Translational Embeddings via Stable Canonical Rules

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

This thesis presents a new uniform method for studying modal companions of superintuitionistic deductive systems and related notions, based on the machinery of stable canonical rules. Using our method, we obtain alternative proofs of classic results in the theory of modal companions, chiefly the Blok-Esakia theorem for both logics and rule systems. We also establish several new results about modal companions, including a generalisation of the Dummett-Lemmon conjecture to rule systems and axiomatic characterisations of modal companions and superintuitionistic fragments in terms of stable canonical rules.

Because stable canonical rules may be developed for any rule system admitting filtration, our method generalises smoothly to richer signatures. We illustrate this via two case studies. Firstly, we study tense companions of bi-superintuitionistic deductive systems. Via straightforward adaptations of the techniques used in the case of modal companions, we obtain a number of new results about tense companions, including an analogue of the Blok-Esakia theorem (which was known for logics but not rule systems), an extension of the Dummett-Lemmon conjecture, and axiomatic characterisations of tense companions and bi-superintuitionistic fragments in terms of stable canonical rules.

Secondly, we study the Kuznetsov-Muravitsky isomorphism between the lattice of extensions of the modal intuitionistic logic KM and the lattice of extensions of provability logic GL. We develop a new, more flexible analogue of stable canonical rules, called *pre-stable canonical rules*, which are based on a non-standard notion of filtration appropriate for KM and GL. Following essentially the same blueprint as in previous cases, we prove an extension of the Kuznetsov-Muravitsky theorem to rule systems, which yields the latter as a corollary, and obtain new axiomatic characterisations of the underlying isomorphisms in terms of pre-stable canonical rules.

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Contents

Introduction

This thesis studies translations between deductive systems in different signatures, and the structure-preserving mappings they induce between lattices of deductive systems. It does so via new uniform techniques based on *stable canonical rules*. This brief section gives an overview of the topics to be discussed, explains what the main contributions of our work are, and gives a guide to the chapters that follow.

Modal Companions

A modal companion of a superintuitionistic logic $\mathtt{L}$ is defined as any normal modal logic $\mathtt{M}$ extending the modal logic $\mathtt{S4}$ of quasi-ordered Kripke frames, such that the *Gödel translation* fully and faithfully embeds $\mathtt{L}$ into $\mathtt{M}$. We recall that the Gödel translation, introduced by Gödel [1933], is the mapping assigning every superintuitionistic formula φ to the modal formula resulting from prefixing a box to every subformula of φ.

That the Gödel translation provides a full and faithful embedding of the intuitionistic propositional calculus $\mathtt{IPC}$ into $\mathtt{S4}$ was conjectured already by Gödel, a conjecture which was later proved by McKinsey and Tarski [1948]. Research into the general notion of a modal companion was sparked by later work due to Dummett and Lemmon [1959], who first applied the Gödel translation to arbitrary superintuitionistic logics. This research line proved remarkably prolific, as the surveys Chagrov and Zakharyaschev [1992] and Wolter and Zakharyaschev [2014] demonstrate. The jewel of this research line is the celebrated *Blok-Esakia theorem*, first proved independently by Blok [1976] via algebraic methods and by Esakia [1976] via duality-theoretic methods. The theorem states that the lattice of superintuitionistic logics is isomorphic to the lattice of normal extensions of Grzegorczyk's modal logic $\mathtt{Grz}$, via the mapping which sends each superintuitionistic logic $\mathtt{L}$ to the normal extension of $\mathtt{Grz}$ by the set of all Gödel translations of formulae in $\mathtt{L}$.

A unified approach to the theory of modal companions of superintuitionistic logics was eventually proposed by Zakharyashchev [1991], using superintuitionistic and modal *canonical formulae*. Roughly, a canonical formula is a formula whose shape syntactically encodes the structure of a finite *refutation pattern*, i.e., a finite (intuitionistic or modal) frame together with a (possibly empty) set of parameters. By applying the technique of *selective filtration*, every formula can be matched with a finite set of finite refutation patterns, in such a way that the conjunction of all the canonical formulae associated with the refutation patterns is equivalent to the original formula. Zakharyashchev obtained a number of known and novel results about modal companions by studying how the Gödel translation affects superintuitionistic canonical formulae. Among these, he confirmed the *Dummett-Lemmon conjecture*, formulated in Dummett and Lemmon [1959], which states that a superintuitionistic logic is Kripke complete iff its weakest modal companion is.

The notion of a modal companion was generalised along two main dimensions. The first dimension concerns signatures: analogues of the notion applying to logics formulated in signatures expanding that of superintuitionistic logics were developed. We mention two examples to be discussed in this thesis, although the list is by no means exhaustive.

Bi-superintuitionistic logics and tense logics. Wolter [1998] generalised the Gödel translation to the language of *bi-superintuitionistic logics*, which expands the language of superintuitionistic logics by a co-implication operator, a sort of order-dual of implication. This translation fully and faithfully embeds bi-intuitionistic logics into *tense logics*, leading to the notion of a *tense companion* of a bi-superintuitionistic logic. A counterpart of the Blok-Esakia theorem was found to hold, relating the lattice of extensions of the bi-intuitionistic propositional calculus `2IPC` to the lattice of normal extensions of `GrzT`, i.e., the least normal tense logic containing the `Grz`-axiom for both modal operators.

Modal superintuitionistic logics and classical modal logics. Earlier, Kuznetsov and Muravitsky [1986] proved that the lattice of normal extension of the intuitionistic provability logic `KM` is isomorphic to the lattice of normal extensions of the Gödel-Löb provability logic `GL`, via a Gödel-style translation. We refer to this result as the *Kuznetsov-Muravitsky theorem*, and to the relevant isomorphism as the *Kuznetsov-Muravitsky isomorphism*. Esakia [2006] introduced and studied modal companions of normal extensions of the *modalised Heyting calculus* `mIPC`, a modal superintuitionistic logic weaker than `KM`. The underlying translation is the same as Kuznetsov and Muravitsky's, and fully and faithfully embeds normal extensions of `mIPC` into normal extensions of `K4`. Esakia derived the Kuznetsov-Muravitsky theorem as a corollary, and announced without proof an analogue of the Blok-Esakia theorem, stating that the lattice of normal extensions of `mIPC` is isomorphic to the lattice of normal extensions of the weak Grzegorczyk logic `wGrz`. This was later proved by Litak [2014]. In a similar vein, Bezhanishvili [2009] studied a Gödel-style translation of *monadic superintuitionistic logics*, namely superintuitionistic logics expanded with a *universal modality*, and proved a counterpart of the Blok-Esakia theorem relating monadic superintuitionistic logics and normal modal logics above the least extension of `Grz` with the universal modality. A more general treatment of modal companions of modal superintuitionistic logics was developed by Wolter and Zakharyaschev [1998, 1997]. In their framework, the modal companions of modal superintuitionistic logics are polymodal rather than monomodal, with one modal operator interpreting the superintuitionistic implication and the others interpreting the superintuitionistic modalities. Among other results, Wolter and Zakharyaschev obtained analogues of the Blok-Esakia theorem for various types of modal superintuitionistic logics, including some non-normal ones.

The second dimension along which the notion of a modal companion was generalised concerns the type of deductive system under investigation: recently, the notion was applied to single- and multi-conclusion *rule systems* or *consequence relations*, rather than logics. Superintuitionistic and modal rule systems are decidedly understudied compared to superintuitionistic and modal logics, although this has begun to change thanks to the work of Kracht [2007]; Jerábek [2009]; Iemhoff [2016]; Bezhanishvili et al. [2016a,b] among others. The study of modal companions of superintuitionistic rule systems was initiated by Jerábek [2009]. He generalised Zakharyaschev's canonical formulae to *canonical rules*, and applied them to extend selected results in the theory of modal companions to rule systems, including an extension of the Blok-Esakia theorem. An algebraic approach generalising Blok's original proof was later pursued by Stronkowski [2018], which led to new results concerning the transfer of structural completeness between superintuitionistic rule systems and their modal companions.

The two research directions just outlined, i.e. that on modal companions of logics in richer signatures and that of modal companions of superintuitionistic rule systems, have yet to cross paths.

Stable Canonical Formulae and Rules

Stable canonical formulae and rules were recently developed in a series of papers by Guram and Nick Bezhanishvili and collaborators (see, e.g., Bezhanishvili et al. 2016a,b; Bezhanishvili and Bezhanishvili 2017) as a simple, more general alternative to Zakharyaschev and Jerabék-style canonical rules and formulae. The basic idea is the same: a stable canonical formula or rule syntactically encodes the semantic structure of a finite refutation pattern. The main difference lies in how such structure is encoded, which affects how refutation patterns are constructed in the process of rewriting a formula (or rule) into a conjunction of stable canonical formulae (or rules). Namely, in the case of stable canonical formulae and rules finite refutation patterns are constructed by taking *filtrations* rather than selective filtrations of countermodels.

Research on stable canonical rules and formulae is still in its infancy. Therefore, it is still unclear to what extent the vast body of theory established via canonical formulae and rules can be recovered or even extended to different settings using stable canonical formulae and rules. On the other hand, because filtration is a considerably more flexible technique than selective filtration, stable canonical formulae and rules present numerous advantages over canonical ones. Most notably, stable canonical rules can axiomatise every modal rule system, whereas canonical rules only axiomatise transitive rule systems. Moreover, as we shall see throughout this thesis, stable canonical rules readily generalise to richer signatures, which is not always the case for their canonical counterparts.

Our Contribution

This thesis presents and applies a new, uniform approach to the study of modal companions and notions in the vicinity. Our approach echoes Jerabék's [2009], and by extension Zakharyaschev's [1991], but employs stable canonical rules instead. We apply our approach to study the following topics:

- Modal companions of superintuitionistic rule systems and logics (Chapter 2).
- Tense companions of bi-superintuitionistic rule systems and logics (Chapter 3).
- The Kuznetsov-Muravitsky isomorphism between the lattice of normal extensions of KM and the lattice of normal extensions of GL, and its generalisation to rule systems (Chapter 4).

Our main contribution to the literature on these and related topics is methodological. We develop a new flexible technique capable of delivering central results in each of these areas in a notably uniform fashion, and with high potential for further generalisation. In pursuing this methodological goal, we obtain alternative proofs of a number of known results, including the Blok-Esakia theorem, Jerábek's [2009] and Wolter's [1998] generalisations thereof, and the Kuznetsov-Muravitsky theorem. We also obtain several new results. Most notably, we generalise Wolter's Blok-Esakia-like theorem and the Kuznetsov-Muravitsky theorem to rule systems, prove analogues of the Dummett-Lemmon conjecture for rule systems both in the superintuitionistic/modal and the bi-superintuitionistic/tense settings, and obtain several new axiomatisation results in terms of stable canonical rules for rule systems in all the signatures under discussion.

Additionally, our work also contributes to the growing literature on stable canonical rules. The first such contribution is the development of new kinds of stable (or stable-like) canonical rules: for bi-superintuitionistic and tense logics on the one hand, and (more significantly) for modal superintuitionistic rule systems over KM and modal rule systems over GL on the other. The second such contribution is that by providing uniform and flexible techniques for developing the theory of modal companions and related notions via stable canonical rules, we demonstrate how the latter are optimally suited to perform a rather significant piece of theoretical work traditionally associated with Zakharyaschev-style canonical formulae. Thus our work highlights an important aspect in which stable canonical rules are at least equally good as their canonical counterparts.

We have tried to make this thesis as self-contained as space permits. As a result, we hope that the present work can also serve the secondary purpose of providing an accessible and reasonably comprehensive introduction to the theory of modal companions and related topics for the non-expert reader.

Guide to Chapters

We give a quick roadmap of the thesis. With the exception of Chapter 1, which covers the basic technical preliminaries needed throughout the thesis, all chapters follow a common blueprint, which we now briefly describe. In each chapter we study deductive systems in a specific pair of signatures $(\imath, \nu)$: for example Chapter 2 studies superintuitionistic and modal deductive systems. Every chapter is sectioned in the following way.

1. The first section reviews chapter-specific preliminaries. These include definitions and basic facts concerning $\imath$- and ν-deductive systems, their algebraic and geometrical semantics, and some duality theory connecting the two.

2. The second section develops the theory of stable canonical rules for the signatures $\imath, \nu$.
3. The third section contains the main results of the chapter. It is organised in roughly the following way, with some variation between chapters.

    3.1 The basic transformations between algebraic and geometric models of $\imath$- and ν-deductive systems are introduced.

    3.2 The most central results concerning the topic of the chapter are proved. In the case of Chapters 2 and 3 these would be a characterisation of the set of modal (resp. tense) companions of a superintuitionistic (resp. bi-superintuitionistic) deductive system, and a Blok-Esakia theorem for the appropriate signature pair. In the case of Chapter 4 these would be the Kuznetsov-Muravitsky theorem and its generalisation to rule systems. This is the part of the chapter where we present and apply the essential components of our techniques.

    3.3 Several axiomatisation results in terms of stable canonical rules are proved. In Chapters 2 and 3 we give axiomatic characterisations of modal (resp. tense) companions and of superintuitionistic (resp. bi-superintuitionistic) fragments. In Chapter 4, we describe the maps underwriting the Kuznetsov-Muravitsky isomorphism for rule systems. We also give concrete examples illustrating these axiomatisation results.

    3.4 In Chapters 2 and 3 we present selected additional results obtained via our methods: an analogue of the Dummett-Lemmon conjecture for rule systems in the appropriate signature, a result concerning filtration, and a result concerning the preservation and reflection of stability (in the sense of Bezhanishvili et al. 2018) between superintuitionistic (resp. bi-superintuitionistic) rule systems and their modal (resp. tense) companions. This part is missing in Chapter 4, partly because–as will become clear–an analogue of the Dummett-Lemmon conjecture in the relevant setting is outside the scope of that chapter, and partly due to reasons of space.

4. The fourth and last section gives a brief review of the results obtained in the current chapter.

Methodological Note

Rule systems, not logics, are the main protagonists of this thesis. We generally apply our techniques to prove results about rule systems first, and subsequently, when possible, obtain results concerning logics as corollaries. In principle, one could follow a more lengthy approach: develop two versions of our main arguments, one using stable canonical formulae and one using stable canonical rules, so to obtain results about logics directly. With this approach one could also prove results about logics which are not present in this thesis, because not straightforwardly implied by the corresponding results about rule systems (e.g. analogues of the Dummett-Lemmon conjecture.) Nonetheless, we feel our indirect method makes more efficient use of the available space and keeps repetition to a minimum.

We also note that we focus on multiple-conclusion rule systems, not single-conclusion ones. Single-conclusion rule systems are discussed at length in Rybakov [1997]; Kracht [2007]. They do not appear here largely for reasons of space. Nonetheless, we are confident that all the results we formulate for both logics and multiple-conclusion rule systems could be extended to single-conclusion rule systems, via essentially the same procedure we use to extract results about logics from results about rule systems.

1 | General Preliminaries

This chapter fixes notational conventions and reviews the background theory needed throughout the thesis. We collect here all definitions and results which all subsequent chapters of the thesis draw on. Preliminary information specific to the topic of a particular chapter is instead presented therein.

§1.1 Functions and Relations

We begin by fixing some notation concerning functions and binary relations. If $f : X \to Y$ is a function and $U \subseteq X$ we write $f[U]$ for the set $\{f(x) : x \in U\}$. Moreover, if f is surjective we let $f^{-1} : \wp(Y) \to \wp(X)$ be the inverse of f, given by $f^{-1}(V) = \{x \in X : f(x) \in V\}$ for all $V \subseteq Y$.

Let X be a set, R a transitive binary relation on X, and $U \subseteq X$. We define:

$$qmax_R(U) := \{x \in U : \text{ for all } y \in U, \text{ if } Rxy \text{ then } Ryx\} \tag{1.1}$$
$$max_R(U) := \{x \in U : \text{ for all } y \in U, \text{ if } Rxy \text{ then } x = y\} \tag{1.2}$$
$$qmin_R(U) := \{x \in U : \text{ for all } y \in U, \text{ if } Ryx \text{ then } Rxy\} \tag{1.3}$$
$$min_R(U) := \{x \in U : \text{ for all } y \in U, \text{ if } Ryx \text{ then } x = y\}. \tag{1.4}$$

The elements of $qmax_R(U)$ and $max_R(U)$ are called *R-quasi-maximal* and *R-maximal* elements of U respectively, and similarly the elements of $qmin_R(U)$ and $min_R(U)$ are called *R-quasi-minimal* and *R-minimal* elements of U respectively. Observe that if R is a partial order then both $qmax_R(U) = max_R(U)$ and $qmin_R(U) = min_R(U)$. Lastly, we say that an element $x \in U$ is *R-passive* in U if for all $y \in X \setminus U$, if Rxy then there is no $z \in U$ such that Ryz. Intuitively, an R-passive element of U is an $x \in U$ such that one cannot "leave" and "re-enter" U starting from x and "moving through" R. The set of all R-passive elements of U is denoted by $pas_R(U)$.

§1.2 Algebraic and Topological Structures

§1.2.1 Universal Algebra

Next, we review some basic concepts and results from universal algebra. The reader may consult Burris and Sankappanavar [2012] for a more comprehensive overview of the following material.

A *signature* is a set ν of *function symbols*, where each $f \in \nu$ is assumed to come with a fixed finite *arity* determined by a map $ar : \nu \to \omega$. Throughout this subsection we let ν denote an arbitrary signature with function symbols $\nu = \{f_i : i \in I\}$. A ν-*algebra* is a tuple $\mathfrak{A} = (A, f_i^{\mathfrak{A}})_{i \in I}$, where A is a set called the *carrier* of $\mathfrak{A}$ and each $f_i^{\mathfrak{A}}$ is an n-ary operation on A for $n = ar(f_i)$. A ν-algebra whose carrier is a singleton is called *trivial*. In practice we will often find it convenient to blur the distinction between function symbols and their corresponding operations, and rely on the same notation for both when context suffices for resolving ambiguities.

We assume that the reader is familiar with the notions of homomorphism, subalgebra, congruence, quotient algebra, direct product, and ultraproduct for ν-algebras. We write $\mathfrak{A} \twoheadrightarrow \mathfrak{B}$ if $\mathfrak{B}$ is a homomorphic image of $\mathfrak{A}$, and $\mathfrak{A} \rightarrowtail \mathfrak{B}$ if $\mathfrak{A}$ is isomorphic to a subalgebra of $\mathfrak{B}$. Moreover, we denote the quotient algebra of $\mathfrak{A}$ by a congruence relation θ by $\mathfrak{A}_{/\theta}$, the direct product of the family $\{\mathfrak{A}_i : i \in I\}$ by $\prod_{i \in I} \mathfrak{A}_i$, and the ultraproduct of the family $\{\mathfrak{A}_i : i \in I\}$ by the ultrafilter U by $\left(\prod_{i \in I} \mathfrak{A}_i\right)_{/U}$. We extend the notions of homomorphic image, subalgebra, direct product and ultraproduct to class operators, using the following notation.

$$
\begin{aligned}
\mathsf{H}\mathcal{K} &:= \{\mathfrak{A} : \mathfrak{B} \twoheadrightarrow \mathfrak{A} \text{ for some } \mathfrak{B} \in \mathcal{K}\} \\
\mathsf{S}\mathcal{K} &:= \{\mathfrak{A} : \mathfrak{A} \rightarrowtail \mathfrak{B} \text{ for some } \mathfrak{B} \in \mathcal{K}\} \\
\mathsf{P}\mathcal{K} &:= \left\{\mathfrak{A} : \mathfrak{A} \cong \textstyle\prod_{i \in I} \mathfrak{A}_i \text{ for some } \{\mathfrak{A}_i : i \in I\} \subseteq \mathcal{K}\right\} \\
\mathsf{P}_{\mathsf{U}}\mathcal{K} &:= \left\{\mathfrak{A} : \mathfrak{A} \cong \left(\textstyle\prod_{i \in I} \mathfrak{A}_i\right)_{/U} \text{ for some } \{\mathfrak{A}_i : i \in I\} \subseteq \mathcal{K} \text{ and ultrafilter } U \subseteq \wp(I)\right\}
\end{aligned}
$$

The study of deductive systems via their algebraic semantics leads naturally to the study of classes of algebras closed under particular combinations of class operators. *Varieties* are perhaps the most well studied among such classes.

Definition 1.1. A class $\mathcal{V}$ of ν-algebras is a *variety* if $\mathsf{H}\mathcal{V} \subseteq \mathcal{V}$, $\mathsf{S}\mathcal{V} \subseteq \mathcal{V}$, and $\mathsf{P}\mathcal{V} \subseteq \mathcal{V}$.

If $\mathcal{K}$ is a class of ν-algebras we let $\mathsf{Var}\mathcal{K}$ be the least variety containing $\mathcal{K}$. Moreover, if $\mathcal{V}$ is a variety, then $\mathcal{K}$ is said to *generate* $\mathcal{V}$ (as a variety) if $\mathcal{V} = \mathsf{Var}\mathcal{K}$.

Theorem 1.2 (Burris and Sankappanavar 2012, Theorem II.9.5)**.** Let $\mathcal{K}$ be a class of ν-algebras. Then $\mathsf{Var}\mathcal{K} = \mathsf{HSP}\mathcal{K}$, and $\mathcal{K}$ is a variety iff $\mathcal{K} = \mathsf{HSP}\mathcal{K}$.

Another noteworthy way of grouping ν-algebras together is to form *universal classes.*

Definition 1.3. A class $\mathcal{U}$ of ν-algebras is a *universal class* if $\mathsf{S}\mathcal{U} \subseteq \mathcal{U}$ and $\mathsf{P}_{\mathsf{U}}\mathcal{U} \subseteq \mathcal{U}$.

Observe that every variety is a universal class. If $\mathcal{K}$ is a class of ν-algebras we let $\mathsf{Uni}\mathcal{K}$ be the least universal class containing $\mathcal{K}$. Moreover, if $\mathcal{U}$ is a universal class, then $\mathcal{K}$ is said to *generate* $\mathcal{U}$ (as a universal class) if $\mathcal{U} = \mathsf{Uni}\mathcal{K}$. Theorem 1.2 has the following analogue for universal classes.

Theorem 1.4 (Burris and Sankappanavar 2012, Theorem V.2.20)**.** Let $\mathcal{K}$ be a class of ν-algebras. Then $\mathsf{Uni}\mathcal{K} = \mathsf{SP}_{\mathsf{U}}\mathcal{K}$, and $\mathcal{K}$ is a universal class iff $\mathcal{K} = \mathsf{SP}_{\mathsf{U}}\mathcal{K}$.

From Theorem 1.4 it is evident that $\mathsf{Uni}\mathcal{K} \subseteq \mathsf{Var}\mathcal{K}$ for every class of ν-algebras $\mathcal{K}$.

§1.2.2 Topology

We now turn to general topological preliminaries. We assume that the reader is familiar with rudimentary notions of topology such as those of topological space, open and closed set, continuous map, basis, open cover, etc. The reader may consult Engelking [1977] for more detailed information.

We denote a topological space by $\mathfrak{X} = (X, \mathcal{O})$, where $\mathcal{O}$ is the set of open sets. We use $\mathsf{Clop}(\mathfrak{X})$ to denote the family of clopen sets of $\mathfrak{X}$. If $f : \mathfrak{X} \to \mathfrak{Y}$ is a continuous surjection between topological spaces, we say that $\mathfrak{Y}$ has the *quotient topology* if the open sets of $\mathfrak{Y}$ are exactly the sets of the form $f[U]$ where U is open in $\mathfrak{X}$.

The topological structures we deal with in this thesis are expansions of structures called *Stone spaces*.

Definition 1.5. A *Stone space* is a topological space $\mathfrak{X} = (X, \mathcal{O})$ satisfying the following conditions.

1. $\mathfrak{X}$ is compact: every open cover has a finite subcover;
2. $\mathfrak{X}$ is Hausdorff: whenever $x \neq y$ there are $U_x, U_y \in \mathcal{O}$ such that $x \in U_x, y \in U_y$ and $U_x \cap U_y = \emptyset$;
3. $\mathfrak{X}$ has a basis of clopens.

Below we list some well-known properties of Stone spaces, which will come useful later.

Proposition 1.6. Let $\mathfrak{X}$ be a Stone space. Then the following conditions hold.

1. If $x \neq y$ then there is $U \in \mathsf{Clop}(\mathfrak{X})$ such that $x \in U$ and $y \notin U$.
2. If $U \subseteq X$ is closed in $\mathfrak{X}$ then U equipped with the subspace topology is again a Stone space.
3. If U, V are closed in $\mathfrak{X}$ and $U \cap V = \emptyset$, then there are $U', V' \in \mathsf{Clop}(\mathfrak{X})$ such that $U \subseteq U', V \subseteq V'$ and $U' \cap V' = \emptyset$.

§1.2.3 Duality

The algebras we deal with in this thesis admit representation via some *order-topological duality*. By this we mean generalisations of the celebrated *Stone duality theorem*.

Theorem 1.7 (Stone duality)**.** The category of Boolean algebras with homomorphisms is dually equivalent to the category of Stone spaces with continuous maps.

We briefly recall the main ingredients of this duality, and refer the reader to Johnstone [1982, ch. 2] for a more comprehensive overview. If $\mathfrak{A}$ is a lattice, a *filter* on $\mathfrak{A}$ is a non-empty set $F \subseteq A$ such that:

- If $a \in F$ and $a \leq b$ then $b \in F$;
- If $a, b \in F$ then $a \wedge b \in F$.

If F is a filter on $\mathfrak{A}$, $F \neq A$ and, additionally, we have that whenever $a \vee b \in F$ it follows that either $a \in F$ or $b \in F$, then F is called *prime*. Now let $\mathfrak{A}$ be a Boolean algebra. Let $Spec(\mathfrak{A})$ be the set of prime filters on $\mathfrak{A}$. The *Stone map* is given by

$$\begin{aligned} \beta : A &\to \wp(Spec(\mathfrak{A})) \\ a &\mapsto \{F \in Spec(\mathfrak{A}) : a \in F\}. \end{aligned}$$

The *Stone topology* on $Spec(\mathfrak{A})$ is the topology $\mathfrak{A}_* = (Spec(\mathfrak{A}), \mathcal{O})$ obtained by taking

$$\{\beta(a) : a \in A\} \cup \{-\beta(a) : a \in A\}$$

as a basis, where $-$ denotes set-theoretic complement in X.[1] Conversely, if $\mathfrak{X}$ is a Stone space we define its dual Boolean algebra as $\mathfrak{X}^* := (\mathsf{Clop}(\mathfrak{X}), \cap, \cup, -, \emptyset, X)$. Then one can prove that for all Boolean algebras $\mathfrak{A}$ we have $\mathfrak{A} \cong \mathfrak{A}_*{}^*$, and that for all Stone spaces $\mathfrak{X}$ we have $\mathfrak{X} \cong \mathfrak{X}^*{}_*$. As for morphisms, we have that a map $h : \mathfrak{A} \to \mathfrak{B}$ is a Boolean homomorphism iff $h^{-1} : \mathfrak{B}_* \to \mathfrak{A}_*$ is a continuous map, and that a map $f : \mathfrak{X} \to \mathfrak{Y}$ is continuous iff $f^{-1} : \mathfrak{Y}^* \to \mathfrak{X}^*$ is a Boolean homomorphism.

§1.3 Deductive Systems

This section covers *deductive systems*, which span both propositional logics and the lesser known rule systems, or multi-conclusion consequence relations. We review their syntax and explain how to interpret them over classes of algebraic or topological structures. The reader may consult Iemhoff [2016] for more information on deductive systems (especially rule systems) in general.

§1.3.1 Syntax

The set $Frm_\nu(X)$ of *formulae* in signature ν over a set of variables X is the least set containing X and such that for every $f \in \nu$ and $\varphi_1, \dots, \varphi_n \in Frm_\nu(X)$ with $n = ar(f)$ we have $f(\varphi_1, \dots, \varphi_n) \in Frm_\nu(X)$. Henceforth we will take $Prop$ to be a fixed arbitrary countably infinite set of variables and write simply Frm_ν for $Frm_\nu(Prop)$. If φ is a formula we write $Sfor(\varphi)$ for the set of subformulae of φ. We occasionally write formulae in the form $\varphi(p_1, \dots, p_n)$ to indicate that the variables occurring in φ are among $p_1, \dots, p_n$. A *substitution* is a map $s : Prop \to Frm_\nu(Prop)$. Every substitution may be extended to a map $\bar{s} : Frm_\nu(Prop) \to Frm_\nu(Prop)$ recursively, by setting $\bar{s}(p) = s(p)$ if $p \in Prop$, and $\bar{s}(f(\varphi_1, \dots, \varphi_n)) = f(\bar{s}(\varphi_1), \dots, \bar{s}(\varphi_n))$.

Definition 1.8. A *logic* over Frm_ν is a set $\mathtt{L} \subseteq Frm_\nu$, such that

$$\varphi \in \mathtt{L} \Rightarrow \bar{s}(\varphi) \in \mathtt{L} \text{ for every substitution } s \qquad \text{(structurality)}$$

[1] In fact $\{\beta(a) : a \in A\}$ gives the same basis because $\beta(\neg a) = -\beta(a)$. We define the Stone topology this way so to use the same definition for all dualities here considered, some of which involve structures where the latter identity fails.

Interesting examples of logics, such as those this thesis deals with, are normally closed under conditions other than structurality. If Γ, Δ are sets of formulae and $\mathscr{S}$ is a set of logics, we write $\Gamma \oplus_{\mathscr{S}} \Delta$ for the least logic in $\mathscr{S}$ extending both Γ, Δ.

For any sets X, Y, write $X \subseteq_\omega Y$ to mean that $X \subseteq Y$ and $|X|$ is finite. A *(multi-conclusion) rule* in signature ν over a set of variables X is a pair (Γ, Δ) such that $\Gamma, \Delta \subseteq_\omega Frm_\nu(X)$. In case $\Delta = \{\varphi\}$ we write Γ/Δ simply as Γ/φ, and analogously if $\Gamma = \{\psi\}$. We use ; to denote union between finite sets of formulae, so that $\Gamma; \Delta = \Gamma \cup \Delta$ and $\Gamma; \varphi = \Gamma \cup \{\varphi\}$. We write $Rul_\nu(X)$ for the set of all rules in ν over X, and simply Rul_ν when $X = Prop$. If Γ/Δ is a rule we write $Sfor(\Gamma/\Delta)$ for the set of all formulae which are subformulae of some $\xi \in \Gamma \cup \Delta$.

Definition 1.9. A *rule system* or *multi-conclusion consequence relation* is a set $\mathtt{S} \subseteq Rul_\nu(X)$ satisfying the following conditions.

1. If $\Gamma/\Delta \in \mathtt{S}$ then $\bar{s}[\Gamma]/\bar{s}[\Delta] \in \mathtt{S}$ for all substitutions s (structurality).
2. $\varphi/\varphi \in \mathtt{S}$ for every formula φ (reflexivity).
3. If $\Gamma/\Delta \in \mathtt{S}$ then $\Gamma; \Gamma'/\Delta; \Delta' \in \mathtt{S}$ for any finite sets of formulae Γ', Δ' (monotonicity).
4. If $\Gamma/\Delta; \varphi \in \mathtt{S}$ and $\Gamma; \varphi/\Delta \in \mathtt{S}$ then $\Gamma/\Delta \in \mathtt{S}$ (cut).

If $\mathscr{S}$ is a set of rule systems and Σ, Ξ are sets of rules, we write $\Xi \oplus_{\mathscr{S}} \Sigma$ for the least rule system in $\mathscr{S}$ extending both Ξ and Σ. A set of rules Σ is said to *axiomatise* a rule system $\mathtt{S} \in \mathscr{S}$ *over* some rule system $\mathtt{S}' \in \mathscr{S}$ if $\mathtt{S}' \oplus_{\mathscr{S}} \Sigma = \mathtt{S}$.

If $\mathtt{S}$ is a rule system we let the set of *tautologies* of $\mathtt{S}$ be the set

$$\mathsf{Taut}(\mathtt{S}) := \{\varphi \in Frm_\nu : /\varphi \in \mathtt{S}\}.$$

By the structurality condition for rule systems, it follows that $\mathsf{Taut}(\mathtt{S})$ is a logic for every rule system $\mathtt{S}$.

§1.3.2 Semantics

We interpret deductive systems over algebras in the same signature. Let $\mathfrak{A}$ be some ν-algebra. A *valuation* on $\mathfrak{A}$ is a map $V : Prop \to A$. Every valuation V on $\mathfrak{A}$ may be recursively extended to a map $\bar{V} : Frm_\nu \to A$, by setting

$$\bar{V}(p) := V(p)$$
$$\bar{V}(f(\varphi_1, \dots, \varphi_n)) := f^{\mathfrak{A}}(\bar{V}(\varphi_1), \dots, \bar{V}(\varphi_n)).$$

A pair $(\mathfrak{A}, V)$ where $\mathfrak{A}$ is a ν-algebra and V a valuation on $\mathfrak{A}$ is called a *model*. A rule Γ/Δ is *valid* on a ν-algebra $\mathfrak{A}$ if the following holds: for any valuation V on $\mathfrak{A}$, if $\bar{V}(\gamma) = 1$ for all $\gamma \in \Gamma$, then $\bar{V}(\delta) = 1$ for some $\delta \in \Delta$. When this holds we write $\mathfrak{A} \models \Gamma/\Delta$, otherwise we write $\mathfrak{A} \not\models \Gamma/\Delta$ and say that $\mathfrak{A}$ *refutes* Γ/Δ. As a special case, a formula φ is valid on a ν-algebra $\mathfrak{A}$ if the rule $/\varphi$ is. We write $\mathfrak{A} \models \varphi$ when this holds, $\mathfrak{A} \not\models \varphi$ otherwise. The notion of validity extends to classes of ν-algebras: $\mathscr{K} \models \Gamma/\Delta$ means that $\mathfrak{A} \models \Gamma/\Delta$ for every $\mathfrak{A} \in \mathscr{K}$, and $\mathscr{K} \not\models \Gamma/\Delta$ means that $\mathfrak{A} \not\models \Gamma/\Delta$ for some $\mathfrak{A} \in \mathscr{K}$. Analogous notation is used for formulae. Finally, if Ξ is a set of formulae or rules and $\mathfrak{A}$ a ν-algebra, $\mathfrak{A} \models \Xi$ means that every formula or rule in Ξ is valid on $\mathfrak{A}$, $\mathfrak{A} \not\models \Xi$ means that some formula or rule in Ξ is not valid on $\mathfrak{A}$, and similarly for classes of ν-algebras.

Write $\mathscr{A}_\nu$ for the class of all ν-algebras. For every deductive system S we define

$$\mathsf{Alg}(\mathsf{S}) := \{\mathfrak{A} \in \mathscr{A}_\nu : \mathfrak{A} \models \mathsf{S}\}.$$

Conversely, if $\mathscr{K}$ is a class of ν-algebras we set

$$\mathsf{ThR}(\mathscr{K}) := \{\Gamma/\Delta \in Rul_\nu : \mathscr{K} \models \Gamma/\Delta\}$$
$$\mathsf{Th}(\mathscr{K}) := \{\varphi \in Frm_\nu : \mathscr{K} \models \varphi\}$$

We also interpret deductive systems over ν-formulae on expansions of Stone spaces dual to ν-algebras, which for the moment we refer to as *ν-spaces*. Precise definitions of these topological sturctures are given in each subsequent chapter. A valuation on a ν-space $\mathfrak{X}$ is a map $V : Prop \to \mathsf{Clop}(\mathfrak{X})$. When working with certain types of Stone space expansions we impose more restrictive conditions on valuations. Every valuation V extends to a full truth function $\bar{V} : Frm_\nu \to \mathsf{Clop}(\mathfrak{X})$ in a unique way, although the exact details on how to do so vary depending on the signature. Given a valuation V on a ν-space $\mathfrak{X}$ and a point $x \in X$, we call $(\mathfrak{X}, V)$ a (global) *model*. A formula φ is *satisfied* on a model $(\mathfrak{X}, V)$ at a point x if $x \in \bar{V}(\varphi)$. In this case we write $\mathfrak{X}, V, x \models \varphi$, otherwise we write $\mathfrak{X}, V, x \not\models \varphi$ and say that the model $(\mathfrak{X}, V)$ *refutes* φ at a point x. A rule Γ/Δ is *valid* on a model $(\mathfrak{X}, V)$ if the following holds: if for every $x \in X$ we have $\mathfrak{X}, V, x \models \gamma$ for each $\gamma \in \Gamma$, then for every $x \in X$ we have $\mathfrak{X}, V, x \models \delta$ for some $\delta \in \Delta$. In this case we write $\mathfrak{X}, V \models \Gamma/\Delta$, otherwise we write $\mathfrak{X}, V \not\models \Gamma/\Delta$ and say that the model $(\mathfrak{X}, V)$ *refutes* φ. A rule Γ/Δ is *valid* on a ν-space $\mathfrak{X}$ if it is valid on the model $(\mathfrak{X}, V)$ for every valuation V on $\mathfrak{X}$, otherwise $\mathfrak{X}$ *refutes* Γ/Δ. We write $\mathfrak{X} \models \Gamma/\Delta$ to mean that Γ/Δ is valid on $\mathfrak{X}$, and $\mathfrak{X} \not\models \Gamma/\Delta$ to mean that $\mathfrak{X}$ refutes Γ/Δ. As in the algebraic case we define validity on models and ν-spaces for a formula φ as validity of the rule $/\varphi$, and write $\mathfrak{X} \models \varphi$ if φ is valid in $\mathfrak{X}$, otherwise $\mathfrak{X} \not\models \varphi$. The notion of validity generalises to classes of ν-spaces, so that if $\mathscr{K}$ is a class of ν-space then $\mathscr{K} \models \Gamma/\Delta$ means $\mathfrak{X} \models \Gamma/\Delta$ for every $\mathfrak{X} \in \mathscr{K}$, and $\mathscr{K} \not\models \Gamma/\Delta$ means $\mathfrak{X} \not\models \Gamma/\Delta$ for some $\mathfrak{X} \in \mathscr{K}$. We extend the present notation for validity to sets of formulae or rules the same way as for algebras.

Write $\mathscr{S}_\nu$ for the class of all ν-spaces. For every deductive system S we define

$$\mathsf{Spa}(\mathsf{S}) := \{\mathfrak{X} \in \mathscr{S}_\nu : \mathfrak{X} \models \mathsf{S}\}.$$

Conversely, if $\mathscr{K}$ is a class of ν-spaces we set

$$\mathsf{ThR}(\mathscr{K}) := \{\Gamma/\Delta \in Rul_\nu : \mathscr{K} \models \Gamma/\Delta\}$$
$$\mathsf{Th}(\mathscr{K}) := \{\varphi \in Frm_\nu : \mathscr{K} \models \varphi\}$$

Throughout the thesis we study the structure of lattices of deductive systems via semantic methods. This is made possible by the following fundamental result, connecting the syntactic types of deductive systems to closure conditions on the classes of algebras validating them. Item 1 is widely known as *Birkhoff's theorem*, after Birkhoff [1935].

Theorem 1.10. For every class $\mathcal{K}$ of ν-algebras, the following conditions hold:

1. $\mathcal{K}$ is a variety iff $\mathcal{K} = \mathsf{Alg}(\mathtt{S})$ for some set of ν-formulae $\mathtt{S}$ Burris and Sankappanavar [2012, Theorem II.11.9];
2. $\mathcal{K}$ is a universal class iff $\mathcal{K} = \mathsf{Alg}(\mathtt{S})$ for some set of ν-rules $\mathtt{S}$ Burris and Sankappanavar [2012, Theorem V.2.20].

In this sense, ν-logics correspond to varieties of ν-algebras, whereas ν-rule systems correspond to universal classes of ν-algebras.

This concludes our general preliminaries. We now begin the study of modal companions via stable canonical rules.

2 | Modal Companions of Superintuitionistic Deductive Systems

This chapter studies the theory of modal companions of superintuitionistic deductive systems via stable canonical rules. Its main purpose is to present our method in detail and show that it performs as expected. After some brief preliminaries (§ 2.1), we present superintuitionistic and modal stable canonical rules (§ 2.2). The main results of this chapter are included in § 2.3 and § 2.3.3. The former uses stable canonical rules to give a characterisation of the set of modal companions of a superintuitionistic deductive system, and proves the Blok-Esakia theorem for both logics and rule systems. The latter provides several new axiomatic characterisations of modal companions and si-fragments in terms of stable canonical rules. The chapter concludes with a collection of further miscellaneous results about modal companions obtained via our methods (§ 2.3.4), including a version of the Dummett-Lemmon conjecture for rule systems.

§2.1 Preliminaries

This section gives a brief overview of the semantic and syntactic structures discussed throughout the present chapter.

§2.1.1 Superintuitionistic Deductive Systems, Heyting Algebras, and Esakia Spaces

We work with the *superintuitionistic signature*,

$$si := \{\wedge, \vee, \to, \bot, \top\}.$$

The set Frm_{si} of superintuitionistic (si) formulae is defined recursively as follows.

$$\varphi ::= p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi.$$

We abbreviate $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$. We let $\mathtt{IPC}$ denote the *intuitionistic propositional calculus*, and point the reader to Chagrov and Zakharyaschev [1997, Ch. 2] for an axiomatisation.

Definition 2.1. A *superintuitionistic logic*, or si-logic for short, is a logic $\mathtt{L}$ over Frm_{si} satisfying the following additional conditions:

1. $\mathtt{IPC} \subseteq \mathtt{L}$;
2. $\varphi \to \psi, \varphi \in \mathtt{L}$ implies $\psi \in \mathtt{L}$ (MP).

A *superintuitionistic rule system*, or si-rule system for short, is a rule system $\mathtt{L}$ over Frm_{si} satisfying the following additional requirements.

1. $/\varphi \in \mathtt{L}$ whenever $\varphi \in \mathtt{IPC}$.
2. $\varphi, \varphi \to \psi / \psi \in \mathtt{L}$ (MP-R).

For every si-logic $\mathtt{L}$ write $\mathbf{Ext}(\mathtt{L})$ for the set of si-logics extending $\mathtt{L}$, and similarly for si-rule systems. Then $\mathbf{Ext}(\mathtt{IPC})$ is the set of all si-logics. It is well known that $\mathbf{Ext}(\mathtt{IPC})$ admits the structure of a complete lattice, with $\oplus_{\mathbf{Ext}(\mathtt{IPC})}$ serving as join and intersection as meet. Clearly, for every $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC})$ there exists a least si-rule system $\mathtt{L}_\mathtt{R}$ containing $/\varphi$ for each $\varphi \in \mathtt{L}$. Hence $\mathtt{IPC}_\mathtt{R}$ is the least rule system. The set $\mathbf{Ext}(\mathtt{IPC}_\mathtt{R})$ is also a lattice when endowed with $\oplus_{\mathbf{Ext}(\mathtt{IPC}_\mathtt{R})}$ as join and intersection as meet. Slightly abusing notation, we refer to these lattices as we refer to their underlying sets, i.e., $\mathbf{Ext}(\mathtt{IPC})$ and $\mathbf{Ext}(\mathtt{IPC}_\mathtt{R})$ respectively. Additionally, we make use of systematic ambiguity and write both $\oplus_{\mathbf{Ext}(\mathtt{IPC})}$ and $\oplus_{\mathbf{Ext}(\mathtt{IPC}_\mathtt{R})}$ simply as $\oplus$, leaving context to clarify which operation is meant.

The following proposition is central for transferring results about si-rule systems to si-logics. Its proof is routine.

Proposition 2.2. The mappings $(\cdot)_\mathtt{R}$ and $\mathsf{Taut}(\cdot)$ are mutually inverse complete lattice isomorphisms between $\mathbf{Ext}(\mathtt{IPC})$ and the sublattice of $\mathbf{Ext}(\mathtt{IPC}_\mathtt{R})$ consisting of all si-rule systems $\mathtt{L}$ such that $\mathsf{Taut}(\mathtt{L})_\mathtt{R} = \mathtt{L}$.

The algebraic models for si deductive systems are *Heyting algebras*.

Definition 2.3. A *Heyting algebra* is a tuple $\mathfrak{H} = (H, \wedge, \vee, \to, 0, 1)$ such that $(H, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and for every $a, b, c \in A$ we have

$$c \leq a \to b \iff a \wedge c \leq b.$$

We let HA denote the class of all Heyting algebras. It is well known that every Heyting algebra satisfies the condition

$$a \to b = \bigvee\{c \in A : a \wedge c \leq b\}$$

for all $a, b \in H$. In case a Heyting algebra $\mathfrak{H}$ is such that $a \vee \neg a = 1$ for every $a \in H$, then $\mathfrak{H}$ is a Boolean algebra with $\neg a := a \to 0$.

It is well known that HA is a variety. If $\mathcal{V} \subseteq \mathsf{HA}$ is a variety (resp: universal class) we write $\mathbf{Var}(\mathcal{V})$ and $\mathbf{Uni}(\mathcal{V})$ respectively for the lattice of subvarieties (resp: of universal subclasses) of $\mathcal{V}$. The connections between $\mathbf{Ext}(\mathtt{IPC})$ and $\mathbf{Var}(\mathsf{HA})$ on the one hand, and between $\mathbf{Ext}(\mathtt{IPC}_\mathtt{R})$ and $\mathbf{Uni}(\mathsf{HA})$ on the other, are as intimate as they come.

Theorem 2.4. The following maps are pairs of mutually inverse dual isomorphisms:

1. $\mathsf{Alg} : \mathbf{Ext}(\mathtt{IPC}) \to \mathbf{Var}(\mathsf{HA})$ and $\mathsf{Th} : \mathbf{Var}(\mathsf{HA}) \to \mathbf{Ext}(\mathtt{IPC})$;
2. $\mathsf{Alg} : \mathbf{Ext}(\mathtt{IPC}_\mathtt{R}) \to \mathbf{Uni}(\mathsf{HA})$ and $\mathsf{ThR} : \mathbf{Uni}(\mathsf{HA}) \to \mathbf{Ext}(\mathtt{IPC}_\mathtt{R})$.

Item 1 is proved in Chagrov and Zakharyaschev [1997, Theorem 7.56], whereas Item 2 follows from Jerábek [2009, Theorem 2.2] by standard techniques.

The geometrical semantics of si-rule systems is based on order-topological structures known as *Esakia spaces*.

Definition 2.5. An *Esakia space* is a tuple $\mathfrak{X} = (X, \leq, \mathcal{O})$ such that $(X, \mathcal{O})$ is a Stone space, $\leq$ is a partial order on X, and

1. ${\uparrow}x := \{y \in X : x \leq y\}$ is closed for every $x \in X$;
2. ${\downarrow}U := \{x \in X : {\uparrow}x \cap U \neq \emptyset\} \in \mathsf{Clop}(\mathfrak{X})$ for every $U \in \mathsf{Clop}(\mathfrak{X})$.

We let Esa denote the class of all Esakia spaces. If $\mathfrak{X}$ is an Esakia space, a $\leq$-closed subset of X is called an *upset*. We let $\mathsf{ClopUp}(\mathfrak{X})$ denote the set of clopen upsets in $\mathfrak{X}$. If $\mathfrak{X}, \mathfrak{Y}$ are Esakia spaces, a map $f : \mathfrak{X} \to \mathfrak{Y}$ is called a *bounded morphism* if for all $x, y \in X$ we have that $x \leq y$ implies $f(x) \leq f(y)$, and $h(x) \leq y$ implies that there is $z \in X$ with $x \leq z$ and $h(z) = y$.

The following result recalls some important properties of Esakia spaces, used throughout the thesis. For proofs the reader may consult Esakia [2019, Lemma 3.1.5, Theorem 3.2.1]. We note that we require valuations on Esakia spaces to range over clopen upsets rather than just clopen sets.

Proposition 2.6. Let $\mathfrak{X} \in \mathsf{Esa}$. Then for all $x, y \in X$ we have:

1. If $x \nleq y$ then there is $U \in \mathsf{ClopUp}(\mathfrak{X})$ such that $x \in U$ and $y \notin U$;
2. For all $U \in \mathsf{Clop}(U)$ and $x \in U$, there is $y \in max_{\leq}(U)$ such that $x \leq y$.

The following definition spells out how to interpret si-rule systems over Esakia spaces and classes thereof.

Definition 2.7. Let $\mathfrak{X}$ be an Esakia space, V a valuation on $\mathfrak{X}$ and $x \in X$. The *satisfaction relation* of the model $(\mathfrak{X}, V, x)$ is defined recursively as follows:

$$\begin{array}{ll}
\mathfrak{X}, V, x \models \bot & \text{never;} \\
\mathfrak{X}, V, x \models \top & \text{always;} \\
\mathfrak{X}, V, x \models p & :\iff x \in V(p); \\
\mathfrak{X}, V, x \models \varphi \wedge \psi & :\iff \mathfrak{X}, V, x \models \varphi \text{ and } \mathfrak{X}, V, x \models \psi; \\
\mathfrak{X}, V, x \models \varphi \vee \psi & :\iff \mathfrak{X}, V, x \models \varphi \text{ or } \mathfrak{X}, V, x \models \psi; \\
\mathfrak{X}, V, x \models \varphi \to \psi & :\iff \text{for all } y \in {\uparrow}x, \text{ if } \mathfrak{X}, V, x \models \varphi \text{ then } \mathfrak{X}, V, x \models \psi;
\end{array}$$

Heyting algebras and Esakia spaces are intimately related by an order-topological duality, established by Esakia [1974].

Theorem 2.8 (Esakia duality)**.** The category of Heyting algebras with homomorphisms is dually equivalent to the category of Esakia spaces with continuous bounded morphisms.

