The Classicality of Epistemic Multilateral Logic

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

Epistemic Multilateral Logic (EML) is a natural deduction system for multilateral modal logic. It has the notable feature that its valid inference patterns on the level of formulae seem to allign with those of classical logic, yet it intuitively invalidates certain classically valid metarules. This raises the issue of the extent to which EML can be thought of as classical. However, it is unclear precisely in which sense EML preserves classical logic on the inferential level and departs from it on the metalevel, as the idiosyncracies of the multilateral language prevent a straightforward comparison. We fix the situation by developing a systematic method for the comparison of multilateral to unilateral logics, and applying it to provide a detailed overview of the different ways in which EML conforms with, departs from and approaches classical logic. Along the way, we contribute to the general literature on higher level inferences by clarifying which notions of higher level validity are available, and how they relate to each other along various dimensions. The final results confirm that EML behaves classically only up to the basic inferential level, but also allow us to prove that this is as close to classicality as one can get within the multilateral framework.

Contents

Introduction

Epistemic Multilateral Logic (EML) is a natural deduction system for multilateral modal logic, recently developed by Incurvati and Schlöder (2019, 2020). Since Frege (1919), logic has traditionally been presented 'unilaterally': only explicitly treating derivations between assertions, as rejections are reduced to assertions of negations. On the other hand, bilateral approaches hold the speech act of rejection as primitive alongside assertion (Rumfitt, 2000; Smiley, 1996). The multilateral framework goes a step further and considers the speech act of *weak* assertion, as opposed to the usual 'strong' assertion, to be primitive as well. Formally, EML's multilaterality means that it treats inferences between signed modal formulae. These are formulae from the standard modal language, prefixed by one of three force markers: $+$, $\oplus$ and $\ominus$, for strong assertion, weak assertion and weak rejection respectively.

EML has the notable feature that its valid inference patterns on the level of formulae seem to allign with those of classical logic, yet it intuitively invalidates certain classically valid metarules. This raises the issue of whether, or to what extent, EML can be thought of as classical. Two broad questions are at play here: (i) which (meta-)logical properties does EML have in common with classical logic, and (ii) to what extent do these warrant regarding EML as classical? The main purpose of this thesis is to answer question (i), through a technical study of EML, and occasionally of Classical Propositional Logic (CPL) (though the latter is of course much better understood already). The final aim is a detailed overview of the ways in which EML approaches, conforms with, or departs from classical logic. To guide the research, it will often be important to critically consider which of EML's properties can be deemed relevant to its classicality. But no stance will ultimately be taken on the application criteria of the 'classical' label, so as to answer question (ii).

Fortunately, we need not undertake the task from scratch, for EML's situation is reminiscent of that of Strict-Tolerant (ST) and Supervaluationist (SV) logic. These unilateral logics are known to behave classically on the inferential level, but not on the level of metainferences, and have thus sparked debates around their supposed classicality (see e.g. (Williams, 2008) on supervaluationism or (Scambler, 2020) on ST and its relatives). This is especially true of ST, which was discovered only in the previous decade (Van Rooij, 2012; Cobreros, Egré, Ripley, & van Rooij, 2012), but has since become the subject of an extensive ongoing investigation, with many authors putting forth different methods and metrics for measuring the degree of similarity between ST and CPL. Although it is generally agreed upon that logics are similar to the extent that they validate the same inferences, and that this is the way to compare ST to classical logic, there is room for interpretation along two dimensions. First of all in the different levels of inference which may be taken into consideration, and secondly in what it is taken to mean for a higher level inference to be valid, as there are several natural alternatives. When it comes to determining a criterion of validity for higher level inferences, our methodology draws significant inspiration from the ST literature. We will examine the validity notions that have been considered in the context of ST, offer some novel generalizations and results on their relative strength, and motivate a choice for the most appropriate approach with respect to EML and its classicality.

Once we are clear on what it means for a meta- or higher level inference to be valid, the question remains how this can be applied to compare EML and CPL. Whereas ST, SV and CPL are all unilateral, allowing for direct comparisons via their common syntax, EML is multilateral. Although the presence of modal operators (or unconventional constants in the cases of ST and SV) can be handled by using schemas rather than individual inferences, the non-embeddable force markers are less easily dealt with. To illustrate, consider the earlier claims that EML's valid inferences *seem to* allign with classical logic, yet it *intuitively* invalidates certain classically valid metarules. The phrasing is imprecise because they are informal observations, and there is no straightforward way of formally understanding them, even given some notion of validity. EML intuitively invalidates classicaly valid metarules like Classical Reductio

$$\frac{A, \neg B \vdash \bot}{A \vdash B}$$

because multilateral modal instances such as

$$\frac{\oplus p, +\neg p \vdash \bot}{\oplus p \vdash +p}$$

are EML-invalid. We recognize the latter as an instance of the former, hence its EML-invalidity seemingly represents a failure of *reductio ad absurdum*, and a departure from classical logic. But strictly speaking, it is not a substitution instance of Classical Reductio, since there is no uniform substitution of B. Below the inference line, B is replaced by $+p$, but above it, $\neg B$ is replaced by $+\neg p$. For a uniform substitution, $\neg B$ would have to be replaced by $\neg + p$. Yet this would be ungrammatical, as the non-embeddable $+$ cannot be placed *after* the negation. This is but one instance of a general issue plaguing the comparison between uni- and multilateral logics. Thus a key intermediate goal of the project will be to construct a general formal method for cross-identifying inference rules of different levels, between unilateral logics like CPL and multilateral ones like EML.

Structurally, the thesis' main body is divided into four chapters. Chapter 1 takes care of some preliminaries, by motivating and presenting EML and the issue of its classicality, and drawing the comparison with the situations of ST and SV. Chapter 2 studies the different notions of validity we ought to consider, and proves some general results about their relations and behaviour. Chapter 3 is concerned with constructing the cross-identification method that overcomes the comparison problem, and reassessing how the validity criteria fit given this solution. Chapter 4 is comprised of the main technical results. This is where we apply the accumulated methods to EML, so as to determine its agreements and disagreements with classical logic, and furthermore investigate if and how EML can be strengthened so as to behave more classically given our metrics. This provides the overview of results that constitutes our central objective.

Chapter 1

Background

1.1 Logical Inferentialism and Multilateralism

Epistemic Multilateral Logic (Incurvati & Schlöder, 2019, 2020) is, as the name suggests, epistemic and multilateral. The former simply means that its syntax includes a $\Diamond$ operator, which is interpreted as an epistemic 'might': $\Diamond A$ is read as 'it *might* be the case that A'. The dual $\Box$ is accordingly read as 'it *must* be the case that'. Multilateralism, on the other hand, is a bit more involved. It is a formal framework, motivated by a more general position in the philosophy of language and logic known as logical inferentialism. To understand the meaning of and motivation for EML, we must first briefly familiarize ourselves with logical inferentialism, and the place of multilateralism within it. This will be taken up in the present section, before EML itself is introduced in the next.

Inferentialism is an umbrella term for positions centered around the idea that the content/meaning of a linguistic expression is determined/explained by the rules governing its proper use within inferences.[1] By contrast, other theories of content may for instance point to the reference of a term as constituting its meaning, which is in turn taken to determine the conceptually secondary inferential roles. The inferentialist claim may be made regarding language in general (e.g. Brandom, 1994; Horwich, 1998), or restricted to some specific class of expressions.

Logical inferentialism is an especially natural variety of the latter type, according to which at least the meaning of *logical* terms (connectives, operators, quantifiers etc.) lies with their legitimate inferential patterns (as opposed to e.g. their Tarskian truth conditions). The rules governing the inferential behaviour of a term can be naturally split up into two sorts: those that tell us when we may derive expressions featuring the term, and those that mark which consequences we are licensed to derive from such expressions. When it comes to logical vocabulary, these correspond quite nicely to introduction and

[1] Precisely what the notion of content or meaning, and the relation of determination or explanation, amounts to in this context depends on the specific subschool of inferentialism. See (Murzi & Steinberger, 2017) for an overview of the main options.

elimination rules (I- and E-rules) in natural deduction. Thus logical inferentialism typically takes shape as the thesis that a logical expression receives its meaning from its I- and E-rules in an appropriate natural deduction system. So to understand what 'and' or '$\wedge$' means, for instance, is just to master the use of

$$(\wedge\mathrm{I})\ \frac{A \quad B}{A \wedge B} \quad \text{and} \quad (\wedge\mathrm{E})\ \frac{A \wedge B}{A}\ \frac{A \wedge B}{B}.$$

It may seem that if logical connectives derive their meaning from their I- and E-rules, rather than the other way around, then any pair of such rules is correct, if only in virtue of itself. For there is no constraint that they must conform to a previously given meaning of the connective. The rules themselves define the content of the connective they govern, and are thus self-justifying. This leads to the most well known challenge for logical inferentialists, due to Prior (1960). Namely, if any pair of I- and E-rules is correct by its own definition, then we have no grounds for excluding the connective 'tonk' from our proof theories, where tonk is defined by the following rules.

$$(\text{tonk I})\ \frac{A}{A\text{tonk}B} \quad (\text{tonk E})\ \frac{A\text{tonk}B}{B}$$

But of course adding tonk immediately trivializes a system, as any B follows from any A by successive application of (tonk I) and (tonk E).

The standard reply is to require rule pairs to be in *harmony*, in the sense that the E-rule allows one to derive no less and no more from an instance of the connective than what is required by the I-rule to derive the instance itself. Harmony can be spelled out in various ways (Dummett, 1991; Prawitz, 1974; Tennant, 1997), such that it disallows problematic connectives like tonk. However, it has been contented (most notably by Dummett (1991)) that in doing so it also disqualifies classical logic, as the standard natural deduction formulations feature apparently disharmonious rules for negation.[2] Intuitionistic logic on the other hand is fully harmonious, and thus logcal inferentialism commits one to rejecting classicality in favour of intuitionism, or so the argument goes.

This conclusion has been resisted on the grounds that classical logic only appears disharmonious because, following Frege (1919), it is standardly presented *unilaterally*: only specifying rules for inference between assertions, since denials or rejections are understood as nothing but assertions of a negation. But one might also take a *bilateral* approach: taking rejection to be primitive alongside assertion, rather than reducible to it (Incurvati & Schlöder, 2017; Incurvati & Smith, 2010; Rumfitt, 2000; Smiley, 1996). Assertion and rejection of a sentence are speech acts, respectively expressing the attitudes of assent/acceptance and dissent/refusal towards the sentence. Inferences between speech acts may be taken as valid if the attitudes expressed by the premises jointly commit one to the attitude expressed by the conclusion (Incurvati &

[2] Though Read (2000) maintains that this assessment rests on a flawed understanding of harmony.

Schlöder, 2017, 2019, 2020).[3] For the bilateral inferentialist, then, the meaning of a logical operator lies not just with the inferential rules governing its assertions, but just as much with those governing its rejections.

The relevance of this is that in a bilateral natural deduction system, where the I- and E-rules for rejections of a given logical form are provided alongside the rules for its assertions, classical logic can be presented harmoniously. The trick is usually that the I- and E-rules for negation specify that assertion (rejection) of $\neg A$ is both derivable from and allows one to derive the rejection (assertion) of A, thus respecting harmony (Rumfitt, 2000; Smiley, 1996). But this means that, though rejection is not reducible to assertion of negation, the two are still logically equivalent. Yet linguistic evidence suggests that the speech act of rejection is often weaker than assertion of the corresponding negation (Dickie, 2010). Refusal to accept A may be based merely on the absence of evidence for it, rather than the presence of evidence to the contrary, which would be required for assent to $\neg A$. Hence (Incurvati & Schlöder, 2017) introduces a bilateral logic treating assertion and *weak* rejection, which is the speech act expressing the attitude of refusal to accept. However, understanding rejection weakly comes at the cost of giving up the above mentioned harmonious I- and E-rules for negation.

This is where multilateralism finally comes in. In the multilateral approach, besides weak rejection and the usual (strong) assertion, we consider a third primitive speech act: weak assertion, which expresses the attitude of refusing to accept the negative (Incurvati & Schlöder, 2019, 2020). The benefits are three-fold. First of all it allows harmony to be recovered, as the I- and E-rules for negation now dictate that a weak assertion (rejection) of $\neg A$ is interderivable with a weak rejection (assertion) of A. Moreover, weak assertion is independently motivated on the basis of linguistic data, in particular as corresponding to the force modifier 'perhaps' in natural language. Finally, it allows the logical inferentialist to give an account of the epistemic 'might', by providing harmonious I- and E-rules for it, as is done in EML. This is significant because, whilst the Boolean connectives tend to be relatively easy cases for inferentialism, explicating the inferential roles of logical operators beyond this has traditionally proven difficult.[4] Let us move on to see how EML achieves these feats.

1.2 Epistemic Multilateral Logic

EML is formulated in the language $\mathcal{L}_{MML}$ of multilateral modal logic. Its well-formed formulae are signed modal formulae, i.e. formulae from the stan-

[3] Though see for instance (Smiley, 1996) or (Restall, 2013) for other bilateralist conceptions of validity.

[4] Incurvati and Schlöder (2021) have furthermore applied the multilateral framework to provide a proof theory for supervaluationist logic, including harmonious rules for the 'definitely' operator, but this is less relevant to the present context.

dard language $\mathcal{L}_{ML}$ of modal logic, prefixed by one of the three force-markers $+$, $\oplus$, $\ominus$ for strong assertion, weak assertion and weak rejection respectively. To be precise: we obtain $FOR(\mathcal{L}_{PL})$ by closing a countably infinite set of propositional letters $Prop = \{p, q, ...\}$ under the operations $\neg$ and $\wedge$, and $FOR(\mathcal{L}_{ML})$ by closing $Prop$ under $\neg$, $\wedge$ and $\Diamond$.[5] Then $FOR(\mathcal{L}_{MML}) := \{+A | A \in FOR(\mathcal{L}_{ML})\} \cup \{\oplus A | A \in FOR(\mathcal{L}_{ML})\} \cup \{\ominus A | A \in FOR(\mathcal{L}_{ML})\} \cup \{\bot\}$.[6]

We use 'A', 'B', ... to denote elements of $FOR(\mathcal{L}_{PL})$ or $FOR(\mathcal{L}_{ML})$, which in the context of multilateral logic are called sentences, and 'φ', 'ψ', ... for elements of $FOR(\mathcal{L}_{MML})$, which are called signed formulae. A multilateral modal logic (MML) is a natural deduction system for deductions between elements of $FOR(\mathcal{L}_{MML})$. Logics in $\mathcal{L}_{PL}$ are called unilateral propositional logics (UPL's).

The rules defining EML can be divided into the I- and E-rules for the connectives, and a few coordination principles governing the interaction between the different speech acts. We start with the former, and in particular with $\wedge$. Strongly asserting a conjunction is, as expected, inferentially equivalent to (meaning it involves the same commitments as) strongly asserting both conjuncts.

$$(+\wedge\mathrm{I})\ \frac{+A \quad +B}{+A \wedge B} \qquad\qquad (+\wedge\mathrm{E})\ \frac{+A \wedge B}{+A}\ \frac{+A \wedge B}{+B}$$

As promised, for $\neg$ we can make use of weak assertion and rejection to provide harmonious I- and E-rules. Since weakly asserting $\neg A$ expresses a refusal to assert A, it is equivalent to weakly rejecting A. Similarly, weak rejection of $\neg A$ is equivalent to weak assertion of A.

$$(\oplus\neg\mathrm{I})\ \frac{\ominus A}{\oplus\neg A} \qquad\qquad (\oplus\neg\mathrm{E})\ \frac{\oplus\neg A}{\ominus A}$$

$$(\ominus\neg\mathrm{I})\ \frac{\oplus A}{\ominus\neg A} \qquad\qquad (\ominus\neg\mathrm{E})\ \frac{\ominus\neg A}{\oplus A}$$

Furthermore, weak assertion allows for harmonious rules treating the epistemic $\Diamond$. For strongly asserting 'it might be that A' is inferentially equivalent to stating 'perhaps A', which amounts to weakly asserting A. Furthermore, 'perhaps it might be that A' just expresses a refusal to commit to $\neg A$, and is thus equivalent to simply 'perhaps A'.

[5] The operators $\vee, \rightarrow, \leftrightarrow$ and $\Box$ are defined in terms of $\neg, \wedge$ and $\Diamond$ as usual.

[6] We may also see $\bot$ appearing in sentence positions (i.e. embedded under a force marker or connective), in which case we use it simply as a shorthand for some arbitrary contradictory sentence such as $p \wedge \neg p$. But its inclusion in $FOR(\mathcal{L}_{MML})$ is furthermore necessary to justify its appearance in signed formula positions. In these cases, $\bot$ may be interpreted as denoting that an inferential dead end has been reached through incompatible commitments (Incurvati & Schlöder, 2017, 2019, 2020).

$$(+\Diamond\mathrm{I})\ \frac{\oplus A}{+\Diamond A} \qquad (+\Diamond\mathrm{E})\ \frac{+\Diamond A}{\oplus A}$$

$$(\oplus\Diamond\mathrm{I})\ \frac{\oplus A}{\oplus\Diamond A} \qquad (\oplus\Diamond\mathrm{E})\ \frac{\oplus\Diamond A}{\oplus A}$$

As for the coordination principles, the following express that it is incoherent to strongly assert A and weakly reject A, and furthermore that if strong assertion (weak rejection) of A is incompatible with ones other commitments, this commits one to weakly rejecting (strongly asserting) A.[7]

$$(\text{Rejection})\ \frac{+A \quad \ominus A}{\perp} \qquad (\mathrm{SR}_1)\ \frac{\begin{matrix}[+A]\\ \vdots \\ \perp\end{matrix}}{\ominus A} \qquad (\mathrm{SR}_2)\ \frac{\begin{matrix}[\ominus A]\\ \vdots \\ \perp\end{matrix}}{+A}$$

Two coordination rules remain. The first of which, (Assertion), merely ensures that a strong assertion commits one to everything that the corresponding weak assertion does. The other, (Weak Inference), is a bit more complex. We mark a (sub)derivation with ϵ if it is *evidence preserving*: it does not use $(+\Diamond\mathrm{E})$ or $(\oplus\Diamond\mathrm{E})$, and all open premises or undischarged hypotheses are signed by $+$.

$$(\text{Assertion})\ \frac{+A}{\oplus A} \qquad (\text{Weak Inference})\ \frac{\oplus A \quad \begin{matrix}[+A]\\ \vdots\ \epsilon \\ +B\end{matrix}}{\oplus B}$$

(Weak Inference) serves to capture that if the available evidence does not rule out A, and furthermore suffices to derive that B is the case from the supposition that A is the case, then it also does not rule out B. The restrictions on the derivation from $+A$ to $+B$ are necessary to ensure that commitment to assent to B is actually derivable from the available evidence, plus the supposition of evidence for A, rather than from any *lack* of evidence. An appeal to lack of certain evidence during a subderivation is witnessed either by the presence of weakly rejected/asserted open premises, as these force-markers indicate lack of evidence for/against the sentence they prefix, or by the application of $\Diamond$ elimination rules, since 'it might be the case that' also indicates a mere lack of evidence to the contrary. We disallow lack of evidence from playing a part because, for instance, if besides being committed to $\oplus A$, we are also committed to $\ominus A$, then the temporary supposition that we are committed to $+A$ immediately leads to incoherent commitments. When supposing that we have more evidence than we might actually have, we must suspend those of our existing commitments which are based on lack of evidence.

[7]SR stands for *Smileian Reductio*, as the proof rules thus labeled are due to Smiley (1996).

EML, then, is the MML defined by the above listed rules. Given $\Gamma \subseteq FOR(\mathcal{L}_{MML})$ and $\psi \in FOR(\mathcal{L}_{MML})$, we write $\Gamma \vdash_{EML} \psi$ if there exists a derivation of ψ from Γ in EML. We report without proof a few of the key results of Incurvati and Schlöder (2019, 2020) regarding EML, starting with a model theory. Define a translation function $\tau : FOR(\mathcal{L}_{MML}) \to FOR(\mathcal{L}_{ML})$ as follows:

$$\tau(\varphi) = \begin{cases} \Box A & \text{if } \varphi = +A \\ \Diamond A & \text{if } \varphi = \oplus A \\ \Diamond\neg A & \text{if } \varphi = \ominus A. \end{cases}$$

Under this translation, EML embeds into S5.

Theorem 1.1 (Incurvati & Schlöder, 2020)**.** *Take arbitrary* $\Gamma \subseteq FOR(\mathcal{L}_{MML})$ *and* $\psi \in FOR(\mathcal{L}_{MML})$*. Then* $\Gamma \vdash_{EML} \psi$ *iff* $\tau[\Gamma] \vDash_{S5} \tau(\psi)$*.*

It should be stressed that this embedding does not constitute an *interpretation* of EML; it is merely a useful technical result. The meaning of the logical terms is conferred on them by the proof rules themselves.

Incurvati and Schlöder (2020) furthermore claim that the strongly asserted fragment of EML 'extends classical logic'. This is of particular importance, because bi- and multilateralism are partially motivated by the intention to provide a harmonious presentation of classical logic, so as to render inferentialism compatible with classicality. In support they provide the following two theorems relating EML to CPL, which they label 'Classicality' and 'Supra-Classicality' respectively.

