Positive modal logic beyond distributivity: duality, preservation and completeness

MSc Thesis *(Afstudeerscriptie)*

written by

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Contents

Abstract

In this thesis, we study positive (non-distributive) logics and their modal extensions by means of duality theory. Our work is inspired by topological dualities for semilattices and lattices established by Jipsen and Moshier (2014). First we construct a choice-free version of this duality using methods of Bezhanishvili and Holliday (2020). Then we establish a Priestley-like duality based on Jipsen & Moshier duality for arbitrary lattices. We call it Principal upset Priestley (PUP) duality. We define a filter completion of a lattice and, using PUP duality, prove by a Sahlqvist style argument that filter completions preserve all inequalities. That allows us to obtain a purely dual proof of a classical result by Baker and Hales (1974).

We also extend PUP duality by adding modal operators and prove preservation under filter completions for it, thus obtaining a modal version of the Baker and Hales theorem. Furthermore, we show that Sahlqvist-like inequalities correspond to first-order sentences, just as in standard modal logic. We also consider a PUP duality with a non-standard modality nabla, which can be seen as a generalization of the orthocomplementation operation on ortholattices. Therefore, our duality specializes to a duality for ortholattices that turns out to be equivalent to the one constructed by Goldblatt (1975) and Bimbó (2007). Finally, we develop deductive systems reflecting the PUP dualities. We introduce general team semantics for these deductive systems and demonstrate how preservation by filter completions implies completeness for this type of semantics.

CHAPTER 1

Introduction

Modal and superintuitionistic logics are among the most well-studied non-classical logics [11, 8, 34]. One of the central tools in investigating these logics is duality theory. Duality provides a bridge between relational and algebraic semantics, allowing to address problems from different perspectives. In particular, Esakia duality [18, 17] connects Heyting algebras and order-topologial spaces called Esakia spaces. Jónsson-Tarski duality connects modal algebras and modal spaces [32, 33, 8, 8, 11]. Notably, in these dualities the dual space is constructed by considering the set of prime filters of the corresponding algebraic structure.

One of the research directions in modal logic and duality theory is the study of negation and implication free fragment of classical modal logic, started in [15]. This fragment is called positive modal logic and the duality theory for it was developed in [10, 22, 14]. Conspicuously, this duality relies on the fact that the underlying lattice of the algebraic models is distributive, as it builds on the Priestley duality for distributive lattices [40]. It has been a challenge to develop a theory for the non-distributive case and the results obtained so far seem to be quite convoluted: they significantly differ from the distributive dualities and appear harder to work with. While the approach used in [45, 28, 23] considers pairs of filter and ideals, in the recent papers by Jipsen and Moshier [37, 38], as well as their generalization to posets [26], the set of filters of a lattice is considered as its dual space, making the new duality more alike the standard ones. Therefore, we choose this work as a starting point for our approach.

Note that Jipsen and Moshier replace prime filters with arbitrary filters due to the failure of Prime Ideal Theorem for non-distributive lattices. Similarly, the space of proper filters is considered in the study of ortholattices by means of duality theory by Goldblatt [25] and Bimbó [6] (see also [36]). Moreover, Holliday uses the space of proper filters for possibility semantics in [30] as well as in his joint work with N. Bezhanishvili [4], where proper filters are employed as the dual space in order to make the duality choice-free. Thus, the chosen approach to consider the space of all filters has already proven to be fruitful.

The aim of this thesis is to develop a duality theory for non-distributive lattices and their modal versions, building on the Jipsen & Moshier duality. We begin in Chapter 3 by establishing a choice-free variant, using the methods from [4], as well as simplyfing dual semantics of the join operation of Jipsen and Moshier. We also develop a Priestley-like version of the Jipsen & Moshier duality, as it is closer to the dualities used in the distributive setting. We call the obtained duality the Principal upset Priestley (PUP) duality, since we use an approach similar to the one used in the Priestley duality, e.g., an analogue of the Priestley separation axiom, but replace clopen upsets with clopen principal upsets. To be precise, our Priestley separation axiom states that if a point x is not below point y, then they are separated by a clopen principal upset. We also call the spaces dual to the lattices PUP spaces. Then we employ PUP duality because we find it easier to work with and closer to the classical Priestley duality for distributive lattices.

In Chapter 4 we give our first application of the PUP duality. Instead of the standard canonical extensions, we consider the completions of lattices defined by the set of all filters. We call these filter completions. These completions are naturally associated with PUP duality and appear to be more useful in our context. We develop a Sahlqvist style argument showing that filter completions preserve all inequalities. As a by-product, we are able to provide a purely dual proof of a result by Baker and Hales [2], stating that ideal completions preserve all inequalities. Not only this demonstrates the utility of PUP duality, but also opens the door for generalization to the modal case, as we show in Chapter 5.

As our goal is to study positive modal logic via PUP duality, we add two modal operators $\Box$ and $\Diamond$ to lattices and a binary relation to their dual PUP spaces. Building on the duality between modal lattices and their corresponding dual spaces, we restrict ourselves to the serial case, as it simplifies the correspondence between relational and algebraic semantics. Then we extend the Sahlqvist style argument, mentioned above, to the modal case and prove that each modal inequality is preserved by filter completions. Moreover, we obtain a first-order correspondence, this time for the Sahlqvist-like inequalities only. Our work is mostly inspired by the work on modal duality with distributive base such as [10] and [22].

One of the key examples of non-distributive lattices are ortholattices. The duality theory for ortholattices was developed in [25] and [6] and includes an operation of orthocomplement $'$. It turns out that $'$ is a natural example of a nabla modality, as examined among others by Gehrke, Nagahashi and Venema [22] in the distributive setting, and therefore triggers a study of a PUP duality for lattices with nabla modality. We establish such a duality in Chapter 6, similarly to the modal case. The advantage of our approach is that it is general enough to encompass the duality for ortholattices, as we show in Section 6.2. In analogy with the cases of lattices and modal lattices, a natural next step would be to investigate filter completions of nabla lattices, but it turns out that these are not well-defined for this case. Hence, we leave it as an open problem to find a natural completion of a nabla lattice.

We close the thesis in Chapter 7 by developing logical systems for positive modal logics beyond distributivity. Using the duality results we are also able to introduce general team semantics and general modal team semantics which generalize the standard team semantics of Hodges [29] and Väänänen [46]. These semantics are related to [5] and [41]. We develop an analogue of the canonical model in our setting and using the preservation results of previous chapters prove that each positive logic is complete for GT-semantics and each positive modal logic is complete for GMT-semantics.

In summary the main original contributions of this thesis are:

- ⋆ A choice-free version of Jipsen & Moshier duality;
- ⋆ A new duality for lattices, based on Jipsen & Moshier and Priestley dualities;
- ⋆ The preservation of inequalities by filter completions of lattices;
- ⋆ A new duality for a special class of lattices with modal operators $\Box$ and $\Diamond$;
- ⋆ The preservation of modal inequalities by filter completions of modal lattices;
- ⋆ The first-order correspondence for Sahlqvist inequalities;
- ⋆ A new duality for lattices with a modal operator ∇;
- ⋆ Deductive systems for lattices, modal lattices and nabla lattices;
- ⋆ Completeness of positive logic with respect to general team semantics;
- ⋆ Completeness of positive modal logic with respect to general modal team semantics.

CHAPTER 2

Preliminaries

In this chapter we introduce basic notions and conventions used in the thesis. Most of them are standard and well known. For convenience we divide the chapter into themed sections.

2.1 Partially ordered sets

First we briefly recall the basics of order theory. For a detailed exposition, see, e.g., [13].

Definition 2.1.1. A *partial order* on a set P is a binary relation $\leqslant$ that is reflexive, antisymmetric and transitive, i.e., for every $a, b, c \in P$ the following conditions hold:

- ⋆ $a \leqslant a$;
- ⋆ if $a \leqslant b$ and $b \leqslant a$, then $a = b$;
- ⋆ if $a \leqslant b$ and $b \leqslant c$, then $a \leqslant c$.

Definition 2.1.2. A *partially ordered set (poset)* is a set P with a partial order $\leqslant$ on it.

Definition 2.1.3. Let P be a poset and $S \subseteq P$. Then

- ⋆ $a \in P$ is an upper bound of S if for each $b \in S$ we have $b \leqslant a$;
- ⋆ $a \in P$ is a lower bound of S if for each $b \in S$ we have $a \leqslant b$.

Definition 2.1.4. Let P be a poset and $S \subseteq P$. Then

- ⋆ an upper bound a of S is the least upper bound of S if for each upper bound b of S we have $a \leqslant b$;
- ⋆ a lower bound a of S is the greatest lower bound of S if for each lower bound b of S we have $b \leqslant a$.

It is easy to see that for each $S \subseteq P$ if a and b are the least upper bounds of S, then $a = b$, as well as if a and b are the greatest lower bounds of S, then $a = b$.

Definition 2.1.5. Let P be a poset.

(i) A subset $S \subseteq P$ is an *upset* (or a cone) if it is upward closed, i.e., for each $a \in S$ if $a \leqslant b$, then $b \in S$.

(ii) A subset $S \subseteq P$ is a *downset* if it is downward closed, i.e., for each $a \in S$ if $a \geqslant b$, then $b \in S$.

Given a partially ordered set P we can construct a new partially ordered set by "reversing" the order, i.e., $a \leqslant^{\partial} b \Leftrightarrow b \leqslant a$. We denote the new partially ordered set by P^{∂} and call it the order dual of P.

2.2 Lattices and Semilattices

There are two equivalent ways to define lattices and semilattices. Both of them are useful, therefore we review both. Showing the equivalence between them is a standard result that can be found, for example, in [13]. In general, for a thorough introduction to the lattice theory we refer to [27, 7, 13].

We consider solely bounded lattices and semilattices with a unit, hence in this thesis "lattice" and "semilattice" always mean bounded lattices and semilattices with a unit. Therefore, we define lattices and semilattices to be bounded from the start. Note that this is not a serious restriction, as every lattice or semilattice can be turned into a bounded one by adding top and bottom elements (or only one in case of a semilattice).

Definition 2.2.1. A *(bounded) lattice* is an algebraic structure $(L, \wedge, \vee, 1, 0)$ satisfying, for all $a, b, c \in L$ the following conditions:

1. $a \wedge a = a$ and $a \vee a = a$,
2. $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$,
3. $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ and $a \vee (b \vee c) = (a \vee b) \vee c$,
4. $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$,
5. $a \wedge 1 = a$ and $a \vee 0 = a$.

Given a lattice L, we define the binary relation $\leqslant$ on it by $a \leqslant b$ if $a \wedge b = a$ or equivalently $a \vee b = b$. Then $\leqslant$ is a partial order on L. It is easy to see that for every $a, b \in L$ the element $a \wedge b$ is the greatest lower bound of a and b and the element $a \vee b$ is the least upper bound of a and b. The elements 1 and 0 turn out to be the greatest and the least elements of L respectively. This observation gives rise to the equivalent definition of a lattice.

Definition 2.2.2. Let L be a poset. Then L is a *(bounded) lattice* if

- ⋆ for every $a, b \in L$ there exist the greatest lower bound of a and b (denoted by $a \wedge b$) and the least upper bound of a and b (denoted by $a \vee b$),
- ⋆ there exist the greatest element 1 and the least element 0.

Then $(L, \wedge, \vee, 1, 0)$ constitutes a lattice in the sense of Definition 2.2.1 and these two definitions describe the same class of structures. We now move to semilattices. Informally speaking, a semilattice is a lattice that has only one operation: a meet $\wedge$ or a join $\vee$. Due to the symmetry, it actually does not matter which one exactly to pick. The only difference appears when we define the order on it and need to choose whether the operation is producing the greatest lower bound or the least upper bound. In this work we consider solely meet-semilattices, but all the results can be generalized to join-semilattices by taking the order duals.

Definition 2.2.3. A *(bounded) semilattice* is an algebraic structure $(L, *, e)$ satisfying, for all $a, b, c \in L$ the following conditions:

1. $a * a = a$,
2. $a * b = b * a$,
3. $a * (b * c) = (a * b) * c$,
4. $a * e = a$.

The element e is called a unit.

There are two ways to define an order on a semilattice L. We can say $a \leqslant_\wedge b$ if $a * b = a$. Then L is called a *meet-semilattice*, since $*$ becomes the meet (or infimum) $\wedge$ of $(L, \leqslant_\wedge)$. On the other hand, we can say $a \leqslant_\vee b$ if $a * b = b$. Then L is called a *join-semilattice*, since $*$ becomes the join (or supremum) $\vee$ of $(L, \leqslant_\vee)$. We can also define meet-semilattices and join-semilattices in the following way.

Definition 2.2.4. Let $(L, \leqslant)$ be a partially ordered set. Then L is a *(bounded) meet-semilattice* if

- ⋆ for every $a, b \in L$ there exists the greatest lower bound of a and b (denoted by $a \wedge b$),
- ⋆ there exists the greatest element 1.

Definition 2.2.5. Let $(L, \leqslant)$ be a partially ordered set. Then L is a *(bounded) join-semilattice* if

- ⋆ for every $a, b \in L$ there exists the least upper bound of a and b (denoted by $a \vee b$),
- ⋆ there exists the least element 0.

Note that given these definitions, a lattice can be defined as a partial order that is both a meet-semilattice and a join-semilattice. Using the algebraic approach, we can define a lattice as a semilattice with respect to an operation $\wedge$ and a semilattice with respect to an operation $\vee$ satisfying the so-called absorption laws $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$, that connect $\wedge$ and $\vee$.

An important property of lattices and semilattices is completeness.

Definition 2.2.6. A lattice L is *complete* if every subset $S \subseteq L$ has the greatest lower bound of S, denoted by $\bigwedge S$, and the least upper bound of S, denoted by $\bigvee S$.

It is easy to check that one can express arbitrary meets in terms of arbitrary joins and vice versa. Then for a lattice to be complete it suffices to either have the greatest lower bound for all subsets S or have the least upper bound for all subsets S. This observation establishes the equivalence between complete lattices and complete semilattices.

Definition 2.2.7.

(i) A meet-semilattice L is *complete* if every subset $S \subseteq L$ has the greatest lower bound of S, denoted by $\bigwedge S$.

(ii) A join-semilattice L is *complete* if every subset $S \subseteq L$ has the least upper bound of S, denoted by $\bigvee S$.

As we mentioned above, if L is a complete meet-semilattice or a complete join-semilattice, it is also a complete lattice.

We also consider finite lattices and semilattices. In this case the situation becomes even simpler. Let L be a finite semilattice and assume for convenience it is a finite meet-semilattice. Then having binary meets implies having arbitrary finite non-empty meets. Moreover, since L has the greatest element, the empty set also has a meet. Therefore, since L is finite, it has arbitrary meets and is a complete lattice. So every finite semilattice is a complete lattice.

Now we define the categories of lattices and semilattices.

Definition 2.2.8. Let L and M be lattices. A map $f : L \to M$ is a *lattice morphism* if it preserves operations and the bounds. To be precise, for every $a, b \in L$:

$f(a \wedge b) = f(a) \wedge f(b)$ and $f(a \vee b) = f(a) \vee f(b)$,

$f(1) = 1$ and $f(0) = 0$.

Definition 2.2.9. Let L and M be semilattices. A map $f : L \to M$ is a *semilattice morphism* if it preserves the operation and the unit. To be precise, for every $a, b \in L$:

$$f(a * b) = f(a) * f(b),$$

$$f(e) = e.$$

It is not hard to show that lattices with lattice morphisms form a category, that we denote by Lat, and semilattices with semilattice morphisms form a category, that we denote by SLat.

The last topic we would like to discuss is the construction of the lattice of filters. First we define filters on meet-semilattices.

Definition 2.2.10. Let L be a meet-semilattice. A non-empty subset $F \subseteq L$ is a *filter* if

- $\star$ for every $a \in F$, if $a \leqslant b$, then $b \in F$;
- $\star$ for every $a, b \in F$ we have $a \wedge b \in F$.

In other words, F is non-empty, upward closed and meet closed.

For every family $\{F_i : i \in I\}$ of filters, the set $\bigcap_{i \in I} F_i$ is also a filter. Note that in case the family is empty, we obtain the filter that coincides with the whole meet-semilattice L. Therefore, the set of all filters forms a complete meet-semilattice with respect to the meet $\cap$ and order $\subseteq$. Hence, it is also a complete lattice. It is easy to show that the join is defined on the set of filters by

$$F \vee G = \{a \in L \mid \exists\, b \in F, c \in G : a \geqslant b \wedge c\}.$$

This observation will play a central role in our work.

We also use the notion of ideal, which is dual to the notion of filter.

Definition 2.2.11. Let L be a join-semilattice. A non-empty subset $F \subseteq L$ is an *ideal* if

- ⋆ for every $a \in F$, if $a \geqslant b$, then $b \in F$;
- ⋆ for every $a, b \in F$ we have $a \vee b \in F$.

In other words, F is non-empty, downward closed and join closed.

By dual arguments, $I(L)$ with the inclusion order also forms a complete lattice. Moreover, for a lattice L we have $I(L) \cong F(L^\partial)$.

When L is not just a semilattice, but a lattice or even a complete lattice, there are important kinds of filters that we want to consider.

Definition 2.2.12. Let L be a lattice. A filter $F \subseteq L$ is *prime* if $F \neq L$ and for every $a, b \in L$, if $a \vee b \in F$, then either $a \in F$ or $b \in F$.

Definition 2.2.13. Let L be a complete lattice. A filter $F \subseteq L$ is *completely prime* if $F \neq L$ and for every family $\{a_i \in L : i \in I\}$, if $\bigvee_{i \in I} a_i \in F$, then there exists $i \in I$ such that $a_i \in F$.

2.3 Topological spaces

Besides lattices, we consider various topological spaces. For more details on basic concepts and facts of general topology we refer to [16]. In this section we define the notion of topological spaces and their particular properties.

Definition 2.3.1. A *topological space* is a set X, with a collection τ of subsets of X, such that

1. $\varnothing, X \in \tau$;
2. $U, V \in \tau \Rightarrow U \cap V \in \tau$;
3. for every family $\{U_i : i \in I\} \subseteq \tau$, we have $\bigcup_{i \in I} U_i \in \tau$.

The collection τ is called the *topology* of X. The elements of τ are called the *open* sets of X.

Note that a topology τ, when ordered by the inclusion relation $\subseteq$, forms a complete lattice. Given a set X there are two main ways to define a topology on it: using bases and using subbases.

Definition 2.3.2. A *base* for the topology τ of a topological space X is a family $B \subseteq \tau$ such that for each $U \in \tau$ there exists a subfamily $B_0 \subseteq B$ with $U = \bigcup B_0$.

Often instead of looking at bases of a given topology, we rather introduce a topology from its base. First note that whenever X is a topological space and B is its base, we have $\bigcup B = X$ and for each $U_1, U_2 \in B$ and $x \in U_1 \cap U_2$, there exists $U_3 \in B$ such that $x \in U_3 \subseteq U_1 \cap U_2$. Now let X be just some set. Let B be a collection of subsets of X such that $\bigcup B = X$ and for each $U_1, U_2 \in B$ and $x \in U_1 \cap U_2$, there exists $U_3 \in B$ such that $x \in U_3 \subseteq U_1 \cap U_2$. Then there is a unique topology τ on X such that B is a base for τ. We call this topology *generated* by B.

Definition 2.3.3. A *subbase* for the topology τ of a topological space X is a family $S \subseteq \tau$ such that each $U \in \tau$ can be written as a union of finite intersections of elements of S. Note that this includes an empty intersection, that is equal to X.

Once again, we can generate a topology on X by considering some family $S \subseteq \tau$ as its subbase. However, there are no restrictions on the set S. So for an arbitrary $S \subseteq \tau$ we call the smallest topology τ containing S the topology *generated* by S. Then S is the subbase for τ.

Next basic notions in topology that we define are closed sets, clopen sets and compact sets.

Definition 2.3.4. Let X be a topological space. A subset $C \subseteq X$ is

(i) *closed* if the subset $X \setminus C$ is open;

(ii) *clopen* if it is at the same time closed and open;

(iii) *compact*, whenever for every family of open sets $\{U_i : i \in I\}$, if $C \subseteq \bigcup_{i \in I} U_i$, then there exists a finite subset $J \subseteq I$ such that $C \subseteq \bigcup_{j \in J} U_j$.

We also call X itself compact if it is compact as a subset of X.

It is easy to prove that when X is compact, each closed subset of X is compact. We will use this result in the next chapters.

We now define the subspace topology. Let X be a topological space and $S \subseteq X$. We turn S into a topological space by putting $\tau_S = \{S \cap U \mid U \in \tau\}$, where τ is the topology on X. Then τ_S constitutes a topology on S, known as the *subspace topology*.

Now we move to more specific notions. We follow here the exposition from [37]. Let x be a point in a topological space X with topology τ. We denote $N(x) = \{U \in \tau : x \in U\}$. Then $N(x)$ is a completely prime filter in the lattice τ of open sets.

Definition 2.3.5. A topological space X is a *Kolmogorov space* or T_0 *space* if for every pair of distinct points of X, at least one of them belongs to an open set not containing the other. Equivalently, a topological space X is a Kolmogorov space if the map $x \mapsto N(x)$ is injective.

For a topological space X we define a binary relation $\sqsubseteq$ on X by $x \sqsubseteq y$ if $N(x) \subseteq N(y)$. Then $\sqsubseteq$ is reflexive and transitive. Moreover, when X is a Kolmogorov space, $\sqsubseteq$ becomes antisymmetric and therefore is a partial order on X. We call it the *specialization order*.

Definition 2.3.6. A topological space X is *sober* if the map $x \mapsto N(x)$ is a bijection between X and the collection of completely prime filters in the lattice of open sets. Therefore, every sober space is a Kolmogorov space.

We also define how to go from a partial order to a topological space. For that we first need to explain what a directed set is.

Definition 2.3.7. Let P be a partially ordered set. A subset $D \subseteq P$ is a *directed set* if for each $a, b \in D$ there exists $c \in D$ such that $a, b \leqslant c$.

Definition 2.3.8. Let P be a partially ordered set. The *Scott topology* on P is defined by saying that open sets are upsets U that are inaccessible by directed joins, i.e., if $\bigvee D$ exists for a directed set D and $\bigvee D \in U$, then $D \cap U \neq \varnothing$.

Let P be a partially ordered set and consider Scott topology on it. Then the specialization order of the obtained topological space coincides with the initial order on P.

In the following chapters we mostly work with lattices and topological spaces that also have a lattice structure. In order to easily differentiate between them, we use different notation.

Let L be a lattice or a semilattice and $a \in L$. We denote by $\uparrow a$ the upset $\{b \in L : a \leqslant b\}$.

Let X be an ordered topological space whose underlying order is a lattice. We tend to denote this order by $\sqsubseteq$, just as we did with the specialization order. For $x \in X$ we denote by $\Uparrow x$ the upset $\{y \in X : x \sqsubseteq y\}$.

2.4 Duality

A big part of our work is devoted to establishing various dualities between classes of lattices and classes of ordered topological spaces. In this section we discuss the definition of a duality and the way we approach it consequently in this thesis. For the background on category theory, see [1] and [35].

Definition 2.4.1. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A *functor* F from $\mathcal{C}$ to $\mathcal{D}$ maps each object C of $\mathcal{C}$ to an object $F(C)$ of $\mathcal{D}$ and each morphism $f : C_1 \to C_2$ of $\mathcal{C}$ to a morphism $F(f) : F(C_1) \to F(C_2)$ of $\mathcal{D}$, so that $F(id_C) = id_{F(C)}$ for each object C of $\mathcal{C}$ and $F(g \circ f) = F(g) \circ F(f)$ for each morphisms $f : C_1 \to C_2$ and $g : C_2 \to C_3$ of $\mathcal{C}$.

Definition 2.4.2. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A *contravariant functor* F from $\mathcal{C}$ to $\mathcal{D}$ maps each object C of $\mathcal{C}$ to an object $F(C)$ of $\mathcal{D}$ and each morphism $f : C_1 \to C_2$ of $\mathcal{C}$ to a morphism $F(f) : F(C_2) \to F(C_1)$ of $\mathcal{D}$, so that $F(id_C) = id_{F(C)}$ for each object C of $\mathcal{C}$ and $F(g \circ f) = F(f) \circ F(g)$ for each morphisms $f : C_1 \to C_2$ and $g : C_2 \to C_3$ of $\mathcal{C}$. Here id_c denotes the identity morphism.

Let $\mathcal{C}$ be a category. Recall that an *identity functor* $\mathbf{I}_{\mathcal{C}}$ from $\mathcal{C}$ to $\mathcal{C}$ is the functor identical on objects and morphisms.

Definition 2.4.3. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. They are said to be *dually equivalent* if there are contravariant functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\varepsilon : FG \to \mathbf{I}_{\mathcal{D}}$ and $\eta : \mathbf{I}_{\mathcal{C}} \to GF$.

In our case, the strategy of proving that categories $\mathcal{C}$ and $\mathcal{D}$ are dually equivalent is going to be as follows. First we define the object part of the contravariant functors, i.e., a map F_0 assigning for each object of $\mathcal{C}$ an object of $\mathcal{D}$ and a map G_0 doing the other way around. Then we construct an isomorphism ε that shows $F_0(G_0(D)) \cong D$ for each object D of $\mathcal{D}$ and an isomorphism η that shows $G_0(F_0(C)) \cong C$ for each object C of $\mathcal{C}$.

We then move to defining the morphism part of the contravariant functors, i.e., a map F_1 assigning for each morphism of $\mathcal{C}$ a reversed morphism of $\mathcal{D}$ and a map G_1 doing vice versa. Then usually it is easy to see that F and G are contravariant functors. Hence, it suffices to demonstrate that ε and η are natural transformations to establish a duality.

One of the most important duality results for our work is the duality for distributive lattices. We are interested in two representations of distributive lattices: via spectral spaces and via Priestley spaces. The first one was established in [44] and the second in [40] and [39] (see also [13, Chapter 11]).

We briefly recall the main definitions and constructions and then explain the connection to our work.

Definition 2.4.4. A lattice L is *distributive* if for every $a, b, c \in L$ we have $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

Equivalently, a lattice L is distributive if for every $a, b, c \in L$ we have $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$. Moreover, the inequalities $a \wedge (b \vee c) \geqslant (a \wedge b) \vee (a \wedge c)$ and $a \vee (b \wedge c) \leqslant (a \vee b) \wedge (a \vee c)$ hold in every lattice. Therefore, a lattice L is distributive if for every $a, b, c \in L$ we have $a \wedge (b \vee c) \leqslant (a \wedge b) \vee (a \wedge c)$, or equivalently if $a \vee (b \wedge c) \geqslant (a \vee b) \wedge (a \vee c)$.

Definition 2.4.5. A *spectral space* is a sober space in which the compact open sets form a base that is closed under finite (including empty) intersections. Note that in case of empty intersection we get that X is compact.

Given a distributive lattice L, let X denote the set of prime filters of L. For each $a \in L$, let $\phi(a) = \{F \in X : a \in F\}$ and generate topology on X with $\phi(a)$. Then X is a spectral space.

On the other hand, for a spectral space X the set of its compact open subsets form a distributive lattice. Then one can extend these correspondences to a full duality in the same fashion as described above, as established by Stone in [44].

The approach suggested by Priestley deals with another class of ordered topological spaces.

Definition 2.4.6. Let X be an ordered topological space with the partial order $\leqslant$ on it. Then X is a *Priestley space* X if X is compact and if $x \nleqslant y$, then there exists a clopen upset U such that $x \in U$ and $y \notin U$.

The second condition is usually called the Priestley separation axiom. As before, for a distributive lattice L we take X to be the set of prime filters of L. But now we generate the topology on X by sets of the forms $\phi(a)$ and $X \setminus \phi(a)$. Topology defined in such a way is usually called a patch topology. Then taking as order the set-theoretic inclusion $\subseteq$, the space X becomes a Priestley space. To go in the other direction, consider an arbitrary Priestley space X. Then the collection of clopen upsets of X forms a distributive lattice and we can extend this construction to a full duality.

Moreover, given a spectral space X we can consider the corresponding Priestley space by defining the patch topology, i.e., generating a topology with compact open sets and their complements, and as the order taking the specialisation order. Then the obtained Priestley space and X are dual to the same distributive lattice. Conversely, given a Priestley space we can take open upsets and get the corresponding spectral topology, which is dual to the same distributive lattice. For details see [12] and [3].

In the next chapter we first discuss a duality for lattices (not necessarily distributive) established by Jipsend and Moshier in [37]. Jipsen & Moshier duality follows rather the Stone approach used for spectral space duality. On the other hand, we modify their duality to a Priestley-like version, which we find easier to work with.

CHAPTER 3

Duality for lattices

There are several dualities for lattices, by Urquhart [45] via doubly ordered topological space, by Hartung [28] via topological contexts, more recently by Gehrke and van Gool [23] via polarities and by Jipsen and Moshier [37] via the spectra of filters. The work in this thesis is based on the Jipsen and Moshier duality, since in our view it resembles the most the dualities used in modal and superintuitionistic logics. Hence, we first recall this duality for lattices and semilattices. Next we use the approach from [4] to obtain a choice-free version of this duality. We also show how to simplify the join operation used in [37]. Finally, we introduce a new topological duality for lattices, inspired by Jipsen & Moshier duality and Priestley duality for distributive lattices.

3.1 Jipsen and Moshier duality

We start by briefly recalling the Jipsen & Mosher duality. A more detailed exposition can be found in [37, Sections 2, 3].

Let X be a topological space. A subset of X is called *saturated* if it is upward closed with respect to the specialization order $\sqsubseteq$. Note that we use $\sqsubseteq$ for specialization order solely in this section. A subset F of X is called a *filter* when

(i) F is non-empty,

(ii) if $x \in F$ and $x \sqsubseteq y$, then $y \in F$,

(iii) if $x, y \in F$, then there exists $z \in F$, such that $z \sqsubseteq x, y$.

We consider the following collections of the subsets of X:

$K(X)$ is the set of compact saturated subsets,

$O(X)$ is the set of open subsets,

$F(X)$ is the set of filters.

For the intersection of the collections we use concatenation of the letters above, e.g., $KOF(X)$ is the collection of compact open filters.

First we define the spaces that will become the duals of semilattices and lattices according to the Jipsen & Moshier duality.

Definition 3.1.1. A topological space X is an *HMS space* if it is sober and $KOF(X)$ is closed under finite (including empty) intersection while also forming a base.

A subset of an HMS space is *F-saturated* if it is an intersection of open sets. Let $FSat(X)$ denote the collection of F-saturated subsets of X. Let $\mathcal{S} \subseteq FSat(X)$. Define $\bigwedge \mathcal{S} = \bigcap \mathcal{S}$ and $\bigvee \mathcal{S} = \bigcap \{F \in OF(X) \mid \bigcup \mathcal{S} \subseteq F\}$. Then $FSat(X)$ is a complete lattice with respect to $\bigwedge$ and $\bigvee$.

Definition 3.1.2. An HMS space X is a *JM space* if $KOF(X)$ forms a sublattice of $FSat(X)$.

In the original paper by Jipsen and Moshier JM spaces are caled BL spaces after bounded lattices. However, we find naming them after Jipsen and Moshier more appropriate.

Proposition 3.1.3.

(i) For every HMS space X, the collection $KOF(X)$ is a meet-semilattice with a meet operation given by the intersection.

(ii) For every JM space X, the collection $KOF(X)$ is a lattice with a meet defined as the intersection and a join defined in the same way as for $FSat(X)$, that is,

$$U \vee V = \bigcap \{F \in OF(X) \mid U \cup V \subseteq F\}.$$

In Section 3.3 we will simplify this semantics of $\vee$.

For a meet-semilattice L, we denote by $Filt(L)$ the space of the filters on L with the Scott topology. It can be shown that in this case specialization order coincides with the standard inclusion order on $Filt(L)$.

Proposition 3.1.4.

(i) For every meet-semilattice L, $Filt(L)$ is an HMS space.

(ii) For every lattice L, $Filt(L)$ is a JM space.

For a meet-semilattice L let $\phi : L \to KOF(Filt(L))$ be defined as

$$\phi(a) = \{F \in Filt(L) \mid a \in F\}.$$

For an HMS space X let $\psi : X \to Filt(KOF(X))$ be defined as

$$\psi(x) = \{U \in KOF(X) \mid x \in U\}.$$

Theorem 3.1.5.

(i) For every meet-semilattice L the map ϕ is the isomorphism between L and $KOF(Filt(L))$.

(ii) For every HMS space X the map ψ is the homeomorphism between X and $Filt(KOF(X))$.

(iii) These constructions restrict to lattices and JM spaces. That is, for a lattice L the map ϕ is also the isomorphism between L and $KOF(Filt(L))$ and for a JM space X the map ψ is the homeomorphism between X and $Filt(KOF(X))$.

The morphism part of the Jipsen & Moshier duality is obtained as follows.

Definition 3.1.6. A map $f : X \to Y$ between HMS spaces is *F-continuous* if for every $U \in KOF(Y)$, the set $f^{-1}[U]$ belongs to $KOF(X)$.

Definition 3.1.7. An F-continuous map $f : X \to Y$ between JM spaces is *F-stable* if for every finite family $\{U_i : i \in I\} \subseteq KOF(Y)$, we have $f^{-1}[\bigvee_{i \in I} U_i] = \bigvee_{i \in I} f^{-1}[U_i]$.

Proposition 3.1.8.

(i) Let $f : L \to M$ be a meet-semilattice homomorphism. Then the map $f^* : Filt(M) \to Filt(L)$ defined by $F \mapsto f^{-1}[F]$ is F-continuous.

(ii) Let $f : L \to M$ be a lattice homomorphism. Then the map $f^* : Filt(M) \to Filt(L)$ defined by $F \mapsto f^{-1}[F]$ is F-stable.

Proposition 3.1.9.

(i) Let $f : X \to Y$ be an F-continuous function. Then the map $f_* : KOF(Y) \to KOF(X)$ defined by $U \mapsto f^{-1}[U]$ is a meet-semilattice homomorphism.

