VARIETIES OF INTERIOR ALGEBRAS

by

W. J. BLOK

Abstract

We study (generalized) Boolean algebras endowed with an interior operator, called (generalized) interior algebras. Particular attention is paid to the structure of the free (generalized) interior algebra on a finite number of generators. Free objects in some varieties of (generalized) interior algebras are determined. Using methods of a universal algebraic nature we investigate the lattice of varieties of interior algebras.

Keywords: (generalized) interior algebra, Heyting algebra, free algebra, $*$-algebra, lattice of varieties, splitting algebra.

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WILLEM JOHANNES BLOK

GEBOREN TE HOORN

Promotor : Prof. Dr. Ph.Dwinger

Coreferent: Prof. Dr. A.S.Troelstra

Druk: Huisdrukkerij Universiteit van Amsterdam

T- 5670

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aan renee

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CONTENTS

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INTRODUCTION

1 Some remarks on the subject and its history

In an extensive paper titled "The algebra of topology", J.C.C. McKinsey and A. Tarski [44] started the investigation of a class of algebraic structures which they termed "closure algebras". The notion of closure algebra developed quite naturally from set theoretic topology. Already in 1922, C. Kuratowski gave a definition of the concept of topological space in terms of a (topological) closure operator defined on the field of all subsets of a set. By a process of abstraction one arrives from topological spaces defined in this manner at closure algebras, just as one may investigate fields of sets in the abstract setting of Boolean algebras. A closure algebra is thus an algebra $(L,(+,.,',{}^{c},0,1))$ such that $(L,(+,.,',0,1))$ is a Boolean algebra, where $+,.,'$ are operations satisfying certain postulates so as to guarantee that they behave as the operations of union, intersection and complementation do on fields of sets and where 0 and 1 are nullary operations denoting the smallest element and largest element of L respectively. The operation ^{c} is a closure operator, that is, ^{c} is a unary operation on L satisfying the well-known "Kuratowski axioms"

$$\begin{array}{ll} \text{(i)} & x \leq x^{c} \\ \text{(ii)} & x^{cc} = x^{c} \\ \text{(iii)} & (x+y)^{c} = x^{c} + y^{c} \\ \text{(iv)} & 0^{c} = 0. \end{array}$$

The present paper is largely devoted to a further investigation of classes of these algebras. However, in our treatment, not the closure operator ^{c} will be taken as the basic operation, but instead the interior operator ^{o}, which relates to ^{c} by $x^{o} = x'^{c\prime}$ and which satisfies the postulates (i)' $x^{o} \leq x$, (ii)' $x^{oo} = x^{o}$, (iii)' $(xy)^{o} = x^{o}y^{o}$ and (iv)' $1^{o} = 1$, corresponding to (i) - (iv). Accordingly, we shall speak of interior algebras rather than closure algebras. The reason for our favouring the interior operator is the following. An important feature in the structure of an interior algebra is the set of closed elements, or, equivalently, the set of open elements. In a continuation of their work on closure algebras, "On closed elements in closure algebras", McKinsey and Tarski showed that the set of closed elements of a closure algebra may be regarded in a natural way as what one would now call a dual Heyting algebra. Hence the set of open elements may be taken as a Heyting algebra, that is, a relatively pseudo-complemented distributive lattice with 0,1, treated as an algebra $(L,(+,.,\to,0,1))$ where $\to$ is defined by $a \to b = \max\{z \mid az \leq b\}$. Therefore, since the theory of Heyting algebras is now well-established, it seems preferable to deal with the open elements and hence with the interior operator such as to make known results more easily applicable to the algebras under consideration.

When they started the study of closure algebras McKinsey and Tarski wanted to create an algebraic apparatus adequate to the treatment of certain portions of topology. They were particularly interested in the question as to whether the interior algebras of all subsets of spaces like the Cantor discontinuum or the Euclidean spaces of any number of dimensions are functionally free, i.e. if they satisfy only those topological equations which hold in any interior algebra. By topological equations we understand those whose terms are expressions involving only the operations of interior algebras. McKinsey and Tarski proved that the answer to this question is in the affirmative: the interior algebra of any separable metric space which is dense in itself is functionally free. Hence, every topological equation which holds in Euclidean space of a given number of dimensions also holds in every other topological space.

However, for a deeper study of topology in an algebraic framework interior algebras prove to be too coarse an instrument. For instance, even a basic notion like the derivative of a set cannot be defined in terms of the interior operator. A possible approach, which was suggested in McKinsey and Tarski [44] and realized in Pierce [70], would be to consider Boolean algebras endowed with more operations of a topological nature than just the interior operator. That will not be the course taken here. We shall stay with the interior algebras, not only because the algebraic theory of these structures is interesting, but also since interior algebras, rather unexpectedly, appear in still another branch of mathematics, namely, in the study of certain non-classical, so-called modal logics.

Algebraic structures arising from logic have received a great deal of attention in the past. As early as in the 19th century George Boole initiated the study of the relationship between algebra and classical propositional logic, which resulted in the development of what we now know as the theory of Boolean algebras, a subject which has been studied very thoroughly. In the twenties and thirties several new systems of propositional logic were introduced, notably the intuitionistic logic, created by Brouwer and Heyting [30], various systems of modal logic, introduced by Lewis (see Lewis and Langford [32]), and many-valued logics, proposed by Post [21] and Lukasiewicz. The birth of these non-classical logics stimulated investigations into the relationships between these logics and the corresponding classes of algebras as well as into the structural properties of the algebras associated with these logics. The algebras turn out to be interesting not only from a logical point of view, but also in a purely algebraic sense, and structures like Heyting algebras, Brouwerian algebras, distributive pseudo-complemented lattices, Post algebras and Lukasiewicz algebras have been studied intensively. The algebras corresponding to certain systems of modal logic have received considerable attention, too, and it was shown in McKinsey and Tarski [48] that the algebras corresponding to Lewis's modal system S4 are precisely the interior algebras, the subject of the present treatise. Although no mention will be made of modal logics anywhere in this paper, it seems appropriate to say a few words about the nature of the connection of interior algebras with these logics, in order to facilitate an interpretation of the mathematical results of our work in logical terms.

2 Relation to modal logic

The vocabulary of the language $\mathcal{L}$ of the classical propositional calculus consists, as usual, of infinitely many propositional variables $p,q,r,\ldots$ and of the symbols for the logical operators: $\vee$ for disjunction, $\wedge$ for conjunction, $\sim$ for negation, the truth symbol 1 and the falsehood symbol 0. From these symbols the formulas (which are the meaningful expressions) are formed in the usual way. Every formula ϕ in $\mathcal{L}$ can be interpreted as an algebraic function $\hat{\phi}_L$ on a Boolean algebra L by letting the variables range over L and by replacing $\vee, \wedge, \sim$ with $+, ., '$ respectively. A formula is called valid (also: a tautology) if $\Phi_{\underline{2}} \equiv 1$, where $\underline{2}$ denotes the two element Boolean algebra. It is well-known that a formula ϕ is a tautology if and only if $\Phi_L \equiv 1$ for every Boolean algebra L. An axiomatization of the classical propositional calculus consists of a recursive set of special tautologies, called axioms, and a finite set of rules of inference, such that the derivable formulas - the theorems of the system - are precisely the tautologies.

The need for a refinement of the somewhat crude classical logic which led to the invention of the several modal logics arose, in particular, in connection with deficiencies felt in the formal treatment of the intuitive notion of implication. In classical propositional logic the implication $p \Rightarrow q$ is treated as an equivalent of $\sim p \vee q$, which leads to theorems like

$$p \Rightarrow (q \Rightarrow p)$$

and

$$(p \Rightarrow q) \vee (q \Rightarrow p)$$

which do not seem to be fully compatible with the intuitive notion of implication. In modal logic the language L is enriched by three logical operators to obtain the language L_M: a binary operator $\prec$, to be read as 'strictly implies' a unary operator $\Box$ for "it is necessary that" and a unary operator $\Diamond$ for " it is possible that". Laws governing $\prec$ are formulated intending to give it the desired properties of intuitive implication while avoiding "paradoxical" theorems like those holding for the usual implication. In many systems $\Box$ is now taken as a primitive operator, in which case $p \prec q$ appears as $\Box (p \Rightarrow q)$ and $\Diamond p$ as $\sim \Box \sim p$. The sense of the formula $\Box p$, to be read as "it is necessary that p", can be indicated as follows. When we assert that a certain proposition is necessary we mean that the proposition could not fail, no matter what the world should happen to be like (to speak in Leibnizian terms: true in all possible worlds). However, there was no unanimity among logicians as to what the 'right' laws governing the modal operators were, as appears from the vast number of modal axiomatic systems which have been proposed. One of the more important systems is S4, introduced by Lewis.

Axioms governing the modal operators of S4 are the following:

(i) $\Box(p \to q) \to (\Box p \to \Box q)$

(ii) $\Box p \to p$

(iii) $\Box p \to \Box\Box p$

These axiom schemas together with some axiomatization of the classical propositional calculus and some rules of inference among which the rule that if α is a theorem of S4 then so is $\Box\alpha$, constitute an axiomatization of S4. The following observation will clarify the relation of this system with the notion of interior algebra.

Let V denote any set of propositional variables and $\Phi(V)$ the set of all modal formulas formed from V by using the logical operators $\vee,\wedge,\sim,\Box,0,1$. Since $\Phi(V)$ is closed under these operators the structure

$$F(V) = (\Phi(V),(\vee,\wedge,\sim,\Box,0,1))$$

is an algebra, referred to as the free algebra of formulas in the language L_M. No algebraic equation formulated in terms of the fundamental operations is identically satisfied in $\Phi(V)$ unless it is a pure tautology of the form $p = p$, so that for example the operations $\vee$ and $\wedge$ are neither commutative nor associative. From an algebraic point of view, $F(V)$ presents but little interest. Let us therefore define a relation $\sim_{S4}$ on $\Phi(V)$ by putting, for $\phi,\psi \in \Phi(V)$

$$\phi \sim_{S4} \psi \quad \text{iff} \quad (\psi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi) \text{ is a theorem of S4.}$$

The relation $\sim_{S4}$ is an equivalence relation on $\Phi(V)$ and in fact, it is a congruence relation, hence we can form the quotient algebra $F(V)_{S4} = F(V)/\sim_{S4}$. We refer to this algebra as the canonical algebra for S4, and as one easily verifies, this algebra proves to be an interior algebra. The theorems of S4 are the formulas in $\Phi(V)$ which belong to the equivalence class containing the truth symbol, 1.

If $(L,(+,.,',^{o},0,1))$ is an arbitrary interior algebra and ϕ is any formula in L_M, then, just as in the Boolean case, ϕ can be interpreted as an algebraic function $\hat{\phi}_L$ on L, where in addition $\Box$ is now replaced by ^{o}. It is easily seen that for any theorem ϕ of S4, $\hat{\phi}_L \equiv 1$ on L. Indeed, the interpretations of the axioms of S4 are valid by the laws (i)' - (iii)' in the definition of interior operator, whereas the rule of inference "if ϕ is a theorem then so is $\Box\phi$" corresponds to the equation $1^0 = 1$. The remaining axioms and rules of inference are classical. Conversely, if ϕ is not a theorem of S4, then $\hat{\phi}_{F(V)_{S4}} \not\equiv 1$ on the interior algebra $F(V)_{S4}$, when V is large enough. We arrive at the conclusion that a modal formula ϕ is a theorem of S4 iff $\hat{\phi}_L \equiv 1$ on every interior algebra L.

Now suppose that S is a logic obtained from S4 by adding some set of axioms A (formulas in L_M) to the axioms of S4. Clearly, for each theorem ϕ of S, $\hat{\phi}_L \equiv 1$ for every interior algebra L in which the interpretation of the formulas of A is valid. And by considering algebras $F(V)_S$ whose definition is similar to that of $F(V)_{S4}$ we infer that the converse holds as well. Hence a formula ϕ in L_M is a theorem of S iff $\hat{\phi}_L \equiv 1$ in the class $\underline{K}$ of interior algebras satisfying the interpretations of the axioms in A. Apparently, such a class $\underline{K}$ is determined by the set of equations $\hat{\phi} = 1$, $\phi \in A$, that means, $\underline{K}$ is an equational class, also called a variety. We conclude that every extension of S4 (of the considered type) is completely determined by a certain subvariety of the variety of interior algebras, and since on the other hand every variety of interior algebras gives rise to such an extension of the system S4, the study of these extensions of S4 reduces wholly to the study of subvarieties of the variety of interior algebras. And varieties of algebras are particularly nice to work with, for example, because they are closed under certain general operations frequently used to construct new algebras from given ones, namely the operations of forming subalgebras, homomorphic images and direct products. By a well-known result due to Birkhoff the varieties are precisely those classes of algebras which have all three of these closure properties.

In spite of the fact that the algebraic interpretation proved to be a useful instrument to study several modal systems, notably in the work of McKinsey and Tarski [48], Dummet and Lemmon [59], Lemmon [66], Bull [66] and Rasiowa (see Rasiowa [74]), it has remained a method neglected by most logicians working in this area. A partial explanation may be found in the invention of a different semantics by Kripke [63] [65], which did provide a manageable tool to investigate modal logics and to create some order in this somewhat chaotic field and which, moreover, is intuitively more appealing than the algebraic interpretation. The results of our work will show that the algebraic approach, primarily because it permits us to invoke powerful methods from universal algebra, is in fact a very succesful one, in as much as it provides a clearer and more complete picture of the pattern formed by the various modal systems of a certain kind. Although we shall restrict ourselves to interior algebras, it seems that the algebraic approach might be fruitful in the study of more general modal systems as well. And it need hardly be observed that the correspondence between certain extensions of a given logical system and subvarieties of the variety of algebras associated with that logic is not limited to modal logics. A similar relation exists, for instance, between the so-called intermediate logics, i.e. the extensions of the intuitionistic propositional calculus, and the subvarieties of the variety of Heyting algebras.

3 The subject matter of the paper

The present work contains, aside from an introductory "Preliminaries", three chapters. The first two deal primarily with the algebraic theory of interior algebras proper, in the last one we concern ourselves with the lattice of subvarieties of the variety of interior algebras.

When investigating the structure of algebras in a given variety it is of particular interest to find an answer to the question as to how the (finitely generated) free objects look. The variety of Heyting algebras is closely related to the variety of interior algebras since, as noticed earlier in this introduction, the lattice of open elements of an interior algebra is a Heyting algebra, and conversely, every Heyting algebra may be obtained as the lattice of open elements of some interior algebra. The structure of the free object on one generator in the variety of Heyting algebras has been known for some time (Rieger [57]) and is easy to visualize (see the diagram on page 32 of this dissertation).

On the other hand, Urquhart [73]'s work shows that the free objects on more than one generator in the variety of Heyting algebras are of a great complexity and extremely difficult to describe. Since the Heyting algebra of open elements of a free interior algebra on a given number of generators is easily seen to contain a free Heyting algebra on the same number of generators as a subalgebra, it seems wise to restrict oneself to the problem of determining the free interior algebra on one generator. Easy as it may be to formulate, this problem proves to be a very difficult one, and indeed, large portions of the first two chapters of our work may be seen as an outgrowth of various attempts to get nearer to its solution.

At several points in the theory of interior algebras it appears that the 0 element of an interior algebra, as a nullary operation, is from an algebraic point of view a slightly disturbing element in that it tends to obscure what really is going on. As a matter of fact, a similar phenomenon occurs in the study of Heyting algebras and for that reason some authors have preferred to work with so-called Brouwerian algebras instead, which are, loosely speaking, Heyting algebras not necessarily possessing a least element. As an illustration, the free Brouwerian algebra on one generator is just the two element Boolean algebra; in the - infinite - free Heyting algebra on one generator the 0 element acts as some special generator, besides the free generator, and it is thus a homomorphic image of the free Brouwerian algebra on two generators. We have therefore introduced in addition to the variety $\underline{B}_i$ of interior algebras the variety $\underline{B}_i^-$ of generalized interior algebras. Here we understand by a generalized interior algebra an algebra $(L,(+,.,\Rightarrow,{}^{o},1))$ such that $(L,(+,.,\Rightarrow,1))$ is a generalized Boolean algebra with a largest element 1 (but possibly without a least element), and such that ^{o} is again an interior operator on L. The set of open elements of a generalized interior algebra is a Brouwerian algebra. The fact that the interior operator on a generalized Boolean algebra is not definable in terms of a closure operator on the same algebra is another explanation for our preference to take the interior operator as the basic notion in the definition of interior algebra, rather than the closure operator. In several respects, the theory of generalized interior algebras develops in a much smoother way than the theory of interior algebras, and it turns out that in the description of the free objects in some varieties of interior algebras, undertaken in the second chapter, the free objects in corresponding varieties of generalized interior algebras serve as a seemingly indispensable auxiliary device.

In the first two sections of Chapter I some basic properties of generalized interior algebras and interior algebras are established, in particular regarding the lattices of open elements. It is shown that every Brouwerian algebra can be embedded as the lattice of open elements of its free Boolean extension, the latter being endowed with a suitable interior operator. This result generalizes a similar theorem by McKinsey and Tarski [44] for Heyting algebras. These (generalized) interior algebras, which, as (generalized) Boolean algebras, are generated by their lattices of open elements, play an important role in our discussion and, therefore, deserve a special name: we shall call them $\star$-algebras. Among the finite interior algebras the $\star$-algebras distinguish themselves by the fact that they are precisely the ones which satisfy, speaking in topological terms, the T_0 separation axiom.

The next four sections are devoted to an investigation of the free objects on finitely many generators in $\underline{B}_i$ and $\underline{B}_i^-$. As it appears, even the free generalized interior algebra on one generator, denoted by $F_{\underline{B}_i^-}(1)$, is of an exceedingly complex structure. For example, it can be seen to have continuously many homomorphic images on the one hand, and to contain as a subalgebra the $\star$-algebra whose lattice of open elements is the free Heyting algebra on n generators, for every natural number n, on the other hand. These facts indicate that the problem to characterize $F_{\underline{B}_i^-}(1)$, let alone $F_{\underline{B}_i}(1)$, will be a difficult one.

In this connection, the question arises what the actual content is of McKinsey and Tarski [44]'s theorem which says that no finitely generated free interior algebra is functionally free. It turns out that as far as the lattice of open elements of the free interior algebra on finitely many generators is concerned, this non-functionally freeness is rather inessential, in the sense that by dropping the 0 as a nullary operation, that is, by regarding this lattice of open elements as a

Brouwerian algebra, it becomes a functionally free Brouwerian algebra. As for $F_{\underline{B}_i}(n)$ itself, the situation is different. We show that there exists an increasing chain of subvarieties $\underline{T}_n^-$, n = 1,2, ... of $\underline{B}_i^-$, defined in a natural way, each of which is properly contained in the next one, such that $F_{\underline{B}_i^-}(n)$ is functionally free in $\underline{T}_n^-$. We infer that $F_{\underline{B}_i^-}(n)$ is not functionally free in $\underline{B}_i^-$ and McKinsey and Tarski's theorem follows as an immediate corollary.

One of the reasons to turn our attention to some special subvarieties of $\underline{B}_i$ and $\underline{B}_i^-$, as we do in Chapter II, is the hope that we might be able to describe the free objects in these smaller varieties and might thus obtain knowledge useful to our original aim, the characterization of free objects in $\underline{B}_i$ and $\underline{B}_i^-$. A natural candidate for such an investigation would be the class of all $\star$-algebras, because $\star$-algebras have many pleasant properties and at the same time form a class which is not too restricted in the sense that still every Heyting algebra or Brouwerian algebra occurs as the lattice of open elements of some (generalized) interior algebra in the class. Unfortunately however, the class of $\star$-algebras is not a variety and does not possess any free objects on one or more generators. Therefore the varieties $\underline{B}_i^\star$ and $\underline{B}_i^{-\star}$ are introduced, defined to be the smallest subvarieties of $\underline{B}_i$ and $\underline{B}_i^-$ respectively, containing all $\star$-algebras. These varieties, which are proper subvarieties of $\underline{B}_i$ and $\underline{B}_i^-$, have a lot in common with the varieties of Heyting algebras and Brouwerian algebras; for example, whereas $\underline{B}_i$ has no non-trivial injectives, $\underline{B}_i^\star$ turns out to have essentially the same injectives as the variety of Heyting algebras has. It is regrettable that a description of $F_{\underline{B}_i^\star}(1)$ is still beyond our reach, but at least we are able to determine the free object on one generator in $\underline{B}_i^{-\star}$, which proves to be an infinite algebra, though one of a fairly simple structure.

In the remaining part of Chapter II we pay attention to some varieties of (generalized) interior algebras which are characterized by the fact that their lattices of open elements belong to a certain variety of Heyting, respectively Brouwerian, algebras. We think of varieties of Heyting, respectively Brouwerian, algebras which satisfy the equation $x \to y + y \to x = 1$, known under the name of relative Stone algebras, and some of their subvarieties. Because of the strong structural properties of the subdirectly irreducibles in these varieties we succeed in giving a characterization of the finitely generated free objects in them.

In the third chapter we shift our interest from the proper algebraic study of (generalized) interior algebras to an investigation of the set Ω of subvarieties of $\underline{B}_i$. The set is partially ordered by the inclusion relation and it is easy to see that this partial order induces a lattice structure on Ω. The trivial variety, that is, the variety containing one-element algebras only, is the 0-element of the lattice, $\underline{B}_i$ itself is the 1-element. The unique equationally complete subvariety of $\underline{B}_i$, the variety generated by the two element interior algebra, is contained in every non-trivial variety and hence is the unique atom of Ω.

Though Ω is fairly simple at the bottom, going up, its structure gets highly complex. An important tool for further investigation is provided by a deep result obtained by B. Jónsson [67] for varieties of algebras whose lattices of congruences are distributive, a requirement met by interior algebras. From his work we obtain as immediate corollaries that the lattice Ω is distributive, that $\underline{B}_i$ does not cover any variety (i.e. no subvariety of $\underline{B}_i$ is an immediate predecessor of $\underline{B}_i$ with respect to the partial order induced by the inclusion relation) and that every variety in Ω is covered by some variety in Ω. But also in the subsequent discussion, where we deal with cardinality problems and examine the property of a variety to be generated by its finite members, Jónsson's lemma continues to serve as the main device, as it does in the discussion of the important notion of a splitting variety. A splitting variety is characterized by the property that it is the largest variety not containing a certain finite subdirectly irreducible algebra. Using the concept of splitting variety we are able to give a satisfactory characterization of the locally finite subvarieties of $\underline{B}_i$, i.e. the subvarieties of $\underline{B}_i$ in which the finitely generated algebras are finite, and to describe some principal ideals of Ω in full detail. More interestingly, it is shown that the variety $\underline{B}_i^*$ is the intersection of two splitting varieties. This result would assume a somewhat more elegant form when treated in the framework of $\underline{B}_i^-$: the variety $\underline{B}_i^{-*}$ is a splitting variety, namely, the largest variety not containing the "smallest" non $*$-algebra, the interior algebra $\underline{2}^2$ whose only open elements are 0,1. In fact, $\underline{B}_i^{-*}$ is the first element of an increasing chain of splitting varieties $\underline{T}_n^-$, $n = 0,1,\ldots$ associatied with the interior algebras $\underline{2}^{2^n+1}$, $n = 0,1,2,\ldots$ whose only open elements are 0,1. The $\underline{T}_n^-$, $n = 1,2,\ldots$ are precisely the varieties mentioned earlier in this introduction, for which the $F_{\underline{B}_i}-(n)$ are functionally free. Equations determining a given splitting variety are easily found, hence these results also settle the problem of finding equations defining the variety $\underline{B}_i^*$. And it is interesting to note that the equation for $\underline{B}_i^*$ we arrive at is well-known among modal logicians. The axiom we have in mind reads $\Box(\Box(\Box p \Rightarrow p) \Rightarrow p) \Rightarrow p$. Thus the algebras in $\underline{B}_i^*$ are the algebras corresponding to the modal logic obtained from S4 by adding this axiom (denoted alternatively S4 Dym, Kl.1, S4 Grz). And our slightly unexpected result that the lattice of subvarieties of the variety of Heyting algebras is isomorphic to the lattice of subvarieties of $\underline{B}_i^*$ means, interpreted in logical terms, that the extensions of S4 containing this axiom as a theorem are precisely those which are determined by their intuitionistic content.

CHAPTER 0

PRELIMINARIES

Section 1. Universal Algebra

In the following we shall give a concise survey of notions and results of universal algebra which will be needed in this paper.

The usual set theoretic notation will be used. In particular, if A is a set, $|A|$ will denote its cardinality. N will denote the set of natural members $\{1,2,3\ldots\}$, Z the set of integers, and N^* the set of nonnegative integers. If $n \in N$, then $\underline{n} = \{0,1,\ldots n-1\}$; ω denotes the order type of the natural numbers, ω^* the order type of the negative integers. Finally, $\subseteq$ is used to denote inclusion, $\subset$ is used to denote proper inclusion.

In order to establish the algebraic notation we shall use we recall the definitions of similarity type and algebra.

1.1 Definition. A similarity type τ is an m-tuple $(n_1, n_2, \ldots n_m)$ of non-negative integers. The order of τ, $o(\tau)$, is m.

For every i, $1 \le i \le o(\tau)$, we have a symbol $\underline{f}_i$ of an n_i-ary operation.

1.2 Definition. An algebra of type τ is a pair (A,F), where A is a non-empty set and $F = (f_1, f_2, \ldots f_{o(\tau)})$ such that for each i, $1 \le i \le o(\tau)$, f_i is an n_i-ary operation on A. f_i is the realization of $\underline{f}_i$ in (A,F).

If there is no danger of confusion, we shall write A for (A,F).

For the notions of subalgebra, homomorphism and isomorphism, direct product, congruence relation and other notions not defined, we refer to Grätzer [68], where also proofs of most of the results to be mentioned in this section may be found.

1.3 Classes of algebras

When talking about a class of algebras we shall always assume that the class consists of algebras of the same similarity type.

Let $\underline{K}$ be a class of algebras. We define:

- $I(\underline{K})$: the class of isomorphic copies of algebras in $\underline{K}$
- $S(\underline{K})$: the class of subalgebras of algebras in $\underline{K}$
- $H(\underline{K})$: the class of homomorphic images of algebras in $\underline{K}$
- $P(\underline{K})$: the class of direct products of non-empty families of algebras in $\underline{K}$.

If $\underline{K} = \{A\}$ we write also $I(A)$, $S(A)$, $H(A)$ and $P(A)$. Instead of $B \in I(A)$ we usually write $B \cong A$ or sometimes $B \underset{\underline{K}}{\cong} A$ to emphasize that B and A are to be considered as algebras in $\underline{K}$.

A class $\underline{K}$ of algebras is called a variety or an equational class if $S(\underline{K}) \subseteq \underline{K}$, $H(\underline{K}) \subseteq \underline{K}$ and $P(\underline{K}) \subseteq \underline{K}$. If $\underline{K}$ consists of

1-element algebras only, then $\underline{K}$ is called a trivial variety.

1.4 Theorem. Let $\underline{K}$ be a class of algebras. The smallest variety containing $\underline{K}$ is $HSP(\underline{K})$.

We write often $V(\underline{K})$ instead of $HSP(\underline{K})$, $V(A)$ if $\underline{K} = \{A\}$, and we call $V(\underline{K})$ the variety generated by $\underline{K}$. If $\underline{K}$ and $\underline{K}'$ are varieties such that $\underline{K} \subseteq \underline{K}'$, then we say that $\underline{K}$ is a subvariety of $\underline{K}'$. If $\underline{K}$ is a variety, $A, B \in \underline{K}$, $f: A \longrightarrow B$ a homomorphism then we shall sometimes call f a $\underline{K}$-homomorphism in order to emphasize that f preserves all operations in A, considered as $\underline{K}$-algebra. If $A \in \underline{K}$, $S \subseteq A$, then $[S]$, or $[S]_{\underline{K}}$ if necessary, will denote the $\underline{K}$-subalgebra generated by S.

An algebra A is said to be a subdirect product of a family of algebras $\{A_s \mid s \in S\}$ if there exists an embedding $f: A \to \prod_{s\in S} A_s$ such that for each $s \in S$ $\pi_s \circ f$ is onto, where π_s is the projection on the s-th co-ordinate. If $\underline{K}$ is a class of algebras then $P_S(\underline{K})$ denotes the class of subdirect products of non-void families of algebras in $\underline{K}$.

An algebra A is called subdirectly irreducible if

(i) $|A| > 1$,

(ii) If A is a subdirect product of $\{A_s \mid s \in S\}$, then $\pi_s \circ f$ is an isomorphism for some $s \in S$.

If $\underline{K}$ is a class of algebras, $\underline{K}_{SI}$ ($\underline{K}_{FSI}$) will denote the class of (finite) subdirectly irreducibles in $\underline{K}$.

A useful characterization of the subdirectly irreducible algebras is the following:

1.5 Theorem. An algebra is subdirectly irreducible iff it has a least non-trivial congruence relation.

A classic result by G. Birkhoff [44] states:

1.6 Theorem. If $\underline{K}$ is a variety, then every algebra in $\underline{K}$ is a subdirect product of subdirectly irreducible algebras in $\underline{K}$. In symbols: if $\underline{K} = V(\underline{K})$, then $\underline{K} = P_S(\underline{K}_{SI})$.

According to theorem 1.6 every variety is completely determined by the subclass of its subdirectly irreducibles. The next theorem shows that even a smaller class will do:

1.7 Theorem. Let $\underline{K}$ be a variety. Then $\underline{K}$ is generated by the class of its finitely generated subdirectly irreducibles.

If $\underline{K}$ happens to be a variety in which every finitely generated algebra is finite (such a variety is called locally finite) then we have $\underline{K} = V(\underline{K}_{FSI})$.

1.8 Identities

1.9 Definition. Let $n \in N^*$. The n-ary polynomial symbols of type τ are defined as follows:

(i) $\underline{x}_1, \underline{x}_2, \ldots \underline{x}_n$ are n-ary polynomial symbols

(ii) if $\underline{p}_1, \underline{p}_2, \ldots \underline{p}_{n_i}$ are n-ary polynomial symbols and $1 \leq i \leq o(\tau)$ then $\underline{f}_i(\underline{p}_1, \underline{p}_2, \ldots \underline{p}_{n_i})$ is an n-ary polynomial symbol

(iii) the n-ary polynomial symbols are exactly those symbols which can be obtained by a finite number of applications of (i) and (ii).

If $\underline{p}$ is an n-ary polynomial symbol of type τ, then $\underline{p}$ induces on every algebra A of type τ a polynomial $p: A^n \longrightarrow A$ defined by:

(i) $\underline{x}_i$ induces the map $(a_1,a_2,\dots a_n) \longmapsto a_i$ for any $a_1,a_2,\dots a_n \in A$, $i = 1,2,\dots n$

(ii) if $\underline{p}_j$ induces p_j, $j = 1,2,\dots n_i$, $1 \le i \le o(\tau)$, then $\underline{f}_i(\underline{p}_1,\underline{p}_2,\dots \underline{p}_{n_i})$ induces $f_i(p_1,p_2,\dots p_{n_i})$

Conversely, every n-ary polynomial $p: A^n \longrightarrow A$ is induced by some polynomial symbol $\underline{p}$ on A. We shall often replace $\underline{x}_1,\underline{x}_2,\underline{x}_3,\dots$ by $\underline{x},\underline{y},\underline{z},\dots$ and usually omit _ from polynomial symbols if no confusion will arise.

1.10 <u>Definition</u>. Let $\underline{p}$, $\underline{q}$ be n-ary polynomial symbols of type τ. $\underline{p} = \underline{q}$ is called an <u>identity</u> or <u>equation</u> and is said to be <u>satisfied</u> in a class $\underline{K}$ of algebras of type τ (we write $\underline{K} \models \underline{p} = \underline{q}$) if for every $A \in \underline{K}$ the induced polynomials p and q are identical, or, equivalently, if $\forall A \in \underline{K}$, $\forall a_1, \forall a_2, \dots \forall a_n \in A$ $p(a_1,a_2,\dots a_n) = q(a_1,a_2,\dots a_n)$. If $\underline{K} = \{A\}$ we say that A satisfies $\underline{p} = \underline{q}$ and write $A \models \underline{p} = \underline{q}$.

If $\underline{K}$ satisfies a set of equations Σ, then so does the variety generated by $\underline{K}$, as identities are preserved under application of H, S and P. If Σ is a set of identities, let Σ^* denote *the class of algebras satisfying the identities in* Σ. The following theorem explains why a variety is also called an "equational class".

1.11 Theorem (Birkhoff [35]). A class of algebras $\underline{K}$ is a variety iff there exists some set of identities Σ such that $\underline{K} = \Sigma^*$.

If $\underline{K} = \Sigma^*$ then Σ is called a base for $\mathrm{Id}(\underline{K})$, where $\mathrm{Id}(\underline{K})$ is the set of identities satisfied by $\underline{K}$. In order to characterize the sets of identities which can be represented as $\mathrm{Id}(\underline{K})$ for some class of algebras $\underline{K}$, let us make the following definition.

1.12 Definition. A set of identities Σ is called closed provided

(i) $\underline{x}_i \equiv \underline{x}_i \in \Sigma$, $i = 1,2,\ldots$

(ii) if $\underline{p} \equiv \underline{q} \in \Sigma$, then $\underline{q} \equiv \underline{p} \in \Sigma$

(iii) if $\underline{p} \equiv \underline{q}$, $\underline{q} \equiv \underline{r} \in \Sigma$, then $\underline{p} \equiv \underline{r} \in \Sigma$

(iv) if $\underline{p}_i \equiv \underline{q}_i \in \Sigma$ for $i = 1,2,\ldots n_j$, then so is $\underline{f}_j(\underline{p}_1,\underline{p}_2,\ldots\underline{p}_{n_j}) \equiv \underline{f}_j(\underline{q}_1,\underline{q}_2,\ldots\underline{q}_{n_j})$

(v) If $\underline{p} \equiv \underline{q} \in \Sigma$, and we get $\underline{p}'$, $\underline{q}'$ from $\underline{p}$, $\underline{q}$ by replacing all occurences of $\underline{x}_i$ by an arbitrary polynomial symbol $\underline{r}$, then $\underline{p}' \equiv \underline{q}' \in \Sigma$.

1.13 Theorem (Birkhoff [35]). A set of identities Σ is closed iff $\Sigma = \mathrm{Id}(\underline{K})$ for some class of algebras $\underline{K}$.

1.14 Corollary. The assignment $\underline{K} \longmapsto \mathrm{Id}(\underline{K})$ establishes a 1 - 1 correspondence between varieties and closed sets of identities. If $\underline{K}$ and $\underline{K}'$ are varieties, then $\underline{K} \subseteq \underline{K}'$ iff $\mathrm{Id}(\underline{K}) \supseteq \mathrm{Id}(\underline{K}')$.

1.15 Definition. A set of identities Σ is called equationally complete if $\underline{x}_1 = \underline{x}_2 \notin \Sigma$ and $\Sigma \subseteq \Sigma'$, $\underline{x}_1 = \underline{x}_2 \notin \Sigma'$ imply $\Sigma = \Sigma'$. An equational class $\underline{K}$ is called equationally complete if $Id(\underline{K})$ is equationally complete.

1.16 Free algebras

1.17 Definition. Let $\underline{K}$ be a class of algebras. $A \in \underline{K}$ is said to be free over $\underline{K}$ if there exists a set $S \subseteq A$ such that

(i) $[S] = A$, i.e. A is generated by S

(ii) If $B \in \underline{K}$, $f: S \longrightarrow B$ a map, then there exists a homomorphism $g: A \longrightarrow B$ such that $g \mid S = f$.

We say that S freely generates A, and we write also $F_{\underline{K}}(S)$ for A. If $|S_1| = |S_2|$ and $F_{\underline{K}}(S_1)$, $F_{\underline{K}}(S_2)$ exist, then $F_{\underline{K}}(S_1) \cong F_{\underline{K}}(S_2)$. Therefore we write also $F_{\underline{K}}(|S|)$ instead of $F_{\underline{K}}(S)$. Note that the homomorphism g in (ii) is necessarily unique.

1.18 Theorem (Birkhoff [35]). Let $\underline{K}$ be a non-trivial variety. Then $F_{\underline{K}}(\underline{m})$ exists for any cardinal $\underline{m} > 0$.

1.19 Corollary. Let $\underline{K}$, $\underline{K}'$ be non-trivial varieties. Then $\underline{K} = \underline{K}'$ iff $F_{\underline{K}}(\aleph_o) \cong F_{\underline{K}'}(\aleph_o)$.

1.20 Corollary (Tarski [46]). A class of algebras $\underline{K}$ is a variety iff it is generated by a suitable algebra.

The last corollary is an immediate consequence of theorems 1.7 and 1.18: if $\underline{K}$ is a variety, then $\underline{K} = HSP(F_{\underline{K}}(\aleph_0))$.

1.21 Definition. If $\underline{K}$ is a variety, and $A \in \underline{K}$, such that $\underline{K} = V(A)$, then A is called functionally free in $\underline{K}$ or characteristic for $\underline{K}$.

Sometimes it will be necessary to consider an algebra generated as free as possible with respect to certain conditions. For our purposes it will be sufficient to restrict ourselves to finitely generated algebras.

1.22 Definition. Let $\underline{K}$ be a class of algebras, and let p_i, q_i, $i \in I$, be n-ary polynomial symbols, $n \in N^*$. The algebra A is said to be freely generated over $\underline{K}$ by the elements $a_1, a_2, \ldots a_n$ with respect to $\Omega = \{p_i = q_i \mid i \in I\}$ if

(i) $[\{a_1, a_2, \ldots a_n\}] = A$ and $A \in \underline{K}$

(ii) $p_i(a_1, a_2, \ldots a_n) = q_i(a_1, a_2, \ldots a_n)$ for $i \in I$

(iii) if $B \in \underline{K}$, $b_1, b_2, \ldots b_n \in B$ such that $p_i(b_1, b_2, \ldots b_n) = q_i(b_1, b_2, \ldots b_n)$ for $i \in I$, then the map $a_j \longrightarrow b_j$, $j = 1, 2, \ldots n$ can be extended to a homomorphism $f\colon A \longrightarrow B$. A will be denoted by $F_{\underline{K}}(n, \Omega)$.

Note that if the homomorphism f exists, it is necessarily unique. If $L \cong F_{\underline{K}}(n, \Omega)$ for some finite set Ω, then L is said to be finitely presentable.

1.23 Theorem. Let $\underline{K}$ be a variety. Then $F_{\underline{K}}(n, \Omega)$ exists for any $n \in N$ and for any Ω.

Note that the elements $a_1, a_2, \ldots a_n$ need not be different. $F_{\underline{K}}(n,\Omega)$ is unique up to isomorphism if it exists and $F_{\underline{K}}(n,\emptyset) \cong F_{\underline{K}}(n)$.

We have seen, that a variety of algebras is characterized (i) by its (finitely generated) subdirectly irreducibles (theorem 1.7), (ii) by a base for the identities satisfied by it (theorem 1.11) and (iii) by its free object on countably many generators. An important reason for our favoring the "subdirectly irreducibles approach" is a result obtained by B.Jónsson, which we shall discuss now.

1.24 Congruence distributive varieties

A class of algebras $\underline{K}$ is called congruence distributive if for all $A \in \underline{K}$ the lattice of congruences of A, denoted by $C(A)$, is distributive. If $\{A_i \mid i \in I\}$ is a non-empty set of algebras, $F \subseteq P(I)$ an ultrafilter on I and $\Theta(F)$ the congruence on $\prod_{i \in I} A_i$ defined by

$$x \equiv y\ \Theta(F) \quad \text{iff} \quad \{i \in I \mid x_i = y_i\} \in F$$

for any $x = (x_i)$, $y = (y_i) \in \prod_{i\in I} A_i$, then $\prod_{i\in I} A_i/\Theta(F)$ is called an ultra-product of $\{A_i \mid i \in I\}$. For properties of ultra-products, see Grätzer [68]. If $\underline{K}$ is a class of algebras, let $P_U(\underline{K})$ denote the class of ultra-products of non-empty families of algebras in $\underline{K}$.

1.25 Theorem (Jónsson [67]). Let $\underline{K}$ be a class of algebras such that $V(\underline{K})$ is congruence distributive. Then $V(\underline{K})_{SI} \subseteq HSP_U(\underline{K})$ and hence $V(\underline{K}) = P_S HSP_U(\underline{K})$.

1.26 Corollary. Let $A_1, A_2, \dots A_n$ be finite algebras and suppose that $V(\{A_1, A_2, \dots A_n\})$ is congruence distributive. Then

$$V(\{A_1, A_2, \dots A_n\})_{SI} \subseteq HS(\{A_1, A_2, \dots A_n\}) .$$

We may regard the class of varieties of a given similarity type as a lattice, the lattice product of varieties $\underline{K}_0$, $\underline{K}_1$ being defined to be the variety $\underline{K}_0 \cap \underline{K}_1$, denoted by $\underline{K}_0 \cdot \underline{K}_1$, the lattice sum $V(\underline{K}_0 \cup \underline{K}_1)$, denoted by $\underline{K}_0 + \underline{K}_1$. One could argue that varieties are not sets, and that one therefore cannot speak of the class of varieties. However, our terminology only intends to be suggestive; we could easily avoid this problem by representing a variety by the set containing one isomorphic copy of each finitely generated subdirectly irreducible belonging to it or, alternatively, by the set of identities satisfied by it.

1.27 Corollary. If $\underline{K}$, $\underline{K}'$ are varieties such that $\underline{K} + \underline{K}'$ is congruence distributive, then $(\underline{K} + \underline{K}')_{SI} = \underline{K}_{SI} \cup \underline{K}'_{SI}$.

1.28 Corollary. If $\underline{K}$ is a congruence distributive variety, then the lattice of subvarieties of $\underline{K}$ is distributive.

1.29 Equational categories

Sometimes we shall use the language of category theory. We shall be concerned with categories $\mathcal{K}$ whose objects are algebras belonging to a certain class $\underline{K}$ of similar algebras, and whose morphisms are all homomorphisms $f : A \longrightarrow B$, where A, B are objects of $\mathcal{K}$.

If $\underline{K}$ is a variety, then K is called an equational category. Note that in equational categories the categorical isomorphisms are precisely the algebraic isomorphisms. Furthermore, the momonorphisms are the 1-1 homomorphisms, but epimorphisms need not be onto. For further details we refer to Balbes and Dwinger [74].

Section 2. Lattices

We assume that the reader is familiar with the basic concepts of lattice theory, for which Balbes and Dwinger [74] or Grätzer [71] may be consulted. In this section we collect some topics which will be of special importance in our work.

2.1 Distributive lattices and (generalized) Boolean algebras

The following varieties will play an important role in our discussion:

$\underline{D}$ the variety of distributive lattices $(L,(+,.))$

$\underline{D}_1$ the variety of distributive lattices with 1 $(L,(+,.,1))$

$\underline{D}_{01}$ the variety of distributive lattices with 0,1 $(L,(+,.,0,1))$

$\underline{B}^-$ the variety of generalized Boolean algebras $(L,(+,.,\Rightarrow,1))$

$\underline{B}$ the variety of Boolean algebras $(L,(+,.,',0,1))$, where $+,\cdot$ denote sum and product respectively, 0 and 1 denote the smallest and largest element of L, $'$ complement, and where $\Rightarrow$ is a binary operation denoting "relative complement": $a \Rightarrow b$ is the complement of a in $[ab,1]$. Thus $\underline{B}^-$ has similarity type $(2,2,2,0)$ whereas $\underline{B}$ has similarity type $(2,2,1,0,0)$. If we wish to emphasize that the operations are supposed to be performed in L, we write also $+_L, \cdot_L, 0_L$ etc. or $+^L, \cdot^L, 0^L$ etc. Equations defining the classes $\underline{D}$, $\underline{D}_1$, $\underline{D}_{01}$ and $\underline{B}$ can be found in Balbes and Dwinger [74]; a system of equations defining $\underline{B}^-$ is e.g.

2.2

(i) usual equations for $\underline{D}_1$

(ii) $(x \Rightarrow y)x = xy$

$x \Rightarrow y + x = 1$.

Note that if $L \in \underline{B}^-$ has a smallest element a, then L can be considered as a Boolean algebra, a being the 0, and for any $x \in L$ $x' = x \Rightarrow a$. Conversely, every Boolean algebra L can be regarded as a generalized Boolean algebra, denoted by L^-, with $x \Rightarrow y = x' + y$ for $x,y \in L$. Often $\underline{B}$ and $\underline{B}^-$ will be treated as subclasses of $\underline{D}_{01}$ and $\underline{D}_1$.

2.3 We recall the notion of <u>free</u> <u>Boolean</u> <u>extension</u>. If $L \in \underline{D}_{01}$ then a free Boolean extension of L is a pair (L_1,f) where $L_1 \in \underline{B}$ and $f\colon L \longrightarrow L_1$ is a 1-1 $\underline{D}_{01}$-homomorphism such that if $L_2 \in \underline{B}$ and $g\colon L \longrightarrow L_2$ is a $\underline{D}_{01}$-homomorphism then there exists a unique $\underline{B}$-homomorphism $h\colon L_1 \longrightarrow L_2$ such that $h \circ f = g$. For every $L \in \underline{D}_{01}$ there exists such a free Boolean extension. In other words: there exists a reflector from $\mathcal{D}_{01}$ to $\mathcal{B}$. The free Boolean extension is unique, essentially; therefore we shall always assume that L is a $\underline{D}_{01}$-subalgebra of L_1 and that f is the inclusion map. The free Boolean extension of L will be denoted $B(L)$. Note that $B(L)$ is $\underline{B}$-generated by L, and if L is a $\underline{D}_{01}$-subalgebra of L_1, $L_1 \in \underline{B}$, then $[L]_{\underline{B}} = B(L)$.

The free generalized Boolean extension of a lattice $L \in \underline{D}_1$ is defined analogously: it is a pair (L_1, f), with $L_1 \in \underline{B}^-$ and $f\colon L \longrightarrow L_1$ a 1-1 $\underline{D}_1$-homomorphism such that whenever $L_2 \in \underline{B}^-$ and $g\colon L \longrightarrow L_2$ is a $\underline{D}_1$-homomorphism there exists a unique $\underline{B}^-$-homomorphism $h\colon L_1 \longrightarrow L_2$ such that $h \circ f = g$. It will be denoted by $B^-(L)$. Note that if L_1 is a $\underline{D}_1$-subalgebra of $L_1 \in \underline{B}^-$, then $[L]_{\underline{B}^-} = B^-(L)$.

2.4 Brouwerian algebras and Heyting algebras

If L is a lattice, $a,b \in L$, then the relative pseudo-complement of a with respect to b (if it exists) is $a \to b = \max\{x \mid ax \le b\}$. A <u>Brouwerian</u> lattice L is a lattice in which $a \to b$ exists for every $a,b \in L$. If L has a 0, L is called a <u>Heyting</u> lattice. The classes of Brouwerian lattices and Heyting lattices give rise to

$\underline{H}^-$ the variety of Brouwerian algebras $(L,(+,\cdot,\to,1))$

and

$\underline{H}$ the variety of Heyting algebras $(L,(+,\cdot,\to,0,1))$.

A system of equations defining $\underline{H}^-$ is

2.5

(i) equations for $\underline{D}_1$

(ii) $x \to x = 1$

$x(x \to y) = xy$

$xy \to z = x \to (y \to z)$

$(x \to y)y = y$.

Equations defining $\underline{H}$ are obtained by adding the identity $x.0 = 0$.

If $L \in \underline{H}$ then we may consider L to be a Brouwerian algebra L^- by disregarding the nullary operation 0 (not the element 0). Conversely if $L \in \underline{H}^-$, $0 \notin L$, then we define $0 \oplus L$ to be the Heyting algebra obtained by adding a smallest element 0 to L with the obvious changes in the definitions of the operations in $\{0\} \cup L$. Also, if $f: L \longrightarrow L_1$ is an $\underline{H}^-$-homomorphism then $\bar{f}: 0 \oplus L \longrightarrow 0 \oplus L_1$ defined by $\bar{f}(0) = 0$, $\bar{f} \mid L = f$ is an $\underline{H}$-homomorphism. Thus the assignment $L \longmapsto 0 \oplus L$, $f \longmapsto \bar{f}$ constitutes a covariant functor $\mathcal{H}^- \longrightarrow \mathcal{H}$.

If $n \in N$ then $\underline{n}$ will be used to denote the Heyting algebra $\{0,1,\ldots n-1\}$ with the operations induced by the usual linear order. Hence $\underline{n}^-$ denotes the corresponding Brouwerian algebra.

If L belongs to one of the varieties introduced, $S \subseteq L$, then $(S]$ and $[S)$ denote the ideal and filter generated by S respectively. Instead of $(\{a\}]$ and $[\{a\})$ we write $(a]$ and $[a)$; $(a]$ is called a principal ideal, $[a)$ a principal filter. If $a,b \in L$ then $[a,b] = \{x \in L \mid a \le x \le b\}$. $\mathcal{I}(L)$ will denote the lattice of ideals of L , $\mathcal{F}(L)$ will denote the lattice of filters of L .

2.6 If $L_1, L_2 \in \underline{H}$, then $L_1 \dagger L_2$ stands for the Heyting algebra which is obtained by putting L_2 "on top of" L_1, identifying 1_{L_1} with 0_{L_2}. Thus $L_1 \dagger L_2$ is a lattice which can be written as $(a] \cup [a)$ for some $a \in L_1 \dagger L_2$, such that $(a] \cong L_1$ and $[a) \cong L_2$ as lattices. Identifying $(a]$ with L_1 and $[a)$ with L_2 we have

$$x \underset{L_1 \dagger L_2}{\longrightarrow} y = \begin{cases} 1 & \text{if } x \in L_1,\ y \in L_2 \\ y & \text{if } x \in L_2,\ y \in L_1 \\ x \underset{L_i}{\rightarrow} y & \text{if } x,y \in L_i \text{ for } i = 1,2. \end{cases}$$

A similar operation can be performed if $L_1 \in \underline{H}^-$. Instead of $L \dagger \underline{2}$ we write also $L \oplus 1$. Recall that if $L \in \underline{H}$ or $L \in \underline{H}^-$ then L is subdirectly irreducible iff $L = L' \oplus 1$ for some $L' \in \underline{H}$, $L' \in \underline{H}^-$ respectively.

CHAPTER I

GENERAL THEORY OF (GENERALIZED) INTERIOR ALGEBRAS

In this chapter we develop a portion of the theory of (generalized) interior algebras. Having established the basic facts in sections 1,2 we devote most of our attention to the finitely generated (free) algebras (sections 3-5), also regarding their functional freeness (section 6). Section 7 closes the chapter with some remarks on free products, injectives and projectives.

Section 1. Generalized interior algebras: definitions and basic properties

In this section generalized interior algebras are defined and some of their basic properties are established. In 1.5 the congruence lattice of a generalized interior algebra is characterized, from which we obtain as a corollary a characterization of the subdirectly irreducible generalized interior algebras as well as the result that the class of generalized interior algebras is congruence distributive, a fact we shall use in the third chapter. After some considerations concerning homomorphic images and subalgebras of generalized interior algebras we prove some important theorems dealing with the relation between generalized interior algebras and their lattices of open elements (1.12-1.18). It is shown that for any Brouwerian algebra L the Boolean extension $B^{-}(L)$ of L can be endowed with an interior operator such that the set of open elements in this algebra is precisely L. These generalized interior algebras have several nice properties and will play an important role in the sequel. For lack of a better name we shall call them $*$-algebras.

1.1 Definition. Let $(L,(+,.,1))$ be a lattice with 1. A unary operation $^{o}: L \longrightarrow L$ is called an interior operator if for all $x,y \in L$

$$\text{(i)} \quad 1^{o} = 1$$

$$\text{(ii)} \quad x^{o} \leq x$$

$$\text{(iii)} \quad x^{oo} = x^{o}$$

$$\text{(iv)} \quad (x.y)^{o} = x^{o}.y^{o}$$

1.2 Definition. A generalized interior algebra is an algebra $(L,(+,.,\Rightarrow,{}^{o},1))$ such that $(L,(+,.,\Rightarrow,1))$ is a generalized Boolean algebra and ^{o} is an interior operator on L.

It is clear that the class of generalized interior algebras is equationally definable: the equations given in 0.2.2 and 1.1 provide an equational base. The variety of generalized interior algebras will be denoted by $\underline{B}_i^{-}$.

A typical example of a generalized interior algebra is the generalized Boolean algebra of all subsets of a topological space whose interior is dense in the space, endowed with the (topological) interior operator. In fact, it can be shown that any generalized interior algebra is isomorphic with a subalgebra of some generalized interior algebra of this kind.

If $L \in \underline{B}_i^-$, then an element x of L is said to be <u>open</u> if $x^o = x$ and the set of open elements is denoted by L^o. Obviously, $L^o = \{x^o \mid x \in L\}$ and it is readily seen that L^o is a $\underline{D}_1$-sublattice of L. Furthermore, L^o is a Brouwerian lattice:

1.3 <u>Theorem</u>. Let $L \in \underline{B}_i^-$ and for $a,b \in L^o$ let $a \to b = (a \Rightarrow b)^o$. Then $(L^o,(+,.,\to,1))$ is a Brouwerian algebra.

<u>Proof</u>. We verify that $(a \Rightarrow b)^o$ is the relative pseudocomplement of a with respect to b in L^o. Indeed, $a(a \Rightarrow b)^o \le a(a \Rightarrow b) = ab \le b$, and if $y \in L^o$, $ay \le b$, then $y \le a \Rightarrow b$, hence $y \le (a \Rightarrow b)^o$.□

The next proposition tells us which $\underline{D}_1$-sublattices of a generalized Boolean algebra can occur as the lattices of open elements associated with some interior operator:

1.4 <u>Theorem</u>. Let $L \in \underline{B}^-$, L_1 a $\underline{D}_1$-sublattice of L. There exists an interior operator ^{o} on L such that $L_1 = L^o$ iff for all $a \in L$ $(a] \cap L_1$ has a largest element.

<u>Proof</u>. (i) $\Rightarrow$ a^o satisfies the requirement.

(ii) $\Leftarrow$ Define for any $x \in L$, $x^o = \max (x] \cap L_1$. Then (i) $1^o = 1$, (ii) $x^o \le x$, (iii) $x^{oo} = x^o$ and (iv) $(xy)^o = \max (xy] \cap L_1 = \max (x] \cap L_1 . \max (y] \cap L_1 = x^o y^o$. □

It follows from the proof of the theorem that the interior operator with the property that $L^o = L_1$ is necessarily unique. Note also that in particular for every generalized Boolean algebra L and every finite $\underline{D}_1$-sublattice of L there exists an interior operator on L such that $L^o = L_1$.

If $L \in \underline{B}_i^-$, $F \in F(L)$, then F is called an <u>open filter</u> if for all $x \in F$, $x^o \in F$. The lattice of open filters of L is denoted by $F_o(L)$. A principal filter $[a)$ is open iff $a^o = a$.

1.5 <u>Theorem</u>. Let $L \in \underline{B}_i^-$. Then

$$\text{(i)} \quad C(L) \cong F_o(L)$$

$$\text{(ii)} \quad F_o(L) \cong F(L^o) .$$

<u>Proof</u>. (i) If $\Theta \in C(L)$, let $F_\Theta = \{x \mid (x,1) \in \Theta\}$. Evidently $F \in F_o(L)$. Conversely, if $F \in F_o(L)$, it is easy to verify, that $\Theta_F = \{(x,y) \mid (x \Rightarrow y)(y \Rightarrow x) \in F\} \in C(L)$. Let $f\colon C(L) \longrightarrow F_o(L)$ be defined by $\Theta \longmapsto F_\Theta$ and $g\colon F_o(L) \longrightarrow C(L)$ by $F \longmapsto \Theta_F$. Then $f \circ g$ and $g \circ f$ are the identity mappings and f,g are both order preserving. Thus f establishes a lattice isomorphism between $C(L)$ and $F_o(L)$.
(ii) Let $f\colon F_o(L) \longrightarrow F(L^o)$ be defined by $F \longmapsto F \cap L^o$, $g\colon F(L^o) \longrightarrow F_o(L)$ by $F \longmapsto [F)$. Again, $f \circ g$ and $g \circ f$ are the identity mappings and f,g are order preserving, hence f,g are isomorphisms.□

1.6 <u>Corollary</u>. Let $L \in \underline{B}_i^-$. Then $C(L) \cong C(L^o)$, where L^o is considered as a Brouwerian algebra.
<u>Proof</u>. If $L \in \underline{H}^-$, then $C(L) \cong F(L)$.□

1.7 <u>Corollary</u>. If $L \in \underline{B}_i^-$, then L is subdirectly irreducible iff L^o is a subdirectly irreducible Brouwerian algebra. Thus $L \in \underline{B}_i^-$ SI iff $L^o \cong L_1 \oplus 1$, where $L_1 \in \underline{H}^-$.
<u>Proof</u>. By 1.6 and 0.1.5. For the second remark, cf. 0.2.6.□

1.8 <u>Corollary</u>. The variety $\underline{B}_i^-$ is congruence-distributive.

We recall that a variety $\underline{K}$ has the <u>congruence extension property</u>

(CEP) if for all $L \in \underline{K}$ and for all $L_1 \in S(L)$, for each $\Theta_1 \in C(L_1)$ there exists a $\Theta \in C(L)$ such that $\Theta \cap L_1^2 = \Theta_1$. If $\underline{K}$ has CEP, then for all $L \in \underline{K}$ $HS(L) = SH(L)$.

1.9 Corollary. $\underline{B}_i^-$ has CEP.

Proof. If $L_1 \in S(L)$, $L \in \underline{B}_i^-$, $\Theta_1 \in C(L_1)$, then $\Theta_{[F_{\Theta_1})}$ is the desired extension.□

If $F \in F_o(L)$, $L \in \underline{B}_i^-$, then the quotient algebra with respect to Θ_F will be denoted by L/F and the canonical projection by $\pi_F\colon L \longrightarrow L/F$. Thus for $x \in L$ $\pi_F(x) = \{y \in L \mid (x \Rightarrow y)(y \Rightarrow x) \in F\}$ and in particular $1_{L/F} = \pi_F(1) = F$. Furthermore, if $h\colon L \longrightarrow L_1$ is a homomorphism, $L, L_1 \in \underline{B}_i^-$, which is onto, then $L/F \cong L_1$, where $F = h^{-1}(\{1\})$.

1.10 Every open filter of a generalized interior algebra is also a subalgebra of it. If $L \in \underline{B}_i^-$, $a \in L^o$, then $L_1 = \{a \Rightarrow x \mid x \in L\}$ is a $\underline{B}^-$-subalgebra of L, but in general not a $\underline{B}_i^-$-subalgebra of L, since not necessarily $(a \Rightarrow x)^o = a \Rightarrow y$ for some $y \in L$. But we can provide L_1 with an interior operator ^{o_1}, by defining for $x \in L$ $(a \Rightarrow x)^{o_1} = a \Rightarrow x^o$. It is a matter of easy verification to check that ^{o_1} is well-defined and that it satisfies the requirements (i)-(iv) of 1.1. The map $h_a\colon L \longrightarrow L_1$ defined by $h_a(x) = a \Rightarrow x$ is now a $\underline{B}_i^-$-homomorphism with kernel $\{x \mid a \Rightarrow x = 1\} = [a)$. If in addition $(a \Rightarrow x)^{o_1} = a \Rightarrow x^o$, then L_1 is even a $\underline{B}_i^-$-subalgebra of L, and h_a a $\underline{B}_i^-$-endomorphism.

Similarly, an arbitrary principal filter $[a)$ of a generalized interior algebra can be endowed with an interior operator ^{o_1} by defining $x^{o_1} = x^o + a$ for arbitrary $x \in [a)$.

1.11 If $L \in \underline{B}_i^-$, $a \in L^o$, then $(a]$ can be made into a generalized interior algebra, too. Indeed, define for $x,y \in (a]$ $x \underset{(a]}{\Rightarrow} y = (x \Rightarrow y).a$ and $x^{o(a]} = x^o$. Then $((a],(+,.,\underset{(a]}{\Rightarrow},{}^{o(a]},a))$ is a generalized interior algebra, and the map $f\colon L \longrightarrow (a]$ defined by $x \longmapsto x.a$ is a $\underline{B}_i^-$-homomorphism. Since $f^{-1}(\{a\}) = [a)$, $(a] \cong L/[a)$. In a similar way we define for $a,b \in L^o$, $a \leq b$, a (generalized) interior algebra $[a,b] = \{x \in L \mid a \leq x \leq b\}$. Note that $(a] \in H(L)$, $[a) \in S(L)$, and $[a,b] \in HS(L)$. It is not difficult to verify that if L has a smallest element 0, then $L \cong (a] \times (a \Rightarrow 0]$ if $a, a \Rightarrow 0 \in L^o$.

To close this section we present some important facts concerning the relation between the classes $\underline{B}_i^-$ and $\underline{H}^-$, which are based on work by McKinsey and Tarski [46].

1.12 Theorem. Let $L,L_1 \in \underline{B}_i^-$, $h\colon L \longrightarrow L_1$ a $\underline{B}_i^-$-homomorphism. Then

(i) $h[L^o] \subseteq L_1^o$.

(ii) $h^o = h \mid L^o\colon L^o \longrightarrow L_1^o$ is an $\underline{H}^-$-homomorphism, and if h is onto, then h^o is onto.

Proof. (i) is obvious.

(ii) We verify that h^o preserves $\rightarrow$: $h^o(a \rightarrow b) = h((a \Rightarrow b)^o) = (h(a) \Rightarrow h(b))^o = h^o(a) \rightarrow h^o(b)$, for any $a,b \in L^o$. If h is onto, $y \in L_1^o$, and $x \in L$ such that $h(x) = y$, then $h^o(x^o) = (h(x))^o = y^o = y$, thus h^o is onto.$\square$

1.13 Corollary. The assignment $O^-\colon B_i^- \longrightarrow H^-$ given by $L \longmapsto L^o$ for $L \in \underline{B}_i^-$, $h \longmapsto h^o$ for $\underline{B}_i^-$-homomorphisms h, is a covariant functor which preserves 1-1 homomorphisms and onto-homomorphisms.

1.14 Theorem. Let $L \in \underline{H}^-$. There exists a unique interior operator on $B^-(L)$ such that $(B^-(L))^o = L$, which is defined as follows: if $a \in B^-(L)$, $a = \prod_{i=1}^{n}(u_i \Rightarrow v_i)$, where $u_i, v_i \in L$, then $a^o = \prod_{i=1}^{n}(u_i \to v_i)$.

In particular, it follows that $\mathcal{O}^-$ is representative.

Proof. Recall that for each $a \in B^-(L)$ there exist $u_i, v_i \in L$, $i=1 \ldots n$, such that $a = \prod_{i=1}^{n}(u_i \Rightarrow v_i)$. Now, if $u, v \in L$ then $\max((u \Rightarrow v] \cap L) = \max\{x \in L \mid xu \leq v\} = u \to v$, and therefore, if $a = \prod_{i=1}^{n}(u_i \Rightarrow v_i)$ then $\max((a] \cap L) = \prod_{i=1}^{n}(u_i \to v_i)$. The theorem follows now from 1.4.□

Henceforth $B^-(L)$ will denote the generalized interior algebra provided with the interior operator as defined in 1.14, for any $L \in \underline{H}^-$.

1.15 Definition. If $L \in \underline{B}_i^-$ is such that $L = B^-(L^o)$ then L is called a $*$-algebra.

1.16 Theorem. Let $L \in \underline{H}^-$, $L_1 \in \underline{B}_i^-$, $h: L \longrightarrow L_1^o$ an $\underline{H}^-$-homomorphism. Then there exists a unique $\underline{B}_i^-$-homomorphism $\bar{h}: B^-(L) \longrightarrow L_1$ such that $\bar{h} \mid L = h$.

Proof. There exists a unique $\underline{B}^-$-homomorphism $\bar{h}: B^-(L) \longrightarrow L_1$, extending h. If $a \in B^-(L)$ then $a = \prod_{i=1}^{n}(u_i \Rightarrow v_i)$, $u_i, v_i \in L$ and $\bar{h}(a^o) = h(\prod_{i=1}^{n}(u_i \to v_i)) = \prod_{i=1}^{n} h(u_i) \longrightarrow h(v_i) = (\prod_{i=1}^{n}(h(u_i) \Rightarrow h(v_i)))^o =$
$= (\bar{h}(a))^o$.□

1.17 Corollary. If $L \in \underline{B}_i^-$, L_1 an $\underline{H}^-$-subalgebra of L^o, then $[L_1]_{\underline{B}_i^-} = B^-(L_1)$.

1.18 Corollary. The assignment $\mathcal{B}^{-} : \mathcal{H}^{-} \longrightarrow \mathrm{B}_{i}^{-}$ given by $L \longmapsto B^{-}(L)$ for $L \in \underline{H}^{-}$ and $h \longmapsto \bar{h}$ for $\underline{H}^{-}$-homomorphisms h, is a covariant functor which preserves 1-1 homorphisms and onto homomorphisms. Furthermore, $\mathcal{B}^{-}$ is full embedding.

In fact, the functor $\mathcal{B}^{-}$ is a left adjoint of the functor $\mathcal{O}^{-}$.

Section 2. Interior algebras: definition, basic properties and relation with generalized interior algebras

Most of the results obtained in section 1 for generalized interior algebras hold mutatis mutandis for interior algebras as well. For future reference we list some of them without proof (2.3-2.17). In the second part of this section we establish a relationship between the classes $\underline{B}_i$ and $\underline{\bar{B}}_i$. It is shown that there exist a full embedding $\mathcal{B}_i^- \longrightarrow \mathcal{B}_i$ and a representative covariant functor $\mathcal{B}_i \longrightarrow \mathcal{B}_i^-$ (2.18).

We start now with the definition of interior algebra.

2.1. Definition. An interior algebra is an algebra $(L,(+,.,',^{o},0,1))$ such that $(L,(+,.,',0,1))$ is a Boolean algebra and ^{o} is an interior operator on L.

The class of interior algebras is determined by the usual equations defining the variety of Boolean algebras together with the equations in 1.1. The variety of interior algebras will be denoted by $\underline{B}_i$.

2.2 Associated with an interior operator ^{o} on a Boolean algebra is a closure operator ^{c}, defined by $x^{c} = x^{\prime o \prime}$ for $x \in L$. It satisfies the identities (i)' $0^{c} = 0$, (ii)' $x \leq x^{c}$, (iii)' $x^{cc} = x^{c}$ and (iv)' $(x + y)^{c} = x^{c} + y^{c}$. In the past, most authors preferred to work with the closure operator; therefore our interior algebras are better known under the name closure algebras. The alternative name "topological Boolean algebras" (used in Rasiowa and Sikorski [63]) finds its origin in the well-known theorem by McKinsey and Tarski, which says that every interior algebra can be embedded in the interior algebra constituted by the Boolean algebra of all subsets of some topological space, provided with the topological interior operator.

Most of the results contained in 2.3-2.15 were published earlier in Blok and Dwinger [74].

2.3 Theorem. Let $L \in \underline{B}_i$, and for $a,b \in L^o$ let $a \to b = (a' + b)^o$. Then $(L^o,(+,.,\to,0,1))$ is a Heyting algebra.

2.4 Theorem. Let $L \in \underline{B}$, L_1 a $\underline{D}_{01}$-sublattice of L. There exists an interior operator ^{o} on L such that $L_1 = L^o$ iff for all $a \in L$ $(a] \cap L_1$ has a largest element.

2.5 Theorem. Let $L \in \underline{B}_i$. Then

(i) $\mathcal{C}(L) \cong \mathcal{F}_o(L)$

(ii) $\mathcal{F}_o(L) \cong \mathcal{F}(L^o)$

2.6 Corollary. Let $L \in \underline{B}_i$. Then $\mathcal{C}(L) \cong \mathcal{C}(L^o)$, where L^o is considered as a Heyting algebra.

2.7 Corollary. If $L \in \underline{B}_i$ then L is subdirectly irreducible iff L^o is a subdirectly irreducible Heyting algebra. Thus $L \in \underline{B}_{iSI}$ iff $L^o \cong L_1 \oplus 1$, where $L_1 \in \underline{H}$.

2.8 Corollary. The variety $\underline{B}_i$ is congruence-distributive.

2.9 Corollary. $\underline{B}_i$ has CEP.

If $L \in \underline{B}_i$, $L_1 \in S(L)$, then $0 \in L_1$. Therefore a proper open filter of L is not a subalgebra of L. If $a,b \in L^o$, $a \le b$, then $[a,b]$ can be made into an interior algebra by defining $x^{'[a,b]} = a + x'.b$ and $x^{o[a,b]} = x^o$ for any $x \in [a,b]$, and

$+,.,0,1$ as usual. Moreover, the mapping $f: L \longrightarrow (a]$ defined by $x \longmapsto x.a$ is a $\underline{B}_i$-homomorphism. Furthermore, if $h: L \longrightarrow L_1$ is an onto $\underline{B}_i$-homomorphism, $L, L_1 \in \underline{B}_i$, and $h^{-1}(\{1\}) = [a)$ for some $a \in L^o$, then $L_1 \cong (a]$.

2.10 Theorem. Let $L \in \underline{B}_i$, $a \in L^o$, $a' \in L^o$. Then $L \cong (a] \times (a'] = [a) \times [a')$.

The connection between $\underline{B}_i$ and $\underline{H}$ is clarified by the next few theorems.

2.11 Theorem. Let $L, L_1 \in \underline{B}_i$, $h: L \longrightarrow L_1$ a $\underline{B}_i$-homomorphism. Then

(i) $h[L^o] \subseteq L_1^o$.

(ii) $h^o = h \mid L^o: L^o \longrightarrow L_1^o$ is an $\underline{H}$-homomorphism, and if h is onto, then h^o is onto.

2.12 Corollary. The assignment $\mathcal{O}: \mathcal{B}_i \longrightarrow \mathcal{H}$ given by $L \longmapsto L^o$, $h \longmapsto h^o$ is a covariant representative functor which preserves 1-1 homomorphisms and onto homomorphisms.

2.13 Theorem. Let $L \in \underline{H}$. There exists a unique interior operator on $B(L)$ such that $(B(L))^o = L$, defined as follows: if $a \in B(L)$, $a = \prod_{i=1}^{n}(u_i' + v_i)$, where $u_i, v_i \in L$, then $a^o = \prod_{i=1}^{n}(u_i \rightarrow v_i)$.

In the sequel, if $L \in \underline{H}$, $B(L)$ will denote the interior algebra provided with this interior operator.

2.14 Definition. If $L \in \underline{B}_i$ is such that $L = B(L^o)$ then L is called a $*$-algebra.

2.15 Theorem. Let $L \in \underline{H}$, $L_1 \in \underline{B}_i$, $h: L \longrightarrow L_1^o$ an $\underline{H}$-homomorphism.

Then there exists a unique $\underline{B}_i$-homomorphism $\bar{h}: B(L) \longrightarrow L_1$ such that $\bar{h} \mid L = h$.

2.16 Corollary. If $L \in \underline{B}_i$, L_1 an $\underline{H}$-subalgebra of L^o, then $[L_1]_{\underline{B}_i} = B(L_1)$.

2.17 Corollary. The assignment $\mathcal{B}: \mathcal{H} \longrightarrow \mathcal{B}_i$ given by $L \longmapsto B(L)$, $h \longmapsto \bar{h}$ is a covariant functor which preserves 1-1 homomorphisms and onto homomorphisms. Furthermore, $\mathcal{B}$ is a full embedding.

Again, the functor $\mathcal{B}$ is a left adjoint of the functor $\mathcal{O}$.

2.18 Relation between $\underline{B}_i$ and $\bar{\underline{B}}_i$

2.19 Definition. Let $L \in \underline{B}_i$. An element $x \in L$ is called a dense element of L if $x^{o'o} = 0$ or, equivalently, if $x^{oc} = 1$. The set of dense elements of L will be denoted by $D(L)$.

2.20 Theorem. Let $L \in \underline{B}_i$. Then $D(L)$ is an open filter of L, and hence $D(L)$ is a $\bar{\underline{B}}_i$-subalgebra of L.

Proof. If $x \in D(L)$, $y \geq x$ then $y^{o'o} \leq x^{o'o} = 0$, hence $y \in D(L)$. Let $x,y \in D(L)$. We want to show that $xy \in D(L)$. Clearly $x^o.y^o.(xy)^{o'o} = x^o.y^o(x^o.y^o)'^o = 0$. Hence $y^o.(xy)^{o'o} \leq x^{o'}$ and therefore $y^o.(xy)^{o'o} \leq x^{o'o} = 0$. This implies $(xy)^{o'o} \leq y^{o'}$, therefore $(xy)^{o'o} \leq y^{o'o} = 0$, and thus $xy \in D(L)$. Finally, if $x \in D(L)$ then $x^o \in D(L)$ so $D(L)$ is an open filter of L and hence a $\bar{\underline{B}}_i$-subalgebra of L.$\square$

In fact, every generalized interior algebra can be obtained as the algebra of dense elements of some interior algebra, as we shall show now.

Note that if $L \in \underline{B}_i^-$ then L is a $\underline{D}_1$-lattice. In accordance with the notation in 0.2.4, $0 \oplus L$ denotes the $\underline{D}_{01}$ lattice $\{0\} \cup L$, 0 being added as a smallest element.

