"A partition calculus in set theory" by Erdős and Rado for readers from the twenty-first century

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

The seminal paper "A partition calculus in set theory" by Paul Erdős and Richard Rado is notoriously hard to read and contains many interesting results hidden behind outdated notation. This thesis therefore aims at a modernisation of the contents of the paper, in order to make the paper more accessible. This entails rewriting theorems and their proofs using modern mathematical notation. Of particular interest is a key result from the paper, the Positive Stepping Up Lemma, and we conjecture that one instance of the lemma, the partition relation $\beth_n^+ \to (\omega+n+1)^r_m$, cannot be improved.

We also adjust the proof of the Negative Stepping Up Lemma in order to prove the implication $\kappa \not\to (\omega^\alpha)^r_m \implies 2^\kappa \not\to (\omega^\alpha)^{r+1}_m$, where κ is an infinite cardinal, α is an ordinal and $r, m < \omega$. We deduce the negative partition relations $\beth_n^+ \not\to (\omega^2)^{n+3}_2$ for all $n < \omega$, providing a bound to the conjecture.

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Contents

Chapter 1

Introduction

In 1930, Frank Ramsey proved in the paper [Ram30] a theorem which is now known as Ramsey's Theorem: colouring r-tuples of an infinite set by finitely many colours always yields an infinite subset such that its r-tuples receive the same colour. The discovery of this theorem initiated the mathematical field of *infinite Ramsey theory*, concerning itself with finding generalisations to Ramsey's Theorem. In essence, Ramsey theory shows that "complete disorder is impossible".

The seminal paper "A partition calculus in set theory" [ER56] by Paul Erdős and Richard Rado, published in 1956, was the first systematic study of the *partition calculus*, a subfield of Ramsey theory. This field is concerned with proving partition relations. In the paper, they proved a key result, called the "Positive Stepping Up Lemma", which is a direct generalisation of Ramsey's theorem. The paper's success and impact on the mathematical community was also partly due to the arrow notation invented by Erdős and Rado, which allows one to succinctly write down partition relations. As András Hajnal said in [HL10, p. 130]: "There are cases in mathematical history when a well-chosen notation can enormously enhance the development of a branch of mathematics and a case in point is the ordinary partition symbol."

In this thesis, we will modernise the paper [ER56] by Erdős and Rado. The paper contains many positive and negative results in the field of partition calculus. However, it is hard to locate results in the paper as the mathematical notation is outdated. We will therefore rewrite the theorems and their proofs using modern mathematical language, in order to enhance the accessibility of the paper. We do not treat the whole paper and we **omit** the final two sections, concerning *canonical partition relations* and *polarised partition relations*. We also **omit** section 3 of the paper, concerning Theorem 1 to Theorem 10, as these are results that were known *before* the publication of [ER56]. In the *Overview* (Section 1.2), one can precisely see which results of [ER56] will be presented in the thesis.

1.1 Structure of the thesis

In **Chapter 2**, we establish some basic properties regarding partition relations. These properties will often be called upon in this thesis, in order to prove partition relations. The material presented there corresponds to Theorem 11 to Theorem 20 of [ER56].

In **Chapter 3**, we treat the bulk of [ER56]: Theorem 23 to Theorem 45. Of particular interest is the Positive Stepping Up Lemma, which states

$$\kappa \to (\beta_n)^r_{n<m} \implies (2^{<\kappa})^+ \to (\beta_n + 1)^{r+1}_{n<m}.$$

Starting with Ramsey's Theorem, $\aleph_0 \to (\omega)^r_m$, we obtain $\beth^+_n \to (\omega + n + 1)^r_m$ after applying the Positive Stepping Up Lemma multiple times. We conjecture that this relation is sharp, i.e., increasing the goal to $\omega + n + 2$ will result in a negative partition relation. In [ER56], this is shown for $n = 0$, and a corollary of a result by Albin Jones combined with a theorem by Erdős-Rado shows this for $n = 1$.

In **Chapter 4**, we investigate some results of [Erd+84]. These results are all cardinal-based partition relations, and we adjust the proofs to prove their order-type variants. We investigate the proof of the Negative Stepping Up Lemma in [Erd+84] and improve the result from cardinals to all additively indecomposable ordinals. In other words, we prove the implication $\kappa \not\to (\omega^\alpha)^r_m \implies 2^\kappa \not\to (\omega^\alpha)^{r+1}_m$. As a consequence we obtain the negative partition relations $\beth^+_n \not\to (\omega^2)^{n+3}_2$ for all $n < \omega$. This result provides a bound to the conjecture for all $n \geq 2$.

1.2 Overview

We will present an overview of the results in [ER56] according to the following template.

Location in [ER56]	Location in thesis
The modernised statement of the result.	

We mention again that this overview is not an exhaustive list of the Erdős-Rado paper. The final two sections of the original paper, concerning *canonical partition relations* and *polarised partition relations*, are **not** presented here. These final two sections entail Theorem 46 to Theorem 51, or alternatively, [ER56, pp. 477-488].

At the beginning of the paper, Erdős and Rado list some classical results that were known *before* the publication of [ER56], e.g., Ramsey's Theorem. These are Theorem 1 to Theorem 10, and we do **not** present these in this overview. These results correspond to [ER56, pp. 431-433].

The remaining results of [ER56] are in this overview. These are Theorem 11 to Theorem 45, or alternatively, [ER56, pp. 433-477].

Theorem 11 — Lemma 2.21

Given order-types L, M, and $r < \omega$ and m any cardinal, then the following statements are equivalent:

$$L \to (M)^r_m,$$
$$L^* \to (M^*)^r_m.$$

Theorem 12 — Observation 2.4 & Remark 2.13

Monotonicity of the partition relation: Given a relation $\kappa \to (\mu_n)^r_{n<m}$, and $\kappa \leq \kappa'$, $m' \leq m$, $\mu'_n \leq \mu_n$ and $\mu_n \geq r$ for all $n < m$. Then $\kappa' \to (\mu'_n)^r_{n<m'}$.

Theorems 13 & 14 — Lemma 2.24

Let $r < \omega$ and let k be any cardinal. Let α be an ordinal and suppose that β_n are initial ordinals for all $n < k$. Then the following statements are equivalent:

$$\alpha \to (\beta_n)^r_{n<k},$$
$$|\alpha| \to (|\beta_n|)^r_{n<k}.$$

Theorem 15 — Lemma 2.22

Let L, M be order-types, $r < \omega$ and k any cardinal. If $L + 1 \to (M + 1)^{r+1}_k$, then $L \to (M)^r_k$.

Theorem 16 — Lemma 2.27

Let $r, m, k_n < \omega$ for all $n < m$, let L, M_n, N_{n_i} be order-types for all $n < m$ and $i < k_n$. Suppose $L \to (M_n)^r_{n<m}$ and also $M_n \to (N_{n_i})^r_{i<k_n}$ for each $n < m$. Then $L \to (N_{n_i})^r_{n<m,i<k_n}$.

Theorem 17 — Lemma 2.26

Let $r < \omega$, let L, M be order-types and let m and k be sets. Suppose $|m| = |k|$, then $L \to (M)^r_m$ if and only if $L \to (M)^r_k$.

Theorem 18 — Lemma 2.28

Let $m, r < \omega$. Let L and M_i be order-types for all $i < m$. Suppose $L \to (M_n)^r_{i<m}$ and let $f\colon [L]^r \to m$ be a partition. Then there are sets $I, J \subseteq m$ with $|I| + |J| > m$ such that for all $i \in I$ and $j \in J$, there is an j-homogeneous set for f of order-type M_i.

Theorem 19 — Lemma 2.29

Let L, M, N be order-types and let δ be the initial ordinal of $|L|$. Suppose $L \to (M, N)^2$. Then at least one of the following four situations must be true: (i) $M < \omega$, (ii) $N < \omega$, (iii) $M, N \leq L$ and $M, N \leq \delta$, or (iv) $M, N \leq L$ and $M, N \leq \delta^*$.

Corollary of Theorems 18 & 19 | Corollary 2.30

For all order-types L,

$$L \not\to (\omega, \omega^*)^2.$$

Theorem 20 | Lemma 2.15

Let $r < \omega$ and let m, k be cardinals.
(i) If κ and μ are cardinals such that $\mu \leq \kappa$ and $\mu < r$, then $\kappa \to (\mu)^r_m$.
(ii) If κ is a cardinal, $\mu_n = r$ for all $n < m$, and ν_i are cardinals for $i < k$. Then $\kappa \to ((\mu_n)_{n<m}, (\nu_i)_{i<k})^r$ is equivalent to $\kappa \to (\nu_i)^r_{i<k}$.

Theorem 21 | Observation 2.17

Suppose $\kappa \to (\mu_n)^r_{n<m}$ holds, then either (i) there is some $n < m$ with $\mu_n < r$ and $\mu_n \leq \kappa$, or (ii) $\mu_n \leq \kappa$ for all $n < m$.

Theorem 22 | Not in thesis

This theorem is a table which gives the value of certain trivial partition relations.

Theorem 23 | Theorem 3.25
Theorem 3.26

For all $n < \omega$ and $\alpha < \omega \cdot 2$,

$$\omega \cdot n \to (n, \alpha)^2,$$
$$\omega \cdot n \not\to (n+1, \omega+1)^2.$$

Theorem 24 | Not in thesis

An application of Theorem 25. If $\alpha < \omega \cdot 4$, then

$$\alpha \not\to (3, \omega \cdot 2)^2,$$
$$\omega \cdot 4 \to (3, \omega \cdot 2)^2.$$

Theorem 25 — Theorem 3.29, Lemma 3.30, Theorem 3.27

Let $2 \leq m < \omega$ and $1 \leq n < \omega$. Suppose that $\ell_0 < \omega$ is the least natural number that has property $P_{m,n}$. Then

$$\omega \cdot \ell_0 \to (m, \omega \cdot n)^2,$$
$$\gamma \not\to (m, \omega \cdot n)^2 \text{ for all } \gamma < \omega \cdot \ell_0.$$

If $\ell < \omega$ is such that $\ell \to (m, m, n)^2$, then $\ell_0 \leq \ell$.

Theorem 26 — Theorem 3.12

If $r \geq 0$, $k > 0$, then

$$\lambda \not\to (\omega_1)^r_k.$$

If, additionally $r \geq 2$, then

$$\lambda \not\to (r+1)^r_{\aleph_0}.$$

Theorem 27 — Theorem 3.13

$$\lambda \not\to (\omega, \omega + 2)^3.$$

Theorem 28 — Theorem 3.14

For $r \geq 4$,

$$\lambda \not\to (r+1, \omega + 2)^r.$$

Theorem 29 — Theorem 3.15

Let L be an order-type with $|L| \geq 2^{\aleph_0}$, then

$$\lambda \not\to (L)^1_2.$$

Theorem 30 — Theorem 3.47

$$2^{\aleph_0} \not\to (\aleph_1)^2_2.$$

Lemma 1 — Proposition 3.17

Let S be a linearly ordered set such that $\operatorname{cf}|S| = \aleph_n$ and $\omega_n, \omega_n^* \not\leq \operatorname{otp} S$. Then for every rational q there is a set $A_q \subset S$ such that

1. $|A_q| = |S|$, and
2. $A_p < A_q$, i.e., for all rationals $p < q$ and $x \in A_p, y \in A_q$ it holds that $x < y$.

Theorem 31 — Theorem 3.22, Corollary 3.20, Lemma 3.18, Lemma 3.24

Let ϕ be a real order-type. Let $\alpha < \omega \cdot 2$, $\beta < \omega^2$ and $\gamma < \omega_1$. Then

$$\begin{aligned} &\phi \to (\alpha)^2_3, \\ &\phi \to (\alpha, \beta)^2, \\ &\phi \to (\omega, \gamma)^2, \\ &\phi \to (4, \alpha)^3. \end{aligned}$$

Theorem 32 — Lemma 3.19, Lemma 3.21

Let ϕ be a real order-type. Assume that $\alpha < \omega \cdot 2$ and $\gamma < \omega_1$. Then

$$\begin{aligned} &\phi \to (\alpha, \gamma \vee \omega \cdot \gamma^*)^2, \\ &\phi \to (\omega + \omega^*, \gamma \vee \gamma^*)^2. \end{aligned}$$

Theorem 33 — Theorem 3.46

Let $\alpha < \omega \cdot 2$. Then

$$\omega_1 \to (\alpha)^2_2.$$

Theorem 34 — Proposition 3.52

Let α, β, γ be ordinals and suppose $\alpha \not\to (\beta, \gamma)^2$. Then there exists a sequence of ordinals $\langle \alpha_\mu \mid \mu < \beta^- \rangle$, such that

$$\begin{aligned} \alpha &\not\to (\alpha_\mu + 1)^1_{\mu < \beta^-}, \text{ and} \\ \alpha_\mu &\not\to (\gamma)^1_{\kappa_\mu}, \end{aligned}$$

where $\kappa_\mu = \prod_{\nu < \mu} |\alpha_\nu|$ for all $\mu < \beta^-$.

Corollary 1 of Theorem 34 | Theorem 3.50

Let κ be an uncountable regular cardinal. Then

$$\kappa \to (\omega + 1, \kappa)^2.$$

Corollary 2 of Theorem 34 | Corollary 3.54

Let κ be a (strongly) inaccessible cardinal. Then for all $\beta < \kappa$,

$$\kappa \to (\beta, \kappa)^2.$$

Lemma 2 | Lemma 3.51

Let $(T, <)$ be a well-ordered set and let $f\colon [T]^2 \to 2$ be a partition. Then there exists a unique set $H \subseteq T$ such that H is 1-homogeneous for f and for all $x \in T \setminus H$ there is some $h \in H$ such that $f(\{h < x\}) = 0$.

Theorem 35 | Theorem 3.7

Let L, M, N be order-types and let $s, r < \omega$. Assume $M \geq r \geq 3$ and $s > (r-1)^2$. Suppose $M, M^* \not\leq L$ and $|L| = |N|$. Then

$$N \not\to (s, M)^r.$$

Theorem 36 | Theorem 3.57 Corollary 3.58

Let γ be an ordinal, and let L_β be order-types for all $\beta < \gamma$. Suppose δ is such that $L_\beta < \delta$ for all $\beta < \gamma$. Define $L = \sum_{\beta<\gamma} L_\beta$. Then $L \not\to (\gamma + 1, \delta)^2$.

Theorem 37 | Theorem 3.55

Let κ be an infinite cardinal, and let ν be the least cardinal such that $\kappa^\nu > \kappa$. Let μ be an ordinal such that $\kappa < \operatorname{cf} \aleph_\mu \leq \aleph_\mu \leq \kappa^\nu$, then

$$\aleph_{\omega_\mu} \not\to (\nu^+, \aleph_{\omega_\mu})^2.$$

Corollary of Theorem 37 | Corollary 3.56

If α is an ordinal with $\aleph_0 < \operatorname{cf} \aleph_\alpha \leq \aleph_\alpha \leq 2^{\aleph_0}$, then

$$\aleph_{\omega_\alpha} \not\to (\aleph_1, \aleph_{\omega_\alpha})^2.$$

Lemma 4 — Lemma 3.1

Let L_0, L_1, M_0, M_1 be order-types. Suppose that $r \geq 2$ and $M_0, M_1^* \not\leq L_0$ and $|L_0| = |L_1|$, then

$$L_1 \not\to (M_n)^r_{n<r!},$$

where $M_n = r + 1$ for all $n \geq 2$.

Theorem 38 — Theorem 3.60

Let γ be an ordinal and $r < \omega$. Let m be some cardinal and let α_β be ordinals for all $\beta < m$. If $\aleph_\gamma \not\to (|\alpha_\beta|)^r_{\beta<m}$, then $\omega_{\gamma+1} \not\to (\alpha_\beta + 1)^{r+1}_{\beta<m}$.

Lemma 5 — Lemma 3.36

Let α be an ordinal and k a cardinal. Suppose that β_n are ordinals for all $n < k$ such that for all $\beta < \alpha$ it holds that $\beta \not\to (\beta_n)^r_{n<k}$. Then $\alpha \not\to (\beta_n + 1)^{r+1}_{n<k}$.

Theorem 39: Positive Stepping Up Lemma — Theorem 3.37

Let κ be an infinite cardinal, let $2 \leq m < \kappa$ be a cardinal and let $r \geq 1$ a natural number. Let β_n be ordinals for all $n < m$. Assume $\kappa \to (\beta_n)^r_{n<m}$. Then $(2^{<\kappa})^+ \to (\beta_n + 1)^{r+1}_{n<m}$.

Corollaries of Positive Stepping Up Lemma — Corollary 3.40

For any $r, m < \omega$,

$$\omega_1 \to (\omega + 1)^r_m,$$
$$(2^{\aleph_0})^+ \to (\omega + 2)^r_m.$$

Erdős-Rado Theorem — Theorem 3.41

For any infinite cardinal κ and any $n < \omega$,

$$\beth_n(\kappa)^+ \to (\kappa^+)^{n+1}_\kappa.$$

Theorem 40 — Not in thesis

Let $r < \omega$, let $2 \leq m < \omega$, and let $0 < r \leq k_i < \omega$ for all $i < m$. Let $R(m, r, (k_i)_{i<m})$ denote the least natural number n such that

$$n \to (k_i)^r_{i<m}.$$

Then

$$R(m, r+1, (k_i + 1)_{i<m}) \leq m^{R(m,r,(k_i)_{i<m})^r}.$$

Theorem 41 — Proposition 3.43, Corollary 3.44

For all $n \in \omega$,

$$\omega_{n+1} \not\to (\omega_n + 2, \omega + 1)^3,$$

and as a special case

$$\omega_1 \not\to (\omega + 2, \omega + 1)^3.$$

Theorem 42 — Theorem 3.4

Let $r \geq 3$, and let L_0, L_1, M_0, M_1 be order-types such that $|L_0| = |L_1|$ and $M_0, M_1^* \not\leq L_0$, and M_0, M_1 are additively indecomposable. Then

$$(r-3) + L_1 \not\to ((r-3) + M_0, (r-3) + M_1)^r.$$

Theorem 43 — Theorem 3.5

Let L, M, N be order-types. Let $r < s < \omega$ and $s \leq M$. If $L \to (M, N)^s$ and $N \to (s)^r_k$, then

$$L \to (M)^r_k.$$

Theorem 44: Erdős-Dushnik-Miller Theorem — Theorem 3.49

For all infinite cardinals κ,

$$\kappa \to (\omega, \kappa)^2.$$

Theorem 45 Theorem 3.8

Let $m > 0$, and let M_n be order-types for all $n < m$. Then

$$\prod_{n<m} M_n \to (M_n)^1_{n<m}.$$

Chapter 2

Preliminaries

Throughout this thesis we will work in $\mathsf{ZFC} = \mathsf{ZF} + \mathsf{AC}$. That is, the usual Zermelo-Fraenkel Axioms (ZF) with the Axiom of Choice (AC). We assume that the reader has familiarity with basic set theory, e.g. ordinals, cardinals and cardinal arithmetic. Most of these concepts can be found in a standard textbook, such as [Jec03].

2.1 Basic definitions

Definition 2.1 (Colourings)**.** Let S be a set, let $r < \omega$ and let m be a cardinal.

1. We define the set of r-element subsets of S as
$$[S]^r := \{X \subseteq S \mid |X| = r\}.$$
We call an element $X \in [S]^r$ an *r-tuple*.
An analogous definition holds if $<$ is an order on S and L is an order-type,
$$[S]^L := \{X \subseteq S \mid \operatorname{otp}(X, <) = L\}.$$
2. A function $f\colon [S]^r \to m$ is said to be an *r-partition of S with m colours.* Alternatively, f is said to be a *m-colouring of* $[S]^r$.
3. Given $f\colon [S]^r \to m$, a set H is *homogeneous for f* if $H \subseteq S$ and f is constant on $[H]^r$. Additionally, if H is homogeneous for f with colour $i \in m$, i.e. $f \restriction [H]^r \equiv i$, then we say that H is *i-homogeneous* for f.

We will sometimes identify $[S]^1$ with S, keeping in mind that there is an implicit bijection $f\colon [S]^1 \to S\colon \{x\} \mapsto x$.

Definition 2.2 (The partition relation)**.** Let κ and μ be cardinals, let $r < \omega$ and m any cardinal. We write
$$\kappa \to (\mu)^r_m$$
if the following statements holds:

"For every set S such that $|S| = \kappa$ and every m-colouring of $[S]^r$, $f\colon [S]^r \to m$, there exists a homogeneous set $H \subseteq S$ for f with cardinality $|H| = \mu$."

We will write $\kappa \not\to (\mu)^r_m$ for the negation of $\kappa \to (\mu)^r_m$.

Remark 2.3. An equivalent formulation of $\kappa \to (\mu)^r_m$ is: "For all sets S with $|S| = \kappa$ and for any partition $[S]^r = \bigcup_{i<m} S_i$ of pairwise disjoint sets, there are a set $H \subseteq S$ with $|H| = \mu$ and some $i < m$ such that $[H]^r \subseteq S_i$." In fact, in [ER56] this definition is used, although the sets did not need to be pairwise disjoint.

Borrowing terminology from J. Larson in [Lar12], given a partition relation $\kappa \to (\mu)^r_m$, we will call κ the *resource*, μ is the *goal*, r is referred to as the *exponent* and, finally, m is called the *colour set* or *colour cardinal*.

Observation 2.4. The notation of the partition relation is particularly useful because it adheres to certain monotonicity principles. If the partition relation $\kappa \to (\mu)^r_m$ holds, then the relation still holds if the resource κ is made larger, or anything on the right side of the arrow (goal or colour set) is made smaller. Slightly harder to observe is that in most cases the exponent can be decreased as well. In particular, the relation between the goal and exponent must be preserved. For a more precise treatment of decreasing the exponent we refer the reader to Lemma 2.22.

Remark 2.5. In our definition of a partition relation we only used finite exponents $r < \omega$. A natural question that arises would be what were to happen if we would let $r \geq \omega$. In ZFC it turns out that partition relations with infinite exponents are always negative. This is because (in ZF) the Axiom of Choice implies that all such partition relations are negative, as can be seen in the following theorem.

Theorem 2.6. *For every cardinal κ, $\kappa \not\to (\omega)^\omega_2$.*

Proof. Proof is from [Kan09, Proposition 7.1]. We may assume that κ is an infinite cardinal. As we assume AC, there is a well-order $\prec$ on $[\kappa]^\omega$. Note that whenever $s \in [\kappa]^\omega$ and $t \in [s]^\omega$, then $t \in [\kappa]^\omega$. Define $f\colon [\kappa]^\omega \to 2$ as follows: for $s \in [\kappa]^\omega$ we set $f(s) = 0$ if for every $t \in [s]^\omega \setminus \{s\}$ we have $s \prec t$, and we set $f(s) = 1$ otherwise.

Let $H \subseteq \kappa$ such that $|H| = \aleph_0$, then in particular $H \in [\kappa]^\omega$. Let x be the $\prec$-least element of $[H]^\omega$. For every $y \in [x]^\omega \setminus \{x\}$ we have $y \in [H]^\omega$, and hence by assumption $x \prec y$, which gives $f(x) = 0$. However, let now $x_0 \subset x_1 \subset ... \subset H$ be any infinite sequence. Assume that for every $n \in \omega$ we also have $f(x_n) = 0$, then this implies that $... \prec x_1 \prec x_0$. This shows that $\prec$ is not a well-order, which is a contradiction. Therefore there is some $n \in \omega$ such that $f(x_n) = 1$, showing that H is not homogeneous for f. $\square$

In this thesis we will often look at a slightly different kind of partition relation, one where the resource and goal can be linear order-types instead of merely cardinals. The definition of this partition relation generalises naturally, but first we give a definition of order-types.

Definition 2.7 (Order-types)**.** Two ordered sets $(W_1, \leq_1)$ and $(W_2, \leq_2)$ are *isomorphic* if there exists an order isomorphism $f\colon (W_1, \leq_1) \to (W_2, \leq_2)$. That is, f is bijective, and both f and its inverse f^{-1} are order preserving. We write $(W_1, \leq_1) \cong (W_2, \leq_2)$ if $(W_1, \leq_1)$ and $(W_2, \leq_2)$ are isomorphic.

Two linearly ordered sets $(W_1, \leq_1)$ and $(W_2, \leq_2)$ *have the same order-type* if $(W_1, \leq_1)$ and $(W_2, \leq_2)$ are isomorphic.

It is easily verified that having the same order-type defines an equivalence relation on the class of linear orders. We say that $(W_1, \leq_1)$ *has order-type L* if L is the representative of the equivalence class of $(W_1, \leq_1)$. For equivalence classes of well-orders, we take as representative the unique ordinal in that equivalence class.

We write $\operatorname{otp}(W, \leq)$ for the order-type of the ordered set $(W, \leq)$ and simply write $\operatorname{otp} W$ if the underlying order is obvious from context.

Definition 2.8 (Sum order)**.** Given two linear orders $(W_1, \leq_1)$ and $(W_2, \leq_2)$, we define the *sum* $(W_1, \leq_1) + (W_2, \leq_2)$ to be the linear order $(W_3, \leq_3)$, where $W_3 = W_1 \uplus W_2$ and $\leq_3$ is defined as follows: given $x, y \in W_3$, we have

$$\begin{aligned} x \leq_3 y \text{ if and only if } & x \in W_1 \text{ and } y \in W_2, \text{ or} \\ & x, y \in W_1 \text{ and } x \leq_1 y, \text{ or} \\ & x, y \in W_2 \text{ and } x \leq_2 y. \end{aligned}$$

If L and M are order-types, then we write $L + M$ for the order-type of the sum of the linear orders L and M.

For order-types L and M, we will write $L \leq M$ if L *embeds into* M and we write $L \not\leq M$ for its negation. We write $L < M$ if $L \leq M$ and $M \not\leq L$. We will denote the order-type of the rationals by η, i.e. $\operatorname{otp}(\mathbb{Q}, <) = \eta$. The order-type of the reals will be denoted by λ, i.e. $\operatorname{otp}(\mathbb{R}, <) = \lambda$.

We will need to make use of the universality of $\mathbb{Q}$. That is, any countable linear order embeds into η.

Theorem 2.9 (Cantor)**.** *Let L be a countable order-type. Then $L \leq \eta$.*

Proof. See [Ros82, Theorem 2.5] for a proof. □

We will also need to make use of the $\aleph_0$-categoricity of the theory of dense linear orders without endpoints. That is, any countable dense linear order without endpoints is isomorphic to η.

Theorem 2.10 (Cantor)**.** *Let L be a countable dense order-type without endpoints. Then $L \cong \eta$.*

Proof. See [Ros82, Theorem 2.8] for a proof. □

Definition 2.11 (Partition relations with order-types)**.** Let L and M be order-types, let $r < \omega$ and m any cardinal. We say that

$$L \to (M)^r_m$$

if the following statements holds:

"For every ordered set $(S, <)$ such that $\operatorname{otp}(S, <) = L$ and every m-colouring of $[S]^r$, $f\colon [S]^r \to m$, there exists a homogeneous set $H \subseteq S$ for f of order-type $\operatorname{otp}(H, <) = M$."

So far, we have only introduced *balanced* partition relations, that is, relations $\kappa \to (\mu)^r_m$ in which there is only one goal μ. In the paper by Erdős-Rado [ER56] there are many interesting results in which there are multiple goals, which are called *unbalanced* partition relations.

Definition 2.12 (Unbalanced partition relation)**.** Let $r < \omega$ and m be any cardinal. Let L and M_n be order-types for every $n < m$. We say that

$$L \to (M_n)^r_{n<m}$$

if the following statement holds:

"For every ordered set $(S, <)$ such that $\operatorname{otp} S = L$ and every colouring $f\colon [S]^r \to m$, there exists some $n < m$ and an n-homogeneous set $H \subseteq S$ for f of order-type $\operatorname{otp}(H, <) = M_n$."

Remark 2.13. For a positive unbalanced partition relation $L \to (M_n)^r_{n<m}$, the colour set may only be decreased (or, alternatively, goals may be removed) if those goals are at least as large as the exponent, i.e. $M_n \geq r$. For example, $4 \to (1, 3, 3)^2$ is trivially positive, but $4 \not\to (3, 3)^2$.

Observation 2.14. It is irrelevant what the order is of the goals for an unbalanced partition relation. In other words, if $\pi\colon m \to m$ is some permutation of m, then

$$L \to (M_n)^r_{n<m} \iff L \to (M_{\pi(n)})^r_{n<m}.$$

Sometimes we will explicitly write out all the goals in a partition relation, and then do not mention the colour set. We usually do this in the case of $m = 2$ and $m = 3$, e.g., we write $L \to (M, N)^r$ instead of $L \to (M, N)^r_2$.