(ii) Let $f : X \to Y$ be an F-stable function. Then the map $f_* : KOF(Y) \to KOF(X)$ defined by $U \mapsto f^{-1}[U]$ is a lattice homomorphism.

Let HMS be the category of HMS spaces with F-continuous functions and let JM be the category of JM spaces with F-stable functions.

Theorem 3.1.10.

(i) HMS *is dually equivalent to* SLat.

(ii) JM *is dually equiavalent to* Lat.

3.2 Choice-free duality

In order to establish the dualities above, Jipsen and Moshier use the well-known duality between distributive lattices and spectral spaces [44]. This duality requires the Prime Ideal Theorem, a property weaker than the Axiom of Choice but still not deducible from ZF. Our first goal is to modify the duality to a completely choice-free version using the same approach as in [4], that is, by adding a sobriety-like condition to HMS spaces.

For a topological space X and a point $x \in X$, we let

$$\psi(x) = \{U \in KOF(X) \mid x \in U\}.$$

Definition 3.2.1. A topological space X is a *choice-free HMS space* if it is sober, $KOF(X)$ forms a base that is closed under finite (including empty) intersections and every filter in $(KOF(X), \subseteq)$ is of the form $\psi(x)$ for some $x \in X$.

Let X be a choice-free HMS space and $\mathcal{S}$ a subset of $KOF(X)$. Then let

$$\bigvee \mathcal{S} = \bigcap \{U \in OF(X) \mid \bigcup \mathcal{S} \subseteq U\}.$$

Definition 3.2.2. A choice-free HMS space X is a *choice-free JM space* if for every finite (including empty) $\mathcal{S} \subseteq KOF(X)$ the set $\bigvee \mathcal{S}$ also belongs to $KOF(X)$.

Proposition 3.2.3.

(i) For every choice-free HMS space X, the collection $KOF(X)$ is a meet-semilattice with a meet operation given by the intersection.

(ii) For every choice-free JM space X, the collection $KOF(X)$ is a lattice with a meet defined as the intersection and a join defined by $\bigvee$.

Proof. Since $KOF(X)$ is closed under finite intersections, $KOF(X)$ is a meet-semilattice. If X is a JM space, then $KOF(X)$ has joins and is bounded by the definition of a JM space. $\boxtimes$

Consider a meet-semilattice L and $Filt(L)$ the set of filters on L. Let

$$\phi(a) = \{F \in Filt(L) \mid a \in F\}.$$

We generate the topology on $Filt(L)$ by the sets $\phi(a)$. As it is easy to check that $\phi(a \wedge b) = \phi(a) \cap \phi(b)$ and every $F \in Filt(L)$ belongs to some $\phi(a)$, the set $\{\phi(a) : a \in L\}$ forms a base of this topology. We denote by X_L the obtained topological space.

Here we use the standard Stone topology on a space of filters (see [43] and [13]). On the other hand, Jipsen and Moshier work with Scott topology instead. We will next show that these two definitions give the same topology and employ the Stone one as a more standard one.

Lemma 3.2.4. *Let L be a meet-semilattice and $Filt(L)$ the set of filters on L. Then $\{\phi(a) : a \in L\}$ is a base of the Scott topology on $(Filt(L), \subseteq)$. Therefore, the Scott topology coincides with the topology generated by $\{\phi(a) : a \in L\}$.*

Proof. Consider the Scott topology on $Filt(L)$. Take an open set U. Let $D = \{a \in L \mid {\uparrow}a \in U\}$. We show that $U = \bigcup_{a \in D} \phi(a)$.

($\subseteq$) Let $F \in U$. Then F is a directed join $\bigcup_{a \in F} {\uparrow}a$. Therefore, there exists $a \in F$, such that ${\uparrow}a \in U$. That means $a \in D$ and $F \in \bigcup_{a \in D} \phi(a)$.

($\supseteq$) Suppose $F \in \phi(a)$ for some $a \in D$. Then ${\uparrow}a \subseteq F$. Since ${\uparrow}a \in U$ and U is upward closed, $F \in U$. $\boxtimes$

Now we also know that the specialization order on X_L coincides with the standard inclusion order on $Filt(L)$ and therefore filters on X_L are the filters in the usual sense.

Proposition 3.2.5.

(i) For every meet-semilattice L, X_L is a choice-free HMS space.

(ii) For every lattice L, X_L is a choice-free JM space.

Proof. First we want to show that $KOF(X_L) = \{\phi(a) : a \in L\}$.

($\supseteq$) By construction of the topology, all $\phi(a)$ are open. It is also easy to see that every $\phi(a)$ is a filter. To see that $\phi(a)$ is compact, suppose $\phi(a) \subseteq \bigcup U_i$ for some collection of open sets U_i. We can write every U_i as a union of the elements of the base and obtain $\phi(a) \subseteq \bigcup_{b \in D} \phi(b)$ for some $D \subseteq L$. Then there is a particular $\phi(b)$ such that ${\uparrow}a \in \phi(b)$. That yields $a \leqslant b$ and therefore $\phi(a) \subseteq \phi(b)$. Hence, every $\phi(a)$ belongs to $KOF(X_L)$.

($\subseteq$) For the other part we need to show that every compact open filter U of X_L is of the form $\phi(a)$. Since U is open, it is a union of elements of the form $\phi(a)$ and since it is compact, this union can be made finite. Therefore, it suffices to show that if $\phi(b) \cup \phi(c)$ is a compact open filter, then there is $a \in L$ such that $\phi(b) \cup \phi(c) = \phi(a)$. Since $\phi(b) \cup \phi(c)$ is a filter and $\uparrow b, \uparrow c$ belong to $\phi(b) \cup \phi(c)$, we have $\uparrow b \cap \uparrow c \in \phi(b) \cup \phi(c)$. Without loss of generality we may assume $\uparrow b \cap \uparrow c \in \phi(b)$. That yields $b \in \uparrow b \cap \uparrow c$ and $c \leqslant b$. Therefore, $\phi(b) \cup \phi(c) = \phi(b)$ and every compact open filter U of X_L is of the form $\phi(a)$.

Next we show that X_L is an HMS space. To prove sobriety, consider a completely prime filter $\mathcal{F}$ on the lattice of open sets $O(X_L)$. Let F be a filter on L, generated by $S = \{a \in L \mid \phi(a) \in \mathcal{F}\}$, that is,

$$F = \bigcup_{a_0, \dots, a_n \in S} \uparrow(a_0 \wedge \dots \wedge a_n).$$

We show that $\mathcal{F} = O(F)$, where $O(F) = \{U \in O(X_L) \mid F \in U\}$, concluding that X_L is sober.

($\subseteq$) Take $U \in \mathcal{F}$. Then $U = \bigcup_{a \in D} \phi(a)$ for some $D \subseteq L$. Since $\mathcal{F}$ is completely prime, there exists $a \in D$ such that $\phi(a) \in \mathcal{F}$ and therefore $a \in S$. By definition of F, $\uparrow a \subseteq F$. At the same time we have $\uparrow a \in U$, so F is also an element of U and $U \in O(F)$.

($\supseteq$) Take an open U, such that $F \in U$. Since $U = \bigcup_{a \in D} \phi(a)$ for some $D \subseteq L$, there exists $a \in D$ such that $F \in \phi(a)$. That means $a \in F$, so there are $a_0, \dots, a_n \in S$ such that $a \in \uparrow(a_0 \wedge \dots \wedge a_n)$. Hence,

$$\phi(a) \supseteq \phi(a_0 \wedge \dots \wedge a_n) = \phi(a_0) \wedge \dots \wedge \phi(a_n) \in \mathcal{F}.$$

So $U \in \mathcal{F}$.

The collection $KOF(X_L)$ is a base, since it coincides with $\{\phi(a) : a \in L\}$, which is a base by definition. It is closed under binary intersections, since $\phi(a) \wedge \phi(b) = \phi(a \wedge b)$. It also has a top element $L = \phi(1)$.

To conclude the semilattice part, we prove that every filter $\mathcal{F}$ on $KOF(X_L)$ is of the form $\psi(F)$ for some $F \in X_L$. Let $F = \{a \in L \mid \phi(a) \in \mathcal{F}\}$. F is upward closed, since $\mathcal{F}$ is upward closed. F is a filter, since $\phi(a) \cap \phi(b) = \phi(a \wedge b)$ and $\mathcal{F}$ is a filter. Then the equality $KOF(X_L) = \{\phi(a) : a \in L\}$ implies the equality $\mathcal{F} = \psi(F)$ as follows.

($\subseteq$) Let $U \in \mathcal{F}$. Then there exists $a \in L$ such that $U = \phi(a)$. Therefore, $a \in F$ and $U = \phi(a) \in \psi(F)$.

($\supseteq$) Let $U \in \psi(F)$. Then $F \in U$ and also there exists $a \in L$ such that $U = \phi(a)$. Hence, $a \in F$ and $U = \phi(a) \in \mathcal{F}$.

Finally, suppose L is a lattice. Then X_L has the least element $\{1\}$, where 1 is the top element of L. Now in order to show that X_L is a choice-free JM space, it suffices to show that $\phi(a \vee b)$ is equal to the join of $\phi(a)$ and $\phi(b)$, that is

$$\phi(a \vee b) = \bigcap \{U \in OF(X_L) \mid \phi(a) \cup \phi(b) \subseteq U\}.$$

($\subseteq$) Take $F \in \phi(a \vee b)$ and a set $U \in OF(Filt(M))$ such that $\phi(a) \cup \phi(b) \subseteq U$. Then $\uparrow a$ and $\uparrow b$ belong to U and since U is a filter, $\uparrow a \cap \uparrow b = \uparrow(a \vee b)$ also belongs to U. On the other hand, $\uparrow(a \vee b) \subseteq F$, so $F \in U$.

($\supseteq$) Note that $\phi(a \vee b)$ is an open filter and $\phi(a) \cup \phi(b) \subseteq \phi(a \vee b)$. Hence, $\phi(a \vee b)$ is one of the U on the right side of the equation and the inclusion holds.

$\boxtimes$

Now we are ready to prove a choice-free analogue of Theorem 3.1.5.

Theorem 3.2.6.

(i) For every meet-semilattice L the map ϕ is the isomorphism between L and $KOF(X_L)$.

(ii) For every HMS space X the map ψ is the homeomorphism between X and $X_{KOF(X)}$.

(iii) These constructions restrict to lattices and JM spaces. That is, for a lattice L the map ϕ is also the isomorphism between L and $KOF(X_L)$ and for a JM space X the map ψ is the homeomorphism between X and $X_{KOF(X)}$.

Proof. (i) Since we showed $KOF(X_L) = \{\phi(a) : a \in L\}$, we already know that ϕ is surjective. Now we show that ϕ is injective. Suppose $a \neq b$. Then either $a \not\leq b$ or $b \not\leq a$. Without loss of generality, let $a \not\leq b$. Then the filter $\uparrow a$ belongs to $\phi(a)$ but not to $\phi(b)$. Therefore, ϕ is injective.

We already noted that $\phi(a \wedge b) = \phi(a) \cap \phi(b)$. Moreover, in case L is a lattice, as we proved in the previous proposition, $\phi(a \vee b) = \phi(a) \vee \phi(b)$. Hence, ϕ is an isomorphism.

(ii) By the definition of a choice-free HMS space the map ψ is surjective. To show injectivity, suppose $x \neq y$. Since X is sober, there exists an open U such that x belongs to U but y does not. Then using that $KOF(X)$ forms a base, we obtain $\psi(x) \neq \psi(y)$. Hence, ψ is injective.

Now we prove that ψ is open. It suffices to show that the image of every $U \in KOF(X)$, i.e., $\psi[U] = \{\psi(x) \mid x \in U\}$ is open in $X_{KOF(X)}$. Consider the map $\phi : KOF(X) \to KOF(X_{KOF(X)})$. We claim that $\psi[U] = \phi(U)$ and therefore $\psi[U]$ is open in $X_{KOF(X)}$.

($\subseteq$) Take $\psi(x)$ for some $x \in U$. Then $\psi(x) \in \phi(U) \Leftrightarrow U \in \psi(x) \Leftrightarrow x \in U$. Hence, $\psi(x) \in \phi(U)$.

($\supseteq$) Take $D \in \phi(U)$. Then $U \in D$. Since ψ is surjective, there exists $x \in X$, such that $D = \psi(x)$. Then $x \in U$ and $D \in \psi[U]$.

Finally, we show that ψ is continuous. It suffices to show that the preimage of every $U \in KOF(X_{KOF(X)})$ is open. Take $U \in KOF(X_{KOF(X)})$. Then there exists $F \in KOF(X)$ such that $U = \phi(F)$. We claim that $\psi^{-1}[U] = F$ and therefore the preimage of U is open.

($\subseteq$) Take $x \in \psi^{-1}[U]$. Then $\psi(x)$ belongs to $U = \phi(F)$, which is equivalent to $x \in F$.

($\supseteq$) Take $x \in F$, Then $F \in \psi(x)$ and $\psi(x)$ belongs to $\phi(F) = U$. Therefore, $x \in \psi^{-1}[U]$.

⊠

We do not need to modify the morphisms part as all the proofs are quite simple and the results are deducible from ZF. The same can be said about establishing the categorical duality.

Let HMS^* be the category of choice-free HMS spaces with F-continuous functions and let JM^* be the category of choice-free JM spaces with F-stable functions.

Theorem 3.2.7.

(i) HMS^* *is dually equivalent to* SLat.

(ii) JM^* *is dually equiavalent to* Lat.

3.3 Simplifying joins

Now we take a closer look at the Jipsen & Moshier duality. One of "deficiencies" of this duality is a non-transparent way of evaluating joins in $KOF(X)$, which is defined as

$$U \vee V = \bigcap\{F \in OF(X) \mid U \cup V \subseteq F\}.$$

We will significantly simplify it to the usual join of two filters. The same arguments hold for choice-free JM spaces.

Let X be a JM space. Then using duality, we can view X as $Filt(L)$ for some lattice L. Therefore, X is a complete lattice and from now on we use $\sqsubseteq$ for its order and $\sqcap$ and $\sqcup$ for its meet and join. We also know that every element of $KOF(X)$ is equal to $\phi(a) = \{x \in X : a \in x\}$ for some $a \in L$.

Theorem 3.3.1. *Let X be a JM space and $U, V \in KOF(X)$. Then the join $U \vee V$ in $KOF(X)$ coincides with the filter generated by U and V in $(X, \sqsubseteq)$, i.e.,*

$$\{x \in X \mid \exists y \in U, z \in V : x \sqsupseteq y \sqcap z\}.$$

Proof. Applying the Jipsen & Moshier duality, there is a lattice L and elements a, b of L, such that $U = \phi(a)$ and $V = \phi(b)$. As proved in the previous section, then $U \vee V = \phi(a \vee b)$. Hence, it suffices to show

$$\phi(a \vee b) = \{x \in X \mid \exists y \in U, z \in V : x \sqsupseteq y \sqcap z\}.$$

($\subseteq$) Suppose $a \vee b$ belongs to some filter F. Then we claim $\uparrow a \sqcap \uparrow b \sqsubseteq F$. Indeed, if $c \geqslant a, b$, then $c \geqslant a \vee b$ and therefore $c \in F$. Since $\uparrow a$ is an element of U and $\uparrow b$ is an element of V, we are done.

($\supseteq$) Suppose $x \sqsupseteq y \sqcap z$, where $y \in U$ and $z \in V$. Then $a \in y$ and $b \in z$ and therefore $a \vee b$ belongs to both y and z. Hence, $a \vee b \in y \sqcap z$ and $a \vee b \in x$. ⊠

The proof above also shows that if $U = \Uparrow x$ and $V = \Uparrow y$, we have $U \vee V = \Uparrow(x \sqcap y)$. Moreover, in the case of a distributive lattices, the join becomes even more simple.

Proposition 3.3.2. *Let L be a distributive lattice and X_L its dual space. Let U and V be elements of $KOF(X_L)$. Then the join $U \vee V$ of U and V is equal to*

$$\{x \in X_L \mid \exists y \in U, z \in V : x = y \sqcap z\}.$$

Proof. We first show that for a distributive lattice L, the complete lattice X_L is also distributive. For that it suffices to show that for every filters $F, G, H \in X_L$, we have $F \sqcap (G \sqcup H) \sqsubseteq (F \sqcap G) \sqcup (F \sqcap H)$. Take $a \in F \sqcap (G \sqcup H)$. Then $a \in F$ and there exist $b \in G$ and $c \in H$ such that $a \geqslant b \wedge c$. Let $b' = a \vee b \in F \sqcap G$ and $c' = a \vee c \in F \sqcap H$. We claim that $a = b' \wedge c'$ and therefore X_L is distributive. Indeed, since L is distibutive, $b' \wedge c' = a \vee (b \wedge c) = a$.

By previous proposition, it is obvious that

$$\{x \in X_L \mid \exists y \in U, z \in V : x = y \sqcup z\} \subseteq U \vee V.$$

Hence, we prove the other direction. Suppose x belongs to $U \vee V$. Then by Proposition 3.3.2 there are $y \in U$ and $z \in V$ such that $x \sqsupseteq y \sqcap z$. Let $y' = x \sqcup y \in U$ and $z' = x \sqcup z \in V$. Then since X_L is distributive, $y' \sqcap z' = x \sqcup (y \sqcap z) = x$. Therefore, x belongs to $\{x \in X_L \mid \exists y \in U, z \in V : x = y \sqcap z\}$. ⊠

3.4 Principal upset Priestley duality

Our aim in this section is to establish a new Priestley-like duality for lattices, based on the Jipsen & Moshier duality. This parallels the dualities for distributive lattices. As is well known, for distributive lattice we have a duality via spectral spaces developed by Stone [44] and a duality via Priestley spaces developed by Priestley [40]. There is a connection between these two dualities. Given a spectral space one can take the patch topology and specialization order to obtain a Piriestley space and given a Priestley space one takes open upsets to obtain a spectral space (see for details [3] and [12]). Then the Jipsen & Moshier duality for lattices can be seen as an analogue of duality via spectral spaces.

We will now develop a duality using a patch topology of a JM space. For that we add an analogue of the Priestley separation axiom and consider clopen principal upsets instead of compact open filters. In fact, we can also take clopen filters, since as shown in Corollary 3.4.8, they coincide with clopen principal upsets. Hence, this is just a terminological difference, but for the reasons discussed in Chapter 4, we prefer to use principal upsets.

Similar approach was used by Bimbó and Golblatt for duality of ortholattices. More concretely, Goldblatt [25] established representation of ortholattices, while Bimbó [6] generalized it to a full duality of the appropriate categories. The latter also required addition of the analogue of the Priestley separation axiom. We discuss this duality in more details in Chapter 6. Another example of using the Priestley separation axiom can be found in [4, Section 10.2]. Bezhanishvili and Holliday relate UV-spaces, choice-free duals of Boolean algebras, to Priestley spaces and develop a notion of a UV-Priestley space.

The duality described in this section, as well as dualities just discussed, is not choice-free as it uses Alexander's subbase lemma.

Definition 3.4.1. Let X be simultaneously a topological space and a lattice with a meet $\sqcap$ and a join $\sqcup$. Let $\sqsubseteq$ be the lattice order on X. Then X is a *Principal upset Priestley space (PUP space)* if

(i) X is compact,

(ii) if $x \not\sqsubseteq y$, then there is a clopen principal upset $\Uparrow z$, such that $x \in \Uparrow z$, while $y \notin \Uparrow z$,

(iii) for every clopen principal upsets $\Uparrow x, \Uparrow y$, the principal upset $\Uparrow(x \sqcap y)$, i.e., the join of $\Uparrow x$ and $\Uparrow y$, is also clopen.

For a lattice L, let $Filt(L)$ be the complete lattice of filters on L. Denote its meet by $\sqcap$ and its join by $\sqcup$ as before. For an element $a \in L$, define $\phi(a) = \{F \in Filt(L) \mid a \in F\}$. Note that $\phi(a)$ is always a principal upset $\Uparrow(\uparrow a)$ on the lattice $Filt(L)$. Moreover, for every $a, b \in L$, it is easy to check that $\phi(a \wedge b) = \phi(a) \cap \phi(b)$. By $\overline{\phi(a)}$ we denote $Filt(L) \setminus \phi(a)$. Consider the topology generated on $Filt(L)$ by $\{\phi(a) : a \in L\} \cup \{\overline{\phi(a)} : a \in L\}$. We denote this topological space by X_L. We also denote by $CPU(X)$ the set of clopen principal upsets of X. Note that due to the way we defined topology on X_L, every $\phi(a)$ belongs to $CPU(X_L)$. It turns out that clopen principal upsets of X_L are exactly the ones of the form $\phi(a)$, as we show in Claim 3.4.2.1.

Proposition 3.4.2. *For every lattice L, X_L is a PUP space.*

Proof. By Alexander's subbase lemma, to prove compactness it suffices to consider a covering $X_L = \bigcup_{i \in I} \phi(a_i) \cup \bigcup_{j \in J} \overline{\phi(b_j)}$. Let F be a filter generated by $\{b_j \mid j \in J\}$. Then $F \notin \overline{\phi(b_j)}$ for each $j \in J$, hence there is a_i, such that $a_i \in F$. That means there are $b_0, \ldots, b_n$, such that $a_i \geqslant b_0 \wedge \ldots \wedge b_n$. Then we claim $X = \phi(a_i) \cup \overline{\phi(b_0)} \cup \ldots \cup \overline{\phi(b_n)}$. Indeed, if a filter contains $b_0, \ldots, b_n$, then it has to contain a_i. Therefore, X is covered by a finite subcover and is compact.

Suppose $x \not\sqsubseteq y$. That means there is an element $a \in x \setminus y$. Hence, $\phi(a)$ is a clopen principal upset, such that $x \in \phi(a)$, while $y \notin \phi(a)$, and we showed the second condition.

For the final condition we need to first prove a helpful claim.

Claim 3.4.2.1. *Let L be a lattice. Then the map $\phi : L \to CPU(X_L)$ is surjective, i.e., each clopen principal upset of X_L is of the form $\phi(a)$. Moreover, if $\Uparrow x = \phi(a)$, then $x = \uparrow a$.*

Proof. Let $\Uparrow x$ be a clopen principal upset of X_L. Then $\Uparrow x = \bigcap_{a \in x} \phi(a)$, since both consist of the filters on L that contain all elements of x. Using that $\Uparrow x$ is clopen and X_L is compact, we obtain a finite set $D \subseteq x$, such that $\Uparrow x = \bigcap_{a \in D} \phi(a)$. As already mentioned, ϕ commutes with meets, therefore $\Uparrow x = \phi(\bigwedge D)$.

For the second part of the claim, suppose $\Uparrow x = \phi(a)$. Then a belongs to x and $\uparrow a \subseteq x$. On the other hand, $\uparrow a$ has to belong to $\Uparrow x$, so $x \subseteq \uparrow a$. Therefore, $x = \uparrow a$. ⊠

Now that the claim has been proven, we continue with the proof of Proposition 3.4.2 and show the last condition. Take two clopen principal upsets $\Uparrow x$ and $\Uparrow y$. Using Claim 3.4.2.1, there are $a, b \in L$, such that $\Uparrow x = \phi(a)$ and $\Uparrow y = \phi(b)$. We claim that $\Uparrow(x \sqcap y)$ is equal to $\phi(a \vee b)$.

($\subseteq$) Suppose $z \sqsupseteq x \sqcap y$. Then since $a \vee b$ belongs to both x and y, we have $a \vee b \in z$. Hence, $\Uparrow(x \sqcap y) \subseteq \phi(a \vee b)$.

($\supseteq$) Suppose $a \vee b \in z$. Take $c \in x \sqcap y$. By Claim 3.4.2.1, $x = \uparrow a$ and $y = \uparrow b$. Then c is above both a and b, hence $c \geqslant a \vee b$ and $c \in z$. So $\phi(a \vee b) \subseteq \Uparrow(x \sqcap y)$.

Therefore, $\Uparrow(x \sqcap y) = \phi(a \vee b)$ and $\Uparrow(x \sqcap y)$ is clopen.

⊠

Proposition 3.4.3. *For every PUP space X, the set of its clopen principal upsets $CPU(X)$ forms a lattice.*

Proof. Take two clopen principal upsets $\Uparrow x$ and $\Uparrow y$. Then their intersection is equal to $\Uparrow(x \sqcup y)$ and is clopen as an intersection of two clopen sets. The least principal upset containing both $\Uparrow x$ and $\Uparrow y$ is $\Uparrow(x \sqcap y)$, which is clopen by condition (iii). It is easy to see that those operations define a lattice. ☒

Now we can establish one half of the duality.

Theorem 3.4.4. *For every lattice L, the map between lattices $\phi : L \to CPU(X_L)$ is an isomorphism.*

Proof. We already showed that ϕ preserves meets and that it is surjective. It clearly preserves joins, since

$$\phi(a) \vee \phi(b) = \Uparrow(\uparrow a \sqcap \uparrow b) = \Uparrow\uparrow(a \vee b) = \phi(a \vee b).$$

Hence, it is only left to prove that ϕ is injective. Suppose $a \neq b$. Then either $a \nleq b$ or $b \nleq a$. Without the loss of generality, we may assume that $a \nleq b$. Then $b \notin \uparrow a$. Hence, $\uparrow a$ is a filter that belongs to $\phi(a) \setminus \phi(b)$ and $\phi(a) \neq \phi(b)$. ☒

Lemma 3.4.5. *For every PUP space X, the collection of clopen principal upsets and their complements forms a subbase of the topology on X.*

Proof. It suffices to show that for every open U and $x \in U$, there are finitely many clopen principal upsets and their complements, the intersection of which contains x and is contained in U.

For each $y \notin U$, either $x \not\sqsubseteq y$ or $y \not\sqsubseteq x$. Hence there is a clopen principal upset or a complement of a clopen principal upset U_y, such that $y \in U_y$, but $x \notin U_y$. Then $\bigcup_{y \in X \setminus U} U_y$ covers $X \setminus U$, which is closed and hence compact. Then there are finitely many U_y covering $X \setminus U$. Hence, a finite intersection of their complements $X \setminus U_y$ contains x and is contained in U. ☒

For a PUP space X and a point $x \in X$ let $\psi(x) = \{\Uparrow t \in CPU(X) : x \in \Uparrow t\}$.

Theorem 3.4.6. *For every PUP space X, the map ψ is a homeomorphism between toplogical spaces X and $X_{CPU(X)}$ and also a lattice isomorphism between them.*

Proof. First note that $\psi(x)$ is indeed a filter for every $x \in X$. For injectivity suppose $x \neq y$, hence either $x \not\sqsubseteq y$ or $y \not\sqsubseteq x$. Without loss of generality, we may assume that $x \not\sqsubseteq y$. Then there is a clopen principal upset $\Uparrow t$, such that $x \in \Uparrow t$ while $y \notin \Uparrow t$. Hence, $\Uparrow t \in \psi(x)$ but $\Uparrow t \notin \psi(y)$. Therefore, $\psi(x) \neq \psi(y)$ and ψ is injective.

Next we show that ψ is surjective. Let F be a filter on $CPU(X)$. Consider $\bigcap F$. It is a closed and therefore a compact upset. We claim that $\bigcap F$ is principal. Suppose it is not. Let $\mathcal{D}$ be the collection of clopen principal upsets U such that $\bigcap F \setminus U \neq \varnothing$. For every $x \in F$ there is $y \in F$, such that $x \not\sqsubseteq y$, so there is a clopen principal upset U with $x \in U$ and $y \notin U$. Therefore, $\mathcal{D}$ is a covering of $\bigcap F$ and we can choose a finite subcovering $\mathcal{C}$. Given $U \in \mathcal{C}$ there is a point $x_U \in \bigcap F \setminus U$. Consider a point $x = \bigsqcap_{U \in \mathcal{C}} x_U$. Then $x \in \bigcap F$ but for each $U \in \mathcal{C}$ we have $x \notin U$, which is a contradiction. Therefore, $\bigcap F$ is principal, i.e., $\bigcap F = \Uparrow x$ for some $x \in X$.

Now we show that $F = \psi(x)$ and that concludes the proof of ψ being surjective. Clearly, $F \subseteq \psi(x)$, since for every $\Uparrow t \in F$, we have $\Uparrow x \subseteq \Uparrow t$. Suppose U is a clopen principal upset such that $x \in U$. Then $\Uparrow x = \bigcap F \subseteq U$. Therefore, U together with the complements of the elements of F constitutes a covering of X. Since X is compact, there is a finite number of elements of F, say $U_1, \ldots, U_n$, such that U together with $\overline{U_1}, \ldots, \overline{U_n}$ constitute a covering. But then $U_1 \cap \ldots \cap U_n \subseteq U$ and since F is a filter, $U \in F$.

Thus we have proved that ψ is a bijection. Then ψ and ψ^{-1} commute with arbitrary unions and intersections. Hence, when proving continuity and openness, it suffices to consider only elements of subbases. We first check continuity and then openness.

Let ϕ be the isomorphism between $CPU(X)$ and $CPU(Filt(CPU(X))$ as considered above. Then the subbase defining topology on $Filt(CPU(X))$ is the collection of the sets of the forms $\phi(U)$ and $\overline{\phi(U)}$. The following simple equations show that ψ is continuous:

$$\psi^{-1}[\phi(U)] = \{x \in X : \psi(x) \in \phi(U)\} = \{x \in X : U \in \psi(x)\} = \{x \in X : x \in U\} = U,$$

$$\psi^{-1}[\overline{\phi(U)}] = \{x \in X : \psi(x) \notin \phi(U)\} = \{x \in X : x \notin U\} = X \setminus U.$$

Now we prove that the map ψ is open. As already shown in Lemma 3.4.5, clopen principal upsets and their complements constitute a subbase of X. Let $U \in CPU(X)$. Then $\psi[U] = \{\psi(x) : x \in U\}$. Since every filter on $CPU(X)$ is of the form $\psi(x)$, we have $\{\psi(x) : x \in U\} = \{F \in Filt(CPU(X)) : U \in F\}$ equal to $\phi(U)$, which is an open set. On the other hand, $\psi[\overline{\phi(U)}] = \{\psi(x) : x \notin U\}$. By the same argument, $\{\psi(x) : x \notin U\} = \{F \in Filt(CPU(X)) : U \notin F\}$ equal to $X \setminus \phi(U)$, which is an open set .

Finally, ψ is meet-preserving since $x_1 \sqcap x_2 \sqsupseteq t$ if and only if $x_1 \sqsupseteq t$ and $x_2 \sqsupseteq t$ and ψ is join-preserving since $x_1 \sqcup x_2 \sqsupseteq t$ if and only if $\Uparrow t \supseteq \Uparrow x_1 \cap \Uparrow x_2$. ⊠

Corollary 3.4.7. *Every PUP space is complete as a lattice.*

Proof. We have proven that every PUP space is isomorphic to a lattice of filters, which is known to be complete. ⊠

For a PUP space X let $CF(X)$ be the collection of all clopen filters on X. As promised, we show that clopen filters are the same as clopen principal upsets.

Corollary 3.4.8. *For every PUP space X, $CPU(X) = CF(X)$.*

Proof. As we know, X is isomorphic to X_L for some lattice L. Hence, it suffices to prove the corollary for X_L. Every principal upset is a filter, so we only need to show $CF(X_L) \subseteq CPU(X_L)$. Let F be a clopen filter on X_L.

Since F is open, it is of the form $\bigcup_{i \in I}(\bigcap_{j_i \leqslant n_i} \phi(a_{j_i}) \cap \bigcap_{k_i \leqslant m_i} \overline{\phi(b_{k_i})})$. Since F is closed and therefore compact, we can restrict ourselves to finite I, i.e.,

$$F = \bigcup_{i \leqslant r}(\bigcap_{j_i \leqslant n_i} \phi(a_{j_i}) \cap \bigcap_{k_i \leqslant m_i} \overline{\phi(b_{k_i})}).$$

Using that ϕ preserves meets, we can denote $a_i = a_{0_i} \wedge \ldots \wedge a_{n_i}$ (and in case there are no a_{j_i}, take $a_i = \top$) and obtain

$$F = \bigcup_{i \leqslant r}(\phi(a_i) \cap \bigcap_{k_i \leqslant m_i} \overline{\phi(b_{k_i})}).$$

We assume that every $\phi(a_i) \cap \bigcap_{k_i \leqslant m_i} \overline{\phi(b_{k_i})}$ is non-empty.

Note that for every $i \leqslant r$, the filter $\uparrow a_i$ has to belong to $\phi(a_i) \cap \bigcap_{k_i \leqslant m_i} \overline{\phi(b_{k_i})}$. Indeed, otherwise there is $b_{k_i} \geqslant a_i$ and hence $\phi(a_i) \cap \bigcap_{k_i \leqslant m_i} \overline{\phi(b_{k_i})}$ is empty.

Then since F is a filter, $\uparrow a_0 \cap \ldots \cap \uparrow a_r = \uparrow(a_0 \vee \ldots \vee a_r)$ also has to belong to F. Hence, there is i, such that $a_i \geqslant a_0 \vee \ldots \vee a_r$. Without loss of generality, let it be a_0. Then we claim that $F = \phi(a_0)$.

($\subseteq$) Suppose $x \in F$. Then there is some $a_i \in x$ and since $a_0 \geqslant a_i$, the element a_0 also belongs to x. Hence, $x \in \phi(a_0)$.

($\supseteq$) Suppose $x \in \phi(a_0)$. Then $\uparrow a_0 \subseteq x$, and since $\uparrow a_0 \in F$, we also have $x \in F$.

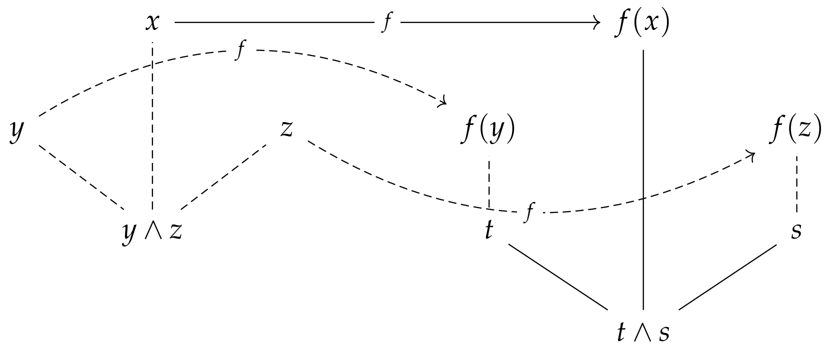
Therefore, F is a clopen principal upset. ⊠

Now we work on the morphism part of the duality.