Proof sketch. The reader may consult Esakia [2019, §3.4] for a modern detailed proof of this result. For objects, the duality works as follows. Given a Heyting algebra $\mathfrak{H} \in \mathsf{HA}$, we construct its dual Esakia space

$$\mathfrak{H}_* = (Spec(\mathfrak{H}), \mathcal{O}, \leq),$$

where $\mathcal{O}$ is the Stone topology (as defined in § 1.2.3) and for every $x, y \in Spec(\mathfrak{H})$

$$x \leq y \iff x \subseteq y.$$

Conversely, given an Esakia space $\mathfrak{X} \in \mathsf{Esa}$ its dual Heyting algebra is

$$\mathfrak{X}^* = (\mathsf{ClopUp}(\mathfrak{X}), \cap, \cup, \to_{\leq}, X, \emptyset),$$

where for all $U, V \in \mathsf{ClopUp}(\mathfrak{X})$ we have

$$U \to_{\leq} V := -{\downarrow}(U \setminus V).$$

Then one can prove that for every $\mathfrak{H} \in \mathsf{HA}$ we have $\mathfrak{H} \cong \mathfrak{H}_*{}^*$, and moreover that for every $\mathfrak{X} \in \mathsf{Esa}$ we have $\mathfrak{X} \cong \mathfrak{X}^*{}_*$. As for morphisms, we have that $h : \mathfrak{H} \to \mathfrak{K}$ is a homomorphism iff $h^{-1} : \mathfrak{K}_* \to \mathfrak{H}_*$ is a continuous bounded morphism, and likewise $f : \mathfrak{X} \to \mathfrak{Y}$ is a continuous bounded morphism iff $f^{-1} : \mathfrak{Y}^* \to \mathfrak{X}^*$ is a homomorphism. □

§2.1.2 Modal Deductive Systems, Modal Algebras, and Modal Spaces

We shall now work in the *modal signature*,

$$md := \{\wedge, \vee, \neg, \Box, \bot, \top\}.$$

The set Frm_{md} of modal formulae is defined recursively as follows.

$$\varphi ::= p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \neg\varphi \mid \Box\varphi.$$

As usual we abbreviate $\Diamond\varphi := \neg\Box\neg\varphi$. Further, we let $\varphi \to \psi := \neg\varphi \vee \psi$ and $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$.

Definition 2.9. A *normal modal logic*, henceforth simply *modal logic*, is a logic $\mathtt{M}$ over Frm_{md} satisfying the following conditions:

1. $\mathtt{CPC} \subseteq \mathtt{M}$, where $\mathtt{CPC}$ is the classical propositional calculus;
2. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \in \mathtt{M}$;
3. $\varphi \to \psi, \varphi \in \mathtt{M}$ implies $\psi \in \mathtt{M}$ (MP);
4. $\varphi \in \mathtt{M}$ implies $\Box\varphi \in \mathtt{M}$ (NEC).

We denote the least modal logic by $\mathtt{K}$. A *normal modal rule system*, henceforth simply *modal rule system*, is a rule system $\mathtt{M}$ over Frm_{md}, satisfying the following additional requirements:

1. $/\varphi \in \mathtt{M}$ whenever $\varphi \in \mathtt{K}$;
2. $\varphi \to \psi, \varphi/\psi \in \mathtt{M}$ (MP-R);
3. $\varphi/\Box\varphi \in \mathtt{M}$ (NEC-R).

If $\mathtt{M}$ is a modal logic let $\mathbf{NExt}(\mathtt{M})$ be the set of modal logics extending $\mathtt{M}$, and similarly for modal rule systems. Obviously, the set of modal logics coincides with $\mathbf{NExt}(\mathtt{K})$. It is well known that $\mathbf{NExt}(\mathtt{K})$ forms a lattice under the operations $\oplus_{\mathbf{NExt}(\mathtt{K})}$ as join and intersection as meet. Clearly, for each $\mathtt{M} \in \mathbf{NExt}(\mathtt{K})$ there is always a least modal rule system $\mathtt{K_R}$ containing $/\varphi$ for each $\varphi \in \mathtt{M}$. Therefore, $\mathtt{K_R}$ is the least modal rule system. The set $\mathbf{NExt}(\mathtt{K_R})$ is also a lattice when endowed with $\oplus_{\mathbf{NExt}(\mathtt{K_R})}$ as join and intersection as meet. With slight abuse of notation, we refer to these lattices as we refer to their underlying sets, i.e. $\mathbf{NExt}(\mathtt{K})$ and $\mathbf{NExt}(\mathtt{K_R})$ respectively. Additionally, we make use of systematic ambiguity and write both $\oplus_{\mathbf{NExt}(\mathtt{K})}$ and $\oplus_{\mathbf{NExt}(\mathtt{K_R})}$ simply as $\oplus$, leaving context to clarify which operation is meant.

We have a modal counterpart of Proposition 2.2.

Proposition 2.10. The mappings $(\cdot)_\mathtt{R}$ and $\mathsf{Taut}(\cdot)$ are mutually inverse complete lattice isomorphisms between $\mathbf{NExt}(\mathtt{K})$ and the sublattice of $\mathbf{NExt}(\mathtt{K_R})$ consisting of all si-rule systems $\mathtt{M}$ such that $\mathsf{Taut}(\mathtt{M})_\mathtt{R} = \mathtt{M}$.

Algebraically, modal logics and modal rule systems are interpreted on *modal algebras*.

Definition 2.11. A *modal algebra* is a tuple $\mathfrak{A} = (A, \wedge, \vee, \neg, \Box, 0, 1)$ such that $(A, \wedge, \vee, \neg, 0, 1)$ is a Boolean algebra and the following equations hold:

$$\Box 1 = 1 \tag{2.1}$$

$$\Box(a \wedge b) = \Box a \wedge \Box b \tag{2.2}$$

We let MA denote the class of all modal algebras. By Theorem 1.10, MA is a variety. We let $\mathbf{Var}(\mathsf{MA})$ and $\mathbf{Uni}(\mathsf{MA})$ be the lattice of subvarieties and the lattice of universal subclasses of MA respectively. We have the following analogue of Theorem 2.4.

Theorem 2.12. The following maps are pairs of mutually inverse dual isomorphisms:

1. $\mathsf{Alg} : \mathbf{NExt}(\mathtt{K}) \to \mathbf{Var}(\mathsf{MA})$ and $\mathsf{Th} : \mathbf{Var}(\mathsf{MA}) \to \mathbf{NExt}(\mathtt{K})$;
2. $\mathsf{Alg} : \mathbf{NExt}(\mathtt{K_R}) \to \mathbf{Uni}(\mathsf{MA})$ and $\mathsf{ThR} : \mathbf{Uni}(\mathsf{MA}) \to \mathbf{NExt}(\mathtt{K_R})$.

Item 2 is proved in Chagrov and Zakharyaschev [1997, Theorem 7.56], whereas Item 2 follows from Bezhanishvili and Ghilardi [2014, Theorem 2.5].

Modal rule systems may also be given geometrical semantics. The central notion here is that of a *modal space*.

Definition 2.13. A *modal space* is a tuple $\mathfrak{X} = (X, R, \mathcal{O})$, such that $(X, \mathcal{O})$ is a Stone space, $R \subseteq X \times X$ is a binary relation, and

1. $R[x] := \{y \in X : Rxy\}$ is closed for every $x \in X$;

2. $R^{-1}(U) := \{x \in X : R[x] \cap U \neq \emptyset\} \in \mathsf{Clop}(\mathfrak{X})$ for every $U \in \mathsf{Clop}(\mathfrak{X})$.

We let Mod denote the class of all modal spaces. If $\mathfrak{X}, \mathfrak{Y}$ are modal spaces, a map $f : \mathfrak{X} \to \mathfrak{Y}$ is called a *bounded morphism* when for all $x, y \in X$, if Rxy then $Rf(x)f(y)$, and $Rf(x)y$ implies that there is $z \in X$ with Rxz and $f(z) = y$.

The following definition spells out how to interpret modal formulae over modal spaces and classes thereof.

Definition 2.14. Let $\mathfrak{X}$ be an modal space, V a valuation on $\mathfrak{X}$ and $x \in X$. The *satisfaction relation* of the model $(\mathfrak{X}, V, x)$ is defined recursively as follows:

$$
\begin{array}{lll}
\mathfrak{X}, V, x \models \bot & & \text{never;} \\
\mathfrak{X}, V, x \models \top & & \text{always;} \\
\mathfrak{X}, V, x \models p & :\iff & x \in V(p); \\
\mathfrak{X}, V, x \models \neg\varphi & :\iff & \mathfrak{X}, V, x \not\models \varphi; \\
\mathfrak{X}, V, x \models \varphi \wedge \psi & :\iff & \mathfrak{X}, V, x \models \varphi \text{ and } \mathfrak{X}, V, x \models \psi; \\
\mathfrak{X}, V, x \models \varphi \vee \psi; & :\iff & \mathfrak{X}, V, x \models \varphi \text{ or } \mathfrak{X}, V, x \models \psi; \\
\mathfrak{X}, V, x \models \Box\varphi & :\iff & \mathfrak{X}, V, y \models \varphi \text{ for all } y \in R[x].
\end{array}
$$

Thus, clearly,

$$\mathfrak{X}, V, x \models \Diamond\varphi :\iff \mathfrak{X}, V, y \models \varphi \text{ for some } y \in R[x].$$

Stone duality between Boolean algebras and Stone spaces generalises smoothly to a topological duality relating modal algebras to modal spaces.

Theorem 2.15 (Modal duality)**.** The category of modal algebras with homomorphisms is dually equivalent to the category of modal spaces with continuous bounded morphisms.

Proof sketch. A detailed proof can be found in Sambin and Vaccaro [1988, §3, §4]. For objects, the duality works as follows. Given a modal algebra $\mathfrak{A} \in \mathsf{MA}$, we construct its dual modal space

$$\mathfrak{A}_* = (Spec(\mathfrak{A}), \mathcal{O}, R_\Box),$$

where $\mathcal{O}$ is the Stone topology (as defined in § 1.2.3) and for every $x, y \in Spec(\mathfrak{A})$

$$R_\Box xy \iff \text{for all } a \in A : \Box a \in x \text{ implies } a \in y.$$

Conversely, given a modal space $\mathfrak{X} \in \mathsf{Mod}$ its dual modal algebra is

$$\mathfrak{X}^* = (\mathsf{Clop}(\mathfrak{X}), \cap, \cup, \Box_R, X, \emptyset)$$

where for all $U \in \mathsf{Clop}(\mathfrak{X})$ we have

$$\Box_R U := \{x \in X : R[x] \subseteq U\}.$$

Then one can prove that for every modal algebra $\mathfrak{H}$ we have $\mathfrak{A} \cong \mathfrak{A}_*{}^*$, and that for every modal space $\mathfrak{X}$ we have $\mathfrak{X} \cong \mathfrak{X}^*{}_*$ As for morphisms, we have that $h : \mathfrak{A} \to \mathfrak{B}$ is a homomorphism iff $h^{-1} : \mathfrak{B}_* \to \mathfrak{A}_*$ is a continuous bounded morphism, and that $f : \mathfrak{X} \to \mathfrak{Y}$ is a continuous bounded morphism iff $f^{-1} : \mathfrak{Y}^* \to \mathfrak{X}^*$ is a homomorphism. $\Box$

In this thesis we are mostly concerned with modal algebras and modal spaces validating one of the following modal logics.

$$\mathtt{K4} := \mathtt{K} \oplus \Box p \to \Box\Box p$$
$$\mathtt{S4} := \mathtt{K4} \oplus \Box p \to p$$

We let $\mathsf{K4} := \mathsf{Alg}(\mathtt{K4})$ and $\mathsf{S4} := \mathsf{Alg}(\mathtt{S4})$. We call algebras in $\mathsf{K4}$ *transitive algebras*, and algebras in $\mathsf{S4}$ *closure algebras*. It is obvious that for every $\mathfrak{A} \in \mathsf{MA}$, $\mathfrak{A} \in \mathsf{K4}$ iff $\Box\Box a \leq \Box a$ for every $a \in A$, and $\mathfrak{A} \in \mathsf{S4}$ iff $\mathfrak{A} \in \mathsf{K4}$ and additionally $\Box a \leq a$ for every $a \in A$. Moreover, it is easy to see that a modal space validates $\mathtt{K4}$ iff it has a transitive relation, and that it validates $\mathtt{S4}$ iff it has a reflexive and transitive relation (see Chagrov and Zakharyaschev 1997, §3.8).

Let $\mathfrak{X} \in \mathsf{Spa}(\mathtt{K4})$. A subset $C \subseteq X$ is called a *cluster* if it is an equivalence class under the relation $\sim$ defined by $x \sim y$ iff both Rxy and Ryx. A cluster is called *improper* if it is a singleton, *proper* otherwise.

We recall some basic properties of $\mathtt{K4}$- and $\mathtt{S4}$-spaces.

Proposition 2.16. Let $\mathfrak{X} \in \mathsf{Spa}(\mathtt{S4})$ and $U \in \mathsf{Clop}(\mathfrak{X})$. Then the following conditions hold:

1. The set $qmax_R(U)$ is closed;
2. If $x \in U$ then there is $y \in qmax_R(U)$ such that Rxy.

Moreover, let $\mathfrak{X} \in \mathsf{Spa}(\mathtt{K4})$ and $U \in \mathsf{Clop}(\mathfrak{X})$. Then the following conditions hold:

3. The structure $(X, R^+, \mathcal{O})$ is a $\mathtt{S4}$-space, where for all $x, y \in X$ we have R^+xy iff Rxy or $x = y$;
4. The set $qmax_R(U)$ is closed;
5. If $x \in U$ then there is $y \in qmax_R(U)$ such that Rxy.

Proof. Properties 1, 2 are proved in Esakia [2019, Theorems 3.2.1, 3.2.3]. Property 3 is obvious, and properties 4, 5 are immediate consequences of 1, 2, and 3. $\Box$

Among extensions of $\mathtt{S4}$, the modal logic $\mathtt{Grz}$ plays a particularly central role in this thesis.

$$\begin{aligned}\mathtt{Grz} : &= \mathtt{K} \oplus \Box(\Box(p \to \Box p) \to p) \to p \\ &= \mathtt{S4} \oplus \Box(\Box(p \to \Box p) \to p) \to p\end{aligned}$$

We let $\mathsf{Grz} := \mathsf{Alg}(\mathtt{Grz})$. It is not difficult to see that Grz coincides with the class of all closure algebras $\mathfrak{A}$ such that for every $a \in A$ we have

$$\Box(\Box(a \to \Box a) \to a) \leq a$$

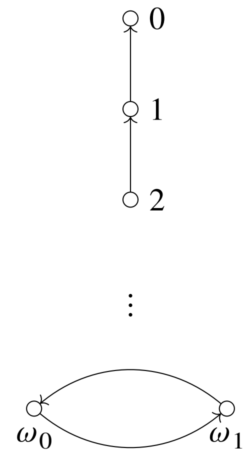
or equivalently,

$$a \leq \Diamond(a \wedge \neg\Diamond(\Diamond a \wedge \neg a)).$$

A poset (X, R) is called *Noetherian* if it contains no infinite R-ascending chain of pairwise distinct points. It is well known that $\mathtt{Grz}$ is complete with respect to the class of Noetherian partially ordered Kripke frames [Chagrov and Zakharyaschev, 1997, Corollary 5.52]. However, in general $\mathtt{Grz}$-spaces may fail to be partially ordered, as is well known.

Example 2.17. Consider the modal space $\mathfrak{X}$, where

- $X = \mathbb{N} \cup \{\omega_0, \omega_1\}$;
- $R = \{(n, m) \in \mathbb{N} \times \mathbb{N} : m \leq n\} \cup \{(\omega_i, x) \in X \times X : i \in \{0, 1\} \text{ and } x \in X\}$;
- $\mathcal{O}$ is given by the basis consisting of all $U \subseteq X$ such that either U is a finite subset of $\mathbb{N}$, or $U = V \cup \{\omega_0, \omega_1\}$ with V a cofinite subset of $\mathbb{N}$, or U is one of the following sets:
$$\{n \in \mathbb{N} : n \text{ is even}\} \cup \{\omega_0\} \qquad \{n \in \mathbb{N} : n \text{ is odd}\} \cup \{\omega_1\}.$$



It is easy to verify that for all $U \in \mathsf{Clop}(\mathfrak{X})$, if $x \in U$ then there is $y \in max_R(U)$ such that Rxy. So let $U \in \mathsf{Clop}(\mathfrak{X})$ and suppose $x \in U$. By the fact just observed there is $y \in max_R(U)$ such that Rxy. Now observe that $y \in max_R(U)$ implies that there is no $z \in X \setminus U$ with Ryz and $R[z] \cap U \neq \emptyset$. This shows $y \in U \setminus R^{-1}(R^{-1}(U) \setminus U)$, whence $x \in R^{-1}(U \setminus R^{-1}(R^{-1}(U) \setminus U))$. Therefore $\mathfrak{X}$ is a $\mathtt{Grz}$-space. But $\mathfrak{X}$ contains a proper cluster, hence it is not partially ordered.

Still, clusters cannot occur just anywhere in a $\mathtt{Grz}$-space, as the following result clarifies.

Proposition 2.18. For every $\mathtt{Grz}$-space $\mathfrak{X}$ and $U \in \mathsf{Clop}(\mathfrak{X})$, the following hold:

1. $qmax_R(U) \subseteq max_R(U)$;
2. The set $max_R(U)$ is closed;
3. For every $x \in U$ there is $y \in pas_R(U)$ such that Rxy;
4. $max_R(U) \subseteq pas_R(U)$.

Additionally, an $\mathtt{S4}$-space $\mathfrak{X}$ is a $\mathtt{Grz}$-space if Item 3 holds for every $U \in \mathsf{Clop}(\mathfrak{X})$.

Proof. Item 1 is proved in Esakia [2019, Theorem 3.5.6]. Item 2 follows from Item 1 and Proposition 2.16. Item 3 is immediate from the $\mathtt{Grz}$-axiom. Item 4 then follows from Proposition 2.16, Item 1, and Item 3. The last part of the Proposition is proved in Esakia [2019, Lemma 3.5.11]. □

Let us say that $U \subseteq X$ *cuts* a cluster $C \subseteq X$ if both $U \cap C \neq \emptyset$ and $U \setminus C \neq \emptyset$. As an immediate consequence of Item 4 in Proposition 2.18 we obtain that for any $U \in \mathsf{Clop}(\mathfrak{X})$, neither $max_R(U)$ nor $pas_R(U)$ cut any clusters in $\mathfrak{X}$.

§2.2 Stable Canonical Rules for Superintuitionistic and Modal Rule Systems

In both the si and the modal cases, the *filtration* technique can be used to construct finite countermodels to a non-valid rule Γ/Δ. Roughly, this construction consists of expanding finitely generated subreducts in a locally finite signature of arbitrary counter-models of Γ/Δ, in such a way that the new operation added to the subreduct agrees with the original one on selected elements. Si and modal *stable canonical rules* are essentially syntactic devices for encoding finite filtrations. The present section briefly reviews this method in both the si and modal case. We point the reader to Bezhanishvili et al. [2016a,b]; Bezhanishvili and Bezhanishvili [2017] and Ilin [2018, Ch. 5] for more in-depth discussion.

§2.2.1 Supertintuitionistic Case

We begin by defining si stable canonical rules.

Definition 2.19. Let $\mathfrak{H} \in \mathsf{HA}$ be finite and $D \subseteq A \times A$. For every $a \in H$ introduce a fresh propositional variable p_a. The *si stable canonical rule* of $(\mathfrak{H}, D)$, is defined as the rule $\eta(\mathfrak{H}, D) = \Gamma/\Delta$, where

$$\begin{aligned}\Gamma =&\{p_0 \leftrightarrow 0\} \cup \{p_1 \leftrightarrow 1\} \cup \\ &\{p_{a\land b} \leftrightarrow p_a \land p_b : a, b \in H\} \cup \{p_{a\lor b} \leftrightarrow p_a \lor p_b : a, b \in H\} \cup \\ &\{p_{a\to b} \leftrightarrow p_a \to p_b : (a, b) \in D\} \\ \Delta =&\{p_a \leftrightarrow p_b : a, b \in H \text{ with } a \neq b\}.\end{aligned}$$

We write si stable canonical rules of the form $\eta(\mathfrak{H}, \emptyset)$ simply as $\eta(\mathfrak{H})$, and call them *stable rules*.

If $\mathfrak{H}, \mathfrak{K} \in \mathsf{HA}$, let us call a map $h : \mathfrak{H} \to \mathfrak{K}$ *stable* if h is a bounded lattice homomorphism, i.e. if it preserves $0, 1, \land$, and $\lor$. If $D \subseteq H \times H$, we say that h satisfies the *bounded domain condition* (BDC) for D if

$$h(a \to b) = h(a) \to h(b)$$

for every $(a, b) \in D$. It is not difficult to check that every stable map $h : \mathfrak{H} \to \mathfrak{K}$ satisfies $h(a \to b) \leq h(a) \to h(b)$ for every $(a, b) \in H$.

Remark 2.20. The BDC was originally called *closed domain condition* in, e.g., Bezhanishvili et al. [2016a]; Bezhanishvili and Bezhanishvili [2017], following Zakharyaschev's terminology for a similar notion in the theory of his canonical formulae. The name *stable domain condition* was later used in Bezhanishvili and Bezhanishvili [2020] to stress the difference with Zakharyaschev's notion. However, this choice may create confusion between the BDC and the property of being a stable map. The terminology used in this thesis is meant to avoid this, while concurrently highlighting the similarity between the geometric version of the BDC, to be presented in a few paragraphs, and the definition of a bounded morphism.

The next two results characterise refutation conditions for si stable canonical rules. For detailed proofs the reader may consult Bezhanishvili et al. [2016b, Proposition 3.2].

Proposition 2.21. For every finite $\mathfrak{H} \in \mathsf{HA}$ and $D \subseteq H \times H$, we have $\mathfrak{H} \nvDash \eta(\mathfrak{H}, D)$.

Proof sketch. Use the valuation $V(p_a) = a$. □

Proposition 2.22. For every $\mathfrak{H}, \mathfrak{K} \in \mathsf{HA}$ with $\mathfrak{H}$ finite, and every $D \subseteq H \times H$, we have $\mathfrak{K} \nvDash \eta(\mathfrak{H}, D)$ iff there is a stable embedding $h : \mathfrak{H} \to \mathfrak{K}$ satisfying the BDC for D.

Proof sketch. ($\Rightarrow$) Assume $\mathfrak{K} \nvDash \eta(\mathfrak{H}, D)$, and take a valuation V on $\mathfrak{K}$ such that $\mathfrak{K}, V \nvDash \eta(\mathfrak{H}, D)$. Define a map $h : \mathfrak{H} \to \mathfrak{K}$ by setting $h(a) = V(p_a)$. Then h is the desired stable embedding satisfying the BDC for D.

($\Leftarrow$) Assume we have a stable embedding $h : \mathfrak{H} \to \mathfrak{K}$ satisfying the BDC for D. By the proof of Proposition 2.21 we know that the valuation V with $V(p_a) = a$ witnesses $\mathfrak{H} \nvDash \eta(\mathfrak{H}, D)$. So put $V(p_a) = h(a)$. □

Si stable canonical rules also have uniform refutation conditions on Esakia spaces. If $\mathfrak{X}, \mathfrak{Y}$ are Esakia spaces, a map $f : \mathfrak{X} \to \mathfrak{Y}$ is called *stable* if $x \leq y$ implies $f(x) \leq f(y)$, for all $x, y \in X$. If $\mathfrak{d} \subseteq Y$ we say that f satisfies the BDC for $\mathfrak{d}$ if for all $x \in X$,

$$\uparrow f(x) \cap \mathfrak{d} \neq \emptyset \Rightarrow f[\uparrow x] \cap \mathfrak{d} \neq \emptyset.$$

If $\mathfrak{D} \subseteq \wp(Y)$ then we say that f satisfies the BDC for $\mathfrak{D}$ if it does for each $\mathfrak{d} \in \mathfrak{D}$. If $\mathfrak{H}$ is a finite Heyting algebra and $D \subseteq H$, for every $(a, b) \in D$ set $\mathfrak{d}_{(a,b)} := \beta(a) \setminus \beta(b)$. Finally, put

$$\mathfrak{D} := \{\mathfrak{d}_{(a,b)} : (a, b) \in D\}.$$

The following result follows straightforwardly from Bezhanishvili and Bezhanishvili [2017, Lemma 4.3].

Proposition 2.23. For every Esakia space $\mathfrak{X}$ and any si stable canonical rule $\eta(\mathfrak{H}, D)$, we have $\mathfrak{X} \nvDash \eta(\mathfrak{H}, D)$ iff there is a continuous stable surjection $f : \mathfrak{X} \to \mathfrak{H}_*$ satisfying the BDC for the family $\mathfrak{D} := \{\mathfrak{d}_{(a,b)} : (a, b) \in D\}$.

In view of Proposition 2.23, when working with Esakia spaces we shall often write a si stable canonical rule $\eta(\mathfrak{H}, D)$ as $\eta(\mathfrak{H}_*, \mathfrak{D})$.

Stable maps and the BDC are closely related to the filtration construction. We recall its definition in an algebraic setting, and state the fundamental theorem underwriting most of its applications.

Definition 2.24. Let $\mathfrak{H}$ be a Heyting algebra, V a valuation on $\mathfrak{A}$, and Θ a finite, subformula closed set of formulae. A (finite) model $(\mathfrak{K}', V')$ is called a *(finite) filtration of* $(\mathfrak{H}, V)$ *through* Θ if the following hold:

1. $\mathfrak{K}' = (\mathfrak{K}, \to)$, where $\mathfrak{K}$ is the bounded sublattice of $\mathfrak{H}$ generated by $\bar{V}[\Theta]$;
2. $V(p) = V'(p)$ for every propositional variable $p \in \Theta$;
3. The inclusion $\subseteq: \mathfrak{H} \to \mathfrak{K}$ is a stable embedding satisfying the BDC for the set

$$\{(\bar{V}'(\varphi), \bar{V}'(\psi)) : \varphi \to \psi \in \Theta\}.$$

Theorem 2.25 (Filtration theorem for Heyting algebras)**.** Let $\mathfrak{H} \in \mathsf{HA}$ be a Heyting algebra, V a valuation on $\mathfrak{H}$, and Θ a a finite, subformula closed set of formulae. If $(\mathfrak{K}', V')$ is a filtration of $(\mathfrak{H}, V)$ through Θ then for every $\varphi \in \Theta$ we have

$$\bar{V}(\varphi) = \bar{V}'(\varphi).$$

Consequently, for every rule Γ/Δ such that $\gamma, \delta \in \Theta$ for each $\gamma \in \Gamma$ and $\delta \in \Delta$ we have

$$\mathfrak{H}, V \models \Gamma/\Delta \iff \mathfrak{K}, V' \models \Gamma/\Delta.$$

A proof of the filtration theorem above follows from, e.g., the proof of Bezhanishvili and Bezhanishvili [2017, Lemma 3.6].

The next result establishes that every si rule is equivalent to finitely many si stable canonical rules. This lemma was proved in Bezhanishvili et al. [2016b, Proposition 3.3], but we rehearse the proof here to illustrate the exact role of filtration in the machinery of stable canonical rules.

Lemma 2.26. For every si rule Γ/Δ there is a finite set Ξ of si stable canonical rules such that for any $\mathfrak{K} \in \mathsf{HA}$ we have $\mathfrak{K} \not\models \Gamma/\Delta$ iff there is $\eta(\mathfrak{H}, D) \in \Xi$ such that $\mathfrak{K} \not\models \eta(\mathfrak{H}, D)$.

Proof. Since bounded distributive lattices are locally finite there are, up to isomorphism, only finitely many pairs $(\mathfrak{H}, D)$ such that

- $\mathfrak{H}$ is at most k-generated as a bounded distributive lattice, where $k = |Sfor(\Gamma/\Delta)|$;
- $D = \{(\bar{V}(\varphi), \bar{V}(\psi)) : \varphi \to \psi \in Sfor(\Gamma/\Delta)\}$, where V is a valuation on $\mathfrak{H}$ refuting Γ/Δ.

Let Ξ be the set of all rules $\eta(\mathfrak{H}, D)$ for all such pairs $(\mathfrak{H}, D)$, identified up to isomorphism.

($\Rightarrow$) Assume $\mathfrak{K} \not\models \Gamma/\Delta$ and take a valuation V on $\mathfrak{K}$ refuting Γ/Δ. Consider the bounded distributive sublattice $\mathfrak{J}$ of $\mathfrak{K}$ generated by $\bar{V}[Sfor(\Gamma/\Delta)]$. Since bounded distributive lattices are locally finite, $\mathfrak{J}$ is finite. Define a binary operation $\rightsquigarrow$ on $\mathfrak{J}$ by setting, for all $a, b \in J$,

$$a \rightsquigarrow b := \bigvee\{c \in J : a \wedge c \leq b\}.$$

Clearly, $\mathfrak{J}' := (\mathfrak{J}, \rightsquigarrow)$ is a Heyting algebra. Define a valuation V' on $\mathfrak{J}'$ with $V'(p) = V(p)$ if $p \in \Theta$, $V'(p)$ arbitrary otherwise. Since $\mathfrak{J}'$ is a sublattice of $\mathfrak{K}$, the inclusion $\subseteq$ is a stable embedding. Now let $\varphi \to \psi \in \Theta$. Then $\bar{V}'(\varphi) \to \bar{V}'(\psi) \in J$. From the fact that $\subseteq$ is a stable embedding it follows that $\bar{V}'(\varphi) \rightsquigarrow \bar{V}'(\psi) \leq \bar{V}'(\varphi) \to \bar{V}'(\psi)$. Conversely, by the properties of Heyting algebras we know that $\bar{V}'(\varphi) \to \bar{V}'(\psi) \wedge \bar{V}'(\varphi) \leq \bar{V}'(\psi)$. But since $\bar{V}'(\varphi) \to \bar{V}'(\psi) \in J$, by the definition of $\rightsquigarrow$ it follows that $\bar{V}'(\varphi) \to \bar{V}'(\psi) \leq \bar{V}'(\varphi) \rightsquigarrow \bar{V}'(\psi)$. Thus $\bar{V}'(\varphi) \rightsquigarrow \bar{V}'(\psi) = \bar{V}'(\varphi) \to \bar{V}'(\psi)$ as desired. We have shown that the model $(\mathfrak{J}', V')$ is a filtration of the model $(\mathfrak{K}, V)$ through $Sfor(\Gamma/\Delta)$, which implies $\mathfrak{J}', V' \not\models \Gamma/\Delta$.

($\Leftarrow$) Assume that there is $\eta(\mathfrak{H}, D) \in \Xi$ such that $\mathfrak{K} \not\models \eta(\mathfrak{H}, D)$. Let V be the valuation associated with D in the sense spelled out above. Then $\mathfrak{H}, V \not\models \Gamma/\Delta$. Moreover $(\mathfrak{H}, V)$ is a filtration of the model $(\mathfrak{K}, V)$, so by the filtration theorem it follows that $\mathfrak{K}, V \not\models \Gamma/\Delta$. $\square$

As an immediate consequence we obtain a uniform axiomatisation of all si-rule systems by means of si stable canonical rules.

Theorem 2.27 (Bezhanishvili et al. 2016b, Proposition 3.4)**.** Any si-rule system $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC_R})$ is axiomatisable over $\mathtt{IPC_R}$ by some set of si stable canonical rules.

Proof. Let $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC_R})$, and take a set of rules Ξ such that $\mathtt{L} = \mathtt{IPC_R} \oplus \Xi$. By Lemma 2.26, for every $\Gamma/\Delta \in \Xi$ there is a finite set $\Pi_{\Gamma/\Delta}$ of si stable canonical rules whose conjunction is equivalent to Γ/Δ. But then $\mathtt{L} = \mathtt{IPC_R} \oplus \bigcup_{\Delta/\Gamma \in \Xi} \Pi_{\Gamma/\Delta}$. □

§2.2.2 Modal Case

We now turn to modal stable canonical rules.

Definition 2.28. Let $\mathfrak{A} \in \mathsf{MA}$ be finite and $D \subseteq A$. For every $a \in A$ introduce a fresh propositional variable p_a. The *modal stable canonical rule* of $(\mathfrak{A}, D)$ is defined as the rule $\mu(\mathfrak{A}, D) = \Gamma/\Delta$, where

$$\begin{aligned}\Gamma = &\{p_{\neg a} \leftrightarrow \neg p_a : a \in A\} \cup \\ &\{p_{a \wedge b} \leftrightarrow p_a \wedge p_b : a, b \in A\} \cup \{p_{a \vee b} \leftrightarrow p_a \vee p_b : a, b \in A\} \cup \\ &\{p_{\Box a} \to \Box p_a : a \in A\} \cup \{\Box p_a \to p_{\Box a} : a \in D\} \\ \Delta = &\{p_a : a \in A \setminus 1\}.\end{aligned}$$

As in the si case, a modal stable canonical rule of the form $\mu(\mathfrak{A}, \emptyset)$ is written simply as $\mu(\mathfrak{A})$ and called a *stable rule*.

If $\mathfrak{A}, \mathfrak{B} \in \mathsf{MA}$ are modal algebras, let us call a map $h : \mathfrak{A} \to \mathfrak{B}$ *stable* if for every $a \in A$ we have $h(\Box a) \leq \Box h(a)$. If $D \subseteq A$, we say that h satisfies the *bounded domain condition* (BDC) for D if $h(\Box a) = \Box h(a)$ for every $a \in D$.

The following two propositions are modal counterparts to Propositions 2.21 and 2.22. Their proofs are similar to the latter's, and can be found in Bezhanishvili et al. [2016a, Lemma 5.3, Theorem 5.4].

Proposition 2.29. For every finite $\mathfrak{A} \in \mathsf{MA}$ and $D \subseteq A$, we have $\mathfrak{A} \not\models \mu(\mathfrak{A}, D)$.

Proposition 2.30. For every $\mathfrak{A}, \mathfrak{B} \in \mathsf{MA}$ with $\mathfrak{A}$ finite, and every $D \subseteq A$, we have $\mathfrak{B} \not\models \mu(\mathfrak{A}, D)$ iff there is a stable embedding $h : \mathfrak{A} \to \mathfrak{B}$ satisfying the BDC for D.

Refutation conditions for modal stable canonical rules on modal spaces are obtained in analogous fashion to the si case. If $\mathfrak{X}, \mathfrak{Y}$ are modal spaces, a map $f : \mathfrak{X} \to \mathfrak{Y}$ is called *stable* if for all $x, y \in X$, we have that Rxy implies $Rf(x)f(y)$. If $\mathfrak{d} \subseteq Y$ we say that f satisfies the BDC for $\mathfrak{d}$ if for all $x \in X$,

$$R[f(x)] \cap \mathfrak{d} \neq \emptyset \Rightarrow f[R[x]] \cap \mathfrak{d} \neq \emptyset.$$

If $\mathfrak{D} \subseteq \wp(Y)$ then we say that f satisfies the BDC for $\mathfrak{D}$ if it does for each $\mathfrak{d} \in \mathfrak{D}$. If $\mathfrak{A}$ is a finite modal algebra and $D \subseteq H$, for every $a \in D$ set $\mathfrak{d}_a := -\beta(a)$. Finally, put $\mathfrak{D} := \{\mathfrak{d}_a : a \in D\}$. The following result is proved in Bezhanishvili et al. 2016a, Theorem 3.6.

Proposition 2.31. For every modal space $\mathfrak{X}$ and any modal stable canonical rule $\mu(\mathfrak{A}, D)$, $\mathfrak{X} \not\models \mu(\mathfrak{A}, D)$ iff there is a continuous stable surjection $f : \mathfrak{X} \to \mathfrak{A}_*$ satisfying the BDC for $\mathfrak{D}$.

In view of Proposition 2.31, when working with modal spaces we may write a modal stable canonical rule $\mu(\mathfrak{A}, D)$ as $\mu(\mathfrak{A}_*, \mathfrak{D})$.

As in the si case, stable maps and the BDC are closely related to the filtration technique.

Definition 2.32. Let $\mathfrak{A}$ be a modal algebra, V a valuation on $\mathfrak{A}$, and Θ a finite, subformula closed set of formulae. A (finite) model $(\mathfrak{B}, V')$ is called a *(finite) filtration of* $(\mathfrak{A}, V)$ *through* Θ if the following conditions hold:

1. $\mathfrak{B} = (\mathfrak{B}', \Box)$, where $\mathfrak{B}'$ is the Boolean subalgebra of $\mathfrak{A}$ generated by $\bar{V}[\Theta]$;
2. $V(p) = V'(p)$ for every propositional variable $p \in \Theta$;
3. The inclusion $\subseteq : \mathfrak{B} \to \mathfrak{A}$ is a stable embedding satisfying the BDC for the set

$$\{\bar{V}(\varphi) : \Box\varphi \in \Theta\}$$

The following result is proved, e.g., in Bezhanishvili et al. [2016a, Lemma 4.4].

Theorem 2.33 (Filtration theorem for modal algebras)**.** Let $\mathfrak{A} \in \mathsf{MA}$ be a modal algebra, V a valuation on $\mathfrak{A}$, and Θ a finite, subformula closed set of formulae. If $(\mathfrak{B}', V')$ is a filtration of $(\mathfrak{A}, V)$ through Θ then for every $\varphi \in \Theta$ we have

$$\bar{V}(\varphi) = \bar{V}'(\varphi).$$

Consequently, for every rule Γ/Δ such that $\gamma, \delta \in \Theta$ for each $\gamma \in \Gamma$ and $\delta \in \Delta$ we have

$$\mathfrak{A}, V \models \Gamma/\Delta \iff \mathfrak{B}, V' \models \Gamma/\Delta.$$

Unlike the si case, filtrations of a given model through a given set of formulae are not necessarily unique when they exist. Depending on which construction is preferred, different properties of the original model may or may not be preserved. In this chapter we mainly deal with closure algebras, whence we are particularly interested in filtrations preserving reflexivity and transitivity. It is very easy to see that any filtration preserves reflexivity. Whilst, in general, the filtration of a transitive model may fail to be transitive, transitive filtrations of transitive models can be constructed in multiple ways. Here we restrict attention to one particular construction.

Definition 2.34. Let $\mathfrak{A} \in \mathsf{S4}$, V a valuation on $\mathfrak{A}$ and Θ a finite, subformula closed set of formula. The *(least) transitive filtration* of $(\mathfrak{A}, V)$ is a pair $(\mathfrak{B}', V')$ with $\mathfrak{B}' = (\mathfrak{B}, \blacksquare)$, where $\mathfrak{B}$ and V' are as per Definition 2.32, and for all $b \in B$ we have

$$\blacksquare b := \bigvee\{\Box a : \Box a \leq \Box b \text{ and } a, \Box a \in B\}$$

It is easy to see that transitive filtrations of transitive models are indeed based on closure algebras (cf. e.g. Bezhanishvili et al. 2016a, Lemma 6.2).

Transitive filtrations provide the necessary countermodels to rewrite modal rules into (conjunctions of) modal stable canonical rules. The following lemma, which is a modal counterpart to Lemma 2.26, explains how.

Lemma 2.35 (Bezhanishvili et al. 2016a, Theorem 5.5)**.** For every modal rule Γ/Δ there is a finite set Ξ of modal stable canonical rules of the form $\mu(\mathfrak{A}, D)$ with $\mathfrak{A} \in \mathsf{S4}$, such that for any $\mathfrak{B} \in \mathsf{S4}$ we have that $\mathfrak{B} \nvDash \Gamma/\Delta$ iff there is $\mu(\mathfrak{A}, D) \in \Xi$ such that $\mathfrak{B} \nvDash \mu(\mathfrak{A}, D)$.

Proof. Since Boolean algebras are locally finite there are, up to isomorphism, only finitely many pairs $(\mathfrak{A}, D)$ such that

- $\mathfrak{A}$ is at most k-generated as a Boolean algebra, where $k = |Sfor(\Gamma/\Delta)|$;
- $D = \{\bar{V}(\varphi) : \Box\varphi \in Sfor(\Gamma/\Delta)\}$, where V is a valuation on $\mathfrak{H}$ refuting Γ/Δ.

Let Ξ be the set of all rules $\mu(\mathfrak{A}, D)$ for all such pairs $(\mathfrak{A}, D)$, identified up to isomorphism. Then we reason as in the proof of Lemma 2.26, using the well-known fact that every model $(\mathfrak{B}, V)$ with $\mathfrak{B} \in \mathsf{S4}$ has a transitive filtration through $Sfor(\Gamma/\Delta)$ to establish the $(\Rightarrow)$ direction. $\square$

Exactly mirroring the si case we apply Lemma 2.35 to obtain the following uniform axiomatisation of modal rule systems extending $\mathtt{S4_R}$.

Theorem 2.36. Every modal rule system $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4_R})$ is axiomatisable over $\mathtt{S4_R}$ by some set of modal stable canonical rules of the form $\mu(\mathfrak{A}, D)$, for $\mathfrak{A} \in \mathsf{S4}$.

§2.3 Modal Companions of Superintuitionistic Deductive Systems via Stable Canonical Rules

We now turn to the main topic of this section. § 2.3.1 reviews the basic ingredients of the theory of modal companions.

§2.3.1 Semantic and Syntactic Mappings

This section reviews semantic transformations between Heyting and closure algebras and the Gödel translation. The results in this section are widely known, and most are collected, e.g., in Esakia [2019, Ch. 3]. Nonetheless, we prefer to include full proofs to help the reader appreciate the details of the constructions under discussion, which should be useful for parsing the new proofs in the subsequent sections of this chapter.

From Heyting to closure algebras and back The first step for turning a given Heyting algebra into a corresponding closure algebra is by constructing its free Boolean extension.

Definition 2.37. Let $\mathfrak{H}$ be a Heyting algebra. The *free Boolean extension of* $\mathfrak{H}$ is the unique (up to isomorphism) Boolean algebra $\mathfrak{B}$ such that for every Boolean algebra $\mathfrak{C}$ and bounded lattice homomorphism $h : \mathfrak{H} \to \mathfrak{C}$ there exists a unique Boolean homomorphism $f : \mathfrak{B} \to \mathfrak{C}$ extending h.

Proposition 2.38. Every Heyting algebra $\mathfrak{H}$ has a unique free Boolean extension.

Proof. We use Esakia duality. Let $\mathfrak{H} \in \mathsf{HA}$ and let $\mathfrak{X} = \mathfrak{H}_*$. For convenience, identify $\mathfrak{H}$ with $\mathfrak{X}^*$. Set $B(H) = \mathsf{Clop}(\mathfrak{X})$. We claim that the algebra

$$B(\mathfrak{H}) = (B(H), \cap, \cup, -, \emptyset, X)$$

is the unique free Boolean extension of $\mathfrak{H}$. To see this, let $\mathfrak{C}$ be some Boolean algebra and $h : \mathfrak{H} \to \mathfrak{C}$ a bounded lattice homomorphism. Extend h to a map $i : \mathsf{Clop}(\mathfrak{H}) \to \mathfrak{C}$ recursively. Set $i(U) := h(U)$ if $U \in \mathsf{ClopUp}(\mathfrak{X})$. Assume that $i(U), i(V)$ have been defined and set

$$\begin{aligned} i(X \setminus U) &:= \neg i(U) \\ i(U \cap V) &:= i(U) \wedge i(V) \\ i(U \cup V) &:= i(U) \vee i(V). \end{aligned}$$

By Esakia duality, every $U \in \mathsf{Clop}(\mathfrak{X})$ is of the form

$$U = \bigcup_{i \leq n} \left(\beta(a_1) \cap \cdots \cap \beta(a_{k_i}) \cap -\beta(b_1) \cap \cdots \cap -\beta(b_{h_i}) \right)$$

with $a_1, \ldots, a_{k_i}, b_1, \ldots, b_{h_i} \in H$. This shows that i is a well-defined map. Now, i is clearly a Boolean homomorphism, and uniqueness is ensured by the construction. □

The construction turning a Heyting algebra into a corresponding closure algebra is completed by augmenting free Boolean extensions of Heyting algebras with a modal operator, as follows.

Definition 2.39. The mapping $\sigma : \mathsf{HA} \to \mathsf{S4}$ assigns every $\mathfrak{H} \in \mathsf{HA}$ to the algebra $\sigma\mathfrak{H} := (B(\mathfrak{H}), \Box)$, where $B(\mathfrak{H})$ is the free Boolean extension of $\mathfrak{H}$ and

$$\Box a := \bigvee \{b \in H : b \leq a\}.$$

Routine reasoning shows that σ is well-defined, that is, for every $\mathfrak{H} \in \mathsf{HA}$ we have that $\sigma\mathfrak{H}$ is indeed a closure algebra.

The next result describes an important normal form property of free Boolean extensions, which we shall appeal to later on.

Proposition 2.40. Let $B(\mathfrak{H})$ be the free Boolean extension of some Heyting algebra $\mathfrak{H}$. Then every element $a \in B(H)$ may be written as

$$\bigwedge_{i \leq n} (\neg a_i \vee b_i)$$

for finitely many $\{a_i, b_i : i \leq n\} \subseteq H$.

Proof. As noted earlier, by Esakia duality for every $a \in B(H)$ we may write $\neg a$ as

$$\begin{aligned} \neg a &= \bigvee_{i \leq n} \left(a_1 \wedge \cdots \wedge a_{k_i} \wedge \neg b_1 \wedge \cdots \wedge \neg b_{h_i} \right) \\ &= \bigvee_{i \leq n} \left(a_1 \wedge \cdots \wedge a_{k_i} \wedge \neg (b_1 \vee \cdots \vee b_{h_i}) \right) \end{aligned}$$

with $a_1, \dots, a_{k_i}, b_1, \dots, b_{h_i} \in H$. Then we let $a_i := a_1 \wedge \cdots \wedge a_{k_i}$ and $b_i = b_1 \vee \cdots \vee b_{h_i}$. Therefore,

$$\begin{aligned} a &= \neg \bigvee_{i \leq n} (a_i \wedge \neg b_i) \\ &= \bigwedge_{i \leq n} (\neg a_i \vee b_i) . \end{aligned}$$

□

Conversely, we also have a method for turning arbitrary closure algebras into corresponding Heyting algebras.

Definition 2.41. The mapping $\rho : \mathsf{S4} \to \mathsf{HA}$ assigns every $\mathfrak{A} \in \mathsf{S4}$ to the algebra $\rho\mathfrak{A} := (O(A), \wedge, \vee, \to)$, where

$$\begin{aligned} O(A) &:= \{a \in A : \Box a = a\} \\ a \to b &:= \Box(\neg a \vee b). \end{aligned}$$

Proposition 2.42. For every $\mathfrak{A} \in \mathsf{S4}$, $\rho\mathfrak{A}$ is a Heyting algebra.

Proof. It is not difficult to see that if $a, b \in O(A)$ then $a \vee b = \Box(a \vee b) \in O(A)$ and $a \wedge b = \Box(a \wedge b) \in O(A)$. Moreover, $1 = \Box 1$ and $0 = \Box 0$, hence $1, 0 \in O(A)$. Thus $\rho\mathfrak{A}$ is a bounded sublattice of $\mathfrak{A}$, hence distributive. To see that $\to$ is a Heyting implication, first observe that for $a, b, c \in O(A)$ we have that $c \leq \Box(\neg a \vee b)$ iff $c \leq \neg a \vee b$, and the latter holds iff $c \wedge a \leq b$ because the formula $\neg x \vee y$ defines a Heyting implication in every Boolean algebra. □

We now give a dual description of the maps σ, ρ on modal and Esakia spaces.

Definition 2.43. If $\mathfrak{X} = (X, \leq, \mathcal{O})$ is an Esakia space we set $\sigma\mathfrak{X} := (X, R, \mathcal{O})$ with $R := \leq$. Let $\mathfrak{Y} := (Y, R, \mathcal{O})$ be an $\mathsf{S4}$-space. For $x, y \in Y$ write $x \sim y$ iff Rxy and Ryx. Define a map $\rho : Y \to \wp(Y)$ by setting $\rho(x) = \{y \in Y : x \sim y\}$. We define $\rho\mathfrak{Y} := (\rho[Y], \leq, \rho[\mathcal{O}])$ where $\rho(x) \leq \rho(y)$ iff Rxy.

Note that σ here is effectively the identity map, though we find useful to distinguish an Esakia space $\mathfrak{X}$ from $\sigma\mathfrak{X}$ notationally in order to signal whether we are treating the space as a model for si or modal deductive systems. On the other hand, the map ρ affects a modal space $\mathfrak{Y}$ by collapsing its R-clusters and endowing the result with the quotient topology. We shall refer to $\rho\mathfrak{Y}$ as the *Esakia skeleton* of $\mathfrak{Y}$, and to $\sigma\rho\mathfrak{Y}$ as the *modal skeleton* of $\mathfrak{Y}$. It is easy to see that the map $\rho : \mathfrak{Y} \to \rho\mathfrak{Y}$ is a surjective bounded morphism which moreover reflects $\leq$.

The following result shows that the algebraic and topological versions of the maps σ, ρ are indeed dual to each other.

Proposition 2.44. The following hold.

1. Let $\mathfrak{H} \in \mathsf{HA}$. Then $(\sigma\mathfrak{H})_* \cong \sigma(\mathfrak{H}_*)$. Consequently, if $\mathfrak{X}$ is an Esakia space then $(\sigma\mathfrak{X})^* \cong \sigma(\mathfrak{X})^*$.
2. Let $\mathfrak{X}$ be an $\mathtt{S4}$ modal space. Then $(\rho\mathfrak{X})^* \cong \rho(\mathfrak{X}^*)$. Consequently, if $\mathfrak{A} \in \mathsf{S4}$, then $(\rho\mathfrak{A})_* \cong \rho(\mathfrak{A}_*)$.

Proof. (1) Let $\mathfrak{H} \in \mathsf{HA}$. By Proposition 2.38 $\sigma\mathfrak{H}$ is isomorphic to $\mathsf{Clop}(\mathfrak{X})$, where $\mathfrak{X} := \mathfrak{H}_*$. Then it suffices to show that $(\sigma\mathfrak{H})_*$ and $\sigma(\mathfrak{X})$ have the same relations. Assume $x \leq y$ and suppose $x \in \Box_R U$ for arbitrary $U \in \mathsf{Clop}(\mathfrak{X})$. Since

$$\Box_R U = \bigcup\{V \in \mathsf{ClopUp}(\mathfrak{X}) : V \subseteq U\}$$

it follows that $x \in V$ for some $V \in \mathsf{ClopUp}(\mathfrak{X})$ with $V \subseteq U$. This means $\beta^{-1}(V) \in x$, and by $x \subseteq y$ in turn $\beta^{-1}(V) \in y$, i.e. $y \in V$ and so $y \in \Box_R U$. Conversely, assume that Rxy and let $U \in \mathsf{ClopUp}(\mathfrak{X})$. Then clearly $\Box_R U = U$. Therefore $y \notin U$ implies $x \notin \Box_R U = U$. So indeed $x \leq y$.

