Intuitionistic General Topology

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INTUITIONISTIC GENERAL TOPOLOGY

ACADEMISCH PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE WISKUNDE EN NATUURWETENSCHAPPEN AAN DE UNIVERSITEIT VAN AMSTERDAM, OP GEZAG VAN DE RECTOR MAGNIFICUS MR. J. VAN DER HOEVEN, HOOGLERAAR IN DE FACULTEIT DER RECHTSGELEERDHEID, IN HET OPENBAAR TE VERDEDIGEN IN DE AULA DER UNIVERSITEIT (TIJDELIJK IN DE LUTHERSE KERK, INGANG SINGEL 411, HOEK SPUI), OP WOENSDAG 15 JUNI 1966, DES NAMIDDAGS TE 4 UUR

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Aan mijn ouders,
Aan mijn vrouw.

VOORWOORD

Op deze plaats wil ik allen dank zeggen die op directe of indirecte wijze mij tot steun geweest zijn in mijn studie en bij het schrijven van dit proefschrift.

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In de formulering van dit voorwoord heb ik platgetreden paden bewandeld; moge dat het geloof in mijn oprechtheid niet verhinderen.

LIST OF NOTATIONS AND CONVENTIONS

1. References are given by indicating chapter, paragraph and section; e.g. 4.3.2 refers to the fourth chapter, third paragraph, second section. In referring to the same chapter, the first number is omitted.

A name (in capitals) followed by a year, and a capital if necessary (e.g. BROUWER 1926 A) refers to the bibliography.

2. Logical symbols: &, $\vee$, $\leftrightarrow$, $\rightarrow$, $\neg$, $\bigwedge$, $\bigvee$.
Quantified variables k, l, m, n, i, j, t always run through the natural numbers; quantified variables ε, δ always run through positive real numbers.
Set theoretic symbols: $\cap$, $\cup$, $\times$ (cartesian product), ^{c} (complementation, 1.2.2), $-$.
$\{X_1, X_2, \ldots\}$, $\{X_i : i \in I\}$ etc. are notations for species. Finite sequences are written as $\langle X_1, \ldots, X_n\rangle$ or $\langle X_i\rangle_{i=1}^{n}$; denumerably infinite sequences $X_1, X_2, \ldots$ are written as $\langle X_i\rangle_{i=1}^{\infty}$ or $\langle X_n\rangle_n$.
Functions or mappings with different domains of definition are considered to be different.
The restriction of a mapping f with domain D to $D' \subset D$ is denoted by $f|D'$. If f is a mapping of D into E, and $F \subset E$, then $f^{-1}F = \{x : fx \in F\}$ is called the counterimage of F.

3. Postulates (alphabetical).
C1-4 3.3.2; C5 3.3.4; D,F 4.1.2; I1-2 3.1.4; I3 3.1.6; I4 3.1.9; I5 3.1.10; I6 3.1.31; K 4.1.2; L1-2 4.2.2; N1-8, N8 ($\mathfrak{B}$) 3.2.1; N9 3.3.2; P 3.3.8; R1-5 3.2.10; S1-2 1.1.5; T 4.1.2; T1-3 1.2.2; T4 1.2.3; T5 1.2.4.

4. Groups of symbols, indexed if necessary, for special purposes.

$\Gamma, \Delta, \ldots$	topological spaces	ρ, ρ' metrics
$\mathfrak{T}, \mathfrak{T}'$	topologies	$A_0, A_\infty, A_n, \ldots$ 3.1.2
θ, θ'	spread laws	P, Q, R, S, T 3.1.2
ϑ, ϑ'	complementary laws	p, q, r points (3.1.13)
#, #'	apartness relations	U, V, W pointspecies (3.1.13)

5. Notations and symbols with a fixed meaning. For symbols of the following list combined with greek capitals for topological spaces Γ, Δ etc. (e.g. φ_Γ, $\Pi(\Delta)$) see 3.1.28.

a) Greek letters (alphabetical).
$\alpha(n)$, $\bar{\alpha}(n)$, α_σ 1.1.3; γ, γ' 3.1.2; ϑ^*, $\bar{\vartheta}$ 3.2.2; φ 3.1.4; ρ (p, V) 1.3.12; $\prod_{i=1}^{n} P_i = \Pi\{P_1, \ldots, P_n\}$ 3.1.3; Π 3.1.9; Π° 3.1.13; Π^* 3.1.35; π_i 3.4.1; $\sum_{i=1}^{n} P_i = \Sigma\{P_1, \ldots, P_n\}$ 3.1.3; Σ 3.1.9; $\langle P_n\rangle_n$ ω Q 3.1.11; $\langle P_n\rangle_n^*$ ω Q, p ω Q 3.1.14.

b) German letters.

$\mathfrak{A}$, $\mathfrak{P}$ 3.1.2; $\mathfrak{A}_i$ 3.3.2; $\mathfrak{A}_\pi$ 3.4.1.

c) Symbols for special spaces.

$\underline{R}$, $\underline{R}^n$, $\underline{R}^\infty$, $\underline{N}$, $\underline{Q}$, $\underline{I}$ 2.1.3; $\underline{H}$ 2.1.4; $\underline{F}$, $\underline{F}^o$ 2.1.5; $\underline{D}(\theta)$ 2.1.6.

d) Various symbols and notations.

$\langle V_o, \mathfrak{T} \rangle$	1.2.2
$\langle V_o, \mathfrak{T}, \# \rangle$	1.2.3
$^-$, V^-, $\underline{U}$	1.2.16
Int V	1.2.18
$\text{Int}^* V$	3.1.21
$U_\varepsilon(p)$, $U(\varepsilon, p)$, $U_\varepsilon(V)$, $U(\varepsilon, V)$	1.3.2
$\langle V_o, \mathfrak{T}(\rho) \rangle$	1.3.3
$Z(.)$	2.1.7
$+, \cdot$	3.1.2
$P \subset Q$	3.1.6
$P \subset V, V \subset P$	3.1.16
$\sim$	3.1.6
$\langle P_n \rangle_n \in Q$	3.1.10
$p \in Q, \langle P_n \rangle_n^* \in Q$	3.1.14
$\underline{\in}, \underline{\in}', \underline{\in}''$	3.1.21
$\langle P_n \rangle_n \# \langle Q_n \rangle_n$	3.1.11, 4.1.13
$p \# q$	3.1.14
$\simeq$	3.1.11, 4.1.13
$\langle P_n \rangle_n^*$	3.1.13, 4.1.13
$[P]$	3.1.16
$\Subset, \Subset', \Subset''$	3.1.18
$\Subset_P$	3.1.29
$\Subset^*$	4.1.4
$\langle \varphi, \underline{\Pi} \rangle$	3.1.25
ϑ^*, ϑ	3.2.2

LIST OF NOTIONS

For a number of intuitionistic notions see the references in 1.1.1.

BIBLIOGRAPHY

G. BIRKHOFF 1948, Lattice theory, Providence, Rh.I (1964).

L.E.J. BROUWER 1919, Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten. Zweiter Teil. Verhandelingen Akad. Amsterdam 12, N^{o} 7.

- 1920, Besitzt jede reelle Zahl eine Dezimalbruchentwickelung? Proc. Akad. Amsterdam 23, p 955-964.
- 1924, Beweis, dass jede volle Funktion gleichmässig stetig ist. Proc. Akad. Amsterdam 27, p 189-193.
- 1924A, Bemerkungen zum Beweise der gleichmässigen Stetigkeit voller Funktionen. Proc. Akad. Amsterdam 27, p 644-646.
- 1926, Ueber Definitionsbereiche von Funktionen. Math. Annalen 97, p 60-76.
- 1926A, Intuitionistische Einführung des Dimensionsbegriffes. Proc. Akad. Amsterdam 29, p 855-864.
- 1926B, Die intuitionistische Form des Heine-Borelschen Theorems. Proc. Akad. Amsterdam 29, p 866-867.
- 1954, Points and spaces. Canadian J. Math. 6, p 1-17.

D. van DALEN 1965, Seminar on Intuitionistic Mathematics, spring semester 1965. Mass. Institute of Technology. Mimeographed.

J.G. DIJKMAN 1952, Convérgentie en divergentie in de intuitionistische wiskunde, Thesis. Amsterdam.

W. FRANZ 1960, Topologie I, Allgemeine Topologie. Berlin.

H. FREUDENTHAL 1936, Zum intuitionistischen Raumbegriff. Compositio Math. 4, p 82-111.

Z. FROLÍK 1960, Generalizations of the G_δ-property of complete metric spaces. Czechoslovak Math. J. 10 (85), p 359-379.

- 1962, Relations of completeness. Czechoslovak Math. J. 12 (87), p 375-381.

A. HEYTING 1956, Intuitionism, an introduction. Amsterdam.

S.C. KLEENE & R. VESLEY 1965, The foundations of intuitionistic mathematics. Amsterdam.

C. KURATOWSKI, 1958, Topologie I, fourth edition. Warszawa.

K. MENGER 1940, Topology without Points. Rice Inst. Pamphlet 27, N^{o}1, p 80-107.

R.L. MOORE 1935, A set of axioms for plane analysis situs. Fund. Math. 25, p 13-28.

K. SCHULTZ 1965, Zum Aufbau der intuitionistischen Topologie. Diplomarbeit (unpublished).

H. de VRIES 1958, Algemene topologie. (Elaborated lecture notes after lectures by J. de Groot.) Amsterdam. Mimeographed.

A. WALD 1932, Ergebnisse eines Math. Kolloquiums, herausgegeben von K. Menger. Vienna 1932, Heft 3, p 6. (only known by reference).

INTRODUCTORY SURVEY

This survey must be understood as a rough draft of the plan of this thesis. Hence we do not aim at careful definitions, but we shall often use classical terminology to deal with intuitionistic notions; in this way it will be easier to grasp for a mathematician unexperienced in intuitionism what we are doing "classically spoken". Even in the formal definitions in this survey some subtleties will be disregarded.

Throughout this thesis we use the notation $\langle V_n \rangle_n$ as an abbreviation for a denumerably infinite sequence $V_1, V_2, \ldots$

The first chapter is mainly introductory; many topological notions are defined, often by definitions literally taken from classical topology. This is a tedious job, but it cannot be avoided, since many classically equivalent definitions represent different notions from an intuitionistic point of view, so we have to stipulate which definition we want to use. E.g. we define a topology by the family of open sets; but we have no reason to assume this definition to be equivalent to a definition by the family of closed sets. For an elucidating example see 2.1.8.

The last paragraph of the first chapter is devoted to the notions "(weakly)located pointset", "relatively located pointsets", and "located system".

These notions are only of importance with respect to closed pointsets; classically, every pointset is located, and every pair of closed pointsets is relatively located, but not so intuitionistically.

Consequently these notions present a typically intuitionistic element in the theory. The notion of a located pointset for example, is introduced to select from all possible pointsets certain pointsets with some constructive features which make them more manageable.

If $\underline{\cup}$ is an operation defined by $V_1 \ \underline{\cup}\ V_2 = (V_1 \cup V_2)^-$, a complete located system $\langle V_n \rangle_n$ can be defined as a family of sets, closed with respect to $\cap$, $\underline{\cup}$, such that every element is located, and every pair of elements is relatively located. The most striking property of a located system is the following:

$$\text{(I)} \quad V_{n_1} \cap V_{n_2} \cap \ldots \cap V_{n_k} = \emptyset \vee V_{n_1} \cap \ldots \cap V_{n_k} \neq \emptyset$$

can be decided constructively.

The second chapter treats separable metric spaces, and, like the first, contains mainly preliminary matters. An analogue of Lindelöf's theorem is obtained by intuitionistic methods.

A metric definition of compacta is given, and some theorems which will be used in the fourth chapter are proved, using the existence of a metric. It is especially important to note that the theorem of Heine-Borel can be proved intuitionistically for these spaces, a result already obtained by Brouwer (BROUWER 1926B).

In the third chapter we start with an axiomatic treatment of topology by introducing so-called intersection spaces (I-spaces).

To describe an I-space, we use a located system $\langle V_n \rangle_n = \mathfrak{C}$, and a set Π of so-called point generators. An element of Π is a centered system (a system with the finite intersection property) $\langle W_n \rangle_n$, $\wedge n(W_n \in \mathfrak{C})$, such that $\bigcap_{n=1}^{k} W_n$ "converges" to a point of the space with increasing k. This method of describing points is analogous to the introduction of real numbers by means of sequences of nested intervals.

Two point generators $\langle W_n \rangle_n$, $\langle W'_n \rangle_n$ are said to coincide (notation $\simeq$) if

(II) $$\wedge n(W_1 \cap \ldots \cap W_n \cap W'_1 \cap \ldots \cap W'_n \neq \emptyset).$$

The points of the space can be identified with the equivalence classes of coinciding point generators.

This method of primarily considering point generators instead of points conforms naturally to the intuitionistic point of view. A point is looked upon as an at any moment unfinished construction; in other words, at any given moment, a point is only known with a certain degree of accuracy (given by an initial segment of a point generator).

A relation $V \Subset W$ between pointsets V, W (analogous to the classical relation $V^- \subset$ Interior W) is defined by:

(III) $$\wedge \langle W_n \rangle_n \in \Pi \vee m(W_1 \cap \ldots \cap W_m \cap V = \emptyset \vee W_1 \cap \ldots \cap W_m \subset W).$$

A topology is introduced by defining $\underline{\in}$ and "open set" as follows:

(IV) $p \underline{\in} V$ iff for every point generator $\langle W_n \rangle_n$ which represents p, for a certain ν $W_1 \cap \ldots \cap W_\nu \Subset V$. V is called open if $V = \{p : p \underline{\in} V\}$.

In the second paragraph of the third chapter a number of postulates is introduced, among others

(V) $V, W \in \mathfrak{C}, V \Subset W \rightarrow \vee U \in \mathfrak{C} (V \Subset U \Subset W)$.

(VI) $V, W \in \mathfrak{C}, V \cap W = \emptyset \rightarrow \vee V' \in \mathfrak{C} \vee W' \in \mathfrak{C} (V \Subset V' \ \& \ W \Subset W' \ \& \ V' \cap W' = \emptyset)$.

(VII) Every point can be represented by a point generator $\langle W_n \rangle_n$ such that $\wedge n(W_{n+1} \Subset W_n)$.

Since $\mathfrak{C}$ is not necessarily closed with respect to complementation followed by closure, (V) and (VI) are in general not equivalent.

We could have required $\mathfrak{C}$ to be closed with respect to complementation followed by closure, but since the complement of a located species is not always located, this is a strengthening of our assumptions. We have preferred not to introduce postulates about complementation in this thesis.

I-spaces which satisfy VII are called IR-spaces. By the introduction of (VII) many simple properties can be proved, e.g. : $\{p\} \Subset V \leftrightarrow p \in V$; the interiors of the elements of $\mathfrak{C}$ constitute a basis for the space.

The postulates of the I-spaces were only just sufficient to introduce a topology; but the introduction of (VII) simplifies the theory considerably. $V \Subset W$ is now classically equivalent to $V^- \Subset$ Interior W, and the IR-spaces are classically equivalent to regular spaces with a countable base.

Other postulates, studied in this paragraph, are the so-called representation postulates.

The notion of a spread is typically intuitionistic. It can be considered as a strongly constructive version of the notion of a set.

An I-space is said to possess a spread representation, if there is a spread with point generators as elements, such that to every point generator of the space a coinciding point generator of the spread can be found.

For spreads very strong methods of proof are available; this accounts for the great importance of postulates concerning the existence of spread representations of certain kinds for I-spaces. The consequences of (VI)-(VII) and the representation postulates are amply discussed in the second paragraph; the results of this study are applied later on, especially in the third paragraph.

In the third paragraph, two axiom systems are introduced. The first system, defining the so-called CIN-spaces, is designed such that:

1) Classically the CIN-spaces coincide with the separable, completely metrizable spaces.
2) All the important results of the second paragraph can be applied to CIN-spaces.

3) A number of very important examples of metrizable spaces can be proved to be CIN-spaces, e.g.
   a) The separable hilbertspace,
   b) The space of all continuous functions on the closed interval, with the topology induced by the metric

$$\rho(f,g) = \sup \{|f(x)-g(x)| : x \in [0,1]\}.$$

   c) Almost all trees (sets of denumerably infinite sequences of natural numbers) with a certain "natural" metric (see 2.1.6), e.g. the topological product of a denumerably infinite sequence of copies of the natural numbers.
   d) All locally compact, metrizable, separable spaces.
   e) The topological product of a denumerable infinity of copies of the real line.

CIN-spaces are defined by means of a sequence of coverings $\langle \mathfrak{A}_i \rangle_i$, such that $\mathfrak{A}_{i+1} \subset \mathfrak{A}_i \subset \mathfrak{C}$ for all i, $\mathfrak{A}_i = \langle V_{i,n} \rangle_n$, and which satisfy some further postulates.
Point generators are sequences $\langle V_{i,n(i)} \rangle_i$, which are centered systems (possess the finite intersection property).

If Γ is a separable metric space, and the sequence of points $\langle p_i \rangle_i$ is dense in Γ, then classically one could take $\mathfrak{A}_i$ to be the set of all closed neighbourhoods $U_r(p_j)^-$, r rational, $r < i^{-1}$. A few of the most interesting properties of CIN-spaces are:

(VIII) If $\langle V_n \rangle_n$ is a covering, then $\langle \text{Interior } V_n \rangle_n$ is a covering too.

(IX) $V \Subset W \leftrightarrow \bigwedge p(p \notin V \vee p \in W)$.

Since the right hand side of this equivalence is classically equivalent to $V \subset W$, this is a remarkable property.

(X) Every mapping defined on a CIN-space into a separable, metric space is continuous.

(XI) A CIN-space is separable and metrizable.

The other axiom system introduced in the third paragraph defines the PIN-spaces, a specialization of the CIN-spaces. Now point generators can be defined explicitly, so Π can be eliminated as a primitive notion of the axiomatic theory. Π is defined by

(XII) $\langle W_n \rangle_n \in \Pi \leftrightarrow \bigwedge n(W_1 \cap \ldots \cap W_n \neq \emptyset)$ & $\bigwedge V \in \mathfrak{C} \bigwedge V' \in \mathfrak{C} (V \cap V' = \emptyset \rightarrow \bigvee n(W_1 \cap \ldots \cap W_n \cap V = \emptyset \vee W_1 \cap \ldots \cap W_n \cap V' = \emptyset))$.

Furthermore (V), (VI) are supposed to hold for PIN-spaces. The resulting axiom system is very strong, but natural. The examples mentioned for CIN-spaces sub (3) c-e are even PIN-spaces.

The fourth paragraph explicitly describes the construction as an I-space of the topological product of a denumerable infinity or a finite set of I-spaces. The most important result is that the topological product of a finite or denumerably infinite sequence of CIN-spaces is again a CIN-space.