Naturally, the definition of the cardinal-based unbalanced partition relation is similar.

More notation and terminology

We say that a partition relation $\kappa \to (\mu)^r_m$ is *positive*, and its negation $\kappa \not\to (\mu)^r_m$ is called *negative*. Given a relation $\kappa \to (\mu)^r_m$, we say that κ is the *resource*, μ the *goal*, r is the *exponent* and m is the *colour set*. A positive partition relation $\kappa \to (\mu)^r_m$ is *sharp* or *tight* if decreasing the resource or increasing the goal, exponent or colour set, results in a negative partition relation. A partition relation $\kappa \to (\mu)^r_m$ is *balanced* if all the goals are equal. The relation $\kappa \to (\mu_n)^r_{n<m}$ is *unbalanced* if some goals are distinct. If some, but not all, of the goals are equal, then we will write, for example, $\kappa \to (\mu, \rho, (\nu)_{m-2})^r$ for the relation $\kappa \to (\mu_n)^r_{n<m}$, where $\mu_0 = \mu, \mu_1 = \rho$ and $\mu_i = \nu$ for all $2 \leq i < m$. The relation $\kappa \to (\mu, \rho \vee \nu)^r$ means that for all $f\colon [S]^r \to 2$ with $|S| = \kappa$, there is either a 0-homogeneous set of cardinality μ, or a 1-homogeneous set of cardinality either ρ or of cardinality ν. All these definitions have an analogous definition for partition relations for order-types.

Given an ordered set $(S, <)$ and an r-tuple $X \in [S]^r$, we write $X = \{x_0 < x_1 < \ldots < x_{r-1}\}$ as shorthand for $X = \{x_0, x_1, \ldots, x_{r-1}\} \wedge x_0 < x_1 < \ldots < x_{r-1}$. Given a colouring $f\colon [S]^r \to m$, some $i \in m$ and $\{x_0, x_1, \ldots, x_{r-1}\} \in [S]^r$, we write $f(\{x_0 < x_1 < \ldots < x_{r-1}\}) = i$ as shorthand for $f(\{x_0, x_1, \ldots, x_{r-1}\}) = i \wedge x_0 < x_1 < \ldots < x_{r-1}$. Given subsets $A, B \subseteq S$, we will write $A < B$ as an abbreviation of "for all $a \in A$ and $b \in B$, we have that $a < b$".

Given sets A and B and integers m, n, we define $[A, B]^{m,n} := \{X \subseteq A \cup B\colon |X \cap A| = m \text{ and } |X \cap B| = n\}$.

2.2 Basic partition results

In this section we will prove many properties of partition relations. All of these results have been known for decades and most can be found in the Erdős-Rado paper [ER56]. To be more precise, Theorem 11 to Theorem 21 (See the Overview in Section 1.2). Throughout the thesis these basic results will often be invoked, and sometimes we will do so without explicitly mentioning. Also, these partition relations will hold if we replace cardinals by order-types and vice versa.

First, we establish some trivial partition relations.

Lemma 2.15. *Let $r < \omega$ and m be any cardinal. If κ and μ are cardinals such that $\mu \leq \kappa$ and $\mu < r$, then $\kappa \to (\mu)^r_m$.*

Proof. Let $|S| = \kappa$ and let $f : [S]^r \to m$ be a colouring. Take any set $H \subseteq S$ with $|H| = \mu$. Note that $[H]^r = \varnothing$ and therefore the function $f \restriction [H]^r$ is the empty function, which is vacuously constant. $\square$

Similarly, all partition relations where the goal does not embed into the resource are trivially negative, unless the colour set m is empty.

Observation 2.16. Let $r < \omega$ and κ be any cardinal and $m > 0$. If μ is a cardinal such that $\mu \not\leq \kappa$, then $\kappa \not\to (\mu)^r_m$.

Observation 2.17. Suppose $\kappa \to (\mu_n)^r_{n<m}$ holds, then either (i) there is some $n < m$ with $\mu_n < r$ and $\mu_n \leq \kappa$, or (ii) $\mu_n \leq \kappa$ for all $n < m$.

We present some classical results of the partition calculus here, because they are referenced in the thesis multiple times. The theorem below is the well-known Ramsey's theorem.

Theorem 2.18 (Ramsey's Theorem, F. Ramsey (1930), [Ram30])**.** *For every $r, k < \omega$ it holds that*

$$\aleph_0 \to (\aleph_0)^r_k .$$

Proof. Proof is from [Jec03, Theorem 9.1]. We can assume $r, k > 0$, else the statement is trivially true.

Base case $r = 1$. If $f : \omega \to k$, then, as $k < \omega$, there is an infinite homogeneous set for f by the Pigeonhole Principle.

Induction step $r + 1$. Assume the statement holds for r. Let $f : [\omega]^{r+1} \to k$ be a k-colouring of $[\omega]^{r+1}$. We will show that there exists an infinite homogeneous set $H \subseteq \omega$ for f, so that the statement holds for $r + 1$.

For each $a \in \omega$ define the function $f_a : [\omega \setminus \{a\}]^r \to k : X \mapsto f(X \cup \{a\})$. By the induction hypothesis, for every a and every infinite $S \subseteq \omega \setminus \{a\}$ there is an infinite homogeneous set $H^S_a \subseteq S$ for f_a. Note that by AC we can choose this set H^S_a. Construct an infinite sequence $\langle a_i \mid i \in \omega \rangle$ as follows: let $S_0 = \omega$ and $a_0 = 0$, and $S_{i+1} = H^S_{a_i}$ and a_{i+1} the least element of S_{i+1} larger than a_i. Note that the set $H^S_{a_i}$ is infinite, so such a_{i+1} exist. For every $i < j$ we have that $a_j \in H^{S_i}_{a_i}$, and hence $[\{a_j \mid j > i\}]^r$ is a homogeneous set for f_{a_i}, with value, say, $g(a_i)$. This defines a function $g : \{a_i \mid i < \omega\} \to k$, and by the Pigeonhole Principle there is an infinite set $H \subseteq \{a_i \mid i < \omega\}$ that is homogeneous for g.

We show that H is homogeneous for f. Let $\{a_{i_0}, a_{i_1}, \dots, a_{i_r}\} \in [H]^{r+1}$ and assume $a_{i_0} < a_{i_1} < \dots < a_{i_r}$, then

$$f(\{a_{i_0}, a_{i_1}, \dots, a_{i_r}\}) = f_{a_{i_0}}(\{a_{i_1}, \dots, a_{i_r}\}) = g(a_{i_0}),$$

which is what we wanted to show. $\square$

As a consequence of Ramsey's theorem, we can also prove a finite version of Ramsey's theorem.

Theorem 2.19 (Finite Ramsey's Theorem)**.** *For all $n, m, r < \omega$, there exists some $\ell < \omega$ such that*

$$\ell \to (n)^r_m. \tag{2.1}$$

Proof. Proof is from [Mar02, Theorem 5.1.2]. Suppose that there are $n, m, r < \omega$ such that for all $\ell < \omega$ we have $\ell \not\to (n)^r_m$. This means for all $\ell < \omega$ there exists a colouring $f_\ell \colon [\ell]^r \to m$ such that there are no homogeneous sets for f_ℓ of size n.

Define for each $\ell < \omega$,

$$T_\ell = \{f \colon [\ell]^r \to m \mid \text{ there is no homogeneous set for } f \text{ of size } n\}.$$

Define the tree $T = \bigcup_{\ell<\omega} T_\ell$, ordered by inclusion. By assumption, each T_ℓ is non-empty and given $h \in T_\ell$ there are only finitely many extensions of h in $T_{\ell+1}$. Therefore T is an infinite finitely branching tree and hence by König's lemma, there is an $\subset$-increasing sequence $\langle g_\ell \mid \ell \in \omega\rangle$ with $g_\ell \in T_\ell$ for each $\ell < \omega$.

Define $g = \bigcup_{\ell<\omega} g_\ell$, then $g \colon [\omega]^r \to m$. By Ramsey's theorem, there exists $X = \{x_0 < \dots < x_{n-1}\}$ such that X is homogeneous for g. Let $s > x_{n-1}$, then X is homogeneous for g_s, contradicting $g_s \notin T_s$. This concludes the proof. $\square$

Definition 2.20 (Dual order-type)**.** For a given order-type L ordered by $\leq$, we write L^* as the *dual order-type* of L, ordered by $\leq^*$. That is, for $x, y \in L$ we have

$$x \leq y \iff y \leq^* x.$$

Lemma 2.21 (Theorem 11)**.** *Given order-types L, M, and $r < \omega$ and m any cardinal, then the following statements are equivalent:*

$$L \to (M)^r_m, \tag{2.2}$$
$$L^* \to (M^*)^r_m. \tag{2.3}$$

Proof. We only show (2.3) $\implies$ (2.2) as the other implication follows analogously.

Given an ordered set $(S, \leq)$ with $\operatorname{otp}(S, \leq) = L$ and a partition $f \colon [S]^r \to k$. By (2.3) there is a homogeneous set $H \subseteq S$ for f with $\operatorname{otp}(H, \leq^*) = M^*$. It follows immediately that $\operatorname{otp}(H, \leq) = M$, which shows (2.2). $\square$

The following lemma shows why, in many cases, the exponent may be decreased for a positive partition relation. Consequently, in many cases the exponent may be increased for negative partition relations.

Lemma 2.22 (Theorem 15)**.** *Let L, M be order-types, $r < \omega$ and k any cardinal. If $L + 1 \to (M + 1)^{r+1}_k$, then $L \to (M)^r_k$.*

Proof. Let $(S, <)$ be an ordered set such that $\mathrm{otp}(S, <) = L$ and let $f\colon [S]^r \to k$ be a colouring. Let x_0 be some set which is not an element of S and define $S_0 \coloneqq S \cup \{x_0\}$. Extend the ordering $<$ of S to an ordering $<'$ of S_0 such that for every $x \in S$, $x <' x_0$. Then $\mathrm{otp}(S_0, <') = L + 1$. Define the colouring $g\colon [S_0]^{r+1} \to k$ such that for $\{y_0 <' \ldots <' y_{r-1} <' y_r\}$ in $[S_0]^{r+1}$ it holds that

$$g(y_0, \ldots, y_{r-1}, y_r) = f(y_0, \ldots, y_{r-1}).$$

By assumption there is a homogeneous set $H_0 = \{h_m \mid m \in M + 1\} \subseteq S_0$ for g such that $\mathrm{otp}(H_0, <') = M+1$. Then it is easy to see that the set $H \coloneqq \{h_m \mid m \in M\} \subseteq S$ is homogeneous for f with $\mathrm{otp}(H, <) = M$. □

Remark 2.23. By applying Lemma 2.21, we can show Lemma 2.22 also holds if we replace $L + 1$ and $M + 1$ by $1 + L$ and $1 + M$.

Lemma 2.22 is particularly useful if $L = \alpha$ and $M = \beta$ are infinite ordinals, because then it holds that $1 + \alpha = \alpha$ and $1 + \beta = \beta$. This means, if the resource and goal are infinite ordinals, we can decrease the exponent of a positive partition relation without altering the resource or the goal.

Lemma 2.24 (Theorem 14)**.** *Let $r < \omega$ and let k be any cardinal. Let α be an ordinal and suppose that β_n are initial ordinals for all $n < k$. Then the following statements are equivalent:*

$$\alpha \to (\beta_n)^r_{n<k}, \tag{2.4}$$

$$|\alpha| \to (|\beta_n|)^r_{n<k}. \tag{2.5}$$

Proof. (2.4) $\implies$ (2.5). Let S be a set with $|S| = |\alpha|$ and let $f\colon [S]^r \to k$ be a colouring. Let $<$ be any order on S such that $\mathrm{otp}(S, <) = \alpha$. Then by (2.4) there is some $n < k$ and some homogeneous set $H \subseteq S$ for f with colour n and $\mathrm{otp}(H, <) = \beta_n$. In particular, $|H| = |\beta_n|$, which shows (2.5).

(2.5) $\implies$ (2.4). Take some ordered set $(S, <)$ such that $\mathrm{otp}(S, <) = \alpha$, and let $g\colon [S]^r \to k$ be an arbitrary colouring. Obviously, $|S| = |\alpha|$, and hence by (2.5) there is some $n < k$ and a homogeneous set $H \subseteq S$ for g with colour n and $|H| = |\beta_n|$. As α is an ordinal, we have that $\mathrm{otp}(H, <) = \gamma$ for some ordinal γ. Since β_n is an initial ordinal, it follows that $\beta_n \leq \gamma$, which shows (2.4). □

Lemma 2.24 shows that we can switch between cardinals and the initial ordinal of that cardinal. We mention that the implication from (2.4) to (2.5) also holds if the β_n are not initial ordinals, and even if α and β_n are merely order-types. Interestingly, Corollary 3.3 shows that (2.5) does not imply (2.4) for all order-types.

One application of Lemma 2.24, together with Ramsey's Theorem, gives the following corollary. We will often refer to this corollary simply as "Ramsey's Theorem".

Corollary 2.25 (Ramsey's Theorem for ordinals)**.** *For all $r, k < \omega$ it holds that*

$$\omega \to (\omega)^r_k.$$

In Remark 2.5 we argued that we only need to define partition relations for finite exponents because AC implies that all partition relations with infinite exponents are negative. Similarly, we only defined the partition $\kappa \to (\mu)^r_m$ for when the colour set m is a cardinal. Naturally, one might wonder if it is interesting if we would let m be an ordinal, or even an order-type. The following result shows that only the cardinality of the colour set m matters.

Lemma 2.26 (Theorem 17)**.** *Let $r < \omega$, let L, M be order-types and let m and k be sets. Suppose $|m| = |k|$, then $L \to (M)^r_m$ if and only if $L \to (M)^r_k$.*

Proof. Assume $m, k \neq \varnothing$, else both partition relations are vacuously true. Suppose $L \to (M)^r_m$, we will show $L \to (M)^r_k$. Let $(S, <)$ be an ordered set with $\operatorname{otp}(S, <) = L$ and let $f\colon [S]^r \to k$ be an arbitrary colouring. As $|m| = |k|$, there is a bijection $g\colon k \to m$. Then $g \circ f\colon [S]^r \to m$, so by assumption there exists a homogeneous set $H \subseteq S$ for $g \circ f$ with colour $n \in m$ and $\operatorname{otp}(H, <) = M$. Then for any $X \in [H]^r$ we have $g \circ f(X) = n$, and hence $f(X) = g^{-1}(n) \in k$. This gives that H is homogeneous for f, thus we have shown $L \to (M)^r_k$. □

The lemma below is the *Transitivity Rule.* This lemma essentially shows that if there are two partition relations agreeing on the exponent, and M is a goal in one partition relation, while it is the resource of the other, then we can substitute the goals of the latter partition relation in the former partition relation. The proof is from [Erd+84, Theorem 9.8].

Lemma 2.27 (Theorem 16)**.** *Let $r, m, k_n < \omega$ for all $n < m$, let L, M_n, N_{n_i} be order-types for all $n < m$ and $i < k_n$. Suppose $L \to (M_n)^r_{n<m}$ and also $M_n \to (N_{n_i})^r_{i<k_n}$ for each $n < m$. Then $L \to (N_{n_i})^r_{n<m, i<k_n}$.*

Proof. Let $I = \{(n, i) \mid n < m \text{ and } i < k_n\}$. Let $(S, <)$ be an ordered set with $\operatorname{otp}(S, <) = L$ and let $f\colon [S]^r \to I$ be a colouring. This gives rise to the colouring $\pi_0 \circ f$, where π_0 is the projection onto the first coordinate. By assumption $L \to (M_n)^r_{n<m}$, so there exists some $n < m$ and a set $X \subseteq S$ which is homogeneous for $\pi_0 \circ f$ with colour n and $\operatorname{otp}(X, <) = M_n$. Similarly, since $M_n \to (N_{n_i})^r_{i<k_n}$, there exists a homogeneous $H \subseteq X$ for $\pi_1 \circ f \restriction [X]^r$ with colour, say, $i < k_n$ and $\operatorname{otp}(H, <) = N_{n_i}$. It follows that $H \subseteq S$ is homogeneous for f with colour (n, i), showing $L \to (N_{n_i})^r_{n<m, i<k_n}$. □

Lemma 2.28 (Theorem 18)**.** *Let $m, r < \omega$. Let L and M_i be order-types for all $i < m$. Suppose $L \to (M_i)^r_{i<m}$ and let $f\colon [L]^r \to m$ be a partition. Then there are sets $I, J \subseteq m$ with $|I| + |J| > m$ such that for all $i \in I$ and $j \in J$, there is an j-homogeneous set for f of order-type M_i.*

Proof. First, we may assume that $M_i \geq r$ for all $i < m$. For suppose otherwise, and $M_i < r$ for some $i < m$. Then the sets $I = \{i\}$ and $J = m$ suffice.

For each $j < m$, define

$$P_j = \{i \in m \mid \text{ there is an } j\text{-homogeneous set for } f \text{ of order-type } M_i\}.$$

Also define $Q_j = m \setminus P_j$.

We want to show that there exists some $J \subseteq m$ such that $|\bigcap_{j \in J} P_j| > m - |J|$, because then the set $I = \bigcap_{j \in J} P_j$ suffices. Then, $m - |\bigcap_{j \in J} P_j| < |J|$. Hence, it is sufficient to show there is some $J \subseteq m$ such that $|\bigcup_{j \in J} Q_j| < |J|$.

Suppose for the sake of contradiction such J does not exist. Then for all $J \subseteq m$, it holds that $|\bigcup_{j \in J} Q_j| \geq |J|$. Then we can pick pairwise distinct elements $i_j \in Q_j$ for all $j < m$.[1] The relation $L \to (M_{i_j})^r_{j<m}$ holds by Observation 2.14. Thus there is some $j < m$ such that there is an j-homogeneous set for f of order-type M_{i_j}, contradicting that $i_j \in Q_j$. □

[1] This result is known as Hall's marriage theorem.

Lemma 2.29 (Theorem 19)**.** *Let L, M, N be order-types and let δ be the initial ordinal of $|L|$. Suppose $L \to (M, N)^2$. Then at least one of the following four situations must be true.*

(i) $M < \omega$, *or*

(ii) $N < \omega$, *or*

(iii) $M, N \leq L$ *and* $M, N \leq \delta$, *or*

(iv) $M, N \leq L$ *and* $M, N \leq \delta^*$.

Proof. Let S be a set and let $<$ and $\ll$ be orders on S such that $\mathrm{otp}(S, <) = L$ and $\mathrm{otp}(S, \ll) = \delta$. Define the partition $f\colon [S]^2 \to 2$ by sending $\{x, y\} \mapsto 0$ if and only if $x < y \iff x \ll y$. Now, we apply Lemma 2.28 in the case where $m = 2$. This gives one of the following four cases.

Case (i). There are a 0-homogeneous set and a 1-homogeneous set for f of order-type M, which we call A and B, respectively. On the one hand, $M = \mathrm{otp}(A, <) = \mathrm{otp}(A, \ll) \leq \mathrm{otp}(S, \ll) = \delta$, and hence M is an ordinal. On the other hand, $M = \mathrm{otp}(B, <) = \mathrm{otp}(B, \gg) \leq \mathrm{otp}(S, \gg) = \delta^*$. This implies $M < \omega$.

Case (ii). There are a 0-homogeneous set and a 1-homogeneous set for f of order-type N. Analogous to case (i). Hence $N < \omega$.

Case (iii). There are 0-homogeneous sets A and B for f of order-type M and N, respectively. Then $M = \mathrm{otp}(A, <) \leq \mathrm{otp}(S, <) = L$ and $M = \mathrm{otp}(A, <) = \mathrm{otp}(A, \ll) \leq \mathrm{otp}(S, \ll) = \delta$. Similarly, $N \leq L$ and $N \leq \delta$.

Case (iv). There are 1-homogeneous sets A and B for f of order-type M and N, respectively. Analogous to case (iii) and this gives $M, N \leq L$ and $M, N \leq \delta^*$. □

Corollary 2.30. *For all order-types L,*

$$L \not\to (\omega, \omega^*)^2. \tag{2.6}$$

Proof. Suppose otherwise, i.e. $L \to (\omega, \omega^*)^2$. Then we apply Lemma 2.29 with $M = \omega$ and $N = \omega^*$. Clearly, cases (i) and (ii) do not hold. Then (iii) or (iv) must hold, hence $M, N \leq \delta$ or $M, N \leq \delta^*$, where δ is the initial ordinal of $|L|$. But as δ is an ordinal, it must be that $\omega \not\leq \delta^*$ and $\omega^* \not\leq \delta$, which gives a contradiction. □

Chapter 3

Modernisation

In this chapter we study a large portion of the Erdős-Rado paper. To be more precise, Theorem 23 to Theorem 45 of [ER56] are presented here.[1] We have partitioned the results into four sections, where in each section we will study partition relations with a certain kind of resource.

In **Section 3.1**, we study partition relations where the resource is an order-type.

In **Section 3.2**, we study the order-type of the reals $\mathbb{R}$, denoted as λ, as resource.

In **Section 3.3**, we study ordinals as resource.

In **Section 3.4**, we study cardinals as resource.

Of course, as ordinals, cardinals and λ are all specific cases of an order-type, Section 3.1 is the most general of all the sections. Any theorem that is proven in that section, can be used in the more specific cases. Similarly, as a cardinal is a specific case of an ordinal (any cardinal is an initial ordinal), any theorem from Section 3.3 can be used in Section 3.4.

As the paper has been published over 65 years ago, many strengthenings of results in [ER56] have been proven since then. We sometimes provide these newer results, but as the literature on partition calculus is so vast, we are not able to show all new results. In fact, if for some result in this thesis no additional improvements are given, this does not imply that no such improvements exist. We refer the reader to an historical exposition of partition calculus by J. Larson in [Lar12] for a more complete overview.

3.1 Order-types

In the first section of this chapter we study partition relations based on order-types. All order-types in this chapter will, in fact, be *linear* order-types. We remark that partition relations based on order-types are the most general in this thesis,[2] and hence any result in this section is also of relevance in subsequent sections.

We begin with an interesting result, with which we will be able to prove that many partition relations are negative.

[1] With the exception of Theorem 40. This result is about *finite* Ramsey Theory and has no further applications in the paper, hence we decided to omit this theorem in the thesis.

[2] There are partition relations for partial orders instead of linear orders, but these will not be treated in this thesis. See [Gal75] for examples of such partition relations.

Lemma 3.1 (Lemma 4). *Let L_0, L_1, M_0, M_1 be order-types. Suppose that $r \geq 2$ and $M_0, M_1^* \not\leq L_0$ and $|L_0| = |L_1|$, then*

$$L_1 \not\to (M_0, M_1, (r+1)_{r!-2})^r.$$

Proof. Throughout we may assume $|M_0|, |M_1| \geq \aleph_0$. Let S be a set and let $<$ be an order on S such that $\operatorname{otp}(S,<) = L_1$. As $|S| = |L_1| = |L_0|$, there is an ordering $\ll$ on S such that $\operatorname{otp}(S, \ll) = L_0$. Given any $X \in [S]^r$, we can index the elements in X such that $X = \{x_0 < x_1 < \ldots < x_{r-1}\}$. There is a unique permutation $\pi\colon r \to r$ such that $x_{\pi(0)} \ll x_{\pi(1)} \ll \ldots \ll x_{\pi(r-1)}$. Note that there are precisely $r!$ permutations of r. Fix an enumeration $\langle \pi_n \mid n < r! \rangle$ of permutations of r, where π_0 is the identity, and $\pi_1 = \pi_0^*$.

Define the $r!$-colouring $f\colon [S]^r \to r!\colon \{x_0 < x_1 < \ldots < x_{r-1}\} \mapsto n$ where π_n is such that $x_{\pi_n(0)} \ll x_{\pi_n(1)} \ll \ldots \ll x_{\pi_n(r-1)}$. Suppose there is an n-homogeneous set H for f with $\operatorname{otp}(H,<) = M_n$, where $M_n = r+1$ for $n \geq 2$. There are three cases to consider.

Case $n = 0$. Then $\operatorname{otp}(H,<) = M_0$ and $f \restriction [H]^r \equiv 0$. As π_0 is the identity and $r \geq 2$, we have in particular for any $x, y \in H$ that $x < y \iff x \ll y$. This means $M_0 = \operatorname{otp}(H,<) = \operatorname{otp}(H,\ll) \leq \operatorname{otp}(S,\ll) = L_0$, which is a contradiction.

Case $n = 1$. Then $\operatorname{otp}(H,<) = M_1$ and $f \restriction [H]^r \equiv 1$. In this case $\pi_1 = \pi_0^*$, which means for any $x, y \in H$ we have $x < y \iff y \ll x$. Therefore $M_1^* = \operatorname{otp}(H,<^*) = \operatorname{otp}(H,\ll) \leq \operatorname{otp}(S,\ll) = L_0$, again a contradiction.

Case $n \geq 2$. Then $\operatorname{otp}(H,<) = r+1$ and $f \restriction [H]^r \equiv n$. In particular $\pi_n \neq \pi_0$ and $\pi_n \neq \pi_1$. Write $H = \{x_0 < x_1 < \ldots < x_{r-1} < x_r\}$ and define $y_k = x_{k+1}$. Then

$$x_{\pi_n(0)} \ll x_{\pi_n(1)} \ll \ldots \ll x_{\pi_n(r-1)},$$
$$y_{\pi_n(0)} \ll y_{\pi_n(1)} \ll \ldots \ll y_{\pi_n(r-1)}.$$

Suppose $x_0 \ll x_1$, then $x_{\pi_n^{-1}(0)} < x_{\pi_n^{-1}(1)}$ and so $y_0 \ll y_1$. This gives $x_1 \ll x_2$. Repeating this argument gives that $x_0 \ll x_1 \ll \ldots \ll x_{r-1}$, and hence $\pi_n = \pi_0$, which is a contradiction. Similarly, if we assume $x_1 \ll x_0$, then by an analogous argument, we get $x_{r-1} \ll \ldots \ll x_1 \ll x_0$, i.e., $\pi_n = \pi_1$, which is also a contradiction.

We conclude that such a homogeneous set H cannot exist, and this concludes the proof. □

The first application of Lemma 3.1 shows that increasing the resource in Ramsey's Theorem to a larger countable ordinal, does not yield larger homogeneous sets.

Corollary 3.2. *For any $\alpha < \omega_1$,*

$$\alpha \not\to (\omega+1, \omega)^2. \tag{3.1}$$

Proof. We may assume α is infinite. Clearly, $\omega+1, \omega^* \not\leq \omega$. As $|\alpha| = |\omega|$, we obtain the desired result by Lemma 3.1. □

As a side note, we mention another application of Lemma 3.1 which shows that the partition relation[3] $\eta \to (\eta, \aleph_0)^2$ cannot be improved to ω. Recall that η denotes the order-type of the rationals.

[3] Theorem 6 in [ER56], see [Jon99, p. 17] for a proof.

Corollary 3.3. $\eta \not\to (\eta, \omega)^2$.

Proof. Obviously, ω is scattered, i.e. $\eta \not\leq \omega$, and ω is well-ordered, which gives $\omega^* \not\leq \omega$. Finally, $|\omega| = |\eta|$, which gives the desired result by Lemma 3.1. $\square$

The following result shows that the colour set $r!$ in Lemma 3.1 can be reduced to 2, given some extra assumptions. In particular, the goals M_0 and M_1 have to be additively indecomposable. An order-type M is said to be *additively indecomposable* if for any additive decomposition $M = N + N'$, it must be that $M \leq N$ or $M \leq N'$. Also, the resource and goal will be slightly increased.

Theorem 3.4 (Theorem 42). *Let $r \geq 3$, and let L_0, L_1, M_0, M_1 be order-types such that $|L_0| = |L_1|$ and $M_0, M_1^* \not\leq L_0$, and M_0, M_1 are additively indecomposable. Then*

$$(r-3) + L_1 \not\to ((r-3) + M_0, (r-3) + M_1)^r. \tag{3.2}$$

Proof. In view of Lemma 2.22, we only need to prove (3.2) for $r = 3$. Let $(S, <)$ be an ordered set with $\operatorname{otp}(S, <) = L_1$. As $|L_0| = |L_1|$, there is an order $\ll$ on S such that $\operatorname{otp}(S, \ll) = L_0$. Given $\{x_0 < x_1 < x_2\} \in [S]^3$, there is a unique permutation $\pi \colon 3 \to 3$ such that $x_{\pi(0)} \ll x_{\pi(1)} \ll x_{\pi(2)}$. A permutation is *even* if it makes an even numbers of *inversions*.[4] The three even permutations of 3 are $(0,1,2) \mapsto (0,1,2), (0,1,2) \mapsto (1,2,0)$ and $(0,1,2) \mapsto (2,0,1)$.

