Symbiosis and Compactness Properties

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written by

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Abstract

We investigate connections between model-theoretic properties of extensions of first-order logic and set-theoretic principles. We build on work of Bagaria and Väänänen in [1], and of Galeotti, Khomskii and Väänänen in [7] and [8], which used the notions of *symbiosis* and *bounded symbiosis* between a logic $\mathcal{L}$ and a predicate of set theory R, respectively, to show that if $\mathcal{L}$ and R are (boundedly) symbiotic, (upwards) Löwenheim-Skolem properties of $\mathcal{L}$ are equivalent to certain (upwards) reflection principles involving R. Similarly, we consider whether under the assumption of symbiosis compactness properties of $\mathcal{L}$ are related to some set-theoretic principle involving R.

For this purpose, we give a thorough introduction to symbiosis and the concepts from abstract model theory and set theory needed in its study. We further give a proof of a characterization of compactness properties of $\mathcal{L}$ in terms of extensions of specific partial orders stated by Väänänen in [17]. We use this and the novel concept of (R, κ)-extensions to formulate a set-theoretic principle which describes that in classes which are definable under the usage of R there exist (R, κ)-extensions with upper bounds for such partial orders. We show that this principle is related to compactness properties of a logic $\mathcal{L}$ symbiotic to R.

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Contents

1 Introduction

The notion of *symbiosis* was introduced by Jouko Väänänen (see e.g. [18]) to compare definability in extensions of first-order logic and in set theory. At the core of this comparison lies the question, whether a class $\mathcal{K}$ of structures in a fixed language is definable in a logic and in set theory. By *language*, here we mean a set of relation, function and constant symbols, and by *structure*, a set equipped with interpretations of these symbols, as in first-order model theory. Now we say that a logic *defines* or *axiomatizes* such a class $\mathcal{K}$, if there is a sentence φ of the logic such that the models of φ are precisely the structures in $\mathcal{K}$. Similarly, a formula in the language of set theory $\Phi(x)$ defines $\mathcal{K}$ if $\Phi(\mathcal{A})$ holds if and only if $\mathcal{A} \in \mathcal{K}$.

To give an example, consider the language $\{<\}$ consisting of one binary relation symbol. Let $\mathcal{K} = \{(A, <^A) \colon <^A$ is a linear order on $A\}$ be the class of all $\{<\}$-structures where $<$ is interpreted to be a linear order. As the axioms of linear orders are expressible by a first-order statement, there is a sentence φ of first-order logic such that

$$(A, <^A) \models \varphi \text{ iff } (A, <^A) \in \mathcal{K}.$$

Similarly, there is a formula in the language of set theory $\Phi(x)$ such that $\Phi((A, <^A))$ holds if and only if $(A, <^A) \in \mathcal{K}$, so $\mathcal{K}$ is definable both in first-order logic and in set theory.

This is not accidental: The logics we deal with in this thesis will be defined in set theory themselves. I.e., we will give a precise formalization in ZFC of what we understand under the term "logic". For this reason, any class that will be definable by a logic, will also be definable in set theory. Contrary to this, not any class definable in set theory has to be axiomatizable in some logic we might be considering. For example, as is well known, the class of all well-orders is *not* axiomatizable by a first-order sentence. It is, on the other hand, definable by a formula in the language of set theory.

However, when defining classes of structures, we can restrict ourselves to use formulas of set theory only up to a certain complexity level Γ. We will see that it indeed does happen, that the classes definable by Γ-formulas are precisely the ones axiomatizable in a logic $\mathcal{L}$. If this is the case, we speak of *symbiosis*. The level Γ on which this occurs often turns out to be the one of so called $\Delta_1(R)$-formulas for a predicate R of set theory. Thus, symbiosis is formulated as a relation between R and $\mathcal{L}$. The logics for which this happens turn out to be extensions of first-order logic.

While the fact that there is equidefinability between such strong logics and set theory is interesting in its own right, in the last years, symbiosis proved to be useful in ways beyond showing this type of equivalences. First Bagaria and Väänänen in [1], and then Galeotti, Khomskii and Väänänen in [8], used the notion of symbiosis to draw systematic connections between model-theoretic properties of $\mathcal{L}$ and set-theoretic principles involving a predicate R symbiotic to $\mathcal{L}$.

That model-theoretic properties of extensions of first-order logic have interesting connections to set theory, is well known: It turns out that often a logic $\mathcal{L}$ having some property is a large cardinal assumption. For example, if κ is an uncountable cardinal and we consider the logic $\mathcal{L}_{\kappa\omega}$, i.e., the extension of first-order logic that allows for infinite conjunctions and disjunctions over sets of formulas of all sizes smaller than κ, then it is not provable in ZFC that a logic of this form satisfies a suitable analogue of the compactness theorem for first-order logic. Under the assumption of the existence of some specific large cardinals however, one can show that there are logics $\mathcal{L}_{\kappa\omega}$ for which such an analogue holds.

The results from [1] and [8] now systematize such connections. In [1] it was shown that a logic $\mathcal{L}$ having certain downwards Löwenheim-Skolem properties, is, under the assumption of $\mathcal{L}$ being symbiotic to R, equivalent to some reflection principle that involves R. Similarly [8] shows under the stronger assumption of so called *bounded* symbiosis between $\mathcal{L}$ and R, that upwards Löwenheim-Skolem properties of $\mathcal{L}$ are equivalent to an upwards directed reflection principle involving R. Both results have applications in large cardinal theory: By using the equivalence between the reflection principles and the Löwenheim-Skolem properties, one can use the former in calculations of the large cardinal strength of the latter.

The goal of this thesis is to continue the investigation of such connections between model-theoretic properties of logics and set-theoretic properties of predicates symbiotic to them. In particular, we will consider compactness properties, as these are the most important properties from model theory not covered in [1] and [8].

We assume that the reader is familiar with model theory of first-order logic and with set theory, roughly to the extent of an introductory graduate level course to the respective topic. On the side of set theory, we further assume that the reader is familiar with concepts such as models of set theory, the Lévy hierarchy, absoluteness and the Reflection Theorem. We will make reference to some results about large cardinals, however they are only brought up in examples and no results or proofs of this thesis require any knowledge of this topic. We do not assume any knowledge about *abstract model theory*, i.e., the study of model theory of general logics, nor about symbiosis.

For this reason, chapter 2 will introduce most of the general preliminaries that are needed: We introduce *regular abstract logics*, which is the standard notion that formalizes what we understand by the term "logic", including examples that we will study in the rest of the thesis. As this is important when being concerned with set-theoretic definability, we will pay extra attention to the complexity of the set-theoretic formulas we are using to define certain notions. Further we will introduce some concepts that are important in the study of symbiosis, e.g., $\Delta_1(R)$-formulas of set theory and the Δ-closure of a logic. We also use this chapter to remind the reader of some basic results from set theory we will need.

In chapter 3 we introduce the notion of symbiosis and study examples of this concept. More specifically, we will study two types of symbiosis. To connect Löwenheim-Skolem properties to reflection principles via symbiosis, it is sufficient to restrict attention to structures in finite vocabularies. Thus for this case, a type of symbiosis is sufficient that deals with set-theoretic definability without any parameters. We will call this *r-symbiosis*. However, when investigating compactness properties, large vocabularies are essential. To talk about those in set theory, one has to allow the use of parameters, something taken care of in what we will call *p-symbiosis*. The latter is hereby a slight variation of the notion of symbiosis in [18].

Chapter 4 is devoted to the study of compactness properties. We will see that most logics are not compact in the sense that first-order logic is. We will thus consider generalizations of the compactness property of first-order logic, most importantly one known as (∞, κ)-compactness for a cardinal κ. In particular, to formulate a set-theoretic principle that is related to compactness properties via symbiosis, we will prove some "abstract" characterization of (∞, κ)-compactness in terms of $\mathcal{L}$-extensions of partial orders $(A, <^A)$ that include upper bounds for $(A, <^A)$. This refers to the notion of an $\mathcal{L}$-embedding we will introduce as well. We relate it to a novel concept we will call (R, λ)-*embedding* for a cardinal λ.

This characterization of compactness and the notion of an (R, λ)-embedding will finally be used in chapter 5 to formulate a set-theoretic principle $\mathrm{EEP}^{\lambda}_{\kappa}(R)$ dependent on a predicate R of set theory and cardinals κ and λ. We will show that this principle is related to (∞, κ)-compactness of a logic symbiotic to R.

As we are dealing with logics which are defined in set theory, it is important to distinguish between formulas in our meta language and those of the logics, the latter being formal objects defined in the meta language. We will denote formulas in the (meta) language of set theory by capitalized Greek letters Φ, Ψ etc. They are built in the usual way using the membership symbol $\in$. Formulas and sentences of the object languages, i.e., the logics we consider, will be denoted by small Greek letters φ, ψ etc. Notice that the latter will always be sets, so formal objects, while the statements of the meta language are not. Of course, as the language of set theory is first-order, and as we will formalize first-order logic in set theory, if we consider any formula of our meta language Φ, there will be a formal analogue of Φ in first-order logic, i.e., a *set* φ representing the meta object Φ. In this case we fix a binary relation symbol E (which is a set as well), and denote this formal analogue φ by Φ_E (or mention that we do this), which is the set that is gained by constructing a formula of first-order logic in the same way that Φ is build up, writing E instead of $\in$. In any case, it should always be clear whether we currently deal with a formula in the meta language or with a set-theoretic object.

If we say that $\mathcal{M}$ is a model of set theory, we mean that $\mathcal{M} = (M, E^M)$ is a class M, equipped with a binary relation E^M on M. By "class" we mean that M can be a proper class, and in this case E^M can be a proper class relation, or that M is a set. We denote membership by $\in$, so in the same way as the symbol used in formulas of the language of set theory. Of course, M is called transitive if for all $x \in M$, we have $x \subseteq M$.

We will be dealing with *many-sorted* logics, i.e., our vocabularies will include sort symbols, and every relation, function and constant symbol will be equipped with a specification of the sorts involved in them. For example, we could have a language $\{s_1, s_2, R\}$ with two sort symbols s_1 and s_2 and a relation symbol R that relates sort s_1 to sort s_2. A structure in that language is then a tuple $\mathcal{A} = (A_1, A_2, R^{\mathcal{A}})$ where A_1 and A_2 are non-empty sets, the *domains in sort* s_1 and s_2, respectively, and $R^{\mathcal{A}} \subseteq A_1 \times A_2$ is a relation between the two domains. We will give a precise definition later.

We are assuming ZFC throughout. Often we will work with models of some finite fragment of ZFC, which we then assume to be sufficiently large to carry out the argument at hand. In this case, by ZFC* we denote such a finite fragment and by ZFC^{-*} we denote a finite fragment of ZFC minus the power set axiom.

2 Formalizing Logics

There are at least two things one could mean by *formalizing logics*. First, one could give a representation of some specific logic in some formal system, say ZFC. And second, one could give a general formal notion of what to understand by the term *logic*. The latter one is necessary for our purposes. As we want to prove results that apply to a wide range of logics, we need to fix what we talk about. Our notion will be what we will call a *regular abstract logic*. It is the standard notion from abstract model theory, as used throughout [4].

The former one is most often skipped. While in logic we want the objects we deal with, like vocabularies, logical symbols or formulas, to be representable in set theory, we often do not care about what concrete sets these objects are. For example, say in first-order logic, a formula is some finite string of symbols and it is clear that we can give some representation of this as a set. In contexts of symbiosis, however, this is crucial. As we are comparing definability in set theory and in logics, it is important to see with what set-theoretic means one can define a logic. More precisely, one considers what complexity the formulas of set theory have that are needed to define the logic, e.g., its satisfaction relation. Thus we want to give at least some idea of which specific representations of logics we deal with and on which complexity level they can be defined. We will try to find a balance here between what we state being precise yet not superfluous. As these will be important for any logic, we will give specific codings of what vocabularies and non-logical symbols, like relation and function symbols, are. Heading to concrete logics, we will still be somewhat precise when it comes to first-order logic, but will mostly refer the reader to [5], where such a definition of first-order logic in set theory has been carried out. The main thing to take away from this is that it can be done in a Δ_1-way in set theory. Finally, for other logics, we will mostly only indicate with which complexity levels we deal with.

We will proceed as follows. In section 1.1, we will introduce the classes of $\Delta_n(R)$, $\Sigma_n(R)$ and $\Pi_n(R)$-formulas for a set-theoretic predicate R, which are a generalized way to measure the complexity of formulas in the language of set theory, the Lévy hierarchy being a special case of it. This generalization will become mostly relevant in later chapters when dealing with symbiosis, but we use this opportunity to give a reminder of the usual results about the Lévy hierarchy. In section 1.2 we will introduce our representations of vocabularies and structures in ZFC. Sections 1.3 and 1.5 give the definition of an abstract regular logic, with the most important example $\mathcal{L}_{\omega\omega}$, our representation of first-order logic in ZFC, being defined in section 1.4. We want to consider some more examples of abstract logics in section 1.6, most of which will accompany us throughout. Getting more specific, we will introduce different notions of *dependence numbers* which measure how large the sentences of a logic can be in section 1.7, as this will be important later. Finally, we will look at the Δ-*closure* of a logic in section 1.8, a concept that we cannot do without when working with symbiosis.

2.1 The Generalized Lévy Hierarchy

The standard and most useful way to measure the complexity of formulas in the language of set theory is the *Lévy hierarchy* of Δ_n, Σ_n and Π_n-formulas introduced in [11]. We assume that the reader is familiar with it and its basic properties. In contexts of symbiosis, a slightly more general concept is used, which is able to give a more fine-grained picture between Δ_n and Δ_{n+1}-formulas, namely the hierarchy of $\Delta_n(R)$, $\Sigma_n(R)$ and $\Pi_n(R)$-formulas for a set-theoretic predicate R. As throughout this thesis, basic properties of formulas in the Lévy hierarchy will be important, we use this opportunity to recall some of these properties, alongside introducing the notions of $\Delta_n(R)$-formulas we need later. The R here, as well as in contexts of symbiosis, refers to a predicate R of set theory. By this we mean the following.

Definition 2.1.1. A *set-theoretic predicate* R is a formula $\Phi(x_1, \dots, x_n)$ in the language of set theory with free variables among $x_1, \dots, x_n$. We will also say $\Phi(x_1, \dots, x_n)$ *defines* R.

We will be somewhat lenient with this definition and speak of the same predicate R for formulas $\Phi(x_1, \dots, x_n)$ and $\Psi(x_1, \dots, x_n)$ whenever $\mathrm{ZFC} \vdash \Phi(x_1, \dots, x_n) \leftrightarrow \Psi(x_1, \dots, x_n)$. Also we will write $R(x_1, \dots, x_n)$ in formulas of set theory.

Examples that will be important throughout this thesis are the predicates $\mathrm{Cd}(x)$ and $\mathrm{Pow}(x, y)$ defined by

$$\mathrm{Cd}(x) \leftrightarrow x \text{ is a cardinal}$$

and

$$\mathrm{Pow}(x, y) \leftrightarrow y \text{ is the power set of } x.$$

As we will most often do, we wrote down the content of the formulas defining Cd and Pow, respectively, in ordinary English. Further we will consider the predicate "x is the empty set" and will denote this as $\emptyset$.

Now the following defines what we will call the *generalized Lévy hierarchy* for a predicate R of set theory. It is defined as the usual Lévy hierarchy, with the only difference being that at the basic level of $\Delta_0(R)$-formulas, besides atomic formulas one is allowed to use the predicate R freely.

Definition 2.1.2. Let R be a set-theoretic predicate. A formula $\Phi(x_1, \dots, x_k)$ is called $\Delta_0(R)$, $\Sigma_0(R)$ and $\Pi_0(R)$ iff it is generated using the following constructions:

1. If Φ is atomic, i.e., $x_i = x_j$ or $x_i \in x_j$.
2. If $\mathrm{ZFC} \vdash R \leftrightarrow \Phi$.
3. If Ψ is $\Delta_0(R)$ and $\Phi = \exists y \in x_i \Psi$ or $\Phi = \forall y \in x_i \Psi$.
4. If Ψ and Θ are $\Delta_0(R)$, and Φ is $\neg\Psi$, $\Psi \wedge \Theta$, $\Psi \vee \Theta$, $\Psi \to \Theta$ or $\Psi \leftrightarrow \Theta$.

Now suppose that for a natural number n, the classes of $\Delta_n(R)$, $\Sigma_n(R)$ and $\Pi_n(R)$-formulas are already defined. Then $\Phi(x_1, \dots, x_k)$ is called

(i) $\Sigma_{n+1}(R)$ iff $\mathrm{ZFC} \vdash \Phi(x_1, \dots, x_k) \leftrightarrow \exists x_i \Psi(x_1, \dots, x_k)$ for a $\Pi_n(R)$-formula Ψ.

(ii) $\Pi_{n+1}(R)$ iff $\mathrm{ZFC} \vdash \Phi(x_1, \dots, x_k) \leftrightarrow \forall x_i \Psi(x_1, \dots, x_k)$ for a $\Sigma_n(R)$-formula Ψ.

(iii) $\Delta_{n+1}(R)$ iff Φ is both $\Sigma_{n+1}(R)$ and $\Pi_{n+1}(R)$.

We say that a class $\mathcal{K}$ is $\Delta_n(R)$-*definable* iff there is a $\Delta_n(R)$-formula $\Phi(x)$ in the language of set theory such that

$$\forall x(\Phi(x) \leftrightarrow x \in \mathcal{K}).$$

We say $\mathcal{K}$ is $\Delta_n(R)$-*definable with parameters* $p_1, \dots, p_n$ iff there is a $\Delta_n(R)$-formula $\Phi(x_1, \dots, n_{n+1})$ and $p_1, \dots, p_n$ are sets such that

$$\forall x(\Phi(p_1, \dots, p_n, x) \leftrightarrow x \in \mathcal{K}).$$

Analogously we define $\Sigma_n(R)$- and $\Pi_n(R)$-definability (with parameters). We also say that $\mathcal{K}$ *is* $\Delta_n(R)$ if it is $\Delta_n(R)$-definable and similar in the other cases.

For the reader who is not familiar with the Lévy hierarchy: It is obtained analogously to the above definition by deleting clause 2, which includes the only usage of R. Notice that if $R = \emptyset$, then the $\Delta_n(\emptyset)$-formulas are precisely the Δ_n-formulas, as being the empty set is Δ_0-definable.

As for the usual Lévy hierarchy, an important usage of these classes of formulas lies in establishing absoluteness results. We assume that the reader is familiar with the notion of absoluteness. Nevertheless we will state it here. To do so, we first have to introduce two other notions.

Definition 2.1.3. If Φ is a formula in the language of set theory and M is a class, we define the *relativization* Φ^M *of* Φ *to* M by relativizing all quantifiers in Φ to M. I.e., we inductively define the relativization by the following rules:

1. If Φ is atomic, then $\Phi^M = \Phi$.
2. If $\Phi = \Psi * \Theta$, then $\Phi^M = \Psi^M * \Theta^M$, where $*$ is one of $\wedge, \vee, \rightarrow$ and $\leftrightarrow$.
3. If $\Phi = \neg\Psi$, then $\Phi^M = \neg(\Psi^M)$.
4. If $\Phi = \exists x \Psi$, then $\Phi^M = \exists x(x \in M \wedge \Psi^M)$.
5. If $\Phi = \forall x \Psi$, then $\Phi^M = \forall x(x \in M \rightarrow \Psi^M)$.

We can think of the relativization of Φ to M to express that the class M "thinks" Φ is true. If $\Phi^M(a)$ holds of some set $a \in M$, then M, as a model of set theory, thinks that Φ really holds of a. Note that if M is a proper class, then it is nothing else than a property of sets defined by a formula $\Theta(x)$, so, e.g., the formula Φ^M in condition 4 above is really $\exists x(\Theta(x) \wedge \Psi^M)$.

Absoluteness really comes into play when talking about transitive $\in$-models.

Definition 2.1.4. Let (M, E^M) be a model of set theory. Then (M, E^M) is called an *$\in$-model* iff $E^M = \in \restriction M$. (M, E^M) is called a *transitive $\in$-model* iff (M, E^M) is an $\in$-model and M is a transitive class.

As in the above E^M is just the membership relation restricted to M, we will often just write $(M, \in)$ for $\in$-models. Now we can introduce absoluteness.

Definition 2.1.5. Let $(M, \in)$ be an $\in$-model and Φ a formula of set theory. We say Φ is

(i) *downwards absolute for* $(M, \in)$ iff $\Phi \to \Phi^M$.

(ii) *upwards absolute for* $(M, \in)$ iff $\Phi^M \to \Phi$.

(iii) *absolute for* $(M, \in)$ iff $\Phi^M \leftrightarrow \Phi$.

If we fix the class $\mathcal{K}$ defined by Φ we also say that $\mathcal{K}$ is (downwards/upwards) absolute for $(M, \in)$.

So a formula Φ is absolute for some model, if Φ relativized to M holds of some $a \in M$, if and only if it holds in the universe. I.e., what M thinks of its elements regarding whether Φ holds of them or not is precisely what is actually true in the set-theoretic universe. Note also that Φ is absolute iff it is downwards and upwards absolute.

The special thing about $\in$-models is, that "their" membership relation corresponds to the real $\in$-relation, i.e., an $\in$-model thinks that a is an element of b iff $a \in b$. One could say $\in$-models are *correct* towards the membership relation. Similarly, we want to define a notion of being R-correct for any predicate R of set theory.

Definition 2.1.6. Let R be a predicate of set theory defined by a formula $\Phi(x_1, \ldots, x_n)$. An $\in$-model $(M, \in)$ is called *R-correct* iff for all $m_1, \ldots, m_n \in M$:

$$\Phi^M(m_1, \ldots, m_n) \text{ iff } R(m_1, \ldots, m_n).$$

So similar to $\in$-models and membership, a model of set theory is R-correct if it interprets the predicate R as it is interpreted in the universe.

The most important result about Δ_0-formulas is that they are absolute for transitive $\in$-models. For a proof of this fact see e.g. [9, Lemma 12.9]. Completely analogously under usage of R-correctness one proves the following.

Proposition 2.1.7. $\Delta_0(R)$-formulas are absolute for transitive R-correct $\in$-models. $\square$

Then the standard proof (compare e.g. [9, p.185]) of Σ_1 and Π_1-formulas being upwards and downwards absolute, respectively, carries over to $\Sigma_1(R)$ and $\Pi_1(R)$. Thus we get

Proposition 2.1.8. $\Sigma_1(R)$ formulas are upwards absolute for transitive R-correct $\in$-models. $\Pi_1(R)$-formulas are downwards absolute for transitive R-correct $\in$-models. $\square$

And finally, of course this implies

Corollary 2.1.9. $\Delta_1(R)$-formulas are absolute for transitive R-correct $\in$-models. $\square$

As Δ_0- and Δ_1-formulas are absolute for all transitive $\in$-models, we also simply say that they are absolute (and drop the qualification "for transitive $\in$-models"). Similarly, if $\mathcal{K}$ is the class defined by a formula which is absolute for transitive $\in$-models, we say that $\mathcal{K}$ is absolute.

Similarly to the above results, one can show that if $\mathcal{K}$ is defined by a $\Delta_0(R)$-formula $\Phi(x, p_1, \ldots, p_n)$ with parameters $p_1, \ldots, p_n$, i.e., $\forall x(x \in \mathcal{K} \leftrightarrow \Phi(x, p_1, \ldots, p_n))$, then $\mathcal{K}$ is absolute for all transitive (R-correct) $\in$-models containing $p_1, \ldots, p_n$. So for all transitive (R-correct) $\in$-models $(M, \in)$, if $p_1, \ldots, p_n \in M$, then

$$\forall x(\Phi^M(x, p_1, \ldots, p_n) \leftrightarrow x \in K).$$

2.2 Vocabularies and Structures

Logics, how we want the term to be understood, as in first-order model theory, take a set of symbols, called *vocabulary, signature* or *language*, and return a set of sentences over that vocabulary, which can either hold or not hold in a structure which interprets the symbols. Often there is no special attention paid to what the symbols are that we talk about: It is sufficient to know that we can form enough of them for all our purposes.

In contexts of symbiosis, what precisely these objects are becomes important, as we want to talk about which classes of structures over a given vocabulary are definable. For example, we sometimes want that for some vocabulary τ, some class of τ-structures is definable by a $\Delta_1(R)$-formula. And for any class of τ-structures to be $\Delta_1(R)$, also τ has to be $\Delta_1(R)$-definable. To be precise about such matters, we will give a specific coding that tells us which sets we understand to be relation, function and constant symbols or vocabularies and we will make sure that those are Δ_1-notions. We will most often not write down the precise Δ_1-formula that defines the concepts we introduce, nor give other explicit proofs of this fact. Instead, in the next lemma we mention properties that are well known to be defined by Δ_0-formulas and note that what we define can be achieved as a combination of those. For a prove of the lemma, see any comprehensive book on set theory, e.g. [9].

Lemma 2.2.1. The following are defined by a Δ_0-formula of set theory, with n and i being any natural numbers:

> $x \subseteq y$, x is an ordered pair, x is an n-tuple, x is the i-th element of the n-tuple y, x is an ordinal, x is a natural number, $x = 0$, $x = 1$, $x = 2$, $\ldots$, $x = \omega$, x is a relation on y, x is a function, x is the domain of a function y, x is transitive. □

With this at hand, we can define what we mean by relation, function and constant symbol. Note that we will use many-sorted logics, so our definition also includes sort symbols. We orientate ourselves at [2] but our definition includes technical differences.

Definition 2.2.2. We introduce the following notions.

(i) For each $n \in \omega$, we call the pair $(0, n)$ a *sort symbol*. We say n is a sort.

(ii) For each set a and $n \in \omega$, we call $(1, (n, a))$ a *constant symbol of sort* n.

(iii) For each set a and for $n_1, \dots, n_i \in \omega$, we call $(2, (n_1, \dots, n_i, a))$ a *relation symbol of arity* i *between the sorts* $n_1, \dots, n_i$.

(iv) For each set a and for $n_1, \dots, n_{i+1} \in \omega$, we call $(3, (n_1, \dots, n_{i+1}, a))$ a *function symbol of arity* i *from the sorts* $n_1, \dots n_i$ *to the sort* n_{i+1}.

(v) We let # be the arity-function, so $\#(R) = n$ for every relation symbol of arity n and $\#(f) = n$ for every function symbol of arity n.

(vi) Further we let conf be the function that returns the *configuration of sorts* of a symbol, i.e.,
   1. if $c = (1, (n, a))$ is a constant symbol, then $\mathrm{conf(c)} = n$.
   2. if $R = (2, (n_1, \dots, n_i, a))$ is a relation symbol then $\mathrm{conf}(R) = (n_1, \dots, n_i)$.
   3. if $f = (3, (n_1, \dots, n_{i+1}, a))$ is a function symbol, then $\mathrm{conf}(f) = (n_1, \dots, n_{i+1})$.

We call $\mathrm{conf}(R)$ and $\mathrm{conf}(f)$ the *configuration of* R and f, respectively.

The sets a in this definition are not supposed to have any bearing on the "meaning" of the symbols we defined, they are simply used to generate enough symbols (class sized many, to be precise) so that we do not run out of them. Notice that contrary to the other symbols, there are only countably many sort symbols. We stick to this, as we won't need to look at structures in more than finitely many sorts. Also notice, that, e.g., a set like $H(\kappa)$ for any infinite cardinal κ, which is closed under taking pairs and contains all the natural numbers, contains $|H(\kappa)|$-many relation, constant and function symbols, respectively.

With symbols at hand, we can define vocabularies.

Definition 2.2.3. A *vocabulary* τ is a set consisting of at least one but only finitely many sort symbols, and of (arbitrarily many) relation, function and constant symbols that only involve the sorts present in τ.

As all symbols are finite tuples and any property specific to a symbol can be read off of it, we easily get using Lemma 2.2.1:

Proposition 2.2.4. Let i and $n_1, \dots, n_{i+1}$ be natural numbers. Then the following hold.

1. The following are definable by a Δ_0-formula:

   x is a sort symbol, x is a relation symbol, x is a function symbol, x is a constant symbol, x is a relation symbol of arity i, x is a function symbol of arity i, x is a relation symbol of arity i with configuration $(n_1, \dots, n_i)$, x is a function symbol of arity i with configuration $(n_1, \dots, n_{i+1})$.

2. The functions # and conf are definable by a Δ_0-formula.

3. "x is a vocabulary" is definable by a Δ_1-formula.

In particular, all the above are absolute for transitive $\in$-models.

Proof. Items 1 and 2 easily follow from Lemma 2.2.1. Similarly, we could show that "x is a set of sort, relation, function and constant symbols" is Δ_0. As being finite is Δ_1, to express that there are finitely many sort symbols in a set is Δ_1, so "x is a vocabulary" is Δ_1. □

Finally, we can define what a τ-structure for a vocabulary τ is.

Definition 2.2.5. Let τ be a vocabulary including the sorts $s_1, \ldots, s_n$. A tuple $\mathcal{A} = (A_1, \ldots, A_n, F)$ is called a *τ-structure* iff $A_1, \ldots, A_n$ are non-empty sets called the *domains for sorts* $s_1, \ldots, s_n$, respectively, and F is a function with domain τ such that

1. if $c = (1, (s_i, a)) \in \tau$ is a constant symbol, then $c^{\mathcal{A}} := F(c) \in A_i$.
2. if $R \in \tau$ is a relation symbol and $\mathrm{conf}(R) = (s_{i_1}, \ldots, s_{i_k})$ is a relation symbol then $R^{\mathcal{A}} := F(R) \subseteq \Pi_{j=1}^{k} A_{s_{i_j}}$.
3. if $f \in \tau$ is a function symbol and $\mathrm{conf}(f) = (s_{i_1}, \ldots, s_{i_{k+1}})$, $f^{\mathcal{A}} := F(f)$ is a function $\Pi_{j=1}^{k} A_{s_{i_j}} \longrightarrow A_{s_{i_{k+1}}}$.

If $s \in \tau$ is a sort symbol and $\mathcal{A}$ a τ-structure we will denote the domain for sort s of $\mathcal{A}$ by A_s.

If $\mathcal{A} = (A_{s_1}, \ldots, A_{s_n}, F)$ is a τ-structure and τ includes sort symbols $s_1, \ldots, s_n$ and, say, a relation symbol R, a function symbol f and a constant symbol c, we will most often write the more customary $(A_{s_1}, \ldots, A_{s_n}, R^{\mathcal{A}}, f^{\mathcal{A}}, c^{\mathcal{A}})$ to denote $\mathcal{A}$, and mean that the values of F are given by $F(R) = R^{\mathcal{A}}, F(f) = f^{\mathcal{A}}$ and $F(c) = c^{\mathcal{A}}$.
Similarly to the above, using Lemmas 2.2.1 and 2.2.4 we get the following.