Definition 3.4.9. Let X and Y be PUP spaces. A map $f : X \to Y$ is a Principal upset Priestley (PUP) morphism, if it is continuous, preserves arbitrary meets and satisfies the following two conditions.

1. If $f(x) \sqsupseteq t \sqcap s$, then there are y and z such that $x \sqsupseteq y \sqcap z$, $f(y) \sqsupseteq t$ and $f(z) \sqsupseteq s$.

   We call this a back condition and illustrate it by the following picture.



2. $f(x) = \top \Leftrightarrow x = \top$, where $\top$ is the top element of X.

Proposition 3.4.10. *Let L and M be lattices and $f : L \to M$ a lattice morphism. Then a map $f^* : X_M \to X_L$ defined by $f^*(F) = f^{-1}[F]$ is a PUP morphism.*

Proof. First of all note that $f^{-1}[F]$ is indeed a filter, since f preserves meets and order. To check continuity it suffices to consider the elements of the subbase. Take some $\phi(a)$. Then $(f^*)^{-1}[\phi(a)] = \{F \in X_M : f^*(F) \in \phi(a)\} = \{F \in X_M : a \in f^{-1}[F]\} = \{F \in X_M : f(a) \in F\} = \phi(f(a))$, which is open. Similarly, for every $\overline{\phi(a)}$, the set $(f^*)^{-1}[\overline{\phi(a)}]$ is equal to an open set $\overline{\phi(f(a))}$. Therefore, f^* is continuous.

Now we claim that f^* preserves arbitrary meets, i.e., that for each family $\{F_i : i \in I\} \subseteq X_M$, we have $f^{-1}[\bigsqcap F_i] = \bigsqcap f^{-1}[F_i]$. Indeed, both consist exactly of $a \in L$, such that $f(a)$ belongs to each F_i.

Next we prove the back condition. Suppose there is a filter F on M and filters G, H on L, such that $f^{-1}[F] \sqsupseteq G \sqcap H$. Consider a set $\Uparrow f[G]$. It is a filter on M, since for each $b_0, b_1 \in M$, such that $b_0 \geqslant f(g_0)$ and $b_1 \geqslant f(g_1)$, where $g_0, g_1 \in G$, we have $b_0 \wedge b_1 \geqslant f(g_0) \wedge f(g_1) = f(g_0 \wedge g_1)$. Similarly, $\Uparrow f[H]$ is a filter. We claim that $F \sqsupseteq \Uparrow f[G] \sqcap \Uparrow f[H]$, $f^*(\Uparrow f[G]) \sqsupseteq G$ and $f^*(\Uparrow f[H]) \sqsupseteq H$. The two latter embeddings are obvious, since for each $g \in G$ we have $f(g) \in \Uparrow f[G]$ and the same for H. So we are left with $F \sqsupseteq \Uparrow f[G] \sqcap \Uparrow f[H]$.

Take $t \in \Uparrow f[G] \sqcap \Uparrow f[H]$, i.e., $t \geqslant f(g)$ for some $g \in G$ and $t \geqslant f(h)$ for some $h \in H$. Consider $g \vee h$. It belongs to $G \sqcap H$ and therefore $f(g \vee h) \in F$. Since f preserves joins, we have $t \geqslant f(g) \vee f(h) = f(g \vee h)$. So $t \in F$ and $F \sqsupseteq \Uparrow f[G] \sqcap \Uparrow f[H]$.

Finally we show the last condition. The top element of X_L is the set L itself and the top element of X_M is M. Clearly, $f^{-1}[M] = L$. Suppose for some filter F on M, we have $f^{-1}[F] = L$. Then $f(0) = 0$ belongs to F and therefore $F = L$. Hence, for each $x \in X_M$, we obtain $f^*(x) = \top \Leftrightarrow x = \top$. ⊠

Proposition 3.4.11. *Let X and Y be PUP spaces and $f : X \to Y$ a PUP morphism. Then a map $f_* : CPU(Y) \to CPU(X)$ defined by $f_*(\Uparrow x) = f^{-1}[\Uparrow x]$ is a lattice morphism.*

Proof. First, note that for a clopen principal upset $\Uparrow x$, the set $f^{-1}[\Uparrow x]$ is indeed clopen, since f is continuous. Let y be the meet of all elements of $f^{-1}[\Uparrow x]$. Then since f preserves arbitrary meets, $f(y)$ is also above x. Take $t \sqsupseteq y$. Then $f(t) \sqsupseteq f(y) \sqsupseteq x$. Hence, $f^{-1}[\Uparrow x] = \Uparrow y$.

Let $\Uparrow x, \Uparrow y$ be clopen principal upsets. Then $f^{-1}[\Uparrow(x \sqcup y)] = f^{-1}[\Uparrow x] \cap f^{-1}[\Uparrow x]$, since $f(z)$ is above $x \sqcup y$ if and only if it is above both x and y. Hence, f_* preserves meets. Now we show that f_* preserves joins, i.e., $f_*(\Uparrow x) \vee f_*(\Uparrow y) = f_*(\Uparrow(x \wedge y))$. Unwrapping definitions, we get

$$f_*(\Uparrow x) \vee f_*(\Uparrow y) = \{t \in X \mid \exists r, s \in X : t \sqsupseteq r \sqcap s, f(r) \sqsupseteq x, f(s) \sqsupseteq y\},$$

$$f_*(\Uparrow(x \sqcap y)) = \{t \in X \mid f(t) \sqsupseteq x \sqcap y\}.$$

Take $t \in f_*(\Uparrow x) \vee f_*(\Uparrow y)$. Then there are $r, s \in X$ such that $t \sqsupseteq r \sqcap s$, $f(r) \sqsupseteq x$, $f(s) \sqsupseteq y$. Since f preserves meets, $f(t) \sqsupseteq f(r) \sqcap f(s) \sqsupseteq x \sqcap y$ and therefore $x \in f_*(\Uparrow(x \sqcap y))$.

Take $t \in f_*(\Uparrow(x \sqcap y))$. Then $f(t) \sqsupseteq x \sqcap y$. Using the first condition, we get r and s such that $t \sqsupseteq r \sqcap s$ and $f(r) \sqsupseteq x$ and $f(s) \sqsupseteq y$. Therefore, x belongs to $f_*(\Uparrow x) \vee f_*(\Uparrow y)$.

Finally, we prove that f_* preserves the bounds. The greatest element of $CPU(Y)$ is Y itself. As $f^{-1}[Y] = X$, the function f_* preserves the top element. The least element of $CPU(Y)$ is $\{\top_Y\}$, where $\top_Y$ is the top element of Y. Then by the second condition, $f^{-1}[\{\top_Y\}] = \{\top_X\}$. ⊠

Let PUP be the category of PUP spaces with PUP morphisms.

Theorem 3.4.12. PUP *is dually equivalent to* Lat.

Proof. We already constructed the functors and the only fact left to check is that the isomorphisms of functors are natural. For that we need to show that the following two diagrams commute.

$$\begin{array}{ccc} L & \xrightarrow{\phi_L} & CPU(X_L) \\ \Big\downarrow f & & \Big\downarrow f_*^* \\ M & \xrightarrow{\phi_M} & CPU(X_M) \end{array}$$

Take $a \in L$. Then we get the following chain of equalities: $f_*^*(\phi_L(a)) = \{F \in X_M : f^*(F) \in \phi_L(a)\} = \{F \in X_M : a \in f^{-1}[F]\} = \{F \in X_M : f(a) \in F\} = \phi_M(f(a))$. Therefore, the first diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\psi_X} & X_{CPU(X)} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_*^*} \\ Y & \xrightarrow{\psi_Y} & X_{CPU(Y)} \end{array}$$

Take $x \in X$. Then we get the following chain of equalities: $f_*^*(\psi_X(x)) = \{U \in CPU(Y) : f^*(U) \in \psi_X(x)\} = \{U \in CPU(Y) : x \in f^{-1}[U]\} = \{U \in CPU(Y) : f(x) \in U\} = \psi_Y(f(x))$. Therefore, the second diagram also commutes. $\boxtimes$

In the remainder of the thesis we will use the duality via PUP spaces to establish some facts about lattice and modal lattices. One of the reasons we chose to work with PUP spaces is that their underlying topological spaces are Stone spaces. From this perspective this choice is similar to preferring to work with Priestley space as opposed to spectral spaces when studying distributive lattices.

CHAPTER 4

Filter completion

We will now use the Principal upset Priestley duality established in the previous chapter to define a filter completion of a lattice that turns out to enjoy an analogue of a Sahlqvist theorem. Then using filter completions we show that every variety of lattices is closed under ideal completions. This result was proved algebraically by Baker and Hales in [2]. Our proof is purely dual and is based on Sambin and Vaccaro's proof of Sahlqvist canonicity [42].

4.1 Positive formulas

We start by explaining which formulas we consider and how to evaluate them on lattices and PUP spaces.

Definition 4.1.1. Let P be a fixed set of of propositional variables. The *positive language* consists of two binary operations $\wedge$ and $\vee$ and two constants $\top$ and $\bot$. The *positive formulas* are formulas obtained by the inductive rule

$$\alpha ::= p \in P \mid \alpha \wedge \beta \mid \alpha \vee \beta \mid \top \mid \bot.$$

We denote the set of all positive formulas by Fm.

Definition 4.1.2. Let L be a lattice. A *valuation* V on a lattice L is a map assigning each $p \in P$ an element of L. Then V naturally extends to all the positive formulas as follows:

$$V(\alpha \wedge \beta) = V(\alpha) \wedge V(\beta),$$

$$V(\alpha \vee \beta) = V(\alpha) \vee V(\beta),$$

$$V(\top) = 1,$$

$$V(\bot) = 0.$$

Definition 4.1.3.

(i) A *positive inequality* is an expression of the form $\alpha \preccurlyeq \beta$, where α and β are positive formulas.

(ii) A *positive equation* is an expression of the form $\alpha \approx \beta$, where α and β are positive formulas.

Definition 4.1.4. Let L be a lattice.

(i) Let $\alpha \preccurlyeq \beta$ be a positive inequality. Then $\alpha \preccurlyeq \beta$ holds in L (denoted by $L \models \alpha \preccurlyeq \beta$) if for every valuation $V : P \to L$ we have $V(\alpha) \leqslant V(\beta)$.

(ii) Let $\alpha \approx \beta$ be a positive equation. Then $\alpha \approx \beta$ holds in L (denoted by $L \models \alpha \approx \beta$) if for every valuation $V : P \to L$ we have $V(\alpha) = V(\beta)$.

Lemma 4.1.5. *For a lattice L and positive formulas α and β,*

$$L \models \alpha \approx \beta \Leftrightarrow L \models \alpha \preccurlyeq \beta \textit{ and } L \models \beta \preccurlyeq \alpha.$$

Proof. Suppose $L \models \alpha \approx \beta$. Take a valuation V on L. Then $V(\alpha) = V(\beta)$, hence also $V(\alpha) \leqslant V(\beta)$. Therefore, $L \models \alpha \preccurlyeq \beta$. Similar, $L \models \beta \preccurlyeq \alpha$.

Now suppose $L \models \alpha \preccurlyeq \beta$ and $L \models \beta \preccurlyeq \alpha$. Take a valuation V on L. Then $V(\alpha) \leqslant V(\beta)$ and $V(\beta) \leqslant V(\alpha)$, hence $V(\alpha) = V(\beta)$. Therefore, $L \models \alpha \approx \beta$. ⊠

Due to this lemma, we concentrate on positive inequalities. Now we define how to evaluate formulas on a PUP space.

Definition 4.1.6. A *valuation* V on a PUP space X is a valuation on the lattice $CPU(X)$. In other words, it is a map assigning for every $p \in P$ a set $V(p) \in CPU(X)$, which is naturally extended to all positive formulas in the following way:

$V(\alpha \wedge \beta) = V(\alpha) \cap V(\beta)$,

$V(\alpha \vee \beta) = \{x \in X \mid \exists y \in V(\alpha), z \in V(\beta)\ x \sqsupseteq y \sqcap z\}$,

$V(\top) = X$,

$V(\bot) = \{\top\}$, where $\top$ is the top element of X.

We denote the relation $x \in V(\alpha)$ by $x \models^V \alpha$.

Note that we define $V(\bot)$ as $\{\top\}$, since $\{\top\}$, which is equal to $\Uparrow\top$, is the smallest clopen principal upset of X.

Definition 4.1.7. Let X be a PUP space.

(i) Let $\alpha \preccurlyeq \beta$ be a positive inequality. We write $X \models \alpha \preccurlyeq \beta$ if for every valuation V on X, $V(\alpha) \subseteq V(\beta)$.

(ii) Let $\alpha \approx \beta$ be a positive equation. We write $X \models \alpha \approx \beta$ if for every valuation V on X, $V(\alpha) = V(\beta)$.

Now it is easy to formulate when does a positive inequality or a positive equation hold in a lattice in terms of its dual PUP space.

Proposition 4.1.8. *Let L be a lattice.*

(i) Let $\alpha \preccurlyeq \beta$ be a positive inequality. Then $L \models \alpha \preccurlyeq \beta$ if and only if $X_L \models \alpha \preccurlyeq \beta$.

(ii) Let $\alpha \approx \beta$ be a positive equation. Then $L \models \alpha \approx \beta$ if and only if $X_L \models \alpha \approx \beta$.

Proof. Suppose $L \models \alpha \preccurlyeq \beta$. Take a valuation V on X_L. Using the isomorphism between L and $CPU(X_L)$ obtain a valuation V' on L, i.e., $V'(\xi) = \phi^{-1}(V(\xi))$. Then $V'(\alpha) \leqslant V'(\beta)$, therefore $V(\alpha) \subseteq V(\beta)$.

For the other direction suppose that for every valuation V on X_L we have $V(\alpha) \subseteq V(\beta)$. Take a valuation V' on L. Using the isomorphism between L and $CPU(X_L)$ obtain a valuation V on X_L, i.e., $V(\xi) = \phi(V'(\xi))$. Then $V(\alpha) \subseteq V(\beta)$ and therefore $V'(\alpha) \leqslant V'(\beta)$. Hence, $L \models \alpha \preccurlyeq \beta$.

The positive equation case follows from Lemma 4.1.5. ⊠

4.2 Filter completion

It turns out that our Principle upset Priestly duality gives rise to a natural kind of completions.

Definition 4.2.1. A *completion* of a lattice L is a pair (e, C), where C is a complete lattice and $e : L \to C$ is a lattice embedding.

Recall that for a lattice L, its order dual lattice is denoted by L^∂.

Definition 4.2.2. Let L be a lattice and $(F(L), \subseteq)$ the complete lattice of its filters. Consider the lattice embedding $\iota : L \to F(L)^\partial$ defined by $\iota(a) = \uparrow a$. Then we call $(\iota, F(L)^\partial)$ the *filter completion* of L.

Note that we need to reverse the order on $F(L)$ since $a \leqslant b \Leftrightarrow \uparrow a \supseteq \uparrow b$. We denote the order on $F(L)^\partial$ by $\sqsubseteq^\partial$ and the meet and join by $\sqcap^\partial$ and $\sqcup^\partial$, as they are indeed the opposites of the lattice structure on X_L.

Let X be a PUP space, $CPU(X)$ the collection of its clopen principal upsets and $PU(X)$ the collection of its principal upsets. Just like $CPU(X)$, the set $PU(X)$ forms a lattice with meet $\Uparrow x \wedge \Uparrow y = \Uparrow(x \sqcup y)$ and join $\Uparrow x \vee \Uparrow y = \Uparrow(x \sqcap y)$. Moreover, since X is complete, $PU(X)$ is also complete. Therefore, $PU(X)$ is a completion of $CPU(X)$.

Let X_L be the dual of a lattice L. Then L is isomorphic to $CPU(X_L)$ and therefore $PU(X_L)$ is a completion of L as well. We call the embedding from L to $PU(X_L)$ defined by the isomorphism ϕ the *principal upset completion*. It turns out to be isomorphic to the filter completion with the isomorphism $\tau : F(L)^\partial \to PU(X_L)$ defined by $\tau(F) = \Uparrow F$.

Proposition 4.2.3. *Let L be a lattice. Then the map $\tau : F(L)^\partial \to PU(X_L)$ defined by $\tau(F) = \Uparrow F$ is an isomorphism and $\tau \circ \iota = \phi$. Therefore, the filter completion is isomorphic to the principal upset completion $PU(X_L)$.*

Proof. The map τ is clearly a bijection. Moreover, $\tau(F \sqcup^\partial G) = \Uparrow(F \sqcup^\partial G) = \Uparrow F \vee \Uparrow G = \tau(F) \vee \tau(G)$ and $\tau(F \sqcap^\partial G) = \Uparrow(F \sqcap^\partial G) = \Uparrow F \cap \Uparrow G = \tau(F) \wedge \tau(G)$. Finally, for every $a \in L$, we have $\tau(\iota(a)) = \tau(\uparrow a) = \Uparrow\uparrow a = \phi(a)$. ⊠

In the previous chapter we discussed that instead of clopen principal upsets we could also use clopen filters. The main reason for our choice of the former is that a natural generalization of clopen principal upsets is the complete lattice of all principal upsets, which as we illustrate below allows us to show strong preservation results.

Another natural completion, thoroughly used in algebraic logic, is canonical extension, see, e.g., [21]. As shown in [37], the canonical extension of a lattice L is isomorphic to the lattice of F-saturated subsets of its dual space. However, F-saturated sets have a much less transparent order-topological structure, so in this thesis we concentrate on filter completions instead. We will show that the topological approach to Sahlqvist canonicty of Sambin and Vacarro [42] and Celani and Jansana [10] can be adapted via PUP duality to filter completions. In Chapter 5 we also generalize it to the modal setting.

Recall the definition of a valuation on a PUP space X from the previous chapter. We can easily extend this notion to the completion $PU(X)$ in the following sense.

Definition 4.2.4. A *principal upset valuation (PU-valuation)* V on a PUP space X is a map assigning for every $p \in P$ a set $V(p) \in PU(X)$, naturally extended to all positive formulas by using the lattice structure on $PU(X)$. To be precise:

$$V(\alpha \wedge \beta) = V(\alpha) \cap V(\beta),$$

$$V(\alpha \vee \beta) = \{x \in X \mid \exists y \in V(\alpha), z \in V(\beta)\ x \sqsupseteq y \sqcap z\},$$

$$V(\top) = X,$$

$V(\bot) = \{\top\}$, where $\top$ is the top element of X.

We denote the relation $x \in V(\alpha)$ by $x \models^V \alpha$. It should be clear from the context if V is a PU-valuation or just a valuation.

Definition 4.2.5. Let X be a PUP space.

(i) Let $\alpha \preccurlyeq \beta$ be a positive inequality. We write $X \models_{PU} \alpha \preccurlyeq \beta$ if for every PU-valuation V on X, $V(\alpha) \subseteq V(\beta)$.

(ii) Let $\alpha \approx \beta$ be a positive equation. We write $X \models_{PU} \alpha \approx \beta$ if for every PU-valuation V on X, $V(\alpha) = V(\beta)$.

Whenever we talk about usual valuations, we denote the relation $\models$ by $\models_{CPU}$, in order to avoid confusion with $\models_{PU}$.

Definition 4.2.6. Let L be a lattice and $\alpha(p_0, \dots, p_n)$ a positive formula. Then for $a_0, \dots, a_n \in L$ we define $\alpha(a_0, \dots, a_n) \in L$ inductively in the following natural way:

- ⋆ if $\alpha = p_i$, then $\alpha(a_0, \dots, a_n) = a_i$;
- ⋆ if $\alpha = \beta \wedge \gamma$, then $\alpha(a_0, \dots, a_n) = \beta(a_0, \dots, a_n) \wedge \gamma(a_0, \dots, a_n)$;
- ⋆ if $\alpha = \beta \vee \gamma$, then $\alpha(a_0, \dots, a_n) = \beta(a_0, \dots, a_n) \vee \gamma(a_0, \dots, a_n)$;
- ⋆ if $\alpha = \top$, then $\alpha(a_0, \dots, a_n) = 1$;
- ⋆ if $\alpha = \bot$, then $\alpha(a_0, \dots, a_n) = 0$.

Lemma 4.2.7. *Fix some $a_1, \dots, a_n$ in a lattice L. Then for every $a, b \in L$ and a positive formula $\alpha(p, p_1, \dots, p_n)$, we have*

$$\alpha(a, a_1, \dots, a_n) \wedge \alpha(b, a_1, \dots, a_n) \geqslant \alpha(a \wedge b, a_1, \dots, a_n).$$

Proof. We prove the lemma by induction on α. For convenience, we write $\xi(x)$ instead of $\xi(x, a_1, \dots, a_n)$.

- ⋆ If α is a propositional letter, $\top$ or $\bot$, the claim is obvious.
- ⋆ If $\alpha = \beta \wedge \gamma$, we obtain $\alpha(a) \wedge \alpha(b) = (\beta(a) \wedge \gamma(a)) \wedge (\beta(b) \wedge \gamma(b)) = (\beta(a) \wedge \beta(b)) \wedge (\gamma(a) \wedge \gamma(b)) \geqslant \beta(a \wedge b) \wedge \gamma(a \wedge b) = \alpha(a \wedge b)$.
- ⋆ If $\alpha = \beta \vee \gamma$, we obtain $\alpha(a) \wedge \alpha(b) = (\beta(a) \vee \gamma(a)) \wedge (\beta(b) \vee \gamma(b)) \geqslant (\beta(a) \wedge \beta(b)) \vee (\gamma(a) \wedge \gamma(b)) \geqslant \beta(a \wedge b) \vee \gamma(a \wedge b) = \alpha(a \wedge b)$.

⊠

Now we prove the intersection lemma, which is is an analogue of the Sambin and Vaccaro intersection lemma [42] and one of the results by Celani and Jansana [10, Theorem 4.2].

Lemma 4.2.8 (The Intersection Lemma)**.** *Let X be a PUP space and consider a lattice $PU(X)$. Then for each positive formula $\alpha(p, p_1, \dots, p_n)$ and principal upsets $\Uparrow x$, $\Uparrow x_1, \dots, \Uparrow x_n$ we have*

$$\alpha(\Uparrow x, \Uparrow x_1, \dots, \Uparrow x_n) = \bigcap\{\alpha(\Uparrow t, \Uparrow x_1, \dots, \Uparrow x_n) \mid \Uparrow t \in CPU(X),\ \Uparrow x \subseteq \Uparrow t\}.$$

Note that here we apply Definition 4.2.6 to the lattice $PU(X)$.

Proof. First of all note that since all our operations are monotone, $\alpha(\Uparrow x, \Uparrow x_1, \dots, \Uparrow x_n) \subseteq \alpha(\Uparrow t, \Uparrow x_1, \dots, \Uparrow x_n)$ for each $\Uparrow x \subseteq \Uparrow t$. Hence,

$$\alpha(\Uparrow x, \Uparrow x_1, \dots, \Uparrow x_n) \subseteq \bigcap\{\alpha(\Uparrow t, \Uparrow x_1, \dots, \Uparrow x_n) \mid \Uparrow t \in CPU(X),\ \Uparrow x \subseteq \Uparrow t\}$$

and the $\subseteq$ part is obvious. As before, we write for convenience $\xi(x)$ instead of $\xi(x, \Uparrow x_1, \dots, \Uparrow x_n)$. We use duality and view X as a dual PUP space to a lattice L. Since we know that all clopen principal upsets are exactly the ones of the form $\phi(a)$, we reformulate the claim as $\alpha(\Uparrow x) = \bigcap_{a \in x} \alpha(\phi(a))$. Note that ϕ is an isomorphism and therefore it commutes with every formula, i.e., $\xi(\phi(a)) = \phi(\xi(a))$. Here on the left side we use ξ for the lattice $CPU(X)$ and on the right side we use ξ for the lattice L in the way described in the Definition 4.2.6. Now we prove the lemma by induction.

- ⋆ Suppose $\alpha(p) = p$. Then $\Uparrow x = \bigcap_{a \in x} \phi(a)$ holds since both are exactly the filters that contain all elements of x.
- ⋆ Suppose $\alpha = \beta \wedge \gamma$. Then using the induction hypothesis, we get $\beta(\Uparrow x) \wedge \gamma(\Uparrow x) = \bigcap_{a \in x} \beta(\phi(a)) \cap \bigcap_{a \in x} \gamma(\phi(a)) = \bigcap_{a \in x}(\beta(\phi(a)) \wedge \gamma(\phi(a)))$.
- ⋆ Suppose $\alpha = \beta \vee \gamma$. We know that $\beta(\phi(a)) \vee \gamma(\phi(a)) = \phi(\beta(a) \vee \gamma(a))$. As one inclusion is obvious, let us show the other one. Suppose $y \in \bigcap_{a \in x}(\beta(\phi(a)) \vee \gamma(\phi(a)))$. Then for each $a \in x$, the element $\beta(a) \vee \gamma(a)$ belongs to y.

  Let t be a filter generated by $\beta(a)$ for all $a \in x$ and s be a filter generated by $\gamma(a)$ for all $a \in x$. Then $t \in \bigcap_{a \in x} \beta(\phi(a))$ and $s \in \bigcap_{a \in x} \gamma(\phi(a))$ and we claim that $t \sqcap s \leqslant y$. Suppose $a \in t \sqcap s$. Then there are $b_1, \dots, b_n \in x$ and $c_1, \dots, c_m \in x$, such that $a \geqslant \beta(b_1) \wedge \dots \wedge \beta(b_n)$ and $a \geqslant \gamma(c_1) \wedge \dots \wedge \gamma(c_m)$. Then surely we also have $a \geqslant \beta(b_1) \wedge \dots \wedge \beta(b_n) \wedge \beta(c_1) \wedge \dots \wedge \beta(c_m)$ and $a \geqslant \gamma(b_1) \wedge \dots \wedge \gamma(b_n) \wedge \gamma(c_1) \wedge \dots \wedge \gamma(c_m)$. Let $d = b_1 \wedge \dots \wedge b_n \wedge c_1 \wedge \dots \wedge c_m \in x$. Using Lemma 4.2.7 we get $a \geqslant \beta(d) \vee \gamma(d)$, which belongs to y. Therefore, $y \in \bigcap_{a \in x} \beta(\phi(a)) \vee \bigcap_{a \in x} \gamma(\phi(a)) = \alpha(\Uparrow x)$.

⋆ If α is equal to $\top$ or $\perp$, the lemma is trivial.

⊠

We finally prove the main theorem of this section.

Theorem 4.2.9. *Let α and β be positive formulas and X a PUP space. Then*

$$X \models_{CPU} \alpha \preccurlyeq \beta \Leftrightarrow X \models_{PU} \alpha \preccurlyeq \beta.$$

Proof. First note that since each valuation is also a PU-valuation, the $\Leftarrow$ part is obvious. Now suppose $X \not\models_{PU} \alpha \preccurlyeq \beta$. Then there is a PU-valuation V on X and $x \in X$, such that $x \models^{V} \alpha$ but $x \not\models^{V} \beta$. Take a family $y_p \in X$, such that $V(p) = \Uparrow y_p$. Using the Intersection Lemma for β, we get a family t_p, such that for every t_p the set $\Uparrow t_p$ is clopen, $\Uparrow y_p \subseteq \Uparrow t_p$ and $x \not\models^{V'} \beta$ under the valuation $V'(p) = \Uparrow t_p$. On the other hand, $x \models^{V'} \alpha$, since $V(p) \subseteq V'(p)$ and all our operations are monotone. So we constructed a valuation, showing that $X \not\models_{CPU} \alpha \preccurlyeq \beta$. ⊠

Theorem 4.2.9 gives us the following result for lattices.

Definition 4.2.10. A class of lattices $\mathcal{C}$ is called a *variety* if there exists a set of positive equations Γ such that $\mathcal{C}$ consists exactly of lattices L such that for every $\alpha \approx \beta \in \Gamma$ we have $L \models \alpha \approx \beta$.

Corollary 4.2.11. *Let L be a lattice and $\alpha \preccurlyeq \beta$ a positive modal inequality. If $L \models \alpha \preccurlyeq \beta$, then $F(L)^{\partial} \models \alpha \preccurlyeq \beta$. Therefore, every variety of lattices is closed under filter completions. Moreover, every variety of lattices is generated by a family of filter completions.*

Proof. By duality $L \models \alpha \preccurlyeq \beta$ if and only if $X_L \models_{CPU} \alpha \preccurlyeq \beta$. From Theorem 4.2.9, that is equivalent to $X_L \models_{PU} \alpha \preccurlyeq \beta$, which by the isomorphism τ holds if and only if $F(L)^{\partial} \models \alpha \preccurlyeq \beta$. ⊠

4.3 Correspondence with ideal completions

The result above could be seen as an analogue of one of the results by Baker and Hales, published in [2]. In the following section we state their result and demonstrate how to derive it from ours.

Definition 4.3.1. Let L be a lattice and $(I(L), \subseteq)$ be the complete lattice of its ideals. Consider the embedding $j : L \to I(L)$ defined by $j(a) = \downarrow a$. Then we call $(j, I(L))$ the *ideal completion*.

Even though the ideal completion is different from the filter completion, they are closely connected. For each lattice L, the ideal completion $I(L)$ is isomorphic to $F(L^{\partial})$, where L^{∂} is the order opposite of L. Therefore, we first establish the connection between properties of a lattice and its order dual.

Definition 4.3.2. Let α be a positive formula. Then define α^{∂} as the formula obtained by changing every $\wedge$ to $\vee$ and vice versa and also changing every 0 to 1 and vice versa.

Lemma 4.3.3. *For every lattice L, a positive equation $\alpha \approx \beta$ holds in L if and only if $\alpha^{\partial} \approx \beta^{\partial}$ holds in L^{∂}.*

Proof. Since $L \simeq L^{\partial\partial}$ and $\alpha = \alpha^{\partial\partial}$, it suffices to show only one direction. Suppose $\alpha \approx \beta$ holds in L. Consider a valuation $V : P \to L^{\partial}$. Let $V' : P \to L$ be the valuation generated by V. By a straightforward induction we see that for every positive formula ξ, the element $V(\xi)$ of L^{∂} and the element $V'(\xi^{\partial})$ of L coincide. Then we know that since $\alpha(V'(p_0), \ldots, V'(p_n)) = \beta(V'(p_0), \ldots, V'(p_n))$ holds in L, $\alpha^{\partial}(V(p_0), \ldots, V(p_n)) = \beta^{\partial}(V(p_0), \ldots, V(p_n))$ has to hold in L^{∂}. ⊠

Next theorem is a result from [2]. Note that Baker and Hales give a different, purely algebraic proof, while ours is solely based on duality theory via Theorem 4.2.9.

Theorem 4.3.4 (Baker and Hales)**.** *Every positive equation that holds in a lattice L likewise holds in $I(L)$. Therefore, every variety of lattices is closed under ideal completions.*

Proof. Suppose $L \models \alpha \preccurlyeq \beta$. Then by Lemma 4.3.3, $L^{\partial} \models \alpha^{\partial} \preccurlyeq \beta^{\partial}$. Then by Corollary 4.2.11, $F(L^{\partial})^{\partial} \models \alpha^{\partial} \preccurlyeq \beta^{\partial}$. But $I(L)$ is isomorphic to $F(L^{\partial}) = F(L^{\partial})^{\partial\partial}$, so by Lemma 4.3.3 $I(L) \models \alpha \preccurlyeq \beta$. ⊠

In this chapter we illustrated the usefulness of the Principal upset Priestley duality by showing in purely dual terms that every variety of lattice is closed under filter and ideal completions. In the next chapter we will extend our investigations to the modal setting and the extended duality for lattices will again be our main tool.

CHAPTER 5

Duality for modal lattices

In this chapter we study lattices enriched with modal operators. We develop duality and Sahlqvist correspondence for these structures. First we define the class of modal PUP spaces dual to modal lattices and establish a duality between them. We show that every inequality is preserved by filter completions and the same holds for the ideal completions, thus obtaining a modal version of the Baker and Hales theorem. We finish the chapter by defining Sahlqvist-like inequalities and proving that they correspond to first-order conditions on modal PUP spaces.

Our work is closely related to the study of positive modal logic, initiated by Dunn in [15]. We are mostly inspired by works of Celani and Jansana [10], but build our duality on the PUP duality developed in the previous chapters whereas [10] builds it over the Priestley duality. Our work can also be seen as a generalization of possibility semantics, studied in [30], to the modal lattices.

5.1 Modal Principal upset Priestley duality

Definition 5.1.1. A lattice L with operators $\Box, \Diamond : L \to L$ is a *modal lattice* if the following axioms hold:

1. $\Box a \leqslant \Diamond a$,
2. $\Diamond(a \wedge b) \geqslant \Diamond a \wedge \Box b$,
3. $\Box(a \wedge b) = \Box a \wedge \Box b$,
4. $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$,
5. $\Box 1 = 1$,
6. $\Diamond 0 = 0$.

In [15] Dunn considered two axioms that connect box and diamond in the positive modal logic setting. We use one of them as the axiom 2 in the numeration above. However, as we work with filters instead of prime filters, our duality argument requires a different second condition. We replaced the other Dunn axiom $\Box(a \vee b) \leqslant \Box a \vee \Diamond b$

with a seriality axiom (axiom 1 on the list). For now it remains open whether axiom 1 can be replaced by a more standard Dunn axiom, see more on this in Chapter 8.

Next we prove a few useful lemmas about filters of modal lattices.

Proposition 5.1.2. *Let L be a modal lattice. For every filter F on L, the set $\Box^{-1}[F] = \{a \in L \mid \Box a \in F\}$ is also a filter.*

Proof. First note that $\Box^{-1}[F]$ is non-empty, as $\Box 1 = 1$ and $1 \in F$.