2.21 Theorem. Let $L \in \underline{B}_i^-$, with interior operator ^{o}, $0 \notin L$. There exists an interior operator ^{o_1} on the Boolean algebra $B(0 \oplus L)$ generated by the $\underline{D}_{01}$ lattice $0 \oplus L$ such that $B(0 \oplus L)^{o_1} = 0 \oplus L^o$ and $D(B(0 \oplus L)) = L$.

Proof. Note that $B(0 \oplus L)$ is the disjoint union of the sets L and $\{x' \mid x \in L\}$. Define for $x \in B(0 \oplus L)$

$$x^{o_1} = \begin{cases} x^o & \text{if } x \in L \\ 0 & \text{if } x' \in L \end{cases}$$

Clearly then $1^{o_1 o_1} = 1$, $x^{o_1} \le x$, $x^{o_1 o_1} = x^{o_1}$ for any $x \in B(0 \oplus L)$. Let $x,y \in B(0 \oplus L)$. If $x,y \in L$, then $(xy)^{o_1} = (xy)^o = x^o y^o = x^{o_1} y^{o_1}$. If $x \notin L$, $y \notin L$, then $(xy)' = x' + y' \in L$, hence $(xy)^{o_1} = 0 = x^{o_1}.y^{o_1}$. If $x \in L$, $y \notin L$, then $(xy)' = x' + y' = x \Rightarrow y' \in L$, hence $(xy)^{o_1} = 0 = x^{o_1} y^{o_1}$. Similarly, if $x \notin L$, $y \in L$. Therefore ^{o_1} is an interior operator on $B(0 \oplus L)$. Furthermore, it follows from the definition of ^{o_1} that $B(0 \oplus L)^{o_1} = 0 \oplus L^o$. Finally, if $u \in L$, then $u^{o_1 '} \notin L$, hence $u^{o_1 ' o_1} = 0$. Thus $L \subseteq D(B(0 \oplus L))$. But if $u \notin L$, then $u^{o_1} = 0$, hence $u^{o_1 ' o_1} = 1$, thus $u \notin D(B(0 \oplus L))$. We conclude that $L = D(B(0 \oplus L))$.□

2.22 Theorem. Let L, $L_1 \in \underline{B}_i$, $f: L \longrightarrow L_1$ a $\underline{B}_i$-homomorphism. Then $f^D = f \mid D(L): D(L) \longrightarrow D(L_1)$ is a $\underline{B}_i^-$-homomorphism. Moreover, if f is onto, then f^D is also onto.

Proof. If $x \in D(L)$, then $(f(x))^{o'o} = f(x^{o'o}) = f(0) = 0$, hence $f(x) \in D(L_1)$. Thus f^D is well-defined. It is obvious that f^D is a

$\underline{B}_i^-$-homomorphism. Next suppose, that $y \in D(L_1)$, and let $x \in L$ be such that $f(x) = y$. Then $f(x^{o'o}) = y^{o'o} = 0$. Thus $f(x + x^{o'o}) = y$, and $(x + x^{o'o})^{o'o} \leq (x^o + x^{o'o})^{'o} = x^{o'o}.x^{o'o'o} = 0$. Hence $x + x^{o'o} \in D(L)$, and $f^D(x + x^{o'o}) = y$.□

2.23 <u>Theorem</u>. Let L, $L_1 \in \underline{B}_i^-$, $f\colon L \longrightarrow L_1$ a $\underline{B}_i^-$-homomorphism. There exists a unique $\underline{B}_i$- homomorphism $\bar{f}\colon B(0 \oplus L) \longrightarrow B(0 \oplus L_1)$ such that $\bar{f} \mid L = f$. If f is onto then so is $\bar{f}$. Here $B(0 \oplus L)$, $B(0 \oplus L_1)$ are understood to be provided with the interior operator as defined in 2.21.

<u>Proof</u>. First extend $f\colon L \longrightarrow L_1$ to a $\underline{D}_{01}$-homomorphism $f'\colon 0 \oplus L \longrightarrow 0 \oplus L_1$ by defining $f' = f \cup \{(0,0)\}$. f' can be considered as a $\underline{D}_{01}$-homomorphism $0 \oplus L \longrightarrow B(0 \oplus L_1)$, hence can be extended uniquely to a $\underline{B}$-homomorphism $\bar{f}\colon B(0 \oplus L) \longrightarrow B(0 \oplus L_1)$. It is a matter of easy verification to show that $\bar{f}(x^{o_1}) = (\bar{f}(x))^{o_1}$ for any $x \in B(0 \oplus L)$, and that $\bar{f}$ is onto iff f is onto.□

2.24 <u>Corollary</u>. $\mathcal{D}\colon \mathcal{B}_i \longrightarrow \mathcal{B}_i^-$ defined by $L \longmapsto D(L)$ and $f \longmapsto f^D$ is a covariant functor, which preserves 1-1 and onto homomorphisms and is representative.

2.25 <u>Corollary</u>. The assignment $L \longmapsto B(0 \oplus L)$, $f \longmapsto \bar{f}$ is a covariant functor from $\mathcal{B}_i^-$ to $\mathcal{B}_i$. It is in fact a full embedding.

<u>Proof</u>. Follows from the fact that $D(B(0 \oplus L)) = L$, for $L \in \underline{B}_i^-$, and by 2.22, 2.23.□

2.26 <u>Remark</u>. We shall often treat $\underline{B}_i$ as a subclass of $\underline{B}_i^-$ by identifying the algebra $L = (L,(+,.,',^o,0,1)) \in \underline{B}_i$ with the algebra $L^- = (L,(+,.,\Rightarrow,^o,1)) \in \underline{B}_i^-$, where for $a,b \in L$ $a \Rightarrow b = a' + b$. If

we want to emphasize that an algebra $L \in \underline{B}_i$ is to be considered an element of $\underline{B}_i^-$ we shall use the notation L^-.
Conversely, every generalized interior algebra L with smallest element a may be looked upon as an interior algebra by letting $0 = a$ and $x' = x \Rightarrow a$ for $x \in L$. Furthermore, if $L_1, L_2 \in \underline{B}_i^-$ both have a smallest element and $h: L_1 \longrightarrow L_2$ is a $\underline{B}_i^-$-homomorphism mapping the smallest element of L_1 upon the smallest element of L_2 then h is a $\underline{B}_i$-homomorphism if we treat L_1, L_2 as indicated.

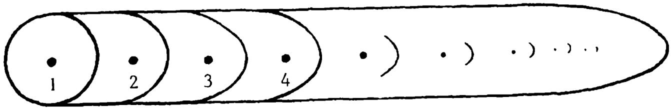
Section 3. Two infinite interior algebras generated by one element

As early as 1922 C. Kuratowski [22] gave an example of a topological space with a subset A, such that there exist $\underline{B}_i$-polynomials $p_0, p_1, \ldots$ with the property that $\forall\, i,j \geq 0 \;\; p_i(A) \neq p_j(A)$ if $i \neq j$. From this result it follows that $F_{\underline{B}_i}(1)$ is infinite, and hence, that $\underline{B}_i$ is not locally finite. The objective of this section is to present two interior algebras, both infinite and generated by one element, which are of a much simpler structure than Kuratowski's example, and which will play a significant role in subsequent sections.

3.1 Let $L \in \underline{B}_i$ be such that

$$L = P(N) \quad \text{and} \quad L^o = \{[1,n] \mid n \in N\} \cup \{\emptyset, N\} ,$$

suggested by the diagram



Let $a = \{2n \mid n \in N\} \in L$. The $\underline{B}_i^-$- subalgebra of L, $\underline{B}_i^-$-generated by by a, $[a]_{\underline{B}_i^-}$, will be denoted by K_∞.

3.2 Theorem. $K_\infty^o \cong \omega + 1$, hence K_∞ is infinite.

Proof. We show that $B(L^o) \subseteq K_\infty$. Define a sequence of $\underline{B}_i^-$-polynomials $p_0, p_1, \ldots$ as follows:

3.3 (i) $p_0(x) = x^o$, $p_1(x) = (x \Rightarrow x^o)^o$

(ii) $p_{2n}(x) = ((x \Rightarrow x^o) \Rightarrow p_{2n-1}(x))^o$,

$p_{2n+1}(x) = (x \Rightarrow p_{2n}(x))^o$.

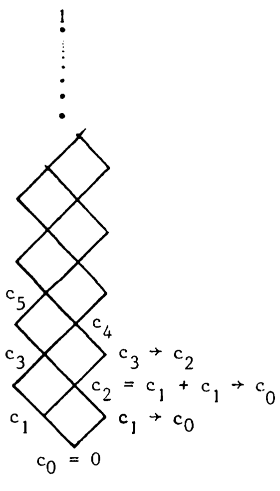
Then $p_0(a) = \emptyset$, $p_1(a) = \{1\}$. We claim that $p_n(a) = [1,n]$, for $n > 1$. Suppose that $p_{2k}(a) = [1,2k]$ for some $k \geq 1$. Then $p_{2k+1}(a) = (a \Rightarrow p_{2k}(a))^o = (\{2n-1 \mid n \in N\} \cup [1,2k])^o = [1,2k+1]$. And if $p_{2k+1}(a) = [1,2k+1]$ for some $k \geq 1$, then $p_{2k+2}(a) = ((a \Rightarrow a^o) \Rightarrow p_{2k+1}(a))^o = (\{2n \mid n \in N\} \cup [1,2k+1])^o =$ $= [1,2k+2]$. Hence $L^o \subset [a]_{\underline{B}_i^-} = K_\infty$, thus $B(L^o) \subseteq K_\infty$ by 2.6.□

In fact, it is not difficult to see that $[B(L^o) \cup \{a\}]_{\underline{B}^-}$ is a $\underline{B}_i^-$-subalgebra of K_∞ , hence $K_\infty = [B(L^o) \cup \{a\}]_{\underline{B}^-} = [B(L^o) \cup \{a\}]_{\underline{B}}$. Note that $B(L^o)$ is,as a Boolean algebra,isomorphic to the Boolean algebra of finite and cofinite subsets of a countable set, and that therefore $a \notin B(L^o)$.

3.4 Since K_∞^o is a well-ordered chain, every open filter of K_∞ is principal, hence every proper homomorphic image of K_∞ is of the form $(\![1,n]\!]$ for some $n \geq 0$. The interior algebra $(\![1,n]\!]$ will be denoted by K_n, $n \geq 0$. Thus $K_n \underset{\underline{B}}{\cong} \underline{2}^n$, $K_n^o \cong \underline{n+1}$. K_n is $\underline{B}_i$-generated by the element $a.[1,n] = \{2k \mid 2k \leq n, k \in N\}$. A remarkable property of the K_n, $n \geq 0$, is that they are generated by their sets of open elements; in symbols, $K_n = B(K_n^o)$. Thus the K_n, $n \geq 0$, are $*$-algebras (cf. 2.14). As $a \in K_\infty \backslash B(K_\infty^o)$, K_∞ itself is not a $*$-algebra.

Our second example of an infinite interior algebra generated by one element is the $*$-algebra, whose lattice of open elements is the free Heyting algebra on one generator, $F_{\underline{H}}(1)$. Rieger [57] was the first to determine the structure of $F_{\underline{H}}(1)$; cf. also Nishimura [60]:

$F_{\underline{H}}(1)$



The generator is c_1. (2)

Let $H_\infty = B(F_{\underline{H}}(1))$ (provided, as usual, with the interior operator of 2.13). H_∞ is a $*$-algebra, obviously, and we have

3.5 Theorem. If c_1 is the generator of $F_{\underline{H}}(1)$, then $H_\infty = [c_1]_{\underline{B}_i}$.

Proof. By 2.11, $[c_1]^o_{\underline{B}_i}$ is an $\underline{H}$-subalgebra of $H^o_\infty = F_{\underline{H}}(1)$. Because $c_1 \in [c_1]^o_{\underline{B}_i}$, $[c_1]^o_{\underline{B}_i} = F_{\underline{H}}(1)$. Therefore

$$H_\infty = B(F_{\underline{H}}(1)) = B([c_1]^o_{\underline{B}_i}) \subseteq [c_1]_{\underline{B}_i} \subseteq H_\infty .\square$$

A set representation of H_∞ is obtained as follows.
Let $L \in \underline{B}_i$ be such, that $L = P(N)$ and
$L^o = \{[1,n] \mid n \in N\} \cup \{[1,n] \cup \{n+2\} \mid n \in N\} \cup \{2\} \cup \{\emptyset,N\}$.
This is a good definition, since the conditions of 2.4 are satisfied. It is easy to see, that $L^o \cong F_{\underline{H}}(1)$, where $\{1\} \in L^o$ corresponds with the generator of $F_{\underline{H}}(1)$. Hence $H_\infty \cong B(L^o)$. $B(L^o)$ consists of the finite and cofinite subsets of N, and is as Boolean algebra generated by the chain $\emptyset \subset \{1\} \subset \{1,2\} \subset \ldots \subset [1,n] \subset \ldots$, which corresponds with the chain $c_0 < c_1 < c_2 < .. < c_n < \ldots$ as indicated in the diagram of $F_{\underline{H}}(1)$. We define a sequence of $\underline{B}_i$-polynomials as follows:

3.6 Definition. $q_0, q_1, \ldots$ are unary $\underline{B}_i$-polynomials such that

(i) $q_0(x) = 0$, $q_1(x) = x$

(ii) for $n \geq 1$ $q_{n+1}(x) = (q_n(x)' + q_{n-1}(x))^o + q_n(x)$

3.7 Theorem. *As a Boolean algebra, H_∞ is isomorphic to the Boolean algebra of finite and cofinite subsets of a countable set. If c_1 is the generator of H^o_∞ , then H_∞ is $\underline{B}$-generated by the chain $q_0(c_1) < q_1(c_1) < .. < q_n(c_1) < ..$, hence for any $x \in H_\infty$ either x or x' can be represented uniquely in the form*
$\sum_{j=1}^{k} q_{i_j}(c_1)'\, q_{i_j+1}(c_1)$ *for some $0 \leq i_1 < \ldots < i_k$, $k \geq 0$.*

Proof. Consider the set representation of H_∞, just given. By the remarks made above, the theorem will follow if we show that $q_n(\{1\}) = [1,n]$, $n \geq 0$. Now $q_0(\{1\}) = \emptyset$, $q_1(\{1\}) = \{1\}$. Suppose $q_n(\{1\}) = [1,n]$, $n > 1$. By definition
$q_{n+1}(\{1\}) = ([n+1,\infty) \cup [1,n-1])^o \cup [1,n] = [1,n-1] \cup \{n+1\} \cup [1,n] =$
$= [1,n+1]$.□

3.8 As can be seen by inspection of the diagram of $F_{\underline{H}}(1)$, all open filters of H_∞ are principal. Hence the proper homomorphic images of H_∞ are of the form $([1,n]]$, $n \geq 0$, which shall be denoted H_n, $n \geq 0$, or of the form $([1,n] \cup \{n+2\}]$, $n \geq 1$. Apparently $H_n \underset{\underline{B}}{\cong} \underline{2}^n$, $H_n^o \cong (c_n] \subseteq F_{\underline{H}}(1)$. The algebras $([1,n] \cup \{n+2\}]$ are isomorphic with $B(H_n^o \oplus 1)$; indeed, $([1,n] \cup \{n+2\}]^o \cong ([1,n]]^o \oplus 1$, and $([1,n] \cup \{n+2\}]$, being a homomorphic image of a $*$-algebra, is a $*$-algebra itself. The next theorem tells us that except for H_∞ these algebras are the only ones which are generated by an open element.

3.9 Theorem. $H_\infty \cong F_{\underline{B}_i}(1,\{x^o = x\})$
Proof. We verify (i), (ii) and (iii) of 0.1.22. H_∞ is generated by the the element c_1, which satisfies $c_1^o = c_1$, and $H_\infty \in \underline{B}_i$. To verify the third requirement, let $L \in \underline{B}_i$, $y \in L$ such that $y^o = y$. Then $y \in L^o \in \underline{H}$, hence there exists an $\underline{H}$-homomorphism $f\colon F_{\underline{H}}(1) \longrightarrow L^o$ satisfying $f(c_1) = y$. By 2.15 f can be extended to a $\underline{B}_i$-homomorphism $\bar{f}\colon B(F_{\underline{H}}(1)) = H_\infty \longrightarrow L$, still satisfying $\bar{f}(c_1) = y$.□

3.10 Corollary. Let $L \in \underline{B}_i$ be generated by an open element x. Then $L \cong H_\infty$, $L \cong H_n$, or $L \cong B(H_n^o \oplus 1)$, for some $n \geq 0$. For all $z \in L$, z or z' can be written as $\sum_{j=1}^{k} q'_{i_j}(x)\, q_{i_j+1}(x)$, for some $k \geq 0$, $0 \leq i_1 < \ldots < i_k$.

Proof. By 3.9, 3.8 and 3.7 . □

Note that H_∞ is the only infinite interior algebra generated by an open element. Theorem 3.9 can be generalized without difficulty to

3.11 Theorem. $B(F_{\underline{H}}(n)) \cong F_{\underline{B}_i}(n,\{x_i^o = x_i \mid i = 1,2,\dots n\})$, for any $n \in N$.

3.12 Whereas K_∞ is $\underline{B}_i^-$-generated by one element, it will follow from considerations in section II.3 that H_∞ is not $\underline{B}_i^-$-generated by any element. However, by slightly modifying the algebra H_∞ we can turn it into an algebra $\underline{B}_i^-$-generated by one element. Indeed, let $L = P(N)$, $L^o = \{\{1\},\{2,3\},\{1,4\},\emptyset,N\} \cup \{[1,n] \mid n \geq 3,\ n \in N\} \cup \{[1,n] \cup \{n+2\} \mid n \geq 3,\ n \in N\}$. L is an interior algebra, and the $\underline{B}_i^-$-subalgebra of L consisting of all finite and cofinite subsets of N is the desired algebra, $\underline{B}_i^-$-generated by one element, which will be denoted H_∞^+. The distinction between H_∞ and H_∞^+ is that the open atom $\{2\} \in H_\infty$ has been replaced by the open set $\{2,3\}$. Further $H_\infty \underset{\underline{B}}{\cong} H_\infty^+$, $H_\infty^o \cong H_\infty^{+o} \cong F_{\underline{H}}(1)$ (see diagram on pg.32). H_∞ is not a $*$-algebra, since $\{2\},\{3\} \in H_\infty^+ \setminus B(H_\infty^{+o})$. H_∞^+ is $\underline{B}_i^-$-generated by its element $\{3,4\}$. Using the polynomials defined in 3.6 we see that $\emptyset = \{3,4\}^o$, $\{1\} = (\{3,4\} \Rightarrow \emptyset)^o$ and $[1,n+1] = q_n(\{1\})$ for $n > 1$, which together with $\{3,4\}$ clearly $\underline{B}^-$-generate H_∞^+. Likewise the homomorphic images of H_∞^+ are $\underline{B}_i^-$-generated by one element; the algebras $([1,n+1]]$ will be denoted H_n^+ for $n > 1$. Hence $H_n^+ \underset{\underline{B}}{\cong} \underline{2}^{n+1}$, $H_n^{+o} \cong (c_n] \subseteq F_{\underline{H}}(1)$, $n > 1$, $n \in N$.

Section 4. Principal ideals in finitely generated free algebras in $\underline{B}_i$ and $\underline{B}_i^-$

In the preceding section we have seen that there are infinite (generalized) interior algebras generated by one element. This implies that $F_{\underline{B}_i}(1)$ as well as $F_{\underline{B}_i^-}(1)$ are infinite. Now we want to obtain some more detailed information concerning these algebras and more generally about $F_{\underline{B}_i}(n)$ and $F_{\underline{B}_i^-}(n)$, $n \in N$. A complete description as the one given of $F_{\underline{H}}(1)$ in section 3 should not be expected: our results will rather show how complicated the structure of even $F_{\underline{B}_i^-}(1)$ and $F_{\underline{B}_i}(1)$ is.

We start with a general theorem on ideals in $F_{\underline{B}_i}(n)$ or $F_{\underline{B}_i^-}(n)$ generated by an open element and deduce some corollaries (4.1-4.11). Having established some facts dealing with covers (4.12-4.17) the most striking one of which is the result that there exists a $u \in F_{\underline{B}_i}(1)^o$ which has $\aleph_o$ open covers, we proceed to show that not every interior algebra generated by one element is isomorphic to a principal ideal of $F_{\underline{B}_i}(1)$ by exhibiting a collection of $2^{\aleph_o}$ non-isomorphic interior algebras generated by one element, which may even be chosen to be subdirectly irreducible (4.18-4.28).

4.1 Theorem. (i) Let $n \in N$. If Ω is a finite set of n-ary $\underline{B}_i$-identities, then there exists a $u \in F_{\underline{B}_i}(n)^o$ such that $(u] \cong F_{\underline{B}_i}(n,\Omega)$.

(ii) If $u \in F_{\underline{B}_i}(n)^o$, then $(u] \cong F_{\underline{B}_i}(n,\{p=1\})$, where p is some n-ary $\underline{B}_i$-polynomial.

Thus an interior algebra L is finitely presentable iff $L \cong (u]$, for some $u \in F_{\underline{B}_i}(n)^o$, and some $n \in N$.

<u>Proof</u>. (i) Any $\underline{B}_i$-identity $p = q$ is equivalent with an identity of the form $r = 1$. Indeed, $p = q$ iff $(p' + q)(p + q') = 1$. Suppose that $\Omega = \{p_i = 1 \mid i = 1\ldots k\}$. If $x_1 \ldots x_n$ are free generators of $F_{\underline{B}_i}(n)$, and $u = \prod_{i=1}^{k} p_i^o(x_1,\ldots x_n)$, then $(u] \cong F_{\underline{B}_i}(n,\Omega)$. Indeed, $(u]$ is generated by the elements $x_1u,\ldots x_nu$, and $p_i(x_1u,\ldots x_nu) = p_i(x_1\ldots x_n).u = u$ for $i = 1,2\ldots k$ since the map $F_{\underline{B}_i}(n) \longrightarrow (u]$ defined by $x \longmapsto x.u$ is a homomorphism. If $L \in \underline{B}_i$, such that $L = [\{b_1,\ldots b_n\}]$, $p_i(b_1\ldots b_n) = 1$ for $i = 1,2\ldots k$, let $g\colon F_{\underline{B}_i}(n) \longrightarrow L$ be a homomorphism such that $g(x_i) = b_i$, $i = 1,2\ldots n$. Then $p_i(x_1,\ldots x_n) \in g^{-1}(\{1\})$, hence $p_i^o(x_1\ldots x_n) \in g^{-1}(\{1\})$ and therefore $[u) \subseteq g^{-1}(\{1\})$. By the homomorphism theorem there exists a homomorphism $\bar{g}\colon (u] \longrightarrow L$ such that the diagram

$$\begin{tikzcd} F_{\underline{B}_i}(n) \arrow[r, "g"] \arrow[d, "\pi"'] & L \\ (u] \arrow[ur, dashed, "\bar{g}"'] & \end{tikzcd}$$

commutes. $\bar{g}$ is the desired homomorphism extending the map $x_iu \longmapsto b_i$, $i = 1,2\ldots n$.

(ii) Let $u \in F_{\underline{B}_i}(n)^o$. Then $u = p(x_1,\ldots x_n)$, for some $\underline{B}_i$-polynomial p, if $x_1\ldots x_n$ are free generators of $F_{\underline{B}_i}(n)$. $(u]$ is generated by the elements $x_1u,\ldots x_nu$, and the generators satisfy $p(x_1u,\ldots x_nu) = p(x_1\ldots x_n).u = u = 1_{(u]}$. The remaining requirement is verified as it was in (i).□

4.2 <u>Remark</u>. The same theorem holds for the varieties $\underline{B}_i^-$, $\underline{H}$, $\underline{H}^-$, and in fact, also for any non-trivial subvariety of $\underline{B}_i$, $\underline{B}_i^-$, $\underline{H}$, $\underline{H}^-$. The proofs are similar to the given one. Though stated for $\underline{B}_i$ only, the following two corollaries apply to the mentioned varieties, too.

4.3 Corollary. If $L \in \underline{B}_i$ is finite and generated by n elements then there is a $u \in F_{\underline{B}_i}(n)^o$ such that $L \cong (u]$.

Proof. Let $L \in \underline{B}_i$ be finite and suppose that $L = [\{a_1,\ldots a_n\}]$. Let p_{a_i} be the n-ary $\underline{B}_i$-polynomial x_i, $i = 1,\ldots n$, and in general let p_x be a $\underline{B}_i$-polynomial such that $p_x(a_1,\ldots a_n) = x$, for each $x \in L$. Let Ω be the collection of equations of the type $p_x + p_y = p_{x+y}$, $p_x \cdot p_y = p_{x.y}$, $p_x' = p_{x'}$ and $p_x{}^o = p_{x^o}$ for $x,y \in L$ and $p_0 = 0$, $p_1 = 1$. Then $L \cong F_{\underline{B}_i}(n,\Omega)$. For if $L_1 = [\{b_1,\ldots b_n\}]$ and $b_1,\ldots b_n$ satisfy Ω then $\{p_x(b_1,\ldots b_n) \mid x \in L\} = L_1$ and the map $f : L \to L_1$ defined by $f(x) = p_x(b_1,\ldots b_n)$ is a homomorphism extending the map $a_i \mapsto b_i$, $i = 1,\ldots n$. Since Ω is finite, the corollary follows.□

4.4 Corollary. If $0 < k \leq n$, then there exists a $u \in F_{\underline{B}_i}(n)^o$ such that $F_{\underline{B}_i}(k) \cong (u]$.

Proof. $F_{\underline{B}_i}(k) \cong F_{\underline{B}_i}(n,\{x_k = x_{k+1}, x_k = x_{k+2}, \ldots x_k = x_n\})$.□

4.5 Corollary. There exists a $u \in F_{\underline{B}_i}(1)^o$, such that $H_\infty \cong (u]$. Hence $F_{\underline{B}_i}(1)$ possesses an infinite number of atoms.

Proof. By 3.9 and 4.1 (i). If $p \in (u]$ is an atom in $(u]$, then p is also an atom in $F_{\underline{B}_i}(1)$; since H_∞ contains infinitely many atoms, so does $F_{\underline{B}_i}(1)$.□

4.6 Corollary. For $n \in N$, there exists a $u \in F_{\underline{B}_i}(n)^o$ such that $B(F_{\underline{H}}(n)) \cong (u]$.

Proof. By 3.11, 4.1 (i).□

4.7 Corollary. For $n \in N$, there exists a $u \in F_{\underline{B}_i^-}(n)^o$, such that $B^-(F_{\underline{H}^-}(n)) \cong (u]$.

Proof. $B^-(F_{\underline{H}^-}(n)) \cong F_{\underline{B}_i^-}(n, \{x_i^o = x_i \mid i = 1 \ldots n\})$.□

Before proceeding to the next result, we need a lemma. It will show that every finitely generated generalized interior algebra has a smallest element and can therefore be treated as an interior algebra (cf. 2.26). Thereupon we prove that the free generalized interior algebra on n generators is $\underline{B}_i$-isomorphic to a principal ideal of the free interior algebra on n generators.

4.8 Lemma. (i) Let $L \in \underline{B}_i^-$ be finitely generated. Then L has a smallest element.

(ii) $F_{\underline{B}_i^-}(n) \underset{\underline{B}_i}{\cong} F_{\underline{B}_i}(n, \{\prod_{i=1}^{n} x_i^o = 0\})$, for any $n \in N$.

Proof. (i) Let $L \in \underline{B}_i^-$, and let $x_1 \ldots x_n$ be generators of L. We claim that $a = \prod_{i=1}^{n} x_i^o$ is the smallest element of L. Obviously, $a \le x_i$, $i = 1 \ldots n$. Let p,q be $\underline{B}_i^-$-polynomials such that $a \le p(x_1, \ldots x_n)$, $a \le q(x_1 \ldots x_n)$. Then $a \le p(x_1 \ldots x_n) + q(x_1 \ldots x_n)$, $a \le p(x_1 \ldots x_n).q(x_1 \ldots x_n)$, $a \le p(x_1 \ldots x_n) \Rightarrow q(x_1 \ldots x_n)$ and $a \le p(x_1 \ldots x_n)^o$. The proof of our claim now follows by induction.

(ii) Let $x_1, \ldots x_n$ be the free generators of $F_{\underline{B}_i^-}(n)$. We shall treat $F_{\underline{B}_i^-}(n)$ as element of $\underline{B}_i$ with smallest element $0 = \prod_{i=1}^{n} x_i^o$. There exists a $\underline{B}_i$-homomorphism $h: F_{\underline{B}_i}(n, \{\prod_{i=1}^{n} x_i^o = 0\}) \longrightarrow F_{\underline{B}_i^-}(n)$ mapping the generators $y_1 \ldots y_n$ of $F_{\underline{B}_i}(n, \{\prod_{i=1}^{n} x_i^o = 0\})$ onto $x_1 \ldots x_n$ respectively. On the other hand, $F_{\underline{B}_i}(n, \{\prod_{i=1}^{n} x_i^o = 0\}) = [\{y_1 \ldots y_n\}]_{\underline{B}_i^-}$, as $0 = \prod_{i=1}^{n} y_i^o \in [\{y_1 \ldots y_n\}]_{\underline{B}_i^-}$. Since h is onto, it follows that

$F_{\underline{B}_i}(n,\{\prod_{i=1}^{n} x_i^o = 0\}) \underset{\bar{\underline{B}}_i}{\cong} F_{\bar{\underline{B}}_i}(n)$, and because $h(0) = 0$,

$F_{\underline{B}_i}(n,\{\prod_{i=1}^{n} x_i^o = 0\}) \underset{\underline{B}_i}{\cong} F_{\bar{\underline{B}}_i}(n)$.□

4.9 <u>Theorem</u>. There exists a $u \in F_{\underline{B}_i}(n)^o$ such that $F_{\bar{\underline{B}}_i}(n) \underset{\bar{\underline{B}}_i}{\cong} (u]$, for any $n \in N$.

<u>Proof</u>. By 4.1 (i), 4.8.□

Conversely, every $F_{\underline{B}_i}(n)$, $n \in N$ is isomorphic to a principal ideal of $F_{\bar{\underline{B}}_i}(n+1)$.

4.10 <u>Lemma</u>. $F_{\underline{B}_i}(n) \underset{\bar{\underline{B}}_i}{\cong} F_{\bar{\underline{B}}_i}(n+1,\{\prod_{i=1}^{n+1} x_i^o = x_{n+1}\})$, for any $n \in N$.

<u>Proof</u>. Let $\{x_1 \ldots x_{n+1}\}$ $\bar{\underline{B}}_i$-generate $F_{\bar{\underline{B}}_i}(n+1,\{\prod_{i=1}^{n} x_i^o = x_{n+1}\})$, such that $\prod_{i=1}^{n} x_i^o = x_{n+1}$, and let $\{y_1 \ldots y_n\}$ $\underline{B}_i$-generate $F_{\underline{B}_i}(n)$. If $y_{n+1} = 0$, then $\{y_1 \ldots y_{n+1}\}$ $\bar{\underline{B}}_i$-generates $F_{\underline{B}_i}(n)$ and $\prod_{i=1}^{n+1} y_i^o = y_{n+1}$, hence there exists a $\bar{\underline{B}}_i$-homomorphism

$$h\colon F_{\bar{\underline{B}}_i}(n+1,\{\prod_{i=1}^{n+1} x_i^o = x_{n+1}\}) \longrightarrow F_{\underline{B}_i}(n)$$

such that $h(x_i) = y_i$, $i = 1,2\ldots n+1$, which is onto. Since h maps the smallest element of $F_{\bar{\underline{B}}_i}(n+1,\{\prod_{i=1}^{n} x_i^o = x_{n+1}\})$ upon the 0 of $F_{\underline{B}_i}(n)$, h is also a $\underline{B}_i$-homomorphism. Finally, $F_{\bar{\underline{B}}_i}(n+1,\{\prod_{i=1}^{n+1} x_i^o = x_{n+1}\})$, regarded as interior algebra, is $\underline{B}_i$-generated by $x_1 \ldots x_n$, therefore h is 1-1 and $F_{\underline{B}_i}(n) \underset{\bar{\underline{B}}_i}{\cong} F_{\bar{\underline{B}}_i}(n+1,\{\prod_{i=1}^{n+1} x_i^o = x_{n+1}\})$.□

4.11 <u>Theorem</u>. There exists a $u \in F_{\bar{\underline{B}}_i}(n+1)^o$ such that $F_{\underline{B}_i}(n) \underset{\bar{\underline{B}}_i}{\cong} (u]$, for any $n \in N$.

<u>Proof</u>. By 4.10, 4.1 (i).□

4.12 Covers in $F_{\underline{B}_i}(n)$

Let L be a partially ordered set, $a,b \in L$. We say that a is covered by b or that b is a cover of a if $a < b$ and there exists no $c \in L$ such that $a < c < b$. If b covers a we write $a \prec b$; if we wish to emphasize that b covers a in L we write also $a \prec b$. Thus the atoms in a lattice are the elements which cover 0.

In 4.5 we concluded that $F_{\underline{B}_i}(1)$ has an infinite number of atoms and in virtue of 4.4 so does $F_{\underline{B}_i}(n)$, $n \in N$. The question arises how many open atoms $F_{\underline{B}_i}(n)$ possesses.

4.13 Theorem. $F_{\underline{B}_i}(n)$ has 2^n open atoms, $n \in N$.

Proof. Let $x_1, x_2, \dots x_n$ be free generators of $F_{\underline{B}_i}(n)$.

Let $x_i^{\varepsilon_i} = \begin{cases} x_i & \text{if } \varepsilon_i = 1 \\ x_i' & \text{if } \varepsilon_i = 2 \end{cases}$. If $f \in \{1,2\}^n$, then $a_f = \prod_{i=1}^{n} x_i^{f(i)o}$ is an open atom. Indeed, a_f is open and $(a_f] = [\{x_1 a_f, \dots x_n a_f\}] = \{0, a_f\} \cong \underline{2}$, since $x_i a_f = a_f$ if $f(i) = 1$, and $x_i a_f = 0$ if $f(i) = 2$, $i = 1 \dots n$. Therefore a_f is an open atom. Conversely, if u is an open atom of $F_{\underline{B}_i}(n)$, then $u \le \prod_{i=1} x_i^{f(i)}$ for some $f \in \{1,2\}^n$, hence $u \le (\prod_{i=1}^{n} x_i^{f(i)})^o = a_f$, thus $u = a_f$. $\square$

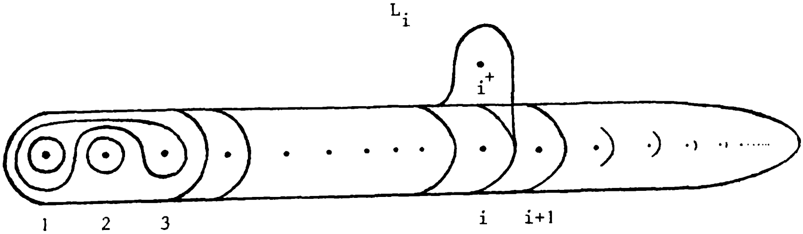
Theorem 4.13 says that the element 0 is covered by 2^n open elements in $F_{\underline{B}_i}(n)$. In particular, the open atoms of $F_{\underline{B}_i}(\{x\})$ are x^o and x'^o. We shall show now that there are open elements in $F_{\underline{B}_i}(1)$ which have infinitely many open covers in $F_{\underline{B}_i}(1)$. Contrast this with the situation in $F_{\underline{H}}(1)$, where every element except 1 has precisely two covers.

We need a rather technical lemma.

4.14 <u>Lemma</u>. Let for $i \in N$ L_i be the interior algebra of all finite and cofinite subsets of $N \cup \{i^+\}$, where $i^+ \notin N$, such that

$$L_i^o = \{\emptyset, N, N \cup \{i^+\}, \{2\}\} \cup \{[1,n],[1,n] \cup \{n+2\} \mid n \in N\}$$
$$\cup \{[1,n] \cup \{i^+\},[1,n] \cup \{n+2\} \cup \{i^+\} \mid n \in N , n \geq i\} ,$$

suggested by the diagram



Let $q_0, q_1, \ldots$ be $\underline{B}_i$-polynomials as defined in 3.6 and let Ω_i consist of the equations

(i) $x_1^o = x_1$

(ii) $x_2 \leq q'_{i+1}(x_1) \cdot q_{i+2}(x_1)$

(iii) $(x_2 + q_{j-1}(x_1) + q'_j(x_1))^o = q_{j-1}(x_1) + q'_j(x_1) \cdot q_{j+1}(x_1)$, $1 \leq j \leq i$

(iv) $(x_2 + q_i(x_1))^o = x_2 + q_i(x_1)$

(v) $x_2'^o = x'_2$

Then $L_i \cong F_{\underline{B}_i}(2, \Omega_i)$, where $\{1\}, \{i^+\}$ are the generators of L_i satisfying Ω_i , for any $i \in N$, $i \geq 2$.

<u>Proof</u>. It easy to verify that L_i is generated by the elements $\{1\}$, $\{i^+\}$, and that these elements satisfy the equations in Ω_i. Let $L \in \underline{B}_i$, $L = [\{y_1, y_2\}]_{\underline{B}_i}$ such that y_1, y_2 satisfy the equations of Ω_i . Let $L' = [[y_1]_{\underline{B}_i}, y_2]_{\underline{B}}$; then $L' \subseteq L$. We claim that $L' = L$. If $w \in L'$, then $w = (y_2 + z_1)(y'_2 + z_2)$, $z_1, z_2 \in [y_1]_{\underline{B}_i}$

(see e.g Grätzer [71] pg. 84). We shall show that $w^o \in L'$.

Now $w^o = (y_2 + z_1)^o . (y_2' + z_2)^o$.

1) $(y_2' + z_2)^o \in L'$. Since $z_2 \in [y_1]_{\underline{B}_i}$, $y_1^o = y_1$, by 3.10 z_2 or z_2' can be written as $\sum_{j=1}^{k} q_{i_j}'(y_1).q_{i_j+1}(y_1)$, $0 \leq i_1 < \dots < i_k$, $k \geq 0$.

a) $z_2 = \sum_{j=1}^{k} q_{i_j}(y_1)\, q_{i_j+1}(y_1)$. If $i_1, i_2, \dots i_k \neq i+1$, then $y_2 z_2 \leq q_{i+1}'(y_1).q_{i+2}(y_1).(\sum_{j=1}^{k} q_{i_j}'(y_1)\, q_{i_j+1}(y_1)) = 0$, by (ii), hence $(y_2' + z_2)^o = y_2'^o = y_2' \in L'$, by (v). If $i_j = i+1$ for some j, $1 \leq j \leq k$, then $y_2 \leq z_2$, hence $(y_2' + z_2)^o \geq (y_2' + y_2)^o = 1 \in L'$.

b) $z_2' = \sum_{j=1}^{k} q_{i_j}'(y_1).q_{i_j+1}(y_1)$. If $i_1, i_2, \dots i_k \neq i+1$, then $y_2 z_2' = 0$, thus $y_2 \leq z_2$ and $(y_2' + z_2)^o \geq (y_2' + y_2)^o = 1 \in L'$. If $i_j = i+1$ for some j, $1 \leq j \leq k$, then $y_2 \leq z_2'$, hence $y_2 z_2 = 0$ and $(y_2' + z_2)^o = y_2'^o = y_2' \in L'$.

2) $(y_2 + z_1)^o \in L'$. If $y_2 \leq z_1$, then $(y_2 + z_1)^o = z_1^o \in L'$. Applying the reasoning of 1), we conclude that if $y_2 \nleq z_1$, then $y_2.z_1 = 0$. Hence $(y_2 + z_1)^o \leq y_2 + y_2'(y_2 + z_1)^o = y_2 + y_2'^o(y_2 + z_1)^o$, again, by (v). Now $y_2'^o(y_2 + z_1)^o \leq y_2'(y_2 + z_1) \leq z_1$, hence $y_2'^o(y_2 + z_1)^o \leq z_1^o$. Thus $(y_2 + z_1)^o \leq y_2 + z_1^o$. If $q_i(y_1) \leq z_1$, then $(y_2 + z_1)^o = (y_2 + q_i(y_1) + z_1)^o \geq (y_2 + q_i(y_1))^o + z_1^o =$ $= y_2 + q_i(y_1) + z_1^o = y_2 + z_1^o$, by (iv), hence $(y_2 + z_1)^o = y_2 + z_1^o \in L'$. If $q_i(y_1) \nleq z_1$, then there is a j_o, $1 \leq j_o \leq i$, such that $q'_{j_o-1}(y_1)\, q_{j_o}(y_1) \nleq z_1$, thus $z_1 \leq q_{j_o-1}(y_1) + q_{j_o}'(y_1)$ and

$$(y_2 + z_1)^o \leq (y_2 + q_{j_o-1}(y_1) + q_{j_o}'(y_1))^o =$$

$$= q_{j_o-1}(y_1) + q_{j_o}'(y_1).q_{j_o+1}(y_1)\,,$$

by $(iii)_{j_o}$, and since $y_2 \leq q_{i+1}'(y_1).q_{i+2}(y_1)$ by (ii), and $j_o \leq i$, it follows that $(y_2 + z_1)^o = z_1^o \in L'$.

Thus we have shown that L' is a $\underline{B}_i$-subalgebra of L ; since L' contains the generators y_1 , y_2 of L it follows that $L = L'$. In order to prove that $L_i \cong F_{\underline{B}_i}(2,\Omega_i)$ it remains to show that the map $\{1\} \mapsto y_1$, $\{i^+\} \mapsto y_2$ can be extended to a homomorphism $L_i \to L$. Since $[\{1\}] \cong H_\infty$, there exists a $\underline{B}_i$-homomorphism $f\colon [\{1\}] \to [y_1]$ such that $f(\{1\}) = y_1$. Then for all $z \in [\{1\}]$, $\{i^+\} \le z$ implies $y_2 \le f(z)$ and $\{i^+\} \ge z$ iff $y_2 \ge f(z)$. It is known (see Grätzer [71] pg. 84) that f can then be extended to a $\underline{B}$-homomorphism $\bar{f}\colon L_i \to L$ such that $\bar{f}(\{i^+\}) = y_2$. Let $w = (\{i^+\} + z_1)(\{i^+\}' + z_2)$, $z_1,z_2 \in [\{1\}]$. Then $\bar{f}(w) = (y_2 + f(z_1))(y_2' + f(z_2))$ and it follows from the preceding arguments that $\bar{f}(w^o) = (\bar{f}(w))^o$. Therefore f is a $\underline{B}_i$-homomorphism and $L_i \cong F_{\underline{B}_i}(2,\Omega_i)$.□

4.15 <u>Theorem</u>. There is a $u \in F_{\underline{B}_i}(1)^o$ which has $\aleph_o$ open covers in $F_{\underline{B}_i}(1)$.

<u>Proof</u>. L_i , $i \ge 2$, $i \in N$ is generated by the single element $x = \{1,i^+\}$, therefore $L_i \cong F_{\underline{B}_i}(1,\Omega_i')$, where Ω_i' consists of the equations of Ω_i with x_1 replaced by x^o , and x_2 by $x\, x^{o\prime}$. Let x be the generator of $F_{\underline{B}_i}(1)$, $f_i\colon F_{\underline{B}_i}(1) \to L_i$, $i \ge 2$, $i \in N$, the homomorphism satisfying $f_i(x) = \{1,i^+\}$. Let $\pi_i\colon L_i \to H_\infty$ be the homomorphism defined by $\pi_i(z) = z.\{i^+\}'$, where we think of H_∞ as being given in the set representation following 3.5. Then for each $i \in N$, $i \ge 2$ $\pi_i \circ f_i\colon F_{\underline{B}_i}(1) \to H_\infty$ with $\pi_i \circ f_i(x) = \{1\}$. Since L_i, $i \ge 2$ and H_∞ are finitely presentable in virtue of 4.14 and 3.9 , there exist by 4.1 (i) $u_i, u \in F_{\underline{B}_i}(1)^o$, $i \ge 2$, such that $(u_i] \cong L_i$, $i \ge 2$, $(u] \cong H_\infty$ and $(u] \subseteq (u_i]$. In fact, if p_i is the atom in $F_{\underline{B}_i}(1)$ corresponding with $\{i^+\}$, then $u = u_i.p_i'$, hence $u \prec u_i$,

$i \geq 2$, $i \in N$. □

A similar result can be obtained for $F_{\underline{B}_i}$-(1). The proof, which we omit, uses a modification of our L_i, based on H_∞^+ instead of H_∞ (cf. 3.12).

4.16 Theorem. There exists a $u \in F_{\underline{B}_i}$-$(1)^o$ which is covered in $F_{\underline{B}_i}$-(1) by $\aleph_o$ open elements.

4.17 Corollary $F_{\underline{B}_i}$-(1) has a subalgebra which has countably many open atoms.

Proof. If $u \in F_{\underline{B}_i}$-(1) has $_o$ open covers then $[u) \subseteq F_{\underline{B}_i}$-(1) is a $\underline{B}_i^-$-subalgebra of $F_{\underline{B}_i}$-(1) having countably many open atoms. □

4.18 $2^{\aleph_o}$ interior algebras generated by one element

As far as principal ideals are concerned, there seems to be a lot of room in $F_{\underline{B}_i}(n)$, $n \in N$. The question arises, whether every n-generated interior algebra is isomorphic to some principal ideal in $F_{\underline{B}_i}(n)$, as is the case for 1-generated Heyting algebras with respect to $F_{\underline{H}}(1)$. We shall answer this question negatively by constructing a family of continuously many pairwise non-isomorphic interior algebras, generated by one element. The algebras will be a generalization of the L_i's employed above.

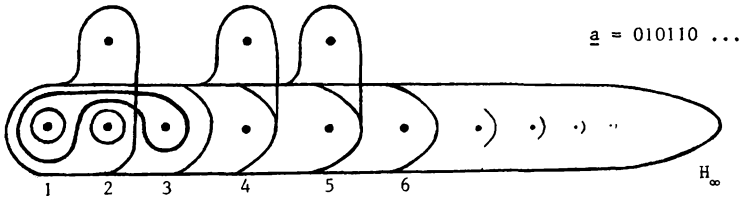
4.19 Let $(a_n)_n = \underline{a}$ be a sequence of 0's and 1's, such that $a_1 = 0$. Let

$$X_{\underline{a}} = \{(n,0) \mid n \in N\} \cup \{(n,1) \mid a_n = 1\} \subseteq N \times \{0,1\}.$$

Let

$$B_{\underline{a}} = \{\emptyset,(2,0)\} \cup \{\{(k,0) \mid k \le n\} \mid n \in N\} \cup$$
$$\cup \{\{(k,0) \mid k \le n\} \cup \{(n+2,0)\} \mid n \in N\} \cup$$
$$\cup \{\{(k,0) \mid k \le n\} \cup \{(n,1)\} \mid n \in N,\ a_n = 1\} .$$

An example is suggested by the diagram:



Let $L = \mathcal{P}(X_{\underline{a}})$, and define an interior operator on L by taking $B_{\underline{a}}$ as a base for the open elements; that is, if $x \in L$ let $x^{o} = \Sigma\ \{y \in B_{\underline{a}} \mid y \le x\}$: then $1^{o} = 1$, $x^{o} \le x$, $x^{oo} = x^{o}$ and $(xy)^{o} = x^{o}y^{o}$ since $B_{\underline{a}}$ is closed under finite products. If $a_n = 1$, let us write n^{+} instead of $(n,1)$ and in general let us write n for $(n,0)$. Let $X^{+}_{\underline{a}} = \{n^{+} \mid n \in N,\ a_n = 1\}$, and let $x_{\underline{a}} = \{1\} \cup X^{+}_{\underline{a}}$. The $\underline{B}_i$-subalgebra of L generated by the element $x_{\underline{a}}$ wil be denoted by $L_{\underline{a}}$.

Since $x^{o}_{\underline{a}} = \{1\}$, $\{1\},X^{+}_{\underline{a}} \in L_{\underline{a}}$. $B((X^{+\prime}_{\underline{a}}]^{o}) \tilde{=} H_\infty$ and $B((X^{+\prime}_{\underline{a}}]^{o}) = [x^{o}_{\underline{a}}]$, hence $B((X^{+\prime}_{\underline{a}}]^{o}) \subseteq L_{\underline{a}}$. By induction we show that $\{n^{+}\} \in L_{\underline{a}}$, if $a_n = 1$. If n_0 is the first $n \in N$ such that $a_{n_0} = 1$, then $\{n_0^{+}\} = ([1,n_0] + x_{\underline{a}})^{o}.[1,n_0]' \in [x_{\underline{a}}]$, and if $\{k^{+}\} \in L_{\underline{a}}$, $k < n$, $a_k = 1$ then

$$x_{\underline{a}} \prod_{\substack{k<n \\ a_k=1}} \{k^{+}\}' \in [x_{\underline{a}}] \text{ and } \{n^{+}\} = ([1,n] + x_{\underline{a}} \prod_{\substack{k<n \\ a_k=1}} \{k^{+}\}')^{o}.[1,n]' \in L_{\underline{a}}.$$

Thus $L_{\underline{a}}$ contains all atoms of L, and also $B_{\underline{a}} \subseteq L^{o}_{\underline{a}}$.

4.20 Lemma. $L_{\underline{a}} \cong L_{\underline{b}}$ iff $\underline{a} = \underline{b}$, for any two sequences $\underline{a} = (a_n)_n$, $\underline{b} = (b_n)_n$ of 0's and 1's, with $a_1 = b_1 = 0$.

Proof. $\Rightarrow$ Let $\varphi: L_{\underline{a}} \to L_{\underline{b}}$ be a $\underline{B}_i$-isomorphism. $\{1\}$ is the unique open atom p in $L_{\underline{a}}$, $L_{\underline{b}}$ satisfying $p^{\prime o \prime o} \neq p$, hence $\varphi(\{1\}) = \{1\}$. Note that if $a_n = 0$ then $[1,n]$ has precisely two open covers, if $a_n = 1$ then $[1,n]$ has precisely three open covers. Suppose $\varphi(\{i\}) = \{i\}$, $1 \le i \le n$. By the remark just made then $a_i = b_i$, $1 \le i \le n$. Since $[1,n+1]$ covers $[1,n]$, $\varphi([1,n+1])$ covers $\varphi([1,n]) = [1,n]$, by the induction hypothesis. But $[1,n+1]$ is the only open cover of $[1,n]$, such that $[1,n+1].\{n\}' = [1,n-1] \cup \{n+1\}$ is open. Hence $\varphi([1,n+1].\{n\}') = \varphi([1,n+1]).\varphi(\{n\})' = \varphi([1,n+1]).\{n\}'$ is open, thus $\varphi([1,n+1]) = [1,n+1]$, and $\varphi(\{n+1\}) = \{n+1\}$, $a_{n+1} = b_{n+1}$. $\square$

4.21 Theorem. There are continuously many non isomorphic interior algebras generated by one element.

Proof. Lemma 4.20 provides them. $\square$

4.22 Corollary. Not every interior algebra generated by one element is isomorphic to a principal ideal $(u]$ for some $u \in F_{\underline{B}_i}(1)^o$ and not every interior algebra generated by one element is finitely presentable.

Proof. Since $F_{\underline{B}_i}(1)^o$ is countable.

4.23 Corollary. Not every open filter in $F_{\underline{B}_i}(1)$ is principal.

Proof. The homomorphic images of $F_{\underline{B}_i}(1)$ correspond in a 1-1 manner with the open filters of $F_{\underline{B}_i}(1)$. By 4.21 there are $2^{\aleph_0}$ open filters, whereas the cardinality of the set of principal open filters is $\aleph_0$. $\square$

4.24 Corollary. There exists an infinite decreasing chain of open elements in $F_{\underline{B}_i}(1)$.

Proof. Let $F \subseteq F_{\underline{B}_i}(1)$ be a non-principal open filter. Let $u_1 \in F$, $u_1 \in F_{\underline{B}_i}(1)^o$. There exists a $v \in F_{\underline{B}_i}(1)^o$, $v \in F$ such that $u_1 \nleq v$. Let $u_2 = u_1 v$, then $u_2 < u_1$, $u_2 \in F_{\underline{B}_i}(1)^o$, $u_2 \in F$. If $u_1, u_2, \ldots u_k \in F_{\underline{B}_i}(1)^o$, $u_1, u_2, \ldots u_k \in F$ such that $u_i < u_j$, $1 \leq j \leq i \leq k$ then there exists a $v \in F_{\underline{B}_i}(1)^o$, $v \in F$ such that $u_k \nleq v$. Then $u_{k+1} = u_k . v < u_k$, $u_{k+1} \in F_{\underline{B}_i}(1)^o$, $u_{k+1} \in F$, and the proof follows by induction.□

We have thus exhibited in $F_{\underline{B}_i}(1)$ an infinite increasing chain of open elements (by 4.5, since H_∞ contains an infinite increasing chain of open elements), an infinite set of incomparable open elements (by 4.19: the u_i, $i \geq 2$, $i \in N$ are incomparable) and an infinite decreasing chain of open elements (by 4.24).

4.25 Corollary. There are $2^{\aleph_0}$ S.I. interior algebras generated by one element.

Proof. We use the notation of 4.19. Let $X^1_{\underline{a}} = X_{\underline{a}} \cup \{\infty\}$, $B^1_{\underline{a}} = B_{\underline{a}} \cup \{X^1_{\underline{a}}\}$, and L^1 the interior algebra $P(X^1_{\underline{a}})$ with $B^1_{\underline{a}}$ as base for the open elements. Let $L^1_{\underline{a}} = [x_{\underline{a}}] \subseteq L^1$. Then $x^o_{\underline{a}} = \{1\}$, thus $X^+_{\underline{a}} \in [x_{\underline{a}}]$, and $\{\infty\} = X^{+'}_{\underline{a}} . X^{+'o'}_{\underline{a}}$ provided that $\underline{a}$ is not the sequence $(a_n)_n$ with $a_n = 0$, for all $n \in N$. Hence $L_{\underline{a}} \cong (\{\infty\}'] \subseteq [x_{\underline{a}}] = L^1_{\underline{a}}$, and $L^{1o}_{\underline{a}} = L^o_{\underline{a}} \oplus 1$, thus $L^1_{\underline{a}}$ is S.I. and generated by one element. From 4.20 it follows that there are $2^{\aleph_0}$ interior algebras $L^1_{\underline{a}}$.□

4.26 Recall that an algebra L is called ($\underline{m}$)-universal for a class $\underline{K}$ of algebras if ($|L| \leq \underline{m}$ and) for every $L_1 \in \underline{K}$, ($|L_1| \leq \underline{m}$), $L_1 \in S(L)$. An interior algebra L will be called a generalized

$(\underline{m})$-universal algebra for a class $\underline{K}$ of interior algebras if $(|L| \le \underline{m}$ and) for all $L_1 \in \underline{K}$, $(|L_1| \le \underline{m})$, there exists a $u \in L^O$ such that $L_1 \in S((u])$ (cf. McKinsey and Tarski [44] pg. 151). In 4.3 we showed that $F_{\underline{B}_i}(n)$ is a generalized universal algebra for all finite interior algebras, generated by n elements.

4.27 Corollary. There does not exist an interior algebra which is $\aleph_0$--universal for $\underline{B}_i$. Neither does there exist an interior algebra, generalized $\aleph_0$-universal for $\underline{B}_i$.

Proof. Suppose $L \in \underline{B}_i$ is generalized $\aleph_0$-universal for $\underline{B}_i$. Since $|L| \le \aleph_0$, there are at most countably many $u \in L^O$, and every $(u]$ has at most countably many subalgebras generated by one element. Therefore it is impossible that every one of the $2^{\aleph_0}$ interior algebras generated by one element can be embedded in some $(u]$, $u \in L^O$.□

The results 4.21-4.27 have their obvious counterparts for $\underline{B}_i^-$, using in the constructions H_∞^+ instead of H_∞. We state two of the results without proof:

4.28 Theorem. There exist $2^{\aleph_0}$ (subdirectly irreducible) algebras in $\underline{B}_i^-$ generated by one element.

4.29 Theorem. There does not exist an $\aleph_0$ - (generalized) universal generalized interior algebra for $\underline{B}_i^-$.

Section 5. Subalgebras of finitely generated free algebras in $\underline{B}_i$ and $\underline{B}_i^-$

We continue the study of finitely generated free (generalized) interior algebras, focussing our attention now to the notion of subalgebra. At this point, the difference between generalized interior algebras and interior algebras becomes remarkable. For example, in 5.6 we show that for each $n \in N$ $F_{\underline{B}_i^-}(n)$ contains a proper subalgebra isomorphic to itself; we were not able to prove such a theorem for $F_{\underline{B}_i}(n)$.

A natural question is whether perhaps for some $n,m \in N$, $n < m$, $F_{\underline{B}_i}(m) \in S(F_{\underline{B}_i}(n))$ or $F_{\underline{B}_i^-}(m) \in S(F_{\underline{B}_i^-}(n))$, as is the case in the variety $\underline{L}$ of lattices where even $F_{\underline{L}}(\aleph_0) \in S(F_{\underline{L}}(3))$.
In the next section we shall answer this question negatively. However, the Brouwerian algebra $F_{\underline{B}_i^-}(1)^o$ has a property which reminds us of this situation. It will be shown, namely, that $F_{\underline{H}^-}(n) \in S(F_{\underline{B}_i^-}(1)^o)$ for each $n \in N$ (theorem 5.11). The description of $F_{\underline{H}}(n)$ given in Urquhart [73] only emphasizes how complicated apparently the structure of even $F_{\underline{B}_i^-}(1)^o$ is.

We start recalling a result from McKinsey and Tarski [46].

5.1 Theorem. $F_{\underline{H}}(n)$ is a subalgebra of $F_{\underline{B}_i}(n)^o$, $n \in N$.
Proof. Let $x_1,x_2,\ldots x_n$ be free generators of $F_{\underline{B}_i}(n)$ and consider $L = [\{x_1^o,x_2^o,\ldots x_n^o\}]_{\underline{H}}$. We claim that $L = F_{\underline{H}}(n)$. Indeed, let $L_1 \in \underline{H}$ and $h: \{x_1^o,\ldots x_n^o\} \longrightarrow L_1$ a map. Define $h_1: \{x_1\ldots x_n\} \longrightarrow B(L_1)$ by $h_1(x_i) = h(x_i^o)$. Let g be the $\underline{B}_i$-homomorphism $F_{\underline{B}_i}(n) \longrightarrow B(L_1)$ extending h_1, then $g(x_i^o) = g(x_i)^o = h_1(x_i)^o = h(x_i^o)^o = h(x_i^o)$ and by 2.11 $g \mid F_{\underline{B}_i}(n)^o: F_{\underline{B}_i}(n)^o \longrightarrow L_1$ is an $\underline{H}$-homomorphism. Hence $g \mid L: L \longrightarrow L_1$ is the desired extension of h. $\square$

Similarly we show:

5.2 <u>Theorem</u>. $F_{\underline{H}^-}(n)$ is a subalgebra of $F_{\underline{B}_i^-}(n)^o$, $n \in N$.

5.3 <u>Corollary</u>. $B(F_{\underline{H}}(n)) \in S(F_{\underline{B}_i}(n))$, $B^-(F_{\underline{H}^-}(n)) \in S(F_{\underline{B}_i^-}(n))$, $n \in N$.

It is not true that $F_{\underline{B}_i}(n)^o \cong F_{\underline{H}}(n)$ or $F_{\underline{B}_i^-}(n)^o \cong F_{\underline{H}^-}(n)$. We have even

5.4 <u>Theorem</u>. $F_{\underline{B}_i}(n)^o$ and $F_{\underline{B}_i^-}(n)^o$ are not finitely generated, $n \in N$.
<u>Proof</u>. The algebra K_∞, introduced in section 3, is $\underline{B}_i^-$-generated by one element. Hence $K_\infty \in H(F_{\underline{B}_i}(n))$, $K_\infty \in H(F_{\underline{B}_i^-}(n))$, for any $n \in N$ and thus by 2.11 $K_\infty^o \in H(F_{\underline{B}_i}(n)^o)$, $K_\infty^o \in H(F_{\underline{B}_i^-}(n)^o)$, $n \in N$. But K_∞^o is an infinite chain, which apparently is not finitely $\underline{H}$- or $\underline{H}^-$-generated.□

5.5 <u>Corollary</u>. $F_{\underline{B}_i}(n)$ and $F_{\underline{B}_i^-}(n)$ contain a subalgebra which is not finitely generated, $n \in N$.
<u>Proof</u>. $B(F_{\underline{B}_i}(n)^o)$ and $B(F_{\underline{B}_i}(n)^o)$ are such subalgebras. Indeed, suppose, for example, that $B(F_{\underline{B}_i}(n)^o)$ is generated by $y_1, y_2, \dots y_n$. There exist $u_1^i, u_2^i, \dots u_{n_i}^i$, $v_1^i, v_2^i, \dots v_{n_i}^i \in F_{\underline{B}_i}(n)^o$, $i = 1 \dots k$, such that $y_i = \sum_{j=1}^{n_i} u_j^i v_j^i$, $i = 1 \dots k$. Then

$$[\{u_j^i, v_j^i \mid j = 1 \dots n_i\ ,\ i = 1 \dots k\}]_{\underline{B}_i} = B(F_{\underline{B}_i}(n)^o)\ ,$$

and hence, by 2.14,

$$[\{u_j^i, v_j^i \mid j = 1 \dots n_i\ ,\ i = 1 \dots k\}]_{\underline{H}} = F_{\underline{B}_i}(n)^o\ ,$$

which would contradict 5.4.□

Next we wish to identify some interesting finitely generated subalgebras, especially in $F_{\underline{B}_i^-}(n)$, $n \in N$.

5.6 <u>Theorem</u>. $F_{\underline{B}_i^-}(n)$ contains a proper subalgebra, isomorphic to $F_{\underline{B}_i^-}(n)$, for any $n \in N$.

<u>Proof</u>. Recall that $F_{\underline{B}_i^-}(n)$ has a smallest element, which shall be denoted by a (cf. 4.8.(i)). Let $x_1 \ldots x_n$ be free generators of $F_{\underline{B}_i^-}(n)$, then $a = \prod_{i=1}^{n} x_i^o$. Let L be the Boolean algebra $\underline{2} \times F_{\underline{B}_i^-}(n)$, provided with an interior operator given by

$$(x,y)^o = \begin{cases} (0,a) & \text{if} \quad x = 0 \\ (1,y^o) & \text{if} \quad x = 1 \end{cases}$$

Note that $L^o \cong \underline{2} + F_{\underline{B}_i^-}(n)^o$. L is generated by the elements $(0,x_1 \Rightarrow a),(1,x_2),\ldots(1,x_n)$. Indeed, $(0,x_1 \Rightarrow a)^o = (0,a)$, $(0,x_1 \Rightarrow a) \Rightarrow (0,a) = (1,x_1)$, and $(\prod_{i=1}^{n}(1,x_i))^o \Rightarrow (0,a) = (0,1)$, and it is clear that the elements $(0,1),(1,x_1)\ldots(1,x_n)$ generate L. Since L is $\underline{B}_i^-$-generated by n elements there exists a surjective homomorphism $f\colon F_{\underline{B}_i^-}(n) \longrightarrow L$. The map $i\colon F_{\underline{B}_i^-}(n) \longrightarrow L$ defined by $i(x) = (1,x)$ is an embedding.

Let $g\colon F_{\underline{B}_i^-}(n) \longrightarrow F_{\underline{B}_i^-}(n)$ be the homomorphism extending a map satisfying $g(x_i) \in f^{-1}(i(x_i))$, $i = 1,2\ldots n$. Then g is an embedding, not onto.□

$$\begin{tikzcd} F_{\underline{B}_i^-}(n) \arrow[r, "i"] \arrow[d, dashed, "g"'] & L \\ F_{\underline{B}_i^-}(n) \arrow[ur, "f"'] & \end{tikzcd}$$

5.7 <u>Corollary</u>. $F_{\underline{B}_i^-}(n)$ contains an infinite decreasing chain of different subalgebras isomorphic to itself.

We have not been able to determine whether or not a proposition similar to 5.6 holds for $F_{\underline{B}_i}(n)$, $n \in N$.

Our next object is to show that $F_{\underline{B}_i^-}(1)^o$ contains even $F_{\underline{H}^-}(n)$

as a subalgebra, for every $n \in N$.

5.8 Lemma. Let $L \in \underline{B}_i$ be $\underline{B}_i$-generated by a finite chain of open elements. Then there exists an $L_1 \in \underline{B}_i$ with the following properties:

(i) L_1 is $\underline{B}_i$-generated by a single element

(ii) there is an $a \in L_1^o$ such that $[a) \cong L$

(iii) L_1 is a $\star$-algebra

(iv) if L is finite then so is L_1.

Proof. Recall that the algebra $H_m \cong (c_m] \subseteq H_\infty$ contains a chain of open elements $0 < c_1 < \ldots < c_m = 1$, and is generated by the element c_1 (see 3.8, 3.10). Let L be $\underline{B}_i$-generated by the open elements $0 = d_1 < d_2 < \ldots < d_m = 1$, for some $m \in N$, $m > 1$. Let L_1 be the Boolean algebra $H_m \times L$, provided with the interior operator ^{o} given by

$$(x,y)^o = (x^o, y^o d_i) \quad \text{where} \quad i = \max\{j \mid c_j \leq x^o\}.$$

^{o} is an interior operator:

(i) $(1,1)^o = (1^o, 1^o.d_m) = (1,1)$

(ii) $(x,y)^o \leq (x,y)$

(iii) $(x,y)^{oo} = (x^o, y^o d_i)^o = (x^{oo}, y^o d_i.d_i) = (x^o, y^o d_i) =$
$= (x,y)^o$, where i is as in the definition.

(iv) $((x,y).(x_1,y_1))^o = (xx_1, yy_1)^o = (x^o x_1^o, y^o y_1^o d_i)$,

where $i = \max\{j \mid c_j \leq x^o x_1^o\}$. On the other hand

$$(x,y)^o.(x_1,y_1)^o = (x^o, y^o.d_k).(x_1^o, y_1^o.d_\ell) = (x^o x_1^o, y^o y_1^o.d_k d_\ell),$$

where $k = \max\{j \mid c_j \leq x^o\}$, $\ell = \max\{j \mid c_j \leq x_1^o\}$. If $k \leq \ell$ then $\max\{j \mid c_j \leq x^o x_1^o\} = k = i$, hence $(x,y)^o.(x_1,y_1)^o = (x^o x_1^o, y^o y_1^o d_i)$, as was to be shown.

Thus $L \in \underline{B}_i$.

(i) L_1 is $\underline{B}_i$-generated by the element $(c_1,1)$. Indeed $(c_1,1)^o = (c_1,0)$, hence $(0,1) = (c_1,0)'.(c_1,1) \in [(c_1,1)]$, and therefore also $(1,0) \in [(c_1,1)]$. Further

$(c_i,0) = q_i((c_1,0)).(1,0) \in [(c_1,1)]$, where q_i is as in 3.6, and $(0,d_i) = (0,1).((c_i,0) + (0,1))^o \in [(c_1,1)]$.

Now if $x \in H_m$, $y \in L$, let p_x,q_y be $\underline{B}_i$-polynomials such that $x = p_x(c_1)$, $y = q_y(d_1,d_2,\ldots d_m)$. Then

$$(x,y) = (1,0).p_x((c_1,0)) + (0,1).q_y((0,d_1),\ldots(0,d_m)) \in [(c_1,1)]_{\underline{B}_i}.$$

(ii) $L \cong [(1,0))$, since for any $y \in L$, $(1,y)^o = (1,y^o)$

(iii) L_1 is $\underline{B}_i$-generated by the open elements $(c_1,0),(1,0)$, $(1,d_1),\ldots(1,d_m)$. Hence $L \cong B([\{(c_1,0),(1,0),(1,d_1)\ldots(1,d_m)\}]_{\underline{H}})$, thus L is a $\star$-algebra.

(iv) *Since H_m is finite, $H_m \times L$ is finite if L is finite.*□

5.9 Lemma. Let $L \in \underline{B}_i^-$ be $\underline{B}_i^-$-generated by a finite chain of open elements. Then there exists an $L_1 \in \underline{B}_i^-$ with the following properties:

(i) L_1 is $\underline{B}_i^-$-generated by a single element

(ii) there is an $a \in L_1^o$ such that $[a) \cong L$, thus L is a subalgebra of L_1

(iii) if L is finite then L_1 is finite.

There exists also an $L_2 \in \underline{B}_i^-$, such that L_2 is a $\star$-algebra, L_2 is $\underline{B}_i^-$-generated by two elements, and L_2 satisfies (ii) and (iii).

Proof. It is easy to see that $L_1 = H_m^+ \times L$ with the interior operator defined as in 5.8 works if L is generated by a chain of open elements $0 = d_1 < \ldots < d_m = 1$. (For the definition of H_m^+ see 3.12). H_m^+ is not a $\star$-algebra, however. In order to save that property, we can use $H_m \times L$, noting that H_m is $\underline{B}_i^-$-generated by the elements $0,c_1$; therefore $H_m \times L$, endowed again with the interior operator of 5.8, is $\underline{B}_i^-$-generated by two elements.□

5.10 Lemma. Let $L \in \underline{H}^-$ or $L \in \underline{H}$ be finitely generated. Then there exists a finite chain which generates L.

Proof. We prove the lemma in case $L \in \underline{H}^-$, proceeding by induction on the number of generators of L. If L is generated by one element, there is nothing to prove. Suppose the theorem has been proved for all $L \in \underline{H}^-$ generated by $m-1$ elements, and let $L = [\{x_1,\dots x_m\}]_{\underline{H}^-}$, $m > 1$. $L_1 = [\{x_1 \dots x_{m-1}\}]_{\underline{H}^-}$ is then $\underline{H}^-$-generated by a chain, say by $D = \{d_0 < d_1 < \dots < d_n\}$, $D \subseteq L_1$. Then L is generated by

$$D' = \{d_0 x_m \le d_0 \le d_0 + d_1 x_m \le \dots \le d_i + d_{i+1} x_m \le \\ \le d_{i+1} \le d_{i+1} + d_{i+2} x_m \le \dots \le d_n \le d_n + x_m\}.$$

Note that if $0 \le i < n$,

$$(d_i + d_{i+1}x_m)(d_i \to d_i x_m) = d_i(d_i \to d_i x_m) + d_{i+1}x_m(d_i \to d_i x_m) = \\ = d_i x_m + d_{i+1} x_m = d_{i+1} x_m .$$

Therefore if $d_i x_m \in [D']_{\underline{H}^-}$, for some i, $0 \le i < n$, then $d_{i+1}x_m = (d_i + d_{i+1}x_m)(d_i \to d_i x_m) \in [D']_{\underline{H}^-}$. Since $d_0 x_m \in [D']_{\underline{H}^-}$, it follows that $d_n x_m \in [D']_{\underline{H}^-}$, hence also

$$x_m = (d_n + x_m)(d_n \to d_n x_m) \in [D']_{\underline{H}^-} .$$

Therefore $x_1, x_2, \dots x_m \in [D']_{\underline{H}^-}$, and $L = [D']_{\underline{H}^-}$. □

5.11 Theorem. $F_{\underline{B}_i^-}(1)^o$ contains $F_{\underline{H}^-}(n)$ as a subalgebra, for every $n \in N$.

Proof. By 5.10 $F_{\underline{H}^-}(n)$ is $\underline{H}^-$-generated by a finite chain of elements, hence $B(F_{\underline{H}^-}(n))$ is $\underline{B}_i^-$-generated by a finite chain of open elements. Let $L_1 \in \underline{B}_i^-$ be the interior algebra whose existence is guaranteed by lemma 5.9, that is, L_1 is $\underline{B}_i^-$-generated by one element and there exists an $a \in L_1^o$, such that $(a] \cong B(F_{\underline{H}^-}(n))$. Hence $F_{\underline{H}^-}(n) \in S(L_1^o)$, $L_1^o \in H(F_{\underline{B}_i^-}(1)^o)$, and because $F_{\underline{H}^-}(n)$ is free, it follows that $F_{\underline{H}^-}(n) \in S(F_{\underline{B}_i^-}(1)^o)$. □

5.12 <u>Corollary</u>. $F_{\underline{B}_i}-(1)$ contains $B(F_{\underline{H}}-(n))$ as a subalgebra, for every $n \in N$.

For $\underline{B}_i$ these results assume the following form.

5.13 <u>Theorem</u>. For each $n \in N$, there exists a $b \in F_{\underline{B}_i}(1)^o$ such that the Heyting algebra $[b)^o$ contains $F_{\underline{H}}(n)$ as a subalgebra.

<u>Proof</u>. By 5.10, $F_{\underline{H}}(n)$ is $\underline{H}$-generated by a finite chain, hence $B(F_{\underline{H}}(n))$ is $\underline{B}_i$-generated by a finite chain of open elements. By 5.8 there exists an algebra $L_1 \in \underline{B}_i$, generated by one element, and containing an element $a \in L_1^o$, such that $[a) \cong B(F_H(n))$. Let $f: F_{\underline{B}_i}(1) \longrightarrow L_1$ be an onto $\underline{B}_i$-homomorphism, and let $b \in f^{-1}(\{a\})$. We may assume that $b = b^o$. Then $\bar{f} = f \mid [b): [b) \to [a)$ is a $\underline{B}_i$-homomorphism. Furthermore, $\bar{f}$ is onto: if $y \in [a)$ let $x \in F_{\underline{B}_i}(1)$ be such that $f(x) = y$. Then $\bar{f}(x + b) = f(x + b) = f(x) + f(b) = y + a =$ $= y$, and $x + b \in [b)$. Since $F_{\underline{H}}(n)$ is free, it follows that $[b)^o$ contains $F_{\underline{H}}(n)$ as a subalgebra.□

5.14 <u>Corollary</u>. For each $n \in N$ there exists a $b \in F_{\underline{B}_i}(1)^o$ such that the interior algebra $[b)$ contains $B(F_{\underline{H}}(n))$ as a subalgebra.