Proposition 2.2.6. "x is a τ-structure" is definable by a Δ_1-formula that uses τ as a parameter. In particular, it is absolute for transitive $\in$-models that contain τ. □

If τ is a vocabulary and $\mathcal{K}$ a class of τ-structures, we say that $\mathcal{K}$ is $\Delta_1(R)$ or $\Delta_1(R)$-*definable* if there is a $\Delta_1(R)$-formula $\Phi(x)$ in the language of set theory such that $\Phi(\mathcal{A})$ holds iff $\mathcal{A} \in \mathcal{K}$ (and similarly for other levels of the generalized Lévy hierarchy). In particular, in contrast to the above proposition, if the vocabulary τ is not itself $\Delta_1(R)$-definable, the class of all τ-structures is not $\Delta_1(R)$ (even though it is Δ_1 with parameters in $\{\tau\}$).
We will often consider *expansions* and *restrictions* of τ-structures $\mathcal{A}$ to super- or subsets of τ, respectively. I.e., if τ is a vocabulary, $\mathcal{A} = (A_{s_1}, \ldots, A_{s_n}, F)$ a τ-structure and $\sigma \subseteq \tau$, the *restriction $\mathcal{A} \upharpoonright \sigma$ of $\mathcal{A}$ to σ* is the σ-structure $(A_{t_1}, \ldots, A_{t_k}, F \upharpoonright \sigma)$ for $\{t_1, \ldots, t_k\} = \{x \in \sigma \colon x \text{ is a sort symbol}\}$. $\mathcal{A} \upharpoonright \sigma$ is the structure that "forgets" about the interpretations of all symbols $\notin \tau$, including all the domains A_s for all sorts $s \notin \sigma$. If $\sigma \supseteq \tau$ and $\mathcal{B}$ is a σ-structure, then $\mathcal{B}$ is called an *expansion of* $\mathcal{A}$ if $\mathcal{B} \upharpoonright \tau = \mathcal{A}$. We also say that τ is an expansion of σ.

If τ is an expansion of σ, say by a relation symbol R, a function symbol f and a constant symbol c, and $\mathcal{A}$ is a σ-structure, we often denote an expansion of $\mathcal{A}$ to a τ-structure by writing $(\mathcal{A}, R^{\mathcal{A}}, f^{\mathcal{A}}, c^{\mathcal{A}})$. By that we mean the τ-structure which expands $\mathcal{A}$ by interpreting R as $R^{\mathcal{A}}$, f as $f^{\mathcal{A}}$ and c as $c^{\mathcal{A}}$.

If $\mathcal{A}$ is a τ-structure, we write A for the union of the domains in all sorts from τ, i.e.,

$$A = \bigcup \{A_s \colon s \in \tau \text{ is a sort symbol}\}.$$

We call $|A|$ the *cardinality of* $\mathcal{A}$ and also write $|\mathcal{A}|$ for this. Notice that if $\sigma \subseteq \tau$, then $|\mathcal{A} \upharpoonright \sigma|$ can be smaller than $|\mathcal{A}|$, if τ contains more than one sort.

If $\mathcal{A}$ and $\mathcal{B}$ are τ-structures and $f : A \longrightarrow B$ a map, the notion of f being an *embedding* and an *isomorphism* are defined in the obvious way. In particular, embeddings respect sort symbols, i.e., if $a \in A_s$, then $f(a) \in B_s$ for every sort symbol $s \in \tau$. We write $f : \mathcal{A} \longrightarrow \mathcal{B}$ to mean that f is an embedding and $\mathcal{A} \cong \mathcal{B}$ to mean that there is an isomorphism between $\mathcal{A}$ and $\mathcal{B}$. If there is an embedding from $\mathcal{A}$ to $\mathcal{B}$, we say that $\mathcal{B}$ is an *extension* of $\mathcal{A}$.

2.3 Abstract Logics

Having vocabularies and structures at hand, we are ready to define what we mean by "logic". What we will call an abstract logic $\mathcal{L}$ will contain two components: First, a function that for every vocabulary τ, returns a class $\mathcal{L}[\tau]$, whose objects we interpret as the $\mathcal{L}$-sentences over τ. Second, a relation $\models_{\mathcal{L}}$ between τ-structures and the elements of $\mathcal{L}[\tau]$, which we interpret as the satisfaction relation of $\mathcal{L}$. All the definitions we will give in this section are the standard ones from [6].

Remember that our specific coding of, e.g., constant symbols involves sets a, in the sense that for every a and sort n, the pair $(1, (n, a))$ is a constant symbol. We said we do not want this specific representation we chose to have bearing on what the symbols mean. To ensure this in the definition of abstract logics, we introduce renamings.

Definition 2.3.1. If τ and σ are vocabularies, a bijection $\rho : \tau \longrightarrow \sigma$ is called a *renaming* if it respects sort symbols and arities in the obvious way. For example, if $s_1, s_2 \in \tau$ are distinct sort symbols, then $\rho(s_1), \rho(s_2) \in \sigma$ are distinct sort symbols or if R is a binary relation symbol with configuration (s_1, s_2), then $\rho(R)$ is a binary relation symbol with configuration $(\rho(s_1), \rho(s_2))$.

Clearly, a renaming ρ and a τ-structure $\mathcal{A}$ induce a $\rho(\tau)$-structure on the domains of $\mathcal{A}$. We will call this structure $\mathcal{A}^\rho$. Now we introduce abstract logics.

Definition 2.3.2. An *abstract logic* $\mathcal{L}$ is a pair consisting of a (class) function that maps every vocabulary τ to a class $\mathcal{L}[\tau]$ called the *class of $\mathcal{L}$-sentences over* τ and a (class) relation $\models_{\mathcal{L}}$ called the *satisfaction relation of* $\mathcal{L}$ such that:

1. If $\mathcal{A} \models_{\mathcal{L}} \varphi$, then $\varphi \in \mathcal{L}[\tau]$ for some vocabulary τ and $\mathcal{A}$ is a τ-structure.

2. If $\sigma \subseteq \tau$ for vocabularies τ and σ, then $\mathcal{L}[\sigma] \subseteq \mathcal{L}[\tau]$.

3. If $\mathcal{A} \cong \mathcal{B}$ for τ-structures $\mathcal{A}$ and $\mathcal{B}$, then for every $\varphi \in \mathcal{L}[\tau]$:
$$\mathcal{A} \models_{\mathcal{L}} \varphi \text{ iff } \mathcal{B} \models_{\mathcal{L}} \varphi.$$
4. If $\varphi \in \mathcal{L}[\sigma]$ and $\sigma \subseteq \tau$ for vocabularies τ and σ, then for every τ-structure $\mathcal{A}$:
$$\mathcal{A} \models_{\mathcal{L}} \varphi \text{ iff } (\mathcal{A} \upharpoonright \sigma) \models_{\mathcal{L}} \varphi.$$
5. If $\rho : \tau \longrightarrow \sigma$ is a renaming, then for all $\varphi \in \mathcal{L}[\tau]$ there is a $\varphi^\rho \in \mathcal{L}[\sigma]$ such that for all τ-structures $\mathcal{A}$:
$$\mathcal{A} \models_{\mathcal{L}} \varphi \text{ iff } \mathcal{A}^\rho \models_{\mathcal{L}} \varphi^\rho.$$

We call 4 the *reduct property* and 5 the *renaming property.* If φ and φ^ρ are the sentences from 5, we say φ and φ^ρ are *equivalent up to renaming.*

This definition is very general and interprets the key components of logics to be the abstract relation $\models_{\mathcal{L}}$ between structures in some vocabulary τ and sentences of the logic over τ. Note that there is no restriction what kind of sets the $\mathcal{L}$-sentences are. To go quickly through the properties we demand of an abstract logic, 1 ensures that the relation $\models_{\mathcal{L}}$ holds between the correct kinds of objects. 2 says that every sentence formed over a small vocabulary σ will also be present when considering an expansion of σ. 3 demands that the logic treats isomorphic structures in exactly the same way. 4 tells us that if we have a sentence φ over some small vocabulary then adding interpretations of additional symbols to some structure, does not change its behaviour towards φ. Finally, 5 ensures that the logic cannot make use of our specific coding of vocabularies.

Note that if τ is a vocabulary, then every $\varphi \in \mathcal{L}[\tau]$ gives rise to a class of τ-structures $Mod^\tau_{\mathcal{L}}(\varphi)$, consisting of all τ-structures φ such that $\mathcal{A} \models_{\mathcal{L}} \varphi$. So
$$Mod^\tau_{\mathcal{L}}(\varphi) = \{\mathcal{A} \colon \mathcal{A} \text{ is a } \tau\text{-structure and } \mathcal{A} \models_{\mathcal{L}} \varphi\}.$$
We call $Mod^\tau_{\mathcal{L}}(\varphi)$ the *model class of* φ. By the isomorphism property, the model class of a sentence φ is always a proper class. Often we will write $\varphi \in \mathcal{L}$, meaning there is a vocabulary τ such that $\varphi \in \mathcal{L}[\tau]$. If $\varphi \in \mathcal{L}[\tau]$, we write $\models_{\mathcal{L}} \varphi$ to indicate that $\mathcal{A} \models_{\mathcal{L}} \varphi$ for all τ-structures $\mathcal{A}$. Often we will drop the subscript and superscript in $\models_{\mathcal{L}}$ and $Mod^\tau_{\mathcal{L}}(\varphi)$ and just write $\models$ and $Mod(\varphi)$. Then it should be clear from the context what is precisely meant.

Model classes allow to compare the strength of logics under some notion of definability:

Definition 2.3.3. Let $\mathcal{L}$ and $\mathcal{L}^*$ be abstract logics. A class $\mathcal{K}$ of τ-structures is called *definable in* or *by* $\mathcal{L}$ iff there is a $\varphi \in \mathcal{L}[\tau]$ with $Mod(\varphi) = \mathcal{K}$.

We say $\mathcal{L}$ is *at least as strong as* $\mathcal{L}^*$ iff for every vocabulary τ and every class $\mathcal{K}$ of τ-structures, if $\mathcal{K}$ is definable in $\mathcal{L}^*$, then $\mathcal{K}$ is also definable in $\mathcal{L}$. In this case we write $\mathcal{L}^* \leq \mathcal{L}$ and also say that $\mathcal{L}$ is an *extension* of $\mathcal{L}^*$.

Note that $\mathcal{L}^* \leq \mathcal{L}$ iff for every vocabulary τ and every $\varphi \in \mathcal{L}^*[\tau]$, there is a $\psi \in \mathcal{L}[\tau]$ with $Mod(\varphi) = Mod(\psi)$. Sometimes we will be lenient and assume that if $\mathcal{L}^* \leq \mathcal{L}$, this ψ is given by φ, so $\mathcal{L}^*[\tau] \subseteq \mathcal{L}[\tau]$ (but also often this will actually be the case when considering a concrete extension of a concrete logic). If $\mathcal{L}^* \leq \mathcal{L}$ and $\mathcal{L} \leq \mathcal{L}^*$, we write $\mathcal{L}^* \equiv \mathcal{L}$. If $\mathcal{L}^* \leq \mathcal{L}$ but $\mathcal{L}^* \not\equiv \mathcal{L}$, we write $\mathcal{L}^* < \mathcal{L}$ and say that $\mathcal{L}$ is a *proper* extension of $\mathcal{L}$. If $\mathcal{K}$ is definable in $\mathcal{L}$, we will also say that it is *axiomatizable* in $\mathcal{L}$ and use these two terms interchangeably.

Similarly to how we assign a class of models to every sentence of $\mathcal{L}$, we can assign a class of $\mathcal{L}$-sentences to every τ-structure $\mathcal{A}$ in the usual way: We call

$$\mathrm{Th}_{\mathcal{L}}(\mathcal{A}) := \{\varphi \in \mathcal{L}[\tau] \colon \mathcal{A} \models_{\mathcal{L}} \varphi\}$$

the *$\mathcal{L}$-theory of $\mathcal{A}$*.

In the following we will want to work with so called *regular abstract logics*, that have some more desirable properties. In particular, those will have first-order logic as a sublogic. To formulate this, first we have to introduce first-order logic as an abstract logic.

2.4 First-Order Logic as an Abstract Logic

In this section we want to consider the abstract logic $\mathcal{L}_{\omega\omega}$ which will be our representation of first-order logic in set theory as an abstract logic. First note that it is clear that if for a vocabulary τ we let $\mathcal{L}_{\omega\omega}[\tau]$ be the set of first-order sentences over τ and $\models_{\mathcal{L}_{\omega\omega}}$ be the usual first-order satisfaction relation, then $\mathcal{L}_{\omega\omega}$ is an abstract logic. Our goal is to be somewhat more precise with what the objects involved are.

Our notion of abstract logic did not include the concept of a (free) variable. Nevertheless, most logics make use of variables as being part of their syntax and as a tool to define their semantics. For this reason we want to fix what a variable is.

Definition 2.4.1. For every sort s and for every set a, we call $x^s_a = (4, (s, a))$ a *variable of sort s*.

Note that similarly to other symbols we defined, "being a variable of sort s" is Δ_0 in set theory. Also again, a set like $H(\kappa)$ that contains all the natural numbers and is closed under taking pairs contains $|H(\kappa)|$-many variables in every sort.

Now it is clear that in a similar way to how we coded variables and sort, relation, function and constant symbols, one can code parentheses, the equality symbol $=$ and the usual logical symbols $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \exists$ and $\forall$ by tuples of, say, natural numbers. As first-order formulas are finite strings of all these symbols, we can conceive every first-order formula as a finite tuple consisting of the codes of all the symbols that it is built from. As not every finite tuple codes a formula of first-order logic, we fix a formula $\mathrm{Form}_{\mathcal{L}_{\omega\omega}}(x, y)$ such that $\mathrm{Form}_{\mathcal{L}_{\omega\omega}}(\varphi, \tau)$ holds iff φ is a tuple coding a first-order formula over the vocabulary τ. If $\mathrm{Form}_{\mathcal{L}_{\omega\omega}}(\varphi, \tau)$ holds we will call φ a *formula of $\mathcal{L}_{\omega\omega}$* or a *formula of first-order logic* over τ. So in the following, when we speak of a first-order formula, we are always referring to a set. It is possible to choose $\mathrm{Form}_{\mathcal{L}_{\omega\omega}}(x, y)$ as a Δ_1-formula of set theory. For a thorough construction and proof of this, consider [5, Chapter 1].

Similarly, we can define the notion of a variable occurring (freely) in a formula of $\mathcal{L}_{\omega\omega}$ in a meaningful (and Δ_1) way.

As we have class-many variables and we in principle could use arbitrary variables in a formula of first-order logic, there are class many formulas of first-order logic over any vocabulary. E.g., for any vocabulary τ and any variable x, we have that $\mathrm{Form}_{\mathcal{L}_{\omega\omega}}(x = x, \tau)$ holds, so $x = x$ is a formula of $\mathcal{L}_{\omega\omega}$ over τ. Note that $x = x$ really is a tuple including the specific codings of x and the equality symbol $=$. Remember that we want $\mathcal{L}_{\omega\omega}[\tau]$ to include only the first-order *sentences* over τ. Now we do not need class-many variables to form all first-order sentences: As any first-order sentence contains only finitely many variables, it is sufficient to restrict ourselves to any infinite set of variables.[1] Thus, to keep $\mathcal{L}_{\omega\omega}[\tau]$ to set size, we simply make the following convention. If $\mathrm{Form}_{\mathcal{L}_{\omega\omega}}(\varphi, \tau)$ holds, then we let $\varphi \in \mathcal{L}_{\omega\omega}[\tau]$ iff

> No variable occurs freely in φ and for all x, if x is a variable occurring in φ, then $x \in H(\omega)$.

Note that thus if $\tau \in H(\kappa)$, then any $\varphi \in \mathcal{L}_{\omega\omega}[\tau]$ is an element of $H(\kappa)$ as well, as φ then is a finite sequence of elements of $H(\kappa) \supseteq H(\omega)$.

We have now introduced the function that gives the $\mathcal{L}_{\omega\omega}$-sentences over any vocabulary τ. The other component of an abstract logic is the satisfaction relation $\models_{\mathcal{L}_{\omega\omega}}$. We can define $(\mathcal{A}, \pi) \models_{\mathcal{L}\omega\omega} \varphi$ holding between τ-structures $\mathcal{A}$, a first-order formula φ over τ and an assignment π from the free variables of φ to elements of $\mathcal{A}$ in the usual inductive way. Then we let $\mathcal{A} \models_{\mathcal{L}_{\omega\omega}} \varphi$ for a sentence $\varphi \in \mathcal{L}_{\omega\omega}[\tau]$ iff $(\mathcal{A}, \emptyset) \models_{\mathcal{L}_{\omega\omega}} \varphi$ for the empty assignment $\emptyset$. It is possible to give a Δ_1-formula $\mathrm{Sat}(x, y, z)$ of set theory such that $\mathrm{Sat}(\mathcal{A}, \varphi, \tau)$ holds iff τ is a vocabulary, $\varphi \in \mathcal{L}_{\omega\omega}[\tau]$ and $\mathcal{A}$ is a τ-structure such that $\mathcal{A} \models_{\mathcal{L}_{\omega\omega}} \varphi$. I.e., the satisfaction relation of first-order logic is Δ_1-definable in set theory. For a proof of this, see e.g. [5, Chapter 1]. In particular we get the following result:

Theorem 2.4.2. "$\mathcal{A} \models_{\mathcal{L}_{\omega\omega}} \varphi$" is absolute for transitive $\in$-models. $\square$

Thus we have formalized first-order logic as an abstract logic inside ZFC. In the following, when we speak of fist-order logic, we will always refer to the abstract logic $\mathcal{L}_{\omega\omega}$.

Notice that if $\Phi(x_1, \ldots, x_n)$ is a formula in the language of set theory (so in our meta-language), we can, now that we have a representation of first-order logic in set theory, fix a binary relation symbol E and variables $y_1, \ldots, y_n$, and give an analogous $\mathcal{L}_{\omega\omega}$-formula $\Phi_E(y_1, \ldots, y_n)$, written using E instead of the membership symbol $\in$. Now if $(M, \in)$ is a set model, one can for elements $m_1, \ldots, m_n$ consider whether $(M, \in) \models_{\mathcal{L}_{\omega\omega}} \Phi_E(m_1, \ldots, m_n)$, i.e., whether the formal satisfaction relation $\models_{\mathcal{L}_{\omega\omega}}$ holds between the sets $(M, \in)$ and $\Phi_E(m_1, \ldots, m_n)$. Remember that we also defined the relativization of Φ to M, i.e., the formula Φ^M from section 2.1. Now one can show that the relativization Φ^M holding is equivalent to $(M, \in)$ being a model of Φ_E (compare e.g. [10, Chapter 4, §10]):

[1] The reader may ask themself, why we introduced class-many variables then. The reason is that we need those when we consider infinitary logics below.

Theorem 2.4.3. Let $(M, \in)$ be a set model, $\Phi(x_1, \dots, x_n)$ be a formula in the language of set theory, and $\Phi_E(y_1, \dots, y_n) \in \mathcal{L}_{\omega\omega}[\{E\}]$ be its formal equivalent constructed as a set in set theory. Then for all $m_1, \dots, m_n \in M$, we have

$$\Phi^M(m_1, \dots, m_n) \text{ iff } (M, \in) \models_{\mathcal{L}_{\omega\omega}} \Phi_E(m_1, \dots, m_n).$$

In particular, if Φ is absolute for $(M, \in)$, then

$$\Phi(m_1, \dots, m_n) \text{ iff } (M, \in) \models_{\mathcal{L}_{\omega\omega}} \Phi_E(m_1, \dots, m_n)$$

and similarly for upwards and downwards absoluteness. In the following, we will in light of Theorem 2.4.3 often switch between considering the relativization Φ^M or the formula Φ_E without further comment, depending on what is more convenient at the time.

Keep in mind that the above theorem is a schema, i.e., for every instance of a formula Φ in the language of set theory we can prove in ZFC that its relativization to $(M, \in)$ is equivalent to Φ_E being satisfied by $(M, \in)$.

2.5 Regular Abstract Logics

Now that we fixed first-order logic as an abstract logic, we are ready to introduce *regular* abstract logics. These will have some desirable properties that one would expect of a somewhat “natural” logic. The properties we will demand for regularity are standard and used throughout abstract model theory. We will again follow [6] in our definitions, up to minor technical details.

First of all, as we are interested in *extensions* of first-order logic, every logic should include $\mathcal{L}_{\omega\omega}$ as a sublogic. Further we demand that it is reasonably closed under boolean operations and usual first-order quantification.

Definition 2.5.1. Let $\mathcal{L}$ be an abstract logic. We say

(i) $\mathcal{L}$ *contains first-order logic* iff $\mathcal{L}_{\omega\omega} \leq \mathcal{L}$.

(ii) $\mathcal{L}$ is *closed under negation* iff for all vocabularies τ and all $\varphi \in \mathcal{L}[\tau]$, there is a sentence $\chi \in \mathcal{L}[\tau]$ such that for all τ-structures $\mathcal{A}$:

$$\mathcal{A} \models_{\mathcal{L}} \chi \text{ iff } \mathcal{A} \not\models_{\mathcal{L}} \varphi.$$

(iii) $\mathcal{L}$ is *closed under conjunctions* iff for all vocabularies τ and all $\varphi, \psi \in \mathcal{L}[\tau]$, there is a sentence $\chi \in \mathcal{L}[\tau]$ such that for all τ-structures $\mathcal{A}$:

$$\mathcal{A} \models_{\mathcal{L}} \chi \text{ iff } \mathcal{A} \models_{\mathcal{L}} \varphi \text{ and } \mathcal{A} \models_{\mathcal{L}} \psi.$$

(iv) $\mathcal{L}$ is *closed under existential quantification* iff for all vocabularies $\tau \cup \{c\}$ including a constant symbol c of sort s with $c \notin \tau$, and all $\varphi \in \mathcal{L}[\tau \cup \{c\}]$, there is a sentence $\chi \in \mathcal{L}[\tau]$ such that for all τ-structures $\mathcal{A}$:

$$\mathcal{A} \models_{\mathcal{L}} \chi \text{ iff there is an } a \in A_s \text{ such that with } c^{\mathcal{A}} = a \text{ we have } (\mathcal{A}, c^{\mathcal{A}}) \models_{\mathcal{L}} \varphi.$$

Property (i) tells us, that for every $\varphi \in \mathcal{L}_{\omega\omega}[\tau]$, there is a $\psi \in \mathcal{L}[\tau]$ such that $Mod(\varphi) = Mod(\psi)$, i.e., every model class definable in first-order logic is definable in $\mathcal{L}$. Using the usual interdefinability between boolean connectives and the existential and universal quantifiers, properties (ii) to (iv) give us that next to negations, conjunctions and existential quantification we have disjunctions, implications and universal quantification. We will write $\neg\varphi$, $\varphi \wedge \psi$ and $\exists x\varphi(x)$ for the sentences χ given by those properties and similar with $\vee$, $\rightarrow$, $\leftrightarrow$ and $\forall$.

For simplicity, we give the other definitions of this section only for the one-sorted case, the general one being an easy adaptation hereof.

Note that first-order logic is closed under relativizations, i.e., if φ is a sentence, then first-order logic can express that a substructure is a model of φ. For example, if $<$ is a binary and P a unary relation symbol and $\varphi = \exists x \exists y(x < y)$, then the relativization of φ to the substructure with domain P can be expressed by $\varphi^P = \exists x \exists y(P(x) \wedge P(y) \wedge x < y)$. Then for any $\{<, P\}$-structure $\mathcal{A}$, if $\mathcal{B}$ is the structure with domain $P^{\mathcal{A}}$, and $<^{\mathcal{B}}$ and $P^{\mathcal{B}}$ are the restrictions of $<^{\mathcal{A}}$ and $P^{\mathcal{A}}$, respectively, then

$$\mathcal{A} \models \varphi^P \text{ iff } \mathcal{B} \models \varphi.$$

Note that this only makes sense if $P^{\mathcal{A}}$ is non-empty as $\mathcal{B}$ must have non-empty domain. Further, to be a proper structure, it should be closed under potential functions defined on $\mathcal{A}$ and include all constants. Thus we introduce the following notion: If $\mathcal{A}$ is a τ-structure and $B \subseteq A$, we say B is *τ-closed* if $B \neq \emptyset$, B is closed under $f^{\mathcal{A}}$ for all function symbols $f \in \tau$ and $c^{\mathcal{A}} \in B$ for all constant symbols $c \in \tau$.

The ability to form relativizations is formalized for arbitrary abstract logics by the following property and used throughout abstract model theory. We thus demand it of a regular abstract logic.

Definition 2.5.2. Let $\mathcal{L}$ be an abstract logic. We say $\mathcal{L}$ has the *relativization property* iff for any $\varphi \in \mathcal{L}[\tau]$ and $\psi \in \mathcal{L}[\sigma \cup \{c\}]$ for a constant symbol $c \notin \tau \cup \sigma$, there is a $\chi \in \mathcal{L}[\tau \cup \sigma]$ such that for all $(\tau \cup \sigma)$-structures $\mathcal{A}$, if $\psi^{\mathcal{A}} = \{c^{\mathcal{A}} \in A \colon (\mathcal{A}, c^{\mathcal{A}}) \models \psi\}$ is τ-closed and $\mathcal{B}$ is the τ-structure with domain $\psi^{\mathcal{A}}$ that is obtained from $(\mathcal{A} \restriction \tau)$ in the obvious way by restricting the interpretations of symbols in τ to $\psi^{\mathcal{A}}$, then

$$\mathcal{A} \models_{\mathcal{L}} \chi \text{ iff } \mathcal{B} \models_{\mathcal{L}} \varphi.$$

Finally, we want to introduce a substitution property that allows to emulate sentences with function and constant symbols by sentences over so called *relational* vocabularies, i.e., only including relation symbols. Note that in first-order logic we can express that a relation symbol is to be interpreted as the graph of a function and thus that if $\varphi \in \mathcal{L}_{\omega\omega}[\{f\}]$ is a sentence using some n-ary function symbol f, we can give a sentence $\psi \in \mathcal{L}_{\omega\omega}[\{R\}]$ with R an $(n+1)$-ary relation symbol such that if $\mathcal{A}$ is an $\{f\}$-structure and $\mathcal{B}$ is an $\{R\}$-structure that is obtained from $\mathcal{A}$ by interpreting $R^{\mathcal{B}}$ as the graph of $f^{\mathcal{A}}$, then

$$\mathcal{A} \models \varphi \text{ iff } \mathcal{B} \models \psi.$$

The following is defined analogously:

Definition 2.5.3. Let $\mathcal{L}$ be an abstract logic. We say $\mathcal{L}$ has the *substitution property* iff for any vocabulary τ, if σ is a vocabulary that is obtained by replacing every n-ary function symbol f by a new $(n+1)$-ary relation symbol R^f and and every constant symbol c by a new 1-ary relation symbol R^c, then for every $\varphi \in \mathcal{L}[\tau]$ there is a $\psi \in \mathcal{L}[\sigma]$ such that for all τ-structures $\mathcal{A}$, if $\mathcal{B}$ is the σ-structure obtained from $\mathcal{A}$ by interpreting every R^f as the graph of $f^{\mathcal{A}}$ and every R^c as the singleton $\{c^{\mathcal{A}}\}$ respectively, then:

$$\mathcal{A} \models_{\mathcal{L}} \varphi \text{ iff } \mathcal{B} \models_{\mathcal{L}} \psi.$$

We will mostly use the substitution property for convenience, as a way to restrict ourselves to the somewhat less complicated case of relational vocabularies.

The above properties define regularity:

Definition 2.5.4. An abstract logic $\mathcal{L}$ is called *regular* iff it contains first-order logic, is closed under negations, conjunctions and existential quantification and has the relativization and substitution properties.

In the following we will restrict ourselves to regular abstract logics and will simply refer to them as *logics*. The reader can convince themself easily that every specific logic we will consider is regular.

2.6 Examples beyond First-Order Logic

We want to give some more examples of (regular abstract) logics. Here we want to focus on logics we will encounter later.

An important class of extensions of $\mathcal{L}_{\omega\omega}$ are logics with additional quantifiers. They are always generated by adding clauses to the typical inductive definitions of formulas of first-order logic and its satisfaction relation. One typical example of this is the *cardinality* quantifier $\mathcal{Q}_\alpha$, defined for each ordinal α, which intuitively says "there are at least $\aleph_\alpha$-many". The logic $\mathcal{L}_{\omega\omega}(\mathcal{Q}_\alpha)$ is obtained by adding to the inductive definition of formulas of first-order logic an extra clause saying:

> If $\varphi(x)$ is a formula and x a variable occurring freely in φ, then $\mathcal{Q}_\alpha x\varphi(x)$ is a formula.

The semantics are then given by adding the following clause to the usual inductive definition of first-order satisfaction:

> $\mathcal{A} \models \mathcal{Q}_\alpha x\varphi(x)$ iff $|\{a \in A \colon \mathcal{A} \models \varphi(a)\}| \geq \aleph_\alpha$.

As by the Löwenheim-Skolem Theorem, $\mathcal{L}_{\omega\omega}$ cannot define $\mathcal{Q}_\alpha$, this gives a proper extension of first-order logic.

Two examples particularly important in symbiosis contexts are the so called *well-foundedness* and the *Härtig* quantifier. The syntax of the logics $\mathcal{L}_{\omega\omega}(\mathrm{WF})$ and $\mathcal{L}_{\omega\omega}(I)$ obtained by adding those to first-order logic can be defined similarly to the above, so we will only give their semantics. The well-foundedness quantifier WF takes two variables and one formula with those variables occurring freely and its meaning is given by

$$\mathcal{A} \models \mathrm{WF}xy\varphi(x,y) \text{ iff } \{(a,b) \in A \times A \colon \mathcal{A} \models \varphi(a,b)\} \text{ is well-founded.}$$

In particular, for a language including a binary relation symbol $<$, the logic $\mathcal{L}_{\omega\omega}(\mathrm{WF})$ can axiomatize the class of all well-founded relations by the sentence $\mathrm{WF}xy(x < y)$. Because this class is not definable in $\mathcal{L}_{\omega\omega}$, again $\mathcal{L}_{\omega\omega}(\mathrm{WF})$ is a proper extension of first-order logic.