Now let $\mathfrak{X}$ be an Esakia space. We have $\mathfrak{X} \cong \mathfrak{X}^*{}_*$, so by the first part we have $\sigma(\mathfrak{X}^*{}_*)^* \cong \sigma(\mathfrak{X}^*)_*{}^* \cong \sigma(\mathfrak{X}^*)$, and we are done.

(2) Let $\mathfrak{X}$ be an $\mathtt{S4}$ modal space. We show that $\rho^{-1} : (\rho\mathfrak{X})^* \to \rho(\mathfrak{X}^*)$ is the desired isomorphism. Recall that for $V \in \mathsf{Clop}(\mathfrak{X})$ we have that V is an upset iff $V = \Box V$. Using that $\rho : X \to \rho[X]$ is a bounded morphism and that $V \in \mathsf{Clop}(\mathfrak{X})$ implies $\rho[V] \in \mathsf{Clop}(\rho\mathfrak{X})$ whenever V does not cut clusters, it follows that the range of ρ^{-1} is $\mathsf{ClopUp}(\mathfrak{X})$. Injectivity is obvious. The fact that ρ^{-1} is a bounded lattice isomorphism follows from the fact that no element $\rho^{-1}(U)$ cuts any cluster. For the same reason we have $\rho^{-1}(-U) = -\rho^{-1}(U)$ and $\rho^{-1}(\downarrow U) = \downarrow\rho^{-1}(U)$ for any $U \in \mathsf{Clop}(\rho\mathfrak{X})$. Therefore $\rho^{-1}(-\downarrow(U - V)) = -\downarrow\rho^{-1}(U) - \rho^{-1}(V)$, hence indeed ρ^{-1} is a Heyting isomorphism. Thus we have shown $(\rho\mathfrak{X})^* \cong \rho(\mathfrak{X}^*)$ as desired.

For the second part, let $\mathfrak{A} \in \mathsf{S4}$. Then $\mathfrak{A} \cong \mathfrak{A}_*{}^*$, so using the first part $\rho(\mathfrak{A})_* \cong \rho(\mathfrak{A}_*{}^*)_* \cong (\rho\mathfrak{A}_*)^*{}_* \cong (\rho\mathfrak{A}_*)$, and we are done. □

The dual description of ρ, σ makes the following result evident.

Proposition 2.45. For every $\mathfrak{H} \in \mathsf{HA}$ we have $\mathfrak{H} \cong \rho\sigma\mathfrak{H}$. Moreover, for every $\mathfrak{A} \in \mathsf{S4}$ we have $\sigma\rho\mathfrak{A} \rightarrowtail \mathfrak{A}$.

Furthermore, we obtain a more accurate approximation of the range of the map σ.

Proposition 2.46. For every $\mathfrak{H} \in \mathsf{HA}$, $\sigma\mathfrak{H}$ is a $\mathtt{Grz}$-algebra.

Proof. By Proposition 2.44 $(\sigma\mathfrak{H})_* = \sigma(\mathfrak{H}_*)$ has the same topology as $\mathfrak{H}_*$. The latter is an Esakia space, therefore by Proposition 2.6, for every $U \in \mathsf{Clop}(\mathfrak{X})$ and $x \in U$ there is $y \in max_{\leq}(U)$ such that $x \leq y$. By Proposition 2.18 it follows that $\sigma(\mathfrak{H}_*)$ is a $\mathtt{Grz}$ space, and so by duality $\sigma\mathfrak{H} \in \mathsf{Grz}$. □

The Gödel Translation The close connection between Heyting and closure algebras just outlined manifests syntactically as the existence of a well-behaved translation of si formulae into modal ones, called the *Gödel translation*, after Gödel [1933].

Definition 2.47 (Gödel translation)**.** The *Gödel translation* is a mapping $T : Frm_{si} \to Frm_{md}$ defined recursively as follows.

$$
\begin{aligned}
T(\bot) &:= \bot \\
T(\top) &:= \top \\
T(p) &:= \Box p \\
T(\varphi \wedge \psi) &:= T(\varphi) \wedge T(\psi) \\
T(\varphi \vee \psi) &:= T(\varphi) \vee T(\psi) \\
T(\varphi \to \psi) &:= \Box(\neg T(\varphi) \vee T(\psi))
\end{aligned}
$$

We extend the Gödel translation from formulae to rules by setting

$$T(\Gamma/\Delta) := T[\Gamma]/T[\Delta].$$

We close this subsection by proving the following key lemma, first proved in Jerábek [2009], which states that the Gödel translation preserves and reflects rule validity between modal algebras and their skeletons.

Lemma 2.48 (Jerábek 2009, Lemma 3.13)**.** For every $\mathfrak{A} \in \mathsf{S4}$ and si rule Γ/Δ,

$$\mathfrak{A} \models T(\Gamma/\Delta) \iff \rho\mathfrak{A} \models \Gamma/\Delta$$

Proof. A simple induction on structure shows that for every si term φ, every modal space $\mathfrak{X}$, every valuation V on $\mathfrak{X}$ and every point $x \in X$ we have

$$\mathfrak{X}, V, x \models T(\varphi) \iff \rho\mathfrak{X}, V_\rho, \rho(x) \models \varphi,$$

where $V_\rho(p) = \rho[V(p)]$ for all $p \in Prop$. Using this equivalence and noting that every valuation V on some Esakia space $\rho\mathfrak{X}$ can be seen as of the form V'_ρ for some valuation V' on $\mathfrak{X}$, the rest of the proof is trivial. □

§2.3.2 Structure of Modal Companions

We now have all the material needed to develop the theory of modal companions via the machinery of stable canonical rules.

Definition 2.49. Let $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC_R})$ be a si-rule system and $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4_R})$ a modal rule system. We say that $\mathtt{M}$ is a *modal companion* of $\mathtt{L}$ (or that $\mathtt{L}$ is the si fragment of $\mathtt{M}$) whenever $\Gamma/\Delta \in \mathtt{L}$ iff $T(\Gamma/\Delta) \in \mathtt{M}$. Moreover, let $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC})$ be a si-logic and $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4})$ a modal logic. We say that $\mathtt{M}$ is a *modal companion* of $\mathtt{L}$ (or that $\mathtt{L}$ is the si fragment of $\mathtt{M}$) whenever $\varphi \in \mathtt{L}$ iff $T(\varphi) \in \mathtt{M}$.

Obviously, $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4_R})$ is a modal companion of $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC_R})$ iff $\mathsf{Taut}(\mathtt{M})$ is a modal companion of $\mathsf{Taut}(\mathtt{L})$, and $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4})$ is a modal companion of $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC})$ iff $\mathtt{M_R}$ is a modal companion of $\mathtt{L_R}$.

Define the following three maps between the lattices $\mathbf{Ext}(\mathtt{IPC_R})$ and $\mathbf{NExt}(\mathtt{S4_R})$.

$$\begin{aligned} \tau : \mathbf{Ext}(\mathtt{IPC_R}) &\to \mathbf{NExt}(\mathtt{S4_R}) \\ \mathtt{L} &\mapsto \mathtt{S4_R} \oplus \{T(\Gamma/\Delta) : \Gamma/\Delta \in \mathtt{L}\} \end{aligned} \qquad \begin{aligned} \sigma : \mathbf{Ext}(\mathtt{IPC_R}) &\to \mathbf{NExt}(\mathtt{S4_R}) \\ \mathtt{L} &\mapsto \mathtt{Grz_R} \oplus \tau \mathtt{L} \end{aligned}$$

$$\begin{aligned} \rho : \mathbf{NExt}(\mathtt{S4_R}) &\to \mathbf{Ext}(\mathtt{IPC_R}) \\ \mathtt{M} &\mapsto \{\Gamma/\Delta : T(\Gamma/\Delta) \in \mathtt{M}\} \end{aligned}$$

Similar mappings are readily defined for lattices of logics.

$$\begin{aligned} \tau : \mathbf{Ext}(\mathtt{IPC}) &\to \mathbf{NExt}(\mathtt{S4}) \\ \mathtt{L} &\mapsto \mathsf{Taut}(\tau \mathtt{L_R}) = \mathtt{S4} \oplus \{T(\varphi) : \varphi \in \mathtt{L}\} \end{aligned} \qquad \begin{aligned} \sigma : \mathbf{Ext}(\mathtt{IPC}) &\to \mathbf{NExt}(\mathtt{S4}) \\ \mathtt{L} &\mapsto \mathsf{Taut}(\sigma \mathtt{L_R}) = \mathtt{Grz} \oplus \{T(\varphi) : \varphi \in \mathtt{L}\} \end{aligned}$$

$$\begin{aligned} \rho : \mathbf{NExt}(\mathtt{S4}) &\to \mathbf{Ext}(\mathtt{IPC}) \\ \mathtt{M} &\mapsto \mathsf{Taut}(\rho \mathtt{M_R}) = \{\varphi : T(\varphi) \in \mathtt{M}\} \end{aligned}$$

Furthermore, extend the mappings $\sigma : \mathsf{HA} \to \mathsf{S4}$ and $\rho : \mathsf{S4} \to \mathsf{HA}$ to universal classes by setting

$$\begin{aligned} \sigma : \mathbf{Uni}(\mathsf{HA}) &\to \mathbf{Uni}(\mathsf{S4}) \\ \mathcal{U} &\mapsto \mathsf{Uni}\{\sigma \mathfrak{H} : \mathfrak{H} \in \mathcal{U}\} \end{aligned} \qquad \begin{aligned} \rho : \mathbf{Ext}(\mathtt{IPC_R}) &\to \mathbf{NExt}(\mathtt{S4_R}) \\ \mathcal{W} &\mapsto \{\rho \mathfrak{A} : \mathfrak{A} \in \mathcal{W}\}. \end{aligned}$$

Finally, introduce a semantic counterpart of τ as follows.

$$\begin{aligned} \tau : \mathbf{Uni}(\mathsf{HA}) &\to \mathbf{Uni}(\mathsf{S4}) \\ \mathcal{U} &\mapsto \{\mathfrak{A} \in \mathsf{S4} : \rho \mathfrak{A} \in \mathcal{U}\} \end{aligned}$$

The goal of this subsection is to establish the following two classic results in the theory of modal companions. Firstly, that for every si-deductive system $\mathtt{L}$, the modal companions of $\mathtt{L}$ are exactly the elements of the interval $\rho^{-1}(\mathtt{L})$ (Theorem 2.54). Secondly, and relatedly, that the syntactic mappings σ, ρ are mutually inverse isomorphism (Theorem 2.55). This last result (restricted to logics) is widely known as the *Blok-Esakia theorem*.

The main problem one needs to deal with in order to prove the results just mentioned consists in showing that the mapping $\sigma : \mathbf{Ext}(\mathtt{IPC_R}) \to \mathbf{NExt}(\mathtt{Grz_R})$ is surjective. We solve this problem by first applying stable canonical rules to show that the semantic mapping $\sigma : \mathbf{Uni}(\mathsf{HA}) \to \mathbf{Uni}(\mathsf{Grz})$ is surjective, and subsequently establishing that the syntactic and semantic versions of σ capture essentially the same transformation. Our key tool is the following technical lemma.

Lemma 2.50. Let $\mathfrak{A} \in \mathsf{Grz}$. Then for every modal rule Γ/Δ, $\mathfrak{A} \models \Gamma/\Delta$ iff $\sigma\rho\mathfrak{A} \models \Gamma/\Delta$.

Proof. ($\Rightarrow$) This direction follows from the fact that $\sigma\rho\mathfrak{A} \rightarrowtail \mathfrak{A}$ (Proposition 2.45).

($\Leftarrow$) We prove the dual statement that $\mathfrak{A}_* \not\models \Gamma/\Delta$ implies $\sigma\rho\mathfrak{A}_* \not\models \Gamma/\Delta$. Let $\mathfrak{X} := \mathfrak{A}_*$. In view of Theorem 2.36 it suffices to consider the case $\Gamma/\Delta = \mu(\mathfrak{B}, D)$, for $\mathfrak{B} \in \mathsf{S4}$ finite. So suppose $\mathfrak{X} \not\models \mu(\mathfrak{B}, D)$ and let $\mathfrak{F} := \mathfrak{B}_*$. Then there is a stable map $f : \mathfrak{X} \to \mathfrak{F}$ satisfying the BDC for $\mathfrak{D} := \{\mathfrak{d}_a : a \in D\}$. We construct a stable map $g : \sigma\rho\mathfrak{X} \to \mathfrak{F}$ which also satisfies the BDC for $\mathfrak{D}$. By the refutation conditions for stable canonical rules, this would show that $\sigma\rho\mathfrak{X} \not\models \mu(\mathfrak{B}, D)$, hence would conclude the proof.

Let $C \subseteq F$ be some cluster. Consider $Z_C := f^{-1}(C)$. As f is continuous, $Z_C \in \mathsf{Clop}(\mathfrak{X})$. Moreover, since f is stable Z_C does not cut any cluster. It follows that $\rho[Z_C]$ is clopen in $\sigma\rho\mathfrak{X}$, because $\sigma\rho\mathfrak{X}$ has the quotient topology. Enumerate $C := \{x_1, \dots, x_n\}$. Then $f^{-1}(x_i) \subseteq Z_C$ is clopen. By Proposition 2.18 we find that $M_i := max(f^{-1}(x_i))$ is closed. Furthermore, as $\mathfrak{X}$ is a $\mathtt{Grz}$ space and every element of M_i is passive in M_i, by Proposition 2.18 again we have that M_i does not cut any cluster. Therefore $\rho[M_i]$ is closed, again because $\sigma\rho\mathfrak{X}$ has the quotient topology. Clearly, $\rho[M_i] \cap \rho[M_j] = \emptyset$ for each $i \neq j$. We shall now find disjoint clopens $U_1, \dots, U_n \in \mathsf{Clop}(\sigma\rho\mathfrak{X})$ with $\rho[M_i] \subseteq U_i$ and $\bigcup_i U_i = \rho[Z_C]$. Let $k \leq n$ and assume that U_i has been defined for all $i < k$. If $k = n$ put $U_n = \rho[Z_C] \setminus (\bigcup_{i<k} U_i)$ and we are done. Otherwise set $V_k := \rho[Z_C] \setminus (\bigcup_{i<k} U_i)$ and observe that it contains each $\rho[M_i]$ for $k \leq i \leq n$. Apply Proposition 1.6, to find for each i with $k < i \leq n$ some $U_{k_i} \in \mathsf{Clop}(\sigma\rho\mathfrak{X})$ with $\rho[M_k] \subseteq U_{k_i}$ and $\rho[M_i] \cap U_{k_i} = \emptyset$. Then set $U_k := \bigcap_{k<i\leq n} U_{k_i} \cap V_k$.

Now define a map

$$
\begin{aligned}
g_C : \rho[Z_C] &\to C \\
z \mapsto x_i &\iff z \in U_i.
\end{aligned}
$$

Note that g_C is relation preserving, evidently, and continuous by construction. Finally, define $g : \sigma\rho\mathfrak{X} \to F$ by setting

$$
g(\rho(z)) := \begin{cases} f(z) & \text{if } f(z) \text{ does not belong to any proper cluster} \\ g_C(\rho(z)) & \text{if } f(z) \in C \text{ for some proper cluster } C \subseteq F. \end{cases}
$$

Now, g is evidently relation preserving. Moreover, it is continuous because both f and each g_C are. Suppose $Rg(\rho(x))y$ and $y \in \mathfrak{d}$ for some $\mathfrak{d} \in \mathfrak{D}$. By construction, $f(x)$ belongs to the same cluster as $g(\rho(x))$, so also $Rf(x)y$. Since f satisfies the BDC for $\mathfrak{D}$, there must be some $z \in X$ such that Rxz and $f(z) \in \mathfrak{d}$. Since $f^{-1}(f(z)) \in \mathsf{Clop}(\mathfrak{X})$, by Proposition 2.18 there is $z' \in max(f^{-1}(f(z)))$ with Rzz'. Then also Rxz' and $f(z') \in \mathfrak{d}$. But from $z' \in max(f^{-1}(f(z))$ it follows that $f(z') = g(\rho(z'))$ by construction, so we have $g(\rho(z')) \in \mathfrak{d}$. As clearly $R\rho(x)\rho(z')$, this concludes the proof. □

Theorem 2.51. Every $\mathcal{U} \in \mathsf{Uni}(\mathsf{Grz})$ is generated by its skeletal elements, i.e., $\mathcal{U} = \sigma\rho\mathcal{U}$.

Proof. By $\sigma\rho\mathfrak{A} \rightarrowtail \mathfrak{A}$ (Proposition 2.45), surely $\sigma\rho\mathcal{U} \subseteq \mathcal{U}$. Conversely, suppose $\mathcal{U} \not\models \Gamma/\Delta$. Then there is $\mathfrak{A} \in \mathcal{U}$ with $\mathfrak{A} \not\models \Gamma/\Delta$. By Lemma 2.50 it follows that $\sigma\rho\mathfrak{A} \not\models \Gamma/\Delta$. This shows $\mathsf{ThR}(\sigma\rho\mathcal{U}) \subseteq \mathsf{ThR}(\mathcal{U})$, which is equivalent to $\mathcal{U} \subseteq \sigma\rho\mathcal{U}$. Hence indeed $\mathcal{U} = \sigma\rho\mathcal{U}$. □

The restriction of Theorem 2.51 to varieties is perhaps best known as the main consequence of the so-called *Blok-lemma*, proved in Blok [1976]. The unrestricted version is explicitly stated and proved in Stronkowski [2018, Lemma 4.4], although it also follows from Jerábek [2009, Theorem 5.5].

We now apply our results to show that the syntactic modal companion maps τ, ρ, σ commute with $\mathsf{Alg}(\cdot)$.

Lemma 2.52 (Jerábek 2009, Theorem 5.9)**.** For each $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC_R})$ and $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4_R})$, the following hold:

$$\mathsf{Alg}(\tau\mathtt{L}) = \tau\mathsf{Alg}(\mathtt{L}) \tag{2.3}$$

$$\mathsf{Alg}(\sigma\mathtt{L}) = \sigma\mathsf{Alg}(\mathtt{L}) \tag{2.4}$$

$$\mathsf{Alg}(\rho\mathtt{M}) = \rho\mathsf{Alg}(\mathtt{M}) \tag{2.5}$$

Proof. (2.3) For every $\mathfrak{A} \in \mathsf{S4}$ we have $\mathfrak{A} \in \mathsf{Alg}(\tau\mathtt{L})$ iff $\mathfrak{A} \models T(\Gamma/\Delta)$ for all $\Gamma/\Delta \in \mathtt{L}$ iff $\rho\mathfrak{A} \models \Gamma/\Delta$ for all $\Gamma/\Delta \in \mathtt{L}$ iff $\rho\mathfrak{A} \in \mathsf{Alg}(\mathtt{L})$ iff $\mathfrak{A} \in \tau\mathsf{Alg}(\mathtt{L})$.

(2.4) In view of Theorem 2.51 it suffices to show that $\mathsf{Alg}(\sigma\mathtt{L})$ and $\sigma\mathsf{Alg}(\mathtt{L})$ have the same skeletal elements. So let $\mathfrak{A} = \sigma\rho\mathfrak{A} \in \mathsf{Grz}$. Assume $\mathfrak{A} \in \sigma\mathsf{Alg}(\mathtt{L})$. Since $\sigma\mathsf{Alg}(\mathtt{L})$ is generated by $\{\sigma\mathfrak{B} : \mathfrak{B} \in \mathsf{Alg}(\mathtt{L})\}$ as a universal class, by Proposition 2.45 and Lemma 2.48 we have $\mathfrak{A} \models T(\Gamma/\Delta)$ for every $\Gamma/\Delta \in \mathtt{L}$. But then $\mathfrak{A} \in \mathsf{Alg}(\sigma\mathtt{L})$. Conversely, assume $\mathfrak{A} \in \mathsf{Alg}(\sigma\mathtt{L})$. Then $\mathfrak{A} \models T(\Gamma/\Delta)$ for every $\Gamma/\Delta \in \mathtt{L}$. By Lemma 2.48 this is equivalent to $\rho\mathfrak{A} \in \mathsf{Alg}(\mathtt{L})$, therefore $\mathfrak{A} = \sigma\rho\mathfrak{A} \in \sigma\mathsf{Alg}(\mathtt{L})$.

(2.5) Let $\mathfrak{H} \in \mathsf{HA}$. If $\mathfrak{H} \in \rho\mathsf{Alg}(\mathtt{M})$ then $\mathfrak{H} = \rho\mathfrak{A}$ for some $\mathfrak{A} \in \mathsf{Alg}(\mathtt{M})$. It follows that for every si rule $T(\Gamma/\Delta) \in \mathtt{M}$ we have $\mathfrak{A} \models T(\Gamma/\Delta)$, and so by Lemma 2.48 in turn $\mathfrak{H} \models \Gamma/\Delta$. Therefore indeed $\mathfrak{H} \in \mathsf{Alg}(\rho\mathtt{M})$. Conversely, for all si rules Γ/Δ, if $\rho\mathsf{Alg}(\mathtt{M}) \models \Gamma/\Delta$ then by Lemma 2.48 $\mathsf{Alg}(\mathtt{M}) \models T(\Gamma/\Delta)$, hence $\Gamma/\Delta \in \rho\mathtt{M}$. Thus $\mathsf{ThR}(\rho\mathsf{Alg}(\mathtt{M})) \subseteq \rho\mathtt{M}$, and so $\mathsf{Alg}(\rho\mathtt{M}) \subseteq \rho\mathsf{Alg}(\mathtt{M})$. □

The result just proved leads straightforwardly to the following, purely semantic characterisation of modal companions.

Lemma 2.53. $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4_R})$ is a modal companion of $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC_R})$ iff $\mathsf{Alg}(\mathtt{L}) = \rho\mathsf{Alg}(\mathtt{M})$.

Proof. ($\Rightarrow$) Assume $\mathtt{M}$ is a modal companion of $\mathtt{L}$. Then we have $\mathtt{L} = \rho\mathtt{M}$. By Lemma 2.52 $\mathsf{Alg}(\mathtt{L}) = \rho\mathsf{Alg}(\mathtt{M})$.

($\Leftarrow$) Assume that $\mathsf{Alg}(\mathtt{L}) = \rho\mathsf{Alg}(\mathtt{M})$. Therefore, by Proposition 2.45, $\mathfrak{H} \in \mathsf{Alg}(\mathtt{L})$ implies $\sigma\mathfrak{H} \in \mathsf{Alg}(\mathtt{M})$. This implies that for every si rule Γ/Δ, $\Gamma/\Delta \in \mathtt{L}$ iff $T(\Gamma/\Delta) \in \mathtt{M}$. □

Having characterised the notion of a modal companion semantically, we may now employ Lemma 2.50 to derive the heart of the theory of modal companions, summarised by the two theorems below.

Theorem 2.54 (Jerábek 2009, Theorem 5.5, Zakharyashchev 1991, Theorem 3)**.** The following conditions hold:

1. For every $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC_R})$, the modal companions of $\mathtt{L}$ form an interval $\{\mathtt{M} \in \mathbf{NExt}(\mathtt{S4_R}) : \tau\mathtt{L} \leq \mathtt{M} \leq \sigma\mathtt{L}\}$.
2. For every $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC})$, the modal companions of $\mathtt{L}$ form an interval $\{\mathtt{M} \in \mathbf{NExt}(\mathtt{S4}) : \tau\mathtt{L} \leq \mathtt{M} \leq \sigma\mathtt{L}\}$.

Proof. (1) In view of Lemma 2.52 it suffices to prove that $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4_R})$ is a modal companion of $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC_R})$ iff $\sigma\mathsf{Alg}(\mathtt{L}) \subseteq \mathsf{Alg}(\mathtt{M}) \subseteq \tau\mathsf{Alg}(\mathtt{L})$.

($\Rightarrow$) Assume $\mathtt{M}$ is a modal companion of $\mathtt{L}$. Then by Lemma 2.53 we have $\mathsf{Alg}(\mathtt{L}) = \rho\mathsf{Alg}(\mathtt{M})$, therefore it is clear that $\mathsf{Alg}(\mathtt{M}) \subseteq \tau\mathsf{Alg}(\mathtt{L})$. To see that $\sigma\mathsf{Alg}(\mathtt{L}) \subseteq \mathsf{Alg}(\mathtt{M})$ it suffices to show that every skeletal algebra in $\sigma\mathsf{Alg}(\mathtt{L})$ belongs to $\mathsf{Alg}(\mathtt{M})$. So let $\mathfrak{A} \cong \sigma\rho\mathfrak{A} \in \sigma\mathsf{Alg}(\mathtt{L})$. Then $\rho\mathfrak{A} \in \mathsf{Alg}(\mathtt{L})$ by Lemma 2.48, so there must be $\mathfrak{B} \in \mathsf{Alg}(\mathtt{M})$ such that $\rho\mathfrak{B} \cong \rho\mathfrak{A}$. But this implies $\sigma\rho\mathfrak{B} \cong \sigma\rho\mathfrak{A} \cong \mathfrak{A}$, and as universal classes are closed under subalgebras, by Proposition 2.45 we conclude $\mathfrak{A} \in \mathsf{Alg}(\mathtt{M})$.

($\Leftarrow$) Assume $\sigma\mathsf{Alg}(\mathtt{L}) \subseteq \mathsf{Alg}(\mathtt{M}) \subseteq \tau\mathsf{Alg}(\mathtt{L})$. It is an immediate consequence of Proposition 2.45 that $\rho\sigma\mathsf{Alg}(\mathtt{L}) = \mathsf{Alg}(\mathtt{L})$, which gives us $\rho\mathsf{Alg}(\mathtt{M}) \supseteq \mathsf{Alg}(\mathtt{L})$. But by construction $\rho\mathsf{Alg}(\mathtt{M}) = \rho\tau\mathsf{Alg}(\mathtt{L})$, hence $\rho\mathsf{Alg}(\mathtt{M}) \subseteq \mathsf{Alg}(\mathtt{L})$. Therefore indeed $\rho\mathsf{Alg}(\mathtt{M}) = \mathsf{Alg}(\mathtt{L})$, so by Lemma 2.53 we conclude that $\mathtt{M}$ is a modal companion of $\mathtt{L}$.

(2) Immediate from Item 1. □

Theorem 2.55 (Blok Esakia theorem)**.** The following conditions hold:

1. The mappings $\sigma : \mathbf{Ext}(\mathtt{IPC_R}) \to \mathbf{NExt}(\mathtt{Grz_R})$ and $\rho : \mathbf{NExt}(\mathtt{Grz_R}) \to \mathbf{Ext}(\mathtt{IPC_R})$ are complete lattice isomorphisms and mutual inverses.
2. The mappings $\sigma : \mathbf{Ext}(\mathtt{IPC}) \to \mathbf{NExt}(\mathtt{Grz})$ and $\rho : \mathbf{NExt}(\mathtt{Grz}) \to \mathbf{Ext}(\mathtt{IPC})$ are complete lattice isomorphisms and mutual inverses.

Proof. (1) It is enough to show that the mappings $\sigma : \mathbf{Uni}(\mathsf{HA}) \to \mathbf{NExt}(\mathsf{Grz})$ and $\rho : \mathbf{NExt}(\mathsf{Grz}) \to \mathbf{Ext}(\mathsf{HA})$ are complete lattice isomorphisms and mutual inverses. Both maps are evidently order preserving, and preservation of infinite joins is an easy consequence of Lemma 2.48. Let $\mathcal{U} \in \mathbf{Uni}(\mathsf{Grz})$. Then $\mathcal{U} = \sigma\rho\mathcal{U}$ by Theorem 2.51, so σ is surjective and a left inverse of ρ. Now let $\mathcal{U} \in \mathbf{Uni}(\mathsf{HA})$. It is an immediate consequence of Proposition 2.45 that $\rho\sigma\mathcal{U} = \mathcal{U}$. Hence ρ is surjective and a left inverse of σ. Thus σ and ρ are mutual inverses, and therefore must both be bijections.

(2) Immediate from Item 1 and Propositions 2.2 and 2.10. □

§2.3.3 Axiomatisation of Modal Companions and Superintuitionistic Fragments via Stable Canonical Rules

In this section we give alternative axiomatic characterisations of the mappings τ, σ, ρ, by describing how applying these mappings affects axiomatisations of rule systems in terms of stable canonical rules. Unless otherwise stated, all results in this section are new.

Our first result concerns the maps τ, σ. Modal rule systems in the range of these maps are axiomatised, over $\mathtt{S4_R}$ and $\mathtt{Grz_R}$ respectively, by Gödel translations of si stable canonical rules. Mirroring a similar result of Zakharyashchev [1991], the next lemma characterises such rules as modal stable canonical rules of finite $\mathtt{Grz}$-algebras.

Lemma 2.56 (Rule translation lemma)**.** For every si stable canonical rule $\eta(\mathfrak{H}, D^{\to})$ and every $\mathfrak{B} \in \mathsf{MA}$ we have

$$\mathfrak{B} \models T(\eta(\mathfrak{H}, D^{\to})) \iff \mathfrak{B} \models \mu(\sigma\mathfrak{H}, D^{\square}),$$

where $D^{\square} := \{\neg a \vee b : (a, b) \in D^{\to}\}$.

Proof. Let $\mathfrak{F} := \mathfrak{H}_*$. Then it is clear that $\sigma\mathfrak{F} = \mathfrak{F}$. By Proposition 2.44 and Lemma 2.48 it suffices to prove that for all modal spaces $\mathfrak{X}$ we have

$$\rho\mathfrak{X} \models \eta(\mathfrak{F}, \mathfrak{D}) \iff \mathfrak{X} \models \mu(\mathfrak{F}, \mathfrak{D})$$

where $\mathfrak{D} = \{\mathfrak{d}_{(a,b)} : (a, b) \in D^{\to}\} = \{\mathfrak{d}_a : a \in D^{\square}\}$.

($\Rightarrow$) Assume $\mathfrak{X} \not\models \mu(\mathfrak{F}, \mathfrak{D})$. Then there is a stable map $f : \mathfrak{X} \to \mathfrak{F}$ satisfying the BDC for $\mathfrak{D}$. Observe that since $\mathfrak{F}$ is a poset, for all $x, y \in X$ we have that $\rho(x) = \rho(y)$ implies $f(x) = f(y)$. Therefore the map

$$\begin{aligned} g : \rho\mathfrak{X} &\to \mathfrak{F} \\ \rho(x) &\mapsto f(x) \end{aligned}$$

is well defined. Evidently, g is relation preserving. To see that it is continuous, let $U \subseteq F$. Now, $f^{-1}(U)$ is clopen since f is continuous, and because f is relation preserving $f^{-1}(U)$ does not cut clusters. Therefore $g^{-1}(U) = \rho[f^{-1}(U)]$ is clopen since $\rho\mathfrak{X}$ has the quotient topology. Let us check that g satisfies the BDC for $\mathfrak{D}$. To this end, let $\mathfrak{d} \in \mathfrak{D}$ and suppose ${\uparrow}g(\rho(x)) \cap \mathfrak{d} \neq \emptyset$ for $\rho(x) \in \rho[X]$. But then $R[f(x)] \cap \mathfrak{d} \neq \emptyset$. Since f satisfies the BDC for $\mathfrak{D}$, there is $y \in X$ with Rxy and $f(y) \in \mathfrak{d}$. Therefore $\rho(x) \leq \rho(y)$, and surely $g(\rho(y)) = f(y) \in \mathfrak{d}$. Thus we have shown $\rho\mathfrak{X} \not\models \eta(\mathfrak{F}, \mathfrak{D})$.

($\Leftarrow$) Assume $\rho\mathfrak{X} \models \eta(\mathfrak{F}, \mathfrak{D})$. Then there is a stable map $g : \rho\mathfrak{X} \to \mathfrak{F}$ satisfying the BDC for $\mathfrak{D}$. Define a map

$$\begin{aligned} f : \mathfrak{X} &\to \mathfrak{F} \\ x &\mapsto g(\rho(x)). \end{aligned}$$

Since g and $\rho : \mathfrak{X} \to \rho\mathfrak{X}$ are both continuous and relation-preserving, so is their composition f. Let us check that f satisfies the BDC for $\mathfrak{D}$. Let $\mathfrak{d} \in \mathfrak{D}$ and assume $R[f(x)] \cap \mathfrak{d} \neq \emptyset$. This is to say ${\uparrow}g(\rho(x)) \cap \mathfrak{d} \neq \emptyset$. Since g satisfies the BDC for $\mathfrak{D}$, there must be some $\rho(y)$ with $\rho(x) \leq \rho(y)$ and $g(\rho(y)) \in \mathfrak{d}$. Since ρ is relation reflecting it follows that Rxy, and surely $f(y) = g(\rho(y)) \in \mathfrak{d}$. Thus we have shown $\mathfrak{X} \not\models \mu(\mathfrak{F}, \mathfrak{D})$. $\square$

Theorem 2.57. Let $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC_R})$ be such that

$$\mathtt{L} = \mathtt{IPC_R} \oplus \{\eta(\mathfrak{A}_i, D_i^{\to}) : i \in I\}.$$

Then we have:

$$\tau\mathtt{L} = \mathtt{S4_R} \oplus \{\mu(\sigma\mathfrak{A}_i, D_i^{\square}) : i \in I\} \tag{2.6}$$

$$\sigma\mathtt{L} = \mathtt{Grz_R} \oplus \{\mu(\sigma\mathfrak{A}_i, D_i^{\square}) : i \in I\}. \tag{2.7}$$

Proof. By definition we have

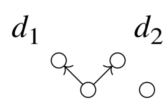
$$\tau\mathtt{L} = \mathtt{S4_R} \oplus \{T(\eta(\mathfrak{A}_i, D_i^{\to})) : i \in I\}$$
$$\sigma\mathtt{L} = \mathtt{Grz_R} \oplus \{T(\eta(\mathfrak{A}_i, D_i^{\to})) : i \in I\}.$$

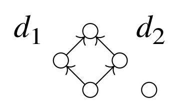
Applying the rule translation lemma to these axiomatisations yields the desired result. □

In particular we obtain that every rule system in $\mathbf{NExt}(\mathtt{Grz_R})$ is axiomatisable by modal stable canonical rules of finite Grz-algebras, since by the Blok Esakia theorem for rule systems every such rule system is the strongest modal companion of some si-rule system. Equation (2.6) of Theorem 2.57 may be compared to Bezhanishvili et al. [2016a, Corollary 5.3], which implies that the weakest modal companion of any si *logic* axiomatisable by *stable* formulae is axiomatisable by stable formulae of finite Grz-algebras.

We illustrate the axiomatisation procedure descibed in Theorem 2.57 with an example. Consider the *Gödel-Dummett Logic* LC, given by

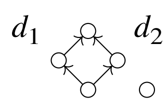
$$\mathtt{LC} := \mathtt{IPC} \oplus (p \to q) \vee (q \to p).$$

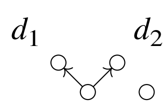
Theorem 2.58. $\mathtt{LC_R} = \mathtt{IPC_R} \oplus \eta\Bigg($ 

$\Bigg) \oplus \eta\Bigg($ 

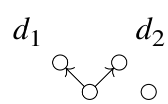
$\Bigg).$

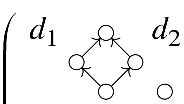
Proof. (⇒) let $\mathfrak{X} \in \mathsf{Esa}$ and suppose that $\mathfrak{X} \not\models \mathtt{LC_R}$. It is easy to see that this implies that there are $x, y, z \in X$ such that $x \leq y$, $x \leq z$ and yet $y \not\leq z \not\leq y$. By Proposition 2.6 there are $U_y, U_z \in \mathsf{ClopUp}$ such that $y \in U_y, z \notin U_y$ and $z \in U_z, y \notin U_z$ and moreover $x \notin U_y \cup U_z$. Distinguish two cases.

- $U_y \cap U_z \neq \emptyset$. Define a map f from $\mathfrak{X}$ to 

by sending the set $U_y \setminus U_z$ to d_1, the set $U_z \setminus U_y$ to d_2, the set $U_y \cap U_z$ to the top element, the set $(R^{-1}(U_y) \setminus U_y) \cup (R^{-1}(U_z) \setminus U_z)$ to the bottom left element, and everything else to the bottom right element. It is easy to see that f so defined is a stable surjection satisfying the BDC for $\{\{d_1\}, \{d_2\}\}$.
- $U_y \cap U_z = \emptyset$. We define a map f from $\mathfrak{X}$ to 

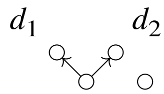
in a similar way as above, but ignoring the case of $U_y \cap U_z$. Again, this yields a stable surjection satisfying the BDC for $\{\{d_1\}, \{d_2\}\}$.

(⇐) Let $\mathfrak{X} \in \mathsf{Esa}$ and suppose that $\mathfrak{X} \not\models \eta\Bigg($ 

$\Bigg)$ or $\mathfrak{X} \not\models \eta\Bigg($ 

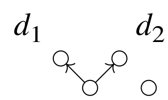
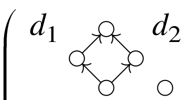
$\Bigg).$

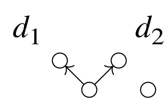
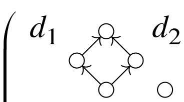
We only do the former case, as the latter is similar. Take a continuous stable surjection f from

$\mathfrak{X}$ to  satisfying the BDC for $\{\{d_1\}, \{d_2\}\}$. Then by the BDC any element x in the f-preimage of the bottom left sees both an element y in the f-preimage of d_1 and an element z in the f-preimage of d_2. However, by stability we have both $y \not\leq z$ and $z \not\leq y$. This shows $\mathfrak{X} \not\models \mathtt{LC_R}$. □

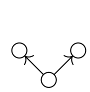
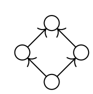
Applying, Theorem 2.57 we obtain the following axiomatisations of the weakest and strongest modal companion of $\mathtt{LC_R}$, which are easily seen to be the modal rule systems $\mathtt{S4.3_R} := \mathtt{S4_R} \oplus \Box(p \to q) \vee \Box(q \to p)$ and $\mathtt{Grz.3_R} := \mathtt{Grz_R} \oplus \Box(p \to q) \vee \Box(q \to p)$.

Corollary 2.59. The following identities hold:

$\mathtt{S4.3_R} = \mathtt{S4_R} \oplus \mu\Big($  $\Big) \oplus \mu\Big($  $\Big)$

$\mathtt{Grz.3_R} = \mathtt{Grz_R} \oplus \mu\Big($  $\Big) \oplus \mu\Big($  $\Big)$.

It is interesting to compare Corollary 2.59 to [Bezhanishvili et al., 2018, Proposition 6.8], which shows that S4.3 is axiomatised, over S4, by the *stable formulae* of the following two frames.

The difference is explained by the fact that refutation conditions for stable canonical formulae are stated in terms of subdirectly irreducible modal algebras, whose duals are topo-rooted [Venema, 2004].

We now turn to the task of giving an axiomatic characterisation of si-fragments in terms of stable canonical rules. To this end, we introduce the notion of a *collapsed* stable canonical rule. We prefer to do so in a geometric setting, so to emphasize the main intuition behind this concept.

Definition 2.60. Let $\mu(\mathfrak{F}, \mathfrak{D})$ be some modal stable canonical rule, with $\mathfrak{F} \in \mathsf{Spa}(\mathtt{S4})$. The *collapsed stable canonical rule* $\eta(\rho\mathfrak{F}, \rho\mathfrak{D})$ is obtained by setting

$$\rho\mathfrak{D} := \{\rho[\mathfrak{d}] : \mathfrak{d} \in \mathfrak{D}\}.$$

Intuitively, $\eta(\rho\mathfrak{F}, \rho\mathfrak{D})$ is obtained from $\mu(\mathfrak{F}, \mathfrak{D})$ by collapsing all clusters in $\mathfrak{F}$ and in the set of domains $\mathfrak{D}$ as well.

Remark 2.61. It is possible to develop collapsed rules in a purely algebraic setting. Let $\mu(\mathfrak{A}, D)$ be some modal stable canonical rule, with $\mathfrak{A} \in \mathsf{S4}$. Algebraically, the collapsed stable canonical $\eta(\rho\mathfrak{A}, \rho D)$ is obtained from the stable canonical rule $\mu(\mathfrak{A}, D)$ as follows. For $a \in A$, take

$$a_\rho := \bigwedge\{b \in B(O(A)) : a \leq b\}.$$

By Proposition 2.40, there is a finite n_{a_ρ} such that a_ρ may be written as

$$\bigwedge_{i \leq n_{a_\rho}} \neg b_i \vee c_i.$$

Let H_a be the set of all such pairs (b_i, c_i) with $i \leq n_{a_\rho}$. Set

$$\rho D := \bigcup_{a \in D} H_a.$$

Then one can prove that for every $\mathfrak{B} \in \mathsf{S4}$ we have $\mathfrak{B} \models \eta(\rho\mathfrak{A}, \rho D)$ iff $\eta(\rho\mathfrak{A}_*, \rho\mathfrak{D})$.

Collapsed rules obey the following refutation condition.

Lemma 2.62 (Rule collapse lemma)**.** For all $\mathfrak{X} \in \mathsf{Spa}(\mathtt{S4})$ and modal stable canonical rule $\mu(\mathfrak{F}, \mathfrak{D})$ with $\mathfrak{F} \in \mathsf{Spa}(\mathtt{S4})$, if $\mathfrak{X} \not\models \mu(\mathfrak{F}, \mathfrak{D})$ then $\rho\mathfrak{X} \not\models \eta(\rho\mathfrak{F}, \rho\mathfrak{D})$.

Proof. Assume $\mathfrak{X} \not\models \mu(\mathfrak{F}, \mathfrak{D})$. Then there is a continuous, relation preserving map $f : \mathfrak{X} \to \mathfrak{F}$ that satisfies the BDC for $\mathfrak{D}$. Consider the map $g : \rho\mathfrak{X} \to \rho\mathfrak{F}$ given by

$$g(\rho(x)) = \rho(f(x)).$$

Now $\rho(x) \leq \rho(y)$ implies Rxy, and since f is relation preserving also $Rf(x)f(y)$, which implies $\rho(f(x)) \leq \rho(f(y))$. So g is relation preserving. Furthermore, again because f is relation preserving we have that for any $U \subseteq F$, the set $f^{-1}(U)$ does not cut clusters, whence $g^{-1}(U) = \rho[f^{-1}(\rho^{-1}(U))]$ is clopen for any $U \subseteq \rho[F]$, as $\rho\mathfrak{X}$ has the quotient topology. Thus g is continuous. Let us check that g satisfies the BDC for $\rho\mathfrak{D}$. Assume that ${\uparrow}g(\rho(x)) \cap \rho[\mathfrak{d}] \neq \emptyset$ for $\mathfrak{d} \in \mathfrak{D}$. Then there is some $\rho(y) \in \rho[F]$ with $\rho(f(x)) \leq \rho(y)$ and $\rho(y) \in \rho[\mathfrak{d}]$. By construction, wlog we may assume that $y \in \mathfrak{d}$. As ρ is relation reflecting it follows that $Rf(x)y$, and so we have that $R[f(x)] \cap \mathfrak{d} \neq \emptyset$. Since f satisfies the BDC for $\mathfrak{D}$ we conclude that $f[R[x]] \cap \mathfrak{d} \neq \emptyset$. So there is some $z \in X$ with Rxz and $f(z) \in \mathfrak{d}$. By definition, $\rho(f(z)) \in \rho[\mathfrak{d}]$. Hence we have shown that $\rho[f[R[x]]] \cap \rho[\mathfrak{d}] \neq \emptyset$, and so g indeed satisfies the BDC for $\mathfrak{D}$. □

By duality and the rule translation lemma, we obtain as an immediate corollary that for all $\mathfrak{X} \in \mathsf{Spa}(\mathtt{S4})$ and modal stable canonical rule $\mu(\mathfrak{F}, \mathfrak{D})$ with $\mathfrak{F} \in \mathsf{Spa}(\mathtt{S4})$, if $\mathfrak{X} \not\models \mu(\mathfrak{F}, \mathfrak{D})$ then $\sigma\rho\mathfrak{X} \not\models \mu(\sigma\rho\mathfrak{F}, \rho\mathfrak{D})$.

It is important to point out that the converse of the rule collapse lemma is false, as the following counter-example testifies.

Example 2.63. Let $\mathfrak{F}$ be a finite modal space consisting of a n-elements clusters, for n finite, and let $\mathfrak{G}$ be a finite modal space consisting of a single reflexive point. Then there is no surjection $f : \mathfrak{G} \to \mathfrak{F}$, on cardinality grounds alone. This is equivalent to $\mathfrak{G} \models \mu(\mathfrak{F})$. However, we have $\rho\mathfrak{F} \cong \mathfrak{G} \cong \rho\mathfrak{G}$, so there clearly is a stable surjection $f : \rho\mathfrak{G} \to \rho\mathfrak{F}$. This, in turn, is equivalent to $\rho\mathfrak{G} \not\models \eta(\rho\mathfrak{F})$.

Observe that by the rule translation lemma, a modal rule Γ/Δ is of the form $T(\Gamma'/\Delta')$ iff it is equivalent to finitely many modal stable canonical rules of the form $\mu(\mathfrak{A}, D)$, with $\mathfrak{A} \in \mathsf{Grz}$. Therefore, by the rule collapse lemma, given a stable canonical axiomatisation of some $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4_R})$ as $\mathtt{M} = \mathtt{S4_R} \oplus \{\mu(\mathfrak{F}_i, \mathfrak{D}_i) : i \in I\}$, the set

$$\{\mu(\mathfrak{F}_i, \mathfrak{D}_i) : i \in I\} \text{ and } \mu(\sigma\rho\mathfrak{F}_i, \rho\mathfrak{D}_i) \in \mathtt{M}\}$$

contains exactly the rules in the given axiomatisation of M which are consequences of some set of translated si rules contained in M. Collapsing these rules yields an axomatisation of $\rho\mathtt{M}$.

Theorem 2.64. Let $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4_R})$ with $\mathtt{M} = \mathtt{S4_R} \oplus \{\mu(\mathfrak{F}_i, \mathfrak{D}_i) : i \in I\}$. Let

$$J := \{i \in I : \mu(\sigma\rho\mathfrak{F}_i, \rho\mathfrak{D}_i) \in \mathtt{M}\}.$$

Then

$$\rho\mathtt{M} = \mathtt{IPC_R} \oplus \{\eta(\rho\mathfrak{F}_i, \rho\mathfrak{D}_i) : i \in J\}.$$

Proof. Let

$$\Xi := \{\mu(\sigma\rho\mathfrak{F}_i, \rho\mathfrak{D}_i) : i \in J\} \cup \{\mu(\mathfrak{F}_i, \mathfrak{D}_i) : i \in I\}.$$

Then by the rule collapse lemma we have

$$\rho\mathtt{M} = \mathtt{IPC_R} \oplus \{\eta(\mathfrak{F}, \mathfrak{D}) : T(\eta(\mathfrak{H}, D)) \in \Xi\},$$

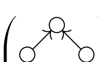
which by the rule translation lemma is equivalent to

$$\begin{aligned} \rho\mathtt{M} &= \mathtt{IPC_R} \oplus \{\eta(\mathfrak{F}, \mathfrak{D}) : \eta(\sigma\mathfrak{F}, \mathfrak{D}) \in \Xi\} \\ &= \mathtt{IPC_R} \oplus \{\eta(\rho\mathfrak{F}_i, \rho\mathfrak{D}_i) : i \in J\}. \end{aligned}$$

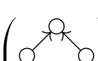
□

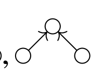
We give two examples describing the axiomatisation procedure obtained in Theorem 2.64. Firstly, consider the rule system obtained by extending $\mathtt{S4_R}$ with the *rule of disjunction*:

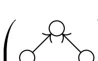
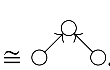
$$\mathtt{Disj} := \mathtt{S4_R} \oplus \Box p \vee \Box q / \{p, q\}.$$

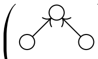
Theorem 2.65 (Bezhanishvili et al. 2016a, Theorem 8.6)**.** $\mathtt{Disj} := \mathtt{S4_R} \oplus \mu$ (○ ○) $\oplus\, \mu$ ().

Via Theorem 2.64 we obtain the following easy corollary.

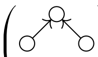
Corollary 2.66. $\rho\mathtt{Disj} = \mathtt{IPC_R} \oplus \eta$(○ ○) $\oplus\, \eta$ ().

Proof. Since both ○ ○,  are cluster-free we have

$\sigma\rho$ (○ ○) $\cong$ ○ ○ $\qquad \sigma\rho$ () $\cong$ .

Therefore $\mu\left(\sigma\rho\left(\circ\quad\circ\right)\right), \mu\left(\sigma\rho\left(\right.\right.$  $\left.\left.\right)\right) \in \mathtt{Disj}$ and in turn by Theorem 2.64 we conclude

$$\rho\mathtt{Disj} = \mathtt{IPC_R} \oplus \eta(\circ\quad\circ) \oplus \eta\left(\quad\right)$$

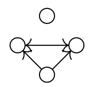


as desired. ☐

Secondly, consider the logic

$$\mathtt{S4.1} := \mathtt{S} \oplus \Box\Diamond p \to \Diamond\Box p.$$

Let $\mathfrak{X}$ be the modal space

d_1  d_2

and put $\mathfrak{D} = \{\{d_1\}, \{d_2\}\}$. We extend Bezhanishvili et al.'s [2016a, Theorem 8.7] axiomatisation of $\mathtt{S4.1}$ in terms of modal stable canonical formulae to an axiomatisation of $\mathtt{S4.1_R}$ in terms of modal stable canonical rules.

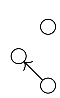
Theorem 2.67. $\mathtt{S4.1_R} = \mathtt{S4_R} \oplus \mu(\mathfrak{X}, \mathfrak{D})$.