The fifth paragraph treats the examples a-c, mentioned before; further, the set of rational numbers with the usual topology is proved to be an IR-space, while (IX) does not hold in this space. Thus the set of rational numbers is an example of an IR-space which is not a CIN-space.

The fourth chapter deals with locally compact, separable metrizable spaces (called LDFTK-spaces).

The first paragraph contains a summary of results from FREUDENTHAL 1936. Furthermore many lemmas and additional theorems are proved, in order to be able to link the theory of the third chapter to the results of Freudenthal, and to prepare the ground for the sequel. Two theorems in this paragraph are of special importance:

(XIII) Every DFTK-space as defined by Freudenthal (the intuitionistic analogue of a compactum) is a PIN-space.

(XIV) If $\{V_1, \dots, V_n\}$ is a covering of a DFTK-space, then a covering $\{V_1', \dots, V_n'\}$, $V_i' \in \mathfrak{C}$, $V_i' \Subset V_i$ for $1 \leqslant i \leqslant n$ can be found.

(XIV) can be looked upon as a specialization of (VIII) to DFTK-spaces, but a separate proof is given.

In the second paragraph the equivalence between a metric characterization and a "purely topological" characterization for LDFTK-spaces is proved. The topological definition characterizes these spaces as PIN-spaces which satisfy special conditions. Let V be the set of all points of an LDFTK-space Γ.

Then Γ is a PIN-space such that

(XV) $V' \in \mathfrak{C} \rightarrow V' = V$ or V' is a DFTK-space.

(XVI) If $V' \in \mathfrak{C}$ is a DFTK-space, a $V'' \in \mathfrak{C}$ can be found such that $V' \Subset V''$, V'' again a DFTK-space.

The third paragraph contains a number of covering theorems for LDFTK-spaces. If we agree to call a covering $\langle W_n \rangle_n$ star-finite if $\{W_i : W_i \cap W_\nu \neq \emptyset\}$ is finite for every ν, then the most important results can be formulated thus:

(XVII) Every open covering has a star-finite refinement consisting of elements of $\mathfrak{C}$.

(XVIII) If $\langle W_n \rangle_n$ is a covering of Γ, $\wedge n(W_n \in \mathfrak{C})$, then there exists a star-finite refinement $\langle W_n' \rangle_n$, $\wedge n(W_n' \in \mathfrak{C})$,

such that $\wedge n \vee m(W'_n \Subset W_m)$.
The fourth paragraph contains a proof of the following theorem:

(XIX) Every LDFTK-space can be metrized by a metric ρ such that every located pointset V of the space has a distance function, i.e. $\rho(p, V)$ is defined for every point p of the space. Γ is also metrically complete with respect to ρ.

It must be remarked that every pointset with a distance function in an arbitrary metric is located in the topology corresponding to this metric, but the converse implication does not hold good (see 2.1.9).

The fifth paragraph treats the topological product of a denumerably infinite sequence of LDFTK-spaces. The final result is:

(XX) If the defining located system $\langle V_n \rangle_n$ for every factor Γ (V is the set of points of Γ) can be chosen in such a way that

$$\wedge n(V_n = V \vee \vee m(V_n \cap V_m = \emptyset \ \& \ V_m \neq \emptyset)),$$

then the product is a PIN-space.

In classical mathematics, one can more or less distinguish set theory in its most general form from topology as a specialization of general set theory. (We are aware, however, of the absence of a sharp borderline.)

In intuitionism, it is much more difficult to make such a distinction; predicates which might be considered as to belong to set theory in its most general form from a classical point of view can be used to describe "typically topological" properties in intuitionism.

(IX) and (X) present striking examples; the intuitionistic theory of connectedness (not treated in this thesis) presents another example.

The contents of this thesis roughly correspond in classical topology to the contents of the first two chapters of de VRIES 1958.

CHAPTER I

TOPOLOGICAL SPACES

1. Intuitionistic notions.

1.1. The following notions due to Brouwer, are defined in HEYTING 1956: species 3.2.1; subspecies 3.2.4; congruency between species, 3.2.4, def.1; detachable, 3.2.4, def.2; infinitely proceeding sequence (ips) 3.1.1; spread, spread law, complementary law, (immediate) descendant, (immediate) ascendant 3.1.2; finitary spread or fan, 3.4.1; for the theory of real numbers see 2.2. The notion of an equivalence relation is defined as usual.

1.2. Definition. A spread X with a spread law θ and a complementary law ϑ is said to have a defining pair $\langle\theta, \vartheta\rangle$. The spread law is identified with the species of admissible finite sequences of natural numbers (cf. HEYTING 1956 3.1.2) A subspread Y of a spread X, defined by $\langle\theta, \vartheta\rangle$, is a spread $\langle\theta', \vartheta'\rangle$ such that θ' is a detachable subspecies of θ, and $\vartheta' = \vartheta | \theta'$.

1.3. Definition. If α is an ips, then the n^{th} component of α is denoted by $\alpha(n)$. The sequence $\langle\alpha(1), \ldots, \alpha(n)\rangle$ is written $\bar{\alpha}(n)$.
Let X be a spread with a defining pair $\langle\theta, \vartheta\rangle$, and let $\sigma \in \theta$, σ a sequence of length n. We suppose ϑ to be the identity. We define a spread element α_σ inductively as follows:
$\bar{\alpha}_\sigma(n) = \sigma$; for $k \geqslant 1$, $\alpha_\sigma(n+k)$ is the least number m, such that $\langle\alpha_\sigma(1), \ldots, \alpha_\sigma(n+k-1), m\rangle \in \theta$.

1.4. Definition. A species X is called secured or inhabitated if $\vee x (x \in X)$.

1.5. Definition. A binary relation # is called a pre-apartness relation in a species V, iff for all $a, b, c \in V$:

S1. $\neg\, a \# a$

S2. $a \# b \rightarrow a \# c \vee b \# c$.

It is easy to see that

(a) $a \# b \rightarrow b \# a$.

(b) If $='$ is defined by $a =' b \leftrightarrow \neg\, a \# b$, then $='$ is an equivalence relation.

If $a =' b \leftrightarrow a = b$ for every $a, b \in V$, then # is called an apartness relation in V. (HEYTING 1956 4.1.1). V is called discrete if $\wedge p \in V \wedge q \in V (p = q \vee p \# q)$.

1.6. Definition. If a mapping ψ from a species X into a species Y satisfies $\psi x = \psi y \leftrightarrow x = y$, then ψ is called a a bi-unique or one-to-one mapping.
Remark. If in a species X an apartness relation # is defined, and if ψ is a one-to-one mapping of X into Y, then in ψX an apartness relation #' can be defined by

$$\psi x \ \#' \ \psi y \leftrightarrow x \# y.$$

1.7. Definition. If ψ is a mapping from a species X into a species Y, and #, #' are apartness relations on X, Y respectively, then ψ is called strongly bi-unique (with respect to #, #') if

$$x \# y \leftrightarrow \psi x \ \#' \ \psi y.$$

1.8. Definition. The notions of finite and denumerably infinite species are defined in HEYTING 1956, 3.2.5. A species X is called quasi-finite if a finite species can be mapped onto X. A species X is called enumerable, if the natural numbers can be mapped onto X.

1.9. Let X be a spread consisting of infinitely proceeding sequences of natural numbers, and let $\equiv$ be an equivalence relation on X. The species of all equivalence classes of X with respect to $\equiv$ can be mapped bi-uniquely onto a species Y by a mapping ψ.
Z is a denumerably infinite species, let us say for the sake of convenience $Z = \underline{N}$, the species of natural numbers. Let P be a property such that

$$\wedge y \in Y \ \vee n(P(y, n)).$$

Intuitionistically this implies the existence (since Y is entirely determined by X and $\equiv$) of a mapping ψ' from X into $\underline{N}$, such that

$$\wedge y \in Y \ \wedge x(x \in \psi^{-1} y \rightarrow P(y, \psi' x)).$$

1.10. The following principle (stated e.g. in BROUWER 1924, 1924A, BROUWER 1926, called Brouwer's principle in KLEENE & VESLEY 1965, I, §7) will be much used in the sequel. It can be stated thus:
If X is a spread with a defining pair $\langle \theta, \vartheta \rangle$, ϑ the identity,

ψ a mapping of X into a denumerably infinite species Y, then there exists a mapping ψ' of θ into $\{0, 1\}$, such that for every $\alpha \in X$ there is exactly one natural number n such that $\psi'\bar{\alpha}(n) = 1$, and $\wedge \alpha \in X \wedge \beta \in X \wedge n(\psi'\bar{\alpha}(n) = 1 \ \& \ \bar{\alpha}(n) = \bar{\beta}(n) \rightarrow \psi\alpha = \psi\beta)$ or in a more informal manner:

There exists a method of computation which for every $\sigma \in \theta$ $(\sigma = \langle i_1, \ldots, i_k \rangle)$ indicates whether $\psi\alpha$ can be determined from $\bar{\alpha}(k)$ if $\bar{\alpha}(k) = \sigma$, or not.

Remark. This principle is often used only in a weaker form:

If X is a spread with a defining pair $\langle \theta, \vartheta \rangle$, and ψ is a mapping of X into a denumerably infinite species Y, then

$$\wedge \alpha \in X \vee n \wedge \beta \in X \ (\bar{\alpha}(n) = \bar{\beta}(n) \rightarrow \psi\alpha = \psi\beta).$$

1.11. Corollary to 1.10 (enumeration principle).

If X is a spread, and ψ a mapping of X into a denumerably infinite species Y, then ψ X is enumerable.

Proof. This follows from the fact that (in the notation of 1.10) $Z = \{\sigma : \sigma \in \theta \ \& \ \psi'\sigma = 1\}$ is detachable and contains at least one element. Therefore Z is enumerable, hence ψX too.

1.12. Theorem. (Fan theorem). If ψ is a mapping of a finitary spread X into the natural numbers, then there is a natural number n, such that for every $\alpha \in X$ $\psi\alpha$ is known from $\bar{\alpha}(n)$.

Proof in HEYTING 1956, 3.4.2 (or in BROUWER 1924, 1924A, 1926, 1954).

1.13. One of the most important consequences of the fan theorem is:

Theorem. A function which is defined everywhere on a closed interval of the real line, has a least upper bound and a greatest lower bound on the interval, and is uniformly continuous. If a function is defined and is positive everywhere on a closed interval, then the greatest lower bound of the function on the interval is positive. (see e.g. HEYTING 1956 3.4.3).

1.14. The intuitionistic notions of a lattice, distributivity, generators, free distributive lattice can be taken from BIRKHOFF 1948. See II, theorem 1; IX, 1; IX, 10.

2. Topological spaces.

2.1. Our intention is to describe in this paragraph a "frame" of fundamental notions, in order to decide what should be called topology.

We try to choose our notions so that they resemble the classical notions as closely as possible (otherwise there would be no reason to call it topology) and at the same time possess a reasonable amount of constructive content.

2.2. Definition. If V_0 is a species, and $\mathfrak{T}$ a certain species of subspecies of V_0 such that

T1. $\emptyset, V_0 \in \mathfrak{T}$,

T2. $V, W \in \mathfrak{T} \rightarrow V \cap W \in \mathfrak{T}$,

T3. The union of an arbitrary species of elements of $\mathfrak{T}$ again belongs to $\mathfrak{T}$,

then $\langle V_0, \mathfrak{T} \rangle$ is called a topological space. $\mathfrak{T}$ is called the topology on V_0; the elements of $\mathfrak{T}$ are called the open species (with respect to $\mathfrak{T}$, or in $\mathfrak{T}$) of V_0. The elements of V_0 are called the points of the space; subspecies of V_0 are called pointspecies. Speaking about a certain space, V^c denotes $V_0 - V$. V^c is called the complement of V (with respect to V_0).

If no confusion is likely to arise, we can also speak of V_0 as a topological space.

We indicate topological spaces by greek capitals Γ, Δ, indexed if necessary.

2.3. Definition. If V_0 is a species with an apartness relation #, then a topological space $\langle V_0, \mathfrak{T} \rangle$ which satisfies

T4. $p \in V \in \mathfrak{T} \ \& \ q \in V^c \rightarrow p \# q$

is called a "topological space with apartness relation", and is indicated by $\langle V_0, \mathfrak{T}, \# \rangle$, if we want to refer explicitly to the apartness relation.

2.4. Remark. We could have defined a topological space by means of the well known axioms of Kuratowski, but a relation between the topology and the apartness relation is most easily expressed in terms of open species. If a topological space with apartness relation satisfies

T5. $p, q \in V_0 \ \& \ p \# q \rightarrow \bigvee W(W \in \mathfrak{T} \ \& \ ((p \in W \ \& \ q \notin W) \vee (p \notin W \ \& \ q \in W)))$

then # can be characterized entirely in terms of open species.

2.5. Definition. A mapping ξ of a space $\langle V_0, \mathfrak{T} \rangle$ into a space $\langle V'_0, \mathfrak{T}' \rangle$ is called continuous if

$$V' \in \mathfrak{T}' \rightarrow \xi^{-1} V' \in \mathfrak{T}.$$

2.6. Definition. A homeomorphism ξ of a topological space

$\langle V_0, \mathfrak{T}\rangle$ onto $\langle V_0', \mathfrak{T}'\rangle$ is a bi-unique mapping of V_0 onto V_0', such that ξ, ξ^{-1} are continuous.
A notion is called a topological notion if it is invariant with respect to homeomorphisms, or to state it more precisely: If R is a predicate for species with n places, R is called a topological notion if for any topological spaces $\langle V_0, \mathfrak{T}\rangle$, $\langle V_0', \mathfrak{T}'\rangle$ which are homeomorphic by a homeomorphism ξ, and for any sequence $\langle V_1, \ldots, V_n\rangle$, $V_i \subset V_0$ for $1 \leqslant i \leqslant n$,

$$R(V_1, \ldots, V_n) \rightarrow R(\xi V_1, \ldots, \xi V_n).$$

2.7. Theorem. If ξ is a bi-unique mapping of a space $\Gamma = \langle V_0, \mathfrak{T}, \#\rangle$ onto a space $\Gamma' = \langle V_0', \mathfrak{T}', \#'\rangle$, and ξ^{-1} is continuous, Γ satisfies T1-5, Γ' satisfies T1-4, then ξ satisfies $x \# y \rightarrow \xi x \#' \xi y$.
Proof. Let $p, q \in V_0$, $p \# q$. A $V \in \mathfrak{T}$ can be found such that $p \in V$, $q \notin V$; $\xi V \in \mathfrak{T}'$, $\xi p \in \xi V$, $\xi q \notin V$; hence $\xi p \#' \xi q$.

2.8. Corollary to 2.7. If ξ is a homeomorphism from a space Γ into a space Γ', and Γ, Γ' satisfy T1-5, then ξ is strongly bi-unique.

2.9. Definition. A subspecies $\mathfrak{C} \subset \mathfrak{T}$ is called a basis for a topological space $\langle V_0, \mathfrak{T}\rangle$, if

$$V \in \mathfrak{T} \rightarrow V = \cup\{W : W \in \mathfrak{C} \;\&\; W \subset V\}.$$

2.10. Theorem.
a) A species $\mathfrak{C}$ of subspecies of V_0 is a basis for a topology on V_0 iff 1) $\cup\{W : W \in \mathfrak{C}\} = V_0$,
2) $W', W'' \in \mathfrak{C} \rightarrow W' \cap W'' = \cup\{W : W \in \mathfrak{C} \;\&\; W \subset W' \cap W''\}$.
b) If $\mathfrak{C}'$, $\mathfrak{C}''$ are two species, satisfying (1), (2) sub (a), then they determine the same topology iff
$p \in W \in \mathfrak{C}' \rightarrow \vee W'(W' \in \mathfrak{C}'' \;\&\; p \in W' \subset W) \;\&$
$p \in W \in \mathfrak{C}'' \rightarrow \vee W'(W' \in \mathfrak{C}' \;\&\; p \in W' \subset W)$
(Hausdorff criterion)

2.11. Definition. If $\langle V_0, \mathfrak{T}\rangle$ is a topological space, $p \in U$, and $\vee W(W \in \mathfrak{T} \;\&\; p \in W \subset U)$, then U is called a neighbourhood of p. If $U \in \mathfrak{T}$, then U is called an open neighbourhood of p.

2.12. Theorem. A mapping ξ from $\Gamma = \langle V_0, \mathfrak{T}\rangle$ into $\Gamma' = \langle V_0', \mathfrak{T}'\rangle$ is continuous iff

$$p \in V_0 \;\&\; \xi p \in W' \in \mathfrak{T}' \rightarrow \vee W(p \in W \in \mathfrak{T} \;\&\; \xi W \subset W').$$

2.13. Definition. If $<V_0, \mathfrak{T}>$ is a topological space, $W_0 \subset V_0$, then $\mathfrak{T}' = \{V \cap W_0 : V \in \mathfrak{T}\}$ is called the relative topology on W_0.

2.14. Definition. A topological space $<V_0, \mathfrak{T}>$ is said to be topologically embedded in a topological space $<V_0', \mathfrak{T}'>$, if there is a bi-unique mapping ξ from V_0 into V_0' such that ξV_0 provided with the relative topology is homeomorphic to $<V_0, \mathfrak{T}>$ by ξ.

2.15. Definition. p is a closure point of a pointspecies V if every neighbourhood of p contains a point of V.
p is a weak closure point of V if the intersection of every neighbourhood of p with V cannot be empty.

2.16. Definition. If V is a pointspecies, then V^- is the species of all closure points of V; V^- is called the closure of V (with respect to, or in, the given topology). V is closed (in the given topology) if $V^- = V$. $^-$ is called the closure operator of the topological space. Sometimes we shall write $V \underline{\cup} W$ for $(V \cup W)^-$.
A pointspecies V is dense in a space $\Gamma = <V_0, \mathfrak{T}>$ if $V^- \supset V_0$.

2.17. Theorem. In every topological space $<V_0, \mathfrak{T}>$ the following assertions are true for all $V, V_1, V_2 \subset V_0$:
$\emptyset^- = \emptyset$; $V_0^- = V_0$; $V \subset V^-$; $V^{--} = V^-$; $V_1 \subset V_2 \rightarrow V_1^- \subset V_2^-$; $(V_1^- \cup V_2^-)^- = (V_1 \cup V_2)^-$; $(V_1 \cap V_2)^- \subset V_1^- \cap V_2^-$.

2.18. Definition. p is an interior point of a pointspecies V of a topological space, if V is a neighbourhood of p. Int V is the species of interior points of V. (Int V is called the interior of V).

2.19. Remark. V is open iff Int V = V.

2.20. Theorem. In every topological space $<V_0, \mathfrak{T}>$ the following assertions are true: Int $\emptyset = \emptyset$; Int $V_0 = V_0$; Int $V \subset V$; Int V = Int Int V; $V_1 \subset V_2 \rightarrow$ Int $V_1 \subset$ Int V_2; Int$(V_1 \cap V_2)$ = Int $V_1 \cap$ Int V_2.