The Härtig quantifier I takes two variables and two formulas and expresses equicardinality in the following way:

$$\mathcal{A} \models Ixy\varphi(x)\psi(y) \text{ iff } |\{a \in A \colon \mathcal{A} \models \varphi(a)\}| = |\{a \in A \colon \mathcal{A} \models \psi(a)\}|.$$

This is not expressible in first-order logic either, so $\mathcal{L}_{\omega\omega} < \mathcal{L}_{\omega\omega}(I)$.

Another important example of a regular logic is second-order logic $\mathcal{L}^2$. We will not give a precise definition, but just note that it is obtained by introducing a new class of variables F_s^n for every natural number n and every sort s, allowing to quantify over n-ary relations of the domain in sort s.

It is clear that analogously to what we indicated for first-order logic, we can define the syntax of the logics considered above by a Δ_1-formula. In contrast, we will see that besides for the case of $\mathcal{L}_{\omega\omega}(\mathrm{WF})$, their semantics (i.e., their satisfaction relation) can*not* be defined by Δ_1-formula.

The third group of logics of importance to us consists of *infinitary* logics. First we define the logic $\mathcal{L}_{\infty\omega}$, which allows for conjunctions and disjunctions of arbitrarily many formulas. For this we introduce new symbols $\bigwedge$ and $\bigvee$, which we assume to be in $H(\omega)$, and add to the inductive definition of formulas known from first-order logic the clause

($*$) If T is any set of formulas, then $\bigvee T$ and $\bigwedge T$ are formulas.

Formally, in our representation of this logic in set theory, we will assume that $\bigvee T$ and $\bigwedge T$ are given by pairs $(\bigvee, T)$ and $(\bigwedge, T)$. Then we let for a vocabulary τ the class $\mathcal{L}_{\infty\omega}[\tau]$ of $\mathcal{L}_{\infty\omega}$-sentences over τ be given by all formulas without free variables. The satisfaction relation is defined in the obvious way by adding the condition:

($**$) If T is a set of formulas with free variables from a set X, and f is an assignment of these variables to elements of A, then $(\mathcal{A}, f) \models_{\mathcal{L}_{\infty\omega}} \bigwedge T$ iff $(\mathcal{A}, f) \models_{\mathcal{L}_{\infty\omega}} \varphi$ for all $\varphi \in T$

and similarly for $\bigvee T$.

Interestingly, the syntax and semantics of $\mathcal{L}_{\infty\omega}$ are definable by a Δ_1-formula of set theory, i.e., there is a Δ_1-predicate $\mathrm{Sat}_{\mathcal{L}_{\infty\omega}}(x, y, z)$ of set-theory such that $\mathrm{Sat}_{\mathcal{L}_{\infty\omega}}(\mathcal{A}, \varphi, \tau)$ holds iff $\mathcal{A}$ is a τ-structure, $\varphi \in \mathcal{L}_{\infty\omega}[\tau]$ and $\mathcal{A} \models_{\mathcal{L}_{\infty\omega}} \varphi$. In particular, the satisfaction relation $\models_{\mathcal{L}_{\infty\omega}}$ is absolute, a fact which we will use often later. For a proof of this, see [3, Chapter III, Section 1].

For every cardinal κ, we want to define a sublogic of $\mathcal{L}_{\infty\omega}$ called $\mathcal{L}_{\kappa\omega}$ that allows for conjunctions and disjunctions over less than κ-many formulas. To define the formulas of $\mathcal{L}_{\kappa\omega}$, this is simply achieved by restricting the condition ($*$) and to refer only to sets T with $|T| < \kappa$. Note that while for $\mathcal{L}_{\infty\omega}$ we cannot avoid $\mathcal{L}_{\infty\omega}[\tau]$ to be a proper class (as conjunctions and disjunctions can have arbitrary length), in formulas of $\mathcal{L}_{\kappa\omega}$ cannot occur more than κ many symbols. Thus we can achieve $\mathcal{L}_{\kappa\omega}[\tau]$ having set-size. For this purpose, similarly to our convention for first-order logic, we say a formula φ of $\mathcal{L}_{\kappa\omega}$ over a vocabulary τ is a *sentence*, i.e., $\varphi \in \mathcal{L}_{\kappa\omega}[\tau]$ iff

> No variable occurs freely in φ and for all x, if x is a variable occurring in φ, then $x \in H(\kappa)$.

We get the following result, which we will make use of later:

Proposition 2.6.1. Let κ be regular, $\tau \in H(\kappa)$ a vocabulary and $\varphi \in \mathcal{L}_{\kappa\omega}[\tau]$. Then $\varphi \in H(\kappa)$.

Proof. We prove this by induction on the structure of φ. As $\tau \in H(\kappa)$ and all the variables are in $H(\kappa)$, this is clear for φ atomic (φ is a finite tuple of elements of $H(\kappa)$). If φ is a negation or a quantified formula, this is clear by the induction hypothesis. Now if $\varphi = (\bigwedge, T)$ for a set of formulas T of size $< \kappa$, then by the induction hypothesis $T \subseteq H(\kappa)$. As $|T| < \kappa$ and κ is regular, also $T \in H(\kappa)$. Thus also $\varphi \in H(\kappa)$ as a finite tuple of elements of $H(\kappa)$. $\square$

To give the semantics of $\mathcal{L}_{\kappa\omega}$, observe that $\mathcal{L}_{\kappa\omega} \subseteq \mathcal{L}_{\infty\omega}$. Thus we simply let $\mathcal{A} \models_{\mathcal{L}_{\kappa\omega}} \varphi$ iff $\mathcal{A} \models_{\mathcal{L}_{\infty\omega}} \varphi$.

$\mathcal{L}_{\infty\omega}$ and $\mathcal{L}_{\kappa\omega}$ are adding infinitary means to the expressive capabilities of first-order logic. Similarly, for an arbitrary logic $\mathcal{L}^*$, we would like to define infinitary extensions $\mathcal{L}^*_{\kappa\omega}$ and $\mathcal{L}^*_{\infty\omega}$. For our concrete examples of logics with additional quantifiers or of second-order logic above, this can simply be achieved by adding appropriate clauses like $(*)$ and $(**)$ to the definitions of their syntax and semantics. But for an arbitrary logic, we have to give some general conditions, which secures that the added infinitary means interact nicely with the other expressive capabilities of the logic. We will not do this here, but simply refer to [16], where this is carried out. It is clear that if the logic $\mathcal{L}^*$ has a finite syntax like $\mathcal{L}_{\omega\omega}$ or our other examples above, which are Δ_1-definable, then we can get the extension $\mathcal{L}^*_{\kappa\omega}$ in a way that gives us an analogue of Proposition 2.6.1.

2.7 Dependence Numbers

In this section we want to introduce a measure of the "size" of a logic, which will become important later. There are various ways to do this (compare [15, Section 2] for a brief discussion), but one of the most important ones is the *dependence number*:

Definition 2.7.1. Let $\mathcal{L}$ be a logic. If such a cardinal exists, we denote by $dep(\mathcal{L})$ the smallest cardinal κ such that for any vocabulary τ and any $\varphi \in \mathcal{L}[\tau]$, there is a subset $\sigma \subseteq \tau$ with $|\sigma| < \kappa$ and $\varphi \in \mathcal{L}[\sigma]$. If such a cardinal does not exist, we let $dep(\mathcal{L}) = \infty$. We call $dep(\mathcal{L})$ the *dependence number of* $\mathcal{L}$.

The intuition behind the dependence number is that if $dep(\mathcal{L}) = \kappa$, then every sentence of $\mathcal{L}$ depends on less than κ many symbols. For logics with finite syntax, of course every sentence depends only on finitely many symbols. Thus we have that $dep(\mathcal{L}_{\omega\omega}) =$

$dep(\mathcal{L}_{\omega\omega}(\mathcal{Q}_\alpha)) = dep(\mathcal{L}_{\omega\omega}(\mathrm{WF})) = dep(\mathcal{L}_{\omega\omega}(I)) = dep(\mathcal{L}^2) = \omega$. For infinitary logics, the picture is a little bit more complicated, but still we have

Proposition 2.7.2. Let κ be a regular cardinal. Then $dep(\mathcal{L}_{\kappa\omega}) = \kappa$.

Proof. We show by induction on the structure of $\varphi \in \mathcal{L}_{\kappa\omega}$ that there are less than κ-many symbols appearing in φ. If φ is atomic, this is trivial. If φ is a finitary boolean combination or a quantified formula, this is obvious by the induction hypothesis. The only interesting case is thus that $\varphi = \bigwedge T$ is a conjunction (or analogously disjunction) over a set of sentences T of size $< \kappa$. By the induction hypothesis, every element of T is a formula involving less than κ many symbols. If for $\psi \in T$, we let S_ψ be the set of symbols appearing in ψ and S the set of symbols appearing in $\bigwedge T$, then clearly $|S| = |\bigcup_{\psi \in T} S_\psi|$. As $|T| < \kappa$ and κ is regular and by the induction hypothesis $|S_\psi| < \kappa$ for every ψ, it follows that $|S| < \kappa$. $\square$

Of course, if κ is singular, the above argument breaks, as then we can have a conjunction of size $< \kappa$ of formulas involving $< \kappa$ many symbols, but still κ many symbols appearing in the whole conjunction:

Example 2.7.3. Consider $\kappa = \aleph_\omega$. Take κ-many constants $\{c_i \colon i < \kappa\}$. Then for every $n \in \omega$,

$$\psi_n := \bigwedge_{i<j<\aleph_n} c_i \neq c_j$$

is a sentence of $\mathcal{L}_{\kappa\omega}$ involving $\aleph_n$ many constants. Also $\psi := \bigwedge_{n\in\omega} \psi_n \in \mathcal{L}_{\kappa\omega}$, as it is a countable conjunction. But ψ involves $\aleph_\omega = \kappa$ many symbols.

For this reason, we will often ignore logics $\mathcal{L}_{\kappa\omega}$ for singular κ.

Notice that if $\mathcal{L}^*$ is any logic with finitary syntax like $\mathcal{L}_{\omega\omega}(\mathcal{Q}_\alpha)$, $\mathcal{L}_{\omega\omega}(\mathrm{WF})$, $\mathcal{L}_{\omega\omega}(I)$ or $\mathcal{L}^2$, then $dep(\mathcal{L}^*_{\kappa\omega})$ is also κ for regular κ. The proof of this goes analogously to the case of $\mathcal{L}^* = \mathcal{L}_{\omega\omega}$ above.

If $dep(\mathcal{L}) = \kappa$, then every $\varphi \in \mathcal{L}[\tau]$ involves less than κ-many symbols, say from a set $\sigma \subseteq \tau$ with $|\sigma| < \kappa$. Clearly there is a renaming $\rho : \sigma \longrightarrow \sigma^*$ for some $\sigma^* \in H(\kappa)$. Then by the renaming property (see Definition 2.3.2), there is a $\varphi^* \in \mathcal{L}[\sigma^*]$ such that φ and φ^* are equivalent up to renaming. I.e., up to renaming, every sentence is built over a vocabulary in $H(\kappa)$.

In the later parts of this thesis, we will often want to look at classes of structures which are $\Delta_1(R)$-definable with parameters from some $H(\kappa)$. In this case it will often be important, that the sentences of a logic themselves are in $H(\kappa)$, as we want to use them as a parameter. The dependence number does not guarantee that this stronger condition is fulfilled, even if the vocabularies considered are from $H(\kappa)$ themselves. The reason for this is that we formulated no restriction on what the sentences $\varphi \in \mathcal{L}$ are. So even over a finite vocabulary, where all symbols are from $H(\omega)$, a logic can have a sentence φ over this vocabulary which is *as a set* very complicated and only from, say, $H(\aleph_{\omega_1})$.

We thus introduce a novel notion, that explicitly demands that certain sentences are from $H(\kappa)$. This can only be achieved in a useful way if the vocabulary itself is from

$H(\kappa)$ (as otherwise a sentence formed over this vocabulary can already involve a set which is not in $H(\kappa)$).

Definition 2.7.4. Let $\mathcal{L}$ be a logic and κ a cardinal. κ is called the *strong dependence number* $dep^*(\mathcal{L})$ *of* $\mathcal{L}$ iff the following two conditions are fulfilled.

(i) $dep(\mathcal{L}) = \kappa$.

(ii) For every vocabulary $\tau \in H(\kappa)$, we have $\mathcal{L}[\tau] \subseteq H(\kappa)$.

The definition secures that if τ is in $H(\kappa)$, then no sentence of $\mathcal{L}$ over τ is not in $H(\kappa)$. Thus we get the following:

Proposition 2.7.5. Let $\mathcal{L}$ be a logic with $dep^*(\mathcal{L}) = \kappa$. If τ is any vocabulary and $\varphi \in \mathcal{L}[\tau]$, there is a $\sigma \subseteq \tau$ with $|\sigma| < \kappa$ and a renaming $\rho : \sigma \longrightarrow \sigma^* \in H(\kappa)$ and a $\varphi^* \in \mathcal{L}[\sigma^*] \cap H(\kappa)$ which equivalent to φ up to renaming.

Proof. Let $\varphi \in \mathcal{L}[\tau]$. By $dep(\mathcal{L}) = \kappa$, take a $\sigma \subseteq \tau$ with $|\sigma| < \kappa$ and $\varphi \in \mathcal{L}[\sigma]$. As above, clearly there is a $\sigma^* \in H(\kappa)$ such that there is a renaming $\rho : \sigma \longrightarrow \sigma^*$. Then by the renaming property there is a $\varphi^* \in \mathcal{L}[\sigma^*]$ which is equivalent to φ up to renaming. By condition (ii) of $dep^*(\mathcal{L}) = \kappa$, we get $\varphi^* \in H(\kappa)$ as demanded. $\square$

So if $dep^*(\mathcal{L}) = \kappa$, every $\varphi \in \mathcal{L}$ is equivalent up to renaming to a sentence in $H(\kappa)$.

For the finitary logics we considered above, it is clear that that we get that $dep^*(\mathcal{L}_{\omega\omega}) = dep^*(\mathcal{L}_{\omega\omega}(\mathcal{Q}_\alpha)) = dep^*(\mathcal{L}_{\omega\omega}(\mathrm{WF})) = dep^*(\mathcal{L}_{\omega\omega}(I)) = dep^*(\mathcal{L}^2) = \omega$.

Moreover we get

Lemma 2.7.6. Let κ be regular. Then $dep^*(\mathcal{L}_{\kappa\omega}) = \kappa$.

Proof. We already showed that $dep(\mathcal{L}_{\kappa\omega}) = \kappa$. Thus it is sufficient to show that $\mathcal{L}_{\kappa\omega}[\tau] \subseteq H(\kappa)$ for $\tau \in H(\kappa)$, i.e., $|trcl(\varphi)| < \kappa$ for all $\varphi \in \mathcal{L}_{\kappa\omega}[\tau]$. We show this by induction on the structure of φ.

If φ is atomic, this is trivial, as $\tau \in H(\kappa)$.[2] If φ is a finitary boolean combination or a quantified formula, this is trivial by the induction hypothesis. The only interesting case is thus $\varphi = (\bigwedge, T)$, where T is a set of sentences of $\mathcal{L}_{\kappa\omega}$ of size $< \kappa$. Without loss of generality assume that $|T| \geq \omega$. Note that $|trcl(\varphi)| = |trcl(T)| = |T \cup \bigcup_{\psi \in T} trcl(\psi)| \leq |T| + |\bigcup_{\psi \in T} trcl(\psi)|$. Now as $|T| < \kappa$ and by the induction hypothesis, for every $\psi \in T$, $|trcl(\psi)| < \kappa$, because κ is regular, also $|\bigcup_{\psi \in T} trcl(\psi)| < \kappa$. So we get that $|trcl(\varphi)| < \kappa + \kappa = \kappa$. $\square$

Clearly, this line of argument can be replicated for any logic with finite syntax, so we get

[2]Notice that for this to be trivial, we have to assume that φ only uses variables from $H(\kappa)$, as otherwise, e.g., if P is a unary predicate in $H(\kappa)$ and x is a variable *not* in $H(\kappa)$, the formula $P(x)$ is not in $H(\kappa)$ either. But since we assumed that every *sentence* of $\mathcal{L}_{\kappa\omega}$ is formed only involving variables in $H(\kappa)$ (see section 1.6) this causes no problems.

Theorem 2.7.7. Let κ be regular and $\mathcal{L}^*$ be any of $\mathcal{L}_{\kappa\omega}(\mathcal{Q}_\alpha)$, $\mathcal{L}_{\kappa\omega}(\mathrm{WF})$, $\mathcal{L}_{\kappa\omega}(I)$ or $\mathcal{L}^2_{\kappa\omega}$. Then $dep^*(\mathcal{L}^*_{\kappa\omega}) = \kappa$. $\square$

We will make extensive use of the strong dependence number of these logics later.

2.8 The Δ-Closure

Intuitively, symbiosis describes an equivalence between $\mathcal{L}$-definability and $\Delta_1(R)$-definability. However, technically this equivalence can only hold for logics which are closed under the so called Δ-closure. This last section of our introductory chapter is devoted to introduce the Δ-closure and its representation as an abstract logic.

Notice that sometimes it is possible for a logic to define a class of τ-structures, when we allow additional symbols not in τ.

Example 2.8.1. Consider the empty language $\tau = \emptyset$. It is well known that the class $\mathcal{K}$ of all infinite τ-structures is not first-order axiomatizable (by a single sentence). However, consider $\tau^* := \{f\} \supseteq \tau$, where f is a unary function symbol. Then the sentence

$$\varphi := \forall x \forall y (f(x) = f(y) \to x = y) \wedge \exists x \forall y (f(y) \neq x)$$

saying that f is an injection but not a surjection has only infinite models. Thus $\mathcal{K} = \{\mathcal{A} \restriction \tau \colon \mathcal{A} \models \varphi\}$. So $\mathcal{K}$ consists of the reducts of all models of φ to τ, i.e., we can define $\mathcal{K}$ when we allow the usage of additional symbols.

In the following, if $\sigma \subseteq \tau$ are vocabularies and $\varphi \in \mathcal{L}[\tau]$, then we will always denote the class containing all the restrictions of models of φ to σ by

$$Mod(\varphi) \restriction \sigma := \{\mathcal{A} \restriction \sigma \colon \mathcal{A} \models \varphi\}.$$

Keep in mind that if τ contains more than one sort symbol, it can happen that $|\mathcal{A} \restriction \sigma| < |\mathcal{A}|$.

Definition 2.8.2. Let $\mathcal{L}$ be a logic, τ a vocabulary and $\mathcal{K}$ a class of τ-structures. $\mathcal{K}$ is called $\Sigma(\mathcal{L})$ or $\Sigma(\mathcal{L})$-*definable* iff there is an expansion τ^* of τ by finitely many symbols and a $\varphi \in \mathcal{L}[\tau^*]$ such that

$$Mod(\varphi) \restriction \tau = \{\mathcal{A} \restriction \tau \colon \mathcal{A} \models \varphi\} = \mathcal{K}.$$

$\mathcal{K}$ is called $\Delta(\mathcal{L})$ or $\Delta(\mathcal{L})$-*definable* iff both $\mathcal{K}$ and the complement of $\mathcal{K}$ are $\Sigma(\mathcal{L})$.[3]

We will now fix the Δ-closure of $\mathcal{L}$. This is an abstract logic whose definable classes are precisely the ones which are $\Delta(\mathcal{L})$-definable.

[3] Often, e.g. in [4], a class $\mathcal{K}$ of τ-structures is called *elementary* or *EC in* $\mathcal{L}$ if it is definable by a τ-sentence over $\mathcal{L}$ and called *projective or PC in* $\mathcal{L}$ if it is defined using additional symbols as in the above definition. In the following we will stick to the terms ($\Sigma(\mathcal{L})$, $\Delta(\mathcal{L})$)-definability though.

Definition 2.8.3. Let $\mathcal{L}$ be a logic. The Δ-*closure* $\Delta(\mathcal{L})$ *of* $\mathcal{L}$ is the logic constructed in the following way.

For a vocabulary τ let κ be the smallest cardinal such that $\tau \in H(\kappa)$. The class of $\Delta(\mathcal{L})$-sentences consists of 4-tuples where $(\varphi_1, \tau_1, \varphi_2, \tau_2) \in \Delta(\mathcal{L})[\tau]$ iff τ_i is finite, $\tau_i \in H(\kappa)$ and $\varphi_i \in \mathcal{L}[\tau \cup \tau_i]$ for $i = 1, 2$ and for all τ-structures $\mathcal{A}$, either there is an expansion $\mathcal{A}_1$ of $\mathcal{A}$ to a $(\tau \cup \tau_1)$-structure and $\mathcal{A}_1 \models_{\mathcal{L}} \varphi_1$ or there is an expansion $\mathcal{A}_2$ of $\mathcal{A}$ to a $(\tau \cup \tau_2)$-structure and $\mathcal{A}_2 \models_{\mathcal{L}} \varphi_2$ (but not both).

The satisfaction relation $\models_{\Delta(\mathcal{L})}$ is defined for a τ-structure $\mathcal{A}$ and for $\varphi \in \Delta(\mathcal{L})[\tau]$ by letting $\mathcal{A} \models_{\Delta(\mathcal{L})} \varphi$ iff $\varphi = (\varphi_1, \tau_1, \varphi_2, \tau_2)$ and there is an expansion $\mathcal{A}_1$ of $\mathcal{A}$ to a $(\tau \cup \tau_1)$-structure such that $\mathcal{A}_1 \models_{\mathcal{L}} \varphi_1$.

Clearly, if $(\varphi_1, \tau_1, \varphi_2, \tau_2) \in \Delta(\mathcal{L})[\tau]$, then $Mod(\varphi_1) \restriction \tau$ and $Mod(\varphi_2) \restriction \tau$ are both $\Sigma(\mathcal{L})$. Further, as we demanded that every τ-structure can either be expanded to a model of φ_1 or to a model of φ_2, $Mod(\varphi_1) \restriction \tau$ and $Mod(\varphi_2) \restriction \tau$ are complimentary, so actually $\Delta(\mathcal{L})$-definable. Thus $\Delta(\mathcal{L})$ can define precisely those classes which are $\Delta(\mathcal{L})$-definable.

The above definition is somewhat overly complicated: We assumed that the vocabularies $\tau_i \supseteq \tau$ come from $H(\kappa)$, where κ is the smallest cardinal with $\tau \in H(\kappa)$. This is of course not necessary in order to achieve that the Δ-closure of $\mathcal{L}$ can define all the $\Delta(\mathcal{L})$-definable classes. We add this clause to secure that the strong dependence number of $\mathcal{L}$ is preserved under the Δ-closure. Similarly, that we allow to only add finitely many symbols in a $\Sigma(\mathcal{L})$-definition of a class is not strictly necessary. But we want the Δ-closure to add as little expressive power as possible to $\mathcal{L}$, thus we stick to this.

The Δ-closure is an actual closure, in the sense that $\Delta(\Delta(\mathcal{L})) \equiv \Delta(\mathcal{L})$. This is easy to see, as anything that is definable in $\Delta(\mathcal{L})$ using additional symbols, was already definable in $\mathcal{L}$ adding additional symbols.

We collect some properties of the Δ-closure in the following

Theorem 2.8.4. Let $\mathcal{L}$ be a regular abstract logic with $dep^*(\mathcal{L}) = \kappa$. Then

1. $\Delta(\mathcal{L})$ is a regular abstract logic.
2. $dep^*(\mathcal{L}) = \kappa$.
3. $\mathcal{L} \leq \Delta(\mathcal{L})$.
4. $\Delta(\Delta(\mathcal{L})) \equiv \Delta(\mathcal{L})$.

Proof. Points 1, 3 and 4 cite [6, Theorem 7.2.4]. For a proof sketch of these parts, see this passage. We only prove point 2, which is unique in referring to our notion of strong dependence number.

Clearly, if $dep(\mathcal{L}) = \kappa$, then also $dep(\Delta(\mathcal{L})) = \kappa$, as a sentence φ of $\Delta(\mathcal{L})$ is a tuple consisting of $\mathcal{L}$-sentences over τ, which at most uses finitely many symbols not in τ.

Further, if $\tau \in H(\kappa)$ and $\varphi = (\varphi_1, \tau_1, \varphi_2, \tau_2) \in \Delta(\mathcal{L})[\tau]$, then $\varphi_i \in \mathcal{L}[\tau \cup \tau_i]$ with $\tau_i \in H(\kappa)$. Therefore also $\tau \cup \tau_i \in H(\kappa)$ and because $dep^*(\mathcal{L}) = \kappa$, thus $\varphi_i \in H(\kappa)$. So φ is a finite tuple of elements of $H(\kappa)$, so itself in $H(\kappa)$. $\square$

The Δ-closure is related to interpolation properties: $\mathcal{L}$ is said to have the Δ-*interpolation property* iff every $\Delta(\mathcal{L})$-definable class is already $\mathcal{L}$-definable.

That this is called an *interpolation property* is explained by the following fact cited from [6, Proposition 3.1.3] which is easy to prove.

Proposition 2.8.5. Let $\mathcal{L}$ be a logic. Then the following are equivalent.

(i) $\mathcal{L}$ has the *Craig-interpolation property*, i.e., for any $\varphi \in \mathcal{L}[\tau]$ and $\psi \in \mathcal{L}[\sigma]$, if $\models \varphi \to \psi$, then there is a $\theta \in \mathcal{L}[\tau \cap \sigma]$ such that $\models \varphi \to \theta$ and $\models \theta \to \psi$.

(ii) For all vocabularies τ, if $\mathcal{K}_0$ and $\mathcal{K}_1$ are disjoint classes of τ-structures and are $\Sigma(\mathcal{L})$, then there is an $\mathcal{L}$-definable class $\mathcal{K}$ of τ-structures with $\mathcal{K}_0 \subseteq \mathcal{K}$ and $\mathcal{K}_1 \cap \mathcal{K} = \emptyset$.

Using the above proposition, the following well known result is easy to prove.

Proposition 2.8.6. Let $\mathcal{L}$ be a logic with the Craig-interpolation property. Then $\mathcal{L} \equiv \Delta(\mathcal{L})$, i.e., $\mathcal{L}$ has the Δ-interpolation property.

Proof. It is sufficient to show that any $\Delta(\mathcal{L})$-definable class is already $\mathcal{L}$-definable. So let $\mathcal{K}$ be $\Delta(\mathcal{L})$, i.e., both $\mathcal{K}$ and its complement $\bar{\mathcal{K}}$ are $\Sigma(\mathcal{L})$. Then by the Craig-interpolation property and Proposition 2.8.5, there is an $\mathcal{L}$-definable class of τ-structures $\mathcal{K}^*$ such that $\mathcal{K} \subseteq \mathcal{K}^*$ and $\bar{\mathcal{K}} \cap \mathcal{K}^* = \emptyset$. But as $\mathcal{K}$ and $\bar{\mathcal{K}}$ are complementary, this can only be the case if $\mathcal{K} = \mathcal{K}^*$, so $\mathcal{K}$ is $\mathcal{L}$-definable. $\square$

Of course, the Craig-interpolation Theorem tells us that first-order logic has the Craig-interpolation property. Therefore we get

Corollary 2.8.7. $\Delta(\mathcal{L}_{\omega\omega}) \equiv \mathcal{L}_{\omega\omega}$, i.e., $\mathcal{L}_{\omega\omega}$ has the Δ-interpolation property. $\square$

Notice that we saw that the class of infinite structures is first-order definable when using additional symbols, so $\Sigma(\mathcal{L}_{\omega\omega})$, but surely it is not first-order definable. The reason for this is that its complimentary class, the class of finite structures, is not $\Sigma(\mathcal{L}_{\omega\omega})$.

3 Symbiosis

Symbiosis was introduced by Jouko Väänänen and compares the definability strength of a logic and set theory, if in the latter we restrict ourselves to a class of formulas of a certain complexity. It turns out that often the same classes of structures can be defined by the Δ-closure of a logic $\mathcal{L}$ and by a $\Delta_1(R)$ formula of set theory. Thus symbiosis is formulated as a relation between $\mathcal{L}$ and a predicate of set theory R.

Notice that our notion of definability in a logic leads to some weird behaviour: The same sentence $\varphi \in \mathcal{L}$ can define different classes, depending on the vocabulary we are considering. As an extreme example, consider the tautology $\varphi = \exists x(x = x)$. (A sentence equivalent to) φ is an element of $\mathcal{L}[\tau]$ for any logic $\mathcal{L}$ and any vocabulary τ. In particular, as φ is a tautology, it holds in every structure, so considered as an element of $\mathcal{L}[\tau]$, it defines the class of all τ-structures. But considered as an element of $\mathcal{L}[\sigma]$ for a different vocabulary σ, it defines the class of all σ-structures. This is somewhat trivial, but something to keep in mind, in particular in view of the technical issues discussed in the paragraph below the next one.

The logics we deal with are set-theoretic objects, so in general, any class that is definable in any logic will also be definable in set theory. The other direction does on the other hand not hold. Consider for example the vocabulary $\{<\}$, consisting of a binary relation symbol and the class of all $\{<\}$-structures where $<$ is interpreted as a well-order:

$$\mathcal{K} = \{(A, <^A) \colon\ <^A \text{ is a well-order on } A\}.$$

As is well known, being a well-order is not axiomatizable in first-order logic, so $\mathcal{L}_{\omega\omega}$ cannot define $\mathcal{K}$. On the other hand, being a well-order is definable by a Δ_1-formula of set theory so in particular by a $\Delta_1(R)$-formula for any predicate R. Thus $\mathcal{L}_{\omega\omega}$ cannot be symbiotic to any R. Stepping forward, $\mathcal{K}$ is definable in $\mathcal{L}_{\omega\omega}(\mathrm{WF})$ thus it seems that $\mathcal{L}_{\omega\omega}(\mathrm{WF})$ is a candidate for being symbiotic with some predicate R and in fact, we will see below that it is symbiotic to the empty predicate.

When considering definability in set theory, in general we have to be very careful whether we talk about being definable *with* or *without* parameters. And when talking about definability of classes of τ-structures, this is especially important for technical reasons: Remember the class $\mathcal{K}$ we considered above. Now notice that $\mathcal{K}$ is *not* the class of all well-orders. Instead, it is the class of all $\{<\}$-structures, where the symbol $<$ is interpreted to be a well-order. And while the class of all well-orders is Δ_1 without parameters, to define $\mathcal{K}$, one additionally has to express, that the elements of $\mathcal{K}$ are $\{<\}$-structures. Now if $<$ is a very complicated set, which is not Δ_1-definable without parameters, then $\mathcal{K}$ is also not Δ_1-definable without parameters, simply because *being a $\{<\}$-structure* is not. Of course, this is some undesirable effect, as if we would do a renaming and instead of the complicated $<$ consider a different binary relation symbol $\prec$, which is a Δ_0-definable set, then the class

$$\mathcal{K}^* = \{(A, \prec^A) \colon\ \prec^A \text{ is a well-order on } A\}.$$

is Δ_1 without parameters. $\mathcal{K}$, on the other hand, is only Δ_1 with parameters in $\{<\}$.