Proof. It suffices to show that for every $\mathtt{S4}$-space $\mathfrak{Y}$, we have $\mathfrak{Y} \models \Box\Diamond p \to \Diamond\Box p$ iff $\mathfrak{Y} \models \mu(\mathfrak{X}, \mathfrak{D})$. Recall that for every $\mathtt{S4}$-space $\mathfrak{Y}$ we have $\mathfrak{Y} \not\models \Box\Diamond p \to \Diamond\Box p$ iff $\mathfrak{Y}$ contains a proper cluster of quasi-maximal points [Chagrov and Zakharyaschev, 1997, p. 82].

($\Rightarrow$) Assume $\mathfrak{Y} \not\models \Box\Diamond p \to \Diamond\Box p$ and let $C \subseteq Y$ be a proper cluster of quasi-maximal points. As C is proper, it can be partitioned into two disjoint clopens, U and V. Define a map $f : \mathfrak{Y} \to \mathfrak{X}$ by sending every element of U to d_1, every element of V to d_2, every element of $R^{-1}(C) \setminus C$ to the point below d_1, d_2, and everything else to the remaining point. It is easy to see that f is a surjective continuous bounded morphism, hence a continuous stable surjection. Moreover, since C is a cluster it follows that f satisfies the BDC for $\{\{d_1\}, \{d_2\}\}$.

($\Leftarrow$) assume $\mathfrak{Y} \not\models \mu(\mathfrak{X}, \mathfrak{D})$. Then there is a continuous stable surjection $f : \mathfrak{Y} \to \mathfrak{X}$ satisfying the BDC for $\{\{d_1\}, \{d_2\}\}$. Take any $x \in f^{-1}(d_1)$. By Proposition 2.16, there is $y \in qmax_R(Y)$ with Rxy. Since f is stable, either $f(y) = d_1$ or $f(y) = d_2$. Wlog, suppose the former holds. We have $R[f(y)] \cap \{d_2\} \neq \emptyset$. Therefore as f satisfies the BDC for $\{d_2\}$, there must be $z \in X$ with Ryz and $f(z) = d_2$. Since $y \in qmax_R(Y)$, it follows that Rzy. Since clearly $y \neq z$, we have found a proper cluster of quasi-maximal points in $\mathfrak{X}$, which yields $\mathfrak{Y} \not\models \Box\Diamond p \to \Diamond\Box p$. ☐

The collapsed rule of $\mu(\mathfrak{X}, \mathfrak{D})$ is $\eta(\rho\mathfrak{X}, \rho\mathfrak{D})$, where $\rho\mathfrak{X}$ is represented as

$\rho(d_1)$ 

and $\rho\mathfrak{D} = \{\{\rho(d_1)\}\}$. Note $\rho(d_1) = \rho(d_2)$. Now, $\sigma\rho\mathfrak{X}$ (which, recall, is just $\rho\mathfrak{X}$ viewed as a modal space) is an $\mathtt{S4}$-modal space. Since it contains no proper cluster, a fortiori it contains no proper cluster of quasi-maximal elements. Therefore $\sigma\rho\mathfrak{X}$ is also an $\mathtt{S4.1}$-modal space. But then the identity map on $\sigma\rho\mathfrak{X}$ witnesses $\sigma\rho\mathfrak{X} \not\models \mu(\sigma\rho\mathfrak{X}, \rho\mathfrak{D})$. By Theorem 2.64, this implies that $\eta(\rho\mathfrak{X}, \rho\mathfrak{D}) \notin \rho\mathtt{S4.1_R}$. Therefore we reach the following result, which was originally proved by Esakia [1979].

Corollary 2.68. $\rho\mathtt{S4.1_R} = \mathtt{IPC_R}$.

We close the present section by establishing an important consequence of the axiomatisation results obtained so far, namely that every modal rule is equivalent, over $\mathtt{Grz_R}$, to a finite conjunction of stable canonical rules of $\mathtt{Grz}$-algebras.

Theorem 2.69. For every modal rule Γ/Δ there is a finite set Ξ of modal stable canonical rules of the form $\mu(\mathfrak{A}, D)$ with $\mathfrak{A} \in \mathsf{Grz}$, such that for any $\mathfrak{B} \in \mathsf{Grz}$ we have that $\mathfrak{B} \not\models \Gamma/\Delta$ iff there is $\mu(\mathfrak{A}, D) \in \Xi$ such that $\mathfrak{B} \not\models \mu(\mathfrak{A}, D)$.

Proof. We prove the dual statement. Let Γ/Δ be a modal rule. By Lemma 2.35 there is a finite set Σ of modal stable canonical rules of the form $\mu(\mathfrak{F}, \mathfrak{D})$ with $\mathfrak{F} \in \mathsf{Spa}(\mathsf{S4})$, such that for every $\mathfrak{X} \in \mathsf{Spa}(\mathsf{S4})$, and so for every $\mathfrak{X} \in \mathsf{Spa}(\mathsf{Grz})$ in particular, we have $\mathfrak{X} \not\models \Gamma/\Delta$ iff there is some $\mu(\mathfrak{F}, \mathfrak{D}) \in \Sigma$ such that $\mathfrak{X} \not\models \mu(\mathfrak{F}, \mathfrak{D})$. Now consider the rule system $\mathtt{Grz_R} \oplus \Sigma$ and note that $\mathtt{Grz_R} \oplus \Sigma = \mathtt{Grz_R} \oplus \Gamma/\Delta$. Let

$$\Xi := \{\mu(\sigma\rho\mathfrak{F}, \rho\mathfrak{D}) : \mu(\mathfrak{F}, \mathfrak{D}) \in \Sigma\} \cap (\mathtt{Grz_R} \oplus \Sigma).$$

By Theorem 2.64 we have

$$\rho(\mathtt{Grz_R} \oplus \Gamma/\Delta) = \mathtt{IPC_R} \oplus \{\eta(\rho\mathfrak{F}, \rho\mathfrak{D}) : \mu(\sigma\rho\mathfrak{F}, \rho\mathfrak{D}) \in \Xi\},$$

which by Theorem 2.57 implies

$$\sigma\rho(\mathtt{Grz_R} \oplus \Gamma/\Delta) = \mathtt{Grz_R} \oplus \Xi.$$

By the Blok-Esakia theorem for rule systems we obtain $\sigma\rho(\mathtt{Grz_R} \oplus \Gamma/\Delta) = \mathtt{Grz_R} \oplus \Gamma/\Delta$. Thus we have shown that for all $\mathfrak{X} \in \mathsf{Spa}(\mathsf{Grz})$ we have $\mathfrak{X} \not\models \Gamma/\Delta$ iff $\mathfrak{X} \not\models \mu(\sigma\rho\mathfrak{F}, \rho\mathfrak{D})$ for some $\mu(\sigma\rho\mathfrak{F}, \rho\mathfrak{D}) \in \Xi$, as desired. □

Observe that the same proof does not generalise to show that every modal rule Γ/Δ is equivalent, *over* $\mathtt{S4_R}$, to some finite set of modal stable canonical rules of $\mathtt{Grz}$-algebras. This is because in general it is not true that $\sigma\rho(\mathtt{S4_R} \oplus \Gamma/\Delta) = \mathtt{S4_R} \oplus \Gamma/\Delta$.

§2.3.4 Additional Results

We close this chapter by collecting selected additional results related to the theory of modal companions obtained via our methods.

The Dummett-Lemmon conjecture A modal or si-rule system is called *Kripke complete* if it is of the form $\mathtt{L} = \{\Gamma/\Delta : \mathscr{K} \models \Gamma/\Delta\}$ for some class of Kripke frames $\mathscr{K}$. It is easy to see that refutation conditions for stable canonical rules work essentially the same way for Kripke frames as they do for Esakia and modal spaces: for every Kripke frame $\mathfrak{X}$ and si stable canonical rule $\eta(\mathfrak{F}, \mathfrak{D})$, $\mathfrak{X} \not\models \eta(\mathfrak{F}, \mathfrak{D})$ iff there is a surjective stable homomorphism $f : \mathfrak{X} \to \mathfrak{F}$ satisfying the BDC for $\mathfrak{D}$, and analogously for the modal case. For details the reader may consult, e.g., Bezhanishvili et al. [2016a]. The mappings σ, τ, ρ also extend to classes of Kripke frames in an obvious way. Finally Lemma 2.48 and the rule translation lemma work for Kripke frames as well, the latter appropriately reformulated to incorporate the refutation conditions for stable canonical rules just stated.

Zakharyashchev [1991, Corollary 2] applied his canonical formulas to prove the *Dummett-Lemmon conjecture* (Dummett and Lemmon 1959), which states that a si-logic is Kripke complete iff its weakest modal companion is. To our knowledge, a proof that the Dummett-Lemmon conjecture generalises to rule systems has not been published. We supply one here, which uses stable canonical rules.

Theorem 2.70 (Dummett-Lemmon conjecture for si-rule systems)**.** For every si-rule system $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC_R})$, we have that $\mathtt{L}$ is Kripke complete iff $\tau\mathtt{L}$ is.

Proof. ($\Rightarrow$) Let $\mathtt{L}$ be Kripke complete. Suppose that $\Gamma/\Delta \notin \tau\mathtt{L}$. Then there is an $\mathtt{S4}$-modal space $\mathfrak{X}$ such that $\mathfrak{X} \not\models \Gamma/\Delta$. By Theorem 2.36, we may assume that $\Gamma/\Delta = \mu(\mathfrak{F}, \mathfrak{D})$ for $\mathfrak{F}$ a preorder. By the rule collapse lemma it follows that $\rho\mathfrak{X} \not\models \eta(\rho\mathfrak{F}, \rho\mathfrak{D})$. As $\sigma\rho\mathfrak{X} \models \tau\mathtt{L}$, by Lemma 2.48 it follows that $\rho\mathfrak{X} \models \mathtt{L}$, and so we conclude $\eta(\rho\mathfrak{F}, \rho\mathfrak{D}) \notin \mathtt{L}$. Since $\mathtt{L}$ is Kripke complete, there is a si Kripke frame $\mathfrak{Y}$ such that $\mathfrak{Y} \not\models \eta(\rho\mathfrak{F}, \rho\mathfrak{D})$. Take a stable map $f : \mathfrak{Y} \to \rho\mathfrak{F}$ satisfying the BDC for $\rho\mathfrak{D}$. Work in $\rho\mathfrak{F}$. For every $x \in \rho[F]$ look at $\rho^{-1}(x)$, let $k = |\rho^{-1}(x)|$ and enumerate $\rho^{-1}(x) = \{x_1, \dots, x_k\}$. Now work in $\mathfrak{Y}$. For every $y \in f^{-1}(x)$ replace y with a k-cluster $y_1, \dots, y_k$ and extend the relation R clusterwise: Ry_iz_j iff either $y = z$ or Ryz. Call the result $\mathfrak{Z}$. Clearly $\mathfrak{Z}$ is a Kripke frame, and moreover $\mathfrak{Z} \models \tau\mathtt{L}$, because $\rho\mathfrak{Z} \cong \mathfrak{Y}$. For convenience, identify $\rho\mathfrak{Z} = \mathfrak{Y}$. For every $x \in \rho[F]$ define a map $g_x : f^{-1}(x) \to \rho^{-1}(x)$ by setting $g_x(y_i) = x_i$ ($i \leq k$). Finally, define $g : \mathfrak{Z} \to \mathfrak{F}$ by putting $g = \bigcup_{x \in \rho[F]} g_x$.

The map g is evidently well defined, surjective, and relation preserving. We claim that moreover, it satisfies the BDC for $\mathfrak{D}$. To see this, suppose that $R[g(y_i)] \cap \mathfrak{d} \neq \emptyset$ for some $\mathfrak{d} \in \mathfrak{D}$. Then there is $x_j \in F$ with $x_j \in \mathfrak{d}$ and $Rg(y_i)x_j$. By construction also $\rho(x_j) \in \rho[\mathfrak{d}]$ and $Rf(\rho(y_i))\rho(x_j)$. As f satisfies the BDC for $\rho\mathfrak{D}$ it follows that there is some $z \in Y$ such that $R\rho(y_i)z$ and $f(z) \in \rho[\mathfrak{d}]$. We may view z as $\rho(z_n)$ where $\rho^{-1}(f(z))$ has cardinality $k \geq n$. Surely Ry_iz_n. Furtheromre, since $f(z) \in \rho[\mathfrak{d}]$ there must be some $m \leq k$ such that $f(z)_m = g(z_m) \in \mathfrak{d}$. By construction Rz_nz_m and so in turn Ry_iz_m. This establishes that g indeed satisfies the BDC for $\mathfrak{D}$. Thus we have shown $\mathfrak{Z} \not\models \mu(\mathfrak{F}, \mathfrak{D})$. It follows that $\tau\mathtt{L}$ is Kripke complete.

($\Leftarrow$) Assume that $\tau(\mathtt{L})$ is Kripke complete. Suppose that $\Gamma/\Delta \notin \mathtt{L}$. Then there is an Esakia space $\mathfrak{X}$ such that $\mathfrak{X} \not\models \Gamma/\Delta$. Therefore $\sigma\mathfrak{X} \not\models T(\Gamma/\Delta)$. Surely $\sigma\mathfrak{X} \models \tau\mathtt{L}$, so $T(\Gamma/\Delta) \notin \tau\mathtt{L}$ and thus there is a Kripke frame $\mathfrak{Y}$ such that $\mathfrak{Y} \models \tau\mathtt{L}$ and $\mathfrak{Y} \not\models T(\Gamma/\Delta)$. But then $\rho\mathfrak{Y} \not\models \Gamma/\Delta$. $\rho\mathfrak{Y}$ is a Kripke frame, and validates $\mathtt{L}$ by Lemma 2.48. Therefore we have shown that $\mathtt{L}$ is indeed Kripke complete. $\square$

Filtration We now discuss the implications of our previous results for the construction of filtrations of models based on Grz-algebras.

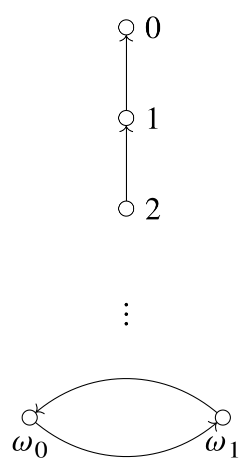
The notion of "admitting filtration" is widely used in the modal logic literature to describe modal deductive systems for which filtration provides "enough" finite countermodels to invalid formulae or rules based on structures validating said deductive system. There are numerous ways to spell out talk of admitting filtration precisely which fit this intuition. Ilin [2018, p. 86] proposes the following two.

Definition 2.71. Let $\mathtt{M} \in \mathbf{NExt}(\mathtt{K_R})$ be a modal rule system.

- $\mathtt{M}$ *weakly admits filtration* if for every modal rule $\Gamma/\Delta \notin \mathtt{M}$ there are $\mathfrak{A} \in \mathsf{Alg}(\mathtt{M})$ and a valuation V on $\mathfrak{A}$ such that there exists a filtration $(\mathfrak{B}', V')$ of the model $(\mathfrak{A}, V)$ through some finite, subformula closed set of formulae Θ containing $Sfor(\Gamma/\Delta)$, where $\mathfrak{B} \in \mathsf{Alg}(\mathtt{M})$.
- $\mathtt{M}$ *strongly admits* if for every $\mathfrak{A} \in \mathsf{Alg}(\mathtt{M})$, valuation V on $\mathfrak{A}$ and finite, subformula closed set of formulae Θ, any filtration $(\mathfrak{B}', V')$ of the model $(\mathfrak{A}, V)$ through Θ is such that $\mathfrak{B}' \in \mathsf{Alg}(\mathtt{M})$.

A quick glance at the proofs of Lemmas 2.26 and 2.35 reveals that both $\mathtt{IPC_R}$ and $\mathtt{S4_R}$ strongly admit filtration. By contrast, it is known that $\mathtt{Grz_R}$ fails to strongly admit filtration, as the following example illustrates.

Example 2.72. Recall the Grz-space from Example 2.17.



Define a valuation V on $\mathfrak{X}$ by setting $V(p)$ as the set of all evens together with ω_0. Then it is easy to see that the least transitive filtration of $(\mathfrak{X}, V)$ through $\{p\}$ is a two-element cluster, which is not a Grz-space.

This example is representative: for any model $(\mathfrak{X}, V)$ with $\mathfrak{X}$ a Grz-space, and for any subformula-closed set of formulae Θ, if there is $\varphi \in \Theta$ such that $\bar{V}(\varphi)$ cuts a cluster in X, then no finite filtration of $(\mathfrak{X}, V)$ through Θ can be based on a Grz-space.

On the other hand, our results suggest that there is *some* sense in which $\mathtt{Grz_R}$ admits filtration. For let Γ/Δ be a modal rule. Then by Theorem 2.69, Γ/Δ is equivalent, over $\mathtt{Grz_R}$, to finitely many modal stable canonical rules of the form $\mu(\mathfrak{A}, D)$, with $\mathfrak{A} \in \mathsf{Grz}$. Therefore, if there is $\mathfrak{B} \in \mathsf{Grz}$ such that $\mathfrak{B} \not\models \Gamma/\Delta$, then $\mathfrak{B} \not\models \mu(\mathfrak{A}, D)$ for one such modal stable canonical rule $\mu(\mathfrak{A}, D)$, whence there is a stable embedding $h : \mathfrak{A} \to \mathfrak{B}$ satisfying the BDC for D. By the proof of Propositions 2.29 and 2.30 we know that the pair $(\mathfrak{A}, D)$ is determined by a model $(\mathfrak{A}, V)$ refuting $\mu(\mathfrak{A}, D)$, and that if we extend V to a valuation W on $\mathfrak{B}$ we obtain a model $(\mathfrak{B}, W)$ such that $(\mathfrak{A}, V)$ is a filtration of $(\mathfrak{B}, W)$ through some finite subformula-closed set of formulae. In general there is no guarantee that this set of formulae includes $Sfor(\Gamma/\Delta)$, but perhaps there is a way of picking the pair $(\mathfrak{A}, V)$ in such a way that this happens to be the case.

This would establish that $\mathtt{Grz_R}$ admits filtration in the following sense, which sits somewhere between the weak and the strong one of Definition 2.71.

Definition 2.73. Let $\mathtt{M} \in \mathbf{NExt}(\mathtt{K_R})$ be a modal rule system. $\mathtt{M}$ *moderately admits filtration* if for every modal rule Γ/Δ and every $\mathfrak{A} \in \mathsf{Alg}(\mathtt{M})$ with $\mathfrak{A} \not\models \Gamma/\Delta$, there is a valuation V on $\mathfrak{A}$ such that there exists a filtration $(\mathfrak{B}, V')$ of the model $(\mathfrak{A}, V)$ through some finite, subformula closed set of formulae Θ containing $Sfor(\Gamma/\Delta)$, where $\mathfrak{B} \in \mathsf{Alg}(\mathtt{M})$.

We prove that $\mathtt{Grz_R}$ does in fact moderately admit filtration, for reasons having to do with the fact that $\mathtt{IPC_R}$, i.e., the si fragment of $\mathtt{Grz_R}$, strongly admits filtration.

Theorem 2.74. $\mathtt{Grz_R}$ moderately admits filtration.

Proof. Let $\mathfrak{A} \in \mathsf{Grz}$, Γ/Δ a modal rule and suppose $\mathfrak{A} \not\models \Gamma/\Delta$. Then by Lemma 2.50 also $\sigma\rho\mathfrak{A} \not\models \Gamma/\Delta$. Take a valuation V on $\sigma\rho\mathfrak{A}$ with $\sigma\rho\mathfrak{A}, V \not\models \Gamma/\Delta$. For every $a \in \bar{V}[Sfor(\Gamma/\Delta)]$ choose finitely many $b_i(a), c_i(a) \in O(A)$ with $i \leq n_a$ such that

$$a = \bigwedge_{i \leq n_a} \neg b_i(a) \vee c_i(a).$$

This can always be done by Proposition 2.40. Now for each $a \in \bar{V}[Sfor(\Gamma/\Delta)]$ and $i \leq n_a$ introduce fresh propositional variables $p_{b_i(a)}, p_{c_i(a)}$ and define a valuation W on $\sigma\rho\mathfrak{A}$ which agrees with V on elements in $Prop \cap Sfor(\Gamma/\Delta)$, and is such that $W(p_{b_i(a)}) = b_i(a)$, $W(p_{c_i(a)}) = c_i(a)$ for each $a \in \bar{V}[Sfor(\Gamma/\Delta)]$ and $i \leq n_a$. Finally, let

$$\Sigma := \{p_{b_i(a)}, p_{c_i(a)} : a \in \bar{W}[Sfor(\Gamma/\Delta)] \text{ and } i \leq n_a\}$$
$$\Theta := Sfor(\Gamma/\Delta) \cup \Sigma$$

let $\mathfrak{D}$ be the least bounded sublattice of $\rho\mathfrak{A}$ generated by $\bar{W}[\Sigma]$. Expand $\mathfrak{D}$ to a Heyting algebra $\mathfrak{H}$ by setting

$$a \rightsquigarrow b := \bigvee \{c \in D : a \wedge c \leq b\}.$$

Reasoning as in Lemma 2.26, we have that the inclusion $\subseteq: \mathfrak{H} \to \rho\mathfrak{A}$ is a bounded lattice embedding. Write $\sigma\mathfrak{H} = (B(\mathfrak{H}), \blacksquare)$.

It sufficies to show that the model $(\sigma\mathfrak{H}, W')$, where W' is any valuation with $W'(p) = W(p)$ for all $p \in Prop \cap \Theta$, is a filtration of $(\sigma\rho\mathfrak{A}, W)$ through Θ. For once this is shown, we may view W as a valuation on $\mathfrak{A}$, and by $\sigma\rho\mathfrak{A} \rightarrowtail \mathfrak{A}$ it will follow that $(\sigma\mathfrak{H}, W')$ is also a filtration of $(\sigma\rho\mathfrak{A}, W)$ through Θ. Firstly, observe that since $\sigma\rho\mathfrak{A}$ is skeletal, $B(\mathfrak{H})$ is isomorphic to the least Boolean subalgebra $\mathfrak{B}'$ of $\sigma\rho\mathfrak{A}$ generated by $\bar{W}[\Theta]$. Next, note that $\blacksquare a \leq \Box a$ for all $a \in B(H)$. Indeed, again because $\sigma\rho\mathfrak{A}$ is skeletal it follows that $\Box a = \bigvee\{b \in O(A) : b \leq a\}$, and since clearly $\blacksquare a \in O(A)$ and $\blacksquare a \leq a$ it follows that $\blacksquare a \leq \Box a$ as desired. This shows that $\subseteq : \sigma\mathfrak{H} \to \sigma\rho\mathfrak{A}$ is a stable embedding. To conclude, let us check that it also satisfies the BDC for $D := \{\bar{W}(\varphi) : \Box\varphi \in \Theta\}$. Let $a \in D$. Then $\Box a = \bar{W}(\Box\varphi) \in H$. As also $\Box a \leq a$, by definition of $\blacksquare$ we infer $\Box a \leq \blacksquare a$. So indeed $\Box a = \blacksquare a$. This concludes the proof. $\square$

It seems that the concept of moderately admitting filtration has not been discussed in the literature. Yet it deserves attention. Moderately admitting filtration is a more flexible notion than its strong counterpart—important modal rule systems which fail to strongly admit filtration still do so moderately. As this section has showed, a case in point is $\mathtt{Grz_R}$. Moreover, such gains in flexibility do not come with excessive costs with respect to strength, as much of the work usually done using the strong sense of admitting filtration can just as well be done using the moderate sense. Representatively, it should be apparent that Theorem 2.74 can be applied to prove via the usual filtration method that $\mathtt{Grz_R}$, and therefore $\mathtt{Grz}$, have the finite model property. This is noteworthy, as existing proofs to the same effect tend to rely on a less standard notion of filtration (e.g. Boolos 1993, pp. 158–9, Esakia 2019, Theorem 3.5.13),[1] to be discussed in Chapter 4.

Stability Stable (si or modal) deductive systems are deductive systems axiomatised by stable rules [Bezhanishvili et al., 2018, 2016a], and are to stable canonical rules and formulas what subframe deductive systems are to Zakharyaschev-Jerábek canonical rules and formulas. Since many interesting si- and modal deductive systems fail to be stable, it is desirable to make the notion of stability more flexible by parametrising it over some base deductive system, as follows.[2]

Definition 2.75. Let $\mathcal{K}, \mathcal{H} \subseteq \mathcal{V}$, for $\mathcal{V} \in \{\mathsf{HA}, \mathsf{MA}\}$. $\mathcal{H}$ is called *$\mathcal{K}$-stable* if for every $\mathfrak{A} \in \mathcal{H}$ and $\mathfrak{B} \in \mathcal{K}$ such that $\mathfrak{B}$ stably embeds into $\mathfrak{A}$, we have $\mathfrak{B} \in \mathcal{H}$. If $\mathtt{L}$ is a si- (resp. modal) deductive system, then a si- (resp. modal) deductive system $\mathtt{L}'$ is called *$\mathtt{L}$-stable* if $\mathsf{Alg}(\mathtt{L}')$ is $\mathsf{Alg}(\mathtt{L})$-stable.

$\mathtt{M}$-stability is usually studied assuming that the base deductive system $\mathtt{M}$ strongly admits filtration. This is because parametrising over a deductive system $\mathtt{M}$ admitting filtration only in very weak senses, if at all, risks trivialising the notion of $\mathtt{M}$-stability, as the conditional definition of

[1] The proof of Esakia 2019, Theorem 3.5.13 is not Esakia's own, but was supplied by the editors and is in fact a revised version of the argument in Boolos 1993, pp. 158–9.

[2] Our definition is somewhat more general than the one in Bezhanishvili et al. [2018], mainly because it does not assume that $\mathtt{L} \subseteq \mathtt{L}'$ nor that $\mathtt{L}$ strongly admits filtration. In both cases our departure from the original definition is motivated by our goal of studying $\mathtt{Grz_R}$-stability.

the latter would be vacuously satisfied rather frequently. However, the proof of Theorem 2.74 should make it apparent that Grz admits a considerable range of filtrations, which makes the notion of $\mathtt{Grz}$-stability worth studying.

Most known results in the theory of $\mathtt{M}$-stable deductive systems where $\mathtt{M}$ strongly admits filtration transfer rather straightforwardly to the case where $\mathtt{M}$ moderately admits filtration. For example, Bezhanishvili et al. [2018, Theorem 3.8] prove that whenever $\mathtt{N}$ is a modal rule system strongly admitting filtration, a modal rule system $\mathtt{M} \in \mathbf{NExt}(\mathtt{N})$ is $\mathtt{N}$-stable iff it is axiomatisable over $\mathtt{N}$ by stable rules of algebras validating $\mathtt{N}$. Essentially the same proof establishes that the claim remains true if "strongly" is replaced with "moderately". Therefore, by Definition 2.32 we have the following result as a special case.

Proposition 2.76. For every modal rule system $\mathtt{M} \in \mathbf{NExt}(\mathtt{Grz_R})$, we have that $\mathtt{M}$ is $\mathtt{Grz_R}$-stable iff it is axiomatisable over $\mathtt{Grz_R}$ by stable rules of $\mathtt{Grz}$-algebras.

Applying Proposition 2.76, we obtain the following preservation theorem concerning stability.

Theorem 2.77. The following conditions hold:

1. For every si-rule system $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC_R})$, we have that $\mathtt{L}$ is $\mathtt{IPC_R}$-stable iff every modal companion of $\mathtt{L}$ is $\mathtt{Grz_R}$-stable;
2. For every modal rule system $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4_R})$, if $\mathtt{M}$ is $\mathtt{Grz_R}$-stable then $\rho\mathtt{M}$ is $\mathtt{IPC_R}$-stable.

Proof. (1) ($\Rightarrow$) Assume that $\mathtt{L}$ is $\mathtt{IPC_R}$-stable and let $\mathtt{M}$ be a modal companion of $\mathtt{L}$. Let $\mathfrak{A} \in \mathsf{Alg}(\mathtt{M})$, $\mathfrak{B} \in \mathsf{Grz}$ finite and suppose there is a stable embedding $h : \mathfrak{B} \to \mathfrak{A}$. Then $\mathfrak{A} \nvDash \mu(\mathfrak{B})$, so by the rule collapse lemma $\rho\mathfrak{A} \nvDash \eta(\rho\mathfrak{B})$ and thus there is a stable embedding $i : \rho\mathfrak{B} \to \rho\mathfrak{A}$. As $\mathtt{L}$ is stable, $\rho\mathfrak{A} \in \mathsf{Alg}(\mathtt{L})$, and $\rho\mathfrak{B} \in \mathsf{HA}$ it follows that $\rho\mathfrak{B} \in \mathsf{Alg}(\mathtt{L})$.

($\Leftarrow$) Suppose that every modal companion of $\mathtt{L}$ is $\mathtt{Grz_R}$-stable. In particular $\sigma\mathtt{L}$ is $\mathtt{Grz_R}$-stable, hence by Proposition 2.76 we have that $\sigma\mathtt{L} = \mathtt{Grz_R} \oplus \{\mu(\mathfrak{A}_i) : i \in I\}$ for some family $\{\mathfrak{A}_i : i \in I\} \subseteq \mathsf{Grz}$. Then by Theorem 2.64 we have $\mathtt{L} = \mathtt{IPC_R} \oplus \{\eta(\rho\mathfrak{A}_i) : i \in I\}$. But then Bezhanishvili et al. [2016b, Proposition 4.5] implies that $\mathtt{L}$ is $\mathtt{IPC_R}$-stable.

(2) Assume that $\mathtt{M}$ is $\mathtt{Grz_R}$-stable. Take $\mathfrak{H} \in \mathsf{Alg}(\rho\mathtt{M})$. Then by Lemma 2.53, $\mathfrak{H} \cong \rho\mathfrak{A}$ for some $\mathfrak{A} \in \mathsf{Alg}(\mathtt{M})$. Let $\mathfrak{K} \in \mathsf{HA}$ be finite and suppose there is a stable embedding $h : \mathfrak{K} \to \rho\mathfrak{A}$. By the proof of Proposition 2.23 and Esakia duality it easily follows that this is also a stable embedding $h : \sigma\mathfrak{K} \to \sigma\rho\mathfrak{A}$. As $\mathtt{M}$ is $\mathtt{Grz_R}$-stable and $\sigma\mathfrak{K} \in \mathsf{Grz}$, it follows that $\sigma\mathfrak{K} \in \mathsf{Alg}(\mathtt{M})$. But then by Lemma 2.53 we have $\rho\sigma\mathfrak{K} \cong \mathfrak{K} \in \mathsf{Alg}(\mathtt{L})$. ☐

Thus, in particular, we obtain that a rule system $\mathtt{M} \in \mathbf{NExt}(\mathtt{Grz_R})$ is $\mathtt{Grz_R}$-stable iff its si fragment is $\mathtt{IPC_R}$-stable. This solves the open problem, left implicit in Bezhanishvili et al. [2018], of characterising the $\mathtt{Grz_R}$-stable rule systems extending $\mathtt{Grz_R}$.

§2.4 Chapter Summary

We summarise the main original contributions of this chapter in the following list.

- We proved the central technical lemma of our strategy (Lemma 2.50), and applied it to characterise the set of modal companions of a superintuitionistic deductive system (Theorem 2.54) and to prove the Blok-Esakia theorem for both rule systems and logics.
- We gave new axiomatic characterisations of the modal companions maps σ, ρ, τ on rule systems via stable canonical rules.
- We used stable canonical rules to generalise the Dummett-Lemmonn conjecture to rule systems (Theorem 2.70).
- We introduced the notion of moderately admitting filtration and proved that $\mathtt{Grz}_{\mathtt{R}}$ moderately admits filtration (Theorem 2.74).
- We introduced the notion of $\mathtt{Grz}_{\mathtt{R}}$-stability and proved a preservation and reflection result describing it.

3 | Tense Companions of Super Bi-intuitionistic Deductive Systems

This chapter applies the techniques presented in Chapter 2 to the study of tense companions of bi-superintuitionistic deductive systems. We begin by reviewing some preliminaries in § 3.1. In § 3.2 we develop tense and bi-superintuitionistic stable canonical rules, which generalise the modal and si stable canonical rules seen in § 2.2. We then apply such rules to extend the results of § 2.3 to the bi-superintuitionistic and tense setting in § 3.3. This section contains the main results of the chapter, which include a characterisation of the set of tense companions of a bi-superintuitionistic deductive system, and extensions of the Blok-Esakia theorem and of the Gödel-Dummett conjecture to the bi-superintuitionistic and tense setting (§ 3.3.2). These results were known for logics (cf. Wolter 1998), but are new for rule systems. We also give new axiomatic characterisations of tense companions and bi-superintuitionistic fragments via stable canonical rules, and illustrate them via concrete examples (§ 3.3.3).

Besides the original results just mentioned, the main contribution of this chapter is showcasing the uniformity of our method across signatures. The majority of results in this chapter are obtained via straightforward generalisations of arguments already seen in Chapter 2. This is a major virtue of our approach, which Zakharyaschev and Jerábek's canonical formulae and rules-based approach does not seem to share to the same extent (§ 3.2.3).

§3.1 Preliminaries

This section briefly reviews definitions and basic facts concerning the structures dealt with in this chapter.

§3.1.1 Bi-superintuitionistic Deductive Systems, bi-Heyting Algebras, and bi-Esakia Spaces

We work in the *bi-superintuitionistic signature*,

$$bsi := \{\wedge, \vee, \to, \leftarrow, \bot, \top\}.$$

The set Frm_{bsi} of bi-superintuitionistic (bsi) formulae is defined recursively as follows.

$$\varphi ::= p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \varphi \leftarrow \varphi$$

We let $\llcorner\varphi := \varphi \leftarrow \top$ and $\varphi \leftrightarrow \psi := (\varphi \to \psi) \land (\psi \to \varphi)$. The *bi-intuitionistic propositional calculus* $\mathtt{2IPC}$ is defined as the least logic over Frm_{bsi} containing $\mathtt{IPC}$, containing the axioms

$$p \to (q \lor (q \leftarrow p)) \qquad (q \leftarrow p) \to \llcorner(p \to q)$$
$$(r \leftarrow (q \leftarrow p)) \to ((p \lor q) \leftarrow p) \qquad \neg(p \leftarrow q) \to (p \to q)$$
$$\neg\llcorner(p \leftarrow p)$$

and such that if $\varphi, \varphi \to \psi \in \mathtt{2IPC}$ then $\psi \in \mathtt{2IPC}$, and if $\varphi \in \mathtt{2IPC}$ then $\neg\llcorner\varphi \in \mathtt{2IPC}$. The logic $\mathtt{2IPC}$ was introduced and extensively studied by Rauszer [1974a,b, 1977], and also investigated by Esakia [1975]. $\mathtt{2IPC}$ is also denoted $\mathtt{biIPC}$ and $\mathtt{HB}$ (for *Heyting-Brower logic*) in the literature.

Definition 3.1. A *bsi-logic* is a logic $\mathtt{L}$ over Frm_{bsi} containing $\mathtt{2IPC}$ and satisfying the following conditions:

- If $\varphi, \varphi \to \psi \in \mathtt{L}$ then $\psi \in \mathtt{L}$ (MP);
- If $\varphi \in \mathtt{L}$ then $\neg\llcorner\varphi \in \mathtt{L}$ (DN).

A *bsi-rule system* is a rule system $\mathtt{L}$ over Frm_{bsi} satisfying the following conditions:

- $\varphi, \varphi \to \psi / \psi \in \mathtt{L}$ (MP-R);
- $\varphi / \neg\llcorner\varphi \in \mathtt{L}$ (DN-R);
- $/\varphi \in \mathtt{L}$ for every $\varphi \in \mathtt{2IPC}$.

If $\mathtt{L}$ is a bsi-logic let $\mathbf{Ext}(\mathtt{L})$ be the set of bsi-logics containing $\mathtt{L}$, and similarly for bsi-rule systems. Then $\mathbf{Ext}(\mathtt{2IPC})$ is the set of all bsi-logics. It is easy to see that $\mathbf{Ext}(\mathtt{2IPC})$ carries a complete lattice, with $\oplus_{\mathbf{Ext}(\mathtt{2IPC})}$ as join and intersection as meet. Observe that for every $\mathtt{L} \in \mathbf{Ext}(\mathtt{2IPC})$ there is a least bsi-rule system containing $/\varphi$ for each $\varphi \in \mathtt{L}$, which we denote by $\mathtt{L}_\mathtt{R}$. Then $\mathtt{2IPC}_\mathtt{R}$ is the least bsi-rule system and $\mathbf{Ext}(\mathtt{2IPC}_\mathtt{R})$ is the set of all bsi-rule systems. Again, it is not hard to verify that $\mathbf{Ext}(\mathtt{2IPC}_\mathtt{R})$ forms a complete lattice with $\oplus_{\mathbf{Ext}(\mathtt{2IPC}_\mathtt{R})}$ as join and intersection as meet. Henceforth we write both $\oplus_{\mathbf{Ext}(\mathtt{2IPC})}$ and $\oplus_{\mathbf{Ext}(\mathtt{2IPC}_\mathtt{R})}$ simply as $\oplus$, leaving context to clarify any ambiguity.

We generalise Proposition 2.2 to the bsi setting.

Proposition 3.2. The mappings $(\cdot)_\mathtt{R}$ and $\mathsf{Taut}(\cdot)$ are mutually inverse complete lattice isomorphisms between $\mathbf{Ext}(\mathtt{2IPC})$ and the sublattice of $\mathbf{Ext}(\mathtt{2IPC}_\mathtt{R})$ consisting of all bsi-rule systems $\mathtt{L}$ such that $\mathsf{Taut}(\mathtt{L})_\mathtt{R} = \mathtt{L}$.

Algebraically, bsi-logics and rule systems are interpreted over expansions of Heyting algebras called *bi-Heyting algebras,* discussed at length in Rauszer [1974a,b, 1977] and more recently in Pedroso De Lima Martins [2021].

Definition 3.3. A *bi-Heyting algebra* is a tuple $\mathfrak{H} = (H, \land, \lor, \to, \leftarrow, 0, 1)$ such that the $\leftarrow$-free reduct of $\mathfrak{H}$ is a Heyting algebras, and such that for all $a, b, c \in H$ we have

$$a \leftarrow b \leq c \iff a \leq b \lor c.$$

Let 2HA denote the class of all bi-Heyting algebras. By Theorem 1.10, 2HA is a variety.

Let $\mathfrak{L} = (L, \wedge, \vee, 0, 1)$ be a bounded lattice. The *order dual* of $\mathfrak{L}$ is the lattice $\bar{\mathfrak{L}} = (L, \vee, \wedge, 1, 0)$, where $\vee$ is viewed as the meet operation and $\wedge$ as the join operation. We have the following elementary but important fact.

Proposition 3.4 (Order duality principle for bi-Heyting algebras)**.** For every bi-Heyting algebra $\mathfrak{H}$, the order dual $\bar{\mathfrak{H}}$ of $\mathfrak{H}$ is a Heyting algebra, where implication is defined, for all $a, b \in H$, by

$$a \leftarrow b := \bigwedge\{c \in H : a \leq b \vee c\}.$$

This observation can be leveraged to establish a number of properties about bi-Heyting algebras via straightforward adaptations of the theory of Heyting algebras. We shall see numerous examples of this strategy in this chapter.

We write $\mathbf{Var}(\mathsf{2HA})$ and $\mathbf{Uni}(\mathsf{2HA})$ respectively for the lattice of subvarieties and of universal subclasses of 2HA. The following result may be proved via the same techniques used to prove Theorem 2.4. A recent self-contained proof of Item 1 may be found in Pedroso De Lima Martins [2021, Theorem 2.8.3].

Theorem 3.5. The following maps are pairs of mutually inverse dual isomorphisms:

1. $\mathsf{Alg} : \mathbf{Ext}(\mathtt{2IPC}) \to \mathbf{Var}(\mathsf{2HA})$ and $\mathsf{Th} : \mathbf{Var}(\mathsf{2HA}) \to \mathbf{Ext}(\mathtt{2IPC})$;
2. $\mathsf{Alg} : \mathbf{Ext}(\mathtt{2IPC}_\mathtt{R}) \to \mathbf{Uni}(\mathsf{2HA})$ and $\mathsf{ThR} : \mathbf{Uni}(\mathsf{2HA}) \to \mathbf{Ext}(\mathtt{2IPC}_\mathtt{R})$.

Geometrically, bsi-logics and rule systems can be interpreted over order-topological structures developed by Esakia [1975], and today known as *bi-Esakia spaces*.

Definition 3.6. A *bi-Esakia space* is an Esakia space $\mathfrak{X} = (X, \leq, \mathcal{O})$, satisfying the following additional conditions:

- ${\downarrow}x$ is closed for every $x \in X$;
- ${\uparrow}[U] \in \mathsf{Clop}(\mathfrak{X})$ whenever $U \in \mathsf{Clop}(\mathfrak{X})$.

We let 2Esa denote the class of all bi-Esakia spaces. For $\mathfrak{X} \in \mathsf{2Esa}$, we write $\mathsf{ClopDown}(\mathfrak{X})$ for the set of clopen downsets in $\mathfrak{X}$. If $\mathfrak{X}, \mathfrak{Y} \in \mathsf{2Esa}$, a map $f : \mathfrak{X} \to \mathfrak{Y}$ is called a *bounded morphism* if for all $x, y \in X$, we have that $x \leq y$ implies that $f(x) \leq f(y)$, and moreover:

- $f(x) \leq y$ implies that there is $z \in X$ with $x \leq z$ and $f(z) = y$;
- $f(x) \geq y$ implies that there is $z \in X$ with $x \geq z$ and $f(z) = y$.

If $\mathfrak{X} = (X, \leq, \mathcal{O})$ is an Esakia space, the *order dual* $\bar{\mathfrak{X}}$ of $\mathfrak{X}$ is the structure $\mathfrak{X} = (X, \geq, \mathcal{O})$, where $\geq$ is the converse of $\leq$. The algebraic order duality principle of Proposition 3.4 has the following geometric counterpart.

Proposition 3.7. For every bi-Esakia space $\mathfrak{X}$, the order dual $\bar{\mathfrak{X}}$ of $\mathfrak{X}$ is an Esakia space.

As in the algebraic case, a number of results from the theory of Esakia spaces can be transferred smoothly to bi-Esakia spaces in virtue of this fact. For example, we may generalise Proposition 2.6 to the following result.

Proposition 3.8. Let $\mathfrak{X} \in \mathsf{2Esa}$. Then for all $x, y \in X$ we have:

1. If $x \nleq y$ then there is $U \in \mathsf{ClopUp}(\mathfrak{X})$ such that $x \in U$ and $y \notin U$;
2. If $y \nleq x$ then there is $U \in \mathsf{ClopDown}(\mathfrak{X})$ such that $x \in U$ and $y \notin U$.

Proof. (1) is just Proposition 2.6, whereas (2) follows from (1) and the order-duality principle. □

Bsi formulae are interpreted over bi-Esakia spaces the same way si formulae are interpreted over Esakia space, except for the following additional clause for co-implication (here $\mathfrak{X} \in \mathsf{2Esa}$, $x \in X$ and V is a valuation on $\mathfrak{X}$).

$$\mathfrak{X}, V, x \models \varphi \leftarrow \psi \iff \text{there is } y \in {\downarrow} x : \mathfrak{X}, V, x \models \varphi \text{ and } \mathfrak{X}, V, x \not\models \psi$$

The order-topological duality holding between Heyting algebras and Esakia spaces generalises smoothly to a duality relating bi-Heyting algebras and bi-Esakia spaces, as shown by Esakia [1975].

Theorem 3.9 (bi-Esakia duality)**.** The category of bi-Heyting algebras with corresponding homomorphisms is dually equivalent to the category of bi-Esakia spaces with continuous bounded morphisms.

We briefly recall the basic elements of bi-Esakia duality. Given a bi-Heyting algebra $\mathfrak{H}$, its dual bi-Esakia space $\mathfrak{H}_*$ is simply the Esakia dual of $\mathfrak{H}$. Conversely, given a bi-Esakia space $\mathfrak{X}$, its dual Heyting algebra is defined as

$$\mathfrak{X}^* = (\mathsf{ClopUp}(\mathfrak{X}), \cap, \cup, \to_{\leq}, \leftarrow_{\leq}, \emptyset, X),$$

where $\to_{\leq}$ is just the implication operation of the Heyting algebra dual to $\mathfrak{X}$, and

$$U \leftarrow_{\leq} V = {\uparrow}(U \setminus V).$$

One can prove that for every $\mathfrak{H} \in \mathsf{2HA}$, the Stone map β witnesses $\mathfrak{H} \cong {\mathfrak{H}_*}^*$, and conversely that for every $\mathfrak{X} \in \mathsf{2Esa}$, the inverse of the stone map $\beta^{-1} : \mathfrak{X} \to {\mathfrak{X}^*}_*$ witnesses $\mathfrak{X} \cong {\mathfrak{X}^*}_*$. Moreover, we have that for any $\mathfrak{H}, \mathfrak{K} \in \mathsf{2HA}$, a map $h : \mathfrak{H} \to \mathfrak{K}$ is a homomorphism iff $h^{-1} : \mathfrak{K}_* \to \mathfrak{H}_*$ is a continuous bounded morphism, and likewise $f : \mathfrak{X} \to \mathfrak{Y}$ is a continuous bounded morphism iff $f^{-1} : \mathfrak{Y}^* \to \mathfrak{X}^*$ is a homomorphism.

§3.1.2 Tense Deductive Systems, Tense Algebras, and Tense Spaces

We now work in the *tense signature*,

$$ten := \{\wedge, \vee, \neg, \Box_F, \Diamond_P, \bot, \top\}.$$

We prefer this signature to one with two primitive boxes to strengthen the connection between bi-Heyting coimplication and backwards looking modalities. As usual, we write $\Diamond_F = \neg\Box_F\neg$ and $\Box_P = \neg\Diamond_P\neg$. The set Frm_{ten} of *tense formulae* is defined recursively as follows:

$$\varphi ::= p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box_F\varphi \mid \Diamond_P\varphi.$$

We introduce *tense deductive systems*. Good references on tense logics include Blackburn et al. [2001, Ch. 1, Ch. 4] and Gabbay et al. [1994]. Tense rule systems have not received much attention in the literature.

Definition 3.10. A *(normal) tense logic* is a logic $\mathtt{M}$ over Frm_{ten} satisfying the following conditions:

1. $\mathtt{S4}_{\Box_F}, \mathtt{S4}_{\Diamond_P} \subseteq \mathtt{M}$, where $\mathtt{S4}_{\heartsuit}$ is the least normal modal logic $\mathtt{S4}$ formulated in the modal signature with modal operator $\heartsuit \in \{\Box_F, \Diamond_P\}$;
2. $\varphi \to \Box_F \Diamond_P \varphi \in \mathtt{M}$;
3. $\varphi \to \psi, \varphi \in \mathtt{M}$ implies $\psi \in \mathtt{M}$ (MP);
4. $\varphi \in \mathtt{M}$ implies $\Box_F \varphi \in \mathtt{M}$ (NEC$_F$);
5. $\varphi \in \mathtt{M}$ implies $\Box_P \varphi \in \mathtt{M}$ (NEC$_P$);

We let $\mathtt{S4.t}$ denote the least normal tense logic. A *(normal) tense rule system* is a rule system $\mathtt{M}$ over Frm_{ten} satisfying the following requirements:

1. $\varphi, \varphi \to \psi / \psi \in \mathtt{M}$ (MP-R);
2. $\varphi / \Box_F \varphi \in \mathtt{M}$ (NEC$_F$-R);
3. $\varphi / \Box_P \varphi \in \mathtt{M}$ (NEC$_P$-R);
4. $/\varphi \in \mathtt{M}$ whenever $\varphi \in \mathtt{S4.t}$.

We note that, for convenience, we are using a somewhat non-standard notion of a tense deductive system by requiring that tense deductive system contain $\mathtt{S4}$. It is more customary to require only that tense deductive system contain $\mathtt{K}$.

If $\mathtt{M}$ is a tense logic let $\mathbf{NExt}(\mathtt{M})$ be the set of normal tense logics containing $\mathtt{M}$, and similarly for tense rule systems. Then $\mathbf{NExt}(\mathtt{S4.t})$ is the set of all tense logics. It is easily checked that $\mathbf{NExt}(\mathtt{S4.t})$ is a complete lattice, with $\oplus_{\mathbf{NExt}(\mathtt{S4.t})}$ as join and intersection as meet. Note that for every $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4.t})$ there is always a least tense rule system containing $/\varphi$ for each $\varphi \in \mathtt{M}$, which we denote by $\mathtt{M}_{\mathtt{R}}$. Then $\mathtt{S4.t}_{\mathtt{R}}$ is the least tense rule system and $\mathbf{NExt}(\mathtt{S4.t}_{\mathtt{R}})$ is the set of all tense rule systems. Again, one can easily verify that $\mathbf{NExt}(\mathtt{S4.t}_{\mathtt{R}})$ forms a complete lattice with $\oplus_{\mathbf{NExt}(\mathtt{S4.t}_{\mathtt{R}})}$ as join and intersection as meet. As usual, we write both $\oplus_{\mathbf{NExt}(\mathtt{S4.t})}$ and $\oplus_{\mathbf{NExt}(\mathtt{S4.t}_{\mathtt{R}})}$ simply as $\oplus$.

We have the following tense counterpart of Proposition 3.2.

Proposition 3.11. The mappings $(\cdot)_{\mathtt{R}}$ and $\mathsf{Taut}(\cdot)$ are mutually inverse complete lattice isomorphisms between $\mathbf{NExt}(\mathtt{S4.t})$ and the sublattice of $\mathbf{NExt}(\mathtt{S4.t}_{\mathtt{R}})$ consisting of all si-rule systems $\mathtt{L}$ such that $\mathsf{Taut}(\mathtt{L})_{\mathtt{R}} = \mathtt{L}$.

We interpret tense logics and rule systems on modal algebra expansions called *tense algebras*, which are extensively discussed in, e.g., Kowalski [1998] and Venema [2007, §8.1].

