duality.

In summary, we have two main dualities for normal modal logic, the "topological" duality of Jónsson and Tarski, and the "discrete" duality of Thomason. There is a tradeoff in the complexity of the structures involved in these dualities. In Jónsson-Tarski duality the

algebra side of the duality is "simple" (finitary), but the frame side is complex (the relational structures need to be equipped with a Stone topology resulting in descriptive Kripke frames). On the other hand, in Thomason duality the frame side of the duality is simple (Kripke frames), while the algebra side is complex (infinitary).

Jónsson-Tarski duality and Thomason duality are related to each other by the following commutative diagram. The diagram on the right is a copy of that on the left, but with the modal algebras and frames presented as algebras and coalgebras for the appropriate functors. The algebraic counterpart of the forgetful functor $\mathcal{U} : \mathsf{DKF} \to \mathsf{KF}$ is given by the canonical extension functor $\sigma : \mathsf{MA} \to \mathsf{CAMA}$ making the diagram on the left commute. (We use $\equiv^{\mathrm{op}}$ to indicate dual equivalence.)

$$
\begin{array}{ccc}
\mathsf{MA} \overset{\equiv^{\mathrm{op}}}{\rule{1.2cm}{0.4pt}} \mathsf{DKF} & \qquad & \mathsf{Alg}(\mathcal{K}) \overset{\equiv^{\mathrm{op}}}{\rule{1.2cm}{0.4pt}} \mathsf{Coalg}(\mathcal{V}) \\
\sigma \downarrow \qquad \downarrow \mathcal{U} & & \sigma \downarrow \qquad \downarrow \mathcal{U} \\
\mathsf{CAMA} \overset{\equiv^{\mathrm{op}}}{\rule{1.2cm}{0.4pt}} \mathsf{KF} & & \mathsf{Alg}(\mathcal{H}) \overset{\equiv^{\mathrm{op}}}{\rule{1.2cm}{0.4pt}} \mathsf{Coalg}(\mathcal{P})
\end{array}
\tag{1.1}
$$

We encounter a similar situation when looking at non-normal modal logic—an extension of classical propositional logic with a unary modal operator that satisfies only the congruence rule. Its algebras are Boolean algebras with an arbitrary endofunction. We call these *neighborhood algebras* and write NA for the category of neighborhood algebras and corresponding homomorphisms. The standard geometric semantics of non-normal modal logic is given by *neighborhood frames*, discovered independently by Scott [47] and Montague [42] (see also [18, 43]). Neighborhood frames can be represented as coalgebras for the double contravariant powerset functor on Set (see, e.g., [57, Example 9.5]). We write NF for the category of neighborhood frames and appropriate morphisms. Došen [20] generalized both Jónsson-Tarski and Thomason dualities to the setting of neighborhood algebras and frames. The former gives rise to the notion of *descriptive neighborhood frames*, and the latter yields *complete atomic neighborhood algebras*. If we write DNF and CANA for the respective categories, then we obtain the following analogue of Diagram (1.1):

$$
\begin{array}{ccc}
\mathsf{NA} & \overset{\equiv^{\mathrm{op}}}{\rule{1.2cm}{0.4pt}} & \mathsf{DNF} \\
\downarrow & & \downarrow \\
\mathsf{CANA} & \overset{\equiv^{\mathrm{op}}}{\rule{1.2cm}{0.4pt}} & \mathsf{NF}
\end{array}
\tag{1.2}
$$

But the analogy is not perfect: in the non-normal case there are two natural versions of the canonical extension, the σ- and π-extensions, that do not coincide. Moreover, neither is functorial. In fact, $\sigma : \mathsf{MA} \to \mathsf{CAMA}$ is a functor because the σ- and π-extensions coincide in the presence of normality. These two extensions have been investigated in the setting of distributive lattices [27, 28], arbitrary lattices [25], and even posets [29, 26]. In Section 6 we will discuss how to obtain functoriality of σ- and π-extensions in some special cases.

An important subcategory of NA is that of *monotone neighborhood frames*. For this subcategory, Hansen [34] and Hansen and Kupke [35] developed an alternative approach to Došen duality. In fact, Hansen's descriptive neighborhood frames are σ-extensions of Došen's descriptive neighborhood frames. While Hansen and Kupke mention a coalgebraic approach to Došen duality, they do not go as far as to prove Došen duality using the coalgebraic approach discussed above. In Section 6.3 we will see how to obtain such a proof from our approach, which we next outline.

Our main goal is to give a uniform (predominantly coalgebraic) approach to dualities for neighborhood frames. We construct an endofunctor $\mathcal{L}$ on the category CABA of complete atomic Boolean algebras (CABAs) that is dual to the double powerset functor. This endofunctor is obtained by modding out the free CABA over a set by the axioms of CANA. The above construction presumes existence of free CABAs. It is well known that free complete Boolean algebras do not exist [24, 32]. However, free CABAs do exist. This follows from the fact that the Eilenberg-Moore algebras of the double contravariant powerset monad are exactly CABAs [54], and that categories of algebras for monads have free objects [2, Proposition 20.7(2)]. A concrete description of free CABAs was given in [10]. We generalize this to any subcategory of CANA axiomatized by one-step axioms in the language of infinitary modal logic. As a consequence, we obtain a uniform approach to duality theory for classes of neighborhood frames, including monotone neighborhood frames, filter frames, and neighborhood contingency frames. Additional correspondence results then give rise to dualities for pretopological spaces, topological spaces, and their various subcategories.

In the second part of the paper we define an analogue of the Vietoris endofunctor on Stone spaces and show that coalgebras for this endofunctor are exactly the descriptive neighborhood frames. This allows us to obtain a coalgebraic proof of the duality between descriptive neighborhood frames and neighborhood algebras. Furthermore, using one-step axioms in the language of finitary modal logic, we restrict this duality to other classes of neighborhood algebras studied in the literature such as normal modal algebras and contingency algebras.

This restriction does not always correspond to known dualities. For example, when restricting neighborhood algebras to monotone Boolean algebra expansions (BAMs) [34, Chapter 7] we do not obtain the duality for monotone modal logic of Hansen and Kupke [35]. In fact, the neighborhood frames underlying the descriptive frames in the duality obtained from our general theory need not be monotone. The descriptive frames used by Hansen and Kupke can be obtained from ours through the theory of canonical extensions. In Section 6 we provide an axiom which guarantees that this construction is functorial. We use this to give an alternative coalgebraic proof of the duality for BAMs in [35].

2. Preliminaries

2.1. **Duality theory for normal modal logics.** Classical modal logic is an extension of classical propositional logic with an additional unary modality $\Box$. The modality $\Box$ is called *normal* if it distributes over finite meets. By the standard interpretation in a Kripke frame (X, R), a state x in X satisfies $\Box\varphi$ if all its R-successors satisfy φ. Clearly a state then satisfies $\Box\varphi \wedge \Box\psi$ iff it satisfies $\Box(\varphi \wedge \psi)$. To interpret a non-normal $\Box$ we need to generalize Kripke semantics to the so-called neighborhood semantics.

Definition 2.1 ([18, 43])**.** A *neighborhood frame* is a pair (X, N) consisting of a set X and a *neighborhood function* $N : X \to \mathcal{PP}X$, where $\mathcal{P}X$ is the powerset of X.

For $x \in X$, we call elements of $N(x)$ the *neighborhoods* of x. A state x in X then satisfies $\Box\varphi$ if the set $[\![\varphi]\!] := \{y \in X \mid y \Vdash \varphi\}$ is a neighborhood of x.

Kripke frames can be thought of as special neighborhood frames. Indeed, if we identify a Kripke frame (X, R) with the neighborhood frame (X, N_R), where $N_R(x) = \{b \subseteq X \mid$

$R[x] \subseteq b\}$, then the interpretation of $\Box\varphi$ in (X, R) and (X, N_R) coincides. (As usual, $R[x] := \{y \in X \mid xRy\}$ denotes the set of R-successors of x.)

The algebraic semantics of normal modal logic is given by *modal algebras*. These are Boolean algebras endowed with a unary function $\boxdot : B \to B$ that preserves finite meets. Together with $\boxdot$-preserving Boolean homomorphisms they form the category MA.

Every Kripke frame (X, R) gives rise to the modal algebra $(\mathcal{P}X, \boxdot_R)$ where $\boxdot_R a := \{x \in X \mid R[x] \subseteq a\}$. By the Jónsson-Tarski representation theorem [37], each modal algebra $(B, \boxdot)$ can be represented as a subalgebra of the modal algebra $(\mathcal{P}X, \boxdot_R)$, where X is the set of ultrafilters of B and R is defined on X by xRy iff $\boxdot a \in x$ implies $a \in y$ for each $a \in B$. In fact, one can endow X with a topology τ such that the Boolean algebra of clopen subsets of (X, τ) is isomorphic to B.

This gives rise to the notion of a *descriptive Kripke frame*, that is, a Kripke frame $(\mathbb{X}, R)$ where $\mathbb{X}$ is a Stone space (a compact Hausdorff zero-dimensional space) and R is a continuous relation on X; see also [57]. Let KF be the category of Kripke frames and bounded morphisms and DKF the category of descriptive Kripke frames and continuous bounded morphisms. We then have the duality for modal algebras and descriptive frames in Theorem 2.2 below. The duality on objects traces back to the 1951 paper by Jónsson and Tarski [37] and was subsequently extended to the categorical duality as we know it now [33, 21, 31].

Theorem 2.2 (Jónsson-Tarski duality)**.** MA *is dually equivalent to* DKF*.*

This theorem generalizes Stone duality [50] between the category BA of Boolean algebras and Boolean homomorphisms and the category Stone of Stone spaces and continuous functions. To obtain a similar duality for KF we recall Tarski duality for complete and atomic Boolean algebras. Let CABA be the category of complete and atomic Boolean algebras and complete Boolean homomorphisms, and let Set be the category of sets and functions. Observe that CABA is a non-full subcategory of BA. Then Tarski duality states that CABA is dually equivalent to Set. On objects, this duality stems from Tarski's 1935 paper [52] and a statement of the full duality can be found in [36, Example 4.6(a)]. The name "Tarski duality" was coined recently in [13]. The duality is obtained by sending a set X to its powerset, conceived of as a CABA, and a CABA B to its set of atoms. On morphisms, a function $f : X \to X'$ is sent to its inverse image, and a complete Boolean homomorphism $h : A \to A'$ to the restriction of its left adjoint to the atoms. Thus, we obtain the contravariant functors $\wp : \mathsf{Set} \to \mathsf{CABA}$ and $at : \mathsf{CABA} \to \mathsf{Set}$ which yield Tarski duality.

Thomason [55] generalized Tarski duality to the category CAMA, whose objects are modal algebras $(B, \boxdot)$ such that B is a CABA and $\boxdot$ is completely multiplicative, i.e., preserves all meets. The morphisms of CAMA are complete modal algebra homomorphisms.

Theorem 2.3 (Thomason duality)**.** CAMA *is dually equivalent to* KF*.*

2.2. **Duality theory for non-normal modal logics.** The algebraic semantics of non-normal classical modal logic is given by pairs $(B, \boxdot)$ where B is a Boolean algebra and $\boxdot : B \to B$ is an arbitrary (not necessarily meet-preserving) function. Došen [20] calls these "modal algebras," but to avoid confusion we will call them *neighborhood algebras*. We write NA for the category of neighborhood algebras and $\boxdot$-preserving Boolean homomorphisms. In addition, we write NF for the category of neighborhood frames and neighborhood morphisms,

where we recall (see, e.g., [43, Definition 2.9]) that a *neighborhood morphism* from (X, N) to (X', N') is a function $f : X \to X'$ such that

$$a' \in N'(f(x)) \quad \text{iff} \quad f^{-1}(a') \in N(x) \tag{2.1}$$

for all $x \in X$ and $a' \subseteq X'$.

Došen [20] generalized Jónsson-Tarski duality and Thomason duality to the setting of neighborhood algebras and frames. The rôle of descriptive Kripke frames is now played by descriptive neighborhood frames (defined in detail in Section 5.1). Together with continuous neighborhood morphisms they comprise the category DNF. Likewise, CAMAs are replaced by *complete atomic neighborhood algebras* (CANAs), i.e., neighborhood algebras $(A, \boxdot)$ such that A is complete and atomic. We write CANA for the category of CANAs and complete neighborhood algebra homomorphisms.

Theorem 2.4 (Došen [20])**.**
  (1) NA *is dually equivalent to* DNF*.*
  (2) CANA *is dually equivalent to* NF*.*

2.3. **The algebra/coalgebra approach.** Recall [2, Definition 5.37] that an *algebra* for an endofunctor $\mathcal{L}$ on a category A is a pair (A, α) consisting of an object $A \in$ A and a morphism $\alpha : \mathcal{L}A \to A$ in A. An $\mathcal{L}$-*algebra morphism* from (A, α) to (A', α') is an A-morphism $h : A \to A'$ such that the following diagram commutes in A:

$$
\begin{array}{ccc}
\mathcal{L}A & \xrightarrow{\mathcal{L}h} & \mathcal{L}A' \\
\alpha \downarrow & & \downarrow \alpha' \\
A & \xrightarrow{h} & A'
\end{array}
$$

We write $\mathsf{Alg}(\mathcal{L})$ for the category of $\mathcal{L}$-algebras and $\mathcal{L}$-algebra morphisms.

Example 2.5.
  (1) It is well known that MA is isomorphic to the category $\mathsf{Alg}(\mathcal{K})$ for the endofunctor $\mathcal{K} : \mathsf{BA} \to \mathsf{BA}$ that sends a Boolean algebra B to the free Boolean algebra generated by the meet-semilattice underlying B ([1], [30, Section 2], [38, Proposition 3.17]).
  (2) As we pointed out in the introduction, free CABAs exist because CABAs are exactly the Eilenberg-Moore algebras of the double contravariant powerset monad [54] (see also [14, Section 5.1]), and categories of Eilenberg-Moore algebras have free objects [2, Proposition 20.7(2)]. In [10] a concrete construction of free CABAs was given and it was shown that CAMA is isomorphic to $\mathsf{Alg}(\mathcal{H})$ for the endofunctor $\mathcal{H} : \mathsf{CABA} \to \mathsf{CABA}$ that sends a CABA B to the free CABA generated by the complete meet-semilattice underlying B.

By an easy adaptation of the proof of [57] and [10], respectively, one can show that:

Proposition 2.6.
  (1) Let $\mathcal{N} : \mathsf{BA} \to \mathsf{BA}$ *be the composition of the forgetful functor* $\mathsf{BA} \to \mathsf{Set}$ *and the free functor* $\mathsf{Set} \to \mathsf{BA}$*. Then*
$$\mathsf{NA} \cong \mathsf{Alg}(\mathcal{N}).$$
  (2) Let $\mathcal{L} : \mathsf{CABA} \to \mathsf{CABA}$ *be the composition of the forgetful functor* $\mathsf{CABA} \to \mathsf{Set}$ *and the free functor* $\mathsf{Set} \to \mathsf{CABA}$*. Then*
$$\mathsf{CANA} \cong \mathsf{Alg}(\mathcal{L}).$$

Proof. If B is a Boolean algebra, then $\mathcal{N}B$ is the free Boolean algebra generated by the set $\{\Box b \mid b \in B\}$. So a homomorphism $\mathcal{N}B \to B$ is uniquely determined by its action on elements of the form $\Box b$. Now given a neighborhood algebra $(B, \boxdot)$, define an $\mathcal{N}$-algebra structure $\alpha_\boxdot : \mathcal{N}B \to B$ via $\alpha_\boxdot(\Box b) = \boxdot b$. Conversely, an $\mathcal{N}$-algebra $\alpha : \mathcal{N}B \to B$ gives rise to the neighborhood algebra $(B, \boxdot_\alpha)$ where $\boxdot_\alpha b = \alpha(\Box b)$. It is easy to verify that these assignments prove the isomorphism of the first item on objects. The verification on morphisms is a routine exercise. The second item can be proven analogously.   □

The dual notion of an algebra is that of a coalgebra. A *coalgebra* for a functor $\mathcal{T} : \mathsf{C} \to \mathsf{C}$ is a pair (X, γ) such that $\gamma : X \to \mathcal{T}X$ is a morphism in C. A $\mathcal{T}$-*coalgebra morphism* from (X, γ) to (X', γ') is a C-morphism $f : X \to X'$ such that the following diagram commutes in C:

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & X' \\
{\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle \gamma'} \\
\mathcal{T}X & \xrightarrow{\ \mathcal{T}f\ } & \mathcal{T}X'
\end{array}
$$

We write $\mathsf{Coalg}(\mathcal{T})$ for the category of $\mathcal{T}$-coalgebras and $\mathcal{T}$-coalgebra morphisms.