Define $f \colon [S]^3 \to 2$ by $\{x_0 < x_1 < x_2\} \mapsto 0$ if and only if the permutation π with $x_{\pi(0)} \ll x_{\pi(1)} \ll x_{\pi(2)}$ is even. We prove this partition shows (3.2). Suppose there is a 0-homogeneous set $H \subseteq S$ for f with $\operatorname{otp}(H, <) = M_0$. Define $B = \{x \in H \mid \forall y \in H(y < x \implies y \ll x)\}$. Define $C \coloneqq H \setminus B$. Let x, y, z denote elements of H. We show a few facts.

1. Suppose $x < y$, $y \in B$, $x \in C$. Since $x \in C$, there is some $z \in H$ such that $z < x$ and $x \ll z$. But then also, $z < y$ and $y \ll z$, contradicting $y \in B$. Therefore $x < y$ and $y \in B$ implies $x \in B$ and $x \ll y$. Also, $x < y$ and $x \in C$ implies $y \in C$. This shows $B < C$ and $\operatorname{otp}(B, <) \leq \operatorname{otp}(B, \ll)$.

2. Suppose $x \in B$, $y \in C$ and $x \ll y$. Then $x \in B$ and $x \ll y$ implies $x < y$. As $y \in C$, there is $z \in H$ with $z < y$ and $y \ll z$. Then $x \ll y \ll z$ and $x < y$ and $z < y$. If it were the case that $x < z$, then the permutation would be odd, which contradicts $\{x, y, z\} \in [H]^3$. Therefore $z < x$, but now $x \ll z$ shows $x \notin B$, also a contradiction. Hence $x \in B$ and $y \in C$ implies $y \ll x$. This implies $C \ll B$.

3. Suppose $x, y \in C$ and $x < y$ and $y \ll x$. As $x \in C$, there is $z \in H$ such that $z < x$ and $x \ll z$. But then $z < x < y$ and $y \ll x \ll z$, showing the permutation is odd and contradicting $\{x, y, z\} \in [H]^3$. Therefore $x, y \in C$ and $x < y$ implies $x \ll y$. This shows $\operatorname{otp}(C, <) \leq \operatorname{otp}(C, \ll)$.

Fact 1 shows that $(B, <)$ is an initial segment of $(H, <)$. Hence $(H, <) = (B, <) + (C, <)$. Define $\operatorname{otp}(B, <) = N_0$ and $\operatorname{otp}(C, <) = N_1$, then $M_0 = N_0 + N_1$. Then the three facts imply that $\operatorname{otp}(H, \ll) \geq N_1 + N_0$. Since M_0 is indecomposable, there is some $i \in 2$ with $M_0 \leq N_i$. We obtain

$$M_0 \leq N_i \leq \operatorname{otp}(H, \ll) \leq \operatorname{otp}(S, \ll) = L_0,$$

[4]An inversion of a permutation π is a pair (i, j) such that $i < j$ and $\pi(i) > \pi(j)$.

for some $i \in 2$. This contradicts that $M_0 \not\leq L_0$.

Suppose now that there is some 1-homogeneous H for f with $\operatorname{otp}(H, <) = M_1$. The proof is nearly analogous to the previous case. If $\{x_0 < x_1 < x_2\} \in [H]^3$, then the permutation π such that $x_{\pi(0)} \ll x_{\pi(1)} \ll x_{\pi(2)}$ is odd. But this means π is an even permutation with respect to $\gg$. Hence, we can replace M_1 by M_1^* and we can completely analogously show that $M_1^* \leq L_0$, which gives a contradiction.

We conclude that $L_1 \not\to (M_0, M_1)^3$. $\square$

Theorem 3.5 (Theorem 43)**.** *Let L, M, N be order-types. Let $r < s < \omega$ and $s \leq M$. If $L \to (M, N)^s$ and $N \to (s)^r_k$, then*

$$L \to (M)^r_k. \tag{3.3}$$

Proof. Let S be a set with $\operatorname{otp} S = L$ and let $f\colon [S]^r \to k$ be a partition. Define the partition $g\colon [S]^s \to 2$ by $g(X) = 0$ if and only if X is homogeneous for f, where $X \in [S]^s$.

We have assumed $L \to (M, N)^s$. Suppose there exists a 1-homogeneous $H \subseteq S$ for g with $\operatorname{otp} H = N$. Since $N \to (s)^r_k$, there exists $X \in [H]^s$ which is homogeneous for $f \restriction [H]^r$, and thus is also homogeneous for f. But this is a contradiction, because $g(X) = 1$, meaning that X cannot be homogeneous for f.

Therefore there is a 0-homogeneous $H \subseteq S$ for g with $\operatorname{otp} H = M$. Fix $X \in [H]^r$, and let $Y \in [H]^r$ be arbitrary such that $X \neq Y$. Such Y exists because $M > r$. Write $X = \{x_0, \ldots, x_{r-1}\}$ and $Y = \{x_m, \ldots, x_{m+r-1}\}$, where $1 \leq m \leq r$. Define $X_n = \{x_n, \ldots, x_{n+r-1}\}$ for all $n \leq m$. Again, $s > r$, so for all $n < m$ there exists $Y_n \in [H]^s$ such that $X_n \cup X_{n+1} \subseteq Y_n$. Note that since H is 0-homogeneous for g, then Y_n is i-homogeneous for f, for some $i < k$.

As Y_n is i-homogeneous for f, it follows immediately that $f(X_n) = f(X_{n+1}) = i$. But then it holds for all $n \leq m$ that $f(X_n) = i$. In particular, $f(X) = f(X_0) = i$ and $f(Y) = f(X_m) = i$. Since Y was arbitrary, we conclude that H is homogeneous for f, showing $L \to (M)^r_k$. $\square$

Lemma 3.6 (Erdős-Szekeres Theorem, 1935)**.** *Let $r < \omega$ and let $s > (r-1)^2$. Let $(S, \ll)$ be an ordered set and suppose that $N = \{n_0, n_1, \ldots, n_{s-1}\} \in [S]^s$. Then there are indices $0 \leq i_0 < i_1 < \ldots < i_{r-1} < s$ such that such that $n_{i_0} \ll n_{i_1} \ll \ldots \ll n_{i_{r-1}}$ or $n_{i_0} \gg n_{i_1} \gg \ldots \gg n_{i_{r-1}}$.*

Steele in his survey [Ste95, p. 114] on the Erdős-Szekeres Theorem[5] credits Seidenberg (1959) for "what is perhaps the slickest and most systematic proof [of the Erdős-Szekeres Theorem]". We present this proof here.

Proof. For every $i < s$, define the pair (a_i, b_i), where a_i is the length of the longest $\ll$-increasing subsequence ending with n_i, and b_i is the length of the longest $\ll$-decreasing subsequence ending with n_i. Given $i < j < s$, one of two cases holds: if $n_i \ll n_j$, then $a_i < a_j$, and if $n_i \gg n_j$, then $b_i < b_j$. In other words, given indices $i \neq j$, we have that $(a_i, b_i) \neq (a_j, b_j)$.

Now, suppose the statement of the theorem is false. Then there are no $\ll$-increasing or $\ll$-decreasing sequences of length r. In particular, for all $i < s$ it holds that $0 < a_i, b_i < r$. This means there are at most $(r-1)^2$ distinct pairs (a_i, b_i), but this contradicts that there must be $s > (r-1)^2$ distinct pairs. This concludes the proof. $\square$

[5] This result is sometimes referred to as the *Ordered Pigeonhole Principle*.

Theorem 3.7 (Theorem 35). *Let L, M, N be order-types and let $s, r < \omega$. Assume $M \geq r \geq 3$ and $s > (r-1)^2$. Suppose $M, M^* \not\leq L$ and $|L| = |N|$. Then*

$$N \not\to (s, M)^r. \tag{3.4}$$

Proof. Let S be a set and let $<$ and $\ll$ be orders on S such that $\operatorname{otp}(S, <) = N$ and $\operatorname{otp}(S, \ll) = L$. Define the partition $f \colon [S]^r \to 2$ by $\{x_0 < x_1 < \ldots < x_{r-1}\} \mapsto 1$ if and only if $x_0 \ll x_1 \ll \ldots \ll x_{r-1}$ or $x_0 \gg x_1 \gg \ldots \gg x_{r-1}$.

Let $H \in [S]^s$ be arbitrary. Since $s > (r-1)^2$, there is as a consequence of the Erdős-Szekeres Theorem some $A \in [H]^r$ such that $f(A) = 1$, which shows H is not 0-homogeneous for f.

Now assume there is a 1-homogeneous $H \subseteq S$ with $\operatorname{otp}(H, <) = M$. Define the sets

$$A \coloneqq \{\{x_0 < x_1 < \ldots < x_{r-1}\} \in [H]^r \mid x_0 \ll x_1 \ll \ldots \ll x_{r-1}\}, \text{ and} \tag{3.5}$$
$$B \coloneqq \{\{x_0 < x_1 < \ldots < x_{r-1}\} \in [H]^r \mid x_0 \gg x_1 \gg \ldots \gg x_{r-1}\} \tag{3.6}$$

Since H is 1-homogeneous for f, it holds that $[H]^r = A \cup B$.

Claim. *$[H]^r = A$ or $[H]^r = B$.*

Proof of claim. Suppose for the sake of contradiction that the claim is false. Then there are sets $X, Y \in [H]^r$ such that

$$X = \{x_0 < x_1 < \ldots < x_{r-1}\} = \{x_0 \ll x_1 \ll \ldots \ll x_{r-1}\}, \tag{3.7}$$
$$Y = \{y_0 < y_1 < \ldots < y_{r-1}\} = \{y_0 \gg y_1 \gg \ldots \gg y_{r-1}\}. \tag{3.8}$$

We suppose that X and Y are chosen in such a way that $n \leq r-1$ is maximal, where $x_i = y_i$ for all $i < n$, but $x_n \neq y_n$. Assume w.l.o.g. that $x_n < y_n$.

First, suppose $n < r-1$. Then define

$$Z \coloneqq \{y_0 < y_1 < \ldots y_{n-1} < x_n < y_n < \ldots < y_{r-2}\} \in [H]^r.$$

By the maximality of n, it cannot be that $Z \in B$, and thus $Z \in A$. Therefore, as also $r \geq 3$, it holds that $y_0 \ll y_{r-2}$, which is a contradiction with (3.8). If, on the other hand, $n = r-1$, then immediately $x_0 = y_0 \gg y_1 = x_1$, contradicting (3.7). This concludes the proof of the claim. ■

If now $[H]^r = A$, then $M = \operatorname{otp}(H, <) = \operatorname{otp}(H, \ll) \leq \operatorname{otp}(S, \ll) = L$, which is a contradiction. Similarly, if $[H]^r = B$, then $M^* = \operatorname{otp}(H, <^*) = \operatorname{otp}(H, \ll) \leq \operatorname{otp}(S, \ll) = L$, which also gives a contradiction. We conclude such H does not exist, and therefore $N \not\to (s, M)^r_2$. □

Given order types M_n, for all $n \leq m < \omega$, we can define an order-type of the product $\prod_{n<m} M_n$, where we order the elements lexicographically according to the orders on M_n.[6]

Theorem 3.8 (Theorem 45). *Let $m > 0$, and let M_n be order-types for all $n < m$. Then*

$$\prod_{n<m} M_n \to (M_n)^1_{n<m}. \tag{3.9}$$

[6] We treat the order-type of the Cartesian product with the lexicographical order in more depth in Section 4.1.

Proof. Let B_n be sets with $\operatorname{otp} B_n = M_n$, and order $S = \prod_{n<m} B_n$ lexicographically by $\prec$.

First, suppose that there is some $n < m$ such that for all $i < n$ there are $x_i \in B_i$ such that for every $x_n \in B_n$, there are some elements $x_k^{x_n} \in B_k$ for $n < k < m$ with $f((x_0, \ldots, x_{n-1}, x_n, x_{n+1}^{x_n}, \ldots, x_{m-1}^{x_n})) = n$. Then the set

$$H = \{(x_0, \ldots, x_{n-1}, x_n, x_{n+1}^{x_n}, \ldots, x_{m-1}^{x_n}) \mid x_n \in B_n\},$$

has order-type M_n and is n-homogeneous for f.

Suppose, on the other hand, that for all $n < m$ and all elements $x_i \in B_i$ with $i < n$, there is some element $g_n(x_0, \ldots, x_{n-1}) \in B_n$ such that for all $x_k \in B_k$ with $n < k < m$ it holds that

$$f((x_0, \ldots, x_{n-1}, g_n(x_0, \ldots, x_{n-1}), x_{n+1}, \ldots, x_{m-1})) \neq n.$$

Then we inductively define elements $y_n = g_n(y_0, \ldots, y_{n-1})$ for all $n < m$. Clearly, for some $n < m$, it must be that $f(y_0, \ldots, y_{n-1}, y_n, y_{n+1}, \ldots, y_{m-1}) = n$, but this contradicts the definition of y_n.

This concludes the proof. □

3.1.1 Baumgartner-Hajnal theorem

Many results in the partition calculus have been proven after the publication of [ER56]. We will give an overview of some results, but we will not go into detail.

The following result by Baumgartner-Hajnal (1973) answered many questions posed by Erdős and Rado.[7] The theorem provides a strengthening for many results in this thesis, such as Theorem 3.46, Theorem 3.22 and Lemma 3.18. The original proof by Baumgartner and Hajnal used a meta-mathematical proof. First, they proved the partition relation assuming Martin's Axiom (MA) and then they used an absoluteness argument to show that the result must be a theorem of ZFC after all.

Two years after the publication of [BH73], Fred Galvin provided a more standard proof, avoiding the use of a meta-mathematical argument. As he says in [Gal75, p. 712]: "While this method [of Baumgartner and Hajnal] works, still one would naturally like to see a direct 'combinatorial' proof".

Theorem 3.9 (Baumgartner-Hajnal Theorem, [BH73], Theorem 1)**.** *Let L be any order-type such that $L \to (\omega)^1_\omega$. Then for any $\gamma < \omega_1$ and any $k < \omega$,*

$$L \to (\gamma)^2_k. \tag{3.10}$$

We will not present a proof in this thesis. The most important uses of the Baumgartner-Hajnal theorem are for the resources ω_1 and λ, where λ is the order-type of the reals.

Central to the partition calculus is to find a potential strengthening, if it exists. Ideally, we want to find proofs that a positive partition relation is *sharp*, i.e. decreasing the resource or increasing the goal, exponent or colour set will give a negative partition relation. One instance of the Baumgartner-Hajnal Theorem is sharp: $\omega_1 \to (\gamma)^2_k$ for all $\gamma < \omega_1$. The exponent cannot be increased by Corollary 3.44, nor

[7] See [Lar12, p. 265] for an historical exposition.

can the resource be decreased by Corollary 3.2. It is also not possible to increase the goal, because as a corollary of Theorem 3.47 we obtain

$$\omega_1 \not\to (\omega_1)^2_2. \tag{3.11}$$

And finally, the colour set can also not be increased because $\alpha \not\to (2)^1_\omega$ for any countable ordinal α, and hence by Lemma 3.36,

$$\omega_1 \not\to (3)^2_\omega. \tag{3.12}$$

Of course, for the sake of completeness, we could decrease the exponent yet again. And as a consequence of $\aleph_1$ being regular, the partition relation is positive when the exponent is 1:

$$\omega_1 \to (\omega_1)^1_\omega. \tag{3.13}$$

3.2 The reals

In this section we study partition relations where the resource is the order-type of the reals. We will denote the order-type of the reals as $\operatorname{otp}\mathbb{R} = \lambda$. We alternate between the usual real line $\mathbb{R}$ and the Cantor space 2^ω, but this does not matter as both $\lambda \leq \operatorname{otp} 2^\omega$ and $\operatorname{otp} 2^\omega \leq \lambda$. Note that $|\lambda| = 2^{\aleph_0}$.

3.2.1 Linear continuum

We give the following definition to make more precise what properties of λ shall be needed.

Definition 3.10 (Linear continuum)**.** A non-empty linearly ordered set $(S, <)$ is a *linear continuum* if it is dense and has the least upper bound property (l.u.b.), that is

1. for all $x, y \in S$ if $x < y$ there exists $z \in S$ such that $x < z < y$, and
2. every non-empty subset $B \subseteq S$ that has an upper bound in S also has a least upper bound in S.

Remark 3.11. In fact, for non-empty linearly ordered sets, having the l.u.b. is equivalent to having the greatest lower bound property, hence every non-empty linear order that has the l.u.b. is *complete*. We give a quick proof below.

Suppose $(S, <)$ is a non-empty linearly ordered set and has the least upper bound property. Let $P \subseteq S$ be a non-empty subset which has a lower bound $\ell \in S$. Then the set of lower bounds $L := \{x \in S \mid \forall p \in P(x < p)\}$ is non-empty as $\ell \in L$. Since $(S, <)$ has the l.u.b. and any $p \in P$ is an upper bound of L, $\sup L$ exists. Finally, it is easy to see that $\sup L = \inf P$, and hence $(S, <)$ has the greatest lower bound property.

Theorem 3.12 (Theorem 26)**.** *If $r \geq 0$, $k > 0$, then*

$$\lambda \not\to (\omega_1)^r_k. \tag{3.14}$$

If, additionally $r \geq 2$, then

$$\lambda \not\to (r+1)^r_{\aleph_0}. \tag{3.15}$$

Proof. (3.14) follows from the fact that ω_1 does not embed into $\mathbb{R}$. For suppose there is an order-preserving function $h\colon \omega_1 \to \mathbb{R}$. As $\mathbb{Q}$ is dense in $\mathbb{R}$, for every $\alpha < \omega_1$ there exists a rational q_α such that $h(\alpha) < q_\alpha < h(\alpha+1)$. Define the mapping $g\colon \omega_1 \to \mathbb{Q}\colon \alpha \mapsto q_\alpha$, which is order-preserving and hence injective. We get a contradiction because ω_1 cannot be injected into a countable set.

In view of Lemma 2.22 it suffices to show (3.15) for $r = 2$. Fix an enumeration $\langle q_i \mid i < \omega\rangle$ of $\mathbb{Q}$. Define the colouring $f\colon [\mathbb{R}]^2 \to \aleph_0$ by

$$\{x < y\} \mapsto \text{ least } i \text{ such that } x < q_i < y.$$

Suppose there exists a set $\{x < y < z\} \in [\mathbb{R}]^3$ that is i-homogeneous for f for some $i \in \omega$. Then it holds that $x < q_i < y < q_i < z$, which is a contradiction. □

Theorem 3.13 (Theorem 27)**.** $\lambda \not\to (\omega, \omega+2)^3_2$.

Proof. Let 2^ω be the Cantor space with the usual linear ordering and topology. That is, for $y, z \in 2^\omega$ we have $y \prec z$ if for the least $n \in \omega$ such that $y(n) \neq z(n)$, then $y(n) < z(n)$. And the basic opens are $N_s = \{x \in 2^\omega \mid s \subset x\}$ for all $s \in 2^{<\omega}$. As $\lambda \leq \operatorname{otp} 2^\omega$, it suffices to show $\operatorname{otp} 2^\omega \not\to (\omega, \omega+2)^3_2$.

Define $f\colon [2^\omega]^3 \to 2$ by $f(\{x \prec y \prec z\}) = 1$ if and only if $n < m$, where n is the least with $x(n) < y(n)$ and m the least with $y(m) < z(m)$. Clearly, there is no 0-homogeneous set for f of order-type ω, else there would be an infinitely decreasing sequence of natural numbers.

Let $H = \{x_\beta \mid \beta < \omega+2\}$ be a set of order-type $\omega+2$. As the Cantor space is compact, the infinite sequence $\{x_n \mid n < \omega\}$ has a converging subsequence with limit, say, $\ell \in 2^\omega$. Suppose $\ell \neq x_\omega$. Then there are x_i, x_m in the converging subsequence such that x_i is closer to x_m, than x_m is to x_ω. In other words, x_i and x_m differ later than x_m and x_ω, i.e. $f(\{x_i \prec x_m \prec x_\omega\}) = 0$. So, H is not 1-homogeneous for f. An analogous result holds if $x_{\omega+1} \neq \ell$. Hence there are no 1-homogeneous sets for f of order-type $\omega+2$. □

Theorem 3.14 (Theorem 28)**.** *For $r \geq 4$,*

$$\lambda \not\to (r+1, \omega+2)^r. \tag{3.16}$$

Proof. It suffices to show the theorem for $r = 4$. Define the colouring $f\colon [(0,1)]^4 \to 2$ by

$$\{x_0 < x_1 < x_2 < x_3\} \mapsto 0 \iff x_2 - x_1 < x_3 - x_2 \text{ and } x_2 - x_1 < x_1 - x_0.$$

Suppose there is set $\{x_0 < x_1 < x_2 < x_3 < x_4\} \subseteq (0,1)$ which is 0-homogeneous for f. Then simultaneously $f(\{x_1, x_2, x_3, x_4\}) = 0$, so that $x_3 - x_2 < x_2 - x_1$, and $f(\{x_0, x_1, x_2, x_3\}) = 0$, and so $x_2 - x_1 < x_3 - x_2$, which gives a contradiction.

Assume that there is a 1-homogeneous set $H \subseteq (0,1)$ for f of order-type $\operatorname{otp} H = \omega + 2$. As $(0,1)$ is complete and H is bounded by $h_{\omega+1}$, there is some subsequence of $\{h_n \mid n < \omega\}$ converging to some limit, say, $\ell \in (0,1)$. Similar to the proof of Theorem 3.13, we assume that $\ell \neq h_\omega$. Let $\varepsilon := \min\{h_1 - h_0, h_\omega - \ell\}$. There exist $m, n < \omega$ with $m > n$ such that $h_m - h_n < \varepsilon/2$, and hence $h_m - h_n < h_1 - h_0 \leq h_n - h_0$ and $h_m - h_n < h_\omega - \ell \leq h_\omega - h_m$, and therefore $f(\{h_0, h_n, h_m, h_\omega\}) = 0$, contradicting our assumption. An analogous argument holds if $\ell \neq h_{\omega+1}$, and this concludes the proof. □

Theorem 3.15 (Theorem 29)**.** *Let L be an order-type with $|L| \geq 2^{\aleph_0}$, then*

$$\lambda \not\to (L)^1_2. \tag{3.17}$$

Proof. Suppose that $L \leq \lambda$, i.e. L embeds into λ, else the result follows trivially because there would be no subset $H \subseteq \mathbb{R}$ with $\operatorname{otp} H = L$. Note that this also implies $|L| = 2^{\aleph_0}$.

Fix some $A \subseteq (0,1)$ such that $\operatorname{otp} A = L$. Let $B \subseteq (0,1)$ with $\operatorname{otp} B = L$ be arbitrary. As A and B have the same order-type, there is an order-isomorphism $f_B \colon A \to B$. We extend f_B to a total function $\tilde{f}_B$ on $(0,1)$ as follows: for all strict lower bounds ℓ of A, we set $\tilde{f}_B(\ell) = 0$, and furthermore for all other $x \in (0,1) \setminus A$,

$$x \mapsto \sup\{f_B(y) \mid y \leq x \wedge y \in A\}.$$

Note that $\tilde{f}_B$ is non-decreasing and that $\tilde{f}_B$ and A uniquely determine B.

As $\tilde{f}_B$ is non-decreasing on $(0,1)$, its set of discontinuous points $D(B)$ is countable. This is because for every $b \in D(B)$ we have that $\tilde{f}_B(b^-) < \tilde{f}_B(b^+)$, where $\tilde{f}_B(b^-) = \lim_{x \to b^-} \tilde{f}_B(x)$ and $\tilde{f}_B(b^+) = \lim_{x \to b^+} \tilde{f}_B(x)$ are the left-sided and right-sided limits, respectively. Hence there exists some rational q_b such that $\tilde{f}_B(b^-) < q_b < \tilde{f}_B(b^+)$. Thus we can construct an injection from $D(B)$ into $\mathbb{Q}$ and so $D(B)$ is countable.

As the rationals are dense in $(0,1)$, we can approximate functions values of f_B on continuous points arbitrarily well using the function values on the rationals. Hence, the function f_B is completely determined by the set $D(B)$, its function values on $D(B)$, and its function values on $\mathbb{Q}$. Therefore there are at most

$$|L^{3 \cdot \aleph_0}| = |(2^{\aleph_0})^{\aleph_0}| = 2^{\aleph_0}$$

such functions. As every B was completely determined by f_B and A, we conclude that are at most $2^{\aleph_0}$ subsets of $(0,1)$ of order-type L.

Let $\kappa \leq 2^{\aleph_0}$ be the amount of subsets of $(0,1)$ of order-type L. Fix an enumeration $\langle B_\gamma \mid \gamma < \kappa \rangle$ of subsets of $(0,1)$ of order-type L. Define inductively for $\gamma < \kappa$,

$$x_\gamma, y_\gamma \in B_\gamma \setminus \{x_\delta, y_\delta \mid \delta < \gamma\} \quad \text{with } x_\gamma \neq y_\gamma.$$

This is possible because for every $\gamma < \kappa$ we have

$$|\{x_\delta, y_\delta \mid \delta < \gamma\}| < 2^{\aleph_0} = |L| = |B_\gamma|.$$

Now define $f \colon (0,1) \to 2$ by sending for all $\gamma < \kappa$, $x_\gamma \mapsto 0$ and $y_\gamma \mapsto 1$, and all other elements to 1 as well.

Take any $H \subseteq (0,1)$ with $\operatorname{otp} H = L$. Then there is some $\gamma < \kappa$ such that $H = B_\gamma$. Then by construction $x_\gamma, y_\gamma \in H$ and $f(x_\gamma) = 0$ and $f(y_\gamma) = 1$, which shows H is not homogeneous for f. □

3.2.2 Real order-types

While proving certain partition results where the resource was λ, the order-type of the continuum, Erdős and Rado noted that only a few properties of λ were needed to prove these theorems. Hence they defined a more general order-type, which we will denote as ϕ throughout this subsection, and proved the partition results with ϕ as resource. Obviously, it follows that all relations in this subsection hold when the resource is λ.

Definition 3.16 (Real order-types)**.** An order-type ϕ is a *real order-type* if $|\phi| > \aleph_0$ and neither ω_1 nor ω_1^* embed into ϕ.

We prove an important proposition, which essentially shows that any set S with real order-type ϕ contains $\mathbb{Q}$-many uncountable sets, where these sets are themselves ordered by the rationals. We prove a slightly stronger result than what we need.

Proposition 3.17 (Lemma 1)**.** *Let S be a linearly ordered set such that* $\operatorname{cf}|S| = \aleph_n$ *and $\omega_n, \omega_n^* \not\leq$* $\operatorname{otp} S$. *Then for every rational q there is a set $A_q \subset S$ such that*

1. $|A_q| = |S|$, *and*
2. $A_p < A_q$ *for all rationals $p < q$, (i.e. $x < y$ for all $x \in A_p$ and $y \in A_q$).*

Proof. Case 1. There is some $A \subseteq S$ with $|A| = |S|$ such that for all $x \in A$ the initial segment of A with respect to x has cardinality strictly less than $|A|$. In other words, for all $x \in A$,

$$|\{y \in A \mid y < x\}| < |A|.$$

Define x_ν inductively for $\nu < \omega_n$, where

$$x_\nu \in A \setminus \bigcup_{\mu<\nu} \{y \in A \mid y \leq x_\mu\}.$$

Note that since $\nu < \omega_n = \operatorname{cf}|A|$ such x_ν exist for all $\nu < \omega_n$. In particular, we have found a strictly $<$-increasing sequence $\langle x_\nu \mid \nu < \omega_n \rangle$ in S. This means that $\omega_n \leq \operatorname{otp} S$, which is a contradiction. Hence such A cannot exist.

Case 2. There is some $A \subseteq S$ with $|A| = |S|$ such that for all $x \in A$ the final segments are sufficiently small, i.e.,

$$|\{y \in A \mid y > x\}| < |A|.$$

Analogous to Case 1, we can show $\omega_n^* \leq \operatorname{otp} S$, which is a contradiction.