2.21. Theorem. If $<V_0, \mathfrak{T}>$ is a topological space, then $V \in \mathfrak{T} \rightarrow (V^c)^- = V^c$.

2.22. Definition. A mapping ξ of a space $\Gamma = <V_0, \mathfrak{T}_1>$ into a space $\Delta = <W_0, \mathfrak{T}_2>$ is called weakly continuous if

$$V \subset V_0 \rightarrow \xi(V^-) \subset (\xi V)^-.$$

2.23. Theorem. a) A mapping ξ of a space Γ into a space Δ is weakly continuous iff the counterimage of a closed set of Δ is always a closed set of Γ.
(b) A continuous mapping is weakly continuous.
Proof. (a) See FRANZ 1960, 5.4.
(b) Let $\Gamma = \langle V_0, \mathfrak{T}\rangle$, $\Delta = \langle V_0', \mathfrak{T}'\rangle$, ξ a continuous mapping from Γ into Δ. Let $V' \subset V_0'$, V' closed in Δ. If p is a closure point of $\xi^{-1}V'$, then ξp is a closure point of V'; for if $\xi p \in W' \in \mathfrak{T}'$, $p \in \xi^{-1}W' \in \mathfrak{T}$; hence there is a q such that $q \in \xi^{-1}W' \cap \xi^{-1}V'$; therefore $\xi q \in W' \cap V'$. We conclude that $\xi p \in V'$, hence $p \in \xi^{-1}V'$. The counterimage of a closed set is closed, hence by (a) ξ is weakly continuous.
Remark. In 2.1.8 a counter example is given to the inverse assertion of (b). This is a new argument in favour of the use of open sets to define a topology.

2.24. Definition. Let $\Gamma_1, \Gamma_2, \ldots$ be a finite or denumerably infinite sequence of topological spaces. $\Gamma_i = \langle V_0^i, \mathfrak{T}_i\rangle$.
We define a topology $\mathfrak{T}$ on the cartesian product $V_0^1 \times V_0^2 \times \ldots = V_0$ as follows. Let $\mathfrak{C}$ be the species of subspecies $V \subset V_0$, such that $V = V^1 \times V^2 \ldots$, $V^i \in \mathfrak{T}_i$, almost all V^i equal to V_0^i. $\mathfrak{C}$ is a basis for $\mathfrak{T}$; $\langle V_0, \mathfrak{T}\rangle$ is called the topological product of the Γ_i.
Remark. That $\mathfrak{C}$ satisfies (1), (2) of 2.10 is proved as usual.

2.25. Definition. Let X be an arbitrary species. If $\{X_i: i \in I\}$ is a family of species such that $\cup\{X_i: i \in I\} \supset X$, then $\{X_i: i \in I\}$ is called a covering of X.
If $\{X_i: i \in I\}$ is a covering of X, then every covering $\{X_i: i \in J\}$, $J \subset I$ of X is called a subcovering of $\{X_i: i \in I\}$.
If $\{X_i: i \in I\}$, $\{Y_j: j \in J\}$ are coverings of X, such that

$$\wedge j(j \in J \rightarrow \vee i(i \in I \ \& \ Y_j \subset X_i)),$$

then $\{Y_j : j \in J\}$ is called a refinement of $\{X_i : i \in I\}$. If $\{X_i : i \in I\}$ is a covering of X, and if $\{X_i : X_i \cap X_k \neq \emptyset\}$ is a quasi-finite species for every k, then the covering is called star-finite. A refinement of a covering which is a star-finite covering is called a star-finite refinement of the original covering.
A covering of a topological space by open sets is called an open covering of the space.

3. Metric spaces.

3.1. Definition. A metric space is a pair $\langle V_0, \rho\rangle$ of a species

V_0 and a non-negative function ρ from $V_0 \times V_0$ into the real numbers, such that for any $x, y, z \in V_0$:

a) $\rho(x,y) = 0 \leftrightarrow x = y$.

b) $\rho(x,y) = \rho(y,x)$.

c) $\rho(x,z) \not> \rho(x,y) + \rho(y,z)$.

ρ is called a metric on V_0.

Remark. In a metric space $\langle V_0, \rho \rangle$ an apartness relation can be introduced by

$$x \# y \leftrightarrow \rho(x,y) > 0.$$

This relation is called the apartness relation of the space.

3.2. Definition. If $\langle V_0, \rho \rangle$ is a metric space, $V \subset V_0$, $\varepsilon > 0$, then $U_\varepsilon(V) = U(\varepsilon, V) = \{q : q \in V_0 \ \& \ \vee p(p \in V \ \& \ \rho(p,q) < \varepsilon)\}$; $U_\varepsilon(p) = U_\varepsilon(\{p\}) = U(\varepsilon, p)$.

3.3. Theorem. With every metric space $\langle V_0, \rho \rangle$ a special topological space $\langle V_0, \mathfrak{T}(\rho) \rangle$ is associated, which satisfies T1-5, and for which $\{U(n^{-1}, p) : p \in V_0,$ n a natural number$\}$ is a basis. The relative topology on a species $V \subset V_0$ corresponds to the restriction of ρ to $V \times V$.

3.4. Definition. A topological space $\langle V_0, \mathfrak{T} \rangle$ is called metrizable if there is a metric space $\langle V_0, \rho \rangle$ such that $\mathfrak{T} = \mathfrak{T}(\rho)$. ρ is called a metric (or an adequate metric) for $\langle V_0, \mathfrak{T} \rangle$.

3.5. Remark. As no confusion is to be expected, we shall sometimes identify $\langle V_0, \rho \rangle$ and $\langle V_0, \mathfrak{T}(\rho) \rangle$ in our notation.

3.6. Theorem. If $\langle V_0, \rho \rangle$, $\langle V_0', \rho' \rangle$ are metric spaces, a mapping ξ from V_0 into V_0' is continuous with respect to $\langle V_0, \mathfrak{T}(\rho) \rangle$ and $\langle V_0', \mathfrak{T}(\rho') \rangle$ iff

$$\wedge y \wedge \varepsilon (y \in V_0 \rightarrow \vee \delta \ (\xi \ U_\delta(y) \subset U \ (\xi(y)))).$$

3.7. Definition. A sequence $\langle p_i \rangle_i \subset V_0$ is a fundamental sequence of a metric space $\langle V_0, \rho \rangle$ if

$$\wedge k \vee l \wedge n \wedge m(n, m > l \rightarrow \rho(p_n, p_m) < 2^{-k}).$$

$\langle p_i \rangle_i$ is said to converge to $p \in V_0$ if $\wedge k \vee l \wedge n(n > l \rightarrow \rho(p, p_n) < 2^{-k})$. p is the limit of the sequence.

3.8. Definition. A metric space is called complete, if every fundamental sequence converges to a limit.

A metrizable topological space $\langle V_0, \mathfrak{T} \rangle$ is called topologically complete, if for a certain metric ρ such that $\mathfrak{T} = \mathfrak{T}(\rho)$, $\langle V_o, \rho\rangle$ is complete.

3.9. Definition. A metric space $\langle V_o, \rho\rangle$ is embedded isometrically in a metric space $\langle V_o', \rho'\rangle$ if there is a bi-unique mapping ξ of V_o into V_o' such that $\rho(x,y) = \rho'(\xi x, \xi y)$. ξ is called an isometrism. If ξ is a mapping onto V_o', we say that $\langle V_o, \rho\rangle$ and $\langle V_o', \rho'\rangle$ are isometric.

3.10. Theorem. Every metric space $\langle V_o, \rho\rangle$ can be embedded isometrically in a complete metric space $\langle V_o', \rho'\rangle$ such that $V_o^- = V_o'$ in $\langle V_o', \mathfrak{T}(\rho')\rangle$.

3.11. Theorem. If ξ is a mapping of a metric space $\langle V_o, \rho\rangle$ into a metric space $\langle V_o', \rho'\rangle$ such that for every sequence $\langle p_i\rangle_i \subset V_o$

$$\lim_{i\to\infty} p_i = p \rightarrow \lim_{i\to\infty} \xi(p_i) = \xi(p)$$

then ξ is a weakly continuous mapping from $\langle V_o, \mathfrak{T}(\rho)\rangle$ into $\langle V_o', \mathfrak{T}(\rho')\rangle$.

3.12. Definition. Let $\langle V_o, \rho\rangle$ be a metric space; if $V \subset V_o$, $p \in V_o$, we say that the distance $\rho(V,p)$ is defined if there exists a real number d such that
a) $q \in V \rightarrow \rho(p,q) \nless d$.
b) For every natural number k there is a $q \in V$ such that $\rho(p,q) < d + 2^{-k}$.

d is denoted by $\rho(p,V)$ and is called the distance between p, V. Diameter V, if it exists, is equal to $\sup\{\rho(p,q): p,q \in V\}$.

3.13. Remark. If $\langle V_o, \rho\rangle$ is a metric space, then the closure operator $^-$ in $\langle V_o, \mathfrak{T}(\rho)\rangle$ is given by $V^- = \{p : \rho(p,V) = 0\}$.

3.14. Theorem. The topological product of a finite or denumerably infinite sequence of metrizable spaces Γ_i, $i = 1,2,\ldots$ is again metrizable.

Proof. We suppose $\Gamma_i = \langle V_o^i, \mathfrak{T}_i\rangle = \langle V_o^i, \mathfrak{T}(\rho_i)\rangle$. If we define $\bar\rho_i(x,y) = \inf\{\rho_i(x,y), 1\}$, $\bar\rho_i$ is a metric such that $\mathfrak{T}(\rho_i) = \mathfrak{T}(\bar\rho_i)$. Then the topological product Γ of $\Gamma_1, \Gamma_2, \ldots$ can be metrized by $\rho(x,y) = \sum_{i=1}^{\infty} 2^{-i}\bar\rho_i(x_i,y_i)$, $x, y \in V_o^1 \times V_o^2 \times \ldots$, $x = \langle x_i\rangle_i$, $y = \langle y_i\rangle_i$.

4. Located pointspecies.

4.1. Definition. A subspecies $V \subset V_0$ of a metric space $\langle V_0, \rho \rangle$ is called metrically located, if $\rho(p, V)$ is defined for every $p \in V_0$.

4.2. A subspecies $V \subset V_0$ is weakly located in a topological space $\langle V_0, \mathfrak{T} \rangle$ if

$$\wedge p \wedge W(p \in W \in \mathfrak{T} \to (\vee q(q \in W \cap V) \vee \vee W'(W' \in \mathfrak{T} \;\&\; p \in W' \subset W \;\&\; W' \cap V = \emptyset))).$$

If V is either secured or empty, then V is called located (in, or with respect to $\langle V_0, \mathfrak{T} \rangle$).

Remark. VAN DALEN 1965, p. 39 gives an analogous definition of "located" for the DFTK-spaces (there called F-spaces) introduced in FREUDENTHAL 1936. In view of a different approach to topology, definition 28 (§6) in SCHULTZ 1965 is also analogous to our definition. Since these definitions are conceived independently of each other, it seems to be a very natural generalization of BROUWER's definitions. See e.g. BROUWER 1919, p. 13; BROUWER 1926A.

4.3. Remarks. a) "weakly located" and "located" are topological notions.

b) For technical reasons, "located" is defined for arbitrary species, but in applications the notion is used for closed pointspecies only. Classically, every pointspecies is located.

4.4. Theorem. If $\mathfrak{C}$ is a basis for the topological space $\langle V_0, \mathfrak{T} \rangle$, then $V \subset V_0$ is weakly located iff

$$\wedge p \wedge W(p \in W \in \mathfrak{C} \to \vee q(q \in W \cap V) \vee \vee W'(W' \in \mathfrak{C} \;\&\; p \in W' \subset W \;\&\; W' \cap V = \emptyset)).$$

Proof. Trivial.

4.5. Corollaries. a) If $\langle V_0, \rho \rangle$ is a metric space, $V \subset V_0$ is located in $\langle V_0, \mathfrak{T}(\rho) \rangle$ iff

$$\wedge p \wedge \varepsilon(\vee q(q \in U_\varepsilon(p) \cap V) \vee \vee \delta(U_\delta(p) \cap W = \emptyset)).$$

b) If $V \subset V_0$ is metrically located in a metric space $\langle V_0, \rho \rangle$, then V is located in $\langle V_0, \mathfrak{T}(\rho) \rangle$.

4.6. Definition. Let $V_1 \subset V_0$, $V_2 \subset V_0$; V_1, V_2 are weakly located in $\langle V_o, \mathfrak{T}\rangle$. V_1, V_2 are called relatively located (with respect to each other) if

$$\wedge p \wedge W(p \in W \in \mathfrak{T} \rightarrow \vee U(p \in U \in \mathfrak{T} \ \& \ (\vee p_1(p_1 \in V_1 \cap U) \ \& \ \vee p_2(p_2 \in V_2 \cap U)) \rightarrow \vee p_3(p_3 \in W \cap V_1 \cap V_2))).$$

4.7. Remarks. a) "relatively located" is a topological notion. b) For technical reasons "relatively located" is defined with respect to arbitrary species, but in applications the notion is used for closed pointspecies only. Classically, every pair of closed pointspecies is relatively located.

4.8. Theorem. $V_1, V_2 \subset V_o$, V_1, V_2 weakly located in the topological space $\langle V_o, \mathfrak{T}\rangle$; $\mathfrak{C}$ a basis for $\mathfrak{T}$. V_1, V_2 are relatively located iff

$$\wedge p \wedge W(p \in W \in \mathfrak{C} \rightarrow \vee U(p \in U \in \mathfrak{C} \ \& \ (\vee p_1(p_1 \in U \cap V_1) \ \& \ \vee p_2(p_2 \in U \cap V_2) \rightarrow \vee p_3(p_3 \in W \cap V_1 \cap V_2))).$$

Proof. Trivial.

4.9. Corollary to 4.8. $V_1 \subset V_o$, $V_2 \subset V_o$, V_1, V_2 weakly located in the topological space $\langle V_o, \mathfrak{T}(\rho)\rangle$, corresponding to the metric space $\langle V_o, \rho\rangle$. V_1, V_2 are relatively located iff

$$\wedge p \wedge \varepsilon \vee \delta(\vee p_1(p_1 \in V_1 \cap U_\delta(p)) \ \& \ \vee p_2(p_2 \in V_2 \cap U_\delta(p)) \rightarrow \vee p_3(p_3 \in U_\varepsilon(p) \cap V_1 \cap V_2)).$$

Remark. This characterization can be considered to be derived from FREUDENTHAL 1936, 7.11, by transformation into a local property. (Compare also BROUWER 1919, p 18).

4.10. Theorem. If V_1, V_2 are weakly located and relatively located in the topological space $\langle V_o, \mathfrak{T}\rangle$, then $V_1 \cap V_2$ is also weakly located in $\langle V_o, \mathfrak{T}\rangle$. (In FREUDENTHAL 1936 7.12 a special case is proved).
Proof. Let $p \in W \in \mathfrak{T}$. There is a $U \in \mathfrak{T}$ such that if $U \cap V_1$, $U \cap V_2$ are secured, then also $W \cap V_1 \cap V_2$ is secured.
On the other hand

$$(U \cap V_1 \text{ is secured}) \vee \vee U_1(p \in U_1 \in \mathfrak{T} \ \& \ U_1 \cap V_1 = \emptyset).$$
$$(U \cap V_2 \text{ is secured}) \vee \vee U_2(p \in U_2 \in \mathfrak{T} \ \& \ U_2 \cap V_2 = \emptyset).$$

Hence

$(\vee p_1(p_1 \in U \cap V_1) \ \& \ \vee p_2(p_2 \in U \cap V_2)) \vee \vee U_1(p \in U_1 \in \mathfrak{T} \ \& \ U_1 \cap V_1 = \emptyset) \vee \vee U_2(p \in U_2 \in \mathfrak{T} \ \& \ U_2 \cap V_2 = \emptyset)$.

We obtain therefore:

$\vee p(p \in W \cap V_1 \cap V_2) \vee \vee U_1(p \in U_1 \in \mathfrak{T} \ \& \ U_1 \cap W \cap V_1 = \emptyset) \vee \vee U_2(p \in U_2 \in \mathfrak{T} \ \& \ U_2 \cap W \cap V_2 = \emptyset)$

Thus

$\vee p(p \in W \cap V_1 \cap V_2) \vee \vee U(p \in U \in \mathfrak{T} \ \& \ U \subset W \ \& \ U \cap V_1 \cap V_2 = \emptyset)$.

4.11. Theorem. a) The union of a quasi-finite species of (metrically) located pointspecies is again (metrically) located.
b) If V_1, V_2, V_3 are weakly located in a topological space $\langle V_0, \mathfrak{T} \rangle$, and V_1, V_2; V_1, V_3 are relatively located, then $V_1, V_2 \cup V_3$ are relatively located.
Proof. (a) trivial.
(b) Let $p \in W \in \mathfrak{T}$. There are W', W'' such that

$$\vee p'(p' \in W' \cap V_1) \ \& \ \vee p''(p'' \in W' \cap V_2) \rightarrow \vee p'''(p''' \in W \cap V_1 \cap V_2)$$
$$\vee p'(p' \in W'' \cap V_1) \ \& \ \vee p''(p'' \in W'' \cap V_3) \rightarrow \vee p'''(p''' \in W \cap V_1 \cap V_3).$$

We put $W''' = W' \cap W''$.

$$\vee p'(p' \in W''' \cap V_1) \ \& \ \vee p''(p'' \in W''' \cap (V_2 \cup V_3)) \rightarrow$$
$$\vee p'(p' \in W''' \cap V_1) \ \& \ \vee p''(p'' \in W''' \cap V_2 \vee p'' \in W''' \cap V_3)$$

Hence $\vee p'''(p''' \in W \cap (V_2 \cup V_3))$.

4.12. Theorem. Let $\langle V_n \rangle_n$ be a sequence of metrically located pointspecies in a metric space $\langle V_0, \rho \rangle$, such that $\wedge i(V_i \subset V_{i+1} \subset U(\varepsilon_i, V_i))$ and $\sum_{i=1}^{\infty} \varepsilon_i < \infty$. Then $V = \bigcup_{i=1}^{\infty} V_i$ is again a metrically located pointspecies.
Proof. We must prove for an arbitrary $p \in V_0$ the existence of $\rho(p, V)$. If $\rho(p, V)$ exists, then $\lim_{n \to \infty} \rho(p, V_n)$ exists, and conversely. $\lim_{n \to \infty} \rho(p, V_n) = \rho(p, V)$. Suppose $\rho(p, V_\nu) = d$, and let $\sum_{i=\nu}^{\infty} \varepsilon_i < \varepsilon$. If $q \in V_\mu$, $\mu \leqslant \nu$, then $\rho(p,q) \nless d$; if $\mu > \nu$, then there are $q_\nu, q_{\nu+1}, \ldots, q_\mu = q$, such that $q_i \in V_i$, $\rho(q_{i+1}, q_i) < \varepsilon_i$ for $\nu \leqslant i < \mu$.