Definition 3.12. A *tense algebra* is a structure $\mathfrak{A} = (A, \wedge, \vee, \neg, \Box_F, \Diamond_P, 0, 1)$, such that both the $\Box_F$-free and the $\Diamond_P$-free reducts of $\mathfrak{A}$ are closure algebras, and $\Box_F, \Diamond_P$ form a residual pair, that is, for all $a, b \in A$ we have the following identity:

$$\Diamond_P a \leq b \iff a \leq \Box_F y.$$

We let Ten denote the class of tense algebras. It is well known that Ten is equationally definable (cf., e.g., Venema 2007, Proposition 8.5), hence a variety by Theorem 1.10. We let $\mathbf{Var}(\mathsf{Ten})$ and $\mathbf{Uni}(\mathsf{Ten})$ be the lattice of subvarieties and of universal subclasses of Ten respectively. The following result can be obtained by similar techniques as Theorem 2.12.

Theorem 3.13. The following maps are pairs of mutually inverse dual isomorphisms:

1. $\mathsf{Alg} : \mathbf{NExt}(\mathtt{S4.t}) \to \mathbf{Var}(\mathsf{Ten})$ and $\mathsf{Th} : \mathbf{Var}(\mathsf{Ten}) \to \mathbf{NExt}(\mathtt{S4.t})$;
2. $\mathsf{Alg} : \mathbf{NExt}(\mathtt{S4.t_R}) \to \mathbf{Uni}(\mathsf{Ten})$ and $\mathsf{ThR} : \mathbf{Uni}(\mathsf{Ten}) \to \mathbf{NExt}(\mathtt{S4.t_R})$.

We now review the geometrical semantics for tense logics and rule systems.

Definition 3.14. A *tense space* is an $\mathtt{S4}$-modal space $\mathfrak{X} = (X, R, \mathcal{O})$, satisfying the following additional conditions:

- $R^{-1}(x)$ is closed for every $x \in X$;
- $R[U] \in \mathsf{Clop}(\mathfrak{X})$ whenever $U \in \mathsf{Clop}(\mathfrak{X})$.

It should be clear from the above definition that tense spaces, like bi-Esakia spaces, also satisfy an order-duality principle.

Proposition 3.15. For every tense space $\mathfrak{X} = (X, R, \mathcal{O})$, its *order dual* $\bar{\mathfrak{X}} = (X, \breve{R}, \mathcal{O})$, where $\breve{R}$ is the converse of R, is an $\mathtt{S4}$-modal space.

If $\mathfrak{X}, \mathfrak{Y}$ are tense spaces, a map $f : \mathfrak{X} \to \mathfrak{Y}$ is called a *bounded morphism* if for all $x, y \in X$, if Rxy then $Rf(x)f(y)$, and moreover for all $x \in X$ and $y \in Y$ the following conditions hold:

- If $Rf(x)y$ then there is $z \in X$ such that Rxz and $f(z) = y$;
- If $Ryf(x)$ then there is $z \in X$ such that Rzx and $f(z) = y$.

The geometrical semantics of tense logics and rule systems over tense spaces is a straightforward generalisation of the semantics of modal logics and rule systems on modal spaces, using the following clauses for interpreting $\Box_F, \Diamond_P$. Here $\mathfrak{X}$ is a tense space, $x \in X$, and V a valuation on $\mathfrak{X}$.

$$\mathfrak{X}, V, x \models \Box_F \varphi \iff \mathfrak{X}, V, y \models \varphi \text{ for all } y \in R[x] \tag{3.1}$$

$$\mathfrak{X}, V, x \models \Diamond_P \varphi \iff \mathfrak{X}, V, y \models \varphi \text{ for some } y \in R^{-1}(x) \tag{3.2}$$

We list some important properties of tense spaces, which are obtained straightforwardly from Proposition 2.16 and the order-duality principle.

Proposition 3.16. Let $\mathfrak{X} \in \mathsf{Spa}(\mathtt{S4.t})$ and $U \in \mathsf{Clop}(\mathfrak{X})$. Then the following conditions hold:

1. The sets $max_R(U)$, $min_R(U)$ are closed;
2. If $x \in U$ then there is $y \in qmax_R(U)$ such that Rxy, and there is $z \in qmin_R(U)$ such that Rzx

The duality between modal algebras and modal spaces extends straightforwardly to a duality relating tense algebras and tense spaces.

Theorem 3.17. The category of tense algebras with homomorphisms is dually equivalent to the category of tense spaces with continuous bounded morphisms.

We sketch the basics of this duality. Given a tense algebra $\mathfrak{A} \in \mathsf{Ten}$, its dual tense space $\mathfrak{A}_*$ is just the dual modal space of the $\lozenge_P$-free reduct of $\mathfrak{A}$. Conversely, given a tense space $\mathfrak{X}$, its dual tense algebra $\mathfrak{X}^*$ is obtained by expanding the modal algebra dual to $\mathfrak{X}$ with the operation

$$\lozenge_{\breve{R}} U := \breve{R}^{-1}(U) = R[U].$$

One can then prove that for every $\mathfrak{A} \in \mathsf{Ten}$ the Stone map β witnesses $\mathfrak{A} = {\mathfrak{A}_*}^*$, and conversely that for every tense space $\mathfrak{X}$ the map $\beta^{-1} : {\mathfrak{X}^*}_* \to \mathfrak{X}$ witnesses $\mathfrak{X} \cong {\mathfrak{X}^*}_*$. Moreover, for any $\mathfrak{A}, \mathfrak{B} \in \mathsf{Ten}$, a map $h : \mathfrak{A} \to \mathfrak{B}$ is a homomorphism iff $h^{-1} : \mathfrak{B}_* \to \mathfrak{A}_*$ is a continuous bounded morphism.

We will pay particular attention to tense algebras and spaces validating the tense logic $\mathtt{Grz.T}$ below.

$$\begin{aligned} \mathtt{Grz.T} := \mathtt{S4.t} &\oplus \Box_F(\Box_F(p \to \Box_F p) \to p) \to p \\ &\oplus p \to \lozenge_P(p \wedge \neg \lozenge_P(\lozenge_P p \wedge \neg p)). \end{aligned}$$

We name this logic $\mathtt{Grz.T}$ rather than $\mathtt{Grz.t}$ to emphasize that the $\mathtt{Grz}$-axiom is required for both operators rather than just for $\Box_F$. We let $\mathsf{Grz.T} := \mathsf{Alg}(\mathtt{Grz.T})$. Clearly, for any $\mathfrak{A} \in \mathsf{Ten}$ we have $\mathfrak{A} \in \mathsf{Grz.T}$ iff every $a \in A$ satisfies both the inequalties

$$\begin{gathered} \Box_F(\Box_F(a \to \Box_F a) \to a) \leq a, \\ a \leq \lozenge_P(a \wedge \neg \lozenge_P(\lozenge_P a \wedge \neg a)). \end{gathered}$$

The following proposition is a counterpart to Proposition 2.18, and is proved straightforwardly using the latter and the order-duality principle.

Proposition 3.18. For every $\mathtt{Grz}$-space $\mathfrak{X}$ and $U \in \mathsf{Clop}(\mathfrak{X})$, the following hold:

1. $qmax_R(U) \subseteq max_R(U)$, and $qmin_R(U) \subseteq min_R(U)$;
2. The sets $max_R(U)$ and $min_R(U)$ is closed;
3. For every $x \in U$ there are $y \in pas_R(U)$ such that Rxy, and $z \in pas_{\breve{R}}(U)$ such that Rzx;
4. $max_R(U) \subseteq pas_R(U)$ and $min_R(U) \subseteq pas_{\breve{R}}(U)$.

Additionally, an $\mathtt{S4.t}$-space $\mathfrak{X}$ is a $\mathtt{Grz.T}$-space if Item 3 holds for every $U \in \mathsf{Clop}(\mathfrak{X})$.

Recall that for $\mathfrak{X}$ a $\mathtt{Grz.T}$-space, a set $U \subseteq X$ is said to *cut* a cluster $C \subseteq X$ when both $U \cap C \neq \emptyset$ and $U \setminus C \neq \emptyset$. As a consequence of Item 4 in Proposition 3.18 above, we obtain in particular that in any $\mathtt{Grz.T}$-space $\mathfrak{X}$, no cluster $C \subseteq X$ can be cut by either of $max_R(U), pas_R(U), min_R(U), pas_{\breve{R}}(U)$ for any $U \in \mathsf{Clop}(\mathfrak{X})$.

§3.2 Stable Canonical Rules for Bi-superintuitionistic and Tense Rule Systems

In this section we generalise the si and modal stable canonical rules from § 2.2 to the bsi and tense setting respectively. While bsi and tense stable canonical rules are not discussed in existing literature, the differences between their theory and that of si and modal stable canonical rules are few and inessential. In particular, all proofs of results in this sections are straightforward adaptations of corresponding results in § 2.2, which is why we omit most of them.

§§ 3.2.1 and 3.2.2 develops bsi and tense stable canonical rules respectively. § 3.2.3 gives a cursory overview of the prospects of generalising Jerábek-style canonical rules to the bsi and tense setting, suggesting said task is less straightforward than it is for stable canonical rules.

§3.2.1 Bi-superintuitionistic Case

We begin by defining bsi stable canonical rules.

Definition 3.19. Let $\mathfrak{H} \in \mathsf{HA}$ be finite and $D^{\to}, D^{\leftarrow} \subseteq A \times A$. For every $a \in H$ introduce a fresh propositional variable p_a. The *bsi stable canonical rule* of $(\mathfrak{H}, D)$, is defined as the rule $\eta_B(\mathfrak{H}, D^{\to}, D^{\leftarrow}) = \Gamma/\Delta$, where

$$\begin{aligned}\Gamma =& \{p_0 \leftrightarrow 0\} \cup \{p_1 \leftrightarrow 1\} \cup \\ & \{p_{a\wedge b} \leftrightarrow p_a \wedge p_b : a, b \in H\} \cup \{p_{a\vee b} \leftrightarrow p_a \vee p_b : a, b \in H\} \cup \\ & \{p_{a\to b} \leftrightarrow p_a \to p_b : (a, b) \in D^{\to}\} \cup \{p_{a\leftarrow b} \leftrightarrow p_a \leftarrow p_b : (a, b) \in D^{\leftarrow}\} \\ \Delta =& \{p_a \leftrightarrow p_b : a, b \in H \text{ with } a \neq b\}.\end{aligned}$$

The notion of a stable map between bi-Heyting algebras is defined exactly as in the Heyting case, i.e., stable maps are simply bounded lattice homomorphisms. We note that for any stable map $h : \mathfrak{H} \to \mathfrak{K}$ with $\mathfrak{H}, \mathfrak{K} \in \mathsf{2HA}$, for any $a \in H$ we also have

$$h(a \leftarrow b) \geq h(a) \leftarrow h(b).$$

Indeed, this is obvious in view of the order-duality principle. If $D \subseteq H \times H$ and $\heartsuit \in \{\to, \leftarrow\}$, we say that h satisfies the $\heartsuit$*-bounded domain condition* (BDC$^{\heartsuit}$) for D if $h(a \heartsuit b) = h(a) \heartsuit h(b)$ for every $(a, b) \in D$. If $D^{\to}, D^{\leftarrow} \subseteq H \times H$, for brevity we say that h satisfies the BDC for $(D^{\to}, D^{\leftarrow})$ to mean that h satisfies the BDC$^{\to}$ for $D^{\to}$ and the BDC$^{\leftarrow}$ for $D^{\leftarrow}$.

The next two results characterise algebraic refutation conditions for bsi stable canonical rules.

Proposition 3.20. For all finite $\mathfrak{H} \in \mathsf{2HA}$ and $D^{\rightarrow}, D^{\leftarrow} \subseteq H \times H$, we have $\mathfrak{H} \not\models \eta_B(\mathfrak{H}, D^{\rightarrow}, D^{\leftarrow})$.

Proposition 3.21. For every $\mathfrak{K} \in \mathsf{2HA}$ and every bsi stable canonical rule $\eta_B(\mathfrak{H}, D^{\rightarrow}, D^{\leftarrow})$, we have $\mathfrak{K} \not\models \eta_B(\mathfrak{H}, D^{\rightarrow}, D^{\leftarrow})$ iff there is a stable embedding $h : \mathfrak{H} \to \mathfrak{K}$ satisfying the BDC for $(D^{\rightarrow}, D^{\leftarrow})$.

We now characterise geometric refutation conditions of bsi stable canonical rules on bi-Esakia spaces. Since bi-Esakia spaces are Esakia spaces, the notion of a stable map applies. Let $\mathfrak{X}, \mathfrak{Y} \in \mathsf{2Esa}$ and $\mathfrak{d} \subseteq Y$. A stable map $f : \mathfrak{X} \to \mathfrak{Y}$ is said to satisfy

- The BDC$^{\rightarrow}$ for $\mathfrak{d}$ if for all $x \in X$ we have

$$\uparrow f(x) \cap \mathfrak{d} \neq \emptyset \Rightarrow f[\uparrow x] \cap \mathfrak{d} \neq \emptyset;$$

- The BDC$^{\leftarrow}$ for $\mathfrak{d}$ if for all $x \in X$ we have

$$\downarrow f(x) \cap \mathfrak{d} \neq \emptyset \Rightarrow f[\downarrow x] \cap \mathfrak{d} \neq \emptyset.$$

If $\mathfrak{D} \subseteq \wp(Y)$, we say that f satisfies the CDC$^{\heartsuit}$ for $\mathfrak{D}$ when it does for each $\mathfrak{d} \in \mathfrak{D}$, where $\heartsuit \in \{\rightarrow, \leftarrow\}$. Given $\mathfrak{D}^{\rightarrow}, \mathfrak{D}^{\leftarrow} \in \wp(Y)$ we write that f satisfies the BDC for $(\mathfrak{D}^{\rightarrow}, \mathfrak{D}^{\leftarrow})$ if f satisfies the CDC$^{\rightarrow}$ for $\mathfrak{D}^{\rightarrow}$ and the CDC$^{\leftarrow}$ for $\mathfrak{D}^{\leftarrow}$. Finally, if $\eta_B(\mathfrak{H}, D^{\rightarrow}, D^{\leftarrow})$ is a bsi stable canonical rule consider $\mathfrak{X} := \mathfrak{H}_*$ and let

$$\mathfrak{D}^{\heartsuit} := \{\mathfrak{d}^{\heartsuit}_{(a,b)} : (a, b) \in D^{\heartsuit}\}$$

where

$$\mathfrak{d}^{\heartsuit}_{(a,b)} := \beta(a) \setminus \beta(b)$$

for $\heartsuit \in \{\rightarrow, \leftarrow\}$.

Proposition 3.22. For any bi-Esakia space $\mathfrak{X}$ and any bsi stable canonical rule $\eta_B(\mathfrak{H}, D^{\rightarrow}, D^{\leftarrow})$, $\mathfrak{X} \not\models \eta_B(\mathfrak{H}, D^{\rightarrow}, D^{\leftarrow})$ iff there is a continuous stable surjection $f : \mathfrak{X} \to \mathfrak{H}_*$ satisfying the BDC for $(\mathfrak{D}^{\rightarrow}, \mathfrak{D}^{\leftarrow})$ defined as above.

In view of Proposition 3.22, in geometric settings we prefer to write a bsi stable canonical rule $\eta_B(\mathfrak{H}, D^{\rightarrow}, D^{\leftarrow})$ as $\eta_B(\mathfrak{H}_*, \mathfrak{D}^{\rightarrow}, \mathfrak{D}^{\leftarrow})$.

We now make the notion of filtration for bi-Heyting algebras presupposed by our bsi stable canonical rules explicit.

Definition 3.23. Let $\mathfrak{H}$ be a bi-Heyting algebra, V a valuation on $\mathfrak{H}$, and Θ a finite, subformula closed set of formulae. A (finite) model $(\mathfrak{K}', V')$ is called a *(finite) filtration of* $(\mathfrak{H}, V)$ *through* Θ if the following hold:

1. $\mathfrak{K}' = (\mathfrak{K}, \rightarrow, \leftarrow)$, where $\mathfrak{K}$ is the bounded sublattice of $\mathfrak{H}$ generated by $\bar{V}[\Theta]$;
2. $V(p) = V'(p)$ for every propositional variable $p \in \Theta$;

3. The inclusion $\subseteq : \mathfrak{K}' \to \mathfrak{H}$ is a stable embedding satisfying the BDC for $(D^{\to}, D^{\leftarrow})$, where
$$D^{\heartsuit} := \{(\bar{V}(\varphi), \bar{V}(\psi)) : \varphi \heartsuit \psi \in \Theta\}$$
for $\heartsuit \in \{\to, \leftarrow\}$.

Theorem 3.24 (Filtration theorem for bi-Heyting algebras)**.** Let $\mathfrak{H} \in \mathsf{2HA}$ be a Heyting algebra, V a valuation on $\mathfrak{H}$, and Θ a a finite, subformula closed set of formulae. If $(\mathfrak{K}', V')$ is a filtration of $(\mathfrak{H}, V)$ through Θ then for every $\varphi \in \Theta$ we have

$$\bar{V}(\varphi) = \bar{V}'(\varphi).$$

Consequently, for every bsi every rule Γ/Δ such that $\gamma, \delta \in \Theta$ for each $\gamma \in \Gamma$ and $\delta \in \Delta$ we have
$$\mathfrak{H}, V \models \Gamma/\Delta \iff \mathfrak{K}', V' \models \Gamma/\Delta.$$

The next lemma is a counterpart to Lemma 2.26.

Lemma 3.25. For every bsi rule Γ/Δ there is a finite set Ξ of si stable canonical rules such that for any $\mathfrak{K} \in \mathsf{2HA}$ we have that $\mathfrak{K} \not\models \Gamma/\Delta$ iff there is $\eta_B(\mathfrak{H}, D^{\to}, D^{\leftarrow}) \in \Xi$ such that $\mathfrak{K} \not\models \eta_B(\mathfrak{H}, D^{\to}, D^{\leftarrow})$.

Proof. The proof is a straightforward generalisation of the proof of Lemma 2.26, using the fact that every finite bounded distributive lattice $\mathfrak{J}$ may be expanded to a bi-Heyting algebra $\mathfrak{J}' = (\mathfrak{J}, \rightsquigarrow, \leftsquigarrow)$ by setting:

$$a \rightsquigarrow b := \bigvee\{c \in J : a \wedge b \leq c\}$$
$$a \leftsquigarrow b := \bigwedge\{c \in J : a \leq b \vee c\}.$$

□

Reasoning as in the proof of Theorem 2.27 we obtain the following axiomatisation result.

Theorem 3.26. Every bsi-rule system $\mathtt{L} \in \mathbf{Ext}(\mathtt{2IPC_R})$ is axiomatisable over $\mathtt{2IPC_R}$ by some set of bsi stable canonical rules.

§3.2.2 Tense Case

We now turn to tense stable canonical rules.

Definition 3.27. Let $\mathfrak{A} \in \mathsf{Ten}$ be finite and $D^{\Box_F}, D^{\Diamond_P} \subseteq A$. For every $a \in A$ introduce a fresh propositional variable p_a. The *tense stable canonical rule* of $(\mathfrak{A}, D^{\Box_F}, D^{\Diamond_P})$, is defined as the rule $\mu_T(\mathfrak{H}, D^{\Box_F}, D^{\Diamond_P}) = \Gamma/\Delta$, where

$$\begin{aligned}\Gamma =&\{p_{a\wedge b} \leftrightarrow p_a \wedge p_b : a, b \in A\} \cup \{p_{a\vee b} \leftrightarrow p_a \vee p_b : a, b \in A\} \cup \{p_{\neg a} \leftrightarrow \neg p_a : a \in A\}\cup\\ &\{\Box_F p_a \to p_{\Box_F a} : a \in A\} \cup \{p_{\Diamond_P a} \to \Diamond_P p_a : a \in A\}\cup\\ &\{p_{\Box_F a} \to \Box_F p_a : a \in D^{\Box_F}\} \cup \{\Diamond_P p_a \to p_{\Diamond_P a} : a \in D^{\Diamond_P}\}\\ \Delta =&\{p_a : a \in A\}.\end{aligned}$$

If $\mathfrak{A}, \mathfrak{B} \in \mathsf{MA}$ are tense algebras, a map $h : \mathfrak{A} \to \mathfrak{B}$ is called *stable* if for every $a \in A$ the following conditions hold:

$$h(\Box_F a) \leq \Box_F h(a) \qquad \Diamond_P h(a) \leq h(\Diamond_P a).$$

If $D \subseteq A$ and $\heartsuit \in \{\Box_F, \Diamond_P\}$, we say that h satisfies the $\heartsuit$*-bounded domain condition* (BDC$^\heartsuit$) for D if $h(\heartsuit a) = \heartsuit h(a)$ for every $a \in D$. If $D^{\Box_F}, D^{\Diamond_P} \subseteq A$, for brevity we say that h satisfies the BDC for $(D^{\Box_F}, D^{\Diamond_P})$ to mean that h satisfies the BDC$^{\Box_F}$ for $D^{\Box_F}$ and the BDC$^{\Diamond_P}$ for $D^{\Diamond_P}$.

We outline algebraic refutation conditions for tense stable canonical rules.

Proposition 3.28. For all finite $\mathfrak{A} \in \mathsf{Ten}$ and $D^{\Box_F}, D^{\Diamond_P} \subseteq A$, we have $\mathfrak{A} \nvDash \mu_T(\mathfrak{A}, D^{\Box_F}, D^{\Diamond_P})$.

Proposition 3.29. For every $\mathfrak{B} \in \mathsf{Ten}$ and any tense stable canonical rule $\mu_T(\mathfrak{A}, D^{\Box_F}, D^{\Diamond_P})$, we have $\mathfrak{B} \nvDash \mu_T(\mathfrak{A}, D^{\Box_F}, D^{\Diamond_P})$ iff there is a stable embedding $h : \mathfrak{A} \to \mathfrak{B}$ satisfying the BDC for $(D^{\Box_F}, D^{\Diamond_P})$.

Tense spaces are modal spaces, therefore the notion of a stable map applies. Let $\mathfrak{X}, \mathfrak{Y}$ be tense spaces. and $\mathfrak{d} \subseteq Y$. A stable map $f : \mathfrak{X} \to \mathfrak{Y}$ is said to satisfy

- The BDC$^{\Box_F}$ for $\mathfrak{d}$ if for all $x \in X$ we have

$$R[f(x)] \cap \mathfrak{d} \neq \emptyset \Rightarrow f[R[x]] \cap \mathfrak{d} \neq \emptyset;$$

- The BDC$^{\Diamond_P}$ for $\mathfrak{d}$ if for all $x \in X$ we have

$$\breve{R}[f(x)] \cap \mathfrak{d} \neq \emptyset \Rightarrow f[\breve{R}[x]] \cap \mathfrak{d} \neq \emptyset.$$

If $\mathfrak{D} \subseteq \wp(Y)$, we say that f satisfies the CDC$^\heartsuit$ for $\mathfrak{D}$ when it does for each $\mathfrak{d} \in \mathfrak{D}$, where $\heartsuit \in \{\Box_F, \Diamond_P\}$. Given $\mathfrak{D}^{\Box_F}, \mathfrak{D}^{\Diamond_P} \in \wp(Y)$ we write that f satisfies the BDC for $(\mathfrak{D}^{\Box_F}, \mathfrak{D}^{\Diamond_P})$ if f satisfies the CDC$^{\Box_F}$ for $\mathfrak{D}^{\Box_F}$ and the CDC$^{\Diamond_P}$ for $\mathfrak{D}^{\Diamond_P}$. Finally, if $\mu_T(\mathfrak{A}, D^{\Box_F}, D^{\Diamond_P})$ is a tense stable canonical rule consider $\mathfrak{X} := \mathfrak{A}_*$ and for $\heartsuit \in \{\Box_F, \Diamond_P\}$ let

$$\mathfrak{D}^\heartsuit := \{\mathfrak{d}_a^\heartsuit : a \in D^\heartsuit\}$$

where for each $a \in A$ we have

$$\mathfrak{d}_a^{\Box_F} := -\beta(a)$$
$$\mathfrak{d}_a^{\Diamond_P} := \beta(a)$$

Proposition 3.30. For any tense space $\mathfrak{X}$ and any tense stable canonical rule $\mu_T(\mathfrak{A}, D^{\Box_F}, D^{\Diamond_P})$, we have $\mathfrak{X} \nvDash \mu_T(\mathfrak{A}, D^{\Box_F}, D^{\Diamond_P})$ iff there is a continuous stable surjection $f : \mathfrak{X} \to \mathfrak{A}_*$ satisfying the BDC for $(\mathfrak{D}^{\Box_F}, \mathfrak{D}^{\Diamond_P})$ defined as above.

In view of Proposition 3.30, in geometric settings we prefer to write a tense stable canonical rule $\mu_T(\mathfrak{A}, D^{\Box_F}, D^{\Diamond_P})$ as $\mu_T(\mathfrak{A}_*, \mathfrak{D}^{\Box_F}, \mathfrak{D}^{\Diamond_P})$.

We now introduce the notion of filtration implicit in tense stable canonical rules. Filtration for tense logics was considered, e.g., in Wolter [1997] from a frame-theoretic perspective. Here we prefer an algebraic approach in line with Chapter 2.

Definition 3.31. Let $\mathfrak{A}$ be a tense algebra, V a valuation on $\mathfrak{A}$, and Θ a finite, subformula closed set of formulae. A (finite) model $(\mathfrak{B}', V')$ is called a *(finite) filtration of* $(\mathfrak{A}, V)$ *through* Θ if the following hold:

1. $\mathfrak{B}' = (\mathfrak{B}, \Box_F, \Diamond_P)$, where $\mathfrak{B}$ is the Boolean subalgebra of $\mathfrak{A}$ generated by $\bar{V}[\Theta]$;
2. $V(p) = V'(p)$ for every propositional variable $p \in \Theta$;
3. The inclusion $\subseteq : \mathfrak{B}' \to \mathfrak{A}$ is a stable embedding satisfying the BDC for $(D^{\Box_F}, D^{\Diamond_P})$, where
$$D^{\heartsuit} := \{\bar{V}(\varphi) : \heartsuit\varphi \in \Theta\}$$
for $\heartsuit \in \{\Box_F, \Diamond_P\}$.

Theorem 3.32 (Filtration theorem for tense algebras)**.** Let $\mathfrak{H} \in \mathsf{2HA}$ be a Heyting algebra, V a valuation on $\mathfrak{H}$, and Θ a finite, subformula closed set of formulae. If $(\mathfrak{K}', V')$ is a filtration of $(\mathfrak{H}, V)$ through Θ then for every $\varphi \in \Theta$ we have

$$\bar{V}(\varphi) = \bar{V}'(\varphi).$$

Consequently, for every tense rule Γ/Δ such that $\gamma, \delta \in \Theta$ for each $\gamma \in \Gamma$ and $\delta \in \Delta$ we have
$$\mathfrak{H}, V \models \Gamma/\Delta \iff \mathfrak{K}', V' \models \Gamma/\Delta.$$

Just like in the $\mathtt{S4}$ case, not every filtration of some model based on a tense algebra is itself based on a tense algebra, because the $\mathtt{S4}$-axiom for either $\Box_F$ or $\Diamond_P$ may not be preserved. However, given any model based on a tense algebra, there is always a method for filtrating it through any finite set of formulae which yields a model based on a tense algebra.

Definition 3.33. Let $\mathfrak{A} \in \mathsf{Ten}$, V a valuation on $\mathfrak{A}$ and Θ a finite, subformula closed set of formula. The (least) *transitive filtration* of $(\mathfrak{A}, V)$ is the pair $(\mathfrak{B}', V')$ with $\mathfrak{B} = (\mathfrak{B}', \blacksquare_F, \blacklozenge_P)$ where $\mathfrak{B}'$ and V' are as per Definition 2.32, and for all $b \in B$ we have

$$\blacksquare_F b := \bigvee\{\Box_F a : \Box_F a \leq \Box_F b \text{ and } a, \Box_F a \in B\}$$
$$\blacklozenge_P b := \bigwedge\{\Diamond_P a : \Diamond_P b \leq \Diamond_P a \text{ and } a, \Diamond_P a \in B\}$$

Via duality, it is not difficult to see that the least transitive filtration of any model based on a tense algebra is again a tense algebra.

At this stage, reasoning as in the proof of Lemma 2.35 using transitive filtrations we obtain the following results.

Lemma 3.34. For every tense rule Γ/Δ there is a finite set Ξ of si stable canonical rules such that for any $\mathfrak{K} \in \mathsf{Ten}$ we have that $\mathfrak{K} \not\models \Gamma/\Delta$ iff there is $\eta_B(\mathfrak{H}, D^{\to}, D^{\leftarrow}) \in \Xi$ such that $\mathfrak{K} \not\models \eta_B(\mathfrak{H}, D^{\to}, D^{\leftarrow})$.

Theorem 3.35. Every tense rule system is axiomatisable over $\mathtt{S4.t_R}$ by some set of tense stable canonical rules.

§3.2.3 Comparison with Jerábek-style Canonical Rules

Our bsi and tense stable canonical rules generalise si and modal stable canonical rules in a way that mirrors the simple and intimate connection existing between Heyting and bi-Heyting algebras on the one hand, and modal and tense algebras on the other, explicated by the order-duality principles. Just like a bi-Heyting algebra is just a Heyting algebra whose order-dual is also a Heyting algebra, so every bsi stable canonical rule is a sort of "independent fusion" between two si stable canonical rules, whose associated Heyting algebras are order-dual to each other. Similarly for the tense case.

Jerábek-style si and modal canonical rules (like Zakharyaschev-style si and modal canonical formulae), by contrast, do not generalise as smoothly to the bsi and tense case. Algebraically, a Jerábek-style si canonical rule may be defined as follows (cf. Bezhanishvili and Bezhanishvili [2009]; Bezhanishvili et al. [2016b]).

Definition 3.36. Let $\mathfrak{H} \in \mathsf{HA}$ be finite and let $D \subseteq H$. The *si canonical rule* of $(\mathfrak{H}, D)$ is the rule $\zeta(\mathfrak{H}, D) = \Gamma/\Delta$, where

$$\begin{aligned}\Gamma :=&\{p_0 \leftrightarrow \bot\}\cup\\ &\{p_{a\wedge b} \leftrightarrow p_a \wedge p_b : a, b \in H\} \cup \{p_{a\to b} \leftrightarrow p_a \to p_b : a, b \in H\}\cup\\ &\{p_{a\vee b} \leftrightarrow p_a \vee p_b : (a, b) \in D\}\\ \Delta :=&\{p_a \leftrightarrow p_b : a, b \in H \text{ with } a \neq b\}.\end{aligned}$$

Generalising the proof of Bezhanishvili et al. [2016b, Corollary 5.10], one can show that every si rule is equivalent to finitely many si canonical rules. The key ingredient in this proof is a characterisation of the refutation conditions for si canonical rules: $\zeta(\mathfrak{H}, D)$ is refuted by a Heyting algebra $\mathfrak{K}$ iff there is a $(\wedge, \to, 0)$-embedding $h : \mathfrak{H} \to \mathfrak{K}$ preserving $\vee$ on elements from D. Because $(\wedge, \to, 0)$-algebras are locally finite, a result known as *Diego's theorem*, one can then reason as in the proof of, e.g., Lemma 2.26 to reach the desired result.

Combining $\zeta(\mathfrak{H}, D)$ and $\zeta(\bar{\mathfrak{H}}, D')$ the same way bsi stable canonical rule combine si stable canonical rules would lead to a definition of the bsi canonical rule $\zeta_B(\mathfrak{H}, D, D')$ as the rule Γ/Δ, with

$$\begin{aligned}\Gamma :=&\{p_0 \leftrightarrow \bot\} \cup \{p_1 \leftrightarrow \top\}\cup\\ &\{p_{a\vee b} \leftrightarrow p_a \vee p_b : a, b \in H\} \cup \{p_{a\to b} \leftrightarrow p_a \to p_b : a, b \in H\}\cup\\ &\{p_{a\wedge b} \leftrightarrow p_a \wedge p_b : a, b \in H\} \cup \{p_{a\leftarrow b} \leftrightarrow p_a \leftarrow p_b : a, b \in H\}\cup\\ &\{p_{a\vee b} \leftrightarrow p_a \vee p_b : (a, b) \in D\}\cup\\ &\{p_{a\wedge b} \leftrightarrow p_a \wedge p_b : (a, b) \in D'\}\\ \Delta :=&\{p_a \leftrightarrow p_b : a, b \in H \text{ with } a \neq b\}.\end{aligned}$$

In this formulation the clauses for D, D' are clearly redundant, so eliminating them yields the same rule. It should be clear enough that if defined this way, the rule $\zeta_B(\mathfrak{H}, D, D')$ is refuted by a bi-Heyting algebra $\mathfrak{K}$ iff there is a bi-Heyting algebra embedding $h : \mathfrak{H} \to \mathfrak{K}$. Since the variety of bi-Heyting algebras is not locally finite, this refutation condition is clearly too strong to deliver a result to the effect that every bsi rule is equivalent to a set of bsi canonical rules. Without such a result, in turn there is no hope of axiomatising every rule system over 2IPC by means of bsi canonical rules.

Similar remarks hold in the tense case, although in this case the details are too complex to do them justice in the limited space we have at our disposal. We limit ourselves to a very rough sketch. Bezhanishvili et al. [2011] show that the proof of the fact that every modal formula is equivalent, over S4, to finitely many modal Zakharyaschev-style canonical formulae of closure algebras rests on an application of Diego's theorem [cf. Bezhanishvili et al., 2011, Main Lemma]. This has to do with how selective filtrations of closure algebras are constructed. Given a closure algebra $\mathfrak{B}$ refuting a rule Γ/Δ, a key step in constructing a finite selective filtration of $\mathfrak{B}$ through $Sfor(\Gamma/\Delta)$ consists in generating a $(\wedge, \to, 0)$-subalgebra of $\rho\mathfrak{A}$ from a finite subset of $O(A)$. This structure is guaranteed to be finite by Diego's theorem. On the most obvious ways of generalising this construction to tense algebras, we would need to replace this step with one of the following:

1. Generate both a $(\wedge, \to, 0)$-subalgebra of $\rho\mathfrak{A}$ and a $(\vee, \leftarrow, 1)$-subalgebra of $\rho\mathfrak{A}$ from a finite subset of $O(A)$;
2. Generate a bi-Heyting subalgebra of $\rho\mathfrak{A}$ from a finite subset of $O(A)$.

On option 1, Diego's theorem and its order dual would guarantee that both the $(\wedge, \to, 0)$-subalgebra of $\rho\mathfrak{A}$ and the $(\vee, \leftarrow, 1)$-subalgebra of $\rho\mathfrak{A}$ are finite. However, it is not clear how one could then combine the two subalgebras into a bi-Heyting algebra, which is required to obtain a selective filtration based on a tense algebra. On option 2, on the other hand, we would indeed obtain a bi-Heyting subalgebra of $\rho\mathfrak{A}$, but not necessarily a finite one, since bi-Heyting algebras are not locally finite.

We realise that the argument sketches just presented are far from conclusive, so we do not go as far as ruling out the possibility that Jerábek-style bsi and tense canonical rules could somehow be developed in such a way as to be a suitable tools for developing the theory of tense companions of bsi-rule systems. What such rules would look like, and in what sense they would constitute genuine generalisations of Jerábek's canonical rules and Zakharyaschev's canonical formulae are interesting questions, but this thesis is not the appropriate space where to pursue them. At this stage we merely wish to stress that answering this sort of questions is a non-trivial matter, whereas generalising stable canonical rules to the bsi and tense setting and applying them to develop the theory of tense companions is a completely routine task. Exactly the same methods used in the si and modal case work equally well in the bsi-tense case. Insofar as similar topics ought to be studied via uniform methods, this highlights one noteworthy aspect in which our strategy seems to fare better than Zakharyaschev and Jerábek's.

§3.3 Tense Companions of Bi-superintuitionistic Rule Systems

We turn to the main topic of this chapter. This section generalises the results of § 2.3 to the bsi-tense setting. As anticipated, this is done using exactly the same techniques seen in the si and modal case, which is one of the main advantages of our method.

In § 3.3.1 we review the relevant transformations between bi-Heyting and tense algebras, and generalise the Gödel translation. In § 3.3.2 we prove the central results of the theory of tense companions, including a genralisation of the Blok-Esakia theorem. The results in this section were proved originally by Wolter [1998] for logics, although their generalisations to rule systems appears to be new. We then give axiomatic characterisations of tense companions and bsi fragments, and illustrate these results with some examples (§ 3.3.3). We close by generalising the Gödel-Dummett conjecture to the bsi-tense setting, and proving additional results about filtrations and stability (§ 2.3.4).

§3.3.1 Semantic and Syntactic Mappings

From bi-Heyting to tense algebras and back We begin by generalising the semantic transformations for turning Heyting algebras into corresponding closure algebras and vice versa, seen in § 2.3, to transformations between bi-Heyting and tense algebras. The results in this section are well known, and the reader may consult Wolter [1998, §7] for a more detailed overview.

Definition 3.37. The mapping $\sigma : \mathsf{2HA} \to \mathsf{Ten}$ assigns every $\mathfrak{H} \in \mathsf{2HA}$ to the algebra $\sigma\mathfrak{H} := (B(\mathfrak{H}), \Box_F, \Diamond_P)$, where $B(\mathfrak{H})$ is the free Boolean extension of $\mathfrak{H}$ and

$$\Box_F a := \bigvee\{b \in H : b \leq a\}$$
$$\Diamond_P a := \bigwedge\{b \in H : a \leq b\}$$

That $\Box_F, \Diamond_P$ are well-defined operations on $B(\mathfrak{H})$ follows from the order-duality principle and the results in the previous chapter. It is easy to verify that $\sigma\mathfrak{H}$ validates the $\mathtt{S4}$ axioms for both $\Box_F$ and $\Diamond_P$. Moreover, for any $a \in B(H)$ clearly $\Diamond_P a \in H$, so $\Box_F \Diamond_P a = \Diamond_P a$. This implies $a \leq \Box_F \Diamond_P a$. Therefore indeed $\sigma\mathfrak{H} \in \mathsf{Ten}$.

The construction transforming a tense algebra into a corresponding bi-Heyting algebra is likewise very similar to that of transforming closure algebras into corresponding Heyting algebras.

Definition 3.38. The mapping $\rho : \mathsf{Ten} \to \mathsf{2HA}$ assigns every $\mathfrak{A} \in \mathsf{Ten}$ to the algebra $\rho\mathfrak{A} := (O(A), \wedge, \vee, \to, \leftarrow, 0, 1)$, where

$$O(A) := \{a \in A : \Box_F a = a\} = \{a \in A : \Diamond_P a = a\}$$
$$a \to b := \Box_F(\neg a \vee b)$$
$$a \leftarrow b := \Diamond_P(a \wedge \neg b).$$

Proposition 3.39. For every $\mathfrak{A} \in \mathsf{Ten}$, the algebra $\rho\mathfrak{A}$ is a bi-Heyting algebra.

Proof. Follows from Proposition 2.42 and the order-duality principle. □

Recall the geometric mappings $\sigma : \mathsf{Esa} \to \mathsf{Spa}(\mathtt{Grz})$ and $\rho : \mathsf{Spa}(\mathtt{S4}) \to \mathsf{Esa}$. Since bi-Esakia spaces are Esakia spaces, and tense spaces are $\mathtt{S4}$-spaces, we may restrict these mappings to $\sigma : \mathsf{2Esa} \to \mathsf{Alg}(\mathtt{Grz.T})$ and $\rho : \mathsf{Spa}(\mathtt{Grz.T}) \to \mathsf{2Esa}$ and obtain geometric counterparts to the algebraic mappings between bi-Heyting and tense algebras defined in the present subsection. Reasoning as in the proof of Proposition 2.44 we find that the algebraic and geometric versions of the maps σ, ρ are indeed dual to each other.

Proposition 3.40. The following hold.

1. Let $\mathfrak{H} \in \mathsf{2HA}$. Then $(\sigma\mathfrak{H})_* \cong \sigma(\mathfrak{H}_*)$. Consequently, if $\mathfrak{X}$ is a bi-Esakia space then $(\sigma\mathfrak{X})^* \cong \sigma(\mathfrak{X})^*$.
2. Let $\mathfrak{X}$ be a tense space. Then $(\rho\mathfrak{X})^* \cong \rho(\mathfrak{X}^*)$. Consequently, if $\mathfrak{A} \in \mathsf{Alg}(\mathtt{S4.t})$, then $(\rho\mathfrak{A})_* \cong \rho(\mathfrak{A}_*)$.

As an easy corollary, we obtain the following analogue of Proposition 2.45.

Proposition 3.41. For every $\mathfrak{H} \in \mathsf{2HA}$ we have $\mathfrak{H} \cong \rho\sigma\mathfrak{H}$. Moreover, for every $\mathfrak{A} \in \mathsf{Ten}$ we have $\sigma\rho\mathfrak{A} \rightarrowtail \mathfrak{A}$.

A Gödelian Translation We extend the Gödel translation of the previous chapter to a translation from bsi formulae to tense ones.

Definition 3.42 (Gödelian translation - bsi to tense)**.** The *Gödelian translation* is a mapping $T : Tm_{bsi} \to Tm_{ten}$ defined recursively as follows.

$$
\begin{aligned}
T(\bot) &:= \bot \\
T(\top) &:= \top \\
T(p) &:= \Box p \\
T(\varphi \wedge \psi) &:= T(\varphi) \wedge T(\psi) \\
T(\varphi \vee \psi) &:= T(\varphi) \vee T(\psi) \\
T(\varphi \to \psi) &:= \Box_F(\neg T(\varphi) \vee T(\psi)) \\
T(\varphi \leftarrow \psi) &:= \Diamond_P(T(\varphi) \wedge \neg T(\psi))
\end{aligned}
$$

An essentially equivalent translation was considered in Wolter [1998], though using $\Box_P$ instead of $\Diamond_P$ to interpret $\leftarrow$.

The following analogue of Lemma 2.48 is proved the same way as the latter.

Lemma 3.43. For every $\mathfrak{A} \in \mathsf{Ten}$ and bsi rule Γ/Δ,

$$\mathfrak{A} \models T(\Gamma/\Delta) \iff \rho\mathfrak{A} \models \Gamma/\Delta$$

We note that Lemma 3.43 does not appear the literature, which only mentions a similar results concerning formulae rather than rules.

§3.3.2 Structure of Tense Companions

We are now ready to generalise Theorem 2.54 and Theorem 2.55 to the bsi-tense setting. We do so in this section. All the results of this section are new inasmuch as they involve rule systems, although their restrictions to logics are known from Wolter [1998].

We begin by formally defining the notion of a *tense companion*.

Definition 3.44. Let $\mathtt{L} \in \mathbf{Ext}(\mathtt{2IPC_R})$ be a bsi-rule system and $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4.t_R})$ a tense rule system. We say that $\mathtt{M}$ is a *tense companion* of $\mathtt{L}$ (or that $\mathtt{L}$ is the bsi fragment of $\mathtt{M}$) whenever $\Gamma/\Delta \in \mathtt{L}$ iff $T(\Gamma/\Delta) \in \mathtt{M}$ for every bsi rule Γ/Δ. Moreover, let $\mathtt{L} \in \mathbf{Ext}(\mathtt{2IPC})$ be a bsi-logic and $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4.t})$ a tense logic. We say that $\mathtt{M}$ is a *tense companion* of $\mathtt{L}$ (or that $\mathtt{L}$ is the bsi fragment of $\mathtt{M}$) whenever $\varphi \in \mathtt{L}$ iff $T(\varphi) \in \mathtt{M}$.

Clearly, $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4.t_R})$ is a modal companion of $\mathtt{L} \in \mathbf{Ext}(\mathtt{2IPC_R})$ iff $\mathsf{Taut}(\mathtt{M})$ is a modal companion of $\mathsf{Taut}(\mathtt{L})$, and $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4.t})$ is a modal companion of $\mathtt{L} \in \mathbf{Ext}(\mathtt{2IPC})$ iff $\mathtt{M_R}$ is a modal companion of $\mathtt{L_R}$.

Define the following three maps between the lattices $\mathbf{Ext}(\mathtt{2IPC_R})$ and $\mathbf{NExt}(\mathtt{S4.t_R})$.

$$\tau : \mathbf{Ext}(\mathtt{2IPC_R}) \to \mathbf{NExt}(\mathtt{S4.t_R}) \qquad \sigma : \mathbf{Ext}(\mathtt{2IPC_R}) \to \mathbf{NExt}(\mathtt{S4.t_R})$$
$$\mathtt{L} \mapsto \mathtt{S4.t_R} \oplus \{T(\Gamma/\Delta) : \Gamma/\Delta \in \mathtt{L}\} \qquad \mathtt{L} \mapsto \mathtt{Grz.T_R} \oplus \tau\mathtt{L}$$

$$\rho : \mathbf{NExt}(\mathtt{S4.t_R}) \to \mathbf{Ext}(\mathtt{2IPC_R})$$
$$\mathtt{M} \mapsto \{\Gamma/\Delta : T(\Gamma/\Delta) \in \mathtt{M}\}$$

These mappings are readily extended to lattices of logics.

$$\tau : \mathbf{Ext}(\mathtt{2IPC}) \to \mathbf{NExt}(\mathtt{S4.t}) \qquad \sigma : \mathbf{Ext}(\mathtt{2IPC}) \to \mathbf{NExt}(\mathtt{S4.t})$$
$$\mathtt{L} \mapsto \mathsf{Taut}(\tau\mathtt{L_R}) = \mathtt{S4.t} \oplus \{T(\varphi) : \varphi \in \mathtt{L}\} \qquad \mathtt{L} \mapsto \mathsf{Taut}(\sigma\mathtt{L_R}) = \mathtt{Grz.T} \oplus \{T(\varphi) : \varphi \in \mathtt{L}\}$$

$$\rho : \mathbf{NExt}(\mathtt{S4.t}) \to \mathbf{Ext}(\mathtt{2IPC})$$
$$\mathtt{M} \mapsto \mathsf{Taut}(\rho\mathtt{M_R}) = \{\varphi : T(\varphi) \in \mathtt{M}\}$$

Furthermore, extend the mappings $\sigma : \mathsf{2HA} \to \mathsf{Ten}$ and $\rho : \mathsf{Ten} \to \mathsf{2HA}$ to universal classes by setting

$$\sigma : \mathbf{Uni}(\mathsf{2HA}) \to \mathbf{Uni}(\mathsf{Ten}) \qquad \rho : \mathbf{Uni}(\mathsf{Ten}) \to \mathbf{Uni}(\mathsf{2HA})$$
$$\mathcal{U} \mapsto \mathsf{Uni}\{\sigma\mathfrak{H} : \mathfrak{H} \in \mathcal{U}\} \qquad \mathcal{W} \mapsto \{\rho\mathfrak{A} : \mathfrak{A} \in \mathcal{W}\}.$$

Finally, introduce a semantic counterpart to τ as follows.

$$\tau : \mathbf{Uni}(\mathsf{2HA}) \to \mathbf{Uni}(\mathsf{Ten})$$
$$\mathcal{U} \mapsto \{\mathfrak{A} \in \mathsf{Ten} : \rho\mathfrak{A} \in \mathcal{U}\}$$

The following lemma is a counterpart to Lemma 2.50. It is proved via essentially the same argument which establishes the latter, though some adaptations are necessary which may be less than completely obvious. For this reason, as well as for the central place this lemma occupies in our strategy, we spell out the proof in some detail.

Lemma 3.45. Let $\mathfrak{A} \in \mathsf{Grz.T}$. Then for every modal rule Γ/Δ, we have $\mathfrak{A} \models \Gamma/\Delta$ iff $\sigma\rho\mathfrak{A} \models \Gamma/\Delta$.

Proof. ($\Rightarrow$) This direction follows from the fact that $\sigma\rho\mathfrak{A} \rightarrowtail \mathfrak{A}$ (Proposition 3.41).

($\Leftarrow$) We prove the dual statement that $\mathfrak{A}_* \not\models \Gamma/\Delta$ implies $\sigma\rho\mathfrak{A}_* \not\models \Gamma/\Delta$. Let $\mathfrak{X} := \mathfrak{A}_*$. In view of Theorem 3.35 it is enough to consider the case $\Gamma/\Delta = \mu_T(\mathfrak{B}, D^{\Box_F}, D^{\Diamond_P})$, for $\mathfrak{B} \in \mathsf{Ten}$ finite. So suppose $\mathfrak{X} \not\models \mu(\mathfrak{B}, D)$ and let $\mathfrak{F} := \mathfrak{B}_*$. Then there is a stable map $f : \mathfrak{X} \to \mathfrak{F}$ satisfying the BDC for $(\mathfrak{D}^{\Box_F}, \mathfrak{D}^{\Diamond_P})$. We construct a stable map $g : \sigma\rho\mathfrak{X} \to \mathfrak{F}$ which satisfies the BDC for $(\mathfrak{D}^{\Box_F}, \mathfrak{D}^{\Diamond_P})$.

Let $C := \{x_1, \dots, x_n\} \subseteq F$ be some cluster and let $Z_C := f^{-1}(C)$. Reasoning as in the proof of Lemma 2.50, we obtain that $\rho[Z_C]$ is clopen, and so is $f^{-1}(x_i)$ for each $x_i \in C$. Now for each $x_i \in C$ let

$$M_i := max_R(f^{-1}(x_i))$$
$$N_i := min_R(f^{-1}(x_i)).$$

By Proposition 3.18, both M_i, N_i are closed, and moreover neither cuts any cluster. Since $\sigma\rho\mathfrak{X}$ has the quotient topology, it follows that both $\rho[M_i]$, $\rho[N_i]$ are closed as well.

For each $x_i \in C$ let $O_i := M_i \cup N_i$. Clearly, $O_i \cap O_j = \emptyset$ for each $i, j \leq n$. Therefore, using Proposition 1.6 to reason as in the proof of Lemma 2.50, there are disjoint clopens $U_1, \dots, U_n \in \mathsf{Clop}(\sigma\rho\mathfrak{X})$ with $\rho[O_i] \subseteq U_i$ and $\bigcup_{i \leq n} U_i = \rho[Z_C]$.

