The Expressive Power of Derivational Modal Logic

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

Alongside the traditional Kripke semantics, modal logic also enjoys a topological interpretation, which is becoming increasingly influential. We present various developments related to the topological derivational semantics based on the Cantor derivative operator. We establish useful characterizations of the validity of the axioms of bounded depth, and prove results of soundness and completeness for many other classical modal logics. We also address the expressivity of the topological μ-calculus, an extension of modal logic with fixpoint operators. We examine the tangled fragments of μ-calculus and show that they are not expressively complete. We also exhibit a large collection of classes of spaces that are definable in μ-calculus, but not in plain modal logic, thus demonstrating the strength of the former.

Acknowledgements

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Most importantly, I would like to dedicate this work to the courage and the endurance shown by my little brother Antoine during his illness and his endless, exhausting treatment.

Contents

Chapter 1

Introduction

Modal logic has proven itself as an invaluable tool for modelling reasoning about various concepts such as knowledge, belief, time, obligations, etc. While it is often associated with the relational semantics introduced by Saul Kripke [Kri63], it also enjoys a topological interpretation which can be traced back to the work of McKinsey and Tarski [MT44]: they proposed to interpret $\Diamond$ as the topological closure operator (hence introducing the *c-semantics*) and proved the celebrated result that **S4** is the logic of any separable metric dense-in-itself space. This was subsequently strengthened by Rasiowa and Sikorski [RS63] who eliminated the separability condition; for a good survey of these results we recommend [vBB07]. Since open sets can naturally be interpreted as *pieces of observation* [Vic96], this approach has recently gained momentum in fields such as formal epistemology [BBÖS19] [Özg17] and learning theory [dBY10].

A less known close kin of the c-semantics is the *derivational semantics*, or *d-semantics*. It is obtained by interpreting $\Diamond$ not as the closure, but as the *derived set* or *derivative* operator which is attributed to Georg Cantor. This variant was also introduced by McKinsey and Tarski, and further investigated by Esakia and others (see e.g., [Esa81][Esa01]). First, it must be noted that it is more expressive than the c-semantics, in the sense that any modally expressible property with respect to the c-semantics, is also modally expressible with respect to the d-semantics. This semantics thus enables a more refined classification of spaces. Further, while the logic of the c-semantics is **S4**, the logic of the d-semantics is **wK4**, as proved by Esakia [Esa01]. Since **wK4** is weaker than **S4**, it has more extensions, and thus more logics can be studied from the perspective of the d-semantics.

In spite of these compelling features, the d-semantics has received much less attention than the c-semantics, and our knowledge of it is largely incomplete: the interpretation of many standard logics is missing, and so are proofs of their completeness. One example is the axiom bd_n (for any natural integer n) which characterizes the Kripke frames that contain no path of length greater than n; in other words, those with *depth bounded by* n. With the topological semantics, bd_n also yields a notion of depth for spaces, and this kind of parameter is of great interest when it comes to classification: spaces with finite depth are generally easier to deal with, and their logics tend to have good properties (e.g., the finite model property). The topological interpretation of the concept of depth, however, is not obvious. This question was solved by Bezhanishvili et al. [BBLB$^+$17] for the c-semantics: they introduced for any space X a number called the *modal Krull dimension* of X, which is smaller than n exactly when X validates bd_n. However, the same problem for the d-semantics remained open prior to our work. We may also mention the axioms .2 and .3 whose relation to extremally disconnected spaces is well known in the c-semantics [vBB07], but whose interpretation in the d-semantics was still unexplored until now. This thesis aims to fill those gaps.

We also hope to identify promising proof techniques in response to the particular challenge of turning Kripke frames into appropriate topological spaces. Here by "appropriate" we mean in a way that preserves the truth of formulas with respect to the relevant semantics. This operation is crucial because it allows one to immediately transfer results of completeness from the Kripke semantics to the topological semantics. While this is straightforward in the c-semantics, the case of the d-semantics presents many difficulties related to reflexive points, that we will explain in detail. Our starting point will be the technique presented in [BBFD21], but as we will see, there are other options, each having its pros and cons.

This inquiry will be revealing of what derivational modal logic can express, but how about what it cannot? Among the proposals that have been made to increase its expressivity, the most powerful is certainly the μ-calculus, which enables self-referencing definitions [Koz83] through fixpoint operators. As a result it embeds many temporal logics, while remaining decidable, and this makes it very appreciated in formal verification. Its topological aspect is also of great interest: for example, one can express the Cantor-Bendixson's *perfect core* of a space in μ-calculus, but not in basic modal logic. The perfect core is the greatest subset which is equal to its own derivative, and it plays a central role in the modelling of epistemic puzzles such as the Surprise Examination Paradox [Par92].

The completeness of the μ-calculus is a notoriously difficult problem, and recent progress has been achieved in [BBFD21] regarding the topological completeness of many axiom systems. Yet we do not know whether the μ-calculus defines more classes of spaces than basic modal logic. If this were not the case, this would somewhat limit the relevance of the μ-calculus, while making modal logic more attractive. Another line of inquiry regards the *tangled closure* and *tangled derivative* operators, based on modalities introduced by Dawar and Otto [DO09], and whose topological meaning was further investigated by Fernández-Duque [FD11]. In the search of a simple, completely expressive fragment of the μ-calculus, they stand as natural candidates. In [BBFD21], it was shown that the tangled closure cannot be expressed in terms of the tangled derivative, but other questions remain: what is the expressivity of the tangled derivative relative to the tangled closure? And what about the two taken together?

We describe below the structure of the thesis and enumerate our main results:

- In chapter 2 we present the mathematical background and notations used in the document.

- In chapter 3, we investigate the d-semantics of $\mathbf{wK4} + \mathsf{bd}_n$, and prove that it is also captured by modal Krull dimension; we also give a proof of completeness. We then consider some characterizations of modal Krull dimension and adapt them into characterizations that are more relevant to the d-semantics. Finally, we investigate how modal Krull dimension is affected by the derivative.

- In chapter 4 we consider diverse extensions of **wK4** adapted from **S4.2** and **S4.3**, and identify the classes of spaces they describe. In particular, the study of axiom .3 will bring to the table an intriguing class of topological spaces called *accumulative*, that we proceed to fully axiomatize. We also provide various completeness results for stronger logics obtained by adding the Löb axiom or bd_n to the previous systems.

- In chapter 5 we investigate the relative expressivity of the tangled closure, the tangled derivative and a new modality called the *hybrid tangled operator*. After that, we address the expressivity of the μ-calculus and exhibit a large collection of axiom systems of the μ-calculus that are *not* reducible to plain modal logic. We also show that these counter-examples are not too pathological, by identifying one which is rather simple (thus not too unintuitive) and showing that it is topologically complete. This backs our claim that there exist "good" logics that lie outside of basic modal logic, and gives legitimacy to μ-calculus. Nonetheless, we also manage to identify a syntactic fragment of the μ-calculus which is as expressive as modal logic.

- In chapter 6, we summarize our contribution and propose directions for future work.

Chapter 2

Background

In this chapter we introduce the material used throughout the document. Though all notations are defined from the ground, elementary knowledge of modal logic, general topology and μ-calculus is strongly recommended. Our main references for modal logic will be [BRV01] and [CZ97]. For topology one may refer to [Rys89]. We present modal logic and some axiom systems in section 2.1, and the Kripke semantics in section 2.2. In section 2.3, we introduce rudiments of general topology as well as topological semantics for modal logic. Finally, section 2.4 is devoted to the modal μ-calculus.

2.1 Modal logic

We start with the syntax:

Definition 2.1. We fix a countable set Prop of atomic propositions. The *basic modal language* $\mathcal{L}$ is generated by the following grammar:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid \Diamond\phi$$

As usual we use the abbreviations $\phi \vee \psi := \neg(\neg\phi \wedge \neg\psi)$, $\Box\phi := \neg\Diamond\neg\phi$ as well as $\bot := p \wedge \neg p$ and $\top := \neg\bot$ for some arbitrary $p \in \mathsf{Prop}$. Let $\phi, \psi_1, \ldots, \psi_n \in \mathcal{L}$ and $p_1, \ldots, p_n \in \mathsf{Prop}$, and let us write $\overline{\psi} := (\psi_1, \ldots, \psi_n)$ and $\overline{p} := (p_1, \ldots, p_n)$. We then write $\phi[\overline{\psi}/\overline{p}]$ the formula ϕ where each ψ_i is substituted for p_i.

The smallest *normal* modal logic is the system **K**, which consists of the following induction rules and axioms:

Name	Axiom/inference rule
	All propositional tautologies
Substitution	From ϕ infer $\phi[\overline{\psi}/\overline{p}]$
K (Distribution)	$\Box(p \to q) \to (\Box p \to \Box q)$
Modus Ponens	From ϕ and $\phi \to \psi$ infer ψ
Necessitation	From ϕ infer $\Box\phi$

In this thesis we will also be interested in the following axioms and axiom systems:

Name	Axiom
T	$p \to \Diamond p$
4	$\Diamond\Diamond p \to \Diamond p$
w4	$\Diamond\Diamond p \to p \vee \Diamond p$
.2	$\Diamond\Box p \to \Box\Diamond p$
.3	$\Box(\Box p \to q) \vee \Box(\Box q \to p)$
gl	$\Box(\Box p \to p) \to \Box p$

$$\begin{aligned}
\mathbf{wK4} &:= \mathbf{K} + \mathsf{w4} \\
\mathbf{K4} &:= \mathbf{K} + \mathsf{4} \\
\mathbf{S4} &:= \mathbf{K} + \mathsf{T} + \mathsf{4} \\
\mathbf{wK4.2} &:= \mathbf{wK4} + \mathsf{.2} \\
\mathbf{wK4.3} &:= \mathbf{wK4} + \mathsf{.3} \\
\mathbf{S4.2} &:= \mathbf{S4} + \mathsf{.2} \\
\mathbf{S4.3} &:= \mathbf{S4} + \mathsf{.3} \\
\mathbf{GL} &:= \mathbf{K} + \mathsf{gl}
\end{aligned}$$

2.2 Kripke semantics

The Kripke semantics consists in interpreting formulas in *Kripke frames*. A Kripke frame is a set of possible worlds along with a relation of accessibility, indicating which worlds can be "seen" from a given world.

Definition 2.2. A *Kripke frame* is a pair $\mathfrak{F} = (W, R)$ with W a set of *possible worlds* and $R \subseteq W^2$ a relation. A *Kripke model* based on $\mathfrak{F}$ is a tuple of the form (W, R, ν) with $\nu : \mathsf{Prop} \to \mathcal{P}(W)$ a *valuation* over $\mathfrak{F}$. If $w \in W$ we also call $(\mathfrak{M}, w)$ a *pointed Kripke model*.

Definition 2.3. Given a Kripke model $\mathfrak{M} = (W, R, \nu)$ and a world $w \in W$, we define by induction on a formula $\phi \in \mathcal{L}$ the satisfiability condition $\mathfrak{M}, w \vDash \phi$:

- $\mathfrak{M}, w \vDash p$ if $w \in \nu(p)$ (given $p \in \mathsf{Prop}$)
- $\mathfrak{M}, w \vDash \neg\phi$ if $\mathfrak{M}, w \nvDash \phi$
- $\mathfrak{M}, w \vDash \phi \wedge \psi$ if $\mathfrak{M}, w \vDash \phi$ and $\mathfrak{M}, w \vDash \psi$
- $\mathfrak{M}, w \vDash \Diamond\phi$ if there exists $u \in W$ such that wRu and $\mathfrak{M}, u \vDash \phi$

We then define $[\![\phi]\!]_{\mathfrak{M}} := \{w \in W \mid \mathfrak{M}, w \vDash \phi\}$ the *extension* of ϕ in $\mathfrak{M}$. If $\mathfrak{M}, w \vDash \phi$ for some $w \in W$ we say that ϕ is *satisfiable* in $\mathfrak{M}$. If ϕ is satisfiable in some model based on $\mathfrak{F}$ we say that ϕ is *satisfiable* in $\mathfrak{F}$.

If $\mathfrak{M}, w \vDash \phi$ for all $w \in W$ we write $\mathfrak{M} \vDash \phi$. If $\mathfrak{M} \vDash \phi$ for any model $\mathfrak{M}$ based on $\mathfrak{F}$ we write $\mathfrak{F} \vDash \phi$ and we say that ϕ is *valid* in $\mathfrak{F}$. We also have a notion of pointwise validity, that is, if $w \in W$ and $\mathfrak{M}, w \vDash \phi$ for all model $\mathfrak{M}$ based on $\mathfrak{F}$, then we write $\mathfrak{F}, w \vDash \phi$.

If $\mathfrak{F} \vDash \phi$ for all frame $\mathfrak{F}$ we write $\vDash \phi$. If $\vDash \phi \leftrightarrow \psi$ we write $\phi \equiv \psi$ and say that ϕ and ψ are *equivalent*. Finally, if Γ is a set of formulas we write $\mathfrak{M}, w \vDash \Gamma$ in case $\mathfrak{M}, w \vDash \phi$ for all $\phi \in \Gamma$, and all of the other notations are adapted similarly.

Here are additional useful properties on frames:

Definition 2.4. Let $\mathfrak{F} = (W, R)$ be a Kripke frame.

- A world $w \in W$ is said to be *reflexive* if wRw, and *irreflexive* if *not* wRw. The frame $\mathfrak{F}$ is reflexive (resp. irreflexive) if every $w \in W$ is reflexive (resp. irreflexive).
- $\mathfrak{F}$ is *transitive* if for all $w, u, v \in W$, wRu and uRv implies wRv.
- $\mathfrak{F}$ is *weakly transitive* if for all $w, u, v \in W$, wRu and uRv implies wRv or $w = v$.
- $\mathfrak{F}$ is *rooted* in $r \in W$ if for all $w \in W$ we have rRw or $r = w$. The world r is then called a *root* of $\mathfrak{F}$.
- $\mathfrak{F}$ is *strongly directed* if for all $w, u, v \in W$, wRu and wRv implies that there exists $t \in W$ such that uRt and vRt.
- $\mathfrak{F}$ is *strongly connected* if for all $w, u, v \in W$, wRu and wRv implies uRv or vRu.
- $\mathfrak{F}$ is *converse well-founded* if there exists no sequence $(w_n)_{n\in\mathbb{N}} \in W^{\mathbb{N}}$ such that $w_n R w_{n+1}$ for all $n \in \mathbb{N}$.

These properties transfer to Kripke models too, e.g., a model $\mathfrak{M}$ based on $\mathfrak{F}$ is reflexive if $\mathfrak{F}$ is reflexive.

These properties are then related to axioms systems through the notion of Kripke completeness:

Definition 2.5. A logic $\mathbf{L}$ *defines* a class of frames $\mathcal{C}$ if $\mathcal{C} = \{\mathfrak{F} \mid \mathfrak{F} \vDash \mathbf{L}\}$. We call $\mathbf{L}$ *sound and complete* with respect to a class of frames $\mathcal{C}$ if for any formula ϕ we have $\mathbf{L} \vdash \phi$ iff $\mathfrak{F} \vDash \phi$ for all $\mathfrak{F} \in \mathcal{C}$. We say that $\mathbf{L}$ is *Kripke complete* if it is sound and complete with respect to the class of frames defined by it.

Theorem 2.6. *Each logic* $\mathbf{L}$ *on the left-hand side of the following table defines the corresponding class of Kripke frames on the right-hand side. All of these logics are also Kripke complete.*

Logic	*Condition on frames*
wK4	*weakly transitive*
K4	*transitive*
S4	*reflexive and transitive*
S4.2	*reflexive, transitive and strongly directed*
S4.3	*reflexive, transitive and strongly connected*
GL	*converse well-founded*

As a consequence of this theorem, weakly transitive frames (resp. models) will often be called **wK4** *frames* (resp. **wK4** *models*) for short, while transitive and reflexive frames (resp. models) will be called **S4** *frames* (resp. **S4** *models*).

A particular case of Kripke completeness is the well known finite model property [BRV01, sec. 2.3 & 3.4]:

Definition 2.7. A logic $\mathbf{L}$ has the *finite model property* if whenever $\mathbf{L} \nvdash \neg\phi$, there exists a finite Kripke frame $\mathfrak{F}$ in which ϕ is satisfiable and such that $\mathfrak{F} \vDash \mathbf{L}$.

We also introduce subframes, which are known for preserving the validity of formulas:

Definition 2.8. A frame (W_0, R_0) is called a *subframe* of (W, R) if $W_0 \subseteq W$ and $R_0 = W_0^2 \cap R$. Note that it is completely characterized by W_0, so we call it the subframe *induced* by W_0. We say that (W_0, R_0) is a *generated subframe* if $w \in W_0$, $u \in W$ and wRu implies $u \in W_0$. Given $X \subseteq W$, the subframe of (W, R) *generated* by X is the subframe (W_0, R_0) induced by the set

$$W_0 := \{w_n \mid w_0, \dots, w_n \in W \text{ and } w_0 \in X \text{ and } w_0 R \dots R w_n\}$$

Then (W_0, R_0) is the smallest generated subframe of (W, R) containing X.

A model $\mathfrak{M}_0 = (W_0, R_0, \nu_0)$ is called a *submodel* of $\mathfrak{M} = (W, R, \nu)$ if (W_0, R_0) is a subframe of (W, R) and for all $p \in \mathsf{Prop}$ we have $\nu_0(p) = \nu(p) \cap W_0$. Again, $\mathfrak{M}_0$ is completely characterized by W_0 and we call it the submodel *induced* by W_0. If (W_0, R_0) is a generated subframe of (W, R), then we say that $\mathfrak{M}_0$ is a *generated submodel* of $\mathfrak{M}$. Given $X \subseteq W$, let us write (W_0, R_0) the subframe of (W, R) generated by X; the submodel of $\mathfrak{M}$ induced by W_0 is then called the submodel *generated* by X.

Proposition 2.9. *Let ϕ be a formula, $\mathfrak{M}$ a Kripke model, $\mathfrak{M}_0$ a generated submodel of $\mathfrak{M}$ and w a world in $\mathfrak{M}_0$. Then $\mathfrak{M}, w \vDash \phi$ if and only if $\mathfrak{M}_0, w \vDash \phi$.*

We conclude this section by introducing chains, a simple kind of Kripke frames that the reader will encounter many times throughout this document:

Definition 2.10. Let $n \in \mathbb{N}$. An *n-chain* is a Kripke frame of the form $\mathfrak{F} = (W, R)$ with $W = \{w_i \mid i \in [\![0, n-1]\!]\}$ and

$$\{(w_i, w_j) \mid 0 \leq i < j \leq n-1\} \subseteq R \subseteq \{(w_i, w_j) \mid 0 \leq i \leq j \leq n-1\}$$

If $R = \{(w_i, w_j) \mid 0 \leq i < j \leq n-1\}$, we write $\mathfrak{F} = \underline{n}$. If $R = \{(w_i, w_j) \mid 0 \leq i \leq j \leq n-1\}$, we write $\mathfrak{F} = \underline{n}^+$.

Visually, an n-chain looks as in figure 2.1. Each world may be either reflexive or irreflexive, so for all n there are 2^n different n-chains (up to isomorphism). We can see that $\underline{n}$ denotes the irreflexive n-chain, and $\underline{n}^+$ the reflexive n-chain.

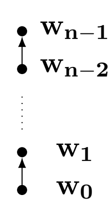


Figure 2.1: An n-chain

2.3 Topological semantics

This section will discuss the c-semantics and the d-semantics for modal logic, but before that we recall some basics of general topology:

Definition 2.11. Let X be a set of *points*. A *topology* on X is a set $\tau \subseteq \mathcal{P}(X)$ satisfying the following:

- $\emptyset \in \tau$ and $X \in \tau$;
- if $\mathcal{U} \subseteq \tau$ then $\bigcup \mathcal{U} \in \tau$;
- if $U_1, \ldots, U_n \in \tau$ then $\bigcap_{i=1}^{n} U_i \in \tau$.

The pair (X, τ) is then called a *topological space*. The elements of τ are called the *open* sets of X. The complement of an open set is called a *closed* set. If $x \in U \in \tau$ then U is called an *open neighbourhood* of x.

A *topological model* based on X is a pair of the form (X, ν) with $\nu : \mathsf{Prop} \to \mathcal{P}(X)$ a valuation. Given $x \in X$ we then call $(\mathfrak{M}, x)$ a *pointed topological model.*

Slightly abusing terminology, we will often keep τ implicit and let X refer to the space (X, τ).

Definition 2.12. Let X be a topological space, $A \subseteq X$ and $x \in X$.

- The point x is said to be an *interior point* of A if there exists an open set U such that $x \in U \subseteq A$. We denote by $\mathsf{Int}(A)$ the set of all interior points of A and call it the *interior* of A.
- The point x is said to be an *adherent point* of A if for any open neighbourhood U of x, we have $U \cap A \neq \emptyset$. We denote by $\mathsf{Cl}(A)$ the set of all adherent points of A and call it the *closure* of A.

In case of ambiguity, the notations $\mathsf{Int}_X(A)$ and $\mathsf{Cl}_X(A)$ may be used to indicate the space wherein these two operators are evaluated.

When working with a space, a topological *base* is often very useful as it simplifies reasoning. A base is essentially a collection of open sets from which the whole topology is generated by arbitrary unions.

Definition 2.13. Let X be a set. A *base* for a topology is a collection $\mathcal{B} \subseteq \mathcal{P}(X)$ such that:

- $\bigcup \mathcal{B} = X$;
- for all $U, V \in \mathcal{B}$ and $x \in U \cap V$, there exists $W \in \mathcal{B}$ such that $x \in W \subseteq U \cap V$.

The set $\tau := \{\bigcup B \mid B \subseteq \mathcal{B}\}$ is then a topology, called the topology *generated* by $\mathcal{B}$.

If a space admits a base $\mathcal{B}$, it is well known that many properties can be reduced to a condition involving the elements of $\mathcal{B}$ only, e.g., we have $x \in \mathsf{Int}(A)$ iff there exists $U \in \mathcal{B}$ such that $x \in U \subseteq A$.

The topological counterpart of subframes is the notion of subspace:

Definition 2.14. Let (X, τ) and (X_0, τ_0) be two topological spaces. We say that (X_0, τ_0) is a *subspace* of (X, τ) if $\tau_0 = \{U \cap X_0 \mid U \in \tau\}$. Note that τ_0 is completely characterized by X_0, so any set $X_0 \subseteq X$ can be called a subspace of X.

A model $\mathfrak{M}_0 = (X_0, \nu_0)$ based on $X_0 \subseteq X$ is then called a *submodel* of $\mathfrak{M} = (X, \nu)$ if for all $p \in \mathsf{Prop}$ we have $\nu_0(p) = \nu(p) \cap X_0$. Again, $\mathfrak{M}_0$ is completely characterized by X_0 and we call it the submodel *induced* by X_0. If X_0 is open in X then we call $\mathfrak{M}_0$ an *open submodel* of $\mathfrak{M}$.

Definition 2.15. Let X be a topological space.

- A point $x \in X$ is said to be *isolated* if $\{x\}$ is open. If $x \in A \subseteq X$ we say that x is *isolated in* A if there exists U open such that $\{x\} = U \cap A$. The space X is called *dense-in-itself* if it contains no isolated point, and *discrete* if all of its points are isolated. The space X is called *scattered* if any subspace of X contains an isolated point.

- We say that X is *extremally disconnected* if $\mathsf{Cl}(U)$ is open for all open set U. It is called *hereditarily extremally disconnected* (or HED for short) if every subspace $Y \subseteq X$ is extremally disconnected.

- We say that X is T_0 if whenever $x, y \in X$, there exists an open set U such that $|U \cap \{x, y\}| = 1$. The space X is called T_1 if for all $x, y \in X$ such that $x \neq y$ there exists an open set U such that that $x \in U$ and $y \notin U$. Finally X is called T_d if every $x \in X$ is isolated in $\mathsf{Cl}(\{x\})$.

The conditions T_0, T_d and T_1 are known as *separation axioms*, though they are not the only ones. It is easy to prove that T_1 is stronger than T_d, and that T_d is stronger than T_0.

We now introduce the *c-semantics*, which consists in interpreting $\Diamond$ as the closure operator:

Definition 2.16. Let $\mathfrak{M} = (X, \nu)$ be a topological model. We define by induction on ϕ the *extension* $[\![\phi]\!]^c_{\mathfrak{M}}$ of ϕ in $\mathfrak{M}$:

- $[\![p]\!]^c_{\mathfrak{M}} := \nu(p)$
- $[\![\neg\phi]\!]^c_{\mathfrak{M}} := X \setminus [\![\phi]\!]^c_{\mathfrak{M}}$
- $[\![\phi \wedge \psi]\!]^c_{\mathfrak{M}} := [\![\phi]\!]^c_{\mathfrak{M}} \cap [\![\psi]\!]^c_{\mathfrak{M}}$
- $[\![\Diamond\phi]\!]^c_{\mathfrak{M}} := \mathsf{Cl}([\![\phi]\!]^c_{\mathfrak{M}})$

Given $x \in X$ we write $\mathfrak{M}, x \vDash_c \phi$ whenever $x \in [\![\phi]\!]^c_{\mathfrak{M}}$. The other notations introduced in definition 2.3 are defined similarly.

We can adapt definition 2.5 to the topological setting:

Definition 2.17. A logic $\mathbf{L}$ *defines* a class of topological spaces $\mathcal{C}$ if $\mathcal{C} = \{X \mid X \vDash_c \mathbf{L}\}$. We call $\mathbf{L}$ *sound and complete* with respect to a class of spaces $\mathcal{C}$ if for any formula ϕ we have $\mathbf{L} \vdash \phi$ iff $X \vDash_c \phi$ for all $X \in \mathcal{C}$. We say that $\mathbf{L}$ is *topologically complete* if it is sound and complete with respect to the class of spaces defined by it.

When it comes to completeness, many results can be easily adapted from the Kripke semantics, by a simple operation which turns a Kripke frame into a space while preserving logical truth:

Definition 2.18. Let $\mathfrak{F} := (W, R)$ be a weakly transitive Kripke frame. A set $U \subseteq W$ is called an *upset* if $w \in U$ and wRu implies $u \in U$. The collection τ_R of all upsets over W is then a topology, and (W, τ_R) is called the topological space *induced* by $\mathfrak{F}$. If $\mathfrak{M} = (W, R, \nu)$ is a Kripke model based on $\mathfrak{F}$, then $((W, \tau_R), \nu)$ is the topological model *induced* by $\mathfrak{M}$.

In this document we will largely abuse terminology and not distinguish a weakly transitive Kripke frame (resp. model) from the topological space (resp. model) induced by it. So the reader should not be surprised when we start talking about the topological properties of such and such frame. This convention is partly motivated by the following property, which states that the Kripke semantics and the c-semantics agree over **S4** models:

Proposition 2.19. *For all reflexive and transitive Kripke model $\mathfrak{M} = (W, R, \nu)$, all world $w \in W$ and all formula ϕ we have*

$$\mathfrak{M}, w \vDash \phi \iff ((W, \tau_R), \nu), w \vDash_c \phi$$

This proposition, combined with theorem 2.6, yields the topological completeness of **S4**.

Theorem 2.20. *In the c-semantics,* **S4** *is sound and complete with respect to the class of all topological spaces.*

We now introduce a slight variant of the closure operator called the *derived set operator*:

Definition 2.21. Let X be a topological space, $A \subseteq X$ and $x \in X$. The point x is said to be a *limit point* of A if for any open neighbourhood U of x, we have $U \cap A \setminus \{x\} \neq \emptyset$. We denote by $\mathsf{d}(A)$ the set of all limit points of A and call it the *derived set* of A. The dual of $\mathsf{d}(\cdot)$ is defined by $\widehat{\mathsf{d}}(A) := X \setminus \mathsf{d}(X \setminus A)$. Again the notations $\mathsf{d}_X(A)$ and $\widehat{\mathsf{d}}_X(A)$ may be used if needed.

We see that for x to be a limit point of A, it is not enough that each of its open neighbourhoods intersects A: the intersection must contain a point *different* from x. So any limit point of A is an adherent point of A, and as a result we have $\mathsf{d}(A) \subseteq \mathsf{Cl}(A)$, but the converse is not true in general. We may then adapt the c-semantics to this new operator, and obtain what we call the *derivational semantics*, or *d-semantics* for short.

Definition 2.22. Let $\mathfrak{M} = (X, \nu)$ be a topological model. We define by induction on ϕ the *extension* $[\![\phi]\!]^d_{\mathfrak{M}}$ of ϕ in $\mathfrak{M}$ by

$$[\![\Diamond\phi]\!]^d_{\mathfrak{M}} := \mathsf{d}([\![\phi]\!]^d_{\mathfrak{M}})$$

and the other cases are as in definition 2.16. Given $x \in X$ we write $\mathfrak{M}, x \vDash_d \phi$ whenever $x \in [\![\phi]\!]^d_{\mathfrak{M}}$. The other notations introduced in definition 2.3 are defined similarly.

Because the d-semantics is the main topic of this thesis, we do not find relevant to make explicit all the time which of the c-semantics or the d-semantics we are talking about, so we will write $\vDash$ for $\vDash_d$ and $[\![\phi]\!]_{\mathfrak{M}}$ for $[\![\phi]\!]^d_{\mathfrak{M}}$ as long as this causes no ambiguity.

When considering **wK4** frames as topological spaces, the Kripke semantics and the d-semantics do *not* coincide in general, though they agree on irreflexive frames:

Proposition 2.23. *For all irreflexive and weakly transitive Kripke model $\mathfrak{M} = (W, R, \nu)$, all world $w \in W$ and all formula ϕ we have*

$$\mathfrak{M}, w \vDash \phi \iff ((W, \tau_R), \nu), w \vDash_d \phi$$

By the method of unraveling [BRV01, sec. 2.1], it is possible to prove that **wK4** is complete for irreflexive and weakly transitive Kripke frames; from proposition 2.23 we then derive the following:

Theorem 2.24. *In the d-semantics,* **wK4** *is sound and complete with respect to the class of all spaces.*

We can adapt definition 2.5 again. The following definition conflicts with definition 2.17, but that is harmless as long as we mention in which semantics we are working.

Definition 2.25. A logic **L** *defines* a class of topological spaces $\mathcal{C}$ if $\mathcal{C} = \{X \mid X \vDash_d \mathbf{L}\}$. We call **L** *sound and complete* with respect to a class of spaces $\mathcal{C}$ if for any formula ϕ we have $\mathbf{L} \vdash \phi$ iff $X \vDash_d \phi$ for all $X \in \mathcal{C}$. We say that **L** is *topologically complete* if it is sound and complete with respect to the class of spaces defined by it.

For instance, in the d-semantics we have just seen that **wK4** is topologically complete. This is also the case for the following logics (see e.g., [vBB07]):

- **K4**, which defines the class of T_d spaces;
- **wK4** $+ \Diamond\top$, which defines the class of dense-in-itself spaces;
- **GL**, which defines the class of scattered spaces.

Part of what makes the d-semantics interesting is that it subsumes the c-semantics. Indeed, it is easy to show that the identity $\mathsf{Cl}(A) = A \cup \mathsf{d}(A)$ holds in any space, and this gives birth to a logical translation:

Definition 2.26. Given $\phi \in \mathcal{L}$ we write $\Box^+\phi := \phi \wedge \Box\phi$ and $\Diamond^+\phi := \phi \vee \Diamond\phi$. The **S4**-*translation* ϕ^+ of a modal formula ϕ is defined inductively as followed:

- $p^+ := p$
- $(\neg\phi)^+ := \neg\phi^+$
- $(\phi \wedge \psi)^+ := \phi^+ \wedge \psi^+$
- $(\Box\phi)^+ := \Box^+\phi^+$

Note that we also have $(\Diamond\phi)^+ \equiv \Diamond^+\phi^+$.