Hence $\rho(q_\mu, q_\nu) \not> \rho(q_\mu, q_{\mu-1}) + \rho(q_{\mu-1}, q_{\mu-2}) + \ldots + \rho(q_{\nu+1}, q_\nu)$
$< \sum_{i=\nu}^{\mu-1} \varepsilon_i < \sum_{i=\nu}^{\infty} \varepsilon_i < \varepsilon$.
Hence $|\rho(p,q) - \rho(p,q_\nu)| < \varepsilon$; we conclude to:
$\rho(p,V_\nu) - \rho(p,V_\mu) < \varepsilon$. Therefore $\lim_{n\to\infty} \rho(p,V_n)$ exists.

4.13. Remark. a) If V is metrically located in a metric space $\langle V_0, \rho\rangle$, then V^- is metrically located.
If V is (weakly) located in a topological space $\langle V_0, \mathfrak{T}\rangle$, then V^- is (weakly) located, since if $p \in W \in \mathfrak{T}$, $q \in W \cap V$, then $q \in W \cap V^-$; and if $p \in W \in \mathfrak{T}$, $W \cap V = \emptyset$, then $W \cap V^- = \emptyset$. (For if $q \in W \cap V^-$, then there would be a $q' \in W \cap V$, because W is a neighbourhood for q.)
b) If V_1, V_2 are weakly located and relatively located, then V_1^-, V_2^-; V_1^-, V_2; V_1, V_2^- are relatively located. (This is seen by the same kind of reasoning as for (a)).

4.14. Definition. A system (species) $\mathfrak{C}$ of subspecies of V_0 is called a located system with respect to a topological space $\langle V_0, \mathfrak{T}\rangle$, if every finite intersection of elements of $\mathfrak{C}$ is again located, and if any two finite intersections W_1, W_2 of elements of $\mathfrak{C}$ are relatively located. A sequence which is a located system is called a located sequence. A located system, closed with respect to $\cap, \underline{\cup}$ is called a complete located system.

4.15. Lemma. $\langle V_0, \mathfrak{T}\rangle$ is a topological space. V, V', V'' are weakly located. V, V'' and V', V'' are relatively located, V'' is closed. Then $(V \cup V')^- \cap V'' = ((V \cap V'') \cup (V' \cap V''))^-$.
Proof. Suppose $p \in W \in \mathfrak{T}$, $p \in (V \cup V')^- \cap V''$. Then there are $W', W'' \in \mathfrak{T}$, $p \in W' \cap W''$ such that

$$\vee q(q \in W' \cap V) \;\&\; \vee q'(q' \in W' \cap V'') \rightarrow \vee q''(q'' \in W \cap V \cap V'')$$
$$\vee q(q \in W'' \cap V') \;\&\; \vee q'(q' \in W'' \cap V'') \rightarrow \vee q''(q'' \in W \cap V' \cap V'').$$

We put $W''' = W'' \cap W'$.
Then there exists a $q \in (V \cup V') \cap W'''$, hence $q \in V \cap W''' \vee q \in V' \cap W'''$; $p \in V'' \cap W'''$.
Therefore an $r \in W$ can be found, such that $r \in V \cap V'' \vee r \in V' \cap V''$.
Hence $r \in W \cap ((V \cap V'') \cup (V' \cap V''))$. Therefore $p \in ((V \cap V'') \cup (V' \cap V''))^-$. Conversely, we suppose $p \in ((V \cap V'') \cup (V' \cap V''))^-$, $p \in W \in \mathfrak{T}$.
Then there is a q, $q \in W \cap V \cap V'' \vee q \in W \cap V' \cap V''$; so $p \in (V \cup V')^-$, $p \in (V'')^- = V''$.

4.16. Lemma. $<V_0, \mathfrak{T}>$ is a topological space. $V, V', V'' \subset V_0$, V', V weakly located and relatively located. Then

$$((V \cap V') \cup V'')^- = (V \cup V'')^- \cap (V' \cup V'')^-.$$

Proof. We apply the rule $(V_1 \cap V_2)^- \subset V_1^- \cap V_2^-$ from 2.17.

$((V \cup V'') \cap (V' \cup V''))^- = ((V \cap V') \cup V'')^- \subset (V \cup V'')^- \cap (V' \cup V'')^-$.

Let $p \in W \in \mathfrak{T}$, $p \in (V \cup V'')^- \cap (V' \cup V'')^-$.
There is a $W' \in \mathfrak{T}$, $p \in W'$, $W' \subset W$ such that

$$\vee q(q \in W' \cap V) \ \& \ \vee q'(q' \in W' \cap V') \rightarrow \vee q''(q'' \in W \cap V \cap V').$$

Now there are $q_1, q_2, q_1 \in W' \cap (V \cup V'')$, $q_2 \in W' \cap (V' \cup V'')$. Consequently there is a $q, q \in W' \cap V'' \vee q \in W \cap V' \cap V$, so $p \in (V'' \cup (V' \cap V))^-$.

4.17. Theorem. If $<V_n>_n$ is a located sequence of closed pointspecies, then the system of pointspecies, obtained by closure of $<V_n>_n$ with respect to $\cap$, $\underline{\cup}$ is again a located system.
Proof. We call the closure $\mathfrak{C}$. Lemmas 4.15, 4.16 imply that for any V_i, V_j, V_k: $(V_i \underline{\cup} V_j) \cap V_k = (V_i \cap V_k) \underline{\cup} (V_j \cap V_k)$
$(V_i \cap V_j) \underline{\cup} V_k = (V_i \underline{\cup} V_k) \cup (V_j \underline{\cup} V_k)$.
These are the distributive laws with respect to $\cap$, $\underline{\cup}$; any element $W \in \mathfrak{C}$ can therefore be written as $W_1 \underline{\cup} W_2 \underline{\cup} \ldots \underline{\cup} W_n$, where the W_i are finite intersections of elements of $<V_n>_n$. Every $W \in \mathfrak{C}$ is therefore located (using 4.11 (a), 4.13 (a)), and if $W, W' \in \mathfrak{C}$ then $W \cap W'$ is located too. If W, W', W'' are located, and W, W'; W, W'' are relatively located, then $W, W' \underline{\cup} W''$ are relatively located by 4.11(b), 4.13 (b). In this way we can prove inductively, that every pair $W, W' \in \mathfrak{C}$ is relatively located.

CHAPTER II

SEPARABLE METRIC SPACES

1. Definitions and examples.

1.1. Definition. A topological space $<V_0, \mathfrak{T}>$ is called separable, if there is an enumerable sequence of points $<p_i>_i$ which is dense in the space.
A metric space $<V_0, \rho>$ is separable if $<V_0, \mathfrak{T}(\rho)>$ is separable. $<p_i>_i$ is called a basic pointspecies or basic species.

1.2. In this paragraph we introduce some special separable metric spaces: $\underline{R}$, $\underline{R}^n$, $\underline{R}^\infty$, $\underline{N}$, $\underline{Q}$, $\underline{F}$, $\underline{H}$, $\underline{D}(\theta)$, $\underline{I}$.
The corresponding topological spaces will be indicated by the same symbols; from the context it will be clear which meaning is intended.

1.3. Definition of $\underline{R}$, $\underline{R}^n$, $\underline{R}^\infty$, $\underline{N}$, $\underline{Q}$, $\underline{I}$.
$\underline{R}$ is the real line, with the usual metric $\rho(x,y) = |x-y|$. $\underline{R}^1 = \underline{R}$; $\underline{R}^n$ is the euclidean n-dimensional space, defined as usual, metrized by $\rho(x,y) = \sup\{|x_i - y_i| : 1 \leqslant i \leqslant n\}$, or by $\rho(x,y) = (\sum_{i=1}^{n} (x_i - y_i)^2)^{1/2}$.
$\underline{R}^\infty$ consists of all denumerably infinite sequences of real numbers, metrized by $\rho(x,y) = \sum_{i=1}^{\infty} 2^{-i} \rho_i(x_i, y_i)$, $\rho_i(x_i, y_i) = \inf\{1, |x_i - y_i|\}$, $x = <x_i>_i$, $y = <y_i>_i$. By 1.3.14, the topological space $\underline{R}^\infty$ is the topological product of a denumerably infinite sequence of spaces homeomorphic to $\underline{R}$. The rationals of $\underline{R}^\infty$ are all infinite sequences with all elements rational, and almost all elements zero. The species of rationals of $\underline{R}^\infty$ is a basic pointspecies.
$\underline{N}$ is the species of natural numbers, $\underline{Q}$ the species of rational numbers, $\underline{I} = [0,1]$ the closed interval; their metrics are obtained by restriction of the metric of $\underline{R}$.

1.4. Definition of $\underline{H}$.
$\underline{H}$ consists of all denumerably infinite sequences of real numbers $<x_i>_i$ such that $\sum_{i=1}^{\infty} x_i^2 < \infty$.
The rationals of $\underline{H}$ are the same as the rationals of $\underline{R}^\infty$, and form a basic pointspecies for $\underline{H}$. $\underline{H}$ is metrized by:

$$\rho(x,y) = (\sum_{i=1}^{\infty} (x_i - y_i)^2)^{1/2}; \quad x = <x_i>_i, \quad y = <y_i>_i.$$

1.5. Definition of $\underline{F}$.
The points of $\underline{F}$ are the functions from $\underline{I}$ into $\underline{R}$; $\underline{F}$ is metrized by

$$\rho(f,g) = \sup\{|f(x) - g(x)|: x \in [0,1]\}.$$

A function $f \in \underline{F}$ is called a rational polygonal function, if there exists a finite species of rational numbers $\{r_1, \dots, r_n\}$, $0 = r_1 < r_2 < \dots < r_n = 1$, such that

$$x \in [r_i, r_{i+1}] \rightarrow f(x) = a_i x + b_i, \quad a_i, b_i \in \underline{Q}, \quad 1 \leqslant i \leqslant n,$$

$$a_i r_{i+1} + b_i = a_{i+1} r_{i+1} + b_{i+1}, \quad 1 \leqslant i \leqslant n.$$

The species of rational polygonal functions will be denoted by $\underline{F}^o$. $\underline{F}^o$ is a basic pointspecies for $\underline{F}$.

1.6. Definition of $\underline{D}(\theta)$.
Suppose θ to be a spread law, ϑ a complementary law which is the identity. The spread $D(\theta)$ with a defining pair $\langle\theta, \vartheta\rangle$ is supposed to be metrized by the following well-known metric: $\alpha, \beta \in \underline{D}(\theta) \rightarrow \rho(\alpha,\beta) = \lim_{n\to\infty} (\mu(\bar{\alpha}(n), \bar{\beta}(n)) + 1)^{-1}$,
where $\mu(\bar{\alpha}(n), \bar{\beta}(n))$ is the least number $m \leqslant n$ such that $\bar{\alpha}(m) = \bar{\beta}(m)$ & $m + 1 \leqslant n \rightarrow \alpha(m+1) \neq \beta(m+1)$.
The species $\{\alpha_\sigma: \sigma \in \theta\}$ is a basic pointspecies for $\underline{D}(\theta)$.

1.7. Definition. We define a special predicate Z.
Z(n) holds iff n is the number of the last decimal of the first sequence of ten consecutive numerals 7 in the decimal representation of π.

1.8. Example.
We define a mapping ξ from $\underline{N}$ into $\underline{Q}$ by:
$\neg Z(n) \rightarrow \xi(n) = n$; $Z(n) \rightarrow \xi(n) = 1 + n^{-1}$.
We remark:
A) ξ is strongly bi-unique, as is readily seen.
B) ξ is continuous, since every subspecies of $\underline{N}$ is open.
C) ξ^{-1} is weakly continuous, since every $V \subset \xi\underline{N}$ is closed (as can be proved by showing $p \in V^- \subset \xi\underline{N} \rightarrow p \in V$).
D) ξ^{-1} cannot be proved to be continuous. For if we define $\neg Z(n) \rightarrow p_n = 1$, $Z(n) \rightarrow p_n = 1 + n^{-1}$, then $\langle p_n \rangle_n$ converges but we cannot prove the same for $\langle \xi^{-1} p_n \rangle_n$.

1.9. Example.
We construct another mapping ξ from $\underline{N}$ into $\underline{N}$:

$$\neg Z(n) \rightarrow \xi(n) = n + 1; \quad Z(n) \rightarrow \xi(n) = 1.$$

ξ is easily verified to be strongly bi-unique, ξ, ξ^{-1} are both continuous.
$V = \{m : m \in \underline{N} \ \& \ m > 2\}$ is a metrically located subspecies of $\underline{N}$. $\overline{\xi}V$ is not any longer metrically located in $\xi\underline{N}$, for
$2 \overline{\in} \xi \underline{N}$, $\neg \vee n(Z(n)) \to \rho(2, \xi V) = 2$, $\vee n(Z(n)) \to \rho(2, \xi V) = 1$; therefore $\rho(2, \xi V)$ cannot be calculated.

The property of being metrically located therefore depends on the metric.

1.10. Some other counterexamples to analogues of classical theorems can be found for example in HEYTING 1956: 5.1.3, after definition 4; 5.1.4, remark after theorem 1; 5.2.1, example.

2. Basic pointspecies and point representations.

2.1. Theorem. If $\langle p_i \rangle_i$ is a basic pointspecies for a metric space $\langle V_0, \rho \rangle$, then a basic pointspecies $\langle p_i^! \rangle_i \subset \langle p_i \rangle_i$ can be found which is discrete.

Proof. We construct a sequence $\langle i_n \rangle_n$ (which may contain repetitions) such that $p_{i_n} = p'_n$ for every n.

We put $i_1 = 1$, and construct the sequence $\langle i_n \rangle_n$ by steps. First step. We choose $i_2 = 1$ or $i_2 = 2$, such that the conditions (a_1) and (b_1) are met:

(a_1) $\rho(p_1, p_2) < 2^{-2} \to i_2 = 1$.

(b_1) $\rho(p_1, p_2) \nless 2^{-1} \to i_2 = 2$.

k^{th} step. We suppose to have constructed $i_1, i_2, \ldots, i_{n(k-1)}$ after the $(k-1)^{th}$ step; $\{i_1, \ldots, i_{n(k-1)}\} \subset \{1, 2, \ldots, k-1\}$. We order the different natural numbers occurring in $\{i_1, \ldots, i_{n(k-1)}\}$ according to increasing magnitude, and call them in this order $j_1, \ldots, j_q$ (hence $s < t \to j_s < j_t$); $q < k$. We order $\{1, 2, \ldots, k\} - \{j_1, \ldots, j_q\}$ after increasing magnitude, and call them $j_{q+1}, \ldots, j_k$. ($j_k = k$).

From $\langle j_1, \ldots, j_k \rangle$ we construct a sequence $\langle j'_1, \ldots, j'_k \rangle$ such that $j'_i = j_i$ for $i \leq q$; after the choice of $j'_1, \ldots, j'_{r-1}$, we choose for $j_r, r > q$, either j'_{r-1} or j_r, such that conditions (a_k), (b_k) are met:

(a_k) If there exists a $t \leq r-1$, such that $\rho(p_{j'_t}, p_{j_r}) < 2^{-k-1}$ then $j'_r = j'_{r-1}$.

(b_k) If for every $t \leq r-1$ $\rho(p_{j'_t}, p_{j_r}) \nless 2^{-k}$ then $j'_r = j_r$.

Then we take $i_{n(k-1)+t} = j'_{q+t}$ for $1 \leq t \leq k-q$. We see that $n(k) = n(k-1) + (k-q)$.

Now we prove the discreteness of $\langle p_{i_n} \rangle_n$ by proving the discreteness of $\langle p_{i_1}, \ldots, p_{i_{n(k)}} \rangle$ for every k.

Suppose already proved the discreteness of $\langle p_{i_1}, \ldots, p_{i_{n(k-1)}} \rangle$.

By conditions (a_k), (b_k) it is clear that $j_r, j_r \notin \{j'_1, \ldots, j'_{r-1}\}$, is included only in $\{j'_1, \ldots, j'_k\}$ if p_{j_r} lies apart from every element of $\{p_{j'_1}, \ldots, p_{j'_k}\}$. Therefore $\{p_{i_1}, \ldots, p_{i_{n(k)}}\}$ is also discrete.

There remains to be proved that $\langle p_{i_n} \rangle_n$ is a basic point-species. To see this we remark that for every p_l with $l < k$ either $p_l \in \{p_{i_1}, p_{i_2}, \ldots, p_{i_{n(k)}}\}$ or there exists an i_t $(1 \leqslant t \leqslant n(k))$, such that $\rho(p_l, p_{i_t}) < 2^{-k+1}$ (for if $l \notin \langle j_1, \ldots, j_q \rangle$ at the k^{th} step, there is a j_r such that $l = j_r$ $(r > q)$; $j_r \notin \langle j'_1, \ldots, j'_k \rangle$ implies $\rho(p_{j_r}, p_{j'_t}) < 2^{-k+1}$ $(t < r)$). Therefore, if we put $p_{i_n} = p'_n$, and $\langle p''_i \rangle_i$ is a converging sequence, $\langle p''_i \rangle_i \subset \langle p_i \rangle_i$, then we are able to find a sequence $\langle p'''_i \rangle_i$, $\langle p'''_i \rangle_i \subset \langle p'_i \rangle_i$, such that for every i $\rho(p'''_i, p''_i) < 2^{-i}$. (We only have to take, if $p''_j = p_l$, a k such that $l < k$, and to apply the preceding considerations.)

2.2. Definition. Let $\langle V_0, \rho \rangle$ be a metric space. We say that $\langle V_0, \rho \rangle$ has a point representation if there is a sequence $\langle p_i \rangle_i \subset V_0$ (the basis of the representation) and a spread with a defining pair $\langle \theta, \vartheta \rangle$ such that

a) $\langle i_1, \ldots, i_k \rangle \in \theta \rightarrow \vartheta \langle i_1, \ldots, i_k \rangle = \langle p_{i_1}, \ldots, p_{i_k} \rangle$.
b) Every spread element converges to a point of V_0.
c) For every $p \in V_0$ there exists a spread element converging to p.