We can now define a map

$$g_C : \rho[Z_C] \to C$$
$$z \mapsto x_i \iff z \in U_i.$$

Clearly, g_C is relation preserving and continuous. Finally, define $g : \sigma\rho\mathfrak{X} \to F$ by setting

$$g(\rho(z)) := \begin{cases} f(z) & \text{if } f(z) \text{ does not belong to any proper cluster} \\ g_C(\rho(z)) & \text{if } f(z) \in C \text{ for some proper cluster } C \subseteq F. \end{cases}$$

Now, g is evidently relation preserving. Moreover, it is continuous because both f and each g_C are. Reasoning as in the proof of Lemma 2.50, we obtain that g satisfies the BDC$^{\Box_F}$ for $\mathfrak{D}^{\Box_F}$. The proof of the fact that g satisfies the BDC$^{\Diamond_P}$ for $\mathfrak{D}^{\Diamond_P}$ is a straightforward adaptation of the latter, using that for all $U \in \mathsf{Clop}(\mathfrak{X})$, if $x \in U$ there is $y \in min_R(U)$ such that Ryx (Proposition 3.18). $\Box$

Theorem 3.46. Every $\mathcal{U} \in \mathsf{Uni}(\mathsf{Grz.T})$ is generated by its skeletal elements, i.e. $\mathcal{U} = \sigma\rho\mathcal{U}$.

Proof. Follows easily from Lemma 3.45, reasoning as in the proof of Theorem 2.51. $\Box$

As in the previous chapter, the next step is to apply Lemma 3.45 to prove that the syntactic tense companion maps τ, ρ, σ commute with $\mathsf{Alg}(\cdot)$, which leads to a purely semantic characterisation of tense companions.

Lemma 3.47. For each $\mathtt{L} \in \mathbf{Ext}(\mathtt{2IPC_R})$ and $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4.t_R})$, the following hold:

$$\mathsf{Alg}(\tau\mathtt{L}) = \tau\mathsf{Alg}(\mathtt{L}) \quad (3.3)$$

$$\mathsf{Alg}(\sigma\mathtt{L}) = \sigma\mathsf{Alg}(\mathtt{L}) \quad (3.4)$$

$$\mathsf{Alg}(\rho\mathtt{M}) = \rho\mathsf{Alg}(\mathtt{M}) \quad (3.5)$$

Proof. The proof of Eq. (3.3) is trivial. To prove Eq. (3.4), in view of Theorem 3.46 it is enough to show that $\mathsf{Alg}(\sigma\mathtt{L})$ and $\sigma\mathsf{Alg}(\mathtt{L})$ have the same skeletal elements. This is proved the same way as Eq. (2.4) in Lemma 2.52. Finally, Eq. (3.5) is proved analogously to Eq. (2.5) in Lemma 2.52, applying Lemma 3.43 instead of Lemma 2.48. □

Lemma 3.48. $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4.t_R})$ is a tense companion of $\mathtt{L} \in \mathbf{Ext}(\mathtt{2IPC_R})$ iff $\mathsf{Alg}(\mathtt{L}) = \rho\mathsf{Alg}(\mathtt{M})$.

Proof. Analogous to Lemma 2.53. □

The main results of this section can now be proved.

Theorem 3.49. The following conditions hold:

1. For every $\mathtt{L} \in \mathbf{Ext}(\mathtt{2IPC_R})$, the modal companions of $\mathtt{L}$ form an interval

$$\{\mathtt{M} \in \mathbf{NExt}(\mathtt{S4.t_R}) : \tau\mathtt{L} \leq \mathtt{M} \leq \sigma\mathtt{L}\};$$

2. For every $\mathtt{L} \in \mathbf{Ext}(\mathtt{2IPC})$, the modal companions of $\mathtt{L}$ form an interval

$$\{\mathtt{M} \in \mathbf{NExt}(\mathtt{S4.t}) : \tau\mathtt{L} \leq \mathtt{M} \leq \sigma\mathtt{L}\}.$$

Proof. Item 1 is proved the same way as Item 1 in Theorem 2.54. Item 2 is immediate from Item 1. □

Theorem 3.50 (Blok-Esakia theorem for bsi- and tense deductive systems)**.** The following conditions hold:

1. The mappings $\sigma : \mathbf{Ext}(\mathtt{2IPC_R}) \to \mathbf{NExt}(\mathtt{Grz.T_R})$ and $\rho : \mathbf{NExt}(\mathtt{Grz.T_R}) \to \mathbf{Ext}(\mathtt{2IPC_R})$ are complete lattice isomorphisms and mutual inverses.
2. The mappings $\sigma : \mathbf{Ext}(\mathtt{2IPC}) \to \mathbf{NExt}(\mathtt{Grz.T})$ and $\rho : \mathbf{NExt}(\mathtt{Grz.T}) \to \mathbf{Ext}(\mathtt{2IPC})$ are complete lattice isomorphisms and mutual inverses.

Proof. Item 1 is proved the same way as Item 1 in Theorem 2.55. Item 2 follows straightforwardly from Item 1 and Propositions 3.2 and 3.11. □

§3.3.3 Axiomatisation of Tense Companions and Bi-superintuitionistic Fragments via Stable Canonical Rules

This section generalises the axiomatisation results of § 2.3.3 to the bsi-tense setting, thus obtaining new axiomatic characterisations of the mappings τ, σ, ρ between bsi and tense rule systems.

We begin by proving a counterpart to the rule translation lemma.

Lemma 3.51 (Rule translation lemma - bsi-tense)**.** For every $\mathfrak{A} \in \mathsf{Ten}$ and any bsi stable canonical rule $\eta_B(\mathfrak{H}, D^{\to}, D^{\leftarrow})$ and we have

$$\mathfrak{A} \models T(\eta_B(\mathfrak{H}, D^{\to}, D^{\leftarrow})) \iff \mathfrak{A} \models \mu_T(\sigma\mathfrak{H}, D^{\Box_F}, D^{\Diamond_P}),$$

where

$$D^{\Box_F} := \{\neg a \vee b : (a, b) \in D^{\to}\}$$
$$D^{\Diamond_P} := \{a \wedge \neg b : (a, b) \in D^{\leftarrow}\}$$

Proof. Let $\mathfrak{F} := \mathfrak{A}_*$. Then, clearly, $\sigma\mathfrak{F} = \mathfrak{F}$. Given Proposition 3.40 and Lemma 3.43, it suffices to prove the dual claim that for all tense spaces $\mathfrak{X}$ we have

$$\rho\mathfrak{X} \models \eta_B(\mathfrak{F}, \mathfrak{D}^{\to}, \mathfrak{D}^{\leftarrow}) \iff \mathfrak{X} \models \mu_T(\mathfrak{F}, \mathfrak{D}^{\Box_F}, \mathfrak{D}^{\Diamond_P}),$$

where $\mathfrak{D}^{\Box_F} := \mathfrak{D}^{\to}$ and $\mathfrak{D}^{\Diamond_P} := \mathfrak{D}^{\leftarrow}$.

($\Rightarrow$) Assume $\mathfrak{X} \not\models \mu_T(\mathfrak{F}, \mathfrak{D}^{\Box_F}, \mathfrak{D}^{\Diamond_P})$. Then there is a stable map $f : \mathfrak{X} \to \mathfrak{F}$ satisfying the BDC for $(\mathfrak{D}^{\Box_F}, \mathfrak{D}^{\Diamond_P})$. Observe that since $\mathfrak{F}$ is a poset, the map

$$g : \rho\mathfrak{X} \to \mathfrak{F}$$
$$\rho(x) \mapsto f(x)$$

is well defined. Reasoning as in the proof of Lemma 2.56, we obtain that g is stable, continuous, and satisfies the BDC$^{\to}$ for $\mathfrak{D}^{\to}$. The proof that g satisfies the BDC$^{\leftarrow}$ for $\mathfrak{D}^{\leftarrow}$ is analogous. It follows that $\rho\mathfrak{X} \not\models \eta_B(\mathfrak{F}, \mathfrak{D}^{\to}, \mathfrak{D}^{\leftarrow})$.

($\Leftarrow$) Assume $\rho\mathfrak{X} \not\models \eta_B(\mathfrak{F}, \mathfrak{D}^{\to}, \mathfrak{D}^{\leftarrow})$. Then there is a stable map $g : \rho\mathfrak{X} \to \mathfrak{F}$ satisfying the BDC for $\mathfrak{D}$. Define a map

$$f : \mathfrak{X} \to \mathfrak{F}$$
$$x \mapsto f(\rho(x)).$$

Since g and $\rho : \mathfrak{X} \to \rho\mathfrak{X}$ are both continuous and relation-preserving, so is their composition f. Reasoning as in the proof of Lemma 2.56 we obtain that f satisfies the BDC$^{\Box_F}$ for $\mathfrak{D}^{\Box_F}$, and the BDC$^{\Diamond_P}$ for $\mathfrak{D}^{\Diamond_P}$ is checked analogously. $\square$

As an immediate corollary of Lemma 3.51 we obtain the following analogue of Theorem 2.57.

Theorem 3.52. For every $\mathtt{L} \in \mathbf{Ext}(\mathtt{2IPC_R})$, if

$$\mathtt{L} = \mathtt{2IPC_R} \oplus \{\eta_B(\mathfrak{H}_i, D_i^{\to}, D_i^{\leftarrow}) : i \in I\}$$

then

$$\tau\mathtt{L} = \mathtt{S4.t_R} \oplus \{\mu_T(\sigma\mathfrak{H}_i, D_i^{\Box_F}, D_i^{\Diamond_P}) : i \in I\}$$
$$\sigma\mathtt{L} = \mathtt{Grz.T_R} \oplus \{\mu_T(\sigma\mathfrak{H}_i, D_i^{\Box_F}, D_i^{\Diamond_P}) : i \in I\}$$

where for each $i \in I$ the sets $D^{\Box_F}$ and $D^{\Box_P}$ are defined as in the statement of the rule translation lemma - bsi-tense.

Proof. Follows from the Blok Esakia theorem for bsi and tense rule systems and Lemma 3.51. □

We now generalise the notion of collapsed stable canonical rules to the tense setting, and use collapsed tense stable canonical rules to obtain an axiomatic characterisations bsi fragments analogous to Theorem 2.64. As in the previous chapter, we introduce collapsed tense stable canonical rules geometrically to facilitate an intuitive understanding of this concept.

Definition 3.53. Let $\mu_T(\mathfrak{F}, \mathfrak{D}^{\Box_F}, \mathfrak{D}^{\Diamond_P})$ be a tense stable canonical rule. The *collapsed tense stable canonical rule* $\eta_B(\rho\mathfrak{F}, \rho\mathfrak{D}^{\to}, \rho\mathfrak{D}^{\leftarrow})$ is defined by setting

$$\rho\mathfrak{D}^{\to} = \{\rho[\mathfrak{d}] : \mathfrak{d} \in \mathfrak{D}^{\Box_F}\}$$
$$\rho\mathfrak{D}^{\leftarrow} = \{\rho[\mathfrak{d}] : \mathfrak{d} \in \mathfrak{D}^{\Diamond_P}\}$$

Lemma 3.54 (Rule collapse lemma - bsi-tense)**.** For every tense space $\mathfrak{X}$ and every tense stable canonical rule $\mu_T(\mathfrak{F}, D^{\Box_F}, D^{\Diamond_P})$, we have that $\mathfrak{X} \not\models \mu_T(\mathfrak{F}, \mathfrak{D}^{\Box_F}, \mathfrak{D}^{\Diamond_P})$ implies $\rho\mathfrak{X} \not\models \eta_B(\rho\mathfrak{F}, \rho\mathfrak{D}^{\to}, \rho\mathfrak{D}^{\leftarrow})$.

Proof. Assume $\mathfrak{X} \not\models \mu_T(\mathfrak{F}, \mathfrak{D}^{\Box_F}, \mathfrak{D}^{\Diamond_P})$. Then there is a continuous stable surjection $f : \mathfrak{X} \to \mathfrak{F}$ satisfying the BDC for $(\mathfrak{D}^{\Box_F}, \mathfrak{D}^{\Diamond_P})$. From the proof of Lemma 2.56, the map $g : \rho\mathfrak{X} \to \rho\mathfrak{F}$ given by

$$g(\rho(x)) = \rho(f(x))$$

is a continuous stable surjection satisfying the BDC$^{\to}$ for $\rho\mathfrak{D}^{\to}$. Dual reasoning shows that g also satisfies the BDC$^{\leftarrow}$ form $\rho\mathfrak{D}^{\leftarrow}$. Thus indeed $\rho\mathfrak{X} \not\models \eta_B(\rho\mathfrak{F}, \rho\mathfrak{D}^{\to}, \rho\mathfrak{D}^{\leftarrow})$. □

Observe that by the rule translation lemma (bsi-tense), a tense rule Γ/Δ is of the form $T(\Gamma'/\Delta')$ iff it is equivalent to finitely many tense stable canonical rules $\mu_T(\mathfrak{F}, \mathfrak{D}^{\Box_F}, \mathfrak{D}^{\Diamond_P})$ with $\mathfrak{F} \in \mathsf{Spa}(\mathtt{Grz.T})$. Using this observation, we may reason as in the proof of Theorem 2.64 to obtain the following result.

Theorem 3.55. Let $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4.t_R})$ with $\mathtt{M} = \mathtt{S4.t_R} \oplus \{\mu_T(\mathfrak{F}_i, \mathfrak{D}_i^{\Box_F}, \mathfrak{D}_i^{\Diamond_P}) : i \in I\}$. Set

$$J := \{i \in I : \mu_T(\sigma\rho\mathfrak{F}_i, \rho\mathfrak{D}_i^{\Box_F}, \rho\mathfrak{D}_i^{\Diamond_P}) \in \mathtt{M}\}.$$

Then

$$\rho\mathtt{M} = \mathtt{2IPC_R} \oplus \{\eta_B(\rho\mathfrak{F}_i, \rho\mathfrak{D}_i^{\to}, \rho\mathfrak{D}_i^{\leftarrow}) : i \in J\}.$$

As a consequence we obtain the following analogue of Theorem 2.69, via similar reasoning as in the proof of the latter.

Theorem 3.56. For every modal rule Γ/Δ there is a finite set Ξ of tense stable canonical rules of the form $\mu_T(\mathfrak{A}, D^{\Box_F}, D^{\Diamond_P})$ with $\mathfrak{A} \in \mathsf{Grz.T}$, such that for any $\mathfrak{B} \in \mathsf{Grz.T}$ we have that $\mathfrak{B} \not\models \Gamma/\Delta$ iff there is $\mu_T(\mathfrak{A}, D^{\Box_F}, D^{\Diamond_P}) \in \Xi$ such that $\mathfrak{B} \not\models \mu(\mathfrak{A}, D)$.

§3.3.4 Examples

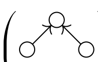
This section gives some examples of axiomatisations of bsi and tense rule systems in terms of stable canonical rules, and applies them to illustrate the axiomatisation procedures proposed in the previous section. We begin by axiomatising some noteworthy bsi-rule systems via stable canonical rules.

Let $\mathfrak{X} \in \mathsf{2Esa}$. An *R-path* is a finite sequence $(x_0, \dots, x_n)$ such that for all $i \leq n$ we have either Rx_ix_{i+1} or $\breve{R}x_ix_{i+1}$. We say that $\mathfrak{X}$ is *connected* if any point can be reached from any other point via an R-path. We call $\mathfrak{X}$ *forward-rooted* if there is $x \in X$ with Rxy for all $y \in X$, *backward-rooted* if there is $x \in X$ with $\breve{R}xy$ for all $y \in X$, and *bi-rooted* if both forward-rooted and backward-rooted. The next result is a counterpart to Bezhanishvili et al. [2016a, Theorem 8.1] in the bsi setting.

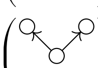
Theorem 3.57. Let $\mathtt{Con}$, $\mathtt{Root}_{\uparrow}$, $\mathtt{Root}_{\downarrow}$, and $\mathtt{Root}_{\updownarrow}$ be respectively the rule systems of all connected, forward-rooted, backward-rooted, and bi-rooted bi-Esakia spaces. Then the following identities hold.

$$\mathtt{Con} = \mathtt{2IPC_R} \oplus \eta_B(\circ \quad \circ) \tag{3.6}$$

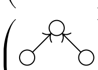
$$\mathtt{Root}_{\uparrow} = \mathtt{2IPC_R} \oplus \eta_B(\circ \quad \circ) \oplus \eta_B\left(\right) \tag{3.7}$$

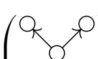


$$\mathtt{Root}_{\downarrow} = \mathtt{2IPC_R} \oplus \eta_B(\circ \quad \circ) \oplus \eta_B\left(\right) \tag{3.8}$$



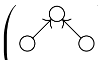
$$\mathtt{Root}_{\updownarrow} = \mathtt{2IPC_R} \oplus \eta_B(\circ \quad \circ) \oplus \eta_B\left(\right) \oplus \eta_B\left(\right) \tag{3.9}$$

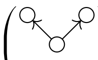


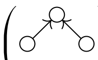
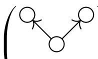


Consequently,

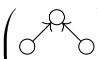
$$\tau\mathtt{Con} = \mathtt{S4.t_R} \oplus \mu_T\,(\circ \quad \circ) \tag{3.10}$$

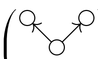
$\tau\mathtt{Root}_\uparrow = \mathtt{S4.t_R} \oplus \mu_T\,(\circ \quad \circ) \oplus \mu_T$ () (3.11)

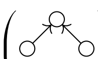
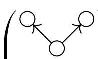
$\tau\mathtt{Root}_\downarrow = \mathtt{S4.t_R} \oplus \mu_T\,(\circ \quad \circ) \oplus \mu_T$ () (3.12)

$\tau\mathtt{Root}_\updownarrow = \mathtt{S4.t_R} \oplus \mu_T\,(\circ \quad \circ) \oplus \mu_T$ () $\oplus\, \mu_T$ () (3.13)

$$\sigma\mathtt{Con} = \mathtt{Grz.T_R} \oplus \mu_T\,(\circ \quad \circ) \tag{3.14}$$

$\sigma\mathtt{Root}_\uparrow = \mathtt{Grz.T_R} \oplus \mu_T\,(\circ \quad \circ) \oplus \mu_T$ () (3.15)

$\sigma\mathtt{Root}_\downarrow = \mathtt{Grz.T_R} \oplus \mu_T\,(\circ \quad \circ) \oplus \mu_T$ () (3.16)

$\sigma\mathtt{Root}_\updownarrow = \mathtt{Grz.T_R} \oplus \mu_T\,(\circ \quad \circ) \oplus \mu_T$ () $\oplus\, \mu_T$ () (3.17)

Proof. (3.6) and (3.7) are proved analogously as items 3, 4 in Bezhanishvili et al. [2016a, Theorem 8.1], and the proof of (3.8) is dual to that of (3.7). Then (3.9) follows immediately. The second part of the theorem follows from the first by a straightforward application of Theorem 3.52. □

A *zig-zag* is an R-path $(x_0, \dots, x_n)$ such that for all $i < n$, if Rx_ix_{i+1} then $\breve{R}x_{i+1}x_{i+2}$ and if $\breve{R}x_ix_{i+1}$ then $Rx_{i+1}x_{i+2}$, whenever x_{i+2} exists. For each finite n we define the finite bi-Esakia $\mathfrak{F}_n^\uparrow$, $\mathfrak{F}_n^\downarrow$ depicted in Figs. 3.1 and 3.2. In both cases the R-path $(x_1, \dots, x_n)$ is meant to be a zig-zag. We say that a bi-Esakia space $\mathfrak{X}$ is n-connected if any point can be reached from any other point by a zig-zag of length n.

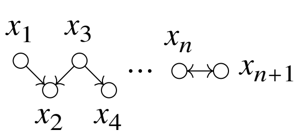


Figure 3.1: $\mathfrak{F}_n^\uparrow$

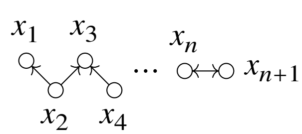


Figure 3.2: $\mathfrak{F}_n^\downarrow$

Theorem 3.58. Let $\mathtt{Con_n}$ be the rule system of all n-connected bi-Esakia spaces. Then

$$\mathtt{Con_n} = \mathtt{2IPC_R} \oplus \eta_B\,(\circ \quad \circ) \oplus \eta_B\,(\mathfrak{F}_{\mathtt{n}}^\uparrow) \oplus \eta_B\,(\mathfrak{F}_{\mathtt{n}}^\downarrow)\,.$$

Consequently,

$$\tau\mathtt{Con_n} = \mathtt{S4.t_R} \oplus \eta_B\,(\circ \quad \circ) \oplus \eta_B\,(\mathfrak{F}_{\mathtt{n}}^\uparrow) \oplus \eta_B\,(\mathfrak{F}_{\mathtt{n}}^\downarrow)$$
$$\sigma\mathtt{Con_n} = \mathtt{Grz.T_R} \oplus \eta_B\,(\circ \quad \circ) \oplus \eta_B\,(\mathfrak{F}_{\mathtt{n}}^\uparrow) \oplus \eta_B\,(\mathfrak{F}_{\mathtt{n}}^\downarrow)\,.$$

Proof. Let $\mathfrak{X} \in \mathsf{2Esa}$ be connected. It suffices to show that $\mathfrak{X}$ is n-connected iff it validates both $\eta_B(\mathfrak{F}_\mathfrak{n}^\uparrow), \eta_B(\mathfrak{F}_\mathfrak{n}^\downarrow)$.

($\Rightarrow$) Suppose that $\mathfrak{X}$ refutes one of $\eta_B(\mathfrak{F}_\mathfrak{n}^\uparrow), \eta_B(\mathfrak{F}_\mathfrak{n}^\downarrow)$, say $\mathfrak{X} \nvDash \eta_B(\mathfrak{F}_\mathfrak{n}^\uparrow)$ (the case $\mathfrak{X} \nvDash \eta_B(\mathfrak{F}_\mathfrak{n}^\downarrow)$ is analogous). Then there is a continuous stable surjection $f : \mathfrak{X} \to \mathfrak{F}_n^\uparrow$. Take arbitrary $x, y \in X$ with $f(x) = x_0$ and $f(y) = x_{n+1}$. We claim that there is no zig-zag of length n or less from x to y. For take any zig-zag $(z_1, z_2 \dots, z_m)$, with $z_1 = x$ and $z_m = y$. Wlog, we may choose $(z_1, z_2 \dots, z_m)$ so that there is no zig-zag from x to y of length strictly shorter than m. Now, note that for all $i \leq m$, if $f(z_i) \neq f(z_{i+1})$ then $f(z_{i+1}) = x_{j+1}$ or $f(z_{i+1}) = x_{j-1}$, where $x_j = f(z_i)$. For if we had, say, $f(z_{i+1}) = x_l$ with $l > j+1$, then by stability x_j and x_l would be such that either Rx_jx_l or $\breve{R}x_lx_j$, which is clearly not the case. By choice of x, y, this implies that $m > n$, as desired.

($\Leftarrow$) Suppose that $\mathfrak{X}$ is not n-connected. Then there are x, y such that every zig-zag connecting x and y has length greater than n. Take one such zig-zag $(z_1, z_2 \dots, z_m)$, with $x = z_1$ and $y = z_m$. Wlog, we may choose $(z_1, z_2 \dots, z_m)$ with the property that any zig-zag of length shorter than m does not have x and y as endpoints. We consider only the case where the first step in this zig-zag is an R-step, i.e. Rz_1z_2 (the case $\breve{R}z_1z_2$ is similar). Observe that for all $i, j \leq m$, if $|i - j| > 1$ then $x_j \notin R[x_i]$ and $x_i \notin R[x_j]$: else we would be able to construct a shorter zig-zag from x to y, as the reader can easily check. We separate $z_1, z_2, \dots, z_m$ by clopen sets $U_1, U_2, \dots, U_m$, such that U_i is an upset iff i is even, a downset otherwise. Let $i \leq m$ and assume inductively that U_j has been defined for all $j \leq i$. Set

$$V_i := X \setminus \left(\bigcup_{j<i} U_j \right).$$

If $i = m$, let $U_i := V_i$. Otherwise, observe that V_i with the subspace topology is again a bi-Esakia space. If i is odd, then $x_i \notin R[x_j]$ for all $j > i$. So by Proposition 3.8, for each $j > i$ there is $Z_{(i,j)} \in \mathsf{ClopDown}$ such that $x_i \in Z_{(i,j)}$ and $x_j \notin Z_{(i,j)}$. Then put

$$U_i := \bigcap_{j>i} Z_{(i,j)}.$$

By construction, $U_1, U_2, \dots, U_m$ is a partition of X. Define a map $f : \mathfrak{X} \to \mathfrak{F}_n^\uparrow$ by setting

$$f(x) := \begin{cases} x_i & \text{if } x \in U_i \text{ and } i \leq n \\ x_{n+1} & \text{otherwise.} \end{cases}$$

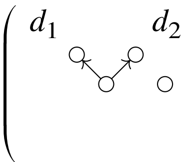
This is clearly a continuous surjection (recall that $n < m$), and by the fact that $U_1, U_2, \dots, U_m$ are disjoint upsets and downsets we obtain that f is in fact a stable map. This establishes $\mathfrak{X} \nvDash \eta_B(\mathfrak{F}_\mathfrak{n}^\uparrow)$.

The second part of the theorem follows immediately by applying Theorem 3.52. $\square$

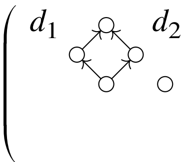
We close with an example of a stable canonical axiomatisation of a tense rule system, and of its image under the ρ-map.

Theorem 3.59. Let $\mathtt{Lin} := \Box_F(\Box_F p \to q) \vee \Box_F(\Box_F q \to p) \oplus \Box_P(\Box_P p \to q) \vee \Box_P(\Box_P q \to p)$. Then

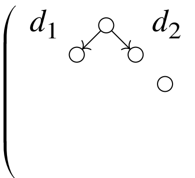
$$\mathtt{Lin_R} = \mathtt{S4.t} \oplus \mu_T$$



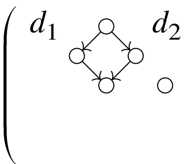
$$\oplus\, \mu_T$$



$$\oplus\, \mu_T$$



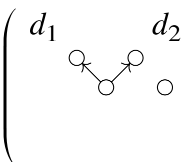
$$\oplus\, \mu_T$$



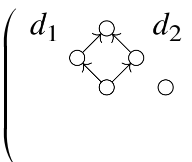
.

Consequently,

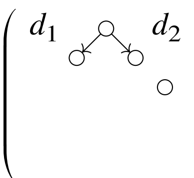
$$\rho\mathtt{Lin_R} = \mathtt{S4.t} \oplus \eta_B$$



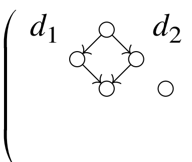
$$\oplus\, \eta_B$$



$$\oplus\, \eta_B$$



$$\oplus\, \eta_B$$



.

Proof. The first part follows by an easy adaptation of the proof of Theorem 2.58. The second part follows immediately by applying Theorem 3.55, noting that all the finite tense spaces depicted above are posets. □

§3.3.5 Additional Results

We close the present chapter with some remarks concerning generalisation of results obtained in § 2.3.4 to the bsi and tense setting. In this case as well, straightforward adaptations of the proofs seen in § 2.3.4 suffice for establishing all results in this section, which again highlights the uniformity of our approach.

Firstly, the construction used to prove the Dummett-Lemmon conjecture for rule systems generalises to a proof of a variant of the conjecture applying to bsi-rule systems and their weakest tense companions.

Theorem 3.60. For every bsi-rule system $\mathtt{L} \in \mathbf{Ext}(\mathtt{2IPC_R})$, $\mathtt{L}$ is Kripke complete iff $\tau\mathtt{L}$ is.

The proof of this result differs from that of Theorem 2.70 only notationally, so we omit it.

Secondly, the definition of moderately admitting filtration can be formulated for tense rule systems in an obvious way. An analogue of Theorem 2.74 holds for $\mathtt{Grz.T_R}$.

Theorem 3.61. $\mathtt{Grz.T_R}$ moderately admits filtration.

Proof sketch. The proof mirrors the proof of Theorem 2.74. We give a rough sketch since in this case the required construction, while very similar, is not exactly the same as in the si-modal case. If $\mathfrak{A} \in \mathsf{Grz.T}$ and $\mathfrak{A} \not\models \Gamma/\Delta$ for some tense rule Γ/Δ, then using Lemma 3.45 we infer $\sigma\rho\mathfrak{A} \not\models \Gamma/\Delta$, which gives a valuation V on $\sigma\rho\mathfrak{A}$ refuting Γ/Δ. For each $a \in \bar{V}[\Gamma/\Delta]$, use Proposition 2.40 to rewrite a as a Boolean compound of finitely many elements in $O(A)$. Use all these elements to generate a bounded distributive lattice $\mathfrak{H}$. The key observation is that analogously to the proof of Theorem 2.74, $\mathfrak{H}$ can be expanded to a bi-Heyting algebra $\mathfrak{H}'$, such that $\sigma\mathfrak{H}' \rightarrowtail \sigma\rho\mathfrak{A}$, by defining implication and co-implication in the usual way. Then reasoning as in the si-modal case one obtains a model $(\sigma\mathfrak{H}', V')$ which is a filtration of (essentially) the model $(\sigma\rho\mathfrak{A}, V)$. Viewing V as a valuation on $\mathfrak{A}$ shows that $(\sigma\mathfrak{H}', V')$ is also a filtration of $(\sigma\mathfrak{H}', V')$. □

Finally, the fact that $\mathtt{Grz.T_R}$ admits filtration in a robust enough sense makes the notion of $\mathtt{Grz.T_R}$-stability, whose definition should be obvious, worth investigating. In this case as well one obtains an analogue of Theorem 2.77.

Theorem 3.62. The following hold:

1. For every si rule system $\mathtt{L} \in \mathbf{Ext}(\mathtt{2IPC_R})$, $\mathtt{L}$ is $\mathtt{2IPC_R}$-stable iff every modal companion of $\mathtt{L}$ is $\mathtt{Grz_R}$-stable;
2. For every modal rule system $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4.t_R})$, if $\mathtt{M}$ is $\mathtt{Grz.T_R}$-stable then $\rho\mathtt{M}$ is $\mathtt{2IPC_R}$-stable.

The proof is again essentially the same as that of Theorem 2.77. To prove (2) one needs to show that a tense rule system is $\mathtt{Grz.T_R}$-stable iff it is axiomatisable by stable rules of $\mathtt{Grz.T}$-algebras, and that a bsi-rule system is $\mathtt{2IPC_R}$-stable iff it is axiomatisable by bsi stable rules. The proofs of both claims are easy adaptations of the analogous claims for modal and si rule systems. Then a straightforward application of the rule translation lemma (bsi-tense) yields the desired result.

§3.4 Chapter Summary

We summarise the main original contributions of this chapter in the following list.

- We generalised si and modal stable canonical rules to bsi and tense stable canonical rules.
- We showed that the central technical lemma of our strategy generalises smoothly to the bsi and tense setting (Lemma 3.45), and applied it to characterise the set of tense companions of a superintuitionistic deductive system (Theorem 2.54) and obtain an analogue of the Blok-Esakia theorem for bsi and tense deductive systems.
- We gave new axiomatic characterisations of the modal companions maps σ, ρ, τ on rule systems via stable canonical rules, extending the axiomatisation results of § 2.3.3.

- Via straightforward generalisations of arguments from § 2.3.4, we obtained an analogue of the Dummett-Lemmonn conjecture for bsi and tense rule systems (Theorem 2.70), proved that $\mathtt{Grz}_{\mathtt{R}}$ moderately admits filtration (Theorem 2.74), and described the notion of $\mathtt{Grz.T}_{\mathtt{R}}$-stability.

4 | The Kuznetsov-Muravitsky Isomorphism for Logics and Rule Systems

In this chapter, we further generalise our techniques to study translational embeddings of (normal) *modal superintuitionistic* rule systems and logics into modal ones. We develop algebra-based rules for modal superintuitionistic rule systems over the intuitionistic provability logic KM, as well as a new kind of algebra-based rules for modal rule systems over the Gödel-Löb provability logic (§ 4.2). We call these *pre-stable canonical rules*. We apply pre-stable canonical rules to prove that the lattice of modal superintuitionistic rule systems (resp. logics) over KM is isomorphic to the lattice of modal rule systems (resp. logics) over GL via a Gödel-style translational embedding (§ 4.3). This result was proved for logics by Kuznetsov and Muravitsky [1986], but appears to be new for rule systems. We close in § 4.3.3 by giving axiomatic characterisations of the underlying isomorphisms in terms of pre-stable canonical rules, mirroring the results of §§ 2.3.3 and 3.3.3.

For reasons of space, this chapter does not pursue the full theory of modal companions of superintuitionistic logics in the sense of either Esakia [2006] or Wolter and Zakharyaschev [1998, 1997], although we are confident that our techniques would work in that setting as well. Because of this, some results Chapters 2 and 3, most notably the Dummett-Lemmon conjecture, have no counterparts in the present chapter.

Besides supplying new results, this chapter further highlights the flexibility and uniformity of our techniques. Standard filtration does not work well for KM and GL, suggesting a different, less standard notion of filtration should be used to generalise stable canonical rules to the present setting. The rest of our approach delivers the desired results despite this different design choice, which shows its flexibility. Moreover, it does so without needing any major changes and accommodations: the proofs of the main results in this chapter follow the basic blueprints of their counterparts from Chapter 2. This, once again, shows the uniformity of our approach.

§4.1 Preliminaries

We begin by briefly reviewing definitions and basic properties of the structures under discussion.

§4.1.1 Intuitionistic Provability, Frontons, and KM-spaces

In this subsection we shall work with the *modal superintuitionistic signature*,

$$msi := \{\land, \lor, \to, \boxtimes, \bot, \top\}.$$

The set Frm_{msi} of *modal superintuitionistic (msi) formulae* is defined recursively as follows.

$$\varphi ::= p \mid \bot \mid \top \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \varphi \to \varphi \mid \boxtimes \varphi$$

where $p \in Prop$.

The logic $\mathtt{IPCK}$ is obtained by extending $\mathtt{IPC}$ by the $\mathtt{K}$-axiom

$$\boxtimes(p \to q) \to (\boxtimes p \to \boxtimes q)$$

and closing under necessitation, that is, requiring that whenever $\varphi \in \mathtt{IPCK}$ then $\boxtimes\varphi \in \mathtt{IPCK}$ as well.

Definition 4.1. A *normal modal superintuitionistic logic*, or msi-logic for short, is a logic $\mathtt{L}$ over Frm_{msi} satisfying the following additional conditions:

1. $\mathtt{IPCK} \subseteq \mathtt{L}$;
2. If $\varphi \to \psi, \varphi \in \mathtt{L}$ then $\psi \in \mathtt{L}$ (MP);
3. If $\varphi \in \mathtt{L}$ then $\boxtimes\varphi \in \mathtt{L}$ (NEC).

A *modal superintuitionistic rule system*, or msi-rule system for short, is a rule system $\mathtt{L}$ over Frm_{msi} satisfying the following additional requirements.

1. $/\varphi \in \mathtt{L}$ whenever $\varphi \in \mathtt{IPCK}$;
2. $\varphi, \varphi \to \psi / \psi \in \mathtt{L}$ (MP-R);
3. $\varphi / \boxtimes \varphi \in \mathtt{L}$ (NEC-R).

If $\mathtt{L}$ is an msi-logic (resp. msi-rule system) we write $\mathbf{NExt}(\mathtt{L})$ for the set of msi-logics (resp. rule systems) extending $\mathtt{L}$. Surely, the set of msi-logics systems coincides with $\mathbf{NExt}(\mathtt{IPCK})$. It is easy to check that $\mathbf{NExt}(\mathtt{IPCK})$ forms a lattice under the operations $\oplus_{\mathbf{NExt}(\mathtt{K})}$ as join and intersection as meet. If $\mathtt{L} \in \mathbf{NExt}(\mathtt{IPCK})$, let $\mathtt{L}_\mathtt{R}$ be the least msi-rule system containing $/\varphi$ for each $\varphi \in \mathtt{L}_\mathtt{R}$. Then $\mathtt{IPCK}_\mathtt{R}$ is the least msi-rule system. The set $\mathbf{NExt}(\mathtt{IPCK}_\mathtt{R})$ of msi-rule systems is also a lattice when endowed with $\oplus_{\mathbf{NExt}(\mathtt{IPCK}_\mathtt{R})}$ as join and intersection as meet. As usual, we refer to these lattices as we refer to their underlying sets, i.e. $\mathbf{NExt}(\mathtt{IPCK})$ and $\mathbf{NExt}(\mathtt{IPCK}_\mathtt{R})$ respectively. We also write both $\oplus_{\mathbf{NExt}(\mathtt{IPCK})}$ and $\oplus_{\mathbf{NExt}(\mathtt{IPCK}_\mathtt{R})}$ simply as $\oplus$, leaving context to resolve ambiguities. Clearly, for every $\mathtt{L} \in \mathbf{NExt}(\mathtt{IPCK})$ we have that $\mathsf{Taut}(\mathtt{L}_\mathtt{R}) = \mathtt{L}$, which establishes the following result.

Proposition 4.2. The mappings $(\cdot)_\mathtt{R}$ and $\mathsf{Taut}(\cdot)$ are mutually inverse complete lattice isomorphisms between $\mathbf{NExt}(\mathtt{IPCK})$ and the sublattice of $\mathbf{NExt}(\mathtt{IPCK}_\mathtt{R})$ consisting of all msi-rule systems $\mathtt{L}$ such that $\mathsf{Taut}(\mathtt{L})_\mathtt{R} = \mathtt{L}$.

Rather than studying $\mathbf{NExt}(\mathtt{IPCK_R})$ in its entirety, we shall focus on the sublattice of $\mathbf{NExt}(\mathtt{IPCK_R})$ consisting of all normal extensions of the rule system $\mathtt{KM_R}$, where $\mathtt{KM}$ is the msi-logic axiomatised as follows.

$$\mathtt{KM} := \mathtt{IPCK} \oplus p \to \boxtimes p \oplus (\boxtimes p \to p) \to p \oplus \boxtimes p \to (q \vee (q \to p)).$$

The logic $\mathtt{KM}$ was introduced in Kuznetsov [1978] (see also Kuznetsov and Muravitsky 1986) and later studied by Esakia [2006]. Its main motivation lies in its close connection with the Gödel-Löb provability logic, to be discussed in the next section. An extensive overview of both the history and theory of $\mathtt{KM}$ may be found in Muravitsky [2014].

The algebraic models for msi-rule systems in $\mathbf{NExt}(\mathtt{KM_R})$ are called *frontons*, and discussed in detailed, e.g., in Esakia [2006]; Litak [2014].

Definition 4.3. A *fronton* is a tuple $\mathfrak{H} = (H, \wedge, \vee, \to, \boxtimes, 0, 1)$ such that $(H, \wedge, \vee, \to, 0, 1)$ is a Heyting algebra and for every $a, b \in H$, $\boxtimes$ satisfies

$$\boxtimes 1 = 1 \tag{4.1}$$
$$\boxtimes(a \wedge b) = \boxtimes a \wedge \boxtimes b \tag{4.2}$$
$$a \leq \boxtimes a \tag{4.3}$$
$$\boxtimes a \to a = a \tag{4.4}$$
$$\boxtimes a \leq b \vee (b \to a) \tag{4.5}$$

We let Frt denote the class of all frontons. By Theorem 1.10, Frt is a variety. We write $\mathbf{Var}(\mathsf{Frt})$ and $\mathbf{Uni}(\mathsf{Frt})$ respectively for the lattice of subvarieties and of universal subclasses of Frt. Item 1 in the following result follows from, e.g., Muravitsky [2014, Proposition 7], whereas Item 2 can be obtained via the techniques used in the proofs of Theorems 2.4 and 2.12.

Theorem 4.4. The following maps are pairs of mutually inverse dual isomorphisms:

1. $\mathsf{Alg} : \mathbf{NExt}(\mathtt{KM}) \to \mathbf{Var}(\mathsf{Frt})$ and $\mathsf{Th} : \mathbf{Var}(\mathsf{Frt}) \to \mathbf{Ext}(\mathtt{KM})$;
2. $\mathsf{Alg} : \mathbf{NExt}(\mathtt{KM_R}) \to \mathbf{Uni}(\mathsf{Frt})$ and $\mathsf{ThR} : \mathbf{Uni}(\mathsf{Frt}) \to \mathbf{NExt}(\mathtt{KM_R})$.

We mention a simple yet important property of frontons, which plays a key role in the development of algebra-based rules for rule systems in $\mathbf{NExt}(\mathtt{KM_R})$.

Proposition 4.5 (cf. Esakia 2006, Proposition 5)**.** Every fronton $\mathfrak{H}$ satisfies the identity

$$\boxtimes a = \bigwedge \{b \vee (b \to a) : b \in H\}.$$

for every $a \in H$.

It follows that for every Heyting algebra $\mathfrak{H}$, there is at most one way of expanding $\mathfrak{H}$ to a fronton, namely by setting

$$\boxtimes a := \bigwedge \{b \vee (b \to a) : b \in H\}$$

Geometrically, msi-rule systems extending $\mathtt{KM_R}$ are interpretable over modal expansions of Esakia spaces, which we term $\mathtt{KM}$-*spaces.*

Definition 4.6. A KM-*space* is a tuple $\mathfrak{X} = (X, \leq, \sqsubseteq, \mathcal{O})$, such that $(X, \leq, \mathcal{O})$ is an Esakia space, and $\sqsubseteq$ is a binary relation on X satisfying the following conditions, where $\Uparrow x := \{y \in X : x \sqsubseteq y\}$ and $\Downarrow x := \{y \in X : y \sqsubseteq x\}$, and $x < y$ iff $x \leq y$ and $x \neq y$:

1. $x < y$ implies $x \sqsubseteq y$;
2. $x \sqsubseteq y$ implies $x \leq y$;
3. $\Uparrow x$ is closed for all $x \in X$;
4. $\Downarrow[U] \in \mathsf{Clop}(\mathfrak{X})$ for every $U \in \mathsf{ClopUp}(\mathfrak{X})$;
5. For every $U \in \mathsf{ClopUp}(\mathfrak{X})$ and $x \in X$, if $x \notin U$ then there is $y \in -U$ such that $x \leq y$ and $\Uparrow y \subseteq U$.

KM-spaces are discussed in Esakia [2006], and more at length in Castiglioni et al. [2010].

The geometrical semantics for msi-rule systems extending $\mathtt{KM_R}$ over KM-spaces is obtained straightforwardly by combining the geometrical semantics of si-rule systems and that of modal rule systems. The relation $\leq$ is used to interpret the implication connective $\to$, and the relation $\sqsubseteq$ is used to interpret the modal operator $\boxtimes$.

If $\mathfrak{X}, \mathfrak{Y}$ are KM-spaces, a map $f : \mathfrak{X} \to \mathfrak{Y}$ is called a *bounded morphism* if for all $x, y \in X$ we have:

- $x \leq y$ implies $f(x) \leq f(y)$;
- $x \sqsubseteq y$ implies $f(x) \sqsubseteq f(y)$;
- $f(x) \leq y$ implies that there is $z \in X$ with $x \leq z$ and $f(z) = y$;
- $f(x) \sqsubseteq y$ implies that there is $z \in X$ with $x \sqsubseteq z$ and $f(z) = y$

We recall some useful properties of KM-spaces, which are proved in Castiglioni et al. [2010] (cf. Proposition 4.8).

Proposition 4.7. For every KM-space $\mathfrak{X}$, the following conditions hold:

1. For every $U \in \mathsf{ClopUp}(U)$ we have $\{x \in X : \Uparrow x \subseteq U\} = U \cup max_{\leq}(-U)$;
2. If $\mathfrak{X}$ is finite, then for all $x, y \in X$ we have $x \sqsubseteq y$ iff $x < y$.

Frontons and KM-spaces are related by a Stone-like duality, obtained by combining Esakia duality with the duality between modal algebras and modal spaces. It was stated in Esakia [2006, 354–5], and proved in detail by Castiglioni et al. [2010, Theorem 4.4].

Theorem 4.8. The category of frontons with corresponding homomorphisms is dually equivalent to the category of KM-spaces with continuous bounded morphisms.

We briefly recall the main ingredients of this duality. Given a fronton $\mathfrak{H} \in \mathsf{Frt}$, its dual KM-space is

$$\mathfrak{H}_* = (Spec(\mathfrak{H}), \mathcal{O}, \leq, \sqsubseteq)$$

where $(\mathsf{Spec}(\mathfrak{H}), \mathcal{O}, \leq)$ is the Esakia dual of the Heyting reduct of $\mathfrak{H}$, and $\sqsubseteq$ is defined by

$$x \sqsubseteq y :\iff \quad \text{for all } a \in H : \boxtimes a \in x \text{ implies } a \in y.$$

Conversely, given a $\mathtt{KM}$-space $\mathfrak{X}$ its dual modal algebra is

$$\mathfrak{X}^* = (\mathsf{Clop}(\mathfrak{X}), \cap, \cup, \to_{\leq}, \boxtimes_{\sqsubseteq}, X, \emptyset)$$

where for all $U, V \in \mathsf{ClopUp}(\mathfrak{X})$ we have

$$\begin{aligned} U \to_{\leq} V &:= -{\downarrow}(U \setminus V) \\ \boxtimes_{\sqsubseteq} U &:= \{x \in X : {\Uparrow} x \subseteq U\} \end{aligned}$$

One can prove that for every $\mathfrak{H} \in \mathsf{Frt}$, the Stone map β witnesses $\mathfrak{H} \cong \mathfrak{H}_*{}^*$, and conversely that for every $\mathfrak{X} \in \mathsf{Spa}(\mathtt{KM})$, the inverse of the stone map $\beta^{-1} : \mathfrak{X} \to \mathfrak{X}^*{}_*$ witnesses $\mathfrak{X} \cong \mathfrak{X}^*{}_*$. Moreover, we have that for any $\mathfrak{H}, \mathfrak{K} \in \mathsf{Frt}$, a map $h : \mathfrak{H} \to \mathfrak{K}$ is a homomorphism iff $h^{-1} : \mathfrak{K}_* \to \mathfrak{H}_*$ is a continuous bounded morphism, and for every $\mathfrak{X}, \mathfrak{Y} \in \mathsf{Spa}(\mathtt{KM})$, a map $f : \mathfrak{X} \to \mathfrak{Y}$ is a continuous bounded morphism iff $f^{-1} : \mathfrak{Y}^* \to \mathfrak{X}^*$ is a fronton homomorphism.

§4.1.2 Classical Provability, Magari Algebras, and GL-spaces

We now work in the modal signature *md* already discussed in Chapter 2. The modal logic $\mathtt{GL}$ is axiomatised by extending $\mathtt{K}$ with the well-known *Löb formula*.

$$\begin{aligned} \mathtt{GL} :=& \mathtt{K} \oplus \Box(\Box p \to p) \to \Box p \\ =& \mathtt{K4} \oplus \Box(\Box p \to p) \to \Box p \end{aligned}$$

The logic $\mathtt{GL}$ was independently discovered by K. Segerberg, G. Boolos, and the Siena logic group led by R. Magari (see Segerberg 1971; Sambin 1974, 1976; Magari 1975b; Sambin and Valentini 1982; Boolos 1980) as a formalisation of the provability predicate of Peano arithmetic. The reader may consult Boolos [1993] (as well as the more recent if less comprehensive Muravitsky [2014]) for an overview of known results concerning $\mathtt{GL}$.

Rule systems in $\mathbf{NExt}(\mathtt{GL_R})$ are interpreted over special kinds of modal algebras known as *Magari algebras*, named after Magari [1975b].

Definition 4.9. A modal algebra $\mathfrak{A}$ is called a *Magari algebra* if it satisfies the identity

$$\Box(\Box a \to a) = \Box a$$

for all $a \in A$.

Magari algebras are also called $\mathtt{GL}$-algebras, e.g. in Litak [2014]. We let Mag denote the variety of all Magari algebras. Clearly, every Magari algebra is a transitive modal algebra, and moreover Mag coincides with the class of all modal algebras satisfying the equation

$$\Diamond a = \Diamond(\Box\neg a \wedge a).$$

The following result is a straightforward consequence of Theorem 2.12.

Theorem 4.10. The following maps are pairs of mutually inverse dual isomorphisms:

1. $\mathsf{Alg} : \mathbf{NExt}(\mathtt{GL}) \to \mathbf{Var}(\mathsf{Mag})$ and $\mathsf{Th} : \mathbf{Var}(\mathsf{Mag}) \to \mathbf{Ext}(\mathtt{GL})$;
2. $\mathsf{Alg} : \mathbf{NExt}(\mathtt{GL_R}) \to \mathbf{Uni}(\mathsf{Mag})$ and $\mathsf{ThR} : \mathbf{Uni}(\mathsf{Mag}) \to \mathbf{NExt}(\mathtt{GL_R})$.

Modal spaces dual to Magari algebras are called GL-*spaces*. GL-spaces display various similarities with Grz-spaces, as the reader can appreciate by comparing the following result with Proposition 2.18.

Proposition 4.11 (cf. Magari 1975a)**.** For every GL-space $\mathfrak{X}$ and $U \in \mathsf{Clop}(\mathfrak{X})$, the following conditions hold:

1. If $x \in max_R(U)$ then $R[x] \cap U = \emptyset$;
2. $max_R(U) \in \mathsf{Clop}(\mathfrak{X})$;
3. If $x \in U$ then either $x \in max_R(U)$ or there is $y \in max_R(U)$ such that Rxy;
4. If $\mathfrak{X}$ is finite then R is irreflexive.