Definition 2.27. Let $\mathfrak{F} = (W, R)$ be a weakly transitive Kripke frame. The *reflexive closure* of R is the relation $R^+ := R \cup \{(w, w) \mid w \in W\}$. We then write $\mathfrak{F}^+ := (W, R^+)$. If $\mathfrak{M} = (W, R, \nu)$ is a weakly transitive Kripke model, we also write $\mathfrak{M}^+ := (W, R^+, \nu)$.

Proposition 2.28. *Let ϕ be a modal formula.*

- *For all pointed* **wK4** *model $(\mathfrak{M}, w)$ we have $\mathfrak{M}, w \vDash \phi^+ \iff \mathfrak{M}^+, w \vDash \phi$.*
- *For all pointed topological model $(\mathfrak{M}, x)$ we have $\mathfrak{M}, x \vDash_d \phi^+ \iff \mathfrak{M}, x \vDash_c \phi$*

As a result, any topological property that can be expressed in the c-semantics (via the formula ϕ), can also be expressed in the d-semantics (via the formula ϕ^+).

We now turn our attention to invariance results. The following proposition shows that open submodels are the topological analogue of generated submodels. In fact, it is easy to see that a generated submodel *is* an open submodel (in the sense of definition 2.18), so this result actually subsumes proposition 2.9.

Proposition 2.29. *Let ϕ be a formula, $\mathfrak{M}$ a topological model, $\mathfrak{M}_0$ an open submodel of $\mathfrak{M}$ and x a point in $\mathfrak{M}_0$. Then $\mathfrak{M}, x \vDash_d \phi$ if and only if $\mathfrak{M}_0, x \vDash_d \phi$.*

Finally, we define the relevant morphisms for both the c-semantics and the d-semantics:

Definition 2.30. A *interior map* from a space X to a space Y is a function $f : X \to Y$ such that:

- if U is open in X, then $f[U]$ is open in Y;
- if U is open in Y, then $f^{-1}[U]$ is open in X.

If f is surjective, we call Y an *interior image* of X.

A *d-morphism* from a space X to a **wK4** frame (W, R) is a function $f : X \to W$ such that:

- f is an interior map;
- for all irreflexive point $w \in W$, the subspace $f^{-1}(w)$ of X is discrete;
- for all reflexive point $w \in W$, we have $f^{-1}(w) \subseteq \mathsf{d}(f^{-1}(w))$.

If f is surjective, we call W a *d-morphic image* of X.

What we mean when we call them morphisms is that they preserve logical validity in the respective semantics:

Proposition 2.31. *Let X be a space.*

1. *If $f : X \to Y$ is an interior map and $X \vDash_c \phi$ then $Y \vDash_c \phi$.*
2. *If $f : X \to W$ is a d-morphism and $X \vDash_d \phi$ then $W \vDash \phi$.*

A convenient characterization of d-morphisms is given by the following theorem:

Theorem 2.32. [BEG05] *A map $f : X \to W$ is a d-morphism iff for all $A \subseteq W$ we have $f^{-1}(R^{-1}A) = \mathsf{d}(f^{-1}A)$.*

We note that unlike interior maps, d-morphisms are only defined when the codomain is a Kripke frame. For our purpose this restriction is not limiting, though we could generalize the definition on the basis of theorem 2.32.

2.4 Mu-calculus

Here we briefly introduce the modal μ-calculus; if needed a more in-depth survey can be found in [BS07]. We also present an axiomatization of the topological μ-calculus.

Definition 2.33. The language $\mathcal{L}_\mu$ of the modal μ-calculus is defined by the following syntax:

$$\phi ::= p \mid \neg p \mid \phi \wedge \psi \mid \phi \vee \psi \mid \Box\phi \mid \Diamond\phi \mid \mu p.\phi \mid \nu p.\phi$$

where $\neg p$ does not occur in formulas of the form $\mu p.\phi$ and $\nu p.\phi$. We also assume without loss of generality that every formula ϕ is *clean*, that is, no bound variable is also a free variable, and for every variable p there is at most one subformula of ϕ of the form $\mu p.\psi$ or $\nu p.\psi$.

This version of μ-calculus is called the *negative normal form* because negations only apply to atomic propositions. Another (equivalent) presentation allows unrestricted negations and only takes μ as primitive; the operator ν is then defined as the dual of μ by

$$\nu p.\phi := \mu p.\neg\phi[\neg p/p]$$

In this thesis we opt for the negative normal form as it better suits our needs. For instance, it will allow us to easily define the ν-free fragment of the μ-calculus in chapter 5, by removing the operator μ from the syntax.

Definition 2.34. Let X be a set, $\nu : \mathsf{Prop} \to X$ a valuation, $p \in \mathsf{Prop}$ and $A \subseteq X$. We define the valuation $\nu[p := A]$ by setting, for all $q \in \mathsf{Prop}$:

$$\nu[p := A](q) := \begin{cases} A & \text{if } q = p \\ \nu(p) & \text{otherwise} \end{cases}$$

Definition 2.35. We extend the Kripke semantics to the μ-calculus as follows: if $\mathfrak{M} = (W, R, \nu)$ is a Kripke model, the extension $[\![\phi]\!]_{\mathfrak{M}}$ of a formula $\phi \in \mathcal{L}_\mu$ in $\mathfrak{M}$ is defined by

$$\begin{aligned}
[\![p]\!]_{\mathfrak{M}} &:= \nu(p) \\
[\![\neg p]\!]_{\mathfrak{M}} &:= W \setminus \nu(p) \\
[\![\phi \wedge \psi]\!]_{\mathfrak{M}} &:= [\![\phi]\!]_{\mathfrak{M}} \cap [\![\psi]\!]_{\mathfrak{M}} \\
[\![\phi \vee \psi]\!]_{\mathfrak{M}} &:= [\![\phi]\!]_{\mathfrak{M}} \cup [\![\psi]\!]_{\mathfrak{M}} \\
[\![\Diamond\phi]\!]_{\mathfrak{M}} &:= \{w \in W \mid \exists u \in [\![\phi]\!]_{\mathfrak{M}}, wRu\} \\
[\![\Box\phi]\!]_{\mathfrak{M}} &:= \{w \in W \mid \forall u \in W, wRu \implies u \in [\![\phi]\!]_{\mathfrak{M}}\} \\
[\![\mu p.\phi]\!]_{\mathfrak{M}} &:= \bigcap\{A \subseteq W \mid [\![\phi]\!]_{W,R,\nu[p:=A]} \subseteq A\} \\
[\![\nu p.\phi]\!]_{\mathfrak{M}} &:= \bigcup\{A \subseteq W \mid A \subseteq [\![\phi]\!]_{W,R,\nu[p:=A]}\}
\end{aligned}$$

This definition scheme applies to the topological semantics too: it suffices to replace $\mathfrak{M}$ by a topological model and to set

$$[\![\Diamond\phi]\!]^c_{\mathfrak{M}} := \mathsf{Cl}([\![\phi]\!]^c_{\mathfrak{M}}) \text{ and } [\![\Box\phi]\!]^c_{\mathfrak{M}} := \mathsf{Int}([\![\phi]\!]^c_{\mathfrak{M}})$$

for the c-semantics and

$$[\![\Diamond\phi]\!]^d_{\mathfrak{M}} := \mathsf{d}([\![\phi]\!]^d_{\mathfrak{M}}) \text{ and } [\![\Box\phi]\!]^d_{\mathfrak{M}} := \widehat{\mathsf{d}}([\![\phi]\!]^d_{\mathfrak{M}})$$

for the d-semantics.

One can prove that $[\![\mu p.\phi]\!]_{\mathfrak{M}}$ and $[\![\nu p.\phi]\!]_{\mathfrak{M}}$ are respectively the *least fixpoint* and the *greatest fixpoint* of the map $A \mapsto [\![\phi]\!]_{W,R,\nu[p:=A]}$, and this is why we call μ and ν *fixpoint operators*. Further, all the notations and concepts (satisfiability, validity, completeness, ...) related to the Kripke and topological semantics can be extended to the μ-calculus. We now introduce the axiom system μ**wK4**:

Definition 2.36. The axiom system μ**wK4** is the extension of **wK4** with the *fixpoint axioms*

$$\phi[\mu p.\phi/p] \to \mu p.\phi$$

$$\nu p.\phi \to \phi[\nu p.\phi/p]$$

and the *induction rules*

$$\text{From } \psi[\phi/p] \to \phi \text{ infer } \mu p.\psi \to \phi$$

$$\text{From } \phi \to \psi[\phi/p] \text{ infer } \phi \to \nu p.\psi$$

A completeness theorem covering μ**wK4** and many of its extensions was established in [BBFD21]. For the moment though, the reader only needs to know this result for μ**wK4**:

Theorem 2.37. [BBFD21] *The logic* μ**wK4** *is Kripke complete, and topologically complete in the d-semantics.*

Our framework is now fully settled, and we are ready to jump into the next chapter devoted to the modal depth of topological spaces.

Chapter 3

Topological Depth and Modal Logic

In this chapter we investigate modal depth in the d-semantics. Depth in the sense of the maximal length of a path is a natural characteristic of Kripke frames, and it is well known to be captured in modal logic by the family $(\mathsf{bd}_n)_{n\in\mathbb{N}}$ of *bounded depth* formulas [CZ97, sec. 3.5]. While the concept of depth has no intuitive topological meaning, the bd_n's still have a topological semantics, so we are necessarily faced with the question of their topological interpretation.

In the c-semantics, this was solved in [BBLB+17], where it was shown that the truth of bd_n corresponds to a parameter called *modal Krull dimension*: roughly summarized, the modal Krull dimension of a space X is the size of a maximal stack of nested non-empty nowhere dense subspaces of X. Though deeply meaningful from an algebraic perspective which is out of the scope of this thesis, this definition has barely anything to do with depth, and one may ask for a more natural characterization. This is provided by another result of [BBLB+17] stating that $X \vDash_c \mathsf{bd}_n$ iff X does not "contain" the reflexive $n+1$-chain (seen as a space). Such a formulation is already much closer to the initial graph-theoretic notion of depth. What "contain" means in this context will be made precise later, but we can already mention that it can be defined it two ways: either by satisfiability of a Jankov-Fine formula, or by the existence of a surjective interior map. All these results are recalled in section 3.1.

In the d-semantics, the interpretation of bd_n is still unknown and one may desire a "derivative modal dimension" capturing their meaning. In section 3.2, we show that in fact modal Krull dimension precisely plays this role: in other words, the c-semantics and the d-semantics of bd_n coincide. We also establish the topological completeness of $\mathbf{wK4}+\mathsf{bd}_n$ in the d-semantics. Though this answers the initial question, this result is not entirely satisfying since it does not provide a characterization that is relevant to the d-semantics. Indeed the aforementioned conditions with reflexive chains either involve satisfiability in the c-semantics, or interior maps, which are the morphisms for the c-semantics. It is then natural to ask for conditions involving truth in the d-semantics or d-morphisms, and this is investigated in section 3.3.

Finally, we briefly study in section 3.4 how taking the derivative of a space affects its modal dimension.

3.1 Background

We start by making explicit the meaning of depth in Kripke frames.

Definition 3.1. Let $\mathfrak{F} = (W, R)$ be a Kripke frame. The *depth* $\mathsf{depth}(\mathfrak{F})$ of $\mathfrak{F}$ is the largest $n \in \mathbb{N} \cup \{\infty\}$ such that there exists a sequence $w_1, \dots, w_n \in W$ satisfying $w_i R w_{i+1}$ and *not* $w_{i+1} R w_i$ for all $i \in [\![1, n-1]\!]$.

The important part to keep in mind is the condition that every point in the sequence does not see its predecessor. This definition thus differs from another, stricter concept of depth which does not include this requirement. As mentioned at the beginning, depth can be measured by means of modal formulas:

Definition 3.2. [CZ97, sec. 3.5] We define by recursion the following formulas:

- $\mathsf{bd}_0 := \bot$
- $\mathsf{bd}_{n+1} := \Diamond(\Box p_{n+1} \wedge \neg \mathsf{bd}_n) \to p_{n+1}$

Proposition 3.3. *Let $\mathfrak{F}$ be a Kripke frame. For all $n \in \mathbb{N}$, we have*

$$\mathfrak{F} \vDash \mathsf{bd}_n \iff \mathsf{depth}(\mathfrak{F}) \leq n$$

In the c-semantics, the bd_n's measure the modal Krull dimension of a topological space:

Definition 3.4. Let X be a topological space. A subspace $Y \subseteq X$ is *nowhere dense* in X if $\mathsf{Int}(\mathsf{Cl}(Y)) = \emptyset$.

Definition 3.5. [BBLB+17] The *modal Krull dimension* $\mathsf{mdim}(X)$ of a space X is defined as follows:

- $\mathsf{mdim}(X) \leq -1$ if $X = \emptyset$
- $\mathsf{mdim}(X) \leq n + 1$ if for all $Y \subseteq X$ nowhere dense in X, we have $\mathsf{mdim}(Y) \leq n$
- $\mathsf{mdim}(X) := \mathsf{inf}\ \{n \in \mathbb{N} \cup \{-1\} \mid \mathsf{mdim}(X) \leq n\}$ with the convention that $\mathsf{inf}\ \emptyset = \infty$

Theorem 3.6. *Let X be a topological space. For all $n \in \mathbb{N}$ we have*

$$X \vDash_c \mathsf{bd}_n \iff \mathsf{mdim}(X) \leq n - 1$$

Since modal dimension is a rather convoluted notion, other characterizations are also helpful, and [BBLB+17] offers a handful of them; we will give special attention to three conditions describing the depth of space as the length of the greatest reflexive chain contained in it. This property can be expressed using interior maps, but another way involves Jankov-Fine formulas, which encode a "pattern" given by a finite Kripke frame:

Definition 3.7. [BRV01, sec. 3.4] Let $\mathfrak{F} = (W, R)$ be a finite rooted **S4** frame. We write $W = \{w_i \mid i \in [\![0, n-1]\!]\}$ where w_0 is a root. The *Jankov-Fine formula* $\chi_{\mathfrak{F}}$ of $\mathfrak{F}$ is the conjunction of the following formulas:

1. q_0
2. $\Box \bigvee_{i=0}^{n-1} q_i$
3. $\Box \neg (q_i \wedge q_j)$ for all $0 \leq i < j < n$

4. $\Box(q_i \to \Diamond q_j)$ for all $i, j \in [\![0, n-1]\!]$ such that $w_i R w_j$

5. $\Box(q_i \to \neg\Diamond q_j)$ for all $i, j \in [\![0, n-1]\!]$ such that *not* $(w_i R w_j)$

We then have the following result (recall that $\underline{n}^+$ denotes the reflexive n-chain):

Theorem 3.8. [BBLB+17] *Let X be a topological space. The following conditions are equivalent for all $n \in \mathbb{N}$:*

1. $\mathsf{mdim}(X) \leq n-1$
2. $X \vDash_c \neg\chi_{\underline{n+1}^+}$
3. $\underline{n+1}^+$ *is not an interior image of* X
4. $\underline{n+1}^+$ *is not an interior image of an open subspace of* X

3.2 Bounded depth and modal Krull dimension

The main goal of this section is to show that given $n \in \mathbb{N}$ and a space X, we have $X \vDash_d \mathsf{bd}_n$ if and only if $\mathsf{mdim}(X) \leq n-1$. To achieve this goal we proceed by induction on n. This will bring us to a point where we have a nowhere dense subspace Y of X, and try to prove $Y \nvDash \mathsf{bd}_n$ iff $X \nvDash \mathsf{bd}_{n+1}$. From left to right, this is done by *shifting up* a valuation: summarized roughly, p_1 is relabelled as p_2, p_2 is relabelled as p_3, etc. From right to left, this is simply achieved by the reverse transformation which is the process of *shifting down* a valuation: p_2 is relabelled as p_1, p_3 is relabelled as p_2, etc. Here is the formal definition of these operations:

Definition 3.9. If $n \in \mathbb{N}$, we write $P_n := \{p_k \mid k \in [\![1, n]\!]\}$. Let X be a topological space.

- Let ν be a valuation over X with domain P_{n+1} and $Y \subseteq X$. The Y-*downshift* of ν is the valuation $\nu_Y^{\downarrow}$ over Y with domain P_n defined by $\nu_Y^{\downarrow}(p_k) := \nu(p_{k+1}) \cap Y$ for all $k \in [\![1, n]\!]$.
- Let $Y \subseteq X$ and ν be a valuation over Y with domain P_n. The Y-*upshift* of ν is the valuation $\nu_Y^{\uparrow}$ with domain P_{n+1} defined by $\nu_Y^{\uparrow}(p_1) := X \setminus Y$ and for all $k \in [\![1, n]\!]$, $\nu_Y^{\uparrow}(p_{k+1}) := \nu(p_k) \cup X \setminus Y$.

Their effect is depicted in figure 3.1[1]. In addition, in the implication from right to left, the subspace Y is not given, so it is our task to define a nowhere dense subspace Y of X with the desired property. We are going to show that the extension of the formula

$$\sigma := \Diamond^+(\neg p_1 \wedge \Diamond\Box p_1)$$

is qualified for being such Y. First, we prove that it is nowhere dense:

Lemma 3.10. *Given a topological model $\mathfrak{M}$, the subspace $[\![\sigma]\!]_{\mathfrak{M}}$ is nowhere dense in X.*

[1]Contrary to what the picture might suggest, the extension of p_n is in general *not* included in the extension of p_{n+1}.

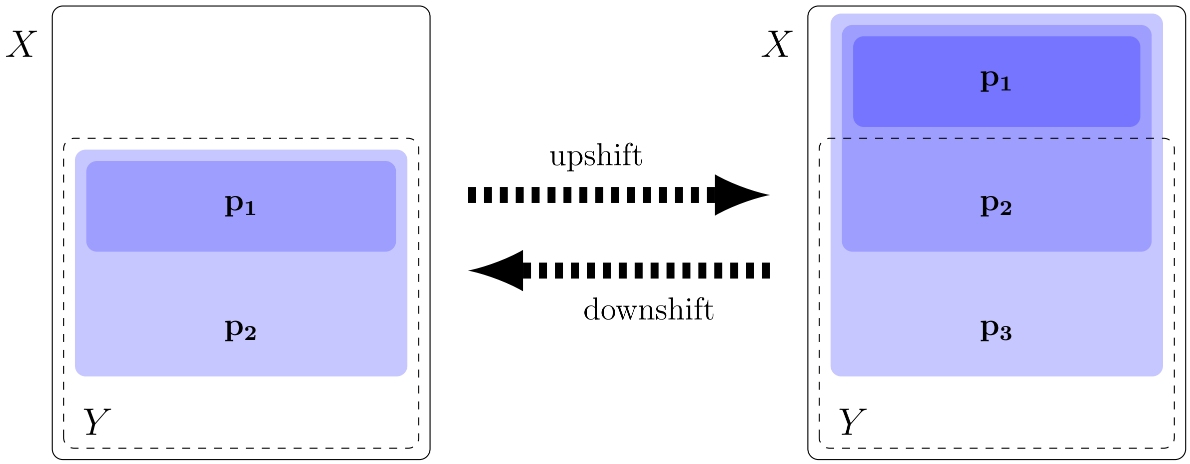


Figure 3.1: Depiction of valuation shifting

Proof. We aim at proving $\mathsf{Int}(\mathsf{Cl}([\![\sigma]\!]_{\mathfrak{M}})) = \emptyset$, or equivalently

$$[\![\Box^+\Diamond^+\Diamond^+(\neg p_1 \wedge \Diamond\Box p_1)]\!]_{\mathfrak{M}} = \emptyset$$

By soundness of **wK4**, it suffices to show

$$\mathbf{wK4} \vdash \neg\Box^+\Diamond^+\Diamond^+(\neg p_1 \wedge \Diamond\Box p_1)$$

which can be reduced to

$$\mathbf{wK4} \vdash \neg\Box^+\Diamond^+(\neg p_1 \wedge \Diamond\Box p_1)$$

So using Kripke completeness, we consider a **wK4** Kripke model $\mathfrak{M}' = (W, R, \nu)$ and $w \in W$ and suppose toward a contradiction that $\mathfrak{M}', w \vDash \Box^+\Diamond^+(\neg p_1 \wedge \Diamond\Box p_1)$. Then there exists $u \in W$ such that wR^+u and $\mathfrak{M}', u \vDash \neg p_1 \wedge \Diamond\Box p_1$. It follows that there exists $v \in W$ such that uRv and $\mathfrak{M}', v \vDash \Box p_1$. By weak transitivity we have wR^+v and therefore $\mathfrak{M}', v \vDash \Diamond^+(\neg p_1 \wedge \Diamond\Box p_1)$, that is, there exists $t \in W$ such that vR^+t and $\mathfrak{M}', t \vDash \neg p_1 \wedge \Diamond\Box p_1$. If $t \neq v$ then vRt and it follows $\mathfrak{M}', t \vDash p_1$, a contradiction. Therefore $t = v$ and we obtain $\mathfrak{M}', v \vDash \neg p_1 \wedge \Diamond\Box p_1$.

The whole reasoning about u can then be applied again to v, giving us the existence of $v' \in W$ such that vRv' and $\mathfrak{M}', v' \vDash \neg p_1 \wedge \Diamond\Box p_1$. This contradicts $\mathfrak{M}', v \vDash \Box p_1$, concluding the proof. $\square$

We now introduce, for all $n \in \mathbb{N}$, the formula

$$\theta_n := \Diamond(\Box p_{n+1} \wedge \neg \mathsf{bd}_n)$$

which is the antecedent of bd_{n+1}, that is, we have $\mathsf{bd}_{n+1} = \theta_n \to p_{n+1}$. Equivalently we have $\neg\mathsf{bd}_{n+1} \equiv \theta_n \wedge \neg p_{n+1}$, so for a topological model (X, ν), the following result will help transferring the falsity of bd_n from X to $Y := [\![\sigma]\!]_{(X,\nu)}$:

Lemma 3.11. *For all $n \geq 1$, we have* $\mathbf{wK4} \vdash \theta_n \to \sigma$.

Proof. We proceed by induction on n. For $n = 1$ this is immediate since $\theta_1 \equiv \Diamond(\Box p_2 \wedge \Diamond\Box p_1 \wedge \neg p_1)$. Suppose that this holds for n. Let $\mathfrak{M} = (W, R, \nu)$ be a **wK4** Kripke model and $w \in W$, and suppose $\mathfrak{M}, w \vDash \theta_{n+1}$. We have

$$\theta_{n+1} \equiv \Diamond(\Box p_{n+2} \wedge \theta_n \wedge \neg p_{n+1})$$

so there exists $u \in W$ such that wRu and $\mathfrak{M}, u \vDash \Box p_{n+2} \wedge \theta_n \wedge \neg p_{n+1}$. In particular $\mathfrak{M}, u \vDash \theta_n$ so there exists $v \in W$ such that uRv and $\mathfrak{M}, v \vDash \Box p_{n+1} \wedge \neg \mathsf{bd}_n$. By weak transitivity we have wR^+v. If $w = v$ then vRu and from $\mathfrak{M}, v \vDash \Box p_{n+1}$ and $\mathfrak{M}, u \vDash \neg p_{n+1}$ we derive a contradiction. Therefore wRv, and it follows $\mathfrak{M}, w \vDash \Diamond(\Box p_{n+1} \wedge \neg \mathsf{bd}_n)$, that is, $\mathfrak{M}, w \vDash \theta_n$; by the induction hypothesis, we then obtain $\mathfrak{M}, w \vDash \sigma$. By Kripke completeness this proves $\mathbf{wK4} \vdash \theta_{n+1} \to \sigma$. $\square$

We are now ready to prove that the operations of shifting up and down have the desired properties:

Lemma 3.12. *Let (X, ν) be a topological model, Y a subspace of X such that $[\![\sigma]\!]_{(X,\nu)} \subseteq Y$ and $n \in \mathbb{N}$. If $X, \nu, x \vDash \neg \mathsf{bd}_{n+1}$ then $x \in Y$ and $Y, \nu_Y^{\downarrow}, x \vDash \neg \mathsf{bd}_n$.*

Proof. By induction on n. For $n = 0$, suppose $X, \nu, x \vDash \neg \mathsf{bd}_1$, i.e., $X, \nu, x \vDash \theta_0 \wedge \neg p_1$. Recall that $\sigma = \Diamond^+(\neg p_1 \wedge \Diamond\Box p_1)$; therefore $x \in [\![\sigma]\!]_{(X,\nu)}$, so $x \in Y$ and the rest is immediate since $\neg \mathsf{bd}_0 \equiv \top$.

Suppose it holds for n and assume $X, \nu, x \vDash \neg \mathsf{bd}_{n+2}$, that is, $X, \nu, x \vDash \theta_{n+1} \wedge \neg p_{n+2}$. Since $n + 1 \geq 1$ we can apply lemma 3.11 to obtain $x \in [\![\sigma]\!]_{(X,\nu)}$ and thus $x \in Y$.

Now consider an open neighbourhood U of x in Y, of the form $U = U' \cap Y$ with U' open. From $X, \nu, x \vDash \theta_{n+1}$ and $x \in U'$ we obtain the existence of some $y \in U'$ such that $y \neq x$ and $X, \nu, y \vDash \Box p_{n+2} \wedge \neg \mathsf{bd}_{n+1}$. Then by the induction hypothesis we obtain $y \in Y$ and $Y, \nu_Y^{\downarrow}, y \vDash \neg \mathsf{bd}_n$. There also exists some open neighbourhood V of y such that $V \setminus \{y\} \subseteq \nu(p_{n+2})$, and thus $Y \cap V \setminus \{y\} \subseteq \nu_Y^{\downarrow}(p_{n+1})$. All in all we have $y \in Y \cap U'$ with $y \neq x$ and $Y, \nu_Y^{\downarrow}, y \vDash \Box p_{n+1} \wedge \neg \mathsf{bd}_n$. Therefore $Y, \nu_Y^{\downarrow}, x \vDash \theta_n$. We also have $x \notin \nu(p_{n+2}) = \nu_Y^{\downarrow}(p_{n+1})$ so finally $Y, \nu_Y^{\downarrow}, x \vDash \theta_n \wedge \neg p_{n+1}$ as desired. $\square$

Lemma 3.13. *Let (X, ν) be a topological model, Y nowhere dense in X, $x \in Y$ and $n \in \mathbb{N}$. If $Y, \nu, x \vDash \neg \mathsf{bd}_n$ then $X, \nu_Y^{\uparrow}, x \vDash \neg \mathsf{bd}_{n+1}$.*

Proof. By induction on n. For $n = 0$, suppose that $x \in \widehat{\mathsf{d}}(\mathsf{d}(Y))$. Then there exists an open neighbourhood U of x such that $U \setminus \{x\} \subseteq \mathsf{d}(Y)$. We note that x is not isolated, otherwise $\{x\} \subseteq Y \subseteq \mathsf{Cl}(Y)$ with $\{x\}$ open, contradicting the fact that Y is nowhere dense. We then prove that $x \in \mathsf{Cl}(Y)$. Indeed, let V be an open neighbourhood of x; since x is not isolated there exists $y \in U \cap V$ such that $y \neq x$. Thus $y \in \mathsf{d}(Y)$, and it follows $V \cap Y \neq \emptyset$ as desired. Therefore $x \in \mathsf{Cl}(Y)$ and since $U \setminus \{x\} \subseteq \mathsf{d}(Y) \subseteq \mathsf{Cl}(Y)$ it follows $U \subseteq \mathsf{Cl}(Y)$ with U non-empty, a contradiction since Y is nowhere dense. Therefore $x \in X \setminus \widehat{\mathsf{d}}(\mathsf{d}(Y)) = \mathsf{d}(\widehat{\mathsf{d}}(X \setminus Y))$, in other words $X, \nu_Y^{\uparrow}, x \vDash \Diamond\Box p_1 \wedge \neg p_1$.

Now suppose that it holds for n, and assume $Y, \nu, x \vDash \neg \mathsf{bd}_{n+1}$, that is, $Y, \nu, x \vDash \theta_n \wedge \neg p_{n+1}$. It is then immediate that $X, \nu_Y^{\uparrow}, x \vDash \neg p_{n+2}$, so we have to show $X, \nu_Y^{\uparrow}, x \vDash \theta_{n+1}$. Let U be an open neighbourhood of x. By assumption there exists $y \in Y \cap U \setminus \{x\}$ such that $Y, \nu, y \vDash \Box p_{n+1} \wedge \neg \mathsf{bd}_n$. Thus there exists an open set W such that $y \in Y \cap W$ and $Y \cap W \setminus \{y\} \subseteq \nu(p_{n+1})$. Now let $z \in W \setminus \{y\}$; if $z \in X \setminus Y$ we have $z \in \nu_Y^{\uparrow}(p_{n+2})$, and otherwise $z \in W \cap Y \subseteq \nu(p_{n+1}) \subseteq \nu_Y^{\uparrow}(p_{n+2})$. Hence $W \setminus \{y\} \subseteq \nu_Y^{\uparrow}(p_{n+2})$. From $Y, \nu, y \vDash \neg \mathsf{bd}_n$ we also get $X, \nu_Y^{\uparrow}, y \vDash \neg \mathsf{bd}_{n+1}$ by the induction hypothesis. All in all we have $y \in U \setminus \{x\}$ and $X, \nu_Y^{\uparrow}, y \vDash \Box p_{n+2} \wedge \neg \mathsf{bd}_{n+1}$. Therefore $X, \nu_Y^{\uparrow}, x \vDash \theta_{n+1}$, and this concludes the proof. $\square$

This provides the desired theorem, which is the direct analogue of theorem 3.6 for the d-semantics:

Theorem 3.14. *Let X be a topological space. Then for all $n \in \mathbb{N}$ we have*

$$\mathsf{mdim}(X) \leq n-1 \iff X \vDash \mathsf{bd}_n$$

Proof. By induction on n. For $n = 0$ this is immediate. Suppose that this holds for n, and assume $\mathsf{mdim}(X) > n$. Then there exists some Y nowhere dense in X such that $\mathsf{mdim}(Y) > n-1$. By the induction hypothesis we have $Y \nvDash \mathsf{bd}_n$, so there exists a valuation ν and $x \in Y$ such that $Y, \nu, x \vDash \neg\mathsf{bd}_n$. Then by lemma 3.13 it follows $X, \nu_Y^{\uparrow}, x \vDash \neg\mathsf{bd}_{n+1}$, and thus $X \nvDash \mathsf{bd}_{n+1}$.