Example 2.7.
(1) Let $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$ be the covariant powerset functor on Set. It is well known (see, e.g., [45, Example 2.1]) that KF is isomorphic to $\mathsf{Coalg}(\mathcal{P})$.
(2) Let $\mathcal{T} : \mathsf{Set} \to \mathsf{Set}$ be the contravariant powerset functor on Set.[1] Then NF is isomorphic to $\mathsf{Coalg}(\mathcal{TT})$ [57, Example 9.5]. We abbreviate $\mathcal{B} := \mathcal{TT}$, so $\mathsf{NF} \cong \mathsf{Coalg}(\mathcal{B})$.
(3) Let $\mathcal{V} : \mathsf{Stone} \to \mathsf{Stone}$ be the Vietoris endofunctor on Stone. We recall (see e.g. [38, Definition 2.5]) that the *Vietoris space* $\mathcal{V}\mathbb{X}$ of a Stone space $\mathbb{X}$ is the set of closed subsets of $\mathbb{X}$ whose topology is generated by the (clopen) subbasis

$$\boxminus a = \{c \in \mathcal{V}\mathbb{X} \mid c \subseteq a\}, \qquad \diamondsuit a = \{c \in \mathcal{V}\mathbb{X} \mid c \cap a \neq \varnothing\},$$

where a ranges over the clopen subsets of $\mathbb{X}$. (Note that $\diamondsuit a = \mathcal{V}\mathbb{X} \setminus \boxminus(\mathbb{X} \setminus a)$, so the topology can alternatively be defined by taking the Boolean closure of $\{\boxminus a \mid a \in clp\mathbb{X}\}$ as a basis.) It is well known that $\mathcal{V}X$ is a Stone space, and that the assignment $\mathcal{V}$ extends to an endofunctor on Stone by setting $\mathcal{V}f : \mathcal{V}\mathbb{X} \to \mathcal{V}\mathbb{X}' : c \mapsto f[c]$ for a continuous function $f : X \to X'$. It is well known that DKF is isomorphic to $\mathsf{Coalg}(\mathcal{V})$; see, e.g., [21, 1, 38].
(4) In Section 5.1 we will define a neighborhood analogue $\mathcal{D}$ of the Vietoris endofunctor such that the category DNF of descriptive neighborhood frames is isomorphic to $\mathsf{Coalg}(\mathcal{D})$.

Since MA is isomorphic to $\mathsf{Alg}(\mathcal{K})$ and DKF is isomorphic to $\mathsf{Coalg}(\mathcal{V})$, a convenient way to obtain Jónsson-Tarski duality is to prove that Stone duality between BA and Stone lifts to a dual equivalence between $\mathsf{Alg}(\mathcal{K})$ and $\mathsf{Coalg}(\mathcal{V})$. This can be done by lifting Stone duality to a *duality of functors*.

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[1] Note that $\mathcal{T} : \mathsf{Set} \to \mathsf{Set}$ is the composition of $\wp : \mathsf{Set} \to \mathsf{CABA}$ with the forgetful functor $\mathsf{CABA} \to \mathsf{Set}$.

Definition 2.8. We call functors $\mathcal{L} : \mathsf{BA} \to \mathsf{BA}$ and $\mathcal{T} : \mathsf{Stone} \to \mathsf{Stone}$ *Stone-duals* if the diagram below commutes up to natural isomorphism.

$$
\begin{array}{ccc}
\mathsf{BA} & \underset{clp}{\overset{uf}{\rightleftarrows}} & \mathsf{Stone} \\
\mathcal{L} \downarrow & & \downarrow \mathcal{T} \\
\mathsf{BA} & \underset{clp}{\overset{uf}{\rightleftarrows}} & \mathsf{Stone}
\end{array}
$$

Here *uf* and *clp* are the contravariant functors that establish Stone duality. That is, *uf* denotes the functor that sends a Boolean algebra to its Stone space of ultrafilters and a homomorphism h to h^{-1}. In the other direction, *clp* is the functor that sends a Stone space to its Boolean algebra of clopen sets and a continuous function f to f^{-1}.

Since algebra and coalgebra are dual concepts, we then obtain

$$\mathsf{Alg}(\mathcal{L}) \equiv^{\mathrm{op}} \mathsf{Coalg}(\mathcal{T}).$$

Thus, Jónsson-Tarski duality follows from the fact that $\mathcal{K}$ and $\mathcal{V}$ are Stone-duals [38].

Similarly, since CAMA is isomorphic to $\mathsf{Alg}(\mathcal{H})$ and KF is isomorphic to $\mathsf{Coalg}(\mathcal{P})$, we can obtain Thomason duality by lifting Tarski duality between CABA and Set to a dual equivalence between $\mathsf{Alg}(\mathcal{H})$ and $\mathsf{Coalg}(\mathcal{P})$.

Definition 2.9. We call $\mathcal{L} : \mathsf{CABA} \to \mathsf{CABA}$ and $\mathcal{T} : \mathsf{Set} \to \mathsf{Set}$ *Tarski-duals* if the diagram below commutes up to natural isomorphism.

$$
\begin{array}{ccc}
\mathsf{CABA} & \underset{\wp}{\overset{at}{\rightleftarrows}} & \mathsf{Set} \\
\mathcal{L} \downarrow & & \downarrow \mathcal{T} \\
\mathsf{CABA} & \underset{\wp}{\overset{at}{\rightleftarrows}} & \mathsf{Set}
\end{array}
$$

As with Stone-dual functors, this gives rise to a duality

$$\mathsf{Alg}(\mathcal{L}) \equiv^{\mathrm{op}} \mathsf{Coalg}(\mathcal{T}).$$

Thomason duality can now be obtained from the fact that $\mathcal{H}$ and $\mathcal{P}$ are Tarski-duals, which is proven in [10, Theorem 5.3].

3. Thomason type dualities for neighborhood frames

In this section we derive Thomason type dualities for classes of neighborhood frames. We focus on classes of CANAs and classes of neighborhood frames that are described by the so-called one-step axioms. As we will see, the corresponding dualities are then obtained as algebra/coalgebra dualities. Indeed, classes of CANAs defined by one-step axioms can be viewed as categories of algebras for some endofunctor on CABA, while the corresponding classes of neighborhood frames as categories of coalgebras for an endofunctor on Set. We can then obtain the desired duality as a *duality of functors* (see Definition 2.9).

Our results generalize those in [10], where a functor duality is proved between the endofunctor $\mathcal{H}$ on CABA whose algebras are CAMAs (Example 2.5(2)) and the powerset functor $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$ (Example 2.7(1)). In Example 3.23 we detail how the results of [10] fit in our general scheme. The algebra/coalgebra dualities we obtain pave the way for investigations of the resulting classes of frames using methods of *coalgebraic logic*, such as developed in e.g. [57, 39].

As running examples, we derive Thomason duality for Kripke frames [55] and Došen duality for neighborhood frames [20, Theorem 12]. Section 4 is dedicated to deriving new dualities for various classes of neighborhood frames using the theory from this section.

3.1. **Infinitary languages.** We will work in a modal language with arbitrary conjunctions. For each cardinal κ, let V_κ be a set of variables of cardinality κ and define $\mathfrak{L}^\kappa$ to be the set of formulae generated by the grammar

$$\varphi ::= v \mid \top \mid \neg\varphi \mid \bigwedge_{i \in I} \varphi_i$$

where $v \in V_\kappa$ and I is some index set of cardinality $< \kappa$. We then define $\mathfrak{L}$ as the proper class that contains the formulae in $\mathfrak{L}^\kappa$ for all cardinals κ.

The study of propositional and first-order languages with infinite conjunctions was pioneered by Scott and Tarski [48] and Tarski [53]. In both these references the size of conjunctions (and quantifiers) is bounded by some cardinal. First-order logics that allow conjunctions and disjunctions over arbitrary sets of formulae have been studied comprehensively; see, e.g., [17], [6, Part C], [7], and the references therein.

We next extend the grammar of $\mathfrak{L}(V)$ with a modal operator.

Definition 3.1. For each cardinal κ, let V_κ be a set of variables of cardinality κ and define $\mathfrak{L}^\kappa_\square$ as the set of formulae generated by the grammar

$$\varphi ::= v \mid \top \mid \neg\varphi \mid \bigwedge_{i \in I} \varphi_i \mid \square\varphi$$

where $v \in V_\kappa$ and I is an index set of cardinality $< \kappa$. We then define $\mathfrak{L}_\square$ as the proper class that contains the formulae in $\mathfrak{L}^\kappa_\square$ for all cardinals κ. We will think of elements of $\mathfrak{L}_\square$ as axioms.

Note that $\mathfrak{L}^\kappa$ is the $\square$-free fragment of $\mathfrak{L}^\kappa_\square$.

Definition 3.2. Let $(A, \square)$ be a CANA. An *assignment* for $(A, \square)$ is a family $\vartheta_\kappa : V_\kappa \to A$, where κ ranges over the cardinals. Every assignment $\vartheta_\kappa : V_\kappa \to A$ can be extended in an obvious way to a map

$$\widehat{\vartheta}_\kappa : \mathfrak{L}^\kappa_\square \to A.$$

An assignment gives rise to a map $\widehat{\vartheta} : \mathfrak{L}_\square \to A$ that sends $\varphi \in \mathfrak{L}^\kappa_\square$ to $\widehat{\vartheta}_\kappa(\varphi) \in A$.

(1) If $\varphi \in \mathfrak{L}_\square$, then we say that $(A, \square)$ *validates* φ and write $(A, \square) \Vdash \varphi$ if $\widehat{\vartheta}(\varphi) = 1$ for every assignment $\vartheta : (V_\kappa)_{\kappa \in \mathsf{Card}} \to A$.

(2) If $\mathrm{Ax} \subseteq \mathfrak{L}_\square$ is a class of axioms, then we say that $(A, \square)$ *validates* Ax and write $(A, \square) \Vdash \mathrm{Ax}$ if $(A, \square) \Vdash \varphi$ for all $\varphi \in \mathrm{Ax}$.

Definition 3.3. If Ax is a class of axioms, then we write $\mathsf{CANA}(\mathrm{Ax})$ for the full subcategory of CANA whose objects validate Ax.

3.2. **One-step axioms.** We next concentrate on the so-called one-step axioms. Intuitively, these are formulae in $\mathfrak{L}_\square$ such that every variable occurs in the scope of precisely one box. We will see that, in case Ax consists solely of one-step axioms, the category $\mathsf{CANA}(\mathrm{Ax})$ is isomorphic to $\mathsf{Alg}(\mathcal{L}_{\mathrm{Ax}})$ for some endofunctor $\mathcal{L}_{\mathrm{Ax}}$ on CABA.

Definition 3.4. For each cardinal κ, let V_κ be a set of variables of cardinality κ and define $(\mathfrak{L}_\square^\kappa)^1$ as the set of formulae generated by the grammar

$$\varphi ::= \square\pi \mid \top \mid \neg\varphi \mid \bigwedge_{i\in I} \varphi_i$$

where $\pi \in \mathfrak{L}(V_\kappa)$ and I is some index set of cardinality $< \kappa$. We then define $(\mathfrak{L}_\square)^1$ as the proper class that contains the formulae in $(\mathfrak{L}_\square^\kappa)^1$ for all cardinals κ. A *one-step axiom* is a formula $\varphi \in (\mathfrak{L}_\square)^1$.

Remark 3.5. When viewed as formulae in the modal language $\mathfrak{L}_\square$, the formulae in $(\mathfrak{L}_\square)^1$ are sometimes referred to as "formulae of modal depth 1" or "rank 1 formulae." This justifies our notation. Observe that every one-step axiom is in particular an axiom in the sense of Section 3.1.

Example 3.6.
  (1) Using standard abbreviations $\bot, \rightarrow, \leftrightarrow$, and $\bigvee$, examples of one-step axioms are $\square v \rightarrow \square(v \vee u)$, $\square v \wedge \square u \leftrightarrow \square(v \wedge u)$, and

$$\bigwedge_{\lambda<\kappa} \square v_\lambda \leftrightarrow \square\left(\bigwedge_{\lambda<\kappa} v_\lambda\right),$$

  where $v, u, v_\lambda \in V_\kappa$ and λ, κ are cardinals.
  (2) On the other hand, the axioms $\square\square v \rightarrow \square v$, $v \rightarrow \square v$, and $\square v \wedge v \leftrightarrow \square\square v$ are not one-step axioms.

The appeal of one-step axioms lies in the fact that they define endofunctors on CABA in a structured way. All of these are subfunctors of $\mathcal{L} : \mathsf{CABA} \rightarrow \mathsf{CABA}$ from Proposition 2.6(2). If φ is a one-step axiom and $\vartheta : (V_\kappa)_{\kappa\in\mathsf{Card}} \rightarrow A$ is an assignment, then ϑ gives rise to a map $\overline{\vartheta} : (\mathfrak{L}_\square)^1 \rightarrow \mathcal{L}A$. Note that this does *not* rely on any CANA-structure on A. Indeed, we can use the fact that φ is of modal depth 1 and define $\overline{\vartheta}$ as follows.

Definition 3.7. For a CABA A and assignment $\vartheta : (V_\kappa)_{\kappa\in\mathsf{Card}} \rightarrow A$ we define $\overline{\vartheta} : (\mathfrak{L}_\square)^1 \rightarrow \mathcal{L}A$ recursively via

$$\overline{\vartheta}(\pi) = \square\widehat{\vartheta}(\pi),$$

for $\pi \in \mathfrak{L}$ (which is well defined because π does not contain any boxes), and

$$\overline{\vartheta}(\top) = 1$$
$$\overline{\vartheta}(\neg\varphi) = \neg\overline{\vartheta}(\varphi)$$
$$\overline{\vartheta}\left(\bigwedge_{i\in I} \varphi_i\right) = \bigwedge_{i\in I} \overline{\vartheta}(\varphi_i)$$

Intuitively, the endofunctor on CABA corresponding to a collection Ax of one-step axioms sends $A \in \mathsf{CABA}$ to the free complete atomic Boolean algebra generated by A modulo (instantiations of) the axioms, i.e., modulo the relations $\overline{\vartheta}(\varphi) = 1$, where $\varphi \in \mathrm{Ax}$ and ϑ is an assignment for A. Before defining this functor, we recall the notion of a complete congruence on a CABA.

Definition 3.8. A *complete congruence* on a CABA A is an equivalence relation $\sim$ on the underlying set such that $a \sim b$ implies $\neg a \sim \neg b$, and $a_i \sim b_i$ for all i in some index set I implies $\bigwedge_{i \in I} a_i \sim \bigwedge_{i \in I} b_i$.

If $\sim$ is a complete congruence on A, then we can define the quotient CABA $A/\sim$. Writing $[a]$ for the equivalence class of a, the CABA-operations on $A/\sim$ are defined by $1 = [1_A]$, $\neg[a] = [\neg a]$, and $\bigwedge[a_i] = [\bigwedge a_i]$. Furthermore, the quotient map $q : A \to A/\sim : a \mapsto [a]$ is a complete homomorphism.

The collection of complete congruences on a CABA is closed under arbitrary intersections. Therefore, if we have a collection R of equations on A (that is, a collection of equations of the form $a = b$, where $a, b \in A$), then we can define the complete congruence generated by R to be the smallest complete congruence $\sim_R$ that contains $a \sim_R b$ for all equations $a = b$ in R. Consequently, $A/\sim_R$ is the largest quotient (= complete homomorphic image) of A in which all equations in R hold.

We are now ready to define an endofunctor $\mathcal{L}_{\mathrm{Ax}}$ on CABA from a collection Ax of one-step axioms.

Definition 3.9. Let Ax be a collection of one-step axioms. For $A \in$ CABA, define $\mathcal{L}_{\mathrm{Ax}} A$ to be the free complete atomic Boolean algebra generated by the set $\{\Box a \mid a \in A\}$ modulo the congruence $\sim_{\mathrm{Ax}}$ generated by $\{\overline{\vartheta}(\varphi) \sim_{\mathrm{Ax}} 1\}$, where φ ranges over the axioms in Ax and ϑ ranges over the assignments $(V_\kappa)_{\kappa \in \mathsf{Card}} \to A$ for the variables in φ. For a complete homomorphism $h : A \to A'$ define $\mathcal{L}_{\mathrm{Ax}} h$ on generators by

$$\mathcal{L}_{\mathrm{Ax}} h(\Box a) = \Box h(a).$$

If Ax consists of a single axiom ax, then we write $\mathcal{L}_{(\mathsf{ax})}$ instead of $\mathcal{L}_{\{(\mathsf{ax})\}}$. Many well-known functors in modal logic can be obtained via Definition 3.9. We give two examples.

Example 3.10. If we work with no axioms (i.e., $\mathrm{Ax} = \varnothing$), then the functor that arises from Definition 3.9 is precisely the functor $\mathcal{L}$ from Proposition 2.6(2).

In our next example we recover the endofunctor $\mathcal{H} :$ CABA $\to$ CABA from [10, Section 4], where $\mathcal{H}$ sends a CABA A to the free CABA generated by the complete meet-semilattice underlying A (see Example 2.5(2)).

Intuitively, the CABA $\mathcal{H} A$ is the free CABA generated by the set $\{\Box a \mid a \in A\}$ modulo $\bigwedge_{b \in X} \Box b = \Box \bigwedge X$ for every $X \subseteq A$. Although this looks like the instantiation of a single one-step axiom, it is not. Indeed, if we define

$$\varphi = \bigwedge_{v \in V_\kappa} \Box v \leftrightarrow \Box \left(\bigwedge V_\kappa \right)$$

then the axiom φ only implies κ-*distributivity*. To remedy this, we work with a class of axioms indexed by the class Card of cardinal numbers. For each $\kappa \in \mathsf{Card}$, define

(C_κ) $\bigwedge\{\Box v \mid v \in V_\kappa\} \leftrightarrow \Box \bigwedge V_\kappa$.