Theorem 1.2 (Incurvati & Schlöder, 2020)**.** *Take arbitrary* $\Gamma \subseteq FOR(\mathcal{L}_{PL})$ *and* $A \in FOR(\mathcal{L}_{PL})$*. Then* $\Gamma \vDash_{CPL} A$ *iff* $\{+\gamma \mid \gamma \in \Gamma\} \vdash_{EML} +A$*.*

Theorem 1.3 (Incurvati & Schlöder, 2020)**.** *Take arbitrary* $\Gamma \subseteq FOR(\mathcal{L}_{PL})$ *and* $A \in FOR(\mathcal{L}_{PL})$*, and* $\eta : Prop \to FOR(\mathcal{L}_{ML})$*. If* $\Gamma \vDash_{CPL} A$ *then* $\{+\eta[\gamma] \mid \gamma \in \Gamma\} \vdash_{EML} +\eta[A]$*.*

These provide a good starting point when it comes to the comparison between EML and CPL. But they are far from the full story. First of all, both are limited to the strongly asserted fragment. This is admittedly the part of EML that allows for the most direct comparison with CPL, since from the multilateral perspective, the latter is just a logic of strong assertion. Yet EML's behaviour more broadly might still be meaningfully compared to that of CPL. For instance, we might be interested in the *structural* properties of $\vdash_{EML}$, such as reflexivity or monotonicity, which are understood to apply to content in general, regardless of force. Such properties of EML would thus be partially determined by its treatment of sentences signed by $\oplus$ and $\ominus$. So in order to assess EML's structural classicality, we will need to find a way to take into account the entirety of the multilateral language.

Furthermore, both of these results only compare EML and CPL on the basic inferential level. However, much is revealed about a logic by considering its valid *metainferences*. The concept of a metainference and its validity criteria will be made precise in sections 2.1 and 2.2 respectively. As we will see, the correct notion of validity for metainferences is an especially controversial issue. But let it suffice for now that a metainference is an inference whose premises and conclusion are themselves regular inferences, and that it is considered valid if one can in some sense conclude the latter given the former. The valid metainferences can thereby capture significant aspects of the nature of validity/derivability in a given logic. For instance, logics may be said to license Cut, Conditional Proof or Contraposition when they validate all metainferences of the following forms respectively:

$$\frac{\Gamma \vdash A \qquad \Gamma, A \vdash B}{\Gamma \vdash B} \qquad \frac{A \vdash B}{\vdash A \to B} \qquad \frac{A \vdash B}{\neg B \vdash \neg A}$$

It seems that if two logics disagree on the level of metainferences, then their accounts of deductive reasoning differ wildly, and so they can not be identified with one another, even if they agree on the standard level inferences in some defined sense. Hence if EML disagrees with CPL on the validity of certain metainferences, this marks a departure from classicality on its part. Moreover, this seems to be the case. Consider the metainferential form of Classical Reductio, the CPL (and classical first order) validity of which is typically taken to capture that classical logic supports *reductio ad absurdum* arguments:

$$\frac{A, \neg B \vdash \bot}{A \vdash B}$$

EML apparently does not validate all metainferences of this shape. There are three broad types of counterexamples to it, utilizing a $\Diamond$, $\oplus$, or $\ominus$, as in the following respective cases.

$$\frac{+\Diamond p, +\neg p \vdash \bot}{+\Diamond p \vdash +p} \qquad \frac{\oplus p, +\neg p \vdash \bot}{\oplus p \vdash +p} \qquad \frac{\ominus\neg p, +\neg p \vdash \bot}{\ominus\neg p \vdash +p}$$

In each of these, the premise inference is EML derivable whilst the conclusion inference is not. For example it is incoherent to strongly assert both 'not p' and 'it might be that p', but the latter is clearly weaker than strongly asserting p. However, besides it being unclear at this point precisely what it means for one of these metainferences to be invalid, it is also unclear in which sense they are instances of the general form of Classical Reductio. We recognize each of them as witnessing that unlike classical logic, EML does not generally support *reductio ad absurdum*. But none of them are uniform substitution instances of Classical Reductio. For in each case, B is replaced by $+p$ below the inference line, but above the inference line $\neg B$ is replaced by $+\neg p$. For a uniform substitution, $\neg B$ would have to be replaced by $\neg + p$. Yet this would be ungrammatical, as the non-embeddable $+$ cannot be placed after the negation. In fact, Classical Reductio as such has no uniform substitution instances for multilateral modal logic at all, nor do most other non-trivial schemas one can formulate for unilateral logic. The result is that even given a specific level of inference and corresponding notion of validity, it is far from clear how one can compare between unilateral and multilateral logics.

All of these problems need to be solved or circumvented in order to come to a complete picture of the supposed classicality of EML. It will be helpful in this regard that some related questions have been debated when it comes to the classicality of SV and ST. In particular, the question of validity for higher level inferences has been extensively discussed in the literature surrounding ST. Since SV and ST are both unilateral, they furthermore offer a context where this issue can be isolated and considered without interference from the additional complications posed by the multilateral setting of EML. So we will briefly introduce these logics, and their metainferential deviation from CPL, in the following section.

1.3 Supervaluationist and Strict-Tolerant Logic

Both SV and ST are three-valued propositional logics,[8] (partially) motivated by the problems surrounding vagueness and the Sorites paradox, which might be taken to arise as a result of assigning every sentence a definite truth-value 1 or 0 (Hyde, 2011). They are thus defined in terms of Strong-Kleene (SK) valuations, which incorporate an 'intermediate' truth-value $\frac{1}{2}$. An SK valuation is a function $sk : FOR(\mathcal{L}_{PL}) \to \{0, \frac{1}{2}, 1\}$ which respects the following Strong-Kleene truth-tables.

	$\neg$
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1

$\wedge$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

Note that a Boolean valuation is just an SK valuation with a range of $\{0, 1\}$.

Supervaluationist logic (Van Fraassen, 1966) is based on the idea that each SK valuation specifies a set of Boolean valuations, determined by the different ways that one can assign bivalent truth-values 1 and 0 to all sentences with value $\frac{1}{2}$, and that something is true on an SK valuation if it is true on each of these Boolean valuations. That is, given an SK valuation sk, a *precisification* of sk is a Boolean valuation v such that, for all $A \in FOR(\mathcal{L}_{PL})$ with $sk(A) \in \{0, 1\}$, $v(A) = sk(A)$. Then $A \in FOR(\mathcal{L}_{PL})$ is said to be *supertrue* on sk if $v(A) = 1$ for each precisification v of sk. The supervaluationist motto is that truth properly understood is supertruth, and so inferential validity is cashed out as preservation of supertruth. Given $\Gamma \subseteq FOR(\mathcal{L}_{PL})$ and $A \in FOR(\mathcal{L}_{PL})$, we say that Γ entails A in SV ($\Gamma \vDash_{SV} A$) just in case A is supertrue in every SK valuation where each $\gamma \in \Gamma$ is supertrue.

[8]Though they also admit of first-order formulations, these will not be of concern to us.

It turns out that SV preserves classical logic on the inferential level, in the sense that $\Gamma \vDash_{SV} A$ iff $\Gamma \vDash_{CPL} A$ (Fine, 1975). But this is not the end of the story. Since supertruth is a central concept to the supervaluationist framework, we would want to be able to speak of it in the object language. This can be done by adding a unary 'definitely' operator D to the syntax. So we let $FOR(\mathcal{L}^D_{PL})$ be the closure of $Prop$ under $\neg$, $\wedge$ and D, and add a semantic clause specifying that DA is true on a precisification of sk just in case A is supertrue on sk. Then DA is supertrue just in case A is supertrue, and $\neg DA$ is supertrue just when A is not supertrue. However, the behaviour of SV no longer appears to be all that classical on the metainferential level.

Recall for instance the classically legitimate metainferential patterns of Conditional Proof and Contraposition.

$$\frac{A \vdash B}{\vdash A \to B} \qquad \frac{A \vdash B}{\neg B \vdash \neg A}$$

Now consider their following respective instances in $FOR(\mathcal{L}^D_{PL})$.

$$\frac{p \vdash Dp}{\vdash p \to Dp} \qquad \frac{p \vdash Dp}{\neg Dp \vdash \neg p}$$

Though again, we have yet to provide precise criteria for metainferential validity, both of these examples seem to be invalidated by SV. Since Dp is supertrue just when p is, it follows that $p \vDash_{SV} Dp$. However, there are SK valuations such that $sk(p) = \frac{1}{2}$. Then there are precisifications where p is false, so $\neg Dp$ is supertrue on sk. But there are also precisifications v where p is true, in which case $p \to Dp$ is false, hence $\nvDash_{SV} p \to Dp$. Furthermore, $\neg p$ is false in v, so not supertrue on sk, despite $\neg Dp$ being supertrue on sk. Therefore $\neg Dp \nvDash_{SV} \neg p$.[9]

Note that the examples in this case, unlike EML's counterexamples to Classical Reductio, are actually direct substitution instances of the corresponding schemas. So SV in the extended language refuses several classically acceptable forms of argument. And indeed, these and similar observations have led to a widespread discussion on the extent to which the SV approach to vagueness rests on a revision of classical logic (Incurvati & Schlöder, 2021; Keefe, 2000a, 2000b; Machina, 1976; Williams, 2008; Williamson, 1994).

Strict-Tolerant Logic (Van Rooij, 2012; Cobreros et al., 2012) is in much the same situation, though it takes a different interpretation of SK valuations. An inference is ST valid if whenever all the premises have truth-value 1, the conclusion has truth-value at least $\frac{1}{2}$. That is to say, given $\Gamma \subseteq FOR(\mathcal{L}_{PL})$ and $A \in FOR(\mathcal{L}_{PL})$, Γ entails A in ST ($\Gamma \vDash_{ST} A$) when for every SK valuation sk, if $sk(\gamma) = 1$ for each $\gamma \in \Gamma$, then $sk(A) \in \{\frac{1}{2}, 1\}$. The idea is that inferences may still be valid even if they slightly 'lessen' the truth-value from premises to conclusion. This allows one to resist the steps leading to the Sorites paradox,

[9]In terms that will be defined only in the next chapter, what we have just shown is that these instances are globally invalid, which also entails that they are locally invalid. SV already locally invalidates Conditional Proof and Contraposition even without the presence of D, but globally invalidates them only after D is added.

and even to extend the language with a transparent truth-predicate without falling prey to the Liar or other such paradox (Cobreros et al., 2012; Cobreros, Egré, Ripley, & Van Rooij, 2013). And all of this is seemingly accomplished without giving up classical logic, for just as with SV, we have $\Gamma \vDash_{ST} A$ iff $\Gamma \vDash_{CPL} A$ for propositional Γ and A (Ripley, 2012).

However, allowing the truth-value to be weakened from premises to conclusion comes at the cost of giving up several classically valid forms of metainferential reasoning. Recall for example the Cut rule.

$$\frac{\Gamma \vdash A \qquad \Gamma, A \vdash B}{\Gamma \vdash B}$$

It may be that some SK valuation assigns 1 to every element of Γ, $\frac{1}{2}$ to A, and 0 to B. In these cases $\Gamma \vdash A$ and $\Gamma, A \vdash B$ are both ST-satisfied, but $\Gamma \vdash B$ is not. Furthermore, suppose the language is expanded with a constant λ for the truth-value $\frac{1}{2}$ (this automatically happens when a transparent truth-predicate is added, for in this case the Liar sentence can be formulated, which must always take value $\frac{1}{2}$ to avoid contradiction). Then substituting λ for A, every $\gamma \in \Gamma$ for a tautology and B for a contradiction results in an instance of Cut where the premise inferences are both ST valid but the conclusion is not.[10] Again, note that like the examples for SV but unlike those for EML, these arguments concern straightforward substitution instances of the schema Cut. For similar reasons, ST invalidates classically legitimate modes of inference such as Transitivity

$$\frac{A \vdash B \qquad B \vdash C}{A \vdash C}$$

as well as certain metaformulations of Modus Ponens, Modus Tollens, Explosion and Disjunctive Syllogism. All of this might be and indeed has been considered quite a departure from classical logic, leading to an extensive debate on the supposed classicality of ST (Barrio, Pailos, & Szmuc, 2020; Barrio, Rosenblatt, & Tajer, 2015; Ripley, 2021; Scambler, 2020).

Before we can begin to investigate such questions for the case of EML, we will first need to clarify how an MML can be compared to classical logic. The next chapter will take the first step towards this end, by illuminating the concepts of inference and validity for levels higher than the standard.

[10]In the terminology of the next chapter, these arguments respectively show that Cut is locally invalid in ST in the regular language, and furthermore globally invalid in the language with λ.

Chapter 2

Classicality by Inferences

2.1 Levels of Inference

We have seen several examples of logics that strongly deviate from the classical account of reasoning, despite full agreement with CPL on the basic inferential level. This establishes that when assessing a logic's classicality, we should not only look at ground level sequents, but also at those of a higher order. It does not suffice to restrict ourselves to the levels of inferences and metainferences, however; we should just as well consider metametainferences, metametametainferences and so on, where each higher order inference deduces a single conclusion inference (of the previous order) from a set of premise inferences (also of the previous order).[11] Such a broad approach is required because, as Barrio et al. (2020) have shown, it is possible for logics to behave identically on the basic and meta-levels whilst diverging on the metametalevel, or more generally to agree on the first n levels despite disagreement on some or all higher levels.

Formally, we study the following infinite hierarchy.

Definition 2.1. *If $\mathcal{L}_X$ is a logical language, the set SEQ_X^n of level n inferences in $\mathcal{L}_X$ is defined recursively for $n > 0$:*

$$SEQ_X^1 := \{(\Gamma, \psi) \mid \Gamma \cup \{\psi\} \subseteq FOR(\mathcal{L}_X)\}.$$

[11] We restrict our attention to single-conclusion inferences throughout this thesis. This choice is often made for the sake of simplicity when nothing hinges on it. However, there are more substantial motivations for it in the present context. As a natural deduction system, EML does not directly treat multi-conclusion inferences. Moreover, due to the non-embeddable nature of the force markers, one cannot simply reinterpret a sequent of the form $\Gamma \Rightarrow^1 \Delta$ as $\Gamma \Rightarrow^1 \bigvee \Delta$. We can of course interpret multi-conclusion sequents in terms of their satisfaction conditions, but dependence on a model-theoretic interpretation of inferences conflicts with inferentialism as well as EML's identification as a proof system. Aside from this, there has been some debate on whether inferentialism is compatible with a mutli-conclusion treatment of inferences even in principal (e.g. Dicher (2020) and Steinberger (2011) argue 'yes' and 'no' respectively), but I do not intend to take a stance on this question here.

$$SEQ_X^{n+1} := \{(\Theta, \Pi) \mid \Theta \cup \{\Pi\} \subseteq SEQ_X^n\}.$$

We will denote $(\Theta, \Pi) \in SEQ_X^n$ either as $\Theta \Rightarrow^n \Pi$ or as

$$\frac{\theta_1, ..., \theta_m}{\Pi}\ n$$

where $\{\theta_i \mid 1 \leq i \leq m\} = \Theta$. The order specifying n is often left out if the inferential level is clear from context.[12] To illustrate, consider the case of $\mathcal{L}_{PL}$. The following are examples of ((meta-)meta-)inferences from SEQ^1_{PL}, SEQ^2_{PL} and SEQ^3_{PL} respectively.

$$p \vee q, \neg q \Rightarrow^1 p \qquad \frac{\neg p \wedge q \Rightarrow r \qquad q \Rightarrow \neg r}{\neg p \Rightarrow r}\ 2 \qquad \frac{\dfrac{p \vee q \Rightarrow p}{\emptyset \Rightarrow r}\ 2 \qquad \dfrac{\neg q \Rightarrow r}{p \Rightarrow \neg r}\ 2}{\dfrac{p \vee q \Rightarrow p \qquad \emptyset \Rightarrow p}{\emptyset \Rightarrow \neg r}\ 2}\ 3$$

Two logics $\mathcal{K}_1$ and $\mathcal{K}_2$ in the same language $\mathcal{L}_X$ are said to agree on level n if they agree on the validity of every member of SEQ_X^n (given some chosen notion of validity for level n inferences). A UPL is said to be classical on level n if it agrees with CPL on level n. There is of course a sense in which two logics are more similar if they agree up to a higher level. And so the higher the level up to which some UPL $\mathcal{K}$ is classical, given a notion of validity, the more classical $\mathcal{K}$ is. This principle lies at the core of the argument that ST and SV are not fully classical because they depart from CPL at level 2 and beyond.

It is important to note why it makes sense to speak of agreement with classical logic *up to* a certain inferential level. The reason is that on all relevant notions of validity, the validities at the higher inferential levels fully determine those at the lower ones. Hence full agreement with classical logic on level n entails full agreement on all levels below, and disagreement on n entails disagreement on all levels above. This observation has two relevant consequences. First of all it means that, when studying the classicality of some logic, one typically need not consider inferences higher than the first level at which it deviates from CPL. As a result, literature on ST and SV does not standardly treat inferences or validity criteria of levels $n > 2$.[13] Yet for the purposes of section 4.2, where we will investigate whether EML can be adapted so as to behave more classically, we need to venture beyond level 2. Furthermore, it means that if a UPL $\mathcal{K}_1$ agrees with classical logic on some inferential level where $\mathcal{K}_2$ deviates from it, then the set of levels on which $\mathcal{K}_1$

[12] When referring to inferential levels, we will use $n, m, ...$ to denote arbitrary elements of the positive integers, rather than the full natural numbers, since we do not have an inferential level 0.

[13] Notable exceptions are (Barrio et al., 2020; Fitting, 2021; Pailos, 2020; Ripley, 2021; Scambler, 2020), which include many relatives of ST that display classical behaviour up to various levels (for various ways of understanding of validity).

behaves classically is strictly bigger than that of $\mathcal{K}_2$, which justifies the claim that $\mathcal{K}_1$ is strictly more classical than $\mathcal{K}_2$ - at least given that criterion of validity.

So every notion of validity for inferences of arbitrary level n provides its own measure of classicality for UPL's, for it allows us to ask up to which level a given UPL behaves classically. The next step is thus to survey the different validity criteria available.

2.2 Local and Global Validity

We should to prefix our coverage of 'validity notions' by clarifying our usage of the expression. We understand it broadly, as designating any criterion intended to capture what it means for an inference of some level to be justified or in good standing according to a logic. Our conception of the term 'logic' is similarly unspecific. For now it is more fruitful not to limit ourselves to stances on what logics are or how they should be primarily understood (e.g. semantically, as natural deduction systems, sequent calculi, equivalence classes of proof systems, and so on). We will typically present validity notions model-theoretically, and comment on if and how they can be characterized in terms of different types of proof systems. We do for the time being restrict ourselves to notions of validity for individual inferences, whereas the next chapter will cover inference schemas expressing general rules.

For ground level inferences, there is of course a single standard criterion of validity. It might receive different characterizations, e.g. as derivability in a proof system or as satisfaction at every model, but these at least pick out the same validities (assuming soundness and completeness). For higher levels of inference, however, several natural but non-equivalent notions of validity present themselves.

These complications first appear at level 2, for which we find two alternative validity criteria in the ST literature: local and global.[14] To define them in some generality, assume for the remainder that we are working with logics $\mathcal{K}$ in $\mathcal{L}_X$ such that the ground level validities can be defined as those $\Gamma \Rightarrow^1 \psi \in SEQ^1_X$ that meet some satisfaction condition with respect to every model $M \in \mathcal{M}$, for some class of structures $\mathcal{M}$.[15] We write $M \vDash_{\mathcal{K}} \Gamma \Rightarrow^1 \psi$ to denote that $\Gamma \Rightarrow^1 \psi$ satisfies this property for $M \in \mathcal{M}$. For CPL, $\mathcal{M}$ is of course the class of Boolean valuations v, and the satisfaction condition is that either

[14] Local validity was the notion of choice in e.g. (Barrio et al., 2020; Cobreros, La Rosa, & Tranchini, 2020; Dicher & Paoli, 2019; Golan, 2021a; Pailos, 2019, 2020). The global criterion is employed in (Barrio et al., 2015; Cobreros et al., 2013; Cobreros, Egré, Ripley, & van Rooij, 2020) amongst others.

[15] Far from every conceivable logic fulfills these assumptions. Many UPL's that do not are featured in (Barrio et al., 2020; Fitting, 2021; Pailos, 2019, 2020; Ripley, 2021; Scambler, 2020). But defining the validity notions more generally would be cumbersome, and more importantly uncalled-for, since the logics considered in this thesis (ST, SV, CPL, EML and EML*) all fit the limits of our mode of presentation.

$v(\gamma) = 0$ for some $\gamma \in \Gamma$ or $v(\psi) = 1$. In case of EML, we may temporarily assume that the S5 embedding provides its canonical semantics. Then $\mathcal{M}$ is the class of pointed Kripke models with equivalence relations, and the satisfaction condition for $(M, w) \in \mathcal{M}$ is that either $(M, w) \not\models \tau(\gamma)$ for some $\gamma \in \Gamma$ or $(M, w) \models \tau(\psi)$, where $\models$ signifies truth at a world.