Conversely, suppose that $X \nvDash \mathsf{bd}_{n+1}$. Then there exists a valuation ν and $x \in X$ such that $X, \nu, x \vDash \neg\mathsf{bd}_{n+1}$. We define $Y := [\![\sigma]\!]_{\mathfrak{M}}$ and then by lemma 3.12 we know that $x \in Y$ and $Y, \nu_Y^{\downarrow}, x \vDash \neg\mathsf{bd}_n$. Thus $Y \nvDash \mathsf{bd}_n$ and by the induction hypothesis we obtain $\mathsf{mdim}(Y) > n-1$. By lemma 3.10, Y is nowhere dense in X and therefore $\mathsf{mdim}(X) > n$. ☐

This being achieved, we address the topological completeness of $\mathbf{wK4}+\mathsf{bd}_n$ (recall definition 2.25 for this term). To obtain this result, we need two ingredients: Kripke completeness of this logic, and a way to transform a Kripke frame into a topological space. The former is provided by [BGJ11] which asserts the finite model property for a class of extensions of **wK4** called *cofinal subframe logics.*

Definition 3.15. [BGJ11] Let $\mathfrak{F} = (W, R)$ be a Kripke frame. A subframe $\mathfrak{F}' = (W', R')$ of $\mathfrak{F}$ is called a *cofinal subframe* of $\mathfrak{F}$ if $w' \in W'$ and $w'Rw$ implies the existence of $u' \in W'$ such that wR^+u'.

Definition 3.16. [BGJ11] Let $\mathbf{L}$ be an extension of $\mathbf{K}$. The logic $\mathbf{L}$ is called *cofinal subframe* if whenever $\mathfrak{F} \vDash \mathbf{L}$ and $\mathfrak{F}'$ is a cofinal subframe of $\mathfrak{F}$, we have $\mathfrak{F}' \vDash \mathbf{L}$.

Theorem 3.17. [BGJ11] *Every extension of* **wK4** *which is a cofinal subframe logic has the finite model property.*

If $\mathfrak{F}$ is a **wK4** frame and $\mathfrak{F}'$ a subframe of $\mathfrak{F}$, it is clear that $\mathfrak{F}'$ is a **wK4** frame too and that $\mathsf{depth}(\mathfrak{F}') \leq \mathsf{depth}(\mathfrak{F})$. Therefore $\mathbf{wK4} + \mathsf{bd}_n$ is a cofinal subframe logic, and from theorem 3.17 it follows that is has the FMP, and thus that it is Kripke complete. As we can see, the FMP is not used in itself but is merely instrumental to completeness. Later in the thesis, other applications of this theorem will fulfill an actual need for the FMP.

The next step is to derive a topological space X from a Kripke frame $\mathfrak{F}$, but taking the space induced by $\mathfrak{F}$ (see definition 2.18) will not suffice. Indeed, we want that $\mathfrak{F} \nvDash \phi$ implies $X \nvDash_d \phi$. If we consider the frame $\mathfrak{F}$ consisting of a single reflexive point, we see that $\mathfrak{F} \nvDash \Box\bot$ whereas $\mathfrak{F} \vDash_d \Box\bot$. This is because the information that the point is reflexive is lost in the process[2]. To remedy this, we duplicate every reflexive point, according to the following pattern:

$$\begin{tikzcd} \mathbf{w} \arrow[loop above] \arrow[r, dashed, Rightarrow] & (\mathbf{w},\mathbf{0}) \arrow[r, bend left] & (\mathbf{w},\mathbf{1}) \arrow[l, bend left] \end{tikzcd}$$

This construction is adapted from [BEG09]. Here is the formal definition:

[2]As witnessed by the fact that the single *irreflexive* point generates the same space.

Definition 3.18. Let $\mathfrak{F} = (W, R)$ be a Kripke frame. We denote by W_r the set of reflexive worlds of W, and W_i the set of irreflexive worlds of W. The *dereflexivation* of $\mathfrak{F}$ is the Kripke frame $\mathfrak{F}_\bullet = (W_\bullet, R_\bullet)$ with:

- $W_\bullet := (W_i \times \{0\}) \cup (W_r \times \{0, 1\})$
- $R_\bullet := \{((w, k), (u, i)) \in {W_\bullet}^2 \mid wRu \text{ and } w \neq u\} \cup \bigcup_{wRw}\{((w, 0), (w, 1)), ((w, 1), (w, 0))\}$

Lemma 3.19. [BEG09] *If $\mathfrak{F}$ is* **wK4** *frame, then the map $\pi : \mathfrak{F}_\bullet \to \mathfrak{F}$ defined by $\pi(w, k) := w$ is a d-morphism.*

In addition, we need to prove that dereflexivation preserves the validity of bd_n:

Lemma 3.20. *If $\mathfrak{F}$ is a* **wK4** *frame such that $\mathsf{depth}(\mathfrak{F}) \vDash \mathsf{bd}_n$, then $\mathfrak{F}_\bullet \vDash \mathsf{bd}_n$.*

Proof. Assume that $\mathfrak{F} \vDash \mathsf{bd}_n$, that is, $\mathsf{depth}(\mathfrak{F}) \leq n$. We prove that $\mathsf{depth}(\mathfrak{F}_\bullet) \leq n$. For suppose there exists $(w_1, k_1), \ldots, (w_{n+1}, k_{n+1}) \in W_\bullet^{n+1}$ such that $(w_i, k_i)R_\bullet(w_{i+1}, k_{i+1})$ and not $(w_{i+1}, k_{i+1})R_\bullet(w_i, k_i)$ for all $i \in [\![1, n]\!]$. It is then clear that we have w_iRw_{i+1} and not $w_{i+1}Rw_i$ for all $i \in [\![1, n]\!]$, contradicting $\mathsf{depth}(\mathfrak{F}) \leq n$. Therefore $\mathsf{depth}(\mathfrak{F}_\bullet) \leq n$, and so $\mathfrak{F}_\bullet \vDash \mathsf{bd}_n$. $\square$

We may then conclude with the final theorem:

Theorem 3.21. *In the d-semantics, the logic* $\mathbf{wK4} + \mathsf{bd}_n$ *is topologically complete.*

Proof. Suppose that $\mathbf{wK4} + \mathsf{bd}_n \nvDash \phi$. We have argued above that $\mathbf{wK4} + \mathsf{bd}_n$ is Kripke complete, so let $\mathfrak{F}$ be a **wK4** frame such that $\mathfrak{F} \vDash \mathsf{bd}_n$ and $\mathfrak{F} \nvDash \phi$. Then by lemma 3.20, we have $\mathfrak{F}_\bullet \vDash \mathsf{bd}_n$ as well, and since $\mathfrak{F}_\bullet$ is irreflexive, proposition 2.23 yields $\mathfrak{F}_\bullet \vDash_d \mathsf{bd}_n$. In addition, π is a d-morphism by lemma 3.19, so from $\mathfrak{F} \nvDash \phi$ it follows $\mathfrak{F}_\bullet \nvDash_d \phi$. $\square$

3.3 Bounded depth and chains

In this section we aim to prove a variant of theorem 3.8 that is meaningful in the d-semantics. To be more precise, we aim to find:

- an analogue of item 2 involving validity in the d-semantics instead of the c-semantics;
- an analogue of items 3 and 4 involving d-morphisms instead of interior maps.

To meet the first goal, we introduce a variant of Jankov-Fine formulas. They are adapted from the *subframe formulas* presented in [CZ97, sec. 9.4], with some syntactic rework to make them more readable.

Definition 3.22. Let $\mathfrak{F} = (W, R)$ be a finite rooted Kripke frame. Let $W := \{w_i \mid i \in [\![0, n-1]\!]\}$ where w_0 is a root. The *subframe formula* $\alpha_\mathfrak{F}$ associated to $\mathfrak{F}$ is the conjunction of the following formulas:

1. q_0
2. $\Box^+\neg(q_i \wedge q_j)$ for all $0 \leq i < j < n$
3. $\Box^+(q_i \to \Diamond q_j)$ for all $i, j \in [\![0, n-1]\!]$ such that w_iRw_j

4. $\Box^+(q_i \to \neg\Diamond q_j)$ for all $i, j \in [\![0, n-1]\!]$ such that $\neg(w_i R w_j)$

In words, $\alpha_{\mathfrak{F}}$ is just like $\chi_{\mathfrak{F}}$ apart from two points: first, the terms of the conjunction are under the scope of $\Box^+$ instead of $\Box$, but this is merely an adjustment to account for the fact that we are no longer restricted to reflexive frames; second, the term $\Box^+ \bigvee_{i=0}^{n-1} q_i$ is not present. This means that satisfiability of $\alpha_{\mathfrak{F}}$ in a space X encodes the presence of the "pattern" given by $\mathfrak{F}$ within *some* subspace of X, instead of necessarily X itself. This is made more precise by the following result:

Proposition 3.23. *Let X be a topological space. Then $\alpha_{\mathfrak{F}}$ is satisfiable in X iff $\mathfrak{F}$ is a d-morphic image of some subspace of X.*

Proof. Suppose that there exists a set $Y \subseteq X$ and a surjective d-morphism $f : Y \to W$. We define a valuation ν by setting, for all $i \in [\![0, n-1]\!]$, $\nu(q_i) := f^{-1}(w_i)$. Since f is surjective, there exists $x \in \nu(q_0)$. We prove that $X, \nu, x \vDash \alpha_{\mathfrak{F}}$.

1. $X, \nu, x \vDash q_0$ by construction.

2. If $0 \le i < j < n$ then $X, \nu, x \vDash \Box^+\neg(q_i \wedge q_j)$ since any point in X has at most one image by f.

3. Suppose $w_i R w_j$ and let $y \in Y$ such that $f(y) = w_i$ (implying $y \in Y$). Then $y \in f^{-1}(R^{-1}\{w_j\})$, so $y \in \mathsf{d}_Y(f^{-1}(\{w_j\}))$ by theorem 2.32, that is, $y \in \mathsf{d}(f^{-1}(\{w_j\})) \cap Y$. Hence $X, \nu, y \vDash \Diamond q_j$. Since X is open it follows that $X, \nu, x \vDash \Box^+(q_i \to \Diamond q_j)$.

4. Suppose $\neg w_i R w_j$ and let $y \in Y$ such that $f(y) = w_i$ (implying $y \in Y$). Then $y \notin f^{-1}(R^{-1}\{w_j\})$, so $y \in \mathsf{d}_Y(f^{-1}(\{w_j\}))$ by theorem 2.32, that is, $y \notin \mathsf{d}(f^{-1}(\{w_j\})) \cap Y$. Hence $X, \nu, y \vDash \neg\Diamond q_j$. Since X is open it follows that $X, \nu, x \vDash \Box^+(q_i \to \neg\Diamond q_j)$.

Conversely, suppose that there exists a valuation ν and a point $x \in X$ such that $X, \nu, x \vDash \alpha_{\mathfrak{F}}$. After swapping boxes and conjunction we can find an open neighbourhood U of x such that every $y \in U$ satisfies formulas 2 to 4. Then, given $y \in U$, if there exists some $i \in [\![0, n-1]\!]$ such that $X, \nu, y \vDash q_i$, it is unique and we set $f(y) := q_i$. We define $Y := \mathsf{Dom}(f)$ and prove that $f : Y \to W$ meets the requirements.

- We show that f is surjective. By construction we have $f(x) = q_0$. Let $i \in [\![1, n-1]\!]$. Since $\mathfrak{F}$ is weakly transitive and rooted in $w_0 \neq w_i$ we have $w_0 R w_i$. Thus $X, \nu, x \vDash q_0 \to \Diamond q_i$, so $X, \nu, x \vDash \Diamond q_i$, so there exists $y \in U \setminus \{x\}$ such that $X, \nu, y \vDash q_i$ and consequently $f(y) = q_i$.

- We show that f is a d-morphism. If $j \in [\![0, n-1]\!]$, we show that $f^{-1}(R^{-1}\{w_j\}) = \mathsf{d}_Y(f^{-1}\{w_j\}))$. For suppose $y \in f^{-1}(R^{-1}\{w_j\})$; writing $w_i := f(y)$, this means that $w_i R w_j$. It follows that $X, \nu, y \vDash q_i \to \Diamond q_j$ and thus $y \in \mathsf{d}(\nu(q_j))$, so finally $y \in \mathsf{d}(\nu(q_j)) \cap Y = \mathsf{d}_Y(\nu(q_j))$. Conversely, suppose $y \notin f^{-1}(R^{-1}\{w_j\})$; then $\neg(w_i R w_j)$, and thus $X, \nu, y \vDash q_i \to \neg\Diamond q_j$ and $y \notin \mathsf{d}(\nu(q_j))$, and therefore $y \notin \mathsf{d}_Y(\nu(q_j))$

  Since W is finite and R^{-1} and d commute with union, it follows that $f^{-1}(R^{-1}A) = \mathsf{d}_Y(f^{-1}A)$ for all $A \subseteq W$. By theorem 2.32 we obtain that f is a d-morphism.

$\square$

Now, our goal is to connect subframe formulas to modal dimension by showing that given a space X we have $X \vDash \mathsf{bd}_n$ iff $X \vDash \neg\alpha_{\mathfrak{F}}$ for all $n+1$-chain $\mathfrak{F}$. We can see that we are now quantifying over all chains instead of considering only the reflexive one, which makes sense with regard to the d-semantics. Here the challenging implication is

$$X \vDash \bigwedge\{\neg\alpha_{\mathfrak{F}} \mid \mathfrak{F} \text{ is a } n+1\text{-chain}\} \implies X \vDash \mathsf{bd}_n$$

and for convenience we will instead prove

$$X \vDash \bigwedge\{\neg\alpha_{\mathfrak{F}} \mid \mathfrak{F} \text{ is a } n+1\text{-chain}\} \implies X \vDash \mathsf{bd}_n^+$$

(we will see that this is sufficient). It will however be easier to work with Kripke frames, which is possible if we move to the syntactic level, and then use Kripke completeness. However, for a reason that will soon be clear, attempting to show

$$\mathbf{wK4} + \{\neg\alpha_{\mathfrak{F}} \mid \mathfrak{F} \text{ is a } n+1\text{-chain}\} \vdash \mathsf{bd}_n^+$$

will not succeed. At some point in the proof we will need fixpoint operators (see section 2.4), and this is why need to work within the stronger system $\mu\mathbf{wK4}$. Thus, the claim we are going to prove is

$$\mu\mathbf{wK4} + \{\neg\alpha_{\mathfrak{F}} \mid \mathfrak{F} \text{ is a } n+1\text{-chain}\} \vdash \mathsf{bd}_n^+ \tag{3.1}$$

Unfortunately, while $\mu\mathbf{wK4}$ is Kripke complete, we do not know whether this is the case for $\mu\mathbf{wK4} + \{\neg\alpha_{\mathfrak{F}} \mid \mathfrak{F} \text{ is a } n+1\text{-chain}\}$. The solution then comes from one crucial observation, namely that only finitely many instances of the $\neg\alpha_{\mathfrak{F}}$'s are needed to derive bd_n^+. Indeed we know that if bd_n^+ is refuted in a frame $\mathfrak{F}_0$, this is because of the presence of a sequence $w_0R^+ \dots R^+w_n$ such that not $w_{i+1}R^+w_i$ for all $i \in [\![0, n-1]\!]$. From this sequence we can then construct an $n+1$-chain $\mathfrak{F}$[3] such that by properly instantiating the q_i's, we can satisfy $\alpha_{\mathfrak{F}}$ in $\mathfrak{F}_0$. This instance is derived from a tuple of formulas essentially describing the structure of $\mathfrak{F}$ (i.e., which points are reflexive) along with some information retrieved from the falsity of bd_n; for this reason it is called the *refutation tuple* associated to $\mathfrak{F}$. Since there are finitely many $n+1$-chains and each chain has only one refutation tuple, we end up as intended with finitely many instances of subframe formulas. Therefore 3.1 can be seen as an implication in $\mu\mathbf{wK4}$.

This also explains why we work with the μ-calculus: if $\mathfrak{F}$ is a $n+1$-chain containing reflexive points, then $\alpha_{\mathfrak{F}}$ contains subformulas of the form $\Box^+(q_i \to \Diamond q_i)$, which imposes a self-referential condition on the formula that is substituted for q_i. When fixpoint operators are available, constructing such a formula is then very easy.

Definition 3.24. Let $n \in \mathbb{N}$ and $\mathfrak{F} = (W, R)$ an n-chain with $W = \{w_k \mid k \in [\![0, n-1]\!]\}$. The *refutation tuple* associated to $\mathfrak{F}$ is the tuple of formulas $t_{\mathfrak{F}} = (\phi_0, \dots, \phi_{n-1})$ defined by, for all $k \in [\![0, n-1]\!]$:

$$\phi_k := \begin{cases} \nu p.(\psi_k \wedge \Diamond p) & \text{if } w_k \text{ is reflexive} \\ \psi_k \wedge \neg\Diamond\psi_k & \text{otherwise} \end{cases}$$

where ψ_k is defined by downward recursion as follows:

[3] Warning: while one might think that the $n+1$-chain induced by $\{w_0, \dots, w_n\}$ is a natural choice for this $\mathfrak{F}$, this is *not* the case in general.

- $\psi_{n-1} := \bigwedge_{1\leq i\leq n-1} \Box^+ p_i$
- for all $k \in [\![0, n-2]\!]$, $\psi_k := \Diamond\phi_{k+1} \wedge \neg p_{n-k-1} \wedge \bigwedge_{n-k\leq i\leq n-1} \Box^+ p_i$

The following lemma then establishes the result explained above:

Lemma 3.25. *Let us write $\overline{q} := (q_0, \dots, q_n)$. We then have*

$$\mu\mathbf{wK4} \vdash \bigwedge\{\Box^+\neg\alpha_{\mathfrak{F}}[t_{\mathfrak{F}}/\overline{q}] \mid \mathfrak{F} \textit{ is a } n+1\textit{-chain and } t_{\mathfrak{F}} = (\phi_0, \dots, \phi_n)\} \to \mathsf{bd}_n^+$$

Proof. By theorem 2.37, it suffices to consider a weakly transitive Kripke model $\mathfrak{M} = (W, R, V)$ and $w \in W$ such that

$$\mathfrak{M}, w \vDash \bigwedge\{\Box^+\neg\alpha_{\mathfrak{F}}[t_{\mathfrak{F}}/\overline{q}] \mid \mathfrak{F} \text{ is a } n+1\text{-chain and } t_{\mathfrak{F}} = (\phi_0, \dots, \phi_n)\}$$

and to prove that $\mathfrak{M}, w \vDash \mathsf{bd}_n^+$. For suppose not. Then there exist $w_0, \dots, w_n$ with $w_0 = w$ such that $\mathfrak{M}, w_0 \vDash \neg p_n$ and $\mathfrak{M}, w_n \vDash \Box^+ p_1$ and for all $k \in [\![1, n-1]\!]$, $w_k R^+ w_{k+1}$ and $\mathfrak{M}, w_k \vDash \neg p_{n-k} \wedge \Box^+ p_{n-k+1}$. We define the formulas $\phi_0, \dots, \phi_n$ and $\psi_0, \dots, \psi_n$ recursively, with the condition that $\mathfrak{M}, w_k \vDash \psi_k$ for all $k \in [\![0, n]\!]$. First we set $\psi_n := \bigwedge_{1\leq i\leq n} \Box^+ p_i$, and if $k \in [\![0, n]\!]$ and ψ_k is defined, we consider two cases:

- if $\mathfrak{M}, w_k \vDash \nu p.(\psi_k \wedge \Diamond p)$ then $\phi_k := \nu p.(\psi_k \wedge \Diamond p)$;
- otherwise we set $\phi_k := \psi_k \wedge \neg\Diamond\psi_k$.

If $k > 0$ we then define $\psi_{k-1} := \Diamond\phi_k \wedge \neg p_{n-k+1} \wedge \bigwedge_{n-k+2\leq i\leq n} \Box^+ p_i$. It is then clear that $(\phi_0, \dots, \phi_n)$ is the refutation tuple associated to some $n+1$-chain $\mathfrak{F}$; we write $u_0, \dots, u_n$ its elements.

We now construct by downward recursion a sequence of worlds $(w'_0, \dots, w'_n)$ satisfying $w_k R^+ w'_k$ and $\mathfrak{M}, w'_k \vDash \phi_k$ for all $k \in [\![0, n]\!]$:

- Since $\mathfrak{M}, w_{n-i+1} \vDash \Box^+ p_i$ for all $i \in [\![1, n]\!]$ and $\mathfrak{M}$ is weakly transitive we have $\mathfrak{M}, w_n \vDash \psi_n$. If $\mathfrak{M}, w_n \vDash \nu p.(\psi_n \wedge \Diamond p)$ we set $w'_n := w_n$ and we are done; otherwise there exists a path $v_1 R \dots R v_m$ with $v_1 = w_n$ and such that $\mathfrak{M}, v_i \vDash \psi_n$ for all $i \in [\![1, m-1]\!]$ and $\mathfrak{M}, v_m \nvDash \Diamond\psi_n$. By weak transitivity we have $w_0 R^+ v_m$ so we set $w'_0 := v_m$ and we are done.
- If $k \in [\![0, n-1]\!]$ we have $w_k R^+ w_{k+1} R^+ w'_{k+1}$ and $\mathfrak{M}, w'_{k+1} \vDash \phi_{k+1}$ by the induction hypothesis. Then by weak transitivity we have $w_k R^+ w'_{k+1}$, but $w_k = w'_{k+1}$ is impossible since $\mathfrak{M}, w_k \vDash \neg p_{n-k}$ and $\mathfrak{M}, w_{k+1} \vDash \Box^+ p_{n-k}$ yields $\mathfrak{M}, w'_{k+1} \vDash p_{n-k}$. Hence $w_k R w'_{k+1}$ and since $\mathfrak{M}, w'_{k+1} \vDash \phi_{k+1}$ by the induction hypothesis, we obtain $\mathfrak{M}, w_k \vDash \Diamond\phi_{k+1}$. We also have $\mathfrak{M}, w_k \vDash \neg p_{n-k}$ and $\mathfrak{M}, w_k \vDash \Box^+ p_i$ for all $i \in [\![n+1-k, n+1]\!]$ by the same argument as above. Therefore $\mathfrak{M}, w_k \vDash \psi_k$ and the construction of w'_k is analogous to the case $k = 0$.

$$\begin{tikzcd}
\Box^+ p_1 & \mathbf{w_n} \arrow[r] & \cdots \arrow[r] & \mathbf{w'_n} & \phi_n \\
\neg p_1, \Box^+ p_2 & \mathbf{w_{n-1}} \arrow[u] \arrow[r] & \cdots \arrow[r] & \mathbf{w'_{n-1}} & \phi_{n-1} \\
\neg p_2, \Box^+ p_3 & \mathbf{w_{n-2}} \arrow[u] \arrow[r] & \cdots \arrow[r] & \mathbf{w'_{n-2}} & \phi_{n-2} \\
 & \vdots \arrow[u] & & & \\
\neg p_n & \mathbf{w_0} \arrow[u] \arrow[r] & \cdots \arrow[r] & \mathbf{w'_0} & \phi_0
\end{tikzcd}$$

We then show that $\mathfrak{M}, w'_0 \models \neg\alpha_{\mathfrak{F}}[t_{\mathfrak{F}}/\overline{q}]$:

1. We know that $\mathfrak{M}, w'_0 \models \phi_0$.
2. If $0 \leq i < j \leq n$ then $\models \phi_j \to \Box^+ p_{n-i}$ whereas $\models \phi_i \to \neg p_{n-i}$, so this proves $\mathfrak{M}, w'_0 \models \Box^+\neg(\phi_i \wedge \phi_j)$.
3. Suppose $0 \leq i \leq j \leq n$.
   - If $i < j$ we show that $\mathfrak{M}, w'_0 \models \Box^+(\phi_i \to \Diamond\phi_j)$. For suppose $\mathfrak{M}, v \models \phi_j$. Since for all $k \in [\![i+1, j]\!]$ we have $\models \phi_k \to \Diamond\phi_{k+1}$ we obtain the existence of a path $v_1 R \ldots R v_m$ with $v_1 = v$ and such that $\mathfrak{M}, v_m \models \phi_j$. By weak transitivity it follows that vR^+v_m and by 2 we cannot have $v = v_m$, so vRv_m, and this proves $\mathfrak{M}, v \models \Diamond\phi_j$.
   - If $i = j$ we suppose u_i reflexive in $\mathfrak{F}$, i.e., $\phi_i = \nu p.(\psi_i \wedge \Diamond p)$. Then $\mathfrak{M}, w'_0 \models \Box^+(\phi_i \to \Diamond\phi_i)$ as an immediate consequence of the fixpoint axiom.
4. Suppose $0 \leq i \leq j \leq n$
   - If $i < j$ we show that $\mathfrak{M}, w'_0 \models \Box^+(\phi_j \to \neg\Diamond\phi_j)$. For suppose $\mathfrak{M}, v \models \phi_j$ and vRu with $\mathfrak{M}, u \models \phi_i$. Then $\mathfrak{M}, v \models \Box^+ p_{n-i}$ whereas $\mathfrak{M}, u \models \neg p_{n-i}$, a contradiction. This proves the claim.
   - If $i = j$, suppose u_i irreflexive in $\mathfrak{F}$, i.e. $\phi_i = \psi_i \wedge \neg\Diamond\psi_i$. It is then obvious that $\mathfrak{M}, w'_0 \models \Box^+(\phi_i \to \neg\Diamond\phi_i)$.

Since $wR^+w'_0$, it follows that $\mathfrak{M}, w \not\models \Box^+\neg\alpha_{\mathfrak{F}}[t_{\mathfrak{F}}/\overline{q}]$, a contradiction. This concludes the proof. $\Box$

Lemma 3.26. *We have*

$$\mu\mathbf{wK4} + \{\neg\alpha_{\mathfrak{F}} \mid \mathfrak{F} \textit{ is a } n+1\textit{-chain}\} \vdash \mathsf{bd}_n^+$$

Proof. Let $\mathfrak{F}'$ be a $n+1$-chain. By the rules of necessitation and substitution we have

$$\mu\mathbf{wK4} + \{\neg\alpha_{\mathfrak{F}} \mid \mathfrak{F} \text{ is a } n\text{-chain}\} \vdash \Box^+\neg\alpha_{\mathfrak{F}}[t_{\mathfrak{F}'}/\overline{q}]$$

and we conclude by lemma 3.25. $\Box$

We are now able to state and prove the main result of this section:

Theorem 3.27. *Let X be a topological space. The following conditions are equivalent for all $n \in \mathbb{N}$:*

1. $\mathsf{mdim}(X) \leq n - 1$
2. $X \vDash \neg\alpha_{\mathfrak{F}}$ *for all* $n+1$*-chain* $\mathfrak{F}$
3. *No* $n+1$*-chain is a d-morphic image of a subspace of* X

Proof. From 3 to 2, suppose that $X \nvDash \neg\alpha_{\mathfrak{F}}$ for some $n+1$-chain $\mathfrak{F}$, that is, $\alpha_{\mathfrak{F}}$ is satisfiable in X. Then by proposition 3.23, there exists a subspace Y of X and a surjective d-morphism from Y to some $\mathfrak{F}$.

From 1 to 3, suppose that there exists a surjective d-morphism f from an open set $Y \subseteq X$ to some $n+1$-chain $\mathfrak{F}$. In particular, f is an interior map, and it can be seen as an interior map from Y to the reflexive $n+1$-chain (since all $n+1$-chains induce the same topological space). Then from theorem 3.8, we obtain $\mathsf{mdim}(X) > n - 1$.

From 2 to 1, suppose that $X \vDash \neg\alpha_{\mathfrak{F}}$ for all $n+1$-chain $\mathfrak{F}$. In other words

$$X \vDash \mu\mathbf{wK4} + \{\neg\alpha_{\mathfrak{F}} \mid \mathfrak{F} \text{ is a } n+1\text{-chain}\}$$

and from lemma 3.26 it follows that $X \vDash \mathsf{bd}_n^+$. Then by proposition 2.28 we obtain $X \vDash_c \mathsf{bd}_n$, and therefore $\mathsf{mdim}(X) \leq n - 1$ by theorem 3.8.

□

Remark 3.28. Items 3 and 4 of theorem 3.8 may suggest that, similarly:

- $\mathsf{mdim}(X) \leq n - 1$ iff no $n+1$-chain is a d-morphic image of X;
- $\mathsf{mdim}(X) \leq n - 1$ iff no $n+1$-chain is a d-morphic image of some open subspace of X.

This differs from $1 \iff 3$ above on the quantification over subspaces of X, that is, we only consider X itself in the first case, and its *open* subspaces in the second case. Yet this is not the case in general: if we take $X := \{0, 1, 2, 3\}$ and $\tau := \{\emptyset, \{1, 2\}, \{3\}, X\}$ (see figure 3.2) we find that $\mathsf{mdim}(X) > 0$ yet no 2-chain is a d-morphic image of X, nor any open subspace of X.

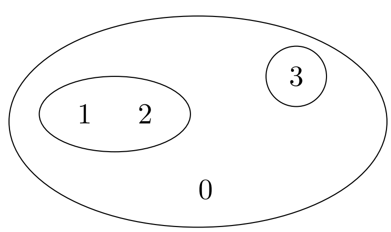


Figure 3.2: A counter-example

3.4 Bounded depth and derivative

We conclude this chapter with a last, modest result: we show that taking the derivative of a space decreases its modal dimension by at most 1. This proposition can be split into two parts: on the one hand, given a space X we have $\mathsf{mdim}(\mathsf{d}(X)) \leq \mathsf{mdim}(X)$, but in fact we will show the stronger claim that $\mathsf{mdim}(Y) \leq \mathsf{mdim}(X)$ whenever Y is a subspace of X; on the other hand, $\mathsf{mdim}(X) \geq \mathsf{mdim}(\mathsf{d}(X)) + 1$. To achieve this goal we essentially reuse the techniques introduced in section 3.2, starting with transformations of valuations:

Definition 3.29. Let $n \in \mathbb{N}$, X a space, $Y \subseteq X$ and ν a valuation over Y. The *Y-stuffing* $\nu_Y^\bullet$ of ν is the valuation over X with domain P_n such that for all $k \in [\![1, n]\!]$, $\nu_Y^\bullet(p_k) := \nu(p_k) \cup X \setminus Y$.

Lemma 3.30. *Let (X, ν) be a topological model, $x \in Y \subseteq X$ and $n \in N$. If $Y, \nu, x \vDash \neg\mathsf{bd}_n$ then $X, \nu_Y^\bullet, x \vDash \neg\mathsf{bd}_n$.*

Proof. By induction on n. For $n = 0$ this is trivial since $\neg\mathsf{bd}_0 \equiv \top$.

Suppose that it holds for n, and assume $Y, \nu, x \vDash \neg\mathsf{bd}_{n+1}$, i.e., $Y, \nu, x \vDash \theta_n \wedge \neg p_{n+1}$. Let U be an open neighbourhood of x. By assumption there exists $y \in U \cap Y \setminus \{x\}$ such that $Y, \nu, y \vDash \Box p_{n+1} \wedge \neg\mathsf{bd}_n$. Thus there exists an open neighbourhood V of y such that $V \cap Y \setminus \{y\} \subseteq \nu(p_{n+1})$. If $z \in V \setminus \{y\}$ we have either $z \in Y$, in which case $z \in \nu(p_{n+1}) \subseteq \nu_Y^\bullet(p_{n+1})$, or $z \notin Y$ which yields $z \in \nu_Y^\bullet(p_{n+1})$. This proves $X, \nu_Y^\bullet, y \vDash \Box p_{n+1}$, and we also have $X, \nu_Y^\bullet, y \vDash \neg\mathsf{bd}_n$ by the induction hypothesis. Since $y \in U \setminus \{x\}$ we obtain $X, \nu_Y^\bullet, x \vDash \theta_n$ and we are done. □

Proposition 3.31. *Let X be a topological space and $Y \subseteq X$ a subspace of X. Then $\mathsf{mdim}(Y) \leq \mathsf{mdim}(X)$.*

Proof. If $\mathsf{mdim}(X) = \infty$ this is obvious. Assume that $n := \mathsf{mdim}(X)$ is finite.