Section 6. Functional freeness of finitely generated algebras in $\underline{B}_i$ and $\underline{B}_i^-$

Recall that an algebra L in a variety $\underline{K}$ is called functionally free in or characteristic for $\underline{K}$ if $V(L) = \underline{K}$. For any variety $\underline{K}$, $F_{\underline{K}}(\aleph_0)$ is functionally free (cf. 0.1.20,0.1.21). If $\underline{L}$ is the variety of lattices, then $F_{\underline{L}}(3)$ is functionally free, since $F_{\underline{L}}(\aleph_0) \in S(F_{\underline{L}}(3))$; $\underline{2}$ is functionally free in $\underline{B}$. McKinsey and Tarski [44] have shown that no finitely generated interior algebra can be functionally free in $\underline{B}_i$. Their proof is based on the fact that the interior algebra with trivial interior operator M_k, where $M_k \underset{\underline{B}}{\cong} \underline{2}^k$, $M_k^o \cong \underline{2}$, does not belong to $V(F_{\underline{B}_i}(n))$, if we choose $k \in N$ large enough. The question therefore comes up whether perhaps $F_{\underline{B}_i}(n)^o$ is characteristic for $\underline{H}$, or, loosely speaking, whether $F_{\underline{B}_i}(n)$ is characteristic for $\underline{B}_i$ as far as the lattices of open elements are concerned. We shall show that this is not the case, that is, $V(F_{\underline{B}_i}(n)^o) \neq \underline{H}$ for all $n \in N$ (theorem 6.4). However, it follows easily from the results of the previous section that $V(F_{\underline{B}_i^-}(1)^o) = \underline{H}^-$. Essentially, this means that the only reason why $F_{\underline{B}_i}(1)^o$ is not characteristic for $\underline{H}$ is the presence of the 0 as a nullary operation.

Immediately the question comes to mind if McKinsey and Tarski's result that $V(F_{\underline{B}_i}(n)) \neq \underline{B}_i$ finds its origin in a similar phenomenon. The second part of this section is devoted to that question. In 6.10 we prove that for all $n \in N$ $V(F_{\underline{B}_i^-}(n)) \neq \underline{B}_i^-$. This shows that the situation here is substantially different: whereas $F_{\underline{B}_i}(n)^o$ is not characteristic for $\underline{H}$ since it is not "general enough" near the 0 which is there as a nullary operation, $F_{\underline{B}_i}(n)$ is "nowhere" characteristic for $\underline{B}_i$. In order to arrive at this result we introduce in 6.7 the rank of triviality of a (generalized) interior algebra, which measures how far the algebra is from being a $*$-algebra. This notion gives rise to a strictly increasing chain $\underline{T}_0^-, \underline{T}_1^-, \ldots$ of subvarieties of $\underline{B}_i^-$ with the property that $\underline{T}_n^- = V(F_{\underline{B}_i^-}(n))$, $n \in N$ (theorem 6.14), which implies that $V(F_{\underline{B}_i^-}(n)) \neq \underline{B}_i^-$.

Before starting our main subject we wish to give some more information concerning the algebras under consideration. First we need a definition.

6.1 <u>Definition</u>. Let $L \in \underline{B}_i$ or $L \in \underline{B}_i^-$. If $L^o \cong \underline{2}$ then the interior operator on L is called a <u>trivial</u> interior operator. The finite interior algebra with k atoms and trivial interior operator will be denoted M_k, $k \in N$. Thus $M_k = \underline{2}^n$, $M_k^o \cong \underline{2}$.

Note that if L is an interior algebra, then the open atoms of L are atoms of the Heyting algebra L^o; the atoms of L^o need not be atoms of L, however.

6.2 <u>Theorem</u>. Let x be a free generator of $F_{\underline{B}_i}(1)$. Then

(i) x^o, x'^o are the only open atoms of $F_{\underline{B}_i}(1)$

(ii) $F_{\underline{B}_i}(1)^o$ has three atoms: x^o, x'^o and an atom a, where $(a] \cong M_2$.

(iii) if $0 \neq u \in F_{\underline{B}_i}(1)^o$, then $x^o \leq u$, $x'^o \leq u$ or $u = a$.

<u>Proof</u>. (i) is a special case of 4.13.

(ii) Let $a = (x^{o'}x'^{o'})^o = x^{o'o}x'^{o'o}$. If b is an atom of the algebra M_2, then $b^{o'o}.b^{'o'o} = 1$. Since x is a free generator it follows that $a \neq 0$. Furthermore $xa \neq 0$, since otherwise $a \le x'^{o}$. Because also $a \le x'^{o'}$, it would follow that $a \le x'^{o}.x'^{o'} = 0$, which is impossible. Similarly $x'a \neq 0$. Finally $(xa)^o = x^o a = 0$, $(x'a)^o = x'^{o}a = 0$, therefore $(a] = [xa]_{\underline{B}_i} = [xa]_{\underline{B}} \underset{\underline{B}}{\cong} \underline{2}^2$, $(a]^o = \{0,a\}$, so $(a] \cong M_2$.

(iii) Let $0 \neq u \in F_{\underline{B}_i}(1)^o$, $x^o \not\le u$, $x'^{o} \not\le u$. Then $u \le (x^{o'}.x'^{o'})^o$, thus $u = a$.□

In particular we see that $F_{\underline{B}_i}(1)^o$ has finitely many atoms and that for every $u \in F_{\underline{B}_i}(1)^o$, $u \neq 0$, there exists an atom in $F_{\underline{B}_i}(1)^o$ contained in u. The same is true in a more general case.

6.3 __Lemma__. Let $L \in \underline{B}_i$ be finitely generated.

(i) L^o has at least one atom.

(ii) L^o has only finitely many atoms.

__Proof__. (i) Let $L = [\{x_1,x_2,\ldots x_n\}]_{\underline{B}_i}$. We may assume that $x_i x_j = 0$ if $1 \le i < j \le n$ and that $\sum_{i=1}^{n} x_i = 1$. Let $A \subseteq \{x_1 \ldots x_n\}$ be a minimal set such that $(\Sigma A)^o \neq 0$. Such a set exists, since $(\Sigma\{x_1,\ldots,x_n\})^o = 1 \neq 0$, and is non-empty. Let $a = (\Sigma A)^o$, then $(a]$ is generated, as interior algebra, by $\{x_i a \mid x_i \in A\}$. Indeed, $(a]$ is generated by $x_1 a, \ldots x_n a$, but if $x_i \notin A$, then $x_i a = x_i(\Sigma A)^o \le x_i.(\Sigma A) = 0$. Let $L_1 = [\{x_i a \mid x_i \in A\}]_{\underline{B}} \subseteq (a]$. If $y \in L_1$, then $y = \Sigma\{x_i a \mid x_i \in A'\}$ for some $A' \subseteq A$. Hence

$$y^o = (\Sigma\{x_i a \mid x_i \in A'\})^o =$$
$$= ((\Sigma\{x_i \mid x_i \in A'\}).a)^o = (\Sigma\{x_i \mid x_i \in A'\})^o.a\ ,$$

thus $y^o = 0$ if $A' \subset A$, and $y^o = a$ if $A' = A$. Hence $L_1 = [\{x_i a \mid x_i \in A\}]_{\underline{B}_i} = (a]$, and $L_1^o = \{0,a\}$. So a is an atom in L^o.

(ii) Let $a \in L^o$ be an atom in L^o. Then $(a]$ is an interior algebra, generated by $x_1a, x_2a, \ldots x_na$, such that $(a]^o = \{0,a\}$. Therefore $[\{x_1a,x_2a,\ldots x_na\}] = [\{x_1a,x_2a,\ldots x_na\}]$, thus $|(a]| \leq 2^{2^n}$. There are only finitely many homomorphisms $L \to M_{2^n}$, and since different atoms in L^o give rise to different homomorphisms $L \to M_{2^n}$, it follows that there are only finitely many atoms in L^o.□

We are now in a position to prove

6.4 <u>Theorem</u>. $F_{\underline{B}_i}(n)^o$ is not characteristic for $\underline{H}$, for any $n \in N$. Hence there exists no finitely generated $L \in \underline{B}_i$, such that L^o is characteristic for $\underline{H}$.

<u>Proof</u>. By 6.3 (ii) we know that $F_{\underline{B}_i}(n)^o$ has finitely many, say k atoms. Note that since for any $u \in F_{\underline{B}_i}(n)^o$ $(u]$ is a finitely generated interior algebra it follows from 6.3 (i) that every $u \in F_{\underline{B}_i}(n)^o$, $u \neq 0$, contains an atom of $F_{\underline{B}_i}(n)^o$. Let $m \in N$ be such that $2^m > k$, and consider the equation $\sum_{f\in\{1,2\}^m} (\prod_{i=1}^m x_i^{f(i)o})'^o = 1$, where

$$x_i^{\varepsilon_i} = \begin{cases} x_i & \text{if } \varepsilon_i = 1 \\ x_i' & \text{if } \varepsilon_i = 2. \end{cases}$$

If $f,g \in \{1,2\}^m$, $f \neq g$, then $\prod_{i=1}^m x_i^{f(i)o} \cdot \prod_{i=1}^m x_i^{g(i)o} = 0$.
If we evaluate the left hand side of the given equation in $F_{\underline{B}_i}(n)$ for any assignment of $x_1,x_2,\ldots x_m$, then one of the terms $\prod_{i=1}^m x_i^{f(i)o}$ will get the value 0, since otherwise we would have $> k$ disjoint non zero open elements, each of which would contain an atom of $F_{\underline{B}_i}(n)^o$, which is impossible. Therefore $\prod_{i=1}^m x_i^{f(i)o} = 0$ for some $f \in \{1,2\}^m$, and hence $\sum_{f\in\{1,2\}^m} (\prod_{i=1}^m x_i^{f(i)o})'^o = 1$, for any assignment of $x_1,x_2,\ldots x_m$ in $F_{\underline{B}_i}(n)$.

Now let $L = B(F_{\underline{B}}(m) \oplus 1)$, $L^o = F_{\underline{B}}(m) \oplus 1 \cong 2^{2^m} \oplus 1$, and let $a_1, a_2, \ldots a_m \in (1_{F_{\underline{B}}(m)}]$ be the free Boolean generators of $(1_{F_{\underline{B}}(m)}]$. In L, $\prod_{i=1}^{m} a_i^{f(i)o}$ is an open atom and $(\prod_{i=1}^{m} a_i^{f(i)o})'^{o} \le 1_{F_{\underline{B}}(m)} < 1$. Hence $\sum_{f \in \{1,2\}^m} (\prod_{i=1}^{m} a_i^{f(i)o})'^{o} \le 1_{F_{\underline{B}}(m)} < 1$, and therefore the identity $\sum_{f \in \{1,2\}^m} (\prod_{i=1}^{m} x_i^{f(i)o})'^{o} = 1$ is not satisfied in L. We conclude that $L \notin V(F_{\underline{B}_i}(n))$.
Now it is easy to see (cf. section II.1), that if $\underline{K} \subseteq \underline{B}_i$ is a variety, then $\underline{K}^o = \{L^o \mid L \in \underline{K}\} \subseteq \underline{H}$ is also a variety. Since $F_{\underline{B}_i}(n)^o \in V(F_{\underline{B}_i}(n))^o$, it follows that $V(F_{\underline{B}_i}(n)^o) \subseteq V(F_{\underline{B}_i}(n))^o$. If $L^o \in V(F_{\underline{B}_i}(n)^o)$ then $L^o \in V(F_{\underline{B}_i}(n))^o$, hence $L = B(L^o) \in V(F_{\underline{B}_i}(n))$, contradicting the conclusion just arrived at. Therefore $L^o \notin V(F_{\underline{B}_i}(n)^o)$, and $F_{\underline{B}_i}(n)^o$ is not functionally free in $\underline{H}$. □

We thus obtained at the same time a new proof of Tarski and McKinsey's result, that $F_{\underline{B}_i}(n)$ is not characteristic for $\underline{B}_i$, for any $n \in N$. The corollary following now is a theorem of McKinsey and Tarski [46].

6.5 <u>Corollary</u>. $F_{\underline{H}}(n)$ is not characteristic for $\underline{H}$, for any $n \in N$.
<u>Proof</u>. By 5.1 and 6.4. □

Roughly speaking, the proof of 6.4 shows that $F_{\underline{B}_i}(n)^o$ is not general enough near the 0 element to be characteristic for $\underline{H}$. The presence of the 0 as a nullary operation seems to be crucial. And indeed, the results of the previous section enable us to state

6.6 <u>Theorem</u>. $F_{\underline{B}_i^-}(1)^o$ is characteristic for $\underline{H}^-$.

Proof. By 5.11, for every $n \in N$ $F_{\underline{H}^-}(n) \in S(F_{\underline{B}_i^-}(1)^o)$. Hence $\underline{H}^- = V(\{F_{\underline{H}^-}(n) \mid n \in N\}) \subseteq V(F_{\underline{B}_i^-}(1)^o) \subseteq \underline{H}^-$.□

This implies that $\underline{H} \subseteq \underline{H}^- = V(F_{\underline{B}_i^-}(1)^o) = V((F_{\underline{B}_i}(1)^o)^-)$.

As mentioned before we still wish to answer the question of the functional freeness of the algebras $F_{\underline{B}_i^-}(n)$ in $\underline{B}_i^-$. It shall be answered in the negative, and in order to arrive at that conclusion we shall show that if $n \in N$ we can choose $k \in N$ large enough such that M_k does not belong to $V(F_{\underline{B}_i^-}(n))$. In doing so we use the same approach Tarski and McKinsey used in their proof of the non-characteristicity of $F_{\underline{B}_i}(n)$ for $\underline{B}_i$, for any $n \in N$. However, it seems not possible to modify their proof so as to make it applicable to the $\underline{B}_i^-$-case. Our argument will be quite different, and it will at the same time provide an elegant proof of their result.

6.7 Let $L \in \underline{B}_i$. The rank of triviality $r_T(L)$ of L is the smallest cardinal number $\underline{m}$ such that there exists a set $X \subseteq L$, $|X| = \underline{m}$, with the property that $L = [B(L^o) \cup X]_{\underline{B}}$. If $L \in \underline{B}_i^-$, $r_T(L)$ is defined similarly, the set X now having the property that $L = [B^-(L^o) \cup X]_{\underline{B}^-}$. If L is a $*$-algebra, that is, $L = B(L^o)$, then apparently $r_T(L) = 0$. If L is an interior algebra with trivial interior operator, then $r_T(L)$ is just the rank of L considered as Boolean algebra.

6.8 Next we define a sequence of varieties $\underline{T}_n^- = V(\{L \in \underline{B}_i^- \mid r_T(L) \leq n\})$ and $\underline{T}_n = V(\{L \in \underline{B}_i \mid r_T(L) \leq n\})$, for $n = 0,1,2,\ldots$. Note that $\underline{T}_n^{(-)}$ may contain algebras L with $r_T(L) > n$; indeed, the algebra K_∞ introduced in 3.1 has $r_T(K_\infty) = 1$, since $K_\infty = [B(K_\infty^o) \cup \{x\}]_{\underline{B}}$

but $K_\infty \neq B(K_\infty^o)$ and $K_\infty \in SP(\{K_n \mid n > 0\})$ as one easily verifies, thus $K_\infty \in \underline{T}_0$

In the proof of the following theorem we generalize a method employed in McKinsey and Tarski [44] to prove that $\underline{B}_i$ is generated by its finite members.

6.9 Theorem. $\underline{T}_n^-$ and $\underline{T}_n$ are generated by their finite members of rank of triviality $\leq n$, for $n = 0,1,2,\ldots$.

Proof. We prove the theorem for $\underline{T}_n$, $n \geq 0$. Suppose that

$$\underline{T}_n \neq V(\{L \in \underline{T}_n \mid L \text{ finite and } r_T(L) \leq n\}).$$

Then for some $\ell \in N$ there exists an ℓ-ary $\underline{B}_i$-polynomial p such that the equation $p(x_1,x_2,\ldots x_\ell) = 1$ is satisfied in

$$V(\{L \in T_n \mid L \text{ finite and } r_T(L) \leq n\})$$

but not in $\underline{T}_n$. Let $L \in \underline{T}_n$, $a_1,a_2,\ldots a_\ell \in L$ such that $p(a_1,a_2,\ldots a_\ell) \neq 1$. We may assume that $r_T(L) \leq n$. Let $q_i(x_1,x_2,\ldots x_\ell)$, $i = 1,2,\ldots m$ be a shortest sequence of $\underline{B}_i$-polynomials satisfying

$q_i(x_1,x_2,\ldots x_\ell) = 0,1$, or x_j for some j, $1 \leq j \leq \ell$, or

$q_i(x_1,x_2,\ldots x_\ell) = q_j(x_1,x_2,\ldots x_\ell) + q_k(x_1,x_2,\ldots x_\ell)$, $j,k < i$, or

$q_i(x_1,x_2,\ldots x_\ell) = q_j(x_1,x_2,\ldots x_\ell) \cdot q_k(x_1,x_2,\ldots x_\ell)$, $j,k < i$, or

$q_i(x_1,x_2,\ldots x_\ell) = q_j(x_1,x_2,\ldots x_\ell)'$, $j < i$, or

$q_i(x_1,x_2,\ldots x_\ell) = q_j(x_1,x_2,\ldots x_\ell)^o$, $j < i$, such that

$q_m(x_1,x_2,\ldots x_\ell) = p(x_1,x_2,\ldots x_\ell)$.

Thus the q_i are the sub-polynomials of p, ordered according to increasing complexity.

Let $b_i = q_i(a_1,a_2,\ldots a_\ell)$, $i = 1,2,\ldots m$. Let $y_1,y_2,\ldots y_n \in L$ be such that $L = [B(L^o) \cup \{y_1,y_2,\ldots y_n\}]_{\underline{B}}$. Every b_i, $i = 1,2,\ldots m$ can be represented in the form $\sum_{j=1}^{2^n} c_{ij}\, y_1^{f_j(1)}\, y_2^{f_j(2)} \ldots y_n^{f_j(n)}$, where $f_1, f_2, \ldots f_{2^n}$ are all possible maps $\{1,2,\ldots n\} \to \{1,2\}$, and

$y_k^{\varepsilon_k} = \begin{cases} y_k & \text{if } \varepsilon_k = 1 \\ y_k' & \text{if } \varepsilon_k = 2 \end{cases}$, and c_{ij}, $j = 1,2,\ldots 2^n$, belongs to $B(L^o)$.

In its turn, every c_{ij} can be written as

$$\sum_{k=1}^{n_{ij}} (u_k^{ij})' . v_k^{ij} , \text{ with } u_k^{ij}, v_k^{ij} \in L^o. \text{ Let}$$

$L_0 = B([\{u_k^{ij}, v_k^{ij} \mid k = 1,2,\ldots n_{ij}, \; i = 1,2,\ldots m, \; j = 1,2,\ldots 2^n\}]_{\underline{D}_{01}})$ and let $L_1 = [L_0 \cup \{y_1, y_2, \ldots y_n\}]_{\underline{B}}$. Since L_1 is finite, we may provide L_1 with an interior operator ^{o_1} by defining $x^{o_1} = \Sigma \{y \in L_1 \mid y \leq x, \; y^o = y\}$. It follows that $[\{u_k^{ij}, v_k^{ij} \mid k = 1,2,\ldots n_{ij}, \; i = 1,2,\ldots m, \; j = 1,2,\ldots 2^n\}]_{D_{01}} \subseteq L_1^{o_1}$, hence $L_1 = [B(L_1^{o_1}) \cup \{y_1, y_2, \ldots y_n\}]_{\underline{B}}$, which implies that $r_T(L_1) \leq n$. Though L_1 in general is not a subalgebra of L, we claim that the value of p evaluated at $a_1, a_2, \ldots a_\ell$ in L_1 equals the value of p evaluated at $a_1, a_2, \ldots a_\ell$ in L, or, in symbols: $p_{L_1}(a_1, a_2, \ldots a_\ell) = p_L(a_1, a_2, \ldots a_\ell)$. Indeed, if $q_{jL_1}(a_1, a_2, \ldots a_\ell) = q_{jL}(a_1, a_2, \ldots a_\ell)$, for all $j < i$, $i \in \{1,2,\ldots m\}$, then $q_{iL_1}(a_1, a_2, \ldots a_\ell) = q_{iL}(a_1, a_2, \ldots a_\ell)$. This is obvious if $q_i(x_1, x_2, \ldots x_\ell) = 0, 1$ or x_j , for some j, $1 \leq j \leq \ell$, and if $q_i = q_j + q_k$, $q_i = q_j . q_k$ or $q_i = q_j'$ for j, $k < i$, since L_1 is a $\underline{B}$-subalgebra of L. If $q_i = q_j^o$, then $q_{iL_1}(a_1, a_2, \ldots a_\ell) = q_{jL_1}(a_1, a_2, \ldots a_\ell)^{o_1} = b_j^{o_1} \leq b_j^o = b_i$, but by the definition of ^{o_1}, $b_i \leq b_j^{o_1}$, hence $b_j^{o_1} = b_i = q_{iL}(a_1, a_2, \ldots a_\ell)$ and therefore $q_{iL_1}(a_1, a_2, \ldots a_\ell) = q_{iL}(a_1, a_2, \ldots a_\ell)$.

Thus $p_{L_1}(a_1, a_2, \ldots a_\ell) \neq 1$, L_1 is finite and $r_T(L_1) \leq n$, contradictory to our assumption.□

6.10 <u>Lemma</u>. Let $L \in \underline{B}_i$ or $L \in \underline{B}_i^-$.

(i) If $L_1 \in H(L)$, then $r_T(L_1) \le r_T(L)$

(ii) If $a \in L^o$, then $r_T([a)) \le r_T(L)$.

<u>Proof</u>. Suppose that $L = [B(L^o) \cup X]_{\underline{B}}$, with $|X| = r_T(L)$.

(i) Let $f\colon L \to L_1$ be an onto $\underline{B}_i$-homomorphism. Then $L_1 = [B(L_1^o) \cup f[X]]_{\underline{B}}$ and $r_T(L_1) \le |f[X]| \le |X| = r_T(L)$.

(ii) Let $a \in L^o$. Then $[a) = [B([a)^o) \cup \{x + a \mid x \in X\}]_{\underline{B}}$. Indeed, if $z \in L$, then $z = \sum_{i=1}^{n} c_i \, \Pi X_i$, where X_i is a finite subset of $X \cup \{x' \mid x \in X\}$, $c_i \in B(L^o)$, $n \in N$. If $z \ge a$, then

$$z = \sum_{i=1}^{n} c_i \, \Pi X_i + a =$$

$$= \sum_{i=1}^{n} (c_i + a).\Pi\{x + a \mid x \in X_i\} \in [B([a)^o) \cup \{x + a \mid x \in X\}]_{\underline{B}},$$

since if $c_i = \sum_{j=1}^{k} u_j' v_j$, $u_j, v_j \in L^o$, then

$$c_i + a = \sum_{j=1}^{k} u_j' v_j + a =$$
$$= \sum_{j=1}^{k} (u_j' + a)(v_j + a) = \sum_{j=1}^{k} (u_j + a)^{'[a)}.(v_j + a) \in B([a)^o).$$

Therefore $r_T([a)) \le |\{x + a \mid x \in X\}| \le |X| = r_T(L)$.□

6.11 <u>Theorem</u>.

(i) $\underline{T}_0 \subset \underline{T}_1 \subset \dots \subset \underline{T}_n \subset \underline{T}_{n+1} \subset \dots \subset \underline{B}_i$, $V(\cup \underline{T}_n) = \underline{B}_i$

(ii) $\underline{T}_0^- \subset \underline{T}_1^- \subset \dots \subset \underline{T}_n^- \subset \underline{T}_{n+1}^- \subset \dots \subset \underline{B}_i^-$, $V(\cup \underline{T}_n^-) = \underline{B}_i^-$.

<u>Proof</u>. It is clear from the definition of $\underline{T}_n$, $\underline{T}_n^-$ that $\underline{T}_n \subseteq \underline{T}_{n+1}$, $\underline{T}_n^- \subseteq \underline{T}_{n+1}^-$, for $n = 0,1,2,\dots$. Furthermore, in McKinsey and Tarski [44] it is shown that $\underline{B}_i$ is generated by its finite members; in a similar way one can show that $\underline{B}_i^-$ is generated by its finite members. Obviously $\underline{B}_{iF} \subseteq \cup \underline{T}_n$, $\underline{B}_{iF}^- \subseteq \cup \underline{T}_n^-$, therefore $\underline{B}_i = V(\cup \underline{T}_n)$, $\underline{B}_i^- = V(\cup \underline{T}_n^-)$. We prove now, that $\underline{T}_n^- \subset \underline{T}_{n+1}^-$, $n = 0,1,2,\dots$. In a similar manner one can show that $\underline{T}_n \subset \underline{T}_{n+1}$, $n = 0,1,2,\dots$. Recall that $M_{2^{n+1}}^-$ denotes the generalized interior algebra with

trivial interior operator and 2^{n+1} atoms (cf. 2.26, 6.1). It is easy to verify that $r_T(M^-_{2^{n+1}}) = n + 1$, therefore $M^-_{2^{n+1}} \in \underline{T}^-_{n+1}$. We claim that $M^-_{2^{n+1}} \notin \underline{T}^-_n$. Since in M^-_k, $\forall\, x_1, x_2, \dots x_{2^k+1}$ $\bigvee_{1 \le i < j \le 2^k+1} x_i \Rightarrow x_j$, it follows that in M^-_k the following equation is satisfied:

$$\sum_{1 \le i < j \le 2^k+1} (x_i \Rightarrow x_j)^o (x_j \Rightarrow x_i)^o = 1$$

and therefore also

$$\Big(\big(\sum_{1 \le i < j \le 2^k+1} (x_i \Rightarrow x_j)^o (x_j \Rightarrow x_i)^o\big) \Rightarrow \prod_{i=1}^{2^k+1} x_i^o\Big)^o = \prod_{i=1}^{2^k+1} x_i^o .$$

Let the left hand side of this equation be called f_k.

The equation $f_{2^n}(x_1, x_2, \dots x_{2^{2^n}+1}) = \prod_{i=1}^{2^{2^n}+1} x_i^o$ is not satisfied in $M^-_{2^{n+1}}$: since $M^-_{2^{n+1}}$ has $2^{2^{n+1}} > 2^{2^n} + 1$ elements we may assign to $x_1, x_2, \dots x_{2^{2^n}+1}$ different values, in which case we obtain $1 = 0$, a contradiction. We claim however, that $f_{2^n}(x_1, x_2, \dots x_{2^{2^n}+1}) = \prod_{i=1}^{2^{2^n}+1} x_i^o$ is identically satisfied in $\underline{T}^-_n$. Suppose not. By 6.9 there exists a finite $L \in \underline{B}^-_i$, $r_T(L) \le n$, $a_1, a_2, \dots a_{2^{2^n}+1} \in L$ such that $f_{2^n}(a_1, a_2, \dots a_{2^{2^n}}) > \prod_{i=1}^{2^{2^n}+1} a_i^o = a$. $[a)$ is a finite generalized interior algebra, and by 6.10 (ii), $r_T([a)) \le n$. Let $b \in L^o$ be such that $a \prec_{L^o} b \le f_{2^n}(a_1, a_2, \dots a_{2^{2^n}})$. Such a b exists, since L is finite, $f_{2^n}(a_1, a_2, \dots a_{2^{2^n}+1})$ is open and $> a$. Then $[a,b] \in H([a))$, hence by 6.10 $r_T([a,b]) \le n$ and since $[a,b]^o = \{a,b\}$, $[a,b] = M^-_k$ for some k, $1 \le k \le 2^n$. By the remark made above, $f_{2^n}(a_1 b, a_2 b, \dots a_{2^{2^n}+1} . b) = a$ in $[a,b]$. But on the other hand $f_{2^n}(a_1 b, a_2 b, \dots a_{2^{2^n}+1} . b) = f_{2^n}(a_1, a_2, \dots a_{2^{2^n}+1}).b = b$. Since $a \neq b$, *we arrived at a contradiction.*

Thus we have found an equation, identically satisfied by $\underline{T}_n^-$, but not by $M^-_{2^{n+1}}$. Therefore $M^-_{2^{n+1}} \notin \underline{T}_n^-$. □

6.12 Corollary. $F_{\underline{B}_i^-}(n)$ is not characteristic for $\underline{B}_i^-$, for any $n \in N$. Likewise, $F_{\underline{B}_i}(n)$ is not characteristic for $\underline{B}_i$, for any $n \in N$.

Proof. Let $x_1, x_2, \dots x_n$ be free generators of $F_{\underline{B}_i^-}(n)$. Then $F_{\underline{B}_i^-}(n) = [B(F_{\underline{B}_i^-}(n)^o) \cup \{x_1, x_2, \dots x_n\}]_{\underline{B}^-}$. Indeed, let f be any $\underline{B}^-$-polynomial, of arity $m \geq 0$, $y_1, y_2, \dots y_m \in B(F_{\underline{B}_i^-}(n)^o) \cup \{x_1, x_2, \dots x_n\}$, then $f(y_1, y_2, \dots y_m)^o \in B(F_{\underline{B}_i^-}(n)^o)$, hence $[B(F_{\underline{B}_i^-}(n)^o) \cup \{x_1, x_2, \dots x_n\}]_{\underline{B}^-} =$
$= [B(F_{\underline{B}_i^-}(n)^o) \cup \{x_1, x_2, \dots x_n\}]_{\underline{B}_i^-} = F_{\underline{B}_i^-}(n)$.
Therefore $r_T(F_{\underline{B}_i^-}(n) \leq n$ and $F_{\underline{B}_i^-}(n) \in \underline{T}_n^-$, hence

$$V(F_{\underline{B}_i^-}(n)) \subseteq \underline{T}_n^- \subset B_i^- .$$

In similar way one shows that

$$V(F_{\underline{B}_i}(n)) \subseteq \underline{T}_n \subset \underline{B}_i . \square$$

6.13 Remark. Later we shall prove, tha $F_{\underline{B}_i^-}(n)^o$ has the property that $\forall\, u,v \in F_{\underline{B}_i^-}(n)^o$, if $u < v$ then there exists a $w \in F_{\underline{B}_i^-}(n)^o$ such that $u \underset{F_{\underline{B}_i^-}(n)^o}{\prec} w \leq v$, that is, that $F_{\underline{B}_i^-}(n)^o$ is strongly atomic (for terminology, see Crawley and Dilworth [74]). Using this observation, it is possible to prove Corollary 6.12 more directly. Similarly for $F_{\underline{B}_i}(n)$.

In the proof of 6.12 we show that $F_{\underline{B}_i^-}(n) \in \underline{T}_n^-$, $F_{\underline{B}_i}(n) \in \underline{T}_n$, for $n \in N$. In the $\underline{B}_i^-$ case, we are able to prove that in fact $F_{\underline{B}_i^-}(n)$ is characteristic for $\underline{T}_n^-$.

6.14 <u>Theorem</u>. $V(F_{\underline{B}_i^-}(n)) = \underline{T}_n^-$, thus $F_{\underline{B}_i^-}(n)$ is characteristic for $\underline{T}_n$, $n \in N$.

<u>Remark</u>. Since by 6.4, $V(F_{\underline{B}_i}(n)^o) \neq \underline{H}$, and on the other hand $\underline{T}_n^o = \underline{H}$, it follows that $V(F_{\underline{B}_i}(n)) \subset \underline{T}_n$.

We need a lemma, which is related to lemmas 5.8 and 5.9.

6.15 <u>Lemma</u>. Let $n \in N$. Let $L \in \underline{B}_i$ be finite such that $r_T(L) \leq n$. There exists an $L_1 \in \underline{B}_i$, also finite, such that L_1 is $\underline{B}_i$-generated by $\leq n$ elements and such that there is an $a \in L^o$ with $[a) \cong L$.

<u>Proof</u>. There exists a chain of open elements in L, say

$$D = \{0 = d_1 < d_2 < \ldots < d_m = 1\},$$

such that $d_i \underset{L^o}{\prec} d_{i+1}$, $i = 1,2,\ldots m-1$. Then $B(L^o) = B(D)$. By 6.10, the interior algebra $[d_i,d_{i+1}]$ is $\underline{B}$-generated by

$$\{d_i,d_{i+1}\} \cup \{x_1^i,x_2^i,\ldots x_n^i\} \subseteq [d_i,d_{i+1}] \text{ for some } x_1^i,x_2^i,\ldots x_n^i,$$

where we may assume that $d_i < x_1^i \leq d_{i+1}$. Let $x_j = \sum_{i=1}^{m} x_j^i.d_i'$, $j = 1,2,\ldots n$. We proceed as in the proof of 5.8: let $L_1 = H_m \times L$, and define an interior operator ^{o} on L_1 by $(x,y)^o = (x^o,y^o.d_i)$, where $i = \max\{j \mid c_j \leq x^o\}$. Then $L_1 \in \underline{B}_i$, and we claim that L_1 is generated by the elements $(c_1,x_1),(0,x_2),\ldots(0,x_n)$. Indeed, $(c_1,x_1)^o = (c_1,0)$, hence

$$(0,x_1) = (c_1,0)'.(c_1,x_1) \in [\{(c_1,x_1),(0,x_2),\ldots(0,x_n)\}],$$

and therefore also

$$(1,x_1')^o = (1,0) \in [\{(c_1,x_1),(0,x_2),\ldots(0,x_n)\}].$$

It follows from our choice of x_1 that $x_1'^o = 0$: suppose that $0 \neq u = u^o \leq x_1' = \prod_{i=1}^{m} (x_1^{i\prime} + d_i)$. Since $(u]$ is finite, there is an atom p of L^o, $p \leq u \leq x_1'$. But there must exist an $i_o \in \{1,2,\ldots m\}$, such that $p = d_{i_o}' d_{i_o+1}$. Then $d_{i_o}' d_{i_o+1} = p \leq x_1' \leq x_1^{i_o\prime} + d_{i_o}$, which implies that $x_1^{i_o} \leq d_{i_o} + d_{i_o+1}'$. But this contradicts our assumption

$d_{i_0} < x_1^{i_0} \leq d_{i_0+1}$. Therefore $x_1^{\prime o} = 0$. We have thus $(1,0)$ and therefore also $(0,1)$ at our disposal; $(c_1,1)$ generates $H_m \times B(L^o)$ according to the proof of 5.8, providing $\{(1,d_i) \mid i = 1,2,\ldots m\}$, which together with $(1,x_1),(1,x_2),\ldots(1,x_n)$ yield $\{1\} \times L$ and thus all of L_1. Obviously $L \cong [(1,0))$, and L_1 is finite.□

6.16 Lemma. Let $n \in N$. Let $L \in \underline{B}_i^-$ such that $r_T(L) \leq n$. There exists an $L_1 \in \underline{B}_i^-$, also finite, such that L_1 is $\underline{B}_i^-$-generated by $\leq n$ elements and such that $L \cong [a)$ for some $a \in L^o$.

Proof. As 6.15, now using H_m^+ .□

Proof of 6.14. Let $n \in N$. Let $L \in \underline{T}_n^-$, L finite, $r_T(L) \leq n$. By lemma 6.16, $L \in SH(F_{\underline{B}_i^-}(n))$. Since $\underline{T}_n^- = V(\{L \in \underline{B}_i^- \mid L$ finite and $r_T(L) \leq n\})$ by 6.9, it follows that $\underline{T}_n^- \subseteq V(F_{\underline{B}_i^-}(n))$. The reverse inclusion holds also, as has been shown in the proof of 6.12.□

Note that it follows from 6.15 that $\underline{T}_n = V(\{[a) \mid a \in F_{\underline{B}_i}(n)^o\})$. In the second chapter we shall study the varieties $\underline{T}_0$ and $\underline{T}_0^-$ in greater detail.

We finish this section with a characterization of the finite interior algebras $[u,v]$, $u,v \in F_{\underline{B}_i^-}(n)^o$, or $u,v \in F_{\underline{B}_i}(n)^o$, $n \in N$.

6.17 Theorem. (i) Let $L \in \underline{B}_i^-$ be finite, $n \in N$. There exist $u,v \in F_{\underline{B}_i^-}(n)^o$ such that $L \cong [u,v]$ iff $r_T(L) \leq n$. In particular, $F_{\underline{B}_i^-}(n)$ is a generalized universal algebra for all finite generalized interior algebras of rank of triviality $\leq n$ (cf. 4.26).

(ii) Let $L \in \underline{B}_i$ be finite, $n \in N$. There exist $u,v \in F_{\underline{B}_i}(n)^o$ such that $L \cong [u,v]$ iff $r_T(L) \leq n$.

Proof. (i) $\Rightarrow$ by 6.10 and proof of 6.12.

$\Leftarrow$ By lemma 6.16 there exists an $L_1 \in \underline{B}_i^-$, finite, $\underline{B}_i^-$-generated by n elements, such that $L \cong [a)$ for some $a \in L_1^o$. By 4.3 for $\underline{B}_i^-$, $L_1 \cong (v]$ for some $v \in F_{\underline{B}_i^-}(n)^o$. If a corresponds with $u \in (v]^o$, then $[u,v] \cong L$.

(ii) Similarly, using 6.15.□

Section 7. Some remarks on free products, injectives and weakly projectives in $\underline{B}_i$ and $\underline{B}_i^-$

We close this chapter with some observations on free products, injectives and weakly projectives in $\underline{B}_i$ and $\underline{B}_i^-$. In 7.3 we note that free products in $\underline{B}_i^-$ and as a matter of fact in every subvariety of $\underline{B}_i^-$ always exist. In $\underline{B}_i$ the free product of any collection of non trivial algebras exists, too (theorem 7.4) but this is not the case in every subvariety of $\underline{B}_i$ (example 7.10 (ii)). There is not much to say about injectives in $\underline{B}_i$ and $\underline{B}_i^-$: there just are no non-trivial ones (theorem 7.12). In the next chapter we shall characterize the injectives in certain subvarieties of $\underline{B}_i$ and $\underline{B}_i^-$.

We do not know yet very much about weakly projectives in $\underline{B}_i$ and $\underline{B}_i^-$. In 7.21, 7.22 we present interesting classes of algebras with that property. It is striking how nice the generalized interior algebras behave compared to the interior algebras here as well as with respect to free products.

7.1 Free products in $\underline{B}_i$ and $\underline{B}_i^-$

We recall the definition of free product in a class $\underline{K}$ of algebras. Let $\{A_i \mid i \in I\} \subseteq \underline{K}$. A is the free product of $\{A_i \mid i \in I\}$ in $\underline{K}$ if

(i) $A \in \underline{K}$

(ii) there exist 1-1 homomorphisms $j_i : A_i \to A$, $i \in I$

(iii) $[\bigcup_{i \in I} j_i[A_i]] = A$

(iv) If $B \in \underline{K}$, and $f_i : A_i \to B$, $i \in I$ are homomorphisms, then there exists a homomorphism $f: A \to B$ such that $f \circ j_i = f_i$, for all $i \in I$.

We shall assume, that the j_i are inclusion maps and thus that the A_i are subalgebras of A and we shall simply write $A = \sum_{i \in I}^{\underline{K}} A_i$ to indicate that A is the free product of $\{A_i \mid i \in I\}$ in $\underline{K}$.
It is known that in $\underline{D}_{01}$ the free product of any collection $\{L_i \mid i \in I\} \subseteq \underline{D}_{01}$, $|L_i| > 1$ for $i \in I$, exists. The same holds for $\underline{B}$. Indeed, if $L \in \underline{D}_{01}$, and $\{L_i \mid i \in I\}$ is a family of $\underline{D}_{01}$--sublattices of L, then L is the free product of $\{L_i \mid i \in I\}$ iff

(a) $[\bigcup_{i \in I} L_i] = L$

(b) If I_1, I_2, are nonvoid finite subsets of I and $a_i \in L_i$, $i \in I_1$, $b_j \in L_j$, $j \in I_2$, $a_i \neq 0$, $b_j \neq 1$, $i \in I_1$, $j \in I_2$, and $\prod_{i \in I_1} a_i \leq \sum_{j \in I_2} b_j$ then there exists an $i \in I_1 \cap I_2$ such that $a_i \leq b_i$ (cf. Grätzer [71]).

The next theorem, which can be found in Pierce and Christensen [59], gives a useful criterium for the existence of free products in a class.

7.2 Theorem. Let $\underline{K}$ be a variety, $\{A_i \mid i \in I\} \subseteq \underline{K}$. The free product of the A_i in $\underline{K}$ exists, provided there exists an $A \in \underline{K}$ and 1-1 homomorphisms $k_i: A_i \longrightarrow A$, $i \in I$.

This can be applied to $\underline{B}_i^-$:

7.3 Theorem. In $\underline{B}_i^-$ and in every subvariety of $\underline{B}_i^-$, free products exist.

Proof. Let $\underline{K} \subseteq \underline{B}_i^-$ be a variety, $\{A_i \mid i \in I\} \subseteq \underline{K}$. Then $\prod_{i\in I} A_i \in \underline{K}$ and $k_j: A_j \longrightarrow \prod_{i\in I} A_i$ defined by $k_j(a) = (a_i)_i$, where $a_i = \begin{cases} 1 & \text{if } i \neq j \\ a & \text{if } i = j \end{cases}$ is a 1-1 homomorphism.□

The problem of the existence of free products in $\underline{B}_i$ and its subvarieties is less simple because of the presence of the 0 as nullary operation. We say that a variety $\underline{K} \subseteq \underline{B}_i$ has free products provided the free product exists in $\underline{K}$ of any collection $\{L_i \mid i \in I\} \subseteq K$, such that $|L_i| > 1$ for all $i \in I$. A similar terminology applies if $\underline{K} = \underline{D}_{01}$, $\underline{K} = \underline{B}$, $\underline{K} \subseteq \underline{H}$ etc.

7.4 Theorem. Let $\{L_i \mid i \in I\} \subseteq \underline{B}_i$, $|L_i| > 1$ for $i \in I$, and suppose that $L = \sum_{i\in I}^{\underline{D}_{01}} L_i$. Then L can be made into an interior algebra such that the interior operator on L extends the interior operators of L_i, $i \in I$.

Proof. Let $L_1 = [\bigcup_{i\in I} L_i^o]_{\underline{D}_{01}}$. We prove that for $a \in L$ $(a] \cap L_1$ has a largest element. By 2.4 it will then follow that the operator on L defined by $a^o = \max(a] \cap L_1$ for any $a \in L$ is an interior operator such that $L^o = L_1$. Since $L = [\bigcup_{i\in I} L_i]_{\underline{D}_{01}}$, $a \in L$ can be written $a = \prod_{j\in J} \sum_{i\in I_j} a_i$ where $\{I_j, j \in J\}$ is a non-void finite collection of non-void finite subsets of I, and $a_i \in L_i$, $i \in I_j$, $j \in J$. Let $a^* = \prod_{j\in J} \sum_{i\in I_j} a_i^o$. Note that $a^* \leq a$, $a^* \in L_1$. Now suppose $b \in L_1$, $b \leq a$. It is to be shown that $b \leq a^*$. b can be written $b = \sum_{k\in K} \prod_{i\in I_k} b_i$, where $\{I_k \mid k \in K\}$ is a non-void finite set of non-void finite subsets of I, and $b_i \in L_i^o$, $i \in I_k$, $k \in K$. Since $b \leq a$, we have $\prod_{i\in I_k} b_i \leq \sum_{i\in I_j} a_i$, for $j \in J$, $k \in K$. It follows that $\prod_{i\in I_k} b_i^o \leq \sum_{i\in I_j} a_i^o$. Indeed, if some $b_i = 0$ or $a_k = 1$ then this is obvious; otherwise we have $b_{i_o} \leq a_{i_o}$ for some $i_o \in I_k \cap I_j$, and therefore $b_{i_o}^o \leq a_{i_o}^o$ and hence $\prod_{i\in I_k} b_i^o \leq \sum_{i\in I_j} a_i^o$. We conclude that $b \leq a^*$, as desired. Finally, it is immediate that if $a \in L_i$ for some $i \in I$, then $a^* = a^o$, thus * extends the original interior operator on L_i, $i \in I$.□

7.5 Corollary. $\underline{B}_i$ has free products.

Proof. By 7.2 and 7.4.□

7.6 Remark. The interior algebra L considered in the proof of 7.4 is in general not the $\underline{B}_i$-free product of the L_i, $i \in I$. Indeed, let $L_1, L_2 \in \underline{B}_i$, $L_1 \cong L_2 \cong M_2$ (see 6.1), and let $L = L_1 \overset{\underline{D}_{01}}{+} L_2$ as in 7.4. Then for $a_1 \in L_1$, $a_2 \in L_2$, we have $(a_1 + a_2)^o = a_1^o + a_2^o$. Let a_1, a_2 be atoms of L_1 and L_2 respectively. Now suppose that $L = L_1 \overset{\underline{B}_i}{+} L_2$, then there exists a $\underline{B}_i$-homomorphism $h: L \longrightarrow L_1$ such that $h(a_1) = h(a_2)'$ and $h(a_1) \neq 0,1$, $h(a_2) \neq 0,1$. But then $h((a_1 + a_2)^o) = h(a_1 + a_2)^o = 1^o = 1$ whereas $h(a_1^o + a_2^o) = h(0) = 0$, a contradiction.

Note also, that it follows from the proof of 7.4 that if $\{L_i \mid i \in I\} \subseteq \underline{H}$, $|L_i| > 1$, $i \in I$, then $L = \sum_{i\in I}^{\underline{D}_{01}} L_i \in \underline{H}$, and that the injections $j_i : L_i \longrightarrow L$ are $\underline{H}$-homomorphisms (this fact was proven earlier in A. Burger [75]), implying that free products in $\underline{H}$ exist as well. Unfortunately, the method employed in the proof of 7.4 will not work for arbitrary subvarieties of $\underline{B}_i$. However, a slight generalization can be obtained. When we say that a class $\underline{K} \subseteq \underline{H}$ is closed under $\underline{D}_{01}$-free products, we mean that if $\{L_i \mid i \in I\} \subseteq \underline{K}$ then $\sum_{i\in I}^{\underline{D}_{01}} L_i \in \underline{K}$ if it exists.

7.7 <u>Corollary</u>. Let $\underline{K} \subseteq \underline{H}$ be a variety such that $\underline{K}$ is closed under $\underline{D}_{01}$-free products. Then $\underline{K}^c = \{L \in \underline{B}_i \mid L^o \in \underline{K}\}$ has free products.

<u>Proof</u>. It is not difficult to see that $\underline{K}^c$ is a variety (cf. II.1). Let $\{L_i \mid i \in I\} \subseteq \underline{K}^c$, such that $|L_i| > 1$, $i \in I$, and let $L = \sum_{i\in I}^{\underline{D}_{01}} L_i$, provided with an interior operator as in 7.4. Then $L^o = \sum_{i\in I}^{\underline{D}_{01}} L_i^o \in \underline{K}$ hence $L \in \underline{K}^c$. Thus $\underline{K}^c$ satisfies the conditions of 7.2 so free products exist in $\underline{K}^c$. □

7.8 <u>Example</u>. The class $\underline{B}^c$ of interior algebras, whose lattices of open elements are Boolean ($\underline{B}^c$ is also called the variety of monadic algebras) has free products, since $\underline{B}$ is closed under $\underline{D}_{01}$-free products.

The next theorem (brought to my attention by prof. J. Berman) is a sharpened version of 7.2:

7.9 Theorem. Let $\underline{K}$ be a variety of algebras and suppose that every collection $\{A_i \mid i \in I\} \subseteq \underline{K}_{SI}$ can be embedded in some $A \in \underline{K}$. Then the free product exists of any collection $\{A_i \mid i \in I\} \subseteq \underline{K}$ satisfying $|A_i| > 1$, $i \in I$.

Proof. Let $A_i \in \underline{K}$, $|A_i| > 1$, $i \in I$. We shall show that there exists an $A \in \underline{K}$ and 1-1 homomorphisms $\ell_i: A_i \longrightarrow A$, $i \in I$. It will follow then from 7.2 that the free product of the A_i exists in $\underline{K}$. For every $i \in I$ there exists a collection $\{B_j \mid j \in J_i\} \subseteq \underline{K}_{SI}$ such that $A_i \in SP\{B_j \mid j \in J_i\}$ by 0.1.6. Let $B \in \underline{K}$ be such that for every $j \in J_i$, $i \in I$ there exists a 1-1 homomorphism $B_j \longrightarrow B$. It follows that $A_i \in SPS(B) \subseteq SP(B)$, say $k_i: A_i \longrightarrow \prod_{s \in S_i} B$ is an embedding. Let $A = \prod_{s \in S} B$, where $S = \bigcup_{i \in I} S_i$, and choose $s_i \in S_i$. Define $\ell_i: A_i \longrightarrow A$ by

$$(\ell_i(a))_s = \begin{cases} (k_i(a))_s & \text{if } s \in S_i \\ \pi_{s_i} \circ k_i(a) & \text{otherwise} \end{cases}$$

ℓ_i is a 1-1 homomorphism for $i \in I$. □

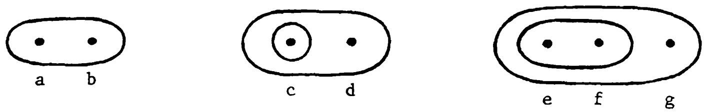
Using 7.9 it can be seen very easily that $\underline{D}_{01}$, $\underline{B}$ and more generally any variety containing only one subdirectly irreducible has free products. Furthermore, classes like those of De Morgan algebras, distributive pseudocomplemented lattices and its subvarieties, and several more are seen to have free products.

7.10 Examples. (i) The variety $V(M_2)$ has free products. Indeed, by 0.1.26, $V(M_2)_{SI} \subseteq HS(M_2) = \{M_0, M_1, M_2\}$ hence $V(M_2)_{SI} = \{M_1, M_2\} \subseteq S(M_2)$. By 7.9, free products exist in $V(M_2)$. Note that the interior algebra $L \underset{\underline{B}}{\cong} M_2 \underset{\underline{D}_{01}}{+} M_2 \cong \underline{2}^4$, with

$L^o = \underline{2} \underset{D_{01}}{+} \underline{2} = \underline{2}$, which came up in the proof of 7.4, does not belong to $V(M_2)$. Theorem 7.9 is used here in an essential way.

Similarly one can show that $V(M_n)$, $n \in N$ has free products.

(ii) We shall present now an example of a subvariety of $\underline{B}_i$ in which free products do not always exist. Let $L_0 = \underline{2}$, $L_1 \cong M_2$, with atoms a,b ; $L_2 \underset{\underline{B}}{\cong} \underline{2}^2$, with atoms c,d and $L_2^o = \{0,c,1\}$; $L_3 \underset{\underline{B}}{\cong} \underline{2}^3$, with atoms e,f,g, $L_3^o = \{0,e+f,1\}$.



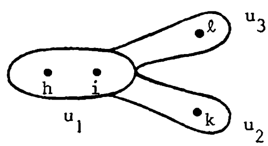
L_1 L_2 L_3

Let $\underline{K}_1 = V(\{L_1,L_2\})$, then $\underline{K}_{1SI} = \{L_0,L_1,L_2\}$ and
$\underline{K}_2 = V(\{L_1,L_2,L_3\})$, then $\underline{K}_{2SI} = \{L_0,L_1,L_2,L_3\}$.

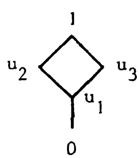
Claim. $L_1 \underset{\underline{K}_1}{+} L_2$ does not exist.

By 7.9 $\underline{K}_2$ has free products, since $\underline{K}_{2SI} \subseteq S(L_3)$. Let
$L_4 = L_1 \underset{\underline{K}_2}{+} L_2$. $L_4 \underset{\underline{B}}{\cong} \underline{2}^4$, say with atoms h,i,k,ℓ and
$L_4^o = \{0,h+i,h+i+\ell,h+i+k,1\}$:

L_4



L_4^o



$u_1 = h + i$
$u_2 = h + i + k$
$u_3 = h + i + \ell$

$i_1 : L_1 \longrightarrow L_4$ is defined by $i_1(a) = h + \ell$, $i_1(b) = i + k$

$i_2 : L_2 \longrightarrow L_4$ is defined by $i_2(c) = h + i$, $i_2(d) = k + \ell$.

i_1 , i_2 are $\underline{B}_i$-embeddings, and $i_1[L_1] \cup i_2[L_2]$ generates L_4 . Furthermore, $L_4 \in S\ P\ (L_3) \subseteq \underline{K}_2$. In order to prove, that L_4 is the free product of L_1 and L_2 in $\underline{K}_2$, it is sufficient to show, that for every two homomorphisms $f_1 : L_1 \longrightarrow L_3$, $f_2 : L_2 \longrightarrow L_3$, there exists a homomorphism $f: L_4 \longrightarrow L_3$, such that $f \circ i_j = f_j$, $j = 1,2$. This can be verified without difficulty. Now suppose $\underline{K}_1$ has free products, and let $L = L_1 \underset{\underline{K}_1}{+} L_2$. Then $L \in H\ (L_4) = \{L_4, L_3, L_1, \underline{1}\}$; but $L_4, L_3 \notin \underline{K}_1$, and $L_2 \notin S(L_1)$, $L_2 \notin S(\underline{1})$. Contradiction.

7.11 <u>Injectives in $\underline{B}_i$ and $\underline{B}_i^-$</u>

Recall that if $\underline{K}$ is a class of algebras, then $A \in \underline{K}$ is <u>injective</u> in $\underline{K}$, if for each monomorphism $f: A_1 \longrightarrow A_2$ and homomorphism $g: A_1 \longrightarrow A$, $A_1, A_2 \in \underline{K}$, there exists a homomorphism $h: A_2 \longrightarrow A$ satisfying $h \circ f = g$.

$$\begin{tikzcd}
A_2 \arrow[dr, dashed, "h"] & \\
A_1 \arrow[u, "f"] \arrow[r, "g"'] & A
\end{tikzcd}$$

As noted before (0.1.29), monic may be replaced by 1-1 in our investigations.

Unlike the classes $\underline{D}_{01}$ and $\underline{B}$, $\underline{B}_i$ and $\underline{B}_i^-$ have no non--trivial injectives. Indeed, suppose $L \in \underline{B}_i$, $|L| > 1$ and L injective. Let $L_1 \in \underline{B}_i$ be such that $|L_1| > |L|$, and $L_1^o = \{0,1\}$. Let $f: \{0,1\} \longrightarrow L_1$ be defined by $f(0) = 0$, $f(1) = 1$, and $g: \{0,1\} \longrightarrow L$ also by $g(0) = 0$, $g(1) = 1$. Let $h: L_1 \longrightarrow L$

be a $\underline{B}_i$-homomorphism such that $h \circ f = g$. Then $h^{-1}(\{1\}) = \{1\}$, so h is 1-1. But $|L_1| > |L|$, a contradiction. A similar argument applies to $\underline{B}_i^-$:

7.12 <u>Theorem</u>. $\underline{B}_i$ and $\underline{B}_i^-$ have no non-trivial injectives.

7.13 <u>Weakly projectives in $\underline{B}_i$ and $\underline{B}_i^-$</u>

If $\underline{K}$ is a class of algebras, then $A \in \underline{K}$ is called <u>weakly projective</u> in $\underline{K}$ if for each onto-homomorphism $f\colon A_1 \twoheadrightarrow A_2$ and homomorphism $g\colon A \to A_2$, $A_1, A_2 \in \underline{K}$, there exists a homomorphism $h\colon A \to A_1$ such that $f \circ h = g$.

$$\begin{tikzcd} & A_1 \arrow[d, "f"] \\ A \arrow[ur, dashed, "h"] \arrow[r, "g"'] & A_2 \end{tikzcd}$$

Since we do not know, at this moment, whether every epic $\underline{B}_i$-homomorphism is onto, we use the notion of weak projectivity rather than that of projectivity.

7.14 <u>Theorem</u>. Let $L \in \underline{B}_i$ be a $\star$-algebra. L is weakly projective in $\underline{B}_i$ iff L^o is weakly projective in $\underline{H}$.

<u>Proof</u>. (i) Suppose $L \in \underline{B}_i$ is weakly projective and $L = B(L^o)$. Let $f\colon L_1 \twoheadrightarrow L_2$ be an onto $\underline{H}$-homomorphism, $L_1, L_2 \in \underline{H}$, and let $g\colon L^o \to L_2$ be an $\underline{H}$-homorphism. Let $f_1\colon B(L_1) \twoheadrightarrow B(L_2)$, $g_1\colon L \to B(L_2)$ be the $\underline{B}_i$-homomorphisms which extend f, g respectively. By assumption and since f_1 is onto, there exists a $\underline{B}_i$-homomorphism $h_1\colon L \to B(L_1)$ with $f_1 \circ h_1 = g_1$. If $h^o = h_1 \mid L^o$, then $f \circ h^o = g$, and h^o

$$\begin{tikzcd} & L_1 \arrow[d, "f"] \\ L^o \arrow[ur, dashed, "h^o"] \arrow[r, "g"'] & L_2 \end{tikzcd}$$

is an $\underline{H}$-homomorphism.

(ii) Let $L \in \underline{B}_i$, $L = B(L^o)$ and suppose that L^o is weakly projective in $\underline{H}$. Let $f\colon L_1 \longrightarrow L_2$ be an onto $\underline{B}_i$-homomorphism and $g\colon L \longrightarrow L_2$ a $\underline{B}_i$-homomorphism. Let $f_1 = f \mid L_1^o$, $g_1 = g \mid L^o$, then there exists an $\underline{H}$-homomorphism $h_1 : L^o \longrightarrow L_1^o$ such that $f_1 \circ h_1 = g_1$. Let $h\colon L = B(L^o) \longrightarrow L_1$ be the $\underline{B}_i$-homomorphism such that $h \mid L^o = h_1$. Then $f \circ h \mid L^o = f_1 \circ h_1 = g_1 = g \mid L^o$, hence, by the uniqueness of the extension, $f \circ h = g$. □

Similarly:

7.15 <u>Theorem</u>. A $*$-algebra $L \in \underline{B}_i^-$ is weakly projective in $\underline{B}_i^-$ iff L^o is weakly projective in $\underline{H}^-$.

Further inspection of the proof of 7.14 shows that the following is true as well:

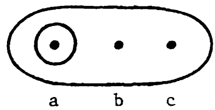
7.16 <u>Theorem</u>. A $*$-algebra $L \in \underline{K}$ is weakly projective in $\underline{K}$ iff L^o is weakly projective in $\underline{K}^o$, for any class $\underline{K} \subseteq \underline{B}_i$, or $\underline{K} \subseteq \underline{B}_i^-$, satisfying $S(\underline{K}) \subseteq \underline{K}$.

The finite weakly projectives in $\underline{H}$ have been characterized in R. Balbes and A. Horn [70]. They showed, that $L \in \underline{H}_F$ is weakly projective iff $L \cong L_0 \dotplus L_1 \dotplus \ldots \dotplus L_n$, for some $n \geq 0$, where $L_n \cong \underline{2}$, $L_i \cong \underline{2}^2$ or $L_i \cong \underline{2}$, $0 \leq i < n$. Thus the $*$-algebras whose lattices of open elements are of this type are weakly projective in $\underline{B}_i$. However, we shall give now an example which shows that these finite interior algebras are not the only finite weakly projectives in $\underline{B}_i$. An important tool will be

7.17 <u>Theorem</u>. Let $\underline{K}$ be a variety. $A \in \underline{K}$ is weakly projective in $\underline{K}$ iff A is a retract of a $\underline{K}$-free algebra.

For a proof of this theorem we refer to Balbes and Dwinger [74].

7.18 <u>Example</u>. Let $L \underset{\underline{B}}{\cong} \underline{2}^3$, $L^o \cong \underline{3}$, with atoms a,b,c and open elements $0,a,1$. This interior algebra will be denoted $M_{1,2}$.



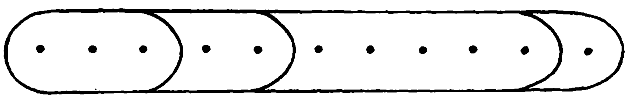
In order to show that $M_{1,2}$ is weakly projective, it suffices to prove that it is a retract of $F_{\underline{B}_i}(1)$ by 7.17. According to 4.3, there exists a $u \in F_{\underline{B}_i}(1)^o$ such that $M_{1,2} \cong (u]$. Let a_1, b_1, c_1 be the atoms of $(u]$, $a_1^o = a_1$. Let $a_2 = (b_1 + c_1)'^o$, $b_2 = b_1$, $c_2 = (a_2 + b_2)'$. Then a_2, b_2, c_2 are disjoint, $a_2 + b_2 + c_2 = 1$. Obviously $a_2^o = a_2$, $b_2^o = 0$. Further $c_2^o.u = 0$, hence $c_2^o \leq (b_1 + c_1)'^o = a_2$, but on the other hand $c_2^o \leq (a_2 + b_2)' \leq a_2'$, hence $c_2^o = 0$. It is also readily seen that $(a_2 + b_2)^o = a_2$, $(a_2 + c_2)^o = a_2$, and finally $(b_2 + c_2)^o = 0$, since $(b_2 + c_2)^o.u = 0$, thus $(b_2 + c_2)^o \leq c_2^o = 0$. Therefore the $\underline{B}_i$- subalgebra of $F_{\underline{B}_i}(1)$ generated by a_2, b_2, and c_2 is isomorphic to $M_{1,2}$.

Moreover, $a_2 u = a_1$, $b_2 u = b_1$, and $c_2 u = c_1$, thus the maps $f: M_{1,2} \longrightarrow F_{\underline{B}_i}(1)$ given by $f(a) = a_2$, $f(b) = b_2$, $f(c) = c_2$ and $g: F_{\underline{B}_i}(1) \longrightarrow M_{1,2}$ given by $g = g_1 \circ \pi_{(u]}$, where $\pi_{(u]}: F_{\underline{B}_i}(1) \longrightarrow (u]$ is defined by $x \longmapsto xu$ and $g_1: (u] \longrightarrow M_{1,2}$

by $g_1(a_1) = a$, $g_1(b_1) = b$, $g_1(c_1) = c$ are $\underline{B}_i$-homomorphisms and $g \circ f$ is the identity on $M_{1,2}$. So $M_{1,2}$ is a retract of $F_{\underline{B}_i}(1)$, hence weakly projective in $\underline{B}_i$ by 7.17 .
In a similar way one shows that $M_{1,2}$ is a retract of $F_{\underline{B}_i^-}(1)$. Hence $M_{1,2}$ is weakly projective in $\underline{B}_i^-$, too.

7.19 Example. *The interior algebras with trivial interior operator and more than two elements, like the M_n , $n \geq 2$ are not weakly projective in $\underline{B}_i$, though their lattices of open elements are weakly projective in $\underline{H}$.* For instance, let $L \underset{\underline{B}}{\cong} \underline{2}^3$, with atoms a,b,c, and $L^o = \{0, a+b, c, 1\} \underset{\underline{B}}{\cong} \underline{2}^2$. Then $(a+b] \cong M_2$, thus $M_2 \in H(L)$, but obviously $M_2 \notin S(L)$. Therefore M_2 is not weakly projective in $\underline{B}_i$.

7.20 Examples 7.18, 7.19 provide the idea underlying the following theorem. $M_{n_1,n_2,\ldots n_k}$ will denote the interior algebra $\underline{2}^n$, where $n = \sum_{i=1}^{k} n_i$, with $M^o_{n_1,n_2,\ldots n_k} \cong \underline{k+1}$, such that if $M^o_{n_1,n_2,\ldots n_k} = \{0 = c_0 < c_1 < \ldots < c_k = 1\}$, then $(c_j]$ has $\sum_{i=1}^{j} n_i$ atoms, $j = 1,2,\ldots k$. Thus the M_n are the finite interior algebras with n atoms and trivial interior operator we met before. For example:



$M_{3,2,5,1}$

7.21 <u>Theorem</u>. Let $k \in N$, $n_1, n_2, \dots n_k \in N$. $M_{n_1,n_2,\dots n_k}$ is weakly projective in $\underline{B}_i$ iff $n_1 = 1$.

<u>Proof</u>. (i) $\Leftarrow$ Let $k \in N$, $n_1, n_2, \dots n_k \in N$, $n_1 = 1$. We prove that $M_{n_1,n_2,\dots n_k}$ is a retract of a free algebra in $\underline{B}_i$.

Let $m \in N$ be such that $M_{n_1,n_2,\dots n_k} \in H(F_{\underline{B}_i}(m))$ and let $u \in F_{\underline{B}_i}(m)^o$ be such that $(u] \cong M_{n_1,n_2,\dots n_k}$. A u with this property exists in virtue of 4.3. Let

$$(u]^o = \{0 = c_0 < c_1 < \dots < c_k = 1\}$$

and $p_1^j, p_2^j, \dots p_{n_j}^j$, $j = 1,2,\dots k$ be the atoms of $(u]$, with $p_i^j \le c_{j-1}' c_j$, $j = 1,2,\dots k$, $i = 1,2,\dots n_j$.

Let $\bar{c}_0 = c_0$, $\bar{c}_j = (c_j + u')^o$, $j = 1,2,\dots k$

$$\overline{p_i^j} = p_i^j, \quad j = 1,2,\dots k, \quad i = 1,2,\dots n_j - 1$$

and $$\overline{p_{n_j}^j} = \left(\sum_{i=1}^{n_j-1} p_i^j\right)' . \overline{c_{j-1}}' . \overline{c_j}, \quad j = 1,2,\dots k.$$

Note that $\overline{c_j} = \overline{c_{j-1}} + \sum_{i=1}^{n_j} \overline{p_i^j}$ and $\overline{p_i^j}.u = p_i^j$.

Define $f: (u] \longrightarrow F_{\underline{B}_i}(m)$ by

$$f(x) = \Sigma \{\overline{p_i^j} \mid p_i^j \le x, \quad j = 1,2,\dots k, \quad i = 1,2,\dots n_j\}.$$

It is clear from the definition of the $\overline{p_i^j}$ that f is a 1-1 $\underline{B}$-homomorphism and that $f(x).u = x$ for all $x \in (u]$. In order to prove that f preserves ^{o}, let $x \in (u]$ such that $x^o = c_\ell$.

1) $1 \le \ell \le k$. Firstly, since $c_\ell = x^o = (f(x).u)^o = f(x)^o.u$ it follows that $f(x)^o \le c_\ell + u'$, hence $f(x)^o \le (c_\ell + u')^o = \bar{c}_\ell$. But also $\bar{c}_\ell \le f(x)^o$ for if this were not the case then there is a j, $1 \le j \le \ell$, and there is an $i \in \{1,2,\dots n_j\}$ such that

$\overline{p_i^j} \not\leq f(x)$, hence $f(x) \leq \overline{p_i^j}'$ and thus $x = f(x).u \leq \overline{p_i^j}'.u = p_i^{j'}.u$, contradicting $p_i^j \leq c_\ell \leq x$. We conclude that $f(x)^o = \overline{c_\ell} = f(x^o)$.

2) $\ell = 0$, i.e. $x^o = 0$. Then $f(x)^o.u = (f(x).u)^o = x^o = 0$, hence $f(x)^o \leq u'^o \leq \overline{c_1}$. But on the other hand, since $n_1 = 1$, $\overline{c_1} = \overline{p_1^1}$ thus $c_1 \not\leq f(x)$ implies $f(x) \leq \overline{c_1}'$. Therefore $f(x)^o \leq \overline{c_1}.\overline{c_1}' = 0$, and we infer $f(x)^o = f(x^o)$.

Now, if $g\colon F_{B_i}(m) \longrightarrow (u]$ is defined by $g(x) = x.u$ then f,g are $\underline{B}_i$-homomorphisms such that $g \circ f$ is the identity on $(u]$. So $(u]$ is a retract of $F_{\underline{B}_i}(m)$ and it follows by 7.17 that $M_{n_1,n_2,\dots n_k} \cong (u]$ is weakly projective in $\underline{B}_i$.

(ii) Suppose $n_1 \neq 1$, $M_{n_1,n_2,\dots n_k}$ is weakly projective. Let $L \underset{\underline{B}}{\cong} \underline{2} \times M_{n_1,n_2,\dots n_k}$, with $(x,y)^o = (x,y^o)$. Then $M_{n_1,n_2,\dots n_k} \cong ((0,1)] \in H(L)$, but $M_{n_1,n_2,\dots n_k} \notin S(L)$, since $M_{n_1,n_2,\dots n_k}$ contains M_2 as a subalgebra, but L apparently does not.□

In $\underline{B}_i^-$ the situation is slightly different. The argument given *in 7.19 to show that* M_2 *is not weakly projective in* $\underline{B}_i$ *does not* remain valid in $\underline{B}_i^-$: indeed, $M_2^- \cong [c)$, which is a $\underline{B}_i^-$-subalgebra of L . In fact we have:

7.22 <u>Theorem</u>. Let $k \in N$, $n_1,n_2,\dots n_k \in N$. Then $M^-_{n_1,n_2,\dots n_k}$ is weakly projective in $\underline{B}_i^-$.

<u>Proof</u>. Let $m \in N$ be such that $M_{n_1,n_2,\dots n_k} \in H(F_{\underline{B}_i^-}(m))$, $u \in F_{\underline{B}_i^-}(m)^o$, $c_j \in F_{\underline{B}_i^-}(m)^o$, $j = 0,1,\dots k$ and $p_i^j \in F_{\underline{B}_i^-}(m)$, $j = 1,2,\dots k$, $i = 1,2,\dots n_j$ as in the proof of 7.21.

Let $\overline{c_j} = (u \Rightarrow c_j)^o$, $j = 0,1,\ldots k$, and for $j = 1,2,\ldots k$

let $\overline{p_i^j} = p_i^j + \overline{c_0}$, $i = 1,2,\ldots n_j-1$,

and $\overline{p_{n_j}^j} = (\overline{c_{j-1}} \Rightarrow \overline{c_0}).\overline{c_j}.((\sum_{i=1}^{n_j-1} p_i^j) \Rightarrow \bar{c}_0)$.

Again, $\overline{c_j} = \overline{c_{j-1}} + \sum_{i=1}^{n_j} \overline{p_i^j}$, and $\overline{p_i^j}.u = p_i^j$.

Define $f: (u] \longrightarrow F_{\underline{B}_i^-}(m)$ by

$$f(x) = \overline{c_0} + \Sigma\{\overline{p_i^j} \mid p_i^j \le x , \ j = 1,2,\ldots k , \ i = 1,2,\ldots n_j\} .$$

It is clear from the definition of $\overline{p_i^j}$ that f is a 1-1 $\underline{B}^-$-homomorphism satisfying $f(x).u = x$. In order to show that f preserves ^{o} , suppose that $x \in (u]$, $x^o = c_\ell$, $0 \le \ell \le k$. Then $f(x)^o.u = (f(x).u)^o = x^o = c_\ell$ hence $f(x)^o \le (u \Rightarrow c_\ell)^o = \overline{c_\ell}$. But also $c_\ell \le f(x)^o$ for if $\ell = 0$ then by definition of f $\overline{c_0} \le f(x)$ hence $\overline{c_0} \le f(x)^o$ and if $\ell > 0$ then $\overline{c_\ell} \not\le f(x)^o$ would imply that for some j, $1 \le j \le \ell$, $i \in \{1,2,\ldots n_j\}$, $\overline{p_i^j} \not\le f(x)$ hence $f(x) \le \overline{p_i^j} \Rightarrow \overline{c_0}$ thus

$$x = f(x).u \le (\overline{p_i^j} \Rightarrow \overline{c_0}).u = (p_i^j \Rightarrow c_0).u ,$$

which however would contradict $p_i^j \le c_\ell \le x$. Thus in all cases $f(x)^o = \overline{c_\ell} = f(x^o)$.

Define $g: F_{\underline{B}_i^-}(m) \longrightarrow (u]$ by $g(x) = x.u$, then f, g are $\underline{B}_i^-$-homomorphisms and $g \circ f$ is the identity on $(u]$. By 7.17 it follows that $(u]$ and hence $M^-_{n_1,n_2,\ldots n_k}$ is weakly projective in $\underline{B}_i^-$.□

CHAPTER II

ON SOME VARIETIES OF (GENERALIZED) INTERIOR ALGEBRAS

In chapter I we have been working in the class of all (generalized) interior algebras, mainly. In order to be able to be somewhat more specific, we shall focus our attention now on (generalized) interior algebras in certain subvarieties of $\underline{B}_i$ and $\underline{B}_i^-$. In section 1 we study the relations between subvarieties of $\underline{B}_i$ and $\underline{H}$, $\underline{B}_i^-$ and $\underline{H}^-$, and finally $\underline{B}_i$ and $\underline{B}_i^-$. In sections 2,3 and 4 an investigation of the variety generated by the (generalized) interior algebras which are $*$-algebras is undertaken, resulting in a characterization of $F_{\underline{B}_i^-*}(1)$ in section 3 and in a characterization of the injectives in $\underline{B}_i^*$ in section 4. Sections 5,6 and 7 are devoted to the study of varieties generated by (generalized) interior algebras whose lattices of open elements are linearly ordered. The main object here is to determine the finitely generated free algebras in some of them.