Let $a \in \Box^{-1}[F]$ and $a \leqslant b$. Then by axiom 3, $\Box a \leqslant \Box b$, so $\Box b \in F$. Hence, $b \in \Box^{-1}[F]$.

Now let $a_1, a_2 \in \Box^{-1}[F]$. Then again by axiom 3, $\Box(a_1 \wedge a_2) = \Box a_1 \wedge \Box a_2 \in F$. Therefore, $a_1 \wedge a_2 \in \Box^{-1}[F]$ and $\Box^{-1}[F]$ is a filter. ⊠

We denote the filter $\Box^{-1}[F]$ by D_F.

Proposition 5.1.3. *Let L be a modal lattice. For each filter F on L, the sets*

$$\uparrow\Box[F] = \{a \in L \mid \exists b \in F : a \geqslant \Box b\},$$

$$\uparrow\Diamond[F] = \{a \in L \mid \exists b \in F : a \geqslant \Diamond b\}$$

are also filters.

Proof. It follows immediately from the definitions that sets $\uparrow\Box[F]$ and $\uparrow\Diamond[F]$ are upward closed and non-empty.

We show that $\uparrow\Box[F]$ is meet closed. Take $a_1, a_2 \in \uparrow\Box[F]$. Then there exist $b_1, b_2 \in F$ such that $a_1 \geqslant \Box b_1$ and $a_2 \geqslant \Box b_2$. Hence, $a_1 \wedge a_2 \geqslant \Box b_1 \wedge \Box b_2 = \Box(b_1 \wedge b_2)$. The element $b_1 \wedge b_2$ belongs to F, and therefore $a_1 \wedge a_2 \in \uparrow\Box[F]$. So $\uparrow\Box[F]$ is a filter.

Now we show that $\uparrow\Diamond[F]$ is meet closed. Take $a_1, a_2 \in \uparrow\Diamond[F]$. Then there exist $b_1, b_2 \in F$ such that $a_1 \geqslant \Diamond b_1$ and $a_2 \geqslant \Diamond b_2$. Hence, $a_1 \wedge a_2 \geqslant \Diamond b_1 \wedge \Diamond b_2$. Since the operator $\Diamond$ is order-preserving, $\Diamond b_1 \wedge \Diamond b_2 \geqslant \Diamond(b_1 \wedge b_2)$. The element $b_1 \wedge b_2$ belongs to F, and therefore $a_1 \wedge a_2 \in \uparrow\Diamond[F]$. So $\uparrow\Diamond[F]$ is a filter. ⊠

Let R be a binary relation on a PUP space X. For each subset $S \subseteq X$, we let

$[R]S = \{x \in X \mid \forall y : \text{ if } xRy \text{ then } y \in S\}$,

$\langle R\rangle S = \{x \in X \mid \exists y : xRy \text{ and } y \in S\}$.

Definition 5.1.4. A *modal PUP space* is a PUP space X with a binary relation R on it, satisfying the following conditions.

1. If $\Uparrow x$ is clopen, then $[R]\Uparrow x$ and $\langle R\rangle\Uparrow x$ are also clopen.
2. For all $x, y \in X$ if $x \not R y$, then there is a clopen principal upset $\Uparrow t$ such that at least one of the following holds:
   (i) $x \in [R]\Uparrow t$ and $y \notin \Uparrow t$,
   (ii) $y \in \Uparrow t$ and $x \notin \langle R\rangle\Uparrow t$.
3. $\sqsubseteq \circ R \subseteq R \circ \sqsubseteq$, i.e., if $x \sqsubseteq yRz$, then there is t, such that $xRt \sqsubseteq z$.
4. $\sqsupseteq \circ R \subseteq R \circ \sqsupseteq$, i.e., if $x \sqsupseteq yRz$, then there is t, such that $xRt \sqsupseteq z$.

5. If for some families $\{x_i \in X : i \in I\}$ and $\{y_i \in X : i \in I\}$ we have $x_i R y_i$ for every i, then $\bigsqcap_{i\in I} x_i \; R \; \bigsqcap_{i\in I} y_i$.

6. If for some family $\{x_i \in X : i \in I\}$ we have $\bigsqcap_{i\in I} x_i R y$, then there exists a family $\{y_i \in X : i \in I\}$, such that $x_i R y_i$ for each i and $\bigsqcap_{i\in I} y_i \sqsubseteq y$.

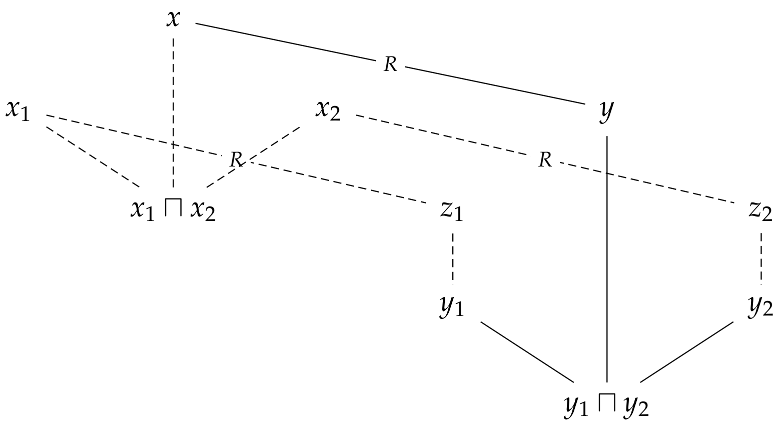
7. If $xRy \sqsupseteq y_1 \sqcap y_2$, then there are x_1, x_2, z_1, z_2, such that

   $x_1 \sqcap x_2 \sqsubseteq x$,

   $x_1 R z_1$ and $x_2 R z_2$,

   $z_1 \sqsupseteq y_1$ and $z_2 \sqsupseteq y_2$.

   We illustrate this condition with the following picture.



8. $xR\top \Leftrightarrow x = \top$, where $\top$ is the top element of X.

Note that every PUP space is a complete lattice, hence conditions 5 and 6 are well-defined. Also conditions 3 and 4 coincide with conditions on frames in Celani and Jansana work [10]. We first prove a couple of useful lemmas about modal PUP spaces.

Lemma 5.1.5. *Let X be a modal PUP space and $\{S_i \subseteq X : i \in I\}$ a family of subsets of X. Then $[R]\bigsqcap_{i\in I} S_i = \bigsqcap_{i\in I}[R]S_i$.*

Proof. ($\subseteq$) Let $x \in [R]\bigsqcap_{i\in I} S_i$ and consider one of S_i. Suppose xRy. Then $y \in \bigsqcap_{i\in I} S_i$ and therefore $y \in S_i$. Hence, $x \in \bigsqcap_{i\in I}[R]S_i$.

($\supseteq$) Let $x \in \bigsqcap_{i\in I}[R]S_i$ and suppose xRy. Then for each S_i we have $y \in S_i$, therefore $y \in \bigsqcap_{i\in I} S_i$. Hence, $x \in [R]\bigsqcap_{i\in I} S_i$. ⊠

Lemma 5.1.6. *Let X be a modal PUP space. Then for every $x \in X$ the sets $[R]{\Uparrow}x$ and $\langle R\rangle{\Uparrow}x$ are both principal upsets.*

Proof. Let ${\Uparrow}x$ be a principal upset. We start with the set $[R]{\Uparrow}x$. Let y be the meet of all elements of $[R]{\Uparrow}x$. Then we claim that y also belongs to $[R]{\Uparrow}x$. Suppose yRz. Then by condition 6, there is a family $\{z_i \in X : i \in I\}$, such that $\bigsqcap_{i\in I} z_i \sqsubseteq z$ and for each $x_i \in [R]{\Uparrow}x$ there is z_i such that $x_i R z_i$. Hence, for each z_i we have $z_i \geqslant x$ and $z \in {\Uparrow}x$. Therefore, $y \in [R]{\Uparrow}x$. Finally we show that $[R]{\Uparrow}x = {\Uparrow}y$. By construction, we already have $[R]{\Uparrow}x \subseteq {\Uparrow}y$. Take $t \sqsupseteq y$ and suppose tRs. By condition 3, there is d such that

$yRd \sqsubseteq s$. Then as we showed above, d is above x and $t \in [R]{\Uparrow}x$. Therefore, $[R]{\Uparrow}x$ is a principal upset.

Now we move to the set $\langle R\rangle{\Uparrow}x$. Let y be the meet of all elements of $\langle R\rangle{\Uparrow}x$. Then by condition 5, $y \in \langle R\rangle{\Uparrow}x$, so there is $z \sqsupseteq x$ such that yRz. By construction, we have $\langle R\rangle{\Uparrow}x \subseteq {\Uparrow}y$. Take $t \sqsupseteq y$. Using condition 4, we obtain an element s such that $tRs \sqsupseteq z \sqsupseteq x$. Therefore, $t \in \langle R\rangle{\Uparrow}x$ and $\langle R\rangle{\Uparrow}x$ is a principal upset. ⊠

Lemma 5.1.7. *Let X be a modal PUP space. For each point $x \in X$, there is a point $y \in X$, such that xRy.*

Proof. Apply condition 3 to $x \sqsubseteq \top R \top$, which holds by condition 8. Then we obtain some $y \in X$ such that $xRy \sqsubseteq \top$. ⊠

Next we establish a duality between modal lattices and modal PUP spaces. Let L be a modal lattice. Let X_L be is dual PUP space and define R_L on it by

$$FR_LG \Leftrightarrow \Box^{-1}[F] \subseteq G \subseteq \Diamond^{-1}[F].$$

Note that the same definition appears in [10]. We first prove three simple lemmas about (X_L, R_L).

Lemma 5.1.8. *Let L be a modal lattice and consider a family of filters $\{F_i : i \in I\}$ of L. Then we have $\Box^{-1}[\bigcap F_i] = \bigcap \Box^{-1}[F_i]$ and $\Diamond^{-1}[\bigcap F_i] = \bigcap \Diamond^{-1}[F_i]$.*

Proof. The lemma follows from elementary set-theoretic fact that for every function the preimage commutes with arbitrary intersections. ⊠

Lemma 5.1.9. *Let L be a modal lattice. Then for every filter $F \in X_L$, we have FR_LD_F.*

Proof. By our definition of D_F, $\Box^{-1}[F] \subseteq D_F$. By the axiom $\Box a \leqslant \Diamond a$, we also have $\Box^{-1}[F] \subseteq \Diamond^{-1}[F]$. Therefore, FR_LD_F. ⊠

Lemma 5.1.10. *Let L be a modal lattice. Then for every $a \in L$, we have*

$$[R_L]\phi(a) = \phi(\Box a) \text{ and } \langle R_L\rangle\phi(a) = \phi(\Diamond a).$$

Proof. We first prove $[R_L]\phi(a) = \phi(\Box a)$.

($\subseteq$) Let $F \in [R_L]\phi(a)$. Then since FR_LD_F, we have $D_F \in \phi(a)$, which means that $a \in \Box^{-1}[F]$. Therefore, $\Box a \in F$ and $F \in \phi(\Box a)$.

($\supseteq$) Let $F \in \phi(\Box a)$, so $\Box a \in F$. Suppose FRG. Then since $\Box^{-1}[F] \subseteq G$, we have $a \in G$. Hence, $G \in \phi(a)$ and $F \in [R_L]\phi(a)$.

Now we prove $\langle R_L\rangle\phi(a) = \phi(\Diamond a)$.

($\subseteq$) Let $F \in \langle R_L\rangle\phi(a)$. Then there is a filter G, such that FR_LG and $a \in G$. By $G \subseteq \Diamond^{-1}[F]$, we have $\Diamond a \in F$. Hence, F is an element of $\phi(\Diamond a)$.

($\supseteq$) Let $F \in \phi(\Diamond a)$, so $\Diamond a \in F$. Consider a filter G, generated by D_F and a, i.e., $G = \{b \in L : \exists d \in D_F\ b \geqslant d \wedge a\}$. Then by construction, $\Box^{-1}[F] \subseteq G$ and $a \in G$. Take $b \geqslant d \wedge a$ for some $d \in D_F$. Then $\Diamond b \geqslant \Diamond(d \wedge a) \geqslant \Box d \wedge \Diamond a$. Since both $\Box d$ and $\Diamond a$ belong to F, we have that $\Diamond b$ also belongs to F and $G \subseteq \Diamond^{-1}[F]$.

⊠

By Claim 3.4.2.1, every clopen principal upset of X_L is of the form $\phi(a)$ for some $a \in L$. Therefore, by Lemma 5.1.10 for every clopen principal upset U, $[R_L]U$ and $\langle R_L \rangle U$ are of the form $\phi(\Box a)$ and $\phi(\Diamond a)$ respectively.

Theorem 5.1.11. *For every modal lattice L, the PUP space X_L with a binary relation R_L is a modal PUP space.*

Proof. We prove the conditions on modal PUP spaces one by one using the same numbering as in Definition 5.1.4.

1. By Lemma 5.1.10, for every clopen principal upset $\Uparrow x$, the sets $[R_L]\Uparrow x$ and $\langle R_L \rangle \Uparrow x$ are of the form $\phi(\Box a)$ and $\phi(\Diamond a)$ respectively and therefore are clopen.

2. Let $F, G \in X_L$ such that $F \not R_L G$. Then there is either $a \in \Box^{-1}[F]$ such that $a \notin G$ or $a \in G$ such that $a \notin \Diamond^{-1}[F]$.

   First suppose there exists $a \in \Box^{-1}[F]$ such that $a \notin G$. Then by Lemma 5.1.10, $F \in [R]\phi(a) = \phi(\Box a)$ while $G \notin \phi(a)$. Hence, $\phi(a)$ is a clopen principal upset which we were looking for.

   Now suppose there exists $a \in G$ such that $a \notin \Diamond^{-1}[F]$. Then by Lemma 5.1.10, $F \notin \langle R \rangle \phi(a) = \phi(\Diamond a)$ while $G \in \phi(a)$. Hence, $\phi(a)$ is a clopen principal upset which we were looking for.

3. Suppose $F \sqsubseteq G R_L H$. As we know, $F R_L D_F$. Let $a \in D_F$. Then $\Box a \in F$, hence $\Box a \in G$. Since $G R_L H$, we have $a \in H$. Therefore, $F R_L D_F \sqsubseteq H$.

4. Suppose $F \sqsupseteq G R_L H$. Consider a filter $D_F \sqcup H$. Then we already have $\Box^{-1}[F] \subseteq D_F \sqcup H$ and $D_F \sqcup H \sqsupseteq H$. Take an element a of $D_F \sqcup H$. Then there are $d \in D_F$ and $h \in H$, such that $a \geqslant d \wedge h$. Therefore, $\Diamond a \geqslant \Diamond(d \wedge h) \geqslant \Box d \wedge \Diamond h$. Since both $\Box d$ and $\Diamond h$ belong to F, we have that $\Diamond a$ also belongs to F. Hence, $D_F \sqcup H \subseteq \Diamond^{-1}[F]$ and $F R_L (D_F \sqcup H) \sqsupseteq H$.

5. Let $\{F_i \in X_L : i \in I\}$ and $\{G_i \in X_L : i \in I\}$ be families of filters of L such that $F_i R_L G_i$. Then $\Box^{-1}[F_i] \subseteq G_i \subseteq \Diamond^{-1}[F_i]$ for every i. By Lemma 5.1.8, we get

   $$\Box^{-1}[\bigcap_{i \in I} F_i] = \bigcap_{i \in I} \Box^{-1}[F_i] \subseteq \bigcap_{i \in I} G_i \subseteq \bigcap_{i \in I} \Diamond^{-1}[F_i] = \Diamond^{-1}[\bigcap_{i \in I} F_i].$$

   Therefore, $\bigsqcap_{i \in I} F_i \, R_L \, \bigsqcap_{i \in I} G_i$.

6. Let $\{F_i \in X_L : i \in I\}$ be a family of filters of L such that $(\bigsqcap_{i \in I} F_i) R_L G$ for some filter G. Consider filters D_{F_i}. Using Lemma 5.1.8, we have $\bigsqcap_{i \in I} D_{F_i} \sqsubseteq G$, and by Lemma 5.1.9 $F_i R_L D_{F_i}$.

7. Suppose $F R_L G \sqsupseteq G_1 \sqcap G_2$. We need to construct filters F_1, F_2, H_1, H_2 such that

   $F_1 \sqcap F_2 \sqsubseteq F$,

   $F_1 R H_1$ and $F_2 R_L H_2$,

   $H_1 \sqsupseteq G_1$ and $H_2 \sqsupseteq G_2$.

Let F_1 and F_2 be the filters $\uparrow\Diamond[G_1]$ and $\uparrow\Diamond[G_2]$ respectively, discussed in Proposition 5.1.3. We first show that $F_1 \sqcap F_2 \sqsubseteq F$. Take $c \in F_1 \sqcap F_2$. Then there are $a_1, \ldots, a_n \in G_1$ and $b_1, \ldots, b_m \in G_2$, such that $c \geqslant \Diamond a_1 \wedge \ldots \wedge \Diamond a_n$ and $c \geqslant \Diamond b_1 \wedge \ldots \wedge \Diamond b_m$. Let $a = a_1 \wedge \ldots \wedge a_n$ and $b = b_1 \wedge \ldots \wedge b_m$. By the monotonicity of $\Diamond$, we get $c \geqslant \Diamond a$ and $c \geqslant \Diamond b$, hence $c \geqslant \Diamond a \vee \Diamond b = \Diamond(a \vee b)$. Since $a \in G_1$ and $b \in G_2$, we have $a \vee b \in G$ and $\Diamond(a \vee b) \in F$. Therefore, $c \in F$ and $F_1 \sqcap F_2 \sqsubseteq F$.

Let $H_1 = G_1 \sqcup D_{F_1}$ and $H_2 = G_2 \sqcup D_{F_2}$. Then it is left to show $F_1 R_L H_1$ and $F_2 R_L H_2$. Moreover, it suffices to prove $H_i \subseteq \Diamond^{-1}[F_i]$. Take $h \in H_i$. Then there are $g \in G_i$ and $d \in D_{F_i}$, such that $h \geqslant g \wedge d$. So we get $\Diamond h \geqslant \Diamond(g \wedge d) \geqslant \Diamond g \wedge \Box d \in F_i$. Hence, $H_1 \subseteq \Diamond^{-1}[F_1]$ and $H_2 \subseteq \Diamond^{-1}[F_2]$.

8. The top element of X_L is L itself. Suppose $F R_L L$. Then $\Box^{-1}[F] \subseteq L \subseteq \Diamond^{-1}[F]$. Therefore, for each $a \in L$ we have $\Diamond a \in F$. In particular, $\Diamond 0 = 0$ belongs to F, and $F = L$. Moreover, $L R_L L$ since $\Box^{-1}[L] \subseteq L \subseteq \Diamond^{-1}[L]$.

$\boxtimes$

Let X be a modal PUP space. By Lemma 5.1.6 and Condition 1, we know that for each clopen principal upset $\Uparrow x$, the sets $[R]\Uparrow x$ and $\langle R\rangle\Uparrow x$ are also clopen principal upsets. Therefore, $[R] : CPU(X) \to CPU(X)$ and $\langle R\rangle : CPU(X) \to CPU(X)$ are operators on the lattice $CPU(X)$.

Theorem 5.1.12. *For every modal PUP space X, the lattice $CPU(X)$ with operators $[R]$ and $\langle R\rangle$ on it is a modal lattice.*

Proof. By PUP duality, we already know that $CPU(X)$ is a lattice. We prove the axioms of modal lattices in the same order as they are listed in Definition 5.1.1.

1. Take a clopen principal upset U and $x \in [R]U$. Using Lemma 5.1.7, construct $y \in X$, such that xRy. Then $y \in U$ and $x \in \langle R\rangle U$. Hence, $[R]U \subseteq \langle R\rangle U$.

2. Take clopen principal upsets U, V and $x \in \langle R\rangle U \cap [R]V$. Then there is $y \in U$ such that xRy and hence $y \in V$. So $xRy \ni (U \cap V)$ and $x \in \langle R\rangle(U \cap V)$. Therefore, $\langle R\rangle U \cap [R]V \subseteq \langle R\rangle(U \cap V)$

3. We get $[R](U \cap V) = [R]U \cap [R]V$ by Lemma 5.1.5.

4. Take clopen principal upsets $\Uparrow s, \Uparrow t$. We want to show $\langle R\rangle(\Uparrow s \vee \Uparrow t) = \langle R\rangle\Uparrow s \vee \langle R\rangle\Uparrow t$. As we know, $\langle R\rangle(\Uparrow s \vee \Uparrow t) = \langle R\rangle(\Uparrow(s \sqcap t))$

   ($\subseteq$) Suppose $x \in \langle R\rangle(\Uparrow(s \sqcap t))$. Then there is $y \sqsupseteq s \sqcap t$, such that xRy. Applying condition 7, we get x_1, x_2, z_1, z_2, such that

   $x_1 \sqcap x_2 \sqsubseteq x$,
   $x_1 R z_1$ and $x_2 R z_2$,
   $z_1 \sqsupseteq s$ and $z_2 \sqsupseteq t$.

   That implies $x_1 \in \langle R\rangle\Uparrow s$ and $x_2 \in \langle R\rangle\Uparrow t$, therefore $x \in \langle R\rangle\Uparrow s \vee \langle R\rangle\Uparrow t$.

   ($\supseteq$) Now for the other direction suppose $x \in \langle R\rangle\Uparrow s \vee \langle R\rangle\Uparrow t$. Then there are x_1, x_2 and $y_1 \sqsupseteq s$, $y_2 \sqsupseteq t$, such that $x_1 \sqcap x_2 \sqsubseteq x$ and $x_1 R y_1$, $x_2 R y_2$. By condition 5, we also have $x_1 \sqcap x_2 \; R \; y_1 \sqcap y_2$. Using condition 4, we obtain $y \sqsupseteq y_1 \sqcap y_2$, such that xRy. Then $y \in \Uparrow(s \sqcap t)$ and $x \in \langle R\rangle\Uparrow(s \sqcap t)$.

5. The top element of $CPU(X)$ is X itself and we have

$$[R]X = \{x \in X \mid \forall y : \text{ if } xRy \text{ then } y \in X\} = X.$$

6. The bottom element of $CPU(X)$ is the set $\{\top\}$, where $\top$ is the top element of X. Note that $\langle R\rangle\{\top\} = \{x \in X : xR\top\}$. Then using condition 8, we obtain $\langle R\rangle\{\top\} = \{\top\}$.

$\boxtimes$

Now we consider our familiar maps ϕ and ψ and show that ϕ preserves modal operators $\Box$ and $\Diamond$, while ψ preserves and reflects the realtion R.

Proposition 5.1.13. *For every modal lattice L, the map $\phi : L \to CPU(X_L)$ preserves the modal operators and hence is an isomorphism.*

Proof. The proposition follows from Lemma 5.1.10 and Theorem 3.4.4. $\boxtimes$

Proposition 5.1.14. *For every modal PUP space X, the map $\psi : X \to X_{CPU(X)}$ preserves and reflects the relation R and hence is an isomorphism.*

Proof. Let us denote the relation on $X_{CPU(X)}$ by R^*. Suppose xRy. We need to show that $[R]^{-1}[\psi(x)] \subseteq \psi(y) \subseteq \langle R\rangle^{-1}[\psi(x)]$. Take $U \in [R]^{-1}[\psi(x)]$. Then $x \in [R]U$, so $y \in U$. Hence, $U \in \psi(y)$. Now take $V \in \psi(y)$, so $y \in V$. Then $x \in \langle R\rangle U$ and $U \in \langle R\rangle^{-1}[\psi(x)]$. Therefore, $\psi(x)R^*\psi(y)$.

Suppose $\psi(x)R^*\psi(y)$. That means exactly that for each clopen principal upset U we have both $x \in [R]U \Rightarrow y \in U$ and $y \in U \Rightarrow x \in \langle R\rangle U$. Therefore, by condition 2, xRy. $\boxtimes$

Now we move to the morphism part.

Definition 5.1.15. Let L and M be two modal lattices. Then a lattice morphism $f : L \to M$ is a *modal lattice morphism* if it preserves $\Box$ and $\Diamond$, i.e., for every $a \in L$ we have

$$\Box f(a) = f(\Box a) \text{ and } \Diamond f(a) = f(\Diamond a).$$

Definition 5.1.16. Let X and Y be two modal PUP spaces. Then a PUP morphism $f : X \to Y$ is *modal*, if the following conditions hold.

1. $xRy \Rightarrow f(x)Rf(y)$.
2. $f(x)Ry \Rightarrow \exists z\, xRz, f(z) \sqsubseteq y$.
3. If $f(x)Ry \sqsupseteq t$, then there is z, such that xRz and $f(z) \sqsupseteq t$.

Proposition 5.1.17. *Let $f : L \to M$ be a modal lattice morphism. Then $f^* : X_M \to X_L$ is a modal PUP morphism.*

Proof. We will prove the Conditions 1-3 in the Definiton 5.1.16 one by one.

1. Suppose FR_MG. Take $a \in \Box^{-1}[f^{-1}[F]]$. Then $f(\Box a) \in F$. Since $f(\Box a) = \Box f(a)$ and $\Box^{-1}[F] \subseteq G$, we have $a \in f^{-1}[G]$. Therefore, $\Box^{-1}[f^{-1}[F]] \subseteq f^{-1}[G]$. Take $b \in f^{-1}[G]$. Then $f(b) \in G$, so $\Diamond f(b) \in F$. Since $\Diamond f(b) = f(\Diamond b)$, we get $b \in \Diamond^{-1}[f^{-1}[F]]$. Therefore, $f^{-1}[G] \subseteq \Diamond^{-1}[f^{-1}[F]]$ and $f^*(F)R_Lf^*(G)$.

2. Suppose $f^*(F)R_L G$. As we know, $FR_M D_F$, so it suffices to show $f^*(D_F) \sqsubseteq G$. Take $a \in f^*(D_F)$. Then $\Box f(a) \in F$. Since $\Box f(a) = f(\Box a)$ and $\Box^{-1}[f^{-1}[F]] \subseteq G$, we have $a \in G$. Therefore, $f^*(D_F) \sqsubseteq G$.

3. Suppose $f^*(F)R_L G \sqsupseteq H$. By Lemma 5.1.3, $\uparrow f[H]$ is a filter. Let $P = D_F \sqcup \uparrow f[H]$. Then it immediately follows that $\Box^{-1}[F] \subseteq P$ and $H \sqsubseteq f^{-1}[P]$. So the only thing left to show is $P \subseteq \Diamond^{-1}[F]$. Take $a \in P$. Then there are $b \in \Box^{-1}[F]$ and $c \geqslant f(d)$, such that $a \geqslant b \wedge c$ and $d \in H$. So we get $\Diamond a \geqslant \Box b \wedge \Diamond f(d) = \Box b \wedge f(\Diamond d)$, which belongs to F. Therefore, $P \subseteq \Diamond^{-1}[F]$.

$\boxtimes$

Proposition 5.1.18. *Let $f : X \to Y$ be a modal PUP morphism. Then $f_* : CPU(Y) \to CPU(X)$ is a modal lattice morphism.*

Proof. Let $\Uparrow t$ be a clopen principal upset on Y. We first prove $\Box f^{-1}[\Uparrow t] = f^{-1}[\Box \Uparrow t]$.

($\subseteq$) Take $x \in \Box f^{-1}[\Uparrow t]$. Suppose $f(x)Ry$. Then there is $z \in X$, such that xRz and $f(z) \sqsubseteq y$. Therefore, $z \in f^{-1}[\Uparrow t]$, hence $f(z) \in \Uparrow t$ and $y \in \Uparrow t$. Therefore, $x \in f^{-1}[\Box \Uparrow t]$.

($\supseteq$) Take $x \in f^{-1}[\Box \Uparrow t]$. Suppose xRy. Then $f(x)Rf(y)$ and therefore $f(y) \in \Uparrow t$. Hence, $y \in f^{-1}[\Uparrow t]$ and $x \in \Box f^{-1}[\Uparrow t]$.

Now we prove that $\Diamond f^{-1}[\Uparrow t] = f^{-1}[\Diamond \Uparrow t]$.

($\subseteq$) Take $x \in \Diamond f^{-1}[\Uparrow t]$. Then there is $y \in f^{-1}[\Uparrow t]$ such that xRy. Hence, $f(x)Rf(y) \sqsupseteq t$, giving us $x \in f^{-1}[\Diamond \Uparrow t]$.

($\supseteq$) Take $x \in f^{-1}[\Diamond \Uparrow t]$. Then there is $y \sqsupseteq t$, such that $f(x)Ry$. Hence, there is also z, such that xRz and $f(z) \sqsupseteq t$. Therefore, $x \in \Diamond f^{-1}[\Uparrow t]$.

$\boxtimes$

Let MLat be the category of modal lattices with modal lattice morphisms and let MPUP be the category of modal PUP spaces with modal PUP morphisms.

Theorem 5.1.19. MPUP *is dually equivalent to* MLat.

Proof. The theorem follows from Theorem 3.4.12 and the results proven in this section.

$\boxtimes$

5.2 Filter completions for modal lattices

It turns out that modal lattices also admit filter completions and Theorem 4.2.9 can be generalized to the modal case. Just as in Chapter 4, we will give a dual characterization of filter completions and then prove the Sahlqvist preservation result. We show that the same result holds for ideal completions that we discussed in Section 4.3. In the next section we also obtain a correspondence result.

Let L be a modal lattice and $(\iota, F(L)^\partial)$ its filter completion. Using Lemma 5.1.3, we define modal operators on $F(L)^\partial$ by $\Box_F F := \uparrow\Box[F]$ and $\Diamond_F F := \uparrow\Diamond[F]$.

We can also turn the ideal completion $(I(L), \subseteq)$ into a modal lattice by defining the modal operators on it by $\Box_I I := \downarrow\Box[I]$ and $\Diamond_I I := \downarrow\Diamond[I]$. Then just as in Section 4.3, we can translate the results of the section into a section for ideal completions.

Proposition 5.2.1. *Let L be a modal lattice. Then $F(L)^\partial$ is a modal lattice and the embedding $\iota : L \to F(L)^\partial$ commutes with modal operators.*

Proof. First we show that $F(L)^\partial$ is a modal lattice by proving the axioms of modal lattices in the same order as they appear in the Definition 5.1.1.

1. Let F be a filter on L. Take $a \in \Diamond_F F = \uparrow\Diamond[F]$. Then there is $b \in F$ such that $a \geqslant \Diamond b$. Since $\Box b \leqslant \Diamond b$, we also have $a \geqslant \Box b$. Therefore, $a \in \Box_F F = \uparrow\Box[F]$ and $\Box_F F \supseteq \Diamond_F F$, giving us $\Box_F F \sqsubseteq^\partial \Diamond_F F$.

2. Let F, G be filters on L. Take $a \in \Diamond_F(F \sqcap^\partial G)$. Then there are $b \in F$, $c \in G$ and $d \geqslant b \wedge c$ such that $a \geqslant \Diamond d$. Hence, $a \geqslant \Diamond(b \wedge c) \geqslant \Diamond b \wedge \Box c$. By definitions of $\Diamond_F$ and $\Box_F$, we have $\Diamond b \in \Diamond_F F$ and $\Box c \in \Box_F G$. Therefore, $a \in \Diamond_F F \sqcap^\partial \Box_F G$ and $\Diamond_F(F \sqcap^\partial G) \sqsupseteq^\partial \Diamond_F F \sqcap^\partial \Box_F G$.

3. Let F, G be filters on L. We prove $\Box_F(F \sqcap^\partial G) = \Box_F F \sqcap^\partial \Box_F G$.

   ($\subseteq$) Take $a \in \Box_F(F \sqcap^\partial G)$. Then there are $b \in F$, $c \in G$ and $d \geqslant b \wedge c$ such that $a \geqslant \Box d$. Hence, $a \geqslant \Box(b \wedge c) = \Box b \wedge \Box c$. By definition of $\Box_F$, we have $\Box b \in \Box_F F$ and $\Box c \in \Box_F G$. Therefore, $a \in \Box_F F \sqcap^\partial \Box_F G$ and $\Box_F(F \sqcap^\partial G) \subseteq \Box_F F \sqcap^\partial \Box_F G$.

   ($\supseteq$) Take $a \in \Box_F F \sqcap^\partial \Box_F G$. Then there are $b \in \Box_F F$ and $c \in \Box_F G$ such that $a \geqslant b \wedge c$. Hence, there are also $d \in F$ and $e \in G$ such that $b \geqslant \Box d$ and $c \geqslant \Box e$. We obtain $a \geqslant \Box d \wedge \Box e = \Box(d \wedge e)$, which implies $a \in \Box_F(F \sqcap^\partial G)$. Therefore, $\Box_F F \sqcap^\partial \Box_F G \subseteq \Box_F(F \sqcap^\partial G)$.

4. Let F, G be filters on L. We prove $\Diamond_F(F \sqcup^\partial G) = \Diamond_F F \sqcup^\partial \Diamond_F G$.

   ($\subseteq$) Take $a \in \Diamond_F(F \sqcup^\partial G)$. Then there is $b \in F \cap G$, such that $a \geqslant \Diamond b$. Hence, a belongs to both $\Diamond_F F$ and $\Diamond_F G$. Therefore, $\Diamond_F(F \sqcup^\partial G) \subseteq \Diamond_F F \sqcup^\partial \Diamond_F G$.

   ($\supseteq$) Take $a \in \Diamond_F F \sqcup^\partial \Diamond_F G$. Then there are $b \in F$ and $c \in G$ such that $a \geqslant \Diamond b$ and $a \geqslant \Diamond c$. Hence, $a \geqslant \Diamond b \vee \Diamond c = \Diamond(b \vee c)$. Since $b \vee c \in F \sqcup^\partial G$, we get $a \in \Diamond_F(F \sqcup^\partial G)$ and $\Diamond_F(F \sqcup^\partial G) \subseteq \Diamond_F(F \sqcup^\partial G)$.

5. Due to the dual order, the top element of $F(L)^\partial$ is the filter $\{1\}$. By definition of $\Box_F$, we have $\Box_F\{1\} = \uparrow\Box 1 = \{1\}$.