There are two ways to deal with these technical nuisances when comparing definability in a logic and in set theory. The original discussions of symbiosis (compare e.g. [18]) allowed for syntactic objects of the logic, i.e., vocabularies and formulas, as parameters, so when looking at a model class $Mod(\varphi)$ of some τ-sentence φ of a logic $\mathcal{L}$, then when considering whether it is $\Delta_1(R)$, one allows for $\Delta_1(R)$-formulas with parameters in $\{\varphi, \tau\}$. The other option is to restrict attention to vocabularies of a certain well-behaved type. When considering correspondences of model-theoretic properties of a logic and of set-theoretic principles and their application to large cardinal theory as in [1] and [7], or when looking at models of set theory which only interpret a binary relation symbol E for elementship with maybe some additional predicates, it is often sufficient to only consider vocabularies with finitely many symbols. And in this case, as for any finite vocabulary there is a renaming to a Δ_0-definable vocabulary, one can get away with considering "unparametrized" versions of symbiosis.

As large vocabularies are important when studying compactness properties, this latter option is not feasible for us. For example, often extensions of first-order logic satisfy compactness properties that only come into play when talking about very large sets of sentences, say of cardinality at least of the size of some large cardinal. But if a logic has finite syntax, like e.g. $\mathcal{L}^2$, such large sets of sentences exist only over vocabularies which are themselves very large. In particular, we cannot do with only finite vocabularies. Another example: In contexts where compactness is applied, we will often want to consider the elementary diagram of some structure $\mathcal{B}$. But then we need to deal with a vocabulary that contains at least $|B|$-many constant symbols, and thus we can neither guarantee the vocabulary to be finite nor Δ_0-definable. Nevertheless, as this simplifies the discussion and is somewhat the standard, we will first consider the unparametrized version of symbiosis with finite vocabularies, to discuss some examples and recent results. In the rest of this thesis, however, we will want to follow the first option in allowing parameters in the definition of symbiosis. We switch to the parametrized version in 3.2. For technical reasons discussed there, our definition will slightly differ from the original one in [18].

3.1 Restricted Symbiosis

In this section we want to look at the unparametrized version of symbiosis announced above. We will restrict all our definitions and the results mentioned to what we will call *restricted vocabularies*.

Definition 3.1.1. We say that a vocabulary τ is *restricted* iff it is finite and Δ_1-definable without parameters, i.e., there is a Δ_1-formula $\Phi(x)$ in the language of set theory such that $\Phi(a) \leftrightarrow a = \tau$.

Some of the presented may also hold in more general contexts, but to unify assumptions made by different authors, e.g., in [1], [7] and [8], we stick to this case. The following is the notion we are studying.

Definition 3.1.2. Let $\mathcal{L}$ be a logic and R a predicate of set theory. We say that $\mathcal{L}$ and R are *r-symbiotic* iff the following two conditions are fulfilled:

(S1r) For any restricted vocabulary τ, if $\mathcal{K}$ is an $\mathcal{L}$-definable class of τ-structures, then $\mathcal{K}$ is $\Delta_1(R)$-definable without parameters.

(S2r) For any restricted vocabulary τ, if $\mathcal{K}$ is a class of τ-structures, which is $\Delta_1(R)$-definable without parameters and closed under isomorphism, then $\mathcal{K}$ is $\Delta(\mathcal{L})$-definable.

We put the attribute r to indicate that we only work with restricted vocabularies. Note that model classes of sentences of a logic are always closed under isomorphism, so we cannot do without the qualification of $\mathcal{K}$ being closed that way in (S2r).

(S2r) is often substituted by a condition involving R-correct models of set theory. For this we define the following class for a vocabulary $\{E\}$, where E is a Δ_0-definable binary relation symbol:

$$\mathcal{Q}_R := \{(M, E^M) \colon (M, E^M) \text{ is isomorphic to a transitive } R\text{-correct } \in\text{-model}\}.$$

We call $\mathcal{Q}_R$ the *class of all transitive R-correct models.* We then get the following result.

Theorem 3.1.3 (Väänänen)**.** Let $\mathcal{L}$ be a logic and R a predicate of set theory. Then the following are equivalent:

(i) For any restricted vocabulary τ, if $\mathcal{K}$ is a class of τ-structures, which is $\Delta_1(R)$-definable without parameters and closed under isomorphism, then $\mathcal{K}$ is $\Delta(\mathcal{L})$-definable.

(ii) $\mathcal{Q}_R$ is $\Delta(\mathcal{L})$-definable.

Proof. See the proof of Proposition 5.1 in [1] under consideration of restricted vocabularies. □

In practice, one often uses the above condition (ii) to verify that a logic and a predicate are (restrictedly) symbiotic.

Three important examples of symbiosis are the following ones:

1. $\mathcal{L}_{\omega\omega}(I)$ is symbiotic with the predicate of being a cardinal $\mathrm{Cd}(x)$.
2. $\mathcal{L}_{\omega\omega}(WF)$ is symbiotic with the predicate of being the empty set $\emptyset$.
3. $\mathcal{L}^2$ is symbiotic with the power set predicate $\mathrm{Pow}(x, y)$.

In particular, exactly the same classes of structures in restricted vocabularies are $\Delta_1(\mathrm{Pow})$ $(\Delta_1(\mathrm{Cd}), \Delta_1)$ without parameters, as are axiomatizable in $\mathcal{L}^2$ $(\mathcal{L}_{\omega\omega}(I), \mathcal{L}_{\omega\omega}(\mathrm{WF}))$.

The proofs of these statements are standard and can be found e.g. in [8, Section 3]. We want to repeat them in the following three propositions to give an impression of how such proofs of r-symbiosis work. First we will give the proof for Cd and $\mathcal{L}_{\omega\omega}(I)$ in detail. For the others we will be more brief as for them it is easier to verify (S2r) and analogous to verify (S1r).

Proposition 3.1.4. $\mathcal{L}_{\omega\omega}(I)$ and Cd are r-symbiotic.

Proof. First we show (S1r). So let $\mathcal{K}$ be an $\mathcal{L}_{\omega\omega}(I)$-axiomatizable class of structures in a restricted vocabulary τ. Then there is a sentence $\varphi \in \mathcal{L}_{\omega\omega}(I)[\tau]$ defining $\mathcal{K}$. Remember that the satisfaction relation of $\mathcal{L}_{\omega\omega}(I)$ can be stated as the first-order satisfaction relation including an additional clause in the inductive definition saying $\mathcal{A} \models_{\mathcal{L}_{\omega\omega}(I)} Ixy\chi(x)\psi(y)$ iff

$$|\{a \in A\colon \mathcal{A} \models_{\mathcal{L}_{\omega\omega}(I)} \chi(a)\}| = |\{a \in A\colon \mathcal{A} \models_{\mathcal{L}_{\omega\omega}(I)} \psi(a)\}|.$$

It is easy to see that having the same cardinality is $\Delta_1(\mathrm{Cd})$, so is absolute for transitive Cd-correct models. Using this and the first-order satisfaction relation being absolute, one sees that "$\mathcal{A} \models_{\mathcal{L}_{\omega\omega}(I)} \varphi$" is absolute for transitive Cd-correct models. Then the following gives a $\Sigma_1(\mathrm{Cd})$ definition of $\mathcal{K} = \mathrm{Mod}(\varphi)$, with ZFC* some large enough finite fragment of ZFC. $\mathcal{A} \in \mathcal{K}$ iff

$$\begin{aligned}&\exists M(M \models \mathrm{ZFC}^* \wedge M \text{ is transitive} \wedge \forall x \in M(\mathrm{Cd}^M(x) \leftrightarrow \mathrm{Cd}(x)) \wedge \\ &\quad \mathcal{A} \in M \wedge M \models \text{“}\mathcal{A} \models_{\mathcal{L}_{\omega\omega}(I)} \varphi\text{”}).\end{aligned}$$

Being transitive is Δ_0, first-order satisfaction $\models$ is Δ_1, the formula $\mathrm{Cd}(x)$ saying that something is a cardinal is trivially $\Delta_1(\mathrm{Cd})$, so this is $\Sigma_1(\mathrm{Cd})$. The forward direction holds, because if "$\mathcal{A} \models_{\mathcal{L}_{\omega\omega}(I)} \varphi$" holds in the universe, this is reflected by some model, and the backwards direction holds because of (upwards) absoluteness of "$\mathcal{A} \models_{\mathcal{L}_{\omega\omega}(I)} \varphi$". Note that "$\forall x \in M(\mathrm{Cd}^M(x) \leftrightarrow \mathrm{Cd}(x))$" says that M is Cd-correct. Note further that because τ is restricted and the syntax of $\mathcal{L}_{\omega\omega}(I)$ is Δ_1-definable, also φ and "being a τ-structure" are Δ_1-definable without parameters. Thus we can do without including φ and τ as a parameter.

Similarly we get a Π_1-definition. $\mathcal{A} \in \mathcal{K}$ iff

$$\begin{aligned}&\forall M([M \models \mathrm{ZFC}^* \wedge M \text{ is transitive} \wedge \forall x \in M(\mathrm{Cd}^M(x) \leftrightarrow \mathrm{Cd}(x)) \wedge \\ &\quad \mathcal{A} \in M] \to \Phi^M(\mathcal{A})).\end{aligned}$$

By similar reasons, this is $\Pi_1(\mathrm{Cd})$. Note that the forward direction holds because of downwards absoluteness of Φ and the backwards direction is clear.

And now we show (S2r) by verifying the equivalent condition from 3.1.3. I.e., we have to show that

$$\mathcal{Q}_{Cd} = \{(M, E^M)\colon (M, E^M) \text{ is isomorphic to a transitive Cd-correct } \in\text{-model}\}$$

is $\Delta(\mathcal{L}_{\omega\omega}(I))$-definable. $\mathcal{M} = (M, E^M) \in \mathcal{Q}_R$ iff it satiesfies the following three conditions:

1. (M, E^M) is well-founded.
2. (M, E^M) is a model of the extensionality axiom.
3. $(M, E^M) \models_{\mathcal{L}_{\omega\omega}(I)} \forall x(\mathrm{Cd}(x) \to \forall yEx\neg Izz(zEy)(zEx))$.

To see this, it is easy to verify that if $\mathcal{M} \in \mathcal{Q}_{Cd}$, it satisfies the three conditions. For the other direction, if 1 and 2 hold, $\mathcal{M}$ is isomorphic to a transitive $\in$-model $\mathcal{N} = (N, \in)$ by Mostowski's Collapsing Theorem (compare e.g. [9, Theorem 6.15]). Note that as $\mathcal{M}$ and $\mathcal{N}$ are isomorphic, also $\mathcal{N}$ satisfies 3, if $\mathcal{M}$ does. But then we can also show $\mathcal{N}$ to be Cd-correct: Because Cd is a Π_1 predicate, it is downwards absolute. Thus we only have to show that if $\mathcal{N}$ thinks something is a cardinal, it actually is. Now if $\mathcal{N}$ satisfies 3, then if $\mathcal{M} \models \mathrm{Cd}(\alpha)$ for some ordinal α, then all elements of α are not in bijection to it. Thus it has to be a cardinal.

The extensionality axiom is expressed by a first-order statement, while the sentence in 3 is one of $\mathcal{L}_{\omega\omega}(I)$. In particular both are $\Delta(\mathcal{L}_{\omega\omega}(I))$ expressible. To show that the the conjunction of the three conditions is $\Delta(\mathcal{L}_{\omega\omega}(I))$-axiomatizable, it is thus sufficient to show that this holds for being well-founded. For this purpose let $\mathcal{K} = \{(M, E^M) \colon E^M \text{ is well-founded}\}$. Suppose the language of $\mathcal{K}$ is $\{s_0, E\}$ where s_0 is a sort symbol and E a binary relation symbol. First we show that the complement $\bar{\mathcal{K}}$ of $\mathcal{K}$, i.e., the class of *not* well-founded structures, is $\Sigma(\mathcal{L}_{\omega\omega}(I))$-definable. For this, take a new unary predicate P in sort s_0 and consider the sentence

$$\varphi := \exists z P(z) \wedge \forall x (P(x) \to \exists y (P(y) \wedge yEx))$$

Then $Mod(\varphi) \restriction \{s_0, E\} = \bar{\mathcal{K}}$: If $\mathcal{A} \in \bar{\mathcal{K}}$, then $E^{\mathcal{A}}$ is not well-founded, so there is a non-empty subset $X \subseteq A$ with no E-least element. Then letting $P^{\mathcal{A}} := X$, we have that $(\mathcal{A}, P^{\mathcal{A}}) \models \varphi$. And if $\mathcal{A} \models \varphi$, clearly $P^{\mathcal{A}}$ is a non-empty subset of A with no $E^{\mathcal{A}}$-least element, so $(A, E^{\mathcal{A}}) \in \bar{\mathcal{K}}$.

And now we show that $\mathcal{K}$ itself is $\Sigma(\mathcal{L}_{\omega\omega}(I))$-definable. This uses the following equivalence, noticed by Per Lindström in [12]. (M, E^M) is well-founded iff

there is a collection $\{X_a \colon a \in M\}$ of sets such that if $aE^M b$ then $|X_a| < |X_b|$.

Add a new sort symbol s_1 and a new binary relation symbol R between sorts s_0 and s_1 and let $\mathcal{K}^*$ be the class of $\{s_0, E, s_1, R\}$-structures $\mathcal{M} = (M_{s_0}, E^{\mathcal{M}}, M_{s_1}, R^{\mathcal{M}})$ such that the following conditions are fulfilled (where with superscripts $0, 1$ we indicate variables in sort s_0, s_1, respectively):

(a) $(M_{s_0}, E^{\mathcal{M}})$ is a partial order.

(b) $\mathcal{M} \models \forall a^0 b^0 (aEb \to \forall x^1((R(a,x) \to R(b,x)) \wedge \neg I y^1 z^1 R(a,y) R(b,z))$.

Now clearly $\mathcal{K}^*$ is $\mathcal{L}_{\omega\omega}(I)$-definable ((a) is a first-order and (b) an $\mathcal{L}_{\omega\omega}(I)$ condition). So it is sufficient to show that $\mathcal{K}^* \restriction \{s_0, E\} = \mathcal{K}$ to see that $\mathcal{K}$ is $\Sigma(\mathcal{L}_{\omega\omega}(I))$. Now if $\mathcal{M} \in \mathcal{K}^*$, then with $X_a := \{b \in M_{s_1} \colon R^{\mathcal{M}}(a,b)\}$ for $a \in M_0$, by (b) we have that if $aE^{\mathcal{M}}b$, then $|X_a| < |X_b|$, so (M_{s_0}, E^M) is well-founded and thus $\mathcal{M} \restriction \{s_0, E\} \in \mathcal{K}$. And if $(M_{s_0}, E^{\mathcal{M}})$ is well-founded, then take sets X_a for $a \in M_{s_0}$ such that $aE^{\mathcal{M}}b \to |X_a| < |X_b|$. Without loss of generality let $X_a \cap X_b = \emptyset$ for $a \neq b$. Then with $M_{s_1} := \bigcup_{a \in M} X_a$ and $R^{\mathcal{M}} = \{(a,x) \colon a \in M_{s_0}, x \in X_a\}$, we have that $\mathcal{M} := (M_{s_0}, E^M, M_{s_1}, R^{\mathcal{M}}) \in \mathcal{K}^*$. $\square$

Proposition 3.1.5. $\mathcal{L}_{\omega\omega}(\mathrm{WF})$ and $\emptyset$ are r-symbiotic.

Proof. First we show (S1r). So let $\mathcal{K}$ be an $\mathcal{L}_{\omega\omega}(\mathrm{WF})$-axiomatizable class of structures in a restricted vocabulary τ. Then there is a sentence $\varphi \in \mathcal{L}_{\omega\omega}(I)[\tau]$ defining $\mathcal{K}$. Similar to above, "$\mathcal{A} \models_{\mathcal{L}_{\omega\omega}(\mathrm{WF})} \varphi$" is absolute for transitive models, as being well-founded is definable by a Δ_1-formula. Thus we can get Σ_1 and Π_1 definitions of $\mathcal{K}$, respectively: $\mathcal{A} \in \mathcal{K}$ iff one of the following holds.

$\exists M(M \models \mathrm{ZFC}^* \wedge M \text{ is transitive} \wedge \mathcal{A} \in M \wedge M \models \text{“}\mathcal{A} \models_{\mathcal{L}_{\omega\omega}(\mathrm{WF})} \varphi\text{”})$.

$\forall M([M \models \mathrm{ZFC}^* \wedge M \text{ is transitive} \wedge \mathcal{A} \in M] \rightarrow M \models \text{“}\mathcal{A} \models_{\mathcal{L}_{\omega\omega}(\mathrm{WF})} \varphi\text{”})$.

To show (S2r), we have to verify that

$$\mathcal{Q}_\emptyset = \{(M, E^M) \colon (M, E^M) \text{ is isomorphic to a transitive } \in\text{-model}\}$$

is axiomatizable in $\Delta(\mathcal{L}_{\omega\omega}(\mathrm{WF}))$. Note that every transitive $\in$-model is $\emptyset$-correct, so we are justified in leaving this out. Again, by Mostowski's Collapsing Theorem, (M, E^M) is isomorphic to a transitive $\in$-model iff it is well-founded and satisfies the extensionality axiom EXT. Thus $(M, E^M) \in \mathcal{Q}_\emptyset$ iff

$$(M, E^M) \models_{\mathcal{L}_{\omega\omega}(\mathrm{WF})} \mathrm{EXT}_E \wedge \mathrm{WF}xy(xEy),$$

which is a $\mathcal{L}_{\omega\omega}(\mathrm{WF})$-sentence. As $\mathcal{L}_{\omega\omega}(\mathrm{WF}) \leq \Delta(\mathcal{L}_{\omega\omega}(\mathrm{WF}))$ the class $\mathcal{Q}_\emptyset$ is thus axiomatisable in the latter. □

Proposition 3.1.6. $\mathcal{L}^2$ and Pow are r-symbiotic.

Proof. First we show (S1r). So let $\mathcal{K}$ be an $\mathcal{L}^2$-axiomatizable class of structures in a restricted vocabulary τ. Then there is a sentence $\varphi \in \mathcal{L}_{\omega\omega}(I)[\tau]$ defining $\mathcal{K}$. Similar to above, "$\mathcal{A} \models_{\mathcal{L}^2} \varphi$" is absolute for transitive Pow-correct models, as those interpret subsets correctly. Thus we can get $\Sigma_1(\mathrm{Pow})$ and $\Pi_1(\mathrm{Pow})$-definitions of $\mathcal{K}$, respectively: $\mathcal{A} \in \mathcal{K}$ iff one of the following holds.

$\exists M(M \models \mathrm{ZFC}^* \wedge M \text{ is transitive} \wedge \forall x \in M(\mathrm{Pow}^M(x) \leftrightarrow \mathrm{Pow(x)}) \wedge \mathcal{A} \in M \wedge M \models \text{“}\mathcal{A} \models_{\mathcal{L}^2} \varphi\text{”})$.

$\forall M([M \models \mathrm{ZFC}^* \wedge M \text{ is transitive} \wedge \forall x \in M(\mathrm{Pow}^M(x) \leftrightarrow \mathrm{Pow(x)}) \wedge \mathcal{A} \in M] \rightarrow M \models \text{“}\mathcal{A} \models_{\mathcal{L}^2} \varphi\text{”})$.

To show (S2r), we have to verify that

$$\mathcal{Q}_{\mathrm{Pow}} = \{(M, E^M) \colon (M, E^M) \text{ is isomorphic to a transitive Pow-correct} \in\text{-model}\}$$

is axiomatizable in $\Delta(\mathcal{L}_{\omega\omega}(\mathrm{WF}))$. It is clear that being well-founded is expressible by a sentence φ of second-order logic, saying that every non-empty subset of the model has an E-minimal element. Thus it is sufficient to take a $\psi \in \mathcal{L}^2$ which is the conjunction of φ, the extensionality axiom and a second-order sentence χ saying that what a model thinks to be the power set of one of its elements, is the actual power set, to get an

$\mathcal{L}^2$-axiomatization of $\mathcal{Q}_R$. It is clear that the latter can be expressed in second-order logic. But to be more precise, consider the following sentence of second-order logic[4]

$$\psi := \varphi \wedge \mathrm{EXT}_E \wedge \chi$$

with

$$\chi := \forall xy(\mathrm{Pow}_E(x,y) \leftrightarrow \forall Y[\forall v(vEy \leftrightarrow Y(v)) \rightarrow \\ \forall Z(\forall v(Z(v) \rightarrow vEx) \rightarrow \exists z(Y(z) \wedge \forall w(wEz \leftrightarrow Z(w))))]).$$

Again, by Mostowski's Collapsing Theorem, (M, E^M) is isomorphic to a transitive $\in$-model iff it is well-founded and satisfies the extensionality axiom. Thus $(M, E^M) \models \varphi \wedge \mathrm{EXT}$ iff it is isomorphic to a transitive $\in$-model $(N, \in)$. So if we can show that the transitive $\in$-models of χ are precisely the Pow-correct models, i.e., that χ expresses that if $(N, \in)$ thinks y is the power set of x, then it really is the power set of x, then we know that ψ axiomatizes $\mathcal{Q}_{\mathrm{Pow}}$. Notice that because Pow is a Π_1 predicate, so downwards absolute, it is sufficient to show that if $(N, \in) \models \mathrm{Pow}_E(x, y)$ for $x, y \in N$, then y is the power set of x. By $(N, \in) \models \chi$ and $(N, \in) \models \mathrm{Pow}_E(x, y)$, we get that

$$(N, \in) \models \forall Y[\forall v(vEy \leftrightarrow Y(v)) \rightarrow \forall Z(\forall v(Z(v) \rightarrow vEx) \rightarrow \exists z(Y(z) \wedge \forall w(wEz \leftrightarrow Z(w))))].$$

This means, that if Y is an (actual) subset of M, which contains precisely what N thinks is an element of the power set of x (i.e., $vEy \leftrightarrow Y(v)$), then we ought to be able to show that Y is the actual power set of x. And this is the case as then for all (actual) subsets Z of N, if Z is an (actual) subset of x (i.e., $Z(v) \rightarrow vEx$), then there is a $z \in Y$ such that z contains all the elements of Z ($wEz \leftrightarrow Z(w)$). $\square$

As indicated earlier, symbiosis allows to show equivalences between model-theoretic properties of a logic $\mathcal{L}$ and of set-theoretic principles, most importantly reflection principles, involving a predicate R symbiotic with $\mathcal{L}$.

First consider the following properties.

Definition 3.1.7. Let $\mathcal{L}$ be a logic and κ a cardinal. We say

(i) $\mathcal{L}$ has the *downwards Löwenheim-Skolem-Tarski property down to* κ iff for every $\varphi \in \mathcal{L}$, if $\mathcal{A} \models \varphi$, then there is a substructure $\mathcal{B} \subseteq \mathcal{A}$ such that $|B| \leq \kappa$ and $\mathcal{B} \models \varphi$.

(ii) κ is called the *downwards Löwenheim-Skolem-Tarski number* $LST(\mathcal{L})$ *of* $\mathcal{L}$ iff κ is the smallest cardinal such that $\mathcal{L}$ has the downwards Löwenheim-Skolem-Tarski property down to κ.

Bagaria and Väänänen showed in [1] that $\mathrm{LST}(\mathcal{L})$ is related to a reflection principle.

[4] In the sentence χ below, we denote first-order quantification with lowercase letters, e.g. $\forall x$, second-order quantification over subsets with capital letters, e.g. $\forall Y$, and the application of a second-order variables to first-order ones by e.g. $Y(x)$. So, e.g., $M \models \forall Y \exists x(Y(x))$ holds iff every subsets of the model M is non-empty (which of course never occurs as $\emptyset \subseteq M$ for all models).

Definition 3.1.8. Let R be a set-theoretic predicate and κ a cardinal. We take $\mathrm{SR}_R(\kappa)$ to be the statement

> For every class $\mathcal{K}$ of structures in a restricted vocabulary τ, if $\mathcal{K}$ is $\Sigma_1(R)$-definable without parameters, then for every $\mathcal{A} \in \mathcal{K}$, there exists $\mathcal{B} \in \mathcal{K}$ with $|B| \leq \kappa$ and a first-order elementary embedding $\mathcal{B} \preccurlyeq \mathcal{A}$.

If such a cardinal exists, the least κ such that $\mathrm{SR}_R(\kappa)$ holds, is called the *structural reflection number of* R and we write $\mathrm{SR}_R = \kappa$ in this case.

To make everything fit in our framework, our definition differs from that in [1] by considering only classes of structures in restricted vocabularies (while there, countable vocabularies are allowed). With this we get the following theorem.

Theorem 3.1.9 (Bagaria & Väänänen [1])**.** Let $\mathcal{L}$ be a logic, R a set-theoretic predicate and κ a cardinal. Assume $\mathcal{L}$ and R to be r-symbiotic. Then $\mathrm{LST}(\mathcal{L}) = \kappa$ iff $\mathrm{SR}_R = \kappa$.

Proof. Theorem 5.5 in [1] under consideration of only restricted vocabularies. □

The concept of symbiosis allows for versatile similar results that show correspondences between model-theoretic properties of a logic and set-theoretic properties of a symbiotic predicate. Next to downwards Löwenheim-Skolem properties as above, this can also be done for the upwards direction.

Definition 3.1.10. Let $\mathcal{L}$ be a logic and κ a cardinal. κ is called the *upwards Löwenheim-Skolem number ULST*($\mathcal{L}$) *of* $\mathcal{L}$ iff it is the smallest cardinal such that

> for any $\varphi \in \mathcal{L}$, if there is a model $\mathcal{A} \models_{\mathcal{L}} \varphi$ with $|A| \geq \kappa$, then for every $\lambda \geq \kappa$ there is a $\mathcal{B}$ with $|B| \geq \lambda$ and such that $\mathcal{A}$ is a substructure of $\mathcal{B}$.

To show the equivalence of upwards Löwenheim-Skolem properties to reflection principles, Galeotti, Khomskii and Väänänen had to introduce a stronger notion called *bounded* symbiosis (compare [8, Definition 4.7]). Notice that the Δ-closure of a logic plays an essential role in the notion of symbiosis. Unfortunately, upwards Löwenheim-Skolem numbers are generally not preserved by the Δ-closure. It is possible to define a stronger version of it, called the *bounded* Δ-operation Δ^B (compare [19, p. 45]), that involves bounds on the enlargement of structures when adding additional (sort) symbols. For reasons of symmetry, one has to introduce similar concepts of *definably bounding functions* F and $\Sigma_1^F(R)$*-formulas* for those functions (compare [8], definitions 4.3 and 4.5). The notion of bounded symbiosis is then defined using those concepts. Also the following upwards reflection principle uses these bounded versions.

Definition 3.1.11. Let R be a set-theoretic predicate and κ a cardinal. κ is called the *bounded upwards structural reflection number* USR_R iff it is the least cardinal such that for every definably bounding F and every $\Sigma_1^F(R)$-definable class $\mathcal{K}$ of structures in a restricted vocabulary τ which is closed under isomorphisms:

> If there is $\mathcal{A} \in \mathcal{K}$ and $|A| \geq \kappa$, then for every $\lambda \geq \kappa$, there is a $\mathcal{B} \in \mathcal{K}$ with $|B| \geq \lambda$ and a first-order elementary embedding $\mathcal{A} \preccurlyeq \mathcal{B}$.

Again, we changed the definition to only talk about restricted vocabularies, to make it fit into our framework. Then the following theorem can be proved.

Theorem 3.1.12 (Galeotti, Khomskii & Väänänen [8]). Let $\mathcal{L}$ be a logic with Δ_0-definable syntax and $dep(\mathcal{L}) = \omega$ and let R be a Π_1-predicate. Assume $\mathcal{L}$ and R to be boundedly symbiotic. Then $\mathrm{ULST}(\mathcal{L}) = \kappa$ iff $\mathrm{USR}_R = \kappa$.

3.2 Parametrized Symbiosis

We mentioned that large vocabularies are essential when dealing with compactness properties. As r-symbiosis only deals with finite vocabularies, we have to introduce a parametrized version of symbiosis which we call *p-symbiosis*. This concept is able to deal with large vocabularies and therefore we will work with it throughout the rest of this thesis.

Definition 3.2.1. Let $\mathcal{L}$ be a logic and R a predicate of set theory. We say $\mathcal{L}$ and R are p-symbiotic iff the following two conditions are fulfilled:

(S1p) If $\mathcal{K}$ is a class of structures in a vocabulary τ and $\mathcal{L}$-definable by $\varphi \in \mathcal{L}[\tau]$, then $\mathcal{K}$ is $\Delta_1(R)$-definable with parameters in $\{\varphi, \tau\}$.

(S2p) $\mathcal{Q}_R$ is $\Delta(\mathcal{L})$-definable.

Historically, when symbiosis was introduced, one was often interested in restricting the syntax of a logic $\mathcal{L}$ to some transitive class A. The result of this restriction is called $\mathcal{L}_A$ and is defined in the same way as $\mathcal{L}$, except that only those vocabularies and sentences of $\mathcal{L}$ are considered, which are in A. Compare [2] for this concept. The original definition of symbiosis with parameters from [18] involves reference to such logics $\mathcal{L}_A$. It is called *symbiosis on A*. But, just like with considering only finite vocabularies, the logic $\mathcal{L}_A$ is too restrictive for our purposes: Again, suppose we want to work with a large set of sentences and for this it is necessary that we consider a large vocabulary τ. Then it might be, that $\tau \notin A$. Or suppose we want to consider the elementary diagram of some structure $\mathcal{B}$. Then if $\mathcal{B}$ is very large, it might be that there are not enough constant symbols in A to formulate $\mathrm{ElDiag}(\mathcal{B})$. As one often considers $A = H(\kappa)$ for some cardinal κ, both cases can easily occur. For this reason, our definition of parametrized symbiosis slightly differs from that in [18] in being formulated generally for a logic $\mathcal{L}$. The difference lies only in condition (S2p): [18] demands $\mathcal{Q}_R$ to be $\Delta(\mathcal{L}_A)$-definable while our condition is demanding $\Delta(\mathcal{L})$-definability. As we want to work with the usual notion of a regular abstract logic, and not with logics of the type $\mathcal{L}_A$, we make this small adjustment.