Suppose $\mathsf{mdim}(Y) > n$. Then, by theorem 3.14 we have $Y \nvDash \mathsf{bd}_{n+1}$, that is, there exists a valuation ν and $x \in \mathsf{d}(X)$ such that $Y, \nu, x \vDash \neg\mathsf{bd}_{n+1}$. Then by lemma 3.30 we obtain $X, \nu_Y^\bullet, x \vDash \neg\mathsf{bd}_{n+1}$ and thus $X \nvDash \mathsf{bd}_{n+1}$. By theorem 3.14 again we get $\mathsf{mdim}(X) > n$, a contradiction. Therefore $\mathsf{mdim}(Y) \leq n$. □

Lemma 3.32. *Let X be a topological space. If $\mathsf{mdim}(\mathsf{d}(X)) \leq n$ then $\mathsf{mdim}(X) \leq n + 1$.*

Proof. By contraposition, suppose $\mathsf{mdim}(X) > n + 1$. Then $X \nvDash \mathsf{bd}_{n+2}$ by theorem 3.14, so there exists a valuation ν and $x \in X$ such that $X, \nu, x \vDash \neg\mathsf{bd}_{n+2}$. It is clear that $[\![\sigma]\!]_{(X,\nu)} \subseteq \mathsf{d}(X)$ so we can apply lemma 3.12 to $Y := \mathsf{d}(X)$ and obtain $x \in \mathsf{d}(X)$ and $\mathsf{d}(X), \nu_Y^\downarrow, x \vDash \neg\mathsf{bd}_{n+1}$. Hence $\mathsf{d}(X) \nvDash \neg\mathsf{bd}_{n+1}$ and it follows that $\mathsf{mdim}(\mathsf{d}(X)) > n$. □

Theorem 3.33. *For all topological space X we have*

$$\mathsf{mdim}(X) - 1 \leq \mathsf{mdim}(\mathsf{d}(X)) \leq \mathsf{mdim}(X)$$

with the convention that $\infty - 1 = \infty$.

Proof. If $\mathsf{mdim}(X)$ is finite then this is an immediate consequence of proposition 3.31 and lemma 3.32. If $\mathsf{mdim}(X) = \infty$, we show that $\mathsf{mdim}(\mathsf{d}(X)) = \infty$ too. If not, we have $\mathsf{mdim}(\mathsf{d}(X)) \leq n$ for some n, so $\mathsf{mdim}(X) \leq n + 1$ by lemma 3.32, a contradiction. This concludes the proof. □

This closes the chapter and leaves us with a good understanding of the derivational meaning of bd_n: in addition to the link with modal Krull dimension and the proof of completeness, we have also established convenient characterizations that allow us to appreciate the differences and similarities between the c-semantics and the d-semantics. These insights will be helpful in the next chapter, wherein we study other axiomatic systems, and plan to combine them with bd_n.

Chapter 4

Extensions of **wK4** and topological completeness

Whereas the logics **S4.2** and **S4.3** have been part of the landscape of the c-semantics for a while, with their well-known connection to extremally disconnected spaces, the interpretation of the axioms .2 and .3 in the d-semantics is still missing. Further, several axioms coincide with .2 or .3 over reflexive and transitive frames, but turn out to be distinct when we step outside of **S4** and only work under **wK4**. This generates some confusion, and preliminary work will be needed in the first place to clarify the situation. We will eventually identify four logics of interest: **wK4.2** and **wK4.3** which correspond directly to **S4.2** and **S4.3**, and $\mathbf{wK4.2}^+$ and $\mathbf{wK4.3}^+$ which are obtained by taking the **S4**-translation of .2 and .3. We can already see that we will obtain the semantics of the last two from proposition 2.28 for free, so we are merely studying them for the sake of comprehensiveness.

More instructive will be the study of the first two systems: in section 4.1, we show that **wK4.2** defines a subclass of the class of extremally disconnected spaces with some forbidden patterns. The case of **wK4.3** is also very fertile, for we observe that from .3 we can derive the simpler axiom $\mathsf{aT} := \Box(p \to \Diamond p)$. While its Kripke semantics is rather unimpressive, its topological semantics will turn out to produce rich results, and this justifies the presence of section 4.2 which is entirely devoted to aT and its relation to a new kind of spaces called *accumulative*. After that, we will show in section 4.3 that **wK4.3** defines the class of spaces that are hereditarily extremally disconnected and accumulative. Finally, we address in section 4.4 a handful of extensions of these logics, obtained by adding the axioms gl and bd_n.

Along this road we will naturally be interested in the topological completeness of these logics. To achieve this goal, we will use the operation of *unfolding* a Kripke frame presented in [BBFD21]. While this strategy works well for **wK4.2** and $\mathbf{wK4.2}^+$, the topology of the unfolded frame turns out to be too coarse as soon as extensions of $\mathbf{wK4} + \mathsf{aT}$ are concerned. To remedy this, we will introduce the operation of *refined unfolding* which generates more open sets and successfully yields the topological completeness of $\mathbf{wK4} + \mathsf{aT}$ and **wK4.3**. For the logic $\mathbf{wK4.3}^+$ we will reuse the technique of dereflexivation, on which we will also rely in section 4.4.

4.1 The logic wK4.2

In this section we adapt **S4.2** to the d-semantics and establish completeness of the resulting logics. The following is well known:

Theorem 4.1. [vBB07] *In the c-semantics,* **S4.2** *defines the class of extremally disconnected spaces, and is topologically complete.*

Recall that (hereditarily) extremally disconnected spaces were introduced in definition 2.15. It may be tempting to focus solely on the axiom $.2 = \Diamond\Box p \to \Box\Diamond p$. But in [CZ97, sec. 3.5] we also find $\mathbf{K4.2} := \mathbf{K4} + \Diamond(p \wedge \Box q) \to \Box(p \vee \Diamond q)$ which defines the class of transitive directed frames. The condition of *directedness* of a Kripke frame $\mathfrak{F} = (W, R)$ is

$$\text{for all } w, u, v \in W, \text{ if } wRu \text{ and } wRv \text{ and } u \neq v \text{ then } \exists t \in W, uRt \text{ and } vRt$$

Recall that the property of *strong directedness* is the same without the condition $u \neq v$. Hence we must consider $\mathbf{wK4} + \Diamond(p \wedge \Box q) \to \Box(p \vee \Diamond q)$ as a candidate to define **wK4.2**. This logic, however, turns out to admit another axiomatization:

Proposition 4.2. *We have* $\mathbf{wK4} + \Diamond(p \wedge \Box q) \to \Box(p \vee \Diamond q) = \mathbf{wK4} + .2^+$*. In addition:*

- *the logic* $\mathbf{wK4} + .2^+$ *defines the class of weakly transitive directed frames and is Kripke complete;*
- *in the d-semantics, the logic* $\mathbf{wK4} + .2^+$ *defines the class of extremally disconnected spaces.*

Proof. We know from [CZ97, sec. 3.5] that $\Diamond(p \wedge \Box q) \to \Box(p \vee \Diamond q)$ defines directedness. Since w4 and $\Diamond(p \wedge \Box q) \to \Box(p \vee \Diamond q)$ are Sahlqvist formulas [BRV01, sec. 4.3], we immediately obtain Kripke completeness of $\mathbf{L} := \mathbf{wK4} + \Diamond(p \wedge \Box q) \to \Box(p \vee \Diamond q)$. In addition, given a Kripke frame $\mathfrak{F}$, we can see that $\mathfrak{F}$ is directed iff $\mathfrak{F}^+$ is strongly directed, so by proposition 2.28 it follows that $.2^+$ defines directedness too.

Since $\mathbf{L}$ and $\mathbf{wK4} + .2^+$ define the same class of frames, and $\mathbf{L}$ is Kripke complete, we obtain $\mathbf{wK4} + .2^+ \subseteq \mathbf{L}$. For the other inclusion, it suffices to prove $\mathbf{L} \vdash .2^+$. Substituting p for q in $\Diamond(p \wedge \Box q) \to \Box(p \vee \Diamond q)$, we obtain $\mathbf{L} \vdash \Diamond\Box^+ p \to \Box\Diamond^+ p$, and from there it is not hard to prove that $\mathbf{L} \vdash \Diamond^+\Box^+ p \to \Box^+\Diamond^+ p$.

Finally, if X is a topological space, we know from proposition 2.28 that $X \vDash_d .2^+$ iff $X \vDash_c .2$, and from theorem 4.1 it follows that $X \vDash_d \mathbf{L}$ iff X is extremally disconnected. $\square$

This being clarified, the choice of the following names is then natural:

$$\mathbf{wK4.2} := \mathbf{wK4} + .2$$

$$\mathbf{wK4.2}^+ := \mathbf{wK4} + .2^+$$

We are now going to investigate the semantics of **wK4.2**. Unsurprisingly, validity of **wK4.2** in a space X implies that X is extremally disconnected, but this condition is not sufficient since the following spaces spaces falsify **wK4.2** even though they are extremally disconnected:

Yet .2 is valid in the following spaces, which are open subspaces of the previous two:

This observation leads us to break any **wK4.2** space into two open subspaces: one called *almost discrete* which gathers all the subspaces of the form X_1' and X_2', and one called *strongly dense-in-itself* which by construction cannot contain X_1 nor X_2. We will then show that this decomposition is actually a complete characterization of the class of spaces defined by **wK4.2**. Here is the definition of these properties:

Definition 4.3. Let X be a topological space. An open set U is said to be *atomic* if it is non-empty and there exists no open set V such that $\emptyset \subset V \subset U$[1]. Then X is called:

- *almost discrete* if every point has an atomic open neighbourhood U such that $|U| \leq 2$;
- *strongly dense-in-itself* if for every non-empty open set U we have $|U| \geq 3$.

Remark 4.4. To understand how this definition is related to discrete and dense-in-itself spaces, it is insightful to reformulate these two conditions as follows:

- X is *discrete* iff every point has an atomic open neighbourhood U such that $|U| \leq 1$;
- X *dense-in-itself* iff for every non-empty open set U we have $|U| \geq 2$.

The link is then clear, and the names are consistent since any discrete space is also almost discrete, and any strongly dense-in-itself space is also dense-in-itself.

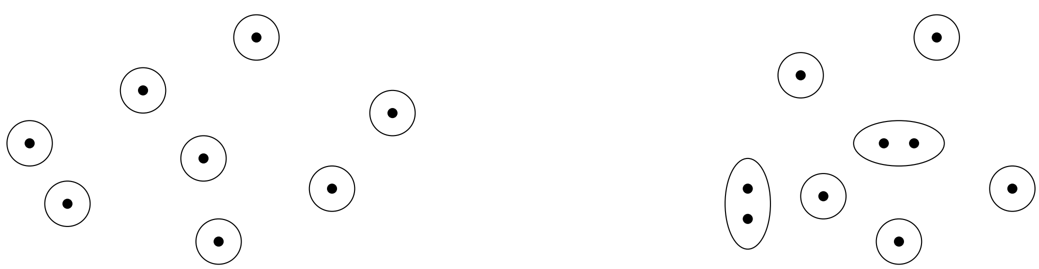


Figure 4.1: A discrete space (left) and an almost discrete space (right)

[1]Here and throughout the whole document, $A \subset B$ means that $A \subseteq B$ and $A \neq B$

The hardest part will be to show that .2 is valid in any strongly dense-itself extremally disconnected space, and to achieve this result a few preliminaries will be needed. We begin by introducing a variant of the notion of extremal disconnectedness:

Definition 4.5. A topological space X is said to be *strongly extremally disconnected* if for all open set U of X, $\mathsf{d}(U)$ is open too.

As suggested by the name, this is a stronger condition than extremal disconnectedness:

Proposition 4.6. *Any strongly extremally disconnected space is also extremally disconnected.*

Proof. Let X be strongly extremally disconnected. If U is open in X then $\mathsf{d}(U)$ is open too, and so is $U \cup \mathsf{d}(U) = \mathsf{Cl}(U)$. $\square$

The converse is not true in general (the space induced by a 2-chain is a counter-example), but the two conditions coincide on dense-in-themselves spaces:

Proposition 4.7. *Any dense-in-itself extremally disconnected space is strongly extremally disconnected.*

Proof. Let X be dense-in-itself and extremally disconnected. If U is open in X, we show that $\mathsf{Cl}(U) = \mathsf{d}(U)$. For consider $x \in \mathsf{Cl}(U)$ and V an open neighbourhood of x. Then $U \cap V$ is also an open neighbourhood of x, and since X is dense-in-itself there must exist $y \in U \cap V$ such that $y \neq x$, so that $x \in \mathsf{d}(U)$. This proves the claim, and since $\mathsf{Cl}(U)$ is open by assumption, $\mathsf{d}(U)$ is open too. $\square$

The class of strongly extremally disconnected spaces then turns out to be defined by a modal formula which without surprise is very similar to .2:

Theorem 4.8. *The logic* $\mathbf{wK4} + (\Diamond\Box^+ p \to \Box\Diamond p)$ *defines the class of strongly extremally disconnected spaces.*

Proof. Let X be a space and suppose $X \vDash \Diamond\Box^+ p \to \Box\Diamond p$. Let U be open in X. By assumption we have $\mathsf{d}(\mathsf{Int}(U)) \subseteq \widehat{\mathsf{d}}(\mathsf{d}(U))$, that is, $\mathsf{d}(U) \subseteq \widehat{\mathsf{d}}(\mathsf{d}(U))$. Then $\mathsf{d}(U) \subseteq \mathsf{d}(U) \cap \widehat{\mathsf{d}}(\mathsf{d}(U)) = \mathsf{Int}(\mathsf{d}(U))$, which means that $\mathsf{d}(U)$ is open.

Conversely, suppose that X is strongly extremally disconnected. If $A \subseteq X$, we know that $\mathsf{d}(\mathsf{Int}(A))$ is open. Thus

$$\mathsf{d}(\mathsf{Int}(A)) = \mathsf{Int}(\mathsf{d}(\mathsf{Int}(A))) \subseteq \widehat{\mathsf{d}}(\mathsf{d}(\mathsf{Int}(A))) \subseteq \widehat{\mathsf{d}}(\mathsf{d}(A))$$

and this proves $X \vDash \Diamond\Box^+ p \to \Box\Diamond p$. $\square$

These results lead to this key proposition:

Proposition 4.9. *Let X be a strongly dense-in-itself space. Then $X \vDash$ **wK4.2** if and only if X is extremally disconnected.*

Proof. Combining proposition 4.7 and theorem 4.8, it suffices to prove that $X \vDash \Diamond\Box p \to \Box\Diamond p$ iff $X \vDash \Diamond\Box^+ p \to \Box\Diamond p$. From left to right, suppose $X \vDash \Diamond\Box p \to \Box\Diamond p$, and let $A \subseteq X$. Then by assumption we obtain $\mathsf{d}(\mathsf{Int}(A)) \subseteq \mathsf{d}(\widehat{\mathsf{d}}(A)) \subseteq \widehat{\mathsf{d}}(\mathsf{d}(A))$.

From right to left, suppose $X \vDash \Diamond\Box^+ p \to \Box\Diamond p$, and let $A \subseteq X$ and $x \in \mathsf{d}(\widehat{\mathsf{d}}(A))$. Then for all open neighbourhood U of x there exists $y_U \in U \setminus \{x\}$ such that $y_U \in \widehat{\mathsf{d}}(A)$. We then set $A' := A \cup \{y_U \mid U \text{ open and } x \in U\}$ and by construction we have $x \in \mathsf{d}(\mathsf{Int}(A'))$. By assumption it follows that $x \in \widehat{\mathsf{d}}(\mathsf{d}(A'))$, that is, there exists an open neighbourhood V of x such that $V \setminus \{x\} \subseteq \mathsf{d}(A')$. Then it suffices to prove that $V \setminus \{x\} \subseteq \mathsf{d}(A)$. For consider $y \in V \setminus \{x\}$ and W an open neighbourhood of y. Then there exists $z \in W \setminus \{y\}$ such that $z \in A'$. If $z \in A$ we are done, otherwise $z = y_U$ for some open neighbourhood U of x. Since $y_U \in \widehat{\mathsf{d}}(A)$ there exists an open neighbourhood T of y_U such that $T \setminus \{y_U\} \subseteq A$. Then $y_U \in W \cap T$ and $W \cap T$ is open, and since X is strongly dense-in-itself, there exists $t, t' \in W \cap T \setminus \{y_U\}$ such that $t \neq t'$. Then we have $t, t' \in W \cap A$, and one of t and t' has to be different from y. Therefore $y \in \mathsf{d}(A)$, and this concludes the proof. $\square$

We are now able to prove the announced decomposition theorem:

Theorem 4.10. *Given a topological space X, we have $X \vDash$ **wK4.2** iff there exist two disjoint open subspaces Y, Z of X such that $X = Y \cup Z$, Y is strongly dense-in-itself and extremally disconnected, and Z is almost discrete.*

Proof. From left to right, suppose $X \vDash$ **wK4.2**. We define

$$Z := \bigcup\{U \mid U \text{ is open and atomic and } |U| \leq 2\}$$

and $Y := X \setminus Z$.

- Z is clearly open.
- We show that Z is almost discrete. If $x \in Z$, then $x \in U$ for some atomic open set U such that $|U| \leq 2$, and we have $U \subseteq Z$. Thus U is also open in Z. Then if $\emptyset \subset V \cap Z \subset U$ with V open, we have $V \cap Z$ open too and this contradicts the fact that U is atomic. Therefore U is atomic in Z.
- We show that Z is closed. First, given any atomic open set V such that $|V| = 2$, we write $V = \{z_0^V, z_1^V\}$, that is, we specify a first and a second element. We then define

  $$A := \{z_0^V \mid V \text{ open and } |V| = 2\}$$

  Now suppose that there exists $x \in \mathsf{Cl}(Z) \setminus Z$. Then for any open neighbourhood U of x there exists an atomic open set V such that $|V| \leq 2$ and $U \cap V \neq \emptyset$. We actually have $V \subseteq U$, otherwise $\emptyset \subset U \cap V \subset U$ contradicts the fact that V is atomic. If $|V| = 1$, then V is of the form $V = \{z\}$, and we have $z \in U \cap \widehat{\mathsf{d}}(A)$ since $V \setminus \{z\} = \emptyset \subseteq A$; we also have $z \neq x$, otherwise $x \in V$ and this contradicts $x \notin Z$. If $|V| = 2$, then $V \setminus \{z_1^V\} = \{z_0^V\} \subseteq A$ and thus $z_0^V \in U \cap \widehat{\mathsf{d}}(A)$; we also have $z_0^V \neq x$ for the same reason as before.

  This proves that $x \in \mathsf{d}(\widehat{\mathsf{d}}(A))$, and by assumption we obtain $x \in \widehat{\mathsf{d}}(\mathsf{d}(A))$, that is, there exists an open neighbourhood W of x such that $W \setminus \{x\} \subseteq \mathsf{d}(A)$. Since $x \notin Z$, there exists $y \in W \setminus \{x\}$, and then since $y \in W$ and $y \in \mathsf{d}(A)$ there exists $z \in W \cap A$ such that $z \neq y$. It follows that $z = z_0^V$ for some atomic open set V such that $|V| = 2$. If $z = x$, then $x \in V$ and this contradicts $x \notin Z$, so we have $z \neq x$. Then since $z \in W$ we obtain $z \in \mathsf{d}(A)$, and from $z \in V$ it follows that $z_1^V \in A$. Therefore there exists an atomic open set V' such that $|V'| = 2$

and $z_1^V = z_0^{V'}$. Then $z_1^V \in V \cap V' \subseteq V$, and since V is atomic it follows that $V \cap V' = V$, that is, $V' \subseteq V$. Again, since V is atomic this gives $V = V'$. Therefore $z_0^V = z_1^V$, a contradiction. Therefore Z is closed, and Y is open.

- We prove that Y is strongly dense-in-itself. For suppose there is an open set U such that $U \cap Y \neq \emptyset$ and $|U \cap Y| \leq 2$. Then there exists $x \in U \cap Y$, and if V is an open neighbourhood of x we prove that $V \cap Z \neq \emptyset$.
  - If $|U \cap V| \leq 2$ then $U \cap V$ cannot be atomic, for otherwise $x \in U \cap V \subseteq Z$. Thus there exists $z \in U \cap V$ such that $\{z\}$ is open, and we then have $z \in Z$.
  - If $|U \cap V| \geq 3$, then from $|U \cap V \cap Y| \leq |U \cap Y| \leq 2$ it follows that $U \cap V \cap Z \neq \emptyset$.

  Therefore $x \in \mathsf{Cl}(Z)$, and since Z is closed it follows that $x \in Z$, a contradiction.
- Finally, since Y is open in X and $X \vDash \Diamond\Box p \to \Box\Diamond p$, we have $Y \vDash \Diamond\Box p \to \Box\Diamond p$ as well. From proposition 4.9 it then follows that Y is extremally disconnected.

From right to left, suppose that there exist two disjoint open subspaces Y, Z of X such that $X = Y \cup Z$, Y is strongly dense-in-itself and extremally disconnected, and Z is almost discrete. Since Y and Z are open, it is enough to prove that $Y \vDash$ **wK4.2** and $Z \vDash$ **wK4.2**. The former immediately stems from proposition 4.9. For the latter, let $x \in Z$ and $A \subseteq Z$ such that $x \in \mathsf{d}(\widehat{\mathsf{d}}(A))$. Let U be an atomic open neighbourhood of x such that $|U| \leq 2$. If $y \in U \setminus \{x\}$, we have $y \in \widehat{\mathsf{d}}(A)$ by assumption, and since U is atomic this straight away entails $U \setminus \{y\} \subseteq A$, that is, $x \in A$. Hence $y \in \mathsf{d}(A)$ (again because U is atomic), and this proves $x \in \widehat{\mathsf{d}}(\mathsf{d}(A))$. $\square$

To prove topological completeness, we use the technique of *unfolding* introduced in [BBFD21] to turn a **wK4** frame into an appropriate topological space[2]. The construction essentially consists in replacing every reflexive point w of a frame by countably many copies of w, and to arrange them all into a dense-it-itself subspace, so that to mimic the reflexivity of w in the d-semantics.

Definition 4.11. Let $\mathfrak{F} = (W, R)$ be a **wK4**-frame. We denote by W^r the set of reflexive worlds of $\mathfrak{F}$, and by W^i the set of irreflexive worlds of $\mathfrak{F}$. We then introduce the *unfolding* of $\mathfrak{F}$ as the space

$$X_{\mathfrak{F}} := (W^r \times \omega) \cup (W^i \times \{\omega\})$$

and the collection $\tau_{\mathfrak{F}} \subseteq \mathcal{P}(X_{\mathfrak{F}})$ of all sets U such that for all $(w, \alpha) \in U$:

1. there exists $n^U_{w,\alpha} < \omega$ such that for all $(u, \beta) \in X_{\mathfrak{F}}$, if wRu, uRw and $\beta \geq n^U_{w,\alpha}$ then $(u, \beta) \in U$;
2. if $(u, \beta) \in X_{\mathfrak{F}}$, wRu and *not* uRw then $(u, \beta) \in U$.

Proposition 4.12. [BBFD21] *The pair $(X_{\mathfrak{F}}, \tau_{\mathfrak{F}})$ is a topological space and the map $\pi : X_{\mathfrak{F}} \to W$ defined by $\pi(w, \alpha) := w$ is a surjective d-morphism.*

As a warm-up, we begin with the topological completeness of **wK4.2**$^+$. The following two lemmas will allow us to transfer the validity of $.2^+$ from a Kripke frame to its unfolding.

Lemma 4.13. *If U is open in $X_{\mathfrak{F}}$ we have* $\mathsf{Cl}(U) = \{(w, \alpha) \in X_{\mathfrak{F}} \mid \exists (u, \beta) \in U, wR^+u\}$.

[2] It should be mentioned that the name "unfolding" is not from the original reference. It is our initiative to baptize the technique as such.

Proof. Let $(w, \alpha) \in \mathsf{Cl}(U)$. The set $V := \{(u, \beta) \in X_{\mathfrak{F}} \mid wR^+u\}$ is clearly an open neighbourhood of (w, α), so $V \cap U \neq \emptyset$ and we are done.

Conversely, suppose that $(w, \alpha) \in X_{\mathfrak{F}}$, $(u, \beta) \in U$ and wR^+u. Let V be an open neighbourhood of (w, α). We consider several cases:

- If wRu and not uRw then $(u, \beta) \in V$ since V is open.
- Otherwise, if $w = u$ and w is irreflexive we obtain $(w, \alpha) = (u, \beta) = (w, \omega)$ so $(u, \beta) \in V$.
- Otherwise, we have wRu and uRw. If u is irreflexive then $\beta = \omega \geq n^V_{w,\alpha}$ so $(u, \beta) \in V$. If u is reflexive, we define $n := \mathsf{max}\ \{n^V_{w,\alpha}, n^U_{u,\beta}\}$; then from uRu it follows that $(u, n) \in V$, and from wRu and uRw it follows that $(u, n) \in U$.

In all cases we find that $U \cap V \neq \emptyset$ and this proves $(w, \alpha) \in \mathsf{Cl}(U)$. □

Lemma 4.14. *If $\mathfrak{F}$ is directed then $X_{\mathfrak{F}}$ is extremally disconnected.*

Proof. Suppose that $\mathfrak{F}$ is directed and let U be open in $X_{\mathfrak{F}}$. To show that $\mathsf{Cl}(U)$ is open, consider $(w, \alpha) \in \mathsf{Cl}(U)$; by lemma 4.13 we know that wR^+u for some $(u, \beta) \in U$.

1. We claim that $n^{\mathsf{Cl}(U)}_{(w,\alpha)} := 0$ satisfies the first condition. Indeed, if we suppose $(v, \gamma) \in X_{\mathfrak{F}}$, wRv and vRw, we obtain vR^+u and thus $(v, \gamma) \in \mathsf{Cl}(U)$.
2. For the second condition, suppose $(v, \gamma) \in X_{\mathfrak{F}}$, wRv and not vRw. Since $\mathfrak{F}$ is directed, we know that $\mathfrak{F}^+$ is strongly directed. Then since wR^+u and wR^+v there exists $t \in W$ such that uR^+t and vR^+t.
   - If tRu then vR^+u and thus $(v, \gamma) \in \mathsf{Cl}(U)$.
   - Otherwise we define $\delta := 0$ if t is reflexive and $\delta := \omega$ otherwise. We have $(u, \beta) \in U$, uRt and not tRu so $(t, \delta) \in U$ since U is open. Then from vR^+t we obtain $(v, \gamma) \in \mathsf{Cl}(U)$.

□

Theorem 4.15. *The logic* $\mathbf{wK4.2}^+$ *is topologically complete.*

Proof. Suppose that $\mathbf{wK4.2} \nvdash \phi$. By proposition 4.2, there exists a directed **wK4** Kripke frame $\mathfrak{F}$ in which ϕ is satisfiable. Since π is a d-morphism from $X_{\mathfrak{F}}$ to $\mathfrak{F}$, we obtain from proposition 2.31 that ϕ is satisfiable in $X_{\mathfrak{F}}$ as well. By lemma 4.14, $X_{\mathfrak{F}}$ is extremally disconnected, and we are done. □

When it comes to **wK4.2**, conditions of validity are more complex, and the first step is to prove a result close enough to theorem 4.10 in the Kripke semantics, so that we may eventually derive a **wK4.2** space from a **wK4.2** frame.

Proposition 4.16. *Let $\mathfrak{F} = (W, R)$ be a* **wK4** *frame. Then $\mathfrak{F} \vDash .2$ if and only if there exist two generated subframes $\mathfrak{F}_0, \mathfrak{F}_1$ of $\mathfrak{F}$ induced respectively by W_0 and W_1 and such that:*

- *$W_0 \cup W_1 = W$ and $W_0 \cap W_1 = \emptyset$;*
- *$\mathfrak{F}_0$ is directed;*

– *every point in $\mathfrak{F}_0$ has a reflexive successor, or has at least two distinct successors;*

– *$\mathfrak{F}_1$ is almost discrete.*

Proof. Suppose that $\mathfrak{F} \vDash .2$. We denote by W_0 the set of points in W that have a reflexive successor, or at least two distinct successors, and $W_1 := W \setminus W_0$. By construction we have $W_0 \cup W_1 = W$ and $W_0 \cap W_1 = \emptyset$. We first prove that the subframes $\mathfrak{F}_0$ and $\mathfrak{F}_1$ induced by W_0 and W_1 are generated subframes of $\mathfrak{F}$:

– Suppose $w \in W_1$ and wRu. If u has a reflexive successor v, then so does w, a contradiction. Note that $w \neq u$ for the same reason. Suppose that u has two distinct successors v_1, v_2. Since u is irreflexive we have $u \notin \{w, v_1, v_2\}$; also $v_i \neq w$ for some $i \in \{1, 2\}$, and thus w has two distinct successors u and v_i, a contradiction. Therefore $u \notin W_0$, and we obtain $u \in W_1$.

– Suppose $w \in W_0$ and wRu. Suppose that $u \notin W_0$. Then u has at most one successor, which is not reflexive. If u has no successor, then $\mathfrak{F}, u \vDash \Box\bot$ so $\mathfrak{F}, w \vDash \Diamond\Box\bot$, so $\mathfrak{F}, w \vDash \Box\Diamond\bot$ since $\mathfrak{F} \vDash .2$. It follows that $\mathfrak{F}, u \vDash \Diamond\bot$, a contradiction. Thus u has a unique successor v. Since u is irreflexive we have $u \neq v$ and $u \neq w$. By the previous item we know that $v \in W_1$, so $v \neq w$ and by weak transitivity it follows that wRv. Then reasoning as before, we know that v has a successor t. We have $t \neq v$ since v is irreflexive. If $u \neq t$ then uRt by weak transitivity and this contradicts the uniqueness of v, so $t = u$. The situation is depicted below:

$$\begin{tikzcd}
\mathbf{v} \arrow[d, bend left] \\
\mathbf{u} \arrow[u, bend left] \\
\mathbf{w} \arrow[u]
\end{tikzcd}$$

We then define a valuation ν by setting $\nu(p) := \{v\}$. It follows that $\mathfrak{F}, \nu, w \vDash \Diamond\Box p$ but $\mathfrak{F}, \nu, w \nvDash \Box\Diamond p$, a contradiction. Therefore $u \in W_0$.

We then proceed to check the remaining conditions:

– By construction, $W_0 \cup W_1 = W$ and $W_0 \cap W_1 = \emptyset$.

– Since $\mathfrak{F} \vDash .2$ we know that $\mathfrak{F}$ is strongly directed, and thus directed. Since $\mathfrak{F}_0$ is a generated subframe of $\mathfrak{F}$, it is directed too.

– The condition on W_0 is satisfied by definition.

– Let $w \in W_1$. If w has no successor then $\{w\}$ is a 1-element upset. Otherwise let u be the unique successor of w. Then we have seen that $u \in W_1$. Reasoning as before, we can prove that u cannot be an endpoint, so it has a unique successor v. Since u is irreflexive we have $v \neq u$, and if $v \neq w$ then wRv by weak transitivity, but this contradicts the uniqueness of v. Therefore $w = v$, so $\{w, u\}$ is a 2-element upset (and $\{u\}$ is not an upset). This proves that $\mathfrak{F}_1$ is almost discrete.