2.3. Remark. (a) If X is a point representation of a metric space $\langle V_0, \rho \rangle$, with a defining pair $\langle \theta, \vartheta \rangle$, and basis $\langle p_i \rangle_i$, and if $\langle p_{i_n} \rangle_n \in X$, there is for every k a sequence $\langle i_1, \ldots, i_v \rangle \in \theta$, such that $\langle i_1, \ldots, i_v, j_{v+1}, \ldots, j_n \rangle \in \theta \longrightarrow \rho(p_{i_v}, p_{j_n}) < 2^{-k}$. (Since 2.2(b) implies that it must be possible to calculate for every $\langle p_{i_n} \rangle_n \in X$ a number m such that for $s, t \geqslant m$ $\rho(p_{i_s}, p_{i_t}) < 2^{-k}$, where m is already known from an initial segment of finite length $\langle p_{i_1}, \ldots, p_{i_r} \rangle$, we may suppose $r \geqslant m$.)
(b) If $\langle V_0, \rho \rangle$, $\langle V_0, \rho' \rangle$ are metric spaces, $\langle V_0, \mathfrak{T}(\rho) \rangle = \langle V_0, \mathfrak{T}(\rho') \rangle$, and $\langle V_0, \rho \rangle$ possesses a point representation, then $\langle V_0, \rho' \rangle$ too.

2.4. Definition. A point representation X with a defining pair $\langle \theta, \vartheta \rangle$, and a basis $\langle p_i \rangle_i$ is called uniform if for every k there exists an n such that

$$m > n \ \& \ \langle i_1, \ldots, i_m \rangle \in \theta \rightarrow \rho(p_{i_n}, p_{i_m}) < 2^{-k}.$$

2.5. Theorem. a) Every metric space with a point representation is separable.
b) Every complete separable metric space possesses a uni-

form point representation.
Proof. (a) is immediate from 2.3(a), since (in the same notation) if $\langle p_{i_n}\rangle_n$ converges to p, then $\rho(p, p_{i_v}) \nleq 2^{-k}$. Hence $\langle p_i\rangle_i$ is a basic pointspecies.
(b) Let $\langle p_i\rangle_i$ be a basic pointspecies for a metric space $\langle V_o, \rho\rangle$. We construct a spread X with a defining pair $\langle \theta, \vartheta\rangle$ such that

A) $\langle \emptyset\rangle \in \theta$; $i \in \underline{N} \rightarrow \langle i\rangle \in \theta$.
B) $\langle i_1, \ldots, i_{k+1}\rangle \,\overline{\in}\, \theta \rightarrow \rho(p_{i_{k+1}}, p_{i_k}) < 3.2^{-k}$.
C) $\langle i_1, \ldots, i_k\rangle \in \theta \;\&\; \rho(p_j, p_{i_k}) < 2^{-k+1} \rightarrow \langle i_1, \ldots, i_k, j\rangle \in \theta$.
D) $\vartheta \langle i_1, \ldots, i_k\rangle = \langle p_{i_1}, \ldots, p_{i_k}\rangle$.

This spread is a representation, since if $p \in V_o$, there is a sequence $\langle p_{i_n}\rangle_n$ such that $\rho(p_{i_n}, p) < 2^{-n}$; hence $\rho(p_{i_n}, p_{i_{n+1}}) \nleq \rho(p, p_{i_n}) + \rho(p, p_{i_{n+1}}) < 2^{-n} + 2^{-n-1} < 2^{-n+1}$. Therefore, by (C), $\langle p_{i_n}\rangle_n \in X$. The uniformity is a consequence of (B).

2.6. Theorem. (Intuitionistic analogue of Lindelöf's theorem).
If $\langle V_o, \rho\rangle$ is a metric space with a point representation, then every open covering of $\langle V_o, \mathfrak{T}(\rho)\rangle$ possesses an enumerable subcovering.
Proof. Let $\langle V_o, \rho\rangle$ be represented by a spread X with a defining pair $\langle \theta, \vartheta\rangle$, basis $\langle p_i\rangle_i$, and let $\{W_i : i \in I\}$ be an open covering of $\langle V_o, \mathfrak{T}(\rho)\rangle$.
To every point p of V_o natural numbers m, k and an element of the covering, W_i, can be found such that $p \in U(2^{-m}, p_k) \subset W_i$. Hence there are functions ψ_1, ψ_2 from X into $\underline{N}$, and a mapping ψ_3 from $\{\langle \psi_1\alpha, \psi_2\alpha\rangle : \alpha \in X\}$ into I, such that if $\alpha \in X$ converges to $p \in V_o$, then

$$p \in U(2^{-\psi_1(\alpha)}, p_{\psi_2(\alpha)}) \subset W_{\psi_3(\psi_1(\alpha), \psi_2(\alpha))}.$$

Since $\{\langle m, k\rangle : m, k \in \underline{N}\}$ is a denumerably infinite species, $\{\langle \psi_1(\alpha), \psi_2(\alpha)\rangle : \alpha \,\overline{\in}\, X\}$ is enumerable, as follows from the application of the enumeration principle to ψ, defined by $\psi(\alpha) = \langle \psi_1(\alpha), \psi_2(\alpha)\rangle$.
Hence $\{W_{\psi_3(\psi_1(\alpha), \psi_2(\alpha))} : \alpha \in X\}$ is an enumerable subcovering of $\{W_i : i \in I\}$.

3. Located compact spaces.

3.1. Definition. A complete metric space is called a metric located compact space (MLC-space) if it has a point representation by means of a finitary spread.

3.2. Definition. An ε-net for a metric space $\langle V_o, \rho\rangle$ is a quasi-finite pointspecies $V_\varepsilon = \{p_1, \ldots, p_n\} \subset V_0$ such that $\rho(p, V_\varepsilon) < \varepsilon$ for every $p \in V_0$.

3.3. Definition. A topological space is called compact, if every open covering includes a quasi-finite subcovering. A space is called ω-compact, if every enumerable covering (not necessarily open) possesses a quasi-finite subcovering.

3.4. Theorem. The following properties are equivalent.
a) $\langle V_0, \rho\rangle$ is an MLC-space.
b) $\langle V_0, \rho\rangle$ is a complete metric space with a point representation by means of a finitary spread with a discrete basis.
c) $\langle V_0, \rho\rangle$ is a complete metric space and possesses an ε-net for every $\varepsilon > 0$.
d) $\langle V_0, \rho\rangle$ is a complete metric space and $\langle V_0, \mathfrak{T}(\rho)\rangle$ is compact.
e) $\langle V_0, \rho\rangle$ is a complete separable metric space and $\langle V_0, \mathfrak{T}(\rho)\rangle$ is ω-compact.
Proof. Our proof follows the scheme: (a) → (d) → (c) → (b) → (a); (a) → (e) → (c).
(a) → (d) was proved in BROUWER 1926B, with a slightly different definition of MLC-space (there called "katalogisiert kompakte Raume"); but the method, which is analogous to the proof of 2.6 (with an application of the fan theorem instead of the enumeration principle) can be transferred without difficulty. (cf. also HEYTING 1956 5.2.2)
(d) → (c). $\{U_\varepsilon(p) : p \in V_0\}$ is an open covering of $\langle V_0, \mathfrak{T}(\rho)\rangle$. Hence there is a quasi-finite subcovering $\{U_\varepsilon(p_1), \ldots, U_\varepsilon(p_n)\}$, $\{p_1, \ldots, p_n\} \subset V_0$. Therefore $\{p_1, \ldots, p_n\}$ is an ε-net for $\langle V_0, \rho\rangle$.
(c) → (b). Let, for every k, $\{q_1^k, \ldots, q_{n(k)}^k\}$ be a 2^{-k-1}-net. We consider the sequence:
$q_1^1, q_2^1, \ldots, q_{n(1)}^1, q_1^2, \ldots, q_{n(2)}^2, q_1^3, \ldots, q_1^k, \ldots, q_{n(k)}^k, \ldots$
We denote the i^{th} member of this sequence by p_i'. $\langle p_i'\rangle_i$ is dense in $\langle V_0, \mathfrak{T}(\rho)\rangle$. We select according to theorem 2.1 a subsequence $\langle p_i\rangle_i \subset \langle p_i'\rangle_i$ which is discrete; $\langle p_i\rangle_i$ is again dense in $\langle V_0, \mathfrak{T}(\rho)\rangle$. As is seen from the definition of $\langle p_i'\rangle_i$, to every k a ν_k can be found such that $\{p_1', \ldots, p_{\nu_k}'\}$ is a 2^{-k-1}-net; we may suppose $\nu_k < \nu_{k+1}$ for every k. To every p_i', $1 \leqslant i \leqslant \nu_k$, a $p_{s(i)}$ can be found, such that $\rho(p_i', p_{s(i)}) < 2^{-k-1}$. Hence $\{p_{s(1)}, \ldots, p_{s(\nu_k)}\}$ is a 2^{-k}-net.

If we put $\mu_k = \sup\{s(i) : 1 \leqslant i \leqslant \nu_k\}$, $\{p_1, \ldots, p_{\mu_k}\}$ is a 2^{-k}-net. Let $\lambda_k = \sup\{\mu_k, \mu_{k-1} + 1\}$; $\{p_1, \ldots, p_{\lambda_k}\}$ is then also a 2^{-k}-net, and $\lambda_{n-1} < \lambda_n$ for every n.

Now we construct a finitary point representation by a modification of the proof of 2.5(b); we retain stipulations

(A), (B), (D), but change (C) into (E):

$$\text{(E)}\ \langle i_1,\ldots,i_k\rangle \in \theta \ \&\ \rho(p_j,p_{i_k}) < 2^{-k+1} \ \&\ 1 \leqslant j \leqslant \lambda_k \rightarrow \langle i_1,\ldots,i_k,j\rangle \in \theta.$$

If $p \in V_0$, there is a sequence $\langle p_{i_n}\rangle_n \subset \langle p_i\rangle_i$, such that $1 \leqslant i_n \leqslant \lambda_n$, $\rho(p_{i_n}, p) < 2^{-n}$. Then $\langle p_{i_n}\rangle_n$ is an element of the constructed spread.
(b) $\rightarrow$ (a) is trivial.
(a) $\rightarrow$ (e) is a straightforward application of the fan theorem. For suppose $\langle V_0, \rho\rangle$ to be represented by a finitary spread X, and let $\langle W_n\rangle_n$ be an enumerable covering of V_0. There exists a function ψ such that if $\alpha \in X$ represents p, then $p \in W_{\psi(\alpha)}$. By the fan theorem ψX is finite, therefore $\langle W_n\rangle_n$ has a quasi-finite subcovering.
(e) $\rightarrow$ (c). $\{U_\varepsilon(p) : p \in V_0\}$ is an open covering of $\langle V_0, \mathfrak{T}(\rho)\rangle$. By 2.5(b), 2.6 there is an enumerable subcovering $\langle U_\varepsilon(p_i)\rangle_i$, $\langle p_i\rangle_i \subset V_0$. Hence there is a quasi-finite subcovering $\{U_\varepsilon(p_{j_1}),\ldots,U_\varepsilon(p_{j_k})\}$; $\{p_{j_1},\ldots,p_{j_k}\}$ is therefore an ε-net for $\langle V_0, \rho\rangle$.

3.5. Definition. If $\langle V_0, \mathfrak{T}\rangle$ is a topological space such that for a certain metric ρ on V_0, $\langle V_0, \mathfrak{T}(\rho)\rangle = \langle V_0, \mathfrak{T}\rangle$, and $\langle V_0, \rho\rangle$ is an MLC-space, then $\langle V_0, \mathfrak{T}\rangle$ is called a located compact (LC-) space.

3.6. Remark. a) If $\langle V_0, \mathfrak{T}(\rho)\rangle$ is an LC-space, then $\langle V_0, \rho\rangle$ is an MLC-space.
b) An LC-space corresponds closely to the definition of "located compact topological space" in BROUWER 1954, as will be clear from comparison of the definition given there with 3.1, 3.4(b); an LC-space is always homeomorphic to a located compact topological space in the sense of BROUWER 1954.

3.7. Theorem. Let $\langle V_0, \rho\rangle$ be an MLC-space. $V \subset V_0$ is located in $\langle V_0, \mathfrak{T}(\rho)\rangle$ iff V is metrically located in $\langle V_0, \rho\rangle$.
Proof. In one direction the implication follows from 1.4.5(b). Let V be weakly located. To every $p \in V_0$ a natural number $k > \nu$ can be found such that F(k, p) holds, where

$$F(k,p) \leftrightarrow (U(2^{-k},p) \cap V = \emptyset \ \vee\ \vee q(q \in U(2^{-k},p) \cap V))$$

$\mathfrak{C} = \{U(2^{-k},p) : p \in V_0 \ \&\ F(k,p)\}$ is an open covering of V_0, hence by 3.4(d) there is a quasi-finite subcovering $\{U_1,\ldots,U_\mu\}$; $U_i \in \mathfrak{C}$ for $1 \leqslant i \leqslant \mu$.
Suppose $U_i \cap V = \emptyset$ for $\lambda < i \leqslant \mu$, $q \in U_i \cap V$ for $1 \leqslant i \leqslant \lambda$.

$\{q_1,\dots,q_\lambda\}$ is a $2^{-\nu}$-net for $\langle V,\rho\rangle$ since $V \subset \bigcup_{i=1}^{\lambda} U_i$.
Hence if $p \in V_0, q \in V$, then $\rho(p,q) \nless$
$\inf\{\rho(p,q_i) : 1 \leqslant i \leqslant \lambda\}-2^{-\nu}$, therefore $\rho(p,V)$ is defined.

3.8. Remark. From the proof of 3.7 follows: if $\langle V_0,\mathfrak{T}\rangle$ is an LC-space, then $V \subset V_0$ is located iff V is weakly located.

3.9. Remark. From the proof of 3.7 and from 3.4(c) it is clear that if $V \subset V_0$ is closed, located, then V (with the relative topology) is an MLC-space, and conversely (Cf. BROUWER 1926A, FREUDENTHAL 1936, 7.5, 7.7).

3.10. Theorem. Let $\langle V_0,\rho\rangle$ be a MLC-space; V_1, V_2 are two located subspecies of V_0. Then the following assertions are equivalent:

a) V_1, V_2 are relatively located.

b) $\wedge\varepsilon\, \vee\delta\, (\vee p\, \vee q\, (\rho(p,q) < \delta \;\&\; p \in V_1 \;\&\; q \in V_2) \rightarrow$
$\vee r(r \in V_1 \cap V_2 \;\&\; \rho(p,r) < \varepsilon))$.

c) $\wedge\varepsilon\, \vee\delta\, (U_\delta(V_1) \cap U_\delta(V_2) \subset U_\varepsilon(V_1 \cap V_2))$.

Proof. (a) $\rightarrow$ (c). Let ε be a fixed real number greater than zero.
To every $p \in V_0$ a $\delta < \varepsilon.2^{-1}$ can be found such that $F(p,\delta)$ holds, where $F(p,\delta) \leftrightarrow$
$\{\vee p(p \in V_1 \cap U_\delta(p)) \;\&\; \vee q(q \in V_2 \cap U_\delta(p)) \rightarrow \vee r(r \in U(2^{-1}\varepsilon,p) \cap V_1 \cap V_2)\}$.
The species $\{U(2^{-1}\delta,p) : p \in V_0 \;\&\; F(p,\delta)\}$ is an open covering of V_0.
By 3.4(d), there exists a finite subcovering $\{U_1,\dots,U_\mu\}$; let $U_i = U(2^{-1}\delta(p_i),p_i)$, $1 \leqslant i \leqslant \mu$. We put $\delta = \inf\{2^{-1}\delta(p_i) : 1 \leqslant i \leqslant \mu\}$. Suppose $q_1 \in V_1 \cap U_\delta(p)$, $q_2 \in V_2 \cap U_\delta(p)$. Then for a certain λ, $1 \leqslant \lambda \leqslant \mu$, $p \in U(2^{-1}\delta(p_\lambda),p_\lambda)$.

$$\rho(q_1,p_\lambda) < 2^{-1}\delta(p_\lambda) + \delta \ngtr \delta(p_\lambda),$$
$$\rho(q_2,p_\lambda) < 2^{-1}\delta(p_\lambda) + \delta \ngtr \delta(p_\lambda).$$

Hence there exists a $q_3 \in U(2^{-1}\varepsilon,p_\lambda) \cap V_1 \cap V_2$.

$$\rho(q_3,p) \ngtr \rho(q_3,p_\lambda) + \rho(p_\lambda,p) < 2^{-1}\varepsilon + 2^{-1}\delta(p_\lambda) \ngtr \varepsilon.$$

Hence $U_\delta(V_1) \cap U_\delta(V_2) \subset U_\varepsilon(V_1 \cap V_2)$.

(c) $\rightarrow$ (b)
Suppose for a certain δ,ε : $U_\delta(V_1) \cap U_\delta(V_2) \subset U_\varepsilon(V_1 \cap V_2)$.
If $p_1 \in V_1$, $p_2 \in V_2$, $\rho(p_1,p_2) < \delta$, then $p_1, p_2 \in U_\delta(V_1) \cap U_\delta(V_2) \subset U_\varepsilon(V_1 \cap V_2)$. Hence there is a $p_3 \in V_1 \cap V_2$ such that $\rho(p_1,p_3) < \varepsilon$, $\rho(p_2,p_3) < \varepsilon$.

(b) $\rightarrow$ (a) is trivial from 1.4.9.

Remark. (b) was given in FREUDENTHAL 1936, 7.11.

3.11. The following theorem is borrowed from FREUDENTHAL 1936.
Theorem. Let $<V_0, \rho>$ be an MLC-space.
a) If $V \subset V_0$ is metrically located in $<V_0, \rho>$, and $\delta, \varepsilon \in \underline{R}$, $0 < \delta < \varepsilon$, there exists a metrically located V' such that $U_\delta(V) \subset V' \subset U_\varepsilon(V)$. (FREUDENTHAL 1936 7.10)
b) If $W, V_1, \ldots, V_k$ are metrically located, $\varepsilon > 0$, then there is a metrically located W', $W \subset W' \subset U_\varepsilon(W)$, such that W' is relatively located with respect to each of $V_1, \ldots, V_k$. (FREUDENTHAL 1936, 7.14)
c) If $W, V_1, \ldots, V_k$ are metrically located, and pairwise relatively located, and if $\{V_1, \ldots, V_k\}$ is closed with respect to intersections, then all intersections constructed from $W, V_1, \ldots, V_k$ are pairwise relatively located. (FREUDENTHAL 1936, 7.15)
Proof. FREUDENTHAL 1936 presupposes another definition for the MLC-space. This does not present any difficulties for the proof of (b),(c), since if (a) is proved, the proofs for (b),(c) in FREUDENTHAL 1936 hold for our definition as well.