Remember that the intuition behind symbiosis between $\mathcal{L}$ and R is, that the same classes are $\Delta_1(R)$ and $\mathcal{L}$-definable. In the above definition, as well as in the definition of symbiosis on A from [18], this is pushed somewhat to the background, as we are dealing with the in some sense more abstract condition (S2p). [18] instead treats the equivalence of definability in $\mathcal{L}$ and by $\Delta_1(R)$-formulas as a theorem we can prove about the notion of symbiosis. We will do the same here. The following is the main theorem on p-symbiosis:

Theorem 3.2.2. Let $\mathcal{L} := \mathcal{L}^*_{\kappa\omega} \geq \mathcal{L}_{\kappa\omega}$ be a logic with $dep^*(\mathcal{L}^*_{\kappa\omega}) = \kappa$ and R a predicate of set-theory. Assume $\mathcal{L}^*_{\kappa\omega}$ and R to be p-symbiotic. Let $\tau \in H(\kappa)$ be a vocabulary. For a class $\mathcal{K}$ of τ-structures, which is closed under isomorphism, the following are equivalent:

(i) $\mathcal{K}$ is $\Delta(\mathcal{L}^*_{\kappa\omega})$-definable.

(ii) $\mathcal{K}$ is $\Delta_1(R)$-definable with parameters in $H(\kappa)$.

Before we prove this theorem, we want to give a few comments, introduce a notation and give a lemma. First, it is possible to prove the same theorem about $\Sigma_1(R)$ and $\Sigma(\mathcal{L}^*_{\kappa\omega})$-definability, but as we will only need the Δ-version and the proof of this is slightly less complicated, we will stick to the variant above. Second, notice that if we chose $\kappa = \omega$, the theorem gives us the definition of r-symbiosis. This is the case, because if τ is in $H(\omega)$, it is restricted and because $\Delta_1(R)$-definability with parameters in $H(\omega)$ coincides with $\Delta_1(R)$-definability: every element of $H(\omega)$ is definable by a $\Delta_1(R)$-formula. Thus we can treat the results from section 3.1 as special cases of p-symbiosis. Third, the combination of $\mathcal{L}^*_{\kappa\omega}$ being an expansion of $\mathcal{L}_{\kappa\omega}$ and $dep^*(\mathcal{L}^*_{\kappa\omega}) = \kappa$ fixes the infinitary means of the logic. On the one hand, $\mathcal{L}^*_{\kappa\omega}$ allows for conjunctions and disjunctions of infinitary but smaller than κ size. On the other hand, the strong dependence number being κ means that the logic cannot have disjunctions of size κ or larger, as every sentence (over a vocabulary in $H(\kappa)$) has to be in $H(\kappa)$ itself. In [18], this fixing of size of formulas is achieved by considering the restriction of the logic $\mathcal{L}^*_{\infty\omega}$ to $H(\kappa)$, i.e., $(\mathcal{L}^*_{\infty\omega})_{H(\kappa)}$, which is given by considering only those sentences of $\mathcal{L}^*_{\infty\omega}$ which are themselves in $H(\kappa)$. In particular, this gets rid of all sentences involving symbols not in $H(\kappa)$. Our approach has the advantage that those are preserved.

The idea behind the proof of Theorem 3.2.2 is that the parameter from $H(\kappa)$ used in the $\Delta_1(R)$-definition of $\mathcal{K}$ is definable in $\mathcal{L}_{\kappa\omega}$. This trick is used in the proof of the main theorem on symbiosis on A in [18] as well. We show how to do this by the following Definition and Lemma.

Definition 3.2.3. Let $a \in H(\kappa)$. For a binary relation symbol E, we inductively define an $\mathcal{L}_{\kappa\omega}$-formula $\delta_a(x)$. If $a = \emptyset$, then let $\delta_\emptyset(x) := \forall y(\neg yEx)$. And if $\delta_b(x)$ is already defined for $b \in a$, then we let

$$\delta_a(x) = \forall y(yEx \leftrightarrow \bigvee_{b \in a} \delta_b(y)).$$

Note that because $a \in H(\kappa)$, all elements of $trcl(\{a\})$ have size $< \kappa$, so all the disjunctions we consider in this definition are of size $< \kappa$, so $\delta_a(x)$ is indeed a formula of $\mathcal{L}_{\kappa\omega}$. Intuitively, $\delta_a(b)$ holds, if the elements of b are precisely the elements of a. Thus $b = a$. We will prove that this intuitive content is the precise one for transitive $\in$-models.

Lemma 3.2.4. Let $a \in H(\kappa)$. Then for all transitive $\in$-models $(M, \in)$ and $c \in M$ the following holds:

$$(M, \in) \models \delta_a(c) \text{ iff } a = c.$$

Proof. Let $(M, \in)$ be transitive and $c \in M$. We prove by $\in$-induction that $(M, \in) \models \delta_a(c)$ iff $a = c$. If $a = \emptyset$, then the assertion is clear, as M is transitive. So suppose $a \in H(\kappa)$ and as induction hypothesis, that for all $b \in a$ and $d \in M$, we have that $M \models \delta_b(d)$ iff $b = d$.

Then if (i) $(M, \in) \models \delta_a(c)$, we have to show that $a = c$. First, if $y \in a$, by the induction hypothesis we have $(M, \in) \models \delta_y(y)$, so $(M, \in) \models \bigvee_{b \in a} \varphi_b(y)$, thus by (i) $(M, \in) \models yEc$, so $y \in c$. And if $y \in c$, then by (i) $(M, \in) \models \bigvee_{b \in a} \delta_b(y)$, so there is a $b \in a$ such that by the induction hypothesis $b = y$. Thus $a = c$.

And now if (ii) $a = c$, with $c \in M$. We have to show that $(M, \in) \models \varphi_a(c)$. First if $(M, \in) \models yEc$, then $y \in c = a$, so by the induction hypothesis $(M, \in) \models \varphi_y(y)$, so $(M, \in) \models \bigvee_{b \in a} \varphi_b(y)$. And if $(M, \in) \models \bigvee_{b \in a} \varphi_b(y)$, then by the induction hypothesis, $y = b$ for some $b \in a$. As $a = c$, thus $y \in c$, so $(M, \in) \models yEc$. $\square$

Now we can give the proof of Theorem 3.2.2.

Proof of Theorem 3.2.2. For the direction from (i) to (ii), let $\varphi \in \mathcal{L}[\tau]$ with $\mathcal{K} = Mod(\varphi)$. Then by condition (S1p) of p-symbiosis, $Mod(\varphi)$ is $\Delta_1(R)$-definable with parameters in $\{\varphi, \tau\}$. We have $\tau \in H(\kappa)$, and together with $dep^*(\mathcal{L}) = \kappa$ also $\varphi \in H(\kappa)$. Thus $Mod(\varphi) = \mathcal{K}$ is $\Delta_1(R)$-definable with parameters in $H(\kappa)$.

Now we prove the direction from (ii) to (i). It goes mostly analogously to the proof of the main theorem on symbiosis on A from [18, Theorem 2.4]. By assumption there is a $p \in H(\kappa)$ and a formula $\Phi(x, y)$ which is $\Delta_1(R)$ in set theory, such that $\mathcal{K}$ is defined by $\Phi(x, p)$, i.e.,

$$\mathcal{A} \in \mathcal{K} \leftrightarrow \Phi(\mathcal{A}, p).$$

For simplicity we assume that there is only one sort symbol $j \in \tau$ and that τ is relational.

Consider a new sort symbol i and binary relation symbol E in sort i. For $a \in H(\kappa)$, take the $\mathcal{L}^*_{\kappa\omega}$ formula $\delta_a(x)$ defined above such that for any transitive $\in$-model (M, E^M) and for all $b \in M$, we have $M \models \varphi_a(b)$ iff $a = b$ (notice that $\mathcal{L}_{\kappa\omega} \leq \mathcal{L}^*_{\kappa\omega}$, so we are justified to consider δ_a as a formula of $\mathcal{L}^*_{\kappa\omega}$). In particular, as $\tau \in H(\kappa)$, we can consider the formula $\varphi_\tau(x)$. Remember that being a structure over some vocabulary σ is expressible by a Δ_1-formula of set theory $s(x, \sigma)$ with parameters in $\{\sigma\}$ (compare Proposition 2.2.6). As $\varphi_\tau(x)$ defines τ, we can use this to formulate an $\mathcal{L}$-formula $\chi(x)$ such that for every transitive $\in$-model $\mathcal{M}$ of set theory and any $a \in M$ we have

$$a \text{ is a } \tau\text{-structure} \leftrightarrow \mathcal{M} \models \chi(a)$$

by letting

$$\chi(x) = \exists t(\varphi_\tau(t) \wedge s(x, t)),$$

written in the vocabulary E. Further, $\tau \subseteq H(\kappa)$, so we can consider $\delta_P(x)$ for every $P \in \tau$.

Now let $\mathcal{K}^*$ be the class of structures $\mathcal{M} = (M_i, E^{\mathcal{M}}, f^{\mathcal{M}}, c_1^{\mathcal{M}}, c_2^{\mathcal{M}}, A_j, \dots)$ in the vocabulary $\tau \cup \{i, E, f, c_1, c_2\}$, where f is a function of the domain M_i with sort i to the domain A_j of sort j and c_1 and c_2 are constant symbols in sort i, which satisfy the following conditions (here the dots indicate the presence of interpretations of the symbols in τ). Remember that $\#(P)$ is the function returning the arity of P.

1. $(M_i, E^{\mathcal{M}}) \in \mathcal{Q}_R$.
2. $\mathcal{M} \models \delta_p(c_2)$.
3. $\mathcal{M} \models \chi(c_1) \wedge c_1 = (m, g) \wedge f \upharpoonright \{y\colon yEm\}$ is a bijection $\wedge$ $\bigwedge_{P\in\tau} \forall a, b(\delta_P(a) \wedge g(a) = b \to \forall x_1 \ldots x_{\#(P)}[(x_1, \ldots, x_{\#(P)})Eb \leftrightarrow P(f(x_1), \ldots, f(x_{\#(P)}))])$.
4. $\mathcal{M} \models \Phi(c_1, c_2)$.

Then $\mathcal{K}^*$ is definable by a $\Delta(\mathcal{L})$-sentence: 1 is expressible by a $\Delta(\mathcal{L})$-sentence by (S2p) of p-symbiosis and 2,3 and 4 are $\mathcal{L}_{\kappa\omega}$-sentences. Note that the conjunctions in 2 and 3, respectively, are of size $< \kappa$, as $p, \tau \in H(\kappa)$. Thus there is a $\psi \in \Delta(\mathcal{L})$ such that $Mod(\psi) = \mathcal{K}^*$.

The intuition is that $\mathcal{M}$ is a model that thinks that c_1 is a τ-structure in $\mathcal{K}$ (by 2 and 4), that by 1 it is R-correct and thus will be right in its judgement about c_1 and that by 3, f is an isomorphism between the elements of c_1 and the τ-part of the model. For this, notice that if $c_1^{\mathcal{M}} = (m, g)$ is a τ-structure, then g is a function with domain τ, returning the interpretations of the symbols in τ. Thus if $\delta_P(a)$ holds and $g(a) = b$, then b is an interpretation of the symbol P and $(x_1, \ldots, x_{\#(P)})Eb \leftrightarrow P(f(x_1), \ldots, f(x_{\#(P)}))$ formulates the condition of f preserving this interpretation.

We now claim that $Mod(\psi) \upharpoonright \tau = \mathcal{K}$.

To show this, first let $\mathcal{A} \in \mathcal{K}$. We have to construct an expansion of $\mathcal{A}$ which satisfies ψ. Because $\mathcal{A} \in \mathcal{K}$, $\Phi(\mathcal{A}, p)$ holds. By the Reflection Theorem (compare e.g. [9, Theorem 12.14]) take an α such that V_α reflects R and $\Phi(x, y)$ and with $\mathcal{A}, p, H(\kappa) \in V_\alpha$. Then $(V_\alpha, \in)$ is transitive and R-correct. Now let

$$\mathcal{M} = (V_\alpha, \in, f^{\mathcal{M}}, c_1^{\mathcal{M}} = \mathcal{A}, c_2^{\mathcal{M}} = p, A_j = A, P^{\mathcal{A}})_{P\in\tau},$$

where $f^{\mathcal{M}}$ is the identity on A and takes any value on the rest of V_α and every $P \in \tau$ is interpreted as in $\mathcal{A}$. As V_α is transitive and R-correct, we have that $(V_\alpha, \in) \in \mathcal{Q}_R$, so $\mathcal{M}$ fulfils 1. Because V_α is absolute for $\Phi(x, y)$ and $\Phi(\mathcal{A}, p)$ holds, also $V_\alpha \models \Phi(\mathcal{A}, p)$. As $c_1^{\mathcal{M}} = \mathcal{A}$ and $c_2^{\mathcal{M}} = p$, thus $\mathcal{M} \models \Phi(c_1, c_2)$, so 4 holds. As $p = c_2^{\mathcal{M}}$ and V_α is a transitive $\in$-model, also 2 holds. Finally 3 holds, as $c_1^{\mathcal{M}} = \mathcal{A}$ is a τ-structure and $f^{\mathcal{M}}$ is the identity on A, so an isomorphism. Thus $\mathcal{M} \models \psi$. As $\mathcal{M} \upharpoonright \tau = \mathcal{A}$, this is what we had to show.

And now let $\mathcal{M} \models \psi$. We have to show that $\mathcal{A} := \mathcal{M} \upharpoonright \tau \in \mathcal{K}$. As $(M, E^{\mathcal{M}}) \in \mathcal{Q}_R$, the structure $\mathcal{M}$ is isomorphic to $\mathcal{N} = (N, \in, f^{\mathcal{N}}, c_1^{\mathcal{N}}, c_2^{\mathcal{N}}, N_j, P^{\mathcal{N}})_{P\in\tau}$, where $(N, \in)$ is transitive and R-correct. By $\mathcal{N} \models 2$, we have that $c_2^{\mathcal{N}} = p$. By $\mathcal{N} \models 3$, we have that $c_1^{\mathcal{N}}$ is a τ-structure which is isomorphic to the τ-part of the model. Because $\mathcal{N} \models \Phi(c_1, c_2)$, we have that $\mathcal{N}$ thinks that $c_2^{\mathcal{N}}$ is in $\mathcal{K}$. As $\mathcal{N}$ is R-correct, it is upwards absolute for $\Sigma_1(R)$-formulas of set theory, in particular for Φ. Thus $\Phi(c_1^{\mathcal{N}}, c_2^{\mathcal{N}})$ really holds. Because

$c_2^{\mathcal{N}} = p$, therefore $c_1^{\mathcal{N}} \in \mathcal{K}$. As $c_1^{\mathcal{N}}$ is isomorphic to the τ-part of $\mathcal{N}$ and $\mathcal{K}$ is closed under isomorphism, we have $\mathcal{N} \upharpoonright \tau \in \mathcal{K}$. Then as $\mathcal{N}$ is isomorphic to $\mathcal{M}$, also $\mathcal{A} = \mathcal{M} \upharpoonright \tau \in \mathcal{K}$.

Because $Mod(\psi) \upharpoonright \tau = \mathcal{K}$, the latter is $\Sigma(\Delta(\mathcal{L}))$. Because $\mathcal{K}$ is $\Delta_1(R)$ with parameters in $H(\kappa)$, also the compliment $\bar{\mathcal{K}}$ of $\mathcal{K}$ is $\Delta_1(R)$ with parameters in $H(\kappa)$ and we can analogously show that $\bar{\mathcal{K}}$ is $\Sigma(\Delta(\mathcal{L}))$. Thus, $\mathcal{K}$ is $\Delta(\Delta(\mathcal{L}))$ and therefore also $\Delta(\mathcal{L})$-definable (as $\Delta(\Delta(\mathcal{L})) \equiv \Delta(\mathcal{L})$ by Theorem 2.8.4). $\square$

Notice that r-symbiosis between $\mathcal{L}$ and R adheres to the intuitive content of symbiosis in fixing an equivalence between a class (in a restricted vocabulary) being $\Delta(\mathcal{L})$ and $\Delta_1(R)$-definable. p-symbiosis, on the other, does not fix such an equivalence of definability alone. Instead, one additionally needs to fix the "infinitary-level" on which the logic and the $\Delta_1(R)$-formulas operate (for the former by its strong dependence number and the size of its conjunctions, for the latter by parameters from which $H(\kappa)$ are allowed).

That p-symbiosis alone does not fix an equivalence of definability as indicated above becomes transparent when looking at examples: Vastly different logics can be p-symbiotic to the same predicate R. To see this, we will now generalize the results on r-symbiotic pairs from the previous section to p-symbiosis.

Proposition 3.2.5. Let κ be a cardinal. Then $\mathcal{L}_{\kappa\omega}(I)$ and Cd are p-symbiotic.

Proof. To verify (S1p), let $\mathcal{K}$ be defined by $\varphi \in \mathcal{L}_{\kappa\omega}(I)[\tau]$. We have to show that $\mathcal{K}$ is $\Delta_1(\mathrm{Cd})$-definable with parameters in $\{\varphi, \tau\}$. Notice that $\mathcal{A} \models_{\mathcal{L}_{\kappa\omega}(I)} \varphi$ iff $\mathcal{A} \models_{\mathcal{L}_{\infty\omega}(I)} \varphi$.[5] As $\models_{\mathcal{L}_{\infty\omega}}$ is absolute (see section 2.6), similarly to the proof of Proposition 3.1.4, we get that "$\mathcal{A} \models_{\mathcal{L}_{\infty\omega}(I)} \varphi$" is absolute for transitive Cd-correct models of set theory, which include φ. Thus the following give $\Sigma_1(\mathrm{Cd})$ and $\Pi_1(\mathrm{Cd})$-definitions of $\mathcal{K}$ with parameters in $\{\varphi, \tau\}$, respectively.

$\mathcal{A}$ is a τ-structure $\wedge \exists M(M \models \mathrm{ZFC}^* \wedge M$ is transitive$\wedge \forall x \in M(\mathrm{Cd}^M(x) \leftrightarrow Cd(x)) \wedge \mathcal{A} \in M \wedge \varphi \in M \wedge M \models$ "$\mathcal{A} \models_{\mathcal{L}_{\infty\omega}(I)} \varphi$").

$\mathcal{A}$ is a τ-structure $\wedge \forall M([M \models \mathrm{ZFC}^* \wedge M$ is transitive$\wedge \forall x \in M(\mathrm{Cd}^M(x) \leftrightarrow Cd(x)) \wedge \mathcal{A} \in M \wedge \varphi \in M] \to M \models$ "$\mathcal{A} \models_{\mathcal{L}_{\infty\omega}(I)} \varphi$").

To verify (S2p), notice that in Proposition 3.1.4 we already showed that $\mathcal{Q}_{\mathrm{Cd}}$ is $\Delta(\mathcal{L}_{\omega\omega}(I))$-definable. Therefore it is also $\Delta(\mathcal{L}_{\kappa\omega}(I))$-definable. $\square$

Proposition 3.2.6. Let κ be a cardinal. Then $\mathcal{L}_{\kappa\omega}(\mathrm{WF})$ and $\emptyset$ are p-symbiotic.

Proof. To verify (S1p), let $\mathcal{K}$ be defined by $\varphi \in \mathcal{L}_{\kappa\omega}(\mathrm{WF})[\tau]$. We have to show that $\mathcal{K}$ is $\Delta_1(\emptyset)$, so Δ_1-definable, with parameters in $\{\varphi, \tau\}$. As above, $\mathcal{A} \models_{\mathcal{L}_{\kappa\omega}(\mathrm{WF})} \varphi$ iff

[5] Note that in contrast to the proofs of r-symbiosis, below we make reference to a different logic than $\mathcal{L}_{\kappa\omega}(I)$ while showing that $\mathcal{K}$ is $\Delta_1(R)$. So the line of argument goes like this: We fix $\mathcal{K}$ as the model class of some $\varphi \in \mathcal{L}_{\kappa\omega}(I)$. We notice that $\mathcal{K}$ is also the model class of some sentence (which happens to be φ itself) in a different logic, namely $\mathcal{L}_{\infty\omega}(I)$. We then use $\mathcal{L}_{\infty\omega}(I)$ to get a $\Delta_1(R)$-definition of $\mathcal{K}$ with parameters in $\{\varphi, \tau\}$. We do this, because to express "$\mathcal{A} \models_{\mathcal{L}_{\kappa\omega}(I)} \varphi$", one has to include κ as a parameter.

$\mathcal{A} \models_{\mathcal{L}_{\infty\omega}(\mathrm{WF})} \varphi$. As $\models_{\mathcal{L}_{\infty\omega}}$ is absolute and being well-founded is Δ_1, similarly to the proof of Proposition 3.2.5, we get that "$\mathcal{A} \models_{\mathcal{L}_{\infty\omega}(\mathrm{WF})} \varphi$" is absolute for transitive $\in$-models of set theory, which include φ. Thus the following give Σ_1 and Π_1-definitions of $\mathcal{K}$ with parameters in $\{\varphi, \tau\}$, respectively.

> $\mathcal{A}$ is a τ-structure $\wedge \exists M(M \models \mathrm{ZFC}^* \wedge M$ is transitive $\wedge\, \mathcal{A} \in M \wedge \varphi \in M \wedge$ $M \models$ "$\mathcal{A} \models_{\mathcal{L}_{\infty\omega}(\mathrm{WF})} \varphi$").
>
> $\mathcal{A}$ is a τ-structure $\wedge \forall M([M \models \mathrm{ZFC}^* \wedge M$ is transitive $\wedge\, \mathcal{A} \in M \wedge \varphi \in M] \rightarrow$ $M \models$ "$\mathcal{A} \models_{\mathcal{L}_{\infty\omega}(\mathrm{WF})} \varphi$").

To verify (S2p), again we already showed in Proposition 3.2.6 that $\mathcal{Q}_\emptyset$ is $\Delta(\mathcal{L}_{\omega\omega}(\mathrm{WF}))$-definable. Therefore it is also $\Delta(\mathcal{L}_{\kappa\omega}(\mathrm{WF}))$-definable. $\square$

Proposition 3.2.7. Let κ be a cardinal. Then $\mathcal{L}^2_{\kappa\omega}$ and Pow are p-symbiotic.

Proof. To verify (S1p), let $\mathcal{K}$ be defined by $\varphi \in \mathcal{L}^2_{\kappa\omega}[\tau]$. We have to show that $\mathcal{K}$ is $\Delta_1(\mathrm{Pow})$-definable with parameters in $\{\varphi, \tau\}$. Once again, notice that $\mathcal{A} \models_{\mathcal{L}^2_{\kappa\omega}} \varphi$ iff $\mathcal{A} \models_{\mathcal{L}^2_{\infty\omega}} \varphi$. As $\models_{\mathcal{L}_{\infty\omega}}$ is absolute, similarly to the proof of Proposition 3.2.5, we get that "$\mathcal{A} \models_{\mathcal{L}^2_{\infty\omega}} \varphi$" is absolute for transitive Pow-correct models of set theory, which include φ. Thus the following give $\Sigma_1(\mathrm{Pow})$ and $\Pi_1(\mathrm{Pow})$-definitions of $\mathcal{K}$ with parameters in $\{\varphi, \tau\}$, respectively.

> $\mathcal{A}$ is a τ-structure $\wedge \exists M(M \models \mathrm{ZFC}^* \wedge M$ is transitive $\wedge \forall x \in M(\mathrm{Pow}^M(x) \leftrightarrow \mathrm{Pow}(x)) \wedge$ $\mathcal{A} \in M \wedge \varphi \in M \wedge M \models$ "$\mathcal{A} \models_{\mathcal{L}^2_{\infty\omega}} \varphi$").
>
> $\mathcal{A}$ is a τ-structure $\wedge \forall M([M \models \mathrm{ZFC}^* \wedge M$ is transitive $\wedge \forall x \in M(\mathrm{Pow}^M(x) \leftrightarrow \mathrm{Pow}(x)) \wedge$ $\mathcal{A} \in M \wedge \varphi \in M] \rightarrow M \models$ "$\mathcal{A} \models_{\mathcal{L}^2_{\infty\omega}} \varphi$").

To verify (S2p), as above notice that in Proposition 3.1.6 we already showed that $\mathcal{Q}_{\mathrm{Pow}}$ is $\Delta(\mathcal{L}^2)$-definable. Therefore it is also $\Delta(\mathcal{L}^2_{\kappa\omega})$-definable. $\square$

As a direct corollary of Theorem 3.2.2 and the above propositions we get

Corollary 3.2.8. Let κ be a regular cardinal. Then we have the following equivalences for a class $\mathcal{K}$ of structures in a vocabulary $\tau \in H(\kappa)$ closed under isomorphism.

1. $\mathcal{K}$ is $\Delta(\mathcal{L}_{\kappa\omega}(\mathrm{WF}))$-definable iff $\mathcal{K}$ is Δ_1 with parameters in $H(\kappa)$.
2. $\mathcal{K}$ is $\Delta(\mathcal{L}_{\kappa\omega}(I))$-definable iff $\mathcal{K}$ is $\Delta_1(\mathrm{Cd})$ with parameters in $H(\kappa)$.
3. $\mathcal{K}$ is $\Delta(\mathcal{L}^2_{\kappa\omega})$-definable iff $\mathcal{K}$ is $\Delta_1(\mathrm{Pow})$ with parameters in $H(\kappa)$.

Proof. By Theorem 3.2.2, it is sufficient to show that for $\mathcal{L}^*$ any of the above, $\mathcal{L}^*$ is an expansion of $\mathcal{L}_{\kappa\omega}$, has strong dependence number κ and is p-symbiotic to its respective predicate. The first is clear, the second is shown in Theorem 2.7.7 in section 2.7 for regular κ. The latter is shown in the propositions above. $\square$

An analogous theorem to the above is shown for symbiosis on A in [18, Corollary 2.5].

We mentioned that very different logics can be p-symbiotic to the same predicate. The examples confirm this: For example, $\mathcal{L}^2$ and $\mathcal{L}^2_{\kappa\omega}$ for κ the first inaccessible κ are both p-symbiotic to Pow, even though the latter logic allows for very large conjunctions while the former is finitary. But this does not mean that $\mathcal{L}^2$ and $\mathcal{L}^2_{\kappa\omega}$ can define the same classes of structures, as p-symbiosis to a predicate alone does not fix what can be defined in a logic. What p-symbiosis gives us, is that $\mathcal{L}^2$ can define all classes of structures in a vocabulary $\tau \in H(\omega)$ which are $\Delta_1(\text{Pow})$-definable. And further, that $\mathcal{L}^2_{\kappa\omega}$ can define all classes of structures in vocabularies $\tau \in H(\kappa)$ which are $\Delta_1(\text{Pow})$ with parameters in $H(\kappa)$.

We want to end this section by proving some results about the status of $\Delta(\mathcal{L})$ whenever $\mathcal{L}$ is p-symbiotic to some R. Notice that p-symbiosis is somewhat of a mix of being a relation between $\mathcal{L}$ and R or between $\Delta(\mathcal{L})$ and R. While Theorem 3.2.2 establishes an equivalence between $\Delta(\mathcal{L})$ and $\Delta_1(R)$-definability, only the condition (S2p) of the definition of p-symbiosis talks about $\Delta(\mathcal{L})$, while (S1p) is formulated for $\mathcal{L}$. Nevertheless we get the following result.

Lemma 3.2.9. Let $\mathcal{L}$ be a logic and R a set-theoretic predicate. Assume $\mathcal{L}$ and R to be p-symbiotic. Then also $\Delta(\mathcal{L})$ and R are p-symbiotic.

Proof. To show p-symbiosis between $\Delta(\mathcal{L})$ and R we have to verify (S1p) and (S2p). Note that the latter follows trivially from (S2p) holding for $\mathcal{L}$ and the fact that $\Delta(\mathcal{L}) \equiv \Delta(\Delta(\mathcal{L}))$. But the former does *not* follow from this fact, as by (S1p) holding of $\mathcal{L}$, we only get that all model classes of $\mathcal{L}$-sentences are definable by $\Delta_1(R)$-formulas with parameters. To show (S1p) for $\Delta(\mathcal{L})$, we further have to show that every model class definable by a $\Delta(\mathcal{L})$-sentence is $\Delta_1(R)$ with parameters.

So let $\varphi \in \Delta(\mathcal{L})[\tau]$ for some vocabulary τ. We have to show that $Mod(\varphi)$ is $\Delta_1(R)$-definable with parameters in $\{\varphi, \tau\}$. As $\varphi \in \Delta(\mathcal{L})[\tau]$, it is a tuple $\varphi = (\varphi_1, \tau_1, \varphi_2, \tau_2) \in \Delta(\mathcal{L})[\tau]$ with τ_i finite and $\varphi_i \in \mathcal{L}[\tau \cup \tau_i]$ for $i = 1, 2$ and for all τ-structures $\mathcal{A}$, either there is an expansion $\mathcal{A}_1$ of $\mathcal{A}$ to a $(\tau \cup \tau_1)$-structure and $\mathcal{A}_1 \models_{\mathcal{L}} \varphi_1$ or there is an expansion $\mathcal{A}_2$ of $\mathcal{A}$ to a $(\tau \cup \tau_2)$-structure and $\mathcal{A}_2 \models_{\mathcal{L}} \varphi_2$. As φ_i and $\tau \cup \tau_i$ can be retrieved from φ and τ by $\Delta_1(R)$-operations, it is sufficient to show that we can define $Mod(\varphi)$ by a $\Delta_1(R)$-formula with parameters in $P := \{\varphi, \tau, \varphi_1, \tau \cup \tau_1, \varphi_2, \tau \cup \tau_2\}$. Let $\mathcal{K} := Mod(\varphi)$. We show that $\mathcal{K}$ is $\Sigma_1(R)$-definable with parameters in P: $\mathcal{A} \in \mathcal{K}$ iff

$$\exists \mathcal{A}^* (\mathcal{A}^* \text{ is a } (\tau \cup \tau_1)\text{-structure and an expansion of the } \tau\text{-structure } \mathcal{A} \text{ and } \mathcal{A}^* \models_{\mathcal{L}} \varphi_1).$$

Because $\tau \cup \tau_1 \in P$, being a $(\tau \cup \tau_1)$-structure and an expansion of a τ-structure $\mathcal{A}$ is $\Delta_1(R)$ with parameters in P. Being a model of the $\mathcal{L}$-sentence φ_1 over the vocabulary $\tau \cup \tau_1$ is $\Delta_1(R)$ with parameters in $\{\varphi_1, \tau \cup \tau_1\} \subseteq P$ as by assumption $\mathcal{L}$ and R are p-symbiotic (so $Mod(\varphi_1)$ is $\Delta_1(R)$-definable with parameteris in $\{\varphi_1, \tau \cup \tau_1\}$). Thus being in $\mathcal{K}$ is $\Sigma_1(R)$ with parameters in P.