Now set

$$\mathcal{C}_\infty = \{(\mathsf{C}_\kappa) \mid \kappa \in \mathsf{Card}\}.$$

Example 3.11. Consider the collection of axioms $\mathrm{Ax} = \mathcal{C}_\infty$. Then the construction of Definition 3.9 yields the functor $\mathcal{H}$ from [10]. Its algebras correspond to complete atomic modal algebras.

Thus, incidentally, the previous example also illustrates the need to allow a proper class of axioms, rather than just a set.

Every $\mathcal{L}_{\mathrm{Ax}}$-algebra (A, α) gives rise to a complete atomic algebra $(A, \boxdot_\alpha)$, where

$$\boxdot_\alpha : A \to A : a \mapsto \alpha(\square a).$$

Furthermore, if $\varphi \in \mathrm{Ax}$, then since $\widehat{\vartheta}(\varphi) \sim_{\mathrm{Ax}} \top$ for all assignments $\vartheta : (V_\kappa)_{\kappa \in \mathsf{Card}} \to A$, we have $(A, \boxdot_\alpha) \Vdash \varphi$. Conversely, if $(A, \boxdot)$ is a CANA and $(A, \boxdot) \Vdash \mathrm{Ax}$, then we can define an $\mathcal{L}_{\mathrm{Ax}}$-algebra structure map $\alpha_\boxdot : \mathcal{L}_{\mathrm{Ax}} A \to A$ on generators by $\alpha_\boxdot(\square a) = \boxdot a$. The fact that $(A, \boxdot) \Vdash \mathrm{Ax}$ implies that $\alpha_\boxdot$ is well defined.

It is easy to see that the two assignments above define a bijection between objects of $\mathsf{Alg}(\mathcal{L}_{\mathrm{Ax}})$ and objects of $\mathsf{CANA}(\mathrm{Ax})$. We can extend this to a natural isomorphism in a standard way.

Theorem 3.12. *If* Ax *is a collection of one-step axioms, then*

$$\mathsf{Alg}(\mathcal{L}_{\mathrm{Ax}}) \cong \mathsf{CANA}(\mathrm{Ax}).$$

Proof sketch. The isomorphism on objects has already been sketched. To prove the isomorphism on morphisms, let (A, α) and (A', α') be two $\mathcal{L}_{\mathrm{Ax}}$-algebras with corresponding CANAs $(A, \boxdot)$ and $(A', \boxdot')$. We claim that a complete homomorphism $h : A \to A'$ is an $\mathcal{L}_{\mathrm{Ax}}$-algebra morphism from (A, α) to (A', α') iff it is a CANA-morphism from $(A, \boxdot)$ to $(A', \boxdot')$.

Note that h is an $\mathcal{L}_{\mathrm{Ax}}$-algebra morphism iff

$$h(\alpha(\square a)) = \alpha'(\mathcal{L}_{\mathrm{Ax}} h(\square a)) \qquad \text{for all } a \in A. \tag{3.1}$$

Since $\alpha(\square a) = \boxdot a$ and $\alpha'(\mathcal{L}_{\mathrm{Ax}} h(\square a)) = \alpha'(\square h(a)) = \boxdot' h(a)$, (3.1) holds iff

$$h(\boxdot a) = \boxdot'(h(a)) \qquad \text{for all } a \in A.$$

In other words, (3.1) holds iff h is a CANA-morphism. $\square$

Going back to the class of axioms $\mathcal{C}_\infty$ from Example 3.11, we see that Theorem 3.12 generalizes [10, Theorem 4.4]:

Example 3.13. Suppose $\mathrm{Ax} = \mathcal{C}_\infty$. Then $\mathcal{L}_{\mathrm{Ax}} = \mathcal{H}$ and $\mathsf{CANA}(\mathrm{Ax}) \cong \mathsf{CAMA}$, so that

$$\mathsf{Alg}(\mathcal{H}) \cong \mathsf{CAMA}.$$

3.3. **A functor duality theorem.** We now classify the atoms of $\mathcal{L}_{\mathrm{Ax}} A$ as subsets of A. This then gives rise to an endofunctor on Set which is a subfunctor of $\mathcal{B}$ of Example 2.7(2) and is dual to $\mathcal{L}_{\mathrm{Ax}}$. We make use of the following notation to characterize the subsets of A we are interested in.

Definition 3.14. Let A be a set and $a \in A$. Define

$$\boxminus a = \{W \subseteq A \mid a \in W\}.$$

We can assign to each one-step formula a subset of $\mathcal{P} A$.

Definition 3.15. Let φ be a one-step axiom, A a set and $\vartheta : V \to A$ an assignment of the variables in φ. Define $\varphi^t[\vartheta]$ to be the subset of $\mathcal{P}A$ given recursively by:

$$(\Box v)^t[\vartheta] = \boxminus \vartheta(v)$$
$$\top^t[\vartheta] = \mathcal{P}A$$
$$(\neg \varphi)^t[\vartheta] = \mathcal{P}A \setminus \varphi^t[\vartheta]$$
$$(\textstyle\bigwedge \varphi_i)^t[\vartheta] = \bigcap \{\varphi_i^t[\vartheta]\}$$

It then follows that:

$$(\textstyle\bigvee \varphi_i)^t[\vartheta] = \bigcup \{\varphi_i^t[\vartheta]\}$$
$$(\varphi \to \psi)^t[\vartheta] = \{W \subseteq A \mid W \in \varphi^t[\vartheta] \Rightarrow W \in \psi^t[\vartheta]\}$$
$$(\varphi \leftrightarrow \psi)^t[\vartheta] = \{W \subseteq A \mid W \in \varphi^t[\vartheta] \Leftrightarrow W \in \psi^t[\vartheta]\}$$

We say that W is a φ-*subset* of A if $W \in \varphi^t[\vartheta]$ for every assignment ϑ of the variables in V. If Ax is a collection of axioms, then we say that W is an Ax-*subset* if W is a φ-subset for all $\varphi \in$ Ax.

The next lemma witnesses the significance of Ax-subsets by proving a bijective correspondence between atoms of $\mathcal{L}_{\mathrm{Ax}}A$ and Ax-subsets of A. Recall that atoms of a CABA A correspond bijectively to complete homomorphisms into the two-element Boolean algebra 2: If $a \in A$ is an atom, then $p_a : A \to 2$, given by $p_a(b) = 1$ iff $a \le b$, defines a complete homomorphism. Conversely, every complete homomorphism p arises in this way, where $a = \bigwedge \{b \in A \mid p(b) = 1\}$.

Lemma 3.16. *Let $A \in$ CABA and let Ax be a collection of axioms. Then the atoms of $\mathcal{L}_{\mathrm{Ax}}A$ correspond bijectively to Ax-subsets of A.*

Proof. We view atoms of $\mathcal{L}_{\mathrm{Ax}}A$ as complete homomorphisms $p : \mathcal{L}_{\mathrm{Ax}}A \to 2$. Since $\mathcal{L}_{\mathrm{Ax}}A$ is defined by generators and relations, p is uniquely determined by its action on the generators of $\mathcal{L}_{\mathrm{Ax}}A$, i.e., the elements of the form $\Box b$ with $b \in A$. Let $W_p \subseteq A$ be the set

$$W_p = \{b \in A \mid p(\Box b) = 1\}. \tag{3.2}$$

Conversely, given a subset $W \subseteq A$, we define $p_W : \mathcal{L}_{\mathrm{Ax}}A \to 2$ on generators by $p_W(\Box b) = 1$ iff $b \in W$.

In order to prove that these assignments are well defined, we use the fact that complete homomorphisms $p : \mathcal{L}_{\mathrm{Ax}}A \to 2$ correspond to complete homomorphisms $p' : \mathcal{L}A \to 2$ whose kernel contains the complete congruence $\sim_{\mathrm{Ax}}$ generated by (instantiations of) the axioms in Ax (see Definition 3.9). Therefore, for $W \subseteq A$, we let $p'_W : \mathcal{L}A \to 2$ be the complete homomorphism defined by $p'_W(\Box b) = 1$ iff $b \in W$.

Claim 3.17. *Let $p : \mathcal{L}A \to 2$ be a complete homomorphism. Then for all one-step formulae φ and assignments $\vartheta : (V_\kappa)_{\kappa \in \mathsf{Card}} \to A$ we have*

$$p(\bar{\vartheta}(\varphi)) = 1 \quad \textit{iff} \quad W_p \in \varphi^t[\vartheta].$$

Proof of claim. We proceed by induction on the complexity of φ. If $\varphi = \Box v$, where v is in one of the V_κ, then we have

$$
\begin{aligned}
p(\overline{\vartheta}(\Box v)) = 1 \quad &\text{iff} \quad p(\Box \vartheta(v)) = 1 \\
&\text{iff} \quad \vartheta(v) \in W_p \\
&\text{iff} \quad W_p \in \boxminus(\vartheta(v)) \\
&\text{iff} \quad W_p \in (\Box v)^t[\vartheta].
\end{aligned}
$$

Let $\varphi = \top$. By definition, $p(\overline{\vartheta}(\varphi)) = 1$ for all complete homomorphisms p. Since $\varphi^t(\vartheta) = \mathcal{P}A$, the result holds for $\varphi = \top$. For negation, we have

$$
\begin{aligned}
p(\overline{\vartheta}(\neg\varphi)) = 1 \quad &\text{iff} \quad p(\neg\overline{\vartheta}(\varphi)) = 1 \\
&\text{iff} \quad p(\overline{\vartheta}(\varphi)) = 0 \\
&\text{iff} \quad W_p \notin \varphi^t[\vartheta] \qquad\qquad\qquad \text{(inductive hypothesis)} \\
&\text{iff} \quad W_p \in \mathcal{P}A \setminus \varphi^t[\vartheta] = (\neg\varphi)^t[\vartheta].
\end{aligned}
$$

Finally, if $\varphi = \bigwedge \varphi_i$ then

$$
\begin{aligned}
p(\overline{\vartheta}(\bigwedge \varphi_i)) = 1 \quad &\text{iff} \quad p(\bigwedge \overline{\vartheta}(\varphi_i)) = 1 \\
&\text{iff} \quad \bigwedge p(\overline{\vartheta}(\varphi_i)) = 1 \\
&\text{iff} \quad W_p \in \varphi_i^t[\vartheta] \quad \text{for all } i \qquad \text{(inductive hypothesis)} \\
&\text{iff} \quad W_p \in \bigcap \varphi_i^t[\vartheta] = (\bigwedge \varphi_i)^t[\vartheta].
\end{aligned}
$$

This completes the proof of the claim. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Now let $p : \mathcal{L}_{\mathrm{Ax}}A \to 2$ be a complete homomorphism. Then composing the quotient map $q : \mathcal{L}A \to \mathcal{L}_{\mathrm{Ax}}A$ with p yields a complete homomorphism

$$
p \circ q : \mathcal{L}A \to 2 \tag{3.3}
$$

whose kernel contains $\sim_{\mathrm{Ax}}$. Moreover, $W_p = W_{p \circ q}$, where $W_{p \circ q}$ is defined as in (3.2) for the complete homomorphism from (3.3), because q sends the generator $\Box a$ of $\mathcal{L}A$ to the equivalence class of $\Box a$ in $\mathcal{L}_{\mathrm{Ax}}A$. Consequently,

$$
W_p \in \varphi^t[\vartheta] \quad \text{iff} \quad W_{p \circ q} \in \varphi^t[\vartheta] \quad \text{iff} \quad (p \circ q)(\overline{\vartheta}(\varphi)) = 1 \quad \text{iff} \quad p(\overline{\vartheta}(\varphi)) = 1.
$$

Since $\overline{\vartheta} \sim_{\mathrm{Ax}} 1$ for all $\varphi \in \mathrm{Ax}$, we have $p(\overline{\vartheta}(\varphi)) = 1$ for all $\varphi \in \mathrm{Ax}$. Using the claim, this proves that W_p is an Ax-subset.

Conversely, a similar computation shows that whenever W is an Ax-subset, then the kernel of $p'_W : \mathcal{L}A \to 2$ contains $\sim_{\mathrm{Ax}}$, and hence p'_W defines a complete homomorphism $p_W : \mathcal{L}_{\mathrm{Ax}}A \to 2$. In addition, for each $b \in A$ we have $p_{W_p}(\Box b) = 1$ iff $b \in W_p$ iff $p(\Box b) = 1$, so $p_{W_p} = p$. Similarly, $b \in W_{p_W}$ iff $p_W(\Box b) = 1$ iff $b \in W$, and hence $W_{p_W} = W$. Thus, these assignments define a bijection. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Guided by Lemma 3.16, we define an endofunctor on Set which we then prove to be the Tarski-dual of $\mathcal{L}_{\mathrm{Ax}}$.

Definition 3.18. Let X be a set and Ax a collection of one-step axioms. Define $\mathcal{B}_{\mathrm{Ax}}X$ to be the set of Ax-subsets of $\mathcal{P}X$. For a function $f : X \to X'$ in Set, define $\mathcal{B}_{\mathrm{Ax}}f$ by

$$
\mathcal{B}_{\mathrm{Ax}}f : \mathcal{B}_{\mathrm{Ax}}X \to \mathcal{B}_{\mathrm{Ax}}X' : W \mapsto \{a' \in \mathcal{P}X' \mid f^{-1}(a') \in W\}.
$$

If Ax consists of a single axiom ax, then we write $\mathcal{B}_{(\mathsf{ax})}$ instead of $\mathcal{B}_{\{(\mathsf{ax})\}}$.

Proposition 3.19. *The assignment $\mathcal{B}_{\mathrm{Ax}}$ is a well-defined endofunctor on* Set.

Proof. Clearly $\mathcal{B}_{\mathrm{Ax}}X$ is a set for every set X. Let $f : X \to X'$ be a function and let $W \in \mathcal{B}_{\mathrm{Ax}}X$. We need to show that $\mathcal{B}_{\mathrm{Ax}}f(W)$ is in $\mathcal{B}_{\mathrm{Ax}}X'$, that is, $\mathcal{B}_{\mathrm{Ax}}f(W)$ is an Ax-subset of $\mathcal{P}X'$. But this follows from the fact that $\mathcal{B}_{\mathrm{Ax}}f(W) \in \boxminus a'$ iff $W \in \boxminus f^{-1}(a')$.

Functoriality of $\mathcal{B}_{\mathrm{Ax}}$ follows from the fact that $\mathcal{B}_{\mathrm{Ax}}$ is a subfunctor of $\mathcal{B}$. $\square$

We are ready to prove the main result of this section.

Theorem 3.20 (Tarski Functor Duality Theorem). *The functors $\mathcal{L}_{\mathrm{Ax}}$ and $\mathcal{B}_{\mathrm{Ax}}$ are Tarski-duals.*

Proof. Define $\xi_X : at(\mathcal{L}_{\mathrm{Ax}}(\wp X)) \to \mathcal{B}_{\mathrm{Ax}}X$ by $p \mapsto W_p$. This defines an isomorphism on objects by Lemma 3.16. We prove that the assignment $\xi = (\xi_X)_{X \in \mathsf{Set}} : at \circ \mathcal{L}_{\mathrm{Ax}} \circ \wp \to \mathcal{B}_{\mathrm{Ax}}$ is natural by showing that for every function $f : X \to X'$ the diagram

$$
\begin{array}{ccc}
at(\mathcal{L}_{\mathrm{Ax}}(\wp X)) & \xrightarrow{\ \xi_X\ } & \mathcal{B}_{\mathrm{Ax}}X \\[4pt]
{\scriptstyle at(\mathcal{L}_{\mathrm{Ax}}(\wp f))}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathcal{B}_{\mathrm{Ax}}f} \\[4pt]
at(\mathcal{L}_{\mathrm{Ax}}(\wp X')) & \xrightarrow{\ \xi_{X'}\ } & \mathcal{B}_{\mathrm{Ax}}X'
\end{array}
$$

commutes. To this end, let $u \in at(\mathcal{L}_{\mathrm{Ax}}(\wp X))$ and let $p : \mathcal{L}_{\mathrm{Ax}}(\wp X) \to 2$ be the corresponding complete homomorphism Furthermore, let $a' \subseteq X'$. Then

$$
\begin{aligned}
a' \in \mathcal{B}_{\mathrm{Ax}}f \circ \xi_X(p) \quad &\text{iff}\quad \xi_X(p)(f^{-1}(a')) = 1 \\
&\text{iff}\quad p(\Box f^{-1}(a')) = 1 \\
&\text{iff}\quad p(\mathcal{L}_{\mathrm{Ax}}(\wp f)(\Box a')) = 1 \\
&\text{iff}\quad at(\mathcal{L}_{\mathrm{Ax}}(\wp f))(p)(\Box a') = 1 \\
&\text{iff}\quad a' \in \xi_{X'} \circ at(\mathcal{L}_{\mathrm{Ax}}(\wp f))(p).
\end{aligned}
$$

This proves the theorem. $\square$

Corollary 3.21. *For every set* Ax *of one-step axioms, we have*

$$\mathsf{Alg}(\mathcal{L}_{\mathrm{Ax}}) \equiv^{\mathrm{op}} \mathsf{Coalg}(\mathcal{B}_{\mathrm{Ax}}).$$

Example 3.22. As the notation suggests, the functor $\mathcal{B}_\varnothing$ defined as in Definition 3.18 using the empty set of axioms is precisely the functor $\mathcal{B}$ from Example 2.7(2). We have seen that its coalgebras are neighborhood frames. Combining this with Example 3.10 and Theorem 3.20 yields that $\mathcal{N}$ and $\mathcal{B}$ are Tarski-duals. As a consequence of Corollary 3.21, we derive Došen's duality [20] for neighborhood frames via

$$\mathsf{CANA} \cong \mathsf{Alg}(\mathcal{N}) \equiv^{\mathrm{op}} \mathsf{Coalg}(\mathcal{B}) \cong \mathsf{NF}.$$

Example 3.23. Let Ax $= \mathcal{C}_\infty$ as in Example 3.13. For a set X, the set $\mathcal{B}_{\mathrm{Ax}}X$ consists of all collections of neighborhoods that are upward closed under inclusion and closed under arbitrary intersections. Such a collection is uniquely determined by its smallest neighborhood, and a straightforward verification shows that $\mathcal{B}_{\mathrm{Ax}}$ is naturally isomorphic to the covariant powerset functor $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$.