For our definition of an MLC-space, (a) is proved thus. Let X be the spread of a point representation as constructed in the proof of (c) $\rightarrow$ (b) from 3.4. If X has a defining pair $<\theta, \vartheta>$, then we construct an X' with a defining pair $<\theta', \vartheta'>$ as follows.
Let $3^{-1}(\varepsilon - \delta) = \varepsilon'$, $3.2^{-k+1} < \inf\{\varepsilon', \delta\}$.
$\{p_1, \ldots, p_{\lambda_k}\}$ is a 2^{-k}-net. We divide $\{p_1, \ldots, p_{\lambda_k}\}$ into two disjoint parts Y, Z such that

$$p_i \in Y \rightarrow \rho(p_i, V) < \varepsilon - \varepsilon'.$$

$$p_i \in Z \rightarrow \rho(p_i, V) > \delta + \varepsilon'.$$

We define $\theta' \subset \theta$ as follows.

$$\theta'_k = \{<i_1, \ldots, i_k> : <i_1, \ldots, i_k> \in \theta \ \& \ p_{i_k} \in Y\}.$$

θ' consists of all descendants and ascendants of elements of θ'_k. ϑ' is the restriction of ϑ to θ'.

If $\lim_{n \to \infty} \rho(p_{i_n}, p) = 0$, $<p_{i_n}>_n \in X'$, then $\rho(p_{i_k}, p) < 3.2^{-k+1} < \varepsilon'$, hence if X' represents a subspecies W of V_0, then $W \subset U_\varepsilon(V)$. If $\rho(p, V) < \delta$, then for a certain i, $1 \leq i \leq \lambda_k$, $\rho(p, p_i) < 2^{-k} < \varepsilon'$, hence $\rho(p_i, V) < \delta + \varepsilon'$, therefore $p_i \in Y$; there is a sequence $<p_{j_n}>_n$, $p_{j_k} = p_i$, $\rho(p_{j_n}, p) \rightarrow 0$. Hence $U_\delta(V) \subset W$. Comparing 3.9, 3.1, 2.5(a), 2.5(b) we see that $W^- = V'$ satisfies all requirements.

3.12. Lemma. We consider pointspecies in an MLC-space. Let $\langle U_i\rangle_i$, $\langle V_i\rangle_i$ be located systems of closed pointspecies, and let $\langle \delta_i\rangle_i$ be a sequence of real numbers greater then zero.
Then it is possible to construct a sequence of (closed) pointspecies, $\langle W_i\rangle_i$, such that $\langle W_i\rangle_i \cup \langle V_i\rangle_i$ is a located system, and

$$\wedge i(U_i \subset W_i \subset U(\delta_i, U_i)).$$

Proof. We proof the assertion by induction.
Suppose already constructed $W_1, \dots, W_{k-1}$, such that $\langle W_i\rangle_{i=1}^{k-1} \cup \langle V_j\rangle_j$ is a located system, and $U_i \subset W_i \subset U(\delta_i, U_i)$ for $1 \leqslant i \leqslant k-1$. Now we construct W_k. We write $\langle W_i\rangle_{i=1}^{k-1} \cup \langle V_j\rangle_j$ as $\langle V_j'\rangle_j$, with $V_i' = W_i$ for $1 \leqslant i < k$, $V_i' = V_{i-k+1}$ for $i \geqslant k$.
We construct a sequence $\langle W_{k,i}\rangle_i$ as follows.
$W_{k,1} = U_k$.
$W_{k,2}$ is a pointspecies, located and relatively located with respect to V_1'. $W_{k,1} \subset W_{k,2} \subset U(2^{-2}\delta_k, W_{k,1})$.
$W_{k,m+1}$ is a located pointspecies, located with respect to every element of $\mathfrak{C}_m$, the species of all finite intersections of elements from $\{V_1', \dots, V_m'\}$, and such that $W_{k,m} \subset W_{k,m+1} \subset U(\varepsilon(m), W_{k,m})$ (using 3.11(b)). $\varepsilon(m)$ is determined thus:
If $V \in \mathfrak{C}_{m-1}$, then V, $W_{k,m}$ are relatively located. Hence there is a δ^V such that (3.10(c)):
$U(\delta^V, V) \cap U(\delta^V, W_{k,m}) \subset U(2^{-m}, V \cap W_{k,m})$. $\varepsilon^m = \inf\{\delta^V : V \in \mathfrak{C}_{m-1}\}$, $\varepsilon(m) = \inf\{2^{-2}\varepsilon^m, 2^{-1}\varepsilon(m-1), \delta_k 2^{-m-1}\}$.

We put $W_k = (\bigcup_{i=1}^{\infty} W_{k,i})^-$. W_k is located, by 1.4.12 and 1.4.13(a).
Since $W_{k,m} \subset W_{k,m+1} \subset U(\delta_k 2^{-m-1}, W_{k,m})$ and $\sum_{n=2}^{\infty} 2^{-n}\delta_k = 2^{-1}\delta_k$, it follows that $\bigcup_{i=1}^{\infty} W_{k,i} \subset U(2^{-1}\delta_k, U_k)$, hence $W_k \subset U(\delta_k, U_k)$.
Finally, we have to show that $W_k \cup \langle V_j'\rangle_j$ is a located system.
To prove this, it is sufficient to prove that W_k is relatively located with respect to any finite intersection of elements of $\langle V_j'\rangle_j$ (as follows from 3.11(c)).
Suppose V is a finite intersection of elements of $\langle V_j'\rangle_j$, so $V \in \mathfrak{C}_n$ for a certain n. Let $n < m$, $2^{-m} < \varepsilon$.
We remark: $\varepsilon(m+1) < 2^{-1}\varepsilon(m)$; $\varepsilon(m+k) < 2^{-1}\varepsilon(m+k-1)$; hence $\sum_{k=0}^{\infty} \varepsilon(m+k) < 2\varepsilon(m) < 2^{-1}\varepsilon^m$.

Therefore, $W_k \subset U(2^{-1}\varepsilon^m, W_{k,m})$.
We obtain
$U(\varepsilon(m),V) \cap U(\varepsilon(m),W_k) \subset U(\varepsilon^m,V) \cap U(2^{-2}\varepsilon^m, U(2^{-1}\varepsilon^m, W_{k,m})) \subset U(\varepsilon^m,V) \cap U(\varepsilon^m, W_{k,m}) \subset U(2^{-m}, V \cap W_{k,m}) \subset U(2^{-m}, V \cap W_k) \subset U(\varepsilon, V \cap W_k)$.
Hence by 3.10(c), W_k, V are relatively located.

3.13. Theorem. (BROUWER 1954, p.17). Every mapping ξ of an LC-space into an LC-space is uniformly continuous. Expressed metrically:

$$\wedge\varepsilon \ \vee\delta \ \wedge y(\xi U_\delta(y) \subset U_\varepsilon(\xi y)).$$

CHAPTER III

INTERSECTION SPACES

1. Definition of intersection spaces.

1.1. In this chapter we want to give an axiomatic treatment of a certain kind of topological spaces, which will be called intersection spaces. The most important feature of this treatment is the characterization of these spaces by means of a species of closed pointspecies with decidable intersection relations, (The pointspecies $V_1, V_2, \ldots, V_n$ have the intersection relation if their intersection contains a point) in the same manner as in FREUDENTHAL 1936.

In this paragraph we shall restrict ourselves to I-spaces, defined by means of a set of postulates, strong enough to ascertain the existence of a topology, but not much more.

In the second paragraph we shall introduce stronger postulates, some of them rather complicated, which are used as tools in proving the theorems about spaces defined in a more graceful manner in the third paragraph.

By this procedure one gets a clearer insight in the import of the different postulates than by starting from the strongest suppositions at once. In this way there are also more possibilities to incorporate a part of the theory in the development of other postulate systems.

1.2. Definitions. We start with a denumerably infinite sequence of formal objects, $A_1, A_2, \ldots$. This sequence is indicated by $\mathfrak{A}$. We construct the free distributive lattice $\mathfrak{P}$ with $\mathfrak{A}$ as a denumerably infinite species of generators, and with a zero-element A_0 and an all-element A_∞; the lattice-operators, join and meet, are written +, •, respectively; often the dot "•" will be omitted, so in this case the meet is denoted by a simple juxtaposition.

$\mathfrak{A}$ is called the lattice basis. If in the sequel we speak of lattice elements without further specification, elements of $\mathfrak{P}$ are meant. Arbitrary elements of $\mathfrak{P}$ will be marked by capitals P, Q, R, S, T, indexed if necessary.

γ indicates a fixed bi-unique mapping of N onto $\mathfrak{P}$; $\gamma^{-1} = \gamma'$.

1.3. We remark that for every two expressions constructed from elements of $\mathfrak{A}$ by means of +, • it can be decided whether they represent the same element of $\mathfrak{P}$ or not.

(See BIRKHOFF 1948, p. 145). We use the following notations for meet and join of a finite number of lattice elements:

$\sum_{i=1}^{n} P_i = \Sigma\{P_1, \ldots, P_n\} = P_1 + \ldots + P_n.$

$\prod_{i=1}^{n} P_i = \Pi\{P_1, \ldots, P_n\} = P_1 \cdot P_2 \cdot \ldots \cdot P_n = P_1 \ldots P_n.$

Every element of $\mathfrak{P}$ can be represented as:

$$\sum_{\sigma_i \epsilon \tau} (\prod_{j \epsilon \sigma_i} A_j) \quad (\tau \text{ finite}) \qquad (*)$$

If we require $\langle \sigma_n \rangle_n$ to be an enumeration of all finite species (ordered in natural order) of natural numbers, without repetitions, and if

$$i \neq j;\ i, j \in \tau \rightarrow \neg\, \sigma_i \subset \sigma_j \ \&\ \neg\, \sigma_j \subset \sigma_i,$$

this representation is unique, i.e. different expressions represent different elements of $\mathfrak{P}$.

1.4. We introduce a mapping φ from $\mathfrak{A} \cup \{A_0, A_\infty\}$ into $\{0, 1\}$, which fulfils the following conditions:

I 1. $\wedge n(\varphi A_n = 1)$, $\varphi A_0 = 0$, $\varphi A_\infty = 1$

I 2. $\varphi A_{n_1} A_{n_2} \ldots A_{n_s} = 1 \ \&\ \{m_1, \ldots, m_t\} \subset \{n_1, \ldots, n_s\} \longrightarrow \varphi A_{m_1} A_{m_2} \ldots A_{m_t} = 1.$

φ can be extended to $\mathfrak{P}$ by stipulating:

$\varphi(P_1 + \ldots + P_n) = 1 \leftrightarrow \vee i(1 \leqslant i \leqslant n \ \&\ \varphi P_i = 1).$

Such an extension is possible in a unique way, as follows from the possibility of representing the lattice elements in a unique way by expressions such as (*) in 1.3.

1.5. Remark. $\wedge n(\varphi A_n A_\infty = 1)$, $\wedge n(\varphi A_n A_0 = 0)$.

1.6. Definition. $P \subset_\varphi Q \leftrightarrow \wedge n(\varphi PA_n = 1 \rightarrow \varphi QA_n = 1).$
$P \sim_\varphi Q \leftrightarrow P \subset_\varphi Q \ \&\ Q \subset_\varphi P.$

Remark. Here and in the sequel we define relations and operations with respect to φ; but in the notation, as long as no confusion can arise, we omit the explicit reference to φ, so we write $P \subset Q$ instead of $P \subset_\varphi Q$, etc..

We postulate:

I 3. $P \subset Q \rightarrow PR \subset QR$, for all P, Q, R.

1.7. A number of very elementary properties of φ and the derived relations $\subset$, $\sim$ are combined in the following theorem.

Theorem. For all P, Q, R, P', Q':

a) $\varphi P = 1 \leftrightarrow \vee n(\varphi PA_n = 1)$; $\wedge n(\varphi PA_n = 0) \leftrightarrow \varphi P = 0$; $P \sim Q \rightarrow (\varphi P = 1 \leftrightarrow \varphi Q = 1)$.

b) $\varphi PQ = 1 \rightarrow \varphi P = 1$; $\varphi P = 0 \rightarrow \varphi PQ = 0$. More general:

$\varphi P_1 \ldots P_n = 1$ & $\{m_1, \ldots, m_t\} \subset \{1, \ldots, n\} \rightarrow$
$\varphi P_{m_1} \ldots P_{m_t} = 1$.

c) $P \subset Q$ & $Q \subset R \rightarrow P \subset R$; $P \subset P$.

d) $\sim$ is an equivalence relation.

e) $P \subset Q \rightarrow P + R \subset Q + R$.

f) $P \sim Q \rightarrow P + R \sim Q + R$ & $PR \sim QR$.

g) $\varphi P = 0 \leftrightarrow P \sim A_0$; $\Lambda n(\varphi PA_n = 1) \leftrightarrow P \sim A_\infty$.

h) $P \sim P'$ & $Q \sim Q'$ & $P \subset Q \rightarrow P' \subset Q'$.

i) $P \subset Q \leftrightarrow PQ \sim P \leftrightarrow P + Q \sim Q$; $P \subset Q$ & $\varphi P = 1 \rightarrow$
$\varphi Q = 1$.

j) $P \subset Q \rightarrow PR \subset Q$ & $P \subset Q + R$; $PR \subset P$; $P \subset P + R$.

k) $P \subset Q$ & $P \subset R \rightarrow P \subset QR$.

l) $P \subset R$ & $Q \subset R \rightarrow P + Q \subset R$.

m) $P \subset Q + R$ & $\varphi PR = 0 \rightarrow P \subset Q$.

Proof: Most of the assertions are trivial consequences of the definitions.

(a) By 1.3 (*) $P = \sum_{\sigma_i \varepsilon \tau} (\prod_{j \varepsilon \sigma_i} A_j)$; $\varphi P = 1$ implies that for a certain σ_μ $\varphi(\prod_{j\varepsilon\sigma_\mu} A_j) = 1$; hence for a certain $\nu \epsilon \sigma_\mu$, $\varphi A_\nu P = 1$.
Conversely, suppose $\varphi PA_\nu = 1$. $PA_\nu = \sum_{\sigma_i \varepsilon \tau} (A_\nu \prod_{j\varepsilon\sigma_i} A_j)$.
$\varphi PA_\nu = 1$ implies that for some μ, $\varphi A_\nu(\prod_{j\varepsilon\sigma_\mu} A_j) = 1$; if $\lambda \epsilon \sigma_\mu$, then $\varphi A_\nu A_\lambda = 1$.

So $\vee n(\varphi PA_n = 1)$. The second assertion of (a) follows from the first by negating both sides. The third assertion is immediate from 1.6, the second and the first assertion.

(b) The second and the third assertion follow from the first.
Let $\varphi PQ = 1$, $P = P_1 + \ldots + P_\nu$, $Q = Q_1 + \ldots + Q_\mu$, $P_1, \ldots, P_\nu$, $Q_1, \ldots, Q_\mu$ meets of elements of $\mathfrak{A}$.
$\varphi PQ = 1 \rightarrow \vee i \; \vee j(\varphi P_i Q_j = 1)$. Suppose $\varphi P_\lambda Q_\sigma = 1$.
By application of I 2 we obtain $\varphi P_\lambda = 1$; hence also $\varphi P = 1$.

(c), (d), (e), (f) are easily verified.

(g) $\varphi P = 0 \leftrightarrow \Lambda n(\varphi PA_n = 0)$ (by (a)).
$\Lambda n(\varphi PA_n = 0)$ & $\Lambda n(\varphi A_0 A_n = 0) \leftrightarrow \Lambda n(\varphi PA_n = 0 \leftrightarrow \varphi A_0 A_n = 0) \leftrightarrow P \sim A_0$.
$P \sim A_\infty \leftrightarrow \Lambda n(\varphi PA_n = 1 \leftrightarrow \varphi A_\infty A_n = 1) \leftrightarrow \Lambda n(\varphi PA_n = 1)$.

(h) is trivial.

(i) If $PQ \sim P$ & $\varphi PA_\nu = 1$, then $\varphi PQA_\nu = 1$, hence by (b) $\varphi QA_\nu = 1$. So $P \subset Q$.
If $P \subset Q$, and $\varphi PQA_\nu = 1$, we have obtained also $\varphi PA_\nu = 1$.
So $PQ \subset P$. By I 3 : $P \subset Q \rightarrow PP \subset QP$, so $PQ \sim P$.
If $P + Q \sim Q$, $\varphi PA_\nu = 1$, we obtain $\varphi(P + Q)A_\nu = 1$, and therefore $\varphi QA_\nu = 1$, hence $P \subset Q$. $P \subset Q$ & $\varphi PA_\nu = 1 \rightarrow \varphi QA_\nu = 1$, and so we have $\varphi(P + Q)A_\nu = 1 \rightarrow \varphi QA_\nu = 1$.
Hence $P + Q \sim Q$, since $\varphi QA_\nu = 1 \rightarrow \varphi(P + Q)A_\nu = 1$ holds trivially for every ν. The second part follows from the first and (b).

(j), (k), (l) are proved by simple verification.
(m) $P \subset Q + R \to P(Q + R) \sim P$, so $PQ + PR \sim P$. $\varphi\, PR = 0$, hence $PR \sim A_0$. By (f) $PQ + PR \sim PQ + A_0 = PQ$; by (d) $PQ \sim P$, and by (i) $P \subset Q$.

1.8. Definition. A sequence of lattice elements $\langle P_n \rangle_n$ is called a centered system if $\wedge n(\varphi\, P_1 \ldots P_n = 1)$.

1.9. We introduce a certain species Π of centered systems, the species of point generators. Π is, just like φ, a primitive notion in our axiomatic theory (i.e. a notion, not defined by means of other notions). If we specialize φ, Π in describing special kinds of topological spaces, Π can be defined explicitly sometimes. We require (splitting axiom):

I 4. $\varphi\, RQ = 0 \;\&\; \langle P_n \rangle_n \in \Pi \to \vee n(\varphi\, P_1 \ldots P_n R = 0 \vee \varphi\, P_1 \ldots P_n Q = 0)$ for all R, Q and $\langle P_n \rangle_n \in \Pi$.

The condition $\varphi\, RQ = 0 \to \vee n(\varphi\, P_1 \ldots P_n R = 0 \vee \varphi\, P_1 \ldots P_n Q = 0)$ for a centered system will be called the splitting condition with respect to R, Q.

The species of all centered systems which fulfil the splitting condition with respect to every pair R, Q and which contain at least one lattice element not equal to A_∞ (splitting systems) will be indicated by Σ.

1.10. Definition. We define a membership relation between a point generator $\langle P_n \rangle_n$ and a lattice element Q by:

$$\langle P_n \rangle_n \in Q \leftrightarrow \wedge m(\varphi\, P_1 \ldots P_m Q = 1).$$

We require

I 5. For every P, $\varphi\, P = 1 \to \vee \langle R_n \rangle_n \in \Pi\, (\langle R_n \rangle_n \in P)$.