Case 3. There is some $A \subseteq S$ with $|A| = |S|$ such that for all $x \in A$, either the initial segment or the final segment with respect to x has cardinality strictly less than $|A|$. Define $A_L \coloneqq \{x \in A \mid |\{y \in A \mid y < x\}| < |A|\}$ and $A_R \coloneqq \{x \in A \mid |\{y \in A \mid y > x\}| < |A|\}$. As $|A| = |A_L| + |A_R|$, we have $|A_L| = |A|$ or $|A_R| = |A|$. If $|A| = |A_L|$, then we follow Case 1, and if $|A| = |A_R|$, we follow Case 2. As both cases give a contradiction we conclude such A does not exist.

So far, we have shown that whenever $A \subseteq S$ is such that $|A| = |S|$, there is some $x \in A$ such that the initial and final segment with respect to x both have cardinality $|A|$. In other words,

$$|\{y \in A \mid y < x\}| = |S| = |\{y \in A \mid y > x\}|.$$

Since $|\{y \in A \mid y < x\}| = |S|$, there is $x' \in A$ such that $|\{y \in A \mid y < x'\}| = |S| = |\{y \in A \mid x' < y < x\}|$. Define

$$A_0 \coloneqq \{y \in A \mid y < x'\}, A_1 \coloneqq \{y \in A \mid x' < y < x\} \text{ and } A_2 \coloneqq \{y \in A \mid x < y\}.$$

Note that $A_0 < A_1 < A_2$.

We "fix" A_1, but we continue this operation for A_0 and A_2, to obtain the sets A_{00}, A_{01}, A_{02} and A_{20}, A_{21}, A_{22}. Generally, for every finite ternary sequence of the form $\langle i_0, \ldots, i_{m-1}, 1\rangle$ with $i_0, \ldots, i_{m-1} \in \{0, 2\}$ we obtain the set $A_{i_0 \ldots i_{m-1} 1}$. The (finite) sequences of this form define the set

$$Q = \{\langle i_0, \ldots, i_{m-1}, 1\rangle \mid i_0, \ldots, i_{m-1} \in \{0, 2\}, m \in \omega\},$$

which when ordered lexicographically is a dense countable linear order without end-points, and hence is isomorphic to the rationals, see Theorem 2.10.

By construction, for all $p \in Q$ we have $|A_p| = |S|$ and for any $q \in Q$ with $p < q$ it holds that $A_p < A_q$, which is what we wanted to show. $\square$

We remark that all the following results in this subsection, with the exception of Lemma 3.21, are strengthened by the Baumgartner-Hajnal theorem. This is because $\phi \to (\omega)^1_\omega$, and hence by Theorem 3.9 we obtain $\phi \to (\gamma)^2_k$, for all $\gamma < \omega_1$ and $k < \omega$.

The main theorem in this section is Theorem 3.22. This theorem heavily depends on Proposition 3.17. First, we prove a lemma which we will need later.

Lemma 3.18 (Theorem 31iii)**.** *Let ϕ be a real order-type and let $\gamma < \omega_1$. Then*

$$\phi \to (\omega, \gamma)^2. \tag{3.18}$$

Proof. Assume w.l.o.g. that $|\phi| = \aleph_1$. Let S be a set with $\operatorname{otp} S = \phi$ and let $f\colon [S]^2 \to 2$ be a colouring. Assume there does not exist a set $B \subseteq S$ with $|B| = \aleph_1$ such that for all $x \in B$ it holds that

$$|\{y \in B \mid f(\{x < y\}) = 0\}| \leq \aleph_0.$$

Then we can inductively define an increasing ω-sequence which will be 0-homogeneous for f. Let $B_0 = S$ and $x_0 \in B_0$ such that

$$|\{y \in B_0 \mid f(\{x_0 < y\}) = 0\}| = \aleph_1.$$

For $n < \omega$ let $x_n \in B_n$ be such that

$$B_{n+1} \coloneqq \{y \in B_n \mid f(\{x_n < y\}) = 0\}$$

is uncountable. Then $\{x_n \mid n < \omega\}$ constitutes the 0-homogeneous set of order-type ω.

Assume now that such B does exist. As $|B| = \aleph_1$ and $\omega_1, \omega_1^* \not\leq B$, by Proposition 3.17 for all rationals q there are $B_q \subseteq B$ such that

1. $|B_q| = \aleph_1$, and
2. $B_q < B_p$ for all rationals $q < p$.

As $(\gamma, <)$ is a countable linear order, there is an order-preserving injection $g\colon (\gamma, <) \to (\mathbb{Q}, <)$, see Theorem 2.9. Inductively define x_δ for $\delta < \gamma$, where

$$x_\delta \in B_{g(\delta)} \setminus \bigcup_{\beta<\delta} \{y \in B_{g(\delta)} \mid f(\{x_\beta < y\}) = 0\}.$$

Note that this is possible because $|B_{g(\delta)}| = \aleph_1$ and for every $\beta < \delta$,

$$|\{y \in B_{g(\delta)} \mid f(\{x_\beta < y\}) = 0\}| \leq \aleph_0,$$

and $\aleph_1$ is regular. Finally, $X \coloneqq \{x_\delta \mid \delta < \gamma\}$ is 1-homogeneous for f of order-type γ. $\square$

Recall that the relation $L \to (M_0, M_1 \vee M_2)^2$ means:

"For all sets S with $\operatorname{otp} S = L$ and every colouring $f\colon [S]^2 \to 2$, there is either a 0-homogeneous set for f of order-type M_0, or a 1-homogeneous set for f of order-type either M_1 or M_2."

Lemma 3.19 (Theorem 32i)**.** *Let ϕ be a real order-type. Assume that $\alpha < \omega \cdot 2$ and $\gamma < \omega_1$, then*

$$\phi \to (\alpha, \gamma \vee \omega \cdot \gamma^*)^2. \tag{3.19}$$

Proof. Let S be a set with $\operatorname{otp} S = \phi$ and assume w.l.o.g. that $|S| = \aleph_1$. As $\alpha < \omega \cdot 2$, we can assume w.l.o.g. that $\alpha = \omega + m$ for some $m < \omega$. We make a case distinction. Assume that whenever $A \subseteq S$ is such that $|A| = \aleph_1$, then there is some $x \in A$ with

$$|\{y \in A \mid f(\{y < x\}) = 0\}| = \aleph_1.$$

Let $A_0 = S$ and $x_0 \in A_0$ such that $A_1 \coloneqq \{y \in A_0 \mid f(\{y < x_0\}) = 0\}$ is uncountable. We can continue this construction to obtain the set $\{x_{m-1} < \ldots < x_0\}$ and the uncountable set $A_m = \{y \in A_{m-1} \mid f(\{y < x_{m-1}\}) = 0\}$. Note that $|A_m| = \aleph_1$ and $\omega_1, \omega_1^* \not\leq \operatorname{otp} A_m$, and hence by Lemma 3.18, $\operatorname{otp} A_m \to (\omega, \gamma)^2$. If there were a 0-homogeneous set $H \subseteq A_m$ for f of order-type ω, then the set $H \cup \{x_{m-1} < \ldots < x_0\}$ would be 0-homogeneous for f of order-type α. If, on the other hand, there were a 1-homogeneous set for f of order-type γ, we would be done as well.

Now we assume that there is some uncountable $A \subseteq S$ such that for all $x \in A$,

$$|\{y \in A \mid f(\{y < x\}) = 0\}| \leq \aleph_0.$$

Again, as $|A| = \aleph_1$ and $\omega_1, \omega_1^* \not\leq \operatorname{otp} A$, we have by Proposition 3.17 that for all rationals q there exist $A_q \subset A$ such that

1. $|A_q| = \aleph_1$, and
2. $A_q < A_p$ for all rationals $q < p$.

As $(\gamma, <)$ is a countable linear order and $(\mathbb{Q}, >)$ is a countable dense linear order without endpoints, there is an order-preserving injection $g\colon (\gamma, <) \to (\mathbb{Q}, >)$, see Theorem 2.9. We inductively define sets P_ν for $\nu < \gamma$. For $\mu < \nu$ assume that $P_\mu \subset A_{g(\mu)}$ with $\operatorname{otp} P_\mu = \omega$ have already been defined. Note that by assumption

$$\{y \in A_{g(\nu)} \mid \exists \mu < \nu \exists x \in P_\mu\, f(\{y < x\}) = 0\}$$

is countable, and hence the set

$$B_\nu = \{y \in A_{g(\nu)} \mid \forall \mu < \nu \forall x \in P_\mu\, f(\{y < x\}) = 1\}$$

is uncountable. By Lemma 3.18 we have $\operatorname{otp} B_\nu \to (\alpha, \omega)^2$. If there exists a 0-homogeneous set $H \subseteq B_\nu$ of order-type $\operatorname{otp} H = \alpha$, we halt the process because we have showed (3.19). Otherwise there is a homogeneous set $P_\nu \subseteq B_\nu$ with colour 1 and $\operatorname{otp} P_\nu = \omega$. This completes the construction of the P_ν. Now, define the set $H = \bigcup_{\nu<\gamma} P_\nu$, which is homogeneous for f with colour 1 and $\operatorname{otp} H = \omega \cdot \gamma^*$, showing (3.19).

We have exhausted all cases and this concludes the proof. □

Corollary 3.20 (Theorem 31ii). *Let ϕ be a real order-type. Let $\alpha < \omega \cdot 2$ and $\beta < \omega^2$, then*

$$\phi \to (\alpha, \beta)^2.$$

Proof. As $\beta < \omega^2$ there is some $m < \omega$ such that $\beta \leq \omega \cdot m$. Let S be a set with $\operatorname{otp} S = \phi$ and let $f \colon [S]^2 \to 2$ be a colouring. By Lemma 3.19, $\phi \to (\alpha, \beta \vee \omega \cdot \beta^*)^2$. Thus there exists some set $H \subseteq S$ such that either H is 0-homogeneous for f of order-type $\operatorname{otp} H = \alpha$, or H is 1-homogeneous for f of either order-type $\operatorname{otp} H = \omega \cdot m$ or of order-type $\operatorname{otp} H = \omega \cdot (\omega \cdot m)^*$. As β embeds into $\omega \cdot m$, β also embeds into $\omega \cdot (\omega \cdot m)^* = \omega \cdot \omega^* \cdot m$, we conclude $\phi \to (\alpha, \beta)^2$. □

Lemma 3.21 (Theorem 32ii). *Let ϕ be a real order-type. Assume $\gamma < \omega_1$. Then*

$$\phi \to (\omega + \omega^*, \gamma \vee \gamma^*)^2. \tag{3.20}$$

Proof. Let S be a set with $\operatorname{otp} S = \phi$ and assume w.l.o.g. that $|S| = \aleph_1$. First we suppose that there exists a set $A \subseteq S$ with $|A| = \aleph_1$ such that for all $x \in A$,

$$|\{y \in A \mid f(\{x < y\}) = 0\}| \leq \aleph_0.$$

By Proposition 3.17, for all rationals q there exists a set $A_q \subseteq A$ such that

1. $|A_q| = \aleph_1$, and
2. $A_q < A_p$ whenever $q < p$.

There exists an order-preserving injection $g \colon (\gamma, <) \to (\mathbb{Q}, <)$. We inductively define elements x_ν for $\nu < \gamma$ by picking

$$x_\nu \in A_{g(\nu)} \setminus \bigcup_{\mu < \nu} \{y \in A \mid f(\{x_\mu < y\}) = 0\}.$$

Note that this is possible because $A_{g(\nu)}$ is uncountable, and $\{y \in A \mid f(\{x_\mu < y\}) = 0\}$ is countable for all $\mu < \nu$. The set $H = \{x_\nu \mid \nu < \gamma\}$ is 1-homogeneous for f of order-type $\operatorname{otp} H = \gamma$. This would show (3.20).

Assume now that there is a set $A \subseteq S$ with $|A| = \aleph_1$ such that for all $z \in A$,

$$|\{y \in A \mid f(\{y < z\}) = 0\}| \leq \aleph_0.$$

As there is an order-preserving injection from $(\gamma, <)$ to $(\mathbb{Q}, >)$, we can find, by an analogous argument, a 1-homogeneous set for f of order-type γ^*. This would also show (3.20).

Finally, suppose that for all sets $A \subseteq S$ with $|A| = \aleph_1$ there are $x, z \in A$ such that

$$|\{y \in A \mid f(\{x < y\}) = 0\}| = \aleph_1, \text{ and} \tag{3.21}$$

$$|\{y \in A \mid f(\{y < z\}) = 0\}| = \aleph_1. \tag{3.22}$$

We inductively define x_n, z_n for $n < \omega$ as follows: let $A_0 = S$ and let $x_0 \in S$ such that $B_0 := \{y \in A_0 \mid f(\{x_0 < y\}) = 0\}$ is uncountable. Then let $z_0 \in B_0$ such that $A_1 := \{y \in B_0 \mid f(\{y < z_0\}) = 0\}$ is uncountable. We then pick $x_1 \in A_1$ such that B_1 is uncountable, etc. After that, we can define the set $H = \{x_0 < x_1 < x_2 < \dots\} \cup \{\dots < z_2 < z_1 < z_0\}$, which has order-type $\omega + \omega^*$. By construction, H is 0-homogeneous for f, hence we have showed (3.20).

We have exhausted all cases and this concludes the proof. □

The following theorem is a key result in this section. Of course, we mention again, that the Baumgartner-Hajnal theorem implies a stronger partition relation. The proof of the following theorem is rather long and complex, using many combinatorial tricks. We advise the reader that, unless they are particularly interested, they can safely skip reading this proof.

Theorem 3.22 (Theorem 31i). *Let ϕ a real order-type. Assume that $\alpha < \omega \cdot 2$. Then*

$$\phi \to (\alpha)^2_3. \tag{3.23}$$

Proof. Assume w.l.o.g. that $|\phi| = \aleph_1$. Let S be a set with $\operatorname{otp} S = \phi$ and let $f\colon [S]^2 \to 3$ be a colouring. Assume for the sake of contradiction that there is no homogeneous set for f of order-type α. As $\alpha < \omega \cdot 2$, we can assume that $\alpha = \omega + m$, for some $m < \omega$. The only properties of S that we will use are that S is uncountable and $\omega_1, \omega_1^* \not\leq \operatorname{otp} S$. Therefore whenever an uncountable subset $A \subseteq S$ has a certain property, we shall assume without loss of generality that S has this property.

By Proposition 3.17 there are uncountable subsets $A_0, A_1 \subseteq S$ such that $A_0 < A_1$. Fix $s_0 \in A_1$, then in particular, there must be some colour $k \in 3$ such that $\{x \in A_0 \mid f(\{x < s_0\}) = k\}$ is uncountable. Continuing with A_0 and repeating this argument $3m$ times, we obtain a set $\{s_{i_{m-1}} < \ldots < s_{i_0}\}$ which is homogeneous for f with, say, colour 0, and $\{x \in S \mid (\forall n < m) f(\{x < s_{i_n}\}) = 0\}$ is uncountable. Note that we can assume without loss of generality that the colour is $k = 0$, because we could reshuffle the colours if necessary.

Suppose that $\{x \in S \mid (\forall n < m) f(\{x < s_{i_n}\}) = 0\}$ has a subset H of order-type ω which is 0-homogeneous for f. Then the set $H \cup \{s_{i_{m-1}} < \ldots < s_{i_0}\}$ is homogeneous for f with order-type α, which is a contradiction. Therefore we can assume without loss of generality that

$$S \text{ has no infinite subset which is 0-homogeneous for } f. \tag{3.24}$$

Suppose now that for all uncountable $A \subseteq S$ there is some $x \in A$ such that

$$|\{y \in A \mid f(\{x < y\}) = 0\}| = \aleph_1.$$

Then we can inductively define sets $A_0 = S$ and $A_{n+1} = \{y \in A_n \mid f(\{h_n < y\}) = 0\}$ and elements $h_n \in A_n$ such that $\{h_n \mid n < \omega\}$ is 0-homogeneous for f of order-type ω, contradicting (3.24). Therefore we may assume without loss of generality that for all $x \in S$,

$$|\{y \in S \mid f(\{x < y\}) = 0\}| \leq \aleph_0. \tag{3.25}$$

Using Proposition 3.17 again, we obtain uncountable subsets $A, B \subseteq S$ such that $A < B$. As $\alpha < \omega_1$, we have the partition relation $\operatorname{otp} A \to (\omega, \alpha)^2$ by Lemma 3.18. By Ramsey's Theorem, $\omega \to (\omega, \omega)^2$, and hence $\operatorname{otp} A \to (\omega, \omega, \alpha)^2$ by Lemma 2.27. By (3.24) and the fact that A has no subset homogeneous for f of order-type α, it must be the case that

$$A \subseteq S \text{ has a subset } P \text{ which is 1-homogeneous for } f \text{ of order-type } \omega. \tag{3.26}$$

Since P is countable, we have by (3.25) that $|\{y \in S \mid (\exists x \in P)(f(\{x < y\}) = 0)\}| \leq \aleph_0$. In particular, $B \setminus \{y \in S \mid (\exists x \in P)(f(\{x < y\}) = 0)\}$ is uncountable, and of course, $P < B$. Therefore we may assume without loss of generality that P is 1-homogeneous for f of order-type ω and

$$f(\{x < y\}) \in \{1, 2\} \text{ for all } x \in P \text{ and } y \in S \setminus P. \tag{3.27}$$

Assume that whenever $Q \subseteq P$ is of order-type ω and $A \subseteq S$ is such that $|A| = \aleph_1$, there is some $x \in A$ such that

$$|\{y \in Q \mid f(\{y < x\}) = 1\}| = \aleph_0. \tag{3.28}$$

We will show that this assumption gives a contradiction. Note that $A = S \setminus P$ is uncountable, and hence by Proposition 3.17 for all rationals q there are $A_q \subseteq A$ such that $|A_q| = \aleph_1$ and $A_q < A_p$ for all rationals $q < p$. There is an order-preserving injection $g\colon (\omega \cdot m, <) \to (\mathbb{Q}, <)$. Let $x_0 \in A_{g(0)}$ such that $P_0 = \{y \in P \mid f(\{y < x_0\}) = 1\}$ is infinite. We define inductively x_ν and P_ν for $\nu < \omega \cdot m$ as follows. Suppose for all $\mu < \nu$ that

1. $x_\mu \in A_{g(\mu)}$,
2. $|P_\mu \setminus P_\rho| < \aleph_0$ for all $\rho < \mu < \nu$, and
3. $P_\mu \subseteq P$.

Fix an enumeration $\langle \delta_n \mid n < \omega \rangle$ of ν. Inductively pick elements for $n < \omega$,

$$y_n \in \bigcap_{j=0}^{n-1} P_{\delta_j} \setminus \{y_0, \ldots, y_{n-1}\}.$$

Set $Y = \{y_n \mid n < \omega\}$, and note that since $Y \subseteq P$ it holds that Y is 1-homogeneous for f and is of order-type ω. By (3.25) the set $A_{g(\nu)} \setminus \{y \in A \mid (\exists \mu < \nu)(f\{x_\mu < y)\} = 0\}$ is uncountable. Therefore, by (3.28), there exists some $x_\nu \in A_{g(\nu)} \setminus \{y \in A \mid (\exists \mu < \nu)(f\{x_\mu < y)\} = 0\}$ such that

$$|\{y \in Y \mid f(\{y < x_\nu\}) = 1\}| = \aleph_0.$$

Let $P_\nu = \{y \in Y \mid f(\{y < x_\nu\}) = 1\}$. We check that P_ν satisfies the properties listed above. Obviously, $P_\nu \subseteq Y \subseteq P$. Also, for $\mu < \nu$ there is some $n < \omega$ such that $\mu = \delta_n$. Then

$$|P_\nu \setminus P_\mu| = |P_\nu \setminus P_{\delta_n}| \leq |Y \setminus P_{\delta_n}| \leq |\{y_0, \ldots, y_n\}| < \aleph_0.$$

This completes the definition of the x_ν and P_ν.

Note that $X = \{x_\mu \mid \mu < \omega \cdot m\}$ has order-type $\omega \cdot m$. By construction of the x_ν, we have that for all any $\mu < \nu < \omega \cdot m$ that $f(\{x_\mu < x_\nu\}) \in \{1, 2\}$, and therefore $f \restriction [X]^2$ is a 2-colouring. By Theorem 3.25 we have $\omega \cdot m \to (m, \alpha)^2$. So, there exists a set $H \subseteq X$ such that either H is 1-homogeneous for f and $\operatorname{otp} H = m$, or H is 2-homogeneous for f and $\operatorname{otp} H = \alpha$. In the latter case we immediately get a contradiction, because we supposed f does not have a homogeneous set of order-type α.

Thus, assume $\operatorname{otp} H = m$ and H is 1-homogeneous for f. We can write $H = \{x_{i_0} < \ldots < x_{i_{m-1}}\}$. For every $n < m$ we have

$$|P_{i_{m-1}} \setminus \{y \in A \mid f(\{y < x_{i_n}\}) = 1\}| \leq |P_{i_{m-1}} \setminus P_{i_n}| < \aleph_0.$$

Therefore $Q = \{y \in P_{i_{m-1}} \mid (\forall n < m)(f(\{y < x_{i_n}\}) = 1)\}$ has order-type ω. As $Q \subseteq P$, we have that $Q \cup \{x_{i_0} < \ldots < x_{i_{m-1}}\}$ is homogeneous for f of order-type α, which is a contradiction. So, our assumption in (3.28) was false. Therefore we can assume that there are $P' \subseteq P$ of order-type ω and uncountable $A \subseteq S$ such that for all $x \in A$

$$|\{y \in P' \mid f(\{y < x\}) = 1\}| < \aleph_0. \tag{3.29}$$

As P' is countable, there are only countably many finite subsets of P'. As A is uncountable, there is an uncountable subset $A' \subseteq A$ such that $\{y \in P' \mid f(\{y < x\}) = 1\}$ is constant for all $x \in A'$. Set $P'' = \{y \in P' \mid f(\{y < x\}) = 2\}$ for some $x \in A'$, which we note by (3.27) is then also constant for $x \in A'$. Note that $\operatorname{otp} P'' = \omega$.

The argument from (3.24) onwards remains valid if we replace S by any uncountable subset $A \subseteq S$. Therefore we have shown that whenever $A \subseteq S$ is uncountable

$$\text{there exists } P, A' \subset A \text{ such that} \tag{3.30}$$

1. A' is uncountable,
2. P is 1-homogeneous for f of order-type ω, and
3. for all $x \in P$ and $y \in A'$, we have $f(\{x < y\}) = 2$.

Using Proposition 3.17 there are $A_0, B_0 \subset S$ such that $|A_0| = \aleph_1 = |B_0|$ and $A_0 < B_0$. There are $P_0, A'_0 \subset A_0$ as in (3.30). Noting that A_0 is uncountable, we can use a repeated application of (3.30), to obtain the sets $P_{n+1}, A'_{n+1} \subset A'_n$, for all $n < \omega$.

Define

$$B_1 \coloneqq B_0 \setminus \{y \in B_0 \mid \exists x \in \bigcup_{n<\omega} P_n(f(\{x < y\}) = 0)\},$$

where we note that $\bigcup_{n<\omega} P_n$ is countable, and hence it follows from (3.25) that $|B_1| = \aleph_1$. Also, the sets P_n have the following properties

1. $f(\{x < y\}) = 2$ when $x \in P_k, y \in P_n$ for $k < n < \omega$, (3.31)
2. $f(\{x < y\}) = 1$ when $x, y \in P_n$ for all $n < \omega$, (3.32)
3. $f(\{x < y\}) \in \{1, 2\}$ when $x \in P_n, y \in B_1$ for all $n < \omega$. (3.33)

Note that (3.31) follows from the fact that $P_n \subset A'_k$ for $k < n$, (3.32) holds because the P_n are homogeneous for f with colour 1.

Fix $n < \omega$ for now. Suppose that there are uncountable $B_2 \subseteq B_1$ and infinite $P' \subseteq P_n$ such that for all $x \in B_2$,

$$\{y \in P' \mid f(\{y < x\}) = 2\} < \aleph_0. \tag{3.34}$$

Then, using the same argument as before, there are only countably many finite subsets of P' and hence there is an uncountable set $B_3 \subseteq B_2$ such that $\{y \in P' \mid f(\{y < x\}) = 2\}$ is constant for all $x \in B_3$. This means in particular that $D = \{y \in P' \mid f(\{y < x\}) = 1\}$ is fixed for all $x \in B_3$, and has order-type ω. Using (3.26) we know there is $Q \subseteq B_3$ which is 1-homogeneous for f of order-type ω. Then the set $D \cup Q$ is homogeneous for f and has order-type $\omega \cdot 2$, which is a contradiction. Thus our assumption in (3.34) was false.

Therefore it is the case for every $n < \omega$ that if $P' \subseteq P_n$ is infinite, then

$$|\{x \in B_1 \colon |\{y \in P' \colon f(\{y < x\}) = 2\}| < \aleph_0\}| \leq \aleph_0. \tag{3.35}$$

The idea is to repeat the whole argument from (3.24), but now to the set B_1 and we "invert" the colors. This entails that instead of using (3.26), we use the relation $\operatorname{otp} B_1 \to (\omega, \alpha, \omega)^2$ to obtain an infinite homogeneous set Q with colour 2, instead of colour 1. Therefore, we will obtain sets $Q_n, B_2 \subseteq B_1$ for $n < \omega$, such that Q_n has order-type ω, B_2 is uncountable, and

1. $f(\{x < y\}) = 1$ when $x \in Q_k, y \in Q_n$ for $k < n < \omega$, (3.36)
2. $f(\{x < y\}) = 2$ when $x, y \in Q_n$ for all $n < \omega$, (3.37)
3. $f(\{x < y\}) \in \{1, 2\}$ when $x \in Q_n, y \in B_2$ for all $n < \omega$. (3.38)

And it also holds for any $n < \omega$ and any infinite $Q' \subseteq Q_n$ that

$$|\{x \in B_2 \colon |\{y \in Q' \colon f(\{y < x\}) = 1\}| < \aleph_0\}| \leq \aleph_0. \tag{3.39}$$

Using Proposition 3.17 we obtain uncountable $C_n \subseteq B_2$ for all $n < \omega$, such that $C_k < C_n$ for all $k < n < \omega$. For any $n < \omega$ and any infinite $P'_n \subseteq P_n$ and $Q'_n \subseteq Q$ we have by (3.35) and (3.39) that there at most $\aleph_0$ elements $x \in B_2$ such that

$$|\{y \in P'_n \colon f(\{y < x\}) = 2\}| < \aleph_0 \text{ and } |\{y \in Q'_n \colon f(\{y < x\}) = 1\}| < \aleph_0. \tag{3.40}$$

Therefore we can inductively define $x_n \in C_n$ such that for all $i < \omega$

$$|\{y \in P_i \colon (\forall k \leq n)(f(\{y < x_k\}) = 2)\}| = \aleph_0, \text{ and} \tag{3.41}$$
$$|\{y \in Q_i \colon (\forall k \leq n)(f(\{y < x_k\}) = 1)\}| = \aleph_0. \tag{3.42}$$

Set $X = \{x_n \mid n < \omega\}$. By Ramsey's Theorem, $\omega \to (\omega)^2_3$, and hence there is some infinite $X' \subseteq X$ which is k-homogeneous for f, where $k \in 3$. Write $X' = \{x_{i_n} \mid n < \omega\}$. If X' were 0-homogeneous, we'd have a contradiction by (3.24). Hence $k = 1, 2$. Choose for all $i < \omega$,

$$y_i \in \{y \in P_i \colon (\forall n \leq i_{m-1})(f(\{y < x_n\}) = 2)\}, \text{ and} \tag{3.43}$$
$$z_i \in \{z \in Q_i \colon (\forall n \leq i_{m-1})(f(\{z < x_n\}) = 1)\}. \tag{3.44}$$

Set $Y = \{y_n \mid n < \omega\}$ and $Z = \{z_n \mid n < \omega\}$.

If $k = 1$, then $Z \cup \{x_{i_0}, x_{i_1}, \ldots, x_{i_{m-1}}\}$ has order-type $\omega + m = \alpha$. By (3.44), (3.36), and since X' is 1-homogeneous for f, the set $Z \cup \{x_{i_0}, x_{i_1}, \ldots, x_{i_{m-1}}\}$ is 1-homogeneous for f. Which gives a contradiction.