6. Due to the dual order, the bottom element of $F(L)^\partial$ is the filter L. By definition of $\Diamond_F$, we have $\Diamond_F L = \uparrow\Diamond[L]$. Since $\Diamond 0 = 0$, we have $0 \in \Diamond[L]$. Therefore, $\uparrow\Diamond[L] = L$.

Now we show that ι preserves $\Box$ and $\Diamond$. We start with $\Box$. Take $a \in L$. We prove $\uparrow\Box a = \uparrow\Box[\uparrow a]$.

($\subseteq$) Consider $b \in \uparrow\Box a$. Then $b \geqslant \Box a$ and since $a \in \uparrow a$, we get $b \in \uparrow\Box[\uparrow a]$.

($\supseteq$) Consider $b \in \uparrow\Box[\uparrow a]$. Then there is $c \in \uparrow a$ such that $b \geqslant \Box c$. Since $c \geqslant a$, we have $\Box c \geqslant \Box a$ and therefore $b \geqslant \Box a$. Thus, $b \in \uparrow\Box a$.

Now we move to $\Diamond$. Take $a \in L$. We prove $\uparrow\Diamond a = \uparrow\Diamond[\uparrow a]$.

($\subseteq$) Consider $b \in \uparrow\Diamond a$. Then $b \geqslant \Diamond a$ and since $a \in \uparrow a$, we get $b \in \uparrow\Diamond[\uparrow a]$.

($\supseteq$) Consider $b \in \uparrow\Diamond[\uparrow a]$. Then there is $c \in \uparrow a$ such that $b \geqslant \Diamond c$. Since $c \geqslant a$, we have $\Diamond c \geqslant \Diamond a$ and therefore $b \geqslant \Diamond a$. Thus, $b \in \uparrow\Diamond a$.

$\boxtimes$

As in the case of (non-modal) lattices, the filter completion of a modal lattice L is isomorphic to the lattice $PU(X_L)$, where X_L is the modal PUP space dual to L and PU is the set of it principal upsets. We know by Lemma 5.1.6 that $[R]$ and $\langle R\rangle$ define operators on the lattice $PU(X)$ for every modal PUP space X.

Proposition 5.2.2. *Let X be a modal PUP space. Then the lattice $PU(X)$ with operators $[R]$ and $\langle R\rangle$ forms a modal lattice.*

Proof. Taking a closer look at the proof of Theorem 5.1.12 we see that the clopenness of considered principal upsets has not been used there. Therefore, we can use the same proof to establish the proposition. $\boxtimes$

Proposition 5.2.3. *Let L be a modal lattice. Then the map $\tau : F(L)^\partial \to PU(X_L)$ defined by $\tau(F) = \Uparrow F$ is a modal isomorphism and $\tau \circ \iota = \phi$. Therefore, the filter completion is isomorphic to the modal lattice $PU(X_L)$ with operators $[R_L]$ and $\langle R_L\rangle$.*

Proof. By Proposition 4.2.3, it suffices to prove that τ preserves the modal operators. We first prove $\tau(\Box_F F) = [R_L]\tau(F)$.

($\subseteq$) Let $G \in \tau(\Box_F F)$. Then $G \sqsupseteq \Box_F F = \uparrow\Box[F]$. Suppose GR_LH, which means $\Box^{-1}[G] \subseteq H \subseteq \Diamond^{-1}[G]$. Take $a \in F$. Then $\Box a \in G$, so $a \in \Box^{-1}[G]$. Hence, $a \in H$ and $F \sqsubseteq H$. Therefore, $H \in \tau(F)$ and $G \in [R_L]\tau(F)$.

($\supseteq$) Let $G \in [R_L]\tau(F)$. Then the filter D_G belongs to $\tau(F)$, i.e., $F \subseteq \Box^{-1}[G]$. Take $a \in \Box_F F$. Then there is $b \in F$ such that $a \geqslant \Box b$. Since b also belongs to $\Box^{-1}[G]$, we have $a \in G$ and $\Box_F F \subseteq G$. Therefore, $G \in \tau(\Box_F F)$.

Now we prove $\tau(\Diamond_F F) = \langle R_L\rangle\tau(F)$.

($\subseteq$) Let $G \in \tau(\Diamond_F F)$. Then $G \sqsupseteq \Diamond_F F = \uparrow\Diamond[F]$. Let $H = F \sqcup D_G$. Since $F \sqsubseteq H$, it suffices to show GR_LH. Moreover, since $\Box^{-1}[G] \subseteq H$, we only need to prove $H \subseteq \Diamond^{-1}[G]$.

Take $a \in H$. Then there are $b \in F$ and $c \in D_G$ such that $a \geqslant b \wedge c$. Hence, $\Diamond a \geqslant \Diamond(b \wedge c) \geqslant \Diamond b \wedge \Box c$. The element $\Box c$ belongs to G and $\Diamond b \in \Diamond[F] \subseteq G$. Therefore, $\Diamond a \in G$ and $H \subseteq \Diamond^{-1}[G]$.

($\supseteq$) Let $G \in \langle R_L\rangle\tau(F)$. Then there is a filter $H \sqsupseteq F$ such that GR_LH. Take $a \in \Diamond_F F$. Then there exists $b \in F$ such that $a \geqslant \Diamond b$. Hence, $b \in H$ and $\Diamond b \in G$. Therefore, $a \in G$ and $\Diamond_F F \sqsubseteq G$.

$\boxtimes$

We finalize the section by generalizing Theorem 4.2.9 to the modal setting.

Definition 5.2.4. Let P be a fixed set of of propositional variables. The *positive modal language* consists of two binary operations $\wedge$ and $\vee$, two constants $\top$ and $\bot$ and two unary operations $\Box$ and $\Diamond$. The *positive modal formulas* are formulas obtained by the inductive rule

$$\alpha ::= p \in P \mid \alpha \wedge \beta \mid \alpha \vee \beta \mid \top \mid \bot \mid \Box\alpha \mid \Diamond\alpha.$$

We denote the set of all positive formulas by Fm.

Next we generalize Sections 4.1 and 4 to the modal setting.

Definition 5.2.5. Let L be a modal lattice. A *modal valuation* V on a modal lattice L is a valuation on a lattice L as described in Definition 4.1.2, extended to all positive modal formulas using the following equations for modal cases:

$$V(\Box\alpha) = \Box V(\alpha),$$

$$V(\Diamond\alpha) = \Diamond V(\alpha).$$

Definition 5.2.6.

(i) A *positive modal inequality* is an expression of the form $\alpha \preccurlyeq \beta$, where α and β are positive modal formulas.

(ii) A *positive modal equation* is an expression of the form $\alpha \approx \beta$, where α and β are positive modal formulas.

Definition 5.2.7. Let L be a modal lattice.

(i) Let $\alpha \preccurlyeq \beta$ be a positive modal inequality. Then $\alpha \preccurlyeq \beta$ holds in L (denoted by $L \models \alpha \preccurlyeq \beta$) if for every modal valuation $V : P \to L$ we have $V(\alpha) \leqslant V(\beta)$.

(ii) Let $\alpha \approx \beta$ be a positive modal equation. Then $\alpha \approx \beta$ holds in L (denoted by $L \models \alpha \approx \beta$) if for every modal valuation $V : P \to L$ we have $V(\alpha) = V(\beta)$.

Lemma 5.2.8. *For a modal lattice L and positive modal formulas α and β,*

$$L \models \alpha \approx \beta \Leftrightarrow L \models \alpha \preccurlyeq \beta \text{ and } L \models \beta \preccurlyeq \alpha.$$

Proof. The proof is analogous to the proof of non-modal case, i.e., Lemma 4.1.5. ⊠

Definition 5.2.9. A *modal valuation* V on a modal PUP space X is a valuation on the PUP space X as described in Definition 4.1.6, extended to all positive modal formulas using the following equations for modal cases:

$$V(\Box\alpha) = [R]V(\alpha),$$

$$V(\Diamond\alpha) = \langle R\rangle V(\alpha).$$

We denote the relation $x \in V(\alpha)$ by $x \models^V \alpha$.

Definition 5.2.10. Let X be a modal PUP space.

(i) Let $\alpha \preccurlyeq \beta$ be a positive modal inequality. We write $X \models_{CPU} \alpha \preccurlyeq \beta$ if for every modal valuation V on X, $V(\alpha) \subseteq V(\beta)$.

(ii) Let $\alpha \approx \beta$ be a positive modal equation. We write $X \models_{CPU} \alpha \approx \beta$ if for every modal valuation V on X, $V(\alpha) = V(\beta)$.

Proposition 5.2.11. *Let L be a lattice.*

(i) Let $\alpha \preccurlyeq \beta$ be a positive modal inequality. Then $L \models \alpha \preccurlyeq \beta$ if and only if $X_L \models_{CPU} \alpha \preccurlyeq \beta$.

(ii) Let $\alpha \approx \beta$ be a positive modal equation. Then $L \models \alpha \approx \beta$ if and only if $X_L \models_{CPU} \alpha \approx \beta$.

Proof. The proof is analogous to the proof of non-modal case, i.e., Lemma 4.1.8. ⊠

Definition 5.2.12. A *modal PU-valuation* V on a modal PUP space X is a PU-valuation on the PUP space X as described in Definition 4.2.4, extended to all positive modal formulas using the following equations for modal cases:

$$V(\Box\alpha) = [R]V(\alpha),$$

$$V(\Diamond\alpha) = \langle R\rangle V(\alpha).$$

We denote the relation $x \in V(\alpha)$ by $x \models^V \alpha$. It should be clear from the context if V is a modal PU-valuation or just a modal valuation.

Definition 5.2.13. Let X be a modal PUP space.

(i) Let $\alpha \preccurlyeq \beta$ be a positive modal inequality. We write $X \models_{PU} \alpha \preccurlyeq \beta$ if for every PU-valuation V on X, $V(\alpha) \subseteq V(\beta)$.

(ii) Let $\alpha \approx \beta$ be a positive modal equation. We write $X \models_{PU} \alpha \approx \beta$ if for every PU-valuation V on X, $V(\alpha) = V(\beta)$.

We will develop now a version of Sahlqvist theory for modal lattices and filter completions and modal PUP spaces, i,e, modal analogues of the results from Section 4.2. This will parallel the duality approach to Sahlqvist theory developed by Sambin and Vaccaro [42].

Definition 5.2.14. Let L be a modal lattice and $\alpha(p_0, \ldots, p_n)$ a positive modal formula. By Definition 4.2.6, we know how to define an element $\alpha(a_0, \ldots, a_n) \in L$ in case of α being positive formula. We extend this definition to all positive modal formulas in the following way:

- ⋆ if $\alpha = \Box\beta$, then $\alpha(a_0, \ldots, a_n) = \Box\beta(a_0, \ldots, a_n)$;
- ⋆ if $\alpha = \Diamond\beta$, then $\alpha(a_0, \ldots, a_n) = \Diamond\beta(a_0, \ldots, a_n)$.

Lemma 5.2.15. *Fix some $a_1, \ldots, a_n$ in a modal lattice L. Then for every $a, b \in L$ and every positive formula $\alpha(p, p_1, \ldots, p_n)$, we have*

$$\alpha(a, a_1, \ldots, a_n) \wedge \alpha(b, a_1, \ldots, a_n) \geqslant \alpha(a \wedge b, a_1, \ldots, a_n).$$

Proof. We prove the result by induction on α, just as in the non-modal case, i.e., Lemma 4.2.7. Hence, we only spell the modal clauses in the proof. As before, for convenience we write $\xi(x)$ instead of $\xi(x, a_1, \ldots, a_n)$.

- ⋆ If $\alpha = \Box\beta$, we obtain $\alpha(a) \wedge \alpha(b) = \Box\beta(a) \wedge \Box\beta(b) = \Box(\beta(a) \wedge \beta(b)) \geqslant \Box(\beta(a \wedge b)) = \alpha(a \wedge b)$.
- ⋆ If $\alpha = \Diamond\beta$, we obtain $\alpha(a) \wedge \alpha(b) = \Diamond\beta(a) \wedge \Diamond\beta(b) \geqslant \Diamond(\beta(a) \wedge \beta(b)) \geqslant \Diamond(\beta(a \wedge b)) = \alpha(a \wedge b)$.

⊠

Next lemma is an analogue of the intersection lemma proved in the previous chapter and parallels the Sambin and Vaccaro intersection lemma [42] and the one result by Celani and Jansana [10, Theorem 4.2].

Lemma 5.2.16 (The modal intersection lemma)**.** *Let X be a modal PUP space and consider a modal lattice $PU(X)$. Then for each formula $\alpha(p, p_1, \dots, p_n)$ and principal upsets $\Uparrow x, \Uparrow x_1, \dots, \Uparrow x_n$ we have*

$$\alpha(\Uparrow x, \Uparrow x_1, \dots, \Uparrow x_n) = \bigcap\{\alpha(\Uparrow t, \Uparrow x_1, \dots, \Uparrow x_n) \mid \Uparrow t \in CPU(X),\ \Uparrow x \subseteq \Uparrow t\}.$$

Note that here we apply Definition 5.2.14 to the modal lattice $PU(X)$.

Proof. First of all note that since all our operations are monotone, $\alpha(\Uparrow x, \Uparrow x_1, \dots, \Uparrow x_n) \subseteq \alpha(\Uparrow t, \Uparrow x_1, \dots, \Uparrow x_n)$ for each $\Uparrow x \subseteq \Uparrow t$. Hence,

$$\alpha(\Uparrow x, \Uparrow x_1, \dots, \Uparrow x_n) \subseteq \bigcap\{\alpha(\Uparrow t, \Uparrow x_1, \dots, \Uparrow x_n) \mid \Uparrow t \in CPU(X),\ \Uparrow x \subseteq \Uparrow t\}$$

and the $\subseteq$ part is obvious. As before, we write for convenience $\xi(x)$ instead of $\xi(x, \Uparrow x_1, \dots, \Uparrow x_n)$. We use duality and view X as a dual modal PUP space to a modal lattice L. Since we know that all clopen principal upsets are exactly the ones of the form $\phi(a)$, we reformulate the claim as $\alpha(\Uparrow x) = \bigcap_{a \in x} \alpha(\phi(a))$. Note that ϕ is an isomorphism and therefore it commutes with every formula, i.e., $\xi(\phi(a)) = \phi(\xi(a))$. Here on the left side we use ξ for the modal lattice $CPU(X)$ and on the right side we use ξ for the modal lattice L in the way described in the Definition 5.2.14. Now we prove the lemma by induction, just as in the non-modal case, i.e., Lemma 4.2.8. Hence, we only spell the modal clauses of the proof.

- ⋆ Suppose $\alpha = \Box\beta$. Then using the induction hypothesis and the fact that $[R]$ commutes with intersections proven in Lemma 5.1.5, we get $[R]\beta(\Uparrow x) = [R]\bigcap_{a \in x} \beta(\phi(a)) = \bigcap_{a \in x}[R]\beta(\phi(a))$.

- ⋆ Suppose $\alpha = \Diamond\beta$. As one inclusion is obvious, let us show the other one. Suppose $y \in \bigcap_{a \in x}\langle R\rangle\beta(\phi(a)) = \bigcap_{a \in x}\phi(\Diamond\beta(a))$. Then for each $a \in x$, we have $\Diamond\beta(a) \in y$.

  Let $z = \uparrow\{\beta(a) : a \in x\}$. Then z is a filter, since $\beta(a) \wedge \beta(b) \geqslant \beta(a \wedge b)$ by Lemma 5.2.15. Also $z \subseteq \Diamond^{-1}[y]$, because if $b \geqslant \beta(a)$, then $\Diamond b \geqslant \Diamond\beta(a)$. Hence, for a filter $s = z \sqcup \Box^{-1}[y]$ we have yRs and $s \in \bigcap_{a \in x}\beta(\phi(a))$. Therefore $y \in \langle R\rangle\bigcap_{a \in x}\beta(\phi(a)) = \langle R\rangle\beta(\Uparrow x)$, due to the induction hypothesis.

⊠

The last statement of the proof is an instance of the so-called Esakia's lemma [18].

Theorem 5.2.17. *Let X be a modal PUP space and α and β be modal positive formulas. Then*

$$X \models \alpha \preccurlyeq_{CPU} \beta \Leftrightarrow X \models \alpha \preccurlyeq_{PU} \beta.$$

Proof. First note that since each modal valuation is also a modal PU-valuation, the $\Leftarrow$ part is obvious. Now suppose $X \not\models_{PU} \alpha \preccurlyeq \beta$. Then there is a modal PU-valuation V on X and $x \in X$, such that $x \models^V \alpha$ but $x \not\models^V \beta$. Take a family $y_p \in X$, such that $V(p) = \Uparrow y$. Using the intersection lemma for β, we get a family t_p, such that for every t_p the set $\Uparrow t_p$ is clopen, $\Uparrow y_p \subseteq \Uparrow t_p$ and $x \not\models^{V'} \beta$ under the valuation $V'(p) = \Uparrow t_p$. On the other hand, $x \models^{V'} \alpha$, since $V(p) \subseteq V'(p)$ and all our operations are monotone. So we constructed a modal valuation, showing that $X \not\models_{CPU} \alpha \preccurlyeq \beta$. ⊠

We also obtain a modal analogue of the Corollary 4.2.11 for modal lattices.

Recall that a class of modal lattices $\mathcal{C}$ is called a *variety* if there exists a set of modal lattice equations Γ such that $\mathcal{C}$ consists exactly of modal lattices L such that for every $\alpha \approx \beta \in \Gamma$ we have $L \models \alpha \approx \beta$.

Corollary 5.2.18. *Let L be a modal lattice and $\alpha \preccurlyeq \beta$ a modal lattice equation. If $L \models \alpha \preccurlyeq \beta$, then $F(L)^{\partial} \models \preccurlyeq \beta$. Therefore, every variety of modal lattices is closed under filter completions. Moreover, every variety of modal lattices is generated by a family of filter completions.*

Proof. By duality $L \models \alpha \preccurlyeq \beta$ if and only if $X_L \models_{CPU} \alpha \preccurlyeq \beta$. From Theorem 5.2.17, that is equivalent to $X_L \models_{PU} \alpha \preccurlyeq \beta$, which by the isomorphism τ holds if and only if $F(L)^{\partial} \models \alpha \preccurlyeq \beta$. ⊠

As mentioned in the beginning of the section, one can translate all the results developed so far to the ideal completion. Then we get a following modal version of Baker and Hales Theorem.

Corollary 5.2.19. *Every modal lattice equation that holds in a modal lattice L likewise holds in $I(L)$. Therefore, every variety of modal lattices L is closed under ideal completions.*

Proof. The proof is similar to the proof of Corollary 5.2.18. ⊠

5.3 First-order correspondence

We also show that there is a first-order correspondence for Sahlqvist sequents following the proof for the Sahlqvist Correspodence Theorem in [10, Section 4.1].

Definition 5.3.1. Let X be a modal PUP space. We denote by $R_{\square}$ the binary relation $R \circ \sqsubseteq$, i.e., $xR_{\square}y$ if there is $z \in X$ such that xRz and $z \sqsubseteq y$.

Lemma 5.3.2. *Let X be a modal PUP space. Then for every upset $S \subseteq X$, we have $[R]S = [R_{\square}]S$.*

Proof. ($\subseteq$) Let $x \in [R]S$. Take y such that $xR_{\square}y$. Then there is $z \sqsubseteq y$ such that xRz and therefore $z \in S$. Since S is an upset, $y \in S$ and $x \in [R_{\square}]S$.

($\supseteq$) Let $x \in [R_{\square}]S$. Take y such that xRy. Then also $xR_{\square}y$ and therefore $y \in S$. Hence, $x \in [R]S$. ⊠

Lemma 5.3.3. *For every modal PUP space X and $z \in X$ the set*

$$R^n_{\square}(z) = \{x \in X \mid \exists x_1, \dots, x_{n-1} : zR_{\square}x_1, x_1R_{\square}x_2, \dots, x_{n-1}R_{\square}x\}$$

is a principal upset.

Proof. Let $y_n = \bigsqcap R^n_{\square}(z)$. We prove $y_n \in R^n_{\square}(z)$ by induction on n.

For $n = 1$, first note that y_1 is also equal to $\bigsqcap R(z)$, where $R(z) = \{x \in X \mid zRx\}$. Then using condition 5 for modal PUP spaces, zRy_1 and $y_1 \in R^1_{\square}(z)$.

Now consider $R^{n+1}_{\square}(z)$. It is equal to $\bigcup\{R(x) : x \in R^n_{\square}(z)\}$. By induction hypothesis, for each $x \in R^n_{\square}(z)$, we have xRs_x, where $s_x = \bigsqcap R(x)$. Then using condition 5 for modal PUP spaces once again, $\bigsqcap R^n_{\square}(z) \ R \ \bigsqcap s_x$, yielding y_nRy_{n+1}. Hence, since $y_n \in R^n_{\square}(z)$, we obtain $y_{n+1} \in R^{n+1}_{\square}(z)$

Since we know $y_n \in R^n_\Box(z)$, there exist $x_1, \dots, x_{n-1} \in X$ such that

$$zR_\Box x_1, x_1 R_\Box x_2, \dots, x_{n-1} R_\Box y_n.$$

Hence, there exists $t \in X$ such that $x_{n-1}Rt$ and $t \sqsubseteq y_n$. Then if we take $s \sqsupseteq y_n$, we get $t \sqsubseteq s$, so $x_{n-1}R_\Box s$ and $s \in R^n_\Box(z)$. Therefore, $\Uparrow y_n = R^n_\Box(z)$ and $R^n_\Box(z)$ is a principal upset. . ⊠

Now we call an $R_\Box$-*term* a finite join of principal upsets of the forms $R^n_\Box(z)$, $\Uparrow z$ and the empty set. Then every $R_\Box$-*term* is either empty or a principal upset.

Lemma 5.3.4. *Each $R_\Box$-term $T(\vec{z})$ corresponds to a first-order formula $\xi_T(x, \vec{z})$ in the language $\{R, \sqsubseteq\}$, i.e., for every modal PUP space X and every $u, \vec{v} \in X$,*

$$u \in T(\vec{v}) \Leftrightarrow X \models \xi_T(u, \vec{v}).$$

Proof. For the $zR_\Box u$ we write a first-order formula $\exists t(zRt \wedge t \sqsubseteq u)$. Using this formula we can also write down $u \in R^n_\Box(z)$.

For the $u \in \Uparrow z$ we write a first-order formula $z \sqsubseteq u$ and $u \in \varnothing$ as $u \neq u$.

Finally, if $u \in T_1(\vec{v}) \Leftrightarrow X \models \xi_{T_1}(u, \vec{v})$ and $u \in T_2(\vec{v}) \Leftrightarrow X \models \xi_{T_2}(u, \vec{v})$, then

$$u \in T_1(\vec{v}) \sqcup T_2(\vec{v}) \Leftrightarrow X \models \exists u_1, u_2(\xi_{T_1}(u_1, \vec{v}) \ \wedge \ \xi_{T_2}(u_2, \vec{v}) \ \wedge \ (u_1 \sqcap u_2) \sqsubseteq u).$$

Note that we need to use $\sqcap$ which is first-order definable from $\sqsubseteq$. ⊠

Moreover, by an easy induction, which we omit here, this result can be generalized to the following proposition.

Proposition 5.3.5. *Let $\alpha(p_0, \dots, p_n)$ be a modal lattice formula and let $T_0(\vec{z}_0), \dots, T_n(\vec{z}_n)$ be $R_\Box$-terms in the variables $\vec{z}_0, \dots, \vec{z}_n$, respectively. Then there is a first-order formula $\xi(x, \vec{z}_0, \dots, \vec{z}_n)$ such that in every modal PUP space X, for every $u, \vec{v}_0, \dots, \vec{v}_n \in X$*

$$u \in \alpha(T_0(\vec{v}_0), \dots, T_n(\vec{v}_n)) \Leftrightarrow X \models \xi(u, \vec{v}_0, \dots, \vec{v}_n).$$

Definition 5.3.6. A positive modal formula is a *Sahlqvist antecedent* if it is built from $\top$, $\bot$ and boxed atoms by applying $\Diamond$ and $\wedge$.

Lemma 5.3.7. *Let $\alpha(p_0, \dots, p_n)$ be a Sahlqvist antecedent and X a modal PUP space. Then there are $R_\Box$-terms $T_0(\vec{x}_0), \dots, T_n(\vec{x}_n)$, and a (possibly empty) conjunction $\theta(x, y_0, \dots, y_k)$ of formulas of the forms xRy, $y = y$ and $y \neq y$, such that for all $\Uparrow t_0, \dots, \Uparrow t_n \in CPU(X)$ and every $u \in X$,*

$$u \in \alpha(\Uparrow t_0, \dots, \Uparrow t_n) \Leftrightarrow X \models \exists y_0, \dots, y_k(\theta(u, y_0, \dots, y_k) \wedge \bigwedge_i T_i(u, y_0, \dots, y_k) \subseteq \Uparrow t_i).$$

Proof. For shortness, let us denote the tuple $y_0, \dots, y_k$ by $\overline{y_i}$. We prove the lemma by induction on α.

- ⋆ If α is p_i, then $u \in \alpha(\Uparrow t_0, \dots, \Uparrow t_n) \Leftrightarrow u \in \Uparrow t_i \Leftrightarrow \Uparrow u \subseteq \Uparrow t_i$.
- ⋆ If α is $\Box^m p_i$, then $u \in \alpha(\Uparrow t_0, \dots, \Uparrow t_n) \Leftrightarrow u \in [R]^n(\Uparrow t_i) \Leftrightarrow R^n_\Box(u) \subseteq \Uparrow t_i$.
- ⋆ If α is $\top$, then $u \in \alpha(\Uparrow t_0, \dots, \Uparrow t_n) \Leftrightarrow u = u$.
- ⋆ If α is $\bot$, then $u \in \alpha(\Uparrow t_0, \dots, \Uparrow t_n) \Leftrightarrow u \neq u$.

- $\star$ If α is $\beta \wedge \gamma$, then using induction hypothesis, we get

$$u \in \beta(\Uparrow t_0, \dots, \Uparrow t_n) \Leftrightarrow \exists\, \overline{y_i}(\theta(u, y_0, \overline{y_i}) \wedge \bigwedge_i T_i(u, \overline{y_i}) \subseteq \Uparrow t_i)$$

and

$$u \in \gamma(\Uparrow t_0, \dots, \Uparrow t_n) \Leftrightarrow \exists\, \overline{y'_i}(\theta'(u, \overline{y'_i}) \wedge \bigwedge_i T'_i(u, y'_0, \dots, y'_k) \subseteq \Uparrow t_i).$$

Since

$$T_i(u, \overline{y_i}) \cup T'_i(u, \overline{y'_i}) \subseteq \Uparrow t_i \Leftrightarrow T_i(u, \overline{y_i}) \vee T'_i(u, \overline{y'_i}) \subseteq \Uparrow t_i,$$

we obtain that $u \in \alpha(\Uparrow t_0, \dots, \Uparrow t_n)$ holds if and only if

$$\exists\, \overline{y_i}, \overline{y'_i}((\theta(u, \overline{y_i}) \wedge \theta'(u, \overline{y'_i}) \wedge \bigwedge_i (T_i(u, \overline{y_i}) \vee T'_i(u, \overline{y'_i})) \subseteq \Uparrow t_i).$$

- $\star$ If α is $\Diamond\beta$, then using induction hypothesis, we get

$$u \in \beta(\Uparrow t_0, \dots, \Uparrow t_n) \Leftrightarrow \exists \overline{y_i}(\theta(u, \overline{y_i}) \wedge \bigwedge_i T_i(u, \overline{y_i}) \subseteq \Uparrow t_i).$$

Hence, $u \in \alpha(\Uparrow t_0, \dots, \Uparrow t_n)$ holds if and only if

$$\exists\, y\, \exists\, \overline{y_i}((\theta(y, \overline{y_i}) \wedge uRy) \wedge \bigwedge_i T_i(y, \overline{y_i}) \subseteq \Uparrow t_i).$$

$\boxtimes$

Now we are ready to prove the main theorem. For a modal lattice inequality $\alpha \preccurlyeq \beta$ we write $X, u \models \alpha \preccurlyeq \beta$ whenever for every modal valuation V on X, if $u \in V(\alpha)$ then $u \in V(\beta)$. We call a positive modal inequality $\alpha \preccurlyeq \beta$ a *Sahlqvist inequality* if α is a Sahlqvist antecedent.

Theorem 5.3.8. *Suppose $\alpha \preccurlyeq \beta$ is a Sahlqvist inequality. Then one can effectively construct a first-order formula $\xi(x)$ in the language $\{\sqsubseteq, R\}$ and with the only variable x free such that for every modal PUP space X and every $u \in X$,*

$$X, u \models \alpha \preccurlyeq \beta \Leftrightarrow X \models \xi(u).$$

Therefore, $X \models_{CPU} \alpha \preccurlyeq \beta \Leftrightarrow X \models \forall u\, \xi(u)$.

Proof. Let $p_0, \dots, p_n$ be variables occurring in α and β. Take modal PUP space X and $u \in X$. By Lemma 5.3.7 there are $R_\Box$-terms $T_0, \dots, T_n$ and a conjunction θ of formulas of the forms xRy, $y = y$ and $y \neq y$ such that

$$X, u \models \alpha \preccurlyeq \beta \Leftrightarrow \forall \Uparrow t_0, \dots, \Uparrow t_n \in CPU(X)(\exists\, \overline{y}(\theta \wedge \bigwedge_i T_i \subseteq \Uparrow t_i) \to u \in \beta(\Uparrow t_0, \dots, \Uparrow t_n)).$$

That is equivalent to

$$\forall y_0, \dots, y_k(\theta \to \forall \Uparrow t_0, \dots, \Uparrow t_n \in CPU(X)(\bigwedge_i T_i \subseteq \Uparrow t_i \to u \in \beta(\Uparrow t_0, \dots, \Uparrow t_n))).$$

Now recall that T_i is always a principal upset. Applying the intersection lemma, we see that the statement above is equivalent to

$$\forall y_0, \dots, y_k(\theta \to u \in \beta(T_0, \dots, T_n)).$$

Finally, as we know that there is a first-order formula $\zeta(x)$ satisfied if and only if $u \in \beta(T_0, \dots, T_n)$, we are done. $\boxtimes$

Note that our algorithm for constructing the first-order correspondent is exactly the same as described in [10, Section 4.1]. Below we provide the obtained first-order translation of some well-known Sahlqvist inequalities.

Example 5.3.9. Consider a Sahlqvist inequality $\Box p \preccurlyeq p$, usually known as reflexivity axiom. First we obtain a first-order correspondent of $\Box p$ using Lemma 5.3.7: $u \in [R]{\Uparrow}t \Leftrightarrow R_\Box(u) \subseteq {\Uparrow}t$. Hence, we have

$$X, u \models \Box p \preccurlyeq p \Leftrightarrow \forall {\Uparrow}t \in CPU(X)(R_\Box(u) \subseteq {\Uparrow}t \to u \in {\Uparrow}t).$$

As $R_\Box(u)$ is a principal upset, we can apply the intersection lemma to get $R_\Box(u) = \bigcap\{{\Uparrow}t \in CPU(X) \mid R_\Box(u) \subseteq {\Uparrow}t\}$. Therefore, $X, u \models \Box p \preccurlyeq p \Leftrightarrow u \in R_\Box(u)$. On the other hand, $u \in R_\Box(u) \Leftrightarrow uR_\Box u \Leftrightarrow \exists v(uRv \wedge v \sqsubseteq u)$. So the standard reflexivity axiom in our case corresponds to the followig condition:

$$\forall u \exists v(uRv \wedge v \sqsubseteq u).$$

Example 5.3.10. Another Sahlqvist inequality to consider is $\Box p \preccurlyeq \Box\Box p$, usually known as transitivity axiom. We already know $u \in [R]{\Uparrow}t \Leftrightarrow R_\Box(u) \subseteq {\Uparrow}t$. Hence, we have

$$X, u \models \Box p \preccurlyeq p \Leftrightarrow \forall {\Uparrow}t \in CPU(X)(R_\Box(u) \subseteq {\Uparrow}t \to u \in [R][R]{\Uparrow}t).$$

Using the intersection lemma in the same way as in the previous example, we get $X, u \models \Box p \preccurlyeq p \Leftrightarrow u \in [R][R]R_\Box(u)$. Finally, $u \in [R][R]R_\Box(u) \Leftrightarrow \forall x, y(uRx \wedge xRy \to uR_\Box y) \Leftrightarrow \forall x, y(uRx \wedge xRy \to (\exists v(uRv \wedge v \sqsubseteq y))$. So the standard transitivity axiom in our case corresponds to the followig condition:

$$\forall u, x, y(uRx \wedge xRy \to (\exists v(uRv \wedge v \sqsubseteq y)).$$

We developed a duality for modal lattices, generalizing PUP duality. We also extended the filter completion construction from Section 4.2 to the modal setting and proved that filter completions preserve inequalities. Finally, we demonstrated that Sahlqvist inequalities correspond to first-order sentences. We will come back to these results in Chapter 7 and show how they relate to positive modal logics. In particular, the preservation results will imply Sahlqvist completeness result for positive modal logics.

CHAPTER 6

Duality for nabla lattices and ortholattices

In this chapter we study modal lattices with an order-reversing modality. This modality has been studied in the context of distributive lattices in [22] and we partially follow this approach. We develop a duality for lattices with an order-reversing modality by extending the PUP duality. Prime examples of lattices with a order-reversing modality are ortholattices. Their duality was developed by Goldblatt [25] and Bimbó [6]. We obtain the Goldblatt-Bimbo duality for ortholattices as a consequence to our duality result.

6.1 Nabla Principal upset Priestley duality

We start by defining a new class of lattices by adding a modality that turns joins into meets.

Definition 6.1.1. A lattice L with an operator $\nabla : L \to L$ on it is a *nabla lattice* if it satisfies the following two conditions.

1. $\nabla(a \vee b) = \nabla a \wedge \nabla b$,
2. $\nabla 0 = 1$.

In [22] this modality was denoted by $\triangleright$. We chose nabla as it is easier to pronounce. Next we show that ∇ is order-reversing.