As a consequence of Theorem 3.20, we obtain that $\mathcal{P}$ is dual to $\mathcal{H}$, a result that was recently established in [10, Theorem 5.3]. Using the well-known fact that $\mathcal{P}$-coalgebras are Kripke frames and the observation that $\mathcal{H}$-algebras are CAMAs, we arrive at Thomason duality:

$$\mathsf{CAMA} \cong \mathsf{Alg}(\mathcal{H}) \equiv^{\mathrm{op}} \mathsf{Coalg}(\mathcal{P}) \cong \mathsf{KF}.$$

4. Applications

In this section we first derive dualities for various types of neighborhood frames using only one-step axioms and Corollary 3.21. This gives rise to Thomason type dualities for monotone neighborhood frames, contingency neighborhood frames, and filter frames. Each of these is an algebra/coalgebra duality.

Next we show how some of these restrict when we invoke further axioms, which are not necessarily one-step axioms. These results can be seen as correspondence results. Most notably, they allow us to obtain McKinsey-Tarski duality for topological spaces (with interior maps) as an easy restriction of the duality for filter frames. An overview of the dual equivalences discussed in this section is given in Table 1.

4.1. One-step dualities.

4.1.1. *Monotone neighborhood frames.* Monotone modal logic is a well-studied branch of modal logic (see, e.g., [18, 34, 35]). The standard semantics for monotone modal logic is given by *monotone neighborhood frames*. Recall that these are neighborhood frames (X, N) such that $N(x)$ is upward closed under inclusion as a subset of $\mathcal{P}X$. We write MF for the full subcategory of NF whose objects are monotone neighborhood frames. It is well known that $\mathsf{MF} \cong \mathsf{Coalg}(\mathcal{U}p\mathcal{P})$, where $\mathcal{U}p\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$ takes a set X to the collection of subsets of $\mathcal{P}X$ that are upward closed under inclusion [34, 35].

The algebraic semantics of monotone modal logic is given by *monotone Boolean algebra expansions* (BAMs for short) [34, Section 7]. A BAM is a neighborhood algebra $(A, \boxdot)$ such that $\boxdot : A \to A$ is a monotone function (that is, $a \leq b$ implies $\boxdot a \leq \boxdot b$). Let BAM be the full subcategory of NA whose objects are BAMs.

Definition 4.1. Let CABAM be the full subcategory of CANA whose objects are also BAMs.

We can view CABAM as a category of algebras for an endofunctor on CABA. To see this, consider the one-step axiom

(M) $\Box(u \wedge v) \to \Box u$

expressing monotonicity. As a consequence of Theorem 3.12, we have:

Corollary 4.2. $\mathsf{CABAM} \cong \mathsf{Alg}(\mathcal{L}_{(\mathsf{M})})$.

For a set X, the (M)-subsets of $\mathcal{P}X$ are precisely the ones that are up-closed under inclusion. It then follows immediately from the definitions that $\mathcal{B}_{(\mathsf{M})}$ defined as in Definition 3.18 coincides with $\mathcal{U}p\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$ from [35, Section 3.1]. As a consequence of Corollary 3.21, we obtain:

Axioms	Algebras Frames	Objects	Location
None	CANA NF	Complete atomic neighborhood algebras Neighborhood frames	Example 3.22
(Cont)	CAContA ContF	Complete atomic contingency algebras Contingency frames	Section 4.1.2
(Conv)	CACA CNF	Complete atomic convex algebras Convex neighborhood frames	Section 4.1.3
(M)	CABAM MF	Complete atomic monotone BA expansions Monotone neighborhood frames	Section 4.1.1
(N), (C)	caMA FF	Modal algebras over CABAs Filter frames	Section 4.1.4
(C_κ)		κ-additive complete atomic modal algebras κ-complete neighborhood frames	Remark 4.7
$\mathcal{C}_\infty$	CAMA KF	Complete atomic modal algebras Kripke frames	Example 3.23
(N), (C), (T)	PreTop$_{\text{int}}$ PreInt	Pretopological spaces Complete atomic pre-interior algebras	Section 4.2
(N), (C), (T), (4)	Top$_{\text{int}}$ Int	Topological spaces Complete atomic interior algebras	Section 4.2

Table 1: Overview of pairs of dual categories.

Theorem 4.3. *The category* MF *of monotone neighborhood frames is dually equivalent to* CABAM.

Proof. As a consequence of Theorem 3.20, the functors $\mathcal{L}_{(\mathsf{M})}$ and $\mathcal{B}_{(\mathsf{M})} = \mathcal{U}p\mathcal{P}$ are dual. Moreover, CABAM $\cong$ Alg$(\mathcal{L}_{(\mathsf{M})})$ by Corollary 4.2, and MF $\cong$ Coalg$(\mathcal{U}p\mathcal{P})$ by [35, Lemma 3.4]. Therefore, Corollary 3.21 implies that Alg$(\mathcal{L}_{(\mathsf{M})}) \equiv^{\text{op}}$ Coalg$(\mathcal{U}p\mathcal{P})$, which proves the theorem. $\square$

4.1.2. *Duality for neighborhood contingency logic.* A formula is called *contingent* if it is possibly true and possibly false. Otherwise it is non-contingent, i.e., it is necessarily true or necessarily false. Neighborhood contingency logic was recently introduced in [23] to reason about contingent formulae, and is investigated further in [3, 22]. The non-contingency modality $\triangle$ is interpreted in a neighborhood frame (X, N) by

$$x \Vdash \triangle\varphi \quad \text{iff} \quad [\![\varphi]\!] \in N(x) \text{ or } X \setminus [\![\varphi]\!] \in N(x),$$

As a consequence of this definition, the interpretation of formulae does not distinguish whether $a \in N(x)$ or $X \setminus a \in N(x)$ or both. Therefore, it suffices to only consider the neighborhood frames (X, N) that satisfy: for all $x \in X$ and $a \subseteq X$,

$$a \in N(x) \quad \text{iff} \quad X \setminus a \in N(x).$$

We call such frames *contingency frames* and write ContF for the full category of NF consisting of contingency frames. Then ContF is isomorphic to the category of $\mathcal{B}_{(\mathsf{Cont})}$-coalgebras, where (Cont) is the axiom

(Cont) $\square v \leftrightarrow \square \neg v$.

Corollary 3.21 implies that $\mathsf{Coalg}(\mathcal{B}_{(\mathsf{Cont})}) \equiv^{\mathrm{op}} \mathsf{Alg}(\mathcal{L}_{(\mathsf{Cont})})$. As a consequence of Theorem 3.12 we can describe the latter category of algebras explicitly as the full subcategory of CANA whose objects are the CANAs $(A, \boxdot)$ satisfying $\boxdot a = \boxdot \neg a$ for all $a \in A$. We call these *complete atomic contingence algebras* and denote the category they form by CAContA. Thus, putting the above together, we obtain:

Theorem 4.4. ContF $\equiv^{\mathrm{op}}$ CAContA.

4.1.3. *Convex frames.* Our next example concerns *convex neighborhood frames*. These are neighborhood frames (X, N) such that $N(x)$ is a convex subset of $\mathcal{P}X$, meaning that for all $x \in X$, if $a, a' \in N(x)$ and $a \subseteq b \subseteq a'$, then $b \in N(x)$. Write CNF for the full subcategory of NF whose objects are convex neighborhood frames.

As we will see in Section 6, convexity is closely related to the question of functoriality of canonical extensions of neighborhood frames. Convexity is captured by the following axiom:

(Conv) $\square(v \wedge v') \wedge \square(v \vee v'') \rightarrow \square v$

Therefore, CNF $\cong \mathsf{Coalg}(\mathcal{B}_{\{(\mathsf{Conv})\}})$. We call the corresponding algebras *complete atomic convex algebras*, and denote by CACA the full subcategory of CANA whose objects are complete atomic convex algebras. Then CACA $\cong \mathsf{Alg}(\mathcal{L}_{\{(\mathsf{Conv})\}})$, and as a consequence of Corollary 3.21, we obtain:

Theorem 4.5. CNF $\equiv^{\mathrm{op}}$ CACA.

4.1.4. *Filter frames.* Finally, we consider filter frames [18, Section 7.2]. These are of interest because they are as close as we can get to topological spaces using only one-step axioms (see below). Recall that a *filter* on a set X is a subset $F \subseteq \mathcal{P}X$ that is closed under finite intersections and upward closed under inclusion. In particular, this implies that $X \in F$ as X is the empty intersection of subsets of X. A *filter frame* is a neighborhood frame (X, N) such that $N(x)$ is a filter on X for each $x \in X$. Let FF be the full subcategory of NF consisting of filter frames.

From the coalgebraic point of view, filter frames are $\mathcal{B}_{\mathrm{Ax}}$-coalgebras, where Ax consists of the axioms

(N) $\square \top$
(C) $\square u \wedge \square v \leftrightarrow \square(u \wedge v)$

We write $\mathcal{B}_\wedge$ for $\mathcal{B}_{\{(\mathsf{N}),(\mathsf{C})\}}$. We then have FF $\cong \mathsf{Coalg}(\mathcal{B}_\wedge)$. As a consequence of Corollary 3.21, we obtain the dual equivalence

$$\mathsf{Coalg}(\mathcal{B}_\wedge) \equiv^{\mathrm{op}} \mathsf{Alg}(\mathcal{L}_\wedge), \tag{4.1}$$

where $\mathcal{L}_\wedge$ abbreviates $\mathcal{L}_{\{(\mathsf{N}),(\mathsf{C})\}}$.

Since (N) and (C) are the axioms that on the algebra side define modal algebras, $\mathcal{L}_\wedge$-algebras are simply modal algebras whose underlying Boolean algebra is complete and atomic. We write caMA for the full subcategory of CANA whose objects are modal algebras based on complete atomic Boolean algebras. By contrast, recall that CAMA denotes the full subcategory of CANA whose objects $(A, \boxdot)$ are such that A is complete and atomic and $\boxdot$ preserves arbitrary meets. Therefore, CAMA is a full subcategory of caMA. Rephrasing (4.1) yields the following generalization of Thomason duality:

Theorem 4.6. $\mathsf{FF} \equiv^{\mathrm{op}} \mathsf{caMA}$.

Remark 4.7. If we require the collection of neighborhoods at each state to be closed under intersections of size $< \kappa$, where κ is some fixed cardinal, then we obtain the κ-*complete neighborhood frames* from [51, Section 4]. This corresponds to the axiom (C_κ), and yields a dual equivalence with κ-*additive complete atomic modal algebras*. This category lies in between CAMA and caMA.

4.2. **Restrictions/correspondence results.** If a collection of frames or algebras is not given by one-step axioms, we can still derive dualities for them from correspondence results for the axioms under consideration, but they are no longer algebra/coalgebra dualities. This allows us to derive other interesting dualities, such as McKinsey-Tarski duality for topological spaces (and interior maps).

Topological spaces are obtained from filter frames by stipulating the reflexivity and transitivity axioms (T) and (4). Adding only (T) results in the more general notion of pre-topological spaces. For a neighborhood frame (X, N) and $a \subseteq X$, let

$$\boxdot_N(a) = \{y \in X \mid a \in N(y)\}.$$

Definition 4.8.
(1) A *pre-topological space* is a filter frame that satisfies

   (Cent) $a \in N(x)$ implies $x \in a$.

(2) A *topological space* is a pre-topological space that satisfies

   (iv) $a \in N(x)$ implies $\boxdot_N(a) \in N(x)$.

Remark 4.9.
(1) The above definition of topological spaces in the language of neighborhood bases is well known (see, e.g., [58, Theorem 4.5]).
(2) The above definition of pre-topological spaces can for example be found in [49, Appendix A].
(3) Neighborhood frames satisfying (Cent) are called centered [40, Section 1.3], hence the abbreviation.

In the language of topological spaces, neighborhood morphisms correspond to maps that are both continuous and open. Such maps are often called *interior maps* [44, Section III.3]. On the other hand, continuous maps are the ones that satisfy only the left-to-right implication of (2.1).

We write $\mathsf{PreTop_{int}}$ for the category of pre-topological spaces and interior maps, and $\mathsf{Top_{int}}$ for its full subcategory consisting of topological spaces. Clearly both $\mathsf{PreTop_{int}}$ and

$\mathsf{Top}_{\mathsf{int}}$ are full subcategories of FF. Moreover, we have:

$$\mathsf{PreTop}_{\mathsf{int}} \cong \mathsf{FF}(\mathsf{Cent}) \quad \text{and} \quad \mathsf{Top}_{\mathsf{int}} \cong \mathsf{FF}(\mathsf{Cent}, \mathsf{iv}),$$

where $\mathsf{FF}(\mathsf{Cent})$ and $\mathsf{FF}(\mathsf{Cent}, \mathsf{iv})$ are the full subcategories of FF whose objects satisfy (Cent) and $(\mathsf{Cent}, \mathsf{iv})$, respectively.

The duals of topological spaces are given by complete atomic interior algebras, and the duals of pre-topological spaces by complete atomic pre-interior algebras.

Definition 4.10.
   (1) A *pre-interior algebra* is a modal algebra $(B, \boxdot)$ that satisfies

   (T) $\boxdot b \leq b$.

   If B is complete and atomic, $(B, \boxdot)$ is a *complete atomic pre-interior algebra*.
   (2) An *interior algebra* is a pre-interior algebra $(B, \boxdot)$ that satisfies

   (4) $\boxdot\boxdot b \leq \boxdot b$.

   If B is complete and atomic, $(B, \boxdot)$ is a *complete atomic interior algebra*.

Remark 4.11.
   (1) The dual concept of interior operator is that of closure operator. Interior algebras were first introduced by McKinsey and Tarski [41] in the language of closure operators and under the name of closure algebras. Rasiowa and Sikorski [44] called these algebras topological Boolean algebras. The name interior algebra is due to Blok [16].
   (2) Generalizing closure on a powerset to pre-closure yields the notion of *Čech closure spaces* [56, Definition 14.A.1]. This provides an alternate language to talk about complete atomic pre-interior algebras.

Complete atomic pre-interior algebras are simply CANAs that satisfy (N), (C), and (T). We write PreInt for the full subcategory of CANA whose objects are pre-interior algebras. Then PreInt is a full subcategory of caMA. Let Int be the full subcategory of PreInt consisting of interior algebras.

Given a neighborhood frame (X, N) a straightforward verification (see, e.g., [18, Section 7.4]) shows that:

(X, N) validates (Cent) iff its dual $(\wp X, \boxdot_N)$ validates (T).
(X, N) validates (iv) iff its dual $(\wp X, \boxdot_N)$ validates (4).

Thus, we arrive at the following duality theorems:

Theorem 4.12. *The dual equivalence from Theorem 4.6 restricts to*

$$\mathsf{PreInt} \equiv^{\mathrm{op}} \mathsf{PreTop}_{\mathsf{int}} \quad and \quad \mathsf{Int} \equiv^{\mathrm{op}} \mathsf{Top}_{\mathsf{int}}.$$

Remark 4.13. The object part of the dual equivalence $\mathsf{Int} \equiv^{\mathrm{op}} \mathsf{Top}_{\mathsf{int}}$ dates back to McKinsey and Tarski [41], and the morphism part to Rasiowa and Sikorski [44, Section III.3]. See [11] for more details.

Remark 4.14. Restricting Theorem 4.12 further gives rise to dualities for the categories of T_0-spaces, T_1-spaces, P-spaces (that is, topological spaces whose topology is closed under

countable intersections), and Alexandrov spaces (topological spaces whose topology is closed under arbitrary intersections), with interior maps as morphisms.

Remark 4.15. Other topology-like spaces are the so-called "generalized topological spaces" of Császár [19]. Proving correspondence results for the relevant axioms gives rise to a duality for such spaces in a similar manner as for pre-topological spaces.