Proof. (1) Suppose otherwise that for some $U \in \mathsf{Clop}(\mathfrak{X})$ there is $x \in max_R(U)$ with $R[x] \cap U \neq \emptyset$. As $x \in max_R(U)$, this is equivalent to the claim that Rxx. Define a valuation V on $\mathfrak{X}$ with $V(p) = U$. Then $\mathfrak{X}, V, x \models \Diamond p$. But since $x \in max_R(U)$ we have $R[x] \cap U = \{x\}$, whence $\mathfrak{X}, V, x \not\models \Diamond(p \wedge \Box\neg p)$. So $\mathfrak{X} \not\models \Diamond p \to \Diamond(p \wedge \Box\neg p)$, contradiction.

(2) By item 1 it follows that

$$max_R(U) = U \cap \Box_R(-U)$$

which immediately implies that $max_R(U) \in \mathsf{Clop}(\mathfrak{X})$ whenever $U \in \mathsf{Clop}(\mathfrak{X})$.

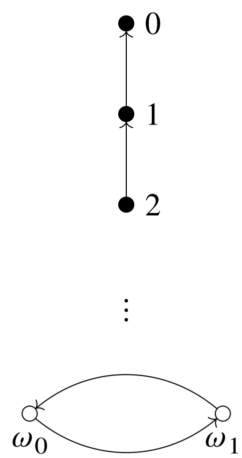
(3) Let $x \in U$ and suppose that $x \notin max_R(U)$. It follows that $R[x] \cap U \neq \emptyset$. Therefore $x \in R^{-1}(U)$, and so we infer $x \in R^{-1}(\Box_R(-U) \cap U)$. This is to say that there is $y \in U$ with Rxy and $R[y] \cap U = \emptyset$. Clearly, the last condition implies that $y \in max_R(U)$.

(4) For all $x \in X$ the set $\{x\}$ must have a maximal element by Item 3, which must be irreflexive by Item 1. Clearly, this element can only be x. $\Box$

GL is well-known to be complete with respect to the class of irreflexive and transitive Kripke frames containing no ascending chain. However, like Grz-spaces, GL-spaces may contain clusters, and a fortiori reflexive points. To see this, consider the following example, which is a variant of Example 2.17. (We adopt the usual convention of drawing $\sqsubseteq$-irreflexive points in black, and $\sqsubseteq$-reflexive points in white.)

Example 4.12. Consider the modal space $\mathfrak{X}$, where

- $X = \mathbb{N} \cup \{\omega_0, \omega_1\}$;
- $R = \{(n, m) \in \mathbb{N} \times \mathbb{N} : m < n\} \cup \{(\omega_i, x) \in X \times X : i \in \{0, 1\}$ and $x \in X\}$;
- $\mathscr{O}$ is given by the basis consisting of all $U \subseteq X$ such that either U is a finite subset of $\mathbb{N}$, or $U = V \cup \{\omega_0, \omega_1\}$ with V a cofinite subset of $\mathbb{N}$, or U is one of the following sets:
$$\{n \in \mathbb{N} : n \text{ is even}\} \cup \{\omega_0\} \qquad \{n \in \mathbb{N} : n \text{ is odd}\} \cup \{\omega_1\}.$$



It is not difficult to see that for every $U \in \mathsf{Clop}(\mathfrak{X})$ we have $max_R(U) \neq \emptyset$. We claim that $\mathfrak{X}$ is a $\mathtt{GL}$-space. For let $U \in \mathsf{Clop}(\mathfrak{X})$, let $x \in X$ and suppose that $x \in R^{-1}(U)$. Then there is $y \in U$ with Rxy. Therefore $U \neq \emptyset$, and so we have $max_R(U) \neq \emptyset$. Observe that $max_R(U) \cap \{\omega_0, \omega_1\} = \emptyset$. Indeed, if U contains exactly one of ω_0, ω_1 then it must also contain elements of $\mathbb{N}$, which lie above both ω_0 and ω_1. If U contains both of ω_0, ω_1, then U is cofinite, and so again must contain elements of $\mathbb{N}$. Since $\mathbb{N}$ is linearly ordered by R and both ω_0, ω_1 see every element of $\mathbb{N}$, it follows that there is $z \in max_R(U)$ such that Ryz, and so Rxz. Since $z \notin \{\omega_0, \omega_1\}$ we have $z \notin R[z]$, therefore $x \in \Box_R(-U)$. Thus we have shown that $x \in R^{-1}(U)$ implies that $x \in R^{-1}(\Box_R(-U) \cap U)$, whence $\mathfrak{X} \in \mathsf{Spa}(\mathtt{GL})$ as desired.

However, in view of Proposition 4.11, clusters cannot occur just anywhere in a $\mathtt{GL}$-space. In particular, they cannot involve R-maximal elements of any clopen.

§4.2 Pre-stable Canonical Rules for Normal Extensions of $\mathtt{KM_R}$ and $\mathtt{GL_R}$

In this section we develop a new kind of algebra-based rules, serving as analogues of stable canonical rules for rule systems in **NExt**$(\mathtt{KM_R})$ and **NExt**$(\mathtt{GL_R})$. These rules encode a notion of filtration weaker than standard filtration, and better suited than the latter to the rule systems under discussion. We call them *pre-stable canonical rules*.

§4.2.1 The $\mathtt{KM_R}$ Case

We have seen notions of filtration for both Heyting and modal algebras. One would hope that combining the latter would yield a suitable notion of filtration for frontons, which could then be used to develop stable canonical rules for rule systems in **NExt**$(\mathtt{KM_R})$. This is in principle possible, but suboptimal. The reason is that with filtrations understood this way, rule systems in **NExt**$(\mathtt{KM_R})$ would turn out to admit very few filtrations. To see this, recall (Proposition 4.7) that in every finite $\mathtt{KM}$-space $\mathfrak{X}$ we have that $x \sqsubseteq y$ iff $x < y$ for all $x, y \in X$. Now let $\mathfrak{X}$ be any $\mathtt{KM}$-space such that there are $x, y \in X$ with $x \neq y$ and $x \sqsubseteq y$. Then any finite image of $\mathfrak{X}$ under a $\sqsubseteq$-preserving map h with $h(x) = h(y)$ would contain a reflexive point, hence would fail to be a $\mathtt{KM}$-space.

We know that every finite distributive lattice has a unique Heyting algebra expansion, and moreover that every finite Heyting algebra has a unique fronton expansion. These constructions lead to a natural method for extracting finite countermodels based on frontons to non-valid msi rules, which we illustrate in the proof of Lemma 4.13. This result, in a somewhat different formulation, was first proved by Muravitsky [1981] via frame-theoretic methods.

Lemma 4.13. For any msi rule Γ/Δ, if $\mathsf{Frt} \nvDash \Gamma/\Delta$ then there is a finite fronton $\mathfrak{H} \in \mathsf{Frt}$ such that $\mathfrak{H} \nvDash \Gamma/\Delta$.

Proof. Assume $\mathsf{Frt} \nvDash \Gamma/\Delta$ and let $\mathfrak{H} \in \mathsf{Frt}$ be a fronton with $\mathfrak{H} \nvDash \Gamma/\Delta$. Take a valuation V with $\mathfrak{H}, V \nvDash \Gamma/\Delta$. Put $\Theta = Sfor(\Gamma/\Delta)$ and set

$$D^{\to} := \{(\bar{V}(\varphi), \bar{V}(\psi)) \in H \times H : \varphi \to \psi \in \Theta\} \cup \{(\bar{V}(\varphi), a) : a \in D^{\boxtimes} \text{ and } \varphi \in \Theta\}$$
$$D^{\boxtimes} := \{\bar{V}(\varphi) \in H : \boxtimes\varphi \in \Theta\}$$

Let $\mathfrak{K}$ be the bounded distributive lattice generated by Θ. For all $a, b \in K$ define

$$a \rightsquigarrow b := \bigvee\{c \in H : a \wedge c \leq b\}$$
$$\boxtimes' a := \bigwedge_{b \in K} b \vee (b \rightsquigarrow a)$$

Obviously $(\mathfrak{K}, \rightsquigarrow)$ is a Heyting algebra, and by Proposition 4.5 it follows that $\mathfrak{K}' := (\mathfrak{K}, \rightsquigarrow, \boxtimes')$ is a fronton. Moreover, the inclusion $\subseteq : \mathfrak{K}' \to \mathfrak{A}$ is a bounded lattice embedding satisfying

$$a \rightsquigarrow b \leq a \to b \qquad \text{for all } (a, b) \in K \times K$$
$$a \rightsquigarrow b = a \to b \qquad \text{for all } (a, b) \in D^{\to}$$
$$\boxtimes' a = \boxtimes a \qquad \text{for all } a \in D^{\boxtimes}.$$

The first two claims are proved the same way as in the proof of Lemma 2.26. For the third claim we reason as follows. Suppose $a \in D^{\boxtimes}$. Then $(b, a) \in D^{\to}$ for every $b \in K$ by construction. Therefore,

$$\boxtimes' a = \bigwedge_{b \in K} b \vee (b \rightsquigarrow a) = \bigwedge_{b \in K} b \vee (b \to a).$$

By the axioms of frontons we have $\boxtimes a \leq b \vee (b \to a)$ for all $b \in H$, hence for all $b \in K$ in particular. Therefore $\boxtimes a \leq \boxtimes' a$. Conversely, for any $a \in K$ we have

$$\begin{aligned} \boxtimes' a &\leq \boxtimes a \vee \boxtimes a \rightsquigarrow a \\ &\leq \boxtimes a \vee \boxtimes a \to a && (\text{by } \boxtimes a \rightsquigarrow a \leq \boxtimes a \to a) \\ &= \boxtimes a. && (\text{by } \boxtimes a \to a = a \leq \boxtimes a) \end{aligned}$$

Let V' be an arbitrary valuation on $\mathfrak{K}'$ with $V'(p) = V(p)$ whenever $p \in Sfor(\Gamma/\Delta) \cap Prop$. Then for every $\varphi \in \Theta$ we have $V(\varphi) = V'(\varphi)$. This is shown easily by induction on the structure of φ. Therefore, $\mathfrak{K}', V' \nvDash \Gamma/\Delta$. □

The proof of Lemma 4.13 motivates an alternative notion of filtration for frontons. Let $\mathfrak{H}, \mathfrak{K} \in \mathsf{Frt}$. A map $h : \mathfrak{H} \to \mathfrak{K}$ is called *pre-stable* if for every $a, b \in H$ we have $h(a \to b) \leq h(a) \to h(b)$. For $a, b \in H$, we say that h satisfies the *$\to$-bounded domain condition* (BDC$^{\to}$) for (a, b) if $h(a \to b) = h(a) \to h(b)$. For $D \subseteq H$, we say that h satisfies the *$\boxtimes$-bounded domain condition* (BDC$^{\boxtimes}$) for D if $h(\boxtimes a) = \boxtimes h(a)$ for every $a \in D$. If $D \subseteq H \times H$, we say that h satisfies the BDC$^{\to}$ for D if it does for each $(a, b) \in D$, and analogously for the BDC$^{\boxtimes}$. Lastly, if $D^{\to} \subseteq H \times H$ and $D^{\boxtimes} \subseteq H$, we say that h satisfies the BDC for $(D^{\to}, D^{\boxtimes})$ if h satisfies the BDC$^{\to}$ for $D^{\to}$ and the BDC$^{\boxtimes}$ for $D^{\boxtimes}$.

Definition 4.14. Let $\mathfrak{H}$ be a fronton, V a valuation on $\mathfrak{H}$, and Θ a finite, subformula closed set of formulae. A (finite) model $(\mathfrak{K}', V')$, with $\mathfrak{K} \in \mathsf{Frt}$, is called a (*finite*) *weak filtration of* $(\mathfrak{H}, V)$ *through* Θ if the following hold:

1. $\mathfrak{K}' = (\mathfrak{K}, \to, \boxtimes)$, where $\mathfrak{K}$ is the bounded sublattice of $\mathfrak{H}$ generated by $\bar{V}[\Theta]$;
2. $V(p) = V'(p)$ for every propositional variable $p \in \Theta$;
3. The inclusion $\subseteq : \mathfrak{B} \to \mathfrak{A}$ is a pre-stable embedding satisfying the BDC$^{\to}$ for the set $\{(\bar{V}(\varphi), \bar{V}(\psi)) : \varphi \to \psi \in \Theta\}$, and satisfying the BDC$^{\boxtimes}$ for the set $\{\bar{V}(\varphi) : \boxtimes\varphi \in \Theta\}$

A straightforward induction on structure establishes the following filtration theorem.

Theorem 4.15 (Filtration theorem for frontons)**.** Let $\mathfrak{H}$ be a fronton, V a valuation on $\mathfrak{H}$, and Θ a a finite, subformula-closed set of formulae. If $(\mathfrak{K}', V')$ is a weak filtration of $(\mathfrak{H}, V)$ through Θ then for every $\varphi \in \Theta$ we have

$$\bar{V}(\varphi) = \bar{V}'(\varphi).$$

Consequently, for every rule Γ/Δ such that $\gamma, \delta \in \Theta$ for each $\gamma \in \Gamma$ and $\delta \in \Delta$ we have

$$\mathfrak{H}, V \models \Gamma/\Delta \iff \mathfrak{K}', V' \models \Gamma/\Delta.$$

We now introduce algebra-based rules for rule systems in $\mathbf{NExt}(\mathtt{KM_R})$ by syntactically encoding weak filtrations as just defined. We call these *pre-stable canonical rules* to emphasize the role of pre-stable maps as opposed to stable maps in their refutation conditions.

Definition 4.16. Let $\mathfrak{H} \in \mathsf{Frt}$ be a finite fronton, and let $D^{\to} \subseteq H \times H$, $D^{\boxtimes} \subseteq H$ be such that $a \in D^{\boxtimes}$ implies $(b, a) \in D^{\to}$ for every $b \in H$. The *stable canonical rule* of $(\mathfrak{H}, D^{\to}, D^{\boxtimes})$, is defined as $\eta_{\boxtimes}(\mathfrak{H}, D^{\to}, D^{\boxtimes}) = \Gamma/\Delta$, where

$$\begin{aligned}
\Gamma :=& \{p_0 \leftrightarrow 0\} \cup \{p_1 \leftrightarrow 1\} \cup \\
& \{p_{a \wedge b} \leftrightarrow p_a \wedge p_b : a \in H\} \cup \{p_{a \vee b} \leftrightarrow p_a \vee p_b : a \in H\} \cup \\
& \{p_{a \to b} \leftrightarrow p_a \to p_b : (a, b) \in D^{\to}\} \cup \{p_{\boxtimes a} \leftrightarrow \boxtimes p_a : a \in D^{\boxtimes}\} \\
\Delta :=& \{p_a \leftrightarrow p_b : a, b \in H \text{ with } a \neq b\}.
\end{aligned}$$

The next two results outline algebraic refutation conditions for msi pre-stable canonical rules. They may be proved with straightforward adaptations of the proofs of similar results seen in earlier chapters.

Proposition 4.17. For every finite fronton $\mathfrak{H}$ and $D^{\to} \subseteq H \times H$, $D^{\boxtimes} \subseteq H$, we have $\mathfrak{H} \not\models \eta_{\boxtimes}(\mathfrak{H}, D^{\to}, D^{\boxtimes})$.

Proposition 4.18. For every $\mathfrak{K} \in \mathsf{Frt}$ and any msi pre-stable canonical rule $\eta_{\boxtimes}(\mathfrak{H}, D^{\to}, D^{\boxtimes})$, we have $\mathfrak{K} \not\models \eta_{\boxtimes}(\mathfrak{H}, D^{\to}, D^{\boxtimes})$ iff there is a pre-stable embedding $h : \mathfrak{H} \to \mathfrak{K}$ satisfying the BDC for $(D^{\to}, D^{\boxtimes})$.

We now give refutation conditions for msi pre-stable canonical rules on KM-spaces. If $\mathfrak{X}, \mathfrak{Y}$ are KM-spaces, a map $f : \mathfrak{X} \to \mathfrak{Y}$ is called *pre-stable* if for all $x, y \in X$, $x \leq y$ implies $f(x) \leq f(y)$. Clearly, if f is pre-stable then for all $x, y \in X$, $x \sqsubseteq y$ implies $f(x) \leq f(y)$. Now let $\mathfrak{d} \subseteq Y$. We say that f *satisfies the BDC*$^{\to}$ *for* $\mathfrak{d}$ if for all $x \in X$,

$$\uparrow[f(x)] \cap \mathfrak{d} \neq \emptyset \Rightarrow f[\uparrow x] \cap \mathfrak{d} \neq \emptyset.$$

We say that f *satisfies the BDC*$^{\boxtimes}$ *for* $\mathfrak{d}$ if for all $x \in X$ the following two conditions hold.

$$\Uparrow[f(x)] \cap \mathfrak{d} \neq \emptyset \Rightarrow f[\Uparrow x] \cap \mathfrak{d} \neq \emptyset \qquad \text{(BDC}^{\boxtimes}\text{-back)}$$

$$f[\Uparrow x] \cap \mathfrak{d} \neq \emptyset \Rightarrow \Uparrow[f(x)] \cap \mathfrak{d} \neq \emptyset \qquad \text{(BDC}^{\boxtimes}\text{-forth)}$$

If $\mathfrak{D} \subseteq \wp(Y)$, then we say that f satisfies the BDC$^{\to}$ for $\mathfrak{D}$ if it does for every $\mathfrak{d} \in \mathfrak{D}$, and similarly for the BDC$^{\boxtimes}$. Finally, if $\mathfrak{D}^{\to}, \mathfrak{D}^{\boxtimes} \subseteq \wp(Y)$, then we say that f satisfies the BDC for $(\mathfrak{D}^{\to}, \mathfrak{D}^{\boxtimes})$ if f satisfies the BDC$^{\to}$ for $\mathfrak{D}^{\to}$ and the BDC$^{\boxtimes}$ for $\mathfrak{D}^{\boxtimes}$.

Let $\mathfrak{H}$ be a finite fronton. If $D^{\to} \subseteq H \times H$, for every $(a, b) \in D^{\to}$ set $\mathfrak{d}^{\to}_{(a,b)} := \beta(a) \setminus \beta(b)$. If $D^{\boxtimes} \subseteq H$, for every $a \in D^{\boxtimes}$ set $\mathfrak{d}^{\boxtimes}_{a} := -\beta(a)$. Finally, put $\mathfrak{D}^{\to} := \{\mathfrak{d}^{\to}_{(a,b)} : (a, b) \in D^{\to}\}$, $\mathfrak{D}^{\boxtimes} := \{\mathfrak{d}^{\boxtimes}_{a} : a \in D^{\boxtimes}\}$.

Proposition 4.19. For every KM-space $\mathfrak{X}$ and any msi stable canonical rule $\eta_{\boxtimes}(\mathfrak{H}, D^{\to}, D^{\boxtimes})$, $\mathfrak{X} \not\models \eta_{\boxtimes}(\mathfrak{H}, D^{\to}, D^{\boxtimes})$ iff there is a continuous pre-stable surjection $f : \mathfrak{X} \to \mathfrak{H}_*$ satisfying the BDC $(\mathfrak{D}^{\to}, \mathfrak{D}^{\boxtimes})$.

Proof. $(\Rightarrow)$ Assume $\mathfrak{X} \not\models \eta_{\boxtimes}(\mathfrak{H}, D^{\to}, D^{\boxtimes})$. Then there is a pre-stable embedding $h : \mathfrak{H} \to \mathfrak{X}^*$ satisfying the BDC for $(D^{\to}, D^{\boxtimes})$. Reasoning as in the proofs of Proposition 2.23 and Proposition 2.31 it follows that there is a pre-stable map $f : \mathfrak{X} \to \mathfrak{H}_*$ satisfying the BDC$^{\to}$ for $\mathfrak{D}^{\to}$ and satisfying the BDC$^{\boxtimes}$-back for $\mathfrak{D}^{\boxtimes}$, namely the map $f = h^{-1}$. Let us check that f satisfies the BDC$^{\boxtimes}$-forth for $\mathfrak{D}^{\boxtimes}$. Let $\mathfrak{d}^{\boxtimes}_{a} \in \mathfrak{D}^{\boxtimes}$. Assume $f[\Uparrow x] \cap \mathfrak{d}^{\boxtimes}_{a} \neq \emptyset$, i.e., that there is $y \in \Uparrow x$ with $f(y) \in \mathfrak{d}^{\boxtimes}_{a}$. So $x \notin \boxtimes_{\sqsubseteq} h(U)$, where $U := -\mathfrak{d}^{\boxtimes}_{a}$. Since h satisfies the BDC$^{\boxtimes}$ for $\mathfrak{d}^{\boxtimes}_{a}$ we have $\boxtimes_{\sqsubseteq} h(U) = h(\boxtimes_{\sqsubseteq} U)$, and so $x \notin h(\boxtimes_{\sqsubseteq} U)$. This implies $f(x) \notin \boxtimes_{\sqsubseteq}(U)$, therefore there must be some $z \in \mathfrak{d}^{\boxtimes}_{a}$ such that $f(x) \sqsubseteq z$, i.e. $\Uparrow[f(x)] \cap \mathfrak{d}^{\boxtimes}_{a} \neq \emptyset$.

$(\Leftarrow)$ Assume that there is a continuous pre-stable surjection $f : \mathfrak{X} \to \mathfrak{H}_*$ satisfying the BDC for $(\mathfrak{D}^{\to}, \mathfrak{D}^{\boxtimes})$. By the proof of Proposition 2.23, $f^{-1} : \mathfrak{H} \to \mathfrak{X}^*$ is a pre-stable embedding satisfying the BDC$^{\to}$ for $D^{\to}$. Let us check that f^{-1} satisfies the BDC$^{\boxtimes}$ for $D^{\boxtimes}$. Let $U \subseteq X$ be such that $U = \beta(a)$ for some $a \in D^{\boxtimes}$, and reason as follows.

$$\begin{aligned} x \notin f^{-1}(\boxtimes_{\sqsubseteq} U) &\iff \Uparrow x \cap f^{-1}(\mathfrak{d}^{\boxtimes}_{a}) \neq \emptyset \\ &\iff \Uparrow[f(x)] \cap \mathfrak{d}^{\boxtimes}_{a} \neq \emptyset && (f \text{ satisfies the BDC}^{\boxtimes} \text{ for } \mathfrak{d}^{\boxtimes}_{a}) \\ &\iff x \notin \boxtimes_{\sqsubseteq} f^{-1}(U). \end{aligned}$$

□

In view of Proposition 4.19, when working with KM-spaces we may write an msi pre-stable canonical rule $\eta_{\boxtimes}(\mathfrak{H}, D^{\to}, D^{\boxtimes})$ as $\eta_{\boxtimes}(\mathfrak{H}_*, \mathfrak{D}^{\to}, \mathfrak{D}^{\boxtimes})$.

We close this subsection by proving that our msi pre-stable canonical rules are expressive enough to axiomatise every rule system in **NExt**($\mathtt{KM_R}$).

Lemma 4.20. For every msi rule Γ/Δ there is a finite set Ξ of msi stable canonical rules such that for any $\mathfrak{K} \in \mathsf{Frt}$ we have $\mathfrak{K} \not\models \Gamma/\Delta$ iff there is $\eta_{\boxtimes}(\mathfrak{H}, D^{\to}, D^{\boxtimes}) \in \Xi$ such that $\mathfrak{K} \not\models \eta_{\boxtimes}(\mathfrak{H}, D^{\to}, D^{\boxtimes})$.

Proof. Since bounded distributive lattices are locally finite there are, up to isomorphism, only finitely many triples $(\mathfrak{H}, D^{\to}, D^{\boxtimes})$ such that

- $\mathfrak{H} \in \mathsf{Frt}$ and $\mathfrak{H}$ is at most k-generated as a bounded distributive lattice, where $k = |Sfor(\Gamma/\Delta)|$;
- There is a valuation V on $\mathfrak{H}$ refuting Γ/Δ, such that

$$D^{\to} = \{(\bar{V}(\varphi), \bar{V}(\psi)) : \varphi \to \psi \in Sfor(\Gamma/\Delta)\} \cup \{(\bar{V}(\varphi), b) : \boxtimes\varphi \in Sfor(\Gamma/\Delta) \text{ and } b \in H\}$$
$$D^{\boxtimes} = \{\bar{V}(\varphi) : \boxtimes\varphi \in Sfor(\Gamma/\Delta)\}.$$

Let Ξ be the set of all msi pre-stable canonical rules $\eta_{\boxtimes}(\mathfrak{H}, D^{\to}, D^{\boxtimes})$ for all such triples $(\mathfrak{H}, D^{\to}, D^{\boxtimes})$, identified up to isomorphism.

($\Rightarrow$) Let $\mathfrak{K} \in \mathsf{Frt}$ and suppose $\mathfrak{H} \not\models \Gamma/\Delta$. Take a valuation V on $\mathfrak{H}$ such that $\mathfrak{K}, V \not\models \Gamma/\Delta$. Then by the proof of Lemma 4.13 there is a weak filtration $(\mathfrak{H}', V')$ of $(\mathfrak{K}, V)$ through $Sfor(\Gamma/\Delta)$, which by the filtration theorem for frontons is such that $\mathfrak{H}', V' \not\models \Gamma/\Delta$. This implies that there is a stable embedding $h : \mathfrak{H}' \to \mathfrak{K}$, which again by the proof of Lemma 4.13 satisfies the BDC for the pair $(\mathfrak{D}^{\to}, \mathfrak{D}^{\boxtimes})$ defined as above. Therefore $\eta_{\boxtimes}(\mathfrak{H}', D^{\to}, D^{\boxtimes}) \in \Xi$ and $\mathfrak{K} \not\models \eta_{\boxtimes}(\mathfrak{H}', D^{\to}, D^{\boxtimes})$.

($\Leftarrow$) Analogous to the same direction in, e.g., Lemma 2.26. □

Theorem 4.21. Every msi-rule system $\mathtt{M} \in$ **NExt**($\mathtt{KM_R}$) is axiomatisable over $\mathtt{KM_R}$ by some set of msi pre-stable canonical rules of the form $\eta_{\boxtimes}(\mathfrak{H}, D^{\to}, D^{\boxtimes})$, where $\mathfrak{H} \in \mathsf{KM}$.

Proof. Analogous to, e.g., Theorem 2.27. □

§4.2.2 The $\mathtt{GL_R}$ Case

Modal stable canonical rules as developed in § 2.2.2 can axiomatise every rule system in **NExt**($\mathtt{GL_R}$) [Bezhanishvili et al., 2016a, Theorem 5.6]. However, modal stable canonical rules differ significantly from msi pre-stable canonical rules: they are based on a different notion of filtration, which is stated in terms of stable rather than pre-stable maps. Consequently, it is unclear how, if at all, one could obtain rule translation and collapse lemmas connecting msi pre-stable canonical rules with modal stable canonical rules, which in turn makes axiomatisation results analogous to those presented in §§ 2.3.3 and 3.3.3 elusive in the present setting.

Another reason why stable canonical rules are unattractive when working with rule systems in $\mathbf{NExt}(\mathtt{GL_R})$ is that $\mathtt{GL_R}$ admits very few filtrations. The situation is similar to the case of $\mathbf{NExt}(\mathtt{KM_R})$. For recall (Proposition 4.11) that finite $\mathtt{GL}$-spaces are strict partial orders. If $\mathfrak{X}$ is a $\mathtt{GL}$-space and $f : \mathfrak{X} \to \mathfrak{Y}$ is a stable map from $\mathfrak{X}$ onto some finite modal space $\mathfrak{Y}$ such that $f(x) = f(y)$ for some $x, y \in X$ with Rxy, then $\mathfrak{Y}$ contains a reflexive point, hence cannot be a $\mathtt{GL}$-space.

In response to this problem, an alternative notion of filtration was introduced in van Benthem and Bezhanishvili [forthcoming], who note that the same technique was used already in Boolos [1993]. We call it *weak filtration*. As usual, we prefer an algebraic definition. If $\mathfrak{A}, \mathfrak{B}$ are modal algebras and $D \subseteq A$, let us say that a map $h : \mathfrak{A} \to \mathfrak{B}$ satisfies the *$\Box$-bounded domain condition* (BDC$^\Box$) for D if $h(\Box a) = \Box(a)$ for every $a \in D$.

Definition 4.22. Let $\mathfrak{B} \in \mathsf{Mag}$ be a Magari algebra, V a valuation on $\mathfrak{B}$, and Θ a finite, subformula closed set of formulae. A (finite) model $(\mathfrak{A}', V')$, with $\mathfrak{A}' \in \mathsf{Mag}$, is called a (*finite*) *weak filtration of* $(\mathfrak{B}, V)$ *through* Θ if the following hold:

1. $\mathfrak{A}' = (\mathfrak{A}, \Box)$, where $\mathfrak{B}$ is the Boolean subalgebra of $\mathfrak{B}$ generated by $\bar{V}[\Theta]$;
2. $V(p) = V'(p)$ for every propositional variable $p \in \Theta$;
3. The inclusion $\subseteq : \mathfrak{A}' \to \mathfrak{B}$ satisfies the BDC$^\Box$ for $D := \{\bar{V}'(\varphi) : \Box\varphi \in \Theta\}$.

Theorem 4.23. Let $\mathfrak{B} \in \mathsf{Mag}$ be a Magari algebra, V a valuation on $\mathfrak{B}$, and Θ a finite, subformula closed set of formulae. Let $(\mathfrak{A}', V')$ be a weak filtration of $(\mathfrak{B}, V)$. Then for every $\varphi \in \Theta$ we have

$$\bar{V}(\varphi) = \bar{V}'(\varphi).$$

Proof. Straightforward induction on the structure of φ. $\Box$

Unlike weak filtrations in the msi setting, modal weak filtrations are not in general unique. We will be particularly interested in weak filtrations satisfying an extra condition, which we will construe as a modal counterpart to pre-stability in the msi setting. For any modal algebra $\mathfrak{A}$ and $a \in A$ we write $\Box^+(a) := \Box a \wedge a$. Let $\mathfrak{A}, \mathfrak{B} \in \mathsf{Mag}$ be Magari algebras. A Boolean homomorphism $h : \mathfrak{A} \to \mathfrak{B}$ is called *pre-stable* if for every $a \in A$ we have $h(\Box^+ a) \leq \Box^+ h(a)$. Clearly, every stable Boolean homomorphism $h : \mathfrak{A} \to \mathfrak{B}$ is pre-stable, since $h(\Box a) \leq \Box h(a)$ implies $h(\Box a \wedge a) = h(\Box a) \wedge h(a) \leq \Box h(a) \wedge h(a)$. A weak filtration $(\mathfrak{A}', V')$ of some model $(\mathfrak{B}, V)$ through some finite, subformula closed set of formulae Θ is called *pre-stable* if the embedding $\subseteq : \mathfrak{A}' \to \mathfrak{B}$ is pre-stable.

If $\mathfrak{A}, \mathfrak{B}$ are modal algebras and $D \subseteq A$, a map $h : \mathfrak{A} \to \mathfrak{B}$ satisfies the *$\Box^+$-bounded domain condition* (BDC$^{\Box^+}$) for D if $h(\Box^+ a) = \Box^+ h(a)$ for every $a \in D$. Note that if $(\mathfrak{A}', V')$ is a filtration of $(\mathfrak{B}, V)$ through some Θ, then for every $D \subseteq A$ the inclusion $\subseteq : \mathfrak{A} \to \mathfrak{B}$ satisfies the BDC$^{\Box^+}$ for D iff it satisfies the BDC$^\Box$ for D. Indeed, since Θ is subformula-closed we have that $\Box^+\varphi \in \Theta$ implies $\Box\varphi \in \Theta$, which gives the "only if" direction, whereas the converse follows from the fact that $\subseteq$ is a Boolean embedding.

Our algebra-based rules encode pre-stable weak filtrations as defined above, and explicitly include a parameter $D^{\Box^+}$, linked to the BDC$^{\Box^+}$, intended as a counterpart to the parameter $D^{\to}$ of msi pre-stable canonical rules. We call these rules *modal pre-stable canonical rules.*

Definition 4.24. Let $\mathfrak{A} \in \mathsf{MA}$ be a finite modal algebra, and let $D^{\Box^+}, D^{\Box} \subseteq A$. Let $\Box^+\varphi := \Box\varphi \wedge \varphi$. The *pre-stable canonical rule* of $(\mathfrak{A}, D^{\Box^+}, D^{\Box})$, is defined as $\mu_+(\mathfrak{A}, D^{\Box^+}, D^{\Box}) = \Gamma/\Delta$, where

$$\begin{aligned}\Gamma :=& \{p_{a\wedge b} \leftrightarrow p_a \wedge p_b : a \in H\} \cup \{p_{a\vee b} \leftrightarrow p_a \vee p_b : a \in H\} \cup \\ & \{p_{\neg a} \leftrightarrow \neg p_a : a \in A\} \cup \{p_{\Box^+ a} \to \Box^+ p_a : a \in a\} \cup \\ & \{\Box^+ p_a \to p_{\Box^+ a} : a \in D^{\Box^+}\} \cup \{p_{\Box a} \leftrightarrow \Box p_a : a \in D^{\Box}\} \\ \Delta :=& \{p_a : a \in A \setminus 1\}.\end{aligned}$$

It is helpful to conceptualise modal pre-stable canonical rules as algebra-based rules for bimodal rule systems in the signature $\{\wedge, \vee, \neg, \Box, \Box^+, 0, 1\}$ (so that $\Box^+$ is an independent operator rather than defined from $\Box$) and containing $\Box^+ p \leftrightarrow \Box p \wedge p$ as an axiom.[1] From this perspective, modal pre-stable canonical rules are rather similar to msi pre-stable canonical rules. We cash out this similarity more precisely in § 4.3.3, in the form of a rule translation lemma.

Using by now familiar reasoning, it is easy to verify that modal pre-stable canonical rules display the intended refutation conditions. For brevity, let us say that a pre-stable map h satisfies the BDC for $(D^{\Box^+}, D^{\Box})$ if h satisfies the BDC$^{\Box^+}$ for $D^{\Box^+}$ and the BDC$^{\Box}$ for $D^{\Box}$.

Proposition 4.25. For every finite modal algebra $\mathfrak{A} \in \mathsf{MA}$ and $D^{\Box^+}, D^{\Box} \subseteq A$, we have $\mathfrak{H} \nvDash \mu_+(\mathfrak{A}, D^{\Box^+}, D^{\Box})$.

Proposition 4.26. For every modal algebra $\mathfrak{B} \in \mathsf{MA}$ and any modal pre-stable canonical rule $\mu_+(\mathfrak{A}, D^{\Box^+}, D^{\Box})$, we have $\mathfrak{B} \nvDash \mu_+(\mathfrak{A}, D^{\Box^+}, D^{\Box})$ iff there is a pre-stable embedding $h : \mathfrak{B} \to \mathfrak{A}$ satisfying the BDC $(D^{\Box^+} D^{\Box})$.

If $\mathfrak{X}$ is any modal space, for any $x, y \in X$ define R^+xy iff Rxy or $x = y$. Let $\mathfrak{X}, \mathfrak{Y}$ be $\mathtt{GL}$-spaces. A map $f : \mathfrak{X} \to \mathfrak{Y}$ is called *pre-stable* if for all $x, y \in X$ we have that R^+xy implies $R^+f(x)f(y)$. If $\mathfrak{d} \subseteq Y$, we say that f satisfies the BDC$^{\Box^+}$ for $\mathfrak{d}$ if for all $x \in X$,

$$R^+[f(x)] \cap \mathfrak{d} \neq \emptyset \Rightarrow f[R^+[x]] \cap \mathfrak{d} \neq \emptyset.$$

Furthermore, we say that f satisfies the BDC$^{\Box}$ for $\mathfrak{d}$ if for all $x \in X$ the following two conditions hold.

$$R[f(x)] \cap \mathfrak{d} \neq \emptyset \Rightarrow f[R[x]] \cap \mathfrak{d} \neq \emptyset \qquad \text{(BDC}^{\Box}\text{-back)}$$
$$f[R[x]] \cap \mathfrak{d} \neq \emptyset \Rightarrow R[f(x)] \cap \mathfrak{d} \neq \emptyset. \qquad \text{(BDC}^{\Box}\text{-forth)}$$

Finally, if $\mathfrak{D} \subseteq \wp(Y)$ we say that f satisfies the BDC$^{\Box^+}$ (resp. BDC$^{\Box}$) for $\mathfrak{D}$ if it does for every $\mathfrak{d} \in \mathfrak{D}$, and if $\mathfrak{D}^{\Box^+}, \mathfrak{D}^{\Box} \subseteq \wp(Y)$ we write that f satisfies the BDC for $(\mathfrak{D}^{\Box^+}, \mathfrak{D}^{\Box})$ if f satisfies

[1] This view of $\mathtt{GL}$ as a bimodal logic is the main insight informing Litak's [2014] strategy for deriving Item 2 of Theorem 4.38 from the theory of polymodal companions of msi-logics as developed by Wolter and Zakharyaschev [1998, 1997]. In that setting, msi formulae are translated into formulae in a bimodal signature, but the two modalities of the latter can be regarded as implicitly interdefinable in logics where one satisfies the Löb formula.

the BDC$^{\Box^+}$ for $\mathfrak{D}^{\Box^+}$ and the BDC$^{\Box}$ for $\mathfrak{D}^{\Box}$. Let $\mathfrak{A}$ be a finite Magari algebra. If $D^{\Box^+} \subseteq A$, for every $a \in D^{\Box^+}$ set $\mathfrak{d}_a^{\Box^+} := -\beta(a)$. If $D^{\Box} \subseteq A$, for every $a \in D^{\Box}$ set $\mathfrak{d}_a^{\Box} := -\beta(a)$. Finally, put $\mathfrak{D}^{\Box^+} := \{\mathfrak{d}_a^{\Box^+} : a \in D^{\Box^+}\}$, $\mathfrak{D}^{\Box} := \{\mathfrak{d}_a^{\Box} : a \in D^{\Box}\}$.

Proposition 4.27. For all $\mathtt{GL}$-spaces $\mathfrak{X}$ and any modal pre-stable canonical rule $\mu_+(\mathfrak{A}, D^{\Box^+}, D^{\Box})$, we have $\mathfrak{X} \not\models \mu_+(\mathfrak{A}, D^{\Box^+}, D^{\Box})$ iff there is a continuous pre-stable surjection $f : \mathfrak{X} \to \mathfrak{A}_*$ satisfying the BDC for $(\mathfrak{D}^{\Box^+}, \mathfrak{D}^{\Box})$.

As usual, in view of Proposition 4.27 we write a modal pre-stable canonical rule $\mu_+(\mathfrak{A}, D^{\Box^+}, D^{\Box})$ as $\mu_+(\mathfrak{A}_*, \mathfrak{D}^{\Box^+}, \mathfrak{D}^{\Box})$ in geometric settings.

We close this section by proving that pre-stable canonical rules axiomatise any rule system in **NExt**$(\mathtt{GL_R})$.

Lemma 4.28. For every modal rule Γ/Δ there is a finite set Ξ of modal pre-stable canonical rules of the form $\mu_+(\mathfrak{A}, D^{\Box^+}, D^{\Box})$ with $\mathfrak{A} \in \mathsf{K4}$, such that for any $\mathfrak{B} \in \mathsf{Mag}$ we have $\mathfrak{B} \not\models \Gamma/\Delta$ iff there is $\mu_+(\mathfrak{A}, D^{\Box^+}, D^{\Box}) \in \Xi$ such that $\mathfrak{B} \not\models \mu_+(\mathfrak{A}, D^{\Box^+}, D^{\Box})$.

Proof. Since Boolean algebras is locally finite there are, up to isomorphism, only finitely many triples $(\mathfrak{A}, D^{\Box^+}, D^{\Box})$ such that

- $\mathfrak{A} \in \mathsf{K4}$ and $\mathfrak{A}$ is at most k-generated as a Boolean algebra, where $k = |Sfor(\Gamma/\Delta)|$;
- There is a valuation V on $\mathfrak{A}$ refuting Γ/Δ, such that

$$D^{\Box^+} = \{\bar{V}(\varphi) : \Box^+\varphi \in Sfor(\Gamma/\Delta)\}$$
$$D^{\Box} = \{\bar{V}(\varphi) : \Box\varphi \in Sfor(\Gamma/\Delta)\}$$

Let Ξ be the set of all modal pre-stable canonical rules $\mu_+(\mathfrak{A}, D^{\Box^+}, D^{\Box})$ for all such triples $(\mathfrak{A}, D^{\Box^+}, D^{\Box})$, identified up to isomorphism.

($\Rightarrow$) Let $\mathfrak{B} \in \mathsf{Mag}$ and suppose $\mathfrak{B} \not\models \Gamma/\Delta$. Take a valuation V on $\mathfrak{B}$ such that $\mathfrak{B}, V \not\models \Gamma/\Delta$. As is well-known, there is a transitive filtration $(\mathfrak{A}', V')$ of $(\mathfrak{B}, V)$ through $Sfor(\Gamma/\Delta)$. Then $\mathfrak{A}' \in \mathsf{K4}$. Moreover, clearly every filtration is a weak filtration, hence so is $(\mathfrak{A}', V')$. Therefore there is a Boolean embedding $h : \mathfrak{A}' \to \mathfrak{B}$ satisfying the BDC for $(D^{\Box^+}, D^{\Box})$, where $D^{\Box^+} := \{\bar{V}'(\varphi) : \Box^+\varphi \in Sfor(\Gamma/\Delta)\}$ and $D^{\Box} := \{\bar{V}'(\varphi) : \Box\varphi \in Sfor(\Gamma/\Delta)\}$. Indeed, it is obvious that h is a Boolean embedding which satisfies the BDC$^{\Box}$ for $D^{\Box}$. The fact that h satisfies the BDC$^{\Box^+}$ follows by noting that, additionally, $\Box\varphi \in Sfor(\Box^+\varphi)$ for every modal formula φ. Lastly, since $(\mathfrak{A}', V')$ is actually a filtration, f is stable, a fortiori pre-stable. Hence we have shown $\mathfrak{B} \not\models \mu_+(\mathfrak{A}, D^{\Box^+}, D^{\Box})$.

($\Leftarrow$) Routine. $\Box$

Theorem 4.29. Every modal rule system $\mathtt{M} \in$ **NExt**$(\mathtt{GL_R})$ is axiomatisable over $\mathtt{GL_R}$ by some set of modal pre-stable canonical rules of the form $\mu_+(\mathfrak{A}, D^{\Box^+}, D^{\Box})$, where $\mathfrak{A} \in \mathsf{K4}$.

§4.3 The Kuznetsov-Muravitsky Isomorphism via Stable Canonical Rules

We are ready for the main topic of this chapter, the Kuznetsov-Muravitsky isomorphism and its extension to rule systems. We apply pre-stable canonical rules to prove this and related results in the vicinity, using essentially the same techniques seen in §§ 2.3 and 3.3.

In § 4.3.1 we recall the relevant transformations between frontons and Magari algebras, and introduce an appropriate Gödel-style translation of msi-formulae into modal ones. In § 4.3.2 we prove the Kuznetsov-Muravitsky isomorphism and its extension to rule systems. We close by giving axiomatic characterisations of the underlying isomorphisms via pre-stable canonical rules, in the style of §§ 2.3.3 and 3.3.3, and by illustrating these results with some examples (§§ 4.3.3 and 4.3.4).

§4.3.1 Semantic and Syntactic Mappings

From Frontons to Magari Algebras and Back We begin by reviewing the constructions for transforming frontons into corresponding Magari algebras and vice versa. The results in this paragraph are known, and recent proofs can be found in, e.g., Esakia [2006]. As usual, we include them in full to give the reader a clearer idea of the transformations under discussion.

The first step for transforming a given fronton $\mathfrak{H} \in \mathsf{Frt}$ into a Magari algebra is to take the free Boolean extension $B(\mathfrak{H})$ of its underlying Heyting algebra. Then we expand $B(\mathfrak{H})$ with a modal operator, as follows.

Definition 4.30. The mapping $\sigma : \mathsf{Frt} \to \mathsf{Mag}$ assigns every $\mathfrak{H} \in \mathsf{Frt}$ to the algebra $\sigma\mathfrak{H} := (B(\mathfrak{H}), \Box)$, where for every $a \in B(H)$ we have

$$\begin{aligned} Ia &:= \bigvee\{b \in H : b \leq a\} \\ \Box a &:= \boxtimes Ia. \end{aligned}$$

Observe that if $a \in H$ then $Ia = a$, and so $\Box a = \boxtimes a$. Consequently, if $a \in H$ also $\Box^+ a = \boxtimes^+ a$.

The construction for turning a Magari algebra into a corresponding fronton is also a simple extension of the construction for turning closure algebras into corresponding Heyting algebras.

Definition 4.31. The mapping $\rho : \mathsf{Mag} \to \mathsf{Frt}$ assigns every Magari algebra $\mathfrak{A} \in \mathsf{Mag}$ to the algebra $\rho\mathfrak{A} := (O(A), \wedge, \vee, \to, \Box, 1, 0)$, where

$$\begin{aligned} O(A) &:= \{a \in A : \Box^+ a = a\} \\ a \to b &:= \Box^+(\neg a \vee b) \\ \boxtimes a &:= \Box a \end{aligned}$$

By unpacking the definitions just presented it is not difficult to verify that the following Proposition holds.

Proposition 4.32. For every $\mathfrak{H} \in \mathsf{Frt}$ we have $\mathfrak{H} \cong \rho\sigma\mathfrak{H}$. Moreover, for every $\mathfrak{A} \in \mathsf{Grz}$ we have $\sigma\rho\mathfrak{A} \rightarrowtail \mathfrak{A}$.

We call a Magari algebra $\mathfrak{A}$ *skeletal* if $\sigma\rho\mathfrak{A} \cong \mathfrak{A}$ holds.

We now give more suggestive dual descriptions of the maps σ, ρ on KM- and GL-spaces, which also make it easier to show that σ, ρ are the intended ranges.

Definition 4.33. If $\mathfrak{X} = (X, \leq, \sqsubseteq, \mathcal{O})$ is a KM-space we set $\sigma\mathfrak{X} := (X, R, \mathcal{O})$, where $R =\sqsubseteq$. Let $\mathfrak{Y} := (Y, R, \mathcal{O})$ be a GL-space. For $x, y \in Y$ write $x \sim y$ iff Rxy and Ryx. Define a map $\rho : Y \to \wp(Y)$ by setting $\rho(x) = \{y \in Y : x \sim y\}$. We define $\rho\mathfrak{Y} := (\rho[Y], \leq_\rho, \sqsubseteq_\rho \rho[\mathcal{O}])$ where $\rho(x) \sqsubseteq_\rho \rho(y)$ iff Rxy and $\rho(x) \leq_\rho \rho(y)$ iff $R^+_\rho \rho(x)\rho(y)$.

Proposition 4.34. The following conditions hold.

1. Let $\mathfrak{H} \in \mathsf{Frt}$. Then $(\sigma\mathfrak{H})_* \cong \sigma(\mathfrak{H}_*)$. Consequently, if $\mathfrak{X}$ is a KM-space then $(\sigma\mathfrak{X})^* \cong \sigma(\mathfrak{X})^*$.
2. Let $\mathfrak{X}$ be a GL-space. Then $(\rho\mathfrak{X})^* \cong \rho(\mathfrak{X}^*)$. Consequently, if $\mathfrak{A} \in \mathsf{Mag}$, then $(\rho\mathfrak{A})_* \cong \rho(\mathfrak{A}_*)$.

Proof. (1) Let $\mathfrak{H} \in \mathsf{Frt}$. By the proof of Proposition 2.38 it suffices to show that $(\sigma\mathfrak{H})_*$ and $\sigma(\mathfrak{H}_*)$ have the same relations. Let $x, y \in Spec(\sigma\mathfrak{H}) = Spec(\mathfrak{H})$, and suppose $R_\Box xy$ (where $R_\Box$ is the modal relation of $(\sigma\mathfrak{H})_*$). Then for every $a \in B(H)$, and so for every $a \in H$ in particular, $\Box a \in x$ implies $a \in y$. But if $a \in H$ then $\Box a = \boxtimes a$, and so we have that for every $a \in H$, $\boxtimes a \in x$ implies $a \in y$, i.e. $R_\boxtimes xy$ (where $R_\boxtimes$ is the relation of $\sigma(\mathfrak{H}_*)$). Conversely, assume that $R_\boxtimes xy$. Then for every $a \in H$, $\boxtimes a \in x$ implies $a \in y$. Let $a \in B(H)$ and suppose $\Box a \in x$. This is to say $\boxtimes Ia \in x$, and so as $Ia \in H$ in turn $Ia \in y$. Since y is upwards closed and $Ia \leq a$, in turn $a \in y$. Whence $R_\Box xy$. The second part is a trivial consequence of the first part.

(2) Let $\mathfrak{X} \in \mathsf{Spa}(\mathtt{GL})$. Reasoning as in the proof of Item 2 in Proposition 2.44, we obtain that $\rho^{-1} : (\rho\mathfrak{X})^* \to \rho(\mathfrak{X}^*)$ is a Heyting algebra isomorphism. Now let $U \in \mathsf{ClopUp}(\rho\mathfrak{X})^*$. Then for any $\rho(x) \in X$ we have that $\rho(x) \in \boxtimes_{\sqsubseteq_\rho} U$ iff $\Uparrow_\rho[\rho(x)] \subseteq U$ iff $R[x] \subseteq \rho^{-1}(U)$ iff $x \in \Box_R(U)$. The second part is a trivial consequence of the first part. $\Box$

Proposition 4.35. For every fronton $\mathfrak{H} \in \mathsf{Frt}$ we have that $\sigma\mathfrak{H}$ is a Magari algebra, and for every Magari algebra $\mathfrak{A} \in \mathsf{Mag}$ we have that $\rho\mathfrak{A}$ is a fronton.

Proof. Let $\mathfrak{H} \in \mathsf{Frt}$. We prove that $\mathfrak{X} := \sigma(\mathfrak{H}_*)$ is a GL-space. Let $\mathfrak{X}' = (X, R^+, \mathcal{O})$. Then it is easy to see that $\mathfrak{X}' \cong \sigma(\mathfrak{K}_*)$, where $\mathfrak{K}$ is the $\boxtimes$-free reduct of $\mathfrak{H}$. Whence $\mathfrak{X}'$ is a Grz-space. Now let $U \in \mathsf{Clop}(\mathfrak{X})$ and $x \in X$. Suppose $x \in R^{-1}(U)$. In other words, there is $y \in U$ such that Rxy. Then also R^+xy. By Proposition 2.18 there is some $z \in max_{R^+}(U)$ such that R^+yz. Clearly, it follows that $z \in max_R(U)$. Thus, by Proposition 4.11, we obtain that $z \notin R[z]$, and so in turn that $R[z] \cap U = \emptyset$. So we have $z \in U \cap \Box_R - U$. By transitivity we find Rxz, hence we have shown that $x \in R^{-1}(U \cap \Box_R - U)$. This implies that $\mathfrak{X}$ is a GL-space, as desired.