Proposition 6.1.2. *For every a, b in a nabla lattice L, if $a \leqslant b$, then $\nabla b \leqslant \nabla a$.*

Proof. Let $a \leqslant b$. Then $\nabla b = \nabla(a \vee b) = \nabla a \wedge \nabla b$. ⊠

Now we define the dual to a nabla lattice.

Definition 6.1.3. Let X be a PUP space with a binary relation R^∇ on X. For a subset $S \subseteq X$ we define

$$[R^\nabla\rangle S = \{x \in X \mid \forall y : (xR^\nabla y \to y \notin S)\}.$$

Then X is a *nabla PUP space* if it satisfies the following conditions.

1. If $\Uparrow x$ is clopen, then $[R^{\nabla}\rangle\Uparrow x$ is clopen.

2. If $x\not R^{\nabla}y$, then there is a clopen upset $\Uparrow t$, such that $x \in [R^{\nabla}\rangle\Uparrow t$ and $y \in \Uparrow t$.

3. $\sqsubseteq \circ R^{\nabla} \circ \sqsupseteq\ \subseteq\ R^{\nabla}$, i.e. if $x \sqsubseteq yR^{\nabla}z \sqsupseteq t$, then $xR^{\nabla}t$.

4. If $\Uparrow t$ is clopen and for some family (possibly empty) of $\{x_i : i \in I\}$, we have $\bigsqcap x_i R^{\nabla} t$, then there is x_i, such that $x_i R^{\nabla} t$.

5. If $xR^{\nabla}(y_1 \sqcap y_2)$, then either $xR^{\nabla}y_1$ or $xR^{\nabla}y_2$.

6. For each x , $x\not R^{\nabla}\top$.

Let L be a nabla lattice and X_L its dual PUP space. For $t \in X_L$ we denote by $\bar{t}$ the set $L \setminus t$. Define R_L^{∇} on X_L by

$$xR_L^{\nabla}y \Leftrightarrow y \subseteq \overline{\nabla^{-1}[x]}.$$

Consider the lattice isomophism $\phi : L \to CPU(X_L)$. Before showing that (X_L, R_L^{∇}) is a nabla PUP space, we demonstrate that ϕ also preserves ∇, where ∇ on $CPU(X_L)$ is defined by $[R_L^{\nabla}\rangle$.

Lemma 6.1.4. *Let L be a nabla lattice. For every $a \in L$, we have $\phi(\nabla a) = [R_L^{\nabla}\rangle\phi(a)$.*

Proof. ($\subseteq$) Suppose ∇a belongs to a filter x and $xR_L^{\nabla}y$. Then since $y \subseteq \overline{\nabla^{-1}[x]}$, the element a does not belong to y. Therefore, $x \in [R_{\nabla}\rangle\phi(a)$.

($\supseteq$) Suppose $x \in [R_{\nabla}\rangle\phi(a)$. Let $y = \uparrow a$. Then certainly $x\not R_{\nabla}y$, which means, there is b, such that $b \in y$ and $\nabla b \in x$. But the former means $a \leqslant b$ and applying Proposition 6.1.2, we get $\nabla b \leqslant \nabla a$. Therefore, $\nabla a \in x$ and $x \in \phi(\nabla a)$. ⊠

Theorem 6.1.5. *For every nabla lattice L, the space (X_L, R_L^{∇}) is a nabla PUP space.*

Proof. We prove the conditions for nabla PUP spaces one by one in the order that they are presented in the Definition 6.1.3

1. If $\Uparrow x$ is clopen, then for some $a \in L$, $\Uparrow x = \phi(a)$. Hence, using Lemma 6.1.4, $[R_L^{\nabla}\rangle\Uparrow x = \phi(\nabla a)$ which is clopen.

2. Suppose $x\not R_L^{\nabla}y$. Then there is a, such that $a \in y$ and $\nabla a \in x$. Hence, $x \in [R_L^{\nabla}\rangle\phi(a) = \phi(\nabla a)$ and $y \in \phi(a)$, so $\phi(a)$ is the desired $\Uparrow t$.

3. Suppose $x \sqsubseteq yR_L^{\nabla}z \sqsupseteq t$. Let $a \in t$. Then a also belongs to z and therefore $a \notin \nabla^{-1}[y]$. So $\nabla a \notin y$ and hence $\nabla a \notin x$. That implies $a \in \overline{\nabla^{-1}[x]}$ and $xR_L^{\nabla}t$.

4. Let $\phi(a)$ be an arbitrary clopen principal upset and suppose $\bigsqcap x_i R_L^{\nabla}\uparrow a$. Then we know that a does not belong to $\nabla^{-1}[\bigsqcap x_i] = \bigcap \nabla^{-1}[x_i]$. Therefore, there is x_i, such that $a \notin \nabla^{-1}[x_i]$. That gives us $x_i R_L^{\nabla}\uparrow a$, since if $b \geqslant a$, then $\nabla b \leqslant \nabla a$.

5. Suppose $xR_L^{\nabla}y_1 \sqcap y_2$, but neither $xR_L^{\nabla}y_1$ nor $xR_L^{\nabla}y_2$. That yields that there are elements a, b, such that $a \in y_1$, $b \in y_2$ and $\nabla a, \nabla b \in x$. Then also $\nabla a \wedge \nabla b \in x$ and therefore $\nabla(a \vee b) \in x$. But the join $a \vee b$ belongs to $y_1 \sqcap y_2$ and $\nabla(a \vee b)$ cannot belong to x, a contradiction.

6. Note that $xR_L^{\nabla}\top \Leftrightarrow L = \top \subseteq \overline{\nabla^{-1}[x]}$. But for each x, we have $\nabla 0 = 1 \in x$, so $\nabla^{-1}[x]$ cannot be empty and for $x\not R_L^{\nabla}\top$.

$\boxtimes$

Now we show how to go from a nabla PUP space to a nabla lattice. Let X be a nabla PUP space. Consider its dual lattice $CPU(X)$. We claim that $[R^{\nabla}\rangle$ is a well-defined operator on it.

Lemma 6.1.6. *Let X be a nabla PUP space and $\Uparrow t$ a clopen principal upset on it. Then the set $[R^{\nabla}\rangle\Uparrow t$ is also a clopen principal upset.*

Proof. The set $[R^{\nabla}\rangle\Uparrow t$ is clopen by Condition 1 for nabla PUP spaces.

Let $x = \bigsqcap [R^{\nabla}\rangle\Uparrow t$. We want to show $x \in [R^{\nabla}\rangle\Uparrow t$. Suppose the opposite. Then there is y such that $xR^{\nabla}y$ and $y \sqsupseteq t$. By condition 3, that also implies $xR^{\nabla}t$. Using condition 4, we get an element $z \in [R^{\nabla}\rangle\Uparrow t$, such that $zR^{\nabla}t$, which is a contradiction.

Now it suffices to prove that $[R^{\nabla}\rangle\Uparrow t$ is an upset. Let $x \in [R^{\nabla}\rangle\Uparrow t$ and $x \sqsubseteq y$. Suppose $yR^{\nabla}z$. Then if $z \sqsupseteq t$, we would have $xR^{\nabla}t$ by condition 3, which is a contradiction. Therefore, $z \notin \Uparrow t$ and $y \in [R^{\nabla}\rangle\Uparrow t$. $\boxtimes$

Note that this proof unlike the similar previous ones, explicitly uses that $\Uparrow t$ is clopen in order to show that $[R^{\nabla}\rangle\Uparrow t$ is principal. To be precise, we use Condition 4, which only works for clopen principal upsets. Therefore, we cannot prove that $PU(X)$ forms a nabla lattice and obtain a principal upset completion the same way as we did before. Indeed, it turns out that the operator $[R^{\nabla}\rangle$ is not always well-defined on $PU(X)$ as shown in the next examples.

Example 6.1.7. Take a well-order $\omega + 1$ denoted by A and its order opposite $(\omega + 1)^{\partial}$ denoted by B. To distinguish them we write n_A for elements in A and n_B for elements in B. We construct a lattice structure on $L = A \sqcup B$ by putting 0_A as the least element and 0_B as the greatest element, as shown in the picture 6.1.

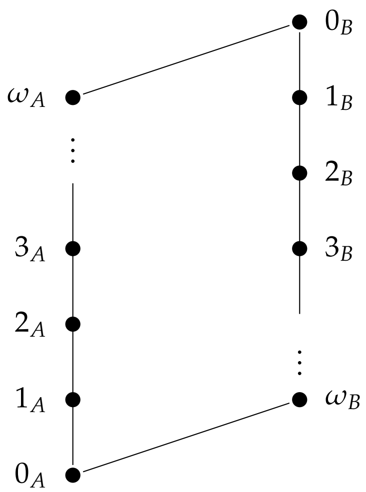


Figure 6.1: Example 6.1.7

We define ∇ on L by $\nabla n_A = n_B$ and $\nabla n_B = n_A$. Then it is easy to check that L is a nabla lattice. Therefore, X_L is a nabla PUP space. Consider a filter $F = \{n_B, \mid n \in \omega\}$.

We show that $[R_L^{\nabla}\rangle{\Uparrow}F$ is not principal by constructing a family of filters $\{G_n \in X_L \mid n \in \mathbb{N}\}$ such that each G_n belongs to $[R_L^{\nabla}\rangle{\Uparrow}F$ but the filter $\bigsqcap_n G_n$ does not.

Let $G_n = {\uparrow}n_A$ for $n \in \mathbb{N}_+$. Suppose $G_n R_L^{\nabla} H$ for some filter H. Then $H \subseteq \overline{\nabla^{-1}[G_n]} = \overline{{\downarrow}n_B}$. Therefore, $F \not\subseteq H$ as $n_B \in F$. Hence, $G_n \in [R_L^{\nabla}\rangle{\Uparrow}F$.

Now consider $\bigsqcap_n G_n = \{\omega_A, 0_B\}$. Then $\bigsqcap_n G_n\ R_L^{\nabla}\ F$, since $F \subseteq \overline{\{0_a, \omega_B\}} = \overline{\nabla^{-1}[\bigsqcap_n G_n]}$. Therefore, $\bigsqcap_n G_n \notin [R_L^{\nabla}\rangle{\Uparrow}F$. Hence, $[R_L^{\nabla}\rangle{\Uparrow}F$ is not principal and $[R_L^{\nabla}\rangle$ is not an operator on $PU(X_L)$.

In the next section we are going to discuss ortholattices as an example of nabla lattices. It turns out that the example above is a non-distributive ortholattice (for definition see 6.2.1). We also provide an example with a distributive lattice, which is however not an ortholattice.

Example 6.1.8. The construction is somewhat similar to the previous one. Let A be a well-order ω and B its order opposite ω^{∂}. Let L be the lattice of an order sum $A + B$, i.e., the set $A \sqcup B$ with the order defined by putting the set B above A. We denote the elements of L is the same fashion as before.

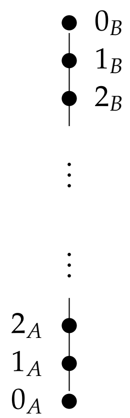


Figure 6.2: Example 6.1.8

We also once again define ∇ as $\nabla n_A = n_B$ and $\nabla n_B = n_A$ and consider X_L. Let F be the filter coinciding with the set B. We define $G_n = {\uparrow}n_A$ for $n \in \mathbb{N}_+$ and show that each G_n belongs to $[R_L^{\nabla}\rangle{\Uparrow}F$ but $\bigsqcap_n G_n$ does not.

Suppose $G_n R_L^{\nabla} H$. Then $H \subseteq \overline{\nabla^{-1}[G_n]} = {\uparrow}n - 1_B$. Therefore, $F \not\subseteq H$ and $G_n \in [R_L^{\nabla}\rangle{\Uparrow}F$.

Now consider $\bigsqcap_n G_n = F$. Since $F \subseteq F = \overline{\nabla^{-1}[F]}$, we have $F R_L^{\nabla} F$ and therefore $\bigsqcap_n G_n = F \notin [R_L^{\nabla}\rangle{\Uparrow}F$. Hence, $[R_L^{\nabla}\rangle{\Uparrow}F$ is not principal and $[R_L^{\nabla}\rangle$ is not an operator on $PU(X_L)$.

We go back to establishing the duality.

Theorem 6.1.9. *Let X be a nabla PUP space. Then the lattice $CPU(X)$ with an operator $[R^{\nabla}\rangle$ is a nabla lattice.*

Proof. We demonstrate the two axioms of nabla lattices one by one.

1. Take two clopen principal upsets $\Uparrow t$ and $\Uparrow s$. We want to show

$$[R^\nabla\rangle \Uparrow (t \sqcap s) = [R^\nabla\rangle \Uparrow t \sqcap [R^\nabla\rangle \Uparrow s.$$

   ($\subseteq$) Let $x \in [R^\nabla\rangle \Uparrow (t \sqcap s)$ and suppose $xR^\nabla y$. Then $y \not\sqsupseteq t \sqcap s$, hence $y \not\sqsupseteq t$ and $y \not\sqsupseteq s$. Therefore, $x \in [R^\nabla\rangle \Uparrow t \sqcap [R^\nabla\rangle \Uparrow s$.

   ($\supseteq$) Let $x \in [R^\nabla\rangle \Uparrow t \sqcap [R^\nabla\rangle \Uparrow s$ and suppose $xR^\nabla y$. Then if $y \sqsupseteq (t \sqcap s)$, we can use conditions 3 and 5 and get that either $xR^\nabla t$ or $xR^\nabla s$, which would be a contradiction. Therefore, $y \notin \Uparrow (t \sqcap s)$ and $x \in [R^\nabla\rangle \Uparrow (t \sqcap s)$.

2. The top element of $CPU(X)$ is X itself and the bottom one is $\{\top\}$, where $\top$ is the top element of X. Then $[R^\nabla\rangle\{\top\} = \{x \in X \mid \forall y : xR^\nabla y \to y \neq \top\}$. By condition 6, that is equal to X.

$\boxtimes$

Finally, we prove that the isomorphisms ϕ and ψ preserve the new structures.

Theorem 6.1.10.

(i) Let L be a nabla lattice. Then the map $\phi : L \to CPU(X_L)$ preserves the operator ∇.

(ii) Let X be a nabla PUP space. Then the map $\psi : X \to X_{CPU(X)}$ preserves and reflects the binary relation R^∇.

Proof. The first part was already shown in Lemma 6.1.4. For the second one denote the relation R^∇ on $X_{CPU(X)}$ by $\tilde{R^\nabla}$.

First suppose $xR^\nabla y$ for $x, y \in X$. We want to show $\psi(x)\tilde{R^\nabla}\psi(y)$, which is equivalent to $\psi(y) \subseteq \overline{[R^\nabla\rangle^{-1}[\psi(x)]}$. Take a clopen principal upset U on X such that $y \in U$. Then since $xR^\nabla y$, we have $x \notin [R^\nabla\rangle U$. Therefore, $U \notin [R^\nabla\rangle^{-1}[\psi(x)]$ and we are done.

Now suppose $x \not{R^\nabla} y$ for $x, y \in X$. Then by condition 2, there exists a clopen upset $\Uparrow t$ such that $x \in [R^\nabla\rangle \Uparrow t$ and $y \in \Uparrow t$. Therefore, $\Uparrow t \in y$ and $\Uparrow t \in [R^\nabla\rangle^{-1}[\psi(x)$. Hence, $\psi(x) \not{\tilde{R^\nabla}} \psi(y)$. $\boxtimes$

Now we move to the morphisms.

Definition 6.1.11. Let L and M be two nabla lattices. Then a lattice morphism $f : L \to M$ is a *nabla lattice morphism* if for each $a \in L$ we have $f(\nabla a) = \nabla f(a)$.

Definition 6.1.12. Let X and Y be nabla PUP spaces. Then a PUP morphism $f : X \to Y$ is a *nabla PUP morphism* if it satisfies the following conditions.

1. $xR^\nabla y \Rightarrow f(x)R^\nabla f(y)$.

2. $f(x)R^\nabla y \Rightarrow \exists z\ (xR^\nabla z, y \sqsubseteq f(z))$.

Proposition 6.1.13. *Let $f : L \to M$ be a nabla lattice morphism. Then $f^* : X_M \to X_L$ is a nabla PUP morphism.*

Proof. We prove the two conditions for a nabla PUP morphism one by one.

1. Suppose $FR^{\nabla}G$. We want to show $f^*(F)R^{\nabla}f^*(G)$, i.e., $f^{-1}[G] \subseteq \overline{\nabla^{-1}[f^{-1}[F]]}$. Take $a \in f^{-1}[G]$. Then $f(a) \in G$ and therefore it belongs to $\overline{\nabla^{-1}[F]}$, i.e., $\nabla f(a) \notin F$. Since f is a nabla lattice morphism, $\nabla f(a) = f(\nabla a)$ and $a \in \overline{\nabla^{-1}[f^{-1}[F]]}$.

2. Suppose $f^*[F]R^{\nabla}G$. Let $H = \uparrow f[G]$. Then since f is meet-preserving, H is a filter and clearly $G \subseteq f^*(H)$. Now it suffices to show $H \subseteq \overline{\nabla^{-1}[F]}$. Take $a \in H$. Then there is $b \in G$, such that $f(b) \leqslant a$. We know that $b \in \overline{\nabla^{-1}[f^{-1}[F]]}$, therefore $f(\nabla b) \notin F$. Since f and ∇ commute, $\nabla f(b)$ also does not belong to F. By Proposition 6.1.2, $\nabla a \leqslant \nabla f(b)$, so $\nabla a \notin F$ and $a \in \overline{\nabla^{-1}[F]}$. Therefore, $H \subseteq \overline{\nabla^{-1}[F]}$ and we are done.

⊠

Proposition 6.1.14. *Let $f : X \to Y$ be a nabla PUP morphism. Then $f_* : CPU(Y) \to CPU(X)$ is a nabla lattice morphism.*

Proof. Let $\Uparrow t$ be a clopen principal upset on Y. Then we need to show

$$[R^{\nabla}\rangle f^{-1}[\Uparrow t] = f^{-1}[\,[R^{\nabla}\rangle \Uparrow t].$$

($\subseteq$) Take $x \in [R^{\nabla}\rangle f^{-1}[\Uparrow t]$. Suppose $f(x)R^{\nabla}y$. Then there is $z \in X$, such that $xR^{\nabla}z$ and $y \sqsubseteq f(z)$. So $z \in f^{-1}[\Uparrow t]$, hence $f(z) \in \Uparrow t$ and y also belongs to $\Uparrow t$. Therefore, $x \in f^{-1}[\,[R^{\nabla}\rangle \Uparrow t]$.

($\supseteq$) Take $x \in f^{-1}[\,[R^{\nabla}\rangle \Uparrow t]$. Suppose $xR^{\nabla}y$. Then $f(x)R^{\nabla}f(y)$ and therefore $f(y) \in \Uparrow t$. Hence, $y \in f^{-1}[\Uparrow t]$ and $x \in [R^{\nabla}\rangle f^{-1}[\Uparrow t]$.

⊠

Let NLat be the category of nabla lattices with nabla lattice morphisms and let NPUP be the category of nabla PUP spaces with nabla PUP morphisms.

Theorem 6.1.15. NPUP *is dually equivalent to* NLat.

Proof. The theorem follows from Theorem 3.4.12 and the results proven in this section.

⊠

6.2 Duality for ortholattices

One of the examples of nabla lattices are ortholattices, studied by Goldblatt in [25] and [24]. Even though in [25], a representation theorem for ortholattices is obtained, the orthogonality spaces considered there do not form a full dual for ortholattices. An attempt to generalize this approach to a functorial duality was made by Bimbó in [6], where she developed a duality between ortholattices and orthospaces, and we base the following chapter on her work. As ortholattices are special cases of nabla lattices, our duality also produces a duality for ortholattices. We show that the duality that we get is almost the duality established by Bimbó, the only difference being that Goldblatt and Bimbó take only proper filters, while we take all the filters of a lattice. Therefore, our spaces are basically orthospaces with an added top element, while ortho spaces are basically our spaces with a deleted top element.

We begin this chapter by giving the definition of an ortholattice, connecting it to nabla lattices and recalling some results from [6].

Definition 6.2.1. An *ortholattice* is a lattice $\langle L; \wedge, \vee, 0, 1,' \rangle$ with bounds $0, 1$ and a unary operation $'$ of orthocomplementation with the following properties:

1. $a \wedge a' = 0$ and $a \vee a' = 1$,
2. $a'' = a$,
3. $a \leqslant b \Rightarrow b' \leqslant a'$.

Proposition 6.2.2. *Every ortholattice is a nabla lattice with $\nabla a = a'$.*

Proof. First we want to prove $(a \vee b)' = a' \wedge b'$. Since $a \leqslant a \vee b$, we have $(a \vee b)' \leqslant a'$. Doing the same for b, we get $(a \vee b)' \leqslant a' \wedge b'$. On the other hand, $a' \wedge b' \leqslant a'$, hence $a = a'' \leqslant (a' \wedge b')'$. Doing the same for b, we get $a \vee b \leqslant (a' \wedge b')'$ and therefore $a' \wedge b' \leqslant (a \vee b)'$.

Now we prove $0' = 1$. As we know, $0 \vee 0' = 1$. But for every a, we have $0 \vee a = a$, therefore $0' = 1$. ☒

Using the duality for nabla lattices, we obtain the nabla PUP spaces dual to ortholattices by restricting to the nabla PUP spaces X such that $CPU(X)$ is an ortholattice. To be precise, we consider nabla PUP spaces X such that for each clopen principal upsets $\Uparrow t$, $\Uparrow s$ on X, the following conditions hold:

1. $\Uparrow t \wedge [R^{\nabla}\rangle \Uparrow t = \{\top\}$,
2. $\Uparrow t \vee [R^{\nabla}\rangle \Uparrow t = X$,
3. $[R^{\nabla}\rangle [R^{\nabla}\rangle \Uparrow t = \Uparrow t$,
4. $\Uparrow t \subseteq \Uparrow s \Rightarrow [R^{\nabla}\rangle \Uparrow s \subseteq [R^{\nabla}\rangle \Uparrow t$.

Proposition 6.2.3. *Every nabla PUP space satisfies the last condition. Therefore, we can omit it.*

Proof. Let $\Uparrow t$, $\Uparrow s$ be clopen principal upsets. Suppose $\Uparrow t \subseteq \Uparrow s$. Then $t \sqsupseteq s$. Take $x \in [R^{\nabla}\rangle \Uparrow s$ and suppose $x R^{\nabla} y$. Then $y \notin \Uparrow s$, so $y \not\sqsupseteq s$. Therefore, also $y \not\sqsupseteq t$ and $y \notin \nabla\rangle \Uparrow s \subseteq [R^{\nabla}\rangle$. Hence, $x \in [R^{\nabla}\rangle \Uparrow t$. ☒

Let us call a nabla PUP space satisfying the conditions above an *ortho-PUP space*. We want to show that ortho-PUP spaces are basically the same as orthospaces defined by Bimbó in [6] as follows.

Let X be a set with a binary relation $\perp$. For a subset $Y \subseteq X$ let

$$Y^* = \{x \in X \mid \forall y \in Y : x \perp y\}.$$

We say that Y is *regular* if $Y = Y^{**}$. By $CU(X)$ we denote the set of clopen upsets and by $RCU(X)$ we denote the set of regular clopen upsets. Note that in [6] the term "cone" is used instead of the term "upset".

Definition 6.2.4. An *orthospace* $\langle X, \leqslant, \perp \rangle$ is a compact topological space with an order $\leqslant$ and a binary irreflexive and symmetric relation $\perp$, satisfying the following conditions.

1. $x \not\leqslant y \Rightarrow \exists O \in RCU(X)\ x \in O \wedge y \notin O$.

2. $x \perp y \wedge x \leqslant z \Rightarrow z \perp y$.
3. $O \in RCU(X) \Rightarrow O^*$ is clopen.
4. $x \perp y \Rightarrow \exists O \in RCU(X)\ x \in O \wedge y \in O^*$.

We prove several easy lemmas about orthospaces.

Lemma 6.2.5. *Let X be an orthospace. Then $\varnothing \in RCU(X)$.*

Proof. The empty set is a upset and clopen by the definition of a topological space. Also $\varnothing^* = X$ and since $\perp$ is irreflexive, $X^* = \varnothing$. Therefore, $\varnothing$ is regular. ⊠

Lemma 6.2.6. *For each set Y in an orthospace X, we have $Y \subseteq Y^{**}$. Therefore, Y is regular if and only if $Y^{**} \subseteq Y$.*

Proof. Take $y \in Y$ and $z \in Y^*$. Then $z \perp y$ and therefore $y \in Y^{**}$. ⊠

Lemma 6.2.7. *For every regular sets O_1, O_2 in an orthospace X, the set $O_1 \cap O_2$ is also regular. Therefore, $O_1 \cap O_2 \in RCU(X)$ and $RCU(X)$ forms a meet-semilattice.*

Proof. It suffices to show $(O_1 \cap O_2)^{**} \subseteq O_1 \cap O_2 = O_1^{**} \cap O_2^{**}$. Take $x \in (O_1 \cap O_2)^{**}$ and $y \in O_1^*$. Then $y \in (O_1 \cap O_2)^*$ and $x \perp y$. Therefore, $x \in O_1^{**}$ and similarly $x \in O_2^{**}$. Hence, $x \in O_1^{**} \cap O_2^{**}$. ⊠

Lemma 6.2.8. *Let X be an orthospace. Then for every $O \in RCU(X)$, the set O^* also belongs to $RCU(X)$.*

Proof. We know that O^* is clopen by Condition 3. In order to show that O^* is a upset, consider $x \in O^*$ and suppose $x \leqslant z$. Take $y \in O^*$. Then $x \perp y$ and by Condition 2 $z \perp y$. Therefore, $z \in O^*$. Finally, to prove that O^* is regular, we claim that for each $Y \subseteq X$, the set Y^* is regular. Indeed, if $s \in Y^{***}$, then for every $r \in Y$, r also belongs to Y^{**} by Lemma 6.2.6, so $s \perp r$ and $s \in Y^*$. Hence, $O^* \in RCU(X)$. ⊠

However, it turns out that Bimbó's definition does not give a complete duality with ortholattices, since it does not guarantee for the map $X \to PrFilt(RCU(X))$ to be surjective, where $PrFilt(L)$ is the set of proper filters on a lattice L. Therefore, we add one more condition to the definition.

Definition 6.2.9. An orthospace X is *ortho-sober* if for each proper filter F on $RCU(X)$, there exists $x \in X$ such that F is equal to $F_x = \{O \in RCU(X) \mid x \in O\}$.

Since orthospaces consist of proper filters of an ortholattice rather than of all the filters, we need to add a top element to an orthospace in order to get an ortho-PUP space and vice versa. We formalize it in the following way.

Take an ortho-PUP space X with a top element $\top$. We define X° as the structure obtained in the following way. The domain of X° is $X \setminus \{\top\}$. The topology and order are those induced by X. We define the binary relation $\perp$ on X° by $x \perp y \Leftrightarrow x \not{R}^{\nabla} y$.

Since we have to work with the top element $\top$ of X, we first explore some of its properties.

Lemma 6.2.10. *Let X be an ortho-PUP space and $\top$ its top element. Then for each x we have $x \not{R}^{\nabla} \top$ and $\top \not{R}^{\nabla} x$.*

Proof. By Condition 6 for nabla PUP spaces, $x \not R^{\nabla} \top$ for each x. Suppose $\top R^{\nabla} x$. Then using duality, it is equivalent to $F \subseteq \overline{\nabla^{-1}[L]}$ for a dual nabla lattice L and a filter on it. Since $\nabla^{-1}[L] = L$, that would mean $F = \varnothing$. Therefore, $\top \not R^{\nabla} x$. We could also prove this straightforwardly from Conditions 3 and 4 for nabla PUP spaces. $\boxtimes$

Now we establish connections between different properties of X and X°.

Lemma 6.2.11. *Let X be an ortho-PUP space. Then for every $Y \subseteq X^{\circ}$, we have*

$$Y^* = [R^{\nabla}\rangle Y \setminus \{\top\}.$$

Proof. ($\subseteq$) Let $x \in Y^*$ and suppose $xR^{\nabla}y$. Then $x \not\perp y$, hence $y \notin Y$. Therefore, $x \in [R^{\nabla}\rangle Y$.

($\supseteq$) Let $x \in [R^{\nabla}\rangle Y \setminus \{\top\}$ and let $y \in Y$. Then $x \not R^{\nabla} y$, hence $x \perp y$. Therefore, $x \in Y^*$ $\boxtimes$

Note that we need to throw out $\top$, since $[R^{\nabla}\rangle Y$ is a subset of X, not X°, while $Y^* \subseteq X^{\circ}$.

It also helps to show beforehand that X° is compact and that the relation $\perp$ is irreflexive and symmetric.

Lemma 6.2.12. *Let X be an ortho-PUP space. Then X° is a compact topological space.*

Proof. As the topology on X° is induced from X, it suffices to prove that $X \setminus \{\top\}$ is compact in X. Moreover, since X is compact, we only need to prove that $X \setminus \{\top\}$ is closed. Using duality for PUP spaces, we can view $\{\top\}$ as $\phi(0)$ and therefore $\{\top\}$ is clopen. Hence, $X \setminus \{\top\}$ is closed and X° is compact. $\boxtimes$

Lemma 6.2.13. *Let X be an ortho-PUP space. Then for every $x \neq \top$, we have $xR^{\nabla}x$. Therefore, the relation $\perp$ on X° is irreflexive.*

Proof. Suppose $x \neq \top$ but $x \not R_{\nabla} x$. Then by Condition 2 for nabla PUP spaces, there is a clopen principal upset $\Uparrow t$ such that $x \in \Uparrow t$ and $x \in [R_{\nabla}\rangle \Uparrow t$. But by Condition 1 on ortho-PUP spaces, this could only be true for the top element. Therefore, $xR^{\nabla}x$. $\boxtimes$

Lemma 6.2.14. *Let X be an ortho-PUP space. Then if $xR^{\nabla}y$, then $yR^{\nabla}x$. Therefore, the relation $\perp$ on X° is symmetric.*

Proof. Suppose $xR^{\nabla}y$, but $y \not R^{\nabla} x$. Then by Condition 2 for nabla PUP spaces, there is a clopen principal upset $\Uparrow t$ such that $y \in [R_{\nabla}\rangle \Uparrow t$ and $x \in \Uparrow t$. By Condition 3 for ortho-PUP spaces, x also belongs to $[R_{\nabla}\rangle [R_{\nabla}\rangle \Uparrow t$. But $xR_{\nabla}y$, which contradicts $y \in [R_{\nabla}\rangle \Uparrow t$. Therefore, $yR^{\nabla}x$. $\boxtimes$

Now we show that regular clopen upsets in X° are the same as clopen principal upsets in X modulo the top element. We denote by $Y^{\top}$ the set $Y \cup \{\top\}$ and by Y° the set $Y \setminus \{\top\}$.

Lemma 6.2.15. *Let X be an ortho-PUP space.*

(i) For each $O \in RCC(X^{\circ})$, we have $O^{\top} \in CPU(X)$

(ii) For each $U \in CPU(X)$, we have $U^{\circ} \in RCC(X^{\circ})$.

Therefore, $RCC(X^\circ) = \{U^\circ \mid U \in CPU(X)\}$.

Proof. (i) Take $O \in RCC(X^\circ)$ and consider $O^\top = O \cup \{\top\}$. We want to show $O^\top \in CPU(X)$. Since O is a clopen subset of X° and $\{\top\}$ is a clopen subset of X, the set $O^\top$ is clopen in X. As O is a upset, $O^\top$ is an upset. It is left to prove that $O^\top$ is principal. For that it suffices to prove that O is principal, i.e., $\bigsqcap O \in O$. Suppose $\bigsqcap O \notin O$, then also $\bigsqcap O \notin O^{**}$ since O is regular. Therefore, there is $y \in O^*$, such that $\bigsqcap O R^\nabla y$. Using Lemma 6.2.11, $y \in [R^\nabla\rangle O$ and for each $z \in O$ we have $y \not R^\nabla z$. Then by Condition 2 for nabla PUP spaces and Lemma 6.2.14, for each $z \in O$ there exists a clopen principal upset $\Uparrow t_z$ such that $y \in \Uparrow t_z$ and $z \in [R^\nabla\rangle \Uparrow t_z$.

Hence, $O \subseteq \bigcup_{z \in O}[R^\nabla\rangle \Uparrow t_z$ and using Lemma 6.2.12 and clopennes of O, we can find a finite set $z_1, \ldots, z_n \in O$ such that $O \subseteq [R^\nabla\rangle \Uparrow t_{z_1} \cup \ldots \cup [R^\nabla\rangle \Uparrow t_{z_n}$. Let $t = t_{z_1} \sqcup \ldots \sqcup t_{z_n}$. Then $y \sqsupseteq t$. Recall that $\bigsqcap O R^\nabla y$, so by Condition 3 for nabla PUP spaces, $\bigsqcap O R^\nabla t$. Since $\Uparrow t = \Uparrow t_{z_1} \cap \ldots \cap \Uparrow t_{z_n}$ is clopen, we can use condition 4 for nabla PUP spaces and find $z \in O$, such that $z R^\nabla t$. But z has to belong to one of $[R^\nabla\rangle \Uparrow t_{z_i}$, giving us $t \notin \Uparrow t_{z_i}$, which is a contradiction.

Therefore, O is principal and $O^\top \in CPU(X)$.

(ii) Now take $U \in CPU(X)$ and consider $U^\circ = U \setminus \{\top\}$. Then U° is already a upset in X°. Moreover, U° is clopen because $U^\circ = U \cap (X \setminus \{\top\})$ and $X \setminus \{\top\}$ is clopen. Hence, the only thing left to show is $(U^\circ)^{**} = U^\circ$.

($\subseteq$) Let $x \in (U^\circ)^{**}$. We claim $x \in [R^\nabla\rangle[R^\nabla\rangle U$. Suppose $x R^\nabla y$. Then since $y \neq \top$, we have $x \not\perp y$ and $y \notin (U^\circ)^*$. So there is $z \in U^\circ$ such that $y \not\perp z$. Then $y R^\nabla z$ and $z \in U$, so y cannot be an element of $[R^\nabla\rangle U$. Therefore, $x \in [R^\nabla\rangle[R^\nabla\rangle U$.
Now we use Condition 3 for ortho-PUP spaces to get $x \in U$ and, since $x \neq \top$, we have $x \in U^\circ$.