1.11. Definition. $\langle P_n \rangle_n$, $\langle Q_n \rangle_n \in \Pi$.
$\langle P_n \rangle_n \# \langle Q_n \rangle_n \leftrightarrow \vee m(\varphi\, P_1 \ldots P_m Q_1 \ldots Q_m = 0)$
$\langle P_n \rangle_n \simeq \langle Q_n \rangle_n \leftrightarrow \wedge m(\varphi\, P_1 \ldots P_m Q_1 \ldots Q_m = 1)$
$\langle P_n \rangle_n\; \omega\; R \leftrightarrow \vee m\, (\varphi\, P_1 \ldots P_m R = 0)$

1.12. Theorem. For all $\langle P_n \rangle_n$, $\langle Q_n \rangle_n$, $\langle R_n \rangle_n \in \Pi$, and all Q, R:
a) $\#$ is a pre-apartness relation.
b) $\neg \langle P_n \rangle_n \# \langle Q_n \rangle_n \to \langle P_n \rangle_n \simeq \langle Q_n \rangle_n$
c) $\langle P_n \rangle_n\; \omega\; Q \;\&\; Q \sim R \to \langle P_n \rangle_n\; \omega\; R$
d) $\neg \langle P_n \rangle_n\; \omega\; Q \leftrightarrow \langle P_n \rangle_n \in Q$; $\neg\neg \langle P_n \rangle_n \in Q \to \langle P_n \rangle_n \in Q$.
e) $\langle P_n \rangle_n\; \omega\; Q \;\&\; \langle R_n \rangle_n \in Q \to \langle P_n \rangle_n \# \langle R_n \rangle_n$;
$\langle P_n \rangle_n \simeq \langle R_n \rangle_n \;\&\; \langle P_n \rangle_n\; \omega\; Q \to \langle R_n \rangle_n\; \omega\; Q$.
f) $\langle P_n \rangle_n\; \omega\; R \;\&\; Q \subset R \to \langle P_n \rangle_n\; \omega\; Q$; $\langle P_n \rangle_n\; \omega\; Q \to \langle P_n \rangle_n\; \omega\; QR$.
g) $\langle P_n \rangle_n\; \omega\; Q \;\&\; \langle P_n \rangle_n\; \omega\; R \leftrightarrow \langle P_n \rangle_n\; \omega\; (Q + R)$.

Proof. (a) The symmetry is immediate. $\langle P_n\rangle_n$ # $\langle Q_n\rangle_n$ implies:
for a certain ν $\varphi P_1 \ldots P_\nu Q_1 \ldots Q_\nu = 0$. By I 4 there exists a μ such that $\varphi P_1 \ldots P_\nu R_1 \ldots R_\mu = 0 \vee \varphi Q_1 \ldots Q_\nu R_1 \ldots R_\mu = 0$. If we take $\lambda = \sup\{\nu, \mu\}$, we obtain:

$$\varphi P_1 \ldots P_\lambda R_1 \ldots R_\lambda = 0 \vee \varphi Q_1 \ldots Q_\lambda R_1 \ldots R_\lambda = 0,$$

hence $\langle P_n\rangle_n \# \langle R_n\rangle_n \vee \langle Q_n\rangle_n \# \langle R_n\rangle_n$.
(b) is immediate from 1.11, (c) is trivial, (d) is immediate from 1.10, 1.11.
(e) For a certain ν $\varphi P_1 \ldots P_\nu Q = 0$; there exists (by I 4) a μ such that $\varphi P_1 \ldots P_\nu R_1 \ldots R_\mu = 0 \vee \varphi R_1 \ldots R_\mu Q = 0$. The second possibility is excluded, since $\langle R_n\rangle_n \in Q$. If $\lambda = \sup\{\nu, \mu\}$, we obtain $\varphi P_1 \ldots P_\lambda R_1 \ldots R_\lambda = 0$, so $\langle P_n\rangle_n \# \langle R_n\rangle_n$. The second assertion is proved in the same manner.
(f) For a certain ν $\varphi P_1 \ldots P_\nu R = 0$. $QR \sim Q \leftrightarrow Q \subset R$ (1.7(i)); $\varphi P_1 \ldots P_\nu QR = 0$. $QR \sim Q \rightarrow P_1 \ldots P_\nu QR \sim P_1 \ldots P_\nu Q$ (1.7(f)), hence $\varphi P_1 \ldots P_\nu Q = 0$ (1.7(a)). The second assertion is immediate from the first.
(g) Let $\varphi P_1 \ldots P_\nu Q = 0$ & $\varphi P_1 \ldots P_\mu R = 0$, $\lambda = \sup(\nu, \mu)$. Then we obtain: $\varphi P_1 \ldots P_\lambda (Q + R) = 0$, hence $\langle P_n\rangle_n \; \omega \; (Q + R)$.

1.13. Definition. By 1.12(b), $\simeq$ is an equivalence relation. The species of equivalence classes of Π will be indicated by Π^o; the elements of Π^o are called points. The equivalence class corresponding to a certain $\langle P_n\rangle_n \in \Pi$ will be written as $\langle P_n\rangle_n^*$.

Lower case letters p, q, r (indexed if necessary) will be used to mark elements of Π^o. Capitals U, V, W (indexed if necessary) will be used to mark pointspecies; other capitals or lower case letters will be introduced occasionally for these purposes.

1.14. Definition. $p, q \in \Pi^o$.

$p \# q \leftrightarrow \bigwedge \langle P_n\rangle_n \in p \; \bigwedge \langle Q_n\rangle_n \in q(\langle P_n\rangle_n \# \langle Q_n\rangle_n)$.

$p \; \omega \; Q \leftrightarrow \bigwedge \langle P_n\rangle_n \in p(\langle P_n\rangle_n \; \omega \; Q)$; $p \in Q \leftrightarrow \neg \, p \; \omega \; Q$.

1.15. Theorem. For all p, q, Q, R:

a) $\bigvee \langle P_n\rangle_n \in p \; \bigvee \langle Q_n\rangle_n \in q(\langle P_n\rangle_n \# \langle Q_n\rangle_n) \rightarrow p \# q$.
b) $\bigvee \langle P_n\rangle_n \in p(\langle P_n\rangle_n \; \omega \; Q) \rightarrow p \; \omega \; Q$.
b) # is an apartness relation between points.
c) $p \in Q \leftrightarrow \bigwedge \langle P_n\rangle_n \in p(\langle P_n\rangle_n \in Q) \leftrightarrow \bigvee \langle P_n\rangle_n \in p(\langle P_n\rangle_n \in Q)$;
$\neg\neg \, p \in Q \leftrightarrow p \in Q$.
d) $p \, \omega \, R \; \& \; Q \subset R \rightarrow p \; \omega \; Q$; $p \; \omega \; Q \rightarrow p \; \omega \; QR$;
$p \, \omega \, Q \; \& \; p \; \omega \; R \rightarrow p \; \omega \; (Q + R)$.

Proof. (a) follows from: $\langle P_n\rangle_n \# \langle Q_n\rangle_n$ & $\langle P_n\rangle_n \simeq \langle P'_n\rangle_n$ & $\langle Q_n\rangle_n \simeq \langle Q'_n\rangle_n \rightarrow \langle P'_n\rangle_n \# \langle Q'_n\rangle_n$ (by 1.12(a),(b), and from 1.12(e)).
(b) is immediate from 1.12(a),(b).
(c) $p \in Q \leftrightarrow \neg\, p\ \omega\ Q \leftrightarrow \neg\, \bigwedge\langle P_n\rangle_n \in p(\langle P_n\rangle_n\ \omega\ Q) \leftrightarrow \neg\, \bigvee\langle P_n\rangle_n \in p(\langle P_n\rangle_n\ \omega\ Q)$ (by a)) $\leftrightarrow \bigwedge\langle P_n\rangle_n \in p\, \neg\, (\langle P_n\rangle_n\ \omega\ Q) \leftrightarrow \bigwedge\langle P_n\rangle_n \in p(\langle P_n\rangle_n \in Q) \leftrightarrow \bigvee\langle P_n\rangle_n \in p(\langle P_n\rangle_n \in Q)$ (by contraposition from the second assertion of 1.12(e)).
$\neg\neg\, p \in Q \leftrightarrow p \in Q$ is immediate from 1.14.
(d) follows from 1.12(f),(g).

1.16. Definition. $[P] = \{p : p \in \Pi^o \ \&\ p \in P\}$.
After proving theorem 1.17(a) we shall be justified in writing $P \subset V$, $V \subset P$ instead of $[P] \subset V$, $V \subset [P]$, since no ambiguity is possible.

1.17. Theorem.
a) For all P,Q : $P \subset Q \leftrightarrow [P] \subset [Q]$.
b) For all P,Q : $[PQ] = [P] \cap [Q]$.
c) For all finite species $\{Q_1,\ldots,Q_\mu\}$:
$[Q_1 + \ldots + Q_\mu]$ congruent $[Q_1] \cup \ldots \cup [Q_\mu]$.
Proof. (a) By contraposition from 1.12(f), first assertion: $\langle R_n\rangle_n \in P$ & $P \subset Q \rightarrow \langle R_n\rangle_n \in Q$, hence $[P] \subset [Q]$. Conversely, we suppose $[P] \subset [Q]$. Let for a certain ν $\varphi PA_\nu = 1$. By I 5 an $\langle R_n\rangle_n \in \Pi$, $\langle R_n\rangle_n \in PA_\nu$ can be found. $PA_\nu \subset A_\nu$ by 1.7(j); hence $[PA_\nu] \subset [A_\nu]$, therefore $\langle R_n\rangle_n \in A_\nu$. Suppose $\varphi QA_\nu = 0$. By I 4 there exists a μ such that

$$\varphi\, R_1\ldots R_\mu Q = 0 \ \vee\ \varphi\, R_1\ldots R_\mu A_\nu = 0.$$

The second possibility is excluded, hence $\varphi\, R_1\ldots R_\mu Q = 0$; but this contradicts $\langle R_n\rangle_n \in Q$, so $\varphi QA_\nu = 1$. We have therefore proved by this argument: $P \subset Q$.
(b) By 1.7(j): $PQ \subset P$ & $PQ \subset Q$. $PQ \subset P$ & $PQ \subset Q \rightarrow [PQ] \subset [P]$ & $[PQ] \subset [Q]$ (by (a)), hence
$[PQ] \subset [P] \cap [Q]$.
Let $r \in [P] \cap [Q]$, $\langle R_n\rangle_n \in r$, then $\bigwedge n(\varphi\, R_1\ldots R_n P = 1$ & $\varphi\, R_1\ldots R_n Q = 1)$. Suppose for a certain ν $\varphi\, R_1\ldots R_\nu PQ = 0$.
$\varphi\, R_1\ldots R_\nu PQ = 0 \rightarrow \varphi(R_1\ldots R_\nu P)\ (R_1\ldots R_\nu Q) = 0$.
By I 4 there exists a μ such that
$\varphi(R_1\ldots R_\mu)\ (R_1\ldots R_\nu P) = 0 \ \vee\ \varphi(R_1\ldots R_\mu)\ (R_1\ldots R_\nu Q) = 0$.
Take $\lambda = \sup\{\mu, \nu\}$. We obtain the disjunction $\varphi\, R_1\ldots R_\lambda P = 0 \ \vee\ \varphi\, R_1\ldots R_\lambda Q = 0$, which contradicts our initial assumptions. Therefore $\bigwedge n(\varphi\, R_1\ldots R_n PQ = 1)$, so $\langle R_n\rangle_n \in PQ$; this implies $r \in [PQ]$. Thus we have proved $[P] \cap [Q] \subset [PQ]$.
(c) $Q_i \subset Q_1 + \ldots + Q_\mu$, $1 \leqslant i \leqslant \mu$ (1.7(j)). By (a) we have the result $[Q_i] \subset [Q_1 + \ldots + Q_\mu]$ for $1 \leqslant i \leqslant \mu$, and consequently $[Q_1] \cup \ldots \cup [Q_\mu] \subset [Q_1 + \ldots + Q_\mu]$.

By induction we obtain from 1.12(g), if $\langle R_n\rangle_n \in \Pi$:
$\langle R_n\rangle_n\ \omega\ Q_1\ \&\ \ldots\ \&\ \langle R_n\rangle_n\ \omega\ Q_\mu \leftrightarrow \langle R_n\rangle_n\ \omega\ (Q_1 + \ldots + Q_\mu)$
We have further

$$\neg\,(\bigwedge_{1\leq i\leq \mu} (\langle R_n\rangle_n\ \omega\ Q_i)) \leftrightarrow \neg\neg \bigvee_{1\leq i\leq \mu} (\langle R_n\rangle_n \in Q_i).$$

Hence $r \in [Q_1 + \ldots + Q_\mu] \rightarrow \neg\neg\, r \in [Q_1] \cup [Q_2] \cup \ldots \cup [Q_\mu]$.

1.18. Definitions.
$V \Subset W \leftrightarrow \bigwedge \langle P_n\rangle_n \in \Pi\ \bigvee m([P_1 \ldots P_m] \cap V = \emptyset \vee [P_1 \ldots P_m] \subset W)$.
$V \Subset' W \leftrightarrow \bigwedge p(p \notin V \vee p \in W)$.
$V \Subset'' W \leftrightarrow \bigwedge \langle P_n\rangle_n \in \Pi\ (\langle P_n\rangle_n^* \in V \rightarrow \bigvee n([P_1 \ldots P_n] \subset W))$.
Remark. $\Subset$, $\Subset''$ are relations depending on φ, Π; explicit reference to φ, Π will be omitted generally. (See also 1.28).
$\Subset$ is called the relation of strong inclusion.
For lattice elements we define relations χ, $\chi = \Subset, \Subset', \Subset''$ by $P\ \chi\ Q \rightarrow [P]\ \chi\ [Q]$.

1.19. Remarks. (a) By the foregoing definition, $P \Subset Q$ can be defined as $P \Subset Q \leftrightarrow \bigwedge \langle R_n\rangle_n \in \Pi\ \bigvee m(\varphi\, R_1 \ldots R_m P = 0 \vee R_1 \ldots R_m \subset Q)$.
b) $V \Subset' W$ is classically equivalent to $V \subset W$.
Some elementary properties of the relations $\Subset$, $\Subset'$, $\Subset''$ are collected in the following theorem.

1.20. Theorem.
a) $V \Subset W \rightarrow V \Subset' W$ b) $V \Subset W \rightarrow V \Subset'' W$.
The following assertions hold for $\chi = \Subset, \Subset', \Subset''$:
c) $V\ \chi\ W \rightarrow V \subset W$.
d) $U\ \chi\ V\ \&\ V \subset W \rightarrow U\ \chi\ W$; $U \subset V\ \&\ V\ \chi\ W \rightarrow U\ \chi\ W$.
e) $U\ \chi\ V\ \&\ U\ \chi\ W \rightarrow U\ \chi\ (V \cap W)$.
f) $U\ \chi\ V \rightarrow U\ \chi\ (V \cup W)$.
g) $U\ \chi\ V \rightarrow (U \cap W)\ \chi\ V$.
h) $P\ \chi\ (Q + R)\ \&\ \varphi\, PR = 0 \rightarrow P\ \chi\ Q$.
The following assertions are also valid:
i) $P \Subset Q\ \&\ R \Subset Q \rightarrow (P + R) \Subset Q$.
j) $P \Subset Q\ \&\ p \in P\ \&\ q \notin Q \rightarrow p\ \#\ q$.
Proof. (a), (b), (c) are trivial; (d) is proved by straightforward verification.
(e) is trivial for $\Subset'$. Let $U \Subset V\ \&\ U \Subset W$, $\langle P_n\rangle_n \in \Pi$. There exist ν, μ such that $[P_1 \ldots P_\nu] \cap U = \emptyset \vee [P_1 \ldots P_\nu] \subset V$ and $[P_1 \ldots P_\mu] \cap U = \emptyset \vee [P_1 \ldots P_\mu] \subset W$. Let $\lambda = \sup\{\nu, \mu\}$. Since $[P_1 \ldots P_\lambda] = [P_1 \ldots P_\nu] \cap [P_1 \ldots P_\mu]$ (1.17(b)) we obtain $[P_1 \ldots P_\lambda] \cap U = \emptyset \vee [P_1 \ldots P_\lambda] \subset (V \cap W)$. Likewise for $\Subset''$.
(f), (g) are immediate consequences of (d).

(h). Let $P \in Q + R$, $\varphi PR = 0$, $\langle S_n \rangle_n \in \Pi$. For certain ν, μ we have $\varphi S_1 \ldots S_\nu P = 0 \vee S_1 \ldots S_\nu \subset Q + R$, $\varphi S_1 \ldots S_\mu P = 0 \vee \varphi S_1 \ldots S_\mu R = 0$. Take $\lambda = \sup\{\nu, \mu\}$. Then $\varphi S_1 \ldots S_\lambda P = 0 \vee (S_1 \ldots S_\lambda \subset Q + R \,\&\, \varphi S_1 \ldots S_\lambda R = 0)$. Therefore $S_1 \ldots S_\lambda P = 0 \vee S_1 \ldots S_\lambda \subset Q$ (1.7(m)). Likewise for $\in''$.
Let $P \in' Q + R$, $\varphi PR = 0$, $s = \langle S_n \rangle_n^* \in \Pi^\circ$. Then $s \notin [P] \vee s \in [Q + R]$. There exists a ν such that $\varphi S_1 \ldots S_\nu P = 0 \vee \varphi S_1 \ldots S_\nu R = 0$; hence $s \notin [P] \vee (s \in [Q + R] \,\&\, \varphi S_1 \ldots S_\nu R = 0)$. Since $s \in [Q + R] \,\&\, s \notin [R] \rightarrow s \in [Q]$ (1.17(c), 1.15(c)) we conclude that $s \notin [P] \vee s \in [Q]$.
(i). Let $P \in Q$ & $R \in Q$, $\langle S_n \rangle_n \in \Pi$. There exist ν, μ such that $\varphi S_1 \ldots S_\nu P = 0 \vee S_1 \ldots S_\nu \subset Q$, $\varphi S_1 \ldots S_\mu R = 0 \vee S_1 \ldots S_\mu \subset Q$.
We take again $\lambda = \sup\{\nu, \mu\}$ and obtain $\varphi S_1 \ldots S_\lambda (P + R) = 0 \vee S_1 \ldots S_\lambda \subset Q$.
(j). Let $P \in Q$, $\langle R_n \rangle_n \in P$, $\langle S_n \rangle_n \notin Q$, $\langle R_n \rangle_n$, $\langle S_n \rangle_n \in \Pi$. There is a ν such that $\varphi S_1 \ldots S_\nu P = 0 \vee S_1 \ldots S_\nu \subset Q$. $S_1 \ldots S_\nu \subset Q$ is impossible, therefore $\varphi S_1 \ldots S_\nu P = 0$. As a consequence $\langle S_n \rangle_n \,\omega\, P$, so $\langle R_n \rangle_n \# \langle S_n \rangle_n$ (1.12(e)).