Section 1. Relations between subvarieties of $\underline{B}_i$ and $\underline{H}$, $\underline{B}_i$ and $\underline{H}^-$, $\underline{B}_i$ and $\underline{B}_i^-$

The purpose of this section is to see how the functors $\mathcal{O}$, $\mathcal{O}^-$, $\mathcal{B}$, $\mathcal{B}^-$, $\mathcal{D}$ and the one introduced in I.2.25 behave with respect to the operations H, S and P. It will follow, in particular, that $\mathcal{O}$, $\mathcal{O}^-$ and $\mathcal{D}$ map varieties onto varieties (1.3, 1.5, 1.12); a useful result, we referred to already once (cf. the proof of I.6.4.). Moreover, in 1.1 we show that $\mathcal{D}$ establishes a 1-1 correspondence between the non-trivial subvarieties of a certain variety $\underline{S} \subset \underline{B}_i$ and the subvarieties of $\underline{B}_i^-$, respecting the inclusion relations. The behaviour of the functors $\mathcal{B}$ and $\mathcal{B}^-$ is not so easy to grasp. The crucial question whether a subalgebra of a $\star$-algebra is itself a $\star$-algebra will be deferred to the next section. There we shall also see that the product of $\star$-algebras need not be a $\star$-algebra. Hence $\mathcal{B}$ fails to map varieties of Heyting algebras upon varieties of interior algebras, and for $\mathcal{B}^-$ a similar statement holds.

If $\underline{K} \subseteq \underline{B}_i$ or $\underline{K} \subseteq \underline{B}_i^-$, then $\underline{K}^o = \mathcal{O}[\underline{K}]$ respectively $\underline{K}^o = \mathcal{O}^-[\underline{K}]$, thus $\underline{K}^o = \{L^o \mid L \in \underline{K}\}$.
Several of the following results are essentially contained in Blok and Dwinger [75].

1.1 Theorem. Let $\underline{K} \subseteq \underline{B}_i$. Then:

(i) $H(\underline{K}^o) = H(\underline{K})^o$

(ii) $S(\underline{K}^o) = S(\underline{K})^o$

(iii) $P(\underline{K})^o = P(\underline{K})^o$

In other words: $\mathcal{O}$ commutes with H, S and P.

<u>Proof</u>. (i) Let $L \in H(\underline{K}^o)$, then there exists $L_1 \in \underline{K}^o$, $f: L_1 \to L$ an $\underline{H}$-homomorphism which is onto. Let $L_2 \in \underline{K}$, such that $L_2^o = L_1$. f can be extended to $\bar{f}: B(L_1) \twoheadrightarrow B(L)$, with $\bar{f}$ an onto $\underline{B}_i$-homomorphism. Since $\underline{B}_i$ has CEP (cf. I.2.9.) there exists an $L_3 \in \underline{B}_i$, $\bar{\bar{f}}: L_2 \to L_3$, $\bar{\bar{f}}$ an onto $\underline{B}_i$-homomorphism such that $\bar{\bar{f}}[B(L_1)] \cong B(L)$. Since $L_3^o = \bar{\bar{f}}[L_2^o] = \bar{\bar{f}}[L_1] \cong L$, it follows that $L \in H(\underline{K})^o$.
Conversely, if $L \in H(K)^o$, then there exist interior algebras L_1, L_2 and an onto homomorphism $f: L_2 \to L_1$ such that $L_1^o = L$, $L_2 \in \underline{K}$. Then $f[L_2^o] = L_1^o$ and by I.2.11 $f \mid L_2$ is an $\underline{H}$-homomorphism, hence $L_1 \in H(\underline{K}^o)$.

(ii) Let $L \in \underline{K}^o$, $L_1 \in S(L)$, $L_2 \in \underline{K}$ such that $L = L_2^o$. By I.2.16 $B(L_1) \in S(L_2)$, thus $L_1 \in S(\underline{K})^o$.
Conversely, if $L \in S(L_1)$ for some $L_1 \in \underline{K}$, then by I.2.11 $L^o \in S(L_1^o)$, hence $L^o \in S(\underline{K}^o)$.

(iii) Let $\{L_i \mid i \in I\} \subseteq \underline{K}$. Then $(\prod_{i\in I} L_i)^o = \prod_{i\in I} L_i^o$ by the definition of product, hence $P(\underline{K}^o) = P(\underline{K})^o$.□

1.2 <u>Corollary</u>. If $\underline{K} \subseteq \underline{B}_i$, then $V(\underline{K}^o) = V(\underline{K})^o$.

1.3 <u>Corollary</u>. If $\underline{K}$ is a variety of interior algebras then $\underline{K}^o$ is a variety of Heyting algebras.

Similarly for $\underline{B}_i^-$ and $\underline{H}^-$:

1.4 <u>Theorem</u>. Let $\underline{K} \subseteq \underline{B}_i^-$. Then

(i) $H(\underline{K}^o) = H(\underline{K})^o$

(ii) $S(\underline{K}^o) = S(\underline{K})^o$

(iii) $P(\underline{K}^o) = P(\underline{K})^o$

Hence the functor O^- commutes with H, S and P.

1.5 Corollary. If $\underline{K} \subseteq \underline{B}_i^-$, then $V(\underline{K}^o) = V(\underline{K})^o$. In particular, if $\underline{K}$ is a variety of generalized interior algebras, then $\underline{K}^o$ is a variety of Brouwerian algebras.

Next we consider the functors $\mathcal{B}$ and $\mathcal{B}^-$ which assign to $L \in \underline{H}$ respectively $L \in \underline{H}^-$ the algebra $B(L)$ respectively $B^-(L)$ (cf. I.1.14 and I.2.13).

1.6 Theorem. Let $\underline{K} \subseteq \underline{H}$.

(i) $H(\mathcal{B}(\underline{K})) = \mathcal{B}(H(\underline{K}))$

(ii) $P_F(\mathcal{B}(\underline{K})) = \mathcal{B}(P_F(\underline{K}))$, where P_F denotes the operation of taking finite products.

Proof. (i) Let $L \in \underline{K}$, $L_1 \in \underline{B}_i$, $f\colon B(L) \twoheadrightarrow L_1$ an onto $\underline{B}_i$-homomorphism. Then $L_1 = f[B(L)] = B(f[L]) = B(L_1^o)$, hence $L_1 \in \mathcal{B}(H(\underline{K}))$ by I.2.11. Conversely, if $L \in \underline{K}$, $L_1 \in \underline{H}$, $f\colon L \twoheadrightarrow L_1$ an onto $\underline{H}$-homomorphism, then there exists by I.2.15 a $\underline{B}_i$-homomorphism $\bar{f}\colon B(L) \twoheadrightarrow B(L_1)$, which is also onto. Hence $B(L_1) \in H(\mathcal{B}(\underline{K}))$.

(ii) Let $\underline{K} \subseteq \underline{H}$, $L_1, L_2 \in \underline{K}$. We prove that

$$B(L_1 \times L_2) = B(L_1) \times B(L_2).$$

Note that since $(B(L_1) \times B(L_2))^o = L_1 \times L_2$ we may consider $B(L_1 \times L_2)$ as a subalgebra of $B(L_1) \times B(L_2)$. Now let

$$x = \left(\sum_{i=1}^{n} u_i' v_i \,,\, \sum_{j=1}^{m} x_j' y_j\right) \in B(L_1) \times B(L_2) ,$$

where $u_i, v_i \in L_1$, $i = 1,2,\ldots n$ and $x_j, y_j \in L_2$, $j = 1,2\ldots m$. Then

$$x = \sum_{i=1}^{n} \sum_{j=1}^{m} (u_i' v_i, x_j' y_j) = \sum_{i=1}^{n} \sum_{j=1}^{m} (u_i, x_j)' \cdot (v_i, y_j) \in B(L_1 \times L_2) .$$

Thus $B(L_1) \times B(L_2) = B(L_1 \times L_2)$. $\square$

In the next section we shall prove that a subalgebra of a $\star$-algebra is again a $\star$-algebra. This will imply that in addition to 1.6 (i), (ii) also $\mathcal{B}(S(\underline{K})) = S(\mathcal{B}(\underline{K}))$ for any class $\underline{K} \subseteq \underline{H}$. Furthermore we shall see that a product of $\star$-algebras need not be a $\star$-algebra, hence $P(\mathcal{B}(\underline{K})) = \mathcal{B}(P(\underline{K}))$ does not hold in general. It follows that if $\underline{K} \subseteq \underline{H}$ is a variety then $\{B(L) \mid L \in \underline{K}\}$ need not be a variety. Therefore we introduce

1.7 Definition. Let $\underline{K} \subseteq \underline{B}_i$. Then $\underline{K}^\star$ will denote the variety

$$V(\{B(L^o) \mid L \in \underline{K}\}) .$$

Likewise, if $\underline{K} \subseteq \underline{B}_i^-$ then $\underline{K}^\star$ will denote

$$V(\{B^-(L^o) \mid L \in \underline{K}\}) .$$

1.8 Theorem. Let $\underline{K} \subseteq \underline{B}_i$ $(\underline{K} \subseteq \underline{B}_i^-)$ be a variety. Then $\underline{K}^\star$ is the smallest variety $\underline{K}_1$ of (generalized) interior algebras satisfying $\underline{K}_1^o = \underline{K}^o$.

Proof. Let $\underline{K}_1 \subseteq \underline{B}_i$ be a variety such that $\underline{K}_1^o = \underline{K}^o$. Then $\mathcal{B}(\underline{K}^o) \subseteq S(\underline{K}_1) = \underline{K}_1$, hence $\underline{K}^\star \subseteq \underline{K}_1$. □

Also, if $\underline{K} \subseteq \underline{H}$ is a variety, then $V(\mathcal{B}(\underline{K}))$ is the smallest variety $\underline{K}_1 \subseteq \underline{B}_i$ such that $\underline{K}_1{}^o = \underline{K}$. A largest variety among the varieties $\underline{K}_1 \subseteq \underline{B}_i$ such that $\underline{K}_1^o = \underline{K}$ does exist, too. If $\underline{K} \subseteq \underline{H}$ is a class, let $\underline{K}^c = \{L \in \underline{B}_i \mid L^o \in \underline{K}\}$, (cf. I.7.7).

1.9 Theorem. If $\underline{K}$ is a variety of Heyting algebras, then $\underline{K}^c$ is a variety of interior algebras.

Proof. (i) $P(\underline{K}^c) \subseteq \underline{K}^c$, obvious

(ii) $S(\underline{K}^c) \subseteq \underline{K}^c$, by I.2.11

(iii) $H(\underline{K}^c) \subseteq \underline{K}^c$, also by I.2.11 . □

Later we shall see that for any non-trivial variety $\underline{K} \subseteq \underline{H}$, $V(\mathcal{B}(\underline{K})) \subset \underline{K}^c$; and obviously, $\underline{K}^c$ is the largest among the varieties $\underline{K}_1$ of interior algebras such that $\underline{K}_1^o = \underline{K}$.

1.10 <u>Remark</u>. If $\underline{K} \subseteq \underline{H}$ such that $V(\underline{K})_{SI} \subseteq \underline{K}$, then $V(\underline{K})^c = V(\underline{K}^c)$. Indeed , $(V(\underline{K})^c)_{SI} \subseteq (V(\underline{K})_{SI})^c \subseteq \underline{K}^c$, thus $V(\underline{K})^c \subseteq V(\underline{K}^c)$. Obviously, $V(\underline{K}^c) \subseteq V(\underline{K})^c$ and the desired equality follows. We do not know if the condition $V(\underline{K})_{SI} \subseteq \underline{K}$ can be omitted; clearly the condition is unnecessary.

1.11 The correspondence between varieties $\underline{K} \subseteq \underline{H}$ and $\underline{K}^c \subseteq \underline{B}_i$ has a nice feature. If Σ is a basis for the set of identities satisfied by $\underline{K}$, or, loosely speaking if Σ is a basis for $\underline{K}$, then we can easily find from Σ a basis for $\underline{K}^c$. We define a translation of $\underline{H}$-identities into $\underline{B}_i$-identities following the line of thinking of McKinsey and Tarski [48]. Let p be an $\underline{H}$-polynomial. The $\underline{B}_i$-transform of p , Tp is given by an inductive definition:

(i) if $p = x_i$, $i = 0,1,\ldots$, then $Tp = x_i^o$

(ii) if $p = q + r$, where q,r are $\underline{H}$-polynomials then

$$Tp = Tq + Tr$$

(iii) if $p = q.r$, where q,r are $\underline{H}$-polynomials then

$$Tp = Tq \,.\, Tr$$

(iv) if $p = q \to r$, where q,r are $\underline{H}$-polynomials then

$$Tp = ((Tq)' + Tr)^o$$

(v) if $p = 0 , 1$ then $Tp = 0 , 1$ respectively.

If $p = q$ is an $\underline{H}$-identity, then $Tp = Tq$ is the $\underline{B}_i$-translation of $p = q$. If Σ is a collection of $\underline{H}$-identities then $T(\Sigma)$ is the collection of $\underline{B}_i$-translations of the identities in Σ.

1.12 <u>Theorem</u>. If $\underline{K} \subseteq \underline{H}$ is a variety determined by a set Σ of $\underline{H}$-identities then $\underline{K}^c$ is determined by $T(\Sigma)$.

<u>Proof</u>. Let $L \in \underline{K}^c$. Then $L^o \in \underline{K}$, thus L^o satisfies every identity in Σ. Now it is easy to show that if $p = q$ is an $\underline{H}$-identity then $L \in \underline{B}_i$ satisfies $Tp = Tq$ iff L^o satisfies $p = q$. Hence our L satisfies every identity in $T(\Sigma)$. Conversely, if L satisfies every identity in $T(\Sigma)$ then L^o satisfies every identity in Σ, hence $L^o \in \underline{K}$ and $L \in \underline{K}^c$. □

The results 1.6 - 1.12 hold, with obvious modifications, also for $\underline{B}_i^-$, $\underline{H}^-$.

Finally in this section we want to investigate the functor $\mathcal{D}$ in relation with H, S and P and we establish a correspondence between subvarieties of $\underline{B}_i$ and subvarieties of $\underline{B}_i^-$, reminiscent of the correspondence Köhler [M] introduced between subvarieties of $\underline{H}$ and $\underline{H}^-$. We shall use the notation introduced in I. 2.18.

1.13 <u>Theorem</u>. Let $\underline{K} \subseteq \underline{B}_i$.

(i) $\mathcal{D}(H(\underline{K})) \subseteq H(\mathcal{D}(\underline{K}))$ and $H(\mathcal{D}(\underline{K})) \subseteq \mathcal{D}(H(\underline{K}))$ if $\underline{K} = S(\underline{K})$

(ii) $\mathcal{D}(S(\underline{K})) = S(\mathcal{D}(\underline{K}))$

(iii) $\mathcal{D}(P(\underline{K})) = P(\mathcal{D}(\underline{K}))$

<u>Proof</u>. (i) Let $L \in \underline{K}$, $f: L \longrightarrow L_1$, $L_1 \in \underline{B}_i$, f an onto $\underline{B}_i$-homomorphism. By I. 2.22 $D(L_1) = f^D[D(L)]$ and f^D is a $\underline{B}_i^-$-homomorphism, hence $D(L_1) \in H(\mathcal{D}(\underline{K}))$. Conversely, if $L \in \underline{K}$, $L_1 \in \underline{B}_i^-$, $f: D(L) \longrightarrow L_1$ an onto $\underline{B}_i^-$-homomorphism then by I. 2.23 $B(0 \oplus L_1) \in H(B(0 \oplus D(L))$. But $B(0 \oplus D(L)) \in S(L)$ since on the one hand it is a $\underline{B}$-subalgebra of L and on the other hand either $x \in D(L)$, implying $x^o \in D(L) \subseteq B(0 \oplus D(L))$, or $x' \in D(L)$,

implying $x^{o} = x^{''o} \leq x^{'o'o} = 0 \in B(0 \oplus D(L))$. Because $S(\underline{K}) = \underline{K}$ it follows that $L_1 = D(B(0 \oplus L_1)) \in \mathcal{D}(H(\underline{K}))$.

(ii) Let $L \in \underline{K}$, $L_1 \in S(L)$. Then

$$D(L_1) = \{x \in L_1 \mid x^{o'o} = 0\} \in S(D(L)) .$$

Conversely, if $L_1 \in S(D(L))$, L_1 non-trivial, then $B(0 \oplus L_1)$ is a subalgebra of L and $L_1 = D(B(0 \oplus L_1)) \in \mathcal{D}(S(L)) \subseteq \mathcal{D}(S(\underline{K}))$.

(iii) Obvious. □

1.14 Corollary. If $\underline{K} \subseteq \underline{B}_i$ then $\mathcal{D}(V(\underline{K})) = V(\mathcal{D}(\underline{K}))$. In particular, if $\underline{K}$ is a variety of interior algebras then $\mathcal{D}(\underline{K})$ is a variety of generalized interior algebras.

As the following corollary shows, every variety of generalized interior algebras can be obtained in this way.

1.15 Corollary. If $\underline{K} \subseteq \underline{B}_i^-$ is a variety then

$$\underline{K}_1 = V(\{B(0 \oplus L) \mid L \in \underline{K}\}) \subseteq \underline{B}_i$$

is a variety such that $\mathcal{D}(\underline{K}_1) = \underline{K}$.

Proof. By 1.14

$$\mathcal{D}(V(\{B(0 \oplus L) \mid L \in \underline{K}\})) = V(\mathcal{D}(\{B(0 \oplus L) \mid L \in \underline{K}\})) = V(\underline{K}) = \underline{K} . \square$$

1.16 Note that if $L \in \underline{B}_i^-$ then $B(0 \oplus L)$ satisfies the equations:

$$\text{(i)} \quad x^{oc'} + x^{oco} = 1$$

$$\text{(ii)} \quad x^{oc} + x^{'oc} = 1 .$$

Let $\underline{S} \subseteq \underline{B}_i$ be the variety defined by (i) and (ii). Apparently $V(\{B(0 \oplus L) \mid L \in \underline{B}_i^-\}) \subseteq \underline{S}$. The reverse inclusion follows from a lemma:

1.17 Lemma. Let $L \in \underline{B}_{i\,SI}$. If L satisfies the equations (i) and (ii), then $L = B(0 \oplus D(L))$.

Proof. Note that $B(0 \oplus D(L))$ may be considered a $\underline{B}$-subalgebra of L, and indeed, even a $\underline{B}_i$-subalgebra: if $x \in B(0 \oplus D(L))$, then either $x \in D(L)$, implying $x^o \in D(L) \subseteq B(0 \oplus D(L))$, or $x' \in D(L)$, in which case $x^o = 0$, since $x^o = x''^o \leq x'^{o'o} = 0$.

It remains to show that $B(0 \oplus D(L)) = L$. Let $x \in L$. Since L is SI, $L^o \cong L_1 \oplus 1$ for some $L_1 \in \underline{H}$, hence, by equation (i), $x^{o'o} = x^{oc'} = 1$ or $x^{oco} = 1$. If $x^{o'o} = 1$, then $x^o = 0$, hence $x^{oc} = 0$, and by equation (ii), $x^{'oc} = 1$, implying that $x' \in D(L)$ and hence that $x \in B(0 \oplus D(L))$. If $x^{oco} = 1$ then $x^{oc} = 1$ and $x \in D(L) \subseteq B(0 \oplus D(L))$. Thus $L = B(0 \oplus D(L))$. □

1.18 Corollary. $\underline{S} = V(\{B(0 \oplus L) \mid L \in \underline{B}_i^-\})$.

Proof. Because $\underline{S}_{SI} \subseteq \{B(0 \oplus L) \mid L \in \underline{B}_i^-\} \subseteq \underline{S}$. □

1.19 Theorem. If $\underline{K} \subseteq \underline{S}$ is a non-trivial variety then

$$V(\{B(0 \oplus D(L)) \mid L \in \underline{K}\}) = \underline{K}.$$

Proof. Since for all $L \in \underline{K}$, L non-trivial, $B(0 \oplus D(L)) \in S(L)$, it follows that $V(\{B(0 \oplus D(L)) \mid L \in \underline{K}\}) \subseteq \underline{K}$. For the converse, let $L \in \underline{K}_{SI}$. Since L satisfies equations (i) and (ii) of 1.16, being a member of $\underline{S}$, it follows from 1.17 that $L = B(0 \oplus D(L))$, hence $\underline{K}_{SI} \subseteq \{B(0 \oplus D(L)) \mid L \in \underline{K}\}$ and $\underline{K} \subseteq V(\{B(0 \oplus D(L)) \mid L \in \underline{K}\})$, in fact, even $\underline{K} = P_S(\{B(0 \oplus D(L)) \mid L \in \underline{K}\})$. □

1.20 Corollary. There exists a 1-1 correspondence between non-trivial subvarieties of $\underline{S}$ and subvarieties of $\underline{B}_i^-$, which respects the inclusion relations.

<u>Proof</u>. If $\underline{K}_1$, $\underline{K}_2$ are two non-trivial subvarieties of $\underline{S}$, $\underline{K}_1 \neq \underline{K}_2$, then $\mathcal{D}(\underline{K}_1)$, $\mathcal{D}(\underline{K}_2)$ are subvarieties of $\underline{B}_i^-$ by 1.14 and $\mathcal{D}(\underline{K}_1) \neq \mathcal{D}(\underline{K}_2)$ since

$$V(\{B(0 \oplus L) \mid L \in \mathcal{D}(\underline{K}_1)\}) = \underline{K}_1 \neq \underline{K}_2 = V(\{B(0 \oplus L) \mid L \in \mathcal{D}(\underline{K}_2)\}).$$

And if $\underline{K} \subseteq \underline{B}_i^-$ is a variety, then $\mathcal{D}(V(\{B(0 \oplus L) \mid L \in \underline{K}\}) = \underline{K}$, where $V(\{B(0 \oplus L) \mid L \in \underline{K}\})$ is a subvariety of $\underline{S}$ by 1.18. It is clear that $\mathcal{D}$ respects the inclusion relations. $\square$

Section 2. The variety generated by all (generalized) interior $\star$-algebras

In our discussion the notion of a $\star$-algebra came up on several occasions. The importance of $\star$-algebras lies in the fact that because of the absence of a "trivial part" they are completely determined by the Heyting-algebra of their open elements, which makes them easier to deal with. In section 1 we already raised the question, if subalgebras of $\star$-algebras are again $\star$-algebras. The first objective of this section is to prove that the answer to this question is affirmative. Having noticed that the variety $\underline{B}_i^\star$ generated by all $\star$-algebras contains non-$\star$-algebras, we proceed to show that the class of finite algebras in $\underline{B}_i^\star$ consists wholly of $\star$-algebras. We conclude the section with some results on free objects in $\underline{B}_i^\star$ and $\underline{B}_i^{-\star}$ which follow easily from similar results for $\underline{B}_i$ and $\underline{B}_i^-$ obtained in Chapter I.

If $L \in \underline{B}_i$ is a $\star$-algebra, i.e., $L = B(L^o)$, then for each $x \in L$ there are $u_0,\ldots u_n$, $v_0,\ldots v_n \in L^o$ such that $x = \sum_{i=0}^{n} u_i^! v_i$. This representation is not unique, however. If $L_1 \in S(L)$ and we wish to show that L_1 is a $\star$-algebra then we have to prove that for any $x \in L_1$ $u_0,\ldots u_n$, $v_0,\ldots v_n \in L_1^o$ can be found such that $x = \sum_{i=0}^{n} u_i^! v_i$. For this purpose we introduce a sequence of unary $\underline{B}_i$-polynomials $s_0, s_1, \ldots$ defined as follows.

2.1 Definition. $s_0, s_1, \ldots$ are unary $\underline{B}_i$-polynomials defined by

(i) $s_0(x) = x'^o$, $s_1(x) = (x'^o + x)^o$

(ii) $s_{2k}(x) = (s_{2k-1}(x) + x')^o$ and $s_{2k+1}(x) = (s_{2k}(x) + x)^o$, for $k \geq 1$.

If L is a $\star$-algebra such that L^o is a chain then any $x \in B(L^o)$, $x \neq 0$, can be written in a unique way as $x = \sum_{i=0}^{n} u_i' v_i$, where $0 \leq u_0 < v_0 < \ldots < u_n < v_n \leq 1$. It is easy to see that in this case $s_0(x) = u_0$, $s_1(x) = v_0$, $s_{2k}(x) = u_k$ and $s_{2k+1}(x) = v_k$, $1 \leq k \leq n$. Therefore $x = \sum_{i=0}^{n} s_{2i}(x)' s_{2i+1}(x) \in [x]_{\underline{B}_i}$. The next lemma shows that the same conclusion holds in the more general case that L is a $\star$-algebra B-generated by some chain $C \subseteq L^o$.

2.2 <u>Lemma</u>. Let $L \in \underline{B}_i$ be a $\star$-algebra such that L is $\underline{B}$-generated by a chain $C \subseteq L^o$, $0, 1 \in C$. Suppose that

$$x = \sum_{i=0}^{n} c_{2i}' c_{2i+1} \in L,$$

with $0 \leq c_0 < c_1 < \ldots < c_{2n+1} \leq 1$, $c_i \in C$, $i = 0,1,\ldots 2n+1$. Then

$$x = \sum_{i=0}^{n} s_{2i}(x)' s_{2i+1}(x).$$

<u>Proof</u>. (i)

$$s_{2i}(x)' s_{2i+1}(x) = s_{2i}(x)'(s_{2i}(x) + x)^o \leq$$
$$\leq s_{2i}(x)'(s_{2i}(x) + x) \leq x$$

for all $i = 0,1,\ldots$ and similarly

$$s_{2i+1}(x)' s_{2i+2}(x) \leq x', \quad \text{for all} \quad i = 0,1,\ldots$$

(ii) Note that $s_i(x) \leq s_{i+1}(x)$, $i = 0,1,\ldots$. With (i) we obtain

$$\sum_{i=0}^{k} s_{2i}(x)' s_{2i+1}(x) \leq x \cdot s_{2k+1}(x), \quad k = 0,1,\ldots$$

We claim that

$$x \cdot s_{2k+1}(x) = \sum_{i=0}^{k} s_{2i}(x)' s_{2i+1}(x), \quad k = 0,1,\ldots$$

This we show by induction:

a) $k = 0$ $\quad x \cdot s_1(x) = x.(x'^{o} + x)^{o} \le x'^{o'}.(x'^{o} + x)^{o} = s_0(x)'.s_1(x)$,

hence $\quad x \cdot s_1(x) = s_0(x)'.s_1(x)$.

b) Now suppose $\quad x \cdot s_{2k-1}(x) = \sum_{i=0}^{k-1} s_{2i}(x)'s_{2i+1}(x)$ for some $k > 0$.

Then

$$\begin{aligned} x.s_{2k+1}(x) &= x.(s_{2k-1}(x) + s_{2k-1}(x)'.s_{2k}(x) + s_{2k}(x)'s_{2k+1}(x)) \\ &\le x.s_{2k-1}(x) + x.x' + s_{2k}(x)'.s_{2k+1}(x) \\ &\le \sum_{i=0}^{k} s_{2i}(x)'s_{2i+1}(x) \ . \end{aligned}$$

Hence the claim follows.

(iii) $\quad c_i \le s_i(x)$, $\quad$ for $\quad i = 0,\ldots 2n+1$.

Indeed, $\quad c_0.x = c_0.\sum_{i=0}^{n} c'_{2i}c_{2i+1} = 0$, $\quad$ hence $\quad c_0 \le x'^{o} = s_0(x)$.

Furthermore, if $\quad c_{2k} \le s_{2k}(x)$, for some $\quad k \ge 0$, $\quad k \le n$, then

$$c_{2k+1} = c_{2k} + c'_{2k}\,c_{2k+1} \le s_{2k}(x) + x \ , \quad \text{hence}$$
$$c_{2k+1} \le (s_{2k}(x) + x)^{o} = s_{2k+1}(x) \ .$$

And if $\quad k < n$,

$$c_{2k+2} = c_{2k+1} + c'_{2k+1}\,c_{2k+2} \le s_{2k+1}(x) + x' \ , \quad \text{thus}$$
$$c_{2k+2} \le (s_{2k+1}(x) + x')^{o} = s_{2k+2}(x) \ .$$

Finally, $\quad x = x \cdot c_{2n+1} \le x \cdot s_{2n+1}(x) = \sum_{i=0}^{n} s_{2i}(x)'s_{2i+1}(x)$.

With (i), we obtain $\quad x = \sum_{i=0}^{n} s_{2i}(x)'s_{2i+1}(x)$, $\quad$ as desired.□

2.3 <u>Theorem</u>. Let $\quad L \in \underline{B}_i$. Then $\quad x \in B(L^{o})$ $\quad$ iff

$$x = \sum_{i=0}^{n} s_{2i}(x)'s_{2i+1}(x) \ , \quad \text{for some} \quad n \ge 0 \ .$$

<u>Proof</u>. $\Longleftarrow$ Obvious, since $\quad s_i(x) \in L^{o}$, for all $x \in L$, $i = 0,1,\ldots$.

$\Longrightarrow$ Let $\quad x \in B(L^{o})$. $\quad$ Then $\quad x = \sum_{i=1}^{m} u'_i v_i$, $\quad u_1,\ldots u_m, v_1,\ldots v_m \in L^{o}$,

$m > 0$. Let $\quad L_1 = B([\{u_1,\ldots u_m , v_1,\ldots v_m\}]_{\underline{H}}) \in S(L)$. $\quad L_1$ is a

$\underline{B}_i$-subalgebra of L and indeed a $\star$-algebra, and $\quad x \in L_1$. Since

L_1^{o} is a countable distributive lattice with $0,1$, there exists a chain $C \subseteq L_1^o$ such that L_1 is $\underline{B}$-generated by C (cf. Balbes and Dwinger [74]). By 2.2, then,

$$x = \sum_{i=0}^{n} s_{2i}(x)' s_{2i+1}(x) \quad \text{for some} \quad n \geq 0 . \square$$

2.4 Corollary. $L \in \underline{B}_i$ is a $\star$-algebra iff for each $x \in L$ there is an $n \geq 0$ such that

$$x = \sum_{i=0}^{n} s_{2i}(x)' s_{2i+1}(x) .$$

The answer to our question if a subalgebra of a $\star$-algebra is itself a $\star$-algebra follows as an easy corollary:

2.5 Corollary. Let $L \in \underline{B}_i$ be a $\star$-algebra, $L_1 \in S(L)$. Then L_1 is a $\star$-algebra. Hence, if $L \in \underline{B}_i$ then L is a $\star$-algebra iff for each $x \in L$ $[x]_{\underline{B}_i}$ is a $\star$-algebra.

Proof. Let $x \in L_1$. Then $x \in B(L^o)$, hence

$$x = \sum_{i=0}^{n} s_{2i}(x)' s_{2i+1}(x) \quad \text{for some} \quad n \geq 0 .$$

But $s_i(x) \in [x]_{\underline{B}_i}^o$, $i = 0,1,\dots$, and $[x]_{\underline{B}_i}^o \subseteq L_1^o$, hence L_1 is a $\star$-algebra. $\square$

In order to establish similar results for $\star$-algebras in $\underline{B}_i^-$ we just adapt the given proofs to the $\underline{B}_i^-$-case. We define a sequence $s_0^-, s_1^-, \dots$ of $\underline{B}_i^-$-polynomials as follows:

2.6 Definition.

(i) $s_0^-(x) = (x \Rightarrow x^o)^o \qquad s_1^-(x) = ((x \Rightarrow x^o)^o + x)^o$

(ii) $s_{2k}^-(x) = (x \Rightarrow s_{2k-1}^-(x))^o$ and

$s_{2k+1}^-(x) = (s_{2k}^-(x) + x)^o$, for $k \geq 1$.

By modifying the proofs of 2.2 - 2.5 we obtain:

2.7 Theorem. Let $L \in \underline{B}_i^-$. Then $x \in B^-(L^O)$ iff

$$x = \sum_{i=0}^{n} (s_{2i}^-(x) \Rightarrow x^O).s_{2i+1}^-(x), \quad \text{for some} \quad n \geq 0.$$

2.8 Corollary. Let $L \in \underline{B}_i^-$ be a $\star$-algebra, $L_1 \in S(L)$. Then L_1 is a $\star$-algebra. Hence if $L \in \underline{B}_i^-$ then L is a $\star$-algebra iff for each $x \in L$ $[x]_{\underline{B}_i^-}$ is a $\star$-algebra.

2.9 In section 1 we have seen that a finite product of $\star$-algebras is a $\star$-algebra. It is now easy to see that a similar statement does not hold for arbitrary products. Consider the interior algebras $K_n \cong ([1,n]] \subseteq K_\infty$, introduced in I. 3.4, and let $L = \prod_{n=1}^{\infty} K_{2n}$. Obviously the K_{2n} are $\star$-algebras. But L fails to be a $\star$-algebra: if $x \in L$ is the element $(\{2\},\{2,4\},\{2,4,6\},\ldots)$ then $s_{2n}(x)'s_{2n+1}(x) = (\emptyset,\emptyset,\ldots,\emptyset,\underset{(n+1)\text{-th coordinate}}{\{2n+2\}},\{2n+2\},\ldots)$.

Clearly, there is no k such that $x = \sum_{n=0}^{k} s_{2n}(x)'s_{2n+1}(x)$.

The remaining part of this section will be devoted to a further study of the variety generated by all (generalized) interior $\star$-algebras. In accordance with the notation introduced in section 1, let $\underline{B}_i^\star = V(\{L \in \underline{B}_i \mid L = B(L^O)\})$ and $\underline{B}_i^{-\star} = V(\{L \in \underline{B}_i^- \mid L = B(L^O)\})$.

As we have seen, $\prod_{n=1}^{\infty} K_{2n}$ is an example of an interior algebra belonging to $\underline{B}_i^\star$ without being a $\star$-algebra.

We recall that $\underline{B}_i^\star$ and $\underline{B}_i^{-\star}$ are precisely the varieties $\underline{T}_0$ respectively $\underline{T}_0^-$ introduced in I. 6.8. By I. 6.9 $\underline{B}_i^\star$ and $\underline{B}_i^{-\star}$ are generated by their finite $\star$-algebras. I. 6.11 guarantees that

$\underline{B}_i^*$ and $\underline{B}_i^{-*}$ are proper subclasses of $\underline{B}_i$ and $\underline{B}_i^-$ respectively. As a matter of fact, $M_2 \notin \underline{B}_i^*$ and $M_2 \notin \underline{B}_i^{-*}$ by virtue of the proof of I. 6.11. Actually, we can describe the finite members of $\underline{B}_i^*$ and $\underline{B}_i^{-*}$ more precisely.

2.10 Lemma. Let $L \in \underline{B}_i$ or $L \in \underline{B}_i^-$ be finite. L is a $\star$-algebra iff for all $u,v \in L^o$ such that $u < v$ there exists a $w \in L^o$ such that $u \prec w \leq v$.

Proof. $\Longrightarrow$ Suppose that L is a finite $\star$-algebra, $u,v \in L^o$, $u < v$. Since L^o is finite, there exists a $w \in L^o$ such that $u \underset{L^o}{\prec} w \leq v$. If $u \not\prec w$, then there are two atoms $a_1,a_2 \in L$, $a_1 \neq a_2$, $a_1 \leq u'w$, $a_2 \leq u'w$. Because L is a $\star$-algebra, there exist $u_1,v_1 \in L^o$ such that $a_1 = u_1'v_1$. Then $a_2 \leq u_1$ or $a_2 \not\leq v_1$. In the former case, $u < (u + u_1)w < w$ and $(u + u_1)w \in L^o$, contradicting $u \underset{L^o}{\prec} w$. In the latter case, $u < (u + v_1)w < w$ and $(u + v_1)w \in L^o$, again contradicting $u \underset{L^o}{\prec} w$. Thus $u \underset{L}{\prec} w$.
$\Longleftarrow$ Let $a \in L$ be an atom, and let $u = \Sigma\{v \in L^o \mid v \leq a'\}$. Then $u < 1$ and $u \in L^o$, hence, by assumption, there exists a $w \in L^o$ such that $u \prec w$. $w \not\leq a'$, therefore $a \leq w$, and since $u \prec w$, $a \leq u'$, it follows that $a = u'w$. Thus every atom of L belongs to $B(L^o)$ and as L is finite we infer that $L = B(L^o)$.$\square$

2.11 Theorem. Let $L \in \underline{B}_i^*$ or $L \in \underline{B}_i^{-*}$ be finite. Then L is a $\star$-algebra.

Proof. Let $L \in \underline{B}_i^*$ be finite and suppose that L is not a $\star$-algebra. By 2.10, there are $u,v \in L^o$ such that $u \underset{L^o}{\prec} v$, but $u \prec v$. Consider $(v]$. If $u = 0$, then $(v] \cong M_k$ for some

$k > 1$ (cf. I. 6.1), hence $M_k \in H(L) \subseteq \underline{B}_i^*$, which is impossible as we have seen in the proof of I. 6.11 . If $u \neq 0$ then M_2 would be a $\underline{B}_i^-$-subalgebra of $(v] \in \underline{B}_i^*$. Since $(v]^- \in \underline{B}_i^{-*}$ as well (cf. I. 2.26) M_2^- would belong to $\underline{B}_i^{-*}$, again in contradiction with I. 6.11 . Hence L is a $*$-algebra. □

Next we want to make some remarks concerning the free objects on finitely many generators in $\underline{B}_i^*$ and $\underline{B}_i^{-*}$. Many of the results of sections I.4 - I.6 for $F_{\underline{B}_i}(n)$ and $F_{\underline{B}_i^-}(n)$, $n > 0$, carry over to $F_{\underline{B}_i^*}(n)$ and $F_{\underline{B}_i^{-*}}(n)$, with some slight modifications. We shall select a few of the more interesting ones.

Firstly, note that since K_∞ is infinite and an element of $\underline{B}_i^*$, $\underline{B}_i^{-*}$, the fact that it is $\underline{B}_i^-$-generated by one element implies that $F_{\underline{B}_i^*}(1)$ and $F_{\underline{B}_i^{-*}}(1)$ are infinite and hence that neither $\underline{B}_i^*$ nor $\underline{B}_i^{-*}$ are locally finite. Furthermore, remark I. 4.2 applies in particular to $\underline{B}_i^*$ and $\underline{B}_i^{-*}$, hence

2.12 <u>Corollary</u>. If $L \in \underline{B}_i^*$ or $L \in \underline{B}_i^{-*}$ is finite, generated by n elements, then there is a $u \in F_{\underline{B}_i^*}(n)^o$, $u \in F_{\underline{B}_i^{-*}}(n)^o$ respectively, such that $L \cong (u]$.

2.13 <u>Corollary</u> *There exists a* $u \in F_{\underline{B}_i^*}(1)^o$ *such that* $H_\infty \cong (u]$.

<u>Proof</u>. By I. 3.9 , $H_\infty \cong F_{\underline{B}_i}(1,\{x^o = x\})$. Since H_∞ is a $*$-algebra, $H_\infty \cong F_{\underline{B}_i^*}(1,\{x^o = x\})$. □

2.14 <u>Theorem</u>. For any $n \in N$ there exists a $u \in F_{\underline{B}_i^*}(n)^o$, a $v \in F_{\underline{B}_i^{-*}}(n + 1)^o$, such that $F_{\underline{B}_i^{-*}}(n) \cong (u]$, $F_{\underline{B}_i^*}(n) \cong (v]$.

<u>Proof</u>. Similar to the proofs of I. 4.9 , I. 4.11 . □

2.15 Theorem. (i) There is a $u \in F_{\underline{B}_i}{*}(1)^o$ which has $\aleph_o$ open covers in $F_{\underline{B}_i}{*}(1)$.

(ii) There is a $u \in F_{\underline{B}_i}{-*}(2)^o$ which has $\aleph_o$ open covers in $F_{\underline{B}_i}{-*}(2)$.

Proof. (i) Compare I. 4.15 : the algebras L_i are $*$-algebras.

(ii) By (i) and 2.14 . □

The next proposition tells us, how many different homomorphic images $F_{\underline{B}_i}{*}(1)$ and $F_{\underline{B}_i}{-*}(1)$ have.

2.16 Theorem. (i) There are $2^{\aleph_o}$ non-isomorphic subdirectly irreducible algebras in $\underline{B}_i^*$ generated by one element.

(ii) There are $2^{\aleph_o}$ non-isomorphic subdirectly irreducible algebras in $\underline{B}_i^{-*}$ generated by two elements.

Proof. (i) If $\underline{a}$ is a sequence of 0's and 1's, then $L_{\underline{a}} \in \underline{B}_i^*$ (cf. I. 4.19) and so is the SI algebra constructed from $L_{\underline{a}}$ in I. 4.25 .

(ii) Follows from (i) and 2.14. □

As far as subalgebras are concerned, I. 5.1 - I. 5.7 could be restated for $F_{\underline{B}_i}{*}(n)$, $F_{\underline{B}_i}{-*}(n)$ without change. Further, lemma I. 5.8 deals exclusively with $*$-algebras, and the algebra L_1 constructed there is obviously $\underline{B}_i^-$-generated by two elements. Therefore we have

2.17 Theorem. (cf. I. 5.11) $F_{\underline{B}_i}{-*}(2)^o$ contains $F_{\underline{H}}{-}(n)$ as a subalgebra, for all $n \in N$. Hence $B(F_{\underline{H}}{-}(n))$ is a subalgebra of $F_{\underline{B}_i}{-*}(2)$, for all $n \in N$.

Similarly we obtain

2.18 Theorem. (cf. I. 5.13). For any $n \in N$, there is a $b \in F_{\underline{B}_i^*}(1)^o$ such that $F_{\underline{H}}(n) \in S([b)^o)$. Hence, for any $n \in N$ there exists a $b \in F_{\underline{B}_i^*}(1)^o$, such that $B(F_{\underline{H}}(n)) \in S([b))$.

In I. 6.6 we have seen that $F_{\underline{B}_i^-}(1)^o$ is characteristic for $\underline{H}^-$. Here we have

2.19 Theorem. (cf. also I. 6.14) $F_{\underline{B}_i^{-*}}(2)$ is characteristic for $\underline{B}_i^{-*}$.

Proof. Let $L \in \underline{B}_i^-$ be a finite $*$-algebra. Then $L^o \in H(F_{\underline{H}^-}(n))$ for some $n \in N$, hence $L = B(L^o) \in H(B(F_{\underline{H}^-}(n))$.
By 2.17 $L \in HS(F_{\underline{B}_i^{-*}}(2)) \subseteq V(F_{\underline{B}_i^{-*}}(2))$.
Since $\underline{B}_i^{-*} = V(\{L \in \underline{B}_i \mid L = B(L^o)$ and L finite$\})$ by I. 6.9, we deduce $\underline{B}_i^{-*} \subseteq V(F_{\underline{B}_i^{-*}}(2))$. The reverse inclusion is trivial.||

The algebras $B(F_{\underline{B}_i}(1)^o)^-$ and $F_{\underline{B}_i^*}(1)^-$ are two more examples of functionally free algebras in $\underline{B}_i^{-*}$. In $\underline{B}_i^*$ the situation is different:

2.20 Theorem. (cf. I. 6.4) $F_{\underline{B}_i^*}(n)$ is not characteristic for $\underline{B}_i^*$ for any $n \in N$. Hence no finitely generated interior algebra in $\underline{B}_i^*$ is characteristic for $\underline{B}_i^*$.

Finally we notice that from I. 5.8, 2.11 and 2.12 follows

2.21 Theorem. $F_{\underline{B}_i^{-*}}(2)$ is a generalized universal algebra for all finite algebras in $\underline{B}_i^{-*}$.

Section 3. The free algebra on one generator in $\underline{B}_i^{-*}$

Contrary to what one might expect, the results obtained in the last section indicate that $F_{\underline{B}_i^*}(n)$, $n \in N$, and $F_{\underline{B}_i^{-*}}(n)$, $n \geq 2$, $n \in N$, are not much less complicated then $F_{\underline{B}_i}(n)$, $F_{\underline{B}_i^-}(n)$, $n \in N$. Not much has been said so far about $F_{\underline{B}_i^{-*}}(1)$, except that it is infinite. The purpose of this section is to provide a characterization of this algebra. We start with a lemma of a universal algebraic nature.

3.1 <u>Lemma</u>. Let $\underline{K}$ be a class of algebras, $A \in V(\underline{K})$ and $S \subseteq A$ such that $[S] = A$. A is freely generated by S in $V(\underline{K})$ iff for all $B \in S(\underline{K})$ and for every map $f\colon S \longrightarrow B$ such that $[f[S]] = B$ there is a homomorphism $\bar{f}\colon A \longrightarrow B$ such that $\bar{f} \mid S = f$.

<u>Proof</u>. $\Longrightarrow$ Obvious.

$\Longleftarrow$ Let $C \in V(\underline{K})$, $C = [S']$ and $f\colon S \longrightarrow S'$ a surjective map. We want to show that there exists a homomorphism $\bar{f}\colon A \longrightarrow C$ such that $\bar{f} \mid S = f$. Since $C \in V(\underline{K}) = HSP(\underline{K})$, there exists a $C' \in SP(\underline{K})$, and a homomorphism $h\colon C' \longrightarrow C$ which is onto. Choose for every $s \in S'$ a $t_s \in h^{-1}(\{s\})$, let $T = \{t_s \mid s \in S'\}$ and let $f'\colon S \longrightarrow T$ be defined by $s \longmapsto t_{f(s)}$. Then $D =$ $= [T] \in SP(\underline{K}) \subseteq P_S S(\underline{K})$ and $h[D] = C$.

Let $\{\underline{B}_i \mid i \in I\} \subseteq S(\underline{K})$ such that $D \in P_S(\{\underline{B}_i \mid i \in I\})$, with projections $\pi_i\colon D \longrightarrow \underline{B}_i$, $i \in I$. Now $\pi_i \circ f'\colon S \longrightarrow \underline{B}_i$ is a map such that $[\pi_i \circ f'[S]] = \underline{B}_i$, hence $\pi_i \circ f'$ can be extended to a homomorphism $f_i\colon A \longrightarrow \underline{B}_i$. Consider the homomorphism

$$\prod_{i \in I} f_i\colon A \longrightarrow \prod_{i \in I} \underline{B}_i .$$

If $s \in S$ then

$$(\prod_{i\in I} f_i)(s) = (f_i(s))_{i\in I} = ((\pi_i \circ f'(s))_{i\in I} = f'(s) ,$$

thus $\prod_{i\in I} f_i \mid S = f'$. Since $h \circ f' = f$, $\bar{f} = h \circ \prod_{i\in I} f_i : A \longrightarrow C$ is a homomorphism satisfying $\bar{f} \mid S = f$. □

3.2 Theorem. Let $\underline{K}$ be a non-trivial class of algebras such that $\underline{K} = S(\underline{K})$. Let $\underline{m}$ be any cardinal number, $\underline{m} > 0$, and $\underline{K}_{\underline{m}} = \{L \in \underline{K} \mid L$ is generated by $\leq \underline{m}$ elements$\}$. Then $F_{V(\underline{K})}(\underline{m}) \in P_S(\underline{K}_{\underline{m}})$.

Proof. Let S be a set such that $|S| = \underline{m}$. For $A \in \underline{K}_{\underline{m}}$, let $\{f_i^A \mid i \in I_A\}$ be the collection of all possible maps $f: S \longrightarrow A$ such that $f[S] = A$. Let $B = \prod_{A\in \underline{K}_{\underline{m}}} \prod_{i\in I_A} A$, and define

$$i: S \longrightarrow B \quad \text{by} \quad i(s) = ((f_i^A(s))_{i\in I_A})_{A\in \underline{K}_{\underline{m}}} .$$

$[i[S]]$ satisfies the condition of 3.1: if $f: i[S] \longrightarrow C$, $C \in \underline{K}$, is a map such that $[f[i[S]]] = C$ then $C \in \underline{K}_{\underline{m}}$ since $|i[S]| = |S|$ and $f \circ i = f_j^C$ for some $j \in I_C$. Thus $\pi_j \mid [i[S]]: [i[S]] \longrightarrow C$ is the desired extension. By 3.1, then,

$$F_{V(\underline{K})}(S) \cong [i[S]] \in P_S(\underline{K}_{\underline{m}}) . \square$$

Since $\underline{B}_i^{-*}$ is generated by its finite $*$-algebras and the class of finite $*$-algebras is closed under subalgebras, we know by 3.2 that $F_{\underline{B}_i^{-*}}(1) \in P_S(\{L \in \underline{B}_i^- \mid L$ is a finite $*$-algebra, $\underline{B}_i^-$-generated by one element$\})$. In the next theorem a characterization of these finite $*$-algebras $\underline{B}_i^-$-generated by one element is given. For the definition of K_n and K_∞, see I. 3.4 and I. 3.1 .

3.3 Theorem. Let $L \in \underline{B}_i^-$ be a finite $*$-algebra, $\underline{B}_i^-$-generated by one element. Then $L \cong K_n^-$ for some $n \geq 0$.

Proof. Let $x \in L$ such that $L = [x]_{\underline{B}_i^-}$. By the proof of I.4.8(i) x^o is the smallest element of L. First we show that L^o is a chain. We assume that $|L| > 1$.

By lemma 2.10 there exists then a $w \in L^o$ such that $x^o \prec w$. We claim that $w = (x \Rightarrow x^o)^o$. In order to prove this we show that if p is a unary $\underline{B}_i^-$-polynomial then $p(x).(x \Rightarrow x^o)^o = (x \Rightarrow x^o)^o$ or $p(x).(x \Rightarrow x^o)^o = x^o$. Since w is an atom and clearly $w \not\leq x$, $w \leq (x \Rightarrow x^o)$ hence $w \leq (x \Rightarrow x^o)^o$. These two facts will imply our claim that $w = (x \Rightarrow x^o)^o$.

We proceed by induction:

(i) $x.(x \Rightarrow x^o)^o = x^o$.

(ii) Suppose the statement is true for unary $\underline{B}_i^-$-polynomials q, r.

(a) If $p(x) = q(x)^o$ then $p(x).(x \Rightarrow x^o)^o = q(x)^o.(x \Rightarrow x^o)^o =$

$$= \begin{cases} (x \Rightarrow x^o)^o & \text{if } q(x).(x \Rightarrow x^o)^o = (x \Rightarrow x^o)^o \text{ and} \\ x^o & \text{if } q(x).(x \Rightarrow x^o)^o = x^o . \end{cases}$$

(b) If $p(x) = q(x).r(x)$ then

$$p(x).(x \Rightarrow x^o)^o = q(x).r(x).(x \Rightarrow x^o)^o =$$

$$= \begin{cases} x^o & \text{if } q(x).(x \Rightarrow x^o)^o = x^o \text{ or } r(x).(x \Rightarrow x^o)^o = x^o \\ (x \Rightarrow x^o)^o & \text{otherwise.} \end{cases}$$

(c) If $p(x) = q(x) + r(x)$ then

$$p(x).(x \Rightarrow x^o)^o = q(x).(x \Rightarrow x^o)^o + r(x).(x \Rightarrow x^o)^o =$$

$$= \begin{cases} x^o & \text{if } q(x).(x \Rightarrow x^o)^o = x^o \text{ and } r(x).(x \Rightarrow x^o)^o = x^o \\ (x \Rightarrow x^o)^o & \text{otherwise.} \end{cases}$$

(d) If $p(x) = q(x) \Rightarrow r(x)$ then

$$p(x).(x \Rightarrow x^o)^o = (q(x) \Rightarrow r(x)).(x \Rightarrow x^o)^o =$$

$$= (q(x) \Rightarrow x^o)(x \Rightarrow x^o)^o + r(x).(x \Rightarrow x^o)^o =$$

$$= \begin{cases} x^o & \text{if } q(x).(x \Rightarrow x^o)^o = (x \Rightarrow x^o)^o \text{ and } r(x).(x \Rightarrow x^o)^o = x^o \\ (x \Rightarrow x^o)^o & \text{if } q(x).(x \Rightarrow x^o)^o = x^o \text{ or } \\ & \quad r(x).(x \Rightarrow x^o)^o = (x \Rightarrow x^o)^o . \end{cases}$$

We conclude that x^o has a unique open cover namely $(x \Rightarrow x^o)^o$ and that hence $L^o \cong 0 \oplus [(x \Rightarrow x^o)^o)^o$. If we can show that the finite $\star$-algebra $[(x \Rightarrow x^o)^o)$ is $\underline{B}_i^-$-generated by one element, then by repeating this reasoning a finite number of times, it follows that L^o is a chain.

Claim: if p is a unary $\underline{B}_i^-$-polynomial, then $p(x) = q(x \Rightarrow x^o)$ or $p(x) = q(x \Rightarrow x^o).b$, where $b = (x \Rightarrow x^o)^o \Rightarrow x^o$, for some $\underline{B}_i^-$-polynomial q . The claim will be proven by induction on the length of p . Notice first that for any $\underline{B}_i^-$-polynomial q , $q(x \Rightarrow x^o) \geq (x \Rightarrow x^o)^o$ (cf. proof of I. 4.8).

(i) $x = ((x \Rightarrow x^o) \Rightarrow (x \Rightarrow x^o)^o).((x \Rightarrow x^o)^o \Rightarrow x^o)$

$= (x \Rightarrow x^o) \Rightarrow (x \Rightarrow x^o)^o.b = q(x \Rightarrow x^o).b$,

with $q(y) = y \Rightarrow y^o$.

(ii) Suppose the claim has been verified for unary $\underline{B}_i^-$-polynomials r , s.

(a) If $p(x) = r(x)^o$ and $r(x) = q(x \Rightarrow x^o)$ for some $\underline{B}_i^-$-polynomial q then $p(x) = q^o(x \Rightarrow x^o)$. If $r(x) = q(x \Rightarrow x^o).b$, then $p(x) = r(x)^o = q(x \Rightarrow x^o)^o.b^o = x^o = (x \Rightarrow x^o)^o.b$.

(b) Suppose $p(x) = r(x).s(x)$. If $r(x) = q_1(x \Rightarrow x^o)$, $s(x) = q_2(x \Rightarrow x^o)$, q_1 , q_2 $\underline{B}_i^-$-polynomials, then $p(x) = q(x \Rightarrow x^o)$, where $q = q_1.q_2$. If $r(x) = q_1(x \Rightarrow x^o).b$, $s(x) = q_2(x \Rightarrow x^o)$, then $p(x) = q_1(x \Rightarrow x^o).b.q_2(x \Rightarrow x^o) = q(x \Rightarrow x^o).b$, where $q = q_1.q_2$. The other two cases are similar.

(c) Suppose $p(x) = r(x) + s(x)$. If $r(x) = q_1(x \Rightarrow x^o).b$, $s(x) = q_2(x \Rightarrow x^o).b$, then $p(x) = q(x \Rightarrow x^o).b$ with $q = q_1 + q_2$. If $r(x) = q_1(x \Rightarrow x^o).b$, $s(x) = q_2(x \Rightarrow x^o)$, then

$$\begin{aligned} p(x) &= q_1(x \Rightarrow x^o).b + q_2(x \Rightarrow x^o) = \\ &= q_1(x \Rightarrow x^o).b + b \Rightarrow x^o + q_2(x \Rightarrow x^o) = \\ &= q_1(x \Rightarrow x^o) + q_2(x \Rightarrow x^o) = q(x \Rightarrow x^o), \end{aligned}$$

with $q = q_1 + q_2$, since $b \Rightarrow x^o = (x \Rightarrow x^o)^o \leq q_2(x \Rightarrow x^o)$ and $b \Rightarrow x^o \leq q_1(x \Rightarrow x^o)$. The remaining two cases are similar.

(d) Suppose $p(s) = r(x) \Rightarrow s(x)$. If $r(x) = q_1(x \Rightarrow x^o)$, $s(x) = q_2(x \Rightarrow x^o)$ then $p(x) = q(x \Rightarrow x^o)$, with $q = q_1 \Rightarrow q_2$; if $r(x) = q_1(x \Rightarrow x^o).b$, $s(x) = q_2(x \Rightarrow x^o)$, then $p(x) = q(x \Rightarrow x^o)$ with $q = q_1 \Rightarrow q_2$; if $r(x) = q_1(x \Rightarrow x^o)$, $s(x) = q_2(x \Rightarrow x^o).b$, then $p(x) = q(x \Rightarrow x^o).b$, with $q = q_1 \Rightarrow q_2$, and finally, if $r(x) = q_1(x \Rightarrow x^o).b$, $s(x) = q_2(x \Rightarrow x^o).b$, then $p(x) = q(x \Rightarrow x^o)$, $q = q_1 \Rightarrow q_2$.

Now, let $y \in [(x \Rightarrow x^o)^o)$. Then $y = p(x)$ for some $\underline{B}_i^-$-polynomial p, hence, by the claim just proven, $y = q(x \Rightarrow x^o)$ or $y = q(x \Rightarrow x^o).b$, for some $\underline{B}_i^-$-polynomial q. But since $(x \Rightarrow x^o)^o \leq y$, $y \not\leq (x \Rightarrow x^o)^o \Rightarrow x^o = b$, hence $y = q(x \Rightarrow x^o)$ for some $\underline{B}_i^-$-polynomial q. Thus we have shown, that the $\underline{B}_i^-$-subalgebra $[(x \Rightarrow x^o)^o)$ of L is $\underline{B}_i^-$-generated by the element $x \Rightarrow x^o$.

Our assertion that L^o is a chain has thus been proven; say $L^o \cong \underline{(n+1)}^-$, $n \geq 0$. As L is a $*$-algebra, $L = B^-(L^o) \cong B^-(\underline{(n+1)}^-)$, hence $L \cong K_n^-$, for some $n \geq 0$. □

3.4 Lemma. For each $n \geq 0$, K_n^- has precisely one $\underline{B}_i^-$-generator.

Proof. Recall that $K_n^- = P(\{1,\ldots n\})$, $K_n^{-o} = \{[1,k] \mid 0 \leq k \leq n\} \cong (n+1)^-$.

We have seen (cf. I. 3.4) that the element $x = \{2k \mid k \in N,\ 2k \leq n\}$ $\underline{B}_i^-$-generates K_n^-. Suppose that $y \subseteq [1,n]$, $y \neq x$ also $\underline{B}_i^-$-generates K_n^-. By the proof of I. 4.8, y^o is the smallest element of K_n^-, hence $y^o = \emptyset$, and consequently $1 \notin y$. Let i_o be the first number such that $i_o \in y$, $i_o + 1 \in y$ or $i_o \notin y$, $i_o + 1 \notin y$. If

$$L = [y.[1\,,\,i_o + 1]]_{\underline{B}_i^-} \subseteq ([1\,,\,i_o + 1]]\ ,$$

then

$$L = [([1\,,\,i_o - 1]] \cup \{i_o\,,\,i_o + 1\}]_{\underline{B}^-}\ ,$$

and one easily verifies, that $\{i_o\} \notin L$. Thus $([1\,,\,i_o + 1]]$ is not $\underline{B}_i^-$-generated by $y.[1\,,\,i_o + 1]$, and since $([1\,,\,i_o + 1]]$ is a homomorphic image of $([1,n]]$ it follows that K_n^- is not generated by y, a contradiction. Hence, if $1 \leq i < n$, then $i \in y$, $i + 1 \notin y$ or $i \notin y$, $i + 1 \in y$; together with $1 \notin x$, $1 \notin y$, this implies that $y = x$. □

Now we are ready for the main result of this section.

3.5 __Theorem__. $F_{\underline{B}_i^-}*(1) \cong K_\infty^-$. The free generator of K_∞^- is

$$x = \{2n \mid n \in N\}\ .$$

__Proof__. Firstly, $K_\infty^- \in \underline{B}_i^{-*}$. Indeed, if $y \neq z$, $y,z \in K_\infty^-$, then there is an atom $\{n\}$ such that $\{n\} \nleq y$, $\{n\} \leq z$ or conversely. Then $y.[1,n] \neq z.[1,n]$. Therefore the homomorphism

$$f\colon K_\infty^- \longrightarrow \prod_{n=1}^{\infty} K_n^-$$

defined by

$$y \longmapsto (y.[1,n])_{n \in N}$$

is an embedding.

The K_n^- are $*$-algebras, hence $K_\infty^- \in P_S(\{K_n^- \mid n \in N\}) \subseteq \underline{B}_i^{-*}$.

In order to show that K_∞^- is freely generated by x, by 3.1 it suffices to prove that if L is a finite $\star$-algebra $\underline{B}_i^-$-generated by an element y, then there exists a homomorphism $h: K_\infty^- \longrightarrow L$ such that $h(x) = y$. But by 3.3, $L \cong K_n^-$, and by 3.4, y corresponds with $x.[1,n]$. Thus the map $z \longmapsto z.[1,n]$ provides the desired homomorphism.□

3.6 Corollary. If $L \in \underline{B}_i^\star$ or $L \in \underline{B}_i^{-\star}$ then for each $x \in L$ there is an $n \geq 0$ such that $[x]_{\underline{B}_i^-} \cong K_n^-$ or $[x]_{\underline{B}_i^-} \cong K_\infty^-$.

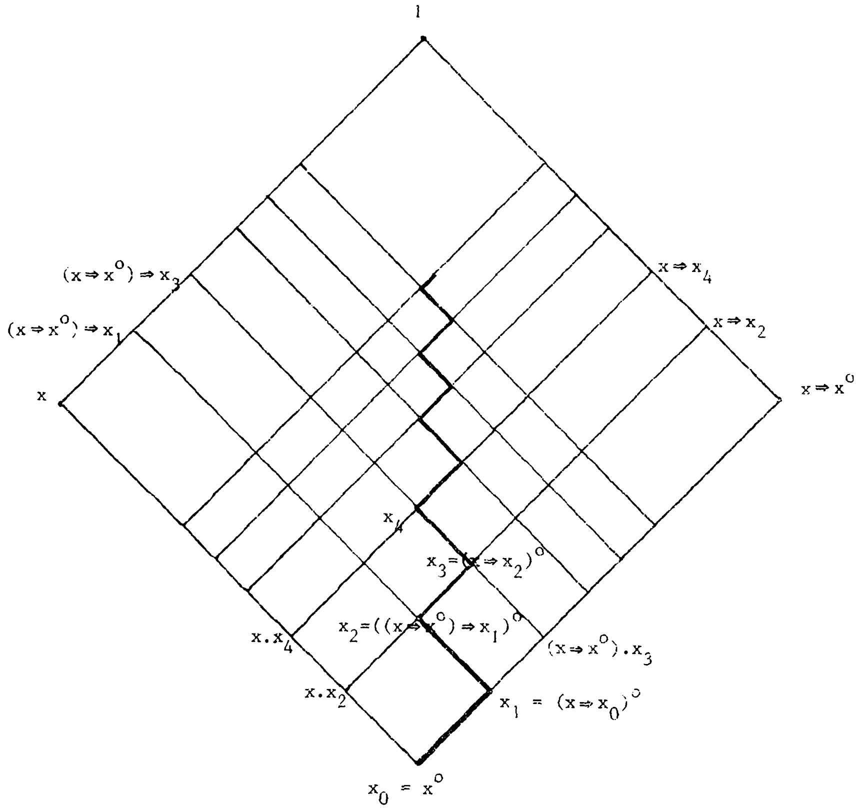
It will be proven later that the converse of this corollary holds as well. Then we shall have at our disposal a nice characterization of algebras belonging to $\underline{B}_i^\star$ or $\underline{B}_i^{-\star}$.

Note that in the proof of 3.5 we only used that $\underline{B}_i^{-\star}$ is generated by its finite $\star$-algebras; not the result 2.11, that each finite algebra in $\underline{B}_i^{-\star}$ is a $\star$-algebra. In fact, this is now an easy consequence:

3.7 Corollary. Let $L \in \underline{B}_i^\star$ or $L \in \underline{B}_i^{-\star}$ be finite. Then L is a $\star$-algebra.

Proof. Let $x \in L$. By 3.6, $[x]_{\underline{B}_i^-} \cong K_n^-$ for some $n \geq 0$, hence $x \in B([x]^o_{\underline{B}_i^-}) \subseteq B(L^o)$. □

The following diagram suggests the more important features of the structure of $F_{\underline{B}_i^{-\star}}(1)$:



$F_{\underline{B}_i}{}^{-*}(1)$

Section 4. Injectives and projectives in $\underline{B}_i^*$ and $\underline{B}_i^{-*}$

In I.7.11 we arrived at the conclusion that in $\underline{B}_i$ and $\underline{B}_i^-$ no non-trivial injectives exist. The reason seemed to be the presence of arbitrarily large interior algebras with trivial interior operator. Since $\underline{B}_i^*$ and $\underline{B}_i^{-*}$ do not contain any interior algebras with trivial interior operator except $\underline{1}$ and $\underline{2}$, non-trivial injectives might be expected to exist in these varieties. As for $\underline{B}_i^*$, this is indeed the case as we prove in 4.10: the injectives in $\underline{B}_i^*$ are the complete so-called discrete interior algebras. This result as well as its proof make the close relationship visible which exists not only between Heyting algebras and $*$-algebras but also between the varieties $\underline{H}$ and $\underline{B}_i^*$. In 4.11 we remark that $\underline{H}^-$ does not have any non-trivial injectives; neither does $\underline{B}_i^{-*}$, as one easily concludes. The section ends with some observations concerning projectives in $\underline{B}_i^*$ and $\underline{B}_i^{-*}$.

4.1 If $\underline{K} \subseteq \underline{B}_i$ is a class such that $S(\underline{K}) \subseteq \underline{K}$ and L is injective in $\underline{K}$ then L^o is injective in $\underline{K}^o$. Indeed, let $L_1, L_2 \in \underline{K}^o$, $g: L_1 \longrightarrow L^o$ and $f: L_1 \longrightarrow L_2$ a monomorphism. g, f can be extended to $\bar{g}: B(L_1) \longrightarrow L$ and $\bar{f}: B(L_1) \longrightarrow B(L_2)$, respectively, where $B(L_1)$, $B(L_2) \in \underline{K}$, and $\bar{f}$ is a monomorphism. By the injectivity of L, there exists a homomorphism $h: B(L_2) \longrightarrow L$ such that $h \circ \bar{f} = \bar{g}$. Hence $h \mid L_2 \circ f = g$ and it follows that L^o is injective in $\underline{K}^o$. Balbes and Horn [70] have shown that the injective Heyting algebras are precisely the complete Boolean algebras. Thus, if $L \in \underline{B}_i^*$ is injective in $\underline{B}_i^*$ then L^o is complete and Boolean.

4.2 Definition. A (generalized) interior algebra L is called discrete if for all $x \in L$, $x^o = x$.

4.3 Theorem. If $L \in \underline{B}_i^*$ is injective in $\underline{B}_i^*$, then L is a complete discrete interior algebra.

Proof. Let $L \in \underline{B}_i^*$ be injective in $\underline{B}_i^*$. Then L^o is injective in $\underline{B}_i^{*o} = \underline{H}$, thus L^o is a complete Boolean algebra as we observed in 4.1. It remains to be shown that $L = L^o$. Suppose $x \in L \setminus L^o$. Then $x^o, x'^{o'} \in L^o$ and $x^o < x < x'^{o'}$. If $a = x^{o'}x'^{o'}$ then $xa \neq 0$, $x'a \neq 0$, and $x^o a = x'^o a = 0$. Therefore $M_2 \cong [xa]_{\underline{B}_i} \subseteq (a]$, and hence, since $a \in L^o$, $M_2 \in SH(L)$. But $M_2 \notin \underline{B}_i^*$ by 2.1i, a contradiction. We conclude that $L = L^o$. □

To establish the converse of theorem 4.3 we shall apply a theorem on Heyting algebras, essentially due to Glivenko [29] (see also Balbes and Dwinger [74]). If $L \in \underline{H}$, $x \in L$, then x is called a regular element if $x = (x \to 0) \to 0$. The set of regular elements of L is denoted by $Rg(L)$.

4.4 Theorem. If $L \in \underline{H}$ then $Rg(L)$ is a Boolean algebra under the operations induced by the partial order of L. The operations are given by:

$$u \underset{Rg(L)}{+} v = ((u + v) \to 0) \to 0$$

$$u \underset{Rg(L)}{\cdot} v = u.v$$

$$u \underset{Rg(L)}{\to} v = ((u \to 0 + v) \to 0) \to 0$$

$$0_{Rg(L)} = 0$$

$$1_{Rg(L)} = 1$$

Moreover, the map $r_L\colon L \longrightarrow Rg(L)$ given by $x \longmapsto (x \to 0) \to 0$ is an $\underline{H}$-homomorphism.

4.5 We want to extend this result to a similar one for algebras in $\underline{B}_i^*$. If $L \in \underline{B}_i$, $x \in L$, then x is called <u>regular</u> if $x^{co} = x$. Hence, if x is regular then $x = x^{oco}$. On the other hand, it is well-known (cf. McKinsey and Tarski [46]) that $(x^{oco})^{co} = x^{oco}$ is an identity in $\underline{B}_i$. Thus the set of regular elements of L, $Rg(L)$, is $\{x^{oco} \mid x \in L\}$. Recall that the set $D(L)$ of dense elements of L is $\{x \in L \mid x^{oc} = 1\}$ (cf. I. 2.19).

4.6 <u>Theorem</u>. Let $L \in \underline{B}_i$. $Rg(L)$ is a Boolean algebra under the operations induced by the partial order of L. In fact, it is a discrete interior algebra, the operations being given by:

$$x \underset{Rg(L)}{+} y = (x+y)^{oco}$$

$$x \underset{Rg(L)}{\cdot} y = x.y$$

$$x^{'Rg(L)} = x^{'oco}$$

$$x^{o Rg(L)} = x$$

$$0_{Rg(L)} = 0$$

$$1_{Rg(L)} = 1 .$$

Moreover, if $L \in \underline{B}_i^*$, then $r_L\colon L \longrightarrow Rg(L)$ defined by $x \longmapsto x^{oco}$ is a $\underline{B}_i$-homomorphism, with kernel $D(L)$. Hence $Rg(L) \cong L / D(L)$.

For the proof of the second part of the theorem we need a lemma.

4.7 <u>Lemma</u>. $\underline{B}_i^*$ satisfies the identity $(x+y)^{oco} = (x^{oco} + y^{oco})^{oco}$ $(*)$.

<u>Proof</u>. Since $\underline{B}_i^*$ is generated by its finite $*$-algebras (see I. 6.9) it suffices to show that every finite $*$-algebra $L \in \underline{B}_i$ satisfies $(*)$. So let $L \in \underline{B}_i$ be a finite $*$-algebra and let $\phi: L^o \longrightarrow Rg(L^o)$ be the $\underline{H}$-homomorphism, which exists by virtue of 4.4. Since $Rg(L^o)$ is Boolean we may regard it as a discrete interior algebra. By I.2.15, there exists a $\underline{B}_i$-homomorphism $\bar{\phi}: L \to Rg(L^o)$, such that $\bar{\phi} \mid L^o = \phi$. Thus, if $x \in L^o$ then $\bar{\phi}(x) = \phi(x) = (x \to 0) \to 0 = x^{'o'o} = x^{co}$, hence for $x \in L$, $\bar{\phi}(x) = (\bar{\phi}(x))^o = \bar{\phi}(x^o) = x^{oco}$. Since $\bar{\phi}$ is a $\underline{B}_i$-homomorphism, then,

$$(x+y)^{oco} = \bar{\phi}(x+y) = \bar{\phi}(x) \underset{Rg(L^o)}{+} \bar{\phi}(y) = x^{oco} \underset{Rg(L^o)}{+} y^{oco} =$$

$$= ((x^{oco} + y^{oco}) \to 0) \to 0 = (x^{oco} + y^{oco})^{oco} \quad .\square$$

Notice that the identity $(*)$ is not valid in $\underline{B}_i$: for example, it is not satisfied by M_2.

<u>Proof of 4.6</u>. Let $L \in \underline{B}_i$. As $Rg(L) = Rg(L^o)$ and $L^o \in \underline{H}$, it follows from 4.4 that $Rg(L)$ is a Boolean algebra under the given operations and indeed, this is a well-known fact (cf. McKinsey and Tarski [46]). Thus $Rg(L)$ provided with the given interior operator is a discrete interior algebra.

Now assume that $L \in \underline{B}_i^*$. The map r_L preserves $\cdot$, because of the well-known identity $(x.y)^{oco} = x^{oco}.y^{oco}$, and r_L preserves $0,1$. Moreover, $r_L(x+y) = (x+y)^{oco} = (x^{oco} + y^{oco})^{oco} = r_L(x) \underset{Rg(L)}{+} r_L(y)$ by 4.7. Thus r_L is a $\underline{D}_{01}$-homomorphism and therefore a $\underline{B}$-homomorphism. Finally, r_L is a $\underline{B}_i$-homomorphism since

$$(r_L(x))^{o_{Rg(L)}} = x^{oco} = r_L(x^o) \ .$$

The last assertion of the theorem follows from

$$r_L^{-1}(\{1\}) = \{x \in L \mid x^{oco} = 1\} = D(L) \ .\square$$

4.8 <u>Theorem</u>. Every complete discrete interior algebra is injective in $\underline{B}_i^*$.