Hence $k = 2$, but then the set $Y \cup \{x_{i_0}, x_{i_1}, \ldots, x_{i_{m-1}}\}$ is 2-homogeneous for f and has order-type α, hence a contradiction.

We conclude that our assumption of f having no homogeneous set of order-type α is false, and hence $\phi \to (\alpha)^2_3$. □

The final result of this section also appeared in [ER56], again with a rather complicated proof. Albin Jones has provided a much simpler proof in [Jon00, Theorem 1], although it relies on an additional assumption.

Theorem 3.23 (Baumgartner-Hajnal, 1973)**.** *$\phi \to ((\omega + m)_n, \omega)^2$ for all $n, m < \omega$.*

We omit the proof of this theorem, but we mention that it follows from the Baumgartner-Hajnal Theorem.[8] Also, Fred Galvin gives a combinatorial proof of the result in [Gal75, Theorem 9].

Lemma 3.24 (Theorem 31iv). *Let ϕ be a real order-type and let $\alpha < \omega \cdot 2$. Then*

$$\phi \to (\alpha, 4)^3. \tag{3.45}$$

Proof. Proof is from [Jon00, Theorem 1].

Let S be a set with $\operatorname{otp} S = \phi$ and let $f\colon [S]^3 \to 2$ be a partition. We can write $\alpha = \omega + m$ for some $m < \omega$. We will show either

(a) There is $A \in [S]^{\omega+m}$ which is 0-homogeneous for f, or

(b) There is $B \in [S]^4$ which is 1-homogeneous for f.

We will make use of the following claim.

Claim. *Suppose there are $x \in S$ and $A \in [S \setminus \{x\}]^{\omega+m}$ such that $f \restriction [\{x\}, A]^{1,2} \equiv 1$, then either (a) or (b) holds.*

Proof of claim. If $A \in [S \setminus \{x\}]^{\omega+m}$ is such that $f \restriction [A]^3 \equiv 0$, then (a) holds. If $f \restriction [A]^3 \not\equiv 0$, then there is a triple $\{a_0, a_1, a_2\} \in [A]^3$ such that $f(\{a_0, a_1, a_2\}) = 1$. Setting $B = \{x, a_0, a_1, a_2\}$ then gives $f \restriction [B]^3 \equiv 1$, showing (b). ■

We may assume throughout that $|S| = \aleph_1$. Now, by Proposition 3.17 there are $R, P \subset S$ such that $\operatorname{otp} R = \phi$, $\operatorname{otp} P = \omega^2$ and $R < P$ (i.e., $r < p$ for all $r \in R$ and $p \in P$). We will focus on the set $R \cup P$. Note that $[R \cup P]^3 = [R]^3 \cup [R, P]^{2,1} \cup [R, P]^{1,2} \cup [P]^3$.

By the Finite Ramsey's Theorem, there exists some $n < \omega$ such that $n \to (m, 4)^3$.

Define for each $r \in R$ the partition $f_r\colon [P]^2 \to 2\colon \{p, p'\} \mapsto f(\{r, p, p'\})$. Using Corollary 3.31, the relation $\omega^2 \to (n, \omega + m)^2$ holds. If there is some $r \in R$ such that there is $A_r \in [P]^{\omega+m}$ with $f_r \restriction [A_r]^2 \equiv 1$, then we are done by the claim.

Therefore suppose for all $r \in R$ there is $D_r \in [P]^n$ with $f_r \restriction [D_r]^2 \equiv 0$. In other words, for all $r \in R$ there is $D_r \in [P]^n$ such that $f \restriction [\{r\}, D_r]^{1,2} \equiv 0$. As $|R| = \aleph_1$ and $|[P]^n| = \aleph_0$, there is some $T \subseteq R$ such that for all $t \in T$ we have $D_t = D$, for some fixed $D = \{d_0, \ldots, d_{n-1}\} \in [P]^n$, and also $\operatorname{otp} T$ is a real order-type. In particular, $f \restriction [T, D]^{1,2} \equiv 0$.

Define the partition $f_D\colon [T]^2 \to n + 1$ by

$$f_D(\{t, t'\}) = \begin{cases} i & \text{if } i < n \text{ is the least such that } f\{t, t', d_i\} = 1, \text{ and} \\ n & \text{otherwise.} \end{cases}$$

Theorem 3.23 gives $\operatorname{otp} T \to ((\omega + m)_n, \omega)^2$. If there is some $i < n$ and $A \in [T]^{\omega+m}$ such that A is i-homogeneous for f_D, then we are done by the claim.

Therefore we may assume there is $C \in [T]^\omega$ such that $f \restriction [C, D]^{2,1} \equiv 0$. Consider the partition $f \restriction [C]^3\colon [C]^3 \to 2$. By Ramsey's theorem, $\omega \to (\omega, 4)^3$. If there is $B \in [C]^4$ such that $f \restriction [B]^3 \equiv 1$, we have shown (b) and are done.

Thus we can assume without loss of generality that there is $E \in [C]^\omega$ which is 0-homogeneous for f. Recall that we chose $n < \omega$ such that $n \to (m, 4)^3$. If

[8] See [BH73, Theorem 1].

there is $B \in [D]^4$ such that B is 1-homogeneous for f, we are done. Therefore we assume there is $F \in [D]^m$ such that $f \restriction [F]^3 \equiv 0$. Define $A = E \cup F$. Note that $E < F$ and hence $\operatorname{otp} A = \omega + m$. Finally, we have already shown that $f \restriction [E]^3 \equiv 0$, $f \restriction [E, F]^{2,1} \equiv 0$, $f \restriction [E, F]^{1,2} \equiv 0$, and $f \restriction [F]^3 \equiv 0$. Hence we have shown A is 0-homogeneous for f and this concludes the proof. □

3.3 Ordinals

In this section we study ordinal-based partition relations. As an ordinal has a well-ordered order-type, we remark that every ordinal-based partition relation is a special case of a partition relation based on an order-type.

3.3.1 Countable ordinals

Theorem 3.25 (Theorem 23i)**.** *For $n < \omega$ and $\alpha < \omega \cdot 2$ it holds that*

$$\omega \cdot n \to (n, \alpha)^2. \tag{3.46}$$

Proof. Let S be a set with $\operatorname{otp} S = \omega \cdot n$ and let $[S]^2 = A \cup B$ be a partition. We can write S as $S = \bigcup_{i<n} S_i$, where $\operatorname{otp} S_i = \omega$ for all $i < n$, and $i < j < n$ implies $S_i < S_j$. Assume for the sake of contradiction that there exists no set $H \subseteq S$ such that either $\operatorname{otp} H = n$ and $[H]^2 \subseteq A$ or $\operatorname{otp} H = \alpha$ and $[H]^2 \subseteq B$.

Note that for every $i < n$ we have essentially a colouring $f_i \colon [S_i]^2 \to 2$. As $|S_i| = \aleph_0$, we have by Ramsey's Theorem that there exists an infinite homogeneous set $K_i \subseteq S_i$ for f_i. By assumption it cannot be that $[K_i]^2 \subseteq A$, and hence it must be that $[K_i]^2 \subseteq B$.

Fix some $i < j < n$. We define an operator $O_{i,j}$ as follows. We look at sets $K \subseteq K_i \cup K_j$ such that $|K \cap K_i| = \aleph_0$ and $[K]^2 \subseteq B$. Note that $K = K_i$ works, so there is at least one such set. As by assumption it must be that $\operatorname{otp} K < \alpha < \omega \cdot 2$, there is in fact such a set K such that $\operatorname{otp} K$ is maximal. Choose such a set K and set

$$O_{i,j}(K_0, K_1, \dots, K_{n-1}) = (L_0, L_1, \dots, L_{n-1}),$$

where $L_i = K \cap K_i$, $L_j = K_j \setminus K$ and $L_m = K_m$ for $m \neq i, j$.

We have $\operatorname{otp} L_j = \omega$, because if $\operatorname{otp} L_j < \omega$, then $\operatorname{otp} K \cap K_j = \omega$. As $\operatorname{otp} K \cap K_i = \omega$ this would imply that $\operatorname{otp} K \geq \omega \cdot 2$, a contradiction because $[K]^2 \subseteq B$. It is even the case that for all $i < n$ we have $\operatorname{otp} L_i = \omega$.

We also have for every $y \in L_j$ that the set $X \coloneqq \{x \in L_i \mid \{x, y\} \in B\}$ has cardinality $|X| < \aleph_0$. For suppose otherwise, then define the set $K' = X \cup (K \cap K_j) \cup \{y\}$. Then $K' \subseteq K_i \cup K_j$ and $|K' \cap K_i| = \aleph_0$. Also $[K']^2 \subseteq B$, because $[K' \setminus \{y\}]^2 \subseteq [K]^2 \subseteq B$, and for every $x \in X$ we have by definition $\{x, y\} \in B$. Also, $[(K \cap K_j) \cup \{y\}]^2 \subseteq [K_j]^2 \subseteq B$. Finally, we have $\operatorname{otp} K' = \operatorname{otp} K + 1$, contradicting the maximality of K.

We now apply the operators $O_{i,j}$ iteratively on the system $(K_0, \dots, K_{n-1})$ for all pairs $i < j < n$. There are $\binom{n}{2}$ such pairs, so we only apply finitely many operators. We denote the end result as $(D_0, \dots, D_{n-1})$. Note we have for all $i < n$ that $\operatorname{otp} D_i = \omega$. Importantly, for every $i < n$ we have for all $y \in \bigcup_{i<j<n} D_j$ that $|\{x \in D_i \colon \{x, y\} \in B\}| < \aleph_0$. Assuming the elements $x_{n-1}, \dots, x_{i+1}$ have been chosen, in this order, then we have

$$|\{x \in D_i \colon \{x, y\} \in B \text{ for some } y \in \{x_{i+1}, \dots, x_{n-1}\}\}| < \aleph_0,$$

hence we can choose an element $x_i \in D_i$ such that for all j with $i < j < n$ we have $\{x_i, x_j\} \in A$. Finally, the set $D = \{x_0, \dots, x_{n-1}\}$ has $\operatorname{otp} D = n$ and $[D]^2 \subseteq A$, which is a contradiction with our assumption that such a set did not exist. $\square$

The following theorem shows that we cannot strengthen Theorem 3.25 by increasing the goal to $n+1$, because the partition relation is already negative for $\omega + 1$.

Theorem 3.26 (Theorem 23ii)**.** *For $n < \omega$ it holds that*

$$\omega \cdot n \not\to (n+1, \omega+1)^2. \tag{3.47}$$

Proof. Define $S \coloneqq \{(m, \ell) \in \omega \times \omega \mid m < n \wedge \ell < \omega\}$ and order S lexicographically by $\prec$. That is, $(m, \ell) \prec (m', \ell')$ if and only if $m < m'$ or $m = m'$ and $\ell < \ell'$. Note that $\operatorname{otp}(S, \prec) = \omega \cdot n$. Let $[S]^2 = A \cup B$ be a partition such that B contains precisely the pairs $\{(m, \ell), (m, \ell')\}$ for all $m < n$ and $\ell, \ell' < \omega$. Suppose there is a set $H = \{(m_0, \ell_0) \prec \dots \prec (m_n, \ell_n)\} \subseteq S$, so $\operatorname{otp}(H, \prec) = n+1$, such that $[H]^2 \subseteq A$. Then in particular we would get $m_0 < \dots < m_n < n$, which is a contradiction. Similarly, suppose $G \subseteq S$ is such that $[G]^2 \subseteq B$ with $\operatorname{otp}(G, \prec) = \omega + 1$. Then G must be of the form $G = \{(m, \ell) \mid \ell < \omega\}$, which does not have order-type $\omega + 1$. We conclude $\omega \cdot n \not\to (n+1, \omega+1)^2$. $\square$

Often we will need to make an assumption before we can prove a partition relation, as is the case in the next theorem.

Theorem 3.27 (Theorem 25)**.** *Let $2 \leq m, n < \omega$ be natural numbers. Assume that $\ell < \omega$ is such that*

$$\ell \to (m, m, n)^2. \tag{3.48}$$

Then

$$\omega \cdot \ell \to (m, \omega \cdot n)^2. \tag{3.49}$$

In fact, we will prove an even stronger result. For this stronger result, we will need to introduce a certain property that natural numbers can have. This property is similar to the ordinary partition relation, but now we want to account for ordered pairs instead of only unordered pairs.

Definition 3.28 ($P_{m,n}$ property)**.** A natural number ℓ is said to have property $P_{m,n}$ if the following holds: for every $h \colon \ell \times \ell \setminus \Delta \to 2$, where $\ell \times \ell$ denotes the ordered pairs of ℓ and $\Delta = \{(d, d) \mid d < \ell\}$ is the diagonal, there is

1. either a set $\{a_0, \dots, a_{m-1}\} \in [\ell]^m$ such that $i < j < m$ implies $h(a_i, a_j) = 0$,
2. or a set $\{a_0, \dots, a_{n-1}\} \in [\ell]^n$ such that for all $i, j < n$ with $i \neq j$ that $h(a_i, a_j) = 1$.

Theorem 3.29 (Theorem 25i). *Let $2 \leq m < \omega$ and $1 \leq n < \omega$. Suppose that $\ell < \omega$ has property $P_{m,n}$. Then*

$$\omega \cdot \ell \to (m, \omega \cdot n)^2. \tag{3.50}$$

Proof. Let $f\colon [\omega \cdot \ell]^2 \to 2$ be a colouring. Note that we can write

$$[\omega \cdot \ell]^2 = \{\{\omega \cdot a + r, \omega \cdot b + s\} \mid a, b < \ell \text{ and } r, s < \omega \text{ and } (a, r) \neq (b, s)\}.$$

Fix an enumeration $\langle p_i \mid i < \ell^2\rangle$ of the ordered pairs $\ell \times \ell$. We assume that the diagonal Δ appears last, i.e. $p_{\ell(\ell-1)+j}$ is the pair (j, j). Define $g\colon [\omega]^2 \to 2^{\ell^2}$ by

$$\{r < s\} \mapsto (x_i)_{i=0}^{\ell^2-1},$$

where $x_i = f(\{\omega \cdot a + r, \omega \cdot b + s\})$ and where $(a, b) = p_i$ is the i-th pair of the enumeration. By Ramsey's Theorem there is a homogeneous set $H \subseteq \omega$ for g with $\operatorname{otp} H = \omega$.

Let $\{r < s\} \in [H]^2$. We can now define $h\colon \ell \times \ell \setminus \Delta \to 2$ by

$$(a, b) \mapsto (g(\{r < s\}))_i = f(\{\omega \cdot a + r, \omega \cdot b + s\}),$$

where $p_i = (a, b)$. Note that by the homogeneity of H it is irrelevant which element $\{r < s\} \in [H]^2$ is chosen. By assumption ℓ has property $P_{m,n}$, which means the following: either there is some set $X_m = \{c_0, \ldots, c_{m-1}\} \subseteq \ell$ such that $i < j < m$ implies $h(c_i, c_j) = 0$, or there is a set $X_n = \{d_0 < \ldots < d_{n-1}\} \subseteq \ell$ such that $h(d_i, d_j) = 1$ for all $i, j < n$ with $i \neq j$. In the first case, define the set $H_m :=$ $\{\omega \cdot c_i + r_i \mid i < m\}$, where $\{r_0 < \ldots < r_{m-1}\} \subseteq H$. It follows that H_m is 0-homogeneous for f and $\operatorname{otp} H_m = m$.

Suppose that there is no 0-homogeneous set for f of order-type m. Then in particular the second case of property $P_{m,n}$ must hold. There is another consequence, namely it holds for all $i < n$ that for $\{r, s\} \in [H]^2$ that $f(\{\omega \cdot d_i + r, \omega \cdot d_i + s\}) = 1$, else the set $\{\omega \cdot d_i + r \mid r \in H\}$ would constitute a 0-homogeneous set for f of order-type ω. We create a partition of $H = \cup_{i<n} H^i$ in n pairwise disjoint sets such that for every $i < n$ we have $\operatorname{otp} H^i = \omega$. Define $H_n := \{\omega \cdot d_i + s \mid i < n \text{ and } s \in H^i\}$. Finally, H_n is 1-homogeneous for f of order-type $\operatorname{otp} H_n = \omega \cdot n$. To see this, take $\{\omega \cdot d_i + r, \omega \cdot d_j + s\} \in [H_n]^2$. If $i = j$ we already saw above that $f(\{\omega \cdot d_i + r, \omega \cdot d_i, s\}) = 1$. If $i \neq j$, then $r \neq s$ and suppose $s < r$ (the case for $r < s$ follows analogously). Let k be the index of the pair (d_j, d_i) in the enumeration. It follows that $1 = h(d_j, d_i) = (g(\{s < r\})_k = f(\{\omega \cdot d_j + s, \omega \cdot d_i + r\})$, which is what we wanted to show. ☐

We are now ready to prove Theorem 3.27.

Proof of Theorem 3.27. In view of Theorem 3.29, it suffices to show that if

$$\ell \to (m, m, n)^2$$

holds, then ℓ has property $P_{m,n}$.

Hence, take an arbitrary function $h\colon \ell \times \ell \setminus \Delta \to 2$. Define the partition $f\colon [\ell]^2 \to$ 3 by

$$\{a < b\} \mapsto \begin{cases} 0 & \text{if } h(a, b) = 0, \\ 1 & \text{if } h(a, b) > h(b, a), \text{ and} \\ 2 & \text{if } h(a, b) = h(b, a) = 1. \end{cases}$$

If there is a 0-homogeneous set for f of order-type m, or there is 2-homogeneous set for f of order-type n, then we are done.

Therefore we may assume we find a 1-homogeneous set $H = \{a_0 < \ldots < a_{m-1}\}$ for f. Define $b_i = a_{m-1-i}$ for all $i < m$, then the set $\{b_0, \ldots, b_{m-1}\}$ has the property that $i < j$ implies $h(b_i, b_j) = 0$. We conclude that ℓ has the property $P_{m,n}$ and this concludes the proof. ☐

The positive partition result from Theorem 3.29 is actually tight, in the sense that if the assumption of ℓ having property $P_{m,n}$ were dropped, the partition relation would be negative.

Lemma 3.30 (Theorem 25ii). *Let $2 \leq m, n < \omega$. Let ℓ_0 denote the least natural number which has property $P_{m,n}$. Then*

$$\gamma \not\to (m, \omega \cdot n)^2 \quad \textit{where } \gamma < \omega \cdot \ell_0. \tag{3.51}$$

Proof. Let $\gamma < \omega \cdot \ell_0$. Then there is some $\ell < \ell_0$ such that $\omega \cdot \ell \leq \gamma < \omega \cdot (\ell + 1)$. By assumption ℓ does not have property $P_{m,n}$, and hence there exists a function $g \colon \ell \times \ell \setminus \Delta \to 2$ such that for all $\{a_0 < \ldots < a_{m-1}\} \subseteq \ell$ there are $i < j < m$ with $g(a_i, a_j) = 1$ and for all $\{b_0 < \ldots < b_{n-1}\} \subseteq \ell$ there are $i, j < n$ where $i \neq j$ with $g(b_i, b_j) = 0$.

Define $f \colon [\gamma]^2 \to 2$ by $f(\{\omega \cdot a + r, \omega \cdot b + s\}) = 0$ if and only if $r < s < \omega$ and $a, b < \ell$, $a \neq b$ and $g(a, b) = 0$. Suppose that there is a set $H \subseteq \gamma$ with $\operatorname{otp} H = m$ such that $f \restriction [H]^2 \equiv 0$. We can write $H = \{\omega \cdot a_0 + r_0 < \ldots < \omega \cdot a_{m-1} + r_{m-1}\}$ where we have for all $i < m$ that $a_i < \ell$. It follows immediately that for all $i < j < m$ we have $g(a_i, a_j) = 0$ and $a_0 < \ldots < a_{m-1}$, which contradicts our assumption.

Suppose, on the other hand, that there is a 1-homogeneous set $H \subseteq \gamma$ for f of order-type $\operatorname{otp} H = \omega \cdot n$. Then we can write $H = \{\omega \cdot b_i + s^i_j \mid b_i < \ell, i < n, j < \omega\}$, where $s^i_j < \omega$ for all $i < n$ and $j < \omega$. It must be that $b_i < \ell$ because $\gamma < \omega \cdot (\ell + 1)$. There are some indices $k_0, \ldots, k_{n-1}$ and $r_0, \ldots, r_{n-1}$ such that $s^0_{k_0} < \ldots < s^{n-1}_{k_{n-1}} < \omega$ and $\omega > s^0_{r_0} > \ldots > s^{n-1}_{r_{n-1}}$. Then for all $i, j < n$ with $i \neq j$, if $i < j$, then it is the case that $g(b_i, b_j) = f(\{\omega \cdot b_i + s^i_{k_i}, \omega \cdot b_j + s^j_{k_j}\}) = 1$, and if $j < i$, then $g(b_j, b_i) = f(\{\omega \cdot b_j + s^j_{r_j}, \omega \cdot b_i + s^i_{r_i}\}) = 1$, which gives a contradiction.

We conclude $\gamma \not\to (m, \omega \cdot n)^2$. ☐

Theorem 3.27 gives us an interesting corollary, which we will need to use in this thesis.

Corollary 3.31. *For all $n, m < \omega$ it holds that*

$$\omega^2 \to (m, \omega \cdot n)^2.$$

Proof. Suppose w.l.o.g. that $m \leq n$ (the case for $n \leq m$ is analogous). By Finite Ramsey's Theorem[9] there exists a natural number ℓ such that $\ell \to (n)^2_3$. Obviously, then $\ell \to (m, m, n)^2$ is true as well. Using Theorem 3.27 we have $\omega \cdot \ell \to (m, \omega \cdot n)^2$ and as $\omega \cdot \ell \leq \omega^2$, we obtain the desired result. ☐

[9] See Theorem 2.19.

At the end of this section, we present some results that were proven after the publication of [ER56], but we will not give the proofs or go into much detail.

In a paper by Ernst Specker, [Spe57], we find a strengthening of Corollary 3.31, and the same paper demonstrates that the result cannot be generalised for all finite powers of ω.[10]

Theorem 3.32 (E. Specker, [Spe57]). *For all $m < \omega$ and $n \geq 3$,*

$$\omega^2 \to (m, \omega^2)^2, \text{ and} \tag{3.52}$$
$$\omega^n \not\to (3, \omega^n)^2. \tag{3.53}$$

The next interesting partition relations concerns ω^ω as resource, and C.C. Chang showed that this relation is positive.

Theorem 3.33 (C.C. Chang, [Cha72], Theorem 1).

$$\omega^\omega \to (\omega^\omega, 3)^2. \tag{3.54}$$

E.C. Milner was able to generalise Chang's result to all finite n, although he did not publish this result.[11] Larson provided a short proof in her PhD.[12]

Theorem 3.34 (E.C. Milner, 1972). *For all $n < \omega$,*

$$\omega^\omega \to (\omega^\omega, n)^2. \tag{3.55}$$

In his PhD thesis (1999), Rene Schipperus studied partition relations where the resource was of the form ω^{ω^β}. The results can be found in [Sch10].

E.C. Milner showed in his PhD thesis an ordinal-variant of the Positive Stepping Up Lemma.

Theorem 3.35 (E.C. Milner, [EM72], p. 501). *Let γ and δ be countable ordinals and let $k < \omega$. If $\omega^\gamma \to (\omega^{1+\delta}, k)^2$, then $\omega^{\gamma+\delta} \to (\omega^{1+\delta}, 2k)^2$.*

3.3.2 General ordinals

Lemma 3.36 (Lemma 5). *Let α be an ordinal and k a cardinal. Suppose that β_n are ordinals for all $n < k$ such that for all $\beta < \alpha$ it holds that $\beta \not\to (\beta_n)^r_{n<k}$. Then $\alpha \not\to (\beta_n + 1)^{r+1}_{n<k}$.*

Proof. Let S be a set such that $\mathrm{otp}(S, <) = \alpha$. For every $x \in S$, define $I_x = \{y \in S \mid y < x\}$, then $\mathrm{otp}(I_x, <) = \beta < \alpha$, for some β. By assumption there is some colouring $f_x \colon [I_x]^r \to k$ such that for all $n < k$ there is no set H of order-type β_n which is n-homogeneous for f_x.

Define $f \colon [S]^{r+1} \to k$ by

$$\{x_0 < x_1 < \ldots < x_{r-1} < x_r\} \mapsto f_{x_r}(\{x_0 < x_1 < \ldots < x_{r-1}\}).$$

If there is a homogeneous set $H = \{h_i \mid i < \beta_n + 1\} \subseteq S$ for f with colour n and $\mathrm{otp}(H, <) = \beta_n + 1$, then the set $\{h_i \mid i < \beta_n\}$ is homogeneous for f_{h_β} of order-type β_n and colour n. This is a contradiction and hence the proof is concluded. $\square$

[10] Cited after [Lar12, p. 219].

[11] Larson credits Milner with this unpublished result in [Lar73, p. 129].

[12] See [Lar73, Theorem 3.1].

3.4 Cardinals

In this section we study partition relations where the resource is a cardinal. If we view a cardinal as its initial ordinal with the ordinal ordering, we observe that every cardinal-based partition relation is a special case of an ordinal-based partition relation. Hence, given a cardinal κ and an ordinal α, we often look at partition relations of the form $\kappa \to (\alpha)^r_m$. This relation means that our resource is the *ordinal* κ with the ordinal ordering $<$ on κ, and there is a homogeneous subset $H \subseteq \kappa$ such that $\mathrm{otp}(H, <) = \alpha$.

3.4.1 Positive Stepping Up Lemma

Here we prove a key result of the Erdős-Rado paper: the "Positive Stepping Up Lemma",[13] and we investigate some of its corollaries. Given some positive partition relation, this result allows one to increase the exponent and goal, at the cost of increasing the resource as well. Interestingly, the well-known Erdős-Rado theorem follows from the Positive Stepping Up Lemma. We mention that the proof below is a modernisation by [Löw19].

Theorem 3.37 (Positive Stepping Up Lemma, [ER56], Theorem 39)**.** *Let κ be an infinite cardinal, let $2 \leq m < \kappa$ be a cardinal and let $r \geq 1$ be a natural number. Let β_n be ordinals for all $n < m$. Assume $\kappa \to (\beta_n)^r_{n<m}$. Then $(2^{<\kappa})^+ \to (\beta_n + 1)^{r+1}_{n<m}$.*

Proof. Take some m-colouring $f\colon [(2^{<\kappa})^+]^{r+1} \to m$ and fix some arbitrary ordinal $\alpha < (2^{<\kappa})^+$. Define a sequence of ordinals below α as follows: $\gamma^\alpha_0 := 0, \gamma^\alpha_1 := 1, \dots, \gamma^\alpha_{r-1} := r-1$. For $\delta < \kappa$, assume that $\gamma^\alpha_0, \dots, \gamma^\alpha_\delta$ are already defined. Let $\gamma > \gamma^\alpha_\delta$ be the least ordinal such that for all indices $i_0 < \dots < i_{r-1} \leq \delta$ it holds that

$$f(\gamma^\alpha_{i_0}, \dots, \gamma^\alpha_{i_{r-1}}, \alpha) = f(\gamma^\alpha_{i_0}, \dots, \gamma^\alpha_{i_{r-1}}, \gamma).$$

If $\gamma = \alpha$ we terminate the recursive definition, $\gamma^\alpha_{\delta+1}$ is undefined and we set $\varrho(\alpha) := \delta + 1$; otherwise $\gamma < \alpha$, then set $\gamma^\alpha_{\delta+1} := \gamma$. If γ^α_δ is defined for all $\delta < \kappa$, we set $\varrho(\alpha) := \kappa$.

Define for every $\alpha < (2^{<\kappa})^+$ the set $A_\alpha := \{\gamma^\alpha_\delta \mid \delta < \varrho(\alpha)\}$. There are two possible cases: in the first case there is some $\alpha < (2^{<\kappa})^+$ such that $|A_\alpha| = \kappa$, and in the second case, $|A_\alpha| < \kappa$ for all $\alpha < (2^{<\kappa})^+$. We shall show that in the first case that for some $n < m$ we can find an n-homogeneous set H for f of order-type $\beta_n + 1$, and that the second case will give a contradiction.