Using φ_2 and τ_2 one similarly shows that being in the complement of $\mathcal{K}$ is $\Sigma_1(R)$ with parameters in P. Thus $\mathcal{K} = Mod(\varphi)$ is $\Delta_1(R)$ with parameters in P.

□

Using this we get the following characterization of $\Delta(\mathcal{L})$ for p-symbiotic logics.

Theorem 3.2.10. Let $\mathcal{L}^*_{\kappa\omega} \geq \mathcal{L}_{\kappa\omega}$ be a logic with $dep^*(\mathcal{L}^*_{\kappa\omega}) = \kappa$ and R a set-theoretic predicate. Assume $\mathcal{L}^*_{\kappa\omega}$ and R to be p-symbiotic. Then $\Delta(\mathcal{L}^*_{\kappa\omega})$ is the strongest logic with strong dependence number κ satisfying (S1p) of p-symbiosis with R. More precisely, if $\mathcal{L}$ is any logic with $dep^*(\mathcal{L}) \leq \kappa$ and which satisfies (S1p) of p-symbiosis with R, then $\mathcal{L} \leq \Delta(\mathcal{L}^*_{\kappa\omega})$.

Proof. $\Delta(\mathcal{L}^*_{\kappa\omega})$ satisfies (S1p) of p-symbiosis by the proposition above and has strong dependence number κ by Theorem 2.8.4. To show what is left, let $\mathcal{L}$ be a logic with $dep^*(\mathcal{L}) \leq \kappa$ which satisfies (S1p) of p-symbiosis. To show that $\mathcal{L} \leq \Delta(\mathcal{L}^*_{\kappa\omega})$, let $\varphi \in \mathcal{L}[\sigma]$. By $dep^*(\mathcal{L}) \leq \kappa$ assume without loss of generality that $|\sigma| < \kappa$. We have to show that there is a $\varphi' \in \Delta(\mathcal{L}^*_{\kappa\omega})[\sigma]$ with $Mod(\varphi) = Mod(\varphi')$. Again, as $dep^*(\mathcal{L}) \leq \kappa$, there is a renaming $\rho : \sigma \longrightarrow \sigma^*$ with $\sigma^* \in H(\kappa)$ and a $\psi \in \mathcal{L}[\sigma^*] \cap H(\kappa)$ which is equivalent up to renaming to φ. Because $\mathcal{L}$ fulfils (S1p) of p-symbiosis, $Mod(\psi)$ is $\Delta_1(R)$-definable with parameters in $\{\psi, \sigma^*\} \subseteq H(\kappa)$. Further $Mod(\psi)$ is a class of structures in a vocabulary in $H(\kappa)$ closed under isomorphism. Thus by the main Theorem 3.2.2 on p-symbiosis, $Mod(\psi)$ is $\Delta(\mathcal{L}^*_{\kappa\omega})$-definable. Therefore there is $\psi' \in \Delta(\mathcal{L}^*_{\kappa\omega})[\sigma^*]$ with $Mod(\psi) = Mod(\psi')$. Now if we consider the renaming $\rho^{-1} : \sigma^* \longrightarrow \sigma$, by the renaming property there is a $\varphi' \in \Delta(\mathcal{L}^*_{\kappa\omega})[\sigma]$ such that φ' is up to renaming equivalent to ψ'. Then $Mod(\varphi') = Mod(\varphi)$. We see: Every model class definable in $\mathcal{L}^*_{\kappa\omega}$ is also $\Delta(\mathcal{L}^*_{\kappa\omega})$-definable, i.e., $\mathcal{L}^*_{\kappa\omega} \leq \Delta(\mathcal{L}^*_{\kappa\omega})$. $\square$

4 Compactness Properties and Abstract Embedding Relations

Abstract logics are well-versed to investigate properties known from model theory of first-order logic. In many cases, such properties can be directly formulated in terms of $\mathcal{L}$-sentences and their model classes. For example, we can define a logic to have the *Löwenheim-Skolem property* iff for every $\varphi \in \mathcal{L}$, if $Mod(\varphi)$ contains an infinite model, then $Mod(\varphi)$ contains a countable model. Or we saw above that there is a natural formulation of the Craig interpolation property for any logic.

Of course, one of the most important properties of first-order logic is stated by the Compactness Theorem: a first-order theory T is satisfiable iff all of its finite subsets are satisfiable. Most extensions of first-order logic do *not* satisfy direct analogues of the Compactness Theorem. Nevertheless, we can formulate weaker compactness properties which can hold of even very strong logics. This chapter is devoted to the study of such compactness properties. Our main focus will lie in proving characterizations of compactness in terms of characterizability of partial orders.

That orderings and compactness are connected is not surprising: A classic application of the Compactness Theorem is to show that first-order logic cannot axiomatize the class of all well-orders or, more narrowly, the class of all well-orders of order type ω, even if we allow definability by a set of sentences. Also some form of backwards direction is known: If a logic $\mathcal{L}$ does not have the compactness property restricted to countable sets of sentences, then there is a countable set of sentences of $\mathcal{L}$, which only has models ordered as ω (compare [6, Proposition 5.2.4]). We will see below that one can get a similar characterization of full compactness is terms of well-orders.

Our main focus though will lie on so called (∞, κ)-compactness for a cardinal κ, which we define in section 4.1. We will prove a theorem due to Jouko Väänänen connecting this kind of compactness to the existence of $\mathcal{L}$-extensions of partial orders in which every subset of size smaller than κ is bounded. This refers to the concept of an $\mathcal{L}$-embedding which generalizes that of an elementary embedding known from model theory of first-order logic. $\mathcal{L}$-embeddings are closely related to (R, λ)-embeddings, a novel notion which we will make use of later. We will devote section 4.2 to study both types of embeddings and their relation. Finally, in section 4.3, we will prove the theorem characterizing (∞, κ)-compactness.

4.1 Compactness Properties

We want to give generalizations of the compactness property of logics:

Definition 4.1.1. Let $\mathcal{L}$ be a logic. $\mathcal{L}$ is *compact* iff for every set T of $\mathcal{L}$-sentences the following holds: If every finite subset of T is satisfiable, then T is satisfiable.

Of course, the Compactness Theorem for first-order logic tells us that $\mathcal{L}_{\omega\omega}$ is compact. But in general, most extensions of first-order logic are *not* compact. To give an easy example, consider the following.

Example 4.1.2. Let $\mathcal{L}_{\omega\omega}(\mathcal{Q}_0)$ be first-order logic expanded by the quantifier "there are infinitely many". This logic can define the class of all finite structures by the sentence

$\varphi := \neg \mathcal{Q}_0 x(x = x)$. Now take for every natural number n the first-order sentence

$$\varphi_n := \bigwedge_{i<j<n} x_i \neq x_j$$

that says that there are at least n-elements, and consider the set $T := \{\varphi_n \colon n \in \omega\} \cup \{\varphi\}$. Clearly, every finite subset of T is satisfiable, but T is not satisfiable. So $\mathcal{L}_{\omega\omega}(\mathcal{Q}_0)$ is not compact.

This example shows that any logic that can define finiteness is not compact. A famous result by Lindström shows that this is not accidental: Any logic that is strictly stronger than first-order logic is either not compact or does not satisfy the Löwenheim-Skolem property (compare [13]). As we are interested in symbiosis, this becomes even more pressing: No logic that is symbiotic with any predicate of set theory is fully compact.

Proposition 4.1.3. Let $\mathcal{L}$ be a logic p-symbiotic to a predicate R. Then $\mathcal{L}$ is not compact.

Proof. We prove below (Lemma 4.1.9) that $\mathcal{L}$ is compact iff $\Delta(\mathcal{L})$ is compact. Thus it is sufficient to show that $\Delta(\mathcal{L})$ is not compact. Assume towards a contradiction that it is compact. Because $\mathcal{L}$ is p-symbiotic with R, the logic $\Delta(\mathcal{L})$ can axiomatize all classes $\mathcal{K}$ in restricted vocabularies which are $\Delta_1(R)$-definable with parameters in $H(\omega)$. Take a binary relation symbol $<\in H(\omega)$. Then the class $\mathcal{K} = \{(A, <^A) \colon <^A$ is a well-order on $A\}$ is Δ_1-definable. Thus it is also $\Delta(\mathcal{L})$-axiomatizable. Say by φ.

Now introduce new constant c_n for $n \in \omega$ and consider the $\Delta(\mathcal{L})$-theory

$$T := \{c_n < c_m \colon n, m \in \omega, m < n\} \cup \{\varphi\}.$$

Now clearly every finite subset of T has a model but T does not have a model, because a model $\mathcal{M}$ of T has to be well-ordered by satisfying φ and has to have an infinite descending chain $c_0^{\mathcal{M}} > c_1^{\mathcal{M}} > c_2^{\mathcal{M}} > \dots$. □

For these reasons, we are interested in weaker compactness properties. Particularly important for us will be the one introduced in point (ii) of the following

Definition 4.1.4. Let $\mathcal{L}$ be a logic, $\kappa \geq \lambda$ cardinals. We define the followig.

(i) A set of T of $\mathcal{L}$-sentences is called λ-*satisfiable* iff every subset $S \subseteq T$ with $|S| < \lambda$ is satisfiable.

(ii) $\mathcal{L}$ is (∞, λ)-*compact* iff every set T of $\mathcal{L}$-sentences that is λ-satisfiable, is itself satisfiable.

(iii) $\mathcal{L}$ is (κ, λ)-*compact* iff every set T of $\mathcal{L}$-sentences with $|T| \leq \kappa$ that is λ-satisfiable, is itself satisfiable.

Compactness properties of logics extending first-order logic often have large cardinal strength, as many other model-theoretic properties of these logics as well. Thus they are important in set theory as they can give interesting characterizations of large cardinals. For instance, the notions of a *weakly* or *strongly* compact cardinal refer back to compactness properties of infinitary logics. Well known is the following characterization:

Theorem 4.1.5. Let κ be an inaccessible cardinal. Then we have the following equivalences:

(i) $\mathcal{L}_{\kappa\omega}$ is (κ, κ)-compact iff κ is weakly compact.

(ii) $\mathcal{L}_{\kappa\omega}$ is (∞, κ)-compact iff κ is strongly compact.

Proof. See [9], Theorem 17.13 and Lemma 20.2. □

One of the most important results in model theory of abstract logics is the following.

Theorem 4.1.6 (Magidor)**.** Let κ be a cardinal. Then the following are equivalent:

(i) κ is the smallest extendible cardinal.

(ii) κ is the smallest cardinal such that $\mathcal{L}^2$ is (∞, κ)-compact.

(iii) κ is the smallest cardinal such that a logic of the form $\mathcal{L}^2_{\kappa\omega}$ is (∞, κ)-compact.

Proof. See [14, Theorem 4]. □

In the following we will occasionally refer back to these examples. They are however not used in any of our main results and only mentioned to give the reader with experience in large cardinals some context.

Clearly, a logic is compact iff it is (∞, ω)-compact. We collect a few more basic properties of the compactness notions we are considering.

Proposition 4.1.7. Let $\mathcal{L}$ be a logic. The following hold.

1. If $\mathcal{L}$ is (κ, λ)-compact and $\gamma \geq \lambda$, then $\mathcal{L}$ is (κ, γ)-compact.
2. If $\mathcal{L}$ is (κ, λ)-compact and $\gamma \leq \kappa$, then $\mathcal{L}$ is (γ, λ)-compact.
3. $\mathcal{L}$ is (∞, λ)-compact iff $\mathcal{L}$ is (κ, λ)-compact for all $\kappa \geq \lambda$.

Proof. All claims are obvious by the definitions. □

Compactness properties are obviously transferred from stronger to weaker logics:

Lemma 4.1.8. Let $\mathcal{L}_0$ and $\mathcal{L}_1$ be logic with $\mathcal{L}_0 \leq \mathcal{L}_1$. If $\mathcal{L}_1$ is (κ, λ)-compact, then so is $\mathcal{L}_0$.

Proof. As for every set of $\mathcal{L}_0$-sentences, there is an equivalent set of $\mathcal{L}_1$-sentences (of the same size), this is obvious. □

We mentioned above that the Δ-closure does not necessarily preserve upwards Löwenheim-Skolem properties. It does, on the other hand, preserve compactness properties, as mentioned without proof in [6, p. 72]. We give a proof of this fact here, in terms of our precise definition of $\Delta(\mathcal{L})$.

Lemma 4.1.9. Let $\mathcal{L}$ be a logic. Then $\mathcal{L}$ is (κ, λ)-compact iff $\Delta(\mathcal{L})$ is (κ, λ)-compact.

Proof. That (κ, λ)-compactness of $\Delta(\mathcal{L})$ implies (κ, λ)-compactness of $\mathcal{L}$ is clear, as $\mathcal{L} \leq \Delta(\mathcal{L})$. For the other direction assume that $\mathcal{L}$ is (κ, λ)-compact. Let $T \subseteq \Delta(\mathcal{L})[\tau]$ with $|T| = \kappa$ be λ-satisfiable. We have to show that T is satisfiable. We have that for every $\varphi \in T$, the model class $Mod(\varphi)$ is $\Delta(\mathcal{L})$, thus there is an expansion $\tau_\varphi \supseteq \tau$ and $\varphi^* \in \mathcal{L}[\tau_\varphi]$ with $Mod(\varphi^*) \upharpoonright \tau = Mod(\varphi)$. Take for every $\varphi \in T$ such a $\varphi^* \in \mathcal{L}[\tau_\varphi]$. For simplicity, assume that none of the τ_φ add any additional sort symbols.[6] Assume without loss of generality that for $\varphi \neq \psi$, we do not add any identical symbols to τ, i.e., $(\tau_\varphi \setminus \tau) \cap (\tau_\psi \setminus \tau) = \emptyset$.[7] Let $\tau^* := \bigcup_{\varphi \in T} \tau_\varphi$ and $T^* := \{\varphi^* \colon \varphi \in T\} \subseteq \mathcal{L}[\tau^*]$. Clearly $|T^*| = |T| = \kappa$.

We claim that T^* is λ-satisfiable: Let $S^* \subseteq T^*$ be of size $< \lambda$. By assumption $S = \{\varphi \colon \varphi^* \in S^*\} \subseteq T$ is satisfiable, as T is λ-satisfiable. So there is a model $\mathcal{A} \models S$. Now for every $\varphi \in S$, there is an expansion of the τ-structure $\mathcal{A}$ to a τ_φ-structure $\mathcal{A}^*_\varphi \models \varphi^*$. Let $\mathcal{A}^*$ be the expansion of $\mathcal{A}$ to a $(\tau \cup \bigcup_{\varphi \in S} \tau_\varphi)$-structure, where every symbol from τ_φ is interpreted as in $\mathcal{A}^*_\varphi$.[8] As for $\varphi \neq \psi$, we have that τ_φ and τ_ψ add only different symbols to τ, this is well-defined. Now $\mathcal{A}^*$ is a model of S^*: If $\varphi^* \in S^*$, then $\mathcal{A}^*_\varphi \models \varphi^*$, so $\mathcal{A}^* \models \varphi^*$ by the reduct property, as $\mathcal{A}^* \upharpoonright \tau_\varphi = \mathcal{A}^*_\varphi$ and $\varphi^* \in \mathcal{L}[\tau_\varphi]$.[9] So S^* is satisfiable.

Hence, by (κ, λ)-compactness of $\mathcal{L}$, we get that T^* is satisfiable. So let $\mathcal{B}^* \models T^*$. Then $\mathcal{B}^* \in Mod(\varphi^*)$ for every $\varphi^* \in T^*$. Now we have that $Mod(\varphi^*) \upharpoonright \tau = Mod(\varphi)$. So $\mathcal{B}^* \upharpoonright \tau \in Mod(\varphi)$ for every $\varphi \in T$. But this means that $\mathcal{B} \upharpoonright \tau \models T$, so T is satisfiable. $\square$

As a direct corollary we get

Corollary 4.1.10. Let $\mathcal{L}$ be a logic. Then $\mathcal{L}$ is (∞, λ)-compact iff $\Delta(\mathcal{L})$ is (∞, λ)-compact.

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[6] If this is not the case and there is a φ such that τ_φ contains an additional sort symbol, the proof gets somewhat more complicated. See the next three footnotes on how to adapt the simpler case to this harder one.

[7] In the more complicated case, instead, we have to make the following assumptions without loss of generality. Note that as different sorts can be emulated by unary predicates, we can assume that every τ_φ adds at most one additional sort symbol. Then assume that all τ_φ add the same sort symbol s (if any). By the substitution property, assume that every τ_φ is relational. Similar to the easier case, assume that for $\varphi \neq \psi$, the sets of new relation symbols in τ_φ and in τ_ψ are disjoint, so $(\tau_\varphi \setminus \tau) \cap (\tau_\psi \setminus \tau) \subseteq \{s\}$. Further assume that if $s \in \tau_\varphi$, then τ_φ contains a new unary predicate symbol P_φ in sort s and that φ^* has the following property: For all τ_φ-structures $\mathcal{B}$, if $\mathcal{P}^{\mathcal{B}}_\varphi$ is the τ_φ-structure obtained from $\mathcal{B}$ by having $P^{\mathcal{B}}_\varphi$ as its domain in sort s (and all other symbols are restricted accordingly), we have

$$\mathcal{B} \models \varphi^* \text{ iff } \mathcal{P}^{\mathcal{B}}_\varphi \models \varphi^*,$$

I.e., all potential quantifiers in sort s of φ^* are relativized to P_φ.

[8] In the harder case, add that the domain in the new sort s is given by $A^*_s := \bigcup_{\varphi \in S} (A^*_\varphi)_s$.

[9] In the harder case, if $s \in \tau_\varphi$, then $\mathcal{A}^* \upharpoonright \tau_\varphi \models \varphi^*$, as $\mathcal{P}^{\mathcal{A}^*_\varphi}_\varphi \models \varphi^*$ and $\mathcal{P}^{\mathcal{A}^* \upharpoonright \tau_\varphi}_\varphi = \mathcal{P}^{\mathcal{A}^*_\varphi}_\varphi$.

Proof. By Lemma 4.1.9 and Proposition 4.1.7, point 3 above. □

We will make use of this corollary to prove our main theorem in chapter 5.

The main goal of this chapter is to give a characterization in terms of $\mathcal{L}$-extensions of partial orders as our main theorem about (∞, λ)-compactness. But first, to formulate this theorem, we have to look at the concept of $\mathcal{L}$-embeddings.

4.2 Embedding Relations

Of course, an embedding $f : \mathcal{A} \longrightarrow \mathcal{B}$ between τ-structures is called *elementary embedding* iff for every first-order formula $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in A$

$$\mathcal{A} \models \varphi(a_1, \ldots, a_n) \text{ iff } \mathcal{B} \models \varphi(f(a_1), \ldots, f(a_n)).$$

As is well known, the existence of an elementary embedding from $\mathcal{A}$ to $\mathcal{B}$ is equivalent to (an expansion of) $\mathcal{B}$ being a model of the *elementary diagram of* $\mathcal{A}$, i.e., the set of all first-order sentences that hold in the structure $(\mathcal{A}, c_a^{\mathcal{A}})_{a\in A}$ over the language τ_A, i.e., τ expanded by constant symbols c_a for every element a of A which in $(\mathcal{A}, c_a^{\mathcal{A}})_{a\in A}$ is interpreted as a itself. The intuition behind the elementary diagram is that it states everything that first-order logic can express about $\mathcal{A}$ when names are ready for all of its elements. This makes it superfluous to talk about formulas with free variables, as the same expressive power can be emulated by these names.

Elementary diagrams thus make it easy to generalize the concept of an elementary embedding to arbitrary logics which do not necessarily include free variables in their syntax. This is the the most standard way to provide such a generalization (compare [15]), which is why we adopt it here.

Definition 4.2.1. Let $\mathcal{L}$ be a logic, τ a vocabulary, $\mathcal{A}$ and $\mathcal{B}$ be τ-structures and $f : \mathcal{A} \longrightarrow \mathcal{B}$ an embedding. Let $|\mathcal{A}| = \lambda$ and let $\{c_i\colon i < \lambda\}$ be a set of λ-many constant symbols which are not in τ. Further let $c_i^{\mathcal{A}} \in A$ be an interpretation of the constants $\{c_i\colon i < \lambda\}$ by elements of $\mathcal{A}$ such that for all $a \in A$, there is an $i < \lambda$ such that $c_i^{\mathcal{A}} = a$ and for all $i, j < \lambda$, if $i \neq j$, then $c_i^{\mathcal{A}} \neq c_j^{\mathcal{A}}$. Then

(i) $\mathrm{Diag}_{\mathcal{L}}(\mathcal{A}) := \mathrm{Th}_{\mathcal{L}}((\mathcal{A}, c_i^{\mathcal{A}})_{i<\lambda})$ is called the $\mathcal{L}$-*diagram* of $\mathcal{L}$.

(ii) f is called an $\mathcal{L}$-*embedding* iff $(\mathcal{B}, f(c_i^{\mathcal{A}}))_{i<\lambda} \models_{\mathcal{L}} \mathrm{Diag}_{\mathcal{L}}(\mathcal{A})$.

(iii) If there is an $\mathcal{L}$-embedding from $\mathcal{A}$ to $\mathcal{B}$, the latter is called an $\mathcal{L}$-*extension* of $\mathcal{A}$.

If f is an $\mathcal{L}$-embedding we write $f : \mathcal{A} \preccurlyeq_{\mathcal{L}} \mathcal{B}$. We also write $\mathcal{A} \preccurlyeq_{\mathcal{L}} \mathcal{B}$ to indicate that there is an $\mathcal{L}$-embedding from $\mathcal{A}$ to $\mathcal{B}$. If we omit the subscript $\mathcal{L}$, we are talking about (first-order) elementary embeddings.

Note that in this definition, being an $\mathcal{L}$-embedding is independent of the choice of the constants c_i by the renaming property. It thus makes sense to talk of *the* $\mathcal{L}$-diagram. As in first-order logic, the $\mathcal{L}$-diagram describes all the properties of a structure expressible in $\mathcal{L}$ when names for all the elements of $\mathcal{A}$ are at hand.

We make the following easy but useful observation.

Lemma 4.2.2. Let $\mathcal{L}_0$ and $\mathcal{L}_1$ be logics with $\mathcal{L}_0 \leq \mathcal{L}_1$. Then every $\mathcal{L}_1$-embedding is also an $\mathcal{L}_0$-embedding.

Proof. Because $\mathcal{L}_0 \leq \mathcal{L}_1$, for every $\varphi \in \mathrm{Diag}_{\mathcal{L}_0}$, there is a $\varphi^* \in \mathcal{L}_1$ which is equivalent to φ. Then if we list the $\mathcal{L}_0$-diagram of some structure $\mathcal{A}$ as $\mathrm{Diag}_{\mathcal{L}_0}(\mathcal{A}) = \{\varphi_i \colon i < \lambda\}$, then $\{\varphi_i^* \colon i < \lambda\} \subseteq \mathrm{Diag}_{\mathcal{L}_1}(\mathcal{A})$. Thus any model of the $\mathcal{L}_1$-diagram of $\mathcal{A}$, will also be a model of the $\mathcal{L}_0$-diagram of $\mathcal{A}$. □

We want to show that if the logic does not allow for too many symbols, one can restrict oneself to adding not too many constants.

Lemma 4.2.3. Let $dep(\mathcal{L}) \leq \kappa$. Let $\mathcal{A}$ and $\mathcal{B}$ be τ-structures and $f : A \longrightarrow B$ a map. Let $\{c_i\}_{i<\kappa}$ be a set of κ-many constants symbols which are not in τ. Then the following are equivalent

(i) f is an $\mathcal{L}$-embedding.

(ii) For all $\gamma < \kappa$, if $c_i^{\mathcal{A}}$ is an interpretation of the constants $\{c_i\}_{i<\gamma}$ by arbitrary elements of A, then for all $\varphi \in \mathcal{L}[\tau \cup \{c_i\}_{i<\gamma}]$:

$$(\mathcal{B}, f(c_i^{\mathcal{A}}))_{i<\gamma} \models \varphi \text{ iff } (\mathcal{A}, c_i^{\mathcal{A}})_{i<\gamma} \models \varphi.$$

Note that in the definition $\mathrm{Diag}_{\mathcal{L}}(\mathcal{A})$ we add $|A|$-many constants, while here we restrict ourselves to κ-many additional constant symbols, independently of the size of $\mathcal{A}$.

Proof. For the direction from (i) to (ii), we have to show that $(\mathcal{B}, f(c_i^{\mathcal{A}}))_{i<\gamma}$ is a model of the $\mathcal{L}$-theory $T := \mathrm{Th}_{\mathcal{L}}((\mathcal{A}, c_i^{\mathcal{A}})_{i<\gamma})$. If $|\mathcal{A}| \geq \gamma$, then $T \subseteq \mathrm{Diag}_{\mathcal{L}}(\mathcal{A})$, so this follows by f being an $\mathcal{L}$-embedding. If $|\mathcal{A}| < \gamma$, then we actually add more constants to τ than we need to formulate the $\mathcal{L}$-diagram. It is clear that those additional constants do not add any more expressive power, so again this follows by f being an $\mathcal{L}$-embedding.

For the direction from (ii) to (i), we have to show that with $\{d_i\}_{i<\lambda}$ the constants used to formulate $\mathrm{Diag}_{\mathcal{L}}(\mathcal{A})$, if $\varphi \in \mathrm{Diag}_{\mathcal{L}}(\mathcal{A})$, then $(\mathcal{B}, f(d_i^{\mathcal{A}}))_{i<\lambda} \models \varphi$. As $dep(\mathcal{L}) = \kappa$, there are less than κ-many, say γ-many, of the d_i actually appearing in φ. Without loss of generality let $\varphi \in \mathcal{L}[\tau \cup \{d_i\}_{i<\gamma}]$. Then let $c_i^{\mathcal{A}} = d_i^{\mathcal{A}}$. By the renaming property there is a $\varphi^* \in \mathcal{L}[\tau \cup \{c_i\}_{i<\gamma}]$ which is equivalent up to renaming to φ. Then as $(\mathcal{A}, d_i^{\mathcal{A}})_{i<\gamma} \models \varphi$ also $(\mathcal{A}, c_i^{\mathcal{A}})_{i<\gamma} \models \varphi^*$. By assumption, this means that $(\mathcal{B}, f(c_i^{\mathcal{A}}))_{i<\gamma} \models \varphi^*$. But as φ and φ^* are equivalent up to renaming, thus $(\mathcal{B}, f(d_i^{\mathcal{A}}))_{i<\gamma} \models \varphi$. But then also $(\mathcal{B}, f(d_i^{\mathcal{A}}))_{i<\lambda} \models \varphi$ by the reduct property, as $\varphi \in \mathcal{L}[\tau \cup \{d_i\}_{i<\gamma}]$ and $(\mathcal{B}, f(d_i^{\mathcal{A}}))_{i<\gamma} = (\mathcal{B}, f(d_i^{\mathcal{A}}))_{i<\lambda} \upharpoonright \tau \cup \{d_i\}_{i<\gamma}$. □

The useful direction in the result above is the one from (ii) to (i), as it allows us to consider fewer than $|A|$-many additional constant symbols. Also it allows us to consider a specific set of constant symbols of our choice. The reader may find this focus on a fixed set of constant symbols somewhat pedantic. After all, the renaming property allows us to disregard which specific ones we currently look at, which is a blessing as it saves us from having to switch around between different sets of symbols via the renaming property like in the above proof. Nevertheless, the result can be useful in contexts where there are not arbitrarily many constant symbols around. For example, suppose you work in some set model of some fragment of ZFC and want to express that something is an $\mathcal{L}$-embedding. As our model will only contain a fixed set of constant symbols, there might just not be enough constant symbols around for it to formulate the full $\mathcal{L}$-diagram. Then by the above proposition, we can restrict our attention to one small set of constant symbols that is available to us.

Now we want to introduce a "set-theoretic" embedding relation. To the author's knowledge this is a novel notion. It will not play any role in the rest of this chapter, but we will use it in chapter 5 to give a refined formulation of our main result. We introduce it here as it is closely related to $\mathcal{L}$-embeddings.

Definition 4.2.4. Let R be a predicate of set theory, κ a cardinal and $\tau \in H(\kappa)$ a vocabulary. Let $\mathcal{A}$ and $\mathcal{B}$ be τ-structures and $f : \mathcal{A} \longrightarrow \mathcal{B}$ an embedding. f is called an (R, κ)-*embedding* iff for every $\sigma := \tau \cup \{c_i : i < \gamma\} \in H(\kappa)$ where the c_i are constant symbols, for every class $\mathcal{K}$ of σ-structures, which is $\Delta_1(R)$-definable with parameters in $H(\kappa)$ and closed under isomorphism, for every interpretation $c_i^{\mathcal{A}}$ of the c_i by elements of $\mathcal{A}$:

$$(\mathcal{A}, c_i^{\mathcal{A}})_{i<\gamma} \in \mathcal{K} \text{ iff } (\mathcal{B}, f(c_i^{\mathcal{A}}))_{i<\gamma} \in \mathcal{K}.$$

If $f : \mathcal{A} \longrightarrow \mathcal{B}$ is an (R, κ)-embedding, we say $\mathcal{B}$ is an (R, κ)-*extension* of $\mathcal{A}$.

As we announced, the following result relates (R, κ)-embeddings to $\mathcal{L}$-embeddings.

Lemma 4.2.5. Let R be a predicate of set-theory, κ a cardinal, $\tau \in H(\kappa)$ a vocabulary, $\mathcal{A}$ and $\mathcal{B}$ be τ-structures and $f : A \longrightarrow B$ an (R, κ)-embedding. Then the following holds:

> For every $\mathcal{L}'$ satisfying (S1p) of p-symbiosis with R and with $dep^*(\mathcal{L}') \leq \kappa$, f is an $\mathcal{L}'$-embedding.