<u>Proof</u>. Suppose that L is a complete discrete interior algebra. Let $L_1, L_2 \in \underline{B}_i^*$, $f\colon L_1 \longrightarrow L_2$ a 1-1 $\underline{B}_i$-homomorphism, $g\colon L_1 \longrightarrow L$ a $\underline{B}_i$-homomorphism. Let $r_{L_i}\colon L_i \longrightarrow Rg(L_i)$, $i = 1,2$ be the maps guaranteed by 4.6. Let $\bar{g}\colon Rg(L_1) \longrightarrow L$ be defined by $\bar{g} = g \mid Rg(L_1)$. Then $\bar{g}(0) = 0$, $\bar{g}(1) = 1$, $\bar{g}(x \underset{Rg(L)}{\cdot} y) = \bar{g}(x.y) = g(x.y) =$
$= g(x).g(y) = \bar{g}(x).\bar{g}(y)$, and $\bar{g}(x \underset{Rg(L)}{+} y) = \bar{g}((x+y)^{oco}) =$
$= g((x+y)^{oco}) = (g(x) + g(y))^{oco} = \bar{g}(x) + \bar{g}(y)$, for any $x,y \in Rg(L_1)$. Since both $Rg(L_1)$ and L are discrete, it follows that $\bar{g}$ is a $\underline{B}_i$-homomorphism satisfying $\bar{g} \circ r_{L_1} = g$. Analogously, there exists a $\bar{f}\colon Rg(L_1) \longrightarrow Rg(L_2)$ such that $\bar{f}$ is a $\underline{B}_i$-homomorphism, $\bar{f} = r_{L_2} \circ f \mid Rg(L_1)$, and $\bar{f} \circ r_{L_1} = r_{L_2} \circ f$. Note that $\bar{f}$ is 1-1.

$$\begin{tikzcd}
L_2 \arrow[dr, "r_{L_2}"] & & \\
& Rg(L_2) \arrow[dr, dashed, "h"] & \\
L_1 \arrow[uu, "f"] \arrow[rr, "g"] \arrow[dr, "r_{L_1}"'] & & L \\
& Rg(L_1) \arrow[uu, "\bar{f}"] \arrow[ur, "\bar{g}"'] &
\end{tikzcd}$$

Since L is injective in the category of Boolean algebras (see Balbes and Dwinger [74]), there exists a $\underline{B}$-homomorphism $h\colon Rg(L_2) \longrightarrow L$ such that $h \circ \bar{f} = \bar{g}$. Since $Rg(L_2)$ and L are discrete h is also a $\underline{B}_i$-homomorphism. Now $h \circ \bar{f} \circ r_{L_1} = \bar{g} \circ r_{L_1}$, thus $h \circ r_{L_2} \circ f = g$, and $h \circ r_{L_2}$ is the sought-for $\underline{B}_i$-homomorphism. □

4.9 Remark. As a matter of fact, we have shown in the proof of 4.8 that the equational category $\underline{B}$ of discrete interior algebras is a reflective subcategory of $\underline{B}_i^*$ and that the reflector preserves monomorphisms.

The promised characterization of the injectives in $\underline{B}_i^*$ follows now from 4.3 and 4.8:

4.10 Corollary. The injectives in $\underline{B}_i^*$ are the complete discrete interior algebras.

4.11 As in 4.1 we see that if $L \in \underline{B}_i^{-*}$ is injective then L^o is injective in $\underline{H}^- = (\underline{B}_i^{-*})^o$. However, let $L \in \underline{H}^-$ be injective in $\underline{H}^-$, $|L| > 1$, $u \in L$, $u \neq 1$. Let $L_1 \in \underline{H}^-$ be a subdirectly irreducible algebra such that $|L_1| > |L|$, with $v \in L_1$ the unique dual atom in L_1. Define $g\colon \underline{2} \longrightarrow L$ by $g(0) = u$, $g(1) = 1$ and $f\colon \underline{2} \longrightarrow L_1$ by $f(0) = v$, $f(1) = 1$. Then both g and f are $\underline{H}^-$-homomorphisms. Suppose that $h\colon L_1 \longrightarrow L$ is a homomorphism such that $h \circ f = g$. Since

$$\begin{tikzcd} L_1 \arrow[dr, dashed, "h"] & \\ \underline{2} \arrow[u, "f"] \arrow[r, "g"'] & L \end{tikzcd}$$

$|L_1| > |L|$ h is not 1-1, hence $h^{-1}(\{1\}) \neq \{1\}$. But then $v \in h^{-1}(\{1\})$. This implies that $1 = h(v) = h \circ f(0) = g(0) = u$, contradicting our assumption that $u \neq 1$. Hence L cannot be injective, unless it is a trivial algebra. We have shown:

4.12 Theorem. $\underline{B}_i^{-*}$ does not have any non-trivial injectives.

Note that 4.11 provides a different proof for the $\underline{B}_i^-$-part of I. 7.12 as well; in fact, this proof works for any variety $\underline{K}$ such that $\underline{B}_i^{-*} \subseteq \underline{K} \subseteq \underline{B}_i^-$.

As for projectives in $\underline{B}_i^*$, I. 7.16 assumes the following elegant form:

4.13 Theorem. The finite weakly projectives in $\underline{B}_i^*$ are the algebras $B(L)$, where $L \cong L_0 \dagger L_1 \dagger \ldots \dagger L_n$ for some $n \geq 0$, $L_n \cong \underline{2}$, $L_i \cong \underline{2}^2$ or $L_i \cong \underline{2}$, $0 \leq i < n$.

Proof. By 2.11 and Balbes and Horn [70]. □

We have not considered the problem of determining the weakly projectives in $\underline{B}_i^{-*}$. Though it is clear that the algebras mentioned in 4.13 are weakly projective in $\underline{B}_i^{-*}$, one might expect that there are more, because of a possible existence of finite algebras, weakly projective in $\underline{H}^-$ but not in $\underline{H}$. We shall confine ourselves therefore to the obvious remark that the finite weakly projectives of $\underline{B}_i^{-*}$ are the algebras $B(L)$, such that L is weakly projective in $\underline{H}^-$.

Section 5. Varieties generated by (generalized) interior algebras whose lattices of open elements are chains

Particularly nice and simple examples of Brouwerian algebras and Heyting algebras are the ones which are totally ordered, i.e., the chains. For any two elements x,y in a chain, $x \to y = 1$ or $y \to x = 1$, which leads to the observation that chains satisfy the equation $(x \to y) + (y \to x) = 1$ (*). The subvarieties $\underset{=}{C}^-$ of $\underset{=}{H}^-$ and $\underset{=}{C}$ of $\underset{=}{H}$ determined by this equation (*) are a natural object of study, and have indeed received considerable attention in the literature, for example in Horn [69, 69 a], Hecht and Katrinäk [72] and Köhler [73].

The remaining three sections will be mainly devoted to an investigation of the free finitely generated objects in some subvarieties of $\underset{=}{C}^c$ and $\underset{=}{C}^{-c}$. In this section we start with some preparatory results.

5.1 $\underset{=}{C}$ and $\underset{=}{C}^-$

It will be useful to give a brief review of the main facts concerning $\underset{=}{C}$ and $\underset{=}{C}^-$. We restrict ourselves to $\underset{=}{C}$ and its subvarieties since the results and arguments for $\underset{=}{C}^-$ are essentially the same.

It is known (see Balbes and Dwinger [74]) that $\underset{=}{C}^-$ is the class of relative Stone algebras and that $\underset{=}{C}$ is the class of relative Stone algebras with 0. Recall that a relative Stone algebra is a distributive lattice such that each interval $[a,b]$ of L is a Stone algebra, i.e. a distributive lattice with pseudocomplementation * satisfying the identity $x^* + x^{**} = 1$. More interesting to us is the fact that the subvarieties of $\underline{C}$ and $\underline{C}^-$ have been characterized, in Hecht and Katrinák [72]. We shall sketch now a simple proof of their results.

5.2 Firstly, if $L \in \underline{C}_{SI}$ then $L \cong L_1 \oplus 1$, where $L_1 \in \underline{C}$, *hence for any* $x,y \in L \in \underline{C}_{SI}$ $x \to y = 1$ *or* $y \to x = 1$ *and thus* $x \leq y$ or $y \leq x$. Therefore $\underline{C}_{SI}$ consists entirely of chains with a penultimate element. If $\underline{n}$, $n \in N$, denotes as usual the chain $\{0 < 1 < \dots < n-1\}$, considered as Heyting algebra, then the finite subdirectly irreducibles in $\underline{C}$ are $\underline{n}$, $n > 1$, $n \in N$. We consider the subvarieties $V(\underline{n})$ of $\underline{C}$, $n \in N$. Since $\underline{m} \in H(\underline{n})$ if $0 \leq m \leq n$, it follows that $V(\underline{m}) \subseteq V(\underline{n})$ for $m \leq n$. Moreover, if $m > n \geq 0$ then $\underline{m} \notin V(\underline{n})$. This can be seen by applying one of Jónsson's results (0.1.26), but also in a more elementary way, by realizing that the identity $\sum_{i=1}^{m} x_i \to x_{i+1} = 1$ is satisfied in $\underline{n}$ but not in $\underline{m}$. As a matter of fact, this identity determines $V(\underline{m-1})$ relative to $\underline{C}$. We conclude that $V(\underline{1}) \subset V(\underline{2}) \subset \dots \subset \underline{C}$.

The variety $\underline{C}$ is locally finite. Indeed, it is sufficient to note that if $L \in \underline{C}_{SI}$ is generated by k elements then $|L| \leq k+2$. If $L_1 \in \underline{C}$ is generated by k elements then by 0.1.6 L_1 is a subdirect product of subdirectly irreducibles in $\underline{C}$. Since there are only finitely many mappings from L_1 onto subdirectly irreducibles L in $\underline{C}$, L_1 is a subalgebra of a finite product of finite algebras, hence finite. Since any variety is generated by its finitely generated free algebras, it follows in particular that $\underline{C}$ is generated by its finite members, and even by its finite subdirectly irreducibles. Hence $\underline{C} = V(\bigcup_{n=1}^{\infty} V(\underline{n}))$. Now, if $\underline{K} \subseteq \underline{C}$ is a variety then $\underline{K}_{SI} \subseteq \{\underline{n} \mid n \leq m\}$ for some $m \geq 1$ since otherwise $\underline{K}_{SI} \supseteq \underline{C}_{FSI}$, as one easily verifies, and hence $\underline{K} = \underline{C}$, in contradiction with our assumption. Thus $\underline{K} = V(\underline{m})$ for some $m \in N$, and the chain of subvarieties $V(\underline{1}) \subset V(\underline{2}) \subset \ldots \subset \underline{C}$ comprises all subvarieties of $\underline{C}$. If $\underline{\omega+1}$ denotes the Heyting algebra of order type $\omega+1$, then obviously $\underline{C} = V(\underline{\omega+1})$.

Summarizing, we have:

5.3 Theorem. (cf. Hecht and Katrinák [72])

(i) The subvarieties of $\underline{C}$ form a chain of type $\omega+1$:

$V(\underline{1}) \subset V(\underline{2}) \subset \ldots \subset \underline{C} = V(\underline{\omega+1})$ and

$V(\underline{n})_{SI} = \{\underline{m} \mid 1 < m \leq n\}$, $n \in N$,

$\underline{C}_{SI} = \{C \oplus 1 \mid C \in \underline{H},\ C \text{ a chain}\}$.

(ii) The subvarieties of $\underline{C}^-$ form a chain of type $\omega+1$:

$V(\underline{1}^-) \subset V(\underline{2}^-) \subset \ldots \subset \underline{C}^- = V((\underline{\omega+1})^-)$ and

$V(\underline{n}^-)_{SI} = \{\underline{m}^- \mid 1 < m \leq n\}$, $n \in N$,

$\underline{C}^-_{SI} = \{C \oplus 1 \mid C \in \underline{H}^-,\ C \text{ a chain}\}$.

Furthermore, both $\underline{C}$ and $\underline{C}^-$ are locally finite.

In section 1 of this chapter we associated with a variety $\underline{K}$ of Heyting algebras the variety $\underline{K}^c = \{L \in \underline{B}_i \mid L^o \in \underline{K}\}$, and similarly with a variety $\underline{K}$ of Brouwerian algebras the variety $\underline{K}^c = \{L \in \underline{B}_i^- \mid L^o \in \underline{K}\}$.

Let now

$$\underline{M} = \underline{C}^c , \qquad \underline{M}^- = \underline{C}^{-c} \qquad \text{and}$$

$$\underline{M}_n = V(\underline{n})^c , \quad \underline{M}_n^- = V(\underline{n}^-)^c , \qquad \text{for} \quad n \in N .$$

By 1.9 $\underline{M}$ and $\underline{M}_n$, $n \in N$ are varieties, and similarly $\underline{M}^-$, $\underline{M}_n^-$, $n \in N$ are varieties.

5.4 Theorem.

(i) The equation $(x^{o'} + y^o)^o + (y^{o'} + x^o)^o = 1$ determines $\underline{M}$ relative to $\underline{B}_i$.
The equation $(x^o \Rightarrow y^o)^o + (y^o \Rightarrow x^o)^o = 1$ determines $\underline{M}^-$ relative to $\underline{B}_i^-$.

(ii) $\underline{M}_{SI} = \{L \in \underline{B}_i \mid L^o$ is a chain with a dual atom$\}$,
$\underline{M}_{SI}^- = \{L \in \underline{B}_i^- \mid L^o$ is a chain with a dual atom$\}$.

(iii) $\underline{M} = V(\underline{M}_{FSI})$ and $\underline{M}^- = V(\underline{M}_{FSI}^-)$
$\underline{M}_n = V(\{L \in \underline{B}_i \mid L$ is finite and $L^o \cong \underline{n}\})$, $n \in N$,
$\underline{M}_n^- = V(\{L \in \underline{B}_i^- \mid L$ is finite and $L^o \cong \underline{n}^-\})$, $n \in N$.

(iv) $\underline{M}_1 \subset \underline{M}_2 \subset \ldots \subset \underline{M}$ and $\underline{M} = V(\bigcup_{n=1}^{\infty} \underline{M}_n)$,
$\underline{M}_1^- \subset \underline{M}_2^- \subset \ldots \subset \underline{M}^-$ and $\underline{M}^- = V(\bigcup_{n=1}^{\infty} \underline{M}_n)$.

Note that it is not claimed - and as a matter of fact it is not true - that the chains of subvarieties of $\underline{M}$ and $\underline{M}^-$ mentioned in (iii) and (iv) comprise all subvarieties of $\underline{M}$ and $\underline{M}^-$ respectively.

Proof. (i) Apply 1.12 to the equation $(x \rightarrow y) + (y \rightarrow x) = 1$, which defines $\underline{C}$ and $\underline{C}^-$ relative to $\underline{H}$ and $\underline{H}^-$ respectively.

(ii) By 5.3, since for any $L \in \underline{B}_i$, $L \in \underline{M}_{SI}$ iff $L^o \in \underline{M}_{SI}^o = \underline{C}_{SI}$. Similarly for $\underline{M}^-$.

(iii) We show that

$$\underline{M}_n^- = V(\{L \in \underline{B}_i \mid L^o \cong \underline{n} \text{ and } L \text{ is finite}\}),\ n \in N.$$

The treatment of the other cases is similar.

According to 0. 1.6, $\underline{M}_n = V(\underline{M}_{n\,SI})$. In virtue of I. 2.7 and 5.3, $L \in \underline{M}_{n\,SI}$ iff $L^o \in V(\underline{n})_{SI}$ iff $L^o \cong \underline{m}$, for some m, $1 < m \leq n$. Assuming $\underline{M}_n = V(\underline{M}_{n\,FSI})$, it is sufficient to note that if L is finite, $L^o \cong \underline{m}$, $1 < m \leq n$, then $L \cong M_{k_1,\ldots k_m}$ (cf. I. 7.20) hence $L \in H(L_1)$, where $L_1 \cong M_{k_1,\ldots k_m, k_{m+1},\ldots k_n}$, $k_{m+1},\ldots k_n$ being arbitrary positive numbers and $L_1^o \cong \underline{n}$, in order to conclude that $\underline{M}_n = V(\underline{M}_{n\,FSI}) \subseteq V(\{L \in \underline{B}_i \mid L \text{ is finite and } L^o \cong \underline{n}\})$. It remains to show that $\underline{M}_n$ is generated by its finite subdirectly irreducibles. Suppose that $\underline{M}_n \neq V(\underline{M}_{n\,FSI})$, then there exists an $L \in \underline{M}_{n\,SI} \setminus V(\underline{M}_{n\,FSI})$ and a $\underline{B}_i$-polynomial p such that the identity $p(x_1,\ldots x_n) = 1$ is satisfied by $V(\underline{M}_{n\,FSI})$ but not by L. Let $a_1,\ldots a_n \in L$ be such that $p(a_1,\ldots a_n) \neq 1$. Apply a simplified version of the method exhibited in I. 6.9, i.e. let $b_1,\ldots b_m$ be the subterms of $p(a_1,\ldots a_n)$, including $a_1,\ldots a_n$, and define on $L_1 = [\{b_1,\ldots b_m\}]_{\underline{B}}$ an interior operator ^{o_1} by

$$x^{o_1} = \Sigma\{y \in L_1 \mid y^o = y \text{ and } y \leq x\}.$$

Since L_1 is finite this is a good definition and it follows that $L_1 \in \underline{M}_{n\,FSI}$, because L_1^o is a chain, the length of which is at most the length of L^o. Furthermore, $p_{L_1}(a_1,\ldots a_n) = p_L(a_1,\ldots a_n) \neq 1$, a contradiction. Thus $\underline{M}_n = V(\underline{M}_{n\,FSI})$ as desired.

(iv) is an immediate consequence of (iii) and 5.3.□

It is not our aim to describe here the lattice of all subvarieties of $\underline{M}$ or $\underline{M}^-$; that will be done afterwards. Now we want to throw some light on the algebras in $\underline{M}$ and $\underline{M}^-$ themselves, in particular on the finitely generated free algebras. In order to do so, we shall have to restrict ourselves occasionally to suitable subvarieties of $\underline{M}$ and $\underline{M}^-$.

The class $\underline{M}_2$ of interior algebras the lattices of open elements of which are Boolean, definable by the equation $x^{oc} = x^o$, deserves some special attention. It is about the only proper subvariety of $\underline{B}_i$, which has been investigated before, and indeed, rather extensively. The interest in the algebras belonging to $\underline{M}_2$ is not surprising if one considers that they form on the one hand a starting point for the notion of cylindric algebra (Henkin, Monk, Tarski [71]), on the other hand for that of a polyadic algebra (Halmos [62]). The algebras in $\underline{M}_2$, which got already some attention from McKinsey and Tarski [48], are known as monadic algebras, and it was probably Halmos who gave them this name.

The subdirectly irreducible monadic algebras are precisely the interior algebras with trivial interior operator (the finite ones among which are our familiar M_n, $n \in N$) and therefore simple, which facilitates the study of monadic algebras greatly. Bass [58] showed that $\underline{M}_2$ is locally finite and he determined the finitely generated free objects in $\underline{M}_2$. See also J. Berman [M].

5.5 Theorem. $F_{\underline{M}_2}(n) \cong \prod_{1 \leq k \leq 2^n} M_k^{\binom{2^n}{k}}$ hence $F_{\underline{M}_2}(n)$ has $2^n.2^{2^n-1}$ atoms, and $F_{\underline{M}_2}(n)^o$ has $2^{2^n}-1$ atoms.

This result is in fact a simple corollary of lemma 3.1 and corollary 6.5, still to follow.

The subvarieties of $\underline{M}_2$ have been characterized by Monk [70]:

5.6 <u>Theorem</u>. The subvarieties of $\underline{M}_2$ form a chain of type $\omega+1$:

$$V(M_0) \subset V(M_1) \subset \dots \subset \underline{M}_2 .$$

<u>*Proof*</u>. Clearly $V(M_k) \subseteq V(M_\ell)$, if $0 \le k \le \ell$.
That $V(M_k) \subset V(M_\ell)$ if $k < \ell$ follows immediately from 0. 1.26 and I. 2.8 , or, alternatively, from the observation that the equation

$$\sum_{1 \le i < j \le 2^k+1} (x_i' + x_j)^o (x_j' + x_i)^o = 1 \quad (*)$$

is valid in $V(M_k)$ but not in $V(M_\ell)$, $\ell > k$. Furthermore, if $\underline{K} \subseteq \underline{M}_2$ is a variety, and $\underline{K}$ would contain an infinite subdirectly irreducible or infinitely many finite subdirectly irreducibles, then one deduces that $\underline{K}_{SI} \supseteq \underline{M}_{2\,FSI}$, hence by 5.4 (iii), $\underline{K} = \underline{M}_2$. Therefore, $\underline{K} = \underline{M}_2$ or $\underline{K}_{SI} \subseteq \{M_k \mid k \le n\}$ for some $n \in N$ implying $\underline{K} = V(M_n)$ for some $n \in N$. It also follows that the equation $(*)$ defines $V(M_k)$ relative to $\underline{M}_2$, $k \ge 0$. □

5.7 In I. 7.8 we have already observed that free products in $\underline{M}_2$ exist. In virtue of I. 7.9 , free products exist in each subvariety $V(M_n)$, $n \in N$, of $\underline{M}_2$. On the other hand, the variety $\underline{M}_2$ has no non-trivial injectives : the proof of I. 7.12 applies to $\underline{M}_2$ as well as it does to $\underline{B}_i$. As for the subvarieties of $\underline{M}_2$, it is known that $V(M_n)$ has no non-trivial injectives if $n \ge 3$, $n \in N$, whereas the injectives of $V(M_1)$, the class of discrete interior algebras, are of course the complete algebras, and the injective objects of $V(M_2)$ are the extensions of M_2 by complete Boolean algebras (cf. Quackenbusch [74]).

The classes $\underline{M}_n$, $n > 2$, $n \in N$, are more difficult to deal with, chiefly because of the presence of many more subdirectly irreducibles. Before starting to try to characterize finitely generated free objects in (subvarieties of) $\underline{M}_n$, $\underline{M}_n^-$, $n > 2$, $n \in N$, we want to establish the important fact that these algebras are finite. Note that since $K_\infty \in \underline{M}$, $K_\infty^- \in \underline{M}^-$, K_∞ and K_∞^- being $\underline{B}_i$-generated by one element, it follows that $\underline{M}$ and $\underline{M}^-$ are not locally finite, unlike the classes $\underline{C} = \underline{M}^o$, and $\underline{C}^- = \underline{M}^{-o}$.

5.8 <u>Theorem</u>. $\underline{M}_n$ and $\underline{M}_n^-$ are locally finite, for each $n \in N$.

<u>Proof</u>. Let $L \in \underline{M}_{n\,SI}$ and suppose that L is generated by k elements, say by $x_1,\dots x_k$. Then

$$L = [\{x_1,\dots x_k\}]_{\underline{B}_i} = [\{x_1,\dots x_k\} \cup L^o]_{\underline{B}}$$

and since $|L^o| \leq n$ it follows that $|L| \leq 2^{2^{k+n}}$. Hence there are only finitely many subdirectly irreducibles in $\underline{M}_n$ generated by k elements all of which are finite, and using 0.1.6 it follows as in 5.2 that every algebra in $\underline{M}_n$ generated by k elements is finite. The proof for $\underline{M}_n^-$ is similar. □

In the next section, $F_{\underline{M}_n^-}(1)$ and $F_{\underline{M}_n}(1)$, $n \in N$, will be determined. As far as the free objects in $\underline{M}_n$ and $\underline{M}_n^-$ on more than one generator are concerned, we shall restrict ourselves to finding the finitely generated free objects in the subvarieties $\underline{M}_n^{-*}$ and $\underline{M}_n^*$ of $\underline{M}_n^-$ and $\underline{M}_n$ respectively. These varieties are still typical for the behaviour of $\underline{M}_n^-$ and $\underline{M}_n$ in the sense that $V(\underline{n}^-) = \underline{M}_n^{-o} = \underline{M}_n^{-*o}$ and $V(\underline{n}) = \underline{M}_n^o = \underline{M}_n^{*o}$ and have at the same time the advantage of possessing only a very limited number of subdirectly irreducibles.

We need a simple lemma on $*$-varieties.

5.9 Lemma. Let $\underline{K} \subseteq \underline{B}_i$ or $\underline{K} \subseteq \underline{B}_i^-$ be a variety and let $\underline{K}_1 \subseteq \underline{K}^o$ be such that $\underline{K}^o = V(\underline{K}_1)$. Then $\underline{K}^* = V(\{B(L) \mid L \in \underline{K}_1\})$.

Proof. $V(\{B(L) \mid L \in \underline{K}_1\})^o = V(\underline{K}_1) = \underline{K}^o$ by 1.2,

hence by 1.8

$$\underline{K}^* \subseteq V(\{B(L) \mid L \in \underline{K}_1\}).$$

On the other hand

$$V(\{B(L) \mid L \in \underline{K}_1\}) \subseteq V(\{B(L^o) \mid L \in \underline{K}\}) = \underline{K}^* . \square$$

5.10 Theorem.

(i) $\underline{M}_n^* = V(K_{n-1})$ and $\underline{M}_n^{-*} = V(K_{n-1}^-)$, $n \in N$.

$\underline{M}_{n\,SI}^* = \{K_m \mid 1 \leq m < n\}$, $\underline{M}_{n\,SI}^{-*} = \{K_m^- \mid 1 \leq m < n\}$, $n \in N$.

(ii) $\underline{M}^* = V(K_\infty)$, $\underline{M}^{-*} = V(K_\infty^-)$.

(iii) The subvarieties of $\underline{M}^*$ are

$$\underline{M}_1^* \subset \underline{M}_2^* \subset \ldots \subset \underline{M}^* .$$

The subvarieties of $\underline{M}^{-*}$ are

$$\underline{M}_1^{-*} \subset \underline{M}_2^{-*} \subset \ldots \subset \underline{M}^{-*} .$$

Proof. (i) The first line follows from 5.9, since $\underline{M}_n^o = V(\underline{n})$, $\underline{M}_n^{-o} = V(\underline{n}^-)$, and $B(\underline{n}) \cong K_{n-1}$, $B(\underline{n}^-) \cong K_{n-1}^-$, $n \in N$. In order to prove the second pair of statements, let $n \in N$ and let $L \in \underline{M}_{n\,SI}^*$. Then $L^o \in V(\underline{n})_{SI}$, hence by 5.3 (ii), $L^o \cong \underline{m}$ for some m, $1 < m \leq n$. If L were not a $*$-algebra then L would contain a finite non $*$-algebra as a subalgebra, in contradiction with 2.11. Thus $L = B(L^o)$ and hence $L \cong K_m$ for some m, $1 \leq m < n$. Similarly for $\underline{M}_n^{-*}$.

(ii) By the proof of 3.5, $\underline{K}_\infty \in SP(\{K_n \mid n > 0\}$, hence $K_\infty \in \underline{M}^\star$. Since by 5.3 (i) $\underline{C} = V(\underline{\omega + 1})$ and $K_\infty^o \cong \omega + 1$, it follows that $\underline{M}^\star = V(B(\underline{\omega + 1})) \subseteq V(K_\infty) \subseteq \underline{M}^\star$, with 5.9. Similarly for $\underline{M}^{-\star}$.

(iii) Note that $\underline{M}^\star_{FSI} = \{K_n \mid n \in N\}$. Let $\underline{K} \subseteq \underline{M}^\star$ be a variety. If $\underline{K} \neq \underline{M}^\star$ then $\underline{K}^o \subset \underline{M}^o = \underline{C}$, hence, by 5.3 (i), $\underline{K}^o = V(\underline{n})$ for some n, $n \in N$. Thus $\underline{K} \subseteq \underline{M}_n$, and by 5.8 $\underline{K}$ is locally finite, implying $\underline{K} = V(\underline{K}_{FSI})$. But

$$\underline{K}_{FSI} \subseteq \underline{M}^\star_{FSI} \cap \underline{M}_{n\,SI} = \{K_m \mid 1 \leq m < n\}$$

and it follows that $\underline{K} = V(K_{m-1})$ for some $m \in N$. Similarly for any variety $\underline{K} \subseteq \underline{M}^{-\star}$. □

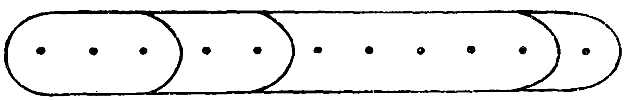
Section 6. Finitely generated free objects in $\underline{M}_n^-$ and $\underline{M}_n$, $n \in N$

Our first goal is to determine $F_{\underline{M}_n^-}(1)$ and $F_{\underline{M}_n}(1)$, $n \in N$ (6.6 and 6.8). We shall use 3.1, and since $\underline{M}^-$ and $\underline{M}$ are generated by their finite members, we therefore first have to find out what the finite subdirectly irreducibles generated by one element are in $\underline{M}$ and $\underline{M}^-$. Some of the lemmas will be formulated in a more general fashion than needed at this point; they will be useful in the characterization of $F_{\underline{M}_n^\star}(k)$ and $F_{\underline{M}_n^{-\star}}(k)$, $n \in N$, $n \geq 2$, $k \in N$, our second object in this section.

If $L = M_{n_1,\ldots n_k}$, $n_1,\ldots n_k$ being arbitrary positive numbers, (cf. I. 7.20 for notation), then the chain of open elements of L will be denoted by $\{0 = c_0 < c_1 < \dots < c_k = 1\}$.

6.1 Lemma. Let $L \in \underline{M}$ or $L \in \underline{M}^-$ be a finite subdirectly irreducible algebra generated by one element. Then $L \cong M_{n_1,\dots n_k}$ respectively $L \cong M^-_{n_1,\dots n_k}$ for some $k \in N$, with $n_1 = \dots = n_{k-1} = 1$, $n_k = 1$ or $n_k = 2$.

Proof. Let $L \in \underline{M}_{FSI}$. Since L is a finite subdirectly irreducible algebra, by 5.4 (ii) $L^o \cong \underline{k}$, for some $k > 0$ and hence $L \cong M_{n_1,\dots n_k}$, where $n_1,\dots n_k$ are positive integers (cf. I.7.20). Suppose now that L is $\underline{B}_i$-generated by one element x and that $n_i > 1$ for some i, $1 \leq i < k$.



$M_{3,2,5,1}$

(i) If $c'_{i-1}c_i \leq x$ or $c'_{i-1}c_i \leq x'$, then

$$[x]_{\underline{B}_i} \subseteq [(c_{i-1}] \cup \{c'_{i-1}c_i\} \cup (c'_i]]_{\underline{D}_{01}} \neq L ,$$

a contradiction.

(ii) If $c'_{i-1}c_i \nleq x$, $c'_{i-1}c_i \nleq x'$, then

$$(x + c_{i-1})^o = (x' + c_{i-1})^o = c_{i-1} ,$$

and it follows that $[x]_{\underline{B}_i} \subseteq [(c_{i-1}] \cup \{x\}]_{\underline{B}}$ thus $[x]^o_{\underline{B}_i} \subseteq (c_{i-1}]^o \cup \{1\}$ and since $i < k$, $c_i \in L \setminus [x]_{\underline{B}_i}$, a contradiction.

We conclude that $n_i = 1$, $i = 1,\dots k-1$. By assumption, $r_T(L) \leq 1$, hence $r_T([c_{k-1})) \leq 1$ by I. 6.10 (ii), thus for some $y \in [c_{k-1})$, $[c_{k-1}) = [\{c_{k-1},1\} \cup \{y\}]_{\underline{B}}$ hence $[c_{k-1}) \underset{\underline{B}}{\cong} \underline{2}^2$ or $[c_{k-1}) \underset{\underline{B}}{\cong} \underline{2}$. So $n_k = 2$ or $n_k = 1$. □

In the following we shall use again the notation introduced in I.2.21, i.e., if $L \in \underline{B}_i^-$, then $B^{(-)}(0 \oplus L)$ will denote the (generalized) interior algebra $\underline{B}^{(-)}$-generated by $0 \oplus L$ satisfying $B^{(-)}(0 \oplus L)^o = 0 \oplus L^o$.

6.2 <u>Lemma</u>. Let $L \in \underline{B}_i^-$, $L = [\{x_1,\dots x_n\}]_{\underline{B}_i^-}$, $n \in N$.

(i) $B^-(0 \oplus L)$ is $\underline{B}_i^-$-generated by any set

$$\{y_1,\dots y_n\} \subseteq B^-(0 \oplus L),$$

where $y_i = x_i$ or $y_i = x_i \Rightarrow 0$, $i = 1,\dots n$, and for at least one i, $1 \le i \le n$, $y_i = x_i \Rightarrow 0$.

(ii) $B(0 \oplus L)$ is $\underline{B}_i$-generated by any set

$$\{y_1,\dots y_n\} \subseteq B(0 \oplus L),$$

where $y_i = x_i$, or $y_i = x_i'$, $i = 1,\dots n$.

<u>Proof</u>. (i) Since there is an i, $1 \le i \le n$ such that $y_i = x_i \Rightarrow 0$, $y_i^o = (x_i \Rightarrow 0)^o = 0$ for some i, $1 \le i \le n$ (cf. I.2.21), hence $0 \in [\{y_1,\dots y_n\}]_{\underline{B}_i^-}$. Let $x_i^* = y_i$ if $y_i = x_i$, $x_i^* = y_i \Rightarrow 0$ if $y_i = x_i \Rightarrow 0$. Then $x_i^* \in [\{y_1,\dots y_n\}]_{\underline{B}_i^-}$ and $x_i^* = x_i$, $i = 1,\dots n$. Hence

$$\{0\} \cup L = \{0\} \cup [\{x_1\dots x_n\}]_{\underline{B}_i^-} \subseteq [\{y_1,\dots y_n\}]_{\underline{B}_i^-},$$

thus $B^-(0 \oplus L) \subseteq [y_1,\dots y_n]_{\underline{B}_i^-}$.

(ii) The assertion follows from the fact, that

$$[\{y_1,\dots y_n\}]_{\underline{B}_i} \supseteq \{0\} \cup [\{y_1,\dots y_n\}]_{\underline{B}_i^-} \supseteq \{0\} \cup L,$$

hence $B(0 \oplus L) = [\{y_1,\dots y_n\}]_{\underline{B}_i}$ for any set $\{y_1,\dots y_n\}$ as given. □

6.3 Lemma. Let $k \in N$, $n_1 = \ldots = n_{k-1} = 1$, $n_k = 1$ or 2.

(i) $M^-_{n_1,\ldots n_k}$ is $\underline{B}^-_i$-generated by one element. If x and y $\underline{B}^-_i$-generate $M^-_{n_1,\ldots n_k}$ then there exists an automorphism

$$\varphi : M^-_{n_1,\ldots,n_k} \longrightarrow M^-_{n_1,\ldots,n_k}$$

such that $\varphi(x) = y$.

(ii) $M_{n_1,\ldots n_k}$ is $\underline{B}_i$-generated by one element. If x and y $\underline{B}_i$-generate $M_{n_1,\ldots n_k}$ then there exists an automorphism

$$\varphi : M_{n_1,\ldots,n_k} \longrightarrow M_{n_1,\ldots,n_k}$$

such that $\varphi(x) = y$ or $\varphi(x') = y$.

Proof. (i) If $k = 1$ then $M_{n_1} = M_1$ or $M_{n_1} = M_2$, in which cases the statement is obvious.

Suppose now that the assertion is true if $k = m \geq 1$. Then

$$M_{n_1,\ldots n_{m+1}} \cong B(0 \oplus M_{n_2,\ldots n_{m+1}})$$

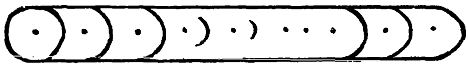
and since by assumption $M_{n_2,\ldots n_{m+1}}$ is generated by one element, it follows by 6.2 that $M_{n_1,\ldots n_{m+1}}$ is generated by one element.

In order to prove the uniqueness of the $\underline{B}^-_i$-generator, let x, y be $\underline{B}^-_i$-generators of $M_{n_1,\ldots n_k}$, $k \geq 1$. If $n_k = 1$ the statement follows from 3.4, so assume $n_k = 2$ and $x \neq y$. Then $x.c_{k-1}$ and $y.c_{k-1}$ are $\underline{B}^-_i$-generators of $(c_{k-1}] \cong M_{n_1,\ldots n_{k-1}} \cong K_{k-1}$, hence $x.c_{k-1} = y.c_{k-1}$ by 3.4. Let a, b be the atoms $\leq c'_{k-1}c_k$, then $x = x.c_{k-1} + a$, $y = y.c_{k-1} + b$ or vice versa.
Let $\varphi : M_{n_1,\ldots n_k} \to M_{n_1,\ldots n_k}$ be the automorphism defined by $\varphi \mid (c_{k-1}] = id \mid (c_{k-1}]$, $\varphi(a) = b$, $\varphi(b) = a$. Then $\varphi(x) = y$.

(ii) Let x be a $\underline{B}_i$-generator of $M_{n_1,\ldots n_k}$. Note that $x^o = 0$

or $x'^{o} = 0$. If $x^{o} = 0$ then x is also a $\underline{B}_i^-$-generator of $M_{n_1,\ldots n_k}$, and if $x'^{o} = 0$ then x' is a $\underline{B}_i^-$-generator of $M_{n_1,\ldots n_k}$. □

Thus the finite subdirectly irreducibles in $\underline{M}^-$ and $\underline{M}$, generated by one element may be pictured as follows:



The next lemma is concerned with the generation of products of (generalized) interior algebras.

6.4 Lemma. Let $L_1, L_2 \in \underline{B}_i$ or $\underline{B}_i^-$ be finite, with smallest element 0, and let $L \subseteq L_1 \times L_2$ be a $\underline{B}_i$- respectively $\underline{B}_i^-$-subalgebra such that $\pi_1[L] = L_1$, $\pi_2[L] = L_2$. If there are no onto homomorphisms $f_1 \colon L_1 \twoheadrightarrow L_3$, $f_2 \colon L_2 \twoheadrightarrow L_3$, $L_3 \in \underline{B}_i$, $L_3 \in \underline{B}_i^-$ respectively, $|L_3| \geq 2$, such that $f_1 \circ \pi_1 \mid L = f_2 \circ \pi_2 \mid L$, then $L = L_1 \times L_2$.

Proof. We restrict ourselves to the $\underline{B}_i$-case.

Let $(a,b) \in L$ be an atom of L. We want to show that it is an atom of $L_1 \times L_2$ as well. Since any atom of $L_1 \times L_2$ is contained in some atom of L, it will follow then that all atoms of $L_1 \times L_2$ belong to L, and hence that $L = L_1 \times L_2$. First note that if $a \neq 0$ then a is an atom of L_1, and similarly, if $b \neq 0$ then b is an atom of L_2. Indeed, since $\pi_1[L] = L_1$, there exist $a_1 \in L_1$,

$b_1 \in L_2$, such that a_1 is an atom of L_1, $a_1 \leq a$, and $(a_1,b_1) \in L$. Hence $0 \neq (aa_1,bb_1) = (a_1,bb_1) \in L$, and $(a_1,bb_1) \leq (a,b)$, thus $a = a_1$ and a is an atom of L_1.

Next we show that $a = 0$ or $b = 0$. Let $F_1 = \pi_1^{-1}(\{1\}) \cap L$, $F_2 = \pi_2^{-1}(\{1\}) \cap L$, $F_1 = [g_1) \subseteq L$, $F_2 = [g_2) \subseteq L$, $F = [g_1g_2) \subseteq L$. Let $f_1\colon {}^{L}/_{F_1} \to {}^{L}/_{F}$, $f_2\colon {}^{L}/_{F_2} \to {}^{L}/_{F}$ be defined in the canonical way. Then $f_1 \circ \pi_1\colon L \to {}^{L}/_{F}$, $f_2 \circ \pi_2\colon L \to {}^{L}/_{F}$, and for $x \in L$ $f_1 \circ \pi_1(x) = x.g_1g_2 = f_2 \circ \pi_2(x)$. By assumption then, ${}^{L}/_{F} \cong \underline{1}$, hence $g_1.g_2 = 0$. Since (a,b) is an atom of L, $(a,b).g_1 = 0$ or $(a,b).g_2 = 0$, hence $\pi_1((a,b)) = 0$ or $\pi_2((a,b)) = 0$, so $a = 0$ or $b = 0$. We infer that (a,b) is an atom of $L_1 \times L_2$. $\square$

Note that this proposition could be stated in a more general setting as well: a lemma of this kind holds for example in any equational class in which the algebras have a distributive lattice structure (possibly with some additional operations, of course).

6.5 <u>Corollary</u>. Let $L_i \in \underline{B}_i$ or $\underline{\bar{B}}_i$, $i = 1,\dots n$ be finite, $L \subseteq \prod_{i=1}^{n} L_i$ a $\underline{B}_i$- respectively $\underline{\bar{B}}_i$-subalgebra such that $\pi_i[L] = L_i$, $i = 1,\dots n$. If for no i, j, $1 \leq i < j \leq n$ there are onto homomorphisms $f_i\colon L_i \to L_0$, $f_j\colon L_j \to L_0$, $L_0 \in \underline{B}_i, \underline{\bar{B}}_i$ respectively, $|L_0| \geq 2$, such that $f_i \circ \pi_i \mid L = f_j \circ \pi_j \mid L$, then $L = \prod_{i=1}^{n} L_i$.

<u>Proof</u>. Let $(a_1,\dots a_n)$ be an atom of L. As in the proof of 6.4 we can show that a_i is an atom of L_i. Suppose $a_i \neq 0$, $b_j \neq 0$, $1 \leq i < j \leq n$. Consider the subalgebra $L' = (\pi_i \times \pi_j)[L] \subseteq L_i \times L_j$. Clearly (a_i,a_j) is an atom of L', and $\pi_i'[L'] = L_i$, $\pi_j'[L'] = L_j$, where π_i', π_j' are the projections from $L_i \times L_j$ to L_i and L_j respectively.

If $L \in \underline{B}_i, \underline{B}_i^-$ respectively, $|L_0| \geq 2$, $f_i: L_i \longrightarrow L_0$, $f_j: L_j \longrightarrow L_0$ are onto homomorphisms such that

$$f_i \circ \pi_i' \mid L' = f_j \circ \pi_j' \mid L' ,$$

then also

$$f_i \circ \pi_i \mid L = f_j \circ \pi_j \mid L ,$$

contradictory to our assumption. Hence we can apply 6.4 to $L' \subseteq L_i \times L_j$ and conclude that $a_i = 0$ or $b_j = 0$, a contradiction.□

Now we are ready to give the characterization of $F_{\underline{M}_n^-}(1)$, $n \geq 2$.

6.6 Theorem. $F_{\underline{M}_n^-}(1) \cong B^-(0 \oplus F_{\underline{M}_{n-1}^-}(1)) \times M_2^-$, for $n \geq 2$, where $F_{\underline{M}_1^-}(1)$ stands for the one element algebra. If x is a free generator of $F_{\underline{M}_{n-1}^-}(1)$, *then* $(x \Rightarrow 0, a)$ *is a free generator of* $F_{\underline{M}_n^-}(1)$, *where* a is an atom of M_2^-.

Proof. (i) For $n = 2$ the assertion takes the form $F_{\underline{M}_2^-}(1) \cong M_1^- \times M_2^-$, $(0,a)$ being a free generator. Indeed, obviously, $[(0,a)]_{\underline{B}_i^-} = M_1^- \times M_2^-$. If $L \in \underline{M}_{2SI}^-$ then $L^0 \cong \underline{2}^-$. If in addition L is generated by one element, then by 6.1 $L \cong M_1^-$ or $L \cong M_2^-$. The $\underline{B}_i^-$-generator of M_1^- is 0, the $\underline{B}_i^-$-generators of M_2^- are a or $a \Rightarrow 0$, a being an atom of M_2^-. The desired homomorphisms are

$$\pi_1: M_1^- \times M_2^- \longrightarrow M_1^-, \quad \text{with} \quad \pi_1((0,a)) = 0 ,$$

$$\pi_2: M_1^- \times M_2^- \longrightarrow M_2^-, \quad \text{with} \quad \pi_2((0,a)) = a , \quad \text{or}$$

$$h \circ \pi_2: M_1^- \times M_2^- \longrightarrow M_2^-, \quad \text{with} \quad h \circ \pi_2((0,a)) = a \Rightarrow 0$$

h being the automorphism of M_2^- interchanging a and $a \Rightarrow 0$. It follows from 3.1 and the fact that $\underline{M}_2^- = V(\underline{M}_{2\,FSI}^-)$ that $F_{\underline{M}_2^-}(1) \cong M_1^- \times M_2^-$ and that $(0,a)$ is a free generator.

(ii) Let $n > 2$. Firstly, we claim that

$$B^-(0 \oplus F_{\underline{M}_{n-1}^-}(1)) \times M_2^- = [(x \Rightarrow 0, a)]_{\underline{B}_i^-},$$

where x is a free generator of $F_{\underline{M}_{n-1}^-}(1)$ and a is an atom of M_2^-. We use lemma 6.4. Since x is a $\underline{B}_i^-$-generator of $F_{\underline{M}_{n-1}^-}(1)$ it follows by 6.2 that $B^-(0 \oplus F_{\underline{M}_{n-1}^-}(1))$ is $\underline{B}_i^-$-generated by $x \Rightarrow 0$. Obviously, $M_2^- = [a]_{\underline{B}_i^-}$. Hence

$$\pi_1[[(x \Rightarrow 0, a)]_{\underline{B}_i^-}] = B^-(0 \oplus F_{\underline{M}_{n-1}^-}(1))$$

and

$$\pi_2[[(x \Rightarrow 0, a)]_{\underline{B}_i^-}] = M_2^-.$$

If $L \in \underline{B}_i^-$, $|L| \geq 2$, $f\colon B^-(0 \oplus F_{\underline{M}_{n-1}^-}(1)) \longrightarrow L$ and $g\colon M_2^- \longrightarrow L$ are onto homomorphisms such that

$$f \circ \pi_1 \mid [(x \Rightarrow 0, a)]_{\underline{B}_i^-} = g \circ \pi_2 \mid [(x \Rightarrow 0, a)]_{\underline{B}_i^-},$$

then on the one hand $L \cong M_2^-$ since $H(M_2^-) = \{M_2^-, \underline{1}\}$, but on the other hand $M_2^- \notin H(B^-(0 \oplus F_{\underline{M}_{n-1}^-}(1)))$ since every non-trivial homomorphic image of $B^-(0 \oplus F_{\underline{M}_{n-1}^-}(1))$ contains an open atom. The reason for this is the fact that if $u \in B^-(0 \oplus F_{\underline{M}_{n-1}^-}(1))^o$, $u \neq 0$, then u contains the open atom x^o (cf. I. 2.21). This is a contradiction, and the claim follows by 6.4.

In order to show that $B^-(0 \oplus F_{\underline{M}_{n-1}^-}(1)) \times M_2^-$ is freely generated by $(x \Rightarrow 0, a)$ in $\underline{M}_n^-$ we apply 3.1. Since $\underline{M}_n^- = V(\underline{M}_{n\,FSI}^-)$ it suffices to prove that for each $L \in \underline{M}_{n\,FSI}^-$ such that $L = [y]_{\underline{B}_i^-}$ for some $y \in L$ there exists a homomorphism

$$f\colon B^-(0 \oplus F_{\underline{M}_{n-1}^-}(1)) \times M_2^- \longrightarrow L$$

such that

$$f((x \Rightarrow 0, a)) = y.$$

Let $L \in \underline{M}_{n\,FSI}^-$ be $\underline{B}_i^-$-generated by one element, say by y. Then $L \cong M_{n_1,\dots n_k}^-$, where $1 \leq k \leq n-1$, $n_1 = \dots = n_{k-1} = 1$ and

$n_k = 1$ or 2, according to 6.1. Furthermore the generator y is unique up to automorphisms of L in virtue of 6.3.

If $k = 1$, then $L \cong M_2^-$ or $L \cong M_1^-$. If $L \cong M_2^-$, then

$$\pi_2 : B^-(0 \oplus F_{\underline{M}_{n-1}^-}(1)) \times M_2^- \longrightarrow M_2^-$$

or π_2 followed by an automorphism of M_2^- is the desired homomorphism (cf. (i) of this proof). If $L \cong M_1^-$ then $L = [0]_{\underline{B}_i^-}$, and the homomorphism

$$B^-(0 \oplus F_{\underline{M}_{n-1}^-}(1)) \times M_2^- \longrightarrow (x^o] \quad ,$$

defined by $(w,z) \longmapsto w.x^o$ is the desired one, since then

$$(x \Rightarrow 0 \,,\, a) \longmapsto (x \Rightarrow 0).x^o = 0 \,.$$

If $k > 1$, then $L \cong B^-(0 \oplus M^-_{n_2,\ldots n_k})$. By 6.2 and 6.3 there exists a $\underline{B}_i^-$-generator y_1 of $M^-_{n_2,\ldots n_k}$ such that $y = y_1 \Rightarrow 0$. Since $M^-_{n_2,\ldots n_k} \in \underline{M}^-_{n-1}$, there exists a homomorphism

$$h : F_{\underline{M}_{n-1}^-}(1) \longrightarrow M^-_{n_2,\ldots n_k}$$

such that

$$h(x) = y_1 \,.$$

By I. 2.23, h can be extended to a homomorphism

$$\bar{h} : B^-(0 \oplus F_{\underline{M}_{n-1}^-}(1)) \longrightarrow B^-(0 \oplus M^-_{n_2,\ldots n_k}) = L \,.$$

Then $\bar{h}(x \Rightarrow 0) = y_1 \Rightarrow 0 = y$, hence

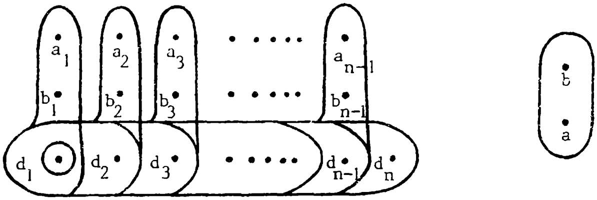
$$\bar{h} \circ \pi_2 : B^-(0 \oplus F_{\underline{M}_{n-1}^-}(1)) \times M_2^- \longrightarrow L$$

is a homomorphism such that

$$\bar{h} \circ \pi_2((x \Rightarrow 0 \,,\, a)) = y \,,$$

as required. □

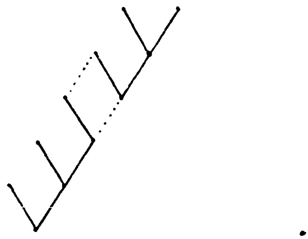
A set representation of $F_{\underline{M}_{n+1}^-}(1)$ is suggested by the diagram:



$B^-(0 \oplus F_{\underline{M}_n^-}(1))$ $\qquad\qquad$ M_2^-

Free generator: $\{b_1, d_2, a_2, b_3, d_4, a_4, b_5, \ldots\} \cup \{a\}$.

The p.o. set of join irreducibles of $F_{\underline{M}_{n+1}^-}(1)^{\circ}$ can be represented as follows:



6.7 <u>Corollary</u>. $F_{\underline{M}_{n+1}^-}(1) \underset{\underline{B}^-}{\cong} \underline{2}^{3n}$, $n \in N$.

<u>Proof</u>. If $n = 1$, $F_{\underline{M}_2^-}(1) \cong M_1^- \times M_2^- \underset{\underline{B}^-}{\cong} \underline{2}^3$. Let for $L \in \underline{B}_i$ or $L \in \underline{B}_i^-$, L finite, At L denote the set of atoms of L . Suppose that $F_{\underline{M}_m^-}(1) \underset{\underline{B}^-}{\cong} \underline{2}^{3m-3}$, $m \geq 1$. Since $F_{\underline{M}_{m+1}^-}(1) \cong B^-(0 \oplus F_{\underline{M}_m^-}(1)) \times M_2^-$,

$$|At(F_{\underline{M}_{m+1}^-}(1))| = 1 + |At(F_{\underline{M}_m^-}(1))| + 2 = 3m . \square$$

The step to the characterization of the free object on one generator in the corresponding varieties $\underline{M}_n$, $n \geq 2$ is only a small one, now.

6.8 Theorem. $F_{\underline{M}_n}(1) \cong (B(0 \oplus F_{\overline{\underline{M}}_{n-1}}(1)))^2 \times M_2$, for $n \geq 2$. If x is a free generator of $F_{\overline{\underline{M}}_{n-1}}(1)$, then (x',x,a) is a free generator of $(B(0 \oplus F_{\overline{\underline{M}}_{n-1}}(1)))^2 \times M_2$, where a is an atom of M_2.

Proof. First we show that (x',x,a) generates $(B(0 \oplus F_{\overline{\underline{M}}_{n-1}}(1)))^2 \times M_2$, using 6.4. Note that both x' and x are $\underline{B}_i$-generators of $B(0 \oplus F_{\overline{\underline{M}}_{n-1}}(1))$, by 6.2. Let $L \in \underline{B}_i$, $|L| \geq 2$, $f,g\colon B(0 \oplus F_{\overline{\underline{M}}_{n-1}}(1)) \longrightarrow L$ onto homomorphisms such that $f(x') = g(x)$. Since $B(0 \oplus F_{\overline{\underline{M}}_{n-1}}(1))$ has a unique open atom x^o, L has one open atom too, say c_1. Let $h\colon L \longrightarrow \underline{2} \cong (c_1]$ be the canonical projection. Because f and g are onto, so are $h \circ f$ and $h \circ g$, hence $0 = h \circ f(x') = h \circ g(x)$, x' being a $\overline{\underline{B}}_i$-generator of $B(0 \oplus F_{\overline{\underline{M}}_{n-1}}(1))$ by 6.2. But on the other hand, $g(x^o) = c_1$, hence $h \circ g(x) = 1$, a contradiction. By 6.4, then, $B(0 \oplus F_{\overline{\underline{M}}_{n-1}}(1))^2$ is generated by (x',x). For the proof of the fact that (x',x,a) generates $B(0 \oplus F_{\overline{\underline{M}}_{n-1}}(1))^2 \times M_2$, we refer to the corresponding part of the proof of 6.6.

In order to show that (x',x,a) freely generates $(B(0 \oplus F_{\overline{\underline{M}}_{n-1}}(1)))^2 \times M_2$, we apply 3.1 again. Let $L \in \underline{M}_{n\,FSI}$ be $\underline{B}_i$-generated by one element, say by y. By 6.1, $L \cong M_{n_1,\dots n_k}$, where $1 \leq k \leq n-1$, $n_1 = \dots = n_{k-1} = 1$, $n_k = 1$ or 2.

(i) If $k = 1$ then $L \cong M_2$ or $L \cong M_1$. If $L \cong M_2$, then $y = a$ or $y = a'$, and the desired homomorphism is

$$\pi_3\colon (B(0 \oplus F_{\overline{\underline{M}}_{n-1}}(1)))^2 \times M_2 \longrightarrow M_2$$

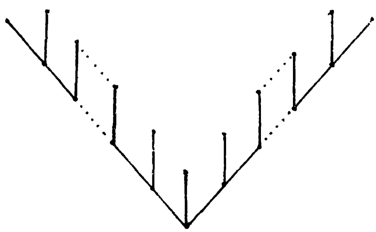
or π_3 followed by an automorphism of M_2. If $L \cong M_1$, then $y = 1$ or $y = 0$. If $y = 1$ let $h\colon B(0 \oplus F_{\underline{M}_{n-1}^-}(1)) \longrightarrow M_1$ be defined by $z \longmapsto 1$ if $z \geq x^o$, $z \longmapsto 0$ if $z \not\geq x^o$. Then h is a homomorphism, and $h \circ \pi_2 : B((0 \oplus F_{\underline{M}_{n-1}^-}(1)))^2 \times M_2 \longrightarrow M_1$ satisfies $h \circ \pi_2((x',x,a)) = 1$, as required. If $y = 0$, then $h \circ \pi_1$ is the desired homomorphism since $h \circ \pi_1((x',x,a)) = 0$.

(ii) $k > 1$. y or y' is a $\underline{B}_i^-$-generator of L. If y is a $\underline{B}_i^-$-generator, then $\bar{h} \circ \pi_1$ is the desired homomorphism, where $\bar{h}$ is as in the $k > 1$ case of (ii) of the proof of 6.6. If y' is a $\underline{B}_i^-$-generator of L, then define in a way analogous to the definition of $\bar{h}$, a homomorphism

$$g\colon B(0 \oplus F_{\underline{M}_{n-1}^-}(1)) \longrightarrow L$$

such that $g(x') = y'$. $g \circ \pi_2$ is the homomorphism we were looking for, since $g \circ \pi_2((x',x,a)) = g(x) = g(x')' = y$. □

The p.o. set of join irreducibles of $F_{\underline{M}_{n+1}}(1)^o$ may be represented as follows:



6.9 Corollary. $F_{\underline{M}_{n+1}}(1) \underset{\underline{B}}{\cong} \underline{2}^{6n-2}$, $n \in N$.

Proof. By 6.7, $B(0 \oplus F_{\underline{M}_n}^-(1)) \underset{\underline{B}}{\cong} \underline{2}^{3n-2}$. □

We announced already earlier that instead of trying to determine $F_{\underline{M}_n^-}(k)$, $F_{\underline{M}_n}(k)$, $n \geq 2$, $k \geq 2$, which seem rather complicated, we shall restrict ourselves to characterizing the simpler but still typical algebras $F_{\underline{M}_n^{-*}}(k)$, $F_{\underline{M}_n^*}(k)$, $n \geq 2$, $k \geq 2$.

6.10 <u>Theorem</u>. $F_{\underline{M}_n^{-*}}(k) \cong (B^-(0 \oplus F_{\underline{M}_{n-1}^{-*}}(k)))^{2^k-1}$, for $n,k \in N$, $n\geq 2$. Here $F_{\underline{M}_1^{-*}}(k)$ stands for the one element algebra.

<u>Proof</u>. Let $k \in N$. $\underline{M}_2^{-*}$ is the class of discrete generalized interior algebras. Thus $F_{\underline{M}_2^{-*}}(k) \cong_{\underline{B}^-} F_{\underline{B}^-}(k) \cong \underline{2}^{2^k-1}$. Next we consider the case $n > 2$. Let $\{y_1,\ldots y_k\}$ be a set of free generators of $F_{\underline{M}_{n-1}^{-*}}(k)$, $\{x_1,\ldots x_k\}$ a set of free generators of $F_{\underline{M}_n^{-*}}(k)$. Let $A \subseteq \{1,\ldots k\}$ be a non-empty set, and let

$$g_A: F_{\underline{M}_n^{-*}}(k) \longrightarrow B^-(0 \oplus F_{\underline{M}_{n-1}^{-*}}(k))$$

be a homomorphism, such that

$$g_A(x_j) = y_j \quad \text{if } j \notin A, \quad j \in \{1,\ldots k\}.$$
$$g_A(x_j) = y_j \Rightarrow 0 \quad \text{if } j \in A.$$

By 6.2, $A \neq \emptyset$ implies that $g_A[\{x_1,\ldots x_k\}]$ $\underline{B}_i^-$-generates $B^-(0 \oplus F_{\underline{M}_{n-1}^{-*}}(k))$. Let

$$g = \prod_{\substack{A\subseteq\{1,\ldots k\}\\ A\neq\emptyset}} g_A : F_{\underline{M}_n^{-*}}(k) \longrightarrow (B^-(0 \oplus F_{\underline{M}_{n-1}^{-*}}(k)))^{2^k-1}$$

(i) g is a homomorphism

(ii) g is $1-1$.

Let $x \in F_{\underline{M}_n^{-*}}(k)$, $x \neq 1$. There exists an $L \in \underline{M}_{n\,SI}^{-*}$ and an onto homomorphism $f: F_{\underline{M}_n^{-*}}(k) \longrightarrow L$ such that $f(x) \neq 1$. By 5.10 (i) $L \cong K_m^-$ for some m, $1 \leq m < n$. Let c_1 be the open atom of L. Then $L \cong B^-(0 \oplus [c_1))$ and $[c_1) \in \underline{M}_{n-1}^{-*}$. Let $h: F_{\underline{M}_{n-1}^{-*}}(k) \longrightarrow [c_1) \subseteq L$

be a homomorphism such that

$$h(y_j) = f(x_j) \quad \text{if} \quad c_1 \leq f(x_j), \quad 1 \leq j \leq k$$

and

$$h(y_j) = f(x_j) \Rightarrow 0 \quad \text{if} \quad c_1 \nleq f(x_j), \quad 1 \leq j \leq k .$$

By I. 2.23 , h can be extended to a homomorphism

$$\bar{h} : B^-(0 \oplus F^{-*}_{\underline{M}_{n-1}}(k)) \longrightarrow B^-(0 \oplus [c_1)) \cong L$$

Let $A \subseteq \{1,\ldots k\}$ be defined as follows:

$$j \notin A \quad \text{iff} \quad c_1 \leq f(x_j) , \quad 1 \leq j \leq k .$$

(*) Since f is onto, there is at least one j, $1 \leq j \leq k$, such that $c_1 \nleq f(x_j)$. Hence $A \neq \emptyset$.

We claim that $\bar{h} \circ g_A = f$. Indeed, if $c_1 \leq f(x_j)$ then $j \notin A$, thus $\bar{h} \circ g_A(x_j) = \bar{h}(g_A(x_j)) = \bar{h}(y_j) = h(y_j) = f(x_j)$.

If $c_1 \nleq f(x_j)$, then $j \in A$, so $\bar{h} \circ g_A(x_j) = \bar{h}(g_A(x_j)) = \bar{h}(y_j \Rightarrow 0) = \bar{h}(y_j) \Rightarrow 0 = h(y_j) \Rightarrow 0 = (f(x_j) \Rightarrow 0) \Rightarrow 0 = f(x_j)$.

Thus $\bar{h} \circ g_A(x_j) = f(x_j)$, $j = 1,\ldots k$, hence $\bar{h} \circ g_A = f$. Since $f(x) \neq 1$, it follows that $g_A(x) \neq 1$, whence $g(x) \neq 1$. So g is 1-1.

$$\begin{tikzcd}
F^{-*}_{\underline{M}_n}(k) \arrow[r, "g"] \arrow[dr, "g_A"] \arrow[ddr, "f"'] & (B^-(0 \oplus F^{-*}_{\underline{M}_{n-1}}(k)))^{2^{k-1}} \arrow[d, "\pi_A"] \\
& B^-(0 \oplus F^{-*}_{\underline{M}_{n-1}}(k)) \arrow[d, "\bar{h}"] \\
& L
\end{tikzcd}$$

(iii) g is onto.

We apply lemma 6.4 again. Let

$$L_1 = [\{g(x_i) \mid i = 1,\ldots k\}]_{\underline{B}_i^-} \subseteq (B^-(0 \oplus F^{-*}_{\underline{M}_{n-1}}(k)))^{2^k-1} .$$

Since $g_A[\{x_1,\ldots x_k\}]$ $\underline{B}_i^-$-generates $B^-(0 \oplus F_{\underline{M}_{n-1}^{-*}}(k))$, it follows that $\pi_A[L_1] = B^-(0 \oplus F_{\underline{M}_{n-1}^{-*}}(k))$. Let $A,B \subseteq \{1,\ldots k\}$, $A,B \neq \emptyset$, $A \neq B$, $L \in \underline{B}_i^-$, $|L| \geq 2$ and $f_1, f_2 : B^-(0 \oplus F_{\underline{M}_{n-1}^{-*}}(k)) \longrightarrow L$ be onto homomorphisms such that $f_1 \circ \pi_A \mid L_1 = f_2 \circ \pi_B \mid L_1$. As in (i) of the proof of 6.8 we may assume that $L \cong \underline{2}$. Suppose that $j \in A \setminus B$. Then $g_A(x_j) = y_j \Rightarrow 0$, $g_B(x_j) = y_j$. Hence $f_1 \circ \pi_A(g(x_j)) = f_1(y_j \Rightarrow 0) = 0$, and $f_2 \circ \pi_B(g(x_j)) = f_2(y_j) = 1$, a contradiction. By 6.4, then, $L_1 = (B^-(0 \oplus F_{\underline{M}_{n-1}^{-*}}(k)))^{2^k-1}$. □

Note that the free object on one generator in $\underline{M}_n^{-*}$ has a particularly simple structure: $F_{\underline{M}_n^{-*}}(1) \cong K_{n-1}^-$.

6.11 Corollary. $F_{\underline{M}_{n+1}^{-*}}(k) \cong_{\underline{B}^-} \underline{2}^{\alpha}$, where $\alpha = (2^k - 1) \cdot \dfrac{(2^k-1)^n - 1}{2^k - 2}$, $n,k \in N$, $k > 1$.

Proof. If $n = 1$, $k > 1$, then $F_{\underline{M}_2^{-*}}(k) \cong \underline{2}^{2^k-1}$ hence the number of atoms is $2^k - 1$.

If the statement is correct for some $n \geq 1$, $k > 1$, then using 6.10 we see that the number of atoms of $F_{\underline{M}_{n+1}^{-*}}(k)$ is

$$(2^k - 1) \cdot \left[(2^k - 1) \cdot \frac{(2^k-1)^{n-1} - 1}{2^k - 2} + 1\right] = (2^k - 1) \cdot \frac{(2^k - 1)^n - 1}{2^k - 2}. \square$$

The free object in $\underline{M}_n^{*}$ on finitely many generators, $n \geq 2$, is only slightly more complicated.

6.12 Theorem. $F_{\underline{M}_n^{*}}(k) \cong B(0 \oplus F_{\underline{M}_{n-1}^{-*}}(k))^{2^k}$, $n, k \in N$, $n \geq 2$.

Again, $F_{\underline{M}_1^{-*}}(k)$ denotes the one element algebra.

Proof. The proof is almost identical to the one just given in 6.10. We omit however the condition that $A \subseteq \{1,\ldots k\}$ be non-empty, and thereby obtain 2^k factors. By 6.2, also $g_\emptyset[\{x_1,\dots x_k\}]$ $\underline{B}_i^-$-generates $B(0 \oplus F_{\underline{M}_{n-1}^-*}(k))$. Furthermore, the remark (*) in the proof of 6.10 becomes irrelevant now, and should be dropped in this case. □

Thus $F_{\underline{M}_n^*}(1) \cong K_{n-1}^2$, $n \geq 2$.

6.13 Corollary. $F_{\underline{M}_{n+1}^*}(k) \underset{\underline{B}}{\cong} \underline{2}^\alpha$ where $\alpha = 2^k \cdot \dfrac{(2^k-1)^n - 1}{2^k - 2}$, $n,k \in N$, $k > 1$.

We wish to emphasize the essential role played by the generalized interior algebras in the discovery of the free interior algebras in $\underline{M}_n^*$, $n \geq 2$. A similar idea has been exploited in P. Köhler [73], where the finitely generated free objects in the classes $\underline{C}^-$ and $\underline{C}$ and their subvarieties are characterized; our proof of 6.10 has been inspired by his work. We mention some of his results:

$$F_{V(\underline{n+1}^-)}(k) \cong \prod_{i=0}^{k-1} (0 \oplus F_{V(\underline{n}^-)}(i))^{\binom{k}{i}} \qquad n,k \in N$$

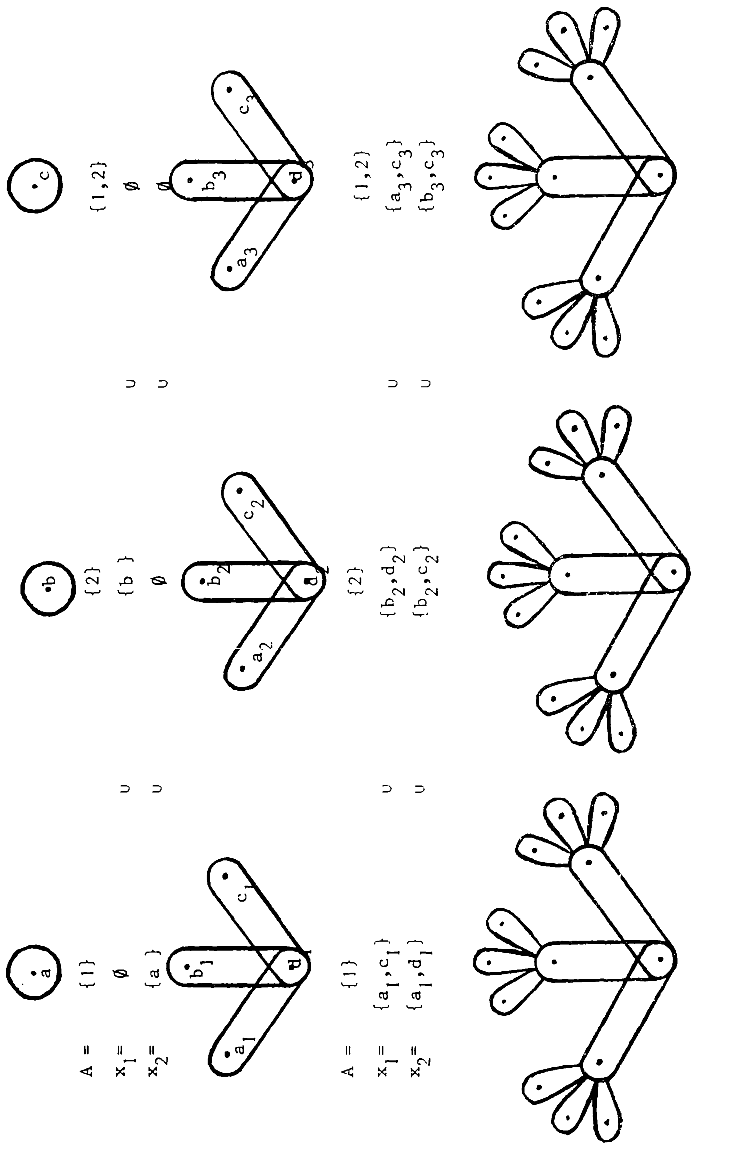
and

$$F_{V(\underline{n+1})}(k) \cong \prod_{i=0}^{k} (0 \oplus F_{V(\underline{n})}(i))^{\binom{k}{i}} \qquad n,k \in N.$$

$F_{V(\underline{1}^-)}(k)$ and $F_{V(\underline{1})}(k)$ are used to denote the one element algebra. See also Horn [69 a].

In 6.10 and 6.12 we have not given the free generators explicitly. However, one can find them easily just following the construction of the proof. We illustrate this with a simple example.

$F_{\underline{M}_n}\text{-}*(2)$ is suggested below for $n = 2,3$ and 4. x_1 and x_2 are the free generators.



Section 7. Free objects in $\underline{M}^-$ and $\underline{M}$

The free objects on k generators, $k \in N$, in $\underline{M}^-$ and $\underline{M}$ can be found in a rather straightforward manner from those in $\underline{M}_n^-$ and $\underline{M}_n$, respectively, $n \geq 2$. We shall discuss the principle behind this in more general terms, so as to be able to employ the results in the sequel as well.

7.1 Let $\underline{K}$ denote a non-trivial variety of (generalized) interior algebras, and let $\underline{K}_1 \subset \underline{K}_2 \subset \ldots \subset \underline{K}$ be a chain of non-trivial subvarieties of $\underline{K}$, such that $V(\bigcup_{n\in N} \underline{K}_n) = \underline{K}$, and such that $\underline{K}_n$ is locally finite, $n \in N$. Note that $\underline{K}$ is then necessarily generated by its finite members. Let $k \in N$. We wish to describe $F_{\underline{K}}(k)$ in terms of the $F_{\underline{K}_n}(k)$, $n \in N$.

Let $x_1, \ldots x_k$ be free generators of $F_{\underline{K}}(k)$, $y_1^n, \ldots y_k^n$ free generators of $F_{\underline{K}_n}(k)$, $n \in N$. By I.4.2, there exists a $u_n \in F_{\underline{K}}(k)^O$, such that $(u_n] \cong F_{\underline{K}_n}(k)$, while $x_i.u_n$ corresponds with y_i^n, for $i = 1, \ldots k$. In fact, there exists a chain of open elements in $F_{\underline{K}}(k)$, $u_1 < u_2 < \ldots < u_n < \ldots$, such that for each $n \in N$ there exists an isomorphism $\varphi_n : (u_n] \to F_{\underline{K}_n}(k)$ with the property that $\varphi_n \circ g_n = f_n$, f_n being the homomorphism $F_{\underline{K}}(k) \to F_{\underline{K}_n}(k)$ satisfying $f_n(x_i) = y_i^n$, $i = 1, \ldots k$, and g_n being the projection $F_{\underline{K}}(k) \to (u_n]$, defined by $x \mapsto x.u_n$. Let $\pi_{nm}: (u_n] \to (u_m]$ for $n \geq m \geq 1$ be the homomorphism defined by $x \mapsto x.u_m$. Then $\pi_{m\ell} \circ \pi_{nm} = \pi_{n\ell}$, for $n \geq m \geq \ell \geq 1$. Thus we have an inverse system $\mathcal{U}_k^{\underline{K}} = \{(u_n], \pi_{nm} \mid n \geq m \geq 1\}$, and the inverse limit $U_k^{\underline{K}} = \varprojlim \mathcal{U}_k^{\underline{K}}$ exists, since the $(u_n]$, $n \geq 1$, are finite (cf. Grätzer [68], pg 131). Recall that

$$U\frac{K}{k} = \{(x_n)_n \in \prod_{n=1}^{\infty} (u_n] \mid \pi_{\ell m}(x_\ell) = x_m,\ \ell \geq m \geq 1\}.$$

Let $\pi_n: U\frac{K}{k} \longrightarrow (u_n]$ be the canonical projection. Let $z_i = (u_1 x_i, u_2 x_i, \ldots)$, $i = 1,\ldots k$. Then $z_i \in U\frac{K}{k}$, and we claim that $F_{\underline{K}}(k) \cong [\{z_1,\ldots z_k\}] \subseteq U\frac{K}{k}$, and hence that $F_{\underline{K}}(k) \in S(U\frac{K}{k})$. In virtue of lemma 3.1 and the fact that $\underline{K} = V(\bigcup_{n=1}^{\infty} \underline{K}_n)$, we only need to show, that every map $z_i \longmapsto a_i$, $i = 1,\ldots k$, to an algebra $L = [\{a_1,\ldots a_k\}]$ belonging to a $\underline{K}_n$ can be extended to a homomorphism from $[\{z_1,\ldots z_n\}]$ to that algebra. But if $L \in \underline{K}_n$, then there exists a homomorphism $h: F_{\underline{K}_n}(k) \longrightarrow L$ such that $h(y_i^n) = a_i$, $i = 1,\ldots k$ and

$$h \circ \varphi_n \circ \pi_n \mid [\{z_1,\ldots z_k\}] : [\{z_1,\ldots z_k\}] \longrightarrow L$$

is the desired homomorphism extending the map $z_i \mapsto a_i$, $i = 1,\ldots k$.