Conversely, suppose that there exists such of decomposition of $\mathfrak{F}$ into $\mathfrak{F}_0$ and $\mathfrak{F}_1$. Since $\mathfrak{F}_0$ and $\mathfrak{F}_1$ are generated subframes of $\mathfrak{F}$, it suffices to prove that $\mathfrak{F}_0 \vDash .2$ and $\mathfrak{F}_1 \vDash .2$, or equivalently that $\mathfrak{F}_0$ and $\mathfrak{F}_1$ are strongly directed. This is clear for $\mathfrak{F}_1$, since it is almost discrete. For $\mathfrak{F}_0$, suppose that $w, u, v \in W_0$ with wRu and wRv. Since $\mathfrak{F}_0$ is directed, there exists $t \in W_0$ such that uR^+t and vR^+t. Then there are four cases:

- If uRt and vRt we are done.
- Suppose that $u = v = t$. Then by assumption t has a successor t' and it follows uRt' and vRt'.
- Suppose that uRt and $t = v$. If t has a reflexive successor t' then uRt' by weak transitivity. Otherwise t has two distinct successors t_1 and t_2, and there exists $i \in \{1, 2\}$ such that $t_i \neq u$. Then uRt_i by weak transitivity.
- If vRt and $t = u$, the reasoning is analogous to the previous case.

□

Theorem 4.17. *The logic* **wK4.2** *is topologically complete.*

Proof. Suppose that $\mathbf{wK4.2} \nvDash \phi$. Since .2 is a Sahlqvist formulas, **wK4.2** is Kripke complete, so there exists a Kripke frame $\mathfrak{F} = (W, R)$ in which ϕ is satisfiable and such that $\mathfrak{F} \vDash \mathbf{wK4.2}$. We then write $\mathfrak{F}_0$ and $\mathfrak{F}_1$ the Kripke frames introduced by proposition 4.16, and consider the subspaces $Y := \pi^{-1}[W_0]$ and $Z := \pi^{-1}[W_1]$ of $X_{\mathfrak{F}}$. We claim that Y satisfy the conditions of theorem 4.10.

- Since $W_0 \cup W_1 = W$ and $W_0 \cap W_1 = \emptyset$, we have $Y \cup Z = X$ and $Y \cap Z = \emptyset$.
- Since π is a d-morphism, it is also an interior map. Since $\mathfrak{F}_0$ and $\mathfrak{F}_1$ are generated subframes of $\mathfrak{F}$, they are open in $\mathfrak{F}$, and therefore Y and Z are open in X.
- Every point in W_1 is irreflexive, so $Z = W_1 \times \{\omega\}$. As a result, the restriction of π to Z is bijective, and thus a homeomorphism. Then, since $\mathfrak{F}_1$ is almost discrete, so is Z.
- Since $\mathfrak{F} \vDash .2$, the frame $\mathfrak{F}$ is strongly directed. Then by lemma 4.14 it follows that $X_{\mathfrak{F}}$ is extremally disconnected, and so is Y since it is open in X.
- We show that Y is strongly dense-it-itself. Let $(w, \alpha) \in Y$ and let U be an open neighbourhood of (w, α) in Y. If w is reflexive, we set $n := 1 + \mathsf{max}\ \{n^U_{w,\alpha}, \alpha\}$, and we obtain $(w, n) \in U$ and $(w, n+1) \in U$ with (w, α), (w, n) and $(w, n+1)$ all different. Otherwise, since w is in W_0 it has two distinct successors u, v both different from w. Since $\omega \geq n^U_{w,\alpha}$, we have $(u, \omega) \in U$ (whether uRw or not) and likewise $(v, \omega) \in U$. Since (w, ω), (u, ω) and (v, ω) are all different, we are done.

Therefore $X_{\mathfrak{F}} \vDash \mathbf{wK4.2}$. Since π is a d-morphism from $X_{\mathfrak{F}}$ to $\mathfrak{F}$ and ϕ is satisfiable in $\mathfrak{F}$, we obtain from proposition 2.31 that ϕ is satisfiable in $X_{\mathfrak{F}}$, and this concludes the proof. □

4.2 The axiom aT and accumulative spaces

If we substitute p for q in the axiom .3, we obtain $\mathsf{aT} := \Box(p \to \Diamond p)$, whose name stands for "almost T" (recall that $\mathsf{T} = p \to \Diamond p$). Whereas this formula is a mere tautology in the c-semantics, it contains a surprising amount of information in the d-semantics, in fact enough to justify taking a detour via this section to focus solely on it. At the level of Kripke frames, its semantics is quite simple:

Definition 4.18. A Kripke frame (W, R) is called *almost reflexive* if for all $w, u \in W$, wRu implies uRu.

Proposition 4.19. *The logic* $\mathbf{wK4} + \mathsf{aT}$ *defines the class of transitive and almost reflexive Kripke frames. It also has the finite model property.*

Proof. If $\mathfrak{F}$ is transitive and almost reflexive then it is clear that $\mathfrak{F} \vDash \mathbf{wK4} + \mathsf{aT}$. Conversely, suppose $\mathfrak{F} \vDash \mathbf{wK4} + \mathsf{aT}$. If wRu, we define a valuation ν by $\nu(p) := \{u\}$; then from $\mathfrak{F}, \nu, w \vDash \Box(p \to \Diamond p)$ it follows that $\mathfrak{F}, \nu, u \vDash p \to \Diamond p$ and thus uRu. Since $\mathfrak{F} \vDash \mathbf{wK4}$ we also know that it is weakly transitive, and we show that it is in fact transitive. Indeed, suppose $wRuRv$. If $w \neq v$ we obtain wRv and we are done. If $w = v$, then since uRv we know that vRv, and it follows that wRv too.

Finally, any cofinal subframe of a transitive and almost reflexive frame is also transitive and almost reflexive, so $\mathbf{wK4} + \mathsf{T}$ is a cofinal subframe logic. By theorem 3.17, this yields the FMP. $\Box$

Not that since $\mathbf{wK4} + \mathsf{aT}$ defines a class of transitive frames, it is, by completeness, equivalent to $\mathbf{K4} + \mathsf{aT}$. Just as $\mathbf{K4} + \mathsf{T}$ is called $\mathbf{S4}$, we then choose to define $\mathbf{aS4} := \mathbf{K4} + \mathsf{aT}$. On the topological side, it is connected to a class of spaces that we call *accumulative*:

Definition 4.20. A space X is said to be *accumulative* if for all $A \subseteq X$ such that $\mathsf{d}(A) \neq \emptyset$, there exists an open set U with $\emptyset \neq A \cap U \subseteq \mathsf{d}(A)$.

As the concept of accumulative spaces might look somewhat unintuitive, we immediately provide an example:

Example 4.21. We call $A \subseteq \mathbb{N}$ *cofinite* if $\mathbb{N} \setminus A$ is finite. We then endow the space $\mathbb{N}$ with the topology $\tau := \{A \mid A \text{ is cofinite}\} \cup \{\emptyset\}$. It is easy to see that for any $A \subseteq \mathbb{N}$, we have $\mathsf{d}(A) \neq \emptyset$ iff $\mathsf{d}(A) = \mathbb{N}$ iff $\mathsf{d}(A)$ is unbounded. Thus $\mathsf{d}(A) \neq \emptyset$ implies $\mathsf{d}(A) \subseteq \mathbb{N}$, and it follows that $(\mathbb{N}, \tau)$ is accumulative.

This example is not fully satisfying though: the property that $\mathsf{d}(A) \neq \emptyset$ implies $\mathsf{d}(A) = \mathbb{N}$ is absurdly strong and not representative of what accumulative spaces look like in general. A more refined range of examples would be welcome, and this can be obtained by generalizing example 4.21 to any pre-order:

Definition 4.22. Let $(X, \preceq)$ be a pre-order. If $x \in X$ we define ${\uparrow}x := \{y \in X \mid x \preceq y\}$. The *cofinite topology* on X is the topology generated by the base

$$\{{\uparrow}x \setminus A \mid x \in X \text{ and } A \text{ finite}\}$$

That the aforementioned sets form a base is quite straightforward to prove: for any $x \in X$ we have $x \in {\uparrow}x$ so they cover all of X, and for any $x, y \in X$, A and B finite and $z \in ({\uparrow}x \setminus A) \cap ({\uparrow}y \setminus B)$

we have $z \in \uparrow z \setminus (A \cup B) \subseteq (\uparrow x \setminus A) \cap (\uparrow y \setminus B)$. The name "cofinite topology" is motivated by the fact that when A is finite, the set $\uparrow x \setminus A$ can be seen as "cofinite in $\uparrow x$". We are then going to prove that X is accumulative whenever X has "almost finite depth" and "almost finite width" in a sense that we define below:

Definition 4.23. Let $(X, \preceq)$ be a pre-order.

- Given $x, y \in X$, we call x and y *incomparable* if $x \not\preceq y$ and $y \not\preceq x$, in which case we write $x \perp y$. A set $A \subseteq X$ is called an *antichain* if for all $x, y \in A$ such that $x \neq y$ we have $x \perp y$. We say that $(X, \preceq)$ has *locally finite width* if for all $x \in X$ and all antichain $A \subseteq X$, the set $\uparrow x \cap A$ is finite. We say that $(X, \preceq)$ is *total* if is contains no antichain of size 2.

- Given an ordinal α, a *descending α-sequence* is a sequence $(x_\xi)_{\xi<\alpha} \in X^\alpha$ such that whenever $\xi < \eta < \alpha$ we have $x_\eta \preceq x_\xi$ and $x_\xi \not\preceq x_\eta$. We call $(X, \preceq)$ *almost well-founded* if it contains no descending $\omega + 1$-sequence.

- If $A \subseteq X$ we call A *infinitely ascending* if for all $x \in A$, the set $\uparrow x \cap A$ is infinite. The set A is called *almost infinitely ascending* if there exists $B \subseteq A$ finite such that $A \setminus B$ is infinitely ascending.

The definition of locally finite width is related to the property of finite width which says that $(X, \preceq)$ has no infinite antichain; what it states instead is that $(X, \preceq)$ has no infinite antichain *above a point*. Likewise, almost well-founded pre-orders are reminiscent of well-founded orders, i.e., those that contain no descending ω-sequence. We prove that the conjunction of these two conditions is a necessary and sufficient conditions for X to be accumulative. First, the following result states that under these two assumptions, any set is "big enough above a point":

Lemma 4.24. *Suppose that $(X, \preceq)$ is almost well-founded and has locally finite width, and let $x \in X$. Then any subset of $\uparrow x$ is almost infinitely ascending.*

Proof. Suppose that there exists and $A \subseteq \uparrow x$ not almost infinitely ascending. We construct by recursion a sequence $(x_n)_{n<\omega} \in A^\omega$ such that for all $n < \omega$, the set $\uparrow x_n \cap A$ is finite. First we select some $x_0 \in A$ such that $\uparrow x_0 \cap A$ is finite. If x_n is defined, we know that for all $k \leq n$, the set $\uparrow x_k \cap A$ is finite, and thus so is

$$B := A \cap \bigcup_{k=0}^{n} \uparrow x_k$$

It follows that $A \setminus B$ is not infinitely ascending, so there exists $x_{n+1} \in A \setminus B$ such that $\uparrow x_{n+1} \cap A \setminus B$ is finite, and then $\uparrow x_{n+1} \cap A$ is finite too.

Now suppose toward a contradiction that there exists $n < \omega$ such that $x_n \perp x_m$ for all $m > n$. Then $A_0 := \{x_m \mid m \geq n\}$ is an antichain and $A_0 \cap \uparrow x = A_0$ is infinite, and this contradicts the fact that $(X, \preceq)$ has locally finite width. Thus we can construct an increasing sequence of integers $(n_k)_{k<\omega}$ as follows: we set $n_0 := 0$ and for all $k < \omega$, we select some $n_{k+1} > n_k$ such that x_{n_k} and $x_{n_{k+1}}$ are comparable. By construction we also have $x_{n_k} \not\preceq x_{n_{k+1}}$ so we are left with $x_{n_{k+1}} \preceq x_{n_k}$. Further we have $\forall n < \omega, x \preceq x_n$. Thus if $x_{n_k} \preceq x$ for some $k < \omega$, it follows that $x_{n_k} \preceq x_{n_k+1}$, a contradiction. Therefore, if we define $y_k := x_{n_k}$ for all $k < \omega$ and $y_\omega := x$ we obtain that $(y_k)_{k \leq \omega}$ is a descending $\omega + 1$-sequence. This contradicts the fact that $(X, \preceq)$ is almost well-founded, and concludes the proof. $\square$

We can then prove the announced equivalence:

Theorem 4.25. *Let $(X, \preceq)$ be a pre-order. Then X (with the cofinite topology) is accumulative if and only if $(X, \preceq)$ is almost well-founded and has locally finite width.*

Proof. Suppose that $(X, \preceq)$ is almost well-founded and has locally finite width. Let $A \subseteq X$ and suppose that $x \in \mathsf{d}(A)$. By lemma 4.24, there exists $B \subseteq A$ finite such that $(\uparrow x \cap A) \setminus B$ is infinitely ascending. We introduce $U := \uparrow x \setminus (B \setminus \{x\})$ which is an open neighbourhood of x. Then we have $U \cap A \neq \emptyset$ and we also prove that $U \cap A \subseteq \mathsf{d}(A)$. For consider $y \in U \cap A$. If $y = x$ then $y \in \mathsf{d}(A)$ immediately, otherwise we have $y \in (\uparrow x \cap A) \setminus B$. Now consider an open neighbourhood of y of the form $\uparrow z \setminus C$ with C finite. We know that $\uparrow y \cap (\uparrow x \cap A) \setminus B$ is infinite and that $C \cup \{y\}$ is finite, so there exists t belonging to the former but not the latter, and it follows that $t \in (\uparrow z \setminus C) \cap A \setminus \{y\}$. Therefore $y \in \mathsf{d}(A)$ and this concludes the proof.

Conversely, suppose that X is accumulative.

- To prove that $(X, \preceq)$ is almost well-founded, suppose toward a contradiction that there exists a descending $\omega+1$-sequence $(x_n)_{n \leq \omega}$. We define $A := \{x_n \mid n < \omega\}$ and claim that $x_\omega \in \mathsf{d}(A)$. Indeed, suppose $x_\omega \in \uparrow x \setminus B$ with B finite. Since A is infinite there exists $n < \omega$ such that $x_n \notin B \cup \{x_\omega\}$, and from $x \preceq x_\omega \preceq x_n$ it follows that $x_n \in A \cap (\uparrow x \setminus B) \setminus \{x_\omega\}$. Therefore $\mathsf{d}(A) \neq \emptyset$, so by assumption there exists an open set U such that $\emptyset \neq A \cap U \subseteq \mathsf{d}(A)$. Thus there exists $x_n \in A \cap U$. Then $V := \uparrow x_n \setminus \{x_k \mid k < n\}$ is an open neighbourhood of x_n, but we have $V \cap A = \{x_n\}$, and this contradicts $x_n \in \mathsf{d}(A)$.

- To prove that $(X, \preceq)$ has locally finite width, suppose toward a contradiction that there exists $x \in X$ and an antichain A such that $\uparrow x \cap A$ is infinite. Then if $x \in \uparrow y \setminus B$ with B finite, there exists $a \in (\uparrow x \cap A) \setminus (B \cup \{x\})$, and it follows that $a \in A \cap (\uparrow y \setminus B) \setminus \{x\}$. This proves $x \in \mathsf{d}(A)$, and by assumption we obtain the existence of an open set U such that $\emptyset \neq U \cap A \subseteq \mathsf{d}(A)$. Then there exists $a \in U \cap A$, but since A is an antichain we have $\uparrow a \cap A = \{a\}$, contradicting $a \in \mathsf{d}(A)$.

$\square$

We also prove two simple properties of accumulative spaces. Proposition 4.26 states that this condition is *hereditary*, i.e., preserved by taking subspaces, and proposition 4.27 states that all accumulative spaces satisfy the separation axiom T_1.

Proposition 4.26. *If X is accumulative then so is any subspace of X.*

Proof. Suppose that X is accumulative and let $Y \subseteq X$. Let $A \subseteq Y$ and assume $\mathsf{d}_Y(A) \neq \emptyset$, i.e., $\mathsf{d}(A) \cap Y \neq \emptyset$. Then $\mathsf{d}(A) \neq \emptyset$, so by assumption there exists an open set U such that $\emptyset \neq U \cap A \subseteq \mathsf{d}(A)$, and since $A \subseteq Y$ we have $(U \cap Y) \cap A = U \cap A \neq \emptyset$. In addition, if $x \in (U \cap Y) \cap A$ we obtain $x \in Y \cap \mathsf{d}(A) = \mathsf{d}_Y(A)$. This proves the claim. $\square$

Proposition 4.27. *Any accumulative space is also T_1.*

Proof. Let X be accumulative. Suppose that X is not T_1, i.e., there exists $x \in X$ such that $\mathsf{d}(\{x\}) \neq \emptyset$. By assumption we obtain the existence of an open set U such that $\emptyset \neq \{x\} \cap U \subseteq \mathsf{d}(\{x\})$. It follows that $x \in \mathsf{d}(\{x\})$, a contradiction. $\square$

Now that we are more familiar with the landscape of accumulative spaces, we can delve into their axiomatization and begin by proving that they are defined by the axiom aT:

Theorem 4.28. *The logic* **aS4** *defines the class of accumulative spaces.*

Proof. Let X be a space and suppose $X \vDash \mathsf{aT}$. Let $A \subseteq X$ such that $\mathsf{d}(A) \neq \emptyset$. Then there exists $x \in \mathsf{d}(A)$, and by assumption there is an open neighbourhood U of x such that $U \setminus \{x\} \subseteq (X \setminus A) \cup \mathsf{d}(A)$. Since $x \in \mathsf{d}(A)$ we have in fact $U \subseteq (X \setminus A) \cup \mathsf{d}(A)$, and this entails $U \cap A \subseteq \mathsf{d}(A)$. Since $x \in U$ and $x \in \mathsf{d}(A)$ we also have $U \cap A \neq \emptyset$.

Conversely, suppose that X is accumulative. Let $\mathfrak{M}$ be a model based on X and $x \in X$, and suppose $\mathfrak{M}, x \nvDash \mathsf{aT}$. Then $\mathfrak{M}, x \vDash \Diamond(p \wedge \Box\neg p)$, that is, $\mathsf{d}(A) \neq \emptyset$ if we write $A := [\![p \wedge \Box\neg p]\!]_{\mathfrak{M}}$. Then by assumption there exists an open set U such that $\emptyset \neq A \cap U \subseteq \mathsf{d}(A)$. So there exists $y \in U \cap A$, and we have $\mathfrak{M}, y \vDash \Box\neg p$, but from $y \in \mathsf{d}(A)$ we also get $\mathfrak{M}, y \vDash \Diamond p$, a contradiction. $\Box$

When it comes to topological completeness, we first notice that we will not be able to resort to the technique of unfolding, because any accumulative space must be T_1 and the spaces obtained by unfolding critically fail to satisfy this requirement. Indeed, if $\mathfrak{F} = (W, R)$ is a Kripke frame and $(w, \alpha), (u, \beta) \in X_{\mathfrak{F}}$ are such that wRu and not uRw, then it is clear that any open neighbourhood of (w, α) is also a neighbourhood of (u, β). In words, the topology of $X_{\mathfrak{F}}$ is too coarse to guarantee enough separation, but this can be fixed by adding more open sets. We first observe that we can endow $X_{\mathfrak{F}}$ with the pre-order $\preceq$ defined by

$$(w, \alpha) \preceq (u, \beta) \iff wR^+u$$

and that the topology of $X_{\mathfrak{F}}$ is generated by

$$\{\uparrow(w, \alpha) \setminus A \mid (w, \alpha) \in X_{\mathfrak{F}} \text{ and } A \text{ finite and for all } (u, \beta) \in A, \text{ not } uRw\}$$

Then if we drop the condition "for all $(u, \beta) \in A$, not uRw", we obtain a finer topology which appears to coincide with the cofinite topology generated by $(X, \preceq)$; the resulting space is called the *refined unfolding* of $\mathfrak{F}$ and denoted by $X^*_{\mathfrak{F}}$. We can already see that this defines a T_1 space, since whenever $(w, \alpha) \neq (u, \beta)$, the open set $\uparrow(w, \alpha) \setminus \{(u, \beta)\}$ separates (u, β) from (w, α). On top of that, we will be able to reuse our results regarding the cofinite topology, and thus easily carry out the proof of completeness. The downside, however, is that the projection π is no longer a d-morphism in most cases. Though the proof still goes through for **aS4** and its extensions, this drastically limits the applicability of the method in general.

Proposition 4.29. *The map $\pi : X^*_{\mathfrak{F}} \to W$ defined by $\pi(w, \alpha) := w$ is a d-morphism if and only if $\mathfrak{F}$ is almost reflexive.*

Proof. Suppose that π is a d-morphism, and let $w, u \in W$ be such that wRu. Suppose toward a contradiction that u is irreflexive. Then we introduce the open set

$$U := \begin{cases} \uparrow(w, 0) \setminus \{(u, \omega)\} & \text{if } w \text{ is reflexive} \\ \uparrow(w, \omega) \setminus \{(u, \omega)\} & \text{otherwise} \end{cases}$$

By assumption, π is an interior map, so $\pi[U]$ is open in $\mathfrak{F}$. Yet $w \in \pi[U]$ and $u \notin \pi[U]$, a contradiction. This proves that $\mathfrak{F}$ is almost reflexive.

Conversely, suppose that $\mathfrak{F}$ is almost reflexive. If $U \subseteq W$ is open, then

$$\pi^{-1}[U] = \bigcup_{w \in U \cap W^r} \uparrow(w,0) \cup \bigcup_{w \in U \cap W^i} \uparrow(w,\omega)$$

so $\pi^{-1}[U]$ is open too. If $U = \uparrow(w,\alpha) \setminus A$ is a base open of X, we clearly have that $\pi[U] \subseteq \{u \in W \mid wR^+u\}$. Also, if wRu, then u is reflexive by assumption and since A is finite there exists $n < \omega$ such that $(u,n) \notin A$, and it follows that $u \in \pi[U]$. So we have either $\pi[U] = \{u \in W \mid wRu\}$ or $\pi[U] = \{u \in W \mid wR^+u\}$, and we prove that in both cases $\pi[U]$ is open in $\mathfrak{F}$. This is clear in the second case, and for the first case suppose that wRu and uRv. By weak transitivity we have either wRv, and we are done, or $w = v$; then we have uRw so w is reflexive by assumption, and it follows that wRv as well. This proves that π is an interior map.

Now let $w \in W$. If w is irreflexive then $\pi^{-1}(w) = \{(w,\omega)\}$ is discrete. If w is reflexive then $\pi^{-1}(w) = \{w\} \times \omega$. If $(w,n) \in \{w\} \times \omega$ we prove that $(w,n) \in \mathsf{d}(\{w\} \times \omega)$. For suppose that $(w,n) \in \uparrow(u,\alpha) \setminus A$ with A finite. Then uR^+w, and we select some $m < \omega$ such that $m \neq n$, $m \neq \alpha$ and $(w,m) \notin A$. It follows that $(w,m) \in (\uparrow(u,\alpha) \setminus A) \cap (\{w\} \times \omega) \setminus \{(w,n)\}$ and we are done. We conclude that π is a d-morphism. $\square$

Theorem 4.30. *The logic* **aS4** *is topologically complete.*

Proof. Suppose that $\mathbf{aS4} \nvdash \neg\phi$. Then by proposition 4.19 there exists a finite transitive and almost reflexive Kripke frame $\mathfrak{F} = (W,R)$ wherein ϕ is satisfiable. That $\mathfrak{F}$ is finite immediately implies that $(X^*_{\mathfrak{F}}, \preceq)$ has locally finite width and is almost well-founded, and from theorem 4.25 it follows that $X^*_{\mathfrak{F}}$ is accumulative. Since $\mathfrak{F}$ is almost reflexive, we also know from proposition 4.29 that π is a d-morphism from $X^*_{\mathfrak{F}}$ to $\mathfrak{F}$. Thus, since ϕ is satisfiable in $\mathfrak{F}$, it is also satisfiable in $X^*_{\mathfrak{F}}$ and we are done. $\square$

4.3 The logic wK4.3

We now move to the axiom .3, which in the c-semantics is the hereditary version of .2:

Theorem 4.31. [BBLBvM15] *In the c-semantics,* **S4.3** *defines the class of HED spaces, and is topologically complete.*

We then find ourselves in a situation similar to that of the beginning of section 4.1. Indeed, in [CZ97, sec. 3.8] we find the definition $\mathbf{S4.3} := \mathbf{S4} + \mathsf{scon}$ with

$$\mathsf{scon} := \Box(\Box p \to q) \vee \Box(\Box q \to p)$$

but it is also noticed that this system is equivalent to $\mathbf{S4} + \mathsf{con}$ with

$$\mathsf{con} := \Box(\Box^+ p \to q) \vee \Box(\Box^+ q \to p)$$

Though scon and con coincide under **S4**, this is not the case in general, so it is worth investigating the semantics of both $\mathbf{wK4} + \mathsf{scon}$ and $\mathbf{wK4} + \mathsf{con}$. In fact we will favour working with the variants $\Box(p \to \Diamond q) \vee \Box(q \to \Diamond p)$ and $\Box(p \to \Diamond^+ q) \vee \Box(q \to \Diamond^+ p)$ obtained by contraposition. The axiom con, however, turns out to coincide with scon^+:

Proposition 4.32. *We have* $\mathbf{wK4} + \mathsf{con} = \mathbf{wK4} + \mathsf{scon}^+$*. In addition:*

– *the logic* $\mathbf{wK4} + \mathsf{con}$ *defines the class of weakly transitive frames* $\mathfrak{F}$ *such that* $\mathfrak{F}^+$ *is strongly connected;*

– *the logic* $\mathbf{wK4} + \mathsf{con}$ *is Kripke complete;*

– *in the d-semantics, the logic* $\mathbf{wK4} + \mathsf{con}$ *defines the class of HED spaces.*

Proof. First, it is clear that $\mathbf{wK4} + \mathsf{scon}^+ \vdash \mathsf{con}$. In addition, given a Kripke frame $\mathfrak{F}$, we know that $\mathfrak{F} \vDash \mathsf{scon}$ iff $\mathfrak{F}$ is strongly connected [CZ97, sec. 3.5], so by proposition 2.28 it follows that $\mathfrak{F} \vDash \mathsf{scon}^+$ iff $\mathfrak{F}^+$ is strongly connected.

We note that con is a Sahlqvist formula and therefore that $\mathbf{wK4} + \mathsf{con}$ is Kripke complete. Thus, to prove $\mathbf{wK4} + \mathsf{con} \vdash \mathsf{scon}^+$, it suffices to show that for any Kripke frame $\mathfrak{F} = (W, R)$, if $\mathfrak{F} \vDash \mathsf{con}$ then $\mathfrak{F}^+$ is strongly connected. According to [CZ97, sec. 3.5], we have $\mathfrak{F} \vDash \mathsf{con}$ iff $\mathfrak{F}$ is *connected*, i.e., for all $w, u, v \in W$, if wRu, wRv and $u \neq v$ then uRv or vRu. It is then easy to see that this condition entails strong connectedness of $\mathfrak{F}^+$, and we are done.

Finally, if X is a topological space, we know from proposition 2.28 that $X \vDash_d \mathsf{scon}^+$ iff $X \vDash_c \mathsf{scon}$, and from theorem 4.31 it follows that $X \vDash_d \mathsf{scon}^+$ iff X is HED. □

This leads us to introduce the following names:

$$\mathbf{wK4.3} := \mathbf{wK4} + \mathsf{scon}$$

$$\mathbf{wK4.3}^+ := \mathbf{wK4} + \mathsf{scon}^+$$

When it comes to $\mathbf{wK4.3}$, we have mentioned earlier that it yields aT. Thus $\mathbf{wK4.3}$ contains $\mathbf{aS4}$ and in particular $\mathbf{K4}$, so it is equivalent to $\mathbf{K4} + \mathsf{scon}$. The following result provides its semantics:

Theorem 4.33. *The logic* $\mathbf{wK4.3}$ *defines the class of HED accumulative spaces.*

Proof. Suppose that $X \vDash \mathsf{scon}$. Then we obviously have $X \vDash \mathsf{con}$ as well, and thus X is HED by proposition 4.32. As mentioned earlier we also have $X \vDash \mathsf{aT}$, and thus X is accumulative by theorem 4.28.

Conversely, suppose that X is HED and accumulative. Let $\mathfrak{M}$ be a model based on X and $x \in X$; we prove that $\mathfrak{M}, x \vDash \mathsf{scon}$. Since X is HED we know that $X \vDash \mathsf{scon}^+$, and by substituting $p \wedge \Box\neg q$ for p and $q \wedge \Box\neg p$ for q we obtain

$$\mathfrak{M}, x \vDash \Box^+(p \wedge \Box\neg q \to \Diamond^+(q \wedge \Box\neg p)) \vee \Box^+(q \wedge \Box\neg p \to \Diamond^+(p \wedge \Box\neg q))$$

Then we assume $\mathfrak{M}, x \vDash \Box^+(p \wedge \Box\neg q \to \Diamond^+(q \wedge \Box\neg p))$ without loss of generality. Thus there exists an open neighbourhood U of x such that $U \subseteq [\![p \wedge \Box\neg q \to \Diamond^+(q \wedge \Box\neg p)]\!]_{\mathfrak{M}}$. Suppose toward a contradiction that $\mathfrak{M}, x \nvDash \Box(p \to \Diamond q)$. Then $\mathfrak{M}, x \vDash \Diamond(p \wedge \Box\neg q)$. If V is an open neighbourhood of x, so is $U \cap V$, and thus there exists $y \in U \cap V \setminus \{x\}$ such that $\mathfrak{M}, y \vDash p \wedge \Box\neg q$. Since $y \in U$ it follows that $\mathfrak{M}, y \vDash \Diamond^+(q \wedge \Box\neg p)$. We cannot have $\mathfrak{M}, y \vDash \Diamond(q \wedge \Box\neg p)$ because this contradicts $\mathfrak{M}, y \vDash \Box\neg q$, so we obtain $\mathfrak{M}, y \vDash q \wedge \Box\neg p$. In particular $\mathfrak{M}, y \vDash p \wedge \Box\neg p$ and this proves $\mathfrak{M}, x \vDash \Diamond(p \wedge \Box\neg p)$.

Writing $A := [\![p \wedge \Box\neg p]\!]_{\mathfrak{M}}$, this yields $\mathsf{d}(A) \neq \emptyset$, so by assumption there exists an open set U such that $\emptyset \neq A \cap U \subseteq \mathsf{d}(A)$. So there exists $y \in A \cap \mathsf{d}(A)$, but then $y \in A$ yields $\mathfrak{M}, y \vDash \Box\neg p$ while $y \in \mathsf{d}(A)$ yields $\mathfrak{M}, y \vDash \Diamond p$, a contradiction. Therefore $\mathfrak{M}, x \vDash \Box(p \to \Diamond q)$ and this concludes the proof. □

In the c-semantics, **S4.2** is the logic of extremally disconnected spaces, while **S4.3** is the logic of HED spaces. Thus **S4.3** is "hereditary **S4.2**", but we do not have this pattern in the d-semantics in the case of **wK4.2** and **wK4.3**. Indeed, the two-element space with the coarse topology is almost discrete and thus .2 is valid in all of its subspaces, but it is not T_1 and therefore .3 is not valid in it. However we are going to show that the converse holds.