We first generalize satisfaction per model to higher level inferences, which then allows us to define global and local validity.[16]

Definition 2.2 (General satisfaction). *Take $M \in \mathcal{M}$ and $\Theta \Rightarrow^n \Pi \in SEQ^n_X$. $\mathcal{K}$-satisfaction of $\Theta \Rightarrow^n \Pi$ by M (written as $M \models_{\mathcal{K}} \Theta \Rightarrow^n \Pi$) is defined recursively for $n > 1$: $M \models_{\mathcal{K}} \Theta \Rightarrow^n \Pi$ iff$_{Def}$*

$$\text{either } M \not\models_{\mathcal{K}} \theta \text{ for some } \theta \in \Theta, \text{ or } M \models_{\mathcal{K}} \Pi.$$

Definition 2.3 (Level 2 local validity). *$\mathcal{K}$ locally validates $\Theta \Rightarrow^2 \Pi \in SEQ^2_X$ ($\models^L_{\mathcal{K}} \Theta \Rightarrow^2 \Pi$) iff$_{Def}$*

$$M \models_{\mathcal{K}} \Theta \Rightarrow^2 \Pi \text{ for every } M \in \mathcal{M}.$$

Definition 2.4 (Level 2 global validity). *$\mathcal{K}$ globally validates $\Theta \Rightarrow^2 \Pi \in SEQ^2_X$ ($\models^G_{CPL} \Theta \Rightarrow^2 \Pi$) iff$_{Def}$*

$$\text{either } M \not\models_{\mathcal{K}} \theta \text{ for some } \theta \in \Theta \text{ and } M \in \mathcal{M}, \text{ or } M \models_{\mathcal{K}} \Pi \text{ for every } M \in \mathcal{M}.$$

When spelled out, the criterion for local validity reads: *for every $M \in \mathcal{M}$, if $M \models_{\mathcal{K}} \theta$ for every $\theta \in \Theta$, then $M \models_{\mathcal{K}} \Pi$.* Moreover, the global definition can be rewritten as: *if for every $M \in \mathcal{M}$, $M \models_{\mathcal{K}} \theta$ for every $\theta \in \Theta$, then also for every $M \in \mathcal{M}$, $M \models_{\mathcal{K}} \Pi$.* So the difference between the two lies in quantifier scope; the local condition is of the form $\forall x(Px \to Qx)$, whereas the global criterion demands that $(\forall x Px) \to (\forall x Qx)$. Thus it is easy to see that local entails global validity at level 2, for every logic. The other direction does not generally hold, as for example

$$\frac{\emptyset \Rightarrow p}{\emptyset \Rightarrow q}\ 2$$

is globally but not locally valid in CPL.

Note also how global validity at level 2 simply amounts to preservation of validity from the premise inferences to the conclusion inference; if all the premise inferences are valid, so is the conclusion. This is significant for two reasons. First, given some sound and complete proof system, validity at level 1 can itself be characterized entirely proof-theoretically. Therefore so can global validity at level 2: it is preservation of derivability. Thus global validity can be defined for a proof system itself, even if no model theory is available, or multiple distinct ones are. In contrast, local validity can only be defined in

[16] These should not be confused for the independent notions of local and global consequence for level 1 inferences, which appears in the context of Kripke semantics for modal logic (see section 1.5 of (Blackburn, De Rijke, & Venema, 2002)), or super- and subvaluationism (see section 5.3 of (Williamson, 1994)).

relation to some specific model theory. If two model theories are both sound and complete for level 1 with respect to the same proof system, and hence with respect to each other, this does not guarantee that they produce the same local validities. This is another lesson that can be drawn from ST and SV. For in their basic propositional formulations, their respective model theories produce the same level 1 validities as CPL, and are hence sound and complete with respect to standard CPL proof systems. Nevertheless, their level 2 local validities are altogether different from those generated by CPL's usual Boolean model theory, as well as from each other.[17]

So it does not strictly speaking make sense to speak of *the* local validities of some proof system, unless something can be identified as *the* model theory for the system.[18] This point has naturally escaped the attention of the ST literature. ST's semantic definition is the original and canonical one, the name 'Strict-Tolerant logic' being a direct reference to it. Several sound and complete proof systems have been developed (see e.g. those in (Barrio et al., 2015) or (Cobreros et al., 2012)), but there is no particular reason to prefer purely syntactic methods when comparing ST to CPL. The case of EML is of course quite different in this respect. Given its inferentialist underpinning, EML should only be identified with the natural deduction system itself. The S5 embedding is a technical tool: useful for studying EML's properties, but not an interpretation of its meaning, certainly not any more so than any alternative model theory (and corresponding set of local validities). Thus when it comes to EML, we must ultimately give preference to validity notions that can be defined by referring purely to the natural deduction system.

Another consequence of global validity being preservation of validity is that, if we want to extend it to levels $n > 2$, we are faced with more choices. The local validity criterion has an obvious generalization to arbitrary inferential levels: satisfaction at every model. But if global validity at level 2 is preservation of level 1 validity, then global validity at level 3 would be preservation of level 2 validity. The question then arises what we mean by level 2 validity here, since there are two distinct options. Therefore global validity itself splits

[17] Another example of this phenomenon can be extracted from Carnap's (1943) categoricity problem. Consider the valuation $v^{\dagger} : FOR(\mathcal{L}_{PL}) \to \{1, 0\}$ such that $v^{\dagger}(A) = 1$ iff $\vDash_{CPL} A$. Carnap pointed out that if we were to expand the class of Boolean valuations with $v^{\dagger}$, and kept the satisfaction condition the same, we would get the same set of valid level 1 inferences. So this alternative model theory would be sound and complete with respect to CPL proof systems. However, the local level 2 validities would be different. For instance, because the following metainference is not satisfied in $v^{\dagger}$ despite being satisfied in every Boolean valuation.

$$\frac{p \Rightarrow q \qquad \neg p \Rightarrow q}{p \vee \neg p \Rightarrow q}\ 2$$

[18] We will sometimes speak of the local (or global$_L$, to be introduced below) validities of an arbitrary MML or other natural deduction system, for which no model theory has been identified at all. In these cases, what is said should be understood as pertaining to *any* sound and complete model theory that may be specified for that proof system.

into two notions on level 3: we can define global$_1$ validity of $\Theta \Rightarrow^3 \Pi$ as preservation of *local* validity from the $\theta \in \Theta$ to Π, and global$_2$ validity of $\Theta \Rightarrow^3 \Pi$ as preservation of *global* validity from the $\theta \in \Theta$ to Π. Combined with the generalization of local validity, this means there are a total of three notions of validity for level 3 inferences. But if global validity is extended to level 4 as preservation of level 3 validities, and there are three notions of level 3 validity, then there are three notions of level 4 global validity. Iterating this process, we see that for every inferential level n, there are n many distinct notions of validity: local validity, plus $n-1$ types of global validity. The increase in global validity criteria at higher levels has apparently not come up in the ST literature either. We mentioned earlier that since ST already departs from classical logic at level 2, validity notions at higher levels are not often considered in that context. The exceptions fall in two categories: (Barrio et al., 2020; Fitting, 2021; Pailos, 2020) merely treat local validity.[19] On the other hand, (Ripley, 2021; Scambler, 2020) work in more general settings, where they have abstracted away from particular validity notions (in our usage of the term 'validity notion', which is different from Scambler's) determining the higher level inferences.[20] Global validity at levels 3 and up has seemingly not been studied before. Yet given the point about proof-theoretic definability, we have reason to do so in some depth.

So far, we have been considering criteria of validity defined per individual level. Recall, however, that we are looking for a validity notion to give substance to the principle that two logics are more similar when they agree on validities up to a higher inferential level. For this purpose we are ultimately interested in general definitions; ones that cover all levels. We could simply pick out for each individual level one of the validity conditions available there, and staple them together to construct a general definition. But this would lead to arbitrary mishmashes, posing radically diverging criteria for different inferential levels. We require a uniform definition, at least on levels $n > 1$, so that we can rightfully call it *a* general notion of inferential validity.

As I take it, there are three general validity notions of sufficient uniformity.

Definition 2.5 (General local validity)**.** *$\mathcal{K}$ locally validates $\Theta \Rightarrow^n \Pi \in SEQ^n_X$ ($\vDash^L_{\mathcal{K}} \Theta \Rightarrow^n \Pi$) iff$_{Def}$*

$$M \vDash_{\mathcal{K}} \Theta \Rightarrow^n \Pi \textit{ for every } M \in \mathcal{M}.$$

Definition 2.6 (General global$_L$ validity)**.** *$\mathcal{K}$ globally$_L$ validates $\Theta \Rightarrow^n \Pi \in SEQ^n_X$ ($\vDash^{GL}_{\mathcal{K}} \Theta \Rightarrow^n \Pi$) iff$_{Def}$ either of the following obtains:*

[19] Fitting does at points speak of 'global validity' for arbitrary levels n, but only with respect to consequence relations that are defined directly at level $n-1$ (thus not fitting in our assumptions about $\mathcal{K}$), so that global validity at level n is just taken to mean preservation of this consequence relation.

[20] Although Ripley's $\mathcal{C}(\widehat{ST})$ and $\mathcal{C}(\widehat{CL})$ are equivalent to what we will call the general local validities of ST and CPL respectively, the former of which is also equivalent to Scambler's $\mathbf{T}_1$.

- *$n = 1$ and $M \vDash_{\mathcal{K}} \Theta \Rightarrow^n \Pi$ for every $M \in \mathcal{M}$.*
- *$n > 1$ and either $M \nvDash_{\mathcal{K}} \theta$ for some $\theta \in \Theta$ and $M \in \mathcal{M}$, or $M \vDash_{\mathcal{K}} \Pi$ for every $M \in \mathcal{M}$.*

Definition 2.7 (General global$_G$ validity)**.** *$\mathcal{K}$ global$_G$ validity of $\Theta \Rightarrow^n \Pi \in SEQ^n_X$ ($\vDash^{GG}_{\mathcal{K}} \Theta \Rightarrow^n \Pi$) is defined recursively for $n > 0$:*

- $\vDash^{GG}_{\mathcal{K}} \Theta \Rightarrow^1 \Pi$ *iff$_{Def}$*

$$M \vDash_{\mathcal{K}} \Theta \Rightarrow^1 \Pi \textit{ for every } M \in \mathcal{M}.$$

- $\vDash^{GG}_{\mathcal{K}} \Theta \Rightarrow^{n+1} \Pi$ *iff$_{Def}$*

$$\textit{either } \nvDash^{GG}_{\mathcal{K}} \theta \textit{ for some } \theta \in \Theta, \textit{ or } \vDash^{GG}_{\mathcal{K}} \Pi.$$

We will henceforth use 'local/global$_L$/global$_G$ validity' to refer to these general versions, unless otherwise specified. So what are their virtues? To start, observe that global$_L$ validity at higher levels just amounts to preservation of local validity. That is, for $n > 1$, $\vDash^{GL}_{\mathcal{K}} \Theta \Rightarrow^n \Pi$ just means that if $\vDash^{L}_{\mathcal{K}} \theta$ for all $\theta \in \Theta$, then also $\vDash^{L}_{\mathcal{K}} \Pi$. We noted earlier that for every inferential level n, there are n many notions of validity - the local one, plus $n - 1$ many global variants - and criteria of $n + 1$ global validity pick one of these to preserve. The uniformity of general global$_L$ validity lies in that it universally selects local validity for level n to preserve at level $n + 1$.

Global$_G$ validity, on the other hand, is uniform because at every level $n+1$, it picks preservation of *itself* for level n, allowing for a recursive definition. It moves from ground level validity, to preservation of ground level validity, to preservation of preservation of ground level validity, and so on. This also means that the definability of ground level validity in purely proof-theoretic terms carries over to the full generality of global$_G$ validity. This property is unique to global$_G$ within the space of general validity definitions laid out above. We have already discussed how proof-theoretic definability is a crucial advantage when it comes to studying EML, and so global$_G$ stands out prominently as the most appropriate criterion under consideration in this section. The other two are strictly speaking irrelevant to the classicality of EML proper. Nevertheless, we will have much to say about them in what remains. Partly because I take it they are interesting in their own right, independently from application to EML, and specifically to those studying ST.[21] But uncovering its relations to these neighbouring notions will also help to illuminate global$_G$ validity itself.

Local validity is in a sense the most uniform among the three. It can be defined with a single clause, because it simply takes the idea of satisfaction at every model from level 1 inferences and applies it across the board. Indeed, Barrio et al. (2020) raise this point as a benefit of local as opposed to global

[21] Cf. footnote 8 of (Cobreros, Egré, et al., 2020).

validity for level 2. Against this, Golan (2021b) argues that we need not presume from the outset that uniformity is better when it comes to generalizing validity, and points out that no specific reason why it is so has been offered. I take it that presumptions in favour of symmetrical accounts are typically justified, but only to the extent that we lack a good explanation for the irregularity in the alternatives. In the case of local versus global validity, I maintain that the latter's asymmetry between the ground- and metalevels is perfectly acceptable, if not desirable. For some asymmetry between levels 1 and higher is present already in the inferential notions themselves. Only the former have *formulae* for their premises and conclusions, whereas all the higher levels have *inferences* of some sort in those places. Formulae and inferences are rather different beasts, so it should be far from surprising that an inference between formulae has different success criteria from an inference between inferences.

In particular, when considering an inference (as opposed to a formula) from a logical perspective, we do not particularly care about whether it is satisfied in some specific model, but rather about whether it is *generally* valid, for this is what can logically justify the inference. In fact, inferential validity is not just the main concept of interest as far as inferences go, but arguably the central object of study in logic altogether. When it comes to formulae we often concern ourselves with their satisfaction per model, rather than their general validity, but only precisely *because* the former allows us to in turn define inferential validity (at level 1). We may generalize model satisfaction to apply to inferences as well, and the resulting notion is interesting enough, but it would be a mistake to think of this as the primary property of inferences for logic to examine. Then, given that validity rather than satisfaction is the main inference property of interest, it seems natural that an inference between inferences is held in good standing when it preserves validity rather than satisfaction. This warrants the asymmetry in the global notions.

A few more comments are in order. Recall our earlier claim that we can measure agreement with classical logic *up to* a certain inferential level, because the set of validities at some level fully determines the sets at all lower levels. We can now see that this indeed follows immediately from the definitions. For local validity, Barrio et al. (2020) amongst others have already observed that level n inferences can simply be thought of as $n+1$ order inferences without premises:

Observation 2.8. *Take* $\Theta \Rightarrow^n \Pi \in SEQ^n_X$.

$$\vDash^L_{\mathcal{K}} \Theta \Rightarrow^n \Pi \textit{ iff } \vDash^L_{\mathcal{K}} \emptyset \Rightarrow^{n+1} (\Theta \Rightarrow^n \Pi).$$

The same thing happens to be true for global$_G$ validity:

Observation 2.9. *Take* $\Theta \Rightarrow^n \Pi \in SEQ^n_X$.

$$\vDash^{GG}_{\mathcal{K}} \Theta \Rightarrow^n \Pi \textit{ iff } \vDash^{GG}_{\mathcal{K}} \emptyset \Rightarrow^{n+1} (\Theta \Rightarrow^n \Pi).$$

However, the exact global$_L$ analogue only holds between levels 1 and 2.

Observation 2.10. *Take* $\Gamma \Rightarrow^1 \psi \in SEQ^1_X$.

$$\vDash^{GL}_{\mathcal{K}} \Gamma \Rightarrow^1 \psi \textit{ iff } \vDash^{GL}_{\mathcal{K}} \emptyset \Rightarrow^2 (\Gamma \Rightarrow^1 \psi).$$

At higher levels it fails. For instance, $(\emptyset \Rightarrow^1 p) \Rightarrow^2 (\emptyset \Rightarrow^1 q)$ is globally$_L$ valid for CPL. But it is not locally valid. Therefore

$$\frac{\emptyset}{(\emptyset \Rightarrow^1 p) \Rightarrow^2 (\emptyset \Rightarrow^1 q)}\ 3$$

is not globally$_L$ valid. Nevertheless, the global$_L$ validities at a level $n > 1$ are determined by those at $n+1$, in a more roundabout manner:

Observation 2.11. *Take* $\Theta \Rightarrow^n \Pi \in SEQ^n_X$ *with* $n > 1$.

$$\vDash^{GL}_{\mathcal{K}} \Theta \Rightarrow^n \Pi \textit{ iff } \vDash^{GL}_{\mathcal{K}} \{\emptyset \Rightarrow^n \theta \mid \theta \in \Theta\} \Rightarrow^{n+1} (\emptyset \Rightarrow^n \Pi).$$

It will turn out helpful that the analogue of observation 2.11 does also hold for local and global$_G$.

Observation 2.12. *Take* $\Theta \Rightarrow^n \Pi \in SEQ^n_X$ *with* $n > 1$.

(i) $\vDash^{L}_{\mathcal{K}} \Theta \Rightarrow^n \Pi$ *iff* $\vDash^{L}_{\mathcal{K}} \{\emptyset \Rightarrow^n \theta \mid \theta \in \Theta\} \Rightarrow^{n+1} (\emptyset \Rightarrow^n \Pi)$.

(ii) $\vDash^{GG}_{\mathcal{K}} \Theta \Rightarrow^n \Pi$ *iff* $\vDash^{GG}_{\mathcal{K}} \{\emptyset \Rightarrow^n \theta \mid \theta \in \Theta\} \Rightarrow^{n+1} (\emptyset \Rightarrow^n \Pi)$.

At $n = 1$, this also holds for local validity, but not for global$_L$ and global$_G$. For examply in CPL, $p \Rightarrow^1 q$ is not globally valid yet $(\emptyset \Rightarrow^1 p) \Rightarrow^2 (\emptyset \Rightarrow^1 q)$ is. But the important point is that for each of the validity criteria and each inferential level n, the validities at n are fully determined by those at $n+1$ in at least one of these ways.

Finally, the question remains how strong the three criteria are with respect to each other at the different levels. This partially depends on the specific logic, but a few relations hold universally. Namely, all three are of course equivalent at the ground level, where they pick out the common notion of validity. Global$_L$ and global$_G$ are furthermore equivalent at level 2, where their criterion amounts to standard global validity as per definition 2.4. Local is at least as strong as global$_L$ validity on all levels, as the difference in their definitions is still merely quantifier scope. These last two facts also mean that local is at least as strong as global$_G$ at level 2.

Further results on relative strength of the criteria can hold for individual logics. In a trivial logic such that every ground level inference is satisfied at every model, the three criteria are equivalent for all n. But no entailments between the notions hold universally, besides those listed in the previous paragraph. In fact, CPL alone offers a counterexample to every other possible entailment. That is to say: for CPL, local validity is *strictly* stronger than global$_L$ at all levels $n > 1$ (hence strictly stronger than global$_G$ at level 2).

Proposition 2.13. *For every* $n > 1$, *there exists* $\Theta \Rightarrow^n \Pi \in SEQ^n_{PL}$ *such that* $\vDash^{GL}_{CPL} \Theta \Rightarrow^n \Pi$ *but* $\nvDash^{L}_{CPL} \Theta \Rightarrow^n \Pi$.

Proof. For $n = 2$, the witness is $(\emptyset \Rightarrow^1 p) \Rightarrow^2 (\emptyset \Rightarrow^1 q)$. This generates witnesses for all higher levels due to observations 2.11 and 2.12 (i). □

Furthermore, global$_G$ is CPL incomparable to (i.e. neither stronger nor weaker than) either of the others at $n > 2$:

Proposition 2.14. *For every $n > 2$, there exists $\Theta \Rightarrow^n \Pi \in SEQ^n_{PL}$ such that $\models^{GG}_{CPL} \Theta \Rightarrow^n \Pi$ but $\not\models^{L}_{CPL} \Theta \Rightarrow^n \Pi$ and $\not\models^{GL}_{CPL} \Theta \Rightarrow^n \Pi$.*

Proof. For $n = 3$, the witness is

$$\frac{\emptyset}{(\emptyset \Rightarrow^1 p) \Rightarrow^2 (\emptyset \Rightarrow^1 q)}\ 3$$

which generates witnesses for all higher levels due to observations 2.11 and 2.12. □

Proposition 2.15. *For every $n > 2$, there exists $\Theta \Rightarrow^n \Pi \in SEQ^n_{PL}$ such that $\models^{L}_{CPL} \Theta \Rightarrow^n \Pi$ and $\models^{GL}_{CPL} \Theta \Rightarrow^n \Pi$ but $\not\models^{GG}_{CPL} \Theta \Rightarrow^n \Pi$.*

Proof. For $n = 3$, the witness is

$$\frac{(\emptyset \Rightarrow^1 p) \Rightarrow^2 (\emptyset \Rightarrow^1 q)}{\emptyset \Rightarrow^2 (p \Rightarrow^1 q)}\ 3$$

which generates witnesses for all higher levels due to observations 2.11 and 2.12. □

This gives us a complete overview of the notions' relative strength when applied to CPL: global$_L$ and global$_G$ are equivalent on levels 1 and 2, after which they are incomparable. Global$_L$ and local are equivalent on level 1, after which local is strictly stronger. Global$_G$ and local are equivalent on level 1, local is strictly stronger on level 2, and after this they are incomparable.

As it happens, the exact same strength relations hold for the criteria when applied to EML, and for parrallel reasons.