($\supseteq$) Let $x \in U^\circ$. Take $y \in (U^\circ)^*$. Then $y \perp x$ and since $\perp$ *issymmetrical* by Lemma 6.2.14, $x \perp y$. Therefore, $x \in (U^\circ)^{**}$.

Hence, $O^\circ \in RCC(X^\circ)$.

⊠

We are finally ready to show that X° is an ortho-sober orthospace.

Proposition 6.2.16. *For every ortho-PUP space X, the structure X° is an ortho-sober orthospace.*

Proof. First note that $\perp$ is irreflexive and symmetric due to Lemma 6.2.13 and Lemma 6.2.14. Now we prove other conditions for orthospaces one by one.

1. Suppose $x \not\sqsubseteq y$. We need to find a regular clopen upset O on X° such that $x \in O$ while $y \notin O$. Using the Priestley separation axiom for a PUP space X, we can find $U \in CPU(X)$ such that $x \in U$ and $y \notin U$. Then by Lemma 6.2.15, $U^\circ \in RCU(X)$ and since $x, y \neq \top$, the set U° works as the desired regular clopen upset.

2. Suppose $x \perp y$ and $x \sqsubseteq z$. We need to prove $z \perp y$. Suppose the opposite, that $z \not\perp y$ and therefore $z R^\nabla y$. Then $x \sqsubseteq z R^\nabla y$, so by Condition 3 for nabla PUP spaces, $x R^\nabla y$, which contradicts $x \perp y$. Hence, $z \perp y$.

3. Let $O \in RCC(X^\circ)$. We need to show that O^* is clopen. By Lemma 6.2.15, $O^\top = O \cup \{\top\} \in CPU(X)$. Hence, by Condition 1 for nabla PUP spaces, $[R^\nabla\rangle O^\top$ is clopen in X. Then since $X \setminus \{\top\}$ is clopen, the set $[R^\nabla\rangle O^\top \setminus \{\top\}$ is also clopen. On the other hand, by Lemma 6.2.11, $O^* = [R^\nabla\rangle O \setminus \{\top\}$. Therefore, it suffices to prove
$$[R^\nabla\rangle O^\top \setminus \{\top\} = [R^\nabla\rangle O \setminus \{\top\}.$$
   ($\subseteq$) Let $x \in [R^\nabla\rangle O^\top \setminus \{\top\}$. Suppose $xR^\nabla y$. Then $y \notin O^\top$ and $y \neq \top$, therefore $y \notin O$. Hence, $x \in [R^\nabla\rangle O \setminus \{\top\}$.

   ($\supseteq$) Let $x \in [R^\nabla\rangle O \setminus \{\top\}$. Suppose $xR^\nabla y$. Then $y \notin O$, therefore $y \notin O^\top$. Hence, $x \in [R^\nabla\rangle O^\top \setminus \{\top\}$.

   Therefore, O^* is clopen.

4. Let $x \perp y$. We need to find a regular clopen upset O such that $x \in O$ and $y \in O^*$. By symmetry of $\perp$ we have $y \not R^\nabla x$ and using Condition 2 for nabla PUP spaces we obtain $U \in CPU(X)$ such that $x \in U$ and $y \in [R^\nabla\rangle U$. Then by Lemma 6.2.15, $U^\circ = U \setminus \{\top\}$ is a regular clopen upset in X°. Then $x \in U^\circ$, since $x \neq \top$, so we only need to show $y \in (U^\circ)^*$.

   Take $z \in U^\circ$. Then z also belongs to U and therefore $y \not R^\nabla z$. Since $z \neq \top$, we also get $y \perp z$. Hence, $y \in (U^\circ)^*$ and U° is the desired regular clopen upset.

Finally, we prove that X° is ortho-sober. Consider a proper filter $\mathcal{F}$ on $RCC(X^\circ)$. Let $\mathcal{F}^\top = \{O \cup \{\top\} \mid O \in \mathcal{F}\}$. Then by Lemma 6.2.15, $\mathcal{F}^\top$ is a proper filter on $CPU(X)$. By duality for nabla PUP spaces, there exists $x \in X$, such that $\mathcal{F}^\top = \psi(x)$. Since $\mathcal{F}^\top$ is proper, $x \neq \top$. Therefore, $\mathcal{F} = F_x = \{O \in RCC(X^\circ) \mid x \in O\}$ and X° is ortho-sober. ⊠

Now we go the other way around and construct an ortho-PUP space from a ortho-sober orthospace. Take a ortho-sober orthospace X. We define $X^\top$ as a structure obtained in the following way. The domain of $X^\top$ is $X \cup \{\top\}$, where $\top$ is a new distinct element. The order is defined by making $\top$ the largest element. The open sets of $X^\top$ are the open sets of X and the open sets of X with added $\top$. We define the binary relation R^∇ by $xR^\nabla y \Leftrightarrow x \not\perp y$ for $x, y \neq \top$. For $\top$ we say that $\top \not R^\nabla x$ and $x \not R^\nabla \top$ for every $x \in X^\top$.

Just as before, we start with several lemmas.

Lemma 6.2.17. *Let X be a ortho-sober orthospace. Then for every $Y \subseteq X^\top$, we have*
$$[R^\nabla\rangle Y = (Y^\circ)^* \cup \{\top\}.$$

Proof. ($\subseteq$) Let $x \in [R^\nabla\rangle Y$ and $x \neq \top$. Take $y \in Y^\circ$. Then $y \in Y$ and therefore $x \not R^\nabla y$. Hence, $x \perp y$ and $x \in (Y^\circ)^*$.

($\supseteq$) First consider the element $\top$. Then there is no element y such that $\top R^\nabla y$ and therefore $\top \in [R^\nabla\rangle Y$ for every $Y \subseteq X^\top$.

Now let $x \in (Y^\circ)^*$. Suppose $xR^\nabla y$. Then since $y \neq \top$, we have $x \not\perp y$ and therefore $y \notin Y^\circ$. Hence, $y \notin Y$ and $x \in [R^\nabla\rangle Y$.

⊠

Lemma 6.2.18. *Let X be a ortho-sober orthospace.*

(i) For each $U \in CPU(X^\top)$ we have $U^\circ \in RCU(X)$.

(ii) For each $O \in RCU(X)$, we have $O^\top \in CPU(X^\top)$.

Therefore, $CPU(X^\top) = \{O^\top \mid O \in RCU(X)\}$.

Proof. (i) Let $\Uparrow t \in CPU(X^\top)$. First consider the case $t = \top$. Then $\Uparrow t = \varnothing^\top$ and since $\varnothing \in RCU(X)$ by Lemma 6.2.5, $\Uparrow t \in \{O^\top \mid O \in RCU(X)\}$.

Now let $t \neq \top$ and consider $(\Uparrow t)^\circ = (\Uparrow t) \setminus \{\top\}$. We want to show $(\Uparrow t)^\circ \in RCU(X)$. By construction of $X^\top$, the set $(\Uparrow t)^\circ$ is a clopen upset. So the only thing left is to show that $(\Uparrow t)^\circ$ is regular.

By Lemma 6.2.6 it suffices to show $((\Uparrow t)^\circ)^{**} \subseteq (\Uparrow t)^\circ$. Let $x \in ((\Uparrow t)^\circ)^{**}$ and suppose $x \notin (\Uparrow t)^\circ$, i.e., $x \not\geqslant t$. Then there is $O \in RCU(X)$ such that $t \in O$ and $x \notin O$. Since $x \notin O = O^{**}$, there exists $y \in O^*$ such that $x \not\perp y$. On the other hand, since $t \in O$ and O is a upset on X, we have $(\Uparrow t)^\circ \subseteq O$ and therefore $y \in ((\Uparrow t)^\circ)^*$. But that contradicts $x \in ((\Uparrow t)^\circ)^{**}$ and $x \not\perp y$.

Hence, $((\Uparrow t)^\circ)^{**} \subseteq (\Uparrow t)^\circ$ and $\Uparrow t \in \{O^\top \mid O \in RCU(X)\}$.

(ii) Let $O \in RCU(X)$. We need to show $O^\top \in CPU(X^\top)$. By construction of $X^\top$, the set $O^\top$ is a clopen upset. Hence, it suffices to prove that $O^\top$ is principal.

If $O = \varnothing$, then $O^\top = \{\top\}$ which is a principal upset $\Uparrow \top$. Otherwise consider a filter $\uparrow O$ on $RCU(X)$. Since $O \neq \varnothing$, the filter $\uparrow O$ is proper. Therefore, since X is ortho-sober, there exists $x \in X$ such that $\uparrow O = \{U \in RCU(X) \mid x \in U\}$. We claim that $O = \Uparrow x$ in X.

($\subseteq$) Let $y \in O$ and suppose $y \not\geqslant x$. Then there exists $U \in RCU(X)$, such that $x \in U$ and $y \notin U$. But since $\uparrow O = \{U \in RCU(X) \mid x \in U\}$, we should have $O \subseteq U$, which contradicts $y \notin U$. Therefore, $y \geqslant x$ and $y \in \Uparrow x$.

($\supseteq$) Let $y \in \Uparrow x$. Since $O \in \{U \in RCU(X) \mid x \in U\}$, we have $x \in O$ and $\Uparrow x \subseteq O$. Therefore, $y \in O$.

Hence, O is a principal upset. Then $O^\top$ is also principal.

$\boxtimes$

Proposition 6.2.19. *For every ortho-sober orthospace X, the structure $X^\top$ is an ortho-PUP space.*

Proof. First of all, we show that $X^\top$ is a complete meet-semilattice and therefore a lattice as well. Take $Y \subseteq X^\top$ and consider $Y^\circ = Y \setminus \{\top\}$. If Y° is empty, then the meet of Y is $\top$. Hence, we can assume that Y° is non-empty and show that it has a meet. Consider a proper filter on $RCU(X)$ defined as $\mathcal{F} = \{O \in RCU(X) : Y^\circ \subseteq O\}$. Then since X is ortho-sober, there exists $x \in X$ such that $\mathcal{F} = F_x$. We claim that x is a meet of Y°.

Let $y \in Y^\circ$. If $x \not\leqslant y$, then there is $O \in RCU(X)$ such that $x \in O$ but $y \notin O$. That implies $O \notin \mathcal{F}$ while $x \in O$, therefore $x \leqslant y$. Consider $z \in X$ such that for every $y \in Y^\circ$ we have $z \leqslant y$. If $z \not\leqslant x$, there should be $O \in RCU(X)$ such that $z \in O$ but $x \notin O$. But $z \in O$ implies $Y^\circ \subseteq O$, obtaining a contradiction. Therefore, $z \leqslant x$ and x is the meet of Y°.

Now we prove that $X^\top$ is a PUP space by considering the corresponding conditions.

1. In order to show that $X^\top$ is compact, suppose $X^\top = \bigcup_{i\in I} U_i$ for open sets U_i in $X^\top$. Then $X = \bigcup_{i\in I} U_i^\circ$. By construction of $X^\top$, each U_i° is open in X. Therefore, there are $U_1, \ldots, U_n$, such that $X = U_1^\circ \cup \ldots \cup U_n^\circ$. There is also U_k such that $\top$ belongs to it. Then $X^\top = U_1 \cup \ldots \cup U_n \cup U_k$. Hence, $X^\top$ is compact.

2. Suppose $x \nleqslant y$. Then $y \neq \top$. If $x = \top$, we can take a clopen principal upset $\{\top\} \in CPU(X^\top)$ to get $x \in \{\top\}$ and $y \notin \{\top\}$.

   Now assume $x \neq \top$. Then we know that there is $O \in RCU(X)$ such that $x \in O$ but $y \notin O$. Using Lemma 6.2.18, we know $O^\top \in CPU(X^\top)$. Then $x \in O^\top$ and $y \notin O^\top$. Hence, the Priestley separation axiom holds.

3. Finally we show that if $\Uparrow x$ and $\Uparrow y$ are clopen, then $\Uparrow(x \wedge y)$ is also clopen. If x or y is equal to $\top$, the claim becomes trivial. Therefore, we assume $x, y \neq \top$. Let $U = (\Uparrow x)^\circ$ and $V = (\Uparrow y)^\circ$. Then both U and V are non-empty. Let $S = (U^* \cap V^*)^*$. By Lemma 6.2.18, $U, V \in RCU(X)$, so by Lemma 6.2.7 and Lemma6.2.8, the set $S = (U^* \cap V^*)^*$ also belongs to $RCU(X)$. Moreover, $S \neq \varnothing$, since otherwise $U^* \cap V^* = X$, hence both U^* and V^* are equal to X and $U, V = \varnothing$. We claim $(\Uparrow(x \wedge y))^\circ = S$. Then $\Uparrow(x \sqcap y)$ is clopen by the definition of topology on $X^\top$ and since S is clopen.

   By Lemma 6.2.18 and since $S \neq \varnothing$, the set $S = (U^* \cap V^*)^*$ is principal, i.e., there exists $t \in X$, such that $S = \Uparrow t$. Then it suffices to show $t = x \wedge y$.

   First suppose $t \ngeqslant x \wedge y$. Then there is $O \in RCU(X)$ such that $x \wedge y \in O$ but $t \notin O$. Therefore, there exists $z \in O*$ such that $t \not\perp z$. Since $\Uparrow x, \Uparrow y \subseteq O$ we get $z \in (\Uparrow x)^* \cap (\Uparrow y)^* = U^* \cap V^*$. But $t \in S$ and $t \not\perp z$, which gives a contradiction. Therefore, $t \geqslant x \wedge y$.

   On the other hand, $x \in S$, since for every $z \in U^* \cap V^*$, we have $z \perp x$. Similarly, $y \in S$. Therefore, $t \leqslant x$ and $t \leqslant y$, so $t = x \wedge y$.

   Hence, $(\Uparrow(x \wedge y))^\circ = S$ and $X^\top$ is a PUP space.

Now we prove the conditions for $X^\top$ being the nabla PUP space.

1. Let $U \in CPU(X^\top)$. We want to show that $[R^\nabla\rangle U$ is clopen. By Lemma 6.2.17, $[R^\nabla\rangle U = (U^\circ)^* \cup \{\top\}$. Since $\{\top\}$ is clopen, it suffices to show that $(U^\circ)^*$ is clopen.

   By Lemma 6.2.18, $U^\circ \in RCU(X)$. Then using Condition 3 for orthospaces, $(U^\circ)^*$ is clopen. Therefore, $[R^\nabla\rangle U$ is clopen.

2. Suppose $x \not R^\nabla y$. First assume $x, y \neq \top$. Then by symmetry $y \perp x$ and by Condition 4 for orthospaces, there exists $O \in RCU(X)$ such that $x \in O^*$ and $y \in O$. Hence, for $O^\top \in CPU(X^\top)$ we have $y \in O^\top$. We need to show $x \in [R^\nabla\rangle O^\top$.

   Suppose $xR^\nabla z$. Then $z \neq \top$ and therefore $x \not\perp z$. Hence, $z \notin O$ and $z \notin O^*$. That means $x \in [R^\nabla\rangle O^\top$ and we are done with this case.

   Now suppose $x = \top$. Then consider a clopen principal upset $X^\top$. We have $y \in X^\top$ and $x \in [R^\nabla\rangle X^\top$, since there is no element z such that $xR^\nabla z$. Hence, this case is also done.

   Finally, suppose $y = \top$. Then consider a clopen principal upset $\{\top\}$. We have $y \in \{\top\}$ and $x \in [R^\nabla\rangle\{\top\}$, since $x \not R^\nabla \top$. Therefore, for every $x \not R^\nabla y$ we can construct $U \in CPU(X^\top)$ such that $x \in [R^\nabla\rangle U$ and $y \in U$.

3. Suppose $x \leqslant y R^{\nabla} z \geqslant t$, but $x \not R^{\nabla} t$. x, y, z and t cannot be $\top$, since otherwise $\top$ would be related to some elements. Then as elements of X, x and t are orthogonal, i.e. $x \perp t$. Hence, by Condition 2 for orthospaces and symmetry of $\perp$, also $y \perp t$ and $y \perp z$, which is a contradiction. Therefore, $x R^{\nabla} z$.

4. Let $\Uparrow t$ be clopen and $\bigwedge x_i R^{\nabla} t$ for some family x_i. Note that the family cannot be empty, since otherwise we have $\top R^{\nabla} t$. Suppose for each x_i we have $x_i \not R^{\nabla} t$. We can assume that no x_i is equal to $\top$, since they cannot all be $\top$ and the meet is the same if you throw all the top elements out of the family. Moreover, $t \neq \top$. Then each x_i is orthogonal to t and therefore by Condition 2 for orthospaces, $x_i \in ((\Uparrow t)^{\circ})^{*}$. By Lemma 6.2.18 and Lemma 6.2.8, $((\Uparrow t)^{\circ})^{*} \in RCU(X)$. Once again by Lemma 6.2.18, $((\Uparrow t)^{\circ})^{*} \cup \{\top\}$ is a principal upset, hence also $((\Uparrow t)^{\circ})^{*}$ is a principal upset. Therefore $\bigwedge x_i \in ((\Uparrow t)^{\circ})^{*}$, which contradicts $\bigwedge x_i R^{\nabla} t$. Thus, there exists x_i such that $x_i R^{\nabla} t$.

5. Suppose $x R^{\nabla} (y_1 \wedge y_2)$ but $x \not R^{\nabla} y_1$ and $x \not R^{\nabla} y_2$. Once again, all elements here have to be not equal to $\top$. Then we know $x \perp y_1$ and $x \perp y_2$. Using Condition 4 for orthospaces, we obtain $O_1, O_2 \in RCU(X)$, such that x belongs to both O_1^{*} and O_2^{*} while $y_1 \in O_1$ and $y_2 \in O_2$. As we know by Lemma 6.2.18, there exist t_1 and t_2 such that $O_i = \Uparrow t_i$ and also as we proved before

$$\Uparrow(t_1 \wedge t_2) = ((\Uparrow t_1)^{*} \cap (\Uparrow t_2)^{*})^{*}.$$

   Then $y_1 \wedge y_2$ belongs to the left side of the equality, while $x \in (\Uparrow t_1)^{*} \cap (\Uparrow t_2)^{*}$. Therefore, $x \perp (y_1 \wedge y_2)$, which is a contradiction. Thus, either $x R^{\nabla} y_1$ or $x R^{\nabla} y_2$.

6. For each x, we have $x \not R_{\nabla} \top$ by definition of R_{∇}.

We now know that $X^{\top}$ is a nabla PUP space. We conclude the proof by showing the three conditions for ortho-PUP spaces.

1. Take $U \in CPU(X^{\top})$. We need to show $U \cap [R^{\nabla}\rangle U = \{\top\}$. First note that $\top \in U$ and $\top \in [R^{\nabla}\rangle U$. On the other hand, if $x \neq \top$, then by irreflexivity of orthogonality, $x R^{\nabla} x$ and therefore x cannot belong to both U and $[R^{\nabla}\rangle U$. Therefore, $U \cap [R^{\nabla}\rangle U = \{\top\}$.

2. Take $\Uparrow t \in CPU(X^{\top})$. We need to show $\Uparrow t \vee [R^{\nabla}\rangle \Uparrow t = X^{\top}$. If $t = \top$, the equation becomes $\{\top\} \vee X^{\top} = X^{\top}$ and therefore holds. Assume $t \neq \top$.

   As we know, $[R^{\nabla}\rangle \Uparrow t$ is also a clopen principal upset, say $\Uparrow s$. Then $\Uparrow t \vee [R^{\nabla}\rangle \Uparrow t = \Uparrow t \vee \Uparrow s = \Uparrow(t \wedge s)$. Let $U = (\Uparrow t)^{\circ}$ and $V = (\Uparrow s)^{\circ}$. We claim $U^{*} = V$.

   ($\subseteq$ Let $x \in U^{*}$. Then $x \perp t$, so $x \not R^{\nabla} t$. We want to show $x \in [R^{\nabla}\rangle \Uparrow t$ and therefore $x \in V$. Suppose $x R^{\nabla} y$ but $y \in \Uparrow t$. Then $x \perp t$ and $t \leqslant y$, hence $x \perp y$, which is a contradiction. Therefore, $x \in V$.

   ($\supseteq$) Let $x \in V$. Then also $x \in [R^{\nabla}\rangle \Uparrow t$. We want to show $x \in U^{*}$, so consider $y \in U$. Then $y \in \Uparrow t$, hence $x \not R^{\nabla} y$. That means $x \perp y$ and therefore, $x \in U^{*}$.

   Now recall the proven equation $(\Uparrow(t \wedge s))^{\circ} = (U^{*} \cap V^{*})^{*}$. Applying $U^{*} = V$ and the previous step, we get $(\Uparrow(t \wedge s))^{\circ} = (V \cap V^{*})^{*} = \varnothing^{*} = X$. Therefore, $\Uparrow t \vee [R_{\nabla}\rangle \Uparrow t = X^{\top}$.

3. Let $U \in CPU(X^{\top})$. We need to show $[R_{\nabla}\rangle[R_{\nabla}\rangle U = U$. By Lemma 6.2.17, we know
$$[R_{\nabla}\rangle[R_{\nabla}\rangle U = (((U^{\circ})^{*} \cup \{\top\})^{\circ})^{*} \cup \{\top\} = (U^{\circ})^{**} \cup \{\top\}.$$
Then since $U^{\circ} \in RCU(X)$, we have $(U^{\circ})^{**} \cup \{\top\} = U^{\circ} \cup \{\top\} = U$.

⊠

Moreover, the defined maps between ortho-sober orthospaces and ortho-PUP spaces are essentially each other inverses.

Theorem 6.2.20. *For every orthospace X, we have $(X^{\top})^{\circ} = X$. For every ortho-PUP space X, we have $(X^{\circ})^{\top} = X$ once we choose the new top element for $(X^{\circ})^{\top}$ to be the original $\top$.*

Proof. Follows immediately from the constructions of X° and $X^{\top}$. ⊠

We showed that even though the duality for nabla lattices is not associated with transparent notion of completion such as a filter completion, it can be used to obtain duality for ortholattices, established in [6]. We now move to the final chapter, which investigated the logics connected to the three dualities defined in Chapters 3, 5 and 6.

CHAPTER 7

Positive modal logic beyond distributivity

In this chapter we introduce logical systems, which are complete for the structures studied in the previous chapters. First we study a positive logic that corresponds to lattices and PUP spaces, then we move to a positive modal logic that corresponds to modal lattices and modal PUP spaces. In the final part of the chapter we investigate a positive nabla logic that corresponds to nabla lattices and nabla PUP spaces. We prove completeness for each considered logic. We also show applying Theorems 4.2.9 and 5.2.17, that every logic axiomatized in the positive or positive modal language is complete for, what we call, general team and general modal team semantics. The name refers to the team semantics of [29] and [46].

7.1 Positive Logic

In this section we work with positive language. We construct logical systems similar to that of [20] (see also [14]) and use pairs of formulas as basic objects. For formulas α and β we call an expression $\alpha \trianglelefteq \beta$ a *consequence pair*.

Definition 7.1.1. A *positive logic* $\mathcal{L}$ is a set of consequence pairs that is closed under substitution and the following axioms and rules:

(reflexivity) $\quad \alpha \trianglelefteq \alpha$

(transitivity) $\quad \dfrac{\alpha \trianglelefteq \beta \quad \beta \trianglelefteq \gamma}{\alpha \trianglelefteq \gamma}$

(conjunction) $\quad \alpha \wedge \beta \trianglelefteq \alpha \quad \alpha \wedge \beta \trianglelefteq \beta \quad \dfrac{\alpha \trianglelefteq \beta \quad \alpha \trianglelefteq \gamma}{\alpha \trianglelefteq \beta \wedge \gamma}$

(disjunction) $\quad \alpha \trianglelefteq \alpha \vee \beta \quad \beta \trianglelefteq \alpha \vee \beta \quad \dfrac{\alpha \trianglelefteq \gamma \quad \beta \trianglelefteq \gamma}{\alpha \vee \beta \trianglelefteq \gamma}$

(top and bottom) $\quad \alpha \trianglelefteq \top \quad \bot \trianglelefteq \alpha$

As semantics for positive logic we use the relation on lattices $L \models \alpha \preccurlyeq \beta$. Let $\mathcal{L}$ be a positive logic. We denote by $L(\mathcal{L})$ the class of all lattices L such that for every $\alpha \trianglelefteq \beta \in \mathcal{L}$ we have $L \models \alpha \preccurlyeq \beta$. Then evidently $\mathcal{L}$ is sound with respect to $L(\mathcal{L})$. Moreover, by the next theorem, $\mathcal{L}$ is also complete for $L(\mathcal{L})$.

Theorem 7.1.2. *Let $\mathcal{L}$ be a positive logic and $\alpha \trianglelefteq \beta$ a consequence pair. For every $L \in L(\mathcal{L})$ we have that $L \models \alpha \preccurlyeq \beta$ implies $\alpha \trianglelefteq \beta \in \mathcal{L}$.*

Proof. Using the standard Lindenbaum-Tarski argument of algebraic logic. For details see, e.g., [19]. ⊠

Now we connect the lattice semantics to the PUP semantics. Let $\mathcal{L}$ be a positive logic. We denote by $PUP(\mathcal{L})$ the set of all PUP spaces X such that for every $\alpha \trianglelefteq \beta \in \mathcal{L}$ we have $X \models \alpha \preccurlyeq \beta$. Then evidently $\mathcal{L}$ is sound with respect to $PUP(\mathcal{L})$. Using completeness for lattices and duality between lattices and PUP spaces, we can also prove completeness for PUP spaces.

Theorem 7.1.3. *Let $\mathcal{L}$ be a positive logic and $\alpha \trianglelefteq \beta$ a consequence pair. For every $X \in PUP(\mathcal{L})$ we have that $X \models \alpha \preccurlyeq \beta$ implies $\alpha \trianglelefteq \beta \in \mathcal{L}$.*

Proof. Suppose for every $X \in PUP(\mathcal{L})$ we have $X \models \alpha \preccurlyeq \beta$. Consider $L \in L(\mathcal{L})$. Then by Proposition 4.1.8, the same inequalities hold in its dual PUP space X_L and therefore $X_L \in PUP(\mathcal{L})$. Hence, $X_L \models \alpha \preccurlyeq \beta$ and $L \models \alpha \preccurlyeq \beta$. We just showed that for every $L \in L(\mathcal{L})$ we have $L \models \alpha \preccurlyeq \beta$. Then by Theorem 7.1.2, $\alpha \trianglelefteq \beta \in \mathcal{L}$. ⊠

Now we introduce general team semantics, a generalization of the team semantics of [29] and [46]. A *general team frame (GT-frame)* is just a complete lattice. However, we think of GT-frames rather as the underlying lattice of a PUP space X, reflecting the way we define valuations on them. Note that each PUP space is indeed a GT-frame. Just like with PUP spaces, for a GT-frame X we define $PU(X)$ as the lattice of principal upsets of X. Note that it is isomorphic to X^∂.

Definition 7.1.4. A *PU-valuation* V on a GT-frame X is a valuation on the lattice $PU(X)$. To be precise, we first take a map $V : P \to PU(X)$ and then extend it to all positive formulas as follows:

$$V(\alpha \wedge \beta) = V(\alpha) \cap V(\beta),$$

$$V(\alpha \vee \beta) = V(\alpha) \vee V(\beta) = \{x \in X \mid \exists\, y \in V(\alpha), z \in V(\beta) : x \geqslant y \wedge z\},$$

$$V(\top) = X,$$

$$V(\bot) = \{\top\} = {\Uparrow}\top, \text{ where } \top \text{ is the top element of } X.$$

Note that when X is a PUP space, a PU-valuation is the same as a PU-valuation, discussed in Section 4.2.

Definition 7.1.5. Let $\alpha \preccurlyeq \beta$ be a positive inequality and X a GT-frame. Then $X \models_{GT} \alpha \preccurlyeq \beta$ if for every PU-valuation V we have $V(\alpha) \subseteq V(\beta)$.

We call this semantics a general team semantics since it generalizes the team semantics of Hodges [29] and Väänänen [46]. We basically replace the powerset frame with a complete lattice and our definition of the semantics of $\vee$ also slightly differs from that of classical team semantics. Note that in the case when X is a powerset frame our semantics coincides with team semantics. A similar version of a generalized team semantics in the intuitionistic setting was developed in [5] with a different motivation. Also [41] defines a team-like semantics for distributive logics which is close in spirit to ours.

Let $\mathcal{L}$ be a positive logic. We denote by $GT(\mathcal{L})$ the set of all GT-frames X such that for every $\alpha \trianglelefteq \beta \in \mathcal{L}$ we have $X \models_{GT} \alpha \preccurlyeq \beta$. Then $\mathcal{L}$ is sound with respect to $GT(\mathcal{L})$. We show it is also complete by constructing an analogue of the canonical model in our setting and applying the Sahlqvist preservation result from Section 4.2. Actually, it is not necessary to use canonical models as for GT-frames completeness follows immediately from preservation and completeness with respect to PUP spaces. But we would like to introduce the notion of a canonical model since it is an important ingredient of a semantical analysis of logical systems. Recall that the canonical model of distributive logics consists of prime theories. Since we are, in general, in a non-distributive setting we will work with all theories.

Fix a positive logic $\mathcal{L}$. First we define a theory of $\mathcal{L}$.

Definition 7.1.6. A *theory* T of $\mathcal{L}$ is a non-empty set of positive formulas such that:

if $\alpha \in T$ and $\alpha \trianglelefteq \beta \in \mathcal{L}$, then $\beta \in T$;

if $\alpha, \beta \in T$, then $\alpha \wedge \beta \in T$.

When $\mathcal{L}$ is clear from the context, we will call theories of $\mathcal{L}$ just theories. Let $X_{\mathcal{L}}$ be the set of all theories of $\mathcal{L}$. We order $X_{\mathcal{L}}$ by inclusion and define topology in the following way. Consider a map χ that maps every formula α to the set $\{T \in X_{\mathcal{L}} : \alpha \in T\}$. Then topology on $X_{\mathcal{L}}$ is generated by sets of the forms $\chi(\alpha)$ and $\overline{\chi(\alpha)}$, where as before $\overline{\chi(\alpha)} = X_{\mathcal{L}} \setminus \chi(\alpha)$.

Then it is easy to see via the standard argument that $X_{\mathcal{L}}$ is dual to the Lindenbaum-Tarski algebra of $\mathcal{L}$. Therefore, $X_{\mathcal{L}}$ is a PUP space and $X_{\mathcal{L}} \in PUP(\mathcal{L})$. Moreover, by using the proven preservation of inequalities by filter completions, i.e., Theorem 4.2.9, we obtain that as a GT-frame $X_{\mathcal{L}}$ is a also a model of $\mathcal{L}$.

Proposition 7.1.7. *The PUP space $X_{\mathcal{L}}$ seen as a GT-frame belongs to $GT(\mathcal{L})$.*

Proof. Consider $\alpha \trianglelefteq \beta \in \mathcal{L}$. By Theorem 4.2.9, $X_{\mathcal{L}} \models \alpha \preccurlyeq \beta \Leftrightarrow X_{\mathcal{L}} \models_{GT} \alpha \preccurlyeq \beta$. Since $X_{\mathcal{L}} \in PUP(\mathcal{L})$, we have $X_{\mathcal{L}} \models \alpha \preccurlyeq \beta$ and therefore $X_{\mathcal{L}} \models_{GT} \alpha \preccurlyeq \beta$. Hence, $X_{\mathcal{L}} \in GT(\mathcal{L})$. $\boxtimes$

Before going further we prove that each $\chi(\alpha)$ is a principal upset. For a set of positive formulas S, let $T(S)$ be the set of all positive formulas α such that there exists a finite subset $D \subseteq S$ with $\bigwedge_{\beta \in D} \beta \trianglelefteq \alpha \in \mathcal{L}$. In case S consists of one formula, we denote $T(\{\gamma\})$ by $T(\gamma)$.

Lemma 7.1.8. *For every set of positive formulas S, the set $T(S)$ is a theory of $\mathcal{L}$. Therefore, $T(S)$ is the smallest theory containing S.*

Proof. First suppose $\alpha \in T(S)$ and $\alpha \trianglelefteq \beta \in \mathcal{L}$. Then there is a finite subset $D \subseteq S$ such that $\bigwedge_{\gamma \in D} \gamma \trianglelefteq \alpha \in \mathcal{L}$. By the rule of transivity, $\bigwedge_{\gamma \in D} \gamma \trianglelefteq \beta \in \mathcal{L}$. Hence, $\beta \in T(S)$.

Now suppose $\alpha, \beta \in T(S)$. Then there are finite subsets $D_0, D_1 \subseteq S$ such that $\bigwedge_{\gamma \in D_0} \gamma \trianglelefteq \alpha$ and $\bigwedge_{\gamma \in D_1} \gamma \trianglelefteq \beta \in$ belong to $\mathcal{L}$. Let $\delta = \bigwedge_{\gamma \in D_0 \cup D_1} \gamma$. Then by the axioms of conjunction, $\delta \trianglelefteq \bigwedge_{\gamma \in D_0} \gamma \in \mathcal{L}$ and $\delta \trianglelefteq \bigwedge_{\gamma \in D_1} \gamma \in \mathcal{L}$. By transitivity, $\delta \trianglelefteq \alpha$ and $\delta \trianglelefteq \beta$ belong to $\mathcal{L}$. Finally, by the rule of conjuction $\delta \trianglelefteq \alpha \wedge \beta$ and $\alpha \wedge \beta \in T(S)$. Therefore, $T(S)$ is a theory of $\mathcal{L}$. ⊠

By this lemma, for every positive formula α, we have $\chi(\alpha) = \Uparrow T(\alpha)$. Therefore, each $\chi(\alpha)$ is a principal upset in $X_{\mathcal{L}}$.