We have proved

7.2 <u>Theorem</u>. $F_{\underline{K}}(k)$ is isomorphic with a subalgebra of an inverse limit $U\frac{K}{k}$ of a chain of order type ω^* of finite algebras, for $k \in N$.

7.3 Next we wish to study the set representation of this inverse limit, thus gaining more insight in the structure of $F_{\underline{K}}(k)$. We discuss the case $\underline{K} \subseteq \underline{B}_i$ only; the results can be transferred to varieties $\underline{K} \subseteq \underline{B}_i^-$ without difficulty. If L is an interior algebra, $a \in L$, then $[\![a]\!]$ will denote the set of atoms $\leq a$. Note that in $F_{\underline{K}}(k)$, $[\![u_m]\!] \subseteq [\![u_n]\!]$, if $1 \leq m \leq n$. Let $X = \bigcup_{n=1}^{\infty} [\![u_n]\!]$, and let $[\![U\frac{K}{k}]\!]$ be the complete Boolean algebra of all subsets of X, as usual with operations $\cup$, $\cap$, $'$, $\emptyset$, X, provided with an interior operator which is determined by:

$A \subseteq X$ is open iff $\forall n \geq 1$ $A \cap [\![u_n]\!] = [\![v]\!]$ for some $v \in (u_n]^{\circ}$. This definition makes sense since the set $[\![U\frac{K}{k}]\!]^{\circ}$ of open elements of $[\![U\frac{K}{k}]\!]$ is clearly a $\underline{D}_{01}$-sublattice of $[\![U\frac{K}{k}]\!]$, and in addition $[\![U\frac{K}{k}]\!]^{\circ}$ is closed under arbitrary unions: if $\{A_i \mid i \in I\} \subseteq [\![U\frac{K}{k}]\!]^{\circ}$, then $\bigcup_{i\in I} A_i \cap [\![u_n]\!] = \bigcup_{i\in I}(A_i \cap [\![u_n]\!]) = \bigcup_{i\in I}[\![v_{i,n}]\!]$, for certain $v_{i,n} \in (u_n]^{\circ}$, $i \in I$, and since $(u_n]$ is finite, it follows that the last union is finite, hence $\bigcup_{i\in I} A_i \cap [\![u_n]\!] = [\![\sum_{i\in I} v_{i,n}]\!]$, where $\sum_{i\in I} v_{i,n} \in (u_n]^{\circ}$, $n \geq 1$. By I.2.4 $[\![U\frac{K}{k}]\!]$ provided with this operator is an interior algebra; in fact, if $A \subseteq X$, then $A^{\circ} = \bigcup_{n=1}^{\infty}(A \cap [\![u_n]\!])^{\circ}$. Observe that $[\![U\frac{K}{k}]\!]^{\circ}$ is also closed under arbitrary intersections: if $\{A_i \mid i \in I\} \subseteq [\![U\frac{K}{k}]\!]^{\circ}$ then $\bigcap_{i\in I} A_i \cap [\![u_n]\!] = \bigcap_{i\in I}(A_i \cap [\![u_n]\!]) = \bigcap_{i\in I}[\![v_{i,n}]\!]$, where $v_{i,n} \in (u_n]^{\circ}$, $i \in I$. The last intersection is finite however, hence $\bigcap_{i\in I} A_i \cap [\![u_n]\!] = [\![\prod_{i\in I} v_{i,n}]\!]$ and $\prod_{i\in I} v_{i,n} \in (u_n]^{\circ}$. Thus $\bigcap_{i\in I} A_i \in [\![U\frac{K}{k}]\!]^{\circ}$.

It is easy to verify that the map $(u_n] \to ([\![u_n]\!]]$ defined by $a \mapsto [\![a]\!]$ establishes an isomorphism between $(u_n]$ and $([\![u_n]\!]]$. We assert that $\varprojlim U\frac{K}{k} = U\frac{K}{k} \cong [\![U\frac{K}{k}]\!]$. Indeed define $\varphi\colon U\frac{K}{k} \to [\![U\frac{K}{k}]\!]$ by $\varphi(a) = \bigcup_{n=1}^{\infty}[\![\pi_n(a)]\!]$. Note that $\varphi(a) \cap [\![u_n]\!] = [\![\pi_n(a)]\!]$. We verify that φ is an isomorphism:

(i) φ is 1 - 1 : if $a \neq b$ then $\pi_n(a) \neq \pi_n(b)$ for some $n \geq 1$, hence $\varphi(a) \cap [\![u_n]\!] = [\![\pi_n(a)]\!] \neq [\![\pi_n(b)]\!] = \varphi(b) \cap [\![u_n]\!]$, thus $\varphi(a) \neq \varphi(b)$.

(ii) φ is onto. Let $A \subseteq X$. If $a = (a_1, a_2, \ldots)$ such that $a_n \in (u_n]$ and $[\![a_n]\!] = A \cap [\![u_n]\!]$, $n = 1,2,\ldots$ then $a \in U\frac{K}{k}$, and $\varphi(a) = A$.

(iii) φ is a $\underline{D}_{01}$-homomorphism. For example, if $a,b \in U\frac{K}{k}$, $a = (a_1,a_2,\dots)$, $b = (b_1,b_2,\dots)$ then

$$\varphi(a.b) = \bigcup_{n=1}^{\infty} [\![\pi_n(a.b)]\!] = \bigcup_{n=1}^{\infty} [\![\pi_n(a).\pi_n(b)]\!] =$$

$$= \bigcup_{n=1}^{\infty} [\![\pi_n(a)]\!] \cap [\![\pi_n(b)]\!] = \bigcup_{n=1}^{\infty} [\![\pi_n(a)]\!] \cap \bigcup_{n=1}^{\infty} [\![\pi_n(b)]\!] =$$

$$= \varphi(a) \cap \varphi(b) .$$

In a similar way it can be shown that φ preserves $+$, 0, 1.

(iv) φ preserves the interior operator:

$$\varphi(a^o) = \bigcup_{n=1}^{\infty} [\![\pi_n(a^o)]\!] = \bigcup_{n=1}^{\infty} [\![\pi_n(a)^o]\!] = \bigcup_{n=1}^{\infty} (\varphi(a) \cap [\![u_n]\!])^o =$$

$= \varphi(a)^o$, by the definition of ^{o} in $[\![U\frac{K}{k}]\!]$.

We conclude

7.4 <u>Theorem</u>. The (complete, atomic) interior algebra $[\![U\frac{K}{k}]\!]$ is isomorphic with $U\frac{K}{k} = \varprojlim\{(u_n], \pi_{nm} \mid n \geq m \geq 1\}$, for any $k \in N$.

According to our previous remarks, it follows that $F_{\underline{K}}(k)$ is isomorphic with the subalgebra of $[\![U\frac{K}{k}]\!]$ generated by the elements $Z_i = \varphi(z_i) = \bigcup_{n=1}^{\infty} [\![x_i u_n]\!]$, $i = 1,\dots k$. The next theorem tells us, that this subalgebra $[\{Z_1,\dots Z_k\}]$ contains at least all "finite" elements of $[\![U\frac{K}{k}]\!]$.

7.5 <u>Theorem</u>. Let $A \in [\![U\frac{K}{k}]\!]$, $k \in N$, be such that $A \subseteq [\![u_m]\!]$ for some $m \in N$. Then $A \in [\{Z_1,\dots Z_k\}]$.

<u>Proof</u>. Let $a \in F_{\underline{K}}(k)$ be such that $A = [\![a]\!]$, where $a \leq u_m$. Let p be a $\underline{B}_i$-polynomial such that $a = p(x_1,\dots x_k)$. In $(u_n]$,

$$p_{(u_n]}(x_1u_n,\dots x_ku_n) = p_{F_{\underline{K}}(k)}(x_1,\dots x_k).u_n = a.u_n .$$

Hence

$$\pi_n(p_{U_k^{\underline{K}}}(z_1,\dots z_k)) = p_{(u_n]}(\pi_n z_1,\dots \pi_n z_k) =$$
$$= p_{(u_n]}(x_1 u_n,\dots x_k u_n) = a u_n .$$

Thus

$$p_{U_k^{\underline{K}}}(z_1,\dots z_k) = (au_1, au_2,\dots) \in U_k^{\underline{K}} .$$

But since $a \le u_m$, we have

$$p_{U_k^{\underline{K}}}(z_1,\dots z_k) = (au_1, au_2,\dots au_{m-1}, a, a,\dots) .$$

Hence

$$p_{[\![U_k^{\underline{K}}]\!]}(Z_1,\dots Z_k) = \varphi(p_{U_k^{\underline{K}}}(z_1,\dots z_k)) = \varphi((au_1,au_2,\dots au_{m-1},a,a,\dots)) =$$
$$= \bigcup_{n=1}^{\infty} [\![\pi_n((au_1,au_2,\dots au_{m-1},a,a,\dots))]\!] = \bigcup_{n=1}^{\infty} [\![au_n]\!] = [\![a]\!] = A . \square$$

7.6 Corollary. $F_{\underline{K}}(k)$ is atomic, for all $k \in N$.

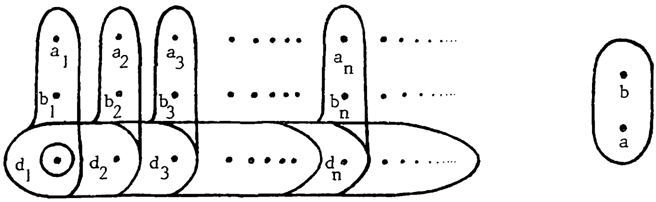
Proof. We show that $[\{Z_1,\dots Z_k\}] \subseteq [\![U_k^{\underline{K}}]\!]$ is atomic. Since $X = \bigcup_{n=1}^{\infty} [\![u_n]\!]$, for every $a \in X$ $\{a\} \subseteq [\![u_m]\!]$ for some $m \in N$. Hence $\{a\} \in [\{Z_1,\dots Z_k\}]$ by 7.5. Thus $[\{Z_1,\dots Z_k\}]$ contains all atoms of $[\![U_k^{\underline{K}}]\!]$; since $[\![U_k^{\underline{K}}]\!]$ is atomic, so is $[\{Z_1,\dots Z_k\}]$. $\square$

7.7 Corollary. $F_{\underline{K}}(k)^{\circ}$ is strongly atomic, i.e., for all $u,w \in F_{\underline{K}}(k)^{\circ}$, if $u < w$ then there is a $v \in F_{\underline{K}}(k)^{\circ}$ such that $u \prec_{F_{\underline{K}}(k)^{\circ}} v \le w$, for every $k \in N$.

Proof. Let $u,w \in F_{\underline{K}}(k)^{\circ}$, $u < w$. Then $[\![u]\!],[\![w]\!] \in [\![U_k^{\underline{K}}]\!]^{\circ}$, and $[\![u]\!] \subset [\![w]\!]$. There exists an $n \in N$ such that $[\![u]\!] \cap [\![u_n]\!] \subset [\![w]\!] \cap [\![u_n]\!]$. Since $(u_n]$ is a finite interior algebra, there exists a $v \in (u_n]^{\circ}$, such that $uu_n \prec_{(u_n]^{\circ}} v \le wu_n \le w$. Then $u \prec_{F_{\underline{K}}(k)^{\circ}} u + v \le w$, and $u + v \in F_{\underline{K}}(k)^{\circ}$. $\square$

In virtue of 5.4 and 5.8 we are allowed to apply the just developed theory in order to obtain a representation of the algebras $F_{\underline{M}}(1)$ and $F_{\underline{M}^-}(1)$ and also of $F_{\underline{M}^*}(k)$, $F_{\underline{M}^{-*}}(k)$, $k \in N$.
The pictures furnish an adequate portrait of these algebras.

$F_{\underline{M}^-}(1) = F \times M_2$



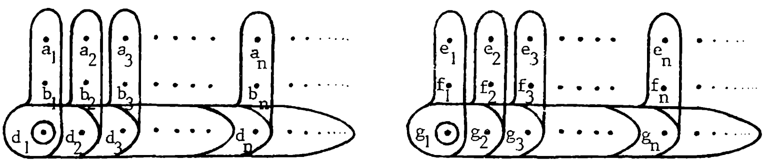
$X = \{a_i, b_i, d_i \mid i \in N\} \cup \{a,b\}$
A base for the open sets of $U_1^{\underline{M}^-}$ consists of the sets:

$\{a,b\}$, $\{d_i \mid i \leq n\}$, $n \in N$, and $\{d_i \mid i \leq n\} \cup \{a_n, b_n\}$, $n \in N$.

Free generator:

$Z = \{a\} \cup \{b_{2i-1} \mid i \in N\} \cup \{d_{2i}, a_{2i} \mid i \in N\}.$

$F_{\underline{M}}(1) = F^2 \times M_2$

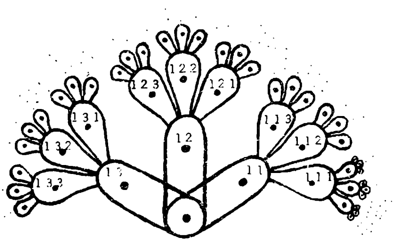


Free generator:

$$Z = \{a\} \cup \{b_{2i-1} \mid i \in N\} \cup \{d_{2i}, a_{2i} \mid i \in N\} \cup$$
$$\cup \{e_{2i-1}, g_{2i-1} \mid i \in N\} \cup \{f_{2i} \mid i \in N\} .$$

In a similar way one obtains the representations of the $F_{\underline{M}}{-}\star(k)$, $F_{\underline{M}}\star(k)$, $k \in N$.

As an illustration, we consider $F_{\underline{M}}{-}\star(2)$:



The atoms of $F_{\underline{M}}{-}\star(2)$ are denoted by the finite sequences consisting of the numbers 1,2, and 3. Thus

$$X = \{a_1 \cdot \ldots a_k \mid k \in N, \quad a_i \in \{1,2,3\}, \quad i = 1,\ldots k\} .$$

A base for the open sets of $U_{\overline{2}}^{\underline{M}^{-\star}}$ consists of the sets:

$$\{a_1 \cdot \ldots \cdot a_\ell \mid 1 \leq \ell \leq k\}, \qquad a_1 \cdot \ldots a_k \in X .$$

In general, the poset of join irreducibles in $F_{\underline{M}}{-}\star(k)^{o}$ may be represented as 2^k-1 copies of a tree with 2^k-1 branches in every node; likewise the poset of join irreducibles of $F_{\underline{M}}\star(k)^{o}$ as 2^k copies of a tree with 2^k-1 branches in every node.

CHAPTER III

THE LATTICE OF SUBVARIETIES OF $\underline{B}_i$

Having studied (generalized) interior algebras in certain subvarieties of $\underline{B}_i^-$ and $\underline{B}_i$ in the chapters I and II we shall turn our attention now to the varieties themselves. We shall deal with several problems. For example, we shall try to obtain information on the lattice Ω of all subvarieties of $\underline{B}_i$ (sections 1 and 8), or, more specifically, on certain principal ideals of this lattice, like the lattice of subvarieties of $\underline{M}$ (sections 5 and 6). We shall investigate some sublattices of the lattice of all subvarieties of $\underline{B}_i$ consisting of varieties having pleasant properties, such as local finiteness (section 4). In these considerations, Jónsson's work (0.1.25 - 0.1.28) will play a central role. Further, in section 2 the problem of finding equations defining a variety which is given in terms of some generating set of algebras will be dealt with. The results obtained there can be used succesfully in the study of the important class of so-called <u>splitting varieties</u> (section 3).

Most of the theory we develop for subvarieties of $\underline{B}_i$ could be carried over to subvarieties of $\underline{B}_i^-$ without difficulty; we shall do so explicitly only if the case seems to be of special interest.

Section 1. General results

As indicated in chapter 0, disregarding set theoretical difficulties, we may consider the class of subvarieties of $\underline{B}_i$ as a lattice. This lattice shall be denoted by Ω. We shall derive now some general results concerning subvarieties of $\underline{B}_i$ and the lattice Ω.

In I.2.8 we mentioned that $\underline{B}_i$ is congruence-distributive. Therefore Jónsson's results 0.1.25 - 0.1.28 can be applied:

1.1 Theorem. For $\underline{K} \subseteq \underline{B}_i$, $V(\underline{K})_{SI} \subseteq HSP_U(\underline{K})$.

1.2 Corollary. If $L_j \in \underline{B}_i$ is finite for $j = 1,\ldots n$, then $V(\{L_1,\ldots L_n\})_{SI} \subseteq HS(\{L_1,\ldots L_n\})$.

1.3 Corollary. If $L_1, L_2 \in \underline{B}_{iFSI}$ then $V(L_1) = V(L_2)$ iff $L_1 \cong L_2$.

1.4 Corollary. If $\underline{K}_0$ and $\underline{K}_1$ are varieties such that $\underline{K}_0, \underline{K}_1 \subseteq \underline{B}_i$, then $(\underline{K}_0 + \underline{K}_1)_{SI} = \underline{K}_{0SI} \cup \underline{K}_{1SI}$.

1.5 Corollary. The lattice Ω of subvarieties of $\underline{B}_i$ is distributive.

In order to exploit 1.1 to the fullest extent, we prove the following lemma, which will serve as an analogue of lemma 5.1 of Jónsson [67].

1.6 Lemma Every interior algebra is a homomorphic image of some subdirectly irreducible interior algebra. In symbols: $\underline{B}_i = H(\underline{B}_{iSI})$.

Proof. Let $L \in \underline{B}_i$, with unit element 1_L and interior operator ^{o}. Then $L \oplus 1 \in \underline{D}_{01}$, where $1 > 1_L$. Let $L_1 = B(L \oplus 1)$ with complementation $'$. We define an interior operator ^{o_1} on L_1 such that $L_1^{o_1} = L^o \oplus 1$, as follows:

$$\text{if } a,b \in L \oplus 1, \text{ then } (a + b')^{o_1} = \begin{cases} (a + b'.\, 1_L)^o & \text{if } b \nleq a \\ 1 & \text{if } b \leq a \end{cases}$$

and if $x = \prod_{i=1}^{n} (a_i + b_i')$, where $a_i, b_i \in L \oplus 1$, $i = 1,\ldots n$, then

$$x^{o_1} = \prod_{i=1}^{n} (a_i + b_i')^{o_1}.$$

It is easy to verify that ^{o_1} is an interior operator on L_1 and that $L_1^{o_1} = L^o \oplus 1$. Thus L_1 is a subdirectly irreducible interior algebra. Let $h: L_1 \to (1_L]$ be defined by $h(x) = x.1_L$. Since $1_L \in L_1^{o_1}$, h is a $\underline{B}_i$-homomorphism. From the definition of ^{o_1} it follows that $(1_L] \cong L$, hence $L \in H(L_1)$.□

Note that it follows from the proof of this lemma that likewise

1.7 Lemma. $\underline{B}_{iF} \subseteq H(\underline{B}_{iFSI})$ and $\underline{B}^*_{iF} \subseteq H(\underline{B}^*_{iFSI})$.

1.8 Corollary. $\underline{B}_i = HSP_U(\underline{B}_{iF}) = HSP_U(\underline{B}_{iFSI})$.

Proof. Since $\underline{B}_i = V(\underline{B}_{iF})$, $\underline{B}_i = H(\underline{B}_{iSI}) = HSP_U(\underline{B}_{iF}) = HSP_U(\underline{B}_{iFSI})$.□

1.9 Corollary. If $\underline{K} \subset \underline{B}_i$ is a variety, then there exists a variety $\underline{K}'$, such that $\underline{K} \prec \underline{K}' \subseteq \underline{B}_i$.

Proof. Let $\underline{K} \subset \underline{B}_i$ be a variety. Since $\underline{B}_i = V(\underline{B}_{iF})$, there exists a finite interior algebra L such that $L \notin \underline{K}$. By 1.2 and 1.4, $(\underline{K} + V(L))_{SI} \subseteq \underline{K}_{SI} \cup HS(L)$, and since every variety is determined by its subdirectly irreducibles, it follows that there are at most finitely many varieties $\underline{K}'$ such that $\underline{K} \subseteq \underline{K}' \subseteq \underline{K} + V(L)$. At least one of these varieties covers $\underline{K}$.□

1.10. Corollary. If $\underline{K}_0$, $\underline{K}_1$ are varieties of interior algebras and $\underline{K}_0 + \underline{K}_1 = \underline{B}_i$, then $\underline{K}_0 = \underline{B}_i$ or $\underline{K}_1 = \underline{B}_i$.

Proof. Suppose that $\underline{K}_0$, $\underline{K}_1$ are varieties such that $\underline{K}_0 \subset \underline{B}_i$, $\underline{K}_1 \subset \underline{B}_i$. Let $L_0 \in \underline{B}_i \backslash \underline{K}_0$, $L_1 \in \underline{B}_i \backslash \underline{K}_1$, and $L \in \underline{B}_{iSI}$ such that $L_0 \times L_1 \in H(L)$. Then $L_0 \in H(L)$, $L_1 \in H(L)$, and thus $L \in \underline{B}_i \backslash \underline{K}_0$ and $L \in \underline{B}_i \backslash \underline{K}_1$. By 1.4, $(\underline{K}_0 + \underline{K}_1)_{SI} = \underline{K}_{0SI} \cup \underline{K}_{1SI}$, therefore $L \notin (\underline{K}_0 + \underline{K}_1)_{SI}$ and hence $\underline{K}_0 + \underline{K}_1 \neq \underline{B}_i$. □

1.11 Corollary. There is no variety $\underline{K} \subseteq \underline{B}_i$, such that $\underline{B}_i$ covers $\underline{K}$.

Proof. If $\underline{K} \subset \underline{B}_i$, $\underline{K}$ a variety, let $L \in \underline{B}_{iF}$ be such that $L \notin \underline{K}$. Then $V(L) \neq \underline{B}_i$, hence by 1.10 $\underline{K} \subset \underline{K} + V(L) \subset \underline{B}_i$.□

The results obtained so far indicate already clearly the strength of 1.1. For the future use of 1.1, let us recall that if $\underline{K}$ is a class of algebras satisfying a first order sentence σ in the language of the algebras, then any $L \in P_U(\underline{K})$ satisfies σ. The first order language L_{B_i}, suitable to speak about algebras in $\underline{B}_i$, contains the following symbols:

(i) variables $\underline{x}$, $\underline{x}_0$, $\underline{x}_1$, ...

(ii) operation symbols $\underline{+}$, $\underline{\cdot}$, $\underline{'}$, $\underline{^C}$, $\underline{0}$, $\underline{1}$

(iii) relation symbols $\underline{=}$, $\underline{\leq}$

(iv) logical connectives $\vee$, $\wedge$, $\sim$, $\Rightarrow$, $\exists$, $\forall$.

As atomic formulas be admitted not only terms connected by $\underline{=}$ or $\underline{\leq}$, but also terms themselves where the term $\underline{p}$ $(\underline{x}_0,\ldots\underline{x}_n)$ is an equivalent of the atomic formula $\underline{p}(\underline{x}_0,\ldots\underline{x}_n) \mathrel{\underline{=}} \underline{1}$. Formulas and sentences will be formed as usual (with their obvious interpretation in interior algebras). Hence the terms are nothing but our $\underline{B}_i$-polynomial symbols. If no confusion is to be expected, we shall write $x_0, x_1, x,y,z,\ldots, + , \cdot, = ,\ldots$ instead of $\underline{x}_0, \underline{x}_1, \underline{x}_2,\ldots, \underline{+} , \underline{\cdot} , \underline{=}$,..., etc. More details regarding these matters can be found in Grätzer [68].

For (generalized) interior algebras L we have that L is subdirectly irreducible iff $L \models \sigma$, where σ is the first order sentence

$$\exists u\ [\ u = u^o \wedge \sim u = 1 \wedge \forall v\ [\ v = v^o \Rightarrow [\ v \leq u \vee v = 1\ ]]]\ .$$

Therefore, if $\underline{K} \subseteq \underline{B}_{iSI}$, then $P_U(\underline{K}) \subseteq \underline{B}_{iSI}$.

Section 2. Equations defining subvarieties of $\underline{B}_i$

If a variety is given in terms of a generating set of algebras, it may be a difficult problem to determine a set of equations which characterizes the variety, i.e. to find a base for the equational theory of the class. Baker [M] has considered this problem for varieties of Heyting algebras generated by a class of algebras which is defined by some set of positive universal sentences in the first order language of Heyting algebras, and also for more general classes. (A sentence is called positive universal, if, when written in prenex form, it contains only universal quantors and the symbols $\vee$ and $\wedge$). Parts of this section are merely an adaptation of Baker's results to our situation.

Firstly we find the identities describing the variety $V(\underline{K})$ for any class $\underline{K}$ of interior algebras given by means of a set of positive universal sentences (2.1- 2.6). In the second half of the section we consider the case where $\underline{K}$ is defined by a set of universal sentences in which the connectives $\Rightarrow$ and $\sim$ may occur but in which not all operations are admitted (2.7-2.12).

2.1 For any formula $\Phi(x_1, x_2,\ldots x_n)$ of $L_{\underline{B}_i}$ without quantifiers let us define the "modal translation" $MT(\Phi)$ of Φ (the reason for this name will be explained later) to be the term, defined by induction on the complexity of Φ in the following way:

(i) If p,q are terms, then

$$MT(p) = p^{o}$$

$$MT(p \leq q) = (p' + q)^{o}$$

$$MT(p = q) = (p' + q)^{o}.(p + q')^{o}, \text{ also written } (p \blacktriangle q)^{o}$$

(ii) Suppose that ϕ and ψ are formulas such that $MT(\phi)$ and $MT(\psi)$ have been defined. Then

$$MT(\phi \vee \psi) = MT(\phi) + MT(\psi)$$

$$MT(\phi \wedge \psi) = MT(\phi).MT(\psi)$$

$$MT(\phi \Rightarrow \psi) = (MT(\phi)' + MT(\psi))^{o}$$

$$MT(\sim\phi) = MT(\phi)'^{o}$$

Note that the formula Φ and the term $MT(\Phi)$ have the same variables. The modal translation of an arbitrary formula in prenex form, containing universal quantors only, say

$$\Phi'(z_1 \ldots z_n) = \forall x_1 \ldots \forall x_k \; \Phi(x_1 \ldots x_k, z_1 \ldots z_n)$$

is $MT(\Phi') = \forall x_1 \ldots \forall x_k \; MT(\Phi)(x_1 \ldots x_k, z_1 \ldots z_n)$.

Now, if σ is a universal sentence in prenex form, then $MT(\sigma)$ will be a universally quantified term, that is, an identity. For example, if σ is the sentence

$$\forall x \forall y \; [x^{o} \leq y^{o} \vee y^{o} \leq x^{o}]$$

then $MT(\sigma)$ is

$$\forall x \forall y \; [(x^{o\prime} + y^{o})^{o} + (y^{o\prime} + x^{o})^{o}]$$

or, just the identity

$$(x^{o\prime} + y^{o})^{o} + (y^{o\prime} + x^{o})^{o} = 1.$$

If Φ is a formula of $L_{\underline{B}_i^-}$, we define $MT(\Phi)$ in a similar way, writing $(p \Rightarrow q)^{o}$ instead of $(p' + q)^{o}$ for terms p,q. If Σ is a set of universal $\underline{B}_i^{(-)}$- sentences, then $MT(\Sigma) = \{MT(\sigma) \mid \sigma \in \Sigma\}$. We say that a sentence or a set of sentences Σ in $L_{\underline{B}_i}$ <u>defines</u> or <u>describes</u> a class $\underline{K}$ of interior algebras, if $\underline{K} = \{L \in \underline{B}_i \mid L \models \Sigma\}$. An identity is just a (very simple) sentence.

2.2 Theorem (cf. 2.1 of Baker [M]).

(i) If $\underline{K} \subseteq \underline{B}_i$ is defined by a positive universal sentence σ of $L_{\underline{B}_i}$ in prenex form then $V(\underline{K})$ is described by $MT(\sigma)$.

(ii) If $\underline{K} \subseteq \underline{B}_i$ is defined by a set Σ of positive universal sentences in prenex form of $L_{\underline{B}_i}$ then $V(\underline{K})$ is described by $MT(\Sigma)$.

(iii) For any $\underline{K} \subseteq \underline{B}_i$, if Σ is a set of positive universal sentences in prenex form in $L_{\underline{B}_i}$ defining $HSP_U(\underline{K})$, then $MT(\Sigma)$ describes $V(\underline{K})$.

Proof. (i) Let $\underline{K}_1 = \{L \in \underline{B}_i \mid L \models MT(\sigma)\}$. We show that $V(\underline{K}) = \underline{K}_1$. σ is supposed to be of the form $\forall x_1 \dots \forall x_k \Phi\ (x_1 \dots x_k)$, where Φ is a quantifier free formula in which only $\vee$ and $\wedge$ occur as logical symbols. We show (I) that $\underline{K} \subseteq \underline{K}_1$, which will imply that $V(\underline{K}) \subseteq \underline{K}_1$ since $\underline{K}_1$ is a variety, and (II) that $\underline{K}_1 \subseteq V(\underline{K})$.

I. $\underline{K} \subseteq \underline{K}_1$

Let $L \in \underline{K}$, then $L \models \sigma$. Let $a_1, \dots a_k \in L$. We prove that $MT(\Phi)(a_1, \dots a_k) = 1$. This will be done by induction on the complexity of Φ.

a) Φ is a term, say p. Then $p(a_1, \dots a_k) = 1$, hence $p(a_1 \dots a_k)^\circ = 1$, thus $MT(\Phi)(a_1, \dots a_k) = 1$.

b) Φ is a formula of the form $p \leq q$, p,q terms. Then $p(a_1, \dots a_k) \leq q(a_1, \dots a_k)$, and hence

$$MT(\Phi)(a_1, \dots a_k) = (p(a_1, \dots a_k)' + q(a_1, \dots a_k))^\circ = 1.$$

c) Φ is a formula of the form $p = q$, where p and q are terms. Similarly.

d) $\Phi = \phi \vee \psi$. Then $\phi(a_1, \dots a_k)$ or $\psi(a_1, \dots a_k)$. Hence, by induction, $MT(\phi)(a_1, \dots a_k) = 1$ or $MT(\Phi)(a_1 \dots a_k) = 1$. Therefore $MT(\Phi)(a_1, \dots a_k) = MT(\phi)(a_1, \dots a_k) + MT(\psi)(a_1, \dots a_k) = 1$.

e) $\Phi = \phi \wedge \psi$. Similarly.

We conclude that $\underline{K} \subseteq \underline{K}_1$, which implies that $V(\underline{K}) \subseteq \underline{K}_1$ since $MT(\sigma)$ is an identity and hence $\underline{K}_1$ is a variety.

II. $\underline{K}_1 \subseteq V(\underline{K})$.

Let $L \in \underline{K}_{1SI}$. Then L satisfies the equation $MT(\sigma) = 1$, where $\sigma = \forall x_1 \ldots \forall x_k \Phi(x_1,\ldots x_k)$. If Φ is a term p or a formula of the form $p \leq q$, $p = q$ where p and q are terms, it is immediate that L satisfies σ.

Next suppose that $\Phi = \phi \vee \psi$. Let $a_1,\ldots a_k \in L$. Then $MT(\Phi)(a_1,\ldots a_k) = MT(\phi)(a_1 \ldots a_k) + MT(\psi)(a_1 \ldots a_k) = 1$. It follows from the definition of the modal translation that $MT(\phi)(a_1,\ldots a_k) \in L^O$ and $MT(\psi)(a_1,\ldots a_k) \in L^O$ Since L is SI, 1 is join-irreducible in L^O and we conclude that $MT(\phi)(a_1,\ldots a_k) = 1$ or $MT(\psi)(a_1,\ldots a_k) = 1$. An even simpler argument works in case $\Phi = \phi \wedge \psi$. In both cases, it follows by induction that $L \models \sigma$. Hence $\underline{K}_{1SI} \subseteq \underline{K}$, and thus $\underline{K}_1 \subseteq V(\underline{K})$, completing this part of the proof.

(ii) The reasoning for individual sentences in (i) applies analogously to sets of sentences: Σ implies $MT(\Sigma)$ for interior algebras, and $MT(\Sigma)$ implies Σ for SI interior algebras. Thus the variety described by $MT(\Sigma)$ is just $V(\underline{K})$.

(iii) $HSP_U(\underline{K})$ can be described by positive universal sentences (cf. Grätzer [68], pg 275), and $V(\underline{K}) = V(HSP_U(\underline{K}))$. The desired result follows by (ii). □

2.3 Corollary. Let $\underline{K} \subseteq \underline{B}_i$ be such that $HS(\underline{K}) = \underline{K}$. If $\underline{K}$ is strictly elementary, i.e. if $\underline{K}$ is definable by a single first-order sentence, then $V(\underline{K})$ is definable by a single identity.

Proof. Since $HS(\underline{K}) = \underline{K}$, $\underline{K}$ is definable by a positive universal sentence σ. By 2.2 $V(\underline{K})$ is described by the single identity $MT(\sigma) = 1$.□

2.4 Corollary. If $L \in \underline{B}_{iF}$, then $V(L)$ has a finite base.

Proof. Let $\underline{K} = HS(L)$. Then $V(\underline{K}) = V(L)$, and since $\underline{K}$ consists of finitely many finite algebras[1] $\underline{K}$ satisfies the hypotheses of 2.3.□

It can be shown, that 2.4 holds for any congruence-distributive variety (Makkai [74]). This is a much deeper result and the proof of it is rather delicate. In general, 2.4 is not true: there exists a 6-element semigroup whose equational theory has no finite base (cf. Lyndon [54]).

2.5 Corollary. The finitely based subvarieties of $\underline{B}_i$ form a sublattice of the lattice of subvarieties of $\underline{B}_i$.

Proof. That the meet of two finitely based subvarieties of $\underline{B}_i$ is finitely based is obvious. If $\underline{K}_1, \underline{K}_2 \subseteq \underline{B}_i$ are finitely based varieties, then $\underline{K}_1 + \underline{K}_2 = V(\underline{K}_1 \cup \underline{K}_2)$, and since $\underline{K}_1 \cup \underline{K}_2$ is a strictly elementary positive universal class, 2.3 applies.□

2.6 Examples. 1) Let $\underline{K} = \{L \in \underline{B}_i \mid L^{\circ} \cong \underline{2}\}$. Then $\underline{K}$ is definable relative to $\underline{B}_i$ by a single positive universal sentence $\forall x[x^{\circ} = 0 \vee x = 1]$, which is equivalent to $\sigma = \forall x[x^{\circ\prime} \vee x]$. Hence $V(\underline{K})$ is described by $MT(\sigma) = \forall x[x^{\circ\prime\circ} + x^{\circ}]$, which is the identity $x^{\circ\prime\circ} + x^{\circ} = 1$, or, $x^{\circ c} = x^{\circ}$. Apparently, $V(\underline{K})$ consists of all interior algebras whose open elements are also closed: $V(\underline{K})$ is the variety of monadic algebras (cf. II.5).

2) Let $\underline{K} = \{L \in \underline{B}_i \mid |L| \le n\}$, where $n \in \mathbb{N}$ is fixed. Then $\underline{K}$ is definable, relative to $\underline{B}_i$, by the single positive universal sentence σ

[1]) An expression like this, here as well as in the sequel, is understood to mean: $\underline{K}$ consists of finitely many algebras, up to isomorphism.

$$\forall x_0 \ldots \forall x_n \bigvee_{0\le i<j\le n} x_i \doteq x_j .$$

Hence $V(\underline{K})$ is described by $MT(\sigma)$:

$$\sum_{0\le i<j\le n} (x_i \Delta x_j)^o = 1.$$

3) Recall that $M_2 \cong_{\underline{B}} \underline{2}^2$, $M_2^o \cong \underline{2}$ (cf. I.6.1). $V(M_2)$ is described by the two identities $x^{oc} = x^o$ and

$$\sum_{0\le i<j\le 4} (x_i \Delta x_j)^o = 1,$$

relative to $\underline{B}_i$.

4) Let $\underline{K} = \{L \in \underline{B}_i \mid L^o \text{ has width } \le m\}$, where $m \in \mathbb{N}$ is fixed. A lattice has width $\le m$ iff it does not contain a totally unordered set of $m + 1$ elements. $\underline{K}$ is desribed by the sentence

$$\bigvee_{\substack{0\le i,j\le m\\ i\ne j}} x_i^o \le x_j^o .$$

Hence $V(\underline{K})$ is described by the identity

$$\sum_{\substack{0\le i,j\le m\\ i\ne j}} (x_i^{o\prime} + x_j^o)^o = 1.$$

5) Let $\underline{K} = \{L \in \underline{B}_i \mid L^o \text{ is a chain of } n \text{ elements}\}$, where $n \in \mathbb{N}$ is fixed. $\underline{K}$ is described by the sentences

$$\forall x \forall y [x^o \le y^o \vee y^o \le x^o]$$

and

$$\forall x_0 \ldots \forall x_n \bigvee_{i=0,\ldots n-1} x_i^o \le x_{i+1}^o$$

$V(\underline{K})$ is definable by the equations

$$(x^{o'} + y^o)^o + (y^{o'} + x^o)^o = 1$$

and

$$\sum_{i=0}^{n-1} (x_i^{o'} + x_{i+1}^o)^o = 1.$$

2.7 Thus far we have solved the problem of finding the identities defining a variety $V(\underline{K})$ of interior algebras in case $\underline{K}$ is a class of interior algebras defined by a set of positive universal sentences. Next we will consider the same problem for classes $\underline{K}$ which are defined by universal sentences in which the connectives $\Rightarrow$ and $\sim$ may occur, but in which the terms contain only the operation symbols $+$, $.$, ^{o} and 0. Let σ be a universal sentence of $L_{\underline{B}_i}$. For the sake of brevity , universal quantors will be omitted. The sentence σ is equivalent to a sentence σ' of the form

$$\text{a)} \quad \bigwedge_{j=1}^{k} \left(\bigvee_{i=1}^{\ell_j} \phi_i \vee \bigvee_{n=1}^{m_j} \sim \psi_n \right),$$

where ϕ_i, ψ_n are atomic formulas of the type p , $p = q$, $p \leq q$, where p , q are $\underline{B}_i$-polynomials.

In its turn, σ' is equivalent to:

$$\text{a)}' \quad \bigwedge_{j=1}^{k} \bigvee_{i=1}^{\ell_j} \left(\left(\bigwedge_{n=1}^{m_j} \psi_n \right) \Rightarrow \phi_i \right)$$

Let σ^* be the positive universal sentence

$$\text{b)} \quad \bigwedge_{j=1}^{k} \bigvee_{i=1}^{\ell_j} \left(\prod_{n=1}^{m_j} MT(\psi_n) \leq MT(\phi_i) \right),$$

where, of course, $MT(\phi_i)$, $MT(\psi_i)$ are now of the form p^o, $(p \;\Delta\; q)^o$ or $(p' + q)^o$.

The following lemma will be useful on several occasions. But first we need a definition (cf. 4.6 of Baker [M]).

2.8 Definition. Let $L \in \underline{B}_i$ or $L \in \underline{B}_i^-$. The algebra L' will be called a principal homomorphic image of L if $L' \cong L/F$ for some principal open filter $F \subseteq L$. If $\underline{K} \subseteq \underline{B}_i$ or $\underline{K} \subseteq \underline{B}_i^-$ then $H_p(\underline{K})$ will denote the class of principal homomorphic images of algebras in $\underline{K}$.

2.9 Lemma. Let $L \in \underline{B}_i$ and let σ be a universal sentence of the form a) or a)' and σ^* as in b) above. The following conditions are equivalent:

(i) $L \models \sigma^*$

(ii) $\forall L' \in HSP_U(L) \quad L' \models \sigma$

(iii) $\forall L' \in H(L) \quad L' \models \sigma$

(iv) $\forall L' \in H_p(L) \quad L' \models \sigma$.

Proof. It suffices to prove the lemma for $k = 1$; the case $k > 1$ will then follow trivially.

(i) $\Rightarrow$ (ii) Since σ^* is a positive universal sentence, $L \models \sigma^*$ implies $L' \models \sigma^*$ for every $L' \in HSP_U(L)$. It remains therefore to be shown that if $L' \models \sigma^*$ for any $L' \in \underline{B}_i$, then $L' \models \sigma$. Suppose $L' \not\models \sigma$ and let $a_1, \ldots a_p \in L'$ be such that $\phi_i(a_1, \ldots a_p)$ is false and $\psi_n(a_1, \ldots a_p)$ is true, for all $i = 1, \ldots \ell_1$, $n = 1, \ldots m_1$. Then $MT(\phi_i)(a_1, \ldots a_p) < 1$ and $MT(\psi_n)(a_1, \ldots a_p) = 1$, for all $i = 1, \ldots \ell_1$, $n = 1, \ldots m_1$, thus

$$\prod_{n=1}^{m_j} MT(\psi_n)(a_1,\dots a_p) \nleq MT(\phi_i)(a_1,\dots a_p)$$

for every $i = 1,\dots \ell_1$, and hence $L' \not\models \sigma^*$.

(ii) $\Rightarrow$ (iii) obvious.

(iii) $\Rightarrow$ (iv) obvious.

(iv) $\Rightarrow$ (i) Suppose $L \not\models \sigma^*$. Then there exist $a_1,\dots a_p \in L$, such that

$$u = \prod_{n=1}^{m_1} MT(\psi_n)(a_1,\dots a_p) \nleq MT(\phi_i)(a_1,\dots a_p),\ i = 1,\dots \ell_1.$$

Since $u \in L^o$, $(u] \in H_p(L)$, and in $(u]$ $MT(\psi_n)(a_1u,\dots a_pu) = 1$ for $n = 1,\dots m_1$, whereas $MT(\phi_i)(a_1u,\dots a_pu) < 1$ for $i = 1,\dots \ell_1$. Thus $\psi_n(a_1u,\dots a_pu)$ would be true in $(u]$ for $n = 1,\dots m_1$, and $\phi_i(a_1u,\dots a_pu)$ would be false in $(u]$ for $i = 1,\dots \ell_1$; hence σ would fail in $(u]$, contradictory to our assumption.□

2.10 Lemma. If only the operations $+, ., ^o, 0$ occur in the sentence σ of 2.7 and 2.9 then for any $L \in \underline{B}_i$, $L \models \sigma$ iff $L \models \sigma^*$.

Proof. We have already shown in the proof of 2.9 (i) $\Rightarrow$ (ii) that for any $L \in \underline{B}_i$, if $L \models \sigma^*$ then $L \models \sigma$. Now suppose $L \models \sigma$. Note that for any $L' \in H_p(L)$, $L' \cong (u]$ for some $u \in L^o$ and $(u]$ is a $(+, ., ^o, 0)$-subalgebra of L. Since σ is a universal sentence, it follows that $(u] \models \sigma$; hence $\forall L' \in H_p(L)$ $L' \models \sigma$. By 2.9 it follows then that $L \models \sigma^*$. □

2.11 Theorem. Let $\underline{K} \subseteq \underline{B}_i$ be a class of algebras defined by a set Σ of universal sentences, the terms of which contain only the operation symbols $+, ., ^o$ and 0. Then $V(\underline{K})$ is defined by $\{MT(\sigma^*) \mid \sigma\in\Sigma\}$, where σ^* is formed as in 2.7.

<u>Proof</u>. We consider the case that Σ consists of one universal sentence of the form a) as presented in 2.7. The general case follows then easily. Let $\underline{K}_1$ be the variety determined by $MT(\sigma^*)$.

(i) Let $L \in \underline{K}$. Then $L \models \sigma$, and since the hypotheses of 2.10 are satisfied $L \models \sigma^*$. Thus certainly $L \models MT(\sigma^*)$, hence $\underline{K} \subseteq \underline{K}_1$ and therefore $V(\underline{K}) \subseteq \underline{K}_1$

(ii) Let $L \in \underline{K}_{1SI}$, then $L \models MT(\sigma^*)$. Since 1 is join irreducible in L^o it follows that $L \models \sigma^*$ (cf. (i)II of the proof of 2.2), and by 2.9, $L \models \sigma$. Thus $\underline{K}_{1SI} \subseteq \underline{K}$, hence $K_1 \subseteq V(\underline{K})$.□

As an illustration we give two examples of an application of 2.11.

2.12 <u>Examples</u>. 1) Let $\underline{K} = \{L \in \underline{B}_i \mid 0 \text{ is meet irreducible in } L^o\}$. $\underline{K}$ is definable relative to $\underline{B}_i$, by the sentence

$$\sigma = \forall x \forall y [x^o y^o = 0 \Rightarrow x^o = 0 \vee y^o = 0].$$

Note that $\underline{K}$ satisfies the hypothesis of 2.11. σ is equivalent to

$$\sigma' = \forall x \forall y [[x^o y^o = 0 \Rightarrow x^o = 0] \vee [x^o y^o = 0 \Rightarrow y^o = 0]].$$

Hence

$$\sigma^* = \forall x \forall y [(x^{o'} + y^{o'})^o \leq x^{o'o} \vee (x^{o'} + y^{o'})^o \leq y^{o'o}].$$

Thus $V(\underline{K})$ is defined by the identity $MT(\sigma^*)$:

$$((x^{o'} + y^{o'})^{o'} + x^{o'o})^o + ((x^{o'} + y^{o'})^{o'} + y^{o'o})^o = 1.$$

However, note that $\underline{K}$ is also defined by the sentence

$$\tau = \forall x [x^o = 0 \vee x^{o'o} = 0].$$

Applying now the method of 2.2 we conclude that $V(\underline{K})$ can also be described by the simpler equation $x^{o'o} + x^{o'o'o} = 1$, or, equivalently, by $x^{oc'} + x^{oco} = 1$.

2) Let σ be the sentence

$$\forall x \forall y \forall z [[z \leq z^{o} \wedge x + y \geq z \wedge xyz = 0 \wedge x^{o} = 0 \wedge y^{o} = 0] \Rightarrow z = 0] .$$

Note that $L \models \sigma$ iff $M_2 \notin SH_p(L)$, for any $L \in \underline{B}_i$. Let $\underline{K} = \{L \in \underline{B}_i \mid L \models \sigma\}$. $\underline{K}$ satisfies the hypothesis of 2.11, hence $V(\underline{K})$ is definable by the equation

$$(z' + z^{o})^{o} . (x + y + z')^{o} . (x' + y' + z')^{o} . x^{o'o} . y^{o'o} \leq z'^{o}$$

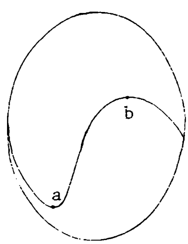
or

$$(z' + z^{o})^{o'} + (x + y + z')^{o'} + (x' + y' + z')^{o'} + x^{oc} +$$
$$y^{oc} + z'^{o} = 1 .$$

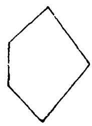
In the next section, a method will be presented which will enable us to find a simpler equation defining this variety.

Section 3. Varieties associated with finite subdirectly irreducibles

3.1 A lattice L is said to be <u>split</u> by a pair of elements (a,b), a, b $\in$ L, if for any c $\in$ L, either a $\leq$ c or c $\leq$ b. Such a pair splits the lattice L into two disjoint intervals, [a) and (b].



McKenzie [72] analyzed the splittings of the lattice Λ of varieties of lattices. It has been known for a long time that the lattice N_5



gives rise to such a splitting of Λ. Indeed, a lattice is modular iff it does not contain a sublattice isomorphic to N_5. Hence, if $\underline{M}$ denotes the variety of modular lattices, then any variety $\underline{K}$ of lattices either contains $V(N_5)$ as a subvariety, or $K \subseteq \underline{M}$. Thus $(V(N_5), \underline{M})$ is a splitting of Λ. There are countably many splitting pairs in Λ, and it can be shown that the first term in each splitting pair is a variety generated by a finite subdirectly irreducible lattice. McKenzie characterizes these lattices, which are called <u>splitting lattices</u>. Although there are countably many splitting lattices, not every finite subdirectly irreducible lattice is splitting: in fact, an effective method is given in McKenzie [72], to determine whether a given finite subdirectly irreducible lattice is splitting or not.

3.2 For an arbitrary variety $\underline{K}$ of algebras the notion of splitting lattice can be generalized to that of a splitting $\underline{K}$-algebra in an obvious way: an algebra $A \in \underline{K}$ is called a <u>splitting algebra</u> if there exists a largest variety $\underline{K}_2 \subseteq \underline{K}$ not containing A, i.e. if $(V(A), \underline{K}_2)$ splits the lattice of subvarieties of $\underline{K}$. If we assume that $\underline{K}$ is congruence distributive and that $\underline{K}$ is generated by its finite members then we can show that here too for every pair $(\underline{K}_1, \underline{K}_2)$ of subvarieties of $\underline{K}$ which splits the lattice of subvarieties of $\underline{K}$ there exists some finite subdirectly irreducible algebra $A \in \underline{K}$ such that $\underline{K}_1 = V(A)$. Indeed, since $\underline{K} = V(\underline{K}_F)$ also $\underline{K} = V(\underline{K}_{FSI})$, hence $\underline{K}_1 \leq \Sigma \{V(A') \mid A' \in \underline{K}_{FSI}\}$. If $\underline{K}_1 \nleq V(A')$ then $V(A') \leq \underline{K}_2$ since the pair $(\underline{K}_1, \underline{K}_2)$ is splitting. But as $\underline{K}_1 \nleq \underline{K}_2$ it follows that $\underline{K}_1 \leq V(A')$ for some $A' \in \underline{K}_{FSI}$. We may apply Jónsson's 0.1.26, $\underline{K}$ being congruence distributive, therefore $\underline{K}_{1SI} \subseteq HS(A')$, say $\underline{K}_{1SI} = \{A_1, \ldots A_n\}$, for some $n \in \mathbb{N}$, A_i finite for $i = 1,\ldots n$. Hence $\underline{K}_1 = \sum_{i=1}^{n} V(A_i)$. Using again the fact that $(\underline{K}_1, \underline{K}_2)$ is a splitting pair we conclude that $\underline{K}_1 \subseteq V(A_i)$ for some i, $i = 1,\ldots n$ and hence that $\underline{K}_1 = V(A_i)$. Apparently, there exists a 1-1 correspondence between the splittings of the lattice of subvarieties of $\underline{K}$ and the finite subdirectly irreducible splitting $\underline{K}$-algebras.

All this applies in particular to the varieties $\underline{H}$ and $\underline{B}_i$. Furthermore, in Jankov [63] an equation ε_L is exhibited for every subdirectly irreducible Heyting algebra L such that for any Heyting algebra L' $L' \models \varepsilon_L$ iff $L \notin V(L')$. This proves that every finite subdirectly irreducible Heyting algebra L is splitting: the pair $(V(L), \{L' \in \underline{H} \mid L' \models \varepsilon_L\})$ is a splitting of the lattice of subvarieties of $\underline{H}$. We want to show now first that a similar result holds for (generalized) interior algebras.

3.3 Theorem. Let $L \in \underline{B}_i$ be a finite subdirectly irreducible algebra. Then L is a splitting interior algebra.

Proof. We have to show that there exists a subvariety $\underline{K}$ of $\underline{B}_i$, such that the pair $(V(L),\underline{K})$ splits the lattice Ω of subvarieties of $\underline{B}_i$. This will be done by finding an equation ε_L (using the method of section 2) which defines the class $\{L' \in \underline{B}_i \mid L \notin V(L')\}$. This is the variety $\underline{K}$ we are looking for.

Let $L' \in \underline{B}_i$. Since L is subdirectly irreducible $L \notin HSP_U(L')$ implies $L \notin V(L')$ and a fortiori $L \notin S(L'')$ for all $L'' \in HSP_U(L')$ implies $L \notin V(L')$. Conversely, $L \notin V(L')$ implies $L \notin S(L'')$ for all $L'' \in HSP_U(L')$, hence

$$L \notin V(L') \quad \text{iff} \quad L \notin S(L'') \text{ for all } L'' \in HSP_U(L').$$

Let $\underset{c\in L}{\exists} x_c$ abbreviate $\exists x_{c_0},\ldots \exists x_{c_n}$ where $c_0,\ldots c_n$ is a complete list without repetitions of the elements of L. Let $c_0 = 0$, $c_n = 1$, c_{n-1} the dual atom of L^o and let σ be the first order sentence

$$\underset{c\in L}{\exists} x_c \bigwedge_{c,d\in L} [x_{c+d} = x_c + x_d \wedge x_{c.d} = x_c . x_d \wedge$$

$$x_{c'} = x_c' \wedge x_{c^o} = x_c^o] \wedge x_{c_0} = 0 \wedge x_{c_n} = 1 \wedge x_{c_{n-1}} \neq 1.$$

Then $L \notin S(L'')$ iff $L'' \models \sim\sigma$. Indeed, if $L \in S(L'')$ then clearly $L'' \models \sigma$ and conversely, if $L'' \models \sigma$ then the positively asserted atomic formulas express that the map $L \to L''$ given by $c \to$ (value of x_c), $c \in L$, is a homomorphism, and the formula $x_{c_{n-1}} \neq 1$ garantees that the homomorphism is $1-1$. Thus $L'' \models \sigma$ iff $L \in S(L'')$. Applying the method given in 2.7 to the sentence $\sim\sigma$ we see that $(\sim\sigma)^*$ is the universally quantified inequality

$$\prod_{c,d\in L} [(x_{c+d} \mathbin{\Delta} x_c + x_d)^o . (x_{c.d} \mathbin{\Delta} x_c . x_d)^o .$$

$$(x_{c'} \mathbin{\Delta} x_c')^o . (x_{c^o} \mathbin{\Delta} x_c^o)^o] . (x_{c_0} \mathbin{\Delta} 0)^o . (x_{c_n} \mathbin{\Delta} 1)^o \leq x_{c_{n-1}}^o .$$

By 2.9, $L' \models (\sim\sigma)^*$ iff for all $L'' \in HSP_U(L')L'' \models \sim\sigma$. Thus $L' \models (\sim\sigma)^*$ iff for all $L'' \in HSP_U(L')$ $L \notin S(L'')$, which by the remark made above implies that $L' \models (\sim\sigma)^*$ iff $L \notin V(L')$. Since $(\sim\sigma)^*$ is equivalent to the identity $\varepsilon_L = MT((\sim\sigma)^*)$, we see that the class $\{L' \in \underline{B}_i \mid L \notin V(L')\}$ is precisely the variety determined by the equation ε_L.□

3.4 Definition. If $L \in \underline{B}_i$ is a finite subdirectly irreducible algebra, then the variety $\{L' \in \underline{B}_i \mid L \notin V(L')\}$ will be denoted by $(\underline{B}_i : L)$ and it will be called the splitting variety associated with L.

3.5 Corollary. Let $L \in \underline{B}_{iFSI}$.

(i) $(\underline{B}_i : L) = \{L' \in \underline{B}_i \mid L \notin SH_p(L')\}$.

(ii) If L is weakly projective then $(\underline{B}_i : L) = \{L' \in \underline{B}_i \mid L \notin S(L')\}$.

Proof. (i) Using the notation of the proof of 3.3, we have that $L' \in (\underline{B}_i : L)$ iff $L' \models (\sim\sigma)^*$. By 2.9 $L' \models (\sim\sigma)^*$ iff $\forall L'' \in H_p(L')$ $L'' \models \sim\sigma$. But $L'' \models \sim\sigma$ iff $L \notin S(L'')$, hence $L' \in (\underline{B}_i : L)$ iff $\forall L'' \in H_p(L')$ $L \notin S(L'')$ iff $L \notin SH_p(L')$.

(ii) is immediate: if $L \in \underline{B}_i$ is weakly projective and $L' \in \underline{B}_i$, $L \in HS(L') = SH(L')$, then also $L \in S(L')$.□

The next corollary is an interesting addition for $\underline{B}_i$ (which likewise holds for $\underline{\bar{B}}_i$) to Jónsson's 1.2:

3.6 Corollary. Let $\underline{K} \subseteq \underline{B}_i$. Then $V(\underline{K})_{FSI} \subseteq SH_p(\underline{K})$.

Proof. Let $\underline{K} \subseteq \underline{B}_i$, $L \in V(\underline{K})_{FSI}$. Suppose that $L \notin SH_p(\underline{K})$. Then $\underline{K} \subseteq \{L' \mid L \notin SH_p(L')\} = (\underline{B}_i : L)$; hence $V(\underline{K}) \subseteq (\underline{B}_i : L)$. But $L \in V(\underline{K})$: a contradiction. □

3.7 We have not paid much attention yet to the property of a variety of being generated by its finite members. We have seen, for example, that $\underline{B}_i$, $\underline{B}_i^*$, $\underline{T}_n$, $n \in \mathbb{N}$, are generated by their finite members (cf. I.6.9). In section 9 we shall give examples of varieties which are not generated by their finite members. At this point corollary 3.6 gives rise to a remark concerning the subvarieties of $\underline{B}_i$ generated by their finite members.

If $\underline{K}_0$, $\underline{K}_1 \subseteq \underline{B}_i$ are varieties such that $\underline{K}_0 = V(\underline{K}_{0F})$, $\underline{K}_1 = V(\underline{K}_{1F})$, then $\underline{K}_0 + \underline{K}_1 = V(\underline{K}_{0F} \cup \underline{K}_{1F})$, hence $\underline{K}_0 + \underline{K}_1$ is also generated by its finite members. Later we shall show (cf. 9.5) that $\underline{K}_0 \cdot \underline{K}_1$ need not be generated by its finite members. We do have however $V(\underline{K}_{0F} \cap \underline{K}_{1F}) \subseteq \underline{K}_0 \cdot \underline{K}_1$, and $V(\underline{K}_{0F} \cap \underline{K}_{1F})$ is certainly the largest variety contained in $\underline{K}_0 \cdot \underline{K}_1$ which is generated by its finite members. Thus, under the partial ordering $\subseteq$, the subvarieties of $\underline{B}_i$ generated by their finite members form a lattice (though not a sublattice of the lattice Ω of all subvarieties of $\underline{B}_i$).

Recall that a subset H of a partially ordered set $(P,\leq)$ is called $\leq$-<u>hereditary</u> if for all $x \in H$ and for all $y \in P$ if $y \leq x$ then $y \in H$. In the following, let $\bar{\underline{B}}_{iFSI}$ denote a set containing precisely one isomorphic copy of each finite subdirectly irreducible interior algebra. We define a relation $\leq$ on $\bar{\underline{B}}_{iFSI}$ by

$$L_1 \leq L_2 \quad \text{iff} \quad L_1 \in HS(L_2)$$

for any $L_1, L_2 \in \bar{\underline{B}}_{iFSI}$. It is easy to verify that $\leq$ is partial ordering on $\bar{\underline{B}}_{iFSI}$.

3.8 <u>Theorem</u>. The subvarieties of $\underline{B}_i$ generated by their finite members form a lattice isomorphic to the set lattice of all $\leq$-hereditary subsets of $(\bar{\underline{B}}_{iFSI},\leq)$.

Proof. We have seen above that the subvarieties of $\underline{B}_i$ generated by their finite members form a lattice, which we shall call Ω_F. Let $H(\underline{\bar{B}}_{iFSI})$ denote the set lattice of all $\leq$-hereditary subsets of $(\underline{\bar{B}}_{iFSI}, \leq)$, set theoretic union being join and set theoretic intersection being meet. Define

$$\Phi : \Omega_F \longrightarrow H(\underline{\bar{B}}_{iFSI})$$

by

$$\underline{K} \longmapsto \underline{K}_{FSI} \cap \underline{\bar{B}}_{iFSI}$$

for any $\underline{K} \in \Omega_F$. It is clear that $\underline{K}_{FSI} \cap \underline{\bar{B}}_{iFSI}$ is $\leq$-hereditary, thus Φ is well-defined. Furthermore, if $\underline{K}_1, \underline{K}_2 \in \Omega_F$, $\underline{K}_1 \neq \underline{K}_2$, then since $\underline{K}_1 = V(\underline{K}_{1FSI} \cap \underline{\bar{B}}_{iFSI})$ and $\underline{K}_2 = V(\underline{K}_{2FSI} \cap \underline{\bar{B}}_{iFSI})$ it follows that $\underline{K}_{1FSI} \cap \underline{\bar{B}}_{iFSI} \neq \underline{K}_{2FSI} \cap \underline{\bar{B}}_{iFSI}$. Thus Φ is 1 - 1. In order to prove that Φ is onto, let us assume that $\underline{K} \subseteq \underline{\bar{B}}_{iFSI}$ is a $\leq$-hereditary subset. In virtue of 3.6 $V(\underline{K})_{FSI} \subseteq HS(\underline{K})$ hence

$$\underline{K} \subseteq V(\underline{K})_{FSI} \cap \underline{\bar{B}}_{iFSI} \subseteq HS(\underline{K}) \cap \underline{\bar{B}}_{iFSI} = \underline{K}.$$

Therefore $\Phi(V(\underline{K})) = \underline{K}$ and Φ is onto. Since Φ and Φ^{-1} are both order preserving it follows that Φ is an isomorphism.□

3.9 Examples. 1) The variety $(\underline{B}_i : M_2)$

The finite members of $(\underline{B}_i : M_2)$ can be described in a simple manner: $L \in (\underline{B}_i : M_2)_F$ iff L is finite and the atoms of L^O are also atoms of L. Indeed, if a is an atom of L^O such that a is not an atom of L, then $|(a]| > 2$, $(a]^O = \{0,a\}$, hence $M_2 \in S((a]) \subseteq SH_p(L)$ and $L \notin (\underline{B}_i : M_2)$. Conversely, if $L \in \underline{B}_{iF} \setminus (\underline{B}_i : M_2)$ then $M_2 \in SH_p(L)$, by 3.5 (i), hence there exists a $u \in L^O$, such that $M_2 \in S((u])$. Let $v \in L^O$ be an atom of L^O such that $v \leq u$, and let $\{0, a, b, u\}$ constitute the copy of M_2 which is a subalgebra of $(u]$. By assumption, v is an atom of L too, hence $v \leq a$ or $v \leq b$. Then $a^o \neq 0$ or $b^o \neq 0$, a contradiction.

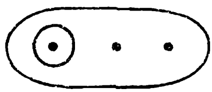
An identity defining $(\underline{B}_i : M_2)$ is easily obtained:

$M_2 \notin S(L)$ iff $L \models \neg\exists x[x^o = 0 \wedge x'^o = 0 \wedge 0 \neq 1]$

iff $L \models \forall x[x^o = 0 \wedge x'^o = 0 \Rightarrow 0 = 1]$.

According to the proof of 3.3, $(\underline{B}_i : M_2)$ is then definable by the identity $x^{o'o}.x'^{o'o} = 0$, or, equivalently, by $x^{oc} + x'^{oc} = 1$.

2) Recall (I.7.18, I.7.20) that $M_{1,2}$ denotes the interior algebra suggested by the diagram:



a b c

that is, $M_{1,2} \underset{\underline{B}}{\tilde{=}} \underline{2}^3$, $M^o_{1,2} \tilde{=} \underline{3}$,
or, more precisely, $M_{1,2} \underset{\underline{B}}{\tilde{=}} P(\{a,b,c\})$, $M^o_{1,2} = \{\emptyset,\{a\}, \{a,b,c\}\}$. We consider now the variety $(\underline{B}_i : M_{1,2})$.

First note that

$(\underline{B}_i : M_{1,2})_F = \{L \in \underline{B}_{iF} \mid$ for all $u, v \in L^o$ if $u'^o = 0$ and $u \prec_{L^o} v$ then $u \prec_L v\}$.
Indeed, if $L \in (\underline{B}_i : M_{1,2})_F$ and there is a $u \in L^o$ such that $u'^o = 0$ and a $v \in L^o$ such that $u \prec_{L^o} v$ but $u \prec_L v$, then $|u'v| > 1$ hence there are $a, b \leq u'v$, such that $a \neq 0$, $b \neq 0$, $ab = 0$ and $a + b = u'v$. It follows that the set $\{u, a, b\}$ is the set of atoms of a $\underline{B}_i$-subalgebra of $(v]$ isomorphic to $M_{1,2}$: $(u + a)^o = (u + b)^o = u$ since $u \prec_{L^o} v$, and $u \neq 0$, $(a + b)^o = 0$ since $u'^o = 0$. Therefore $M_{1,2} \in SH(L)$, contradicting $L \in (\underline{B}_i : M_{1,2})$. Conversely, if $L \in \underline{B}_{iF} \setminus (\underline{B}_i : M_{1,2})$ then $M_{1,2} \in SH(L) = HS(L)$, hence, by I.7.21 $M_{1,2} \in S(L)$. Let $u \in L^o$ be the element corresponding to the open element of $M_{1,2}$ different from

0,1, and let $v \in L^o$ be such that $u \underset{L^o}{\prec} v$. Then $u'^o = 0$ and $u \underset{L}{\nprec} v$.

In order to find identities defining $(\underline{B}_i : M_{1,2})$ note that

$$M_{1,2} \notin S(L) \text{ iff } L \models \sigma_1 \text{ or } L \models \sigma_2,$$

where σ_1 is the universal sentence equivalent to

$$\sim \exists x[x^{oc} = 1 \wedge (x' + x^o)^o \leq x^o \wedge x^o \neq 1]$$

and σ_2 is the universal sentence equivalent to

$$\sim \exists x[x^{oc} = 1 \wedge (x' + x^o)^o \leq x \wedge x^o \neq 1]$$

According to 2.7, σ_1^* is

$$\forall x[x^{oco}.((x' + x^o)^{o'} + x^o)^o \leq x^o]$$

and σ_2^* is

$$\forall x[x^{oco}.((x' + x^o)^{o'} + x)^o \leq x^o]$$

which is equivalent to σ_3:

$$\forall x[x^{oco}.((x' + x^o)^{o'} + x)^o \leq x].$$

It follows by the proof of 3.3 that examples of equations defining $(\underline{B}_i : M_{1,2})$ are

$$MT(\sigma_1^*) : ((x' + x^o)^{o'} + x^o)^{o'} + x^{oco'} + x^o = 1$$

$$MT(\sigma_2^*) : ((x' + x^o)^{o'} + x)^{o'} + x^{oco'} + x^o = 1$$

and $$MT(\sigma_3) : ((x' + x^o)^{o'} + x)^{o'} + x^{oco'} + x = 1.$$

3) Next we consider the variety $(\underline{B}_i : M_2) \cap (\underline{B}_i : M_{1,2})$. We claim that

$$((\underline{B}_i : M_2) \cap (\underline{B}_i : M_{1,2}))_F =$$

$$\{L \in \underline{B}_{iF} \mid \text{for all } u,v \in L^o, \text{ if } u \underset{L^o}{\prec} v \text{ then } u \underset{L}{\prec} v\}$$

Indeed, let $L \in ((\underline{B}_i : M_2) \cap (\underline{B}_i : M_{1,2}))_F$ and let $u, v \in L^o$ such that $u \underset{L^o}{\prec} v$. If $u'^o.v = 0$ then it follows that $u \underset{L}{\prec} v$ from the fact that $(v] \in (\underline{B}_i : M_{1,2})_F$ and from what has been said in example 2). If $u'^o.v \neq 0$ then $u \underset{L^o}{\prec} v$ implies that $u'v$ is an atom of L^o. Since $L \in (\underline{B}_i : M_2)_F$ it follows from example 1) that $u \underset{L}{\prec} v$. The converse is obvious from examples 1) and 2).

One easily deduces from II.2.10 that this means that $((\underline{B}_i : M_2) \cap (\underline{B}_i : M_{1,2}))_F = \underline{B}^*_{iF}$. If we would know that $(\underline{B}_i : M_2) \cap (\underline{B}_i : M_{1,2})$ is generated by its finite members it would follow that $(\underline{B}_i : M_2) \cap (\underline{B}_i : M_{1,2}) = \underline{B}^*_i$, since $\underline{B}^*_i$ is so generated.

An equation defining the variety $(\underline{B}_i : M_2) \cap (\underline{B}_i : M_{1,2})$ is easily obtained. Note that $L \in (\underline{B}_i : M_2) \cap (\underline{B}_i : M_{1,2})$ iff $M_2 \notin SH_p(L)$ and $M_{1,2} \notin SH_p(L)$. Now for any $L' \in \underline{B}_i$, $M_2 \notin SH_p(L')$ and $M_{1,2} \notin S(L')$ iff $L' \models \sigma_1$ or $L' \models \sigma_2$, where σ_1 is the universal sentence equivalent to

$$\sim\exists x[(x' + x^o)^o \leq x^o \wedge x^o \neq 1]$$

and σ_2 is the universal sentence equivalent to

$$\sim\exists x[(x' + x^o)^o \leq x \wedge x^o \neq 1].$$

By 2.9, then, $M_2 \notin SH_p(L)$ and $M_{1,2} \notin SH_p(L)$ iff $L \models \sigma^*_1$ or $L \models \sigma^*_2$, where $\sigma_1{}^*$ is

$$\forall x[((x' + x^o)^{o'} + x^o)^o \leq x^o]$$

and σ^*_2 is

$$\forall x[((x' + x^o)^{o'} + x)^o \leq x^o].$$

The last sentence is equivalent to σ_3:

$$\forall x[((x' + x^o)^{o'} + x)^o \leq x].$$

Since $\sigma_1{}^*$, $\sigma_2{}^*$ and σ_3 are equivalent to the equations

$$((x' + x^o)^{o'} + x^o)^{o'} + x^o = 1$$

$$((x' + x^o)^{o'} + x)^{o'} + x^o = 1$$

and $((x' + x^o)^{o'} + x)^o + x = 1$,

respectively, everyone of these equations defines the variety $(\underline{B}_i : M_2) \cap (\underline{B}_i : M_{1,2})$.

4) In I.6.8 we introduced the chain of subvarieties $\underline{B}_i^* = \underline{T}_0 \subset \underline{T}_1 \subset \ldots \subset \underline{B}_i$. By I.6.9 these varieties are generated by their finite members of rank of triviality $\leq n$. It is not difficult to show that $L \in \underline{B}_{iF}$ has $r_T(L) \leq n$ iff for all $u, v \in L^o$ if $u \underset{L^o}{\prec} v$ then $u'v$ has $\leq 2^n$ atoms. Apparently $L \in \underline{B}_{iF}$ has $r_T(L) \leq n$ iff $L \in (\underline{B}_i : M_{1,2^n+1}) \cap (\underline{B}_i : M_{2^n+1})$. Hence $\underline{T}_n \subseteq (\underline{B}_i : M_{1,2^n+1}) \cap (\underline{B}_i : M_{2^n+1})$ and if we would know that $(\underline{B}_i : M_{1,2^n+1}) \cap (\underline{B}_i : M_{2^n+1})$ is generated by its finite members it would follow that $\underline{T}_n = (\underline{B}_i : M_{1,2^n+1}) \cap (\underline{B}_i : M_{2^n+1})$. By means of the methods employed in examples 1,2 and 3 it is also possible to obtain equations defining the varieties $(\underline{B}_i : M_{1,2^n+1}) \cap (\underline{B}_i : M_{2^n+1})$, for $n \in \mathbb{N}$.

3.10 The problem if every splitting variety and every finite intersection of splitting varieties is generated by its finite members is unsolved yet. In the preceeding examples we have seen that a positive answer to this question, at least for those cases, has interesting consequences: it provides a new characterization of $\underline{B}_i^*$ and of the varieties $\underline{T}_n$, $n \in \mathbb{N}$, and thereby also equations defining these varieties. In the next section we introduce a chain of splitting varieties which are even locally finite, so certainly generated by their finite members. It is also possible to show that the varieties $(\underline{B}_i : M_n)$, $(\underline{B}_i : M_{1,n})$ and $(\underline{B}_i : M_n) \cap (\underline{B}_i : M_{1,n})$ are generated by their finite members. The proofs are tedious, however, and since the most interesting conclusion, namely,

$$\underline{B}_i^* = (\underline{B}_i : M_2) \cap (\underline{B}_i : M_{1,2}),$$

will be derived independently in section 7 we have chosen not to include these results at this point.