Proposition 2.16. *For every $n > 1$, there exists $\Theta \Rightarrow^n \Pi \in SEQ^n_{MML}$ such that $\models^{GL}_{EML} \Theta \Rightarrow^n \Pi$ but $\not\models^{L}_{EML} \Theta \Rightarrow^n \Pi$.*

Proof. For $n = 2$, the witness is $(\emptyset \Rightarrow^1 +p) \Rightarrow^2 (\emptyset \Rightarrow^1 +q)$. This generates witnesses for all higher levels due to observations 2.11 and 2.12 (i). □

Proposition 2.17. *For every $n > 2$, there exists $\Theta \Rightarrow^n \Pi \in SEQ^n_{MML}$ such that $\models^{GG}_{EML} \Theta \Rightarrow^n \Pi$ but $\not\models^{L}_{MML} \Theta \Rightarrow^n \Pi$ and $\not\models^{GL}_{EML} \Theta \Rightarrow^n \Pi$.*

Proof. For $n = 3$, the witness is

$$\frac{\emptyset}{(\emptyset \Rightarrow^1 +p) \Rightarrow^2 (\emptyset \Rightarrow^1 +q)}\ 3$$

which generates witnesses for all higher levels due to observations 2.11 and 2.12. □

Proposition 2.18. *For every $n > 2$, there exists $\Theta \Rightarrow^n \Pi \in SEQ^n_{MML}$ such that $\vDash^L_{EML} \Theta \Rightarrow^n \Pi$ and $\vDash^{GL}_{EML} \Theta \Rightarrow^n \Pi$ but $\nvDash^{GG}_{EML} \Theta \Rightarrow^n \Pi$.*

Proof. For $n = 3$, the witness is

$$\frac{(\emptyset \Rightarrow^1 +p) \Rightarrow^2 (\emptyset \Rightarrow^1 +q)}{\emptyset \Rightarrow^2 (+p \Rightarrow^1 +q)}\ 3$$

which generates witnesses for all higher levels due to observations 2.11 and 2.12. □

This concludes our study of the validity notions for individual inferences. Before closing the chapter, let us mention a few notions of metainferential validity which we choose not to cover in depth, despite some prominence in the literature. To start, given a sequent calculus treating derivations between ground level inferences, a metainference $\Theta \Rightarrow^2 \Pi$ is said to be externally valid if Π is derivable in the system after adding every $\theta \in \Theta$ as an axiom (Barrio et al., 2015; Cobreros, Egré, et al., 2020). Although its proof-theoretic nature may seem like a benefit, there are several reasons for omitting external validity.

First of all, EML is not a sequent calculus. While it is not too difficult to define a sequent calculus which is sound and complete at level 1, in the sense that it derives $\Gamma \Rightarrow^1 \psi$ just in case $\Gamma \vdash_{EML} \psi$, it is hard to see the relevance of the external validities on such a system.[22] The relation between this set of validities and EML proper would be weak. Many different sequent calculi can be sound and complete at level 1, and their respective external validities might be wildly different, as demonstrated by Dicher and Paoli (2019) for ST. Since there is no motivation for taking one particular such system as the canonical EML sequent calculus, there is no non-arbitrary way to identify the set of external EML validities. One might argue that a specific sequent calculus is canonical because its external logic is sound and complete also on level 2, in the sense that the external validities are precisely the globally or locally valid ones. But in that case, comparing the external validities of EML to those of CPL just amounts to comparing their global or local validities, so it brings nothing new to the table.

Moreover, it is unclear how external validity should be generalized to levels $n > 2$. We could recursively define global$_E$ validity for $n > 2$ as preservation of external validity at 3, preservation of preservation of external validity at 4 and so on. However, this involves a significant and unfounded asymmetry between the levels up to 2 and the higher ones. Alternatively, we can imagine higher and higher level proof systems for MML's, treating derivations between level 2 inferences, then level 3 inferences and so on, as (Da Ré & Pailos, 2021; Golan, 2021a) do for several ST-related logics. But then at every step we

[22] A calculus with this property was constructed in the early stages of research on this project, but left out of the thesis because of its inapplicability as argued for here.

are faced anew with the problem of identifying a canonical such system, and accompanying set of external validities, if this is not done in reference to sound- and completeness for the global or local validities at the corresponding level.

In (Da Ré, Szmuc, & Teijeiro, 2021; Humberstone, 1996; Teijeiro, 2019) it is explored how external validity (which they call 'derivability') for a sequent calculus can be related to a model-theoretic level 2 notion dubbed 'absolute global validity'. Perhaps absolute global validity could be adapted to apply to EML's S5 embedding, and even generalized to higher levels. However, as Da Ré, Szmuc and Teijeiro admit, it enjoys no independent motivation as a condition for the logical justification of metainferences; it is contrived purely to demonstrate how external validity can (under certain conditions) be characterized semantically. Hence the irrelevance of external validity to our present purposes carries over to absolute global validity.

We have also opted to leave out negative notions of validity, i.e. criteria capturing how an inference can be in *bad* standing according to a logic. For instance, Scambler (2020) argues that anti-validity can serve as a significant measure of similarity between logics. Anti-validity is most easily understood as a counterpart to local validity. Whereas an inference of some level is locally valid if it is satisfied in every model, it is anti-valid if it is satisfied in no model whatsoever. It is not discussed in depth because, like local and global$_L$ validity, it can only be defined in terms of a specific model theory, but unlike these positive validity notions, anti-validity is not an alternative to global$_G$ validity, and so there is not much insight to be gained from contrasting them.[23]

With our selection of validity criteria made, it remains to be seen how they might be applied to the comparison between uni- and multilateral logics. This will be the topic of the next chapter.

[23] Nevertheless, for those interested, but in terms that will only be introduced later: when it comes to anti-validity, EML is classical up to level 2 and strictly subclassical beyond that, whilst EML* is classical on every level. These are relatively easy consequences of corollary 4.13 and theorem 4.21 respectively, because an inference schema $\Lambda \Rightarrow^n \Omega$ is anti-valid iff Ω is anti-valid whilst every $\lambda \in \Lambda$ is locally valid.

Chapter 3

Classicality by Rules

3.1 The Comparison Problem

The developments of the previous chapter allow one to compare two logics in the same language, by comparing the validities they accord particular inferences in that language at various levels for various notions of validity. But as we have noted, the language of MML's is rather different from that of any classical logic. The differences are such that there are no individual inferences which can be formulated in both: $SEQ^n_{PL} \cap SEQ^n_{MML} = \emptyset$ for each n, and so no individual inferences of any level are available to check for agreement between e.g. EML and CPL.[24]

A natural move is to instead compare the logics by whether they validate the same general inference *rules* at different levels. Inference rules are conditions specifying that every individual inference of a certain logical type is valid, and logics can agree (or disagree) on such rules even if they are formulated in different languages, because this just means different specific inferences will fall under that type. Depending on ones preferred account of logic, the rules satisfied by some $\mathcal{K}$ may be taken to capture how $\mathcal{K}$ treats derivation, truth, or commitment, for content of various logical forms. General rules can be expressed by *schemas*, containing formula variables which can range over the formulae of different languages when the schema is assessed for different logics. Schemas do not always receive a formal definition, as we intuitively know how to identify and read them. But since we will need to quantify over them, to assess agreement with classical logic at different levels, we require some precision. A general treatment can be set up as follows.[25]

[24] Although this section is formulated specifically in relation to multilateral modal logic, I take it most if not all of what is said applies in parallel to other bi- or multilateral languages.

[25] Defining schemas as independent syntactic objects, as done here, rather than as sets of inferences, means that many intuitively identical schemas strictly speaking turn out distinct, e.g. $A_1 \Rightarrow A_1$ and $A_2 \Rightarrow A_2$. However, we will henceforth treat schemas as identical if they can be obtained from one another through uniform substitution of (fresh) variables, i.e. if they have precisely the same instances, since in this case they are for all intends and purposes the same. So we speak of schemas as of the equivalence classes they represent,

Definition 3.1 (Boolean schemas)**.** *Take a countably infinite set of formula variables* $\mathcal{A} = \{A_1, A_2, A_3, ...\}$*, and let* $\mathcal{A}^*$ *be the closure of* $\mathcal{A}$ *under the Boolean operations* $\neg$ *and* $\wedge$*. The set* BSC^n *of level* n *Boolean schemas is defined recursively for* $n > 0$*:*

$$BSC^1 := \{(\Delta, \chi) \mid \Delta \cup \{\chi\} \subseteq \mathcal{A}^*\}.$$

$$BSC^{n+1} := \{(\Lambda, \Omega) \mid \Lambda \cup \{\Omega\} \subseteq BSC^n\}.$$

As before, we will usually denote $(\Lambda, \Omega) \in BSC^n$ as $\Lambda \Rightarrow^n \Omega$ or

$$\frac{\lambda_1, ..., \lambda_m}{\Omega}\ n$$

where $\{\lambda_i \mid 1 \leq i \leq m\} = \Lambda$, and occassionally refer to schemas of arbitrary level as simply 'schemas', to level 1 schemas as 'ground level schemas', to level 2 schemas as 'metaschemas' and so on.

A Boolean schema can be assessed for validity (given some notion of validity for individual inferences) with respect to logics in any language $\mathcal{L}_X$, as long as $FOR(\mathcal{L}_X)$ is closed under the Boolean operations, by reference to substitution functions $\sigma : \mathcal{A} \to FOR(\mathcal{L}_X)$. For such functions, if $\chi \in \mathcal{A}^*$ we write $\sigma[\chi]$ to denote $\chi[\sigma(A)/A]_{A \in \mathcal{A}}$, if Δ is a set such that $\sigma[-]$ is defined for all its elements we write $\sigma[\Delta]$ to denote $\{\sigma[\delta] \mid \delta \in \Delta\}$, and if $\Lambda \Rightarrow^n \Omega \in BSC^n$ we write $\sigma[\Lambda \Rightarrow^n \Omega]$ to denote $\sigma[\Lambda] \Rightarrow^n \sigma[\Omega]$. In this case $\sigma[\Lambda \Rightarrow^n \Omega]$ is called an *instance* of $\Lambda \Rightarrow^n \Omega$. The point is of course that if $FOR(\mathcal{L}_X)$ is closed under Boolean operations, then $\sigma[\Lambda \Rightarrow^n \Omega] \in SEQ^n_X$, allowing for the following definition:

Definition 3.2 (Boolean schema validity)**.** *Let* $\mathcal{K}$ *be a logic in* $\mathcal{L}_X$ *such that* $FOR(\mathcal{L}_X)$ *is closed under Boolean operations, let* s *be a notion of validity for* n *level inferences, and take* $\Lambda \Rightarrow^n \Omega \in BSC^n$*.* $\mathcal{K}$ *s-validates* $\Lambda \Rightarrow^n \Omega$ *(written as* $\vDash^s_{\mathcal{K}} \Lambda \Rightarrow^n \Omega$*) iff*$_{Def}$

$$\vDash^s_{\mathcal{K}} \sigma[\Lambda \Rightarrow^n \Omega] \text{ for every } \sigma : \mathcal{A} \to FOR(\mathcal{L}_X).$$

So a level n schema is global$_G$ valid for CPL, for example, when all of its instances are. Thus we can take agreement on the validity of all Boolean schemas (on a given inferential level and notion of inferential validity) as a measure of similarity between logics. In particular, we can use agreement with CPL on the validity of Boolean schemas of increasingly high level as a measure of classicality, even for logics formulated in a language other than $\mathcal{L}_{PL}$, as long as the well-formed formulae are closed under Boolean operations. This is what allows us to identify the non-classicality of intuitionistic logics in all manner of languages, for example, by their failure to validate the Boolean schema $\emptyset \Rightarrow^1 A \vee \neg A$.

under the equivalence relation determined by uniform substitution/sameness of instances.

Note that this is a proper extension of the approach discussed in the previous sections - whereby logics are compared according to the validity of individual inferences - in the sense that for two logics formulated in the same language $\mathcal{L}_X$, agreement on all of SEQ^n_X entails agreement on all of BSC^n (though not necessarily vice-versa). Moreover, this extension is particularly appropriate for the purpose of evaluating classicality. The two quintessential examples of classical logic, CPL and Classical First-Order Logic (CFOL), are equivalent in terms of Boolean schemas. Furthermore, Boolean schemas promise to provide the strongest schematic method for capturing the agreement between them, giving expression to every existing commonality in their general inference rules. For the full grammar of $\mathcal{L}_{PL}$ is included in the definition of the schemas themselves, so that any more encompassing framework will include schemas that cannot be assessed with respect to CPL. Thus we can identify the classically valid inference rules as: those Boolean schemas validated by CPL (or equivalently CFOL). Consequently, agreement with CPL/CFOL up to higher level Boolean schemas on some validity notion can serve as a measure of classicality extended to alternative languages.[26] The behaviour of schemas with respect to our different notions of inferential validity will be examined in the next section, but another problem requires our attention first.

In the case of MML's, the condition on Boolean schema assessment - that the well-formed formulae of the language must be closed under Boolean operations - presents a serious obstacle. For although the *sentences* of $\mathcal{L}_{MML}$ are thus closed, the well-formed formulae $FOR(\mathcal{L}_{MML})$ consist in the *signed* formulae, as it is inferences between signed formulae that MML's are ultimately concerned with. But since force markers can not be embedded under connectives, the set of signed formulae lacks the required closure properties. Consider for instance the classical Boolean schema of Conjunction Elimination (CE): $A_1 \wedge A_2 \Rightarrow^1 A_1$. If we indeed interpret A_1, A_2 as variables ranging over signed formulae, the schema's EML-validity depends on the EML-validity of inferences like $(\oplus p) \wedge (\oplus q) \Rightarrow^1 \oplus p$. But force markers cannot be embedded under conjunction, thus $(\oplus p) \wedge (\oplus q)$ is ungrammatical. In general, it is not the case that $\sigma[\Lambda \Rightarrow^n \Omega] \in SEQ^n_{MML}$ if $\sigma : \mathcal{A} \to FOR(\mathcal{L}_{MML})$.

So we may need some creativity to apply Boolean schemas to MML's. We might, for instance, try to let the A's range over sentences (meaning elements of $FOR(\mathcal{L}_{ML})$) instead. Yet in that case the EML-validity of CE depends on that of e.g. $p \wedge q \Rightarrow^1 p$. This is an inference between unsigned formulae, therefore multilateral logics have nothing to say on its validity *as such.* Substituting an element of BSC^n with unsigned sentences for the A_i results in an element of SEQ^n_{ML}, not SEQ^n_{MML}. We could try to prefix force markers to all sentences after substitution has taken place, thereby moving from SEQ^n_{ML} to SEQ^n_{MML}, and use the resulting inferences to determine Boolean schema validity. But there are many ways to assign force markers to a given unsigned inference, e.g.

[26]Given this schematic equivalence between CPL and CFOL, we will measure classicality in reference only to the former, for reasons of simplicity.

three markers and two sentences means six possible assignments for $p \wedge q \Rightarrow^1 p$, so the question arises which such 'signings' of any substitution instance are relevant to the validity of the schema. There are several possible answers to consider. All of them fail, but it is illustrative to explore why, as this will help us design our eventual alternative.

First of all, we might simply require that *all* signings of all substitution instances of a Boolean schema must be valid, in order for the schema itself to be valid. So e.g. an MML would validate CE iff it validates $\circ_1(\sigma(A_1) \wedge \sigma(A_2)) \Rightarrow^1 \circ_2\sigma(A_1)$ for every $\sigma : \mathcal{A} \to FOR(\mathcal{L}_{ML})$ and every $\circ_1, \circ_2 \in \{+, \oplus, \ominus\}$. But this leads to unreasonable results. For instance, it means that EML invalidates the Boolean schema of Reflexivity, $A \Rightarrow^1 A$, because it invalidates $+p \Rightarrow^1 \ominus p$. But we seek to measure whether a given MML validates the same general rules as CPL, and this counterexample doesn't constitute a violation of the rule that Reflexivity expresses for CPL, namely (from the inferentialist perspective) that one is committed to the things one is committed to, or (proof-theoretically) that every zero-step derivation is permitted. This rule *does* hold for EML. It is clear that if we adopt the method of considering all signings relevant, then Boolean schemas like Reflexivity place demands for their validity on MML's which are no longer analogous to the demands they place on classical logic. So it is not well suited for evaluating the classicality of MML's.

The Reflexivity example suggests that perhaps we should only consider 'egalitarian' signings: those obtained by prefixing the same force marker to every sentence in the inference. Then a Boolean schema, when assessed for MML validity, essentially generates three different subschemas to be validated: one for each force marker. So an MML would validate Reflexivity iff it validates $+\sigma(A) \Rightarrow^1 +\sigma(A)$, $\oplus\sigma(A) \Rightarrow^1 \oplus\sigma(A)$ and $\ominus\sigma(A) \Rightarrow^1 \ominus\sigma(A)$ for each $\sigma : \mathcal{A} \to FOR(\mathcal{L}_{ML})$. Thus inferences like $+p \Rightarrow^1 \ominus p$ would no longer be relevant. Yet this approach has its own problems: it is simultaneously too inclusive and too limited. It is too inclusive because while e.g. CE no longer requires that an MML validate inferences like $+(p \wedge q) \Rightarrow^1 \ominus p$, it would still require validation of $\ominus(p \wedge q) \Rightarrow^1 \ominus p$. In EML, of course, rejecting a conjunction does not commit one to rejecting the first conjunct in particular; it may be merely the second conjunct that is doubted. But again, it doesn't seem right to call this a departure from classicality. In a sense, rejecting $p \wedge q$ turns out weaker than rejecting p precisely because EML *agrees* with classical logic that a conjunction need not be entailed by its first conjunct. This does not violate the rule that Conjunction Elimination expresses for CPL, namely that the converse entailment does hold.

This may suggest that we should only consider the strongly assered subschema. Such an approach would capture the similarities and dissimilarities between an MML and CPL to essentially the same extent that Incurvati and Schlöder (2020) compare EML and CPL on the inferential level in theorems 1.2 and 1.3, although generalized to higher levels. However, taking this as the full story would imply that for example $\ominus p \Rightarrow^1 \ominus p$ no longer needs to be valid in order for Reflexivity to be satisfied. Yet if an MML fails $\ominus p \Rightarrow^1 \ominus p$, then unlike classical logic its derivation relation simply is not reflexive. So this approach cannot capture all of an MML's potential deviations from classicality.

In fact, for similar reasons even the more general egalitarian signings approach is too limited in some cases. Consider the classical metaschema Transitivity:

$$\frac{A_1 \Rightarrow A_2 \qquad A_2 \Rightarrow A_3}{A_1 \Rightarrow A_3}\ 2$$

Like Reflexivity, Transitivity expresses a rule that should hold under weak assertions and rejections just as well as under strong assertions. In line with this observation, we indeed recognize that the three subschemas generated by egalitarian signing should all be valid if Transitivity is to hold for an MML. So egalitarian signings aren't asking too much here. But instead it seems we are asking too little, as Transitivity then does not require that e.g. the subschema

$$\frac{+A_1 \Rightarrow \ominus A_2 \qquad \ominus A_2 \Rightarrow \oplus A_3}{+A_1 \Rightarrow \oplus A_3}\ 2$$

is valid. An MML failing this subschema is clearly violating the general metarule expressed by Transitivity, but could still validate the Boolean schema of Transitivity if we adopt the egalitarian signings approach.

It is worth a small aside to consider in detail why it is that the subschema $\ominus(A_1 \wedge A_2) \Rightarrow^1 \ominus A_1$ should not be relevant to the validity of Conjunction Elimination, whilst $\ominus A \Rightarrow^1 \ominus A$ and

$$\frac{+A_1 \Rightarrow \ominus A_2 \qquad \ominus A_2 \Rightarrow \oplus A_3}{+A_1 \Rightarrow \oplus A_3}\ 2$$

are clearly required for the validity of Reflexivity and Transitivity. The former fact is explained by the observation that, in general, it is not a sensible demand of classicality for MML's that connectives should display their classical behaviour with respect to every force marker. Under strong assertion they should, for from the multilateral perspective, classical logic is just a logic of strong assertion. Hence a schema expressing a rule about how operators behave in classical logic is really expressing a rule about how the operators behave *under strong assertion*, and so if a schema is to express the analogous rule for MML, it should only concern strongly asserted signings. That CPL is a logic of strong assertion also means that the other force markers, in sheer virtue of being distinct from +, are bound to treat content differently from classical logic, manifested in a different interaction with the connectives. Reflexivity and Transitivity, on the other hand, do not express a rule about the treatment of content of any particular logical form, as witnessed by the absence of operator application/alleviation over the course of these schemas. Hence the fact that different force markers treat particular types of content differently from strong assertion does not lead us to expect that they fail Reflexivity or Transitivity. These schemas instead state rules about the structure of commitment preservation in general, and so should be respected by all force markers equally, if classicality is to be maintained. The takeaway of this detour is that Boolean schemas (or more precisely those specific parts of them) in which operators interact with sentences through application or alleviation over the course of the schema should not be required to hold under the scope of force markers other than $+$, but Boolean schemas (or those parts of them) where this does not occur should.