Now we demonstrate that $X_{\mathcal{L}}$ is indeed a canonical model. For that we consider a special valuation V_c on $X_{\mathcal{L}}$ defined by $V_c(p) = \{T \in X_{\mathcal{L}} : p \in T\}$. Then $V_c(p) = \chi(p)$ and since $\chi(p) \in PU(X_{\mathcal{L}})$, the valuation V_c is well-defined.

Lemma 7.1.9 (Truth Lemma). *For every positive formula α and $T \in X_{\mathcal{L}}$, we have*

$$T \models^{V_c} \alpha \Leftrightarrow \alpha \in T.$$

Proof. We prove the Truth Lemma by induction on α. By definition of V_c, the propositional case is already resolved.

⋆ Suppose $\alpha = \beta \wedge \gamma$. Then

$$T \models^{V_c} \alpha \Leftrightarrow T \in V_c(\beta \wedge \gamma) \Leftrightarrow T \in V_c(\beta) \cap V_c(\gamma).$$

Applying induction hypothesis, we get that $T \models^{V_c} \alpha$ holds if and only if $\beta \in T$ and $\gamma \in T$. We claim that is equivalent to $\beta \wedge \gamma \in T$ and therefore the case of conjunction is proved.

($\Rightarrow$) Suppose $\beta, \gamma \in T$. Then since T is a theory of $\mathcal{L}$, $\beta \wedge \gamma \in T$.

($\Leftarrow$) Suppose $\beta \wedge \gamma \in T$. By conjunction axioms, $\beta \wedge \gamma \trianglelefteq \beta$ and $\beta \wedge \gamma \trianglelefteq \gamma$ belong to $\mathcal{L}$. Therefore, $\beta, \gamma \in T$.

⋆ Suppose $\alpha = \beta \vee \gamma$. Then we need to prove $T \in V_c(\beta) \vee V_c(\gamma) \Leftrightarrow \beta \vee \gamma \in T$.

($\Rightarrow$) Let $T \in V_c(\beta) \vee V_c(\gamma)$. Then there exist theories $T_0 \in V_c(\beta)$ and $T_1 \in V_c(\gamma)$ such that $T_0 \sqcap T_1 \sqsubseteq T$. By induction hypothesis, $\beta \in T_0$ and $\gamma \in T_1$. Since $\beta \trianglelefteq \beta \vee \gamma$ and $\gamma \trianglelefteq \beta \vee \gamma$ belong to $\mathcal{L}$, $\beta \vee \gamma \in T_0 \cap T_1$ and therefore $\beta \vee \gamma \in T$.

($\Leftarrow$) Let $\beta \vee \gamma \in T$. Consider theories $T(\beta)$ and $T(\gamma)$. By induction hypothesis, $T(\beta) \in V_c(\beta)$ and $T(\gamma) \in V_c(\gamma)$. Therefore, it suffices to prove that $T(\beta) \sqcap T(\gamma) \sqsubseteq T$. Let $\delta \in T(\beta) \sqcap T(\gamma)$. Then $\beta \trianglelefteq \delta$ and $\gamma \trianglelefteq \delta$ belong to $\mathcal{L}$. Therefore by the disjunction rule, $\beta \vee \gamma \trianglelefteq \delta \in \mathcal{L}$ and $\delta \in T$. Hence, $T(\beta) \sqcap T(\gamma) \sqsubseteq T$.

⋆ Suppose $\alpha = \top$. Then $T \models^{V_c} \top$ always holds. On the other hand, since every theory T is non-empty and $\alpha \trianglelefteq \top \in \mathcal{L}$, we also always have $\top \in T$.

⋆ Suppose $\alpha = \bot$. Then $T \models^{V_c} \bot$ if and only if T is the top element of $X_{\mathcal{L}}$. The top element of $X_{\mathcal{L}}$ is the inconsistent theory of all formulas, which surely contains $\bot$. On the other hand, since $\bot \trianglelefteq \alpha \in \mathcal{L}$, each theory that contains $\bot$ also contains all the formulas.

⊠

Finally, we use canonical model $X_{\mathcal{L}}$ to prove the completeness of $\mathcal{L}$ with respect to GT-frames.

Theorem 7.1.10 (General Team Completeness for positive logic)**.** *Let α and β be positive formulas and $\mathcal{L}$ a positive logic. For every $X \in GT(\mathcal{L})$ we have that $X \models_{GT} \alpha \preccurlyeq \beta$ implies $\alpha \trianglelefteq \beta \in \mathcal{L}$. Therefore, $\mathcal{L}$ is GT-complete.*

Proof. Suppose that for every $X \in GT(\mathcal{L})$ we have $X \models_{GT} \alpha \preccurlyeq \beta$. Then by Proposition 7.1.7 $X_{\mathcal{L}} \in GT(\mathcal{L})$ and $X_{\mathcal{L}} \models_{GT} \alpha \preccurlyeq \beta$. Consider the valuation $V_c(p) = \{T : p \in T\}$ on $X_{\mathcal{L}}$. Then we obtain $V_c(\alpha) \subseteq V_c(\beta)$. By Lemma 7.1.9, that implies $\beta \in T(\alpha)$ meaning $\alpha \trianglelefteq \beta \in \mathcal{L}$, which completes the proof. ⊠

We have thus proved that a positive logic is always GT-complete. Note that this is in contrast with standard modal logic where we require a logic to be axiomatized by Sahlqvist formulas in order to deduce its Kripke completeness by the same methods.

7.2 Positive Modal Logic

Positive modal logic has been studied in e.g., as [15, 9, 31]. In this section we develop a new logical system corresponding to our notion of modal lattices. In particular, we will generalize the results from the previous section to the modal case. Moreover, we apply Theorem 5.3.8 to show that a positive logic axiomatized by Sahlqvist inequalities is complete with respect to first-order definable structures. We work with positive modal language and consider consequence pairs with positive modal formulas.

Definition 7.2.1. A *positive modal logic* $\mathcal{L}$ is a set of consequence pairs that is closed under substitution and axioms and rules of positive logic as well as the following.

(□) $$\frac{\alpha \trianglelefteq \beta}{\Box\alpha \trianglelefteq \Box\beta}$$

(◊) $$\frac{\alpha \trianglelefteq \beta}{\Diamond\alpha \trianglelefteq \Diamond\beta}$$

(seriality) $\Box\alpha \trianglelefteq \Diamond\alpha$

(Dunn) $\Diamond\alpha \wedge \Box\beta \trianglelefteq \Diamond(\alpha \wedge \beta)$

(□ & ∧) $\Box\alpha \wedge \Box\beta \trianglelefteq \Box(\alpha \wedge \beta)$

(◊ & ∨) $\Diamond(\alpha \vee \beta) \trianglelefteq \Diamond\alpha \vee \Diamond\beta$

(□ & ⊤) $\top \trianglelefteq \Box\top$

(◊ & ⊥) $\Diamond\bot \trianglelefteq \bot$

The reason for adding the additional seriality axiom is to match modal lattices with the seriality axiom defined in Chapter 5.

Just as in the case of positive logic, we use as semantics for positive modal logic the relation on modal lattices $L \models \alpha \preccurlyeq \beta$. Let $\mathcal{L}$ be a positive modal logic. We denote by $ML(\mathcal{L})$ the class of all modal lattices L such that for every $\alpha \trianglelefteq \beta \in \mathcal{L}$ we have $L \models \alpha \preccurlyeq \beta$. Similarly to Section 7.1 we prove soundness and completeness for this semantics.

Theorem 7.2.2. *Let $\mathcal{L}$ be a positive modal logic and $\alpha \trianglelefteq \beta$ a consequence pair. For every $L \in ML(\mathcal{L})$ we have that $L \models \alpha \preccurlyeq \beta$ implies $\alpha \trianglelefteq \beta \in \mathcal{L}$.*

Proof. The standard Lindenbaum-Tarski argument, just as in Theorem 7.1.2. $\boxtimes$

Now we connect the modal lattice semantics to the modal PUP semantics. Let $\mathcal{L}$ be a positive modal logic. We denote by $MPUP(\mathcal{L})$ the set of all modal PUP spaces X such that for every $\alpha \trianglelefteq \beta \in \mathcal{L}$ we have $X \models \alpha \preccurlyeq \beta$. We prove the completeness with respect to modal PUP spaces using the modal PUP duality.

Theorem 7.2.3. *Let $\mathcal{L}$ be a positive modal logic and $\alpha \trianglelefteq \beta$ a consequence pair. For every $X \in MPUP(\mathcal{L})$ we have that $X \models \alpha \preccurlyeq \beta$ implies $\alpha \trianglelefteq \beta \in \mathcal{L}$.*

Proof. Same argument as in Theorem 7.1.3. $\boxtimes$

Before switching to team semantics, we apply the Theorem 5.3.8 about first-order correspondence of Sahlqvist inequalities to show that if a logic is axiomatized by Sahlqvist consequence pairs, then it is complete with respect to a first-order definable class of modal PUP spaces.

Definition 7.2.4. A consequence pair $\alpha \trianglelefteq \beta$ is a *Sahlqvist consequence pair* if α is built from $\top$, $\bot$ and boxed atoms by applying $\Diamond$ and $\wedge$.

Definition 7.2.5. A positive modal logic $\mathcal{L}$ is *Sahlqvist* if there exists a set S of Sahlqvist consequence pairs such that $\mathcal{L}$ is the smallest positive modal logic containing S.

Theorem 7.2.6. *Every Sahlqvist positive modal logic $\mathcal{L}$ is complete with respect to a class of modal PUP spaces that is first-order definable in the language $\{\sqsubseteq, R\}$.*

Proof. We know that $\mathcal{L}$ is complete with respect to $MPUP(\mathcal{L})$. Hence, it suffices to show that $MPUP(\mathcal{L})$ can be first-order defined in the language $\{\sqsubseteq, R\}$.

Let S be the set of Sahlqvist consequence pairs such that $\mathcal{L}$ is the smallest general positive modal logic containing S. Then applying Theorem 5.3.8 for each $\alpha \trianglelefteq \beta \in S$ we get a set Ξ of first-order formulas in the language $\{\sqsubseteq, R\}$. We claim that a modal PUP space X satisfies each formula of Ξ if and only if $X \in MPUP(\mathcal{L})$. Then as we know by Theorem 7.2.3 that $\mathcal{L}$ is complete with respect to $MPUP(\mathcal{L})$, it proves the theorem.

First suppose that a modal PUP space X satisfies all formulas Ξ. Then by Theorem 5.3.8, it satisfies all inequalities in S. As each modal PUP space admits the axioms and rules of positive modal logic, X has to satisfy all inequalities in the logic $\mathcal{L}$ generated by S.

Now suppose $X \in MPUP(\mathcal{L})$. Then for each $\alpha \trianglelefteq \beta \in S \subseteq \mathcal{L}$, we have $X \models \alpha \preccurlyeq \beta$. Therefore, by Theorem 5.3.8, X satisfies all formulas of Ξ. $\boxtimes$

Now we introduce general modal team semantics in the same way as in the previous section.

Definition 7.2.7. A *general modal team frame (GMT-frame)* is a complete lattice X with a binary relation R on it, satisfying Conditions 3, 4, 5, 6 for modal PUP spaces. That is:

(C3) $\sqsubseteq \circ R \subseteq R \circ \sqsubseteq$, i.e., if $x \sqsubseteq yRz$, then there is t, such that $xRt \sqsubseteq z$.

(C4) $\sqsupseteq \circ R \subseteq R \circ \sqsupseteq$, i.e., if $x \sqsupseteq yRz$, then there is t, such that $xRt \sqsupseteq z$.

(C5) If for some families $\{x_i \in X : i \in I\}$ and $\{y_i \in X : i \in I\}$ we have $x_i R y_i$ for every i, then $\bigsqcap_{i\in I} x_i \; R \; \bigsqcap_{i\in I} y_i$.

(C6) If for some family of $\{x_i \in X : i \in I\}$ we have $\bigsqcap_{i\in I} x_i R y$, then there is a family $\{y_i \in X : i \in I\}$, such that $x_i R y_i$ for each i and $\bigsqcap_{i\in I} y_i \sqsubseteq y$.

Note that each modal PUP space is indeed a GMT-frame. Just like with GT-frames, for a GT-frame X we define $PU(X)$ as the lattice of principal upsets of X. Note that it is isomorphic to X^∂. We show that $[R]$ and $\langle R\rangle$ are operators on $PU(X)$.

Lemma 7.2.8. *Let X be a GMT-frame. Then for every $\Uparrow x \in PU(X)$, the sets $[R]\Uparrow x$ and $\langle R\rangle\Uparrow x$ are also principal upsets.*

Proof. The proof is similar to the proof of Lemma 5.1.6.

Let $\Uparrow x$ be a principal upset. We start with the set $[R]\Uparrow x$. Let y be the meet of all elements of $[R]\Uparrow x$. Then we claim that y also belongs to $[R]\Uparrow x$. Suppose yRz. Then by condition 6, there is a family $\{z_i \in X : i \in I\}$, such that $\bigsqcap_{i\in I} z_i \sqsubseteq z$ and for each $x_i \in [R]\Uparrow x$ there is z_i such that $x_i R z_i$. Hence, for each z_i we have $z_i \geqslant x$ and $z \in \Uparrow x$. Therefore, $y \in [R]\Uparrow x$. Finally we show that $[R]\Uparrow x = \Uparrow y$. By construction, we already have $[R]\Uparrow x \subseteq \Uparrow y$. Take $t \sqsupseteq y$ and suppose tRs. By Condition 3, there is d such that $yRd \sqsubseteq s$. Then as we showed above, d is above x and $t \in [R]\Uparrow x$. Therefore, $[R]\Uparrow x$ is a principal upset.

Now we move to the set $\langle R\rangle\Uparrow x$. Let y be the meet of all elements of $\langle R\rangle\Uparrow x$. Then by condition 5, $y \in \langle R\rangle\Uparrow x$, so there is $z \sqsupseteq x$ such that yRz. By construction, we have $\langle R\rangle\Uparrow x \subseteq \Uparrow y$. Take $t \sqsupseteq y$. Using condition 4, we obtain an element s such that $tRs \sqsupseteq z \sqsupseteq x$. Therefore, $t \in \langle R\rangle\Uparrow x$ and $\langle R\rangle\Uparrow x$ is a principal upset. ⊠

Then we are able to define PU-valuations on GMT-frames.

Definition 7.2.9. A *modal PU-valuation* V on a GMT-frame X is a PU-valuation on a GT-frame X, as described in Definition 7.1.4, extended to all positive modal formulas using the following equations for modal cases:

$$V(\Box\alpha) = [R]V(\alpha),$$

$$V(\Diamond\alpha) = \langle R\rangle V(\alpha).$$

Note that when X is a modal PUP space, a modal PU-valuation is the same as a modal PU-valuation, discussed in Section 5.2.

Definition 7.2.10. Let $\alpha \preccurlyeq \beta$ be a positive modal inequality and X a GMT-frame. Then $X \models_{GMT} \alpha \preccurlyeq \beta$ if for every modal PU-valuation V we have $V(\alpha) \subseteq V(\beta)$.

Let $\mathcal{L}$ be a positive modal logic. We denote by $GMT(\mathcal{L})$ the set of all GMT-frames X such that for every $\alpha \trianglelefteq \beta \in \mathcal{L}$ we have $X \models_{GMT} \alpha \preccurlyeq \beta$. Then $\mathcal{L}$ is sound with respect to $GMT(\mathcal{L})$. We show it is also complete using the same strategy as in the previous section.

We construct the canonical model in the same way as we did before. We turn the PUP space $X_{\mathcal{L}}$ of theories of $\mathcal{L}$ into a modal PUP space by saying that TRS if and only if for each positive modal formula α we have

$$\Box\alpha \in T \Rightarrow \alpha \in S \text{ and } \alpha \in S \Rightarrow \Diamond\alpha \in T.$$

Then as $X_{\mathcal{L}}$ is a dual modal PUP space to the Lindenbaum-Tarski algebra, $X_{\mathcal{L}} \in MPUP(\mathcal{L})$. Moreover, as in the previous section, by preservation of inequalities by filter completions, $X_{\mathcal{L}}$ as a GMT-frame is a model of $\mathcal{L}$.

Proposition 7.2.11. *The modal PUP space $X_{\mathcal{L}}$ seen as a GMT-frame belongs to $GMT(\mathcal{L})$.*

Proof. Consider $\alpha \trianglelefteq \beta \in \mathcal{L}$. By Theorem 5.2.17, $X_{\mathcal{L}} \models \alpha \preccurlyeq \beta \Leftrightarrow X_{\mathcal{L}} \models_{GMT} \alpha \preccurlyeq \beta$. Since $X_{\mathcal{L}} \in MPUP(\mathcal{L})$, we have $X_{\mathcal{L}} \models \alpha \preccurlyeq \beta$ and therefore $X_{\mathcal{L}} \models_{GMT} \alpha \preccurlyeq \beta$. Hence, $X_{\mathcal{L}} \in GMT(\mathcal{L})$. ⊠

For a subset of positive modal formulas S we construct as before the smallest theory $T(S)$ containing S, as well as show that the sets $\chi(\alpha)$ are principal upsets. In order to prove the modal analogues of the Intersection Lemma and GT-completeness, we need one technical lemma.

Lemma 7.2.12. *Let T be a theory of $\mathcal{L}$ and let $S = \{\alpha : \Box\alpha \in T\}$. Then S is a theory of $\mathcal{L}$ and TRS.*

Proof. First we show that S is a theory of $\mathcal{L}$. Suppose $\alpha \in S$ and $\alpha \trianglelefteq \beta \in \mathcal{L}$. Then $\Box\alpha \in T$. By box axiom we have $\Box\alpha \trianglelefteq \Box\beta \in \mathcal{L}$, hence $\Box\beta \in T$. Therefore, $\beta \in S$.

Now suppose $\alpha, \beta \in S$. Then $\Box\alpha, \Box\beta \in T$ and $\Box\alpha \wedge \Box\beta \in T$. By axiom of box and conjunction, we also get $\Box(\alpha \wedge \beta) \in T$. Therefore, $\alpha \wedge \beta \in S$ and S is a theory.

Next we prove that TRS. The implication $\Box\alpha \in T \Rightarrow \alpha \in S$ follows immediately from the definition of S. Suppose $\alpha \in S$. Then $\Box\alpha \in T$. By seriality axiom, $\Box\alpha \trianglelefteq \Diamond\alpha \in \mathcal{L}$ and then $\Diamond\alpha \in T$. Therefore, TRS. ⊠

Now consider a special valuation V_c on $X_{\mathcal{L}}$ defined by $V_c(p) = \{T \in X_{\mathcal{L}} : p \in T\}$. Then $V_c(p) = \chi(p)$ and since $\chi(p) \in PU(X_{\mathcal{L}})$, the valuation V_c is well-defined.

Lemma 7.2.13 (Truth Lemma)**.** *For every positive modal formula α and $T \in X_{\mathcal{L}}$, we have*

$$T \models^{V_c} \alpha \Leftrightarrow \alpha \in T.$$

Proof. We prove the Truth Lemma by induction on α, just as in Theorem 7.1.9. Therefore, we only discuss the modal clauses here.

- ⋆ Suppose $\alpha = \Box\beta$. Then $T \models^{V_c} \Box\beta \Leftrightarrow T \in [R]V_c(\beta)$. We show $T \in [R]V_c(\beta) \Leftrightarrow \Box\beta \in T$.
  - ($\Rightarrow$) Let $T \in [R]V_c(\beta)$. Consider the theory $S = \{\alpha : \Box\alpha \in T\}$ from Lemma 7.2.12. Then TRS, so $S \in V_c(\beta)$. Using induction hypothesis, we get $\beta \in S$. Therefore, $\Box\beta \in T$.
  - ($\Leftarrow$) Let $\Box\beta \in T$ and suppose TRS for some theory S. Then by definition of R, we have $\beta \in S$. Therefore, $T \in [R]V_c(\beta)$
- ⋆ Suppose $\alpha = \Diamond\beta$. Then $T \models^{V_c} \Diamond\beta \Leftrightarrow T \in \langle R\rangle V_c(\beta)$. We show $T \in \langle R\rangle V_c(\beta) \Leftrightarrow \Diamond\beta \in T$.
  - ($\Rightarrow$) Let $T \in \langle R\rangle V_c(\beta)$. Then there exists a theory $S \in V_c(\beta)$ such that TRS. Since $\beta \in S$, we have $\Diamond\beta \in T$.

($\Leftarrow$) Let $\Diamond\beta \in T$. By Lemma 7.2.12, TRS for $S = \{\gamma : \Box\gamma \in T\}$. Let $U = S \vee T(\beta)$, that is $U = T(S \cup \{\beta\})$. We claim TRU.

The implication $\Box\gamma \in T \Rightarrow \beta \in U$ follows immediately since $S \subseteq U$. Now suppose $\gamma \in U$. That means there exists a finite subset $D \subseteq S$ such that $\beta \wedge \bigwedge_{\delta\in D} \delta \trianglelefteq \gamma \in \mathcal{L}$. Note that we include β as $\bigwedge_{\delta\in D} \delta \trianglelefteq \gamma \in \mathcal{L}$ implies $\beta \wedge \bigwedge_{\delta\in D} \delta \trianglelefteq \gamma \in \mathcal{L}$.

By monotonicity of diamond, $\Diamond(\beta \wedge \bigwedge_{\delta\in D} \delta) \trianglelefteq \Diamond\gamma \in \mathcal{L}$ and by Dunn's axiom, $\Diamond\beta \wedge \Box(\bigwedge_{\delta\in D} \delta) \trianglelefteq \Diamond(\beta \wedge \bigwedge_{\delta\in D} \delta) \in \mathcal{L}$. Since for each $\delta \in D$ we have $\Box\delta \in T$, by applying box and conjunction axiom we obtain $\Box(\bigwedge_{\delta\in D} \delta) \in T$. Then since $\Diamond\beta \in T$, we have $\Diamond\beta \wedge \Box(\bigwedge_{\delta\in D} \delta) \in T$ and therefore $\Diamond\gamma \in T$. Hence, TRU and since $U \in V_c(\beta)$ by induction hypothesis, we get $T \in \langle R\rangle V_c(\beta)$.

$\boxtimes$

Finally, we use the canonical model $X_{\mathcal{L}}$ to prove the completeness of $\mathcal{L}$ with respect to GMT-frames.

Theorem 7.2.14 (General Team Completeness for positive modal logic)**.** *Let α and β be positive modal formulas and $\mathcal{L}$ a positive modal logic. For every $X \in GMT(\mathcal{L})$ we have that $X \models_{GMT} \alpha \preccurlyeq \beta$ implies $\alpha \trianglelefteq \beta \in \mathcal{L}$. Therefore, $\mathcal{L}$ is GMT-complete.*

Proof. The proof is analogous to the non-modal case, i.e., Theorem 7.1.10. $\boxtimes$

We conjecture that the analogue of Theorem 7.2.6 holds for the GMT-semantics. Validating this conjecture would require establishing Sahlqvist correspondence for GMT-frames.

7.3 Positive Nabla Logic

For the final section we consider a logical system for nabla PUP spaces. We do everything analogous to previous chapters, but since principal upsets do not form a nabla lattice (see Examples 6.1.7 and 6.1.8), we cannot define General Team semantics for nabla logics. We work with positive nabla language and consider consequence pairs with positive nabla formulas.

Definition 7.3.1. A *positive nabla logic* $\mathcal{L}$ is a set of consequence pairs that is closed under substitution and axioms and rules of positive logic as well as the following.

(i) $$\frac{\alpha \trianglelefteq \beta}{\nabla\beta \trianglelefteq \nabla\alpha}$$

(ii) $\nabla\alpha \wedge \nabla\beta \trianglelefteq \nabla(\alpha \vee \beta)$

(iii) $\top \trianglelefteq \nabla\bot$

Just as in case of positive logic, we use as semantics for positive nabla logics the relation on nabla lattices $L \models \alpha \preccurlyeq \beta$. Similar to Section 7.1 we prove soundness and completeness for this semantics.

Let $\mathcal{L}$ be a positive nabla logic. We denote by $NL(\mathcal{L})$ the set of all nabla lattices L such that for every $\alpha \trianglelefteq \beta \in \mathcal{L}$ we have $L \models \alpha \preccurlyeq \beta$. Similar to Section 7.1 we prove soundness and completeness for this semantics.

Theorem 7.3.2. *Let $\mathcal{L}$ be a positive nabla logic and $\alpha \trianglelefteq \beta$ a consequence pair. For every $L \in NL(\mathcal{L})$ we have that $L \models \alpha \preccurlyeq \beta$ implies $\alpha \trianglelefteq \beta \in \mathcal{L}$.*

Proof. The standard Lindenbaum-Tarski argument, just as in Theorem 7.1.2. ⊠

Now we connect the nabla lattice semantics to the nabla PUP semantics. Let $\mathcal{L}$ be a positive nabla logic. We denote by $NPUP(\mathcal{L})$ the set of all nabla PUP spaces X such that for every $\alpha \trianglelefteq \beta \in \mathcal{L}$ we have $X \models \alpha \preccurlyeq \beta$. We prove the completeness with respect to nabla PUP spaces using the nabla PUP duality.

Theorem 7.3.3. *Let $\mathcal{L}$ be a positive nabla logic and $\alpha \trianglelefteq \beta$ a consequence pair. For every $X \in NPUP(\mathcal{L})$ we have that $X \models \alpha \preccurlyeq \beta$ implies $\alpha \trianglelefteq \beta \in \mathcal{L}$.*

Proof. Same argument as in Theorem 7.1.3. ⊠

In this section we presented three types of logical systems were presented, corresponding to the three dualities established in previous chapters. We also demonstrated how our results can be translated into logical framework. In the next, concluding chapter, we summarize the contributions of this thesis and discuss some open questions for the future work.

CHAPTER 8

Conclusion and future work

In this thesis we developed a basic theory of positive modal logic beyond distributivity by means of duality theory. Building on [37] and [4] we have established three new dualities: for lattices, modal lattices and nabla lattices. For lattices and modal lattices we defined the filter completions and proved by Sahlqvist style argument that filter completions preserve all inequalities. Moreover, we demonstrated how Sahlqvist-like inequalities correspond to first-order conditions. We also showed that nabla duality gives rise to a duality for ortholattices that is equivalent to the one established in [25] and [6]. Finally, we constructed logical systems corresponding to lattices, modal lattices and nabla lattices, introduced general team semantics and proved completeness for these logical systems with respect to general team semantics.

As far as we are aware this thesis provides a first systematic semantic study of positive (non-distributive) modal logics. A natural direction for future research is to develop this theory further. Below we underline several directions one could take for future research.

First of all, one could continue studying properties of considered algebraic structures via dualities established in this thesis. For example, one could try to define a filtration for (modal) PUP spaces in order to prove the finite model property for lattices, modal lattices or nabla lattices. Alternatively, one could develop a notion of bisimulation for PUP spaces and find out under which conditions modal equivalence implies it. We could also study various interesting types of lattices, such as lattices of logics and lattices of varieties using our dualities. Moreover, one could try to establish new dualities based on ours, for example, by adding new modalities.

One of the natural problems, arising from our work, is removing the extra seriality condition on modal lattices. This would probably require diverging from the Celani and Jansana approach [10], taken in this work.

Another area of research could be studying natural completions of nabla lattices. As far as we know, for ortholattices no completions with ”nice” properties has been discovered, and therefore such research would be of great value.

Moving to the study of logics, one could investigate multiple properties of positive modal logics (as well as of positive nabla logics) via duality; for instance, whether these systems enjoy the interpolation and uniform interpolation properties as well as studying their admissibility problem. Finally, there is a broad spectrum of possible future research directions considering the introduced general team semantics and general modal team semantics. This includes among others the finite model property, bisimulations, the analogues of the van Benthem and Goldblatt-Thomason theorems.

Bibliography

[1] S. Awodey. *Category Theory*. Oxford University Press, Inc., USA, 2nd edition, 2010.

[2] K.A. Baker and A.W. Hales. From a lattice to its ideal lattice. *Algebra Universalis 4*, pages 250–258, 1974.

[3] G. Bezhanishvili, N. Bezhanishvili, D. Gabelaia, and A. Kurz. Bitopological duality for distributive lattices and heyting algebras. *Mathematical Structures in Computer Science*, 20(3):359–393, 2010.

[4] N. Bezhanishvili and W. H. Holliday. Choice-free Stone duality. *The Journal of Symbolic Logic, 85(1)*, pages 109–148, 2020.

[5] N. Bezhanishvili and F. Yang. Intermediate logics and de Jongh formulas for team semantics. Manuscript, 2021.

[6] K. Bimbó. Functorial duality for ortholattices and De Morgan lattices. *Logica Universalis, 1*, pages 311–333, 2007.

[7] G. Birkhoff. *Lattice Theory*. American Mathematical Society, Providence, 3rd edition, 1967.

[8] P. Blackburn, M. Rijke, and Y. Venema. *Modal Logic*. Cambridge University Press, 2001.

[9] S. Celani and R. Jansana. A new semantics for positive modal logic. *Notre Dame Journal of Formal Logic*, 38(1):1–18, 1997.

[10] S. Celani and R. Jansana. Priestley duality, a Sahlqvist theorem and a Goldblatt-Thomason theorem for positive modal logic. *Logic Journal of the IGPL, Volume 7, Issue 6*, pages 683–715, 1999.

[11] A. Chagrov and M. Zakharyaschev. *Modal Logic*. Clarendon Press, 1997.

[12] W. H. Cornish. On H. Priestley's dual of the category of bounded distributive lattices. *Matematički Vesnik*, 12(27)(60):329–332, 1975.

[13] B. Davey and H. Priestley. *Introduction to Lattices and Order (2nd ed.)*. Cambridge: Cambridge University Press, 2002.

[14] J. de Groot. Positive monotone modal logic. *Studia Logica*, 109(4):829–857, 2021.

[15] J. M. Dunn. Positive modal logic. *Studia Logica: An International Journal for Symbolic Logic, vol. 55, no. 2*, pages 301–317, 1995.

[16] R. Engelking. *General topology*. PWN, Warsaw, 1977.

[17] L. Esakia. *Leo Esakia on Duality in Modal and Intuitionistic Logics*. Springer Netherlands, 2014.

[18] L. Esakia. *Heyting Algebras: Duality Theory*. Springer Publishing Company, Incorporated, 1st edition, 2019.

[19] J. Font. *Abstract Algebraic Logic - An Introductory Textbook*. College Publications, 2016.

[20] J. M. Font and R. Jansana. *General Index*, pages 147–152. Lecture Notes in Logic. Cambridge University Press, 2 edition, 2017.

[21] M. Gehrke and J. Harding. Bounded lattice expansions. *Journal of Algebra*, 238:345–371, 2001.

[22] M. Gehrke, H. Nagahashi, and Y. Venema. A Sahlqvist theorem for distributive modal logic. *Annals of Pure and Applied Logic*, 131(1):65–102, 2005.

[23] M. Gehrke and S. J. van Gool. Distributive envelopes and topological duality for lattices via canonical extensions. *Order 31*, pages 435–461, 2014.

[24] R. I. Goldblatt. Semantic analysis of orthologic. *J Philos Logic 3*, pages 19–35, 1974.

[25] R. I. Goldblatt. The Stone space of an ortholattice. *Bulletin of the London Mathematical Society, Volume 7, Issue 1*, pages 45–48, 1975.

[26] L. González and R. Jansana. A topological duality for posets. *Algebra universalis*, 76, 2016.

[27] G. Grätzer. *General Lattice Theory, second edition*. Springer Science & Business Media, 1998.

[28] G. Hartung. A topological representation of lattices. *Algebra Universalis 29*, pages 273–299, 1992.

[29] W Hodges. Compositional semantics for a language of imperfect information. *Logic Journal of the IGPL, Volume 5, Issue 4*, pages 539–563, 1997.

[30] W. Holliday. Possibility frames and forcing for modal logic. 2015. Manuscript.

[31] R. Jansana. Full models for positive modal logic. *Mathematical Logic Quarterly*, 48(3):427–445, 2002.

[32] B. Jónsson and A. Tarski. Boolean algebras with operators. Part I. *American Journal of Mathematics*, 73(4):891–939, 1951.

[33] B. Jónsson and A. Tarski. Boolean algebras with operators. Part II. *American Journal of Mathematics*, 74(1):127–162, 1952.

[34] M. Kracht. *Tools and Techniques in Modal Logic*. Elsevier, 1999.

[35] S. MacLane. *Categories for the Working Mathematician*. Springer-Verlag, New York, 1971. Graduate Texts in Mathematics, Vol. 5.

[36] J. McDonald and K. Yamamoto. Choice-free duality for orthocomplemented lattices. https://arxiv.org/abs/2010.06763, 2021.

[37] M. A. Moshier and P. Jipsen. Topological duality and lattice expansions, I: A topological construction of canonical extensions. *Algebra universalis 71.2*, pages 109–126, 2014.

[38] M. A. Moshier and P. Jipsen. Topological duality and lattice expansions, II: Lattice expansions with quasioperators. *Algebra universalis*, 71, 2014.

[39] H. A. Priestley. Ordered topological spaces and the representation of distributive lattices. *Proceedings of the London Mathematical Society*, s3-24(3):507–530, 1972.

[40] H.A. Priestley. Representation of distributive lattices by means of ordered Stone spaces. *Bulletin of the London Mathematical Society, 2*, pages 186–190, 1970.

[41] V. Punčochář. A generalization of inquisitive semantics. *Journal of Philosophical Logic*, 45(4):399–428, 2016.

[42] G. Sambin and V. Vaccaro. A new proof of Sahlqvist's theorem on modal definability and completeness. *The Journal of Symbolic Logic*, 54:992–999, 1989.

[43] M. H. Stone. The theory of representation for boolean algebras. *Transactions of the American Mathematical Society Vol. 40, No. 1*, pages 37–111, 1936.

[44] M. H. Stone. Topological representations of distributive lattices and Brouwerian logics. *Časopis pro pěstování matematiky a fysiky 067.1*, pages 1–25, 1938.

[45] A. Urquhart. A topological representation theory for lattices. *Algebra Universalis 8*, pages 45–58, 1978.

[46] J. Väänänen. Dependence logic: A new approach to independence friendly logic. *Cambridge: Cambridge University Press*, 2007.