Section 4. Locally finite and finite varieties

In this section we shall characterize and study the locally finite subvarieties of $\underline{B}_i$ andthe so-called finite subvarieties of $\underline{B}_i$, i.e. the varieties which are generated by a finite algebra. The notion of local finiteness has been considered at several points already. In II.5.8 we learned that the varieties contained in $\underline{M}_n$, $n \in \mathbb{N}$, are locally finite whereas a variety like $\underline{M}^*$ is not, since it contains, unlike the $\underline{M}_n$, $n \in \mathbb{N}$, the algebra $F_{\underline{B}_i}{-*}(1) \cong K_\infty$ (c.f. II.3.5). It will turn out that as far as the local finiteness of a variety $\underline{K} \subseteq \underline{B}_i$ is concerned, the presence of the algebra K_∞ is decisive: $\underline{K}$ is locally finite iff $K_\infty \notin \underline{K}$ (4.2). In order to prove this we shall introduce the chain of varieties $(\underline{B}_i : K_n)$, where K_n, as before, is the finite interior algebra with n atoms whose lattice of open elements is an (n+1)-element chain (cf. I.3.4). This chain of varieties provides a measure for the height of the lattices of open elements of the interior algebras: $(\underline{B}_i : K_n)$ contains the interior algebras whose lattices of open elements have height $\leq n$. It will serve as a tool in the second part of this section, where we shall characterize the finite subvarieties of $\underline{B}_i$ as being the varieties which themselves have only finitely many subvarieties (4.7).

The chain $(\underline{B}_i : K_n)$, $n \in \mathbb{N}$ was earlier introduced in Blok and Dwinger [74]; it is closely related to the chain $(\underline{H} : \underline{n})$, $n \in \mathbb{N}$, $n \geq 2$, investigated by Hosoi [67], Ono [70], Day [M] et al..

4.1 Theorem.

(i) $(\underline{B}_i : K_n) = \{L \in \underline{B}_i \mid K_n \notin S(L)\}$, $n \in \mathbb{N}$.

(ii) $(\underline{B}_i : K_n) \subseteq (\underline{B}_i : K_m)$, for $n, m \in \mathbb{N}$, $n \leq m$ and

$\underline{B}_i = \Sigma\,\{(\underline{B}_i : K_n) \mid n \in \mathbb{N}\}$.

(iii) $(\underline{B}_i : K_{n+1})_{SI} = \{L \in \underline{B}_i \mid L^o \cong L_1^o \oplus 1,\ L_1 \in (\underline{B}_i : K_n)\}$,

$n \in \mathbb{N}$.

(iv) $(\underline{B}_i : K_n)$ is locally finite for each $n \in \mathbb{N}$.

Proof (i) Let $n \in \mathbb{N}$. Since K_n is a finite subdirectly irreducible, $(\underline{B}_i : K_n)$ is a variety by 3.3. By I.7.21, K_n is weakly projective in $\underline{B}_i$; according to 3.5 (ii), then, $(\underline{B}_i : K_n) = \{L \in \underline{B}_i \mid K_n \notin S(L)\}$.

(ii) Let $n, m \in \mathbb{N}$, $n \leq m$, $L \in (\underline{B}_i : K_n)$. Then $K_n \notin S(L)$ by (i), hence $K_m \notin S(L)$ since $K_n \in S(K_m)$; thus $(\underline{B}_i : K_n) \subseteq (\underline{B}_i : K_m)$. To prove the second statement, note that since $\underline{B}_i = V(\underline{B}_{iF})$, it suffices to show that $\underline{B}_{iF} \subseteq \bigcup\{(\underline{B}_i : K_n) \mid n \in \mathbb{N}\}$. Now, if $L \in \underline{B}_{iF}$, say $|L| = 2^n$, for some $n \in \mathbb{N}$, then certainly $K_{n+1} \notin S(L)$, so $L \in (\underline{B}_i : K_{n+1})$.

(iii) Let $L \in (\underline{B}_i : K_{n+1})_{SI}$, $n \in \mathbb{N}$, $L^o = L_1 \oplus 1$ for some $L_1 \in \underline{H}$. As $K_{n+1} \notin S(L)$, $\underline{n+2} \notin S(L^o)$ hence $\underline{n+1} \notin S(L_1)$. Thus $K_n \notin S(B(L_1))$ and hence $B(L_1) \in (\underline{B}_i : K_n)$. Since $L^o = B(L_1)^o \oplus 1$ it follows that $(\underline{B}_i : K_n)_{SI} \subseteq \{L \in \underline{B}_i \mid L^o \cong L_1^o \oplus 1,\ L_1 \in (\underline{B}_i : K_n)\}$. By a reverse argument, if $L_1 \in (\underline{B}_i : K_n)$, $n \in \mathbb{N}$, $L \in \underline{B}_i$ such that $L^o \cong L_1^o \oplus 1$, then $L \in (\underline{B}_i : K_{n+1})_{SI}$.

(iv) $(\underline{B}_i : K_1)$ being the trivial class, the statement is true for $n = 1$. Suppose now that it has been proven for some $n \geq 1$. Let $L \in (\underline{B}_i : K_{n+1})_{SI}$, and suppose $L = [\{x_1,\ldots x_k\}]_{\underline{B}_i}$, where $k \in N$ is fixed. By (iii), $L^o = L_1^o \oplus 1$, where $L_1 \in (\underline{B}_i : K_n)$, and since $1_{L_1} \in L_1^o \subseteq L^o$, $1_{L_1} \in L^o$. Let $L_2 = [(1_{L_1}] \cup \{x_1,\ldots x_k\}]_{\underline{B}}$. We show that $L_2 = L$. Indeed, if $a \in L_2$, $a \neq 1$, then $a^o \in L_1^o \subseteq L_2$, and if $a = 1$, then $a^o = 1$. Hence L_2 is a $\underline{B}_i$- subalgebra of L containing $x_1,\dots x_k$, thus $L_2 = L$. Furthermore, $(1_{L_1}]^o = L_1^o$, where $L_1 \in (\underline{B}_i : K_n)$, hence $(1_{L_1}] \in (\underline{B}_i : K_n)$. Since $(1_{L_1}]$ is a $\underline{B}_i$-homomorphic image of L, $(1_{L_1}]$ is $\underline{B}_i$-generated by $x_1 . 1_{L_1},\dots x_k . 1_{L_1}$. In case $n = 1$ $(1_{L_1}] = \{0\}$ and $|\, L \,| = |\, [\{0\} \cup \{x_1,\dots x_k\}]_{\underline{B}} \,| \leq 2^{2^k}$. In case $n > 1$, $|\, L \,| = |\, [(1_{L_1}] \cup \{x_1,\dots x_k\}]_{\underline{B}} \,| \leq 2^{2^K}$ where $K = |\, F_{(\underline{B}_i:K_n)}(k) \,| + k$ (recall that by the induction hypothesis, $F_{(\underline{B}_i:K_n)}(k)$ is finite). In either case, $|\, L \,| \leq N$, N being a fixed integer. It follows that every member of $(\underline{B}_i : K_{n+1})$ which is generated by k elements is a subalgebra of a finite product of finite subdirectly irreducible algebras, and is therefore finite.□

We noticed already that $(\underline{B}_i : K_1) = \{L \in \underline{B}_i \mid \underline{2} \notin S(L)\}$ is the trivial class. $(\underline{B}_i : K_2)_{SI} = \{L \in \underline{B}_i \mid L^o \cong L_1^o \oplus 1,\ L_1 \in (\underline{B}_i : K_1)\} = \{L \in \underline{B}_i \mid L^o \cong \underline{2}\}$, so $(\underline{B}_i : K_2)$ is generated by the interior algebras with trivial interior operator, and because of local finiteness of $(\underline{B}_i : K_2)$, even by the finite ones: $M_1, M_2, \dots$. Thus $(\underline{B}_i : K_2)$ coincides with the class of monadic algebras, $\underline{M}_2$ (cf. II.5).

4.2 It follows from the definition of $(\underline{B}_i : K_n)$ that $(\underline{B}_i : K_n) = (\underline{H} : \underline{n+1})^c$, $n \in \mathbb{N}$ (see II.1.9). Results 4.1(i), (ii) and (iii) can therefore also be derived from Day [M] and II.1.9. An equation defining $(\underline{B}_i : K_n)$ can thus be found from the well-known equation defining $(\underline{H} : \underline{n+1})$, $n \in \mathbb{N}$, which was first given by McKay [68].

Let $p_1 = x_1$

$$p_{n+1} = ((x_{n+1} \to p_n) \to x_{n+1}) \to x_{n+1} \text{ for } n \geq 1.$$

Then $(\underline{H} : \underline{n+1})$ is defined by the equation $p_n = 1$. Using the "translation" described in II.1.11 we obtain equations defining $(\underline{B}_i : K_n)$, $n \in \mathbb{N}$. Indeed, let

$$q_1 = x_1^o$$

$$q_{n+1} = (((x_{n+1}^{o'} + q_n)^{o'} + x_{n+1}^o)^{o'} + x_{n+1}^o)^o \text{ for } n \geq 1.$$

$(\underline{B}_i : K_n)$ is defined by the equation $q_n = 1$.

Using the ideas presented in section 3, we can find still another equation defining $(\underline{B}_i : K_n)$, in only one variable. We use the notation established in II.2. It follows from the results of II.2, that $K_n \notin S(L)$ iff $L \models \sigma$, where $\sigma = \sim\exists x[s_{n-1}(x)'\cdot s_n(x) \neq 0]$. Thus the identity $s_{n-1}(x) + s_n(x)' = 1$ defines $(\underline{B}_i : K_n)$, $n \in \mathbb{N}$, as well.

Locally finite subvarieties of $\underline{B}_i$ may be characterized now in several ways:

4.3 <u>Theorem</u>. Let $\underline{K} \subseteq \underline{B}_i$ be a variety. The following are equivalent:

(i) $F_{\underline{K}}(1)$ is finite

(ii) $F_{\underline{B}_i^{-*}}(1) \notin \underline{K}$

(iii) $\underline{K} \subseteq (\underline{B}_i : K_n)$, for some $n \in \mathbb{N}$

(iv) $\underline{K}$ is locally finite.

<u>Proof</u>. (i) $\Rightarrow$ (ii) is obvious since $F_{\underline{B}_i^{-*}}(1)$ is infinite (cf. II.3.5).

(ii) $\Rightarrow$ (iii) Suppose $\underline{K} \nsubseteq (\underline{B}_i : K_n)$, for all $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$ there is an $L \in \underline{K}$ such that $K_n \in S(L)$; but since $F_{\underline{B}_i^{-*}}(1) \in SP\{K_n \mid n \in \mathbb{N}\}$, (cf. proof of II.3.5), this would imply that $F_{\underline{B}_i^{-*}}(1) \in \underline{K}$.

(iii) $\Rightarrow$ (iv). This is 4.1.(iv).

(iv) $\Rightarrow$ (i). Obvious. □

4.4 Corollary. The locally finite subvarieties of $\underline{B}_i$ form a sublattice of the lattice of subvarieties of $\underline{B}_i$. Moreover, if $\underline{K}$ is a locally finite subvariety of $\underline{B}_i$ and $\underline{K}'$ is a subvariety of $\underline{B}_i$ which covers $\underline{K}$, then $\underline{K}'$ is locally finite.

Proof. The first statement follows easily from the characterization of locally finite varieties, given in 4.3 (iii).

In order to prove the second statement, assume that $\underline{K} \subseteq \underline{B}_i$ is a locally finite variety. Then $\underline{K} \subseteq (\underline{B}_i : K_n)$, for some $n \in \mathbb{N}$. Now, if $\underline{K}'$ is a subvariety of $\underline{B}_i$ such that $\underline{K} \prec \underline{K}'$ then $\underline{K}' \subseteq (\underline{B}_i : K_{n+1})$. Indeed, suppose not; then there exists an $L \in \underline{K}'$ satisfying $L \notin (\underline{B}_i : K_{n+1})$, hence $K_{n+1} \in S(L) \subseteq \underline{K}'$. Since $K_n \notin \underline{K}$, $K_{n+1} \notin V(\underline{K} \cup \{K_n\}) = \underline{K} + V(K_n)$, we would have $\underline{K} < \underline{K} + V(K_n) < \underline{K}'$, a contradiction. Thus $\underline{K}' \subseteq (\underline{B}_i : K_{n+1})$, and hence $\underline{K}'$ is locally finite by 4.1.(iv). □

In II.2.9 we have seen that a variety generated by $*$-algebras may contain algebras which are not $*$-algebras. In the next corollary we characterize the subvarieties of $\underline{B}_i$ for which such a thing cannot happen. For notation, see II.1.7.

4.5 Corollary. Let $\underline{K} \subseteq \underline{B}_i$ be a variety such that $\underline{K} = \underline{K}^*$. Then $\underline{K}$ consists of $*$-algebras iff $\underline{K}$ is locally finite.

Proof. If $\underline{K}$ is locally finite then $F_{\underline{B}_i^-*}(1) \notin \underline{K}$ by 4.3. Let $L \in \underline{K}$ be arbitrary, $x \in L$, $x \neq 1$. Then $[x]_{\underline{B}_i^-} \in \underline{B}_i^{-*}$, and $[x]_{\underline{B}_i^-}$ is a proper homomorphic image of $F_{\underline{B}_i^-*}(1)$. Hence $[x]_{\underline{B}_i^-} \cong K_n$, for some $n \in \mathbb{N}$ (cf. II.3.6), so L is a $*$-algebra (use II.2.5).
Conversely, if $\underline{K}$ is not locally finite, then by 4.3 $F_{\underline{B}_i^-*}(1) \in \underline{K}$; since $F_{\underline{B}_i^-*}(1) \cong K_\infty^-$ by II.3.5, $F_{\underline{B}_i^-*}(1)$ is not a $*$-algebra. □

4.6 In particular, the classes $(\underline{B}_i : K_n)^*$, $n \in \mathbb{N}$, consist of $\star$-algebras. Note that $(\underline{B}_i : K_2)^*$ is the variety of discrete interior algebras. In Blok en Dwinger [74], elegant equations defining the varieties $(\underline{B}_i : K_n)^*$, $n \in \mathbb{N}$, were obtained. Equations defining $(\underline{B}_i : K_n)^*$, $n \in \mathbb{N}$, can also be found by noting that

$$(\underline{B}_i : K_n)^* = (\underline{B}_i : K_n) \cap (\underline{B}_i : M_2) \cap (\underline{B}_i : M_{1,2})$$

and by using 4.2 and 3.9 3).

The locally finite subvarieties of $\underline{B}_i$ seem to be rather close to the bottom of the lattice Ω. However, these classes may still have infinitely many subvarieties. For example, the variety $(\underline{B}_i : K_2)$ of monadic algebras has according to II.5.6 infinitely many subvarieties. We want to restrict our attention now to subvarieties of $\underline{B}_i$ which are characterized by the fact that they have only finitely many subvarieties. These will turn out to be precisely the finite subvarieties of $\underline{B}_i$, i.e., the subvarieties of $\underline{B}_i$ which are generated by a finite algebra. By 2.4, finite varieties are always finitely based; their theories are decidable. By Jónsson's 1.2, if $\underline{K}$ is a finite variety of interior algebras then $\underline{K}_{SI}$ is a finite set of finite subdirectly irreducibles; and conversely, every finite collection of finite subdirectly irreducibles generates a finite variety: if $L_1,\dots L_n \in \underline{B}_{iFSI}$, then $\prod_{k=1}^{n} L_k$ is finite and $V(\prod_{k=1}^{n} L_k) = V(\{L_1,\dots L_n\})$. It follows, that the finite subvarieties of $\underline{B}_i$ form a sublattice of the lattice Ω of subvarieties of $\underline{B}_i$, and in fact, using the notation of 3.8, we have:

4.7 <u>Theorem</u>. The finite subvarieties of $\underline{B}_i$ form a sublattice of Ω isomorphic to the set lattice of finite $<$-hereditary subsets of $(\underline{B}_{iFSI}, <)$.

<u>Proof</u>. Restrict in the proof of 3.8 the map Φ to the sublattice of Ω_F consisting of finite subvarieties of $\underline{B}_i$.□

It is immediate that every finite variety has only finitely many subvarieties. To establish the converse, we need a lemma.

4.8 <u>Lemma</u>. Let $\underline{K} \subseteq \underline{B}_i$ be a locally finite variety. Then $\underline{K}$ contains an infinite subdirectly irreducible algebra iff $\underline{K}$ contains infinitely many distinct finite subdirectly irreducibles.

<u>Proof</u>. $\Rightarrow$ Let $L \in \underline{K}$ be an infinite subdirectly irreducible algebra. If $x_1, \ldots x_k \in L$, $k \in \mathbb{N}$, then $L' = [\{x_1, \ldots x_k\}]_{\underline{B}_i} \subseteq L$ is finite, hence $L' \neq L$. Moreover L' is subdirectly irreducible, since $u = \Sigma\{x \in L'^{o} \mid x \neq 1\} < 1$, $u = u^{o}$ and for any $y \in L'^{o}\backslash\{1\}$, $y \leq u$. By induction one can construct a sequence of subalgebras $L_1 \subset L_2 \subset \ldots \subset L$, such that L_i is finitely generated, hence finite, and such that L_i is subdirectly irreducible, $i \in \mathbb{N}$. Thus $\{L_i \mid i \in \mathbb{N}\}$ is an infinite set of finite subdirectly irreducibles in $\underline{K}$, which are distinct because they have distinct cardinality.

$\Leftarrow$ If $\underline{K}$ contains an infinite number of distinct finite subdirectly irreducibles, we take a non-trivial ultraproduct of these. That is an infinite algebra in $\underline{K}$ which is subdirectly irreducible since $P_U(\underline{B}_{iSI}) \subseteq \underline{B}_{iSI}$, according to a remark made at the end of section 1.□

4.9 Theorem. A variety $\underline{K} \subseteq \underline{B}_i$ has finitely many subvarieties iff $\underline{K}$ is generated by some finite algebra.

Proof. $\Leftarrow$ is an immediate consequence of 1.2.

$\Rightarrow$ Suppose that $\underline{K} \subseteq \underline{B}_i$ is a variety which has finitely many subvarieties. There exists an $n \in \mathbb{N}$ such that $K_n \notin \underline{K}$, otherwise we would have

$$V(K_1) \subset V(K_2) \subset \ldots \subset \underline{K}.$$

Therefore $\underline{K} \subseteq (\underline{B}_i : K_n)$, for some $n \in \mathbb{N}$, implying that $\underline{K}$ is locally finite. Since $\underline{K}$ has only finitely many subvarieties $\underline{K}$ contains only finitely many finite subdirectly irreducibles. Applying 4.8, we conclude that there are no non-finite subdirectly irreducibles; hence $\underline{K} = V(\{L_1,\ldots L_n\})$ for some $n \in \mathbb{N}$, where $\{L_1,\ldots L_n\} \subseteq \underline{K}_{FSI}$. Thus $\underline{K} = V(\prod_{k=1}^{n} L_k)$ and $\prod_{k=1}^{n} L_k$ is a finite algebra in $\underline{K}$.□

4.10 Corollary. Let $\underline{K}$, $\underline{K}'$ be subvarieties of $\underline{B}_i$, such that $\underline{K}$ is a finite variety and $\underline{K} \prec \underline{K}'$. Then $\underline{K}'$ is a finite variety.

Proof. The lattice $(\underline{K}']$, being a sublattice of Ω, is distributive (cf. 1.5); the length of a maximal chain in $(\underline{K}']$ is therefore independent of the choice of the chain. Since $(\underline{K}]$ is finite the length of a maximal chain in $(\underline{K}]$ will be m, for some $m \in \mathbb{N}^*$. Thus $(\underline{K}']$ contains a maximal chain of length m + 1. Hence $(\underline{K}']$ is a finite lattice, and by 4.9 $\underline{K}'$ is a finite variety.□

While investigating the structure of the lattice of finite subvarieties of $\underline{B}_i$, the question arises how to find all successors of a given finite variety. The next theorem deals with this problem.

4.11 <u>Theorem</u>. Let $\underline{K} \subseteq \underline{B}_i$ be a finite variety. There are only finitely many subvarieties of $\underline{B}_i$ which cover $\underline{K}$.

<u>Proof</u>. Suppose that $\underline{K} \subseteq \underline{B}_i$ is a finite variety and that $\underline{K}' \subseteq \underline{B}_i$ is a variety such that $\underline{K} \prec \underline{K}'$. By 4.10, $\underline{K}'$ is a finite variety, and by 4.7 $| \underline{K}'_{FSI} \setminus \underline{K}_{FSI} | = 1$, say $\underline{K}'_{FSI} = \underline{K}_{FSI} \cup \{L\}$. Suppose that $\underline{K}_{FSI} = \{L_1, \ldots L_n\}$, for some $n \geq 0$.

Let

$$n_0 = \min \{n \mid \underline{K} \subseteq (\underline{B}_i : K_n)\}$$

and

$$k_0 = \min \{k \mid L_1, \ldots L_n \text{ are generated by } \leq k \text{ elements}\}.$$

We claim that $L \in H(F_{(\underline{B}_i : K_{n_0+1})}(k_0 + 1))$. Since $F_{(\underline{B}_i : K_{n_0+1})}(k_0 + 1)$ is finite, this will imply that there are only finitely many subvarieties of $\underline{B}_i$ covering $\underline{K}$.

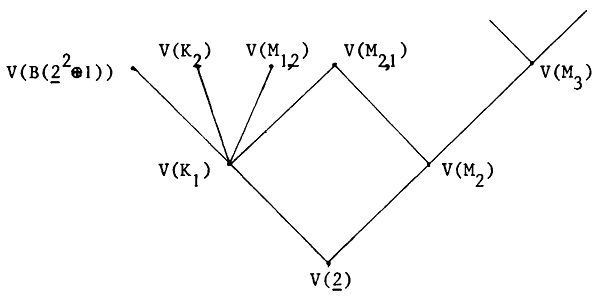
(i) Suppose $L \notin (\underline{B}_i : K_{n_0+1})$. Then $K_{n_0}, K_{n_0+1} \in S(L)$ whereas by our choice of n_0, $K_{n_0} \notin \underline{K}$. Hence $\underline{K} < \underline{K} + V(K_{n_0}) < \underline{K} + V(K_{n_0+1}) \leq \underline{K}'$, a contradiction. Thus $L \in (\underline{B}_i : K_{n_0+1})$.

(ii) Suppose L is not generated by $k_0 + 1$ elements. Let $\ell > k_0 + 1$ be the smallest number such that L is generated by ℓ elements, say by $a_1, \ldots a_\ell$. Consider $L' = [\{a_1, a_2, \ldots a_{\ell-1}\}]_{\underline{B}_i}$. Then $L' \subset L_1$ and L' is not generated by $\leq k_0$ elements, since otherwise L' would be generated by $\leq k_0 + 1$ elements. Furthermore, L' is subdirectly irreducible, being a finite subalgebra of a subdirectly irreducible algebra. Therefore $\underline{K} < \underline{K} + V(L') < \underline{K} + V(L) = \underline{K}'$, a contradiction.□

The proof of 4.11 provides an effective method to find the successors of a given finite variety: if $\underline{K} = V(L)$, $L \in \underline{B}_{iF}$, k_0, n_0 as in the proof of 4.9, then we need only to check which of the finitely many finite subdirectly irreducible homomorphic images of the finite algebra $F_{(\underline{B}_i:K_{n_0+1})}(k_0+1)$ gives rise to a variety $\underline{K}'$ covering $\underline{K}$. In practice there is an obstacle, however: the algebras $F_{(\underline{B}_i:K_n)}(k)$ have not yet been determined. Their structure is very complicated, and their cardinality is fastly growing for increasing n,k.

We hardly need to observe, that the converse of 4.11 is not true: a subvariety $\underline{K} \subset \underline{B}_i$, having but finitely many successors, need not be finite. Indeed, $(\underline{B}_i : K_n)$, $n \geq 2$, $n \in \mathbb{N}$, is not finite, but has only one cover, namely, $(\underline{B}_i : K_n) + V(K_n)$. For if $\underline{K} \subseteq \underline{B}_i$ is a variety such that $\underline{K} \supset (\underline{B}_i : K_n)$ then $\underline{K} \nsubseteq (\underline{B}_i : K_n)$, hence $K_n \in \underline{K}$. Therefore $(\underline{B}_i : K_n) + V(K_n) \subseteq \underline{K}$. On the other hand, obviously $(\underline{B}_i : K_n) + V(K_n) > (\underline{B}_i : K_n)$.

In order to give a (very modest) idea of the structure of the lattice of finite subvarieties of $\underline{B}_i$, we present the following picture of the poset of join irreducibles of the lattice near its bottom.



Note that the trivial class is covered by precisely one variety: the class of discrete interior algebras, the only equationally complete variety of interior algebras (cf. 0. 1. 15). The finite subvarieties of $\underline{B}_i$ correspond 1 - 1 with the finite hereditary subsets of this poset.

In the final part of this section we show how the chain $(\underline{B}_i : K_n)$, $n \in \mathbb{N}$ of subvarieties of $\underline{B}_i$ can be used to obtain information concerning free objects. Indeed, according to 4.1, $\underline{B}_i$ is the lattice sum of the locally finite varieties $(\underline{B}_i : K_n)$, $n \in \mathbb{N}$. Hence $F_{\underline{B}_i}(k)$, $k \in \mathbb{N}$ is isomorphic with a subalgebra of the complete atomic interior algebra $U_k^{\underline{B}_i}$ (cf. II.7.1 - II.7.4), and we have:

4.12 <u>Theorem</u>. (cf. II.7.6) $F_{\underline{B}_i}(k)$ is atomic, for all $k \in \mathbb{N}$.

and

4.13 <u>Theorem</u>. (cf. II.7.7) $F_{\underline{B}_i}(k)^O$ is strongly atomic, for all $k \in \mathbb{N}$.

4.14 <u>Remark</u>. Theorem 4.13 implies that $F_{\underline{H}}(k)$ is strongly atomic for all $k \in \mathbb{N}$. By I.4.6 $B(F_{\underline{H}}(k))$ is isomorphic to a principal ideal of $F_{\underline{B}_i}(k)$; hence $F_{\underline{H}}(k)$ is isomorphic to a principal ideal of $F_{\underline{B}_i}(k)^O$. Since $F_{\underline{B}_i}(k)^O$ is strongly atomic by 4.13, so is $F_{\underline{H}}(k)$. Needless to say, 4.12 and 4.13 are equally valid for $F_{\underline{B}_i^-}(k)$ and $F_{\underline{B}_i^-}(k)^O$, and therefore $F_{\underline{H}^-}(k)$ is also strongly atomic for all $k \in \mathbb{N}$.

The next corollary is a counterpart to an earlier result (I.4.15), which stated that there exists an open element in $F_{\underline{B}_i}(1)^O$ which has countably many open covers.

4.15 <u>Corollary</u>. Let $u \in F_{\underline{B}_i}(k)^o$ be such that $(u]$ is finite. Then there are only finitely many open elements covering u in $F_{\underline{B}_i}(k)^o$.

<u>Proof</u>. Let $(u] \in (\underline{B}_i : K_n)$, for some $n \in \mathbb{N}$, and suppose $u \prec_{F_{\underline{B}_i}(k)^o} v$, $v \in F_{\underline{B}_i}(k)^o$. Then $(v] \in (\underline{B}_i : K_{n+1})$, as one easily verifies, and $(v] \in H(F_{(\underline{B}_i : K_{n+1})}(k))$. Now $F_{(\underline{B}_i : K_{n+1})}(k) \cong (w]$ for some unique $w \in F_{\underline{B}_i}(k)^o$, and $v \leq w$. Since $(w]$ is finite, the corollary is proven.□

Since $\underline{B}_i^* = \sum_{n \in \mathbb{N}} (\underline{B}_i : K_n)^*$, it follows in a similar way that $F_{\underline{B}_i^*}(k)$ is atomic and that $F_{\underline{B}_i^*}(k)^o$ is strongly atomic. Note that if $u,v \in F_{\underline{B}_i^*}(k)^o$, $u \prec_{F_{\underline{B}_i^*}(k)^o} v$ then also $u \prec_{F_{\underline{B}_i^*}(k)} v$, hence $u'v$ is an atom.

Conversely, if $a \in F_{\underline{B}_i^*}(k)$ is an atom, then, in accordance with the results of II.7, there exists a $u \in F_{\underline{B}_i^*}(k)^o$ such that $a \leq u$ and $(u] \cong F_{(\underline{B}_i : K_n)^*}(k)$ for some $n \in \mathbb{N}$. Hence $(u]$ is finite, and it follows that $a = v'w$ for some $v,w \in (u]^o \subseteq F_{\underline{B}_i^*}(k)^o$.

<u>Section 5. The lattice of subvarieties of $\underline{M}$</u>

The purpose of the present and the next section is to give a detailed description of two principal ideals of Ω, $(\underline{M}]$ and $((\underline{B}_i : K_3)]$.

These sublattices are of a relatively simple structure; in particular, both are countable and consist wholly of varieties which are generated by their finite members.

In II.5 we started the investigation of the variety $\underline{M}$ consisting of all those interior algebras, whose lattices of open elements are relatively Stone, and some of its subvarieties, like $\underline{M}_n$, $n \in \mathbb{N}$. Several of the results we are going to present now were obtained earlier in the context of modal logics by Bull [66] and Fine [71]. They studied modal logics which are "normal extentions" of the modal logic called S.4.3. The lattice of these extensions of S.4.3 is the dual of the lattice of subvarieties of $\underline{M}$, we are about to consider. Fine investigated this lattice using the so-called Kripke semantics (cf. Kripke[63]). Our methods, being of an algebraic nature, are quite different and seem to give additional insight into some of the problems. Furthermore, we shall be able to present for any subvariety of $\underline{M}$ the equation defining it. We shall close this section with some facts concerning covers of certain varieties in $(\underline{M}]$.

In II.5.4(iii) we have seen, that though the variety $\underline{M}$ is not locally finite, it is generated by its finite subdirectly irreducibles. In the next theorem it will be shown that any subvariety of $\underline{M}$ is generated by its finite subdirectly irreducibles. Bull [66] discovered this fact, and our proof is similar to his.

5.1 <u>Theorem</u>. Let $\underline{K} \subseteq \underline{M}$ be a variety. Then $\underline{K}$ is generated by its finite members.

Proof. Suppose not. Let $L \in \underline{K}_{SI} \setminus V(\underline{K}_F)$ and let p be a $\underline{B}_i$-polynomial such that the equation $p = 1$ is satisfied in $V(\underline{K}_F)$ but not in L. Let $a_1, \ldots a_n \in L$ such that $p(a_1, \ldots a_n) \neq 1$. Let L_1 be the Boolean algebra $\underline{B}$-generated by all terms occurring in $p(a_1, \ldots a_n)$. (For a more precise formulation, see the proof of I.6.9). Define an interior operator ^{o_1} on L_1 as follows. If $a \in L_1$ then

$$a^{o_1} = \Pi\{b \in L_1 \mid b^o \geq a^o\}.$$

This is a good definition since L_1 is finite. Obviously $1^{o_1} = 1$, $a^{o_1} \leq a$, and $a^{o_1 o_1} = a^{o_1}$. Furthermore, if $a,b \in L_1$, then $a^o \leq b^o$ or $a^o \geq b^o$ since L is subdirectly irreducible. If $a^o \leq b^o$, then $(ab)^{o_1} = \Pi\{c \in L_1 \mid c^o \geq (ab)^o\} = \Pi\{c \in L_1 \mid c^o \geq a^o b^o\} = \Pi\{c \in L_1 \mid c^o \geq a^o\} = a^{o_1}$. Also, if $a^o \leq b^o$ then $a^{o_1} \leq b^{o_1}$, hence $(ab)^{o_1} = a^{o_1} = a^{o_1} b^{o_1}$. Similarly if $a^o \geq b^o$. Hence L_1 is a finite interior algebra. Furthermore, if $a \in L_1$ and $a^o \in L_1$ then $a^o = a^{o_1}$. Therefore $p_{L_1}(a_1, \ldots a_n) = p_L(a_1, \ldots a_n) \neq 1$. We shall show now that $L_1 \in S(L)$ hence $L_1 \in \underline{K}_F$. This will contradict our assumption that $p = 1$ holds in $\underline{K}_F$.

Let $L_1^o = \{x^o \mid x \in L_1\} = \{0 = c_0 < c_1 < \ldots < c_n < c_{n+1} = 1\} \subseteq L$. Note that $L_1^o \not\subseteq L_1$, in general. Let $A_i = \{a \in \mathrm{At}\, L_1 \mid a'^o = c_i\}$, $i = 0, \ldots n$, where At L_1 denotes the set of atoms of L_1. Then $A_i \neq \emptyset$ and $A_i \cap A_j = \emptyset$ if $0 \leq i \neq j \leq n$, and $\bigcup_{i=0}^{n} A_i = \mathrm{At}\, L_1$. Note that if $a \in A_i$ then $a \leq c_i'$ and $a \not\leq c_{i+1}'$, $i = 0, \ldots n$. Choose one atom a_i from every set A_i, $i = 0, \ldots n$. We define a map $\phi : L_1 \to L$ by the following rule: if $a \in A_i$, $i = 0, 1, \ldots n$, let

$$\phi(a) = \begin{cases} a.c_{i+1} & \text{if } a \neq a_i \\ (c_i + \sum\limits_{\substack{a \in A_i \\ a \neq a_i}} a)'.c_{i+1} & \text{if } a = a_i \end{cases}$$

and if $x \in L_1$ is arbitrary then

$$\phi(x) = \Sigma\{\phi(a) \mid a \in \mathrm{At}\, L_1 \text{ and } a \leq x\}$$

We observed already that if $a \in A_i$ then $a \nleq c'_{i+1}$ and hence $a.c_{i+1} \neq 0$ and also $(c_i + \sum_{\substack{a \neq a_i \\ a \in A_i}} a)'.\ c_{i+1} \geq a_i.\ c_{i+1} \neq 0$. Furthermore, if $a,b \in A_i$ such that $a \neq b$ then $\phi(a).\phi(b) = 0$ and $\sum_{a \in A_i} \phi(a) = \sum_{\substack{a \neq a_i \\ a \in A_i}} a.c_{i+1} + (c_i + \sum_{\substack{a \in A_i \\ a \neq a_i}} a)'.\ c_{i+1} = c'_i c_{i+1}$. Since $c'_i.c_{i+1}.\ c'_j.\ c_{j+1} = 0$ if $0 \leq i \neq j \leq n$ and $\sum_{i=0}^{n} c'_i c_{i+1} = 1$ it follows that for any $a,b \in \mathrm{At}\, L_1$ $\phi(a).\phi(b) = 0$ and $\sum_{a \in \mathrm{At}\, L_1} \phi(a) = 1$. Thus the set $\{\phi(a) \mid a \in \mathrm{At}\, L_1\}$ is the set of atoms of a $\underline{B}$-subalgebra of L and ϕ is a $1-1$ $\underline{B}$-homomorphism from L_1 to L.

In order to establish that ϕ is in fact a $\underline{B}_i$-homomorphism let us note that it is sufficient to prove that for any $a \in \mathrm{At}\, L_1$ $\phi(a'^{o_1}) = \phi(a)'^{o}$. It follows from the definition of ^{o_1} that for $a,b \in L_1$ $a^o \geq b^o$ iff $a^{o_1} \geq b^{o_1}$. Therefore, if $a \in A_i$, $0 \leq i \leq n$ and $p \in \mathrm{At}\, L_1$ then

$$p \nleq a'^{o_1} \text{ iff } p'^{o_1} \geq a'^{o_1} \text{ iff } p'^{o} \geq a'^{o} \text{ iff } p \in A_j, \text{ for some } j,$$

where $i \leq j \leq n$.

Hence if $a \in A_i$ then $a'^{o_1} = \sum_{k=0}^{i-1} \Sigma A_k$ and $\phi(a'^{o_1}) = \sum_{k=0}^{i-1} \sum_{a \in A_k} \phi(a) = c_i$. On the other hand, if $a \in A_i$ then

$$\phi(a)'^{o} = (a.c_{i+1})'^{o} = (a' + c'_{i+1})^{o} = c_i \text{ if } a \neq a_i$$

and

$$\phi(a_i)'^{o} = ((c_i + \sum_{\substack{a \in A_i \\ a \neq a_i}} a)'.c_{i+1})'^{o}$$

$$= (c_i + \sum_{\substack{a \in A_i \\ a \neq a_i}} a + c'_{i+1})'^{o}$$

$$\leq (c_i + a'_i + c'_{i+1})^{o} = (a'_i + c'_{i+1})^{o} = c_i,$$

and since $\phi(a_i) \leq c'_i\ c_{i+1}$ also $\phi(a_i)'^{o} \geq c_i$, hence $\phi(a_i)'^{o} = c_i$. It follows that $\phi(a'^{o}) = \phi(a)'^{o}$, for all $a \in \mathrm{At}\, L_1$, hence ϕ is a $\underline{B}_i$-embedding.□

5.2 It follows from 5.1 that every subvariety of $\underline{M}$ is generated by its finite subdirectly irreducibles. The finite subdirectly irreducibles of $\underline{M}$ are of the form $M_{n_0,\ldots n_{k-1}}$, up to isomorphism, where $n_0,\ldots n_{k-1}$ and k are positive integers, (cf. I.7.20 and II. 5. 4). Hence every finite subdirectly irreducible algebra in $\underline{M}$ can be represented by a finite sequence of positive integers. Conversely, each finite non-empty sequence of positive integers determines a finite subdirectly irreducible algebra in $\underline{M}$, which is unique up to isomorphism. Let $\bar{M}$ denote the set of all finite non-empty sequences of positive integers. Let for $x,y \in \bar{M}$, $x \leq y$ iff $M_x \in HS(M_y)$ (cf. 3.8). It is not difficult to see that $\leq$ is a partial ordering on $\bar{M}$. We define a map Φ from the lattice of subvarieties of $\underline{M}$ to the lattice of hereditary subsets of $\bar{M}$, by putting

$$\Phi(\underline{K}) = \{x \in \bar{M} \mid M_x \in \underline{K}\}$$

for any variety $\underline{K} \subseteq \underline{M}$. Just as in the proof of 3.8 we can show that the map Φ is an isomorphism. We conclude:

5.3 Theorem. The lattice of subvarieties of $\underline{M}$ is isomorphic to the lattice of hereditary subsets of the partially ordered set $(\bar{M}, \leq)$.

The next theorem gives a more practical characterization of the relation $\leq$ on $\bar{M}$.

5.4 Theorem. Let $x,y \in \bar{M}$, $x = n_0,n_1,\ldots n_{k-1}$, $y = m_0,m_1,\ldots m_{\ell-1}$. Then $x \leq y$ iff there exist $0 = i_0 < i_1 <\ldots< i_{k-1} \leq \ell - 1$ such that $n_j \leq m_{i_j}$, $j = 0, 1,\ldots k - 1$.

Proof. (i) $\Rightarrow$ Suppose $x,y \in \bar{M}$, $x \le y$. Then by definition of $\le$, $M_x \in HS(M_y)$, hence there exists a $z \in \bar{M}$, such that $M_x \in H(M_z)$, $M_z \in S(M_y)$. But then $z = x,n_k,\dots n_{p-1} = n_0,n_1,\dots n_{p-1}$ for some $p \ge k$. Thus, if we can show that there are $0 = i_0 < i_1 < \dots < i_{p-1} \le \ell - 1$ such that $n_j \le m_{i_j}$, $j = 0,1,\dots p-1$ then it will follow a fortiori that there are $0 = i_0 < i_1 < \dots < i_{k-1} \le \ell - 1$ such that $n_j \le m_{i_j}$, $j = 0,1,\dots k-1$.

Let $M^o_{n_0,n_1,\dots n_{p-1}} = \{0 = c_0 < c_1 < \dots < c_p = 1\}$ and $M^o_{m_0,m_1,\dots m_\ell} = \{0 = d_0 < d_1 < \dots < d_\ell = 1\}$, $i : M_z \to M_y$ a $\underline{B}_i$-embedding. Since $i(M^o_z) \subseteq M^o_y$ and i is order preserving , there exist $0 = i_0 < i_1 < \dots < i_{p-1} < \ell$ such that $i(c_j) = d_{i_j}$, $j = 0, 1,\dots p-1$. Let $a^j_1,\dots a^j_{n_j}$ be the atoms in M_z satisfying $a^j_k \le c^!_j c_{j+1}$, $k = 1,\dots n_j$, $j = 0,1,\dots p-1$. The $i(a^j_k)$, $k = 1,\dots n_j$, $j = 0,1,\dots p-1$ are disjoint elements in M_y and if $n_j > 1$ then $(d_{i_j} + i(a^j_k))^o = (i(c_j + a^j_k))^o = i((c_j + a^j_k)^o) = i(c_j) = d_{i_j}$. Hence $i(a^j_k).d^!_{i_j}.d_{i_j+1} \ne 0$, for $k = 1,\dots n_j$, $j = 0,1,\dots p-1$. Therefore $d^!_{i_j}.d_{i_j+1}$ contains at least n_j atoms and it follows that $n_j \le m_{i_j}$ for $j = 0,1,\dots p-1$.

(ii) $\Leftarrow$ Let $0 = i_0 < i_1 < \dots < i_{k-1} < i_k = \ell$ be such that $n_j \le m_{i_j}$, $j = 0,1,\dots k-1$.

Let $M^o_x = \{0 = c_0 < c_1 < \dots < c_k = 1\}$ and $M^o_y = \{0 = d_0 < d_1 < \dots < d_\ell = 1\}$, as above. We define a map $i : M_x \to M_y$ as follows. Let $i(c_j) = d_{i_j}$, $j = 0, 1,\dots k$. If $a^j_1,\dots a^j_{n_j}$ are the atoms of M_x, $\le c^!_j c_{j+1}$, $b^j_1,\dots b^j_{m_{i_j}}$ the atoms of M_y, $\le d^!_{i_j} d_{i_j+1}$, then let

$$i(a^j_r) = b^j_r,\ r = 1,\dots n_j-1$$

and

$$i(a^j_{n_j}) = (d_{i_j} + \sum_{r=1}^{n_j-1} b^j_r)'.d_{i_{j+1}}, \; j = 0, 1,\ldots k-1.$$

Since $n_j \leq m_{i_j}$, this is possible and $i(a^j_r) \neq 0$, $r = 1,\ldots n_j$, and it is clear, that the $i(a^j_r)$, $r = 1,\ldots n_j$, $j = 0,1,\ldots k-1$ are disjoint and $\sum_{j,r} i(a^j_r) = 1$. Thus the map i defined by

$$i(z) = \Sigma\{i(a) \mid a \text{ is an atom in } M_x,\ a \leq z\}$$

for any $z \in M_x$ is a $\underline{B}$-embedding. In order to show that i is a $\underline{B}_i$-embedding, let $z \in M_x$, $z^o = c_j$, $0 \leq j \leq k$. The case $j = k$, i.e. $c_j = z = 1$ being trivial, let us suppose that $j < k$. Then there exists an atom $a \in M_x$, $a \leq c_j'\, c_{j+1}$ such that $a \nleq z$. By the definition of i, $i(a) \nleq i(z)$, and since $i(a).d_{i_j+1} \neq 0$, $i(z)^o \leq d_{i_j}$. On the other hand, it is obvious that that $i(z)^o \geq d_{i_j}$, implying that $i(z)^o = d_{i_j} = i(c_j) = i(z^o)$. We have now shown that $M_x \in S(M_y)$, and thus that $x \leq y$. □

In the next lemma a useful property of the partially ordered set $(\bar{M}, \leq)$ is established. The technical proof requires only a slight modification of the proof of theorem 5 in Fine [71], and will therefore be omitted.

5.5 Lemma. If $x_1, x_2,\ldots,x_n,\ldots$ is a sequence of elements of $\bar{M}$, then there exists a subsequence $x_{i_1}, x_{i_2},\ldots,x_{i_n},\ldots$ such that $x_{i_j} \leq x_{i_{j+1}}$, $j = 1, 2,\ldots$. In particular, every set of mutually incomparable elements in $\bar{M}$ is finite.

5.6 Now let $\bar{K} \subseteq \bar{M}$ be a hereditary subset, such that $\bar{M} \setminus \bar{K} \neq \emptyset$.

Let

$$A = \{x \in \bar{M} \setminus \bar{K} \mid \text{there is no } y \in \bar{M} \setminus \bar{K} \text{ such that } y \neq x \text{ and } y \leq x\}.$$

Since every element $z \in \bar{M}$ has finitely many predecessors in $\bar{M}$, for every $z \in \bar{M} \setminus \bar{K}$ there is an $a \in A$ such that $a \leq z$, and conversely, if $z \in \bar{M}$, $a \in A$ such that $a \leq z$, then $z \notin \bar{K}$, since $\bar{K}$ is hereditary. Thus

$$x \in \bar{K} \quad \text{iff} \quad \forall a \in A\, [a \nleq x].$$

It is obvious from the definition of A that A consists of mutually incomparable elements. Hence by lemma 5.5 A is finite. The following theorem is now an easy consequence:

5.7 Theorem. Every proper subvariety $\underline{K}$ of $\underline{M}$ is a finite intersection of varieties of the form $(\underline{M} : L)$, $L \in \underline{M}_{FSI}$.

Proof. Let $\underline{K} \subset \underline{M}$ be a variety, $\bar{K} = \Phi(\underline{K})$ (compare 5.2). Then $\bar{K}$ is a proper hereditary subset of $\bar{M}$. According to the remarks above, there is a finite set $A \subseteq \bar{M}$ such that $x \in \bar{K}$ iff $\forall a \in A\, [a \nleq x]$. Hence $\{M_x \mid x \in \bar{K}\} = \{M_x \mid M_a \notin HS(M_x) \text{ if } a \in A\} = \bigcap_{a \in A} (\underline{M} : M_a)_{FSI} = (\bigcap_{a\in A} (\underline{M} : M_a))_{FSI}$. Since $\underline{K} = V(\{M_x \mid x \in \bar{K}\})$, it follows that $\underline{K} = \bigcap_{a \in A} (\underline{M} : M_a)$.□

In the proof of 3.3 we exhibited for any $L \in \underline{B}_{iFSI}$ an equation $\varepsilon_L = 1$ defining the variety $(\underline{B}_i : L)$. Since $(\underline{M} : L) = (\underline{B}_i : L) \cap \underline{M}$ for any $L \in \underline{M}_{FSI}$, $(\underline{M} : L)$ is determined by the equation $\varepsilon_L = 1$ relative to $\underline{M}$.

5.8 Corollary. Every subvariety of $\underline{M}$ is determined relative to $\underline{M}$ by a single equation. More precisely, if $\underline{K} = \bigcap_{i=1}^{n} (\underline{M} : L_i)$, $L_1, \ldots L_n \in \underline{M}_{FSI}$, then relative to $\underline{M}$, $\underline{K}$ is defined by the equation $\prod_{i=1}^{n} \varepsilon_{L_i} = 1$.

By a result of Harrop [58] , it follows from the corollary and 5.1 that the equational theory of every subvariety $\underline{K}$ of $\underline{M}$ is decidable.

5.9 Corollary. (cf. Fine [71]) $\underline{M}$ has $\aleph_0$ subvarieties.

Proof. In II.5.4 we have seen that $\underline{M}$ has at least $\aleph_0$ subvarieties. By 5.7 every subvariety of $\underline{M}$ is determined by a finite set of finite subdirectly irreducibles. Hence $\underline{M}$ has at most $\aleph_0$ subvarieties.□

In order to get some more insight in the structure of the lattice of subvarieties of $\underline{M}$, we prove a lemma dealing with the successor relation in the partially ordered set $(\bar{M}, \leq)$.

5.10 Lemma. Let $x,y \in \bar{M}$, where $x = n_0, n_1, \ldots n_{k-1}$ and $y = m_0, m_1, \ldots m_{\ell-1}$. Then $x \prec y$ iff

(i) $\ell = k + 1$ and there are $0 = i_0 < i_1 \ldots < i_{k-1} \leq \ell - 1$ such that $n_j = m_{i_j}$, $j = 0,1,\ldots k-1$ and $m_i = 1$ if $i \neq i_j$, $j = 0,1,\ldots k-1$.

(ii) $\ell = k$ and there is a j_0, $0 \leq j_0 \leq k-1$ such that $n_j = m_j$ if $j \neq j_0$, $j \in \{0,1,\ldots k-1\}$ and $m_{j_0} = n_{j_0} + 1$.

Proof. $\Leftarrow$ is straightforward

$\Rightarrow$ It is obvious that $k \leq \ell \leq k + 1$.

(i) Suppose that $\ell = k + 1$. There are $0 = i_0 < i_1 \ldots < i_{k-1} \leq \ell - 1$

such that $n_j \leq m_{i_j}$. If $n_{j_0} < m_{i_{j_0}}$ for some j_0, $0 \leq j_0 \leq k-1$, however, then

$$x = n_0,n_1,\ldots n_{k-1} < m_0,m_1,\ldots m_{i_{j_0}}-1,\ldots m_{\ell-1} < m_0,m_1,\ldots m_{\ell-1} = y,$$

contradicting $x \prec y$. Hence $n_j = m_{i_j}$ $j = 0,1,\ldots k-1$. If $\{1,2,\ldots \ell-1\} \setminus \{i_1,i_2,\ldots i_{k-1}\} = \{i\}$, and $m_i > 1$, then

$$x = n_0,n_1,\ldots n_{k-1} < m_0,m_1,\ldots m_{i-1},\ 1,\ m_{i+1},\ldots m_{\ell-1}$$

$$< m_0,m_1,\ldots m_i,\ldots m_{\ell-1} = y$$

again contradicting $x \prec y$. Hence $m_i = 1$.

(ii) Suppose that $\ell = k$. There is at least one j_0, such that $n_{j_0} < m_{j_0}$. If $m_{j_0} > n_{j_0} + 1$, then

$$x = n_0,\ n_1,\ldots n_{j_0},\ldots n_{k-1} < n_0,n_1,\ldots n_{j_0}+1,\ldots n_{k-1}$$

$$< m_0,m_1,\ldots m_{j_0},\ldots m_{\ell-1} = y,$$

a contradiction. Thus $m_{j_0} = n_{j_0} + 1$. Finally, if $j_0, j_1 \in \{0,1,\ldots k-1\}$, $j_0 \neq j_1$ such that $n_{j_0} < m_{j_0}$, $n_{j_1} < m_{j_1}$ then likewise

$$x = n_0,n_1,\ldots n_{k-1} < n_0,n_1,\ldots m_{j_0},\ldots n_{j_1},\ldots n_{k-1} < m_0,m_1,\ldots m_{\ell-1} = y,$$

a contradiction. The implication thus follows.□

5.11 <u>Corollary</u>. Let $x \in \bar{M}$, $x = n_0,n_1,\ldots n_{k-1}$. If a is the number of indices among $1,2,\ldots k-1$ such that $n_i = 1$, then x has $2k - a$ covers in $(\bar{M}, \leq)$.

<u>Proof</u>. x has k different covers of length k, by 6.8. The covers of x of length $k+1$ are

$$n_0,1,n_1,\ldots n_{k-1},\quad n_0,n_1,1,n_2,\ldots n_{k-1},\quad \ldots,\quad n_0,n_1,\ldots n_{k-1},1.$$

Now

$$n_0,n_1,\ldots n_{i-1},n_i,1,\ n_{i+1},\ldots n_{k-1} = n_0,n_1,\ldots n_{i-1},1,n_i,n_{i+1},\ldots n_{k-1}$$

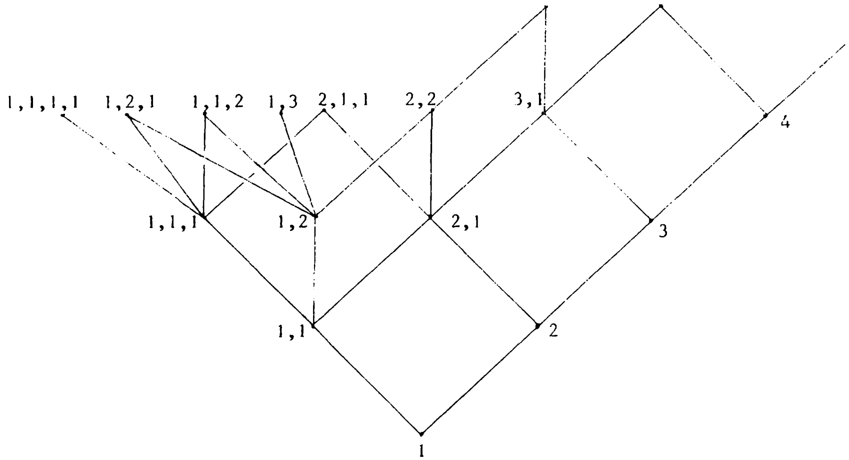
iff $n_i = 1$ for any $i \in \{1,2,\dots k-1\}$ and hence

$$n_0, n_1, \dots n_i, 1, \dots n_{k-1} = n_0, n_1, \dots n_j, 1, \dots n_{k-1}, \quad 0 \le i < j \le k-1$$

iff $n_{i+1} = \dots = n_j = 1$.

Therefore there are $k - a$ covers of length $k + 1$. The total number of covers is thus $2k - a$.□

The lower part of the poset $\bar{M}$ is suggested in the following diagram:



Evidently, this poset is not a lattice. It does have several nice properties, however. As we noted before, any $x \in \bar{M}$ has finitely many predecessors. One can show, that all maximal chains of predecessors of an element x contain the same number of elements.

Indeed, if $x = n_0, n_1, \ldots n_{k-1} \in \bar{M}$, then a maximal chain of predecessors of x contains $k - 1 + \sum_{i=0}^{k-1} (n_i - 1) = \sum_{i=0}^{k-1} n_i - 1$ elements.

Section 6. The lattice of subvarieties of $(\underline{B}_i : K_3)$

As mentioned earlier (see the remark preceding 4.2), $(\underline{B}_i : K_2)$ is the variety of monadic algebras and the lattice of subvarieties of $(\underline{B}_i : K_2)$ is the chain

$$V(M_0) \subset V(M_1) \subset \ldots \subset V(M_n) \subset \ldots \subseteq (B_i : K_2)$$

where M_n, as usual, denotes the interior algebra with n atoms and with trivial interior operator (cf. II.5.6). The next layer, $(\underline{B}_i : K_3)$, will occupy us in this section.

6.1 In 4.1 we showed that $(\underline{B}_i : K_3)$ is locally finite - hence every subvariety of $(\underline{B}_i : K_3)$ is locally finite and therefore generated by its finite members and even by its finite subdirectly irreducibles. It follows also from 4.1 that $L \in (\underline{B}_i : K_3)_{FSI}$ iff L is finite and $L^o \cong L_1 \oplus 1$ for some finite Boolean algebra L_1. Now, if $x = n_0, n_1, \ldots n_k$ is a non-empty sequence of positive integers, let $N_x \in (\underline{B}_i : K_3)_{FSI}$ be an interior algebra with $\sum_{i=0}^{k} n_i$ atoms such that N_x^o has k atoms, say $u_1,\dots u_k$ with $|\,At(u_j]\,| = n_j$, $j = 1,\dots k$, where $At(u_j]$ denotes as usual the set of atoms $\le u_j$. Note that $|\,At((\sum_{i=1}^{k} u_i)']\,| = n_0$, or, in other words, if $N_x^o = L_1 \oplus 1$, then $|\,At(1_{L_1}' .1]\,| = n_0$. Clearly, for any such sequence x, N_x is unique up to isomorphism.

6.2 <u>Lemma</u>. Let $x = n_0,n_1,\dots n_k$, $y = m_0,m_1,\dots m_\ell$, k, $\ell \ge 0$, n_i, $m_j > 0$. Then $N_x \cong N_y$ iff $k = \ell$, $n_0 = m_0$, and $n_1,\dots n_k$ is a permutation of $m_1,\dots m_\ell$.

<u>Proof</u>. Obvious.□

6.3 From now on in this section we shall consider only sequences $x = n_0,n_1,\dots n_k$, $k \ge 0$, $n_i > 0$ such that $n_1 \le n_2 \le \dots \le n_k$. The set of all such sequences will be denoted by $\bar{N}$. Then for $x,y \in \bar{N}$ $N_x \cong N_y$ iff $x = y$. Define a relation $\le$ on $\bar{N}$ by stipulating $x \le y$ iff $N_x \in HS(N_y)$. It is easy to verify that $\le$ is a partial ordering and as in 5.3 we have

6.4 <u>Theorem</u>. The lattice of subvarieties of $(\underline{B}_i : K_3)$ is isomorphic to the lattice of hereditary subsets of the partially ordered set $(\bar{N}, \le)$.

In the next few lemmas we give a more intrinsic description of the partial order on $\bar{N}$.

6.5 Lemma. Let $x,y \in \bar{N}$, $x = n_0,n_1,\dots n_k$, $y = m_0,m_1,\dots m_\ell$. Then $N_x \in H(N_y)$ iff

1) $x = y$

or

2) $k = 0$, $n_0 = m_i$ for some i, $1 \le i \le \ell$.

Proof. $\Leftarrow$ Obvious.

$\Rightarrow$ Let $h : N_y \to N_x$ be an onto homomorphism. If h is an isomorphism, then $x = y$. Hence assume that h is not $1-1$. Since N_x is subdirectly irreducible, it follows that $\ell \ge 1$ and that $N_x \cong (b]$ for some atom b of N_x^o. Therefore $k = 0$ and $n_0 = m_i$, where $m_i = |\,At(b]\,|$, for some i, $1 \le i \le \ell$.□

6.6 Lemma. Let $x,y \in \bar{N}$, $x = n_0,n_1,\dots n_k$, $y = m_0,m_1,\dots m_\ell$. Then $N_x \in S(N_y)$ iff

1) $k,\ell \ge 1$ and $i_0 = 0 < i_1 = 1 < \dots < i_k \le \ell$ such that

$n_j \le m_{i_j}$, $j = 0,1,\dots k$

or

2) $k = 0$, $\ell \ge 1$ and $n_0 \le m_i$, for all i, $1 \le i \le \ell$

or

3) $k = \ell = 0$ and $n_0 \le m_0$.

Proof. Let $a_1,\dots a_k$ be the atoms of N_x^o, $b_1,\dots b_\ell$ the atoms of N_y^o, $p_1^j,\dots p_{n_j}^j$ the atoms of $N_x, \le a_j$, $q_1^j,\dots q_{m_j}^j$ the atoms of N_y, $\le b_j$, $p_1^0,\dots p_{n_0}^0$ the remaining atoms of N_x, $q_1^0,\dots q_{m_0}^0$ the remaining atoms if N_y.

$\Leftarrow$ 1) Suppose $k \geq 1$, $\ell \geq 1$. Define $f : N_x \to N_y$ by

$$f(p_r^j) = \sum_{i_j \leq i < i_{j+1}} q_r^i , \quad r = 1,\dots n_j - 1$$

and

$$f(p_{n_j}^j) = \Big(\sum_{r=1}^{n_j-1} \sum_{i_j \leq i < i_{j+1}} q_r^i\Big)' . \sum_{i_j \leq i < i_{j+1}} b_i .$$

Further, let

$$f(p_i^0) = q_i^0, \; i = 1,\dots n_0 - 1$$

and

$$f(p_{n_0}^0) = \Big(\sum_{i=0}^{n_0-1} q_i^0\Big)' . \Big(\sum_{i=1}^{\ell} b_i\Big)' .$$

Now if $z \in N_x$, let

$$f(z) = \Sigma \{f(p) \mid p \text{ is an atom}, \; p \leq z\}.$$

Then f is a $\underline{B}_i$-embedding.

2) Suppose $k = 0$, $\ell \geq 1$. Define $f : N_x \to N_y$ by

$$f(p^0_j) = \sum_{i=1}^{\ell} q_j^i , \quad 1 \leq j \leq n_0 - 1$$

and

$$f(p_{n_0}^0) = \Big(\sum_{j=1}^{n_0-1} \sum_{i=1}^{\ell} q_j^i\Big)' .$$

Again, f induces a $\underline{B}_i$-embedding.

3) Suppose $k = \ell = 0$. Define $f : N_x \to N_y$ by $f(p_j^0) = q_j^0$, $1 \leq j \leq n_0 - 1$, $f(p_{n_0}^0) = \Big(\sum_{j=1}^{n_0-1} q_j^0\Big)'$. f induces a $\underline{B}_i$-embedding.

$\Rightarrow$ Let $f : N_x \to N_y$ be a $\underline{B}_i$-embedding.

1) Suppose that $k, \ell \geq 1$. Let $I_j = \{i \in \{1,\dots \ell\} \mid b_i \leq f(a_j)\}$, $j = 1,\dots k$. Note that $I_j \neq \emptyset$, $j = 1,\dots k$, $\bigcup_{j=1}^{k} I_j = \{1,\dots \ell\}$ and $j \neq j'$ implies $I_j \cap I_{j'} = \emptyset$. Furthermore, if $i \in I_j$, then $b_i \leq f(a_j)$, hence $n_j \leq m_i$. Indeed, $f(p_r^j). b_i \neq 0$, $r = 1,\dots n_j$, since otherwise $0 = f(0) = f((p_r^{j'} . a_j)^o) = f(p_r^{j'} . a_j)^o \geq b_i \neq 0$, and $f(p_r^j). f(p_{r'}^j) = 0$ if $r \neq r'$. Let $\ell_j = \min \{i \mid i \in I_j\}$.

Then $n_j \leq m_{\ell_j}$. Now, if $\{r_j \mid j = 1,\dots k\} \subseteq \{1,\dots \ell\}$, with $j \neq j'$ implying $r_j \neq r_{j'}$, such that $n_j \leq m_{r_j}$, $j = 1,\dots k$, then if $j < j'$, but $r_j > r_{j'}$, still $n_j \leq m_{r_{j'}}$ and $n_{j'} \leq m_{r_j}$. For $n_j \leq n_{j'} \leq m_{r_{j'}}$ and $n_{j'} \leq m_{r_{j'}} \leq m_{r_j}$. Therefore we can re-arrange the ℓ_j, $j = 1, ,\dots k$, to obtain a sequence i_j, $j = 1,\dots k$ satisfying $n_j \leq m_{i_j}$, $j = 1,\dots k$ and $i_1 < i_2 < \dots < i_k \leq \ell$. There is a $j \in \{1,\dots k\}$ such that $1 \in I_{j_\ell}$ and if $1 \in I_j$ then $\ell_j = 1$. Hence $i_1 = 1$. Finally, $f(\sum_{j=1}^{k} a_j) = \sum_{j=1}^{\ell} b_j$, hence $f((\sum_{j=1}^{k} a_j)') = (\sum_{j=1}^{\ell} b_j)'$. Since $n_0 = |\,At\,((\sum_{j=1}^{k} a_j)']\,|$, $m_0 = |\,At((\sum_{j=1}^{\ell} b_j)']\,|$, it follows that $n_0 \leq m_0$. Thus the sequence $i_0 = 0 < i_1 = 1 < i_2 < \dots < i_k \leq \ell$ satisfies the requirements.

2) Suppose $k = 0$, $\ell \geq 1$. If $f(p_r^0).\, b_i = 0$, for some r, i, $1 \leq r \leq n_0$, $1 \leq i \leq \ell$ then $0 = f(0) = f((p_r^{0'})^0) = f(p_r^0)'^0 \geq b_i \neq 0$, a contradiction. Since $r \neq r'$ implies $f(p_r^0).\, f(p_{r'}^0) = 0$ and for each $i \in \{1,\dots \ell\}$, $b_i \leq \sum_{r=1}^{n_0} f(p_r^0)$ it follows that $n_0 \leq m_i$, $i = 1,\dots \ell$.

3) Suppose $k,\ell = 0$. Then obviously $n_0 \leq m_0$. □

6.7 <u>Lemma</u>. Let $x,y \in \bar{N}$, $x = n_0,n_1,\dots n_k$, $y = m_0,m_1,\dots m_\ell$. Then $x \leq y$ iff

1) $k,\ell \geq 1$ and there are $i_0 = 0 < i_1 = 1 < i_2 < \dots < i_k \leq \ell$ such that $n_j \leq m_{i_j}$, $j = 0,1,\dots k$.

or 2) $k = 0$, $\ell \geq 1$ and $n_0 \leq m_i$, for some i, $1 \leq i \leq \ell$.

or 3) $k = 0$, $\ell = 0$ and $n_0 \leq m_0$.

<u>Proof</u>. By the definition of $\leq$ and lemmas 6.5, 6.6.□

An important feature of the partially ordered set $(\bar{M}, \leq)$ is that every set of incomparable elements is finite, as was shown in lemma 5.5. $(\bar{N}, \leq)$ shares this property with $(\bar{M}, \leq)$ and that this is so can be proven in the same way, except for some minor points.

6.8 Lemma. If $x_1, x_2, \ldots$ is a sequence of elements of $\bar{N}$ then there exists a subsequence $x_{i_1}, x_{i_2}, \ldots$ such that $x_{i_j} \leq x_{i_{j+1}}$, $j = 1,2,\ldots$. In particular, every set of imcomparable elements of $\bar{N}$ is finite.

Proof. Let $x_i = n_0^i, n_1^i, \ldots n_{k_i}^i$ $i = 1,2,\ldots$. If $k_i = 0$ for infinitely many i, then this subsequence has a subsequence satisfying the requirement. Hence we may assume that $k_i > 0$, $i = 1,2,\ldots$. By thinning we may assume that $n_0^i \leq n_0^{i+1}$, $i = 1,2,\ldots$. Let $y_i = n_1^i, n_2^i, \ldots n_{k_i}^i$. By considering the sequence y_i, $i = 1,2,\ldots$ as a sequence in $(\bar{M}, \leq)$ and by applying lemma 5.5 we can find a subsequence y_{i_j}, $j = 1,2,\ldots$ with $y_{i_j} \leq y_{i_{j+1}}$, $j = 1,2,\ldots$ in $(\bar{M}, \leq)$. Then x_{i_j}, $j = 1,2,\ldots$ (where $x_{i_j} = n_0^{i_j}, y_{i_j}$) is a subsequence of x_i, $i = 1,2,\ldots$ satisfying $x_{i_j} \leq x_{i_{j+1}}$ in $(\bar{N}, \leq)$.□

We are now in a position to prove

6.9 Theorem. Let $\underline{K} \subset (\underline{B}_i : K_3)$ be a variety. Then

$$\underline{K} = \bigcap_{L \in A} (\underline{B}_i : K_3) : L)$$

for some finite set $A \subseteq (\underline{B}_i : K_3)_{FSI}$.

Proof. Let $\Phi(\underline{K}) = \{x \in \bar{N} \mid N_x \in \underline{K}_{FSI}\}$. Then $\Phi(\underline{K})$ is a proper hereditary subset of $\bar{N}$ and if $\bar{A} = \{x \in \bar{N} \setminus \Phi(\underline{K}) \mid$ there is no $y \in \bar{N} \setminus \Phi(\underline{K})$ such that $y \neq x$ and $y \leq x\}$ then $\bar{A}$ is finite (and non-empty) since it consists of non-comparable elements and $\{N_x \mid N_x \in \underline{K}_{FSI}\} = \bigcap_{a \in \bar{A}} ((\underline{B}_i : K_3) : N_a)_{FSI}$. Hence, if $A = \{N_x \mid x \in \bar{A}\}$, then

$$\underline{K} = V(\{N_x \mid N_x \in \underline{K}_{FSI}\}) = \bigcap_{L \in A} ((\underline{B}_i : K_3) : L). \square$$

6.10 Corollary. Every subvariety of $(\underline{B}_i : K_3)$ is determined by a single equation. More precisely, if $\underline{K} = \bigcap_{i=1}^{n} ((\underline{B}_i : K_3) : L_i)$, $L_1, \ldots L_n \in (\underline{B}_i : K_3)_{FSI}$, then relative to $(\underline{B}_i : K_3)$ $\underline{K}$ is determined by the equation $\prod_{i=1}^{n} \varepsilon_{L_i} = 1$.

6.11 Corollary. $(\underline{B}_i : K_3)$ has $\aleph_0$ subvarieties.

Proof. Since $(\underline{B}_i : K_2) \subseteq (\underline{B}_i : K_3)$ and $(\underline{B}_i : K_2)$ has the $\aleph_0$ subvarieties $V(M_n)$, $n = 1,2,\ldots$ it follows that $|((\underline{B}_i : K_3)]| \geq \aleph_0$. Using 6.9 we conclude that $|((\underline{B}_i : K_3)]| = \aleph_0$. $\square$

We close this section with some remarks concerning the successor relation in $(\bar{N}, \leq)$.

6.12 Lemma. Let $x,y \in \bar{N}$, $x = n_0,n_1,\ldots n_k$, $y = m_0,m_1,\ldots m_\ell$. Then $x \prec y$ iff one of the following is true:

1) $k = 0$, $\ell = 1$: $n_0 = m_1$, $m_0 = 1$.
2) $k > 0$, $\ell = k + 1$: if $r = \max\{i \mid 1 \leq i \leq k,\ n_i = n_1\}$ then $m_0 = n_0$, $m_1 = m_2 = \ldots = m_{r+1} = n_1$, $m_{j+1} = n_j$ for $r < j \leq k$.

3) $\ell = k$ and a) $n_0 + 1 = m_0$, $n_i = m_i$, $i = 1,2,\dots k$

b) $n_i < n_{i+1}$ for some i, $1 \le i < k$ and $n_j = m_j$, $0 \le j < i$, $n_i + 1 = m_i$, $n_j = m_j$, $i < j \le k$

c) $n_j = m_j$, $0 \le j < k$, $n_k + 1 = m_k$.

Proof. $\Leftarrow$ straightforward.

$\Rightarrow$ It is obvious that $\ell = k + 1$ or $\ell = k$.

1) $\ell = k + 1$. If $k = 0$ then one easily sees that $n_0 = m_1$, $m_0 = 1$. Suppose $k > 0$. Since $x \le y$, there is a sequence $i_0 = 0 < i_1 = 1 < i_2 < \dots < i_k \le \ell$ such that $n_j \le m_{i_j}$, $j = 0,1,\dots k$. Suppose $n_{j_0} < m_{i_{j_0}}$ for some j_0, $0 \le j_0 \le k$. We may assume $n_{j_0+1} > n_{j_0}$ if $0 < j_0 < k$. Then

$$n_0,n_1,\dots n_k < n_0,n_1,\dots n_{j_0-1},n_{j_0}+1,n_{j_0+1},\dots,n_k < m_0,m_1,\dots m_\ell$$

a contradiction. Hence $n_j = m_{i_j}$, $j = 0,1,\dots k$. Let $r = \max\{i \mid 1 \le i \le k, n_i = n_1\}$ and suppose $m_{r+1} \ne n_1$. Then $m_{r+1} > n_1$, hence

$$n_0,n_1,\dots n_k < n_0,n_1,\dots n_r,n_r,n_{r+1},\dots n_k < m_0,m_1,\dots m_r, m_{r+1},m_{r+2},\dots m_\ell.$$

a contradiction. Hence $m_{r+1} = n_1$ and $m_1 = m_2 = \dots = m_{r+1} = n_1$.

2) $\ell = k$. Since $x \le y$, $n_i \le m_i$ for $i = 0,1,\dots k$. Obviously $n_i < m_i$ for at most on i, and then $m_i = n_i+1$. The only i's for which this can occur are $i = 0$, the $i \in \{1,2,\dots \ell - 1\}$ such that $n_{i+1} > n_i$ and $i = k$.□

6.13 Corollary. Let $x \in \bar{N}$, $x = n_0,n_1,\dots n_k$. If a is the number of indices i among $\{1,2,\dots k - 1\}$ such that $n_{i+1} > n_i$, then x has 2 covers if $k = 0$ and x has $3 + a$ covers if $k > 0$.

Proof. If $k = 0$, then x has two covers by 6.12 1) and 3a). If $k > 0$ then x has one cover of the form given in 6.12 2), and $2 + a$ covers as given in 3). Hence, if $k > 0$ then x has $3 + a$ covers. □

Section 7. The relation between the lattices of subvarieties of $\underline{B}_i$ and $\underline{H}$

So far our study of the lattice of varieties of interior algebras has been rather limited, in the sense that we restricted ourselves mainly to varieties generated by their finite members or even to locally finite varieties. Next we want to turn our attention to problems of a more general nature, concerning the structure of the lattice Ω as a whole, and certain interesting subsets of Ω .

In II.1 we established some relations between varieties of interior algebras and varieties of Heyting algebras. The first object of this section is to formulate some corollaries to these results for the lattice Ω of subvarieties of $\underline{B}_i$ and the lattice Σ of subvarieties of $\underline{H}$. We will thus obtain a better insight in the structure of Ω and moreover we shall be able to carry over known results on Σ to Ω directly. The corresponding results for the lattice Ω^- of subvarieties of $\underline{B}_i^-$ and for the lattice Σ^- of subvarieties of $\underline{H}^-$ will not be mentioned explicitly; they follow easily. We want to recall however the 1-1 order-preserving correspondence between non-trivial subvarieties of the variety $\underline{S} \subseteq \underline{B}_i$, where $\underline{S}$ is determined by the equations $x^{oc'} + x^{oco} = 1$ and $x^{oc} + x^{'oc} = 1$, and the subvarieties of $\underline{B}_i^-$, established in II.1.20. It shows that

7.1 Theorem. Ω^- is isomorphic to the sublattice $[V(\underline{2}), \underline{S}]$ of Ω, where $V(\underline{2})$ is the variety of discrete interior algebras, determined by the equation $x = x^{o}$ (cf. II.4.2), and $\underline{S}$ is as above.