The situation is as follows: we need a way to assess an MML's agreement with the classically valid inference rules of different levels. Boolean schemas capture the notion of a classically valid inference rule. But attempts to directly evaluate Boolean schemas for validity with respect to MML's all result in grave distortions of the demands that the schemas originally expressed. It seems we would do better to formulate an independent schematic framework to capture the notion of an MML valid inference rule, so that it is well-suited to accommodate the quirks of multilateralism. If we can furthermore find a systematic way to pair these schemas with Boolean ones expressing the same rule, we will be in position to compare MML's to classical logic. This is what we will undertake in the remainder of this section.

One can easily set up schemas for modal logics by simply taking a set of formulae variables, closing it under Boolean *and* modal operators, and proceeding as in the Boolean case. But for MML's, there is the question of force markers. The preceding discussion brought to light how certain inference rules in MML can and should hold only with respect to specific speech acts. We wish to be general in our framework, so it should allow for the specification of force markers where necessary, to express these rules. On the other hand there are structural rules like Reflexivity and Transitivity, where the interest is precisely in their holding across all speech acts. So it should be possible to leave force markers out of the schemas in some places. But where force markers are specified, the variables must range over sentences, and where they are not, they must range over signed formula. Therefore we require schemas defined with two different types of variables. So we arrive at the following set up.

Definition 3.3 (Multilateral schemas)**.** *Take countably infinite sets of signed formula variables $\Phi = \{\varphi_1, \varphi_2, ...\}$, and sentence variables $\mathcal{A} = \{A_1, A_2, ...\}$. Let $\mathcal{A}^{\#}$ be the closure of $\mathcal{A}$ under the operators $\neg$, $\wedge$ and $\Diamond$. Let $\mathcal{A}^{\circ}$ be the set of $\mathcal{A}^{\#}$ elements prefixed with $+, \oplus$ or $\ominus$. The set MSC^n of level n multilateral schemas is defined recursively for $n > 0$:*

$$MSC^1 := \{(\Delta, \chi) \mid \Delta \cup \{\chi\} \subseteq \Phi \cup \mathcal{A}^{\circ}\}.$$

$$MSC^{n+1} := \{(\Lambda, \Omega) \mid \Lambda \cup \{\Omega\} \subseteq MSC^n\}.$$

Definition 3.4 (Multilateral schema validity)**.** *Let $\mathcal{K}$ be an MML, let s be a notion of validity for n level inferences, and take $(\Lambda \Rightarrow^n \Omega) \in MSC^n$. $\mathcal{K}$ s-validates $\Lambda \Rightarrow^n \Omega$ (written as $\vDash^s_{\mathcal{K}} \Lambda \Rightarrow^n \Omega$) iff$_{Def}$*

$\vDash^s_{\mathcal{K}} \mu[\Theta \Rightarrow^n \Pi]$ *for every* $\mu = \mu_\Phi \cup \mu_{\mathcal{A}}$ *with* $\mu_\Phi : \Phi \to FOR(\mathcal{L}_{MML})$ *and* $\mu_{\mathcal{A}} : \mathcal{A} \to FOR(\mathcal{L}_{ML})$.

Note that we henceforth use 'φ' only for signed formula variables (elements of Φ). We use 'χ' to denote schematic formulae/sentences; the objects appearing in formula or sentence position in schemas, i.e. the elements of $\mathcal{A}^*, \mathcal{A}^\#$, $\mathcal{A}^\circ$ or their unions with Φ. We will reserve 'ψ' for signed formulae themselves (elements of $FOR(\mathcal{L}_{MML})$).

Multilateral schemas are able to express general rules like Reflexivity as $\varphi \Rightarrow^1 \varphi$, but also rules about interactions between particular force markers and operators, as in $\ominus A_1 \to A_2 \Rightarrow^1 +\Diamond(A_1 \wedge \neg A_2)$. Furthermore, we can express rules which combine the two, describing the interaction of a particular force marker with some operator, but dictating that it should hold in the context of any type of speech act. For instance, Multilateral Reductio (MR)

$$\frac{\varphi, +\neg A \Rightarrow \bot}{\varphi \Rightarrow +A}\ 2$$

expresses that if strong assertion of $\neg A$ is incoherent given *some* other attitude, that attitude commits one to strongly asserting A. This schema allows us to finally capture in formal terms one of the observations motivating this project: that EML fails *reductio ad absurdum*, due to metainferences such as

$$\frac{\oplus p, +\neg p \Rightarrow \bot}{\oplus p \Rightarrow +p}\ 2$$

being EML invalid (both locally and globally). Because this is an instance of MR, namely for $\mu_\Phi(\varphi) = \oplus p$ and $\mu_{\mathcal{A}}(A) = p$.

To furthermore make precise why this marks a departure from classicality, however, we need to relate MR to a Boolean schema expressing *reductio*. When it comes to the cross-identification of rules expressed by Boolean and Multilateral schemas, we should observe that many Multilateral schemas state rules which cannot be interpreted by Boolean schemas at all. This includes all those containing modal operators, like the EML valid $+\Diamond\Diamond A \Rightarrow +\Diamond A$. This is not a defect, however; the question whether the rule expressed by this schema is valid for CPL or CFOL simply does not make sense, and so we cannot hope to compare MML's to classical logic on these types of inference rules. A similar point applies to Multilateral schemas with built-in $\oplus$ or $\ominus$ signs. No Boolean schema can express anything analogous to $\ominus\neg A_1 \Rightarrow^1 \oplus A_1$. But this is because as we have noted, CPL is just a logic of strong assertion, and therefore doesn't have anything to say about how $\oplus$ and $\ominus$ interact with connectives, each other, or even with $+$. This means that all of these schemas can and should be ignored when comparing an MML to classical logic. We are left with the following.

Definition 3.5 (Classical expressibility)**.** *Take countably infinite sets of signed formula variables* $\Phi = \{\varphi_1, \varphi_2, ...\}$*, and sentence variables* $\mathcal{A} = \{A_1, A_2, ...\}$.

Let $\mathcal{A}^*$ be the closure of $\mathcal{A}$ under the operators $\neg$ and $\wedge$. Let $\mathcal{A}^+$ be the set of $\mathcal{A}^*$ elements prefixed with $+$. The set CSC^n of level n classically expressible multilateral schemas is defined recursively for $n > 0$:

$$CSC^1 := \{(\Delta, \chi) \mid \Delta \cup \{\chi\} \subseteq \Phi \cup \mathcal{A}^+\}.$$

$$CSC^{n+1} := \{(\Lambda, \Omega) \mid \Lambda \cup \{\Omega\} \subseteq CSC^n\}.$$

Clearly $CSC^n \subseteq MSC^n$, so the definition of validity for general multilateral schemas carries over.

Since unilateral logic is logic of strong assertion, we can obtain a Boolean schema expressing the same rule as some classically expressible schema by simply removing the $+$'s and the distinction between the two variable types.

Definition 3.6 (Unilateralization). *Given $\Lambda \Rightarrow^n \Omega \in CSC^n$, let $U(+\chi)$ be χ for every $+\chi \in \mathcal{A}^+$ appearing in $\Lambda \Rightarrow^n \Omega$, and pick fresh and distinct $U(\varphi) \in \mathcal{A}$ for every signed formulae variable $\varphi \in \Phi$ appearing in $\Lambda \Rightarrow^n \Omega$. The unilateralization of $\Lambda \Rightarrow^n \Omega$ is $U[\Lambda \Rightarrow^n \Omega]$.*

The result of unilateralization is a Boolean schema that expresses a unilateral analogue of what the multilateral schema expressed. For example, MR is classically expressible, and its unilateralization is Classical Reductio:

$$\frac{A_1, \neg A_2 \Rightarrow \bot}{A_1 \Rightarrow A_2}\ 2$$

The fact that EML fails MR whilst CPL validates its unilateralization thus marks a difference between the metainferential rules validated by the logics.[27] Note that every Boolean schema is the unilateralization of at least some classically expressible multilateral one, usually of multiple distinct ones. That is to say, if we think of U as a function from CSC^n to BSC^n, it is total and surjective, but not injective. Therefore, to formulate what it means for an MML to validate more/less/the same inference rules as CPL at some level n, we can do so by quantifying over CSC^n.

Definition 3.7 (MML Subclassicality). *Let $\mathcal{K}$ be an MML and s a notion of validity. $\mathcal{K}$ is subclassical on level n given s iff$_{Def}$*

[27] Note that we need not rely on the appearance of signed formula variables to demonstrate EML's invalidation of *reductio*, for as mentioned above, the diaprity already occurs within the strongly asserted fragment. The multilateral schema

$$\frac{+A_1, +\neg A_2 \Rightarrow \bot}{+A_1 \Rightarrow +A_2}\ 2$$

also unilateralizes to Classical Reductio, and it too fails validity in EML, due to instances like

$$\frac{+\Diamond p, +\neg p \Rightarrow \bot}{+\Diamond p \Rightarrow +p}\ 2$$

being (globally and locally) invalid. But the earlier schema MR captures in more generality the ways in which EML fails *reductio*.

If $\vDash^s_{\mathcal{K}} \Lambda \Rightarrow^n \Omega$ *for some* $\Lambda \Rightarrow^n \Omega \in CSC^n$*, then* $\vDash^s_{CPL} U[\Lambda \Rightarrow^n \Omega]$.

Definition 3.8 (MML Superclassicality)**.** *Let* $\mathcal{K}$ *be an MML and* s *a notion of validity.* $\mathcal{K}$ *is superclassical on level* n *given* s *iff*$_{Def}$

If $\vDash^s_{CPL} U[\Lambda \Rightarrow^n \Omega]$ *for some* $\Lambda \Rightarrow^n \Omega \in CSC^n$*, then* $\vDash^s_{\mathcal{K}} \Lambda \Rightarrow^n \Omega$.

Definition 3.9 (MML Classicality)**.** *Let* $\mathcal{K}$ *be an MML and* s *a notion of validity.*

- $\mathcal{K}$ *is classical on level* n *given* s *iff*$_{Def}$ $\mathcal{K}$ *is subclassical and superclassical on level* n *given* s.
- $\mathcal{K}$ *is strictly subclassical on level* n *given* s *iff*$_{Def}$ $\mathcal{K}$ *is subclassical but not superclassical on level* n *given* s.
- $\mathcal{K}$ *is strictly superclassical on level* n *given* s *iff*$_{Def}$ $\mathcal{K}$ *is superclassical but not subclassical on level* n *given* s.
- $\mathcal{K}$ *is incomparable to classical logic on level* n *given* s *iff*$_{Def}$ $\mathcal{K}$ *is neither subclassical nor superclassical on level* n *given* s.

One might object that, since multiple non-equivalent multilateral schemas can unilateralize to the same Boolean schema, then this Boolean schema can not express a rule that is equivalent to *all* of the original multilateral schemas. For example,

$$\frac{\varphi_1 \Rightarrow \varphi_2 \quad \varphi_2 \Rightarrow \varphi_3}{\varphi_1 \Rightarrow \varphi_3}\ 2$$

expresses general transitivity, whilst

$$\frac{+A_1 \Rightarrow +A_2 \quad +A_2 \Rightarrow +A_3}{+A_1 \Rightarrow +A_3}\ 2$$

only demands transitivity for strong assertion. Meanwhile

$$\frac{\varphi_1 \Rightarrow +A \quad +A \Rightarrow \varphi_2}{\varphi_1 \Rightarrow \varphi_2}\ 2$$

expresses something in between. But the unilateralization of all three is just Transitivity:

$$\frac{A_1 \Rightarrow A_2 \quad A_2 \Rightarrow A_3}{A_1 \Rightarrow A_3}\ 2$$

However, all such cases are of the same sort, which is such that it does not undermine our approach. These cases only occur because φ's and non-embedded $+A$'s both become non-embedded A's after unilateralization. So take arbitrary $\Omega \in BSC^n$ and consider $\mathcal{C} = \{\Omega' \in CSC^n \mid U[\Omega'] = \Omega\}$. Any two elements of $\mathcal{C}$ can be obtained from each other by uniformly substituting appearances of $\varphi_i \in \Phi$ for non-embedded appearances of $+A_i$ for fresh $A_i \in \mathcal{A}$, and/or vice-versa. $\mathcal{C}$ has a particular element $\mathbb{C}$ in which no non-embedded $+A$'s appear, as all these positions are taken up by φ's. This is the proper multilateral analogue of Ω. For as argued above, when some A_i appears only non-embedded in a Boolean schema, its behaviour according to the rule expressed by the schema does not concern the interaction of strong assertion with content of a particular form, but rather a structural property of commitment altogether. Hence our identification of

$$\frac{\varphi_1 \Rightarrow \varphi_2 \qquad \varphi_2 \Rightarrow \varphi_3}{\varphi_1 \Rightarrow \varphi_3}\, 2$$

as the multilateral expression of Transitivity.

But putting a fresh φ in place of a $+A$ can only make the demands of the schema stronger, as it strictly enlarges the set of instances. Hence $\mathbb{C}$ is also the strongest element of $\mathcal{C}$. Therefore, the fact that other elements of $\mathcal{C}$ also happen to unilateralize to Ω does not interfere with the conditions for (sub- or super)classicality at level n, as per definitions 3.7 to 3.9, because $\mathbb{C}$ is s-valid for $\mathcal{K}$ iff every element of $\mathcal{C}$ is.

3.2 Schematic Validity

Chapter 2 demonstrated some important aspects of the behaviour of the various validity notions for individual inferences, with respect to each other and at different levels. Section 3.1 discussed how we must rely on inference schemas instead of individual inferences, and how validity notions can be lifted to apply to schemas. Thus the question arises how the validity notions behave when interpreted as notions of schematic validity. Three issues are of recurring interest: which of the notions are proof-theoretically characterizable, (how) do the validities at a given level determine those at all lower levels per notion, and how strong are the criteria with respect to each other at different levels for different logics? The present section will deal with these in turn.

The first is rather straightforward. Since a schematic validity criterion is defined purely in terms of the corresponding notion of validity for individual instances, the former is proof-theoretically definable if and only if the latter is. Hence only global$_G$ schematic validity is definable in solely syntactic terms.

The answer to the second question is not too surprising either: per validity notion, the validities of level n schemas are determined by those at the higher levels in precisely the same ways as for individual inferences. To be precise, the following are direct consequences of the corresponding observations 2.8 to 2.12.

Observation 3.10. *Take $\Lambda \Rightarrow^n \Omega \in BSC^n$ and let $\mathcal{K}$ be a logic in language $\mathcal{L}_X$ closed under Boolean operations. Alternatively, take $\Lambda \Rightarrow^n \Omega \in MSC^n$ and let $\mathcal{K}$ be an MML. Either way the following hold:*

(i) $\vDash^L_{\mathcal{K}} \Lambda \Rightarrow^n \Omega$ *iff* $\vDash^L_{\mathcal{K}} \emptyset \Rightarrow^{n+1} (\Lambda \Rightarrow^n \Omega)$ *iff* $\vDash^L_{\mathcal{K}} \{\emptyset \Rightarrow^n \lambda \mid \lambda \in \Lambda\} \Rightarrow^{n+1} (\emptyset \Rightarrow^n \Omega)$.

(ii) ***(a)*** *If* $n = 1$*, then* $\models^{GL}_{\mathcal{K}} \Lambda \Rightarrow^n \Omega$ *iff* $\models^{GL}_{\mathcal{K}} \emptyset \Rightarrow^{n+1} (\Lambda \Rightarrow^n \Omega)$*.*

(b) *If* $n > 1$*, then* $\models^{GL}_{\mathcal{K}} \Lambda \Rightarrow^n \Omega$ *iff* $\models^{GL}_{\mathcal{K}} \{\emptyset \Rightarrow^n \lambda \mid \lambda \in \Lambda\} \Rightarrow^{n+1} (\emptyset \Rightarrow^n \Omega)$*.*

(iii) ***(a)*** $\models^{GG}_{\mathcal{K}} \Lambda \Rightarrow^n \Omega$ *iff* $\models^{GG}_{\mathcal{K}} \emptyset \Rightarrow^{n+1} (\Lambda \Rightarrow^n \Omega)$*.*

(b) *If* $n > 1$*, then* $\models^{GG}_{\mathcal{K}} \Lambda \Rightarrow^n \Omega$ *iff* $\models^{GG}_{\mathcal{K}} \{\emptyset \Rightarrow^n \lambda \mid \lambda \in \Lambda\} \Rightarrow^{n+1} (\emptyset \Rightarrow^n \Omega)$*.*

Note also that $\Lambda \Rightarrow^n \Omega \in CSC^n$ iff $\emptyset \Rightarrow^{n+1} (\Lambda \Rightarrow^n \Omega) \in CSC^{n+1}$ iff $\{\emptyset \Rightarrow^n \lambda \mid \lambda \in \Lambda\} \Rightarrow^{n+1} (\emptyset \Rightarrow^n \Omega) \in CSC^{n+1}$. All of this has the convenient effect that for MML's, we can still speak of classicality *up to* a certain level.

Proposition 3.11. *Let* $\mathcal{K}$ *be an MML, such that* $\mathcal{K}$ *is locally/globallly*$_L$*/globally*$_G$ *subclassical (superclassical) on level* $n{+}1$*. Then* $\mathcal{K}$ *is locally/globallly*$_L$*/globally*$_G$ *subclassical (superclassical) on level* n*.*

Proof. We give the argument for global$_L$ subclassicality, where a case distinction is required.

- Suppose $n = 1$. Take arbitrary $\Delta \Rightarrow^1 \chi \in CSC^1$.
  If $\models^{GL}_{\mathcal{K}} \Delta \Rightarrow^1 \chi$
  then $\models^{GL}_{\mathcal{K}} \emptyset \Rightarrow^2 (\Delta \Rightarrow^1 \chi)$
  then $\models^{GL}_{CPL} U[\emptyset \Rightarrow^2 (\Delta \Rightarrow^1 \chi)]$
  then $\models^{GL}_{CPL} \emptyset \Rightarrow^2 U[\Delta \Rightarrow^1 \chi]$
  then $\models^{GL}_{CPL} U[\Delta \Rightarrow^1 \chi]$.

- Suppose $n > 1$. Take arbitrary $\Lambda \Rightarrow^n \Omega \in CSC^n$.
  If $\models^{GL}_{\mathcal{K}} \Lambda \Rightarrow^n \Omega$
  then $\models^{GL}_{\mathcal{K}} \{\emptyset \Rightarrow^n \lambda \mid \lambda \in \Lambda\} \Rightarrow^{n+1} (\emptyset \Rightarrow^n \Omega)$
  then $\models^{GL}_{CPL} U[\{\emptyset \Rightarrow^n \lambda \mid \lambda \in \Lambda\} \Rightarrow^{n+1} (\emptyset \Rightarrow^n \Omega)]$
  then $\models^{GL}_{CPL} \{\emptyset \Rightarrow^n U[\lambda] \mid \lambda \in \Lambda\} \Rightarrow^{n+1} (\emptyset \Rightarrow^n U[\Omega])$
  then $\models^{GL}_{CPL} \{U[\lambda] \mid \lambda \in \Lambda\} \Rightarrow^n U[\Omega]$
  then $\models^{GL}_{CPL} U[\Lambda \Rightarrow^n \Omega]$.

The proofs for local and global$_G$ subclassicality are analogous (although no case distinctions are necessary). The arguments for superclassicality are obtained by reversing the direction of all the "if-then" statements in the corresponding subclassicality proofs. □

Corollary 3.12. *Let* $\mathcal{K}$ *be an MML, such that* $\mathcal{K}$ *is locally/globallly*$_L$*/globally*$_G$ *classical on level* $n{+}1$*. Then* $\mathcal{K}$ *is locally/globallly*$_L$*/globally*$_G$ *classical on level* n*.*

More wondrous are the results on relative strength. For individual inferences, we showed that the following hold with respect to both CPL and EML: global$_L$ and global$_G$ are equivalent on levels 1 and 2, after which they are incomparable. Global$_L$ and local are equivalent on level 1, after which local is strictly stronger. Global$_G$ and local are equivalent on level 1, local is strictly stronger on level 2, and after this they are incomparable.

Matters are rather different for schemas. Per logic, every entailment that holds for individual inferences does thereby automatically hold for schemas as well. But some further entailments may hold only schematically. In (Da Ré et al., 2021) and (Teijeiro, 2019) it is shown how besides local entailing global validity at level 2 as it did for individual inferences, schematically the converse entailment also holds, at least for a certain class of logics including CPL and ST. We adopt their basic strategy, apply it specifically to CPL, and generalize the result to *all* levels for both global$_L$ *and* global$_G$ validity. The idea is that given any $\sigma : \mathcal{A} \to FOR(\mathcal{L}_{PL})$ and Boolean valuation v, we can 'hard-code' the values that v assingns to σ substitutions of schematic formulae (i.e. elements of $\mathcal{A}^*$) into a new substitution function $\sigma^v : \mathcal{A} \to FOR(\mathcal{L}_{PL})$, defined as

$$\sigma^v(A) = \begin{cases} \top & \text{if } v(\sigma(A)) = 1 \\ \bot & \text{if } v(\sigma(A)) = 0. \end{cases}$$

Lemma 3.13. *Take arbitrary $\sigma : \mathcal{A} \to FOR(\mathcal{L}_{PL})$, $\chi \in \mathcal{A}^*$ and Boolean valuations v and v'. Then $v'(\sigma^v[\chi]) = v(\sigma[\chi])$.*

Proof. By induction on the complexity of χ. □

Since v' is completely arbitrary and unrelated to σ and v in the previous lemma, it has the following effect: whether the σ instance of a schema is *satisfied* at v fully determines whether the schema's σ^v instance is *valid*, both locally and globally$_G$. This is explicated in the following two propositions respectively.