Let $\mathfrak{A} \in \mathsf{Mag}$. We show that $\mathfrak{X} := \rho(\mathfrak{A}_*)$ is a KM-space. Conditions 1, 2, and 4 of Definition 4.6 are obvious. For condition 3, it follows from Theorem 4.8 that

$$\{x\} = \bigcap_{a \in x} \beta(a)$$

is closed, and since by assumption $R[x]$ is closed we have that $\Uparrow x = \{x\} \cup R[x]$ is closed as well. For condition 5, let $U \in \mathsf{ClopUp}(\mathfrak{X})$ and $x \in X$. Suppose $x \notin U$. Then $x \in -U$, so by Proposition 4.11 either $x \in max_R(-U)$ or there is $y \in max_R(-U)$ with Rxy. If the former, it follows that $R[x] \subseteq U$, and as $x \leq x$ we are done. If the latter, it follows that $R[y] \subseteq U$, and as $x \leq y$ we are done. □

A Gödelian Translation We now show how to translate msi formulae into modal formulae in a way which suits our current goals. The main idea, already anticipated when developing msi stable canonical rules, is to conceptualise rule systems in $\mathbf{NExt}(\mathtt{GL_R})$ as stated in a signature containing two modal operators $\Box, \Box^+$, so to use $\Box$ to translate $\boxtimes$ and $\Box^+$ to translate $\to$. This leads to the following Gödelian translation function.

Definition 4.36. The Gödelian translation $T : Tm_{msi} \to Tm_{md}$ is defined recursively as follows.

$$
\begin{aligned}
T(\bot) &:= \bot \\
T(\top) &:= \top \\
T(p) &:= \Box p \\
T(\varphi \wedge \psi) &:= T(\varphi) \wedge T(\psi) \\
T(\varphi \vee \psi) &:= T(\varphi) \vee T(\psi) \\
T(\varphi \to \psi) &:= \Box^+(\neg T(\varphi) \vee T(\psi)) \\
T(\boxtimes \varphi) &:= \Box T(\varphi)
\end{aligned}
$$

The translation T above was originally proposed by Kuznetsov and Muravitsky [1986], and is perhaps most comprehensively studied in Wolter and Zakharyaschev [1998, 1997]. Our presentation contains a revised clause for the case of $T(\boxtimes\varphi)$, which was originally defined as

$$T(\boxtimes\varphi) := \Box^+\Box T(\varphi).$$

However, it is not difficult to verify that $\mathsf{Mag} \models \Box p \leftrightarrow \Box^+\Box p$, which justifies our revised clause. As usual, we extend the translation T from terms to rules by setting

$$T(\Gamma/\Delta) := T[\Gamma]/T[\Delta].$$

The following key lemma describes the semantic behaviour of $T(\cdot)$ in terms of the map ρ.

Lemma 4.37. For every $\mathfrak{A} \in \mathsf{Mag}$ and si rule Γ/Δ,

$$\mathfrak{A} \models T(\Gamma/\Delta) \iff \rho\mathfrak{A} \models \Gamma/\Delta$$

Proof. A simple induction on structure shows that for every si term φ, every modal space $\mathfrak{X}$, every valuation V on $\mathfrak{X}$ and every point $x \in X$ we have

$$\mathfrak{X}, V, x \models T(\varphi) \iff \rho\mathfrak{X}, \rho[V], \rho(x) \models \varphi.$$

Using this equivalence and noting that every valuation V on some $\mathtt{KM}$-space $\rho\mathfrak{X}$ can be seen as of the form $\rho[V']$ for some valuation V' on $\mathfrak{X}$, the rest of the proof is trivial. □

§4.3.2 The Kuznetsov-Muravitsky Theorem

We are now ready to state and prove the main result of the present chapter. Extend the mappings $\sigma : \mathsf{Frt} \to \mathsf{Mag}$ and $\rho : \mathsf{Mag} \to \mathsf{Frt}$ by setting

$$\begin{aligned} \sigma : \mathbf{Uni}(\mathsf{Frt}) &\to \mathbf{Uni}(\mathsf{Mag}) & \rho : \mathbf{Uni}(\mathsf{Mag}) &\to \mathbf{Uni}(\mathsf{Frt}) \\ \mathcal{U} &\mapsto \mathsf{Uni}\{\sigma\mathfrak{H} : \mathfrak{H} \in \mathcal{U}\} & \mathcal{W} &\mapsto \{\rho\mathfrak{A} : \mathfrak{A} \in \mathcal{W}\}. \end{aligned}$$

Now define the following two syntactic counterparts to σ, ρ between $\mathbf{NExt}(\mathtt{KM_R})$ and $\mathbf{NExt}(\mathtt{GL_R})$.

$$\begin{aligned} \sigma : \mathbf{NExt}(\mathtt{KM_R}) &\to \mathbf{NExt}(\mathtt{GL_R}) & \rho : \mathbf{NExt}(\mathtt{GL_R}) &\to \mathbf{NExt}(\mathtt{KM_R}) \\ \mathtt{L} &\mapsto \mathtt{GL_R} \oplus \{T(\Gamma/\Delta) : \Gamma/\Delta \in \mathtt{L}\} & \mathtt{M} &\mapsto \{\Gamma/\Delta : T(\Gamma/\Delta) \in \mathtt{M}\} \end{aligned}$$

These maps easily extend to lattices of logics, by setting:

$$\begin{aligned} \sigma : \mathbf{NExt}(\mathtt{KM}) &\to \mathbf{NExt}(\mathtt{GL}) & \rho : \mathbf{NExt}(\mathtt{GL}) &\to \mathbf{NExt}(\mathtt{KM}) \\ \mathtt{L} &\mapsto \mathsf{Taut}(\sigma\mathtt{L_R}) = \mathtt{GL} \oplus \{T(\varphi) : \varphi \in \mathtt{L}\} & \mathtt{M} &\mapsto \mathsf{Taut}(\rho\mathtt{M_R}) = \{\varphi : T(\varphi) \in \mathtt{M}\} \end{aligned}$$

The goal of this subsection is to establish the following result.

Theorem 4.38 (Kuznetsov-Muravitsky theorem)**.** The following conditions hold:

1. $\sigma : \mathbf{NExt}(\mathtt{KM_R}) \to \mathbf{NExt}(\mathtt{GL_R})$ and $\rho : \mathbf{NExt}(\mathtt{GL_R}) \to \mathbf{NExt}(\mathtt{KM_R})$ are mutually inverse complete lattice isomorphisms.
2. $\sigma : \mathbf{NExt}(\mathtt{KM}) \to \mathbf{NExt}(\mathtt{GL})$ and $\rho : \mathbf{NExt}(\mathtt{GL}) \to \mathbf{NExt}(\mathtt{KM})$ are mutually inverse complete lattice isomorphisms.

Similarly to the previous chapters, the main difficulty to overcome here consists in showing that $\sigma : \mathbf{NExt}(\mathtt{KM_R}) \to \mathbf{NExt}(\mathtt{GL_R})$ is surjective. We approach this problem by applying our pre-stable canonical rules, following a similar blueprint as that used in the previous chapters. The following lemma is a counterpart of Lemma 2.50. Its proof is similar to the latter's, thanks to the similarities existing between $\mathtt{Grz}$- and $\mathtt{GL}$-spaces.

Lemma 4.39. Let $\mathfrak{A} \in \mathsf{Mag}$. Then for every modal rule Γ/Δ we have $\mathfrak{A} \models \Gamma/\Delta$ iff $\sigma\rho\mathfrak{A} \models \Gamma/\Delta$.

Proof. $(\Rightarrow)$ This direction follows from the fact that $\sigma\rho\mathfrak{A} \rightarrowtail \mathfrak{A}$ (Proposition 4.32).

$(\Leftarrow)$ We prove the dual statement that $\mathfrak{A}_* \not\models \Gamma/\Delta$ implies $\sigma\rho\mathfrak{A}_* \not\models \Gamma/\Delta$. Let $\mathfrak{X} := \mathfrak{A}_*$. In view of Theorem 4.29 it suffices to consider the case $\Gamma/\Delta = \mu_+(\mathfrak{B}, D^{\Box^+}, D^{\Box})$, for $\mathfrak{B} \in \mathsf{K4}$ finite. So suppose $\mathfrak{X} \not\models \mu_+(\mathfrak{B}, D^{\Box^+}, D^{\Box})$ and let $\mathfrak{F} := \mathfrak{B}_*$. Then there is a pre-stable map $f : \mathfrak{X} \to \mathfrak{F}$ satisfying the BDC for $(\mathfrak{D}^{\Box^+}, \mathfrak{D}^{\Box})$. We construct a pre-stable map $g : \sigma\rho\mathfrak{X} \to \mathfrak{F}$ which also satisfies the BDC for $(\mathfrak{D}^{\Box^+}, \mathfrak{D}^{\Box})$.

Let C be a cluster in $\mathfrak{F}$. Consider $Z_C := f^{-1}(C)$. As f is continuous, Z_C is clopen. Moreover, since f is pre-stable Z_C does not cut any cluster. It follows that $\rho[Z_C]$ is clopen in $\rho\mathfrak{X}$, because $\rho\mathfrak{X}$ has the quotient topology.

Enumerate $C := \{x_1, \dots, x_n\}$. Then $f^{-1}(x_i) \subseteq Z_C$ is clopen. By Proposition 4.11, we have that $M_i := max_R(f^{-1}(x_i))$ is clopen. Furthermore, as every element of M_i is maximal in M_i, by Proposition 4.11 again we have that M_i does not cut any cluster. Therefore $\rho[M_i]$ is clopen, because $\rho\mathfrak{X}$ has the quotient topology. Clearly, $\rho[M_i] \cap \rho[M_j] = \emptyset$ for each $i \neq j$. Therefore there are disjoint clopens $U_1, \dots, U_n$ with $\rho[M_i] \subseteq U_i$ and $\bigcup_i U_i = \rho[Z_C]$. Just take $U_i := \rho[M_i]$ if $i \neq n$, and

$$U_n := \rho[Z_C] \setminus \left(\bigcup_{i<n} U_i\right).$$

Now define

$$g_C : \rho[Z_C] \to C$$
$$g_C(z) = x_i \iff z \in U_i$$

Note that g_C is relation preserving, evidently, and continuous by construction. Finally, define $g : \sigma\rho\mathfrak{X} \to F$ by setting

$$g(\rho(z)) := \begin{cases} f(z) & \text{if } f(z) \text{ does not belong to any proper cluster} \\ g_C(\rho(z)) & \text{if } f(z) \in C \text{ for some proper cluster } C \subseteq F \end{cases}$$

Now, g is evidently pre-stable. Moreover, it is continuous because both f and each g_C are. Let us check that g satisfies the BDC for $(\mathfrak{D}^{\Box^+}, \mathfrak{D}^{\Box})$.

- (BDC$^{\Box^+}$) This may be shown reasoning the same way as in the proof of Lemma 2.50.
- (BDC$^{\Box}$-back) Let $\mathfrak{d} \in \mathfrak{D}^{\Box}$ and $\rho(x) \in \rho[X]$. Suppose that $R[g(\rho(x))] \cap \mathfrak{d} \neq \emptyset$. Let $U := f^{-1}(f(x))$. Then $x \in U$, so by Proposition 4.11 either $x \in max_R(U)$ or there exists $x' \in max_R(U)$ such that Rxx'. We consider the former case only, the latter is analogous. Since $x \in max_R(U)$, by construction we have $g(\rho(x)) = f(x)$. Thus $R[f(x)] \cap \mathfrak{d} \neq \emptyset$. Since f satisfies the BDC for $\mathfrak{d}$, it follows that there is $y \in X$ such that Rxy and $f(y) \in \mathfrak{d}$. As $x \in max_R(U)$ we must have $f(x) \neq f(y)$. Now let $V := f^{-1}(f(y))$. As $y \in V$, by Proposition 4.11 either $y \in max_R(V)$ or there exists some $y' \in max_R(V)$ such that Ryy'. Wlog, suppose the former. Consequently, $f(y) = g(\rho(y))$. But then we have shown that $R\rho(x)\rho(y)$ and $g(\rho(y)) \in \mathfrak{d}$, i.e. $g[R[\rho(x)]] \cap \mathfrak{d} \neq \emptyset$.
- (BDC$^{\Box}$-forth) Let $\mathfrak{d} \in \mathfrak{D}^{\Box}$ and $\rho(x) \in \rho[X]$. Suppose that $g[R[\rho(x)]] \cap \mathfrak{d} \neq \emptyset$. Observe that $g[R[\rho(x)]] \cap \mathfrak{d} \neq \emptyset$ is equivalent to $R[\rho(x)] \cap g^{-1}(\mathfrak{d}) \neq \emptyset$. Therefore there is some $y \in \mathfrak{d}$ such that $R[\rho(x)] \cap g^{-1}(y) \neq \emptyset$. By Proposition 4.11 there is $z \in max_R(g^{-1}(y))$ with $R_\rho \rho(x)\rho(z)$. Observe that since g is pre-stable, $R^+g(\rho(x))g(\rho(z))$, whence if $g(\rho(x)) \neq g(\rho(z))$ in turn $Rg(\rho(x))g(\rho(z))$ and we are done. So suppose otherwise that $g(\rho(x)) = g(\rho(z))$. Distinguish two cases
  - *Case 1*: $y \notin R[y]$. Then y cannot belong to a proper cluster, so by construction $f(x) = g(\rho(x))$ and $f(z) = g(\rho(z))$. From $R\rho(x)\rho(z)$ it follows that Rxz, whence $R[x] \cap f^{-1}(\mathfrak{d}) \neq \emptyset$ Since f satisfies the BDC-forth for $\mathfrak{d}$, there must be some $u \in \mathfrak{d}$ with $Rf(x)u$ and $f(u) \in \mathfrak{d}$. Then also $Rg(\rho(x))u$, i.e. $R[g(\rho(x))] \cap \mathfrak{d} \neq \emptyset$ as desired.

– *Case 2*: $y \in R[y]$. But then $Rg(\rho(x))y$. This shows $R[g(\rho(x))] \cap \mathfrak{d} \neq \emptyset$ as desired.

□

Proposition 4.40. Every universal class $\mathcal{U} \in \mathbf{Uni}(\mathsf{Mag})$ is generated by its skeletal elements, i.e. $\mathcal{U} = \sigma\rho\mathcal{U}$.

Proof. Analogous to Theorem 2.51, but applying Lemma 4.39 instead of Lemma 2.50. □

We now apply Lemma 4.39 to characterise the maps $\sigma : \mathbf{NExt}(\mathtt{KM_R}) \to \mathbf{NExt}(\mathtt{GL_R})$ and $\rho : \mathbf{NExt}(\mathtt{KM_R}) \to \mathbf{NExt}(\mathtt{GL_R})$ in terms of their semantic counterparts.

Lemma 4.41. For each $\mathtt{L} \in \mathbf{Ext}(\mathtt{KM_R})$ and $\mathtt{M} \in \mathbf{NExt}(\mathtt{GL_R})$, the following hold:

$$\mathsf{Alg}(\sigma\mathtt{L}) = \sigma\mathsf{Alg}(\mathtt{L}) \tag{4.6}$$

$$\mathsf{Alg}(\rho\mathtt{M}) = \rho\mathsf{Alg}(\mathtt{M}) \tag{4.7}$$

Proof. (4.6) By Theorem 2.51 it suffices to show that $\mathsf{Alg}(\sigma\mathtt{L})$ and $\sigma\mathsf{Alg}(\mathtt{L})$ have the same skeletal elements. So let $\mathfrak{A} = \sigma\rho\mathfrak{A} \in \mathsf{Mag}$. Assume $\mathfrak{A} \in \sigma\mathsf{Alg}(\mathtt{L})$. Since $\sigma\mathsf{Alg}(\mathtt{L})$ is generated by $\{\sigma\mathfrak{B} : \mathfrak{B} \in \mathsf{Alg}(\mathtt{L})\}$ as a universal class, by Proposition 4.32 and Lemma 4.37 we have $\mathfrak{A} \models T(\Gamma/\Delta)$ for every $\Gamma/\Delta \in \mathtt{L}$. But then $\mathfrak{A} \in \mathsf{Alg}(\sigma\mathtt{L})$. Conversely, assume $\mathfrak{A} \in \mathsf{Alg}(\sigma\mathtt{L})$. Then $\mathfrak{A} \models T(\Gamma/\Delta)$ for every $\Gamma/\Delta \in \mathtt{L}$. By Lemma 4.37 this is equivalent to $\rho\mathfrak{A} \in \mathsf{Alg}(\mathtt{L})$, therefore $\sigma\rho\mathfrak{A} = \mathfrak{A} \in \sigma\mathsf{Alg}(\mathtt{L})$.

(4.7) Let $\mathfrak{H} \in \mathsf{Frt}$. If $\mathfrak{H} \in \rho\mathsf{Alg}(\mathtt{M})$ then $\mathfrak{H} = \rho\mathfrak{A}$ for some $\mathfrak{A} \in \mathsf{Alg}(\mathtt{M})$. It follows that for every rule $T(\Gamma/\Delta) \in \mathtt{M}$ we have $\mathfrak{A} \models T(\Gamma/\Delta)$, and so by Lemma 4.37 in turn $\mathfrak{H} \models \Gamma/\Delta$. Therefore indeed $\mathfrak{H} \in \mathsf{Alg}(\rho\mathtt{M})$. Conversely, for all rules Γ/Δ, if $\rho\mathsf{Alg}(\mathtt{M}) \models \Gamma/\Delta$ then by Lemma 4.37 $\mathsf{Alg}(\mathtt{M}) \models T(\Gamma/\Delta)$, hence $\Gamma/\Delta \in \rho\mathtt{M}$. Thus $\mathsf{ThR}(\rho\mathsf{Alg}(\mathtt{M})) \subseteq \rho\mathtt{M}$, and so $\mathsf{Alg}(\rho\mathtt{M}) \subseteq \rho\mathsf{Alg}(\mathtt{M})$. □

We are now ready to prove the main result of this section.

Theorem 4.38 (Kuznetsov-Muravitsky theorem). The following conditions hold:

1. $\sigma : \mathbf{NExt}(\mathtt{KM_R}) \to \mathbf{NExt}(\mathtt{GL_R})$ and $\rho : \mathbf{NExt}(\mathtt{GL_R}) \to \mathbf{NExt}(\mathtt{KM_R})$ are mutually inverse complete lattice isomorphisms.
2. $\sigma : \mathbf{NExt}(\mathtt{KM}) \to \mathbf{NExt}(\mathtt{GL})$ and $\rho : \mathbf{NExt}(\mathtt{GL}) \to \mathbf{NExt}(\mathtt{KM})$ are mutually inverse complete lattice isomorphisms.

Proof. (1) It suffices to show that the two mappings $\sigma : \mathbf{Uni}(\mathsf{Frt}) \to \mathbf{Uni}(\mathsf{Mag})$ and $\rho : \mathbf{Uni}(\mathsf{Mag}) \to \mathbf{Uni}(\mathsf{Frt})$ are complete lattice isomorphisms and mutual inverses. Both maps are evidently order preserving, and preservation of infinite joins is an easy consequence of Lemma 4.37.

Let $\mathcal{U} \in \mathbf{Uni}(\mathsf{Mag})$. Then $\mathcal{U} = \sigma\rho\mathcal{U}$ by Proposition 4.40, so σ is surjective and a left inverse of ρ. Now let $\mathcal{U} \in \mathbf{Uni}(\mathsf{Frt})$. It follows immediately from Proposition 4.32 that $\rho\sigma\mathcal{U} = \mathcal{U}$. Therefore ρ is surjective and a left inverse of σ. But then σ and ρ are mutual inverses, whence both bijections.

(2) Follows immediately from Item 1 and Proposition 4.2. □

§4.3.3 Axiomatic Characterisation of the Maps σ, ρ

In this subsection we show how to transform a pre-stable canonical axiomatisation of any $\mathtt{L} \in$ **NExt**$(\mathtt{KM_R})$ into a pre-stable canonical axiomatisation of $\rho\mathtt{L}$ (Theorem 4.43), and conversely how to extract a pre-stable canononical axiomatisation of $\rho\mathtt{M}$ out of a pre-stable canonical axiomatisation of any $\mathtt{M} \in$ **NExt**$(\mathtt{GL_R})$ (Theorem 4.46).

Theorem 4.38 implies that every rule system in **NExt**$(\mathtt{GL_R})$ is axiomatisable by Gödelian translations of msi pre-stable canonical rules. Following a by now familiar recipe, we characterise such rules as modal pre-stable canonical rules of skeletal Magari algebras.

Lemma 4.42 (Rule translation lemma - pre-stable)**.** Let $\eta_{\boxtimes}(\mathfrak{H}, D^{\to}, D^{\boxtimes})$ be an msi pre-stable canonical rule. Then for every Magari algebra $\mathfrak{A} \in \mathsf{Mag}$ we have

$$\mathfrak{A} \models T(\eta_{\boxtimes}(\mathfrak{H}, D^{\to}, D^{\boxtimes})) \iff \mathfrak{A} \models \mu_{+}(\sigma\mathfrak{H}, D^{\Box^{+}}, D^{\Box})$$

where $D^{\Box^{+}} := \{\neg a \vee b : (a, b) \in D^{\to}\}$ and $D^{\Box} = D^{\boxtimes}$.

Proof. Let $\mathfrak{F} := \mathfrak{H}_{*}$. By Proposition 4.34 and Lemma 4.37 it suffices to prove that for all $\mathtt{GL}$-spaces $\mathfrak{X}$ we have

$$\rho\mathfrak{X} \models \eta_{\boxtimes}(\mathfrak{F}, \mathfrak{D}^{\to}, \mathfrak{D}^{\boxtimes}) \iff \mathfrak{X} \models \mu_{+}(\sigma\mathfrak{F}, \mathfrak{D}^{\Box^{+}}, \mathfrak{D}^{\Box})$$

where $\mathfrak{D}^{\Box^{+}} = \mathfrak{D}^{\to}$ (note that if $(a, b) \in D^{\to}$ then $\beta(a) \setminus \beta(b) = -\beta(\neg a \vee b)$) and $\mathfrak{D}^{\Box} = \mathfrak{D}^{\boxtimes}$.

($\Rightarrow$) Assume that $\mathfrak{X} \not\models \mu_{+}(\sigma\mathfrak{F}, \mathfrak{D}^{\Box^{+}}, \mathfrak{D}^{\Box})$. Then there is a continuous pre-stable surjection $f : \mathfrak{X} \to \sigma\mathfrak{F}$ satisfying the BDC for $(\mathfrak{D}^{\Box^{+}}, \mathfrak{D}^{\Box})$. Observe that for all $x, y \in X$, if Rxy and Ryx then $R^{+}f(x)f(y)$ and $R^{+}f(y)f(x)$ by pre-stability, whence $f(x) = f(y)$ since $\sigma\mathfrak{F}$ is a poset. Therefore we may define a map $g : \rho\mathfrak{X} \to \mathfrak{F}$ by setting $g(\rho(x)) = f(x)$. Clearly, g is pre-stable: if $\rho(x), \rho(y) \in \rho[X]$ then $\rho(x) \leq \rho(y)$ implies $R^{+}xy$, which by the pre-stability of f implies $R^{+}f(x)f(x)$, whence $R^{+}g(\rho(x))g(\rho(y))$. Let us check that g satisfies the BDC for $(\mathfrak{D}^{\to}, \mathfrak{D}^{\boxtimes})$.

- (BDC$^{\to}$) Let $\mathfrak{d} \in \mathfrak{D}^{\to}$. Then $\mathfrak{d} \in \mathfrak{D}^{\Box^{+}}$. Let $\rho(x) \in \rho[X]$, and suppose that ${\uparrow}[g(\rho(x))] \cap \mathfrak{d} \neq \emptyset$. Then also $R^{+}[f(x)] \cap \mathfrak{d} \neq \emptyset$, and since f satisfies the BDC$^{\Box^{+}}$ for $\mathfrak{D}^{\Box^{+}}$ it follows that $f[R^{+}(x)] \cap \mathfrak{d} \neq \emptyset$. This implies $g[{\uparrow}\rho(x)] \cap \mathfrak{d} \neq \emptyset$ because $\rho : \mathfrak{X} \to \rho\mathfrak{X}$ preserves R^{+}.
- (BDC$^{\boxtimes}$-back) Analogous to the (BDC$^{\to}$).
- (BDC$^{\boxtimes}$-forth) Let $\mathfrak{d} \in \mathfrak{D}^{\boxtimes}$. Then $\mathfrak{d} \in \mathfrak{D}^{\Box}$. Let $\rho(x) \in \rho[X]$ and suppose that $g[{\Uparrow}\rho(x)] \cap \mathfrak{d} \neq \emptyset$. As ρ reflects R it follows that $f[R[x]] \cap \mathfrak{d} \neq \emptyset$. Since f satisfies the BDC$^{\Box}$ for $\mathfrak{D}^{\Box}$, we infer $R[f(x)] \cap \mathfrak{d} \neq \emptyset$, which clearly implies $R[g(\rho(x))] \cap \mathfrak{d} \neq \emptyset$.

($\Leftarrow$) Assume that $\rho\mathfrak{X} \not\models \eta_{\boxtimes}(\mathfrak{F}, \mathfrak{D}^{\to}, \mathfrak{D}^{\boxtimes})$. Then there is a continuous pre-stable surjection $g : \rho\mathfrak{X} \to \mathfrak{F}$ satisfying the BDC for $(\mathfrak{D}^{\to}, \mathfrak{D}^{\boxtimes})$. Define a map $f : \mathfrak{X} \to \sigma\mathfrak{F}$ by setting $f(x) = g(\rho(x))$ for every $x \in X$. Since both $\rho : \mathfrak{X} \to \rho\mathfrak{X}$ and g are continuous, ρ preserves R^{+}, and g is pre-stable, it follows that f is continuous and pre-stable. Moreover, we claim that f satisfies the BDC for $(\mathfrak{D}^{\Box^{+}}, \mathfrak{D}^{\Box})$.

- (BDC$^{\Box^+}$) Let $\mathfrak{d} \in \mathfrak{D}^{\Box^+}$. Then $\mathfrak{d} \in \mathfrak{D}^{\rightarrow}$. Let $x \in X$, and suppose that $R^+[f(x)] \cap \mathfrak{d} \neq \emptyset$. Then also $R^+[g(\rho(x))] \cap \mathfrak{d} \neq \emptyset$, and since g satisfies the BDC$^{\rightarrow}$ for $\mathfrak{D}^{\rightarrow}$ it follows that $g[\uparrow \rho(x)] \cap \mathfrak{d} \neq \emptyset$. This implies $f[R^+[x]] \cap \mathfrak{d} \neq \emptyset$ because $\rho : \mathfrak{X} \to \rho\mathfrak{X}$ reflects R^+.
- (BDC$^{\Box}$-back) Analogous to the (BDC$^{\Box^+}$).
- (BDC$^{\Box}$-forth) Let $\mathfrak{d} \in \mathfrak{D}^{\Box}$. Then $\mathfrak{d} \in \mathfrak{D}^{\boxtimes}$. Let $x \in X$ and suppose that $f[R^+[x]] \cap \mathfrak{d} \neq \emptyset$. As ρ preserves R it follows that $g[\Uparrow \rho(x)] \cap \mathfrak{d} \neq \emptyset$. Since g satisfies the BDC$^{\boxtimes}$ for $\mathfrak{D}^{\boxtimes}$, we infer $R[g(\rho(x))] \cap \mathfrak{d} \neq \emptyset$, which obviously implies $R[f(x)] \cap \mathfrak{d} \neq \emptyset$.

$\Box$

Theorem 4.43. For every $\mathtt{L} \in \mathbf{NExt}(\mathtt{KM_R})$, if

$$\mathtt{L} = \mathtt{KM_R} \oplus \{\eta_{\boxtimes}(\mathfrak{H}_i, D_i^{\rightarrow}, D_i^{\boxtimes}) : i \in I\}$$

then

$$\sigma\mathtt{L} = \mathtt{GL_R} \oplus \{\mu_+(\sigma\mathfrak{H}_i, D_i^{\Box^+}, D_i^{\Box}) : i \in I\}$$

where for each $i \in I$ the sets $D^{\Box^+}$ and $D^{\Box}$ are defined as in the statement of the rule translation lemma (pre-stable).

Proof. By definition have

$$\sigma\mathtt{L} = \mathtt{GL_R} \oplus \{T(\eta_{\boxtimes}(\mathfrak{H}_i, D_i^{\rightarrow}, D_i^{\boxtimes})) : i \in I\},$$

and in turn by the rule translation lemma (pre-stable) we obtain

$$\sigma\mathtt{L} = \mathtt{GL_R} \oplus \{\mu_+(\sigma\mathfrak{H}_i, D_i^{\Box^+}, D_i^{\Box})) : i \in I\}.$$

$\Box$

Next, we generalise the notion of a collapsed rule to cover modal pre-stable canonical rules, and use it to show how to turn axiomatisations of rule systems in $\mathbf{NExt}(\mathtt{GL_R})$ in terms of pre-stable canonical rules of $\mathtt{K4}$-algebras into axiomatisations in terms of pre-stable canonical rules of Magari algebras. We prefer a geometrical presentation of collapsed rules for its intuitiveness.

The first task is extending the map $\rho : \mathsf{Spa}(\mathtt{GL}) \to \mathsf{Spa}(\mathtt{KM})$ to a map $\rho : \mathsf{Spa}(\mathtt{K4}) \to \mathsf{Spa}(\mathtt{KM})$. We only do the finite case, as this will suffice for present purposes. So let $\mathfrak{F} = (F, R) \in \mathsf{Spa}(\mathtt{K4})$. We let $\rho\mathfrak{F} := (\rho[F], \leq_\rho, \sqsubseteq_\rho)$, where $(\rho[F], \leq_\rho)$ is the skeleton of the preorder (F, R^+), and $\sqsubseteq_\rho$ is the irreflexive reduct of $\leq_\rho$. By Proposition 4.7, $\rho\mathfrak{F}$ is indeed a $\mathtt{KM}$-space. Now we can define collapsed pre-stable canonical rules.

Definition 4.44. Let $\mathfrak{F} \in \mathsf{Spa}(\mathtt{K4})$ be finite and $\mathfrak{D}^{\Box^+}, \mathfrak{D}^{\Box} \subseteq \wp(F)$. The *collapsed* pre-stable canonical rule $\eta_{\boxtimes}(\rho\mathfrak{F}, \rho\mathfrak{D}^{\rightarrow}, \rho\mathfrak{D}^{\boxtimes})$ is obtained from the rule $\mu_+(\mathfrak{F}, \mathfrak{D}^{\Box^+}, \mathfrak{D}^{\Box})$ by setting

$$\rho\mathfrak{D}^{\rightarrow} := \{\rho[\mathfrak{d}] : \mathfrak{d} \in \mathfrak{D}^{\Box^+}\}$$
$$\rho\mathfrak{D}^{\boxtimes} := \{\rho[\mathfrak{d}] : \mathfrak{d} \in \mathfrak{D}^{\Box}\}$$

Refutation conditions for collapsed modal pre-stable canonical rules are similar to those for collapsed modal stable canonical rules.

Lemma 4.45 (Rule collapse lemma - pre-stable)**.** For all $\mathfrak{X} \in \mathsf{Spa}(\mathtt{GL})$ and modal stable canonical rule $\mu(\mathfrak{F}, \mathfrak{D})$, $\mathfrak{X} \nvDash \mu_+(\mathfrak{F}, \mathfrak{D}^{\Box^+}, \mathfrak{D}^{\Box})$ implies $\rho\mathfrak{X} \nvDash \eta_{\boxtimes}(\rho\mathfrak{F}, \rho\mathfrak{D}^{\to}, \rho\mathfrak{D}^{\boxtimes})$, where $\mathfrak{D}^{\to} = \mathfrak{D}^{\Box^+}$ and $\mathfrak{D}^{\boxtimes} = \mathfrak{D}^{\Box}$.

Proof. Assume $\mathfrak{X} \nvDash \mu_+(\mathfrak{F}, \mathfrak{D}^{\Box^+}, \mathfrak{D}^{\Box})$. Then there is a pre-stable map $f : \mathfrak{X} \to \mathfrak{F}$ satisfying the BDC for $(\mathfrak{D}^{\Box^+}, \mathfrak{D}^{\Box})$. Define a map $g : \rho\mathfrak{X} \to \rho\mathfrak{F}$ by setting

$$g(\rho(x)) = \rho(f(x)).$$

Clearly, g is pre-stable. Moreover, because f is pre-stable we find that for every $U \subseteq F$, the set $f^{-1}(U)$ does not cut clusters. It follows that $g^{-1}(U) = \rho[f^{-1}(\rho^{-1}(U))]$ is clopen for every $U \subseteq \rho[F]$, because $\rho\mathfrak{X}$ has the quotient topology. So g is continuous.

It remains to be checked that g satisfies the BDC for $(\rho\mathfrak{D}^{\to}, \rho\mathfrak{D}^{\Box})$.

- (BDC$^{\to}$) Let $\rho[\mathfrak{d}] \in \rho\mathfrak{D}^{\to}$. Then $\mathfrak{d} \in \mathfrak{D}^{\Box^+}$. Let $\rho(x) \in \rho[X]$ and suppose ${\uparrow}[g(\rho(x))] \cap \rho[\mathfrak{d}] \neq \emptyset$. Then there is some $\rho(y) \in \rho[\mathfrak{d}]$ such that $g(\rho(x)) \leq \rho(y)$. By the definition of $\rho[\mathfrak{d}]$, wlog we may assume that $y \in \mathfrak{d}$. But then it follows that $R^+[f(x)] \cap \mathfrak{d} \neq \emptyset$, so since f satisfies the BDC$^{\Box^+}$ for $\mathfrak{D}^{\Box^+}$ we conclude $f[R^+[x]] \cap \mathfrak{d} \neq \emptyset$. So there is $z \in R^+[x]$ such that $f(z) \in \mathfrak{d}$. Then $g(\rho(z)) = \rho(f(z)) \in \rho[\mathfrak{d}]$, and since $\rho : \mathfrak{X} \to \rho\mathfrak{X}$ preserves R^+ we also obtain $\rho(x) \leq \rho(y)$. Hence we have shown $g[{\uparrow}\rho(x)] \cap \rho[\mathfrak{d}] \neq \emptyset$.
- (BDC$^{\boxtimes}$-back) Analogous to (BDC$^{\to}$).
- (BDC$^{\boxtimes}$-forth) Let $\rho[\mathfrak{d}] \in \rho\mathfrak{D}^{\boxtimes}$. Then $\mathfrak{d} \in \mathfrak{D}^{\Box}$. Let $\rho(x) \in \rho[X]$ and suppose $g[{\Uparrow}[\rho(x)]] \cap \rho[\mathfrak{d}] \neq \emptyset$. Then there is $\rho(y) \in [{\Uparrow}\rho(x)]$ such that $g(\rho(y)) \in \rho[\mathfrak{d}]$. Since $\rho : \mathfrak{X} \to \rho\mathfrak{X}$ reflects R we find that Rxy, and by the definition of g and the fact that $\rho^{-1}(\rho(y)) \cap \mathfrak{d} \neq \emptyset$ we may assume wlog that $f(y) \in \mathfrak{d}$. But then we have $f[R[x]] \cap \mathfrak{d} \neq \emptyset$, and so as f satisfies the BDC$^{\Box}$ for $\mathfrak{D}^{\Box}$ it follows that $R[f(x)] \cap \mathfrak{d} \neq \emptyset$. Then there is $z \in \mathfrak{d}$ such that $Rf(x)z$. Then clearly $\rho(z) \in \rho[\mathfrak{d}]$, and by the definition of g we have $g(\rho(x)) \sqsubseteq \rho(z)$. Hence we have shown ${\Uparrow}[g(\rho(x))] \cap \rho[\mathfrak{d}] \neq \emptyset$.

□

Now note that by the rule translation lemma (pre-stable), a modal rule Γ/Δ is of the form $T(\Gamma'/\Delta')$ iff it is equivalent to finitely many modal pre-stable canonical rules of the form $\mu_+(\mathfrak{X}, D^{\Box^+}, D^{\Box})$ with $\mathfrak{X} \in \mathsf{Spa}(\mathtt{GL})$. Using this observation we can derive the following axiomatisation of si fragments of modal rule systems in $\mathbf{NExt}(\mathtt{GL_R})$, reasoning as in the proof of Theorem 2.64.

Theorem 4.46. Let $\mathtt{M} \in \mathbf{NExt}(\mathtt{GL_R})$ with $\mathtt{M} = \mathtt{GL_R} \oplus \{\mu_+(\mathfrak{X}_i, \mathfrak{D}_i^{\Box^+}, \mathfrak{D}_i^{\Box}) : i \in I\}$. Let

$$J := \{i \in I : \mu_+(\sigma\rho\mathfrak{X}_i, \rho\mathfrak{D}_i^{\Box^+}, \rho\mathfrak{D}_i^{\Box}) \in \mathtt{M}\}.$$

Then

$$\rho\mathtt{M} = \mathtt{KM_R} \oplus \{\eta_{\boxtimes}(\rho\mathfrak{F}_i, \rho\mathfrak{D}_i^{\to}, \rho\mathfrak{D}_i^{\boxtimes}) : i \in J\}$$

where $\mathfrak{D}^{\to} = \mathfrak{D}^{\Box^+}$ and $\mathfrak{D}^{\boxtimes} = \mathfrak{D}^{\Box}$.

From Theorem 4.46 we obtain the following analogue of Theorems 2.69 and 3.56, using essentially the same proof.

Theorem 4.47. For every modal rule Γ/Δ there is a finite set Ξ of modal pre-stable canonical rules of the form $\mu_+(\mathfrak{A}, D^{\Box^+}, D^{\Box})$ with $\mathfrak{A} \in \mathsf{Mag}$, such that for any $\mathfrak{B} \in \mathsf{Mag}$ we have that $\mathfrak{B} \not\models \Gamma/\Delta$ iff there is $\mu_+(\mathfrak{A}, D^{\Box^+}, D^{\Box}) \in \Xi$ such that $\mathfrak{B} \not\models \mu_+(\mathfrak{A}, D^{\Box^+}, D^{\Box})$.

§4.3.4 Examples

We conclude this chapter by providing some examples of pre-stable canonical axiomatisations of rule systems in $\mathbf{NExt}(\mathtt{KM_R})$ and $\mathbf{NExt}(\mathtt{GL_R})$, and of their images under the maps σ, ρ. We begin with an axiomatisation of a rule system in $\mathbf{NExt}(\mathtt{KM_R})$.

The *reflexivity rule* is the rule $\boxtimes p/p$. Muravitskiy [1985] proved that there exist continuum-many normal msi-logics extending $\mathtt{KM}$ where this rule is admissible. We axiomatise the least extension of $\mathtt{KM}$ where this rule is derivable.

Theorem 4.48. $\mathtt{KM_R} \oplus \boxtimes p/p = \mathtt{KM_R} \oplus \eta_{\boxtimes}\left(\overset{d}{\bullet}\!\rightarrow\!\bullet, \emptyset, \{\{d\}\}\right)$. Consequently, $\sigma(\mathtt{KM_R} \oplus \boxtimes p/p) = \mathtt{GL_R} \oplus \mu_+\left(\overset{d}{\bullet}\!\rightarrow\!\bullet, \emptyset, \{\{d\}\}\right)$

Proof. ($\subseteq$) Let $\mathfrak{X} \in \mathsf{Spa}(\mathtt{KM})$ and suppose $\mathfrak{X} \not\models \boxtimes p/p$. Then there is a valuation V on $\mathfrak{X}$ such that $\mathfrak{X}, V \models \boxtimes p$ and yet there is $x \in X$ with $\mathfrak{X}, V, x \not\models p$. Define a map f from $\mathfrak{X}$ to $\overset{d}{\bullet}\!\rightarrow\!\bullet$ by sending each element of $V(p)$ to the rightmost element, and each element of $-V(p)$ to d. Clearly f is continuous and surjective. Since $V(p) \in \mathsf{ClopUp}$ we have that f is pre-stable. It should also be evident that f satisfies the BDC$^{\boxtimes}$-back for $\{d\}$. Moreover, note that if $x, y \in -V(p)$ then it is not possible to have $x \sqsubseteq y$, because $\mathfrak{X}, V, x \models \boxtimes p$ and $y \notin V(p)$. Therefore we obtain that f also satisfies the BDC$^{\boxtimes}$-forth for $\{d\}$. Thus $\mathfrak{X} \not\models \eta_{\boxtimes}\left(\overset{d}{\bullet}\!\rightarrow\!\bullet, \emptyset, \{\{d\}\}\right)$, as desired.

($\supseteq$) Let $\mathfrak{X} \in \mathsf{Spa}(\mathtt{KM})$ and suppose $\mathfrak{X} \not\models \eta_{\boxtimes}\left(\overset{d}{\bullet}\!\rightarrow\!\bullet, \emptyset, \{\{d\}\}\right)$. Take a continuous pre-stable surjection f from $\mathfrak{X}$ to $\overset{d}{\bullet}\!\rightarrow\!\bullet$ satisfying the BDC$^{\boxtimes}$ for $\{d\}$. Observe that the f-preimage of the rightmost element, call it U, is a $\sqsubseteq$-upset: this follows from the fact that f satisfies the BDC$^{\boxtimes}$-forth for $\{d\}$. For the same reason, every $x \in f^{-1}(d)$ is such that ${\Uparrow} x \subseteq U$. These two claims together imply that the valuation V given by $V(p) = U$ witnesses $\mathfrak{X} \not\models \boxtimes p/p$.

The second part of the theorem follows from the first by an application of Theorem 4.43. $\Box$

As the proof just given illustrates, an appropriate selection of BDC$^{\boxtimes}$ parameters allows one to have control over the area of a given $\mathtt{KM}$-space where the relevant pre-stable map is required to preserve $\sqsubseteq$. This can make the search for axiomatisations easier than it is when working with stable maps.

Next, we consider a rule system in $\mathbf{NExt}(\mathtt{GL_R})$. If $\mathfrak{X}$ is a $\mathtt{GL}$-space, the *depth* of a point $x \in X$ is defined recursively as follows: every $x \in max_R(X)$ has depth 0, and if $x \notin max_R(X)$ then its depth is the maximum depth of any point in $R[x]$ plus 1. We define the *depth* of $\mathfrak{X}$ as the maximum depth of any $x \in X$.

Theorem 4.49. Let $\mathtt{BD_n}$ be the rule system of all $\mathtt{GL}$-space of depth strictly less than n. Then

$$\mathtt{BD_n} = \mathtt{GL_R} \oplus \mu_+\left(\overset{d_n}{\bullet}\!\rightarrow\!\overset{d_{n-1}}{\bullet}\!\rightarrow \cdots \rightarrow\!\overset{d_0}{\bullet}, \emptyset, \{\{d_{n-1}\}, \dots, \{d_0\}\}\right).$$

Consequently,

$$\rho\mathtt{BD_n} = \mathtt{KM_R} \oplus \eta_{\boxtimes}\left(\overset{d_n}{\bullet}\!\rightarrow\!\overset{d_{n-1}}{\bullet}\!\rightarrow \cdots \rightarrow\!\overset{d_0}{\bullet}, \emptyset, \{\{d_{n-1}\}, \dots, \{d_0\}\}\right).$$

Proof. Let ϱ denote the rule $\mu_+\left(\overset{d_n}{\bullet}\!\rightarrow\!\overset{d_{n-1}}{\bullet}\!\rightarrow \cdots \rightarrow\!\overset{d_0}{\bullet}, \emptyset, \{\{d_{n-1}\}, \dots, \{d_0\}\}\right)$. To prove the first part, it suffices to show that for all $\mathfrak{X} \in \mathsf{Spa}(\mathtt{GL})$, $\mathfrak{X}$ has depth strictly less than n iff $\mathfrak{X} \models \varrho$.

($\Rightarrow$) Assume $\mathfrak{X} \not\models \varrho$. Then there is a continuous pre-stable surjection f from $\mathfrak{X}$ to the space $\overset{d_n}{\bullet}\!\rightarrow\!\overset{d_{n-1}}{\bullet}\!\rightarrow \cdots \rightarrow\!\overset{d_0}{\bullet}$ satisfying the BDC$^{\Box}$ for the family $\{\{d_{n-1}\}, \dots, \{d_0\}\}$. Take $x_0 \in f^{-1}(d_n)$. By the BDC$^{\Box}$-back, there must be some $x_{n-1} \in f^{-1}(d_{n-1})$ such that Rx_nx_{n-1}. Iterating this reasoning we construct a sequence $(x_n, x_{n-1}, \dots, x_0)$ such that Rx_ix_{i-1} for all $i < n$, showing that $\mathfrak{X}$ has depth at least n.

($\Leftarrow$) Assume $\mathfrak{X}$ has depth at least n. Then there is at least one point with depth i for each $i \leq n$. Define a map f from X to $\overset{d_n}{\bullet}\!\rightarrow\!\overset{d_{n-1}}{\bullet}\!\rightarrow \cdots \rightarrow\!\overset{d_0}{\bullet}$ as follows: for all $i \leq n$, send all points of depth i to d_i, and send everything else to d_n. It is easy to see that this map is a well defined continuous pre-stable surjection satisfying the BDC$^{\Box}$ for the family $\{\{d_{n-1}\}, \dots, \{d_0\}\}$. The only non-trivial point to verify is the BDC$^{\Box}$-forth: any two $x, y \in X$ both of depth n or greater are mapped to d_n, but the case where Rxy does not cause problems since we do not require that f satisfy the BDC$^{\Box}$ for d_n. Moreover, note that it is impossible for two $x, y \in X$ to both be such that Rxy and have the same (finite) depth.

The second part of the theorem follows immediately from the first and Theorem 4.46. $\square$

Observe that by very similar reasoning one could prove that $\mathtt{BD_n}$ is axiomatised over $\mathtt{GL_R}$ by the rule

$$\mu_+\left(\overset{d_n}{\circ}\!\rightarrow\!\overset{d_{n-1}}{\bullet}\!\rightarrow \cdots \rightarrow\!\overset{d_0}{\bullet}, \emptyset, \{\{d_{n-1}\}, \dots, \{d_0\}\}\right).$$

In this case the map f constructed in the proof of direction ($\Leftarrow$) would actually be stable. Applying Theorem 4.46 and then Theorem 4.43 to this result would yield the axiomatisation stated in Theorem 4.49.

§4.4 Chapter Summary

We summarise the main original contributions of this chapter in the following list.

- We developed pre-stable canonical rules for rule systems in $\mathsf{NExt}(\mathtt{KM}_R)$ and $\mathsf{NExt}(\mathtt{GL}_\mathtt{R})$.
- We generalised our main technical lemma to the present setting (Lemma 3.45), and applied it to establish the Kuznetsov-Muravitsky theorem and its generalisation to rule systems. Once again, our proofs where smooth generalisations of arguments seen in previous chapters.
- We gave new axiomatic characterisations of the σ, ρ on rule systems via pre-stable canonical rules, extending the axiomatisation results of §§ 2.3.3 and 3.3.3.

Conclusions and Future Work

This thesis presented a new approach to research on modal companions and notions in the vicinity. Our techniques have proved effective. With only minor adaptations to a fixed collection of techniques, we provided a uniform treatment of the theories of modal and tense companions, and of the Kuznetsov-Muravitsky isomorphism. We both offered alternative proofs of classic theorems, such as the Blok-Esakia theorem, and established new results, most notably analogues of the Blok-Esakia theorem and the Dummett-Lemmon conjecture.

Furthermore, our work also contributed to the theory of stable canonical rules. We generalised si and modal stable canonical rules to the bsi and tense setting, and introduced pre-stable canonical rules for both msi and modal rule systems. More significantly, our application of stable (and pre-stable) canonical rules highlights an important aspect in which the latter perform at least as well as Zakharyaschev-Jerábek canonical formulae and rules.

Above all, however, the significance of our work is methodological. The techniques presented in this thesis are based on a blueprint easily applicable across signatures. Stable canonical rules can be formulated for any class of algebras which admits a locally finite expandable reduct in the sense of Ilin [2018, Ch. 5], and once stable canonical rules are available there is a clear recipe for adapting our strategy to the case at hand. We propose that further research be done in this direction, in particular addressing the following topics.

Firstly, for reasons of space we have not addressed the full theory of modal companions of msi deductive systems, as developed in Wolter and Zakharyaschev [1998, 1997]. We are confident that our techniques can recover several of the main known results in this area, and generalise them to rule systems. We hope that further work will confirm this.

Secondly, de Groot et al. [2021] recently proved an analogue of the Blok-Esakia theorem for extensions of the *Heyting-Lemmon logic*, which expands superintuitionistic logic with a strict implication connective. Our techniques could be applied to generalise this result to rule systems, and more generally to develop a rich theory of modal companions of deductive systems over the Heyting-Lemmon logic.

Thirdly, and most ambitiously, Goldblatt [1974] formulated a Gödel-style translation giving a full and faithful embedding of the propositional logic $\mathtt{O}$ of all *ortholattices* into the *Browerian modal logic* $\mathtt{B} = \mathtt{K} \oplus \Box p \to p \oplus p \to \Box\Diamond p$. To the best of our knowledge, the theory of modal companions of extensions of $\mathtt{O}$ (which include quantum logics) has not been developed, and in particular it is unknown whether Goldblatt's tranlsation gives rise to an analogue of the Blok-Esakia theorem. If a suitable expandable locally finite reduct of ortholattices can be found, stable canonical rules for rule systems over $\mathtt{O}$ can be developed, and a clear strategy for attacking the problem just mentioned becomes available.

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