Proof. Take a set of constant symbols $\{c_i : i < \kappa\} \in H(\kappa)$ which are not in τ. Note that because $\tau \in H(\kappa)$, we have $|\tau| < \kappa$ and also that in $H(\kappa)$ are $|H(\kappa)| \geq \kappa$-many constant symbols, so this is possible. By Lemma 4.2.3 and because $dep^*(\mathcal{L}') \leq \kappa$, it is sufficient to show that for any $\gamma < \kappa$ and any interpretation $c_i^{\mathcal{A}}$ of the c_i for $i < \gamma$ by elements of $\mathcal{A}$, if we let $\sigma := \tau \cup \{c_i : i < \gamma\}$, then for any $\varphi \in \mathcal{L}'[\sigma]$:

$$(\mathcal{B}, f(c_i^{\mathcal{A}}))_{i<\gamma} \models \varphi \text{ iff } (\mathcal{A}, c_i^{\mathcal{A}})_{i<\gamma} \models \varphi.$$

Notice that $\sigma \in H(\kappa)$, as $\tau \in H(\kappa)$ and $\{c_i : i < \gamma\} \in H(\kappa)$. By $\mathcal{L}'$ fulfilling (S1p) of p-symbiosis, the class $Mod(\varphi)$ is $\Delta_1(R)$-definable with parameters in $\{\varphi, \sigma\}$. Because $dep^*(\mathcal{L}') \leq \kappa$ and $\sigma \in H(\kappa)$, also $\varphi \in H(\kappa)$. Thus $Mod(\varphi)$ is $\Delta_1(R)$-definable with parameters in $H(\kappa)$. Then as f is an (R, κ)-embedding,

$$(\mathcal{B}, f(c_i^{\mathcal{A}}))_{i<\gamma} \in Mod(\varphi) \text{ iff } (\mathcal{A}, c_i^{\mathcal{A}})_{i<\gamma} \in Mod(\varphi).$$

□

Under assumption of symbiosis of R to a logic, the connection to $\mathcal{L}$-embeddings is even tighter:

Corollary 4.2.6. Let $\mathcal{L} := \mathcal{L}^*_{\kappa\omega} \geq \mathcal{L}_{\kappa\omega}$ be a logic with $dep^*(\mathcal{L}) = \kappa$. Let R be a predicate of set theory and assume R and $\mathcal{L}$ to be p-symbiotic. Let $\mathcal{A}$ and $\mathcal{B}$ be τ-structures for a vocabulary $\tau \in H(\kappa)$. Then the following are equivalent for a map $f : A \longrightarrow B$:

(i) f is a $\Delta(\mathcal{L})$-embedding.

(ii) f is an (R, κ)-embedding.

(iii) For every $\mathcal{L}'$ satisfying (S1p) of p-symbiosis with R and with $dep^*(\mathcal{L}') \leq \kappa$, f is an $\mathcal{L}'$-embedding.

Proof. First we assume (i) and show (ii). Let $\mathcal{K}$ be a class of σ-structures where $\sigma = \tau \cup \{c_i \colon i < \gamma\} \in H(\kappa)$ and which is $\Delta_1(R)$-definable with parameters in $H(\kappa)$, closed under isomorphism, and with $(\mathcal{A}, c_i^{\mathcal{A}})_{i<\gamma} \in \mathcal{K}$. We have to show that $(\mathcal{B}, f(c_i^{\mathcal{A}}))_{i<\gamma} \in \mathcal{K}$. Note that also the complement of $\mathcal{K}$ is $\Delta_1(R)$ with parameters in $H(\kappa)$ and so it is sufficient to show this one direction. By the main theorem on p-symbiosis, $\mathcal{K}$ is $\Delta(\mathcal{L})$-definable. So there is $\varphi \in \Delta(\mathcal{L})[\sigma]$ such that $Mod(\varphi) = \mathcal{K}$. As f is a $\Delta(\mathcal{L})$-embedding and $(\mathcal{A}, c_i^{\mathcal{A}})_{i<\gamma} \models \varphi$, also $(\mathcal{B}, f(c_i^{\mathcal{A}}))_{i<\gamma} \models \varphi$, so $(\mathcal{B}, f(c_i^{\mathcal{A}}))_{i<\gamma} \in Mod(\varphi) = \mathcal{K}$.

We already showed that (ii) implies (iii). Thus it is sufficient to show that (iii) implies (i). However, this is trivial because, if we assume (iii), as $\Delta(\mathcal{L})$ is such a logic satisfying (S1p) of p-symbiosis with R and $dep^*(\mathcal{L}) = \kappa$ (compare Lemma 3.2.9 and Theorem 2.8.4), the map f is a $\Delta(\mathcal{L})$-embedding. $\square$

4.3 Compactness and $\mathcal{L}$-Extensions of Partial Orders

To formulate a set-theoretic property of some predicate R that is related to compactness of a logic symbiotic to R, we want a reformulation of compactness that allows for an easier "translation" into set-theoretic terms than the original definition. Thus we will prove a characterization of (∞, κ)-compactness of $\mathcal{L}$ in terms of $\mathcal{L}$-extensions with upper bounds of partial orders.

For this we make the following (standard) conventions.

Definition 4.3.1. By a *partial order* we always mean a strict partial order. Let $(A, <^A)$ and $(B, <^B)$ be partial orders, $f : A \longrightarrow B$ a map. We say

(i) $(A, <^A)$ is *unbounded* iff for all $a \in A$ there is a $b \in A$ such that $a <^A b$.

(ii) $(A, <^A)$ is *directed* iff for all $a, b \in A$ there is a $c \in A$ such that $a <^A c$ and $b <^A c$.

(iii) f is an *embedding* iff for all $a_1, a_2 \in A$

$$a_1 <^A a_2 \text{ iff } f(a_1) <^A f(a_2).$$

(iv) A subset $X \subseteq A$ is said to *have an upper bound* iff there is a $b \in A$ such that for every $a \in X$, we have $a <^A b$.

(v) A subset $X \subseteq A$ is *cofinal (in A)* iff for all $a \in A$ there is a $b \in X$ such that $a <^A b$.

(vi) If $f : A \longrightarrow B$ is an embedding, we say $(B, <^B)$ *contains an upper bound for* $(A, <^A)$ iff $f(A)$ has an upper bound in B. The element $b \in B$ such that $a <^B b$ for all $a \in f(A)$ is called *upper bound for A.*

We will also speak of $<^A$ being unbounded etc. and mean that $(A, <^A)$ is unbounded etc.

Notice that if f is an $\mathcal{L}$-embedding for any logic $\mathcal{L}$ between partially ordered structures $(A, <^A)$ and $(B, <^B)$, then f is an embedding. If $(B, <^B)$ is an $\mathcal{L}$-extension of $(A, <^A)$ and $b \in B$ is an upper bound for A, we say that $(B, <^B)$ is an *$\mathcal{L}$-extension with an upper bound.*

With this at hand we can state the following theorem due to Väänänen [17, Theorem 1].

Theorem 4.3.2 (Väänänen)**.** Let $\mathcal{L}$ be a logic. Then the following are equivalent.

(i) $\mathcal{L}$ is compact.

(ii) If τ is a vocabulary including a binary relation symbol $<$, $\mathcal{A} = (A, <^{\mathcal{A}}, \dots)$ is a τ-structure and $<^{\mathcal{A}}$ a directed partial order, then there is a τ-structure $\mathcal{B}$ and an $\mathcal{L}$-embedding $f : \mathcal{A} \preccurlyeq_{\mathcal{L}} \mathcal{B}$ such that $(B, <^{\mathcal{B}})$ contains an upper bound for $(A, <^{\mathcal{A}})$.

(iii) If τ is a vocabulary including a binary relation symbol $<$, $\mathcal{A} = (A, <^{\mathcal{A}}, \dots)$ is a τ-structure and $<^{\mathcal{A}}$ an unbounded well-order, then there is a τ-structure $\mathcal{B}$ and an $\mathcal{L}$-embedding $f : \mathcal{A} \preccurlyeq_{\mathcal{L}} \mathcal{B}$ such that $(B, <^{\mathcal{B}})$ contains an upper bound for $(A, <^{\mathcal{A}})$.

We omit the proof here, as it is similar to that of the more general Theorem 4.3.3 below which gives a characterization of (∞, κ)-compactness. Theorem 4.3.3 can be found without proof in [17, Theorem 10]. To formulate it, for a set A we write $[A]^{<\kappa}$ to denote the set of all $< \kappa$ sized subsets of A. The equivalence of (i) and (ii) in Theorem 4.3.2 is a special case of the one below as a partial order $(A, <^A)$ is directed iff every $A_0 \in [A]^{<\omega}$ has an upper bound, and because compactness is (∞, ω)-compactness.

Theorem 4.3.3 (Väänänen)**.** Let $\mathcal{L}$ be a logic and κ a regular cardinal. Then the following are equivalent.

(i) $\mathcal{L}$ is (∞, κ)-compact.

(ii) If τ is a vocabulary including a binary relation symbol $<$ and $\mathcal{A} = (A, <^{\mathcal{A}}, \dots)$ is a τ-structure with $<^{\mathcal{A}}$ a partial order in which every every $A_0 \in [A]^{<\kappa}$ has an upper bound, then there is a τ-structure $\mathcal{B}$ and an $\mathcal{L}$-embedding $f : \mathcal{A} \preccurlyeq_{\mathcal{L}} \mathcal{B}$ such that $(B, <^{\mathcal{B}})$ contains an upper bound for $(A, <^{\mathcal{A}})$.

Proof. First assume (i) and let $\mathcal{A}$ be a τ-structure such that $<^{\mathcal{A}}$ is a partial order in which every $A_0 \in [A]^{<\kappa}$ has an upper bound. Take for every $a \in A$ a new constant symbol c_a and an additional new constant symbol c. Let $c_a^{\mathcal{A}} = a$. Then $\mathrm{Diag}_{\mathcal{L}}(\mathcal{A}) = Th_{\mathcal{L}}((A, c_a^{\mathcal{A}})_{a \in A})$. Now consider the set of $\mathcal{L}$-sentences

$$T := \mathrm{Diag}_{\mathcal{L}}(\mathcal{A}) \cup \{c_a < c \colon a \in A\}.$$

T is κ-satisfiable, as any subset of T of size smaller than κ is satisfied by an expansion of $\mathcal{A}$: If $S \subseteq T$ and $|S| < \kappa$, then there is a subset $X \subseteq A$ with $|X| < \kappa$ such that $S \subseteq \mathrm{Diag}_{\mathcal{L}}(\mathcal{A}) \cup \{c_a < c \colon a \in X\}$. By assumption on $<^{\mathcal{A}}$ there is an upper bound $a^* \in A$ for X, so $a <^{\mathcal{A}} a^*$ for all $a \in X$. Then letting $c^{\mathcal{A}} := a^*$ gives an expansion of $\mathcal{A}$ that is a model of S.

By (∞, κ)-compactness of $\mathcal{L}$, T is therefore satisfiable and has a model. Call this model $\mathcal{B}$. Then as $\mathcal{B} \models \mathrm{Diag}_{\mathcal{L}}(\mathcal{A})$ we have $\mathcal{A} \preccurlyeq_{\mathcal{L}} \mathcal{B}$, say via the map f. And then $f(a) = f(c_a^{\mathcal{A}}) = c_a^{\mathcal{B}} <^{\mathcal{B}} c^{\mathcal{B}}$ for all $a \in A$, so $c^{\mathcal{B}}$ is an upper bound for A.

And for the other direction assume (ii). We have to show that $\mathcal{L}$ is (∞, κ)-compact, i.e., that $\mathcal{L}$ is (λ, κ)-compact for any $\lambda \geq \kappa$. So let T be any set of $\mathcal{L}$-sentences of size λ, say over the vocabulary τ, which is κ-satisfiable. By the substitution property we assume without loss of generality that τ is relational and for simplicity that it is in one sort. We have to show that T is satisfiable.

Consider $P := [T]^{<\kappa}$ partially ordered by strict inclusion $\subsetneq$. We claim that all $P_0 \in [P]^{<\kappa}$ have an upper bound. So let $P_0 \subseteq P$ with $|P_0| < \kappa$. Then consider $q := \bigcup P_0$. Clearly $q \subseteq T$. Because κ is regular, as every element of P has size $< \kappa$ and P_0 itself has size $< \kappa$, also $|q| < \kappa$. Therefore $q \in P$. As $|T| = \lambda \geq \kappa$, there is $\varphi \in T \setminus q$. Then $q^* := q \cup \{\varphi\} \in P$ and clearly $p \subseteq q \subsetneq q^*$ for every $p \in P_0$. So q^* is an upper bound for P_0.

Note that every $p \in P$ is a subset of T with size smaller than κ, so by κ-satisfiability of T we can fix models $\mathcal{A}_p \models p$. Assume without loss of generality that the A_p are all disjoint from another and also disjoint from P.

Now take a new vocabulary σ disjoint from τ and consisting of a unary predicate P', two binary predicates U and $<$, constant symbols c_p for every $p \in P$ and for every n-ary relation symbol $R \in \tau$ an $(n+1)$-ary relation symbol R^*. So

$$\sigma = \{O, U, <\} \cup \{c_p \colon p \in P\} \cup \{R^* \colon R \in \tau\}.$$

Let $\mathcal{A}$ be the σ-structure which is specified in the following way.

1. $A = P \cup \bigcup_{p \in P} A_p$.
2. $a \in P'^{\mathcal{A}}$ iff $a = p$ for some $p \in P$, i.e., $P'^{\mathcal{A}} = P$.
3. $c_p^{\mathcal{A}} = p$ for all $p \in P$.
4. $<^{\mathcal{A}}$ is a partial order constructed in the following way: If $a \in \bigcup_{p \in P} A_p$, then $a <^{\mathcal{A}} b$ iff $b \in P$. And if $p, q \in P$, then $p <^{\mathcal{A}} q$ iff $p \subsetneq q$, so $<^{\mathcal{A}} \restriction P$ is the ordering of P by strict inclusion we fixed above.
5. $U^{\mathcal{A}}(a, b)$ iff $a = p$ for some $p \in P$ and $b \in A_p$.
6. $(a, b_1, \ldots, b_n) \in (R^*)^{\mathcal{A}}$ iff $a = p$ for some $a \in P$ and $(b_1, \ldots, b_n) \in R^{\mathcal{A}_p}$.

Now if $\mathcal{D}$ is an arbitrary σ-structure and $d \in D$, we can define a τ-structure $\mathcal{D}^d$ with domain $U^{\mathcal{D}}(d,-) = \{x \in D \colon U^{\mathcal{D}}(d,x)\}$ by letting

$$(b_1, \ldots, b_n) \in R^{\mathcal{D}^d} \text{ iff } (d, b_1, \ldots, b_n) \in (R^*)^{\mathcal{D}}$$

for every $R \in \tau$. In particular, by the construction of $\mathcal{A}$, for $p \in P$ we have $\mathcal{A}^p = \mathcal{A}_p$ (note that $p \in A$).

By the relativization property, for every $\varphi \in T$ there is a relativization $\varphi^*(x)$ of φ such that if $\mathcal{D}$ is any σ-structure, $d \in D$ and $\mathcal{D}^d$ is defined as above:[10]

$$\mathcal{D} \models \varphi^*(d) \text{ iff } \mathcal{D}^d \models \varphi.$$

Note that for every $\varphi \in T$, the set $\{\varphi\} \in P$, so there is the constant $c_{\{\varphi\}} \in \sigma$. Thus for φ we can consider the sentence

$$\psi_\varphi := \forall x(P'(x) \wedge c_{\{\varphi\}} \leq x \to \varphi^*(x)).$$

Then $\mathcal{A} \models \psi_\varphi$ for all $\varphi \in T$: If $p \in P'^{\mathcal{A}}$ and $c^{\mathcal{A}}_{\{\varphi\}} = \{\varphi\} \leq^{\mathcal{A}} p$, then $\{\varphi\} \subseteq p$, so $\varphi \in p$. Then we get $\mathcal{A}^p = \mathcal{A}_p \models \varphi$ by choice of $\mathcal{A}_p$ as a model of $p \ni \varphi$ and therefore $\mathcal{A} \models \varphi^*(p)$.

Now $<^{\mathcal{A}}$ is a partial order such that every $A_0 \in [A]^{<\kappa}$ has an upper bound: Because for $a \in A \setminus P$, we have $a <^{\mathcal{A}} \emptyset \in A$, it is sufficient to show that $A_0 \in [P]^{<\kappa}$ has an upper bound. But because $<^{\mathcal{A}} \restriction P = \subsetneq$ we already showed this above. Therefore by (ii), as $<^{\mathcal{A}}$ is a partial order as demanded in this condition, there is a σ-structure $\mathcal{B}$ with $f : \mathcal{A} \preccurlyeq_{\mathcal{L}} \mathcal{B}$ and with an upper bound for $<^{\mathcal{A}}$, i.e., there is a $b \in B$ such that for $a \in A$ we have $f(a) <^{\mathcal{B}} b$. Because $\mathcal{A} \models \forall x \exists y(P'(y) \wedge x < y)$ also $\mathcal{B}$ is a model of this sentence and thus we can chose $b \in P'^{\mathcal{B}}$. Now for every $\varphi \in T$, because $\mathcal{A} \models \psi_\varphi$, also $B \models \psi_\varphi$. But thus as $b \in P'^{\mathcal{B}}$ and $c^{\mathcal{B}}_{\{\varphi\}} = f(c^{\mathcal{A}}_{\{\varphi\}}) <^{\mathcal{B}} b$, we have $\mathcal{B} \models \varphi^*(b)$. Hence $\mathcal{B}^b \models \varphi$ for all $\varphi \in T$ and therefore $\mathcal{B}^b$ is a model of T. □

[10] For illustration: If φ is, e.g., a first-order sentence formulated over a binary relation symbol $R \in \tau$, say $\varphi = \forall y \exists z(R(y,z))$, then $\varphi^*(x)$ is obtained by replacing every quantification by a relativization to $U(x,-)$ and R by the 3-ary relation symbol R^*. Then we obtain

$$\varphi^*(x) = \forall y(U(x,y) \to \exists z(U(x,z) \wedge (R^*(x,y,z)))).$$

5 $\mathcal{L}$-extensions in $\Sigma_1(R)$ Classes of Partial Orders

Our goal is to prove a theorem, that, analogously to the results by Bagaria and Väänänen about downwards Löwenheim-Skolem numbers from [1] and by Galeotti, Khomskii and Väänänen from [8] about upwards Löwenheim Skolem numbers and reflection principles, respectively, shows a systematic connection between (∞, κ)-compactness of a logic $\mathcal{L}$ and some set-theoretic property involving a predicate R symbiotic to $\mathcal{L}$. We will not be able to formulate a principle that allows for a full equivalence to (∞, κ)-compactness. Instead, we will show that if $\mathcal{L}$ is a logic extending $\mathcal{L}_{\lambda\omega}$ and with strong dependence number λ, then under assumption of p-symbiosis between $\mathcal{L}$ and R and (∞, κ)-compactness of $\mathcal{L}$, we get that any $\Sigma_1(R)$-definable class $\mathcal{K}$ of structures over a vocabulary in $H(\lambda)$ contains (R, λ)-extensions with upper bounds of each partial order in $\mathcal{K}$, in which all subsets of size smaller than κ are bounded. We argue that this result is, in some sense, optimal and cannot be improved to a full equivalence without major changes in formulation of the set-theoretic principle.

5.1 Main Theorems

In this section we prove our main results which are Theorem 5.1.1 and Corollary 5.1.3.

Theorem 5.1.1. Let $\mathcal{L}$ be a logic with $\mathcal{L} := \mathcal{L}^*_{\lambda\omega} \geq \mathcal{L}_{\lambda\omega}$ and $dep^*(\mathcal{L}) = \lambda$. Let R be a predicate of set theory and $\kappa \geq \lambda$ a regular cardinal and assume $\mathcal{L}$ to be p-symbiotic with R. Let $\mathcal{L}$ be (∞, κ)-compact. Then the following statement holds:

> For every class of structures $\mathcal{K}$ in a vocabulary $\tau \in H(\lambda)$, if $\mathcal{K}$ is $\Sigma_1(R)$-definable with parameters in $H(\lambda)$ and $\mathcal{A} = (A, <^{\mathcal{A}}, \dots) \in \mathcal{K}$, if $<^{\mathcal{A}}$ is a partial order such that every $A_0 \in [A]^{<\kappa}$ has an upper bound, then there is $\mathcal{B} \in \mathcal{K}$ and a map $f : A \longrightarrow B$ such that
>
> 1. For every logic $\mathcal{L}'$ which satisfies (S1p) of p-symbiosis with R and for which $dep^*(\mathcal{L}) \leq \lambda$ the map f is an $\mathcal{L}'$-embedding $\mathcal{A} \preccurlyeq_{\mathcal{L}'} \mathcal{B}$.
> 2. $(B, <^{\mathcal{B}})$ contains an upper bound for $(A, <^{\mathcal{A}})$.

Proof. Let $\mathcal{K}$ be a class of τ-structures, where $\tau \in H(\lambda)$, which is $\Sigma_1(R)$-definable with parameters in $H(\lambda)$. So there is $\Phi(x, y)$ defining $\mathcal{K}$ with a parameter $p \in H(\lambda)$. I.e., for every $\mathcal{A}$, we have

$$\mathcal{A} \in K \leftrightarrow \Phi(\mathcal{A}, p).$$

Now let $\mathcal{A} = (A, <^{\mathcal{A}}, \dots) \in \mathcal{K}$ with $<^{\mathcal{A}}$ a partial order such that every $A_0 \in [A]^{<\kappa}$ has an upper bound. For simplicity we assume the vocabulary τ to be relational and in one sort. We have to find a $\mathcal{B} \in \mathcal{K}$ which contains an upper bound for $<^{\mathcal{A}}$ and in which $\mathcal{A}$ $\mathcal{L}'$-embeds for all logics $\mathcal{L}'$ which satisfy (S1p) of p-symbiosis with R and that have strong dependence number at most λ.

Take a new binary relation symbol E. Recall that in Definition 3.2.3, for $b \in H_\lambda$ we introduced the $\mathcal{L}_{\lambda\omega}$-formula $\delta_b(x)$, written in the language $\{E\}$, which defines b. I.e., for every transitive $\in$-model (M, E^M) and every $a \in M$ we have

$$(M, E^{\mathcal{M}}) \models \delta_b(a) \text{ iff } b = a.$$

Using this to get the formula $\delta_\tau(x)$, as in Theorem 3.2.2 we can define an $\mathcal{L}$-formula $\chi(x)$, which intuitively says "x is a τ-structure". Because $<\in \tau \in H(\lambda)$ and $p \in H(\lambda)$, we can further consider the formulas $\delta_<(x)$ and $\delta_p(x)$ defining $<$ and p, respectively.

Now consider the class $\mathcal{K}^*$ of well-founded structures $\mathcal{M} = (M, E^{\mathcal{M}}, c_1^{\mathcal{M}}, c_2^{\mathcal{M}}, c_3^{\mathcal{M}}, c_4^{\mathcal{M}}, \prec^{\mathcal{M}})$ in the vocabulary $\{E, c_1, c_2, c_3, c_4, \prec\}$, where the c_i are constants and $\prec$ is a new binary relation symbol, and which satisfy the following conditions below, where all set-theoretic formulas are written using E and ZFC^{-*} is some large enough fragment of ZFC minus the power set axiom as usual:

1. $\mathcal{M} \models \mathrm{ZFC}^{-*}$.
2. $(M, E^{\mathcal{M}}) \in \mathcal{Q}_R$.
3. $\mathcal{M} \models \chi(c_1) \wedge$ "$c_1 = (a, b) \wedge a = c_2 \wedge \exists x(\delta_<(x) \wedge b(x) = c_3)$".
4. $\mathcal{M} \models \delta_p(c_4)$.
5. $\mathcal{M} \models \Phi(c_1, c_4)$.
6. $\mathcal{M} \models$ " $\prec^{\mathcal{M}}$ is a partial order s.t. $\{x \colon xEc_2\}$ is cofinal in it".
7. $\mathcal{M} \models \forall x, y((x, y)Ec_3 \leftrightarrow x \prec y)$.

Intuitively, $\mathcal{M} \in \mathcal{K}^*$ is a transitive R-correct model (by 1 and 2), in which c_1 is a τ-structure with domain c_2, while c_3 is the interpretation of $<$ (by 3).[11] Further, c_4 is the parameter p and $\mathcal{M}$ thinks $c_1 \in \mathcal{K}$ (by 4 and 5). Finally, on the elements of c_2, i.e., the domain of the τ-structure c_1, the partial order $\prec$ corresponds to the ordering $<$ from c_1, and the elements of c_2 are cofinal in $\prec$ (by 6 and 7).

$\mathcal{K}^*$ is $\Delta(\mathcal{L})$-definable, as the conditions are either expressed by first-order sentences (1, 5, 6 and 7), by $\mathcal{L}_{\lambda\omega}$-sentences (3 and 4), or are $\Delta(\mathcal{L})$-definable by p-symbiosis (2). Also that (M, E^M) is well-founded is $\Delta(\mathcal{L})$-definable: Because being well-founded is Δ_1, for a binary relation symbol $E^* \in H(\omega)$ the class of all well-founded $\{E^*\}$-structures is Δ_1-definable without parameters. Thus it is also $\Delta(\mathcal{L})$-definable by the main theorem on p-symbiosis 3.2.2. Then by the renaming property, also the class of all well-founded $\{E\}$-structures is $\Delta(\mathcal{L})$. So there is $\psi \in \Delta(\mathcal{L})$ with $Mod(\psi) = \mathcal{K}^*$.

We construct a structure $\mathcal{M}$ which will be in $\mathcal{K}^*$. By the Reflection Theorem (compare e.g. [9, Theorem 12.14]), let α be a (limit) ordinal such that V_α reflects R and Φ,

[11] To give some intuition about the content of 3, notice that if c_1 is a τ-structure (by $\chi(c_1)$), then it is a tuple (a, b), where a is the domain of the structure and b is a function with domain τ returning the interpretations of the symbols in τ. Thus we can express that c_2 is the domain of c_1 (by "$a = c_2$"). And as τ contains $<$, in particular b returns the interpretation of $<$ in c_1 and we can express that c_3 is this interpretation (by "$\exists x(\delta_<(x) \wedge b(x) = c_3)$").

$V_\alpha \models \mathrm{ZFC}^{-*}$ and with $\mathcal{A}, H(\lambda) \in V_\alpha$. In particular, by the choice of α we have that V_α is R-correct and that Φ is absolute for V_α. Then let

$$\mathcal{M} = (V_\alpha, \in, c_1^{\mathcal{M}} = \mathcal{A}, c_2^{\mathcal{M}} = A, c_3^{\mathcal{M}} = <^{\mathcal{A}}, c_4^{\mathcal{M}} = p, \prec^{\mathcal{M}}),$$

where $\prec^{\mathcal{M}}$ is a partial order of V_α with $\prec^{\mathcal{M}} \restriction A = <^{\mathcal{A}}$, in which A is cofinal and in which every subset of V_α of size $< \kappa$ contains an upper bound. For precision, for example, let $\prec^{\mathcal{M}}$ be defined in the following way. For $a, b \in A$, let $a \prec^{\mathcal{M}} b$ iff $a <^{\mathcal{A}} b$. And for $a \in V_\alpha \setminus A$, let $a \prec^{\mathcal{M}} b$ iff $b \in A$.

Then clearly $\mathcal{M} \models \mathrm{ZFC}^{-*}$. As V_α is R-correct and transitive, also $(V_\alpha, \in) \in \mathcal{Q}_R$. We have that Φ is absolute for V_α. Thus because $\Phi(\mathcal{A}, p)$ holds, also $\mathcal{M} \models \Phi(c_1, c_4)$, as $c_1^{\mathcal{M}} = \mathcal{A}$ and $c_4^{\mathcal{M}} = p$ (notice that $p \in H(\lambda) \in V_\alpha$ and V_α is transitive, so $p \in V_\alpha$). Because of the latter facts, and because $c_2^{\mathcal{M}} = A$ and $c_3^{\mathcal{M}} = <^{\mathcal{A}}$, also $\mathcal{M}$ fulfils 3 and 4. Finally, we chose $\prec^{\mathcal{M}}$ in such a way that $\mathcal{M}$ fulfils 6 and 7: To see that $c_2^{\mathcal{M}} = A$ is cofinal in $\prec^{\mathcal{M}}$, note that if $a \in V_\alpha \setminus A$, then $a \prec^{\mathcal{M}} b$ for any $b \in A$ and if $a \in A$, then because $|\{a\}| < \kappa$, there is $b \in A$ with $a <^{\mathcal{A}} b$ and thus also $a \prec^{\mathcal{M}} b$. Therefore $\mathcal{M} \in \mathcal{K}^*$.

We further claim that $\prec^{\mathcal{M}}$ is a partial order such that every $A_0 \in [V_\alpha]^{<\kappa}$ has an upper bound in V_α. Because for $a \in V_\alpha \setminus A$, we have $a \prec^{\mathcal{M}} b$ for any $b \in A$, it is sufficient to consider $A_0 \subseteq A$. But such an A_0 has an upper bound in $A \subseteq V_\alpha$ by assumption on $<^{\mathcal{A}}$ and the fact that $\prec^{\mathcal{A}} \restriction A = <^{\mathcal{A}}$. Because $\mathcal{L}$ is (∞, κ)-compact, by Lemma 4.1.9 $\Delta(\mathcal{L})$ is (∞, κ)-compact as well and therefore by Theorem 4.3.3 there is $\mathcal{N}$ which contains an upper bound for $\prec^{\mathcal{M}}$ and a $\Delta(\mathcal{L})$-embedding $f : \mathcal{M} \preccurlyeq_{\Delta(\mathcal{L})} \mathcal{N}$. By Mostowski's Collapsing Theorem, consider the transitive collapse $\pi(\mathcal{N}) =: \bar{\mathcal{N}}$. As $\mathcal{M} \in \mathcal{K}^*$, we have $\mathcal{M} \models \psi$. As $\mathcal{M} \preccurlyeq_{\Delta(\mathcal{L})} \mathcal{N}$, also $\mathcal{N} \models \psi$. As $\mathcal{N}$ and $\bar{\mathcal{N}}$ are isomorphic, thus $\bar{\mathcal{N}} \models \psi$ and $\bar{\mathcal{N}} \in \mathcal{K}^*$. Let $\mathcal{B} := c_1^{\bar{\mathcal{N}}}$. We claim that $\mathcal{B}$ is the structure we are looking for. This is shown by the following claims.

Claim 1: $\mathcal{B} \in \mathcal{K}$.

As $\bar{\mathcal{N}}$ is a transitive $\in$-model (by 2 and as it is a transitive collapse), we have that $c_4^{\bar{\mathcal{N}}} = p$ (by 4). As $\bar{\mathcal{N}}$ is R-correct (again by 2), and Φ is a $\Sigma_1(R)$-formula, it is thus upwards absolute for $\bar{\mathcal{N}}$. As $\bar{\mathcal{N}} \models \Phi(c_1, c_4)$ (by 5) and $c_1^{\bar{\mathcal{N}}} = \mathcal{B}$ and $c_4^{\bar{\mathcal{N}}} = p$, thus $\Phi(\mathcal{B}, p)$ really holds, and thus $\mathcal{B} \in \mathcal{K}$.