Proposition 3.14. *Take arbitrary $\sigma : \mathcal{A} \to FOR(\mathcal{L}_{PL})$, Boolean valuation v and $\Lambda \Rightarrow^n \Omega \in BSC^n$. Then $v \vDash_{CPL} \sigma[\Lambda \Rightarrow^n \Omega]$ iff $\vDash^L_{CPL} \sigma^v[\Lambda \Rightarrow^n \Omega]$.*

Proof. We take arbitrary Boolean valuation v' and show that $v \vDash_{CPL} \sigma[\Lambda \Rightarrow^n \Omega]$ iff $v' \vDash_{CPL} \sigma^v[\Lambda \Rightarrow^n \Omega]$. We proceed by induction over n.

- The base case ($n = 1$) follows from the previous lemma.
- Assume as induction hypothesis that the claim holds for all of BSC^{n-1}.
  $v \vDash_{CPL} \sigma[\Lambda \Rightarrow^n \Omega]$ iff
  $v \nvDash_{CPL} \sigma[\lambda]$ for some $\lambda \in \Lambda$ or $v \vDash_{CPL} \sigma[\Omega]$ iff
  $v' \nvDash_{CPL} \sigma^v[\lambda]$ for some $\lambda \in \Lambda$ or $v' \vDash_{CPL} \sigma^v[\Omega]$ iff
  $v' \vDash_{CPL} \sigma^v[\Lambda \Rightarrow^n \Omega]$. □

Proposition 3.15. *Take arbitrary $\sigma : \mathcal{A} \to FOR(\mathcal{L}_{PL})$, Boolean valuation v and $\Lambda \Rightarrow^n \Omega \in BSC^n$. Then $v \vDash_{CPL} \sigma[\Lambda \Rightarrow^n \Omega]$ iff $\vDash^{GG}_{CPL} \sigma^v[\Lambda \Rightarrow^n \Omega]$.*

Proof. We prove by induction over n.

- The base case is equivalent to that of proposition 3.14.

- Assume as induction hypothesis that the proposition holds for $n-1$.
  $v \vDash_{CPL} \sigma[\Lambda \Rightarrow^n \Omega]$ iff
  $v \nvDash_{CPL} \sigma[\lambda]$ for some $\lambda \in \Lambda$ or $v \vDash_{CPL} \sigma[\Omega]$ iff
  $\nvDash^{GG}_{CPL} \sigma^v[\lambda]$ for some $\lambda \in \Lambda$ or $\vDash^{GG}_{CPL} \sigma^v[\Omega]$ iff
  $\vDash^{GG}_{CPL} \sigma^v[\Lambda \Rightarrow^n \Omega]$. □

As it turns out, this means that if σ and v together constitute a counterexample to the local validity of some Boolean schema, then σ^v is a counterexample to the global$_L$ and global$_G$ validity of that same schema. Hence if a schema is global$_L$ or global$_G$ valid, it must be locally valid as well. This is worked out in turn in the following two propositions.

Proposition 3.16. *Take arbitrary* $\Lambda \Rightarrow^n \Omega \in BSC^n$*. If* $\vDash^{GL}_{CPL} \Lambda \Rightarrow^n \Omega$*, then* $\vDash^{L}_{CPL} \Lambda \Rightarrow^n \Omega$*.*

Proof. Suppose for contraposition that $\nvDash^{L}_{CPL} \Lambda \Rightarrow^n \Omega$. Then there is some $\sigma : \mathcal{A} \to FOR(\mathcal{L}_{PL})$ and Boolean valuation v such that $v \nvDash_{CPL} \sigma[\Lambda \Rightarrow^n \Omega]$. This means $v \vDash_{CPL} \sigma[\lambda]$ for all $\lambda \in \Lambda$ and $v \nvDash_{CPL} \sigma[\Omega]$. Hence by proposition 3.14, $\vDash^{L}_{CPL} \sigma^v[\lambda]$ for all $\lambda \in \Lambda$ but $\nvDash^{L}_{CPL} \sigma^v[\Omega]$. Therefore $\nvDash^{GL}_{CPL} \sigma^v[\Lambda \Rightarrow^n \Omega]$ and hence $\nvDash^{GL}_{CPL} \Lambda \Rightarrow^n \Omega$. □

Proposition 3.17. *Take arbitrary* $\Lambda \Rightarrow^n \Omega \in BSC^n$*. If* $\vDash^{GG}_{CPL} \Lambda \Rightarrow^n \Omega$*, then* $\vDash^{L}_{CPL} \Lambda \Rightarrow^n \Omega$*.*

Proof. Suppose for contraposition that $\nvDash^{L}_{CPL} \Lambda \Rightarrow^n \Omega$. Then there is some $\sigma : \mathcal{A} \to FOR(\mathcal{L}_{PL})$ and Boolean valuation v such that $v \nvDash_{CPL} \sigma[\Lambda \Rightarrow^n \Omega]$. Hence by proposition 3.15, $\nvDash^{GG}_{CPL} \sigma^v[\Lambda \Rightarrow^n \Omega]$. Therefore $\nvDash^{GG}_{CPL} \Lambda \Rightarrow^n \Omega$. □

So both global$_L$ and global$_G$ entail local validity for CPL schemas. Of course local still entails global$_L$ for schemas, as it did for individual inferences. However, it does not entail global$_G$.

Proposition 3.18. *For every* $n > 2$*, there exists* $\Lambda \Rightarrow^n \Omega \in BSC^n$ *such that* $\vDash^{L}_{CPL} \Lambda \Rightarrow^n \Omega$ *and* $\vDash^{GL}_{CPL} \Lambda \Rightarrow^n \Omega$ *but* $\nvDash^{GG}_{CPL} \Lambda \Rightarrow^n \Omega$*.*

Proof. For $n = 3$, the witness is

$$\frac{(\emptyset \Rightarrow^1 A_1) \Rightarrow^2 (\emptyset \Rightarrow^1 A_2)}{\emptyset \Rightarrow^2 (A_1 \Rightarrow^1 A_2)}\ 3$$

which generates witnesses for all higher levels due to observation 3.10. □

This is quite the turn of events. In the ST literature, a recurring objection against global as opposed to local validity at level 2 is that it is too weak a criterion (see e.g. (Barrio et al., 2020; Dicher & Paoli, 2019)). This is partly what prompts Teijeiro (2019) to demonstrate that moving to schemas induces equivalence at level 2. Now we have seen that this equivalence remains if we generalize to higher levels via the global$_L$ route, whereas if we opt for global$_G$ we even get a strictly *stronger* criterion, at least with respect to CPL.

While on the subject, let us take a detour to discuss another objection to global validity which, although not pertaining to the strength of the criteria, also loses its force when considered for schemas. According to Golan (2021b), global validity is inappropriate as a notion for (level 2) validity because it is not preserved under substitution. Some individual metainferences are globally valid, but do not remain so if we uniformly substitute each propositional letter for a formula. For instance $(\emptyset \Rightarrow^1 p) \Rightarrow^2 (\emptyset \Rightarrow^1 q)$ is globally valid (in CPL) because its premise inference is not valid, yet if we substitute p for a tautology and q for itself, the result is globally invalid. But logic should be formal, Golan argues, and a necessary requirement for this formality is that "the principles of logic hold under uniform substitution" (Golan, 2021b, p. 5). So global validity is rejected on the grounds that it compromises the formality, and hence logicality, of metainferences.

Golan does not exactly specify what is meant by 'principles' or 'matters' of logic, so as to outline the application range of the formality condition. Many notions commonly used in logic, and of a distinctly logical character, are not preserved under substitution. For instance: satisfiability, consistency, or invalidity. But let us suppose that Golan intends to demand this type of formality specifically of positive validity notions, as this seems a reasonable ask. It should suffice to point out that schematic global validity does remain under substitution. If we take a schema and uniformly substitute appearances of variables $A \in \mathcal{A}$ for schematic formulae (meaning elements of $\mathcal{A}^*$ for Boolean or $\mathcal{A}^\#$ for multilateral schemas), any instance of the resulting schema is an instance of the original, so validity is preserved.

Besides this, I take it Golan's overall objection fails even as applied to individual inferences. The titular central claim of (Golan, 2021b) is that there is no tenable notion of global validity. The standard formulation fails substitution, and no successful fixes are available. However, the argument overlooks the option of adding substitution preservation directly to the definition. We might call an individual level 2 inference global-substitutionally valid iff every uniform substitution instance preserves validity from premisses to conclusion. This way of cashing out global validity has already been employed in e.g. (Barrio et al., 2015; Cobreros, Egré, et al., 2020). Given that formality is a core tenet of logicality, it seems plausible to represent this in validity conditions. Then the only difference between global and local/level 1 validity is that for the latter we do not need to *explicitly* add substitution preservation to the definition, because it already follows from the simpler definitions usually given. Hence Golan's objection reduces to a point about simplicity of definitions, which can hardly serve as a defeating blow against global validity.[28]

[28]Given this defense of the global-substitutional interpretation, the reader may wonder why it was not discussed as an option for individual inference validity in section 2.2. The reason is that in reading individual inferences substitutionally, one is really treating them as schemas, and thereby blurring the line between the two object types. Our eventual leap to schemas already accomplished everything that reading individual inferences substitutionally would have, since every instance of a schema is global-substitutionally valid iff every

But let us return now to the relative strength of our schematic validity criteria, as we have yet to compare them with respect to EML. It turns out the above proof strategy does not apply in this case. The schematic local-global collapse does not occur for EML even at level 2. In fact the relative strength of all three notions with respect to EML is exactly the same for schemas as it is for individual inferences.

Proposition 3.19. *For every $n > 1$, there exists $\Lambda \Rightarrow^n \Omega \in MSC^n$ such that $\models^{GL}_{EML} \Lambda \Rightarrow^n \Omega$ and $\models^{GG}_{EML} \Lambda \Rightarrow^n \Omega$ but $\not\models^{L}_{EML} \Lambda \Rightarrow^n \Omega$.*

Proof. For $n = 2$, the witness is

$$\dfrac{+A \Rightarrow \bot \qquad +\neg A \Rightarrow \bot}{\emptyset \Rightarrow \bot}\ 2$$

which generates witnesses for all higher levels due to observation 3.10. The witness at $n = 2$ is locally invalid because there are S5 models which satisfy neither $\Box p$ nor $\Box\neg p$ for some p, hence satisfy both premise inferences for substitution $\mu(A) = p$, yet clearly these do not satisfy the conclusion sequent. However, it is globally valid because there exists no $\mu(A) \in FOR(\mathcal{L}_{ML})$ such that both $+\mu(A)$ and $+\neg\mu(A)$ are incoherent. If it did, soundness would entail that $\neg\Box\mu(A)$ and $\neg\Box\neg\mu(A)$ are both S5 theorems. But this is impossible because S5 has single-world models, in which either $\Box\mu(A)$ or $\Box\neg\mu(A)$ must be true depending on whether $\mu(A)$ itself is true, by reflexivity. □

Furthermore, global$_G$ is incomparable to either of the others at $n > 2$:

Proposition 3.20. *For every $n > 2$, there exists $\Lambda \Rightarrow^n \Omega \in MSC^n$ such that $\models^{GG}_{EML} \Lambda \Rightarrow^n \Omega$ but $\not\models^{GL}_{EML} \Lambda \Rightarrow^n \Omega$ and $\not\models^{L}_{EML} \Lambda \Rightarrow^n \Omega$.*

Proof. For $n = 3$, the witness is

$$\dfrac{\emptyset}{\dfrac{+A \Rightarrow \bot \qquad +\neg A \Rightarrow \bot}{\emptyset \Rightarrow \bot}\ 2}\ 3$$

which generates witnesses for all higher levels due to observation 3.10. □

Proposition 3.21. *For every $n > 2$, there exists $\Lambda \Rightarrow^n \Omega \in MSC^n$ such that $\models^{L}_{EML} \Lambda \Rightarrow^n \Omega$ and $\models^{GL}_{EML} \Lambda \Rightarrow^n \Omega$ but $\not\models^{GG}_{EML} \Lambda \Rightarrow^n \Omega$.*

Proof. For $n = 3$, the witness is

$$\dfrac{(\emptyset \Rightarrow^1 +A_1) \Rightarrow^2 (\emptyset \Rightarrow^1 +A_2)}{\emptyset \Rightarrow^2 (+A_1 \Rightarrow^1 +A_2)}\ 3$$

which generates witnesses for all higher levels due to observation 3.10. □

instance is globally valid *simpliciter*, and in fact more, through the addition of signed formula variables to multilateral schemas. To facilitate a clear discussion of the comparison problem in section 3.1, it was moreover conductive to keep inferences and schemas sharply distinct.

Hence for EML, the following still hold after moving to schemas: global$_L$ and global$_G$ are equivalent on levels 1 and 2, after which they are incomparable. Global$_L$ and local are equivalent on level 1, after which local is strictly stronger. Global$_G$ and local are equivalent on level 1, local is strictly stronger on level 2, and after this they are incomparable. Note also that the strength relations are the same if we restrict attention to the classically expressible schemas, since the witnesses provided for the last three propositions all fall within this category.

To close, let us highlight two more helpful effects of the preceding results: that for MML's, global$_L$ subclassicality entails local subclassicality, whilst on the other hand local superclassicality entails global$_L$ superclassicalty. These follow in particular from proposition 3.16, and the fact that schematically local still entails global$_L$ validity for every logic.

Proposition 3.22. *Let $\mathcal{K}$ be an MML and n an inferential level.*

(i) If $\mathcal{K}$ is globally$_L$ subclassical at n, then it is locally subclassical at n.

(ii) If $\mathcal{K}$ is locally superclassical at n, then it is globally$_L$ superclassical at n.

Proof.

(i) Take arbitrary $\Lambda \Rightarrow^n \Omega \in CSC^n$ such that $\vDash^L_{\mathcal{K}} \Lambda \Rightarrow^n \Omega$. Then $\vDash^{GL}_{\mathcal{K}} \Lambda \Rightarrow^n \Omega$ since local entails global$_L$ validity. Therefore $\vDash^{GL}_{CPL} U[\Lambda \Rightarrow^n \Omega]$ by global$_L$ subclassicality. Hence $\vDash^L_{CPL} U[\Lambda \Rightarrow^n \Omega]$ by proposition 3.16.

(ii) Take arbitrary $\Lambda \Rightarrow^n \Omega \in CSC^n$ such that $\vDash^{GL}_{CPL} U[\Lambda \Rightarrow^n \Omega]$. Then $\vDash^L_{CPL} U[\Lambda \Rightarrow^n \Omega]$ by proposition 3.16. Therefore $\vDash^L_{\mathcal{K}} \Lambda \Rightarrow^n \Omega$ by local superclassicality. Hence $\vDash^L_{\mathcal{K}} \Lambda \Rightarrow^n \Omega$ since local entails global$_L$ validity. $\square$

Chapter 4

Classicality Applied

4.1 The Classicality of Epistemic Multilateral Logic

The work of the preceding chapters has finally put us in position to formally study the classicality of EML. We will utilize two results of (Incurvati & Schlöder, 2020), which we mentioned in chapter 1.2 but reiterate here for convenience.

Theorem 4.1 (Incurvati & Schlöder, 2020)**.** *Take arbitrary* $\Gamma \Rightarrow^1 B \in SEQ^1_{PL}$*. Then* $\vDash_{CPL} \Gamma \Rightarrow^1 B$ *iff* $\vDash_{EML} \{+\gamma \mid \gamma \in \Gamma\} \Rightarrow^1 +B$*.*

Theorem 4.2 (Incurvati & Schlöder, 2020)**.** *Take arbitrary* $\Gamma \Rightarrow^1 B \in SEQ^1_{PL}$ *and* $\eta : Prop \to FOR(\mathcal{L}_{ML})$*. If* $\vDash_{CPL} \Gamma \Rightarrow^1 B$ *then* $\vDash_{EML} \{+\eta[\gamma] \mid \gamma \in \Gamma\} \Rightarrow^1 +\eta[B]$*.*

These provide some handle on the relation between EML and CPL at the ground level. We also require the following lemma about $\mu^\sigma_{\mathcal{A}} : \mathcal{A} \to FOR(\mathcal{L}_{ML})$, which is defined given some $\sigma : \mathcal{A} \to FOR(\mathcal{L}_{PL})$ as simply $\mu^\sigma_{\mathcal{A}}(A) = \sigma(A)$.

Lemma 4.3. *Take arbitrary* $\sigma : \mathcal{A} \to FOR(\mathcal{L}_{PL})$ *and* $\Lambda \Rightarrow^n \Omega \in CSC^n$ *such that no signed formulae variables from* Φ *appear in* $\Lambda \Rightarrow^n \Omega$*. Then* $\vDash^{GG}_{EML} \mu^\sigma_{\mathcal{A}}[\Lambda \Rightarrow^n \Omega]$ *iff* $\vDash^{GG}_{CPL} \sigma[U[\Lambda \Rightarrow^n \Omega]]$*.*

Proof. We prove by induction over n.

- For the base case, observe that $\mu^\sigma_{\mathcal{A}}[\Lambda \Rightarrow^1 \Omega] = \{+\lambda \mid \lambda \in \sigma[U[\Lambda]]\} \Rightarrow^1 +\sigma[U[\Omega]]$. Thus the result follows by theorem 4.1.
- Induction step: by the recursive definition of global$_G$ validity. □

Combining these results allows us to determine for which inferential levels EML is (globally$_G$) subclassical and/or superclassical, giving us the next three theorems.

Theorem 4.4. *EML is globally$_G$ subclassical at every level* n*.*

Proof. We take arbitrary $\Lambda \Rightarrow^n \Omega \in CSC^n$ such that $\vDash^{GG}_{EML} \Lambda \Rightarrow^n \Omega$, and show that $\vDash^{GG}_{CPL} U[\Lambda \Rightarrow^n \Omega]$. We make a case distinction based on whether any signed formula variables from Φ appear in $\Lambda \Rightarrow^n \Omega$.

(i) Suppose none do. Take arbitrary $\sigma : \mathcal{A} \to FOR(\mathcal{L}_{PL})$. $\vDash^{GG}_{EML} \Lambda \Rightarrow^n \Omega$ means $\vDash^{GG}_{EML} \mu^{\sigma}_{\mathcal{A}}[\Lambda \Rightarrow^n \Omega]$. Hence $\vDash^{GG}_{CPL} \sigma[U[\Lambda \Rightarrow^n \Omega]]$ by lemma 4.3. Therefore $\vDash^{GG}_{CPL} U[\Lambda \Rightarrow^n \Omega]$ since σ was arbitrary.

(ii) Suppose some do. Pick fresh and distinct $A_\varphi \in \mathcal{A}$ for every $\varphi \in \Phi$ appearing in $\Lambda \Rightarrow^n \Omega$. Let $\Lambda' \Rightarrow^n \Omega'$ be the schema obtained from $\Lambda \Rightarrow^n \Omega$ by replacing every occurrence of every $\varphi \in \Phi$ with $+A_\varphi$. Every instance of $\Delta' \Rightarrow^1 \chi'$ is an instance of $\Lambda \Rightarrow^n \Omega$, so $\vDash^{GG}_{EML} \Lambda \Rightarrow^n \Omega$ entails $\vDash^{GG}_{EML} \Lambda' \Rightarrow^n \Omega'$. Therefore $\vDash^{GG}_{CPL} U[\Lambda' \Rightarrow^n \Omega']$ by case (i). But $U[\Lambda' \Rightarrow^n \Omega'] = U[\Lambda \Rightarrow^n \Omega]$, so $\vDash^{GG}_{CPL} U[\Lambda \Rightarrow^n \Omega]$. ☐

Theorem 4.5. *EML is superclassical at level 1.*

Proof. Take arbitrary $\Delta \Rightarrow^1 \chi \in CSC^1$ such that $\vDash_{CPL} U[\Delta \Rightarrow^1 \chi]$. Take arbitrary $\mu = \mu_{\mathcal{A}} \cup \mu_\Phi$ with $\mu_{\mathcal{A}} : \mathcal{A} \to FOR(\mathcal{L}_{ML})$ and $\mu_\Phi : \Phi \to FOR(\mathcal{L}_{MML})$. Given any $\sigma : \mathcal{A} \to FOR(\mathcal{L}_{PL})$ and $\eta : Prop \to FOR(\mathcal{L}_{ML})$, successively applying them to $U[\Delta \Rightarrow^1 \chi]$ and signing every sentence with $+$ will result in an inference which is EML-valid, by theorem 4.2 and the assumption that $\vDash_{CPL} U[\Delta \Rightarrow^1 \chi]$. Thus the idea is to prove that $\vDash_{EML} \mu[\Delta \Rightarrow^1 \chi]$ by defining σ and η such that the aforementioned inference must be EML-equivalent to $\mu[\Delta \Rightarrow^1 \chi]$, i.e. such that the following diagram commutes up to EML-equivalence.

$$\begin{tikzcd} CSC^1 \arrow[r, "U[-]"] \arrow[d, "\mu[-]"'] & BSC^1 \arrow[d, "\sigma[-]"] \\ SEQ^1_{MML} & SEQ^1_{PL} \arrow[l, "+\eta[-]"] \end{tikzcd}$$

To this end, take any injective $\sigma : \mathcal{A} \to Prop$. Define $\eta : Prop \to FOR(\mathcal{L}_{ML})$ such that $\eta(\sigma(A)) = \mu_{\mathcal{A}}(A)$ for every $A \in \mathcal{A}$, and $\eta(\sigma(U(\varphi))) = \tau(\mu_\Phi(\varphi))$ for any $\varphi \in \Phi$. By assumption, $\vDash_{CPL} \sigma[U[\Delta \Rightarrow^1 \chi]]$. Hence by theorem 4.2, $\vDash_{EML} \{+\eta[\sigma[U(\delta)]] \mid \delta \in \Delta\} \Rightarrow^1 +\eta[\sigma[U(\chi)]]$.