Lemma 4.34. *If X is extremally disconnected and accumulative then $X \vDash$* **wK4.2***.*

Proof. We define

$$Z := \bigcup\{U \mid U \text{ is open and atomic and } |U| \leq 2\}$$

and $Y := X \setminus Z$. The proof that Z is open and almost discrete, and that Y is strongly dense-in-itself, can be directly imported from the proof of theorem 4.10; however we also need to prove that Z is closed and that Y is extremally disconnected. First, given any atomic open set U such that $|U| = 2$, we write $U = \{z_0^U, z_1^U\}$, that is, we specify a first and a second element. We then define

$$Z^\bullet := \{z \in Z \mid z \text{ is isolated}\}$$

$$Z_0 := \{z_0^U \mid U \text{ is open and atomic and } |U| \leq 2\}$$

$$Z_1 := \{z_1^U \mid U \text{ is open and atomic and } |U| \leq 2\}$$

and it is clear that $Z = Z^\bullet \cup Z_0 \cup Z_1$. Now suppose that there exists $x \in \mathsf{Cl}(Z) \setminus Z$. Then $x \in \mathsf{d}(Z) = \mathsf{d}(Z^\bullet) \cup \mathsf{d}(Z_0) \cup \mathsf{d}(Z_1)$, so there exists $Z' \in \{Z^\bullet, Z_0, Z_1\}$ such that $x \in \mathsf{d}(Z')$. By assumption we then obtain the existence of an open set V such that $\emptyset \neq Z' \cap V \subseteq \mathsf{d}(Z')$. Then there exists $z \in Z' \cap V$ and we have three cases:

- $Z' = Z^\bullet$ is an immediate contradiction since $z \in \mathsf{d}(Z^\bullet)$ and z is isolated.
- If $Z' = Z_0$, then $z = z_0^U$ for some atomic open set U such that $|U| = 2$. From $z_0^U \in U$ and $z_0^U \in \mathsf{d}(Z_0)$ we then obtain $z_1^U \in Z_0$, so $z_0^U = z_1^{U'}$ for some U'. But then $\emptyset \neq U \cap U' \subseteq U$, and since U is atomic it follows that $U \cap U' = U$. Hence $U \subseteq U'$ and since U' is atomic we get $U = U'$. Therefore $z_0^U = z_1^U$, a contradiction.
- The case $Z' = Z_1$ is symmetric.

We thus conclude that Z is closed.

Consequently, Y is open. Since X is extremally disconnected, we have $X \vDash .2^+$, and therefore $Y \vDash .2^+$ as well. Hence Y is extremally disconnected. We can finally apply theorem 4.10 and obtain $X \vDash$ **wK4.2**. □

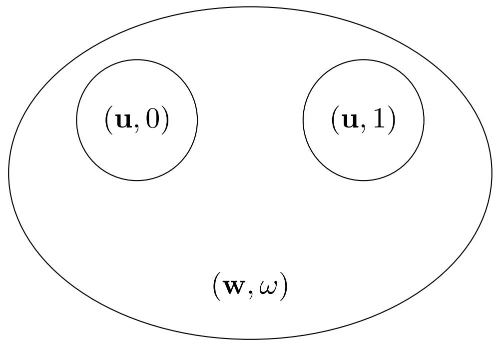
Proposition 4.35. *If $X \vDash$* **wK4.3** *then for any subspace Y of X we have $Y \vDash$* **wK4.2***.*

Proof. Suppose that $X \vDash$ **wK4.3**. By theorem 4.33 we know that X is HED and accumulative. If Y is a subspace of X, it is then extremally disconnected and also accumulative by proposition 4.26. By lemma 4.34 we then obtain $Y \vDash$ **wK4.2**.

□

We now address the topological completeness of **wK4.3**$^+$. Once again the method of unfolding will not succeed here; consider indeed the following frame:

The subspace $Y := \{(w, \omega), (u, 0), (u, 1)\}$ of its unfolding $X_{\mathfrak{F}}$ is depicted below:



We can see that it is not HED since $\{(u, 0)\}$ is open but $\mathsf{Cl}(\{(u, 0)\}) = \{(u, 0), (w, \omega)\}$ is not. The problem here stems from the possibility to separate the elements of $\pi^{-1}(u)$ from each other with open sets, so this time we need a coarser space. This is why we resort again to the method of dereflexivation presented in section 3.2. For a reminder of the formal construction, we refer to definition 3.18.

Lemma 4.36. *If $\mathfrak{F}^+$ is weakly transitive and strongly connected then $\mathfrak{F}_\bullet$ is HED.*

Proof. By a straightforward case distinction, we can see that if $\mathfrak{F}^+$ is strongly connected then so is $\mathfrak{F}_\bullet^+$. Therefore $\mathfrak{F}_\bullet^+ \vDash \mathbf{S4.3}$, so $\mathfrak{F}_\bullet \vDash_c \mathbf{S4.3}$, so $\mathfrak{F}_\bullet$ is HED by theorem 4.31. □

Theorem 4.37. *The logic* $\mathbf{wK4.3}^+$ *is topologically complete.*

Proof. Suppose that $\mathbf{wK4.3}^+ \nvdash \phi$. Then by proposition 4.32, ϕ is satisfiable in a $\mathbf{wK4}$ frame $\mathfrak{F}$ such that $\mathfrak{F}^+$ is strongly connected. By lemma 4.36 we then know that $\mathfrak{F}_\bullet$ is HED. Then since ϕ is satisfiable in $\mathfrak{F}$ and π is a d-morphism by lemma 3.19, it is also satisfiable in $X_{\mathfrak{F}}^*$ and we are done. □

We now address the topological completeness of **wK4.3**, which is essentially a continuation of the work laid out in section 4.2; the only extra ingredient we need is a way to preserve the validity of **wK4.3** by refined unfolding. First, let us establish Kripke completeness:

Proposition 4.38. *The logic* **wK4.3** *defines the class of transitive and strongly connected Kripke frames. It is also has the finite model property.*

Proof. We know that scon defines strong connectedness [CZ97, sec 3.5] and $\mathsf{w4}$ defines weak transitivity, and that weak transitivity along with almost reflexivity yields transitivity (see the proof of proposition 4.19), so the proof is routine. Finally, any cofinal subframe of a transitive and strongly connected frame is also transitive and strongly connected, so again we obtain the FMP from theorem 3.17. □

Lemma 4.39. *If $(X, \preceq)$ is a total pre-order then the space X (with the cofinite topology) is HED.*

Proof. Let $(X, \preceq)$ be a total pre-order, $Y \subseteq X$ and $\uparrow x \setminus A$ a base open set. We show that

$$U := \mathsf{Cl}_Y((\uparrow x \setminus A) \cap Y) = Y \cap \mathsf{Cl}((\uparrow x \setminus A) \cap Y)$$

is open. Given $y \in U$, we show that the open neighbourhood $Y \cap \uparrow y \setminus (A \setminus \{y\})$ of y is included in U. Indeed, let $z \in Y \cap \uparrow y \setminus (A \setminus \{y\})$. If $z = y$ it is immediate that $z \in U$ so we can assume $y \prec z$, and it follows that $z \notin A$. Since $\preceq$ is total, there are two cases:

- If $x \preceq z$ then we have $z \in Y \cap (\uparrow x \setminus A)$ and $z \in U$ follows immediately.

- Otherwise $x \not\preceq z$ and $z \preceq x$. Suppose $z \in Y \cap (\uparrow t \setminus B)$ with B finite; we can assume $B \subseteq \uparrow t$. Then suppose toward a contradiction that $Y \cap (\uparrow x \setminus A) \cap (\uparrow t \setminus B) = \emptyset$. Since $t \preceq z \preceq x$ this boils down to $Y \cap (\uparrow x \setminus A) \setminus B = \emptyset$, and thus $Y \cap (\uparrow x \setminus A) \subseteq B$. If $y \notin B$ then $Y \cap (\uparrow y \setminus B)$ is an open neighbourhood of y in Y which does not intersect $\uparrow x \setminus A$, and this contradicts $y \in U$. Therefore $y \in B$, and it follows that $t \preceq y$. Then $Y \cap \uparrow t \setminus (B \setminus \{y\})$ is an open neighbourhood of y in Y, so there exists

$$u \in Y \cap (\uparrow x \setminus A) \cap (\uparrow t \setminus (B \setminus \{y\}))$$

but since $Y \cap (\uparrow x \setminus A) \setminus B = \emptyset$ the only possibility is $u = y$. In particular $y \in \uparrow x$, and from $y \preceq z$ it follows that $x \preceq z$, a contradiction.

□

Theorem 4.40. *The logic* **wK4.3** *is topologically complete.*

Proof. Suppose that **wK4.3** $\nvdash \phi$. Then by proposition 4.38, ϕ is satisfiable in a finite transitive and strongly connected frame $\mathfrak{F} = (W, R)$. We can also assume that $\mathfrak{F}$ is rooted, so for all $w, u \in W$ we have either wR^+u or uR^+w. As a result, the order $\preceq$ on $X^*_{\mathfrak{F}}$ is total, and because $\mathfrak{F}$ is finite it is also almost well-founded and has locally finite width. Then from theorem 4.25 and lemma 4.39 it follows that $X^*_{\mathfrak{F}}$ is HED and accumulative. In addition, since $\mathfrak{F} \vDash$ **wK4.3** we also have $\mathfrak{F} \vDash$ **aS4**, so $\mathfrak{F}$ is almost reflexive, and from proposition 4.29 we obtain that π is a d-morphism. Then since ϕ is satisfiable in $\mathfrak{F}$, it is also satisfiable in $X^*_{\mathfrak{F}}$ and we are done. □

4.4 Additional axioms

In this section we consider other extensions of **wK4** for which completeness results can be obtained effortlessly. More precisely, we address the axiom bd_n that we have already seen in chapter 3, and the axiom gl. Recall that gl defines the class of converse well-founded Kripke frames, and in the d-semantics the class of scattered spaces [vBB07]. Most of our results are obtained by dereflexivation, so we need a handful of related invariance results. Some of them are already known, and the remaining ones are proved in the following lemma:

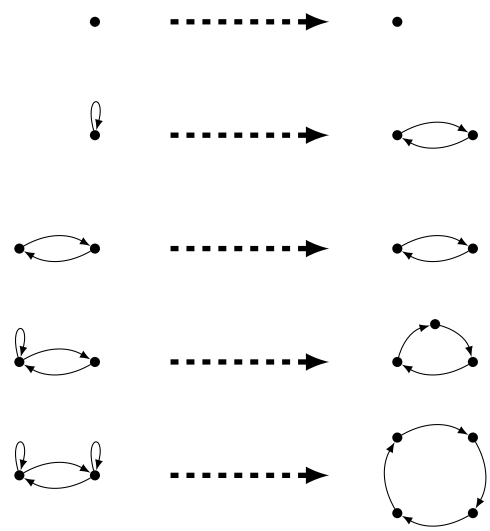
Lemma 4.41. *Let $\mathfrak{F} = (W, R)$ be a* **wK4** *frame.*

1. *If $\mathfrak{F}$ is directed then $\mathfrak{F}_\bullet$ is extremally disconnected.*

2. *If $\mathfrak{F} \vDash$* **wK4.2** *then $\mathfrak{F}_\bullet \vDash_d$* **wK4.2**.

Proof. 1. Clearly, if $\mathfrak{F}$ is directed, then so is $\mathfrak{F}_\bullet$. Therefore $\mathfrak{F}_\bullet^+ \vDash \mathbf{S4.2}$, so $\mathfrak{F}_\bullet \vDash_c \mathbf{S4.2}$, so $\mathfrak{F}_\bullet$ is extremally disconnected by theorem 4.1.

2. Suppose that $\mathfrak{F} \vDash \mathbf{wK4.2}$. Let $\mathfrak{F}_0 = (W_0, R_0)$ and $\mathfrak{F}_1 = (W_1, R_1)$ be the generated subframes of $\mathfrak{F}$ obtained by proposition 4.16. Then $\mathfrak{F}_0$ is directed, and so is $\mathfrak{F}_{0\bullet}$. Further, let $(w, n) \in W_{0\bullet}$. If w has two distinct successors u and v, then $(u, 0)$ and $(v, 0)$ are distinct successors of (w, n). Otherwise, w has a reflexive successor u, in which case $(u, 0)$ and $(u, 1)$ are two distinct successors of (w, n). Since $\mathfrak{F}_1$ is almost discrete, the shape of $\mathfrak{F}_{1\bullet}$ is completely described by the following case distinction:



In the first three patterns, almost discreteness is preserved. In the last two, all the resulting worlds have two distinct successors, so we can group them with the worlds of $\mathfrak{F}_{0\bullet}$ to form $\mathfrak{F}'_0$. We then define $\mathfrak{F}'_1$ as the complement of $\mathfrak{F}'_0$, and we have seen that all the conditions of proposition 4.16 are satisfied. Therefore $\mathfrak{F}_\bullet \vDash \mathbf{wK4.2}$, and since $\mathfrak{F}_\bullet$ is irreflexive it follows $\mathfrak{F}_\bullet \vDash_d \mathbf{wK4.2}$.

□

We are now ready to state the completeness theorem, which covers all the mentioned combinations excepted $\mathbf{wK4.3} + \mathsf{bd}_n$:

Theorem 4.42. *In the d-semantics, the following logics are topologically complete:*

$$\begin{array}{ll} \mathbf{wK4.2}^+ + \mathsf{bd}_n & \mathbf{GL.2}^+ := \mathbf{wK4.2}^+ + \mathsf{gl} \\ \mathbf{wK4.2} + \mathsf{bd}_n & \mathbf{GL.2} := \mathbf{wK4.2} + \mathsf{gl} \\ \mathbf{wK4.3}^+ + \mathsf{bd}_n & \mathbf{GL.3}^+ := \mathbf{wK4.3}^+ + \mathsf{gl} \\ & \mathbf{GL.3} := \mathbf{wK4.3} + \mathsf{gl} \end{array}$$

Proof. First, it is easy to check that all of these logics are cofinal subframe logics. By theorem 3.17, they are thus Kripke complete. For $\mathbf{wK4.2}^+ + \mathsf{bd}_n$, $\mathbf{wK4.2} + \mathsf{bd}_n$ and $\mathbf{wK4.3}^+ + \mathsf{bd}_n$, the result follows from lemma 3.20, lemma 4.36, lemma 4.41 and the usual proof scheme. For the extensions of $\mathbf{GL}$, we notice that a converse well-founded frame is necessarily irreflexive, so it suffices to apply proposition 2.23. □

The logic $\mathbf{wK4.3} + \mathsf{bd}_n$ is absent from the list above, because dereflexivation does not yield T_1 spaces, so for the reasons already explained it is not adapted for $\mathbf{wK4.3}$. We also notice that the system $\mathbf{GL.3}$ is so strong that it actually collapses to a trivial logic:

Lemma 4.43. *We have* $\mathbf{aS4} + \mathsf{gl} = \mathbf{GL.3} = \mathbf{K} + \Box\bot$.

Proof. Let $\mathfrak{F} = (W, R)$ be a Kripke frame such that $\mathfrak{F} \vDash \mathbf{aS4} + \mathsf{gl}$. Then $\mathfrak{F}$ is almost reflexive, but also irreflexive since it validates gl. Therefore $R = \emptyset$, and it follows that $\mathfrak{F} \vDash \Box\bot$. By Kripke completeness, this proves that $\mathbf{aS4} + \mathsf{gl} \vdash \Box\bot$. Conversely, it is clear that $\mathsf{w4}$, aT, $.3$ and gl are derivable in $\mathbf{K} + \Box\bot$, and this concludes the proof. $\Box$

Remark 4.44. Due to the simplicity of dereflexivation, one may wonder why it was not used to prove completeness of $\mathbf{wK4.2}$ and $\mathbf{wK4.2}^+$. This is because the operation of unfolding has other good properties that we did not mention, e.g., it turns transitive frames into T_d spaces [BBFD21], so it is best if it remains the standard method whenever this is possible.

At this point we have managed to describe a large zoology of logics, and have acquired a satisfying grasp of the techniques involved to generate spaces from Kripke frames. We now move to the topic of fixpoint logics.

Chapter 5

Topological mu-calculus and expressivity

The success of the μ-calculus naturally raises the question of whether other logics can compete with its expressive power. By "expressive power" we actually mean two different notions: expressivity with respect to *formulas*, and expressivity with respect to *logics*. The former is the strongest and most well known, it says that a fragment $\mathcal{L}'$ of $\mathcal{L}_\mu$ is *as expressive as* $\mathcal{L}_\mu$ if for any formula ϕ in $\mathcal{L}_\mu$ there exists a formula $\psi \in \mathcal{L}'$ such that $\phi \equiv \psi$. For instance, the *tangled derivative* operator (introduced by Dawar and Otto [DO09]) is known to be as expressive as $\mathcal{L}_\mu$ when restricted to T_d spaces [DO09], but not in general since it does not capture the *tangled closure* operator [BBFD21]. This raises the question of whether the tangled closure is more expressive than the tangled derivative, and if not, whether the combination of the two is enough to obtain a completely expressive fragment. In section 5.1, we give a negative answer to both of these questions. We prove this by introducing the *hybrid modality* which combine the tangled closure and the tangled derivative in a more expressive operator.

The other notion of expressivity involves equivalence of formulas as whether they induce the same logic, instead of usual equivalence. This is a weaker condition, since for instance the two formulas $p \to \Diamond p$ and $\Box p \to p$ define **KT** even though they are not equivalent. A less trivial example is $\mu\mathbf{wK4} + \mu p.\Box p = \mathbf{GL}$ [vB06], which defines the class of scattered spaces in two different ways, one involving fixpoint operators and the other using only basic modal logic. We have already mentioned the results of [BBFD21] which establish topological completeness for many extensions of $\mu\mathbf{wK4}$ with basic modal axioms. This raises the question of whether any axiom of the μ-calculus is reducible to a basic modal axiom, in which case the landscape of logics over μ-calculus would simply collapse. We show in section 5.2 that this is true for a fragment called the *ν-free fragment*, but fails in general; we also make sure to present the best counter-example we can produce, thus obtaining several refinements of the answer.

Slightly abusing notations, we will interpret the relation of logical equivalence $\equiv$ as equivalence *modulo* **wK4**, that is, we write $\phi \equiv \psi$ whenever $\mathfrak{M}, w \vDash \phi \leftrightarrow \psi$ for all pointed **wK4** model $(\mathfrak{M}, w)$. We will also use the term *path* with the following meaning:

Definition 5.1. Let $\mathfrak{F} = (W, R)$ be a Kripke frame. A *path* in $\mathfrak{F}$ (starting from w_0) is a sequence $(w_n)_{n<\alpha}$ with $\alpha \leq \omega$ and such that $0 < n < \alpha$ implies $w_{n-1} R w_n$. If $\alpha < \omega$ this path is said to be of *length* α, otherwise it is called an infinite path.

5.1 The tangled fragments

In this section we investigate the expressive power of the tangled closure and derivative relatively to each other and to the new hybrid modality.

Definition 5.2. Let $\{\phi_1, \ldots, \phi_n\} \subseteq \mathcal{L}_\mu$ be a finite set of formulas. We define:

- the *tangled derivative* $\Diamond_\infty\{\phi_1, \ldots, \phi_n\} := \nu p. \bigwedge_{i=1}^n \Diamond(\phi_i \wedge p)$
- the *tangled closure* $\Diamond_\infty^+\{\phi_1, \ldots, \phi_n\} := \nu p. \bigwedge_{i=1}^n \Diamond^+(\phi_i \wedge p)$
- the *hybrid tangled operator* $\Diamond_\infty^\bullet\{\phi_1, \ldots, \phi_n\} := \nu p. \bigvee_{j=1}^n \left(\Diamond^+(\phi_j \wedge p) \wedge \bigwedge_{i \neq j} \Diamond(\phi_i \wedge p)\right)$

We then define:

- $\mathcal{L}_{\Diamond_\infty}$ the basic modal language extended with $\Diamond_\infty$;
- $\mathcal{L}_{\Diamond_\infty^+}$ the basic modal language extended with $\Diamond_\infty^+$;
- $\mathcal{L}_{\Diamond_\infty, \Diamond_\infty^+}$ the basic modal language extended with $\Diamond_\infty$ and $\Diamond_\infty^+$;
- $\mathcal{L}_{\Diamond_\infty^\bullet}$ the basic modal language extended with $\Diamond_\infty^\bullet$.

The formula $\Diamond_\infty^\bullet\{\phi_1, \ldots, \phi_n\}$ says that every time the ν operator is unfolded, the formula $\Diamond(\phi_i \wedge p)$ must be satisfied for all $i \in [\![1, n]\!]$, *except* for possibly *one* i, for which we may have $\phi_i \wedge p$ instead. The hybrid tangled operator is thus a sort of a mix of the tangled closure and the tangled derivative, hence its name. In fact, we can prove that it subsumes both of them:

Proposition 5.3. *The modalities $\Diamond^+$ and $\Diamond_\infty^+$ can be expressed in $\mathcal{L}_{\Diamond_\infty^\bullet}$.*

Proof. Let $\Gamma = \{\phi_1, \ldots, \phi_n\}$ be a finite set of formulas. For all $i \in [\![1, n]\!]$ we introduce some ϕ_i' distinct from ϕ_i but equivalent to it (for instance $\phi_i' := \phi_i \wedge \top$). Then each term of the disjunction in $\Diamond_\infty^\bullet\{\phi_1, \phi_1', \ldots, \phi_n, \phi_n'\}$ contains at least one of $\Diamond(\phi_i \wedge p)$ or $\Diamond(\phi_i' \wedge p)$, and it follows that

$$\Diamond_\infty^\bullet\{\phi_1, \phi_1', \ldots, \phi_n, \phi_n'\} \equiv \Diamond_\infty\{\phi_1, \ldots, \phi_n\}$$

For the tangled closure, we define

$$\phi := \bigvee_{(I_1,\ldots,I_m) \text{ is a partition of } [\![1,n]\!]} \Diamond_\infty^\bullet \left\{ \bigwedge_{i \in I_j} \phi_i \mid 1 \leq j \leq m \right\}$$

and we show that $\phi \equiv \Diamond_\infty^+\Gamma$. Since μ**wK4** is sound and complete with respect to the class of **wK4** frames, and has the finite model property [BBFD21], it suffices to prove that $\mathfrak{M}, w \vDash \phi \leftrightarrow \Diamond_\infty^+\Gamma$ for any finite weakly transitive Kripke model $\mathfrak{M} = (W, R, \nu)$ and $w \in W$. That $\mathfrak{M}, w \vDash \phi$ implies $\mathfrak{M}, w \vDash \Diamond_\infty^+\Gamma$ is easy to check. Conversely, suppose that $\mathfrak{M}, w \vDash \Diamond_\infty^+\Gamma$. We define

$$W_{\mathsf{max}} := \{u \in [\![\Diamond_\infty^+\Gamma]\!]_\mathfrak{M} \mid \text{for all } v \in [\![\Diamond_\infty^+\Gamma]\!]_\mathfrak{M}, \text{ if } uR^+v \text{ then } vR^+u\}$$

Suppose that $W_{\mathsf{max}} = \emptyset$. Then we can construct an infinite path $(w_k)_{k\in\mathbb{N}}$ within $[\![\Diamond_\infty^+\Gamma]\!]_\mathfrak{M}$ by setting $w_0 := w$ and for all $k \in \mathbb{N}$, taking some $w_{k+1} \in [\![\Diamond_\infty^+\Gamma]\!]_\mathfrak{M}$ such that $w_k R^+ w_{k+1}$ and *not* $w_{k+1} R^+ w_k$

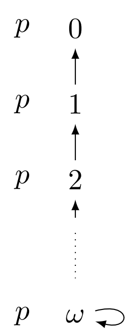
(which exists by assumption). If $k < m$, we have $w_k \neq w_m$, for otherwise we obtain $w_m R^+ w_{m-1}$ by weak transitivity, a contradiction. We thus end up with infinitely many worlds, contradicting the fact that $\mathfrak{M}$ is finite. Therefore, there exists $u \in W_{\mathsf{max}}$. From $\mathfrak{M}, u \vDash \Diamond_\infty^+ \Gamma$, it follows that for all $i \in [\![1, n]\!]$ there exists $w_i \in W$ such that $\mathfrak{M}, w_i \vDash \phi_i \wedge \Diamond_\infty^+ \Gamma$ and $u R^+ w_i$. Then by construction of W_{max} we have $w_i R^+ u$ as well. By weak transitivity it follows that $w_i R^+ w_j$ for all $i, j \in [\![1, n]\!]$. However the w_i's are not necessarily pairwise distinct, so we write $\{w_1, \ldots, w_n\} = \{u_1, \ldots, u_m\}$ with the u_i's being pairwise distinct. This induces a partition $(I_1, \ldots, I_m)$ of $[\![1, n]\!]$, where

$$I_j := \{i \in [\![1, n]\!] \mid w_i = u_j\}$$

for all $j \in [\![1, m]\!]$. Then given $j \in [\![1, m]\!]$ we have $\mathfrak{M}, u_j \vDash \bigwedge_{i \in I_j} \phi_i$, and whenever $j' \neq j$ we have $u_j R u_{j'}$ and thus $\mathfrak{M}, u_j \vDash \Diamond \bigwedge_{i \in I_{j'}} \phi_i$. Therefore $\mathfrak{M}, w \vDash \Diamond_\infty^\bullet \left\{ \bigwedge_{i \in I_j} \phi_i \mid 1 \leq j \leq m \right\}$ and this concludes the proof. □

In [BBFD21] it was proven that $\Diamond_\infty^+$ is not expressible in $\mathcal{L}_{\Diamond_\infty}$, and thus that $\mathcal{L}_{\Diamond_\infty}$ is not as expressive as $\mathcal{L}_\mu$. Reusing the same technique we easily obtain the symmetric result, that is, $\Diamond_\infty$ is not expressible in $\mathcal{L}_{\Diamond_\infty^+}$. This can also be seen as a warm-up for the next result. To achieve this, we introduce the Kripke model $\mathfrak{M} = (\omega + 1, R, \nu)$ with:

- $R := \{(\beta, \alpha) \mid \alpha < \beta \leq \omega\} \cup \{(\omega, \omega)\}$
- ν the valuation defined by $\nu(p) := \omega + 1$ and $\nu(q) := \emptyset$ for any other variable q



Lemma 5.4. *For every formula $\phi \in \mathcal{L}_{\Diamond_\infty^+}$, there exists $n_\phi < \omega$ such that $n_\phi \leq \alpha, \beta \leq \omega$ implies $\mathfrak{M}, \alpha \vDash \phi \iff \mathfrak{M}, \beta \vDash \phi$.*

Proof. By induction on ϕ:

- For an atomic proposition q, we simply have $n_q := 0$.
- If this holds for ϕ, then it is clear that $n_{\neg\phi} := n_\phi$ works for $\neg\phi$.
- If this holds for $\phi \wedge \psi$, then it is clear that $n_{\phi\wedge\psi} := \mathsf{max}\, \{n_\phi, n_\psi\}$ works for $\phi \wedge \psi$.
- Suppose that this holds for ϕ. We set $n_{\Diamond\phi} := n_\phi + 1$. Suppose that $n_{\Diamond\phi} \leq \alpha, \beta \leq \omega$ and that $\mathfrak{M}, \alpha \vDash \Diamond\phi$. Then there exists $\xi \leq \omega$ such that $\alpha R \xi$ and $\mathfrak{M}, \xi \vDash \phi$. If $\xi < \beta$ we are done, otherwise $n_\phi < \beta \leq \xi$. Then by the induction hypothesis, $\mathfrak{M}, \xi \vDash \phi$ entails $\mathfrak{M}, n_\phi \vDash \phi$, and therefore $\mathfrak{M}, \beta \vDash \Diamond\phi$.

– Suppose that this holds for $\phi_1, \dots, \phi_n$ and let $\Gamma := \{\phi_1, \dots, \phi_n\}$. We set $n_{\Diamond^+_\infty \Gamma} := \mathsf{max}\ \{n_\gamma \mid \gamma \in \Gamma\}$. Suppose that $n_{\Diamond^+_\infty \Gamma} \leq \alpha, \beta \leq \omega$ and that $\mathfrak{M}, \alpha \vDash \Diamond^+_\infty \Gamma$. We define $\alpha_0 := \mathsf{min}\ \{\xi \leq \omega \mid \mathfrak{M}, \xi \vDash \Diamond^+_\infty \Gamma\}$.

  For all $i \in [\![1, n]\!]$, there exists ξ_i such that $\alpha_0 R^+ \xi_i$ and $\mathfrak{M}, \xi_i \vDash \phi_i \wedge \Diamond^+_\infty \Gamma$. Then $\xi_i = \alpha_0$ by minimality of α_0, and therefore $\mathfrak{M}, \alpha_0 \vDash \bigwedge \Gamma$. If $n_{\Diamond^+_\infty \Gamma} < \alpha_0$, then by the induction hypothesis we obtain $\mathfrak{M}, n_{\Diamond^+_\infty \Gamma} \vDash \bigwedge \Gamma$ and thus $\mathfrak{M}, n_{\Diamond^+_\infty \Gamma} \vDash \Diamond^+_\infty \Gamma$, contradicting the minimality of α_0. Therefore $\alpha_0 \leq n_{\Diamond^+_\infty \Gamma} \leq \beta$, and it follows that $\mathfrak{M}, \beta \vDash \Diamond^+_\infty \Gamma$.

□

Proposition 5.5. *No formula in $\mathcal{L}_{\Diamond^+_\infty}$ is equivalent to $\Diamond_\infty\{p\}$.*

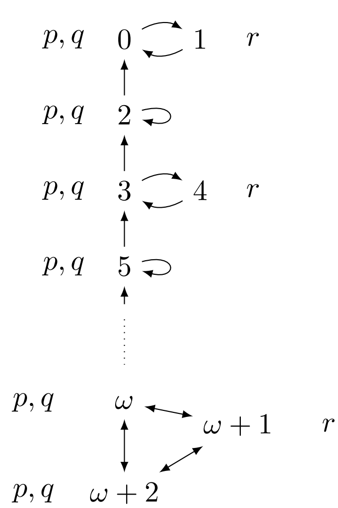
Proof. Suppose that there is such a formula ϕ. Let n_ϕ be the integer given by lemma 5.4. Since $\mathfrak{M}, \omega \vDash \Diamond_\infty\{p\}$ we obtain $\mathfrak{M}, n_\phi \vDash \Diamond_\infty\{p\}$ as well, contradicting the absence of infinite path from n_ϕ. □

Now that we know that $\mathcal{L}_{\Diamond_\infty}$ and $\mathcal{L}_{\Diamond^+_\infty}$ are not as expressive as $\mathcal{L}_\mu$, we may be tempted to combine the two into $\mathcal{L}_{\Diamond_\infty, \Diamond^+_\infty}$ with the hope of obtaining a fragment as expressive as $\mathcal{L}_\mu$. We are going to prove that this conjecture fails, and more precisely that the hybrid tangled operator $\Diamond^\bullet_\infty$ cannot be expressed in $\mathcal{L}_{\Diamond_\infty, \Diamond^+_\infty}$. To this end we introduce the Kripke model $\mathfrak{M} := (\omega + 3, R, \nu)$ with:

- $R := \{(\beta, \alpha) \mid \alpha < \beta < \omega + 3\} \cup \{(n, n) \mid n < \omega \text{ and } n \equiv_3 2\}$
  $\cup \{(\alpha, \alpha + 1) \mid \alpha < \omega + 3 \text{ and } \alpha \equiv_3 0\} \cup \{(\omega, \omega + 2), (\omega + 1, \omega + 2)\}$
- ν the valuation defined by $\nu(p) := \nu(q) := \{\alpha < \omega + 3 \mid \alpha \equiv_3 0 \text{ or } \alpha \equiv_3 2\}$,
  $\nu(r) := \{\alpha < \omega + 3 \mid \alpha \equiv_3 1\}$ and $\nu(p') = \emptyset$ for any other variable p'.

with the relation $\equiv_3$ being the classical equivalence modulo 3 extended to ordinals:

Definition 5.6. If $n, m, k \in \mathbb{N}$ we write $n \equiv_k m$ whenever there exists $q \in \mathbb{Z}$ such that $n = m + kq$. Given $\alpha, \beta \in \{0, \omega\}$ we also write $\alpha + n \equiv_k \beta + m$ whenever $n \equiv_k m$.