Case 1. There is some $\alpha < (2^{<\kappa})^+$ such that $|A_\alpha| = \kappa$. Define the function $\widehat{f}\colon [\kappa]^r \to m$ by

$$\widehat{f}(i_0, \dots, i_{r-1}) := f(\gamma^\alpha_{i_0}, \dots, \gamma^\alpha_{i_{r-1}}, \alpha).$$

By the assumption $\kappa \to (\beta_n)^r_{n<m}$, there is some n-homogeneous set H for $\widehat{f}$ of order-type β_n, for some $n < m$. Then $H \cup \{\alpha\}$ has order-type $\beta_n + 1$ and we claim that it is n-homogeneous for f. Every $(r+1)$-tuple of $H \cup \{\alpha\}$ that contains α clearly has colour n, since $f(\gamma^\alpha_{i_0}, \dots, \gamma^\alpha_{i_{r-1}}, \alpha) = \widehat{f}(i_0, \dots, i_{r-1}) = n$. Also, for every $(r+1)$-tuple not containing α, we have for $i_0 < \ldots < i_{r-1} < i_r$,

$$f(\gamma^\alpha_{i_0}, \dots, \gamma^\alpha_{i_{r-1}}, \gamma^\alpha_{i_r}) = f(\gamma^\alpha_{i_0}, \dots, \gamma^\alpha_{i_{r-1}}, \alpha) = \widehat{f}(i_0, \dots, i_{r-1}) = n,$$

[13] This result is [ER56, Theorem 39].

where the first equality follows by definition of $\gamma^\alpha_{i_r}$.

Case 2. $|A_\alpha| < \kappa$ for every $\alpha < (2^{<\kappa})^+$. For any two ordinals $\alpha, \beta < (2^{<\kappa})^+$ we say that α and β are *equivalent*, if $\varrho(\alpha) = \varrho(\beta)$ and for all indices $i_0 < \ldots < i_{r-1} < \varrho(\alpha)$,

$$f(\gamma^\alpha_{i_0}, \ldots, \gamma^\alpha_{i_{r-1}}, \alpha) = f(\gamma^\beta_{i_0}, \ldots, \gamma^\beta_{i_{r-1}}, \beta).$$

Note that the equivalence class of some ordinal $\alpha < (2^{<\kappa})^+$ is completely determined by the value $\varrho(\alpha) < \kappa$ and by the function

$$\tilde{f} \colon [\varrho(\alpha)]^r \to m \colon (i_0, \ldots, i_{r-1}) \mapsto f(\gamma^\alpha_{i_0}, \ldots, \gamma^\alpha_{i_{r-1}}, \alpha).$$

Note that there are at most $|m^{[\varrho(\alpha)]^r}| \leq 2^{<\kappa}$ such functions. Therefore there are at most $|\sum_{\mu<\kappa} m^{|\mu|}| = 2^{<\kappa}$ equivalence classes and so there must be an equivalence class of size $(2^{<\kappa})^+$.

We show that any two equivalent ordinals must be the same. For the sake of contradiction suppose that α and β are equivalent and $\alpha < \beta$. By an inductive argument we can show that for every $\delta < \varrho(\alpha) = \varrho(\beta)$, $\gamma^\alpha_\delta = \gamma^\beta_\delta$. Now, for all indices $i_0 < \ldots < i_{r-1} < \varrho(\beta)$, we have by definition of the equivalence

$$f(\gamma^\beta_{i_0}, \ldots, \gamma^\beta_{i_{r-1}}, \beta) = f(\gamma^\alpha_{i_0}, \ldots, \gamma^\alpha_{i_{r-1}}, \alpha) = f(\gamma^\beta_{i_0}, \ldots, \gamma^\beta_{i_{r-1}}, \alpha),$$

which shows that α could have taken the role of $\gamma^\beta_{\varrho(\beta)}$ and gives us a contradiction. □

There are some incredibly interesting applications of the Positive Stepping Up Lemma, which we will demonstrate here. It will be useful to define notation for iterated exponentiation.

Definition 3.38 (Beth function)**.** Let κ be any cardinal. By recursion on the ordinals, we define the *beth function* as

$$\beth_0(\kappa) = \kappa, \qquad \beth_{\alpha+1}(\kappa) = 2^{\beth_\alpha(\kappa)}, \qquad \beth_\alpha(\kappa) = \sup\{\beth_\beta(\kappa) \mid \beta < \alpha\} \text{ for } \alpha \text{ limit.}$$

In the case where $\kappa = \aleph_0$, we simply write $\beth_\alpha = \beth_\alpha(\aleph_0)$.

Theorem 3.39. *For any $r, m, n < \omega$,*

$$\beth_n^+ \to (\omega + n + 1)^r_m. \tag{3.56}$$

Proof. Start with the relation $\aleph_0 \to (\omega)^r_m$, which is true by Ramsey's Theorem. Then iteratively apply the Positive Stepping Up Lemma $n+1$ times. Note that for all $n < \omega$, $(2^{<\beth_n^+})^+ = (2^{\beth_n})^+ = \beth_{n+1}^+$. □

In particular, we obtain the following two relations.

Corollary 3.40. *For all $r, m < \omega$,*

$$\omega_1 \to (\omega + 1)^r_m, \text{ and} \tag{3.57}$$

$$(2^{\aleph_0})^+ \to (\omega + 2)^r_m. \tag{3.58}$$

As promised, the well-known Erdős-Rado Theorem also follows from the Positive Stepping Up Lemma.

Theorem 3.41 (Erdős-Rado Theorem). *For any infinite cardinal κ and any $n < \omega$,*

$$\beth_n(\kappa)^+ \to (\kappa^+)^{n+1}_\kappa.$$

Proof. Proof by induction on n.

Base case $n = 0$. $\kappa^+ \to (\kappa^+)^1_\kappa$ is true because $\kappa < \mathrm{cf}(\kappa^+)$.

Assume for $n \geq 1$ that $\beth_{n-1}(\kappa)^+ \to (\kappa^+)^n_\kappa$ is true. As

$$\begin{aligned} 2^{<\beth_{n-1}(\kappa)^+} &= 2^{\beth_{n-1}(\kappa)} \\ &= \beth_n(\kappa), \end{aligned}$$

the result follows from the Positive Stepping Up Lemma. □

Remark 3.42. We actually obtain a slightly stronger partition relation for the Erdős-Rado Theorem, namely the relation

$$\beth_n(\kappa)^+ \to (\kappa^+ + n)^{n+1}_\kappa.$$

One reason why some applications of the Positive Stepping Up Lemma are so interesting, is because they seem to be sharp. We are able to show that the relations of Corollary 3.40 cannot be improved, i.e. increasing the goal will change the parity of the partition relation. In Chapter 4, we will conjecture that Theorem 3.39 is sharp. Unfortunately, we have not been able to prove nor disprove this. However, we have been successful in establishing a bound: there are no homogeneous sets of order-type ω^2. For these results, see Chapter 4.

For now, we want to show the relation $\omega_1 \to (\omega+1)^r_m$ is sharp. We have already seen a different result where there are no homogeneous sets of order-type $\omega + 2$, namely Theorem 3.13. This theorem gave us $\operatorname{otp} 2^\omega \not\to (\omega + 2)^3_2$, where 2^ω is the Cantor space.[14] Even though the Cantor space has the cardinality of the continuum, Theorem 3.13 does not imply $2^{\aleph_0} \not\to (\omega + 2)^3_2$, nor even the weaker $\omega_1 \not\to (\omega + 2)^3_2$, because ω_1 does not embed into Cantor space. However, [ER56, Theorem 41] implies this negative partition relation.

Proposition 3.43 (Theorem 41). *For all $n < \omega$,*

$$\omega_{n+1} \not\to (\omega_n + 2, \omega + 1)^3. \tag{3.59}$$

Proof. Clearly $\omega_n + 1, \omega^* \not\leq \omega_n$. For any $\beta < \omega_{n+1}$ it holds that $|\beta| \leq |\omega_n|$ and hence by Lemma 3.1 we have $\beta \not\to (\omega_n + 1, \omega)^2$. Then immediately by Lemma 3.36 it follows that $\omega_{n+1} \not\to (\omega_n + 2, \omega + 1)^3$. □

Corollary 3.44. $\omega_1 \not\to (\omega + 2, \omega + 1)^3$.

In fact, Theorem 4.9 provides the strengthening $\omega_1 \not\to (\omega + 2, \omega)^3$. Obviously, Corollary 3.44 implies $\omega_1 \not\to (\omega+2)^3_2$. It is not clear whether we can strengthen this relation to the resource $2^{\aleph_0}$, i.e. whether $2^{\aleph_0} \not\to (\omega + 2)^3_2$ is true. It turns out that this relation is indeed negative, as we shall show in Corollary 4.10.

The partition relation from Corollary 3.44 is negative when $r = 3$. Changing the exponent in the partition relations gives interesting results. If we increase the exponent, then one of the goals can be reduced to a natural number and the partition relation will remain negative. Curiously, if we reduce the exponent to $r = 2$, then the partition relation is positive, as we prove in Theorem 3.46.

[14]Recall that $\operatorname{otp} 2^\omega \equiv \lambda$, i.e. $\operatorname{otp} 2^\omega \leq \lambda$ and $\operatorname{otp} 2^\omega \geq \lambda$.

Lemma 3.45. *There is a natural number*[15] $\ell < \omega$ *such that for all* $4 \leq r < \omega$,

$$\omega_1 \not\to (\omega + 2, \ell + r - 4)^r. \tag{3.60}$$

Proof. In view of Lemma 2.22, it suffices to show (3.60) for $r = 4$. First, by Corollary 3.44, we have $\omega_1 \not\to (\omega + 2)^3_2$. Then, by Finite Ramsey's Theorem, there is some $\ell < \omega$ such that $\ell \to (4)^3_2$. Finally, using Theorem 3.5, we obtain $\omega_1 \not\to (\omega + 2, \ell)^4$. $\square$

Theorem 3.46 (Theorem 33)**.** *Let* $\alpha < \omega \cdot 2$, *then* $\omega_1 \to (\alpha)^2_2$.

Proof. Let S be a set such that $\operatorname{otp} S = \omega_1$ and let $f\colon [\omega_1]^2 \to 2$. As $\alpha < \omega \cdot 2$, we can assume w.l.o.g. that there is some $k < \omega$ such that $\alpha = \omega + k$. Assume for the sake of contradiction that there is no homogeneous set for f which has order-type $\alpha = \omega + k$. By the Dushnik-Miller Theorem, $\omega_1 \to (\omega, \omega_1)^2$. As f cannot have an uncountable homogeneous set, there must be a 0-homogeneous set $P \subseteq S$ for f with $\operatorname{otp} P = \omega$.

Suppose first, that whenever $P' \subseteq P$ is infinite, there is an uncountable set $A \subseteq S$ such that for every $x \in A$,

$$|\{y \in P' \mid f(\{y < x\}) = 0\}| = \aleph_0. \tag{3.61}$$

Let $x_0 \in S$ such that $|\{y \in P \mid f(\{y < x_0\}) = 0\}| = \aleph_0$. Call this set P_0. Let $\nu < \omega_1$ and assume that x_μ, P_μ are defined for all $\mu < \nu$, such that

1. $|P_\mu \setminus P_\rho| < \aleph_0$, for all $\rho < \mu < \nu$, and
2. $P_\mu \subseteq \{y \in P \mid f(\{y < x_\mu\}) = 0\}$ has order-type ω, for all $\mu < \nu$.

Note that $\nu < \omega_1$ is countable, so fix an enumeration $\langle \delta_n \mid n < \omega \rangle$ of ν. Inductively choose elements y_n such that for every $n < \omega$,

$$y_n \in \bigcap_{i \leq n} P_{\delta_i} \setminus \{y_0, \ldots, y_{n-1}\}.$$

Set $P' = \{y_n \mid n < \omega\}$, which is then infinite and contained in P. Note that, since $\operatorname{otp} S = \omega_1$, it must be that for every $x \in S$, $|\{y \in S \mid y < x\}| \leq \aleph_0$. By assumption there is an uncountable set A such that for every $x \in A$, (3.61) holds. Therefore, we can pick

$$x_\nu \in A \setminus \bigcup_{\mu < \nu} \{y \in S \mid y \leq x_\mu\}.$$

Set $P_\nu = \{y \in P' \mid f(\{y < x_\nu\}) = 0\}$, and since $x_\nu \in A$, $\operatorname{otp} P_\nu = \omega$. Also, note that for any $\mu < \nu$, there is $n < \omega$ such that $\mu = \delta_n$, and then

$$|P_\nu \setminus P_\mu| = |P_\nu \setminus P_{\delta_n}| \leq |P' \setminus P_{\delta_n}| \leq |\{y_0, \ldots, y_{n-1}\}| < \aleph_0.$$

This concludes our definition of x_ν and P_ν for all $\nu < \omega_1$.

Set $X := \{x_\delta \mid \delta < \omega_1\}$, which we note has order-type ω_1 by construction. Again, by the Dushnik-Miller Theorem, $\omega_1 \to (\omega, \omega_1)^2$. There cannot be a homogeneous set of order-type ω_1, therefore there are indices $i_0 < \ldots < i_{m-1} < \omega_1$ such that

[15] Erdős and Rado calculate on [ER56, p. 474] that $\ell = 2^{26}$ suffices.

$f \restriction [\{x_{i_0} < \ldots < x_{i_{m-1}}\}]^2 \equiv 0$. Note that for any $j < m$, $|P_{i_{m-1}} \setminus P_{i_j}| < \aleph_0$, and hence

$$Q = \{y \in P_{i_{m-1}} \mid (\forall j \leq m-1)\, f(\{y < x_{i_j}\}) = 0\}$$

has order-type ω. Then the set $Q \cup \{x_{i_0} < \ldots < x_{i_{m-1}}\}$ is homogeneous for f and has order-type $\omega + m = \alpha$. This is a contradiction.

Therefore we now assume that there is $P' \subseteq P$ such that

$$|\{x \in S \colon |\{y \in P' \colon f(\{y < x\}) = 0\}| = \aleph_0\}| \leq \aleph_0. \tag{3.62}$$

This means there is uncountable $A \subseteq S$ such that for all $x \in A$,

$$|\{y \in P' \mid f(\{y < x\}) = 0\}| < \aleph_0. \tag{3.63}$$

As there are only countably many finite subsets of P', there is an uncountable set $A' \subseteq A$ such that $E \coloneqq \{y \in P' \mid f(\{y < x\}) = 0\}$ is constant for all $x \in A'$. Set $P'' = P' \setminus E$, which has order-type ω. Again, there is uncountable $A'' \subseteq A'$ such that $P'' < A''$. Note that so far we have proven that,

1. $\forall y \in P'', x \in A''$, $f(\{y < x\}) = 1$,
2. P'' is 0-homogeneous for f of order-type ω,
3. A'' is uncountable.

Since A'' is uncountable, we can repeat the construction above, this time on A'' instead of S. This will give us sets P_n such that for every $n \in \omega$,

1. P_n is homogeneous for f with colour 0 and order-type ω,
2. $\forall y \in P_j, x \in P_n$, $f(\{y < x\}) = 1$, whenever $j < n < \omega$.

By (3.62), there is for every $n < \omega$ a countable set Q_n, such that for every $x \in S \setminus Q_n$, the set $\{y \in P_n \mid f(\{y < x\}) = 0\}$ is finite. Note that $\bigcup_{n<\omega} Q_n$ is countable, and hence there is an uncountable set $B \subseteq S \setminus \bigcup_{n<\omega} Q_n$ such that $\bigcup_{n<\omega} P_n < B$. By Dushnik-Miller, $\omega_1 \to (\omega_1, \omega)^2$, and as we cannot have a homogeneous set of order-type ω_1, there is a set $D \in [B]^k$ homogeneous for f with colour 1. Since $D \subseteq B$, it follows that for every $n < \omega$ that the set

$$\{y \in P_n \mid (\exists x \in D) f(\{y < x\}) = 0\} \text{ is finite.}$$

Therefore we can pick $y_n \in \{y \in P_n \mid (\forall x \in D) f(\{y < x\}) = 1\}$. Put $Y = \{y_n \mid n < \omega\}$, then $Y \cup D$ is homogeneous for f and has order-type $\omega + k$. This gives a contradiction with our assumption, and therefore $\omega_1 \to (\omega + k)^2_2$. □

Returning to the Erdős-Rado theorem, we obtain in particular the relation $\beth_1^+ \to (\omega_1)^2_2$. The classical result by Sierpińsky shows that reducing the resource to $\beth_1$ will produce a negative partition relation. Remarkably, this result shows that a direct generalisation of Ramsey's Theorem already fails at the first uncountable cardinal.

Theorem 3.47 (Sierpińsky, 1933)**.**

$$2^{\aleph_0} \not\to (\aleph_1)^2_2. \tag{3.64}$$

Proof. Fix a well-order $<^*$ on $\mathbb{R}$ and let $<$ be the usual ordering. Define the colouring $f\colon [\mathbb{R}]^2 \to 2$ by

$$\{x <^* y\} \mapsto 0 \iff x < y.$$

Suppose there is a set $H \subseteq \mathbb{R}$ homogeneous for f with colour, say, 1. Then $>$ is a well-order on H, and for every $h \in H$ there is a rational q_h between h and its $>$-successor h'. This gives an injection of H into $\mathbb{Q}$, and hence H is at most countable. Therefore there is no uncountable homogeneous set for f. □

The technique employed in Theorem 3.47, where one considers distinct orders on a set, is also used in Lemma 3.1. As a consequence, Sierpińsky's result can be improved to all ordinals of cardinality $\beth_1$. In particular, this shows that the first instance of the Erdős-Rado Theorem cannot be strengthened by reducing the resource.

Lemma 3.48. *For all $\beta < (\beth_1)^+$, $\beta \not\to (\omega_1)^2_2$.*

Proof. Since $\omega_1, \omega_1^* \not\leq \lambda$ and $|\lambda| = 2^{\aleph_0}$, the result follows by Lemma 3.1. □

3.4.2 Erdős-Dushnik-Miller theorem

In 1941, one of the first partition relations concerning uncountable cardinals was proven by Dushnik and Miller. They credit[16] Paul Erdős for help with the proof, and in particular for proving the case for when κ is a singular cardinal. And thus, the result is now known as the Erdős-Dushnik-Miller Theorem. In [ER56, Theorem 44], Erdős and Rado present a proof of this theorem, which we will omit. See [Jec03, Theorem 9.7] for a proof.

Theorem 3.49 (Erdős-Dushnik-Miller Theorem, [DM41])**.** *For all infinite cardinals κ,*

$$\kappa \to (\aleph_0, \kappa)^2. \tag{3.65}$$

Erdős and Rado proved a strengthening of the Erdős-Dushnik-Miller theorem and were able to increase the order-type of the homogeneous sets from ω to $\omega + 1$. However, they were only able to prove this for uncountable regular cardinals.

Theorem 3.50 (Theorem 34)**.** *Let κ be an uncountable regular cardinal. Then*

$$\kappa \to (\omega + 1, \kappa)^2. \tag{3.66}$$

In Corollary 3.58 we show that (3.66) fails for all infinite cardinals with countable cofinality. In [SS00], Saharon Shelah and Lee J. Stanley show that it is consistent for singular cardinals κ with uncountable cofinality that $\kappa \to (\omega+1, \kappa)^2$. They claim on [SS00, p. 259] that it is consistent for (3.66) to fail for singular cardinals with uncountable cofinality.

To prove Theorem 3.50 we will need to establish some other results first.

Lemma 3.51 (Lemma 2)**.** *Let $(T, <)$ be a well-ordered set and let $f\colon [T]^2 \to 2$ be a partition. Then there exists a unique set $H \subseteq T$ such that H is 1-homogeneous for f and for all $x \in T \setminus H$ there is some $h \in H$ such that $f(\{h < x\}) = 0$.*

[16] See the footnote on [DM41, p. 606].

Proof. We can assume $T \neq \emptyset$. Let κ be a cardinal such that $\kappa > |T|$. We define the set H by induction on κ. Suppose $\langle h_\beta \mid \beta < \alpha \rangle$ is already defined for some $\alpha < \kappa$.

Case 1. If there is $x \in T \setminus \{h_\beta \mid \beta < \alpha\}$ such that for all $\beta < \alpha$, $f(\{x, h_\beta\}) = 1$, then set h_α to be the $<$-least such x in $T \setminus \{h_\beta \mid \beta < \alpha\}$.

Case 2. If for all $x \in T \setminus \{h_\beta \mid \beta < \alpha\}$ there is $\beta < \alpha$ such that $f(\{x, h_\beta\}) = 0$, we set $h_\alpha = h_0$.

As $|T| < \kappa$ there is a least $\alpha < \kappa$ such that $\alpha > 0$ and $h_\alpha = h_0$. Define $H = \{h_\beta \mid \beta < \alpha\}$. By construction, H is 1-homogeneous for f. Also, if $x \in T \setminus H$, then by definition there is a least $\beta < \alpha$ such that $f(\{h_\beta, x\}) = 0$. Then, for all $\gamma < \beta$, we have $f(\{h_\gamma, x\}) = 1$, and by definition of h_β it must be that $h_\beta < x$.

To see that H is unique, let $H' \subseteq T$ be a set that also satisfies the properties. Assume $x \in T$ is the least element in the symmetric difference $H \triangle H'$. Suppose w.l.o.g. that $x \notin H'$ (the case for $x \notin H$ is analogous). Then by assumption there exists $y \in H'$ such that $f(\{y < x\}) = 0$. As x was the $<$-least, it must be that $y \in H$. But as H is 1-homogeneous for f, it follows that $x \notin H$, contradicting that $x \in H \triangle H'$. Therefore $H \triangle H' = \emptyset$, showing $H = H'$ and thus H is unique. □

The following proposition will be pivotal to prove Theorem 3.50, although the proof is rather technical. Recall that β^- denotes the predecessor of β, i.e. it is the ordinal γ where $\gamma + 1 = \beta$ if β is a successor ordinal, and $\beta^- = \beta$ if β is a limit ordinal.

Proposition 3.52 (Theorem 34)**.** *Let α, β, γ be ordinals and suppose $\alpha \not\to (\beta, \gamma)^2$. Then there exists a sequence of ordinals $\langle \alpha_\mu \mid \mu < \beta^- \rangle$, such that*

$$\alpha \not\to (\alpha_\mu + 1)^1_{\mu < \beta^-}, \text{ and} \tag{3.67}$$

$$\alpha_\mu \not\to (\gamma)^1_{\kappa_\mu}, \tag{3.68}$$

for all $\mu < \beta^-$, where $\kappa_\mu = \prod_{\nu < \mu} |\alpha_\nu|$.

Proof. Let $(S, <)$ be an ordered set with $\operatorname{otp}(S, <) = \alpha$ and let $f \colon [S]^2 \to 2$ be a colouring witnessing $\alpha \not\to (\beta, \gamma)^2$. Note that α is an ordinal, hence $(S, <)$ is well-ordered. Let ρ be an ordinal such that $|\rho| > |\alpha|$. Fix $x \in S$, we define a sequence $\langle \gamma^x_\mu \mid \mu < \rho \rangle$ by transfinite recursion on ρ. Let $\mu < \rho$ and suppose the sequence $\langle \gamma^x_\nu \mid \nu < \mu \rangle$ is already defined such that

1. $\gamma^x_\nu \in S$ for all $\nu < \mu$, and
2. $f(\{\gamma^x_\nu < x\}) = 0$ if $\nu < \mu$ and $\gamma^x_\nu \neq x$.

We define γ^x_μ as follows. If $\gamma^x_\nu = x$ for some $\nu < \mu$, then we let $\gamma^x_\mu = x$. Otherwise, $\gamma^x_\nu \neq x$ for all $\nu < \mu$, and in this case we define the set T consisting precisely of the elements $y \in S$ such that $f(\{\gamma^x_\nu < y\}) = 0$ for all $\nu < \mu$. Note that by assumption $x \in T$, so T is non-empty. There exists a unique set $H \subseteq T$ satisfying the properties of Lemma 3.51. Then $H \subseteq T$ is 1-homogeneous for $f \restriction [T]^2$, and hence also 1-homogeneous for f. Since f witnesses $\alpha \not\to (\beta, \gamma)^2$, it holds that $\operatorname{otp}(H, <) < \gamma$. If $x \in H$, then we set $\gamma^x_\mu = x$. If $x \notin H$, then there exists some $z \in H$ such that $f(\{z < x\}) = 0$ and we set γ^x_μ to be the $<$-least such $z \in H$. This completes the definition of $\langle \gamma^x_\mu \mid \mu < \rho \rangle$.

For every $x \in S$, there is a $<$-least $\sigma(x) < \rho$ such that $\gamma^x_{\sigma(x)} = x$. Otherwise, the sequence $\langle \gamma^x_\mu \mid \mu < \rho \rangle$ would be strictly increasing sequence in $(S, <)$, where

$\operatorname{otp}(S,<) = \alpha < \rho$, giving a contradiction. Therefore, for fixed $x \in S$, we obtain a set $A = \{\gamma^x_\mu \mid \mu \leq \sigma(x)\}$, which has order-type $\sigma(x)+1$. The set A is 0-homogeneous for f by construction, and since f witnesses $\alpha \not\to (\beta,\gamma)^2$, we have $\sigma(x)+1 < \beta$. In other words, $\sigma(x) < \beta^-$.

Define for every $\mu < \beta^-$, $M_\mu = \{\gamma^x_\mu \mid x \in S \text{ and } \sigma(x) \geq \mu\}$. Clearly, $S = \cup_{\mu<\beta^-} M_\mu$, and we can define a partition $g\colon S \to \beta^-$ by sending $x \in S$ to the least $\mu < \beta^-$ such that $x \in M_\mu$. Define $\alpha_\mu = \operatorname{otp}(M_\mu,<)$ for all $\mu < \beta^-$, then g witnesses $\alpha \not\to (\alpha_\mu+1)^1_{\mu<\beta^-}$. Hence we have shown (3.67).

Using Lemma 3.51, we obtain a unique set $H(S)$ which is 1-homogeneous for f and for all $x \in S \setminus H(S)$ there is $h \in H(S)$ with $f(\{h < x\}) = 0$. By construction and the uniqueness of $H(S)$, we obtain $M_0 \subseteq H(S)$. Therefore $\alpha_0 = \operatorname{otp} M_0 < \gamma$ and since $\kappa_0 = 1$ by definition, we obtain $\alpha_0 \not\to (\gamma)^1_{\kappa_0}$.

Fix $0 < \mu < \beta^-$. We want to write M_μ as a union of "sufficiently small" sets. We know $z \in M_\mu$ if and only if there is $x \in S$ with $\sigma(x) \geq \mu$ and $\gamma^x_\mu = z$. In particular, if $z \in M_\mu$ there are $y_\nu \in M_\nu$ for all $\nu < \mu$ such that $\gamma^x_\nu = y_\nu$. This gives us

$$M_\mu = \bigcup_{\langle y_\nu \mid \nu<\mu\rangle \in \prod_{\nu<\mu} M_\nu} \{\gamma^x_\mu \mid x \in S \text{ and } \sigma(x) \geq \mu \text{ and } \forall \nu < \mu(\gamma^x_\nu = y_\nu)\}.$$

Now, given some $\langle y_\nu \mid \nu < \mu\rangle \in \prod_{\nu<\mu} M_\nu$, we will show that the set $\{\gamma^x_\mu \mid \sigma(x) \geq \mu \text{ and } \forall \nu < \mu(\gamma^x_\nu = y_\nu)\}$ has order-type strictly less than γ. To see this, consider the set

$$T = \{y \in S \mid f(\{y_\nu < y\}) = 0 \text{ for all } \nu < \mu\}.$$

Let $H \subseteq T$ be the set we obtain from Lemma 3.51. If $x \in S$ is now such that $\sigma(x) \geq \mu$ and $\gamma^x_\nu = y_\nu$ for all $\nu < \mu$, then $\gamma^x_\mu \in H$, by definition of the γ^x_μ. Therefore

$$\{\gamma^x_\mu \mid x \in S \text{ and } \sigma(x) \geq \mu \text{ and } \forall \nu < \mu(\gamma^x_\nu = y_\nu)\} \subseteq H.$$

Finally, H is 1-homogeneous for f and therefore $\operatorname{otp}(H,<) < \gamma$. Define $\kappa_\mu = \prod_{\nu<\mu} |\alpha_\nu|$. Also, define the colouring $h\colon M_\mu \to \kappa_\mu$ by sending $y \in M_\mu$ to the $<^*$-least element $\langle y_\nu \mid \nu < \mu\rangle \in \prod_{\nu<\mu} M_\nu$ such that there are $x \in S$ with $\sigma(x) \geq \mu$ with $\gamma^x_\nu = y_\nu$ for all $\nu < \mu$ (here $<^*$ is some fixed well-order on κ_μ). Then h witnesses $\alpha_\mu \not\to (\gamma)^1_{\kappa_\mu}$, showing (3.68). This concludes the proof. □

We are now ready to prove Theorem 3.50.