It remains to show that this is equivalent to just $\vDash_{EML} \mu[\Delta \Rightarrow^1 \chi]$. To this end it suffices to point out that $+\eta[\sigma[U(\delta)]]$ is EML-interderivable with $\mu[\delta]$ for every $\delta \in \Delta \cup \{\chi\}$. For every $\delta \in \Delta \cup \{\chi\}$, either $\delta \in \mathcal{A}^+$ or $\delta \in \Phi$. In the former case, $+\eta[\sigma[U(\delta)]] = +\mu_{\mathcal{A}}[\delta] = \mu[\delta]$. In the second case, $+\eta[\sigma[U(\delta)]] = +\tau(\mu_\Phi(\delta)) = +\tau(\mu[\delta])$ with $\mu[\delta] \in FOR(\mathcal{L}_{MML})$, and $+\tau(\psi)$ is interderivable with ψ for every $\psi \in FOR(\mathcal{L}_{MML})$.

Therefore $\vDash_{EML} \{+\eta[\sigma[U(\delta)]] \mid \delta \in \Delta\} \Rightarrow^1 +\eta[\sigma[U(\chi)]]$ entails $\vDash_{EML} \mu[\Delta \Rightarrow^1 \chi]$. Hence $\vDash_{EML} \Delta \Rightarrow^1 \chi$ since μ was arbitrary. ☐

Theorem 4.6. *EML is not globally$_G$ superclassical at any level $n > 1$.*

Proof. At $n = 2$,

$$\frac{\varphi, +\neg A \Rightarrow \bot}{\varphi \Rightarrow +A}\ 2$$

is not globally EML valid due to instances like

$$\frac{\oplus p, +\neg p \Rightarrow \bot}{\oplus p \Rightarrow +p}\ 2$$

but its unilateralization

$$\frac{A_1, \neg A_2 \Rightarrow \bot}{A_1 \Rightarrow A_2}\ 2$$

is globally CPL valid. From this the higher levels follow by proposition 3.11. □

Putting the pieces together, the full story is thus:

Corollary 4.7. *Given global$_G$ validity, EML is classical at level 1 and strictly subclassical at every level $n > 1$.*

We've argued above how global$_G$ validity is the most appropriate criterion for comparing EML to classical logic, since it is the only one that can be defined in terms of the natural deduction system we call 'EML'. Hence corollary 4.7 is the key result of this section, and the vindication of the introduction's informal observation that EML behaves classically on the inferential level but weaker than classical logic on higher levels.

However, seeing as we have come all this way with the local and global$_L$ validities of EML's S5 embedding, we may as well go the distance and see how they compare to classical logic, even though they are not strictly speaking the local/global$_L$ validities of EML itself. As it turns out, whether we assume global$_G$, local or global$_L$ validity does not matter for the classicality of EML: given the latter two, EML is also classical at level 1 and strictly subclassical beyond this. To demonstrate local and global$_L$ subclassicality we make use of another construction: given Boolean valuation v and $\sigma : \mathcal{A} \to FOR(\mathcal{L}_{PL})$, we define $\mu_{\mathcal{A}}^{\sigma^v} : \mathcal{A} \to FOR(\mathcal{L}_{ML})$ as

$$\mu_{\mathcal{A}}^{\sigma^v}(A) = \begin{cases} p \vee \neg p & \text{if } v(\sigma(A)) = 1 \\ p \wedge \neg p & \text{if } v(\sigma(A)) = 0. \end{cases}$$

This 'hard-codes' into a modal substitution the values that v assigns to σ substitutions of formulae. Hence *truth* in v of a σ substitution of some schematic formula determines *theoremhood* of the corresponding $\mu_{\mathcal{A}}^{\sigma^v}$ substitution, in the sense of lemma 4.8.

Lemma 4.8. *Take arbitrary $\sigma : \mathcal{A} \to FOR(\mathcal{L}_{PL})$, Boolean valuation v, and pointed S5 model (M, w). Then $M, w \vDash \tau(\mu_{\mathcal{A}}^{\sigma^v}[\chi])$ iff $v \vDash_{CPL} \sigma[U(\chi)]$ for every $\chi \in \mathcal{A}^+$.*

Proof. It is easy to show by induction on the complexity of B that $\vDash_{S5} \mu_{\mathcal{A}}^{\sigma^v}[B]$ iff $v \vDash_{CPL} \sigma[B]$ for any $B \in \mathcal{A}^*$. But $\vDash_{S5} \mu_{\mathcal{A}}^{\sigma^v}[B]$ iff $\vDash_{S5} \Box\mu_{\mathcal{A}}^{\sigma^v}[B]$, therefore $\vDash_{S5} \Box\mu_{\mathcal{A}}^{\sigma^v}[B]$ iff $v \vDash_{CPL} \sigma[B]$. Furthermore, every $\chi \in \mathcal{A}^+$ is $+B$ for some $B \in \mathcal{A}^*$, hence $B = U(\chi)$ and $\Box\mu_{\mathcal{A}}^{\sigma^v}[B] = \tau(\mu_{\mathcal{A}}^{\sigma^v}[\chi])$. So $\vDash_{S5} \tau(\mu_{\mathcal{A}}^{\sigma^v}[\chi])$ iff $v \vDash_{CPL} \sigma[U(\chi)]$ and $\vDash_{S5} \neg\tau(\mu_{\mathcal{A}}^{\sigma^v}[\chi])$ iff $v \vDash_{CPL} \neg\sigma[U(\chi)]$. Thus for arbitrary (M, w) we have $M, w \vDash \tau(\mu_{\mathcal{A}}^{\sigma^v}[\chi])$ iff $v \vDash_{CPL} \sigma[U(\chi)]$. □

Moreover, this link between truth of a σ substitution in v and theoremhood in EML extends to link *satisfaction* in v of a σ substitution of a schema with (local) *validity* of the corresponding $\mu_{\mathcal{A}}^{\sigma^v}$ substitution, as in the following lemma.

Lemma 4.9. *Take arbitrary $\sigma : \mathcal{A} \to FOR(\mathcal{L}_{PL})$, Boolean valuation v and $\Lambda \Rightarrow^n \Omega \in CSC^n$ such that no signed formulae variables from Φ appear in $\Lambda \Rightarrow^n \Omega$. Then $\vDash_{EML}^{L} \mu_{\mathcal{A}}^{\sigma}[\Lambda \Rightarrow^n \Omega]$ iff $v \vDash_{CPL} \sigma[U[\Lambda \Rightarrow^n \Omega]]$.*

Proof. We take arbitrary pointed S5 model (M, w) and show by induction on n that $M, w \vDash_{EML} \mu_{\mathcal{A}}^{\sigma}[\Lambda \Rightarrow^n \Omega]$ iff $v \vDash_{CPL} \sigma[U[\Lambda \Rightarrow^n \Omega]]$.

- For the base case:
  $M, w \vDash_{EML} \mu_{\mathcal{A}}^{\sigma}[\Lambda \Rightarrow^1 \Omega]$ iff
  $M, w \nvDash \tau(\mu_{\mathcal{A}}^{\sigma^v}[\lambda])$ for some $\lambda \in \Lambda$ or $M, w \vDash \tau(\mu_{\mathcal{A}}^{\sigma^v}[\Omega])$ iff (by lemma 4.8)
  $v \nvDash_{CPL} \sigma[U(\lambda)]$ for some $\lambda \in \Lambda$ or $v \vDash_{CPL} \sigma[U(\Omega)]$ iff
  $v \vDash_{CPL} \sigma[U[\Lambda \Rightarrow^1 \Omega]]$.
- Induction step: by the recursive definition of satisfaction. □

The result is that given any σ and v representing a counterexample to a rule being locally or globally$_L$ valid in CPL (recall that these are equivalent by proposition 3.16), $\mu_{\mathcal{A}}^{\sigma^v}$ provides a counterexample to the rule being locally or globally$_L$ valid in EML, leading to the next two theorems.

Theorem 4.10. *EML is globally$_L$ subclassical at every level n.*

Proof. We take arbitrary $\Lambda \Rightarrow^n \Omega \in CSC^n$ such that $\vDash_{EML}^{GL} \Lambda \Rightarrow^n \Omega$, and show that $\vDash_{CPL}^{GL} U[\Lambda \Rightarrow^n \Omega]$. We make a case distinction based on whether any signed formula variables from Φ appear in $\Lambda \Rightarrow^n \Omega$.

(i) Suppose none do. Take arbitrary $\sigma : \mathcal{A} \to FOR(\mathcal{L}_{PL})$ and Boolean valuation v. $\vDash_{EML}^{GL} \Lambda \Rightarrow^n \Omega$ means $\vDash_{EML}^{GL} \mu_{\mathcal{A}}^{\sigma}[\Lambda \Rightarrow^n \Omega]$. Therefore either $\nvDash_{EML}^{L} \mu_{\mathcal{A}}^{\sigma}[\lambda]$ for some $\lambda \in \Lambda$ or $\vDash_{EML}^{L} \mu_{\mathcal{A}}^{\sigma}[\Omega]$. Hence by lemma 4.9 either $v \nvDash_{CPL} \sigma[U[\lambda]]$ for some $\lambda \in \Lambda$ or $v \vDash_{CPL} \sigma[U[\Omega]]$. Therefore $v \vDash_{CPL} \sigma[U[\Lambda \Rightarrow^n \Omega]]$. Hence $\vDash_{CPL}^{L} U[\Lambda \Rightarrow^n \Omega]]$ since σ and v were arbitrary. Therefore $\vDash_{CPL}^{GL} U[\Lambda \Rightarrow^n \Omega]]$ since local entails global$_L$ validity.

(ii) As case (ii) of theorem 4.4. □

Theorem 4.11. *EML is locally subclassical at every level n.*

Proof. By theorem 4.10 and proposition 3.22 (i). □

When it comes to superclassicality, local as well as global$_L$ fare the same as global$_G$ validity, and for essentially the same reason.

Theorem 4.12. *EML is not locally or globally$_L$ superclassical at any level $n > 1$.*

Proof. At $n = 2$, the counterexamples are the same as for theorem 4.6, and the higher levels follow by proposition 3.11. □

Combining these leads to the following overview, precisely mirroring corollary 4.7 on global$_G$ validity.

Corollary 4.13. *Given local or global$_L$ validity, EML is classical at level 1 and strictly subclassical at every level $n > 1$.*

This completes the application of the three schematic validity notions to EML and its S5 embedding, as compared to CPL. On each, EML is classical on level 1 but weaker on all higher levels. Given these results, it is natural to wonder whether we can do any better. Barrio et al. (2020) have demonstrated how, starting from ST, it is possible to construct an infinite sequence of logics that behaves classically up to higher and higher levels. That is, for every inferential level, they define a logic which is locally classical precisely up to that level, and strictly subclassical beyond. So perhaps EML too can be strengthened to match classical logic up to levels beyond the first. If it can, how much must be added to EML to get classicality up to level 2, up to level 3, ..., or even at every level? This is relevant to the classicality of EML itself too, for it may serve as a measure of the distance between EML and classical logic. The next section will address these questions, and demonstrate the sense in which EML is only one rule away from full classicality.

4.2 EML*

We are in search of variants of EML that behave classically up to higher levels, ideally every level. The obvious initial strategy is to consider those classically valid level 2 rules that fail in EML, and seeing what might be added to validate them, in the hopes of attaining classicality at least up to level 2. But it is not obvious that this approach can get us very far at all. It might be that EML's failures towards superclassicality at level 2 are of such a complexity and/or variety that they can not be easily fixed by adding a few inference rules to the proof theory. Moreover, although we have proven EML's non-superclassicality at levels $n > 2$ by proving it at 2, from which the higher levels could then be inferred, it does not follow that fixing the disparity at level 2 thereby also fixes it at any $n > 2$. Yet if we have to keep manually adding rules for every level we wish to climb, we can never hope to reach general superclassicality. Finally, there is always the worry that by adding rules in pursuit of superclassicality at some level, we strengthen the system in such a way that we wind up losing subclassicality, at that same level or lower down, which would defeat the purpose entirely. We can even conceive that level n superclassicality, when combined with the inference rules of EML itself, *entails* non-subclassicality for some level n or lower.

However, we will find that none of these obstacles arise, and the path from EML to full classicality is remarkably short. By adding a single rule to the natural deduction system, addressing the specific example we have focused on for EML's superclassicality failure at level 2, we immediately arrive at an MML that is classical at every inferential level. This example, recall, was that of *reductio*: EML fails to globally (or locally) validate the schema of Multilateral Reductio

$$\frac{\varphi, +\neg A \Rightarrow \bot}{\varphi \Rightarrow +A}\ 2$$

because instances such as

$$\frac{\oplus p, +\neg p \Rightarrow \bot}{\oplus p \Rightarrow +p}\ 2$$

are globally (and locally) invalid. But the unilateralization of this schema is Classical Reductio

$$\frac{A_1, \neg A_2 \Rightarrow \bot}{A_1 \Rightarrow A_2}\ 2$$

which is globally (and locally) CPL valid. The global validity of Multilateral Reductio corresponds directly to the presence or derivability of the following natural deduction rule.

$$(+\neg\mathrm{E})\ \frac{\begin{matrix}[+\neg A] \\ \vdots \\ \bot\end{matrix}}{+A}$$

So we let EML* be the system obtained by supplementing EML with $(+\neg\mathrm{E})$.

EML* is locally, globally$_L$ and most importantly globally$_G$ classical at every inferential level. To demonstrate this, it will be useful that EML* can be alternatively characterized by strengthening an existing rule of EML, instead of adding a new one. Namely, we can remove the restrictions on (Weak Inference). Let EML** be the natural deduction system obtained from EML by replacing (Weak Inference) with the following:

$$(\text{Weak Inference}^*)\ \frac{\oplus A \qquad \begin{matrix}[+A] \\ \vdots \\ +B\end{matrix}}{\oplus B}$$

EML* and EML** are equally strong, since in the presence of EML's other inference rules, (Weak Inference*) entails $(+\neg\mathrm{E})$ and vice-versa. That is, $(+\neg\mathrm{E})$ is derivable in EML** via this proof tree:

$$\dfrac{\dfrac{\begin{array}{c}[+A]^1\\ \vdots\\ \bot\end{array}\qquad \dfrac{[\ominus\neg A]^2}{\oplus A}\,(\ominus\neg\mathrm{E})}{\bot}\,(\text{Weak Inference}^*)^1}{+\neg A}\,(\mathrm{SR}_2)^2$$

On the other hand, EML* derives (Weak Inference*) as follows:

$$\begin{array}{c}\dfrac{\dfrac{\dfrac{\oplus A}{\ominus\neg A}\,(\ominus\neg\mathrm{I})\qquad [+\neg A]^1}{\bot}\,(\text{Rejection})}{+A}\,(+\neg\mathrm{E})^1\\ \vdots\\ \dfrac{+B}{\oplus B}\,(\text{Assertion})\end{array}$$

I will henceforth use ‘EML*’ to refer indiscriminately to either one of these systems, for in the remainder we will be concerned with the set of derivable inferences they produce, rather than the particular rules they use to do so. But it is convenient that we no longer have to worry about restrictions on the use of $\Diamond$E rules within larger derivations.

Observe how our proof that $(+\neg\mathrm{E})$ derives (Weak Inference*) works by deriving $+A$ from $\oplus A$. This gives a preview of the way in which EML* achieves full classicality, namely through a complete lateral and modal collapse: $\oplus A$ and $+\Diamond A$ are EML*-equivalent to simply $+A$, and $\ominus A$ to simply $+\neg A$. Thus every signed formula is equivalent to a strongly asserted propositional formula, which behave highly classically already in EML (cf. theorem 4.1). That they behave *fully* classically in EML*, however, now remains to be proven.

We start by providing a semantics. Given the same τ translation as in the model theory of EML, EML* is sound and complete for the class of reflexive single world Kripke models; those of the form $(W = \{w\}, R = \{(w, w)\}, V)$. We will refer to these as T1 models, and write $\vDash_{T1}$ for their semantic consequence relation.

Theorem 4.14 (EML* Soundness)**.** *Take arbitrary* $\Gamma \Rightarrow^1 \psi \in SEQ^1_{\mathcal{L}_{MLL}}$. *If* $\Gamma \vdash_{EML^*} \varphi$ *then* $\tau[\Gamma] \vDash_{T1} \tau(\psi)$.

Proof. By induction on the length of derivations. The steps for rules present in EML follows from EML's soundness for S5, because all T1 models are S5 models. It remains to show either (Weak Inference*) or $(+\neg\mathrm{E})$, and we opt for the former.

For (Weak Inference*), suppose that $\tau[\Gamma], \Box A \vDash_{T1} \Box B$ and $\tau[\Gamma] \vDash_{T1} \Diamond A$. On T1 models, $\Diamond A$ and $\Box A$ are both true just in case A is. So $\tau[\Gamma] \vDash_{T1} \Diamond A$ means $\tau[\Gamma] \vDash_{T1} \Box A$. Therefore $\tau[\Gamma], \Box A \vDash_{T1} \Box B$ means $\tau[\Gamma] \vDash_{T1} \Box B$, which means $\tau[\Gamma] \vDash_{T1} \Diamond B$ as desired. □

For completeness we require some facts:

Lemma 4.15. *Let $\Gamma \subseteq FOR(\mathcal{L}_{MLL})$ be a maximally $\vdash_{EML^*}$ consistent set.*

(i) $+\neg A \in \Gamma$ *iff* $+A \notin \Gamma$

(ii) $+\Diamond A \in \Gamma$ *iff* $+A \in \Gamma$

Proof.

(i) (Left-to-right) is obvious because $+A, +\neg A \vdash_{EML^*} \bot$, which was already derivable for EML.
(Right-to-left) follows directly from the fact that $\Gamma, +A \vdash_{EML^*} \bot$ entails $\Gamma \vdash_{EML^*} +\neg A$, which is shown by the following deduction:

$$\dfrac{\dfrac{\begin{matrix}[+A]^1\\ \vdots \\ \bot\end{matrix} \qquad \dfrac{[\ominus\neg A]^2}{\oplus A}\,(\ominus\neg\mathrm{E})}{\bot}\,(\text{Weak Inference}^*)^1}{+\neg A}\,(\mathrm{SR}_2)^2$$

(ii) (Left-to-right) follows from $+\Diamond A \vdash_{EML^*} +A$, which is derived as follows:

$$\dfrac{\dfrac{\dfrac{[+A]^1[\ominus A]^2}{\bot}\,(\text{Rejection}) \qquad \dfrac{+\Diamond A}{\oplus A}\,(+\Diamond\mathrm{E})}{\bot}\,(\text{Weak Inference}^*)^1}{+A}\,(\mathrm{SR}_2)^2$$

(Right-to-left) follows from $+A \vdash_{EML^*} +\Diamond A$, which was already derivable for EML. □

Using these we can prove completeness by constructing a model for every consistent set:

Theorem 4.16 (EML* Completeness)**.** *Let $\Gamma \Rightarrow^1 \varphi \in SEQ^1_{\mathcal{L}_{MLL}}$. If $\tau[\Gamma] \vDash_{T1} \tau(\varphi)$ then $\Gamma \vdash_{EML^*} \varphi$.*

Proof. Let Γ be an $\vdash_{EML^*}$ consistent set, and construct a maximally consistent superset Γ' by the usual procedure. Consider the T1 model M with valuation $V(w) = \{p \in Prop \mid +p \in \Gamma'\}$. We prove that $M \vDash A$ iff $+A \in \Gamma'$, by induction over the complexity of A.

- Base case and $\wedge$ are trivial.
- For $\neg$, by Lemma 4.15 (i) we have $+\neg A \in \Gamma'$ iff $+A \notin \Gamma'$ iff $M \nvDash A$ iff $M \vDash \neg A$.
- For $\Diamond$, by Lemma 4.15 (ii) we have $+\Diamond A \in \Gamma'$ iff $+A \in \Gamma'$ iff $M \vDash A$ iff $M \vDash \Diamond A$.

So $M \vDash A$ for all $+A \in \Gamma$. Therefore $M \vDash \Box A$ hence $M \vDash \tau(+A)$ for all $+A \in \Gamma$. Since every formula is EML*-equivalent to a strongly asserted one, we can assume without loss of generality that Γ is fully strongly asserted. Hence $M \vDash \tau[\Gamma]$. $\Box$

Given this model theory, it should be expected that EML* turns out very close to classical logic. T1 models are fully determined by their valuation $V(w)$, so in a sense they are just Boolean valuations with some extra fluff that only serves to determine that $\Diamond A$ and $\Box A$ are true just in case A is true. Let us work this out in detail.