The findings of this section can be summarized in the following diagram. The horizontal arrows indicate full inclusions of categories. The vertical arrows denote dual equivalences, and are labelled with the relevant theorem or example.

$$\begin{array}{ccccccccccccc}
\mathsf{KF} & \hookrightarrow & \mathsf{Top} & \hookrightarrow & \mathsf{PreTop} & \hookrightarrow & \mathsf{FF} & \longrightarrow & \mathsf{MF} & \longrightarrow & \mathsf{CNF} & \longrightarrow & \mathsf{NF} & \longleftarrow & \mathsf{ContF} \\
\updownarrow{\scriptstyle\text{Exm. }3.23} & & \updownarrow{\scriptstyle\text{Thm. }4.12} & & \updownarrow{\scriptstyle\text{Thm. }4.12} & & \updownarrow{\scriptstyle\text{Thm. }4.6} & & \updownarrow{\scriptstyle\text{Thm. }4.3} & & \updownarrow{\scriptstyle\text{Thm. }4.5} & & \updownarrow{\scriptstyle\text{Exm. }3.22} & & \updownarrow{\scriptstyle\text{Thm. }4.4} \\
\mathsf{CAMA} & \hookrightarrow & \mathsf{Int} & \hookrightarrow & \mathsf{PreInt} & \hookrightarrow & \mathsf{caMA} & \hookrightarrow & \mathsf{CABAM} & \hookrightarrow & \mathsf{CACA} & \hookrightarrow & \mathsf{CANA} & \hookleftarrow & \mathsf{CAContA}
\end{array}$$
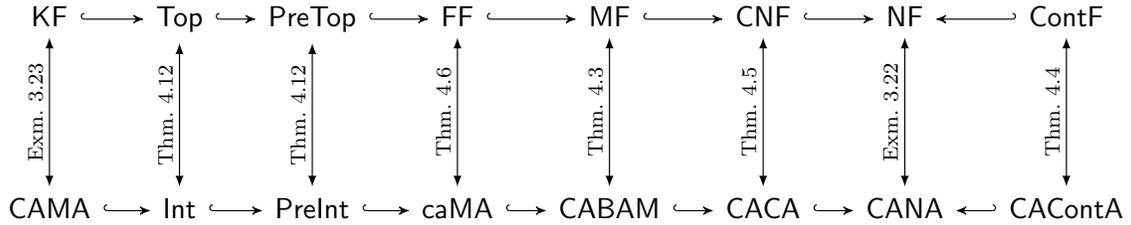
5. Jónsson-Tarski type dualties

In this section we derive categorical dualities for classes of neighborhood algebras that are not necessarily complete and atomic. While this simplifies the algebraic side of our story, it requires extra structure on the frame side of the duality: we now have to work with *descriptive* neighborhood frames [20]. For this we work with one-step axioms in the standard modal language with finitary connectives. As corollaries we derive Jónsson-Tarski duality for modal algebras and Došen duality for neighborhood algebras.

Our main contribution is to define an analogue of $\mathcal{B}$ on Stone spaces by modifying the celebrated Vietoris construction. As a result, we obtain a new endofunctor on Stone and show that the category of coalgebras for this endofunctor is isomorphic to the category of descriptive neighborhood frames of [20, Section 2]. The Vietoris space of a Stone space $\mathbb{X}$ is embeddable in this new hyperspace of $\mathbb{X}$ as a closed subspace.

5.1. **Descriptive neighborhood frames.** We start by recalling the definition of a descriptive neighborhood frame, first introduced by Došen [20, Section 2]. However, to be in line with standard practice in modal logic, our definition of *general* frames deviates slightly from that of Došen in that Došen's definition requires that all neighborhoods are admissible, while we view this as an additional *tightness* condition. So our *tight general frames* correspond to Došen's *general frames*.

Subsequently, we define a new endofunctor on Stone generalizing the Vietoris endofunctor, and show that the category of descriptive neighborhood frames can be viewed as the category of coalgebras for this endofunctor.

Definition 5.1.
(1) A *general (neighborhood) frame* is a tuple (X, N, A) consisting of a neighborhood frame (X, N) and a Boolean subalgebra A of $\mathcal{P}X$ such that A is closed under $\boxdot_N : \mathcal{P}X \to \mathcal{P}X$ given by
$$\boxdot_N a = \{x \in X \mid a \in N(x)\}.$$
(2) A general frame (X, N, A) is *tight* if $N(x) \subseteq A$ for all $x \in X$.

(3) A general frame (X, N, A) is *differentiated* if for all distinct $x, y \in X$ there is $a \in A$ such that $x \in a$ and $y \notin a$; and *compact* if whenever $A' \subseteq A$ has the finite intersection property, then $\bigcap A' \neq \varnothing$.

(4) A *descriptive (neighborhood) frame* is a general frame that is differentiated, compact, and tight.

(5) A *general frame morphism* from (X, N, A) to (X', N', A') is a neighborhood frame morphism $f : (X, N) \to (X', N')$ such that $f^{-1}(a') \in A$ for all $a' \in A'$. We denote the category of descriptive frames and general frame morphisms by DNF.

Let (X, N, A) be a descriptive frame. As usual, we can generate a topology τ_A on X using A as a base. The space (X, τ_A) is compact and Hausdorff because (X, N, A) is compact and differentiated. Moreover, it is zero-dimensional because A is closed under complements, and hence (X, τ_A) is a Stone space. A subset $a \subseteq X$ is in A iff it is clopen in (X, τ_A), so we can recover A from the Stone topology.

Since $A = clp\mathbb{X}$ and $N(x) \subseteq A$, it makes sense to define a functor $\mathcal{D}$ on Stone that sends $\mathbb{X} \in \mathsf{Stone}$ to $\mathcal{P}(clp\mathbb{X})$. The choice of topology on $\mathcal{P}(clp\mathbb{X})$ is motivated by the desideratum to turn $\mathcal{D}\mathbb{X}$ into a Stone space, and is a generalization of the Vietoris topology.

Definition 5.2. The *$\mathcal{D}$-hyperspace* $\mathcal{D}\mathbb{X}$ of a Stone space $\mathbb{X}$ is the space $\mathcal{P}(clp\mathbb{X})$ whose topology is generated by the clopen subbase

$$\boxminus a = \{W \in \mathcal{P}(clp\mathbb{X}) \mid a \in W\}, \qquad \diamondsuit a = \{W \in \mathcal{P}(clp\mathbb{X}) \mid X \setminus a \notin W\},$$

where a ranges over the clopen subsets of $\mathbb{X}$. For a continuous function $f : \mathbb{X} \to \mathbb{X}'$ between Stone spaces, define $\mathcal{D}f : \mathcal{D}\mathbb{X} \to \mathcal{D}\mathbb{X}'$ by

$$\mathcal{D}f(W) = \{a' \in clp\mathbb{X}' \mid f^{-1}(a') \in W\}.$$

Lemma 5.3. *The assignment $\mathcal{D}$ defines an endofunctor on* Stone.

Proof. To see that $\mathcal{D}$ is well defined, we first show that $\mathcal{D}\mathbb{X}$ is a Stone space. Zero-dimensionality of $\mathcal{D}\mathbb{X}$ follows from the fact that it is generated by a base that is closed under complementation. (Indeed, for all $a \in clp\mathbb{X}$ we have $\mathcal{D}\mathbb{X} \setminus \boxminus a = \diamondsuit(X \setminus a)$ and $\mathcal{D}\mathbb{X} \setminus \diamondsuit a = \boxminus(X \setminus a)$.) To see that $\mathcal{D}\mathbb{X}$ is Hausdorff, suppose that $W, W' \in \mathcal{D}\mathbb{X}$ are distinct. Then there must be an $a \in clp\mathbb{X}$ such that either $a \in W$ and $a \notin W'$, or $a \notin W$ and $a \in W'$. In either case $\boxminus a$ and $\diamondsuit(X \setminus a)$ provide two disjoint open subsets of $\mathcal{D}\mathbb{X}$ separating W and W'.

For compactness, by the Alexander subbase theorem, it suffices to prove that every open cover consisting of subbasic (cl)opens has a finite subcover. So suppose

$$\mathcal{D}\mathbb{X} = \bigcup_{a \in A} \boxminus a \cup \bigcup_{b \in B} \diamondsuit b, \tag{5.1}$$

where $A, B \subseteq clp\mathbb{X}$. Consider

$$W = \{X \setminus b \mid b \in B\} \in \mathcal{D}\mathbb{X}.$$

By construction, this is in none of the $\diamondsuit b$, so there must be $a' \in A$ such that $W \in \boxminus a'$. But this means $a' = X \setminus b'$ for some $b' \in B$. Consequently, if V is an arbitrary element of $\mathcal{D}\mathbb{X}$ such that $V \notin \boxminus a'$, then $V \in \mathcal{D}\mathbb{X} \setminus \boxminus a' = \diamondsuit(X \setminus a') = \diamondsuit b'$. Therefore,

$$\mathcal{D}\mathbb{X} = \boxminus a' \cup \diamondsuit b',$$

so we have found a finite subcover of the cover in (5.1). Thus, $\mathcal{D}\mathbb{X}$ is a Stone space.

Finally, we show that $\mathcal{D}$ is well defined on morphisms. Let $f : \mathbb{X} \to \mathbb{X}'$ be a morphism in Stone. In order to prove that $\mathcal{D}f$ is continuous it suffices to show that $(\mathcal{D}f)^{-1}(\boxminus a')$ is clopen in $\mathcal{D}\mathbb{X}$ for all $a' \in clp\mathbb{X}'$. (The case for diamonds follows by working with complements.) So let $a' \in clp\mathbb{X}'$. Then

$$(\mathcal{D}f)^{-1}(\boxminus a') = \{W \in \mathcal{D}\mathbb{X} \mid a' \in \mathcal{D}f(W)\} = \{W \in \mathcal{D}\mathbb{X} \mid f^{-1}(a') \in W\} = \boxminus f^{-1}(a'),$$

which is clopen in $\mathcal{D}\mathbb{X}$. Consequently, $\mathcal{D}$ is well defined. Functoriality follows from the fact that $\mathcal{U}\mathcal{D}$ is a subfunctor of $\mathcal{B}\mathcal{U}$, where $\mathcal{B} = \mathcal{PP}$ and $\mathcal{U} : \mathsf{Stone} \to \mathsf{Set}$ is the forgetful functor. $\qquad\square$

Theorem 5.4. $\mathsf{DNF} \cong \mathsf{Coalg}(\mathcal{D})$.

Proof. For isomorphism on objects, if (X, N, A) is a descriptive neighborhood frame and τ_A is the topology on X generated by A, then N is a function from $\mathbb{X} = (X, \tau_A)$ to $\mathcal{D}\mathbb{X}$ which is continuous because A is closed under $\boxdot_N$ defined in Definition 5.1(1). Thus, $(\mathbb{X}, N)$ is a $\mathcal{D}$-coalgebra.

Conversely, a $\mathcal{D}$-coalgebra $(\mathbb{X}, \gamma)$ gives rise to the descriptive neighborhood frame $(X, N, clp\mathbb{X})$, where X is the set underlying $\mathbb{X}$ and N is defined by $N(x) = \gamma(x)$. Continuity of γ entails that $clp\mathbb{X}$ is closed under $\boxdot_N$.

The isomorphism on morphisms follows from a straightforward verification. $\qquad\square$

Remark 5.5. The Vietoris functor $\mathcal{V}$ is a subfunctor of $\mathcal{D}$ via the natural transformation $\zeta : \mathcal{V} \to \mathcal{D}$ defined on components by

$$\zeta_{\mathbb{X}} : \mathcal{V}\mathbb{X} \to \mathcal{D}\mathbb{X} : c \mapsto \{a \in clp\mathbb{X} \mid c \subseteq a\}.$$

This gives rise to a functor $\widehat{\zeta} : \mathsf{Coalg}(\mathcal{V}) \to \mathsf{Coalg}(\mathcal{D})$, defined on objects by sending $(\mathbb{X}, \gamma) \in \mathsf{Coalg}(\mathcal{V})$ to

$$\mathbb{X} \xrightarrow{\ \gamma\ } \mathcal{V}\mathbb{X} \xrightarrow{\ \zeta_{\mathbb{X}}\ } \mathcal{D}\mathbb{X}$$

and on morphisms by $\widehat{\zeta}f = f$. Specifically, if $(\mathbb{X}, \gamma)$ is a $\mathcal{V}$-coalgebra, then the corresponding $\mathcal{D}$-coalgebra is given by $(\mathbb{X}, N)$, where $N(x) = \{a \in clp\mathbb{X} \mid \gamma(x) \subseteq a\}$. The descriptive neighborhood frames lying in the image of $\widehat{\zeta}$ are precisely those descriptive frames validating (N) and (C); see also Example 5.16.

5.2. **Functor dualities.** Like we did in Section 3.3, we can use one-step axioms to prove that certain quotient functors of $\mathcal{N}$ from Proposition 2.6(1) are Stone-dual to subfunctors of $\mathcal{D}$.

Since $\mathcal{N}$ is an endofunctor on BA, we can only take quotients with finitary axioms. These are axioms that have finite conjunctions and disjunctions. Since this implies that axioms can only contain a finite number of variables, it suffices to work with a countable set of variables. Thus, we will work with the standard modal language, viewed as a sublanguage of $\mathfrak{L}_\square$.

Definition 5.6. A *finitary axiom* is a formula in the language $\mathfrak{L}_\square^\omega$ (defined as in Definition 3.1). A *finitary one-step axiom* is a formula in the language $(\mathfrak{L}_\square^\omega)^1$.

We define assignments and satisfaction of these axioms in neighborhood algebras as in Definition 3.2. That is, an assignment for a neighborhood algebra $(B, \boxdot)$ is a function $\vartheta : V_\omega \to B$ and extends uniquely to a map $\widehat{\vartheta} : \mathfrak{L}_\square^\omega \to B$, and we say $(B, \boxdot) \Vdash \varphi$ if $\widehat{\vartheta}(\varphi) = 1$ for every assignment ϑ.

Furthermore, the assignment ϑ gives rise to a map $\overline{\vartheta} : (\mathfrak{L}_\square^\omega)^1 \to \mathcal{N}B$ which interprets finitary one-step axioms in $\mathcal{N}B$ in the same manner as in Definition 3.4.

Definition 5.7. Let Ax be a collection of finitary one-step axioms. For $B \in \mathsf{BA}$, define $\mathcal{N}_{\mathrm{Ax}}B$ to be the free Boolean algebra generated by $\{\square b \mid b \in B\}$ modulo the congruence relation $\sim_{\mathrm{Ax}}$ generated by $\{\overline{\vartheta}(\varphi) \sim_{\mathrm{Ax}} 1\}$, where φ ranges over Ax and ϑ over the assignments $V_\omega \to B$. For a homomorphism $h : B \to B'$ define $\mathcal{N}_{\mathrm{Ax}}h$ on generators by $\mathcal{N}_{\mathrm{Ax}}h(\square b) = \square h(b)$. Then $\mathcal{N}_{\mathrm{Ax}}$ defines a functor $\mathsf{BA} \to \mathsf{BA}$.

Example 5.8. Again, well-known functors can be obtained via this procedure. Of course, if we take $\mathrm{Ax} = \varnothing$, then we get $\mathcal{N}_\varnothing = \mathcal{N}$. Similarly, the axioms

(N) $\square\top$
(C) $\square v \wedge \square v' \leftrightarrow \square(v \wedge v')$

give rise to the endofunctor on BA whose algebras are normal modal algebras.

Like in Section 3, for each $\mathcal{N}_{\mathrm{Ax}}$ we can define a dual functor $\mathcal{D}_{\mathrm{Ax}}$. The functor $\mathcal{D}_{\mathrm{Ax}}$ arises as a subfunctor of $\mathcal{D}$. In particular, this means that for every Stone space $\mathbb{X}$, the space $\mathcal{D}_{\mathrm{Ax}}\mathbb{X}$ is a subspace of $\mathcal{D}\mathbb{X}$.

Definition 5.9. Let $\mathbb{X}$ be a Stone space and B its dual Boolean algebra of clopens. For a finitary one-step formula φ and an assignment $\vartheta : V_\omega \to B$ of the variables we define the clopen neighborhood $\varphi^t[\vartheta]$ recursively by

$$(\square v)^t[\vartheta] = \boxminus v(\vartheta(a))$$
$$\top^t[\vartheta] = \mathcal{P}B$$
$$(\neg\varphi)^t[\vartheta] = \mathcal{P}B \setminus \varphi^t[\vartheta]$$
$$(\varphi \wedge \psi)^t[\vartheta] = \varphi^t[\vartheta] \cap \psi^t[\vartheta]$$

Let φ be a finitary one-step axiom. Then we call $W \in \mathcal{D}\mathbb{X}$ a φ-subset of A if $W \in \varphi^t[\vartheta]$ for every assignment ϑ of the variables V_ω. If Ax is a collection of finitary one-step axioms, then we say that W is an Ax-*subset* if W is a φ-subset for all axioms $\varphi \in \mathrm{Ax}$.

The next lemma is an analogue of Lemma 3.16 in the setting of descriptive frames.