Proof of Theorem 3.50. Suppose $\kappa \not\to (\omega+1,\kappa)^2$. Then by Proposition 3.52 there exists a sequence of ordinals $\langle \alpha_n \mid n < \omega\rangle$ satisfying

$$\kappa \not\to (\alpha_n+1)^1_{n<\omega}, \text{ and} \tag{3.69}$$
$$\alpha_n \not\to (\kappa)^1_{\kappa_n}, \tag{3.70}$$

for all $n < \omega$, where $\kappa_n = \prod_{m<n} |\alpha_m|$.

We show $\alpha_n < \kappa$ for all $n < \omega$. By definition $\kappa_0 = 1$ is the cardinality of the empty product. Thus immediately, $\alpha_0 < \kappa$ by $\alpha_0 \not\to (\kappa)^1_1$. Let $n \geq 1$ and suppose now that $\alpha_m < \kappa$ for all $m < n$. Then

$$\kappa_n = \prod_{m<n} |\alpha_m| = \max\{|\alpha_m| \colon m < n\} < \kappa.$$

Then by (3.70) we have $\alpha_n \not\to (\kappa)^1_{\kappa_n}$, which immediately gives $\alpha_n < \kappa$, because $\kappa_n < \mathrm{cf}(\kappa) = \kappa$.

Now, by (3.69), there is a partition $\kappa = \bigcup_{n<\omega} A_n$, such that $\operatorname{otp} A_n \leq \alpha_n$ for all $n < \omega$. In particular,

$$\kappa \leq \sum_{n<\omega} |\alpha_n| < \kappa,$$

because $|\alpha_n| < \kappa$ for all $n < \omega$ and $\mathrm{cf}(\kappa) > \aleph_0$. This is a contradiction and hence we conclude $\kappa \to (\omega+1, \kappa)^2$. □

A straightforward modification of the proof above gives us the following result.

Theorem 3.53. *Let κ and ν be infinite cardinals such that $\kappa^\mu \leq \kappa$ for all $\mu < \nu$. Then $\kappa^+ \to (\nu+1, \kappa^+)^2$.*

Theorem 3.50 should be compared to the result by Todorčević [Tod86, Theorem 2], in which he showed that for any cardinal κ of uncountable cofinality, there is a c.c.c. forcing which adds a witness to $\kappa \not\to (\omega+2, \kappa)^2$.

We find another corollary of Proposition 3.52.

Corollary 3.54. *Let κ be a (strongly) inaccessible cardinal.*[17] *Then for all $\beta < \kappa$,*

$$\kappa \to (\beta, \kappa)^2. \tag{3.71}$$

Proof. Suppose for the sake of contradiction that there is some ordinal $\beta < \kappa$ such that $\kappa \not\to (\beta, \kappa)^2$. Using Proposition 3.52, we obtain a sequence of ordinals $\langle \alpha_\mu \mid \mu < \beta^- \rangle$ such that

$$\kappa \not\to (\alpha_\mu + 1)^1_{\mu<\beta^-}, \text{ and} \tag{3.72}$$

$$\alpha_\mu \not\to (\kappa)^1_{\kappa_\mu}, \tag{3.73}$$

for all $\mu < \beta^-$, where $\kappa_\mu = \prod_{\nu<\mu} |\alpha_\nu|$.

We prove by induction that $|\alpha_\mu| < \kappa$ for all $\mu < \beta^-$. Suppose for some $\mu < \beta^-$ it holds for all $\gamma < \mu$ that $|\alpha_\gamma| < \kappa$. Define $\nu = \sum_{\gamma<\mu} |\alpha_\gamma|$, then $\nu < \kappa$ by the regularity of κ. Then $|\kappa_\mu| \leq \nu^{|\mu|} \leq 2^{\nu\cdot|\mu|} < \kappa$, since κ is a strong limit. By (3.73), we can write $|\alpha_\mu| = \sum_{\nu<\kappa_\mu} |\rho_\nu|$ for ordinals $\rho_\nu < \kappa$ for all $\nu < \kappa_\mu$. Then, again by the regularity of κ, $|\alpha_\mu| < \kappa$. This concludes the proof by induction. Hence, $|\alpha_\mu| < \kappa$ for all $\mu < \beta^-$.

Now, by (3.72) and the regularity of κ, we obtain $\kappa = \sum_{\mu<\beta^-} |\alpha_\mu| < \kappa$. This is a contradiction. We therefore conclude that $\kappa \to (\beta, \kappa)^2$. □

Theorem 3.55 (Theorem 37)**.** *Let κ be an infinite cardinal, and let ν be the least cardinal such that $\kappa^\nu > \kappa$. Let μ be an ordinal such that $\kappa < \mathrm{cf}\,\aleph_\mu \leq \aleph_\mu \leq \kappa^\nu$, then*

$$\aleph_{\omega_\mu} \not\to (\nu^+, \aleph_{\omega_\mu})^2. \tag{3.74}$$

Proof. As $\kappa^\kappa > \kappa$, we have by minimality of ν that $\nu \leq \kappa$. Define F to be the set of functions form ν to κ, then $|F| = \kappa^\nu$. Let $\prec$ be the lexicographic ordering on κ^ν. Then $\kappa^+, (\nu^+)^* \not\leq \operatorname{otp}(F, \prec)$ by [ER53, Lemma 2].

[17] This means that $\kappa > \aleph_0$, κ is regular, and for all cardinals $\nu < \kappa$ it holds that $2^\nu < \kappa$. Of course, if $\kappa = \aleph_0$, the result holds by Ramsey's Theorem.

Fix some $X \subseteq F$ with $|X| = \aleph_\mu$. Let $f\colon X \to \omega_\mu$ be an injection, then $f[X]$ is unbounded in ω_μ. Define $S = \{(x, \alpha) \mid x \in X \text{ and } \alpha < \omega_{f(x)}\}$, and write $\operatorname{otp}(S, \lll) = L$, where $\lll$ is the lexicographic ordering on S (w.r.t. the lexicographic ordering $\prec$ on X, and the usual ordering $<$ on $\omega_{f(x)}$). Then, on the one hand,

$$|S| = \sum_{x \in X} \aleph_{f(x)} \leq \sum_{x \in X} \aleph_{\omega_\mu} \leq |X| \cdot \aleph_{\omega_\mu} = \aleph_{\omega_\mu}.$$

On the other hand, for all $\rho < \aleph_{\omega_\mu}$ there is some $\tau < \omega_\mu$ such that $\rho < \aleph_\tau$. As $f[X]$ is unbounded in ω_μ, there is some $x \in X$ with $f(x) > \tau$. Therefore $\rho < \aleph_{f(x)} \leq |S|$. Hence $|L| = |S| = \aleph_{\omega_\mu}$.

Claim. *$\omega_{\omega_\mu}, (\nu^+)^* \not\leq L$.*

Proof of claim. First we show $\omega_{\omega_\mu} \not\leq L$. Let $S_1 \subseteq S$ be arbitrary such that $\operatorname{otp}(S_1, \lll)$ is well-ordered. Define the projection $X_1 = \pi(S_1) = \{x \in X \mid \exists \alpha < \omega_\mu (x, \alpha) \in S_1\}$, then $\operatorname{otp}(X_1, \prec)$ is well-ordered. As $\kappa^+ \not\leq \operatorname{otp}(F, \prec)$, it holds that $\operatorname{otp}(X_1, \prec) < \kappa^+$ and hence $|X_1| \leq \kappa < \operatorname{cf} \aleph_\mu$. Therefore, $\sum_{x \in X_1} |f(x)| < \aleph_\mu$ and thus $\beta := \cup_{x \in X_1} f(x) < \omega_\mu$. Then

$$|S_1| \leq \sum_{x \in X_1} \aleph_{f(x)} \leq \sum_{x \in X_1} \aleph_\beta = |X_1| \cdot \aleph_\beta < \aleph_{\omega_\mu}.$$

Since S_1 was an arbitrary well-ordered subset of S, we conclude that $\omega_{\omega_\mu} \not\leq L$. Note that $L = \operatorname{otp}(S, \lll)$.

It remains to show $(\nu^+)^* \not\leq L$. For this, let $S_2 \subseteq S$ be an arbitrary subset such that $\operatorname{otp}(S_2, \lll^*)$ is well-ordered. Define $X_2 = \pi(S_2) = \{x \in X \mid \exists \alpha < \omega_\mu (x, \alpha) \in S_2\}$, then $\operatorname{otp}(X_2, \prec^*)$ is well-ordered. As $(\nu^+)^* \not\leq \operatorname{otp}(F, \prec)$ and $\operatorname{otp}(X_2, \prec) \leq \operatorname{otp}(F, \prec)$, we obtain $\operatorname{otp}(X_2, \prec) < (\nu^+)^*$ and hence $\operatorname{otp}(X_2, \prec^*) < \nu^+$. In particular, $|X_2| \leq \nu$. For $x \in X_2$, define $N_x = \{\beta < \omega_\mu \mid (x, \beta) \in S_2\}$, then $\operatorname{otp}(N_x, <)$ is well-ordered. But, since $\operatorname{otp}(S_2, \lll^*)$ is well-ordered, it follows that $\operatorname{otp}(N_x, >)$ is also well-ordered. This can only happen if $|N_x| < \aleph_0$. Therefore

$$|S_2| = \sum_{x \in X_2} |N_x| \leq |X_2| \cdot \aleph_0 \leq \nu.$$

This gives $\operatorname{otp}(S_2, \lll) < (\nu^+)^*$, and since S_2 was arbitrary, we get $(\nu^+)^* \not\leq L$. This concludes the proof of the claim. ■

Now we can apply Lemma 3.1: since $|L| = \aleph_{\omega_\mu}$ and using the claim we obtain the result $\aleph_{\omega_\mu} \not\to (\aleph_{\omega_\mu}, \nu^+)^2$. □

Corollary 3.56. *If α is an ordinal with $\aleph_0 < \operatorname{cf} \aleph_\alpha \leq \aleph_\alpha \leq 2^{\aleph_0}$, then*

$$\aleph_{\omega_\alpha} \not\to (\aleph_1, \aleph_{\omega_\alpha})^2. \tag{3.75}$$

Theorem 3.57 (Theorem 36). *Let γ be an ordinal, and let L_β be order-types for all $\beta < \gamma$. Suppose δ is such that $L_\beta < \delta$ for all $\beta < \gamma$. Define $L = \sum_{\beta < \gamma} L_\beta$. Then $L \not\to (\gamma + 1, \delta)^2$.*

Proof. Define $f\colon [L]^2 \to 2$ by $\{x, y\} \mapsto 1$ if and only if $\{x, y\} \in [L_\beta]^2$ for some $\beta < \gamma$. If there is some $H \subseteq L$ that is 0-homogeneous for f, then for all $\beta < \gamma$, it holds that $|H \cap L_\beta| \leq 1$ and hence $\operatorname{otp} H < \gamma + 1$. Suppose, on the other hand, H is 1-homogeneous for f. Then there is a unique $\beta < \gamma$ such that $[H]^2 \subseteq [L_\beta]^2$, which implies $\operatorname{otp} H < \delta$. □

Corollary 3.58. *Let κ be an infinite cardinal, then $\kappa \not\to (\operatorname{cf}\kappa + 1, \kappa)^2$.*

Proof. By definition of cofinality, there is a sequence $\langle \kappa_\beta \mid \beta < \operatorname{cf}\kappa \rangle$ of ordinals below κ and $\kappa = \sum_{\beta < \operatorname{cf}\kappa} \kappa_\beta$. Then apply Theorem 3.57. □

As $\operatorname{cf}\aleph_\omega = \omega$, this immediately yields the following partition relation.

Corollary 3.59. $\aleph_\omega \not\to (\omega + 1, \aleph_\omega)^2$.

Theorem 3.60 (Theorem 38)**.** *Let γ be an ordinal and $r < \omega$. Let m be some cardinal and let α_β be ordinals for all $\beta < m$. If $\aleph_\gamma \not\to (|\alpha_\beta|)^r_{\beta<m}$, then $\omega_{\gamma+1} \not\to (\alpha_\beta + 1)^{r+1}_{\beta<m}$.*

Proof. Let $\delta < \omega_{\gamma+1}$, then $|\delta| \leq \aleph_\gamma$. Then by the assumption $|\delta| \not\to (|\alpha_\beta|)^r_{\beta<m}$, and hence $\delta \not\to (\alpha_\beta)^r_{\beta<m}$. Then using Lemma 3.36 we obtain the desired result. □

Chapter 4

Sharpness of Positive Stepping Up Lemma

In this chapter, we investigate some results from [Erd+84]. This compendium written by Erdős, Hajnal, Máté, and Rado, is about cardinal-based partition relations. Of course, as remarked before, cardinals are very specific order-types, and we want to generalise these results by placing them in the context of order-types.

Of particular importance is the "Negative Stepping Up Lemma" in [Erd+84, Theorem 24.1]. We will adjust its proof and prove the implication $\kappa \not\to (\omega^\alpha)^r_m \implies 2^\kappa \not\to (\omega^\alpha)^{r+1}_m$. This in turn will provide us with the relation $\beth^+_n \not\to (\omega^2)^{n+3}_2$. We relate this result to an instance of the Positive Stepping Up Lemma, $\beth^+_n \to (\omega + n + 1)^r_m$, and conjecture that this partition relation is sharp.

4.1 Some results in the context of order-types

We will study some results from [Erd+84, Chapter 5], and we place these results in the context of order-types.

Definition 4.1 (Discrepancy)**.** Let α be an ordinal and let f and g be distinct functions from α to m, where m is some arbitrary set. Then the *discrepancy* $\delta(f,g)$ is defined to be

$$\delta(f,g) = \min\{\xi \in \alpha \mid f(\xi) \neq g(\xi)\}.$$

Observation 4.2. If f, g, h are pairwise distinct functions from α to m, then $\delta(f,g) < \delta(g,h)$ implies $\delta(f,g) = \delta(f,h)$.

Definition 4.3 (Cartesian product)**.** Let α be an ordinal and assume that we have a linearly ordered set $(A_\gamma, <_\gamma)$ for every $\gamma < \alpha$. Define the lexicographical ordering $\prec$ on $\prod_{\gamma<\alpha} A_\gamma$ with respect to the orderings $<_\gamma$. That is, for distinct $f, g \in \prod_{\gamma<\alpha} A_\gamma$, we let $f \prec g$ if and only if $f(\xi) <_\xi g(\xi)$, where $\xi = \delta(f,g)$.

Similarly, given order-types $L_\gamma = \mathrm{otp}(A_\gamma, <_\gamma)$ for all $\gamma < \alpha$, we can define $L = \prod_{\gamma<\alpha} L_\gamma$ as the order-type of $(\prod_{\gamma<\alpha} A_\gamma, \prec)$.

Observation 4.4. Let $(A_\gamma, <_\gamma)$ be ordered sets and let $\prec$ be the lexicographic ordering on the Cartesian product $A = \prod_{\gamma<\alpha} A_\gamma$. Suppose $f, g, h \in A$ are such that $f \prec g \prec h$. If $\delta(f,g) \leq \delta(g,h)$, then $\delta(f,g) = \delta(f,h)$.

Theorem 4.5 (Theorem 19.3, [Erd+84])**.** *Given an ordinal $\alpha > 0$ and order-types L_γ for every $\gamma < \alpha$. Let $\prec$ be the lexicographical ordering of the Cartesian product $L = \prod_{\gamma<\alpha} L_\gamma$. Suppose that κ is a regular cardinal, and suppose that there exists a subset $B \subseteq L$ such that $|B| = \kappa$ and $\prec$ well-orders B.*

Then there exists a strictly $\prec$-increasing sequence $\langle f_\beta \mid \beta < \kappa\rangle$ in B, and there exists a non-decreasing sequence $\langle \xi_\beta \mid \beta < \kappa\rangle$ of ordinals less than α with the additional property that

$$\delta(f_\mu, f_\nu) = \xi_\mu \tag{4.1}$$

whenever $\mu < \nu < \kappa$.
An analogous result holds if we replace $\prec$ by the reverse lexicographic ordering $\succ$.

Proof. We may assume that $\mathrm{otp}(B, \prec) = \kappa$. We construct by transfinite recursion on κ a sequence $\langle B_\nu \mid \nu < \kappa\rangle$ of non-empty final segments of B. Define $B_0 = B$. Suppose for some $0 < \nu < \kappa$ we have defined non-empty final segments B_μ of B for all $\mu < \nu$ such that for $\eta < \mu < \nu$, we have $B \supseteq B_\eta \supseteq B_\mu$. Then let $B'_\nu = \bigcap_{\mu<\nu} B_\mu$. Then B'_ν is a final segment of B, and by the regularity of $|B|$, we have that B'_ν is non-empty. Now, let $f_\nu = \min_\prec\{f \in B \mid f \in B'_\nu\}$ and define $\xi_\nu = \min_<\{\delta(f_\nu, f) \mid f \in B'_\nu \setminus \{f_\nu\}\}$. Then define $B_\nu = \{f \in B'_\nu \setminus \{f_\nu\} \mid \delta(f, f_\nu) = \xi_\nu\}$. Note $B_\nu \subseteq B_\mu$.

Clearly, B_ν is non-empty. To see that B_ν is a final segment of B, let $f \in B_\nu$ and let $g \in B$ be such that $f \prec g$. Then as $f \in B'_\nu$ and B'_ν is a final segment of B, we have $g \in B'_\nu$. Then $f_\nu \prec f \prec g$. If it were the case that $\delta(f, g) < \xi_\nu = \delta(f_\nu, f)$, then by Observation 4.2, we have $\delta(g, f_\nu) < \xi_\nu$. As $g \in B'_\nu$, this gives a contradiction with the minimality of ξ_ν. Therefore $\delta(f_\nu, f) \leq \delta(f, g)$, and hence by Observation 4.4 we obtain $\delta(f_\nu, g) = \xi_\nu$. This gives $g \in B_\nu$, and hence B_ν is a final segment of B. This concludes the definition of $\langle B_\nu \mid \nu < \kappa\rangle$.

By construction, we see that $\langle f_\nu \mid \nu < \kappa\rangle$ is a strictly $\prec$-increasing sequence. Also, given $\mu < \nu < \kappa$, we have $f_\nu \in B_\mu$ and hence $\delta(f_\mu, f_\nu) = \xi_\mu$, showing (4.1). Finally, to see $\langle \xi_\nu \mid \nu < \kappa\rangle$ is non-decreasing, let $\eta < \mu < \nu < \kappa$. Then $\xi_\eta = \delta(f_\eta, f_\mu) = \delta(f_\eta, f_\nu) \leq \delta(f_\mu, f_\nu) = \xi_\mu$. ☐

Definition 4.6. Given order-types L, M, N, R, we say that L *establishes* the relation $M \not\to (N, R)^2$ if $|L| = |M|$ and $N, R^* \not\leq L$. Note that Lemma 3.1 shows that this definition is well-defined.

Theorem 4.7 (Theorem 19.6, [Erd+84])**.** *Let $\alpha > 0$ be an ordinal and let $L_\gamma, M_\gamma, N_\gamma$ and R_γ be order-types for all $\gamma < \alpha$ such that L_γ establishes the relation $M_\gamma \not\to (N_\gamma, R_\gamma)^2$. Put $M = \prod_{\gamma<\alpha} M_\gamma$ as in Definition 4.3. Let ρ and σ regular cardinals and assume*

$$\rho \to (N_\gamma)^1_{\gamma<\alpha}, \text{ and} \tag{4.2}$$

$$\sigma \to (R_\gamma)^1_{\gamma<\alpha}. \tag{4.3}$$

Then there is an order-type establishing the relation

$$M \not\to (\rho, \sigma)^2. \tag{4.4}$$

Proof. Let $L = (\prod_{\gamma<\alpha} L_\gamma, \prec)$, where $\prec$ is the lexicographical ordering. Then, as $|L_\gamma| = |M_\gamma|$ for all $\gamma < \alpha$, clearly $|L| = |M|$. We will show that L is the order-type establishing the relation (4.4).

Suppose for the sake of contradiction that $\rho \cong (B, \prec)$ where $(B, \prec) \subseteq L$ is a subordering. As $|B| = \rho$ is regular and $\prec$ well-orders B, we have by Theorem 4.5 a strictly $\prec$-increasing sequence $\langle f_\beta \mid \beta < \rho \rangle$ in B, and there exists a non-decreasing sequence $\langle \xi_\beta \mid \beta < \rho \rangle$ of ordinals less than α such that (4.1) holds.

Define the partition $g \colon \rho \to \alpha$ by $g(\gamma) = \zeta$ if and only if $\xi_\gamma = \zeta$. By (4.2), there exists some set $\chi_\gamma \subseteq \rho$ such that $\chi_\gamma \cong N_\gamma$ and $g \restriction \chi_\gamma$ is constant. This means for all $\eta, \lambda \in \chi_\gamma$ that $\delta(f_\eta, f_\lambda) = \zeta$, for some fixed ζ. As the sequence $\langle f_\eta \mid \eta \in \chi_\gamma \rangle$ is strictly increasing in the lexicographical ordering, the sequence $\langle f_\eta(\zeta) \mid \eta \in \chi_\gamma \rangle$ is strictly increasing in L_γ. However, we assumed that $\chi_\gamma \cong N_\gamma \not\leq L_\gamma$, which gives us a contradiction. Hence $\rho \not\leq L$.

The case for $\sigma^* \not\leq L$ is analogous. Hence we conclude that the order-type L establishes the relation $M \not\to (\rho, \sigma)^2$. □

Theorem 4.8 (Theorem 21.1, [Erd+84])**.** *Let γ be a cardinal and let $\alpha_\xi, \beta_{\xi,\nu}, \tau, \rho_\nu$ be ordinals for all $\xi < \tau$ and all $\nu < \gamma$. Assume*

$$\alpha_\xi \not\to (\beta_{\xi,\nu})^2_{\nu<\gamma}, \text{ and} \tag{4.5}$$

$$\tau \not\to (\rho_\nu)^2_{\nu<\gamma}. \tag{4.6}$$

For all $\nu < \gamma$, put

$$\sigma_\nu = \sup\{\operatorname{otp}\left(\sum_{\xi \in Y} \zeta_\xi\right) + 1 \mid Y \subseteq \tau \wedge \operatorname{otp} Y < \rho_\nu \wedge \forall \xi < \tau\, [\zeta_\xi < \beta_{\xi,\nu}]\}. \tag{4.7}$$

Then

$$\sum_{\xi<\tau} \alpha_\xi \not\to (\sigma_\nu)^2_{\nu<\gamma}. \tag{4.8}$$

(Note that τ is an ordinal, and hence any subset $Y \subseteq \tau$ is isomorphic to an ordinal. Using that all $\beta_{\xi,\nu}$ are also ordinals, we have for any sequence $\langle \zeta_\xi \mid \xi \in Y \rangle$ with $\zeta_\xi < \beta_{\xi,\nu}$ that $(\sum_{\xi \in Y} \zeta_\xi) + 1$ is isomorphic to an ordinal and hence there exists a supremum, i.e., σ_ν is well-defined.)

Proof. For every $\xi < \tau$ there is a partition $f_\xi \colon [\alpha_\xi]^2 \to \gamma$ witnessing (4.5). Similarly, there is a partition $f \colon [\tau]^2 \to \gamma$ witnessing (4.6). We can assume without loss of generality that all α_ξ are pairwise disjoint. Define the partition $f' \colon [\sum_{\xi<\tau} \alpha_\xi]^2 \to \gamma$ by $f'(\{x, y\}) = f_\xi(\{x, y\})$ if $x, y \in \alpha_\xi$ for some unique $\xi < \tau$ and let $f'(\{x, y\}) = f(\{\xi, \eta\})$ if $x \in \alpha_\xi$ and $y \in \alpha_\eta$ where $\xi \neq \eta$.

Suppose that $H \subseteq \sum_{\xi<\tau} \alpha_\xi$ is ν-homogeneous for f' for some $\nu < \gamma$. Then, by definition of f', we have for every $\xi < \tau$ that the set $H \cap \alpha_\xi$ is homogeneous for f_ξ. As f_ξ witnesses (4.5), it must be that $\operatorname{otp} H \cap \alpha_\xi < \beta_{\xi,\nu}$. Similarly, the set

$$Y = \{\xi < \tau \mid H \cap \alpha_\xi \neq \emptyset\}$$

is homogeneous for f. Therefore $Y < \rho_\nu$ by (4.6). By definition of σ_ν it follows that $\sigma_\nu \not\leq \operatorname{otp} H$. As H was an arbitrary homogeneous set for f', we conclude that are no homogeneous sets for f' into which σ_ν embeds, and this concludes the proof. □

4.2 A curious pattern emerges

In this section we return to the question whether one corollary of the Positive Stepping Up Lemma, namely Theorem 3.39, is sharp. In Corollary 3.44 we had already shown that $\beth_0^+ \not\to (\omega+2)^3_2$. After investigating the consequences of a result by Albin Jones, and combining it with an earlier result of Erdős-Rado, we observe that the pattern continues for at least one more relation, i.e. $\beth_1^+ \not\to (\omega+3)^4_2$.

Theorem 4.9 (Albin L. Jones (2000), [Jon00], Theorem 2)**.** *Let L be an order-type and let κ be an infinite cardinal. If $L \not\to (\omega)^1_{2^\kappa}$, then $L \not\to (\kappa+2,\omega)^3$.*

Proof. Let $e\colon L \to 2^\kappa$ be a witness of $L \not\to (\omega)^1_{2^\kappa}$. As ω is regular, it follows for every $B \in [L]^\omega$ there is $C \in [B]^\omega$ such that $e \restriction C$ is injective.

We view 2^κ as the set of functions from κ to 2. Recall that, given $s,t \in 2^\kappa$, we define the *discrepancy* $\delta(s,t)$ as the least $\xi < \kappa$ such that $s(\xi) \neq t(\xi)$, if it exists, and let $\delta(s,t) = \kappa$ otherwise. Define the partition $f\colon [L]^2 \to \kappa+1$ by $\{x,y\} \mapsto \delta(e(x),e(y))$.

Define the partition of triples $g\colon [L]^3 \to 2$ by

$$g(\{x<y<z\}) = \begin{cases} 0 & \text{if } e \text{ is injective on } \{x,y,z\} \text{ and } f(\{x,y\}) < f(\{y,z\}), \text{ and} \\ 1 & \text{if } e \text{ is not injective on } \{x,y,z\} \text{ or } f(\{x,y\}) \geq f(\{y,z\}). \end{cases}$$

We show that g is the partition which proves $L \not\to (\kappa+2,\omega)^3$.

Claim. *There is no 0-homogeneous $H \subseteq L$ for g with $\operatorname{otp} H = \kappa+2$.*

Proof of claim. Suppose for the sake of contradiction that such $H = \{h_\gamma \mid \gamma < \kappa+2\}$ exists. We observe immediately that $e \restriction H$ is injective. In particular, $e(h_\kappa) \neq e(h_{\kappa+1})$ and hence $\delta(e(h_\kappa), e(h_{\kappa+1})) = \xi < \kappa$. For any $\mu < \nu < \kappa$ we have $f(\{h_\mu, h_\nu\}) < f(\{h_\nu, h_\kappa\})$, and by Observation 4.2, $f(\{h_\mu, h_\kappa\}) = f(\{h_\mu, h_\nu\}) < f(\{h_\nu, h_\kappa\})$. Note that $f(\{h_\mu, h_\kappa\}) < f(\{h_\kappa, h_{\kappa+1}\}) = \xi < \kappa$. Hence, the sequence $\langle f(\{h_\mu, h_\kappa\}) \mid \mu < \kappa\rangle$ is a strictly increasing sequence of ordinals below ξ which has length κ, which gives a contradiction. ■

Claim. *There is no 1-homogeneous $H \subseteq L$ for g with $\operatorname{otp} H = \omega$.*

Proof of claim. Again, for the sake of contradiction assume such $H \in [L]^\omega$ exists. By the remark above there is $B \in [H]^\omega$ such that $e \restriction B$ is injective. Consider the colouring $h\colon [B]^3 \to 2$ by

$$h(\{x<y<z\}) = \begin{cases} 0 & \text{if } f(\{x,y\}) > f(\{y,z\}), \text{ and} \\ 1 & \text{if } f(\{x,y\}) = f(\{y,z\}). \end{cases}$$

By definition of g and since B is 1-homogeneous for g, the colouring h is well-defined.