Given Boolean valuation v, let M^v be the TI model $(\{w\}, \{(w, w)\}, V)$ with $V(w) = \{p \in Prop \mid v(p) = 1\}$. Note that this construction is conversible: given T1 model $M = (\{w\}, \{(w, w)\}, V)$, let v^M be the Boolean valuation such that $v^M(p) = 1$ iff $p \in V(w)$. Then $M^{v^M} = M$ and $v^{M^v} = v$. Furthermore, let $\nabla : \mathcal{L}_{ML} \rightarrow \mathcal{L}_{PL}$ simply remove all modal operators. It is easily shown (by induction on complexity of A) that $M^v \vDash A$ iff $v(\nabla(A)) = 1$ for any $A \in FOR(\mathcal{L}_{ML})$. This gets us the following correspondence between satisfaction in M^v and v of certain pairs of individual inferences from $SEQ^n_{\mathcal{L}_{MML}}$ and $SEQ^n_{\mathcal{L}_{PL}}$ respectively.

Lemma 4.17. *Take arbitrary $\Theta \Rightarrow^n \Pi \in SEQ^n_{\mathcal{L}_{MLL}}$ and Boolean valuation v. Then $M^v \vDash_{EML^*} \Theta \Rightarrow^n \Pi$ iff $v \vDash_{CPL} \nabla[\tau[\Theta \Rightarrow^n \Pi]]$.*

Proof. By induction on n.

- Base case: $M^v \vDash_{EML^*} \Theta \Rightarrow^1 \Pi$ iff
  $M^v \nvDash \tau(\theta)$ for some $\theta \in \Theta$ or $M^v \vDash \tau(\Pi)$ iff
  $v \nvDash \nabla(\tau(\theta))$ for some $\theta \in \Theta$ or $v \vDash \nabla(\tau(\Pi))$ iff
  $v \vDash_{CPL} \nabla[\tau[\Theta \Rightarrow^1 \Pi]]$.
- Induction step: by the recursive definition of satisfaction. $\Box$

The preceding lemma also induces a correspondence in the global$_G$, global$_L$ and local validity of these inference pairs according to EML* and CPL.

Lemma 4.18. *Take arbitrary $\Theta \Rightarrow^n \Pi \in SEQ^n_{\mathcal{L}_{MLL}}$. Then $\vDash^{GG}_{EML^*} \Theta \Rightarrow^n \Pi$ iff $\vDash^{GG}_{CPL} \nabla[\tau[\Theta \Rightarrow^n \Pi]]$.*

Proof. By induction over n.

- Base case: by the base case of lemma 4.17.
- Induction step: by the recursive definition of global$_G$ validity. $\Box$

Lemma 4.19. *Take arbitrary $\Theta \Rightarrow^n \Pi \in SEQ^n_{\mathcal{L}_{MLL}}$. Then $\vDash^{GL}_{EML^*} \Theta \Rightarrow^n \Pi$ iff $\vDash^{GL}_{CPL} \nabla[\tau[\Theta \Rightarrow^n \Pi]]$.*

Proof. By lemma 4.17. $\Box$

Lemma 4.20. *Take arbitrary* $\Theta \Rightarrow^n \Pi \in SEQ^n_{\mathcal{L}_{MLL}}$. *Then* $\vDash^L_{EML^*} \Theta \Rightarrow^n \Pi$ *iff* $\vDash^L_{CPL} \nabla[\tau[\Theta \Rightarrow^n \Pi]]$.

Proof. By lemma 4.17. □

Finally, this forces agreement on the validities of *rules* between EML* and CPL, because for any $\Lambda \Rightarrow^n \Omega \in CSC^n$ and its unilateralization $U[\Lambda \Rightarrow^n \Omega]$, every instance of the one is paired with some instance of the other, under the pairing of individual inferences we have been discussing. Namely, given any $\sigma : \mathcal{A} \to FOR(\mathcal{L}_{PL})$, we define $\mu^\sigma = \mu_\Phi \cup \mu_\mathcal{A}$ with $\mu_\Phi : \Phi \to FOR(\mathcal{L}_{MML})$ and $\mu_\mathcal{A} : \mathcal{A} \to FOR(\mathcal{L}_{ML})$ such that $\mu_\Phi(\varphi) = +\sigma(U(\varphi))$ and $\mu_\mathcal{A}(A) = \sigma(A)$. Then $\nabla[\tau[\mu^\sigma[\Lambda \Rightarrow^n \Omega]]] = \sigma[U[\Lambda \Rightarrow^n \Omega]]$.

On the other hand, given any $\mu = \mu_\Phi \cup \mu_\mathcal{A}$ with $\mu_\Phi : \Phi \to FOR(\mathcal{L}_{MML})$ and $\mu_\mathcal{A} : \mathcal{A} \to FOR(\mathcal{L}_{ML})$, we define a partial $\sigma^\mu : \mathcal{A} \to FOR(\mathcal{L}_{PL})$ such that for all $A \in \mathcal{A}$ appearing in $U[\Lambda \Rightarrow^n \Omega]$,

$$\sigma^\mu(A) := \begin{cases} \nabla(\tau(\mu_\Phi(\varphi))) & \text{if } A = U(\varphi) \text{ for } \varphi \in \Phi \text{ appearing in } \Lambda \Rightarrow^n \Omega \\ \nabla(\mu_\mathcal{A}(A)) & \text{if } A \text{ itself appears in } \Lambda \Rightarrow^n \Omega. \end{cases}$$

Then $\sigma^\mu[U[\Lambda \Rightarrow^n \Omega]] = \nabla[\tau[\mu[\Lambda \Rightarrow^n \Omega]]]$.

Thus counterexamples to the EML*-validity of a rule correspond one-to-one with counterexamples to the CPL-validity of the same rule, given any validity criterion.

Theorem 4.21. *EML* is globally*$_G$*/globally*$_L$*/locally classical at every inferential level* n.

Proof. We do the proof for global$_G$ validity, as the others are analogous. We take arbitrary $\Lambda \Rightarrow^n \Omega \in CSC^n$ and show $\vDash^{GG}_{EML^*} \Lambda \Rightarrow^n \Omega$ iff $\vDash^{GG}_{CPL} U[\Lambda \Rightarrow^n \Omega]$.

> (Left-to-right) Suppose for contraposition that $\nvDash^{GG}_{CPL} U[\Lambda \Rightarrow^n \Omega]$. Then $\nvDash^{GG}_{CPL} \sigma[U[\Lambda \Rightarrow^n \Omega]]$ for some $\sigma : \mathcal{A} \to FOR(\mathcal{L}_{PL})$. Hence $\nvDash^{GG}_{CPL} \nabla[\tau[\mu^\sigma[\Lambda \Rightarrow^n \Omega]]]$. Therefore $\nvDash^{GG}_{EML^*} \mu^\sigma[\Lambda \Rightarrow^n \Omega]$ by lemma 4.18. Hence $\nvDash^{GG}_{EML^*} \Lambda \Rightarrow^n \Omega$.
>
> (Right-to-left) Similar, though using σ^μ where μ is the counterexample to $\vDash^{GG}_{EML^*} \Lambda \Rightarrow^n \Omega$. □

This establishes the central claim of this final section: that EML is but one metarule away from full classicality. However, there is an additional reason why these results are significant, namely the special place EML* takes up in the space of EML strengthenings.

First of all, EML* is the minimal superclassicality extension of EML. By a superclassicality extension of EML we mean an MML which is (i) as strong as EML and (ii) locally, globally$_L$ *or* globally$_G$ superclassical up to a higher level than EML.[29] EML* is the minimal element of this class, in the sense that any MML with properties (i) and (ii) is at least as strong as EML*. In other words, if we wish to strengthen EML to perform 'better' on any one of the superclassicality measures, we must strengthen it to at least EML*. That EML* meets (i) follows immediately from its definition. Property (ii) has been demonstrated by theorem 4.21. To see that it is minimal, observe that if any MML $\mathcal{K}$ meets condition (ii), then $\mathcal{K}$ must be either globally or locally superclassical at level 2. The latter entails the former by proposition 3.22, so we may assume $\mathcal{K}$ is globally superclassical at level 2. This means $\mathcal{K}$ must globally validate Multilateral Reductio, and more generally the schema

$$\frac{\varphi_1, ..., \varphi_m, +\neg A \Rightarrow \bot}{\varphi_1, ..., \varphi_m \Rightarrow +A}\ 2$$

for any $m \in \mathbb{N}$. Hence $(+\neg\text{E})$ is derivable in $\mathcal{K}$. If $\mathcal{K}$ furthermore meets (i), then it derives every inference rule in the definition of EML*, and so must be at least as strong.[30]

This also means that EML* is the minimal classicality extension of EML, in the sense that any MML which is as strong as EML, and classical (locally, globally$_L$ or globally$_G$) up to a strictly higher level than EML, must be at least as strong as EML*. For being classical on a higher level than EML requires being superclassical on level 2. So we have added only the absolute minimum in derivational strength that must be added in order to increase (super)classicality on any one of the validity notions by even a single level.

Furthermore, EML* is even the *only* classicality extension of EML, if we restrict attention to logics whose level 1 validities are closed under uniform substitution. As mentioned before, it is a plausible criterion of formality for logics that if an inference is valid, then the result of uniformly substituting its

[29]For the sake of precision: a natural deduction system is said to be *as strong as* another in the same language if for any derivation in the latter, there exists a derivation in the former with the same premises and conclusion.

[30]It might be that whilst $\mathcal{K}$ globally validates all instances of

$$\frac{\varphi_1, ..., \varphi_m, +\neg A \Rightarrow \bot}{\varphi_1, ..., \varphi_m \Rightarrow +A}\ 2$$

and hence effectively derives $(+\neg\text{E})$, some instance of the schema is only globally $\mathcal{K}$ valid because $\varphi_1, ..., \varphi_m \Rightarrow +A$ is derivable using $\Diamond$E rules, whilst the corresponding instance of $\varphi_1, ..., \varphi_m, +\neg A \Rightarrow \bot$ is derivable without them. In this case the assumption that $\mathcal{K}$ is as strong as EML apparently does not license the conclusion that $\mathcal{K}$ derives (Weak Inference) as stated, but only a version of Weak Inference with an additional constraint disallowing the use of $(+\neg\text{E})$ in that part of the preceding derivation where $\Diamond$E rules are disallowed. Since we did not put such a constraint on (Weak Inference) in the definition of EML*, it might seem mistaken to conclude that $\mathcal{K}$ must be at least as strong as EML*. However, as we have seen, $(+\neg\text{E})$ along with some rules of EML - crucially not inclding (Weak Inference) - allows for the derivation of (Weak Inference*), which has no resitrictions on the use of $\Diamond$E rules at all. Hence such uses of Weak Inference are permitted in EML* and any other $\mathcal{K}$ meeting (i) and (ii).

propositional letters for sentences should also be valid. If this were not the case even for level 1 inferences, then the logic would determine validity based on matters other than logical form alone. So let a *formal* classicality extension of EML be any MML such that (i) it is as least as strong as EML, (ii) it is (locally, globally$_L$ or globally$_G$) classical up to a higher level than EML and (iii) its level 1 validities are closed under uniform substitution.

Theorem 4.22. *Any formal classicality extension of EML is precisely as strong as EML*.*

Proof. Suppose $\mathcal{K}$ is a formal classicality extension of EML. By properties (i) and (ii), $\mathcal{K}$ is at least as strong as EML*.

Suppose for contradiction that $\mathcal{K}$ is strictly stronger. Then there exists $\Gamma \Rightarrow^1 \psi \in SEQ^1_{MML}$ such that $\vDash_{\mathcal{K}} \Gamma \Rightarrow^1 \psi$ but $\nvDash_{EML^*} \Gamma \Rightarrow^1 \psi$. Any element of $FOR(\mathcal{L}_{MML})$ is EML*- and hence $\mathcal{K}$-interderivable with a strongly asserted propositional sentence (in particular any ψ is interderivable with $+\nabla[\tau(\psi)]$). So we can assume without loss of generality that $\Gamma \subseteq \{+A | A \in FOR(\mathcal{L}_{PL})\}$ and $\psi \in \{+A | A \in FOR(\mathcal{L}_{PL})\}$.

Now let $\Delta \Rightarrow^1 \chi \in CSC^1$ be the result of uniformly substituting every propositional letter in $\Gamma \Rightarrow^1 \psi$ with a distinct $A \in \mathcal{A}$. Then $\vDash_{\mathcal{K}} \Delta \Rightarrow^1 \chi$ by property (iii), because $\vDash_{\mathcal{K}} \Gamma \Rightarrow^1 \psi$ and every instance of $\Delta \Rightarrow^1 \chi$ is a uniform substitution instance of $\Gamma \Rightarrow^1 \psi$. However, $\nvDash_{EML^*} \Gamma \Rightarrow^1 \psi$ and therefore $\nvDash_{EML^*} \Delta \Rightarrow^1 \chi$. Hence $\nvDash_{CPL} U[\Delta \Rightarrow^1 \chi]$ by theorem 4.21. But then we have both $\vDash_{\mathcal{K}} \Delta \Rightarrow^1 \chi$ and $\nvDash_{CPL} U[\Delta \Rightarrow^1 \chi]$, so $\mathcal{K}$ is not subclassical at level 1, which contradicts property (ii). □

So starting from EML, if we want to increase (super)classicality by any level on any of the validity notions, we have to go at least to the strength of EML*, and furthermore, we cannot go a bit beyond this without immediately losing (sub)classicality at level 1. So EML* is really the only game in town when it comes to strengthening EML to increase classicality.

It should also be noted, however, that EML* is rather unsatisfying as a multilateral modal logic. All the distinction between strong and weak assertion is lost, since $+A$ is interderivable with $\oplus A$, as is all the difference between rejecting a sentence and asserting its negation, since $\ominus A$ is interderivable with $+\neg A$. Finally, the epistemc 'might' has lost all meaning, as the $\Diamond$ is completely transparent. EML* is both harmonious - if presented via the strengthening of (Weak Inference) rather than the addition of ($+\neg$E) - and fully classical, but so are previously known bilateral systems (Rumfitt, 2000; Smiley, 1996). The advantages of the multilateral approach, namely to give a harmonious account of *weak* rejection and the epistemic 'might', have all been forfeited.

Conclusion

The research in this thesis was motivated by the observations that, in providing a harmonious treatment of weak assertion and the epistemic $\Diamond$, EML seemingly manages to preserve classicality for standard inferences, but not for the higher inferential levels. However, it was unclear precisely in which sense this is the case. Our main purpose was to fix this situation, by providing an overview of formal results detailing how EML conforms with or departs from classical logic at different inferential levels.

In order to achieve this, we first needed to get clear on what it means for an inference of level $n > 1$ to be valid in the first place, either in CPL or EML. The ST literature offered two ways of understanding validity at level 2, namely local and global validity. The former has an obvious known generalization to higher levels. We demonstrated how the latter can be generalized in various ways, and assessed the strength of these options compared to each other and the local variant, with respect to various logics. We argued that the global$_G$ variant is the most appropriate criterion when it comes to MML's, due to its proof-theoretic characterizability, and diffused the existing objections to the plausibility of global validity notions. Specifically, the charges are that global validity is (i) not sufficiently uniform in its treatment of the different inferential levels, (ii) too weak a criterion, causing it to overgenerate validities and (iii) not closed under uniform substitutions. The first of these was met by establishing a justification for the asymmetry, whilst the other two disappear upon moving to schemas (or once we read individual inferences substitutionally).

The move to schemas was necessitated by the fact that EML and CPL are formulated in different languages. However, we encountered the impossibility of defining a single type of schema that can be properly assessed for validity with respect to both multi- and unilateral logics. In order to overcome this obstacle, we developed and motivated a systematic method for cross-identifying the inference rules expressed by multilateral schemas and unilateral Boolean schemas. After doing so, we were finally in position to formally study the similarities and differences between EML and classical logic. The result was a precise expression of the sense in which EML is classical on the basic inferential level, but weaker than classical logic beyond this.

It also turned out, though, that the only way to make EML more classical is through giving up all of its virtues, by inducing the complete multilateral and modal collapse that characterizes EML*. This tells us that all of EML's departures from classicality are direct consequences of the mere choice to read rejection weakly, and include weak assertion and epistemic $\Diamond$. The takeaway is that whilst EML is not entirely classical, it is as close to classical as is possible within the multilateral framework. If one wishes to treat $\ominus$, $\oplus$ and $\Diamond$, and wishes to keep them in character, EML is as classical as it gets.

As for future research, there are a few promising avenues suggested by our developments. First of all is the application of the novel global$_L$ and global$_G$ validity criteria to logics other than CPL and EML. For example we may wonder how strong the notions are with respect to each other and local validity at different logics, such as ST and SV. Though their application to the classicality of these logics is very straightforward. For both SV and ST, in the standard propositional language their classicality on the inferential level implies full global$_G$ classicality. The difference in local validity at level 2 means ST and SV will be globally$_L$ classical precisely up to level 2. If we consider SV and ST in the languages supplemented with the operator D or the constant λ respectively, for a schematic comparison with CPL, then they will be globally classical only up to level 1. But when it comes to logics whose classical behaviour seems to extend to higher levels, such as many of those discussed in (Barrio et al., 2020; Fitting, 2021; Pailos, 2020; Ripley, 2021; Scambler, 2020), global$_L$ and global$_G$ validity may prove to be useful tools for explicating the higher level differences and similarities.

Furthermore, the general idea of unilateralization may serve as a method for comparison between unilateral and bi- or multilateral logics well beyond our applications of it here. In particular, it lends itself to the assessment of classicality for logics other than EML. One very natural candidate in this regard would be Supervaluationist Multilateral Logic (SML), an alternative application of the multilateral framework, developed in (Incurvati & Schlöder, 2021). It is formulated in a language different from $\mathcal{L}_{MML}$, and we certainly cannot expect to assess multilateral schemas as we have defined them here for SML validity. But it would be routine to set up an analogous notion of supervaluationist multilateral schemas, and corresponding method of unilateralization, at which point one could apply global$_G$ validity (or some other preferred criterion) to work out in detail the relation between SML and classical logic.

Finally, even within the more narrow topic of EML's comparison to classical logic, there are still some options left unexplored. We have motivated the choice to leave out alternative validity notions, such as absolute global validity and antivalidity, usually on the basis that they can only be defined in terms of a specific model theory rather than the natural deduction system itself. This is a solid justification to the extent that we are interested in EML only for its inferentialist purposes. But there may well be other reasons to consider EML, or properly speaking, its S5 embedding, in which case multilateral schemas and their unilateralization may also be applied to investigate classicality via these criteria.

Another potential approach, which we have not mentioned at all thus far, is to consider what we may call the contravalidity of schemas. Given a notion of validity for individual inferenes, a schema is contravalid if none of its instances are valid. This presents a schematic level analogue to antivalidity. The specific notion of global$_G$ contravalidity would be of particular interest to EML's classicality, as like global$_G$ validity itself, it can be characterized entirely in terms of the proof system EML. Thus a comparison of EML and CPL on their globally$_G$ contravalid schemas might reveal connections or disparities that can be genuinely attributed to EML itself, rather than merely to its model theory. If one wishes to compare EML and CPL also in terms of a negative notion of validity, this would certainly be the first place to look.

Contravalidity need not be the only option in this regard though. Another might be simply to consider invalidtities. Rosenblatt (2021) argues that bilateralists should treat invalidity independently of and as on a par with validity itself. The obvious argument for not explicitly including invalidities when comparing different logics is that doing so does not grant any insight, beyond what is already gained by comparing their validities. For two logics in the same language agree on all validities of a given level n iff they agree on all invalidities at n. Similarly, an MML validates the same inference rules as CPL at some level iff it invalidates the same inference rules as CPL. In general, since the set of invalidities of a logic at some level for some notion of validity is merely the complement of the corresponding set of validities, any difference between the invalidities of two logics corresponds to a difference in their validities and vice-versa.

This line of reasoning does presume, however, that the set of invalidities of a logic is indeed the complement of the set of validities, as there are neither gluts nor gaps between validity and invalidity. Rosenblatt maintains that such gluts and gaps can not be excluded on the bilateral framework. Though his arguments are aimed specifically at a particular brand of bilateralism, attributed to Restall (2013) and Ripley (2013). On these accounts, inference takes place between sentences, rather than signed formulae, but validity itself is cashed out in terms of bilateral attitudes. Namely, an inference is valid (invalid) if accepting the premises is incoherent (coherent) with rejecting the conclusion. Rosenblatt's arguments, then, are to the effect that in some cases it might be both coherent and incoherent, or neither coherent nor incoherent, to accept the premises whilst resisting the conclusion. Thus the bilateralist should provide seperate proof rules for deduction between invalidities.

The multilateral approach is rather different from this particular strain of bilateralism, in that the attitudes are incorporated directly into the language and by extension into the inferences, whilst validity is understood as commitment preservation between these attitudes. Nevertheless, it might be worth exploring the extent to which the force of Rosenblatt's arguments carries over to multilateralism, and what the proof rules for EML's invalidities would look like. Then multilateral schemas and their unilateralization can again be applied to compare the invalidities to those of CPL.

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