Lemma 5.10. *Ultrafilters of $\mathcal{N}_{\mathrm{Ax}}B$ correspond bijectively to* Ax-*subsets of B.*

Proof. The bijection is established by sending an ultrafilter of $\mathcal{N}_{\mathrm{Ax}}B$ (viewed as a homomorphism $p : \mathcal{N}_{\mathrm{Ax}}B \to 2$) to the set

$$W_p = \{b \in B \mid p(\square b) = 1\} \subseteq B, \tag{5.2}$$

and by sending any Ax-subset $W \subseteq B$ to the map $p_W : \mathcal{N}_{\mathrm{Ax}}B \to 2$ defined on generators by $p_W(\square b) = 1$ iff $b \in W$. The remainder of the proof can be obtained from the proof of Lemma 3.16 by replacing "$\mathcal{L}$" with "$\mathcal{N}$", "$\mathcal{L}_{\mathrm{Ax}}$" with "$\mathcal{N}_{\mathrm{Ax}}$", "complete homomorphism" with "homomorphism", and "complete congruence" with "congruence". $\square$

We can now define the functor $\mathcal{D}_{\mathrm{Ax}}$ in a similar manner to $\mathcal{B}_{\mathrm{Ax}}$ from Definition 3.18.

Definition 5.11. For a Stone space $\mathbb{X}$, let $\mathcal{D}_{\mathrm{Ax}}\mathbb{X}$ be the subspace of $\mathcal{D}\mathbb{X}$ whose objects are Ax-subset of $clp\mathbb{X}$. For a continuous function $f : \mathbb{X} \to \mathbb{X}'$ let $\mathcal{D}_{\mathrm{Ax}}f$ be the restriction of $\mathcal{D}f$ to $\mathcal{D}_{\mathrm{Ax}}\mathbb{X}$.

While for any set X the set $\mathcal{B}_{\mathrm{Ax}}X$ is automatically a subset of $\mathcal{B}X$, in our current setting it is not guaranteed that $\mathcal{D}_{\mathrm{Ax}}\mathbb{X}$ is a Stone subspace of $\mathcal{D}\mathbb{X}$. The next proposition ensures that this is indeed the case.

Proposition 5.12. $\mathcal{D}_{\mathrm{Ax}}$ *is an endofunctor on* Stone.

Proof. Let $\mathbb{X}$ be a Stone space. In order to show that $\mathcal{D}_{\mathrm{Ax}}\mathbb{X}$ is again a Stone space it suffices to prove that it is a closed subspace of $\mathcal{D}\mathbb{X}$. Note that $\varphi^t[\vartheta]$ is a clopen subset of $\mathcal{D}\mathbb{X}$ for any finitary one-step axiom φ and assignment $\vartheta : V_\omega \to clp\mathbb{X}$. By definition, the set underlying $\mathcal{D}_{\mathrm{Ax}}\mathbb{X}$ is given by

$$\bigcap\{\varphi^t[\vartheta] \mid \varphi \in \mathrm{Ax}, \vartheta : V_\omega \to clp\mathbb{X}\} \subseteq \mathcal{D}\mathbb{X}. \tag{5.3}$$

Since this is the intersection of clopen subsets of $\mathcal{D}\mathbb{X}$, it is a closed subset of $\mathcal{D}\mathbb{X}$.

That $\mathcal{D}_{\mathrm{Ax}}$ is well defined on morphisms can be proven as in Proposition 3.19, and functoriality of $\mathcal{D}_{\mathrm{Ax}}$ follows from functoriality of $\mathcal{D}$. $\qquad\square$

The next theorem is an analogue of Theorem 3.20 in the setting of descriptive frames.

Theorem 5.13 (Stone Functor Duality Theorem)**.** *Let* Ax *be a collection of finitary one-step axioms. Then the functors* $\mathcal{N}_{\mathrm{Ax}}$ *and* $\mathcal{D}_{\mathrm{Ax}}$ *are Stone-duals.*

Proof. We view ultrafilters as homomorphisms, so for example when we say $p \in uf(\mathcal{N}_{\mathrm{Ax}}(clp\mathbb{X}))$ we view p as a homomorphism $\mathcal{N}_{\mathrm{Ax}}(clp\mathbb{X}) \to 2$. For a Stone space $\mathbb{X}$, define the function $\xi_\mathbb{X} : uf(\mathcal{N}_{\mathrm{Ax}}(clp\mathbb{X})) \to \mathcal{D}_{\mathrm{Ax}}\mathbb{X}$ by $p \mapsto W_p$ (defined as in (5.2)). This yields a bijection on objects by Lemma 5.10.

To see that $\xi_\mathbb{X}$ is continuous, recall that the topology on $\mathcal{D}\mathbb{X}$ is generated by $\boxminus a$ and $\diamondsuit a$, where a ranges over the clopens of $\mathbb{X}$. So the topology on $\mathcal{D}_{\mathrm{Ax}}\mathbb{X}$ is generated by

$$\boxminus a \cap \mathcal{D}_{\mathrm{Ax}}\mathbb{X}, \qquad \diamondsuit a \cap \mathcal{D}_{\mathrm{Ax}}\mathbb{X},$$

where a ranges over the clopen subsets of $\mathbb{X}$. Also, recall that for any Boolean algebra B, the topology on $uf B$ is generated by set of the form $\beta(a) = \{p \in uf B \mid p(a) = 1\}$, where $a \in B$. Continuity of $\xi_\mathbb{X}$ now follows from the fact that

$$\xi_\mathbb{X}^{-1}(\boxminus a \cap \mathcal{D}_{\mathrm{Ax}}(\mathbb{X})) = \{p \in uf(\mathcal{N}_{\mathrm{Ax}}(clp\mathbb{X})) \mid p(\Box a) = 1\} = \beta(\Box a)$$

is open in $uf(\mathcal{N}_{\mathrm{Ax}}(clp\mathbb{X}))$, and similarly $\xi_\mathbb{X}^{-1}(\diamondsuit a) = \beta(\neg\Box\neg a)$. Since $\xi_\mathbb{X}$ is a continuous bijection between Stone spaces, it is a homeomorphism, i.e., an isomorphism in Stone.

Finally, we prove that the assignment $\xi = (\xi_\mathbb{X})_{\mathbb{X}\in\mathsf{Stone}} : uf \circ \mathcal{N}_{\mathrm{Ax}} \circ clp \to \mathcal{D}_{\mathrm{Ax}}$ is natural by showing that for every continuous function $f : \mathbb{X} \to \mathbb{X}'$ the diagram

$$
\begin{array}{ccc}
uf(\mathcal{N}_{\mathrm{Ax}}(clp\mathbb{X})) & \xrightarrow{\ \xi_\mathbb{X}\ } & \mathcal{D}_{\mathrm{Ax}}\mathbb{X} \\
{\scriptstyle uf(\mathcal{N}_{\mathrm{Ax}}(clpf))}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathcal{D}_{\mathrm{Ax}}f} \\
uf(\mathcal{N}_{\mathrm{Ax}}(clp\mathbb{X}')) & \xrightarrow{\ \xi_{\mathbb{X}'}\ } & \mathcal{D}_{\mathrm{Ax}}\mathbb{X}'
\end{array}
$$

commutes. To this end, let $p \in \mathit{uf}(\mathcal{N}_{\mathrm{Ax}}(\mathit{clp}\mathbb{X}))$ and $a' \in \mathit{clp}\mathbb{X}'$. Then

$$
\begin{aligned}
a' \in \mathcal{D}_{\mathrm{Ax}} f \circ \xi_{\mathbb{X}}(p) \quad &\text{iff}\quad \xi_{\mathbb{X}}(p)(f^{-1}(a')) = 1 \\
&\text{iff}\quad p(\Box f^{-1}(a')) = 1 \\
&\text{iff}\quad p(\mathcal{N}_{\mathrm{Ax}}(\mathit{clp}f)(\Box a')) = 1 \\
&\text{iff}\quad \mathit{uf}(\mathcal{N}_{\mathrm{Ax}}(\mathit{clp}f))(p)(\Box a') = 1 \\
&\text{iff}\quad a' \in \xi_{\mathbb{X}'} \circ \mathit{uf}(\mathcal{N}_{\mathrm{Ax}}(\mathit{clp}f))(p).
\end{aligned}
$$

This proves the theorem. $\qquad\square$

Corollary 5.14. *For every set* Ax *of finitary one-step axioms, we have*
$$
\mathsf{Alg}(\mathcal{N}_{\mathrm{Ax}}) \equiv^{\mathrm{op}} \mathsf{Coalg}(\mathcal{D}_{\mathrm{Ax}}).
$$

Some well-known dualities are instantiations of Corollary 5.14.

Example 5.15. If we take $\mathrm{Ax} = \varnothing$, then we recover Došen's duality for descriptive neighborhood frames [20, Theorem 6]:
$$
\mathsf{NA} \cong \mathsf{Alg}(\mathcal{N}) \equiv^{\mathrm{op}} \mathsf{Coalg}(\mathcal{D}) \cong \mathsf{DNF}.
$$

Example 5.16. We can derive Jónsson-Tarski duality for normal modal algebras as follows. As we pointed out in Remark 5.5, $\mathcal{V} = \mathcal{D}_{\{(\mathsf{N}),(\mathsf{C})\}}$. Moreover, it is immediate from the definition of $\mathcal{N}_{\{(\mathsf{N}),(\mathsf{C})\}}$ that $\mathsf{MA} \cong \mathsf{Alg}(\mathcal{N}_{\{(\mathsf{N}),(\mathsf{C})\}})$. Therefore,
$$
\mathsf{MA} \cong \mathsf{Alg}(\mathcal{N}_{\{(\mathsf{N}),(\mathsf{C})\}}) \equiv^{\mathrm{op}} \mathsf{Coalg}(\mathcal{D}_{\{(\mathsf{N}),(\mathsf{C})\}}) \cong \mathsf{Coalg}(\mathcal{V}) \cong \mathsf{DKF}.
$$

Corollary 5.14 also gives rise to the notion of a descriptive contingency frame, as shown in the next example.

Example 5.17. Since (Cont) is a finitary axiom, it gives rise to the notions of contingency neighborhood algebras, descriptive contingency neighborhood frames, and a duality between them. A *contingency neighborhood algebra* is a neighborhood algebra $(B, \boxdot)$ such that $\boxdot b = \boxdot \neg b$. We write CNA for the full subcategory of NA of contingency neighborhood algebras. A *descriptive contingency neighborhood frame* is a $\mathcal{D}_{(\mathsf{Cont})}$-coalgebra. Explicitly we can describe these as tuples (X, N, A) such that (X, N, A) is a descriptive neighborhood frame and (X, N) is a contingency neighborhood frame. Writing DCNF for the full subcategory of DNF whose objects are descriptive contingency neighborhood frames, we obtain:
$$
\mathsf{CNA} \cong \mathsf{Alg}(\mathcal{N}_{(\mathsf{Cont})}) \equiv^{\mathrm{op}} \mathsf{Coalg}(\mathcal{D}_{(\mathsf{Cont})}) \cong \mathsf{DCNF}.
$$

5.3. **Forgetting incorrectly.** Just like (Cont) in Example 5.17, the axiom (M) is also finitary. Therefore, we get functors $\mathcal{N}_{(\mathsf{M})}$ and $\mathcal{D}_{(\mathsf{M})}$ (shorthand for $\mathcal{N}_{\{(\mathsf{M})\}}$ and $\mathcal{D}_{\{(\mathsf{M})\}}$) such that $\mathsf{Alg}(\mathcal{N}_{(\mathsf{M})}) \equiv^{\mathrm{op}} \mathsf{Coalg}(\mathcal{D}_{(\mathsf{M})})$. Moreover, we have
$$
\mathsf{Alg}(\mathcal{N}_{(\mathsf{M})}) \cong \mathsf{BAM}.
$$

Thus, one may think that the coalgebras in $\mathsf{Coalg}(\mathcal{D}_{(\mathsf{M})})$ give a suitable notion of descriptive monotone frames. However, care is needed: if we take a $\mathcal{D}_{(\mathsf{M})}$-coalgebra and forget about the topology, we do *not* obtain a $\mathcal{B}_{(\mathsf{M})}$-coalgebra. Indeed, since only clopen sets are allowed to serve as neighborhoods of a state, the collection of neighborhoods at a state need not be upward closed under inclusion. We give an example of this phenomenon.

Example 5.18. Let $X = \mathbb{N} \cup \{\infty\}$ and generate a topology τ on X by the finite subsets of $\mathbb{N}$ and cofinite sets containing ∞. Thus, $\mathbb{X} = (X, \tau)$ is the one-point compactification of the discrete space $\mathbb{N}$. Clearly $\mathbb{X}$ is a Stone space. Define $\gamma : \mathbb{X} \to \mathcal{D}_{(\mathsf{M})}\mathbb{X}$ by

$$\gamma(x) = clp\mathbb{X}.$$

Then $(\mathbb{X}, \gamma)$ is a $\mathcal{D}_{(\mathsf{M})}$-coalgebra. However, (X, γ) does not define a $\mathcal{B}_{(\mathsf{M})}$-coalgebra because $\mathbb{N}_{\mathrm{odd}} = \{x \in \mathbb{N} \mid x \text{ is odd }\}$ is not clopen, hence it is not in $\gamma(x)$, while both $\varnothing \in \gamma(x)$ and $\varnothing \subseteq \mathbb{N}_{\mathrm{odd}}$.

A similar problem occurs with descriptive convex frames. In fact, Example 5.18 also shows that if we forget about the topological structure of a $\mathcal{D}_{(\mathsf{Conv})}$-coalgebra, then we do not necessarily end up with a $\mathcal{B}_{(\mathsf{Conv})}$-coalgebra because $\varnothing \subseteq \mathbb{N}_{\mathrm{odd}} \subseteq X$ and $\varnothing, X \in \gamma(x)$. One way to remedy this is by looking at canonical extensions. Such an approach was carried out in [34] for the special case of monotone frames. We explore this idea in the next section.

6. Canonical extensions

In this section we discuss σ- and π-extensions of (descriptive) neighborhood algebras. In general, these extensions are not functorial. However, we show that adding the convexity and co-convexity axioms yields functoriality of σ- and π-extensions, respectively. We also show that adding these axioms allows us to view descriptive frames as categories of coalgebras. We use this to give an alternative coalgebraic proof of the duality for monotone Boolean algebra expansions of [35].

6.1. σ- and π-**exensions.** There is an obvious forgetful functor $\mathsf{DNF} \to \mathsf{NF}$. Although this forgetful functor is conveniently simple, it has an undesirable property: it "leaves gaps." For example, if (X, N, A) is a monotone descriptive frame, then (X, N) is *not* generally a monotone frame. Indeed, monotonicity now only holds with respect to admissible sets (clopens). Similarly, if (X, N, A) is a normal descriptive frame and R_N is the relation that arises from N via $R_N(x) = \bigcap N(x)$, then we would prefer that the underlying neighborhood frame be normal as well. This would be the case if (X, N) satisfied $a \in N(x)$ iff $R_N[x] \subseteq a$ for each $a \subseteq X$. Again, since $N(x)$ contains only admissible subsets of X, this need not be the case.[2]

To remedy this, we explore alternatives to the forgetful functor. These are dual versions of the well-known σ- and π-extensions from the theory of canonical extensions (see, e.g., [37, 27, 25, 28]):

Definition 6.1. Let A be a CABA and B a Boolean subalgebra of A.

(1) B is called *dense* in A if every element in A is the join of meets of elements in B.
(2) We say that B is *compact* in A if for all sets $S, T \subseteq B$ with $\bigwedge S \leq \bigvee T$ in A, there exist finite $S' \subseteq S$ and $T' \subseteq T$ such that $\bigwedge S' \leq \bigvee T'$.
(3) A *canonical extension* of a Boolean algebra B is a pair (B^σ, e) where B^σ is a CABA and $e : B \to B^\sigma$ is a Boolean embedding such that $e[B]$ is dense and compact in B^σ.

---

[2]For example, let $\mathbb{X} = \mathbb{N} \cup \{\infty\}$ be the one-point compactification of $\mathbb{N}$ with the discrete topology and X the set underlying $\mathbb{X}$. For each $x \in X$ let $N(x)$ be the collection of co-finite subsets of X containing ∞. Then it is easy to see that $(X, N, clp\mathbb{X})$ is a descriptive neighborhood frame, that $R_N(x) = \{\infty\}$, but that $\{\infty\} \notin N(x)$.

Remark 6.2. It is well known [37] that the canonical extension of a Boolean algebra is unique up to isomorphism, and can explicitly be described as the powerset of the dual Stone space of B. Thus, we often speak of B^σ as *the* canonical extension of B and view B as sitting inside B^σ.

Let $\boxdot : B \to B$ be a endofunction (not necessarily a homomorphism). Then we can extend $\boxdot$ to a function $B^\sigma \to B^\sigma$ in several ways. Two well-known extensions are the σ- and π-extensions. To define these, we recall the notions of closed and open elements of B^σ [37, Definition 1.20].