Lemma 5.7. *For every formula $\phi \in \mathcal{L}_{\Diamond_\infty,\Diamond_\infty^+}$, there exists $n_\phi < \omega$ such that $n_\phi \leq \alpha, \beta < \omega + 3$ and $\alpha \equiv_3 \beta$ implies $\mathfrak{M}, \alpha \vDash \phi \iff \mathfrak{M}, \beta \vDash \phi$.*

Proof. By induction on ϕ:

- For an atomic proposition p', we simply have $n_{p'} := 0$.
- If this holds for ϕ, then it is clear that $n_{\neg\phi} := n_\phi$ works for $\neg\phi$.
- If this holds for $\phi \wedge \psi$, then it is clear that $n_{\phi\wedge\psi} := \mathsf{max}\ \{n_\phi, n_\psi\}$ works for $\phi \wedge \psi$.
- Suppose that this holds for ϕ. We set $n_{\Diamond\phi} := n_\phi + 3$. Suppose that $n_{\Diamond\phi} \leq \alpha, \beta < \omega + 3$ and that $\mathfrak{M}, \alpha \vDash \Diamond\phi$. Then there exists $\xi < \omega + 3$ such that $\alpha R \xi$ and $\mathfrak{M}, \xi \vDash \phi$. If $\xi < \beta$ we are done, otherwise $n_\phi \leq \beta \leq \xi$. Let n be the integer in $\{n_\phi, n_\phi + 1, n_\phi + 2\}$ satisfying $n \equiv_3 \xi$; since $n_\phi + 3 \leq \beta$, we have $n < \beta$ and thus $\beta R n$. Then by the induction hypothesis, $\mathfrak{M}, \xi \vDash \phi$ entails $\mathfrak{M}, n \vDash \phi$, and therefore $\mathfrak{M}, \beta \vDash \Diamond\phi$.
- Suppose that this holds for $\phi_1, \ldots, \phi_n$ and let $\Gamma := \{\phi_1, \ldots, \phi_n\}$. We set $n_{\Diamond_\infty^+\Gamma} := \mathsf{max}\ \{n_\gamma \mid \gamma \in \Gamma\} + 2$. Suppose that $n_{\Diamond_\infty^+\Gamma} \leq \alpha, \beta < \omega + 3$ and that $\mathfrak{M}, \alpha \vDash \Diamond_\infty^+\Gamma$. We define $\alpha_0 := \mathsf{min}\ \{\xi < \omega + 3 \mid \mathfrak{M}, \xi \vDash \Diamond_\infty^+\Gamma\}$.
  - Suppose $\alpha_0 < \omega$. For all $i \in [\![1, n]\!]$, there exists ξ_i such that $\alpha_0 R^+ \xi_i$ and $\mathfrak{M}, \xi_i \vDash \phi_i \wedge \Diamond_\infty^+\Gamma$. Then $\xi_i \in \{\alpha_0, \alpha_0 + 1\}$ by minimality of α_0.
    Let k be the integer in $\{n_{\Diamond_\infty^+\Gamma} - 2, n_{\Diamond_\infty^+\Gamma} - 1, n_{\Diamond_\infty^+\Gamma}\}$ satisfying $k \equiv_3 \alpha_0$. Suppose that $\alpha_0 > n_{\Diamond_\infty^+\Gamma}$. Then for all $i \in [\![1, n]\!]$ we have $\alpha_0, k \geq n_{\phi_i}$, as well as $\mathfrak{M}, \alpha_0 \vDash \phi_i$ or $\mathfrak{M}, \alpha_0 + 1 \vDash \phi_i$, so $\mathfrak{M}, k \vDash \phi_i$ or $\mathfrak{M}, k + 1 \vDash \phi_i$ by the induction hypothesis. We consider three cases:
    - · If $\alpha_0 \equiv_3 0$, we have $k \equiv_3 0$ as well so $kRk + 1$, and therefore $\mathfrak{M}, k \vDash \Diamond_\infty^+\Gamma$.
    - · The case $\alpha_0 \equiv_3 1$ cannot occur because then $\mathfrak{M}, \alpha_0 - 1 \vDash \Diamond_\infty^+\Gamma$ as well, contradicting the minimality of α.
    - · If $\alpha_0 \equiv_3 2$ then $\neg\alpha_0 R^+ \alpha_0 + 1$ so $\xi_i = \alpha_0$. Therefore $\mathfrak{M}, k \vDash \bigwedge\Gamma$ and it follows that $\mathfrak{M}, k \vDash \Diamond_\infty^+\Gamma$.

    Since $k < \alpha_0$, this contradicts the minimality of α_0. Hence $\alpha_0 \leq n_{\Diamond_\infty^+\Gamma} \leq \beta$, so $\beta R^+ \alpha_0$ and it follows that $\mathfrak{M}, \beta \vDash \Diamond_\infty^+\Gamma$.
  - Suppose $\alpha_0 \geq \omega$. Then for all $i \in [\![1, n]\!]$ there exists ξ_i such that $\mathfrak{M}, \xi_i \vDash \phi_i \wedge \Diamond_\infty^+\Gamma$. By minimality of α_0 we have $\xi_i \in \{\omega, \omega + 1, \omega + 2\}$, and since ω and $\omega + 2$ are bisimilar we can assume $\xi_i \in \{\omega, \omega + 1\}$. Then, if we take $k < \omega$ such that $k \geq n_{\Diamond_\infty^+\Gamma}$ and $k \equiv_3 0$, we obtain by the induction hypothesis that either $\mathfrak{M}, k \vDash \phi_i$ or $\mathfrak{M}, k + 1 \vDash \phi_i$. Therefore $\mathfrak{M}, k \vDash \Diamond_\infty^+\Gamma$, contradicting the minimality of α_0.
- Suppose that this holds for $\phi_1, \ldots, \phi_n$ and let $\Gamma := \{\phi_1, \ldots, \phi_n\}$. We set $n_{\Diamond_\infty\Gamma} := \mathsf{max}\ \{n_\gamma \mid \gamma \in \Gamma\} + 3$. Suppose that $n_{\Diamond_\infty\Gamma} \leq \alpha, \beta < \omega + 3$ and that $\mathfrak{M}, \alpha \vDash \Diamond_\infty\Gamma$. We define $\alpha_0 := \mathsf{min}\ \{\xi < \omega + 3 \mid \mathfrak{M}, \xi \vDash \Diamond_\infty\Gamma\}$.
  - Suppose $\alpha_0 < \omega$. For all $i \in [\![1, n]\!]$, there exists ξ_i such that $\alpha_0 R \xi_i$ and $\mathfrak{M}, \xi_i \vDash \phi_i \wedge \Diamond_\infty\Gamma$. Then $\xi_i \in \{\alpha_0, \alpha_0 + 1\}$ by minimality of α_0. If $\alpha_0 \equiv_3 2$ then $\xi_i = \alpha_0$. Otherwise $\alpha_0 \equiv_3 0$ and $\xi_i = \alpha_0 + 1$; we also have $\mathfrak{M}, \alpha_0 + 1 \vDash \Diamond_\infty\Gamma$ so there exists $\xi_i' < \alpha_0 + 1$ such that $\mathfrak{M}, \xi_i' \vDash \phi_i \wedge \Diamond_\infty\Gamma$; again by minimality of α_0 we obtain $\xi_i' = \alpha_0$.

Let k be the integer in $\{n_{\Diamond_\infty\Gamma}-3, n_{\Diamond_\infty\Gamma}-2, n_{\Diamond_\infty\Gamma}-1\}$ satisfying $k \equiv_3 \alpha_0$. Suppose that $\alpha_0 \geq n_{\Diamond_\infty\Gamma}$.

· If $\alpha_0 \equiv_3 0$ then for all $i \in [\![1,n]\!]$ we have $\alpha_0, k \geq n_{\phi_i}$, as well as $\mathfrak{M}, \alpha_0 \vDash \phi_i$ and $\mathfrak{M}, \alpha_0+1 \vDash \phi_i$, so $\mathfrak{M}, k \vDash \phi_i$ and $\mathfrak{M}, k+1 \vDash \phi_i$ by the induction hypothesis.

· If $\alpha_0 \equiv_3 2$ then for all $i \in [\![1,n]\!]$ we have $\alpha_0, k \geq n_{\phi_i}$, as well as $\mathfrak{M}, \alpha_0 \vDash \phi_i$, so $\mathfrak{M}, k \vDash \phi_i$ by the induction hypothesis.

In both cases we obtain $\mathfrak{M}, k \vDash \Diamond_\infty\Gamma$, and since $k < \alpha_0$ this contradicts the minimality of α_0. Hence $\alpha_0 < n_{\Diamond_\infty\Gamma} \leq \beta$, so $\beta R \alpha_0$ and it follows that $\mathfrak{M}, \beta \vDash \Diamond_\infty\Gamma$.

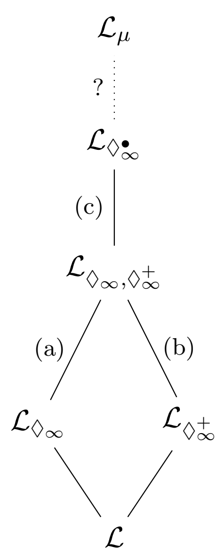
◦ Suppose $\alpha_0 \geq \omega$. Then in particular $\mathfrak{M}, \omega+1 \vDash \Diamond_\infty\Gamma$, so for all $i \in [\![1,n]\!]$ there exists ξ_i such that $\omega+1 R \xi_i$ and $\mathfrak{M}, \xi_i \vDash \phi_i \wedge \Diamond_\infty\Gamma$. By minimality of α we must have $\xi_i \in \{\omega, \omega+2\}$, and since ω and $\omega+2$ are bisimilar we can assume $\xi_i = \omega+2$. Then, if we take $k < \omega$ such that $k \geq n_{\Diamond_\infty\Gamma}$ and $k \equiv_3 2$, we obtain by the induction hypothesis that $\mathfrak{M}, k \vDash \phi_i$. Therefore $\mathfrak{M}, k \vDash \Diamond_\infty\Gamma$, contradicting the minimality of α_0.

□

Proposition 5.8. *No formula in $\mathcal{L}_{\Diamond_\infty, \Diamond_\infty^+}$ is equivalent to $\Diamond_\infty^\bullet\{p,q,r\}$.*

Proof. Suppose that there is such a formula ϕ. Let n_ϕ be the integer given by lemma 5.7, and consider some $k < \omega$ such that $k \geq n_\phi$ and $k \equiv_3 0$. Since $\mathfrak{M}, \omega \vDash \Diamond_\infty^\bullet\{p,q,r\}$ we obtain $\mathfrak{M}, k \vDash \Diamond_\infty^\bullet\{p,q,r\}$ as well, a contradiction. □

These results are summarized in figure 5.1, where a language $\mathcal{L}_1$ is placed above a language $\mathcal{L}_2$ whenever $\mathcal{L}_1$ is more expressive than $\mathcal{L}_2$. The question that naturally arises now is whether $\mathcal{L}_{\Diamond_\infty^\bullet}$ is as expressive as the μ-calculus, but we leave this problem to future work.



(a) from [BBFD21]
(b) from propositions 5.3 and 5.5
(c) from proposition 5.8

Figure 5.1: Relative expressivity of the tangled fragments

5.2 Logic-wise expressivity

As mentioned at the beginning of the chapter, the question under discussion is whether for all $\phi \in \mathcal{L}_\mu$ there exists $\psi \in \mathcal{L}$ such that $\mu\mathbf{wK4} + \phi = \mu\mathbf{wK4} + \psi$, that is, whether every axiom of the μ-calculus can be rid of its fixpoint operators and reduced to a plain modal axiom. We will favour, however, a more semantic conception.

Proposition 5.9. *Let $\phi, \psi \in \mathcal{L}_\mu$.*

- *If $\mu\mathbf{wK4} + \phi = \mu\mathbf{wK4} + \psi$ then $\{\mathfrak{F} \mid \mathfrak{F} \vDash \mu\mathbf{wK4} + \phi\} = \{\mathfrak{F} \mid \mathfrak{F} \vDash \mu\mathbf{wK4} + \psi\}$.*
- *If $\mu\mathbf{wK4} + \phi$ and $\mu\mathbf{wK4} + \psi$ are Kripke complete, then $\{\mathfrak{F} \mid \mathfrak{F} \vDash \mu\mathbf{wK4} + \phi\} = \{\mathfrak{F} \mid \mathfrak{F} \vDash \mu\mathbf{wK4} + \psi\}$ implies $\mu\mathbf{wK4} + \phi = \mu\mathbf{wK4} + \psi$.*
- *If $\mu\mathbf{wK4} + \phi = \mu\mathbf{wK4} + \psi$ then $\{X \mid X \vDash \mu\mathbf{wK4} + \phi\} = \{X \mid X \vDash \mu\mathbf{wK4} + \psi\}$.*
- *If $\mu\mathbf{wK4} + \phi$ and $\mu\mathbf{wK4} + \psi$ are topologically complete, then $\{X \mid X \vDash \mu\mathbf{wK4} + \phi\} = \{X \mid X \vDash \mu\mathbf{wK4} + \psi\}$ implies $\mu\mathbf{wK4} + \phi = \mu\mathbf{wK4} + \psi$.*

Proof. Straightforward. $\square$

This proposition shows that in general, the equality of two logics is a stronger condition than the equality of their classes of frames or spaces. This motivates the following definition:

Definition 5.10. A class of spaces (resp. Kripke frames) $\mathcal{C}$ is *modally definable* if there exists a modal formula ϕ such that for all space X (resp. for all frame $\mathfrak{F}$) we have

$$X \in \mathcal{C} \iff X \vDash \phi$$

(resp. $\mathfrak{F} \in \mathcal{C} \iff \mathfrak{F} \vDash \phi$).

Let ϕ be a formula of the μ-calculus. In general, rather than "Is the axiom ϕ reducible to a basic modal axiom?", the question we are going to address is "Is the class of spaces/frames defined by ϕ, modally definable too?". There are two reasons for this choice: first, a negative answer to the latter will bring by contraposition a negative answer to the former, and since most of our results will be negative this is essentially the strongest choice; second, this approach lends itself to many refinements since we can add any restriction we want to the kind of spaces we consider (for example we can quantify over finite spaces only), and thus derive interesting auxiliary results.

5.2.1 The ν-free μ-calculus

We first consider the fragment of the μ-calculus wherein the ν operator does not occur, and prove that it does not define more classes of spaces than basic modal logic. This result is not only interesting in itself, it also guides us in the process of finding an axiom of the μ-calculus which is not reducible to an axiom of basic modal logic, by telling what kind of formula can *not* be such a counter-example.

Definition 5.11. The language $\mathcal{L}^0_\mu$ of the *ν-free μ-calculus* is defined by the following grammar:

$$\phi ::= p \mid \neg p \mid \phi \wedge \psi \mid \phi \vee \psi \mid \Box\phi \mid \Diamond\phi \mid \mu p.\phi$$

where $\neg p$ does not occur in formulas of the form $\mu p.\phi$.

We recall that the extension of a formula of the form $\mu p.\phi$ in a topological model (X, ν) is defined as

$$[\![\mu p.\phi]\!]_{X,\nu} := \bigcap\{A \subseteq X \mid [\![\phi]\!]_{X,\nu[p:=A]} \subseteq A\}$$

Therefore, if $x \in X$ we have

$$X, \nu, x \vDash \mu p.\phi \text{ iff } \forall A \subseteq X, ([\![\phi]\!]_{X,\nu[p:=A]} \subseteq A) \implies x \in A$$

We can then observe that the universal quantification over the subsets of X is, implicitly, nothing more than a quantification over the possible valuations of p. This is precisely the kind of quantification that validity of formulas allows to capture, and this leads to the following translation:

Definition 5.12. We define the function $\mathsf{tr}_\mu : \mathcal{L}^0_\mu \to \mathcal{L}$ by induction as follows:

- $\mathsf{tr}_\mu(p) := p$
- $\mathsf{tr}_\mu(\neg p) := \neg p$
- $\mathsf{tr}_\mu(\phi \wedge \psi) := \mathsf{tr}_\mu(\phi) \wedge \mathsf{tr}_\mu(\psi)$
- $\mathsf{tr}_\mu(\phi \vee \psi) := \mathsf{tr}_\mu(\phi) \vee \mathsf{tr}_\mu(\psi)$
- $\mathsf{tr}_\mu(\Box\phi) := \Box\mathsf{tr}_\mu(\phi)$
- $\mathsf{tr}_\mu(\Diamond\phi) := \Diamond\mathsf{tr}_\mu(\phi)$
- $\mathsf{tr}_\mu(\mu p.\phi) := \Box^+(\mathsf{tr}_\mu(\phi) \to p) \to p$

Recall that formulas of the μ-calculus are assumed to be clean, so each formula of the form $\mu p.\phi$ comes with its own variable p. Our goal is then to prove that ϕ and $\mathsf{tr}_\mu(\phi)$ define the same class of spaces. One direction is obtained by a stronger claim:

Lemma 5.13. *For all $\phi \in \mathcal{L}^0_\mu$ we have $\vDash \phi \to \mathsf{tr}_\mu(\phi)$.*

Proof. By induction on ϕ. This is straightforward for Boolean and modal formulas, so we only treat the fixpoint operator.

Let $\mathfrak{M} = (X, \nu)$ be a pointed topological model, $x \in X$ and assume $\mathfrak{M}, x \vDash \mu p.\phi$ and $\mathfrak{M}, x \vDash \Box^+(\mathsf{tr}_\mu(\phi) \to p)$. By the induction hypothesis we have $[\![\phi]\!]_{\mathfrak{M}} \subseteq [\![\mathsf{tr}_\mu(\phi)]\!]_{\mathfrak{M}}$, and thus $\mathfrak{M}, x \vDash \Box^+(\phi \to p)$. Then there exists a neighbourhood U of x such that $U \subseteq [\![\phi \to p]\!]_{\mathfrak{M}}$ and we set $A := (X \setminus U) \cup [\![p]\!]_{\mathfrak{M}}$.

Let $y \in [\![\phi]\!]_{X,\nu[p:=A]}$. If $y \in X \setminus U$, then $y \in A$ immediately. Otherwise $y \in U$, and it is easy to prove by induction on ϕ that $U \cap [\![\phi]\!]_{X,\nu[p:=A]} = U \cap [\![\phi]\!]_{\mathfrak{M}}$. Since $y \in U \cap [\![\phi]\!]_{X,\nu[p:=A]}$, and $U \subseteq [\![\phi \to p]\!]_{\mathfrak{M}}$, we obtain $y \in [\![p]\!]_{\mathfrak{M}} \subseteq A$. This proves $[\![\phi]\!]_{X,\nu[p:=A]} \subseteq A$, so $x \in A$ by assumption. Since $x \in U$ it follows that $x \in [\![p]\!]_{\mathfrak{M}}$ and this concludes the proof. $\Box$

For the other direction, we will need to transform a model of $\mathsf{tr}_\mu(\phi)$ into a model of ϕ. This is obtained by tweaking a valuation in a way that makes any formula of the form $\mu p.\psi$ coextensive with p:

Definition 5.14. Let (X, ν) be a topological model and $\phi \in \mathcal{L}^0_\mu$. We define a valuation $[\nu]_\phi$ as follows: for all subformula of ϕ of the form $\mu p.\psi$, we set $[\nu]_\phi(p) := [\![\mu p.\psi]\!]_{X,\nu}$, and for any other $q \in \mathsf{Prop}$ we set $[\nu]_\phi(q) := \nu(q)$.

Lemma 5.15. *Let (X, ν) be a pointed topological model and $\phi \in \mathcal{L}^0_\mu$. If $X, [\nu]_\phi, x \vDash \mathsf{tr}_\mu(\phi)$ then $X, \nu, x \vDash \phi$.*

Proof. By induction on ϕ. Again, this is straightforward for Boolean and modal formulas.

Suppose that $X, [\nu]_{\mu p.\phi}, x \vDash \Box^+(\mathsf{tr}_\mu(\phi) \to p) \to p$. We write $A := [\![\mu p.\phi]\!]_{X,\nu}$ and then the fix-point equation gives $A = [\![\phi]\!]_{X,\nu[p:=A]}$. By the induction hypothesis we then have $[\![\mathsf{tr}_\mu(\phi)]\!]_{X,[\nu[p:=A]]_\phi} \subseteq A$, and $[\nu[p := A]]_\phi = [\nu]_{\mu p.\phi}$ by construction, so $[\![\mathsf{tr}_\mu(\phi)]\!]_{X,[\nu]_{\mu p.\phi}} \subseteq A$. Hence $X, [\nu]_{\mu p.\phi}, x \vDash \Box^+(\mathsf{tr}_\mu(\phi) \to p)$, and by assumption it follows that $X, [\nu]_{\mu p.\phi}, x \vDash p$. Therefore $X, \nu, x \vDash \mu p.\phi$. □

We can now conclude with the desired result:

Theorem 5.16. *For any formula $\phi \in \mathcal{L}^0_\mu$, the class $\{X \mid X \vDash \mu\mathbf{wK4} + \phi\}$ is modally definable.*

Proof. Given $\phi \in \mathcal{L}^0_\mu$, we prove that ϕ and $\mathsf{tr}_\mu(\phi)$ define the same class of spaces. Indeed, if $X \vDash \phi$ then $X \vDash \mathsf{tr}_\mu(\phi)$ follows immediately from lemma 5.13. Conversely, suppose $X \vDash \mathsf{tr}_\mu(\phi)$. Then for any valuation ν we have $X, [\nu]_\phi \vDash \mathsf{tr}_\mu(\phi)$, and by lemma 5.15 we obtain $X, \nu \vDash \phi$. Therefore $X \vDash \phi$. □

5.2.2 The case of full μ-calculus

The goal of this section is to show that in the general case, the μ-calculus is more expressive than basic modal logic. Thanks to the previous section, we know that the counter-example we are looking for should contain the operator ν. It turns out that a whole family of formulas of the form $\phi \vee \nu p.\Diamond p$ will yield the desired result. It is easy to see that given a pointed Kripke model $(\mathfrak{M}, x)$, we have $\mathfrak{M}, x \vDash \nu p.\Diamond p$ if and only if there exists a infinite path starting from x.

Our final result will be the following:

Theorem 5.17. *Let $\phi \in \mathcal{L}$ and suppose that for all $n \in \mathbb{N}$ there exists a* **wK4** *frame $\mathfrak{F}_n = (W_n, R_n)$ and $r_n \in W_n$ such that:*

- *$\mathfrak{F}_n$ is rooted in r_n and $\mathfrak{F}_n, r_n \nvDash \phi \vee \nu p.\Diamond p$;*
- *$\mathfrak{F}_n$ contains a path of length n.*

Then there is no formula $\psi \in \mathcal{L}$ such that $\mu\mathbf{wK4} + \phi \vee \nu p.\Diamond p = \mu\mathbf{wK4} + \psi$.

Further, suppose that for all $w \in W_n \backslash \{r_n\}$ we have $\mathfrak{F}_n, w \vDash \phi$. Then $\{\mathfrak{F} \mid \mathfrak{F} \vDash \mu\mathbf{wK4}+\phi\vee\nu p.\Diamond p\}$ and $\{\mathfrak{F} \mid \mathfrak{F}$ is transitive and $\mathfrak{F} \vDash \mu\mathbf{wK4} + \phi \vee \nu p.\Diamond p\}$ are not modally definable. In addition:

- *If every $\mathfrak{F}_n$ is finite, then $\{\mathfrak{F} \mid \mathfrak{F}$ is finite and $\mathfrak{F} \vDash \mu\mathbf{wK4}+\phi\vee\nu p.\Diamond p\}$ is not modally definable.*
- *If every $\mathfrak{F}_n$ is irreflexive then*

$$\{\mathfrak{F} \mid \mathfrak{F} \text{ is irreflexive and } \mathfrak{F} \vDash \mu\mathbf{wK4} + \phi \vee \nu p.\Diamond p\}$$

$$\{\mathfrak{F} \mid \mathfrak{F} \text{ is irreflexive and transitive and } \mathfrak{F} \vDash \mu\mathbf{wK4} + \phi \vee \nu p.\Diamond p\}$$

$$\{X \mid X \vDash \mu\mathbf{wK4} + \phi \vee \nu p.\Diamond p\}$$

$$\{X \mid X \text{ is } T_d \text{ and } X \vDash \mu\mathbf{wK4} + \phi \vee \nu p.\Diamond p\}$$

are not modally definable.

– *If every $\mathfrak{F}_n$ is irreflexive and finite then*

$$\{\mathfrak{F} \mid \mathfrak{F} \textit{ is irreflexive and finite and } \mathfrak{F} \vDash \mu\mathbf{wK4} + \phi \vee \nu p.\Diamond p\}$$

$$\{X \mid X \textit{ is finite and } X \vDash \mu\mathbf{wK4} + \phi \vee \nu p.\Diamond p\}$$

are not modally definable.

From now on, we fix a formula ϕ and a family of frames $(\mathfrak{F}_n)_{n\in\mathbb{N}}$ satisfying the assumptions of theorem 5.17. For all $n \in \mathbb{N}$, we assume that $W_n \cap \omega = \emptyset$. We start with an elementary observation:

Claim 5.18. *For all $n \in \mathbb{N}$, $\mathfrak{F}_n$ is transitive.*

Proof. For suppose not. Then there exist $n \in \mathbb{N}$ and $w, u \in W_n$ such that wR_nu, uR_nw and not wR_nw. This entails $\mathfrak{F}_n, w \vDash \nu p.\Diamond p$ and then $\mathfrak{F}_n, r_n \vDash \nu p.\Diamond p$ since r_n is a root. Therefore $\mathfrak{F}_n, r_n \vDash \phi \vee \nu p.\Diamond p$, a contradiction. ☐

Given a subframe $\mathfrak{F} = (W, R)$ of $\mathfrak{F}_n$, we define the **wK4** frames $\mathfrak{A}_n^{\mathfrak{F}} = (W_A, R_A)$ and $\mathfrak{B}_n^{\mathfrak{F}} = (W_B, R_B)$ by:

- $W_A := W_n \cup \omega$
- $R_A := R_n \cup \{(r_n, k) \mid k \in \omega\} \cup \{(m, k) \mid 0 \leq m < k < \omega\} \cup \{(k, w) \mid k \in \omega \text{ and } w \in W\}$
- $W_B := W_n \cup \{0, 1\}$
- $R_B := R_n \cup \{(r_n, 0), (r_n, 1), (0, 1), (1, 0)\} \cup \{(k, w) \mid k \in \{0, 1\} \text{ and } w \in W\}$

In words, $\mathfrak{A}_n^{\mathfrak{F}}$ is the frame $\mathfrak{F}_n$ endowed with an infinite branch starting from the root, and whose every element sees all the worlds of $\mathfrak{F}$. The frame $\mathfrak{B}_n^{\mathfrak{F}}$ is constructed similarly, but with a two-element loop instead of a branch. The two frames are depicted in figure 5.2.

If $\mu\mathbf{wK4} + \phi \vee \nu p.\Diamond p = \mu\mathbf{wK4} + \psi$ for some modal formula ψ, then by construction ψ should be refuted at $\mathfrak{F}_n, r_n$ for any n, but not at $\mathfrak{A}_n^{\mathfrak{F}}, r_n$ or $\mathfrak{B}_n^{\mathfrak{F}}, r_n$ since in both of them there is an infinite path starting from the root. Yet we will prove that if n is big enough and $\neg\psi$ is satisfiable at $\mathfrak{F}_n, r_n$ then it is also satisfiable at $\mathfrak{B}_n^{\mathfrak{F}}, r_n$ for some $\mathfrak{F}$, leading to a contradiction[1]. The proof is very technical, but we can sketch the main lines of our strategy. First, it is clear that transferring the satisfiability of a diamond formula (i.e., of the form $\Diamond\theta$) or a Boolean formula from $\mathfrak{F}_n, r_n$ to $\mathfrak{B}_n^{\mathfrak{F}}, r_n$ is immediate, so the challenge really comes from box formulas (of the form $\Box\theta$). The difficulty here is that a box formula may contain other box subformulas, which themselves contains their own box subformulas, and so on. However, since n may be arbitrarily large, we can always consider a frame with an arbitrarily long path. By means of a tricky pigeonhole argument, we will then be able to show that somewhere on this path, if $\Box\theta$ is satisfied, then so is θ (when $\Box\theta$ is any subformula of $\neg\psi$). Then, transferring the truth of $\Box\theta$ to the two elements of the loop in $\mathfrak{B}_n^{\mathfrak{F}}$ will be straightforward.