Now, by Ramsey's Theorem, $\omega \to (\omega,4)^3$. Hence, either

(a) there is $C \in [B]^\omega$ such that $h \restriction [C]^3 \equiv 0$, or

(b) there is $D \in [B]^4$ such that $h \restriction [D]^3 \equiv 1$.

If (a) holds, then $\langle f(\{c_n, c_{n+1}\}) \mid n \in \omega \rangle$, where $C = \{c_0 < c_1 < \dots\}$, is a strictly decreasing sequence of ordinals of length ω, which is a contradiction. Alternatively, if (b) holds and such $D = \{x < y < z < w\}$ exists, then $f(\{x, y\}) = f(\{y, z\}) = f(\{x, z\})$. This gives us three pairwise distinct functions $e(x), e(y), e(z) \in 2^\kappa$ such that they are pairwise different at some point $\xi < \kappa$, which is not possible since these functions map to 2. Hence we reach a contradiction. ■

This shows that there cannot be homogeneous set for g of the appropriate order-type, and this concludes the proof. □

Corollary 4.10. *Let α be an ordinal such that $\alpha < (2^{\aleph_0})^+$, then*

$$\alpha \not\rightarrow (\omega + 2, \omega)^3. \tag{4.9}$$

Proof. As $\alpha < (2^{\aleph_0})^+$ we have $|\alpha| \leq 2^{\aleph_0}$ and we can define the colouring $f\colon [\alpha]^1 \to \alpha\colon \{\beta\} \mapsto \beta$, which witnesses $\alpha \not\rightarrow (\omega)^1_\alpha$. Then $\alpha \not\rightarrow (\omega)^1_{2^{\aleph_0}}$, hence the result follows by Theorem 4.9. □

This result gives, in particular, the negative partition relation $2^{\aleph_0} \not\rightarrow (\omega + 2, \omega)^3$, which improves Corollary 3.44.

Theorem 4.11.

$$\beth_1^+ \not\rightarrow (\omega + 3, \omega + 1)^4. \tag{4.10}$$

Proof. By Corollary 4.10 we have for every ordinal $\alpha < (2^{\aleph_0})^+$ the negative relation $\alpha \not\rightarrow (\omega + 2, \omega)^3$. Then Lemma 3.36 gives us the desired result. □

The theorem above demonstrates that the goal of the relation $(2^{\aleph_0})^+ \rightarrow (\omega + 2)^r_k$ from Corollary 3.40 cannot be increased.

A pattern of partition relations seems to emerge, showing the sharpness of the positive partition relations of Theorem 3.39. Here r and k are natural numbers.

1. $\beth_0^+ \rightarrow (\omega + 1)^r_k$.
2. $\beth_0^+ \not\rightarrow (\omega + 2)^3_2$.
3. $\beth_1^+ \rightarrow (\omega + 2)^r_k$.
4. $\beth_1^+ \not\rightarrow (\omega + 3)^4_2$.
5. $\beth_2^+ \rightarrow (\omega + 3)^r_k$.

Of course, the Positive Stepping Up Lemma guarantees that the pattern continues for the positive partition relations. In general, the relation $\beth_n^+ \rightarrow (\omega + n + 1)^r_k$ is true by Theorem 3.39. However, it is not clear whether all such positive partition relations are sharp, i.e., whether increasing the goal results in a negative partition relation.

It is natural to conjecture that this pattern continues, and hence we propose that the following partition relation is negative.

Conjecture 4.12. *For all $n < \omega$,*

$$\beth_n^+ \not\rightarrow (\omega + n + 2)^{n+3}_2. \tag{4.11}$$

Note that we have already shown that this conjecture holds for both $n = 0$ and $n = 1$.

Before we move on to the next section and provide a bound for the conjecture, we want to make a few notes about the exponent of the partition relation in Conjecture 4.12. If we compare (4.11) to the Erdős-Rado Theorem, which gives $\beth_n^+ \to (\aleph_1)_\omega^{n+1}$, we see that the partition relation is positive with exponent $n+1$, even with $\aleph_1$ as goal. However, what happens if we decrease the exponent in (4.11) to $n+2$? In other words, why don't we conjecture $\beth_n^+ \not\to (\omega+n+2)_2^{n+2}$? In this case we also have a conclusive answer. Starting with a corollary of the Baumgartner Hajnal Theorem, we have the positive partition relation $\beth_0^+ \to (\gamma)_2^2$, where $\gamma < \omega_1$. Applying the Positive Stepping Up Lemma iteratively, we obtain for all $n < \omega$, $\beth_n^+ \to (\gamma)_2^{n+2}$. This result shows that the exponent in Conjecture 4.12 cannot be decreased, unless the partition relation becomes positive.

As a final side note, the goal in the relation $\beth_1^+ \to (\gamma)_2^3$ cannot be increased to $\omega_1 + 1$: immediate from Lemma 3.36 and Lemma 3.48 the following result is true. (Alternatively, it follows from Theorem 3.47 and Theorem 3.60.)

Lemma 4.13.

$$\beth_1^+ \not\to (\omega_1+1)^3. \tag{4.12}$$

4.3 Negative Stepping Up Results

Our main goal in this section is studying the implication

$$\kappa \not\to (\lambda)_m^{r-1} \implies 2^\kappa \not\to (\lambda)_m^r. \tag{4.13}$$

In the case that κ and λ are infinite cardinals and $r \geq 4$, this implication is true and the result is known as the "Negative Stepping Up Lemma".[1]

Ideally, we would be able to prove $\kappa \not\to (\alpha)_m^{r-1} \implies 2^\kappa \not\to (\alpha+1)_m^r$, in the case where κ is a cardinal and α is any infinite ordinal. This would immediately solve Conjecture 4.12. Unfortunately, we have not been successful in proving this implication for arbitrary infinite ordinals. We have, however, been able to adjust the proof of the Negative Stepping Up Lemma and prove the implication in the case that α is an additively indecomposable ordinal. This novel result provides a bound to Conjecture 4.12. We thank Luke Gardiner for his help with this result.

Throughout we will fix a well-order $<^*$ on 2^κ, where 2^κ is viewed as the set of functions from κ to 2. We will denote by $\prec$ the lexicographic order on 2^κ.[2] The main idea is that if there are $<^*$-increasing sequences in 2^κ, then we can find sufficiently large $<$-increasing sequences of their *discrepancies*. If that is the case, then if $\langle I_n \mid n < m\rangle$ is a partition of $[\kappa]^{r-1}$ witnessing $\kappa \not\to (\lambda)_m^{r-1}$, there exists a partition $\langle J_n \mid n < m\rangle$ of $[2^\kappa]^r$ such that if $\lambda \leq \operatorname{otp} J_n$, then $\lambda \leq \operatorname{otp} I_n$. Hence this partition of $[2^\kappa]^r$ witnesses $2^\kappa \not\to (\lambda)_m^r$.

[1] See [Erd+84, Section 24].

[2] Recall that the lexicographical order on 2^κ is defined as $f \prec g$ if and only if $f(\xi) < g(\xi)$, where $\xi = \min\{\eta < \kappa \mid f(\eta) \neq g(\eta)\}$.

Now, we will need a bunch of definitions. As said before, the goal is to find subsets of 2^κ on which the well-order $<^*$ and lexicographic order $\prec$ agree. Given a sequence $x_0 <^* x_1 <^* \ldots <^* x_{r-1}$ of 2^κ, put

$$\eta(x_0 <^* x_1 <^* \ldots <^* x_{r-1}) = (\eta(x_0, x_1), \eta(x_1, x_2), \ldots, \eta(x_{r-2}, x_{r-1})), \tag{4.14}$$

where $\eta(x, y) = 0$ if $x <^* y \iff x \prec y$, and $\eta(x, y) = 1$ otherwise.

Let $1 \leq s \leq r-1$ and $k_0, k_1, \ldots, k_{s-1} \in 2$, define

$$K(k_0, k_1, \ldots, k_{s-1}) = \{u \in [2^\kappa]^r \mid \eta(u) \restriction s = (k_0, k_1, \ldots, k_{s-1})\}. \tag{4.15}$$

In particular, if $s = r-1$, then we put for $i = 0, 1$,

$$K_i = K(i, i, \ldots, i), \tag{4.16}$$

and also

$$K = K_0 \cup K_1. \tag{4.17}$$

We also want to extend Definition 4.1, so we define

$$\delta(x_0, x_1, \ldots, x_{r-1}) = (\delta(x_0, x_1), \delta(x_1, x_2), \ldots, \delta(x_{r-2}, x_{r-1})). \tag{4.18}$$

For distinct ordinals δ_0, δ_1, define $\zeta(\delta_0, \delta_1) = 0$ if $\delta_0 < \delta_1$ and let $\zeta(\delta_0, \delta_1) = 1$ if $\delta_0 > \delta_1$. Similarly, for a tuple of distinct ordinals such that $\delta_i \neq \delta_{i+1}$ we define

$$\zeta(\delta_0, \delta_1, \ldots, \delta_{r-2}) = (\zeta(\delta_0, \delta_1), \zeta(\delta_1, \delta_2), \ldots, \zeta(\delta_{r-3}, \delta_{r-2})). \tag{4.19}$$

Finally, for $1 \leq s \leq r-2$, and $k_0, k_1, \ldots, k_{s-1} \in 2$, we define

$$P(k_0, k_1, \ldots, k_{s-1}) = \{u \in K \mid \zeta(\delta(u)) \restriction s = (k_0, k_1, \ldots, k_{s-1})\}. \tag{4.20}$$

and also for $s = r-2$ and $i = 0, 1$ we put

$$P_i = P(i, i, \ldots, i), \tag{4.21}$$

and

$$P = P_0 \cup P_1. \tag{4.22}$$

Lemma 4.14 (Lemma 23.12, [Erd+84])**.** *Let $r \geq 3$, let κ be a cardinal and let α be an ordinal. Let $I \subseteq [\kappa]^{r-1}$ and put*

$$I^* = \{u \in P_0 \mid \delta(u) \in I\}. \tag{4.23}$$

Assume that $[H]^r \subseteq I^*$ for some $\emptyset \neq H \subseteq 2^\kappa$ where by assumption $\mathrm{otp}(H, <^*) = \alpha$. *Then there is $X \subseteq \kappa$ with* $\mathrm{otp}(X, <) = \alpha^-$ *such that $[X]^{r-1} \subseteq I$.*

Here α^- is the ordinal β such that $\beta + 1 = \alpha$, if it exists, and is α otherwise.

Proof. We may assume that $|H| \geq r$ and write $H = \{h_\gamma \mid \gamma < \alpha\}$ where $\alpha = \mathrm{otp}(H, <^*)$. (Recall that $<^*$ is a fixed well-order on 2^κ). For ordinals γ such that $\gamma + 1 < \alpha$ we let

$$\delta_\gamma = \delta(h_\gamma, h_{\gamma+1}),$$

where the function δ is defined in Definition 4.1. Set

$$X = \{\delta_\gamma \mid \gamma + 1 < \alpha\}.$$

First we show that $\operatorname{otp}(X, <) = \alpha^-$. It obviously suffices to show for all $\gamma < \gamma' < \alpha^-$ that $\delta_\gamma < \delta_{\gamma'}$. By the assumption $[H]^r \subseteq I^* \subseteq P_0$, it follows that

$$\zeta(\delta(\{h_\gamma, h_{\gamma+1}, h_{\gamma'}\})) = \zeta(\delta(h_\gamma, h_{\gamma+1}), \delta(h_{\gamma+1}, h_{\gamma'})) = 0.$$

Also

$$\zeta(\delta(\{h_{\gamma+1}, h_{\gamma'}, h_{\gamma'+1}\})) = \zeta(\delta(h_{\gamma+1}, h_{\gamma'}), \delta(h_{\gamma'}, h_{\gamma'+1})) = 0.$$

In other words, $\delta_\gamma < \delta(h_{\gamma+1}, h_{\gamma'}) < \delta_{\gamma'}$. Note that we assumed $\gamma + 1 < \gamma'$, because if $\gamma + 1 = \gamma'$, we could just leave out the term $\delta(h_{\gamma+1}, h_{\gamma'})$. In particular, we obtain $\delta_\gamma < \delta_{\gamma'}$, showing that $\operatorname{otp}(X, <) = \alpha^-$.

It remains to show that $[X]^{r-1} \subseteq I$. Given $\xi_0 < \ldots < \xi_{r-2} < \alpha^-$, we want to show $\{\delta_{\xi_0} < \ldots < \delta_{\xi_{r-2}}\} \in I$. Let $0 \leq i \leq r-2$ and suppose that $\xi_i + 1 < \xi_{i+1}$. As $[H]^r \subseteq P_0$, we have $\delta(h_{\xi_i}, h_{\xi_i+1}) < \delta(h_{\xi_i+1}, h_{\xi_{i+1}})$ and hence by Observation 4.2, $\delta(h_{\xi_i}, h_{\xi_i+1}) = \delta(h_{\xi_i}, h_{\xi_{i+1}})$. If $\xi_i + 1 = \xi_{i+1}$, then $\delta(h_{\xi_i}, h_{\xi_i+1}) = \delta(h_{\xi_i}, h_{\xi_{i+1}})$ obviously holds as well. Now, writing $\xi_{r-1} = \xi_{r-2} + 1$, we obtain

$$\begin{aligned}\{\delta_{\xi_i} \mid i < r-1\} &= \{\delta(h_{\xi_i}, h_{\xi_i+1}) \mid i < r-1\} \\ &= \{\delta(h_{\xi_i}, h_{\xi_{i+1}}) \mid i < r-1\} \\ &= \delta(\{h_{\xi_i} \mid i < r\}).\end{aligned}$$

As $\{h_{\xi_i} \mid i < r\} \in [H]^r \subseteq I^*$, we have by definition of I^* that $\{\delta_{\xi_i} \mid i < r-1\} \in I$. This gives us $[X]^{r-1} \subseteq I$, which is what we wanted to show. □

Lemma 4.15. *Let α be any ordinal, and suppose $(X, <)$ is an ordered set such that $\operatorname{otp}(X, <) = \omega^\alpha$. Then for any non-empty final segment $(Y, <)$ of $(X, <)$, it holds that $\operatorname{otp}(Y, <) = \omega^\alpha$.*

Proof. First, we prove by transfinite induction on α that every power of ω is additively indecomposable. That is, if $\beta, \gamma < \omega^\alpha$, then $\beta + \gamma < \omega^\alpha$.

If $\alpha = 0$, then $\omega^\alpha = \omega^0 = 1$. Clearly, $0 + 0 < 1$.

If $\alpha = \delta + 1$, then $\omega^\alpha = \omega^{\delta+1} = \omega^\delta \cdot \omega$. For $\beta, \gamma < \omega^\alpha$, there is some $n < \omega$ such that $\beta, \gamma < \omega^\delta \cdot n$. Then $\beta + \gamma \leq \omega^\delta \cdot n + \omega^\delta \cdot n = \omega^\delta \cdot (n + n) < \omega^\delta \cdot \omega = \omega^\alpha$.

If α is a limit ordinal, then for $\beta, \gamma < \omega^\alpha$ there is some $\alpha' < \alpha$ such that $\beta, \gamma < \omega^{\alpha'}$. In that case, $\beta + \gamma \leq \omega^{\alpha'} + \omega^{\alpha'} = \omega^{\alpha'} \cdot 2 \leq \omega^{\alpha'} \cdot \omega = \omega^{\alpha'+1} < \omega^\alpha$. This concludes the proof by induction.

Now, we let $(Y, <)$ be a non-empty final segment of $(X, <)$. Suppose it were the case that $\operatorname{otp}(Y, <) < \omega^\alpha$. Then there are $\beta, \gamma < \omega^\alpha$ with $\operatorname{otp}(X \setminus Y, <) = \beta$ and $\operatorname{otp}(Y, <) = \gamma$, but $\beta + \gamma = \omega^\alpha$. This is a contradiction, because ω^α is additively indecomposable. Therefore we conclude that $\operatorname{otp}(Y, <) = \omega^\alpha$. □

Lemma 4.16 (Lemma 23.5, [Erd+84])**.** *Let κ be an infinite cardinal, let $X \subseteq 2^\kappa$ and assume $\operatorname{otp}(X, <^*) = \omega^\alpha$, where $\alpha > 0$ is any ordinal. Assume that (i) $[X]^r \cap K(0,1) = \varnothing$ or (ii) $[X]^r \cap K(1,0) = \varnothing$. Then there is a set $Y \subseteq X$ with $\operatorname{otp}(Y, <^*) = \omega^\alpha$ such that $[Y]^r \subseteq K_0$ or $[Y]^r \subseteq K_1$.*

Proof. Assume for the sake of contradiction that no such Y exists. We first prove a claim

Claim. *There are elements $x_0 <^* x_1 <^* x_2 <^* x_3$ such that $x_0 \prec x_1 \succ x_2 \prec x_3$.*

Proof of claim. For every $x \in X$ there are $y, z \in X$ and $y', z' \in X$ such that

$$x \leq^* y <^* z \text{ and } y \prec z, \tag{4.24}$$
$$x \leq^* y' <^* z' \text{ and } y' \succ z'. \tag{4.25}$$

Suppose not and let $x \in X$ be a counterexample, then define the set $Y = \{x' \in X \mid x \leq^* x'\}$. Clearly, $(Y, <^*)$ is a non-empty final segment of $(X, <^*)$ and by Lemma 4.15 we obtain $\operatorname{otp}(Y, <^*) = \omega^\alpha$. As Y is contained in either K_0 or K_1, we obtain a contradiction.

Now let $x_0, z_1 \in X$ with $x_0 <^* z_1$ and $x_0 \prec z_1$. Then let $y_1, z_2 \in X$ with $z_1 \leq^* y_1 <^* z_2$ with $y_1 \succ z_2$. Define $x_1 = \max_\prec\{y_1, z_1\}$, then $x_0 <^* x_1$ and $x_0 \prec x_1$. Also, $x_1 \succ z_2$.

Pick $y_2, x_3 \in X$ with $z_2 \leq^* y_2 <^* x_3$ and $y_2 \prec x_3$. Let $x_2 = \min_\prec\{y_2, z_2\}$. Then $x_1 <^* x_2$ and $x_1 \succ x_2$. Also, $x_2 \prec x_3$. This proves the claim ■

Finally, $\{x_0, x_1, x_2, \ldots\} \in [X]^r \cap K(0,1)$ and $\{x_1, x_2, x_3, \ldots\} \in [X]^r \cap K(1,0)$, contradicting (i) or (ii), respectively. Hence such Y exists. □

Lemma 4.17 (Lemma 23.9, [Erd+84])**.** *Let $r \geq 4$, let κ be an infinite cardinal, let $X \subseteq 2^\kappa$ and let $\alpha > 0$ be any ordinal such that* $\operatorname{otp}(X, <^*) = \omega^\alpha$. *Suppose $[X]^r \subseteq K_0$ or $[X]^r \subseteq K_1$. Assume (i) $[X]^r \cap P(0,1) = \varnothing$ or (ii) $[X]^r \cap P(1,0) = \varnothing$. Then there exists $Y \subseteq X$ with* $\operatorname{otp}(Y, <^*) = \omega^\alpha$ *such that $[Y]^r \subseteq P_0$.*

Proof. We first prove a claim.

Claim. *Suppose $x_0 <^* x_1 <^* \ldots <^* x_{s-1}$ are such that*

$$\zeta(\delta_i, \delta_{i+1}) \neq \zeta(\delta_{i+1}, \delta_{i+2}), \tag{4.26}$$

for all $i \leq s-4$. Then $s \leq 4$.

Proof of claim. Suppose $s \geq 5$ and $x_0 <^* x_1 <^* x_2 <^* x_3 <^* x_4$ constitutes a counterexample. If $\zeta(\delta_0, \delta_1) < \zeta(\delta_1, \delta_2) > \zeta(\delta_2, \delta_3)$, then $\delta_0 < \delta_1 > \delta_2 < \delta_3$. Then $\{x_0, x_1, x_2, x_3, \ldots\} \in [X]^r \cap P(0,1)$ or $\{x_1, x_2, x_3, x_4, \ldots\} \in [X]^r \cap P(1,0)$, giving a contradiction with (i) or (ii), respectively.

Similarly, if $\zeta(\delta_0, \delta_1) > \zeta(\delta_1, \delta_2) < \zeta(\delta_2, \delta_3)$, we get $\delta_0 > \delta_1 < \delta_2 > \delta_3$. In this case $\{x_0, x_1, x_2, x_3, \ldots\} \in [X]^r \cap P(1,0)$ or $\{x_1, x_2, x_3, x_4, \ldots\} \in [X]^r \cap P(0,1)$, also giving a contradiction with (ii) or (i), respectively. ■

Now let such $s \leq 4$ be maximal (note $s \geq 3$ always holds) and define $x = x_{s-3}$, $y = x_{s-2}$ and $z = x_{s-1}$. Note that since $[X]^r \subseteq K_0$ or $[X]^r \subseteq K_1$, we have $x \prec y \prec z$ or $x \succ y \succ z$. This gives $\delta(x,y) \neq \delta(y,z)$, hence either (a) $\delta(x,y) > \delta(y,z)$ or (b) $\delta(x,y) < \delta(y,z)$. Then by maximality of s, for all $z \leq^* z_0 <^* z_1$, either

$$\text{(a) } \textbf{not } \delta(x,y) > \delta(y,z_0) < \delta(z_0,z_1), \text{ or} \tag{4.27}$$
$$\text{(b) } \textbf{not } \delta(x,y) < \delta(y,z_0) > \delta(z_0,z_1). \tag{4.28}$$

We show case (a) is impossible. For suppose otherwise, then for all $z_0 \in X$ with $z <^* z_0$ we have

$$\delta(y, z) > \delta(z, z_0) = \delta(y, z_0).$$

Therefore, $\delta(x, y) > \delta(y, z_0)$. Continuing, we obtain by (a) for all $z_1 \in X$ with $z_0 <^* z_1$ that $\delta(y, z_0) > \delta(z_0, z_1) = \delta(y, z_1)$.

Picking an $<^*$-increasing sequence $\langle z_n \mid n < \omega \rangle$ gives us

$$\delta(y, z_0) > \delta(y, z_1) > \delta(y, z_2) > \ldots,$$

which is an infinitely decreasing sequence of ordinals, and hence gives us a contradiction.

So, assume (b) holds. Let $z_0, z_1, z_2 \in X$ be arbitrary such that $z <^* z_0 <^* z_1 <^* z_2$. Then firstly, $\delta(x, y) < \delta(y, z) < \delta(z, z_0)$, hence $\delta(x, y) < \delta(y, z_0) = \delta(y, z)$. As $\delta(x, y) < \delta(y, z_0)$, it must be by (b) that $\delta(x, y) < \delta(y, z_0) < \delta(z_0, z_1)$. Then $\delta(x, z_0) = \delta(x, y)$ and so $\delta(x, z_0) < \delta(z_0, z_1)$. Therefore, in view of the maximality of s,

$$\delta(z_0, z_1) < \delta(z_1, z_2).$$

Define $Y = \{z' \in X \mid z <^* z'\}$, then we showed that $[Y]^r \subseteq P_0$. Also, $(Y, <^*)$ is a non-empty final segment of $(X, <^*)$, and hence by Lemma 4.15 it holds that $\operatorname{otp}(Y, <^*) = \omega^\alpha$. This concludes the proof. □

We are now ready to prove the Negative Stepping Up Lemma where the goal is an additively indecomposable ordinal.

Theorem 4.18 (Negative Stepping Up Lemma, [Erd+84])**.** *Suppose $r \geq 3$ and m is any cardinal. Let $\alpha > 0$ be any ordinal and suppose that κ is an infinite cardinal. Assume $\kappa \not\to (\omega^\alpha)^r_m$. Then $2^\kappa \not\to (\omega^\alpha)^{r+1}_m$.*

Proof. Let $[\kappa]^r = \bigcup_{n<m} I_n$ be the partition witnessing $[\kappa]^r \not\to (\omega^\alpha)^r_m$.

Define a partition $[2^\kappa]^{r+1} = \bigcup_{n<m} J_n$ as follows. For $n \geq 2$, we let $J_n = I^*_n$, where I^*_n is defined as in Lemma 4.14, and

$$J_1 = K(0, 1) \cup P(0, 1) \cup I^*_1,$$

and

$$J_0 = [2^\kappa]^{r+1} \setminus J_1.$$

Note that $J_n \cap P_0 = I^*_n$ for all $n < m$.

Suppose there is $X \subseteq 2^\kappa$ is such that $\operatorname{otp}(X, <^*) = \omega^\alpha$ and $[X]^{r+1} \subseteq J_0$. Then $[X]^{r+1} \cap K(0, 1) = \varnothing$, hence by Lemma 4.16 there is some $Y \subseteq X$ with $\operatorname{otp}(Y, <^*) = \omega^\alpha$ and $[Y]^{r+1} \subseteq K_0$ or $[Y]^{r+1} \subseteq K_1$. By Lemma 4.17, there is $Z \subseteq Y$ with $\operatorname{otp}(Z, <^*) = \omega^\alpha$ and $[Z]^{r+1} \subseteq P_0$. But this means $[Z]^{r+1} \subseteq I^*_0$. Using Lemma 4.14, we find a homogeneous set of order-type ω^α in I_0, which is a contradiction.

Similarly, suppose $X \subseteq 2^\kappa$ such that $\operatorname{otp}(X, <^*) = \omega^\alpha$ and $[X]^{r+1} \subseteq J_1$. Then $[X]^{r+1} \subseteq K \cup K(0, 1)$, and thus $[X]^{r+1} \cap K(1, 0) = \varnothing$. This gives by Lemma 4.16 some $Y \subseteq X$ with $[Y]^{r+1} \subseteq K_0$ or $[Y]^{r+1} \subseteq K_1$ and $\operatorname{otp}(Y, <^*) = \omega^\alpha$. Then $[Y]^{r+1} \subseteq P(0, 1) \cup P_0$, hence $[Y]^{r+1} \cap P(1, 0) = \varnothing$. Thus by Lemma 4.17 there is $Z \subseteq Y$ with $[Z]^{r+1} \subseteq P_0$ and $\operatorname{otp}(Z, <^*) = \omega^\alpha$. Therefore $[Z]^{r+1} \subseteq I^*_1$ and so we find a homogeneous set of order-type ω^α in I_1, a contradiction.

If there is some $X \subseteq 2^\kappa$ with $\operatorname{otp}(X, <^*) = \omega^\alpha$ and $[X]^{r+1} \subseteq J_n = I^*_n$ for $n \geq 2$, then immediately by Lemma 4.14 there is a homogeneous set of order-type ω^α in I_n, which is a contradiction. We conclude $2^\kappa \not\to (\omega^\alpha)^{r+1}_m$. □

Finally, we are able to establish a limitative result for Conjecture 4.12.

Corollary 4.19. *For all $n < \omega$,*

$$\beth_n \not\to (\omega^2)^{n+2}_2. \tag{4.29}$$

Proof. We prove this by induction on n, we can skip $n = 0$ as this case is trivially negative. For $n = 1$ we obtain by Corollary 4.10 the negative relation $\beth_1 \not\to (\omega+2)^3_2$. Hence $\beth_1 \not\to (\omega^2)^3_2$.

For the inductive step, supposing we have $\beth_n \not\to (\omega^2)^{n+2}_2$, we apply Theorem 4.18 to obtain $\beth_{n+1} \not\to (\omega^2)^{n+2+1}_2$. □

In particular, from Corollary 4.19 we can deduce the following theorem.

Theorem 4.20. *For all $n < \omega$,*

$$\beth^+_n \not\to (\omega^2)^{n+3}_2, \tag{4.30}$$

This gives a bound to Conjecture 4.12.

Future work could be directed towards investigating Conjecture 4.12, which states that increasing the goal in the relation $\beth^+_n \to (\omega + n + 1)^r_m$ results in a negative partition relation. It is unknown where the threshold is, but we know that if α_n is the least ordinal such that $\beth^+_n \not\to (\alpha_n)^{n+3}_m$, then $\omega + n + 2 \leq \alpha_n \leq \omega^2$.

We suspect that the bound ω^2 can be decreased. In particular, if the conjecture were proven to be true, then we would have a uniform bound of $\omega \cdot 2$, i.e., for all $n < \omega$, $\beth^+_n \not\to (\omega \cdot 2)^{n+3}_m$.

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