We say that $x \in B^\sigma$ is *closed* if it is a meet of elements from B, and x is *open* if it is a join of elements from B. We write $\mathcal{K}_B$ and $\mathcal{O}_B$ for the sets of closed and open elements of B^σ, respectively. The σ- and π-extensions of $\boxdot : B \to B$ are now defined by

$$\boxdot^\sigma x = \bigvee \left\{ \bigwedge \{\boxdot b \mid b \in B \text{ and } c \le b \le d\} \mid c \in \mathcal{K}_B, d \in \mathcal{O}_B, c \le x \le d \right\}$$

$$\boxdot^\pi x = \bigwedge \left\{ \bigvee \{\boxdot b \mid b \in B \text{ and } c \le b \le d\} \mid c \in \mathcal{K}_B, d \in \mathcal{O}_B, c \le x \le d \right\}$$

These give maps $\sigma, \pi : \mathsf{NA} \to \mathsf{CANA}$, which in turn give rise to $\Sigma, \Pi : \mathsf{DNF} \to \mathsf{NF}$ by composing as follows:

$$\Sigma = at \circ \sigma \circ clp \quad \text{and} \quad \Pi = at \circ \pi \circ clp.$$

Thus we have the following diagram:

$$
\begin{array}{ccc}
\mathsf{NA} & \underset{clp}{\overset{uf}{\rightleftarrows}} & \mathsf{DNF} \\
{\scriptstyle\sigma}\Big\downarrow\Big\downarrow{\scriptstyle\pi} & & {\scriptstyle\Sigma}\Big\downarrow\Big\downarrow{\scriptstyle\Pi} \\
\mathsf{CANA} & \underset{\wp}{\overset{at}{\rightleftarrows}} & \mathsf{NF}
\end{array}
\tag{6.1}
$$

We point out that $\sigma, \pi : \mathsf{NA} \to \mathsf{CANA}$ are not necessarily functors (see [12, Example 3.4 and Remark 3.5]), and hence neither are Σ and Π. We shall temporarily ignore this issue and focus solely on the action of σ and π on objects.

We next define Σ and Π explicitly. For this we need the notions of closed and open elements of (X, N, A). These are defined to be the closed and open sets of the topological space $\mathbb{X} = (X, \tau_A)$, and denoted by $\mathcal{K}_A$ and $\mathcal{O}_A$, respectively. Finally, for $c, d \in \mathcal{P}X$ define $[c, d] = \{e \in \mathcal{P}X \mid c \subseteq e \subseteq d\}$.

Definition 6.3. Let (X, N, A) be a descriptive neighborhood frame.

(1) Define the σ-extension of N by

$$N^\sigma(x) = \{e \in \mathcal{P}X \mid \exists c \in \mathcal{K}_A, d \in \mathcal{O}_A \text{ with } c \subseteq e \subseteq d \text{ and } [c, d] \cap A \subseteq N(x)\},$$

and set $\Sigma(X, N, A) = (X, N^\sigma)$.

(2) Define the π-extension of N by

$$N^\pi(x) = \{e \in \mathcal{P}X \mid \forall c \in \mathcal{K}_A, d \in \mathcal{O}_A \text{ with } c \subseteq e \subseteq d \text{ we have } [c, d] \cap A \cap N(x) \ne \varnothing\},$$

and set $\Pi(X, N, A) = (X, N^\pi)$.

Proposition 6.4. *The following diagrams commute on objects, up to natural isomorphism.*

$$
\begin{array}{ccc}
\text{NA} \underset{clp}{\overset{uf}{\rightleftarrows}} \text{DNF} & \qquad & \text{NA} \underset{clp}{\overset{uf}{\rightleftarrows}} \text{DNF} \\
\sigma \downarrow \qquad \downarrow \Sigma & & \pi \downarrow \qquad \downarrow \Pi \\
\text{CANA} \underset{\wp}{\overset{at}{\rightleftarrows}} \text{NF} & & \text{CANA} \underset{\wp}{\overset{at}{\rightleftarrows}} \text{NF}
\end{array}
$$

Proof. Let (X, N, A) be a descriptive neighborhood frame. Recall that $\Box_N a = \{x \in X \mid a \in N(x)\}$ for $a \in A$. Then $(A, \Box_N)$ is the dual modal algebra of (X, N, A). It is well known that the canonical extension of A is $\wp X$. Write (X, N') for the neighborhood frame dual to $(A^\sigma, \Box_N^\sigma)$. We aim to show that for all $x \in X$ and $e \subseteq X$ we have $e \in N'(x)$ iff $e \in N^\sigma(x)$. By definition,

$$
\Box_N^\sigma e = \bigcup \left\{ \bigcap \{\Box_N a \mid a \in A, c \subseteq a \subseteq d\} \mid c \in \mathcal{K}_A, d \in \mathcal{O}_A, c \subseteq e \subseteq d \right\}.
$$

Thus, $e \in N'(x)$ iff $x \in \Box_N^\sigma e$, which happens iff there are closed c and open d such that $c \subseteq e \subseteq d$ and $a \in [c, d] \cap A$ implies $x \in \Box_N a$. The latter means that we have $a \in N(x)$ for all such a, and hence c and d witness the fact that $e \in N^\sigma(x)$.

A similar reasoning proves the statement for π-extensions. $\qquad \square$

The extensions Σ and Π are closely related. To see this, we need the notion of *dual* (descriptive) neighborhood frames.

Definition 6.5. For a neighborhood frame (X, N), define

$$
N^c : X \to \mathcal{PP}X : x \mapsto \{a \subseteq X \mid a \notin N(x)\}.
$$

We call $(X, N)^c := (X, N^c)$ the *complement* of (X, N).

If (X, N, A) is a descriptive neighborhood frame, then we define

$$
N_A^c : X \to \mathcal{PP}X : x \mapsto \{a \in A \mid a \notin N(x)\}.
$$

It is easy to see that (X, N_A^c, A) is again a descriptive frame. We call (X, N_A^c, A) the *complement* of (X, N, A) and denote it by $(X, N, A)^c$.

It is easy to see that $((X, N)^c)^c = (X, N)$. Moreover, $(\cdot)^c : \text{NF} \to \text{NF}$ defines an involution, where $f^c = f$ for a neighborhood morphism f. Similar statements hold for descriptive frames and the descriptive complement.

Proposition 6.6. *Let (X, N, A) be a descriptive neighborhood frame. Then*

$$
\Pi(X, N, A) = (\Sigma(X, N_A^c, A))^c.
$$

Consequently, $\Sigma(X, N, A) = (\Pi(X, N_A^c, A))^c$.

Proof. Note that $(\Sigma(X, N_A^c, A))^c = (X, ((N_A^c)^\sigma)^c)$. We need to prove that for all $x \in X$ and $e \subseteq X$ we have

$$
e \in N^\pi(x) \quad \text{iff} \quad e \in ((N_A^c)^\sigma)^c(x). \tag{6.2}
$$

We do so by unravelling the definitions.

For $x \in X$ and $e \subseteq X$ we have $e \in ((N_A^c)^\sigma)^c(x)$ iff $e \notin (N_A^c)^\sigma(x)$. In other words, $e \in ((N_A^c)^\sigma)^c(x)$ iff we can find no closed $c \in \mathcal{K}_A$ and open $d \in \mathcal{O}_A$ such that $c \subseteq e \subseteq d$ and $[c, d] \cap A \subseteq N^c(x)$. Therefore, $e \in ((N_A^c)^\sigma)^c(x)$ iff for all $c \in \mathcal{K}_A$ and $d \in \mathcal{O}_A$ such that $c \subseteq e \subseteq d$ we have $[c, d] \cap A \cap N(x) \neq \varnothing$. But this is exactly the definition of $e \in N^\pi(x)$, so (6.2) holds. $\qquad \square$

6.2. **σ- and π-descriptive frames.** As we have seen in Section 5.3, we do not always have a forgetful functor $\mathcal{U} : \mathsf{Coalg}(\mathcal{D}_{\mathrm{Ax}}) \to \mathsf{Coalg}(\mathcal{B}_{\mathrm{Ax}})$. In particular, recall that convexity and monotonicity are not preserved. This is, in part, solved by replacing $\mathcal{U}$ with either Σ or Π.

If (X, N, A) is monotone, then the definition of N^σ simplifies to

$$N^\sigma(x) = \{e \in \mathcal{P}X \mid \exists c \in \mathcal{K}_A \text{ with } c \subseteq e \text{ and } [c, X] \cap A \subseteq N(x)\}.$$

It is then easy to see that the neighborhood frame $\Sigma(X, N, A) = (X, N^\sigma)$ is monotone as well. Next suppose (X, N, A) is convex and there are $e, e' \in N^\sigma(x)$ and $e'' \subseteq X$ such that $e \subseteq e'' \subseteq e'$. Then by definition of N^σ we have closed sets c, c' and open sets d, d' such that $e \in [c, d]$, $e' \in [c', d']$, $[c, d] \cap A \subseteq N(x)$ and $[c', d'] \cap A \subseteq N(x)$. Since $c \subseteq e \subseteq e'' \subseteq e' \subseteq d'$, convexity of (X, N, A) implies $[c, d'] \cap A \subseteq N(x)$. This, in turn, witnesses the fact that $e'' \in N^\sigma(x)$. Therefore, (X, N^σ) is a convex neighborhood frame.

Thus, on objects we have the following well-defined assignments:

$$\Sigma : \mathsf{Coalg}(\mathcal{D}_{(\mathsf{M})}) \to \mathsf{Coalg}(\mathcal{B}_{(\mathsf{M})}) \qquad \text{and} \qquad \Sigma : \mathsf{Coalg}(\mathcal{D}_{(\mathsf{Conv})}) \to \mathsf{Coalg}(\mathcal{B}_{(\mathsf{Conv})}).$$

From the connection between Σ and Π discussed in Proposition 6.6 we get that

$$\Pi : \mathsf{Coalg}(\mathcal{D}_{(\mathsf{M})}) \to \mathsf{Coalg}(\mathcal{B}_{(\mathsf{M})}) \qquad \text{and} \qquad \Pi : \mathsf{Coalg}(\mathcal{D}_{(\mathsf{CoConv})}) \to \mathsf{Coalg}(\mathcal{B}_{(\mathsf{CoConv})})$$

are well-defined assignments as well. Here (CoConv) denotes the co-convexity axiom

$$(\mathsf{CoConv}) \quad \Box v \to \Box(v \wedge v') \vee \Box(v \vee v'')$$

On neighborhood frames (X, N) this corresponds to $N^c(x)$ being convex for all $x \in X$.

We can turn the assignments Σ and Π into "proper" forgetful functors by incorporating the additional neighborhoods that arise from Σ or Π into the notion of a descriptive frame. This yields two alternative definitions of descriptive frames: σ-descriptive and π-descriptive frames. In what follows we will focus on σ-descriptive frames, leaving the dual treatment of π-descriptive frames to the reader.

Definition 6.7. A σ-*descriptive neighborhood frame* is a general neighborhood frame (X, N, A) that is differentiated and compact (see Definition 5.1) and satisfies the following modification of the tightness condition: for all $x \in X$ and $e \in \mathcal{P}X$

$$e \in N(x) \quad \text{iff} \quad \exists c \in \mathcal{K}_A, d \in \mathcal{O}_A \text{ with } c \subseteq e \subseteq d \text{ and } [c, d] \cap A \subseteq N(x).$$

We write DNF^σ for the category of σ-descriptive neighborhood frames and general neighborhood morphisms.

If (X, N, A) is a descriptive neighborhood frame, then (X, N^σ, A) is a σ-descriptive neighborhood frame. Conversely, given a σ-descriptive neighborhood frame (X, N, A), setting $N_A(x) = N(x) \cap A$ yields a descriptive neighborhood frame (X, N_A, A). It is straightforward to see that these two assignments give rise to a bijective correspondence between objects in DNF and objects in DNF^σ. Thus, the following diagram commutes on objects, where $\mathcal{U}$ is the forgetful functor that does not add any neighborhoods:

$$
\begin{array}{ccc}
\mathsf{DNF} & \xleftrightarrow{\ 1\text{-}1\ } & \mathsf{DNF}^\sigma \\
& {\scriptstyle \Sigma}\searrow \quad \swarrow{\scriptstyle \mathcal{U}} & \\
& \mathsf{NF} &
\end{array}
$$

While in descriptive neighborhood frames (Definition 5.1) the tightness condition stipulates that all neighborhoods are admissible sets, σ-tightness allows non-admissible sets to act as neighborhoods too.

Remark 6.8. Let DNF^π be the category of π-descriptive frames, defined analogously to Definition 6.7. That is, a π-descriptive frame is a general neighborhood frame that is differentiated and compact and satisfies for all $x \in X$ and $e \subseteq X$:

$$e \in N(x) \quad \text{iff} \quad \forall c \in \mathcal{K}_A, d \in \mathcal{O}_A \text{ with } c \subseteq e \subseteq d \text{ we have } [c, d] \cap A \cap N(x) \neq \varnothing.$$

We have that (X, N, A) is σ-descriptive iff (X, N_A^c, A) is π-descriptive. Moreover, $f : (X, N, A) \to (X', N', A')$ is a general morphism between σ-descriptive frames iff f is a general morphism between (X, N_A^c, A) and $(X', (N')_A^c, A')$. Thus, we obtain an isomorphism between DNF^σ and DNF^π.

Write $\mathsf{DNF}^\sigma(\mathrm{Ax})$ for the full subcategory of DNF such that the axioms in Ax are satisfied when interpreting the variables v as clopens. Write also $\mathcal{U}_{\mathrm{Ax}} : \mathsf{DNF}^\sigma(\mathrm{Ax}) \to \mathsf{Coalg}(\mathcal{B}_{\mathrm{Ax}})$ for the functor that sends a σ-descriptive neighborhood frame to its underlying frame (viewed as a coalgebra). If $\mathcal{U}_{\mathrm{Ax}}$ is well defined, then it is automatically a functor, because the additional (non-admissible) neighborhoods of σ-descriptive frames ensure that every morphism in $\mathsf{DNF}^\sigma(\mathrm{Ax})$ is in particular a neighborhood morphism between the underlying neighborhood frames.

While the introduction of σ-descriptive frames ensures that $\mathcal{U}_{\mathrm{Ax}} : \mathsf{DNF}^\sigma(\mathrm{Ax}) \to \mathsf{Coalg}(\mathcal{B}_{\mathrm{Ax}})$ becomes a functor, it only moves the problem of functoriality elsewhere. Indeed, we are not guaranteed that the categories $\mathsf{Alg}(\mathcal{N}_{\mathrm{Ax}})$ is dual to $\mathsf{DNF}^\sigma(\mathrm{Ax})$.

In Section 6.3 we will prove that whenever Ax implies (Conv), then

(1) $\mathsf{DNF}^\sigma(\mathrm{Ax})$ is a category of coalgebras for an endofunctor $\mathcal{D}_{\mathrm{Ax}}^\sigma$ on Stone; and
(2) $\mathcal{N}_{\mathrm{Ax}}$ is dual to $\mathcal{D}_{\mathrm{Ax}}^\sigma$.

Combined, these give the dual equivalence $\mathsf{Alg}(\mathcal{N}_{\mathrm{Ax}}) \equiv^{\mathrm{op}} \mathsf{Coalg}(\mathcal{D}_{\mathrm{Ax}}^\sigma) \cong \mathsf{DNF}^\sigma(\mathrm{Ax})$.

6.3. **When are Σ and Π functors?** In this section we give a sufficient condition for $\mathcal{D}_{\mathrm{Ax}}^\sigma$ to be a functor.

Definition 6.9. Let $\mathbb{X}$ be a Stone space with the underlying set X. We write $\mathcal{K}\mathbb{X}$ and $\mathcal{O}\mathbb{X}$ for the closed and open sets of $\mathbb{X}$ and define $\mathcal{D}^\sigma \mathbb{X}$ to be the space consisting of $W \subseteq \mathcal{P}X$ that satisfy

$$e \in W \quad \text{iff} \quad \exists c \in \mathcal{K}\mathbb{X}, d \in \mathcal{O}\mathbb{X} \text{ with } c \subseteq e \subseteq d \text{ and } [c, d] \cap clp\mathbb{X} \subseteq W.$$

The topology on $\mathcal{D}^\sigma \mathbb{X}$ is generated by the clopen subbase

$$\boxminus a = \{W \mid a \in W\}, \qquad \diamondsuit a = \{W \mid X \setminus a \notin W\},$$

where a ranges over $clp\mathbb{X}$.

Using the fact that elements of $\mathcal{D}^\sigma \mathbb{X}$ are determined uniquely by the clopens of $\mathbb{X}$ they contain, combined with assignments similar to the ones in the paragraph following Definition 6.7, it is easy to see that $\mathcal{D}^\sigma \mathbb{X}$ is homeomorphic to $\mathcal{D}\mathbb{X}$. Therefore, $\mathcal{D}^\sigma \mathbb{X}$ is a Stone space. Moreover, $W \in \mathcal{D}^\sigma \mathbb{X}$ is an Ax-subset (see Definition 5.9) iff $W \cap clp\mathbb{X} \in \mathcal{D}\mathbb{X}$ is an Ax-subset. Thus, we define:

Definition 6.10. For a Stone space $\mathbb{X}$, let $\mathcal{D}_{\mathrm{Ax}}^{\sigma}\mathbb{X}$ be the subspace of $\mathcal{D}^{\sigma}\mathbb{X}$ whose elements are Ax-subsets. For a continuous function $f : \mathbb{X} \to \mathbb{X}'$, we define

$$\mathcal{D}_{\mathrm{Ax}}^{\sigma}f : \mathcal{D}_{\mathrm{Ax}}^{\sigma}\mathbb{X} \to \mathcal{D}_{\mathrm{Ax}}^{\sigma}\mathbb{X}' : W \mapsto \{w' \subseteq \mathbb{X}' \mid f^{-1}(w') \in W\}.$$

Since $\mathcal{D}_{\mathrm{Ax}}^{\sigma}\mathbb{X}$ is homeomorphic to $\mathcal{D}_{\mathrm{Ax}}\mathbb{X}$, we have that $\mathcal{D}_{\mathrm{Ax}}^{\sigma}$ sends a Stone space to a Stone space. Furthermore, this implies that $\mathcal{D}_{\mathrm{Ax}}^{\sigma}$ is naturally isomorphic to $\mathcal{D}_{\mathrm{Ax}}$ *whenever the former is well defined*. We prove that it is well defined when Ax implies (Conv).