Proof. Recall that $\underline{M}_2^{\ *} = (\underline{B}_i : K_2)^* = V(\underline{2})$. If $\underline{K} \subseteq \underline{B}_i$ is a nontrivial variety then $\underline{2} \in \underline{K}$, hence $V(\underline{2}) \subseteq \underline{K}$ (that is, $V(\underline{2})$ is the unique atom of Ω, contained in every element of Ω and hence the only equationally complete subvariety of $\underline{B}_i$).The theorem follows now immediately from II.1.20. □

7.2 In order to establish the desired relations between Ω and Σ let us define mappings γ and ρ as follows:

$$\gamma : \Omega \to \Sigma \text{ by } \gamma(\underline{K}) = \underline{K}^{o} \text{ for } \underline{K} \in \Omega$$

and

$$\rho : \Sigma \to \Omega \text{ by } \rho(\underline{K}) = \underline{K}^{c} \text{ for } \underline{K} \in \Sigma.$$

By II.1.3 and II.1.9 γ as well as ρ are well-defined. By Jónsson's results we know that Σ and Ω are complete distributive lattices (cf. section 1).

7.3 Theorem. γ is a complete surjective $\underline{D}_{01}$-homomorphism.

Proof. Since 0_Ω is the variety of trivial interior algebras, $\gamma(0_\Omega) = 0_\Sigma$, the variety of trivial Heyting algebras. Also $\gamma(1_\Omega) = \gamma(\underline{B}_i) = \underline{H} = 1_\Sigma$.

If $\{\underline{K}_i \mid i \in I\} \subseteq \Omega$ then

$$\gamma(\textstyle\sum_{i\in I}\underline{K}_i) = \gamma(V(\bigcup_{i\in I}\underline{K}_i)) = V(\bigcup_{i\in I}\underline{K}_i)^{o} \overset{(*)}{=} V(\bigcup_{i\in I}\underline{K}_i^{o}) = \sum_{i\in I}\rho(\underline{K}_i)$$

where the equality $*$ follows from II.1.2. Further γ also preserves arbitrary meet:

$$\gamma(\textstyle\prod_{i\in I}\underline{K}_i) = \gamma(\bigcap_{i\in I}\underline{K}_i) = (\bigcap_{i\in I}\underline{K}_i)^{o} = \bigcap_{i\in I}\underline{K}_i^{o} = \prod_{i\in I}\gamma(\underline{K}_i).$$

Finally, γ is onto since for any $\underline{K} \in \Sigma$, $\gamma(\underline{K}^{c}) = \underline{K}^{co} = \underline{K}$.□

7.4 Theorem. ρ is a $\underline{D}_{01}$-embedding. Furthermore, $\gamma \circ \rho = \mathrm{id} \mid \Sigma$, hence Σ is a retract of Ω.

Proof. Obviously, $\rho(0_\Sigma) = 0_\Omega$, $\rho(1_\Sigma) = 1_\Omega$. Let $\underline{K}_1, \underline{K}_2 \in \Sigma$. Then

$$\rho(\underline{K}_1 + \underline{K}_2)_{SI} = ((\underline{K}_1 + \underline{K}_2)^c)_{SI} = ((\underline{K}_1 + \underline{K}_2)_{SI})^c$$

since an interior algebra L is subdirectly irreducible iff L^o is a subdirectly irreducible Heyting algebra. By 1.4

$$((\underline{K}_1 + \underline{K}_2)_{SI})^c = (\underline{K}_{1SI} \cup \underline{K}_{2SI})^c = \underline{K}^c_{1SI} \cup \underline{K}^c_{2SI} = \rho(\underline{K}_1)_{SI} \cup \rho(\underline{K}_2)_{SI}.$$

Therefore

$$\rho(\underline{K}_1 + \underline{K}_2) = V(\rho(\underline{K}_1 + \underline{K}_2)_{SI}) =$$
$$= V(\rho(\underline{K}_1)_{SI} \cup \rho(\underline{K}_2)_{SI}) = \rho(\underline{K}_1) + \rho(\underline{K}_2).$$

Also

$$\rho(\underline{K}_1 \cdot \underline{K}_2) = \rho(\underline{K}_1 \cap \underline{K}_2) = (\underline{K}_1 \cap \underline{K}_2)^c$$
$$= \underline{K}_1^c \cap \underline{K}_2^c = \rho(\underline{K}_1) \cap \rho(\underline{K}_2) = \rho(\underline{K}_1) . \rho(\underline{K}_2).$$

To prove the second statement of the theorem we note that for $\underline{K} \in \Sigma$, $\gamma \circ \rho(\underline{K}) = \gamma(\underline{K}^c) = \underline{K}^{co} = \underline{K}$. □

Note that ρ actually preserves arbitrary meet. We do not know at the present time if ρ preserves also arbitrary join.

The map ρ assigns to a variety $\underline{K}$ of Heyting algebras the largest variety $\underline{K}' \subseteq \underline{B}_i$ such that $\underline{K}'^o = \underline{K}$. The smallest variety with this property is the variety $V(\{B(L) \mid L \in \underline{K}\}) \subseteq \underline{B}_i$. Therefore:

7.5 Consider the map

$$\rho^* : \Sigma \longrightarrow (\underline{B}_i^*] \subseteq \Omega$$

defined by

$$\rho^*(\underline{K}) = V(\{B(L) \mid L \in \underline{K}\}) = \rho(\underline{K})^* \quad \text{for} \quad \underline{K} \in \Sigma.$$

It is obvious that ρ^* is 1-1 and that ρ^* preserves 0,1 and (arbitrary) sums. However, the question if ρ^* is onto amounts to the following problem: if $\underline{K}$ is a subvariety of $\underline{B}_i^*$ does it follow that $\underline{K}$ is generated by its $*$- algebras (as is the case for $\underline{B}_i^*$ itself, by definition)? This is reminiscent of the question whether every subvariety of $\underline{B}_i$ is generated by its finite members, $\underline{B}_i$ itself so being generated. This last question will be considered in the next section and will get a negative answer. The more surprising it is that the problem we are dealing with now will be solved in a positive manner. The next lemma is the key result.

7.6 Lemma. Let L be a countable interior algebra satisfying the equation $((x' + x^{o})^{o'} + x^{o})^{o'} + x^{o} = 1$. Suppose that L_1 is a subalgebra of L such that $L^{o} \subseteq L_1 \subseteq L = [L_1 \cup \{x\}]_{\underline{B}}$ for some $x \in L$. Then $L \in SP_U(L_1)$.

Proof. Enumerate the elements of $L_1 : a_1, a_2, \ldots$. Let

$$i \ : L_1 \to \prod_{i=1}^{\infty} L_1$$

be defined by

$$i(b) = \bar{b} = (b,b,\ldots),$$

for $b \in L_1$. Let $\bar{x} \in \prod_{i=1}^{\infty} L_1$ be the element

$$\bar{x} = (\Sigma\, \{a_i \mid a_i \leq x,\ i \leq n\})_{n=1}^{\infty}.$$

Let F be an arbitrary non-principal ultrafilter on $\mathbb{N}$ and

$$h : \prod_{i=1}^{\infty} L_1 \to \prod_{i=1}^{\infty} L_1 / F = L_2$$

be the canonical homomorphism. Finally define $f : L_1 \to L_2$ by $f = h \circ i$. The map f is then a $\underline{B}_i$-homomorphism which is 1-1.

$$\begin{tikzcd} L_1 \arrow[r, hook] \arrow[d, "i"'] \arrow[dr, "f"] & L \arrow[d, dashed, "\bar{f}"] \\ \prod_{i=1}^{\infty} L_1 \arrow[r, "h"] & L_2 \end{tikzcd}$$

We claim that $f : L_1 \to L_2$ can be extended to a $\underline{B}$-homomorphism $\bar{f} : L \to L_2$ by defining $\bar{f}(x) = h(\bar{x})$. In virtue of a well-known lemma (see Grätzer [71], pg. 84) it suffices to prove for $a \in L_1$

1) if $a \leq x$ then $f(a) \leq h(\bar{x})$

2) if $a \geq x$ then $f(a) \geq h(\bar{x})$.

Suppose that $a = a_k$, for some $k \in \mathbb{N}$.

1) If $a \leq x$, then for $n \geq k$ $\bar{x}_n = \Sigma\{a_i \mid a_i \leq x,\ i \leq n\} \geq a_k = a$ Since F is a non-principal ultrafilter on $\mathbb{N}$, it follows that $\{n \in \mathbb{N} \mid \bar{x}_n \geq a\} \in F$. Hence $h(\bar{x}) \geq h(\bar{a}) = h \circ i\ (a) = f(a)$.

2) If $a \geq x$ then $\bar{x}_n \leq a$ for each $n \in \mathbb{N}$, hence $h(\bar{x}) \leq h(\bar{a}) = f(a)$.

Thus $\bar{f} : L \to L_2$ is a Boolean extension of f and

$$\bar{f}[L] = [f[L_1] \cup \{h(\bar{x})\}]_{\underline{B}} = L_3 \subseteq L_2.$$

Now we shall show that for any $z \in L$ $(\bar{f}(z))^{\circ} = \bar{f}(z^{\circ})$. This will imply at the same time that L_3 is a $\underline{B}_i$-subalgebra of L_2

and that $\bar{f} : L \to L_3$ is a $\underline{B}_i$-homomorphism. Since $\bar{f} \mid L^o = f \mid L_1^o$ is 1-1, it follows that $\bar{f}$ is a $\underline{B}_i$-embedding and hence that $L \in S(L_2) \subseteq SP_U(L_1)$.

If $z \in L$, then $z = (x + y_1) \,.\, (x' + y_2)$, where y_1, $y_2 \in L_1$. It suffices to show for $y \in L_1$ that $\bar{f}((x + y)^o) = (\bar{f}(x + y))^o$ and that $\bar{f}((x' + y)^o) = (\bar{f}(x' + y))^o$, since then

$$\bar{f}(z^o) = \bar{f}((x + y_1)^o.\ (x' + y_2)^o) = \bar{f}((x + y_1)^o) \,.$$
$$\bar{f}((x' + y_2)^o) = (\bar{f}(x + y_1))^o.\ (\bar{f}(x' + y_2))^o = (\bar{f}(z))^o.$$

1) Note that $(x + y)^o = ((x + y)^o\ y' + y)^o$ since $(x + y)^o \leq ((x + y)^o\ y' + y)^o \leq (x + y)^o$ because $(x + y)^o y' \leq x$. Since $L^o \subseteq L_1$ $(x + y)^o\ y' \in L_1$, say $(x + y)^o.\ y' = a_k$, for some $k \in \mathbb{N}$. Then for $n \geq k$

$$(x + y)^o = ((x + y)^o y' + y)^o \leq (\bar{x}_n + \bar{y}_n)^o = (\bar{x} + \bar{y})_n^o \leq (x + y)^o.$$

Therefore $h(\overline{(x + y)^o}) = h((\bar{x} + \bar{y})^o) = (h(\bar{x} + \bar{y}))^o$, and hence $\bar{f}((x + y)^o) = (\bar{f}(x + y))^o$.

2) Let $v = (x' + y)^o$, and $u = (xy' + v)^o = ((x' + y)' + (x' + y)^o)^o$. Then $v'u \leq xy'$ and

$$(u' + v)^o = (((x' + y)' + (x' + y)^o)^{o\prime} + (x' + y)^o)^o =$$
$$= (x' + y)^o$$

since L satisfies the equation $((x' + x^o)^{o\prime} + x^o)^{o\prime} + x^o = 1$. Since $v'u \in B(L^o) \subseteq L_1$, there is a $k \in \mathbb{N}$ such that $v'u = a_k$. Then for $n \geq k$ $a_k \leq \bar{x}_n\ \bar{y}_n'$. Hence for $n \geq k$ $a_k' \geq \bar{x}_n' + \bar{y}_n$ and

$$(x' + y)^o = (u' + v)^o = a_k'^o \geq (\bar{x}_n' + \bar{y}_n)^o \geq (x' + y)^o.$$

Thus $(h(\bar{x}' + \bar{y}))^o = h((\bar{x}' + \bar{y})^o) = h(\overline{(x' + y)^o})$ and $\bar{f}((x' + y)^o) = (\bar{f}(x' + y))^o$.

This completes the proof of the fact that $\bar{f}$ is a $\underline{B}_i$-homomorphism.□

7.7 Theorem. Let $\underline{K} \subseteq (\underline{B}_i : M_2) \cap (\underline{B}_i : M_{1,2})$ be a variety. Then $\underline{K} = \underline{K}^*$.

Proof. Let $x_1, x_2, \ldots$ be free generators of $F_{\underline{K}}(\aleph_0)$ and let $L_n = [B(F_{\underline{K}}(\aleph_0)^o) \cup \{x_1, \ldots x_n\}]_{\underline{B}}$, $n = 0,1,2,\ldots$. Then $\underline{K}$ is generated by the L_n, $n = 1,2\ldots$ since $F_{\underline{K}}(n) \in S(L_n)$, $n = 1,2,\ldots$ and $\underline{K}$ is generated by the $F_{\underline{K}}(n)$, $n = 1,2,\ldots$ (see 0.1.7). By induction we show that $L_n \in \underline{K}^*$, for $n = 0,1,2,\ldots$

1) $L_0 = B(F_{\underline{K}}(\aleph_o)^o) \in \underline{K}^*$, by definition of $\underline{K}^*$.

2) Suppose that $L_n \in \underline{K}^*$. In 3.9 3) it was established that the variety $(\underline{B}_i : M_2) \cap (\underline{B}_i : M_{1,2})$ is defined by the equation $((x' + x^o)^{o'} + x^o)^{o'} + x^o = 1$, hence L_{n+1} satisfies this equation. Furthermore, $L_{n+1} = [L_n \cup \{x_{n+1}\}]_{\underline{B}}$, $L^o_{n+1} = F_{\underline{K}}(\aleph_o)^o \subseteq L_n$ and $|L_{n+1}| = \aleph_o$. By lemma 7.6 it follows that $L_{n+1} \in SP_U(L_n) \subseteq \underline{K}^*$.□

7.8 Corollary. $\underline{B}_i^* = (\underline{B}_i : M_2) \cap (\underline{B}_i : M_{1,2})$.

Proof. By 3.9 we know that $\underline{B}_i^* \subseteq (\underline{B}_i : M_2) \cap (\underline{B}_i : M_{1,2})$. According to 7.7 $(\underline{B}_i : M_2) \cap (\underline{B}_i : M_{1,2}) = ((\underline{B}_i : M_2) \cap (\underline{B}_i : M_{1,2}))^* \subseteq \underline{B}_i^*$.□

This corollary does not only provide a nice geometric characterization of $\underline{B}_i^*$: it enables us to derive an equation defining $\underline{B}_i^*$, too.

7.9 Corollary. The variety $\underline{B}_i^*$ is characterized by the equation $((x' + x^o)^{o'} + x^o)^{o'} + x^o = 1$.

Of course, the other equations mentioned in 3.9 3) will do equally well. Observe that it follows from 7.8 and I.6.9 that $(\underline{B}_i : M_2) \cap (\underline{B}_i : M_{1,2})$ is generated by its finite members (cf. 3.9.4)).

Now we can also say more about the map ρ^* introduced in 7.5:

7.10 Theorem. The map $\rho^* : \Sigma \to (\underline{B}_i^*]$ is a $\underline{D}_{01}$-isomorphism.

Proof. It is obvious that ρ^* is 1-1, and by 7.7 ρ^* is onto. If $\underline{K}_1, \underline{K}_2 \in \Sigma$, then $\rho^*(\underline{K}_1 + \underline{K}_2) = V(\{B(L) \mid L \in \underline{K}_1 + \underline{K}_2\}) = \rho^*(\underline{K}_1) + \rho^*(\underline{K}_2)$, and $\rho^*(\underline{K}_1 . \underline{K}_2) = \rho^*(\underline{K}_1).\rho^*(\underline{K}_2)$ since $\rho^*(\underline{K}_1).\rho^*(\underline{K}_2) \subseteq \underline{B}_i^*$ hence by 7.7 $\rho^*(\underline{K}_1).\rho^*(\underline{K}_2) = V(\{B(L^o) \mid L \in \rho^*(\underline{K}_1).\rho^*(\underline{K}_2)\}) = V(\{B(L) \mid L \in \underline{K}_1 . \underline{K}_2\}) = \rho^*(\underline{K}_1.\underline{K}_2)$.□

The assignment $\underline{K} \mapsto \underline{K}^*$ for $\underline{K} \in \Omega$ proves to be a very nice one. Indeed, $\underline{K}^* = \underline{K} . \underline{B}_i^*$ and we have

7.11 Corollary. The map $^* : \Omega \to (\underline{B}_i^*]$ defined by $\underline{K} \mapsto \underline{K}^*$ is a complete surjective $\underline{D}_{01}$-homomorphism.

Proof. If $\underline{K} \in \Omega$, then $\underline{K}^* = \rho^* \circ \gamma(\underline{K})$. The corollary then follows from 7.10 and 7.3.□

It follows that Ω is a disjoint union of intervals $[\underline{K}, \underline{K}^{oc}]$, $\underline{K} \in (\underline{B}_i^*]$, the interval $[\underline{K}, \underline{K}^{oc}]$ being the preimage of $\underline{K}$ under the mapping *. In the study of Ω two important aspects can be distinguished: on the one hand the lattice $(\underline{B}_i^*]$ which is just Σ, on the other hand the lattices $[\underline{K}, \underline{K}^{oc}]$, $\underline{K} \in (\underline{B}_i^*]$, consisting of varieties which do not differ in the lattices of open elements of their algebras. This description gives us the opportunity to separate to a certain degree the "Heyting-aspect" from the "trivial" aspect.

The first part of 7.12 merely repeats in the language of equations what essentially was stated in 7.7: that every subvariety of $(\underline{B}_i : M_2) \cap (\underline{B}_i : M_{1,2})$ is completely determined by the Heyting-algebras of open elements of the interior algebras contained in it. For the notation in 7.12, see II.1.11.

7.12 <u>Theorem</u>. Let $\underline{K} \subseteq \underline{B}_i^*$ be a variety. Suppose that $\underline{K}^o$ is determined by a set Σ of $\underline{H}$-equations.

(i) $\underline{K}$ is determined by $T(\Sigma)$ together with the equation $(((x' + x^o)^{o'} + x^o)^{o'} + x^o = 1$

(ii) $\underline{K}^o$ is finitely based iff $\underline{K}$ is finitely based iff $\underline{K}^{oc}$ is finitely based.

<u>Proof</u>. (i) Since $\underline{K} = \underline{K}^* = \underline{K}^{oc*} = \underline{K}^{oc} . \underline{B}_i^*$ the assertion follows from II.1.12 and 7.9.

(ii) By II.1.12, (i), and the compactness theorem. □

The next theorem will be useful later.

7.13 <u>Theorem</u>. Let $\underline{K} \subseteq \underline{B}_i$ be a variety.

(i) $\underline{K}^o$ is generated by its finite members iff $\underline{K}^*$ is generated by its finite members.

(ii) If $\underline{K}^{oc}$ is generated by its finite members, then so is $\underline{K}^o$.

Proof. (i) Note that $\underline{K}^{o}{}_{F} = (\underline{K}^{*}{}_{F})^{o}$. If $\underline{K}^{o} = V(\underline{K}^{o}{}_{F})$ then $\underline{K}^{o} = V((\underline{K}^{*}{}_{F})^{o}) = V(K^{*}{}_{F})^{o}$ by II.1.2, therefore $\underline{K}^{*} = V(\{B(L) \mid L \in \underline{K}^{o}\}) \subseteq V(\underline{K}^{*}_{F}) \subseteq \underline{K}^{*}$.

Conversely, if $\underline{K}^{*} = V(\underline{K}^{*}{}_{F})$ then $\underline{K}^{o} = \underline{K}^{*o} = V(\underline{K}^{*}{}_{F})^{o} = V((\underline{K}^{*}{}_{F})^{o}) = V(\underline{K}^{o}{}_{F})$, again by II.1.2.

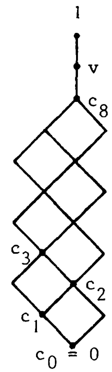
(ii) Follows from II.1.2.□

We do not know if the converse of (ii) holds as well. However, if we require $\underline{K}^{o}$ to be locally finite then it is not difficult to show that $\underline{K}^{oc}$ is generated by its finite members.

Section 8. On the cardinality of some sublattices of Ω

The purpose of this section is to determine the cardinality of certain sublattices of Ω. Since any subvariety of $\underline{B}_i$ is determined by a subset of the (countable) set of all $\underline{B}_i$-equations there are at most $2^{\aleph_0}$ subvarieties of $\underline{B}_i$. On the other hand, it follows from 7.4 that $|\,\Omega\,| \geq |\,\Sigma\,|$. In Jankov [68] it was proved that $\underline{H}$ has $2^{\aleph_0}$ subvarieties, thus $|\,\Omega\,| = |\,\Sigma\,| = 2^{\aleph_0}$. As a matter of fact, even $|\,(B_i^*]\,| = |\,\Sigma\,| = 2^{\aleph_0}$. We start with a simple example of a collection of continuously many subvarieties of $\underline{H}$ (taken from Blok [M]) and adapt it in order to obtain a collection of continuously many subvarieties of $\underline{H}^-$ (8.7), thus providing a proof of the fact that also the lattice Ω^- of subvarieties of $\underline{B}_i^-$ has the cardinality of the continuum. As a by-product we obtain examples of subvarieties of $\underline{B}_i$ and $\underline{B}_i^-$ which are not finitely based. After some remarks on the cardinality of the classes $(\underline{B}_i : K_n)^*$ and $(\underline{B}_i : K_n)$, $n = 1,2,\ldots$ we turn our attention to the cardinality of the intervals $[\underline{K}, \underline{K}^{oc}]$, $\underline{K} \in (\underline{B}_i^*]$ (8.13 - 8.17).

8.1 Let $G_n = (c_n] \dotplus \underline{3}$, $n = 1,2,\ldots,$ where $(c_n]$ is a principal ideal of $F_{\underline{H}}(1)$ and $\underline{3} = \{0 < v < 1\}$ (for notation see I.3). Hence $G_0 \cong \underline{3}$, $G_1 \cong \underline{4}$, $G_2 \cong \underline{2}^2 \dotplus \underline{3}$ and G_8 is suggested by the diagram :



Note that if $n \geq 3$, then c_1 satisfies $c_1 \to 0 \neq 0$ and $(c_1 \to 0) \to 0 \neq c_1$ in G_n, and also in $F_{\underline{H}}(1)$. We shall use the notation Gen (x), where

$$\text{Gen } (x) \quad \text{iff} \quad x \to 0 \neq 0 \quad \text{and} \quad (x \to 0) \to 0 \neq x.$$

Then in G_n, $n \geq 3$, and in $F_{\underline{H}}(1)$, Gen(c_1) and in fact, c_1 is the only element x in G_n, $n \geq 3$, and in $F_{\underline{H}}(1)$ such that Gen (x).

8.2 <u>Lemma</u>. If $n, m \in \mathbb{N}$, $n,m \geq 3$, $n \neq m$, then $G_n \notin SH(G_m)$.

<u>Proof</u>. Let $n, m \in \mathbb{N}$, $n, m \geq 3$, $n \neq m$ and let $L_1 \in H(G_m)$ and $i : G_n \to L_1$ be an $\underline{H}$-embedding. $L_1 \cong (c_k]$, $L_1 \cong (c_k] \oplus 1$ or $L_1 = G_m$ and we may assume that $3 \leq k \leq m$. Since c_1 is the only element x of L_1 satisfying Gen(x), it follows that $i(c_1) = c_1$. Let p_k be a unary $\underline{H}$-polynomial, $k = 0, 1, 2, \ldots$ with (i) $p_0(x) = 0$, $p_1(x) = x$

(ii) $p_{k+1}(x) = p_k(x) + (p_k(x) \to p_{k-1}(x))$, for $k \geq 1$

Since $c_n = p_n(c_1) = p_{n+1}(c_1) < p_{n+2}(c_1) = 1$ in G_n, it follows that $p_n(c_1) = p_{n+1}(c_1) < p_{n+2}(c_1) = 1$ in L_1. Because $n \neq m$, this can only be true if $L_1 \cong (c_n] \oplus 1$. But this is impossible since then $i(v) = c_n$ of $i(v) = 1$, contradicting the fact that i is an embedding.□

8.3 Theorem. (Jankov [68]). $\underline{H}$ has $2^{\aleph_0}$ subvarieties.

Proof. For $A \subseteq \mathbb{N} \setminus \{1,2\}$ let $\underline{K}_A = V(\{G_n \mid n \in A\})$. It is known (Jankov [63]) that each finite subdirectly irreducible Heyting algebra L is splitting, and that the splitting variety $(\underline{H} : L) = \{L' \in \underline{H} \mid L \notin SH(L')\}$ (this result can also be deduced from our 3.3 and 3.5). Hence by the lemma $\underline{K}_A \subseteq (\underline{H} : G_n)$ if $n \notin A$, implying that $G_n \notin \underline{K}_A$ if $n \notin A$. Therefore there are as many subvarieties $\underline{K}_A$ of $\underline{H}$ as there are subsets of $\mathbb{N} \setminus \{1,2\}$.□

8.4 Note that the G_n, $n \in \mathbb{N}$, are $\underline{H}$-generated by 2 elements. Hence $\underline{K}_A = V(F_{\underline{K}_A}(2))$ and it follows that there are $2^{\aleph_0}$ non-isomorphic Heyting algebras generated by 2 elements. In I.3 we have seen that there are only countably many non-isomorphic Heyting algebras generated by one element. Contrast this with I.4.21.

8.5 Corollary. $\underline{B}_i$ and $\underline{B}_i^*$ have $2^{\aleph_0}$ subvarieties.

Proof. By 7.10, $\Sigma \cong (\underline{B}_i^*] \subseteq \Omega$. The statement follows from 8.3.□

By a slight modification of the given example we can show that also $\underline{H}^-$ and therefore $\underline{B}_i^{-*}$ and $\underline{B}_i^-$ have $2^{\aleph_0}$ subvarieties. Let $F_n = \underline{2}^3 \dagger G_n$, $n = 0,1,2\dots$.

8.6 Lemma. Let $n, m \in \mathbb{N}$, $n, m \geq 3$, $n \neq m$. Then $F_n^- \notin SH(F_m^-)$.

Proof. Let $n, m \in \mathbb{N}$, $n, m \geq 3$ and $n \neq m$. Let $L_1 \in H(F_m^-)$ and $i : F_n^- \to L_1$ be an $\underline{H}$-embedding. We may assume that $L_1 \cong (\underline{2}^3 \dagger (c_k])^-$, $L_1 \cong (\underline{2}^3 \dagger (c_k] \oplus 1)^-$ or $L_1 \cong F_m^-$ with $3 \leq k \leq m$. It is easily seen that the three atoms of F_n^- have to be mapped upon the three atoms of L_1. Therefore $i(c_0) = c_0$ and it follows that $i \mid G_n : G_n \to [c_0) \subseteq L_1$ is a $\underline{B}_i$-homomorphism. But then $G_n \in SH(G_m)$, in contradiction with 8.2.□

8.7 Theorem. $\underline{H}^-$ has $2^{\aleph_0}$ subvarieties.

Proof. Similar to the proof of 8.3.□

8.8 Corollary. $\underline{B}_i^-$ and $\underline{B}_i^{-*}$ have $2^{\aleph_0}$ subvarieties.

Proof. Since $|\Sigma^-| \leq |(\underline{B}_i^{-*}]|$,by 8.7. □

As there are only countably many varieties of given finite type which are finitely based (i.e. which are determined by a finite set of equations) it follows from 8.3 and 8.5 (and likewise from 8.7 and 8.8) that there are varieties of Heyting and interior algebras (respectively Brouwerian and generalized interior algebras) that are not finitely based. In order to give an example of such a variety, let

$$\underline{J}_n = V(\{G_k \mid k \in \mathbb{N},\ k \neq n\} \cup (HS(G_n) \setminus \{G_n\})),\ n \in \mathbb{N}$$

and let $\underline{J} = \bigcap_{n=1}^{\infty} \underline{J}_n$.

8.9 Theorem. $\underline{J}$ is not finitely based.

Proof. Let Γ_n be an equational base for $\underline{J}_n$, $n \in \mathbb{N}$, that is, $\underline{J}_n = \{L \in \underline{H} \mid L \models \Gamma_n\}$. Then $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$ is a base for $\underline{J}$. Suppose that $\underline{J}$ has a finite base. By the compactness theorem, there exists then a finite set $\Gamma_0 \subseteq \Gamma$ which is a base for $\underline{J}$. There are $n_0,\dots n_\ell \in \mathbb{N}$ such that $\Gamma_0 \subseteq \bigcup_{i=1}^{\ell} \Gamma_{n_i}$ and therefore $\underline{J} = \bigcap_{i=1}^{\ell} \underline{J}_{n_i}$. But if $k \neq 1,2,n_0,\dots n_\ell$ then $G_k \in \bigcap_{i=1}^{\ell} \underline{J}_{n_i}$ though $G_k \notin \underline{J}$, a contradiction.□

8.10 Corollary. $\rho^*(\underline{J})$ and $\underline{J}^c$ are subvarieties of $\underline{B}_i$ which are not finitely based.

Proof. By 7.12 (ii).□

In a similar way one could give examples of subvarieties of $\underline{H}^-$ and $\underline{B}_i^-$ which are not finitely based, using the F_n^- instead of the G_n.

The variety $\underline{J}$ has still another interesting property. In section 1 we have seen that every $\underline{K} \in \Omega$ is covered by some $\underline{K}' \in \Omega$. The variety $\underline{J}$ is an example of an element of Σ having countably many covers in Σ.

8.11 Theorem. $\underline{J}$ is covered in Σ by the countably many varieties $\underline{J} + V(G_n)$, $n = 3,4,\dots$.

Proof. By Jónsson's 0.1.27 and by definition of $\underline{J}$,

$$(\underline{J} + V(G_n))_{SI} = \underline{J}_{SI} \cup \{G_n\} \supset \underline{J}_{SI}$$

for $n \in \mathbb{N}$, $n \geq 3$. Hence $\underline{J} \prec \underline{J} + V(G_n)$ for $n \in \mathbb{N}$, $n \geq 3$. And if $n \neq m$, $n, m \in \mathbb{N}$, $n, m \geq 3$, then $\underline{J} + V(G_n) \neq \underline{J} + V(G_m)$ since

$$(\underline{J} + V(G_n))_{SI} = \underline{J}_{SI} \cup \{G_n\} \neq \underline{J}_{SI} \cup \{G_m\} = (\underline{J} + V(G_m))_{SI}.\square$$

In fact, if $\underline{N} = V(\{G_n \mid n \in \mathbb{N}\})$ then one can easily verify that the sublattice $[\underline{J}, \underline{N}]$ of Σ is isomorphic to the Boolean lattice of all subsets of a countable set. The atoms of $[\underline{J}, \underline{N}]$ are the varieties $\underline{J} + V(G_n)$, $n = 3,4,\ldots$.

8.12 Corollary. $\rho^*(\underline{J})$ is covered in Ω by the countably many varieties $\rho^*(\underline{J}) + V(B(G_n))$, $n = 3,4,\ldots$.

Proof. Obious from 7.10 and 8.8. It can also be shown directly without difficulty, though.$\square$

Having seen that $\underline{B}_i^*$ has $2^{\aleph_0}$ subvarieties, we want to say a few words about the cardinality of the lattices $((\underline{B}_i : K_n)^*]$ and $((\underline{B}_i : K_n)]$, $n \in \mathbb{N}$. We have already observed that $(\underline{B}_i : K_1)$ is the trivial variety, that $(\underline{B}_i : K_2)^*$ is the class of discrete interior algebras - the unique atom of Ω - and that $(\underline{B}_i : K_2)$ is the class of monadic algebras, the lattice of subvarieties of which is a countable chain of order type $\omega + 1$. In section 6 we have given a description of the lattice $((\underline{B}_i : K_3)]$; in particular, we showed that it is of countable cardinality. The lattice

$((\underline{B}_i : K_3)^*]$, which is isomorphic to $((\underline{H} : \underline{4})]$, is particularly simple: it is the countable chain of ordertype $\omega + 1$:

$$(\underline{B}_i : K_1) \subset (\underline{B}_i : K_2)^* = V(N_1) \subset$$
$$V(N_{11}) \subset \ldots \subset V(N_{11\ldots1}) \subset \ldots \subset (\underline{B}_i : K_3)^*.$$

Recently A.V. Kuznetsov [74] announced that he has shown that the next layer, $((\underline{H} : \underline{5})]$ has $2^{\aleph_0}$ elements. The same holds then for $((\underline{B}_i : K_4)^*]$ and a fortiori for $((\underline{B}_i : K_4)]$. In the context of modal logics a proof of this fact can be found in Fine [74].

So far in this section, our results concerning the cardinality of certain sublattices of Ω were mainly consequences of corresponding results on the cardinality of sublattices of Σ . The question arises what can be said about the cardinality of the intervals $[\underline{K}, \underline{K}^{oc}]$, $\underline{K} \in (\underline{B}_i^*]$. That is, how many subvarieties $\underline{K}$ of $\underline{B}_i$ can there be having a common $\underline{K}^o$?
We start with a simple theorem.

8.13 Theorem. Let $\underline{K} \subseteq \underline{B}_i^*$ be a non-trivial variety. Then $[\underline{K}, \underline{K}^{oc}]$ contains infinitely many varieties.

Proof. Consider the chain of varieties

$$\underline{K} \subset \underline{K} + V(M_2) \subset \underline{K} + V(M_3) \subset \ldots \subset \underline{K} + V(M_n) \subset \ldots \subset \underline{K}^{oc}.$$

Indeed, for any $n \in \mathbb{N}$, $\underline{K}^o \subseteq (\underline{K} + V(M_n))^o = V(\underline{K}^o \cup \{\underline{2}\}) = \underline{K}^o$ since $\underline{2} \in \underline{K}$, $\underline{K}$ being non-trivial. Hence $\underline{K} \subseteq \underline{K} + V(M_n) \subseteq \underline{K}^{oc}$. Furthermore $(\underline{K} + V(M_n))_{SI} = \underline{K}_{SI} \cup \{M_k \mid 1 \leq k \leq n\}$ by 1.4, and $M_k \notin \underline{K}$ if $k > 1$, since $\underline{K} \subseteq \underline{B}_i^*$. Thus

$$\underline{K} + V(M_n) \subset \underline{K} + V(M_{n+1}), \quad n = 1,2,\ldots . \square$$

In section 6 we have seen that $|[(\underline{B}_i : K_2)^*, (\underline{B}_i : K_2)]| = \aleph_0$ and that even $|[(\underline{B}_i : K_3)^*, (\underline{B}_i : K_3)]| = \aleph_0$. From the next theorem it will follow, given Kuznetsov's result, that $|[(\underline{B}_i : K_4)^*, (\underline{B}_i : K_4)]| = 2^{\aleph_0}$.

8.14 <u>Theorem</u>. Let $\underline{K} \subseteq \underline{B}_i^*$ be a variety having $2^{\aleph_0}$ subvarieties generated by their finite members. Then $|[\underline{K}, \underline{K}^{oc}]| = 2^{\aleph_0}$.

<u>Proof</u>. Let $\underline{K}$ be a subvariety of $\underline{B}_i^*$ and let $\{\underline{K}_i \mid i \in I\}$ be a collection of $2^{\aleph_0}$ subvarieties of $\underline{K}$, all generated by their finite members. For $L \in \underline{K}_{FSI}$ let $L^+ \in \underline{B}_{iFSI}$ be such that $L^{+o} = L^o$ but $L^+ \not\cong L$. Note that L is a $\star$-algebra while L^+ is not a $\star$-algebra. Let $\underline{V}_i = V(\underline{K} \cup \{L^+ \mid L \in \underline{K}_{iFSI}\})$. Then $\underline{K} \subseteq \underline{V}_i \subseteq \underline{K}^{oc}$. Suppose that $i \neq j$, $i, j \in I$, and say $L \in \underline{K}_{iFSI} \setminus \underline{K}_{jFSI}$. Then $L^+ \in \underline{V}_i$ but $L^+ \notin \underline{V}_j$. Indeed, $\underline{V}_{jFSI} \subseteq \underline{K}_{FSI} \cup HS(\{L^+ \mid L \in \underline{K}_{jFSI}\})$ by 1.4 and 3.6. But $L^+ \notin \underline{K}_{FSI}$ since L^+ is not a $\star$-algebra, and if $L^+ \in HS(L_1^+)$ for some $L_1 \in \underline{K}_{jFSI}$, then $L^o \in HS(L_1^o)$ hence $L = B(L^o) \in HS(B(L_1^o)) = HS(L_1)) \subseteq \underline{K}_{jF}$, a contradiction.□

By Kuznetsov's result, the varieties $\underline{K}$ which contain $(\underline{B}_i : K_4)^*$ (all of whose subvarieties are generated by their finite members since $(\underline{B}_i : K_4)$ is locally finite) satisfies the requirements of the theorem, hence for those varieties we have $|[\underline{K}^*, \underline{K}^{oc}]| = 2^{\aleph_0}$.

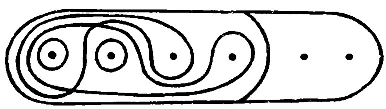
The condition of 8.14 is not necessary, however. We shall now give an example of a variety $\underline{K} \subseteq \underline{B}_i^*$ which has only countably many subvarieties; nevertheless $[\underline{K}, \underline{K}^{oc}]$ has the power of the continuum.

8.15 Let P_n be the interior algebra defined for $n \in \mathbb{N}$ in the following way:

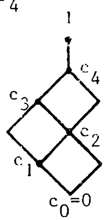
$$P_n \underset{\underline{B}}{\cong} \underline{2}^{n+2}, \quad P_n^o = (c_n] \oplus 1,$$

where $(c_n]$ denotes a principal ideal of $F_{\underline{H}}(1)$ (see I.3), such that $(c_n'] \cong \underline{2}^2$. Then $(c_n] \cong H_n$. The algebra P_4 is suggested in the diagrams:

P_4



P_4^o



8.16 <u>Lemma</u>. Let $n, m \in \mathbb{N}$, $n, m \geq 3$, $n \neq m$. Then $P_n \notin SH(P_m)$.

<u>Proof</u>. Suppose that $P_n \in SH(P_m)$. Note that every homomorphic image of P_m which is different from P_m is a homomorphic image of $(c_m] \cong H_m$ and therefore a $\star$-algebra. Since P_n is not a $\star$-algebra and by II.2.5 subalgebras of $\star$-algebras are

$\star$-algebras we may assume that $P_n \in S(P_m)$. Let $i : P_n \to P_m$ be a $\underline{B}_i$-embedding. Then $i \mid P_n^o : P_n^o \to P_m^o$ is an $\underline{H}$-embedding and since c_1 is the only element x of P_n^o and P_m^o which satisfies $\mathrm{Gen}(x)$ (see 8.1) we conclude that $i(c_1) = c_1$. Since P_m^o is $\underline{H}$-generated by c_1 it follows that $i(P_n^o) = P_m^o$ and hence that $n = m$, a contradiction. □

Recall that $H_\infty = B(F_{\underline{H}}(1))$ (cf. I.3).

8.17 <u>Theorem</u>. The interval $[V(H_\infty),\ V(F_{\underline{H}}(1))^c]$ contains $2^{\aleph_0}$ varieties.

<u>Proof</u>. For $A \subseteq \mathbb{N} \setminus \{1,2\}$ such that A is infinite let $\underline{K}_A = V(\{P_n \mid n \in A\})$. Then $\underline{K}_A^o = V(\{P_n^o \mid n \in A\}) = V(F_{\underline{H}}(1))$, hence $\underline{K}_A \in [V(H_\infty),\ V(F_{\underline{H}}(1))^c]$. If $m \notin A$, $m \in \mathbb{N} \setminus \{1,2\}$, then $P_m \notin \underline{K}_A$ by 8.16 and 3.6. Since there are $2^{\aleph_0}$ infinite subsets of $\mathbb{N} \setminus \{1,2\}$, the theorem follows.□

Using arguments similar to those employed before we can see that $[V(H_\infty),\ V(F_{\underline{H}}(1))^c]$ contains a sublattice isomorphic to the Boolean lattice of all subsets of a countable set and hence also contains varieties covered by infinitely many varieties.

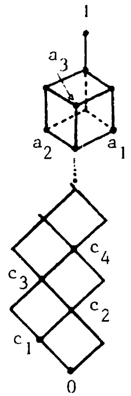
Section 9. Subvarieties of $\underline{B}_i$ not generated by their finite members

The property of a variety to be generated by its finite members is an informative one as we have seen already several times. For example, theorem 3.8 gave a satisfactory description of the lattice of subvarieties of $\underline{B}_i$ which are generated by their finite members, further refined in the description of the lattices $(\underline{M}]$ and $((\underline{B}_i : K_3)]$ of sections 5 and 6. Also, if a variety $\underline{K}$ has this property and moreover $\underline{K}$ is determined by a finite number of equations, then one can decide in a finite number of steps if a given equation is satisfied by $\underline{K}$ or not. Another nice feature of varieties $\underline{K}$ generated by their finite members is the fact that the locally finite varieties $(\underline{B}_i : K_n) \cap \underline{K}$, $n = 2,3,\ldots$, generate $\underline{K}$, which makes the results of II.7 concerning the finitely generated free objects $F_{\underline{K}}(m)$, $m = 1,2,\ldots$ applicable.

As announced before, it appears that not every subvariety of $\underline{B}_i$ is generated by its finite members. Varieties which lack this property are much more difficult to handle and give rise to several problems which are not settled yet. The purpose of this section is to give some examples of varieties which are not generated by their finite members and to consider some of the problems related to them.

To begin with, let us note that if $\underline{K} \subseteq \underline{B}_i$ is a variety which is generated by its finite members then so is $\underline{K}^o$. Indeed, by II.1.2, if $\underline{K} = V(\underline{K}_F)$ then $\underline{K}^o = V((\underline{K}_F)^o) = V((\underline{K}^o)_F)$. Hence, if we can find a variety $\underline{K}$ of Heyting algebras which is not generated by its finite members then the interval $[\rho^*(\underline{K}),\rho(\underline{K})]$ consists completely of subvarieties of $\underline{B}_i$ not generated by their finite members. We shall briefly sketch now an example of a collection of $2^{\aleph_0}$ varieties of Heyting algebras not generated by their finite members.

9.1 Let $X = F_{\underline{H}}(1) \dotplus \underline{2}^3 \dotplus \underline{2}$. The generator of $F_{\underline{H}}(1)$ will as usual be denoted by c_1; the atoms of $\underline{2}^3$ by a_1, a_2, a_3.



9.2 Theorem. $V(X)$ is not generated by its finite members.

Proof. First note that c_1 is the only element x of X satisfying Gen(x) (for notation, see 8.1). By 3.6, $V(X)_{FSI} \subseteq HS(X)$. Now suppose that $L \in V(X)_{FSI}$ and let $d \in L$ be such that Gen(d).

Then there are $L_1 \in S(X)$ and an onto-homomorphism $h : L_1 \to L$. If $h(y) = d$, $y \in L_1$, then necessarily Gen(y), hence by the remark above, $y = c_1$. Thus $c_1 \in L_1$ and therefore $F_{\underline{H}}(1) = [c_1]_{\underline{H}} \subseteq L_1$. Since L is finite it follows that $h^{-1}(\{1\}) = [c_n)$ or $h^{-1}(\{1\}) = [c_{n+1} \to c_n)$ for some $n \in \mathbb{N}$, $n \geq 3$ and we conclude that $L \cong (c_n]$ or $L = (c_n] \oplus 1$ for some $n \in \mathbb{N}$, $n \geq 3$. In particular, L contains no three mutually incomparable elements. Thus the class $V(X)_{FSI}$ satisfies the sentence

$$\exists x \mathrm{Gen}(x) \Rightarrow \forall x_1 \, \forall x_2 \, \forall x_3 \bigvee_{1 \leq i, j \leq 3} x_i \leq x_j$$

which is equivalent to the positive universal sentence σ

$$\forall x \, \forall x_1 \, \forall x_2 \, \forall x_3 \, [x \to 0 = 0 \vee (x \to 0) \to 0 = x \vee \bigvee_{1 \leq i, j \leq 3} x_i \leq x_j].$$

The sentence σ is preserved under the operations H, S and P_U (see Grätzer [68], pg.275). If $V(X)$ would be generated by its finite members then by 1.1 $X \in HSP_U(V(X)_{FSI})$. But X clearly does not satisfy σ, hence $X \notin HSP_U(V(X)_{FSI})$, thus $V(X)$ is not generated by its finite members.□

9.3 <u>Theorem</u>. There are $2^{\aleph_0}$ varieties of Heyting algebras which are not generated by their finite members.

<u>Proof</u>. Let for $A \subseteq \mathbb{N} \setminus \{1,2\}$ $\underline{K}_A$ be the variety of Heyting algebras defined in the proof of 8.3. By 1.4, $(\underline{K}_A + V(X))_{FSI} = \underline{K}_{A\,FSI} \cup V(X)_{FSI}$. It is immediate that $\underline{K}_{A\,FSI}$ satisfies the sentence σ given in the proof of 9.2. Hence $X \notin V((\underline{K}_A + V(X))_{FSI})$, thus $\underline{K}_A + V(X)$ is not generated by its finite members. Let $A, B \subseteq \mathbb{N} \setminus \{1,2\}$, $A \neq B$,

say $n_0 \in A \setminus B$. In the proof of 9.2 we have seen that if $L \in V(X)_{FSI}$ possesses an element d such that Gen(d), then $L \cong (c_n]$ or $L \cong (c_n] \oplus 1$ for some $n \in \mathbb{N}$, $n \geq 3$. Hence $G_{n_0} \notin V(X)_{FSI}$. By the proof of 8.3, $G_{n_0} \notin \underline{K}_{B\ FSI}$. Hence $G_{n_0} \notin \underline{K}_{B\ FSI} \cup V(X)_{FSI} = (\underline{K}_B + V(X))_{FSI}$, and thus $G_{n_0} \notin \underline{K}_B + V(X)$. Since $G_{n_0} \in \underline{K}_A + V(X)$, it follows that $\underline{K}_A + V(X) \neq \underline{K}_B + V(X)$, and hence that there are $2^{\aleph_0}$ different subvarieties of $\underline{B}_i$ not generated by their finite members.□

We observed already that these varieties of Heyting algebras not generated by their finite members give rise to whole intervals of varieties of interior algebras not generated by their finite members. The question arises if there are varieties $\underline{K} \subseteq \underline{B}_i$ such that $\underline{K}^o$ is generated by its finite members while $\underline{K}$ itself is not. In the next theorem we show that this phenomenon does indeed occur.

9.4 Let $Y \in \underline{B}_i$ be such that $Y^o = F_{\underline{H}}(1) \oplus 1$, $(1_{F_{\underline{H}}(1)}] \cong H_\infty$ and $(1'_{F_{\underline{H}}(1)}] \underset{\underline{B}}{\cong} \underline{2}^2$. Let as usual $c_1 \in Y$ denote the generator of $F_{\underline{H}}(1)$. Now $V(Y)^o = V(Y^o) = V(F_{\underline{H}}(1) \oplus 1)$ is generated by its finite members since $F_{\underline{H}}(1) \oplus 1 \in SP_U(\{(c_n] \mid n \in \mathbb{N}\}) \subseteq V(V(Y^o)_F)$. However:

9.4 Theorem. $V(Y)$ is not generated by its finite members.

Proof. Let $L \in V(Y)_{FSI}$ be such that there is a $d \in L^{o}$ satisfying $Gen(d)$ in L^{o} (cf. 8.1), or equivalently, such that $d^{o'o} \neq 0$ and $d^{o'o'o} \neq d^{o}$. By 3.6, $L \in HS(Y)$. Let therefore $L_1 \in S(Y)$ and $h : L_1 \to L$ be an onto homomorphism. Let $y \in L_1$ be such that $h(y) = d$. Then $h(y^{o}) = d$ and $Gen(y^{o})$ in L_1^{o}. Since c_1 is the only element x of Y^{o} satisfying $Gen(x)$ it follows that $c_1 = y^{o} \in L_1$. Hence $[c_1]_{\underline{B}_i} = H_\infty \subseteq L_1$ and we infer that $L \cong H_n$ or $L \cong B((c_n] \oplus 1)$ for some $n \in \mathbb{N}$, $n \geq 3$, so in particular, L is a $\star$-algebra. It follows that in $V(Y)_{FSI}$ the sentence σ

$$\exists x\ Gen(x^{o}) \Rightarrow \forall y[((y' + y^{o})^{o'} + y^{o})^{o'} + y^{o} = 1]$$

is satisfied. The consequence of σ expresses the fact that the algebra under consideration belongs to $\underline{B}_i^{\star}$ (cf. 7.9), which clearly is true for L. The sentence σ is equivalent to the positive universal sentence

$$\forall x \forall y\ [x^{o'o} = 0 \vee x^{o'o'o} = x^{o} \vee ((y' + y^{o})^{o'} + y^{o})^{o'} + y^{o} = 1].$$

Because σ is positive universal, it is preserved under the operations H, S and P_U. But $Y \notin \underline{B}_i^{\star}$ since $M_{1,2} \in S(Y)$, and there is an $x \in Y$ such that $Gen(x)$, hence Y does not satisfy σ. Therefore $Y \notin HSP_U(V(Y)_{FSI})$ and hence $Y \notin V(V(Y)_F)$. □

Using the algebra Y we can obtain a whole bunch of varieties having the same property:

9.5 Theorem. There are $2^{\aleph_0}$ varieties $\underline{K}$ of interior algebras such that $\underline{K}^o$ is generated by its finite members while $\underline{K}$ itself is not.

Proof. For $A \subseteq \mathbb{N} \setminus \{1,2\}$ let $\underline{K}_A = V(\{Y\} \cup \{B(G_n) \mid n \in A\})$ where the G_n are as defined in 8.1. By 1.4, $\underline{K}_{A\ FSI} = V(Y)_{FSI} \cup V(\{B(G_n) \mid n \in A\})_{FSI}$. Because the $B(G_n)$ are $\star$-algebras, $V(\{B(G_n) \mid n \in A\}) \subseteq \underline{B}_i^{\star}$ and therefore surely satisfies the sentence σ given in the proof of 9.4. Hence $\underline{K}_{A\ FSI}$ satisfies σ, and therefore $Y \notin V((\underline{K}_A)_F)$. Further, if $n \geq 3$, $n \in \mathbb{N}$, then $B(G_n) \notin V(Y)_{FSI}$ since $G_n \notin V(Y^o)$ (cf. proof of 9.4), and if $n \notin A$ then $B(G_n) \notin V(\{B(G_n) \mid n \in A\})_{FSI}$, by the proof of 8.3. Therefore if $A, B \subseteq \mathbb{N} \setminus \{1,2\}$, $A \neq B$, then $\underline{K}_A \neq \underline{K}_B$. Finally,

$$\underline{K}_A^o = V(\{Y^o\} \cup \{G_n \mid n \in A\})$$
$$= V(\{(c_n] \mid n \in \mathbb{N}\} \cup \{G_n \mid n \in A\}),$$

hence $\underline{K}_A^o$ is generated by its finite members.□

In 3.7 the question arose if the intersection of two varieties which are generated by their finite members is also generated by its finite members - if this would be true the subset of subvarieties of $\underline{B}_i$ which are generated by their finite members - which is a proper subset by 9.1 - 9.5 - would be a sublattice of Ω. We are now ready to give an example showing that the answer to this question is negative.

9.6 First we present two varieties of Heyting algebras, both generated by their finite members, such that their intersection is not generated by its finite members. Let

$$\underline{K}_1 = V(\{(c_{2n}] \dagger \underline{2}^3 \dagger \underline{2} \mid n = 1,2,\dots\}) \subseteq \underline{H}$$

and

$$\underline{K}_2 = V(\{(c_{2n-1}] \dagger \underline{2}^3 \dagger \underline{2} \mid n = 1,2,\dots\}) \subseteq \underline{H}.$$

Here $(c_n]$, as usual, denotes a principal ideal in $F_{\underline{H}}(1)$ (see I.3). Note that by definition $\underline{K}_1$, $\underline{K}_2$ are generated by their finite members.

9.7 Theorem. $\underline{K}_1 \cap \underline{K}_2$ is not generated by its finite members.

Proof. Let F be a non-principal ultrafilter on $\mathbb{N}$, and let $L = \prod_{n\in\mathbb{N}} (c_{2n}] \dagger \underline{2}^3 \dagger \underline{2}/F$. It is easy to see (using the properties of ultraproducts) that $X = F_{\underline{H}}(1) \dagger \underline{2}^3 \dagger \underline{2} \in S(L) \subseteq \underline{K}_1$. Similarly $X \in \underline{K}_2$. Let $L \in (\underline{K}_1 \cap \underline{K}_2)_{FSI} = \underline{K}_{1FSI} \cap \underline{K}_{2FSI}$ such that L satisfies $\exists x\, \mathrm{Gen}(x)$. By 3.6, $L \in HS((c_{2n}] \dagger \underline{2}^3 \dagger \underline{2})$ and $L \in HS((c_{2k-1}] \dagger \underline{2}^3 \dagger \underline{2}$ for some $n, k \in \mathbb{N}$. Let $L_1 \in S((c_{2n}] \dagger \underline{2}^3 \dagger \underline{2})$ and let $h : L_1 \to L$ be an onto homomorphism. Let $d \in L$ be such that $\mathrm{Gen}(d)$ and $y \in L_1$ such that $h(y) = d$. Then $\mathrm{Gen}(y)$, hence $n \geq 2$ and $y = c_1$ since c_1 is the only element x of $(c_{2n}] \dagger \underline{2}^3 \dagger \underline{2}$ such that $\mathrm{Gen}(x)$, $n \geq 2$. If h is $1-1$ then it follows that $L \cong (c_{2n}] \dagger \underline{2}^{\ell} \dagger \underline{2}$ $\ell = 0,1,2,3$. If h is not $1-1$, then since L is SI we conclude that $L \cong (c_p] \dagger \underline{2}$ for some p, $3 \leq p \leq 2n$. In the same way we show that since $L \in HS((c_{2k-1}] \dagger \underline{2}^3 \dagger \underline{2})$ for some $k \in \mathbb{N}$, it follows that $k \geq 2$ and that $L \cong (c_{2k-1}] \dagger \underline{2}^{\ell} \dagger \underline{2}$, $\ell = 0,1,2,3$, or $L \cong (c_p] \dagger \underline{2}$ for some p, $3 \leq p \leq 2k-1$. In order to satisfy both requirements we must conclude that $L = (c_p] \dagger \underline{2}$ for some

$p \in \mathbb{N}$, $p \geq 3$. So $(\underline{K}_1 \cap \underline{K}_2)_{FSI}$ satisfies the sentence σ

$$\exists x \; Gen(x) \Rightarrow \forall x_1 \; \forall x_2 \; \forall x_3 \bigvee_{1 \leq i,\, j \leq 3} x_i \leq x_j .$$

As σ is a positive universal sentence, it is also satisfied by $HSP_U((\underline{K}_1 \cap \underline{K}_2)_{FSI})$. But σ is not valid in X, so we may infer that $X \notin V((\underline{K}_1 \cap \underline{K}_2)_{FSI})$. Since $X \in \underline{K}_1 \cap \underline{K}_2$, we have proved that $\underline{K}_1 \cap \underline{K}_2 \neq V((\underline{K}_1 \cap \underline{K}_2)_{FSI})$.□

9.8 <u>Corollary</u> $\rho^*(\underline{K}_1)$ and $\rho^*(\underline{K}_2)$ are subvarieties of $\underline{B}_i$ which are generated by their finite members though their intersection is not.

<u>Proof</u>. By 9.7 and 7.13 (i).□

The results of this section imply that the representation of the lattice of subvarieties of $\underline{B}_i$ which are generated by their finite members as a certain lattice of subsets of a countable set does not provide a description of the lattice Ω of all subvarieties of $\underline{B}_i$. The question comes up what the smallest cardinality is for which there exists a set X of that cardinality such that Ω can be embedded as a set lattice in the lattice of all subsets of X. It is not difficult to see that this cardinality is just the cardinality of the "set" of varieties $\underline{K} \subseteq \underline{B}_i$ which are strictly join irreducible in Ω (an element x in a lattice L is called strictly join irreducible if $x = \sum_{i \in I} a_i$ implies $x = a_i$ for some $i \in I$, for any set $\{a_i \mid i \in I\} \subseteq L$). Note that such a variety is always generated by a single subdirectly irreducible. If this subdirectly irreducible is not finite then the variety cannot be generated by its finite members. Our example $V(X)$ of 9.1 is an example of a strictly join irreducible element of Σ and $\rho^*(V(X))$ is a strictly join irreducible of Ω : both are not generated by a finite algebra. The problem to characterize the subdirectly irreducibles in $\underline{H}$ or $\underline{B}_i$ which generate strictly join irreducible varieties in Σ respectively Ω is unsettled yet; we donot even know how many there are.

REFERENCES

Baker, K.A.

[M] Equational axioms for classes of Heyting algebras (manuscript).

Balbes, R. and Dwinger, Ph.

[74] Distributive lattices, University of Missouri Press (1974).

Balbes, R. and Horn, A.

[70] Injective and projective Heyting algebras, Trans. Amer. Math. Soc. 148 (1970), pp. 549-559.

Bass, H.

[58] Finite monadic algebras, Proc. Amer. Math. Soc. 9 (1958), pp. 258-268.

Berman, J.

[M] Algebras with modular lattice reducts and simple subdirectly irreducibles (manuscript).

Birkhoff, G.

[35] On the structure of abstract algebras, Proc. Cambridge Philos. Soc. 31 (1935), pp. 433-454.

[44] Subdirect unions in universal algebra, Bull. Amer. Math. Soc. 50 (1944), pp. 764-768.

Blok, W.J.

[M] $2^{\aleph_0}$ varieties of Heyting algebras not generated by their finite members (to appear in Alg. Universalis).

Blok, W.J. and Dwinger, Ph.

[75] Equational classes of closure algebras I, Ind. Math. 37 (1975), pp. 189-198.

Bull, R.A.

[66] That all normal extensions of S4.3 have the finite model property, Zeitschr. f. math. Logik und Grundlagen d. Math. 12 (1966), pp. 341-344.

Burger, A.

[75] Contributions to the topological representation of bounded distributive lattices, Doctoral dissertation, University of Illinois at Chicago Circle, 1975.

Crawley, P. and Dilworth, R.P.

[73] Algebraic theory of lattices, Prentice-Hall, Englewood Cliffs, N.J., 1973.

Christensen, D.J. and Pierce, R.S.

[59] Free products of α-distributive Boolean algebras, Math. Scand. 7, (1959), pp. 81-105.

Day, A.

[M] Varieties of Heyting algebras I (manuscript).

Dummet, M.A.E. and Lemmon, E.J.

[59] Modal logics between S4 and S5, Zeitschr. f. math. Logik und Grundlagen d. Math. 5 (1959), pp. 250-264.

Fine, K.

[71] The logics containing S4.3, Zeitschr. f. math. Logik und Grundlagen d. Math. 17 (1971), pp. 371-376.

[74] An ascending chain of S4 logics, Theoria 40 (1974), pp. 110-116.

Glivenko, V.

[29] *Sur quelques points de la logique de M. Brouwer, Bull. Acad.* des Sci. de Belgique, 15 (1929), pp. 183-188.

Grätzer, G.

[68] Universal algebra, Van Nostrand, Princeton, 1968.

[71] Lattice theory: First concepts and distributive lattices, W.H. Freeman Co., San Francisco, 1971.

Halmos, P.R.

[62] Algebraic Logic, Chelsea Publ. Co., New York, 1962.

Harrop, R.

[58] On the existence of finite models and decision procedures for propositional calculi, Proc. Cambridge Philos. Soc. 54 (1958), pp. 1-13.

Hecht, T. and Katrinák, T.

[72] Equational classes of relative Stone algebras, Notre Dame J. of Formal Logic 13 (1972), pp. 248-254.

Henkin, L., Monk, J.D., Tarski, A.

[71] Cylindric algebras, Part I, North-Holland Publ.Co., Amsterdam, 1971.

Heyting, A.

[30] Die formalen Regeln der intuitionistischen Logik, Sitzungsberichte der Preussischen Akademie der Wissenschaften, Phys. mathem. Klasse (1930), pp. 42-56.

Horn, A.

[69] Logic with truth values in a linearly ordered Heyting algebra, J. Symbolic Logic 34 (1969), pp. 395-408.

[69a] Free L-algebras, J. Symbolic Logic 34 (1969), pp. 475-480.

Hosoi, T.

[67] On intermediate logics I, J. Fac. Sci. Univ. Tokyo, Sec. I, 14 (1967), pp. 293-312.

Jankov, V.A.

[63] The relationship between deducibility in the intuitionistic propositional calculus and finite implicational structures, Sov. Math. Dokl. 4 (1963), pp. 1203-1204.

[68] Constructing a sequence of strongly independent superintuitionistic propositional calculi, Sov. Math. Dokl. 9 (1968), pp. 806-807.

Jónsson, B.

[67] Algebras whose congruence lattices are distributive, Math. Scand. 21 (1967), pp. 110-121.

Köhler, P.

[73] Freie endlich erzeugte Heyting-Algebren, Diplomarbeit, Justus Liebig Universität, Giessen 1973.

[M] Freie S-Algebren (manuscript).

Kripke, S.A.

[63] Semantical analysis of modal logic I. Normal modal propositional calculi, Zeitschr. f. math. Logik und Grundlagen d. Math. 9 (1963), pp. 67-96.

[65] Semantical analysis of modal logic II. Non-normal propositional calculi. In Addison, Henkin and Tarski (eds.). The theory of models, North-Holland Publ. Co., Amsterdam, 1965, pp. 206-220.

Kuratowski, C.

[22] L'opération $\bar{A}$ de l'analysis situs, Fund. Math. 3 (1922), pp. 182-199.

Kuznetsov, A.V.

[74] On superintuitionistic logics, I.C.M., Vancouver, 1974.

Lemmon, E.J.

[66] Algebraic semantics for modal logics I, II, J. Symbolic Logic 31 (1966), pp. 46-65, pp. 196-218.

Lewis, C.I. and Langford, C.H.

[32] Symbolic logic, The Century Co., New York and London, 1932.

Lyndon, R.C.

[54] Identities in finite algebras, Proc. Amer. Math. Soc. 5 (1954), pp. 8-9.

Makkai, M.

[73] A proof of Baker's finite-base theorem on equational classes generated by finite elements of congruence distributive varieties. Alg. Universalis 3 (1973), pp. 174-181.

McKay, G.C.

[68] The decidability of certain intermediate propositional logics, J. Symbolic Logic 33 (1968), pp. 258-264.

McKenzie, R.

[72] Equational bases and non-modular lattice varieties, Trans. Amer. Math. Soc. 174 (1972), pp. 1-43.

McKinsey, J.C.C. and Tarski, A.

[44] The algebra of topology, Ann. of Math. 45 (1944), pp. 141-191.

[46] On closed elements in closure algebras, Ann. of Math. 47 (1946), pp. 122-162.

[48] Some theorems about the sentential calculi of Lewis and Heyting, J. Symbolic Logic 13 (1948), pp. 1-15.

Monk, J.D.

[70] On equational classes of algebraic versions of logic I, Math. Scand. 27 (1970), pp. 53-71.

Nishimura, I.

[60] On formulas of one variable in intuitionistic propositional calculus, J. Symbolic logic 25 (1960), pp. 327-331.

Ono, H.

[70] Kripke models and intermediate logics, Publ. RIMS, Kyoto Univ. 6 (1970), pp. 461-476.

Pierce, R.S.

[70] Topological Boolean algebras, Proc. of the conference on universal algebra, Queen's papers in pure and applied mathematics 25 (1970), pp. 107-130.

Post, E.L.

[21] Introduction to a general theory of elementary propositions, Amer. J. Math 43 (1921), pp. 163-185.

Quackenbusch, R.W.

[74] Structure theory for equational classes generated by quasi-primal algebras, Trans. Amer. Math. Soc. 187 (1974), pp. 127-145.

Rasiowa, H.

[74] An algebraic approach to non-classical logics, North-Holland Publ. Co., Amsterdam, 1974.

Rasiowa, H. and Sikorski, R.

[63] The mathematics of metamathematics, Warszawa, 1963.

Rieger, L.

[57] Zametka o t. naz. svobodnyh algebrah s zamykanijami, Czechoslovak Math. J. 7 (1957), pp. 16-20.

Tarski, A

[46] A remark on functionally free algebras, Ann. of Math. 47 (1946), pp. 163-165.

Urquhart, A.

[73] Free Heyting algebras, Alg. Universalis 3 (1973), pp. 94-97.

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SAMENVATTING

In dit proefschrift wordt de theorie ontwikkeld van wat wij "inwendige algebra's" noemen, dat zijn Boole algebras voorzien van een extra één-plaatsige operatie ^{o} (de inwendige-operator) die aan de Kuratowskiaxiomas voldoet, d.w.z. $x^o \leq x$, $x^{oo} = x^o$, $(xy)^o = x^o y^o$ en $1^o = 1$. Toen McKinsey en Tarski in 1944 de studie van inwendige algebras aanvingen, was hun bedoeling, een algebraïsch apparaat te scheppen dat geschikt was om een deel van de verzamelings-theoretische topologie te behandelen. Voor ons echter is het feit interessanter dat inwendige algebra's juist de algebraïsche modellen zijn van de modale logica S4, geïntroduceerd door Lewis. Dat geeft in het bijzonder aanleiding tot speciale aandacht voor variëteiten van inwendige algebra's.

De hoofdstukken I en II zijn grotendeels gewijd aan het onderzoek van de algebraïsche structuur van eindig voortgebrachte vrije objecten in variëteiten van inwendige algebra's. Het blijkt nuttig om naast inwendige algebra's ook gegeneraliseerde inwendige algebra's te beschouwen; dat zijn gegeneraliseerde Boole algebra's met een grootste element, voorzien van een inwendige-operator. Een speciale rol wordt voorts gespeeld door de inwendige algebra's, $\star$-algebras genaamd, die, als Boole algebra, voortgebracht worden door hun tralie van open elementen. In hoofdstuk I blijkt hoe ingewikkeld zelfs de vrije inwendige algebra op één voortbrenger is; in het tweede hoofdstuk worden de vrije eindig voortgebrachte objecten in zekere deelvariëteiten gekarakteriseerd.

Hoofdstuk III is gewijd aan een onderzoek van het tralie van alle variëteiten van inwendige algebra's. Resultaten van universeel algebraïsche aard verkregen door B. Jónsson [67] verschaffen ons een doelmatig instrument om de structuur van het tralie nader te leren kennen. Gebruik makende van het begrip splitsingsvariëteit karakteriseren we de locaal eindige deelvariëteiten en geven een gedetailleerde beschrijving van enkele interessante hoofdidealen van het tralie. We tonen aan dat het tralie van deelvariëteiten van de variëteit voortgebracht door alle $*$-algebra's isomorf is met het tralie van variëteiten van Heyting algebra's. Besloten wordt met enige overwegingen betreffende kardinaliteitsproblemen en betreffende variëteiten die niet voortgebracht worden door hun eindige algebra's.

Subject index

Index of symbols

P	power set
$\underline{K}_F$	finite members of $\underline{K}$

Stellingen
bij het proefschrift
"Varieties of interior algebras"

1. Een topologische ruimte X heet lokaal homogeen als er voor elke $x \in X$ willekeurig kleine omgevingen U van x bestaan zó dat voor elke $y \in U$ er een autohomeomorfisme van de ruimte X is dat de identiteit is buiten U en dat x op y afbeeldt.
Zij nu X volledig metrizeerbaar en lokaal homogeen. Als A,B twee aftelbaar dichte deelverzamelingen van X zijn (X is dus separabel) dan bestaat er een autohomeomorfisme van X dat A op B afbeeldt.
Gevolg: $X \setminus A$ is homeomorf met $X \setminus B$. In het bijzonder, als $p \in X \setminus A$, dan is $X \setminus A$ homeomorf met $X \setminus A \setminus \{p\}$.

(met J. de Groot, niet gepubliceerd)

2. Gebruik makende van de topologische representatie van distributieve tralie's met 0,1 kan men een doorzichtig bewijs leveren van een stelling van Balbes die zegt dat het centrum van het vrije product van eindig veel distributieve tralie's met 0,1 juist het vrije product van de centra van deze tralie's is. Bovendien laat dit bewijs zich gemakkelijk aanpassen teneinde de beperking "eindig veel" te kunnen laten vervallen.

Balbes, R. The center of the free product of distributive lattices, PAMS 29 (1971) pp. 434-436

Blok, W.J. The center of the coproduct of distributive lattices with 0,1. Nieuw Archief voor Wiskunde 22 (1974) pp. 166-169

3. Een Post algebra P wordt gekarakteriseerd door een Boole deeltralie B en een totaal geordend (eindig) deeltralie C. Hier zijn

B en C uniek bepaald. Er geldt, dat P als tralie isomorf is met het tralie van continue functies van de Boolese ruimte corresponderend met B naar C, waar C voorzien is van de discrete topologie. Gegeneraliseerde Post algebra's, zoals geïntroduceerd door Chang en Horn, kunnen - als tralie - gerepresenteerd worden door het tralie (X,C) van continue functies van een Boolese ruimte X naar een totaal geordende discrete ruimte C. Als (X,C) en (X',C') twee van dergelijke representaties van eenzelfde gegeneraliseerde Post algebra zijn, dan geldt dat X homeomorf is met X' en dat C isomorf is met C'. Echter, de representatie hier is niet uniek in de zin dat als φ: (X,C) $\to$ (X,C') een isomorfisme is er noodzakelijk een homeomorfisme ψ en een isomorfisme h zouden bestaan zódat

$$\begin{array}{ccc} X & \xrightarrow{f} & C \\ \downarrow{\psi} & & \downarrow{h} \\ X' & \xrightarrow{\varphi(f)} & C' \end{array}$$

commuteert voor elke f $\in$ (X,C).

Blok, W.J. Generalized Post Algebras
doctoraalscriptie U.v.A. 1972

4. Vele enigszins moeizaam verkregen resultaten betreffende intermediaire en modale logica's zijn directe toepassingen van enkele stellingen uit de universele algebra.

cf. T. Hosoi, H. Ono, Intermediate propositional logics, J. Tsuda College 5 (1973) pp. 67-82

5. K. Fine heeft een voorbeeld gegeven van een modale logica die niet door zijn Kripke modellen wordt gekarakteriseerd, d.w.z. een logica met "graad van onvolledigheid" $\geq$ 2. Men kan bewijzen dat de graad van onvolledigheid van de klassieke logica, beschouwd als uitbreiding van het modale grondsysteem K, $2^{\aleph_0}$ bedraagt.

Fine, K. An incomplete logic containing S4, Theoria 40 (1974) pp. 23-29

6. Zij $\underline{K}$ een variëteit van algebra's van eindig type, waarvan alle algebra's een onderliggende structuur hebben van Boolese algebra's. Het vrije object in $\underline{K}$ op aftelbaar veel voortbrengers is, als Boolese algebra, isomorf met de vrije Boolese algebra op aftelbaar veel voortbrengers. In het bijzonder geldt $F_{\underline{B}_i}(\aleph_0) \underset{\underline{B}}{\simeq} F_{\underline{B}}(\aleph_0)$.

7. Een variëteit heet <u>bijna</u> <u>eindig</u> als zij zelf oneindig veel deelvariëteiten heeft maar elke echte deelvariëteit eindig is, i.e. slechts eindig veel deelvariëteiten heeft.
Er zijn twee bijna eindige variëteiten van Brouwerse algebra's: de variëteit voortgebracht door alle lineair geordende Brouwerse algebra's en de variëteit voortgebracht door de algebra's $\underline{2}^n \oplus 1$, $n = 1,2,\ldots$.
Er zijn drie bijna eindige variëteiten van gegeneraliseerde inwendige algebra's: de variëteit voortgebracht door alle *-algebra's waarvan de open verzamelingen een lineair geordende Brouwerse algebra vormen, de variëteit voortgebracht door de algebra's $B(\underline{2}^n \oplus 1)$, $n = 1,2,\ldots$ en de variëteit der monadische gegeneraliseerde inwendige algebra's. Deze drie variëteiten corresponderen met de drie dimensies hoogte, breedte en "trivialiteit" van een gegeneraliseerde inwendige algebra. Zowel in het geval van de Brouwerse algebra's als in dat der gegeneraliseerde inwendige algebra's geldt dat elke niet eindige variëteit een bijna eindige omvat.

8. Men kan op de collectie van deelvariëteiten van $\underline{B}_i^-$ een vermenigvuldiging definiëren door aan deelvariëteiten $\underline{K}_1$ en $\underline{K}_2$ toe te kennen de klasse van alle extensies van algebra's uit $\underline{K}_1$ met behulp van algebra's uit $\underline{K}_2$, i.e. $\underline{K}_1 \cdot \underline{K}_2 = \{L \in \underline{B}_i^- \mid$ er is een open filter F in L, zó dat $F \in \underline{K}_1$, $L/F \in \underline{K}_2\}$. Aldus verkrijgt men een halfgroep. De idempotenten van deze halfgroep zijn de variëteiten $(\underline{B}_i^- : \underline{M}_n)$, $n = 0,1,2,\ldots$ en $\underline{B}_i^-$. De lokaal eindige deelvariëteiten van $\underline{B}_i^-$ vormen een vrije onderhalfgroep van continue machtigheid.

9. De promotieplechtigheid in z'n huidige vorm dient afgeschaft te worden. Een zinvolle vervanging lijkt een voordracht over enkele aspecten van het proefschrift, begrijpelijk voor een publiek, zo breed dat het tenminste het merendeel der vakgenoten omvat.