First, we recall that the *negative normal form* (or NNF for short) for modal logic is the syntax generated by the following grammar:

$$\phi ::= p \mid \neg p \mid \phi \wedge \phi \mid \phi \vee \phi \mid \Box\phi \mid \Diamond\phi$$

It is well known that for any modal formula, there exists an equivalent formula in NNF. We also introduce the notion of *type* of a possible world, but restricted to the box subformulas of a given formula:

[1] The same result with $\mathfrak{A}_n^{\mathfrak{F}}, r_n$ will follow for free. We have introduced both of $\mathfrak{A}_n^{\mathfrak{F}}$ and $\mathfrak{B}_n^{\mathfrak{F}}$ because each of them has its own particularities that will be useful when proving the various results asserted in theorem 5.17

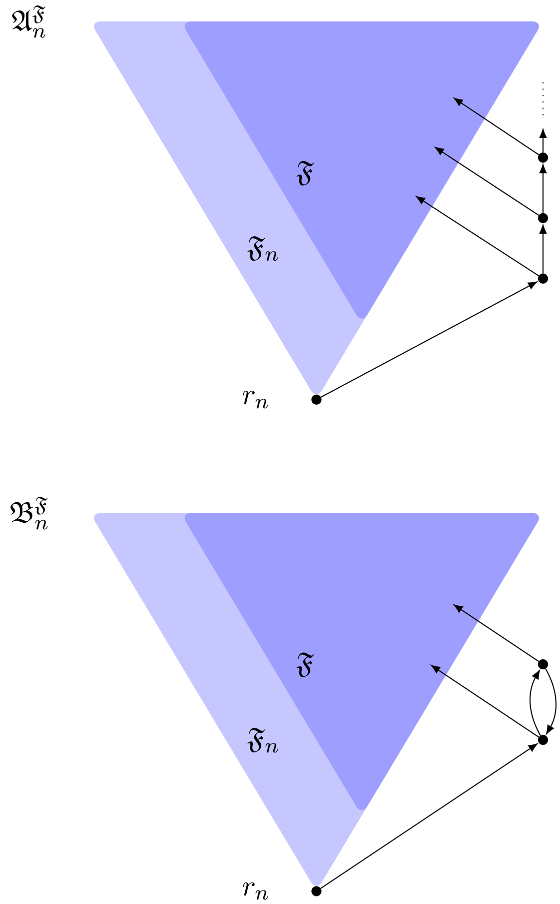


Figure 5.2: The frames $\mathfrak{A}_n^{\mathfrak{F}}$ and $\mathfrak{B}_n^{\mathfrak{F}}$

Definition 5.19. Let ϕ be a modal formula. We write $\psi \trianglelefteq \phi$ whenever ψ is a subformula of ϕ. We also call the *box size* $|\phi|_\Box$ of ϕ the number of subformulas of ϕ of the form $\Box\psi$. If $\mathfrak{M}$ is a Kripke model and w a world in $\mathfrak{M}$, we define the *box type* of w relative to ϕ as the set

$$t^{\phi}_{\mathfrak{M}}(w) := \{\Box\psi \mid \Box\psi \trianglelefteq \phi \text{ and } \mathfrak{M}, w \vDash \Box\psi\}$$

As explained above, the following result allows to transfer the satisfiability of box formulas as soon as the parameter n is large enough:

Claim 5.20. *Let ϕ be a modal formula in NNF and $n \geq 2^{|\phi|_\Box} + 1$. Suppose that there exists a valuation ν over $\mathfrak{F}_n$ such that $\mathfrak{F}_n, \nu, r_n \vDash \Box\phi$. Then there exists a generated subframe $\mathfrak{F}$ of $\mathfrak{F}_n$ and a valuation ν' over $\mathfrak{B}_n^{\mathfrak{F}}$ such that $\mathfrak{B}_n^{\mathfrak{F}}, \nu', r_n \vDash \Box\phi$, and ν and ν' coincide over $\mathfrak{F}_n$.*

Proof. First, we know that $\mathfrak{F}_n$ contains a path $(w_i)_{i \in [\![0,n-1]\!]}$ of length n. By construction there are $2^{|\phi|_\Box}$ different box types relative to ϕ. Thus, by the pigeonhole principle, there exists $i, j \in \mathbb{N}$ such that $0 \leq i < j \leq n-1$ and $t^{\phi}_{\mathfrak{M}}(w_i) = t^{\phi}_{\mathfrak{M}}(w_j)$. We then denote by $\mathfrak{F}$ the subframe of $\mathfrak{F}_n$ generated by w_j and define a valuation ν' over $\mathfrak{B}_n^{\mathfrak{F}}$ by setting, for all $p \in \mathsf{Prop}$:

$$\nu'(p) := \begin{cases} \nu(p) \cup \{0,1\} & \text{if } w_j \in \nu(p) \\ \nu(p) & \text{otherwise} \end{cases}$$

So ν and ν' coincide over $\mathfrak{F}_n$, and ν' is defined over 0 and 1 so that those points satisfy the same atomic propositions as w_j. We then prove by induction on $\psi \trianglelefteq \phi$ that $\mathfrak{F}_n, \nu, w_j \vDash \psi$ implies $\mathfrak{B}_n^{\mathfrak{F}}, \nu', 0 \vDash \psi$ and $\mathfrak{B}_n^{\mathfrak{F}}, \nu', 1 \vDash \psi$:

- If ψ is of the form $\psi = p$ or $\psi = \neg p$ with $p \in \mathsf{Prop}$ this is just true by construction.
- If ψ is of the form $\psi = \psi_1 \wedge \psi_2$, then $\mathfrak{F}_n, \nu, w_j \vDash \psi_1 \wedge \psi_2$ implies $\mathfrak{F}_n, \nu, w_j \vDash \psi_1$ and $\mathfrak{F}_n, \nu, w_j \vDash \psi_2$ and it suffices to apply the induction hypothesis. If ψ is of the form $\psi = \psi_1 \vee \psi_2$, then $\mathfrak{F}_n, \nu, w_j \vDash \psi_1 \vee \psi_2$ implies $\mathfrak{F}_n, \nu, w_j \vDash \psi_1$ or $\mathfrak{F}_n, \nu, w_j \vDash \psi_2$ and the result follows in the same way.
- Suppose that ψ is of the form $\psi = \Diamond\psi_0$ and $\mathfrak{F}_n, \nu, w_j \vDash \psi$. Then since ν and ν' coincide over $\mathfrak{F}_n$, we have $\mathfrak{B}_n^{\mathfrak{F}}, \nu', w_j \vDash \psi$ as well. By transitivity it follows $\mathfrak{B}_n^{\mathfrak{F}}, \nu', 0 \vDash \psi$ and $\mathfrak{B}_n^{\mathfrak{F}}, \nu', 1 \vDash \psi$.
- Suppose that ψ is of the form $\psi = \Box\psi_0$ and that $\mathfrak{F}_n, \nu, w_j \vDash \psi$. Then since $t^{\phi}_{\mathfrak{M}}(w_i) = t^{\phi}_{\mathfrak{M}}(w_j)$, we have $\mathfrak{F}_n, \nu, w_i \vDash \psi$ as well. Since $w_i R_n w_j$ it follows $\mathfrak{F}_n, \nu, w_j \vDash \psi_0$, and then $\mathfrak{B}_n^{\mathfrak{F}}, \nu', 0 \vDash \psi_0$ and $\mathfrak{B}_n^{\mathfrak{F}}, \nu', 1 \vDash \psi_0$ by the induction hypothesis. Since ν and ν' coincide over $\mathfrak{F}_n$ we also have $\mathfrak{B}_n^{\mathfrak{F}}, \nu', w_j \vDash \Box^+\psi_0$. All in all we obtain $\mathfrak{B}_n^{\mathfrak{F}}, \nu', 0 \vDash \Box\psi_0$ and $\mathfrak{B}_n^{\mathfrak{F}}, \nu', 1 \vDash \Box\psi_0$ as desired.

Now observe that since $w_i R_n w_j$ we must have $w_j \neq r_n$, otherwise we would obtain $r_n R_n r_n$ by transitivity. Thus $r_n R_n w_j$ and from $\mathfrak{F}_n, \nu, r_n \vDash \Box\phi$ we obtain $\mathfrak{F}_n, \nu, w_j \vDash \phi$, and then $\mathfrak{B}_n^{\mathfrak{F}}, \nu', 0 \vDash \phi$ and $\mathfrak{B}_n^{\mathfrak{F}}, \nu', 1 \vDash \phi$. Since ν and ν' coincide over $\mathfrak{F}_n$, we conclude that $\mathfrak{B}_n^{\mathfrak{F}}, \nu', r_n \vDash \Box\phi$. □

We can then extend the result to any modal formula:

Claim 5.21. *Let ϕ be a modal formula. There exists $n \in \mathbb{N}$ such that if ϕ is satisfiable at $\mathfrak{F}_n, r_n$, then there exists a generated subframe $\mathfrak{F}$ of $\mathfrak{F}_n$ such that ϕ is satisfiable at $\mathfrak{A}_n^{\mathfrak{F}}, r_n$ and $\mathfrak{B}_n^{\mathfrak{F}}, r_n$.*

Proof. Applying the theorem of disjunctive normal form for propositional logic, and using the fact that $\Box$ and $\wedge$ commute, we can assume that ϕ is of the form $\phi = \bigvee_{i=1}^m \sigma_i$ with, for all $i \in [\![1, m]\!]$,

$$\sigma_i = \Box\psi_i \wedge \rho \wedge \bigwedge_{j=1}^{m_i} \Diamond\theta_{i,j}$$

where ρ is a propositional formula. Note that since $\Box\top$ is a tautology, we can always assume the presence of $\Box\psi_i$. We also suppose that ψ_i is presented in NNF. We then define

$$n := 1 + \mathsf{max}\ \{2^{|\psi_i|_\Box} \mid 1 \leq i \leq m\}$$

and assume that there exists a valuation ν such that $\mathfrak{F}_n, \nu, r_n \vDash \phi$. Then there exists $i \in [\![1, m]\!]$ such that $\mathfrak{F}_n, \nu, r_n \vDash \sigma_i$. It follows that $\mathfrak{F}_n, \nu, r_n \vDash \Box\psi_i$ with $n \geq 2^{|\psi_i|_\Box} + 1$, so by claim 5.20 there exists a subframe $\mathfrak{F}$ of $\mathfrak{F}_n$ and a valuation ν' over $\mathfrak{B}_n^{\mathfrak{F}}$ such that $\mathfrak{B}_n^{\mathfrak{F}}, \nu', r_n \vDash \Box\psi_i$, and ν and ν' coincide over $\mathfrak{F}_n$. It is then clear that $\mathfrak{B}_n^{\mathfrak{F}}, \nu', r_n \vDash \sigma_i$, and thus $\mathfrak{B}_n^{\mathfrak{F}}, \nu', r_n \vDash \phi$.

This proves that ϕ is satisfiable at $\mathfrak{B}_n^{\mathfrak{F}}, r_n$. Now consider the application f which maps every $n \in \omega$ to n mod 2, and every $w \in W_n$ to w itself. Then f clearly defines a bounded morphism from $\mathfrak{A}_n^{\mathfrak{F}}$ to $\mathfrak{B}_n^{\mathfrak{F}}$ with $f(r_n) = r_n$, and we conclude that ϕ is satisfiable at $\mathfrak{A}_n^{\mathfrak{F}}, r_n$. □

We are now ready to prove theorem 5.17:

Proof. Suppose toward a contradiction that there is a formula $\theta \in \mathcal{L}$ such that $\mu\mathbf{wK4} + \phi \vee \nu p.\Diamond p = \mu\mathbf{wK4} + \theta$. Let n be the integer obtained by applying claim 5.21 to $\neg\theta$. By assumption, $\phi \vee \nu p.\Diamond p$ is not valid at $\mathfrak{F}_n, r_n$, so $\neg\theta$ is satisfiable at $\mathfrak{F}_n, r_n$. Then by claim 5.21 there exists a generated subframe $\mathfrak{F}$ of $\mathfrak{F}_n$ such that $\neg\theta$ is satisfiable at $\mathfrak{B}_n^{\mathfrak{F}}, r_n$. Yet $\mathfrak{B}_n^{\mathfrak{F}}$ contains an infinite path starting from r_n, so $\mathfrak{B}_n^{\mathfrak{F}}, r_n \vDash \phi \vee \nu p.\Diamond p$ and thus $\mathfrak{B}_n^{\mathfrak{F}}, r_n \vDash \theta$, a contradiction. In addition, if $\mathfrak{F}_n, w \vDash \phi$ for all $w \neq r_n$, then $\phi \vee \nu p.\Diamond p$ is valid in all of $\mathfrak{B}_n^{\mathfrak{F}}$, so we obtain the stronger result that $\{\mathfrak{F} \mid \mathfrak{F} \vDash \phi \vee \nu p.\Diamond p\}$ is not modally definable.

By the same reasoning, we can show that $\{\mathfrak{F} \mid \mathfrak{F}$ is transitive and $\mathfrak{F} \vDash \phi \vee \nu p.\Diamond p\}$ is not modally definable. To that end it suffices to replace $\mathfrak{B}_n^{\mathfrak{F}}$ by $\mathfrak{A}_n^{\mathfrak{F}}$, which is transitive since $\mathfrak{F}_n$ is transitive by claim 5.18. Likewise, we can prove all the other variants:

- If $\mathfrak{F}_n$ is finite, so is $\mathfrak{B}_n^{\mathfrak{F}}$.
- If $\mathfrak{F}_n$ is irreflexive, then so are $\mathfrak{B}_n^{\mathfrak{F}}$ and $\mathfrak{A}_n^{\mathfrak{F}}$. By proposition 2.23, the Kripke semantics and the d-semantics will then agree over $\mathfrak{B}_n^{\mathfrak{F}}$ and $\mathfrak{A}_n^{\mathfrak{F}}$, and this yields the result for topological spaces. Since $\mathfrak{A}_n^{\mathfrak{F}}$ is transitive, we have $\mathfrak{A}_n^{\mathfrak{F}} \vDash \mathbf{K4}$ and thus $\mathfrak{A}_n^{\mathfrak{F}}$ is a T_d space.
- If $\mathfrak{F}_n$ is irreflexive and finite, so is $\mathfrak{B}_n^{\mathfrak{F}}$.

$\square$

Theorem 5.17 remains a very general result, and it is worth instantiating it with several examples. The following proposition shows the existence of infinitely many non-modally definable classes of spaces:

Corollary 5.22. *Let $m \in \mathbb{N}$. The class of spaces X such that*

$$X \vDash (\Diamond^+\Box^+ q \to \Box^+\Diamond^+ q) \vee \Box^m \bot \vee \nu p.\Diamond p$$

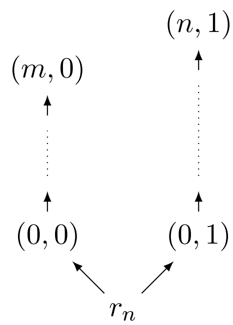
is not modally definable.

Proof. It suffices to prove that the assumptions of theorem 5.17 are satisfied for $\phi := (\Diamond^+\Box^+ q \to \Box^+\Diamond^+ q) \vee \Box^m \bot$. We note that given a frame $\mathfrak{F} = (W, R)$ we have $\mathfrak{F} \vDash \phi \vee \nu p.\Diamond p$ iff for all $w \in W$ one of the following holds:

- for all $u, v \in W$ such that wR^+u and wR^+v, there exists $t \in W$ such that uR^+t and vR^+t;
- there exists no path of length $m+1$ starting from w;
- there exists an infinite path starting from w.

We thus define, for all $n \in \mathbb{N}$, a frame $\mathfrak{F}_n := (W_n, R_n)$ with:

- $W_n := \{r_n\} \cup [\![0, m]\!] \times \{0\} \cup [\![0, n]\!] \times \{1\}$
- $R_n := \{(r_n, w) \mid w \in [\![0, m]\!] \times \{0\} \cup [\![0, n]\!] \times \{1\}\} \cup$
  $\{((k, 1), (k', 1)) \mid 0 \leq k < k' \leq m\} \cup \{((k, 0), (k', 0)) \mid 0 \leq k < k' \leq n\}$



We can see that the $\mathfrak{F}_n$'s fulfil the conditions of theorem 5.17, and are irreflexive, so we are done. $\square$

In addition, we can notice that every $\mathfrak{F}_n$ is finite, so all the sub-results of theorem 5.17 are actually satisfied. This makes $.2^+ \vee \Box^m\bot \vee \nu p.\Diamond p$ an optimal example in some sense, though we can aim at even better. We are going to focus on the case $m = 0$, which reduces to $.2^+ \vee \nu p.\Diamond p$. We first show that it has an intuitive topological interpretation.

Proposition 5.23. *Let ϕ be a modal formula and X a topological space. Then $X \vDash \phi \vee \nu p.\Diamond p$ if and only if there exist two disjoint subspaces Y and Z of X such that $X = Y \cup Z$, $Y \vDash \phi$ and Z is dense-in-itself.*

Proof. From left to right, assume that $X \vDash \phi \vee \nu p.\Diamond p$. We set $Z := \{x \in X \mid X, x \vDash \nu p.\Diamond p\}$ and $Y := X \setminus Z$.

- The fixpoint equation immediately gives $Z = \mathsf{d}(Z)$, so Z is dense-in-itself. We also note that $\mathsf{Cl}(Z) = Z \cup \mathsf{d}(Z) = Z$, so Z is closed and Y is open.
- Let $x \in Y$ and ν be a valuation over Y. We have $X, \nu, x \vDash \phi \vee \nu p.\Diamond p$ and by construction, $X, \nu, x \nvDash \nu p.\Diamond p$, so $X, \nu, x \vDash \phi$. Since Y is open we obtain $Y, \nu, x \vDash \phi$. Therefore $Y \vDash \phi$.

From right to left, suppose that such a decomposition $X = Y \cup Z$ exists. Let $x \in X$ and ν be a valuation over X.

- Suppose that $x \in Z$. Since Z is dense-in-itself we have $Z \subseteq \mathsf{d}(Z)$, so $Z \subseteq [\![\nu p.\Diamond p]\!]_{X,\nu}$. Therefore $X, \nu, x \vDash \nu p.\Diamond p$.
- Suppose that $x \in Y$. If $x \notin \mathsf{Int}(Y)$, then $x \in \mathsf{Cl}(Z)$ and since $x \notin Z$ it follows that $x \in \mathsf{d}(Z)$. We have seen that $X, \nu, z \vDash \nu p.\Diamond p$ for all $z \in Z$, so $X, \nu, x \vDash \Diamond\nu p.\Diamond p$, and then the fixpoint equation gives $X, \nu, x \vDash \nu p.\Diamond p$. Otherwise we have $x \in \mathsf{Int}(Y)$. Since $Y \vDash \phi$ and $\mathsf{Int}(Y)$ is open in Y, we have $\mathsf{Int}(Y) \vDash \phi$. Then $\mathsf{Int}(Y), \nu, x \vDash \phi$ and since $\mathsf{Int}(Y)$ is open, we finally get $X, \nu, x \vDash \phi$.

In all cases we obtain $X, \nu, x \vDash \phi \vee \nu p.\Diamond p$ and this concludes the proof. $\square$

Remark 5.24. As made clear by the proof of the left-to-right implication, we can also assume that Y is scattered and Z is perfect (i.e., closed and dense-in-itself).

From chapter 4, we know that $.2^+$ defines the class of extremally disconnected spaces. We thus obtain the following result:

Corollary 5.25. *The class of spaces that can be written as the disjoint union of an extremally disconnected subspace and a dense-in-itself subspace is not modally definable.*

We can make this example even stronger by showing that $\mu\mathbf{wK4} + .2^+ \vee \nu p.\Diamond p$ is Kripke and topologically complete. To this end, we use the canonical model of this logic and apply the technique of the final model introduced in [BBFD21]. In fact, we will be able to prove the result for any cofinal subframe logic.

Definition 5.26. Let $\mathbf{L}$ be an extension of $\mathbf{K}$. The *canonical model* of $\mathbf{L}$ is the model $\mathfrak{M} = (\Omega, R, \nu)$ with:

- Ω the set of maximal $\mathbf{L}$-consistent subsets of $\mathcal{L}$;
- $R := \{(\Gamma, \Delta) \mid \Box\phi \in \Gamma \implies \phi \in \Delta\}$;
- $\nu(p) := \{\Gamma \in \Omega \mid p \in \Gamma\}$.

The so-called *Truth Lemma* then establishes an equivalence between truth and membership at the worlds of $\mathfrak{M}$, i.e., $\mathfrak{M}, \Gamma \vDash \phi$ iff $\phi \in \Gamma$. Combined with the Lindenbaum's Lemma, this yields completeness of $\mathbf{L}$ with respect to its canonical model [BRV01, sec. 4.2]. When it comes to the μ-calculus, the canonical model is defined in the same way, but the Truth Lemma then fails to hold. In this case the technique consists in restricting oneself to an appropriate cofinal subframe (see definition 3.15). More precisely, given a consistent formula θ, one can construct a finite set of formulas Σ containing θ and with some closure properties, and a so-called Σ-*final* cofinal submodel of $\mathfrak{M}$, wherein θ is satisfiable and the Truth Lemma holds. This tool is used to prove Kripke completeness of $\mu\mathbf{wK4}$ and, in fact, of any logic of the form $\mu\mathbf{wK4} + \phi$ where $\phi \in \mathcal{L}$ and $\mathbf{wK4} + \phi$ is a *cofinal subframe* logic (see definition 3.16). Note that this result is limited to extensions of $\mu\mathbf{wK4}$ with *basic* modal axioms. The theorem we are going to prove is thus a novelty since it asserts completeness of a family of axioms involving a fixpoint operator.

Theorem 5.27. *Let ϕ be a modal formula such that $\mathbf{wK4} + \phi$ is cofinal subframe and canonical. Then $\mu\mathbf{wK4} + \phi \vee \nu p.\Diamond p$ is Kripke complete and topologically complete.*

Proof. We write $\mathbf{L} := \mu\mathbf{wK4} + \phi \vee \nu p.\Diamond p$ and $\mathbf{L}_0 := \mu\mathbf{wK4} + \phi$. Suppose that $\mathbf{L} \nvdash \theta$ and let Σ be a finite set of formulas containing ϕ and with the relevant closure properties. We introduce:

- $\mathfrak{M} = (\Omega, R, \nu)$ the canonical model of $\mathbf{L}$, and $\mathfrak{F} = (\Omega, R)$ the underlying frame;
- $\mathfrak{M}_\Sigma = (\Omega_\Sigma, R_\Sigma, \nu_\Sigma)$ the Σ-final submodel of $\mathfrak{M}$, and $\mathfrak{F}_\Sigma = (\Omega_\Sigma, R_\Sigma)$ the underlying frame;
- $\mathfrak{M}_0 = (\Omega_0, R_0, \nu_0)$ the canonical model of $\mathbf{L}_0$, and $\mathfrak{F}_0 = (\Omega_0, R_0)$ the underlying frame.

We know that $\mathfrak{F}_\Sigma$ is a cofinal subframe of $\mathfrak{F}$. In addition we have $\mathbf{L} \subseteq \mathbf{L}_0$, so for any maximal consistent set Γ such that $\mathbf{L}_0 \subseteq \Gamma$ we also have $\mathbf{L} \subseteq \Gamma$; it is also clear that R and R_0 coincide over Ω_0, so $\mathfrak{F}_0$ is a subframe of $\mathfrak{F}$. We then introduce

$$\Omega' := \{\Gamma \in \Omega_\Sigma \mid \mathfrak{M}_\Sigma, \Gamma \vDash \neg\nu p.\Diamond p\}$$

and Ω' then induces a subframe $\mathfrak{F}' = (\Omega', R')$ of $\mathfrak{F}$, and this is actually a generated subframe: indeed, if $\Gamma \in \Omega'$ and $\Gamma R_\Sigma^+ \Delta$, then since $\mathfrak{M}_\Sigma, \Gamma \vDash \neg\nu p.\Diamond p$ we have $\mathfrak{M}_\Sigma, \Delta \vDash \neg\nu p.\Diamond p$ too and thus $\Delta \in \Omega'$.

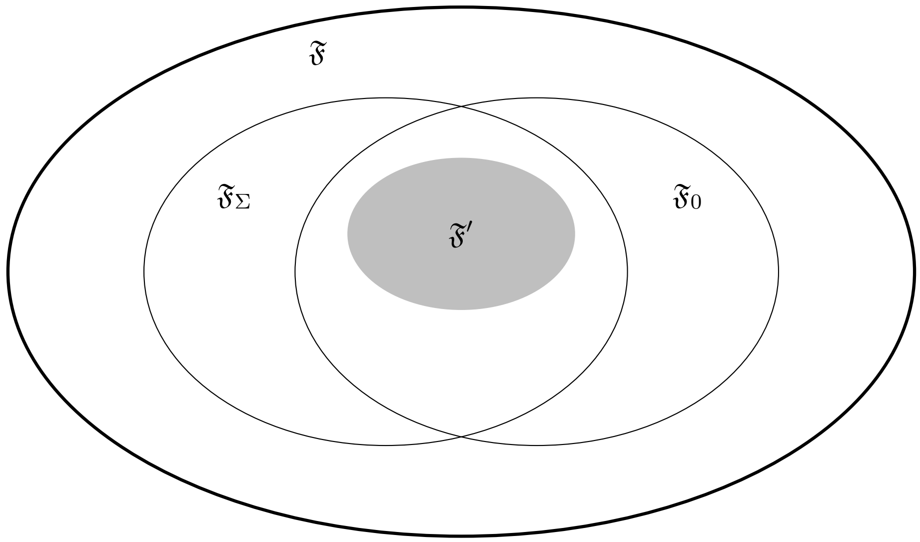


Figure 5.3: The canonical frame of $\mathbf{L}$ and its subframes

Also, if $\Gamma \in \Omega'$ then $\mathfrak{M}_\Sigma, \Gamma \vDash \neg\nu p.\Diamond p$, and we obtain $\neg\nu p.\Diamond p \in \Gamma$ by the Truth Lemma. If ϕ' is any formula obtained from ϕ by substitution, we have $\phi' \vee \nu p.\Diamond p \in \Gamma$ and $\neg\nu p.\Diamond p \in \Gamma$, so we deduce $\phi' \in \Gamma$. Therefore $\mathbf{L}_0 \subseteq \Gamma$, and we obtain $\Gamma \in \Omega_0$. This proves that $\mathfrak{F}'$ is a subframe of $\mathfrak{F}_0$.

Now, suppose $\Gamma \in \Omega'$, $\Delta \in \Omega_0$ and $\Gamma R \Delta$. Since $\mathfrak{F}_\Sigma$ is cofinal in $\mathfrak{F}$, there exists $\Lambda \in \Omega_\Sigma$ such that $\Delta R^+ \Lambda$. By weak transitivity it follows that $\Gamma R^+ \Lambda$, and since $\mathfrak{F}'$ is a generated subframe of $\mathfrak{F}_\Sigma$ it follows that $\Lambda \in \Omega'$. Therefore $\mathfrak{F}'$ is a cofinal subframe of $\mathfrak{F}_0$. Since L_0 is canonical, we have $\mathfrak{F}_0 \vDash \phi$, and since L_0 is cofinal subframe it follows that $\mathfrak{F}' \vDash \phi$ as well.

Now let $\nu_\bullet$ be a valuation over Ω_Σ and $\Gamma \in \Omega_\Sigma$.

- If $\Gamma \in \Omega'$, let $(\Omega', R', \nu'_\bullet)$ be the submodel of $(\Omega_\Sigma, R_\Sigma, \nu_\bullet)$ induced by Ω'. We know that $(\Omega', R', \nu'_\bullet), \Gamma \vDash \phi$, and since $\mathfrak{F}'$ is a generated subframe of $\mathfrak{F}_\Sigma$, it follows that $(\Omega_\Sigma, R_\Sigma, \nu_\bullet), \Gamma \vDash \phi$.

- Otherwise we have $\mathfrak{M}_\Sigma, \Gamma \vDash \nu p.\Diamond p$, but since $\nu p.\Diamond p$ contains no free variable, its truth value does not depend on the valuation ν_Σ, and thus $(\Omega_\Sigma, R_\Sigma, \nu_\bullet), \Gamma \vDash \nu p.\Diamond p$.

Therefore $\mathfrak{F}_\Sigma \vDash \phi \vee \nu p.\Diamond p$. As mentioned earlier, θ is satisfiable in $\mathfrak{M}_\Sigma$ and this proves Kripke completeness.

For topological completeness, we use again the technique of unfolding presented in chapter 4. We introduce the spaces $X := X_{\mathfrak{F}_\Sigma}$, $Y := \pi^{-1}[\Omega']$ and $Z := X \setminus Y$. We prove that Y and Z satisfy the conditions of proposition 5.23.

- We know that $\mathfrak{F}'$ is a generated subframe of $\mathfrak{F}_\Sigma$, so Ω' is open, and since π is an interior map, so is Y. As a result, it is clear that Y is homeomorphic to $X_{\mathfrak{F}'}$. In addition, since $\mathfrak{F}' \vDash \neg\nu p.\Diamond p$, the frame $\mathfrak{F}'$ is irreflexive, so $X_{\mathfrak{F}'} = \Omega' \times \{\omega\}$ is in bijection via π to Ω'. Since π is also an interior map, it follows that $X_{\mathfrak{F}'}$ is homeomorphic to $\mathfrak{F}'$; we also know that the d-semantics and the Kripke semantics coincide over $\mathfrak{F}'$, and therefore $Y \vDash \phi$.

– Suppose toward a contradiction that there exists $(\Gamma, \alpha) \in Z$ isolated in Z. Then $(\Gamma, \alpha) \in \widehat{\mathsf{d}}(X \setminus Z) = \widehat{\mathsf{d}}(\pi^{-1}[\Omega'])$, and since π is a d-morphism, it follows that $(\Gamma, \alpha) \in \pi^{-1}[\widehat{\mathsf{d}}(\Omega')]$ by theorem 2.32, that is, $\Gamma \in \widehat{\mathsf{d}}(\Omega')$. Since $(\Gamma, \alpha) \in Z$ we have $\Gamma \notin \Omega'$, so $\mathfrak{M}_\Sigma, \Gamma \vDash \nu p.\Diamond p$ and thus there exists $\Delta \in \Omega_\Sigma$ such that $\Gamma R_\Sigma \Delta$ and $\mathfrak{M}_\Sigma, \Delta \vDash \nu p.\Diamond p$. If $\Gamma = \Delta$, then Γ is reflexive, and since π is a d-morphism we obtain $(\Gamma, \alpha) \in \mathsf{d}(\pi^{-1}(\Gamma)) \subseteq \mathsf{d}(Z)$, a contradiction. Therefore $\Gamma \neq \Delta$, but since $\Gamma \in \widehat{\mathsf{d}}(\Omega')$ it follows that $\Delta \in \Omega'$, a contradiction. This proves that Z is dense-in-itself.

It follows that $X \vDash \phi \vee \nu p.\Diamond p$. We know that θ is satisfiable in $\mathfrak{F}_\Sigma$, and since π is a d-morphism it follows that θ is satisfiable in X as well. This concludes the proof. $\square$

In particular, since $\mathbf{wK4} + .2^+$ is cofinal subframe, we obtain that $\mu\mathbf{wK4} + .2^+ \vee \nu p.\Diamond p$ is Kripke and topologically complete. We thus obtain a strong non-modal axiom in support of our claim that the μ-calculus is *really* more expressive than modal logic. This closes this chapter which overall leans heavily in favour of the strength of the μ-calculus. The only exception is the proof that axioms expressed in the ν-free fragment are reducible to modal axioms, but this is not too concerning since this language is not very large.

Chapter 6

Conclusion

We presented various advances in the derivational semantics for modal logic. Our results essentially cover the semantics of several classical logics, and limitations of the expressivity of modal logic in comparison with the μ-calculus.

In chapter 3, we elucidated the derivational meaning of the axioms of bounded depth, and proved a number of useful characterizations of depth that are appropriate for the d-semantics. These results show the relevance of topological depth for the derivational framework, and will hopefully lead to the apparition of this parameter in future classifications.

In chapter 4, we presented a handful of soundness and completeness theorems for logics based on the axioms .2, .3 and many variants thereof. In fact, they are known to be related to the axioms of *bounded width* bw_n (with $n \in \mathbb{N}$) [CZ97, sec. 3.5], so in some way they talk about the width of spaces, and thus accompany very well our work on bounded depth. More precisely, .3 is merely equivalent to bw_1, so a natural line of research would be to generalize our results to bw_n in general. In addition, a key concept encountered in our work was the operation of *unfolding* a Kripke frame. Though powerful, it is also very technical: we have seen more than one way to do it, and doing it right requires precision. This demonstrates the richness of the method, and may thus be a source of inspiration to future work.

Lastly, we investigated in chapter 5 the expressive power of the topological μ-calculus. Our work finally discards the tangled closure and tangled derivative as candidates to expressive completeness, even when they are taken in tandem. We achieved this by introducing a hybrid of the two which turns out to subsume both of them, but this is not over, since we are now faced with the question of whether this new modality is expressively complete. And if not, can we strengthen it again so that to obtain an expressively complete operator? The story may go on for a long time. We also proved that many axioms of the μ-calculus cannot be reduced to basic modal axioms, and found in particular a simple and topologically complete example of such an axiom. We would like to conclude by stressing that this result is not a simple curiosity: we believe that it really legitimates the study of the topological μ-calculus and we hope that it will raise interest in the subject.

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