Theorem 6.11. *Let* Ax *be a set of finitary one-step axioms such that* Ax *implies* (Conv). *Then* $\mathcal{D}_{\mathrm{Ax}}^{\sigma}$ *defines an endofunctor on* Stone.

Proof. We have already seen that $\mathcal{D}_{\mathrm{Ax}}^{\sigma}$ is well defined on objects, and functoriality is straightforward because it is a subfunctor of $\mathcal{B}$. Therefore, we only have to prove that for every continuous function $f : \mathbb{X} \to \mathbb{X}'$, the assignment $\mathcal{D}_{\mathrm{Ax}}^{\sigma}f : \mathcal{D}_{\mathrm{Ax}}^{\sigma}\mathbb{X} \to \mathcal{D}_{\mathrm{Ax}}^{\sigma}\mathbb{X}'$ is a well-defined continuous function.

Continuity follows from the fact that $(\mathcal{D}_{\mathrm{Ax}}^{\sigma}f)^{-1}(\boxminus a') = \boxminus f^{-1}(a')$ for all $a' \in clp\mathbb{X}'$. So all that is left is to prove that $\mathcal{D}_{\mathrm{Ax}}^{\sigma}f(W) \in \mathcal{D}_{\mathrm{Ax}}^{\sigma}\mathbb{X}'$ for all $W \in \mathcal{D}_{\mathrm{Ax}}^{\sigma}\mathbb{X}$. That is, we need to show that for all $e' \subseteq X'$,

$$e' \in \mathcal{D}_{\mathrm{Ax}}^{\sigma}f(W) \quad \text{iff} \quad \exists c' \in \mathcal{K}\mathbb{X}', d' \in \mathcal{O}\mathbb{X}' \tag{6.3}$$
$$\text{with } c' \subseteq e' \subseteq d' \text{ and } [c', d'] \cap clp\mathbb{X}' \subseteq \mathcal{D}_{\mathrm{Ax}}^{\sigma}f(W).$$

First assume $e' \in \mathcal{D}_{\mathrm{Ax}}^{\sigma}f(W)$. Then $f^{-1}(e') \in W$ and so there exist $c \in \mathcal{K}\mathbb{X}$ and $d \in \mathcal{O}\mathbb{X}$ such that $c \subseteq f^{-1}(e) \subseteq d$ and $[c, d] \cap clp\mathbb{X} \subseteq W$. Define $c' = f[c]$ and $d' = \mathbb{X}' \setminus f[\mathbb{X} \setminus d]$. Since f is a continuous function between Stone spaces, it sends closed sets to closed sets, so $c' \in \mathcal{K}\mathbb{X}'$ and $d' \in \mathcal{O}\mathbb{X}'$. Furthermore, we claim that $c' \subseteq e' \subseteq d'$. The first inclusion is obvious. For the second, if $x' \notin d'$, then there is $x \in X \setminus d$ such that $f(x) = x'$. But then $x \notin d$, so $x \notin f^{-1}(e')$, and hence $x' = f(x) \notin e'$.

We claim that c' and d' witness that the right-hand side of (6.3) holds. Let $a' \in clp\mathbb{X}'$ such that $c' \subseteq a' \subseteq d'$. Then $c \subseteq f^{-1}(a')$ by definition of c'. Furthermore, $f^{-1}(d') \subseteq d$. To see this, $x \notin d$ implies $x \in \mathbb{X} \setminus d$, so $f(x) \in f[\mathbb{X} \setminus d]$. Therefore, $f(x) \notin \mathbb{X}' \setminus f[\mathbb{X} \setminus d] = d'$, and hence $x \notin f^{-1}(d')$. Thus, $f^{-1}(a') \subseteq d$. By assumption this implies that $f^{-1}(a') \in W$. Consequently, $a' \in \mathcal{D}_{\mathrm{Ax}}^{\sigma}f(W)$.

Next we prove the converse of (6.3). Suppose $e' \subseteq X'$ is such that the right-hand side holds. Denote the relevant closed and open subsets of $\mathbb{X}'$ witnessing this by c' and d'. We aim to show that $f^{-1}(e') \in W$. To prove this, it suffices to show that there exist a closed and open subsets c and d of $\mathbb{X}$ such that $c \subseteq f^{-1}(e') \subseteq d$ and $[c, d] \cap clp\mathbb{X} \subseteq W$.

Take $c = f^{-1}(c')$ and $d = f^{-1}(d')$. By continuity of f these are closed and open, respectively. Now let $a \in clp\mathbb{X}$ be such that $c \subseteq a \subseteq d$. To prove that $a \in W$, we construct $b'_1, b'_2 \in clp\mathbb{X}'$ such that $b'_1, b'_2 \in \mathcal{D}_{\mathrm{Ax}}^{\sigma}f(W)$ and $f^{-1}(b'_1) \subseteq a \subseteq f^{-1}(b'_2)$. Convexity of W then implies that $a \in W$.

To construct b'_1, since c' is closed, we have that $c' = \bigcap\{k' \in clp\mathbb{X}' \mid c' \subseteq k'\}$. Therefore,

$$c = f^{-1}(c') = \bigcap\left\{f^{-1}(k') \mid k' \in clp\mathbb{X}' \text{ and } c' \subseteq k'\right\} \subseteq a.$$

Since a is clopen, compactness yields a finite number $k'_1, \ldots, k'_m \in clp\mathbb{X}'$ from the intersection such that $c \subseteq f^{-1}(k'_1) \cap \cdots \cap f^{-1}(k'_m) \subseteq a$. A similar compactness argument yields a finite number of clopens $k'_{m+1}, \ldots, k'_n$ from the intersection such that $k'_{m+1} \cap \cdots \cap k'_n \subseteq d'$. Setting $b'_1 := k'_1 \cap \cdots \cap k'_n$ gives an element in $clp\mathbb{X}'$ such that $f^{-1}(b'_1) \subseteq a$. Moreover, we have $b'_1 \in \mathcal{D}_{\mathrm{Ax}}^{\sigma}f(W)$ because $c' \subseteq b'_1 \subseteq d'$.

Finally, to construct b_2', since d' is open, we have $d' = \bigcup\{k' \in clp\mathbb{X}' \mid k' \subseteq d'\}$. A similar argument to the above yields b_2' satisfying all the required properties. This proves that $a \in W$, hence $f^{-1}(e') \in W$, completing the proof. $\qquad\square$

The definition of descriptive monotone frames from [35] coincides with the definition of σ-descriptive neighborhood frames satisfying monotonicity. Let us write HDMF for the category of such descriptive monotone frames, as defined in Section 2.4.2 of *op. cit.* The algebraic semantics of monotone modal logic is given by *monotone Boolean algebra expansions* (BAMs), and we write BAM for the full subcategory of NA whose objects are BAMs [35, Section 2.4.1]. As a consequence of Theorem 6.11 we now obtain the duality from [35, Theorem 2.11] as an algebra/coalgebra duality.

Corollary 6.12 (Hansen-Kupke). BAM $\equiv^{\mathrm{op}}$ HDMF.

Proof. One can prove that BAM $\cong$ Alg$(\mathcal{N}_{(\mathsf{M})})$ in the same way as in Proposition 2.6(1). Moreover, HDMF $\cong$ Coalg$(\mathcal{U}p\mathcal{V})$ [35, Theorem 3.12], where $\mathcal{U}p\mathcal{V}$ is the endofunctor on Stone defined in Definition 3.9 of *op. cit.* There is a natural isomorphism $\nu : \mathcal{D}_{(\mathsf{M})}^{\sigma} \to \mathcal{U}p\mathcal{V}$ given on components by

$$\nu_{\mathbb{X}} : \mathcal{D}_{(\mathsf{M})}^{\sigma}\mathbb{X} \to \mathcal{U}p\mathcal{V}\mathbb{X} : W \mapsto \{c \in W \mid c \text{ is closed in } \mathbb{X}\}.$$

This is injective because if $W, V \in \mathcal{D}_{(\mathsf{M})}^{\sigma}$ are distinct, then there exists $a \in clp\mathbb{X}$ such that $a \in W$ and $a \notin V$, or $a \notin W$ and $a \in V$. Since clopen sets are in particular closed, this implies that $\nu_{\mathbb{X}}(W) \neq \nu_{\mathbb{X}}(V)$, so $\nu_{\mathbb{X}}$ is injective. Moreover, it is surjective. To see this, observe that for all $W \in \mathcal{U}p\mathcal{V}\mathbb{X}$ the set $W^{\uparrow} = \{e \subseteq \mathbb{X} \mid \exists c \in W \text{ such that } c \subseteq e\}$ is in $\mathcal{D}_{(\mathsf{M})}^{\sigma}\mathbb{X}$ and satisfies $\nu_{\mathbb{X}}(W^{\uparrow}) = W$. So ν is a bijective continuous function, hence a homeomorphism.

Combining Theorems 5.13 and 6.11 yields that $\mathcal{N}_{(\mathsf{M})}$ is dual to $\mathcal{U}p\mathcal{V}$. The composition

$$\mathsf{BAM} \cong \mathsf{Alg}(\mathcal{N}_{(\mathsf{M})}) \equiv^{\mathrm{op}} \mathsf{Coalg}(\mathcal{U}p\mathcal{V}) \cong \mathsf{HDMF}$$

proves the desired duality. $\qquad\square$

Corollary 6.13. *Let* Ax *be a collection of finitary one-step axioms that implies* (Conv). *Then*

$$\mathsf{Alg}(\mathcal{N}_{\mathrm{Ax}}) \equiv^{\mathrm{op}} \mathsf{Coalg}(\mathcal{D}_{\mathrm{Ax}}^{\sigma}).$$

Proof. Combine Corollary 5.14, Theorem 6.11, and the fact that $\mathcal{D}_{\mathrm{Ax}}$ is naturally isomorphic to $\mathcal{D}_{\mathrm{Ax}}^{\sigma}$ to obtain

$$\mathsf{Alg}(\mathcal{N}_{\mathrm{Ax}}) \equiv^{\mathrm{op}} \mathsf{Coalg}(\mathcal{D}_{\mathrm{Ax}}) \cong \mathsf{Coalg}(\mathcal{D}_{\mathrm{Ax}}^{\sigma}). \qquad\square$$

Corollary 6.14. *Let* Ax *be a collection of finitary one-step axioms that implies* (Conv). *If the restriction* Σ_{Ax} *of* Σ *to* Coalg$(\mathcal{D}_{\mathrm{Ax}})$ *lands in* Coalg$(\mathcal{B}_{\mathrm{Ax}})$, *then* Σ_{Ax} *is a functor.*

Proof. If Σ_{Ax} is well defined, then so is $\mathcal{U}_{\mathrm{Ax}}$, and hence $\mathcal{U}_{\mathrm{Ax}}$ is a functor (as we pointed out after Remark 6.8). Since Σ_{Ax} can be obtained as the composition of functors

$$\mathsf{Coalg}(\mathcal{D}_{\mathrm{Ax}}) \xrightarrow{\ \cong\ } \mathsf{Coalg}(\mathcal{D}_{\mathrm{Ax}}^{\sigma}) \xrightarrow{\ \mathcal{U}_{\mathrm{Ax}}\ } \mathsf{Coalg}(\mathcal{B}_{\mathrm{Ax}})$$
$$\underbrace{\hspace{7cm}}_{\Sigma_{\mathrm{Ax}}}$$

it is a functor as well. $\qquad\square$

We can now obtain relatively easily the commuting diagrams relating the Jónsson-Tarski type and Thomason type dualities for a large class of modal logics. These are analogues of the diagrams for basic normal modal logic depicted in (1.1). We formulate this as a general statement, and then instantiate it to basic monotone modal logic. Observe that this still requires a preservation result reminiscent of a Sahlqvist theorem, proving that validity of axioms on a σ-descriptive neighborhood frame implies validity of the axioms on the underlying neighborhood frame (see, e.g., [46] or [15, Section 5.6]). We leave the search for such theorems to future research.

Theorem 6.15. *Let* Ax *be a collection of finitary one-step axioms that implies* (Conv). *Suppose that for every Stone space* $\mathbb{X}$, *every* Ax-*neighborhood W of $\mathcal{D}\mathbb{X}$ is also an* Ax-*neighborhood of $\mathcal{B}X$, where X is the set underlying $\mathbb{X}$. Then the following diagram commutes*

$$
\begin{array}{ccccc}
\mathsf{Alg}(\mathcal{N}_{\mathrm{Ax}}) & \overset{\equiv^{\mathrm{op}}}{=\!=\!=} & \mathsf{Coalg}(\mathcal{D}_{\mathrm{Ax}}) & \overset{\cong}{=\!=\!=} & \mathsf{Coalg}(\mathcal{D}_{\mathrm{Ax}}^{\sigma}) \\
{\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\Sigma_{\mathrm{Ax}}} & & \\
\mathsf{Alg}(\mathcal{L}_{\mathrm{Ax}}) & \overset{\equiv^{\mathrm{op}}}{=\!=\!=} & \mathsf{Coalg}(\mathcal{B}_{\mathrm{Ax}}) & \overset{\mathcal{U}_{\mathrm{Ax}}}{\longleftarrow} &
\end{array}
$$

Example 6.16. As we have seen, examples of such Ax are Ax $= \{(\mathsf{Conv})\}$ and Ax $= \{(\mathsf{M})\}$. The latter yields the following commuting diagrams, where all edges are functors:

$$
\begin{array}{ccccc}
\mathsf{Alg}(\mathcal{N}_{(\mathsf{M})}) & \overset{\equiv^{\mathrm{op}}}{=\!=\!=} & \mathsf{Coalg}(\mathcal{D}_{(\mathsf{M})}) & \overset{\cong}{=\!=\!=} & \mathsf{Coalg}(\mathcal{D}_{(\mathsf{M})}^{\sigma}) \\
{\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\Sigma_{(\mathsf{M})}} & & \\
\mathsf{Alg}(\mathcal{L}_{(\mathsf{M})}) & \overset{\equiv^{\mathrm{op}}}{=\!=\!=} & \mathsf{Coalg}(\mathcal{B}_{(\mathsf{M})}) & \overset{\mathcal{U}_{(\mathsf{M})}}{\longleftarrow} &
\end{array}
\qquad
\begin{array}{ccccc}
\mathsf{BAM} & \overset{\equiv^{\mathrm{op}}}{=\!=\!=} & \mathsf{Coalg}(\mathcal{D}_{(\mathsf{M})}) & \overset{\cong}{=\!=\!=} & \mathsf{HDMF} \\
{\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\Sigma_{(\mathsf{M})}} & & \\
\mathsf{CABAM} & \overset{\equiv^{\mathrm{op}}}{=\!=\!=} & \mathsf{MF} & \overset{\mathcal{U}}{\longleftarrow} &
\end{array}
$$

7. Conclusions

We have given a general coalgebraic approach to Thomason type dualities for neighborhood frames and Jónsson-Tarski type dualities for neighborhood algebras. Furthermore, we have investigated the relationship between the two types of dualities via the theory of canonical extensions. We list several potential avenues for future research.

Infinitary modal logic: The Thomason type dualities from Sections 3 and 4 provide dualities for algebraic and geometric semantics for *infinitary* modal logic. While some interesting investigations have been conducted by Baltag [4, 5], obtaining a more general coalgebraic approach towards infinitary modal logic (also using the results of this paper) remains open.

Endofunctors as left adjoints: In [10] the functor $\mathcal{H}$ is obtained as the left adjoint of the forgetful functor from CABA to CSL—the category of complete meet-semilattices. In a similar way we can obtain the functor whose coalgebras are monotone neighborhood frames as the left adjoint of the forgetful functor CABA $\to$ Pos, and the functor whose coalgebras are filter frames arises as the left adjoint of CABA $\to$ SL, where SL is the category of (not necessarily complete) meet-semilattices. An interesting direction for future work is to investigate the connection between presentations of classes of algebras via adjoints of forgetful functors and via one-step axioms of infinitary logic.

When are Σ and Π functors? A different approach: In Section 6.3 we proved that in presence of the convexity axiom, Σ defines a functor. Another approach towards the same goal is by modifying the morphisms between (descriptive) neighborhood frames. If we define a σ-morphism to be a function between neighborhood frames that satisfies only the left-to-right implication in (2.1), then replacing neighborhood morphisms with σ-morphisms ensures that Σ is a functor without adding any additional axioms. Considering such morphisms is natural for a number of reasons. For example, in the special case of topological spaces they simply correspond to continuous maps. They also generalize the stable morphisms studied in [8, 9] to the setting of (descriptive) neighborhood frames. We plan to investigate this topic further in a sequel to this paper.

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