Claim 2: With $e := (\pi \circ f) \restriction A$, we have $e : \mathcal{A} \preccurlyeq_{\mathcal{L}'} \mathcal{B}$ for every logic $\mathcal{L}'$ satisfying (S1p) of p-symbiosis with R and with $dep^*(\mathcal{L}') \leq \lambda$.

We show that e is a $\Delta(\mathcal{L})$-embedding. Then by Lemma 4.1.8, e is also an $\mathcal{L}'$-embedding for all logics $\mathcal{L}' \leq \Delta(\mathcal{L})$. By Theorem 3.2.10, if $\mathcal{L}'$ satisfies (S1p) of p-symbiosis with R and has strong dependence number at most λ, we have $\mathcal{L}' \leq \Delta(\mathcal{L})$. Thus, to show that e is a $\Delta(\mathcal{L})$-embedding is sufficient to prove the claim.

To show this, clearly e is a function with domain A and range $\subseteq B$, as for $a \in A$, we have $\mathcal{M} \models aEc_2$ (as $c_2^{\mathcal{M}} = A$), so by f being an elementary embedding and π an isomorphism $\bar{\mathcal{N}} \models e(a)Ec_2$, so $e(a) \in B$ (as $c_2^{\bar{\mathcal{N}}}$ is the domain of $\mathcal{B}$ by 3). Now let $\gamma < \lambda$ and $\{d_i\}_{i<\gamma} \in H(\lambda)$ be a set of constant symbols from $H(\lambda)$ that do not appear in τ.

Notice that since $|\tau| < \lambda$ and $H(\lambda)$ contains $|H(\lambda)|$-many constant symbols, that such a set of constants exists. Let $\sigma := \tau \cup \{d_i : i < \gamma\}$. Then $\sigma \in H(\lambda)$. As $dep(\Delta(\mathcal{L})) = \lambda$, by Lemma 4.2.3 it is sufficient to show that for all $\varphi \in \Delta(\mathcal{L})[\sigma]$, if $d_i^{\mathcal{A}}$ is an interpretation of the d_i by elements of A, then

$$(\mathcal{A}, d_i^{\mathcal{A}})_{i<\gamma} \models \varphi \text{ iff } (\mathcal{B}, e(d_i^{\mathcal{A}}))_{i<\gamma} \models \varphi.$$

So let $d_i^{\mathcal{A}}$ for $i < \gamma$ be such an interpretation. Then $(\mathcal{A}, d_i^{\mathcal{A}})_{i<\gamma} \in \mathcal{M}$, as $\mathcal{M}$ contains all the constants d_i and all the elements $d_i^{\mathcal{A}} \in A$ and it is a model of enough of set theory. Similarly $(\mathcal{B}, e(d_i^{\mathcal{A}}))_{i<\gamma} \in \bar{\mathcal{N}}$. Also notice that since $\sigma \in H(\lambda)$ and because of $dep^*(\mathcal{L}) = \lambda$, also $\varphi \in H(\lambda)$. Further "$(\mathcal{A}, d_i^{\mathcal{A}})_{i<\gamma} \models_{\Delta(\mathcal{L})} \varphi$" and "$(\mathcal{B}, e(d_i^{\mathcal{A}})_{i<\gamma} \models_{\Delta(\mathcal{L})} \varphi$" are absolute for transitive R-correct models of set theory which contain φ and σ, because $\Delta(\mathcal{L})$ is p-symbiotic with R by Lemma 3.2.9 as $\mathcal{L}$ is, and thus being in $Mod(\varphi)$ is $\Delta_1(R)$ with parameters in $\{\sigma, \varphi\}$ by (S1p) of p-symbiosis. In particular this absoluteness holds for $\mathcal{M}$ and $\bar{\mathcal{N}}$: Because $H(\lambda) \in M$ and $\varphi \in H(\lambda)$, also $\varphi \in M$, as M is transitive. If $m \in M$, then $m = \varphi$ iff $\mathcal{M} \models \delta_\varphi(m)$ iff $\bar{\mathcal{N}} \models \delta_\varphi(\pi(f(m)))$ iff $\pi(f(m)) = \varphi$. Because $\mathcal{M} \models \delta_\varphi(\varphi)$, thus $\pi(f(\varphi)) = \varphi$ so $\varphi \in \bar{\mathcal{N}}$. Similarly one sees that $\sigma \in \mathcal{M}$ and $\sigma \in \bar{\mathcal{N}}$. Therefore we get the following chain of equivalences:

$$\begin{aligned}(\mathcal{A}, d_i^{\mathcal{A}})_{i<\gamma} \models_{\Delta(\mathcal{L})} \varphi &\Leftrightarrow \mathcal{M} \models \text{“}(\mathcal{A}, d_i^{\mathcal{A}})_{i<\gamma} \models_{\Delta(\mathcal{L})} \varphi\text{”} \\ &\Leftrightarrow \bar{\mathcal{N}} \models \text{“}(\mathcal{B}, e(d_i^{\mathcal{A}})_{i<\gamma} \models_{\Delta(\mathcal{L})} \varphi\text{”} \\ &\Leftrightarrow (\mathcal{B}, e(d_i^{\mathcal{A}})_{i<\gamma} \models_{\Delta(\mathcal{L})} \varphi.\end{aligned}$$

Here the first and last equivalence hold because of the absoluteness, and the middle one because π and f are an isomorphism and a $\Delta(\mathcal{L})$-elementary embedding, respectively (and $\mathcal{B} = c_1^{\bar{\mathcal{N}}} = \pi(f(c_1^{\mathcal{M}})) = \pi(f(\mathcal{A})))$.

Claim 3: $(B, <^{\mathcal{B}})$ contains an upper bound for $(A, <^{\mathcal{A}})$.

$(N, \prec^{\mathcal{N}})$ contains an upper bound for $(M, \prec^{\mathcal{M}})$. As $\mathcal{N} \cong \bar{\mathcal{N}}$, also $(\bar{N}, \prec^{\bar{\mathcal{N}}})$ contains an upper bound for $(M, \prec^{\mathcal{M}})$. By condition 6, $(B, \prec^{\bar{\mathcal{N}}}\restriction B)$ is cofinal in $(\bar{N}, \prec^{\bar{\mathcal{N}}})$ and $(A, \prec^{\mathcal{M}}\restriction A)$ is cofinal in $(M, \prec^{\mathcal{M}})$, so also $(B, \prec^{\bar{\mathcal{N}}}\restriction B)$ contains an upper bound for $(A, \prec^{\mathcal{M}}\restriction A)$. Now by condition 7, $\prec^{\bar{\mathcal{N}}}\restriction B = <^{\mathcal{B}}$ and $\prec^{\mathcal{M}}\restriction A = <^{\mathcal{A}}$, so $(B, <^{\mathcal{B}})$ contains an upper bound for $(A, <^{\mathcal{A}})$.

□

The theorem shows that by (∞, κ)-compactness of $\mathcal{L}$, we get $\mathcal{L}'$-extensions with upper bounds of specific partial orders in $\Sigma_1(R)$-definable classes for a large class of logics. We showed in Corollary 4.2.6 that, in presence of p-symbiosis between R and $\mathcal{L}$, condition 1 can be equivalently substituted by either of the following two.

1a. $f : \mathcal{A} \preccurlyeq_{\Delta(\mathcal{L})} \mathcal{B}$.

1b. f is an (R, λ)-embedding.

While condition 1a only makes sense in contexts of R being symbiotic to some $\mathcal{L}$, conditions 1 and 1b are also interesting when this is not the case. And while 1 still has a tight connection with model-theoretic definability in referring to $\mathcal{L}'$-embeddings, 1b gets rid of this connection altogether and only makes reference to set-theoretic concepts. Because we wanted to formulate a “set-theoretic” principle connected to compactness properties, we will thus spell this out explicitly, using condition 1b.

Definition 5.1.2. Let R be a set-theoretic predicate and λ and κ be cardinals. We take $\mathrm{EEP}^{\lambda}_{\kappa}(R)$ to be the statement:

> For every class of structures $\mathcal{K}$ in a vocabulary $\tau \in H(\lambda)$, if $\mathcal{K}$ is $\Sigma_1(R)$-definable with parameters in $H(\lambda)$ and $\mathcal{A} = (A, <^{\mathcal{A}}, \dots) \in \mathcal{K}$, if $<^{\mathcal{A}}$ is a partial order where every $A_0 \in [A]^{<\kappa}$ has an upper bound, then there is $\mathcal{B} \in \mathcal{K}$ and a map $f : A \longrightarrow B$ such that
>
> 1b. f is an (R, λ)-embedding.
>
> 2. $(B, <^{\mathcal{B}})$ contains an upper bound for $(A, <^{\mathcal{A}})$.

We call the smallest cardinal κ such that $\mathrm{EEP}^{\lambda}_{\kappa}(R)$ holds, the λ-*end-extension number* of R.

Notice that we did *not* show that $\mathrm{EEP}^{\lambda}_{\kappa}(R)$ is equivalent to the condition on R proved in Theorem 5.1.1. In Corollary 4.2.6 we show that f being an (R, λ)-embedding is equivalent to it being an $\mathcal{L}'$-embedding for logics satisfying (S1p) of p-symbiosis with R and with strong dependence number at most λ *only* under the assumption of p-symbiosis between R and a logic extending $\mathcal{L}_{\lambda\omega}$ and with strong dependence number λ. But the property $\mathrm{EEP}^{\lambda}_{\kappa}(R)$ has no connection back to the notion of symbiosis itself, so it is interesting to consider on its own.

However, as in the situation of Theorem 5.1.1 we *do* work under the assumption of p-symbiosis, we indeed can equivalently substitute condition 1 in Theorem 5.1.1 by condition 1b. Thus we have proved the following

Corollary 5.1.3. Let $\mathcal{L} := \mathcal{L}^*_{\lambda\omega}$ be a logic with $\mathcal{L} \geq \mathcal{L}_{\lambda\omega}$ and $dep^*(\mathcal{L}) = \lambda$. Let R be a predicate of set theory and $\kappa \geq \lambda$ a regular cardinal and assume $\mathcal{L}$ to be p-symbiotic with R. Let $\mathcal{L}$ be (∞, κ)-compact. Then $\mathrm{EEP}^{\lambda}_{\kappa}(R)$ holds. □

Thus under assumption of p-symbiosis between $\mathcal{L}$ as above and R, if $\mathcal{L}$ is (∞, κ)-compact, every class $\mathcal{K}$ that is $\Sigma_1(R)$-definable with parameters in $H(\lambda)$ contains an (R, λ)-extensions with an upper bound for every partial order in $\mathcal{K}$ which has only bounded subsets of size $< \kappa$.

That we do not only talk about elementary embeddings, but stronger ones, is a striking difference between $\mathrm{EEP}^{\lambda}_{\kappa}(R)$ and the reflection principles SR_R and USR_R. Of course, we could substitute condition 1 in the theorem by yet another condition:

1c. f is a (first-order) elementary embedding.

Because every $\mathcal{L}'$-embedding for any (regular) logic is also an elementary embedding, the resulting theorem would be implied by the one we proved. But of course 1c would lead to a weaker statement, as an elementary embedding does not have to be, say, a

$\Delta(\mathcal{L})$-embedding for some strong logic. Thus we chose to give the theorem in the strong version presented above.

With regards to which cardinals we consider, the strongest version of the theorem is achieved by choosing λ the largest we can without violating its assumptions, i.e., $\lambda = \kappa$ when considering logics which are (∞, κ)-compact. Then we get the statement $\mathrm{EEP}^{\kappa}_{\kappa}(R)$. To give an example, consider the following

Corollary 5.1.4. Let κ be the least extendible cardinal. Then $\mathrm{EEP}^{\kappa}_{\kappa}(\mathrm{Pow})$ holds.

Proof. $\mathcal{L}^2_{\kappa\omega}$ is (∞, κ)-compact and has strong dependence number κ (see Theorem 4.1.6 and Proposition 2.7.7). Moreover, it is p-symbiotic with Pow. Thus by Corollary 5.1.3 above, $\mathrm{EEP}^{\kappa}_{\kappa}(\mathrm{Pow})$ holds. $\square$

This corollary tells us that if we have a class of partial orders with the described property, which is $\Sigma_1(\mathrm{Pow})$ with parameters in $H(\kappa)$ for κ the least extendible, then it contains $\mathcal{L}'$-extensions with upper bounds for every $\mathcal{L}'$ which fulfils (S1p) of p-symbiosis and has strong dependence number at most κ; or alternatively, that it contains $\Delta(\mathcal{L}^2_{\kappa\omega})$-extensions, or (R, κ)-extensions with upper bounds (depending on whether we chose to consider condition 1, 1a or 1b).

5.2 Discussion: Backwards Direction, Optimality, Applications and Comparison to other Results

Our goal was to find a set-theoretic principle involving some predicate R, which is related to (∞, κ)-compactness of a logic $\mathcal{L}$ which is symbiotic to R. $\mathrm{EEP}^{\lambda}_{\kappa}(R)$ is such a principle. In contrast to the principles SR_R and USR_R, we do not have a "backwards direction", i.e., a proof of a compactness property of $\mathcal{L}$ under the assumption of $\mathrm{EEP}^{\lambda}_{\kappa}(R)$. In this last section, we want to give some more perspectives on these results, why a backwards direction fails, and show that our result is still optimal in the sense that any strengthening of $\mathrm{EEP}^{\lambda}_{\kappa}(R)$ would lead to inconsistency. We will further notice an interesting difference between compactness and Löwenheim-Skolem properties. Finally, we will argue what applications our results can have.

The main restriction of $\mathrm{EEP}^{\lambda}_{\kappa}(R)$ is that it only talks about classes of partial orders over vocabularies in $H(\lambda)$. On the other hand, the equivalent formulation of (∞, κ)-compactness in terms of extensions of partial orders from Theorem 4.3.3, does not have such a restriction. Thus, it seems, to prove an equivalence of a principle as the above one to (∞, κ)-compactness would require to also consider classes of structures over arbitrary vocabularies.

Also the proof of Theorem 4.3.3 supports this: To prove that a set of sentences $T \subseteq \mathcal{L}[\tau]$ is satisfiable, one builds a partially ordered structure over a vocabulary which contains at least $|T|$-many symbols, and considers an $\mathcal{L}$-extension of this structure. Thus one really needs arbitrarily large vocabularies in this proof.

One might try to consider proving a similar statement to $\mathrm{EEP}^{\lambda}_{\kappa}(R)$, which gets rid of the limitation to vocabularies in $H(\lambda)$ in the following way. Let $\mathrm{EEP}^{\lambda}_{\kappa}(R)^*$ be the statement:

For every class of structures $\mathcal{K}$ in an arbitrary vocabulary τ, if $\mathcal{K}$ is $\Sigma_1(R)$-definable with parameters in $\{\tau\}$ and $\mathcal{A} = (A, <^{\mathcal{A}}, \dots) \in \mathcal{K}$, if $<^{\mathcal{A}}$ is a partial order in which every $A_0 \in [A]^{<\kappa}$ has an upper bound, then there is $\mathcal{B} \in \mathcal{K}$ and a map $f : A \longrightarrow B$ such that

1. For every logic $\mathcal{L}'$ which satisfies (S1p) of p-symbiosis with R and for which $dep^*(\mathcal{L}) \leq \lambda$, $f : \mathcal{A} \preccurlyeq_{\mathcal{L}'} \mathcal{B}$.
2. $(B, <^{\mathcal{B}})$ contains an upper bound for $(A, <^{\mathcal{A}})$.

Note that to make sense of a class of structures in an arbitrary vocabulary being $\Sigma_1(R)$-definable, one is forced to the usage of τ as a parameter, as otherwise it might be that no class of τ-structures is $\Sigma_1(R)$-definable, for instance, if τ itself is not $\Sigma_1(R)$.

With this at hand, we get the following "theorem" showing that from $\mathrm{EEP}^{\lambda}_{\kappa}(R)^*$ we can trivially prove the "backwards direction" of Theorem 5.1.1, i.e., (∞, κ)-compactness of a logic $\mathcal{L}^*_{\lambda\omega} \geq \mathcal{L}_{\lambda\omega}$ which is p-symbiotic to R under assumption of $\mathrm{EEP}^{\lambda}_{\kappa}(R)^*$. While the theorem does indeed hold, it is not of any use, as we will show below that $\mathrm{EEP}^{\lambda}_{\kappa}(R)^*$ is inconsistent.

Theorem 5.2.1. Let $\mathcal{L} := \mathcal{L}^*_{\lambda\omega} \geq \mathcal{L}_{\lambda\omega}$ and $dep^*(\mathcal{L}) = \lambda$. Let R be a predicate of set theory and assume $\mathrm{EEP}^{\lambda}_{\kappa}(R)^*$ holds for a regular cardinal $\kappa \geq \lambda$. Then we can prove $\mathcal{L}$ to be (∞, κ)-compact.

Proof. By Theorem 4.3.3, it is sufficient to show that for every structure $\mathcal{A} = (A, <^{\mathcal{A}}, \dots)$ in an arbitrary vocabulary τ containing a binary relation symbol $<$, if $<^{\mathcal{A}}$ is a partial order which has an upper bound for any $A_0 \in [A]^{<\kappa}$, then there is a $\mathcal{B}$ containing an upper bound for $\mathcal{A}$ and such that $\mathcal{A} \preccurlyeq_{\mathcal{L}} \mathcal{B}$. Now by $\mathrm{EEP}^{\lambda}_{\kappa}(R)^*$, there is such a $\mathcal{B}$ containing an upper bound for $\mathcal{A}$ and such that $\mathcal{A} \preccurlyeq_{\mathcal{L}'} \mathcal{B}$ for every logic satisfying (S1p) of p-symbiosis with R and with $dep^*(\mathcal{L}') \leq \lambda$. Because $\mathcal{L}$ is such a logic satisfying these two conditions, also $\mathcal{A} \preccurlyeq_{\mathcal{L}} \mathcal{B}$. $\square$

As we announced, the above is superfluous, because $\mathrm{EEP}^{\lambda}_{\kappa}(R)^*$ is inconsistent.

Proposition 5.2.2. $\mathrm{EEP}^{\lambda}_{\kappa}(R)^*$ is inconsistent for any cardinals λ and κ and any R.

Proof. The following example of the class $\mathcal{K}$ defined below is taken from [7], where it was used to show that a similar strengthening of the bounded upwards reflection principle to arbitrary vocabularies is inconsistent. Assume $\mathrm{EEP}^{\lambda}_{\kappa}(R)^*$. Consider a vocabulary τ consisting of one binary relation symbol $<$ and κ many constant symbols $\{c_i \colon i < \kappa\}$. Notice that $|\tau| = \kappa$, so $\tau \notin H(\kappa)$. Consider the class of τ-structures

$$\mathcal{K} = \{\mathcal{A} \colon \mathcal{A} \text{ is a } \tau\text{-structure and } \forall x \in A \exists c \in \tau (c \text{ is a constant symbol} \wedge a = c^{\mathcal{A}})\}.$$

I.e., $\mathcal{K}$ is the class of τ-structures where every element is the interpretation of a constant. Clearly, $\mathcal{K}$ is $\Sigma_1(R)$-definable with parameter τ. But now consider any $\mathcal{A} \in \mathcal{K}$ where $<^{\mathcal{A}}$ is a partial order in which any $A_0 \in [A]^{<\kappa}$ has an upper bound. Then by $\mathrm{EEP}^{\lambda}_{\kappa}(R)^*$ there is a $\mathcal{B} \in \mathcal{K}$ containing an upper bound for $\mathcal{A}$ and with $f : \mathcal{A} \preccurlyeq_{\mathcal{L}'} \mathcal{B}$ for every logic $\mathcal{L}'$ which satisfies (S1p) of p-symbiosis with R and for which $dep^*(\mathcal{L}') \leq \lambda$. Consider any such logic, for instance $\mathcal{L}_{\omega\omega}$. So $\mathcal{B}$ is an elementary extension of $\mathcal{A}$. But then $f(c^{\mathcal{A}}) = c^{\mathcal{B}}$ for all $c \in \tau$. And because $\mathcal{B} \in \mathcal{K}$, for all $b \in B$ there is $c \in \tau$ with $c^{\mathcal{B}} = b$. But then f is an isomorphism, so $\mathcal{A} \cong \mathcal{B}$. Therefore $\mathcal{B}$ cannot contain an upper bound for $\mathcal{A}$. We see that $\mathrm{EEP}^{\lambda}_{\kappa}(R)^*$ is inconsistent. $\square$

The above shows that our result is, in some sense, optimal. For, if we allow structures over vocabularies τ from any $H(\lambda)$ with $\lambda > \kappa$, the above argument could be replicated (because we only needed to express that every element of a structure is one of κ-many constants). I.e., similarly one can show:

Proposition 5.2.3. If $\lambda > \kappa$, then $\mathrm{EEP}^{\lambda}_{\kappa}(R)$ is inconsistent.

Proof. As $\lambda > \kappa$, we can take a vocabulary $\tau \in H(\lambda)$ that contains κ-many constant symbols. Then the class $\mathcal{K}$ as in the above proof is $\Sigma_1(R)$-definable with parameters in $H(\lambda)$ and we can analogously show that it cannot contain extensions of partial orders with upper bounds. So $\mathrm{EEP}^{\lambda}_{\kappa}(R)$ does not hold. $\square$

So the principle $\mathrm{EEP}^{\lambda}_{\kappa}(R)$ in its current form cannot be improved to a formulation that allows to consider vocabularies from any $H(\gamma)$, where $\gamma > \kappa$, without becoming inconsistent.

Our discussion mentioned two interesting differences when comparing (∞, κ)-compactness and the principle $\mathrm{EEP}^{\lambda}_{\kappa}(R)$ on the one side, and the upwards/downwards Löwenheim-Skolem numbers $\mathrm{LST}(\mathcal{L})$ and $\mathrm{ULST}(\mathcal{L})$ and the reflection principles SR_R and USR_R on the other side: First, the theorems presented here talk about much stronger embedding relations than SR_R and USR_R, where only elementary embeddings show up. And second, in *our* case, to prove a “backwards direction”, i.e., proving (∞, κ)-compactness of $\mathcal{L}$ from some principle analogous to $\mathrm{EEP}^{\lambda}_{\kappa}(R)$ for a symbiotic predicate R, seems to require getting rid of only talking about some restricted class of vocabularies. Contrary to this, in the case of USR_R and $\mathrm{ULST}(\mathcal{L})$, the restriction to finite vocabularies is indeed sufficient to show an equivalence.

We want to underline both facts here: For the first, in light of Theorem 4.3.3, if we would use a set-theoretic principle holding of R that only talks about elementary embeddings, it seems hopeless that one could ever prove (∞, κ)-compactness of a logic $\mathcal{L}$ symbiotic to R under assumption of this principle. For proving (∞, κ)-compactness from the existence of $\mathcal{L}$-extensions with upper bounds of partial orders, it seems really needed that one uses that the structure $\mathcal{A}$ and its extension $\mathcal{B}$ satisfy the same $\mathcal{L}$-sentences. And for this to hold, one needs the embedding to be an $\mathcal{L}$-embedding and not only an elementary one. The formulation of $\mathrm{ULST}(\mathcal{L})$ on the other hand needs no such strong embeddings.

And for the second, the reason that we can restrict ourselves to finite vocabularies when considering $\mathrm{ULST}(\mathcal{L})$ and USR_R is (at least for logics with finite dependence number), that there we are only interested in models of single sentences: We want to know, for every sentence $\varphi \in \mathcal{L}$: if $Mod(\varphi)$ contains a model of at least size κ, does it already contain arbitrarily large models? But if the dependence number of $\mathcal{L}$ is ω, then φ will only contain finitely many symbols. So it is sufficient to consider only finite vocabularies.

Compactness, on the other hand, really needs to consider large vocabularies, since if we restrict our attention to finite vocabularies, interesting compactness phenomena often do not even come into play: Consider, for example, second-order logic $\mathcal{L}^2$. The smallest cardinal κ for which $\mathcal{L}^2$ is (∞, κ)-compact is the first extendible. Now to make use of (∞, κ)-compactness for a set T of second-order sentences, this set has to be at least of size κ, as otherwise it is trivial that T is satisfiable if it is κ-satisfiable. However, over a finite vocabulary, there are only countably many second-order sentences. In fact, for (∞, κ)-compactness of second-order logic to even come into play, one has to consider vocabularies of at least size κ where κ is the first extendible.[12] Thus, often to even notice that there is some compactness phenomenon going on for some logic, one really has to consider large vocabularies.

It is interesting, that upwards Löwenheim-Skolem properties and compactness properties differ in what sizes of vocabularies one has to consider and what types of embedding relations appear in set-theoretic principles related to them via symbiosis: Clearly, the upwards Löwenheim-Skolem number $\mathrm{ULST}(\mathcal{L})$ of a logic is related to the logic being (∞, κ)-compact. It is easy to show that the latter implies that $\mathrm{ULST}(\mathcal{L}) \leq \kappa$. The precise relation in the other direction is still open. That in compactness properties large vocabularies and strong embeddings seem to be really needed, suggests some kind of more fundamental difference between (∞, κ)-compactness of $\mathcal{L}$ and $\mathrm{ULST}(\mathcal{L})$ and that the former is really stronger than the latter.

Towards applicability, $\mathrm{EEP}^{\lambda}_{\kappa}(R)$ could be used to calculate lower bounds for the *compactness number* of a logic $\mathcal{L}$ extending $\mathcal{L}_{\lambda\omega}$ and with strong dependence number λ. I.e., lower bounds for the smallest cardinal κ for which $\mathcal{L}$ is (∞, κ)-compact:

Proposition 5.2.4. Let $\mathcal{L} := \mathcal{L}^*_{\lambda\omega} \geq \mathcal{L}_{\lambda\omega}$ be a logic with $dep^*(\mathcal{L}) = \lambda$. Assume $\mathcal{L}$ is p-symbiotic to R and that $\kappa \geq \lambda$ is the smallest cardinal such that $\mathrm{EEP}^{\lambda}_{\kappa}(R)$ holds. Then $\mathcal{L}$ is not (∞, γ)-compact for any $\gamma < \kappa$.

Proof. If $\mathcal{L}$ would be (∞, γ)-compact, by Corollary 5.1.3 we would have $\mathrm{EEP}^{\lambda}_{\gamma}(R)$ which contradicts minimality of κ. $\square$

Thus, one could calculate lower bounds for cardinals κ for which a logic of the form $\mathcal{L}_{\lambda\omega}(\mathrm{WF})$ or $\mathcal{L}_{\lambda\omega}(I)$ is (∞, κ)-compact, by considering $\mathrm{EEP}^{\lambda}_{\kappa}(\emptyset)$ and $\mathrm{EEP}^{\lambda}_{\kappa}(\mathrm{Cd})$. The precise compactness numbers of these logics are, to the author's knowledge, unknown, even in the case $\lambda = \omega$ and contrary to the case of second-order logic.

[12] Notice that our result is restricted to vocabularies in $\tau \in H(\lambda)$ where $\lambda \leq \kappa$. But as we are able to consider infinitary logics such as $\mathcal{L}^2_{\kappa\omega}$, even over finite vocabularies there are sets of sentences of logics we consider of at least size κ.

References

[1] J. Bagaria and J. Väänänen. On the symbiosis between model-theoretic and set-theoretic properties of large cardinals. In *J. Symb. Log.*, 81(2):584–604, 2016.

[2] J. Barwise. Implicit definability and compactness in infinitary languages. In J. Barwise *The Syntax and Semantics of Infinitary Languages*, Lecture Notes in Mathematics, pp. 1-35. Springer, Berlin, Heidelberg, New York, 1968.

[3] J. Barwise. *Admissible Sets and Structures.* Perspectives in Mathematical Logic. Springer, Berlin, Heidelberg, 1975.

[4] J. Barwise and S. Feferman (eds.). *Model-Theoretic Logics.* Perspectives in Mathematical Logic. Springer, New York, 1985.

[5] K. J. Devlin. *Constructibility.* Perspectives in Mathematical Logic. Springer, Berlin, Heidelberg, 1984.

[6] H.-D. Ebbinghaus. Extended Logics: The General Framework. In J. Barwise and S. Feferman (eds.) *Model-Theoretic Logics.* Perspectives in Mathematical Logic, pp. 25 - 76. Springer, New York, 1985.

[7] L. Galeotti. *The theory of generalised real numbers and other topics in logic.* ILLC Dissertation Series DS-2019-04. Amsterdam, 2019.

[8] L. Galeotti, Y. Khomskii and J. Väänänen. Bounded Symbiosis and Upwards Reflection. Preprint, 2021.

[9] T. Jech. *Set Theory.* The Third Millennium Edition, revised and expanded. Corrected 4th printing. Springer, Berlin, Heidelberg, New York, 2006.

[10] K. Kunen. *Set Theory. An Introduction to Independence Proofs.* North-Holland, Amsterdam, 1980.

[11] A. Lévy. *A Hierarchy of Formulas in Set Theory.* Memoirs of the American Mathematical Society, 57, Providence, 1965.

[12] P. Lindström. First Order Predicate Logic with Generalized Quantifiers. In *Theoria*, 32: 186-195, 1966.

[13] P. Lindström. On Extensions of Elementary Logic. In *Theoria*, 35:1-11, 1969.

[14] M. Magidor. On the Role of Supercompact and Extendible Cardinals in Logic. In *Israel J. Math.*, 10: 147-157, 1971.

[15] J. A. Makowsky. Compactness, Embeddings and Definability. In J. Barwise and S. Feferman (eds.). *Model-Theoretic Logics.* Perspectives in Mathematical Logic, pp. 645 - 716. Springer, New York, 1985.

[16] M. Nadel. $\mathscr{L}_{\omega_1\omega}$ and Admissible Fragments. In J. Barwise and S. Feferman (eds.) *Model-Theoretic Logics*. Perspectives in Mathematical Logic, pp. 271 - 316. Springer, New York 1985.

[17] J. Väänänen. On the Compactness Theorem. In S. Miettinen and J. Väänänen (editors), *Proceedings of the Symposiums on Mathematical Logic*, Reports from the department of philosophy, University of Helsinki, vol 2, pp. 62-68, 1977.

[18] J. Väänänen. Abstract logic and set theory. I. Definability. *Logic Colloquium '78* (Mons, 1978), volume 97 of Stud. Logic Foundations Math., pp. 391–421. North-Holland, Amsterdam, New York, 1979.

[19] J. Väänänen. Δ-extensions and Hanf-numbers. In *Fund. Math.*, 115(1):43-55, 1983.