1.21. Definition.
$r \underline{\in} P \leftrightarrow \Lambda \langle R_n \rangle_n \in r \vee m (R_1 \ldots R_m \in P)$
$r \underline{\in} V \leftrightarrow \vee R (r \underline{\in} R \subset V)$
$\mathrm{Int}^* V = \{r : r \underline{\in} V\}$; V is called open, if $V = \{p : p \underline{\in} V\}$.
The notions $r \underline{\in}' P$, $r \underline{\in}'' P$, $r \underline{\in}' V$, $r \underline{\in}'' V$, $\in'$-open, $\in''$-open are defined analogously.

1.22. Theorem. For all r, V, W:
$r \underline{\in} V \,\&\, r \underline{\in} W \rightarrow r \underline{\in} V \cap W$
and likewise for $\underline{\in}'$, $\underline{\in}''$.
Proof. Let $\langle R_n \rangle_n^* = r$. There are S_1, S_2, ν, μ such that
$R_1 \ldots R_\nu \in S_1 \subset V \,\&\, R_1 \ldots R_\mu \in S_2 \in W$.
If $\lambda = \sup\{\nu, \mu\}$, the following assertion is also true (by 1.20(e), 1.7(k)).
$R_1 \ldots R_\lambda \in S_1 S_2 \subset V \cap W$.
Hence $r \underline{\in} V \cap W$. Analogously for $\underline{\in}'$, $\underline{\in}''$.

1.23. Theorem.
a) The open subspecies of Π° constitute a topology with apartness relation on $[A_\infty] = \Pi^\circ$.
b) The $\in'$-open ($\in''$-open) subspecies of Π° constitute a topology on $\Pi^\circ = [A_\infty]$.
Proof. The union of a species of open ($\in'$-, $\in''$-open) point-species is again open ($\in'$-, $\in''$-open). This is trivial. Let $V_1, \ldots, V_\mu$ be open pointspecies. $V = V_1 \cap V_2 \cap \ldots \cap V_\mu$. If $p \in V$, we have also $p \in V_i$, $1 \leqslant i \leqslant \mu$. Since V_i is open, $p \underline{\in} V_i$ for $1 \leqslant i \leqslant \mu$. Therefore $R_1, \ldots, R_\mu$ can be found

such that $p \underline{\in} R_i \subset V_i$, $1 \leqslant i \leqslant \mu$. By 1.22 we obtain $p \underline{\in} R_1 \ldots R_\mu \subset V$. Hence V is open too.
$\emptyset$ and Π^o are open in a trivial way.
If we replace $\underline{\in}$ by $\underline{\in}'$ or $\underline{\in}''$, the argument can be repeated without changes.
To see that condition T4 is fulfilled for open sets, we may argue as follows. Let $p \in V$, $q \notin V$, V open. $p \underline{\in} V$, so there exists an R such that $p \underline{\in} R \subset V$. Let $\langle P_n \rangle_n \in p$. For a certain ν, $P_1 \ldots P_\nu \Subset R$. $p \in P_1 \ldots P_\nu$, $q \notin R$, hence by 1.20(j) $p \# q$.

1.24. Remark. If no further specification is given, in the sequel the topology associated with a pair $\langle \varphi, \Pi \rangle$ as introduced before will always be supposed to be the topology of the open species in the sense of definition 1.21.

1.25. Definition. A topological space which has been defined by means of a pair $\langle \varphi, \Pi \rangle$ such that the postulates I1-5 hold (with the notion of open according to 1.21) is called an abstract intersection space (in short: abstract I-space).
Any space homeomorphic to an abstract I-space is called an intersection space (I-space).
The expression "the (abstract) I-space $\langle \varphi, \Pi \rangle$" means the abstract I-space, defined by a function φ (with a domain of definition $\mathfrak{P}$) and a species Π such that I1-5 are satisfied.
The empty species will also be called an I-space.
In the proofs of theorems on I-spaces we suppose the I-space to be abstract in most cases. This can be done without losing generality. The trivial case of the empty space will be disregarded in proofs.
In many statements the qualification "abstract" is omitted, if it is sufficiently clear from the statement itself whether it is about abstract I-spaces (namely if the statement refers to notions defined for abstract I-spaces only, such as lattice elements).
The same convention applies to the notions "abstract IR-space", "abstract PIN-space" etc., to be introduced in the sequel.

1.26. Definition. Let $\Gamma = \langle V, \mathfrak{T} \rangle$ be a topological space, and $\langle V_n \rangle_n$ a located system of pointspecies of Γ, with at least one $V_n \neq \emptyset$. Suppose $V_\nu \neq \emptyset$. $\langle V'_n \rangle_n$ is defined by: $V'_k = V_k$ if $V_k \neq \emptyset$, $V'_k = V_\nu$ if $V_k = \emptyset$.
A mapping ψ is defined on a free distributive lattice $\mathfrak{P}$ with $\mathfrak{A} = \langle A_n \rangle_n$ as a set of generators by:

$$\begin{cases} \psi A_n = V'_n,\ \psi A_o = \emptyset,\ \psi A_\infty = V, \\ \psi(P+Q) = \psi P \mathbin{\underline{\cup}} \psi Q,\ \psi PQ = \psi P \cap \psi Q. \end{cases}$$

ψ is called a standard mapping (with respect to the located system $\langle V_n \rangle_n$).
If we define φ on $\mathfrak{P}$ by $\varphi P = 1 \leftrightarrow \psi P \neq \emptyset$, then φ is said to be defined from ψ.

If φ satisfies I1-3 and a species Π of centered systems can be found such that I4-5 are satisfied, and $\langle \varphi, \Pi \rangle$ is homeomorphic to Γ by a mapping ξ which satisfies

$$\langle P_n \rangle_n \in \Pi \ \& \ \xi \langle P_n \rangle_n^* = q \rightarrow \bigcap_{n=1}^{\infty} \psi P_n = \{q\},$$

then $\langle V_n \rangle_n$ is said to constitute an I-basis for Γ.

1.27. Remark. The translation of statements about abstract I-spaces (IR-spaces, PIN-spaces etc.) into statements about I-spaces (IR-spaces, PIN-spaces etc.) can be effectuated by means of the preceding definition without difficulty.

1.28. If for the sake of clarity we want to discern various notions for different spaces Γ, Δ,... we use notations such as φ, φ_Δ, $\Subset_\Gamma$, γ_Γ, $^{*}\Gamma$, $\mathfrak{P}(\Gamma)$, $\mathfrak{A}(\Gamma)$, $\Pi(\Gamma)$ etc.

1.29. Remark. Let $\Gamma = \langle \varphi, \Pi \rangle$ be an I-space. To every pointspecies $[P] \in \Pi^\circ$ corresponds in a natural way an I-space Δ', if $[P]$ is provided with the relative topology. If $\mathfrak{A}(\Gamma) = \langle A_n \rangle_n$, then $\langle [A_n P] \rangle_n$ is an I-basis for Δ'. Let ψ be a standard mapping with respect to the located system $\langle [PA_n] \rangle_n$, and let φ_Δ be defined from ψ. $\Pi(\Delta)$ can be defined by:

$\langle P_n \rangle_n \in \Pi(\Delta) \leftrightarrow \vee \langle R_n \rangle_n \in \Pi \wedge n([PR_n] = \psi P_n \ \& \ \langle R_n \rangle_n \in P)$.

Then Δ is homeomorphic to Δ'.
Therefore Δ' can be dealt with by considering $\{PQ : Q \in \mathfrak{P}\}$, $\{\langle PR_n \rangle_n : \langle R_n \rangle_n \in \Pi \ \& \ \langle R_n \rangle_n \in P\}$ instead of $\mathfrak{P}(\Delta)$, $\Pi(\Delta)$.

Speaking about the subspace $[P]$, $P \in \mathfrak{P}(\Gamma)$ we always mean $[P]$, provided with the relative topology. Likewise we use notations such as $\Subset_P$, to indicate strong inclusion in the subspace $[P]$.

We see that for all $P, Q, R \in \mathfrak{P}(\Gamma)$:

$P \Subset_\Gamma Q \rightarrow PR \Subset_R QR$.

1.30. Lemma. In an I-space the following assertions are valid for all P, Q, R, S:
a) $P \Subset_S Q \Subset R \ \& \ P, Q, R \subset S \rightarrow P \Subset Q$;
b) $P \Subset_Q R \ \& \ P \Subset Q \rightarrow P \Subset R$.
Proof. (a) The I-space is supposed to be defined by $\langle \varphi, \Pi \rangle$.
Let $\langle T_n \rangle_n \in \Pi$. There exist μ, ν such that

$\varphi T_1 \ldots T_\mu Q = 0 \vee T_1 \ldots T_\mu \subset R$

$\varphi T_1 \ldots T_\nu SP = 0 \vee T_1 \ldots T_\nu S \subset Q$.

Take $\lambda = \sup\{\mu, \nu\}$.

Then $\varphi\ T_1 \ldots T_\lambda Q = 0 \vee T_1 \ldots T_\lambda \subset R$
$\varphi\ T_1 \ldots T_\lambda SP = 0 \vee T_1 \ldots T_\lambda S \subset Q$.
$P \subset S$, hence $T_1 \ldots T_\lambda SP \sim T_1 \ldots T_\lambda P$. $P \Subset R$, therefore $\varphi\ T_1 \ldots T_\lambda P = 0 \vee T_1 \ldots T_\lambda \subset R$. If $T_1 \ldots T_\lambda \subset R$, then $T_1 \ldots T_\lambda S \sim T_1 \ldots T_\lambda$. Thus we obtain $\varphi\ T_1 \ldots T_\lambda P = 0 \vee T_1 \ldots T_\lambda \subset Q$, and we have proved $P \Subset Q$.
(b) is proved by analogous methods.

1.31. There are many possibilities for introducing a topology on a species Π°, defined with respect to φ, Π, if I1-5 hold. We mention a few of them, besides the possibilities already contained in the substitution of $\Subset'$, $\Subset''$ for $\Subset$.
We assume $\Subset$ to be defined in the sense of definition 1.18. Three possibilities of defining $\underline{\in}$ are:
a) Definition 1.21
b) $r \underline{\in} R \leftrightarrow \vee \langle P_n \rangle_n \in r\ \vee m(P_1 \ldots P_m \Subset R)$;
$r \underline{\in} V \leftrightarrow \vee R(r \underline{\in} R \subset V)$.
c) $r \underline{\in} V \leftrightarrow \{r\} \Subset V$; $r \underline{\in} R \leftrightarrow \{r\} \Subset [R]$.
Two possibilities of defining the notion of an open species are:
d) V is open if $V = \{p : p \underline{\in} V\}$.
e) $\mathrm{Int}^* V = \{p : p \underline{\in} V\}$. The species Int $[P]$ constitute a basis for the open sets.
By combination we obtain six possible ways of introducing a topology: a-d, a-e, b-d, b-e, c-d, c-e. The combinations a-d, a-e, c-d, c-e produce a topology without any difficulty; the combinations b-d, b-e produce a topology if we add the postulate

I 6. $\langle P_n \rangle_n \in \Pi\ \&\ \langle P_n \rangle_n \in Q \rightarrow \vee \langle R_n \rangle_n \in \Pi\ (\wedge n(R_{n+1} = P_n)\ \&\ R_1 = Q)$, for all $\langle P_n \rangle_n, Q$.

1.32. Theorem. Suppose Γ, Δ to be abstract I-spaces with the same lattice $\mathfrak{P}$ and defining function φ, defined by pairs $\langle \varphi, \Pi(\Gamma) \rangle$, $\langle \varphi, \Pi(\Delta) \rangle$ respectively.
If a) $\Pi(\Gamma) \subset \Pi(\Delta)$
b) $\wedge \langle P_n \rangle_n \in \Pi(\Delta)\ \vee \langle Q_n \rangle_n \in \Pi(\Gamma)\ (\langle P_n \rangle_n \simeq \langle Q_n \rangle_n\ \&\ \wedge k\ \vee l(P_1 \ldots P_l \subset Q_1 \ldots Q_k))$
then Γ and Δ are homeomorphic, and $P \Subset_\Gamma Q \leftrightarrow P \Subset_\Delta Q$.
Proof. $\Pi^\circ(\Gamma)$ can be mapped bi-uniquely in a natural way onto $\Pi^\circ(\Delta)$, as is seen from supposition (b); for if $\langle P_n \rangle_n^{*\Gamma} \in \Pi^\circ(\Gamma)$, we can define a mapping ψ by:

$$\psi \langle P_n \rangle_n^{*\Gamma} = \langle P_n \rangle_n^{*\Delta}.$$

$P \Subset_\Delta Q \rightarrow P \Subset_\Gamma Q$ is trivial. Let $P \Subset_\Gamma Q$. If $\langle R_n \rangle_n \in \Pi(\Delta)$, there exists a $\langle S_n \rangle_n \in \Pi(\Gamma)$, such that $\langle R_n \rangle_n \simeq \langle S_n \rangle_n$ and $\wedge k\ \vee l(R_1 \ldots R_l \subset S_1 \ldots S_k)$.

There is a μ such that $\varphi\ PS_1\ldots S_\mu = 0 \vee S_1\ldots S_\mu \subset Q$, and there is a ν such that $R_1\ldots R_\nu \subset S_1\ldots S_\mu$. Hence $\varphi\ PR_1\ldots R_\nu = 0 \vee R_1\ldots R_\nu \subset Q$. So $P \Subset_\Delta Q$.
If $r \underline{\in}_\Delta P$, then also $r \underline{\in}_\Gamma P$. Let us suppose $r \underline{\in}_\Gamma P$. If we take $\langle R_n\rangle_n$, $\langle S_n\rangle_n$ to be the same as before, there are ν,μ such that $S_1\ldots S_\nu \Subset_\Gamma P$, $R_1\ldots R_\mu \subset S_1\ldots S_\nu$. So $R_1\ldots R_\mu \Subset_\Gamma P$, hence $R_1\ldots R_\mu \Subset_\Delta P$. Thus we have proved: $r \underline{\in}_\Gamma P \rightarrow r \underline{\in}_\Delta P$. Therefore Γ, Δ must be homeomorphic.

1.33. Definition. We introduce three types of transformations of elements of Σ. Transformations of one of these types are marked by symbols ϕ, $\phi_1, \ldots, \phi_\nu, \ldots$
a) $\langle P_n\rangle_n \in \Sigma$. ϕ is a transformation of $\langle P_n\rangle_n$ of the first type if $\phi\langle P_n\rangle_n = \langle Q_n\rangle_n$, $\wedge n(Q_n = P_{m_n+1}\, P_{m_n+2} \ldots P_{m_{n+1}})$, $\wedge i(m_{i+1} > m_i)$, $m_1 = 0$.
b) $\langle P_n\rangle_n \in \Sigma$. ϕ is a transformation of $\langle P_n\rangle_n$ of the second type if $\phi\langle P_n\rangle_n = \langle Q_n\rangle_n$, $Q_n = P_{f(n)}$, f a bi-unique mapping of the natural numbers onto the natural numbers.
c) $\langle P_n\rangle_n \in \Sigma$. ϕ is a transformation of $\langle P_n\rangle_n$ of the third type if $\phi\langle P_n\rangle_n = \langle Q_n\rangle_n$, and if there exists a sequence $\langle R_i\rangle_i$, such that $\wedge i\ \wedge n(\varphi\ P_1\ldots P_n R_i = 1)$, and a sequence of non-negative integers $\langle m_i\rangle_i$, $\wedge n\ \vee i(m_i > n)$, and such that for all i, $m_i+i < n \leqslant m_{i+1}+i \rightarrow Q_n = P_{n-i}$, $n \leqslant m_1 \rightarrow Q_n = P_n$, $n = m_i+i \rightarrow Q_n = R_i$. Less precise: $\langle Q_n\rangle_n = \langle P_1, \ldots, P_{m_1}, R_1, P_{m_1+1}, \ldots, P_{m_2}, R_2, P_{m_2+1}, \ldots, P_{m_3}, R_3, \ldots\rangle$.

1.34. Remark. For transformations of the first and second kind it is immediately clear that the transformed sequence again belongs to Σ. For transformations of the third kind we may argue as follows. Suppose already proved $\wedge n(\varphi\ P_1\ldots P_n R_1\ldots R_\nu) = 1$ and suppose for a certain μ $P_1\ldots P_\mu R_1\ldots R_\nu R_{\nu+1} = 0$. By I 4 there is a λ such that
$$\varphi(P_1\ldots P_\lambda)\,(P_1\ldots P_\mu R_1\ldots R_\nu) = 0 \vee \varphi(P_1\ldots P_\lambda)R_{\nu+1} = 0.$$
Both possibilities are excluded, therefore
$\varphi\ P_1\ldots P_\mu R_1\ldots R_{\nu+1} = 1$.
Hence $\wedge i\ \wedge n(\varphi\ P_1\ldots P_n R_1\ldots R_i = 1)$.

1.35. Definition. If Γ is an I-space, defined by $\langle\varphi, \Pi\rangle$, we indicate by Π^* the subspecies of $\Sigma(\Gamma)$, obtained by closure from Π with respect to transformations of the first, second, and third kind.

1.36. Corollary to 1.32. If Γ is an I-space with a defining pair $\langle\varphi, \Pi\rangle$, then every Δ, defined by $\langle\varphi, \Pi(\Delta)\rangle$ such that $\Pi \subset \Pi(\Delta) \subset \Pi^*$, is homeomorphic to Γ.
Proof. It is immediate from 1.33 that the conditions of 1.32 are fulfilled.

1.37. Definition and remark. $\{P_i : i \in I\}$ is called a covering of an abstract I-space Γ, if $\{[P_i] : i \in I\}$ covers Γ. If $\langle P_n \rangle_n$ is a star-finite covering, it is always possible to construct a covering $\langle Q_n \rangle_n$ such that $\wedge n(Q_n \sim P_n)$, and such that for every ν $\{Q_i : \varphi\, Q_i Q_\nu = 1\}$ is a finite species. (This fact is easily verified, and since it is somewhat laborious to write down the proof is omitted.)

1.38. Theorem. Let Γ be an I-space. If $P_i \Subset Q_i$ for $1 \leqslant i \leqslant n$, then $\{Q_1, \ldots, Q_n\}$ covers $P_1 + \ldots + P_n$.
Proof. Let $\langle R_n \rangle_n \in P_1 + \ldots + P_n$. A ν can be found such that $\varphi\, P_i R_1 \ldots R_\nu = 1 \rightarrow R_1 \ldots R_\nu \subset Q_i$ for $1 \leqslant i \leqslant n$. Since for a certain λ, $1 \leqslant \lambda \leqslant n$, $\varphi\, P_\lambda R_1 \ldots R_\nu = 1$, we conclude that $R_1 \ldots R_\nu \subset Q_\lambda$, therefore $\langle R_n \rangle_n^* \in Q_\lambda$.

2. Representation and separation postulates.

2.1. In this paragraph we consider some representation and separation postulates and their implications for I-spaces.
To begin with, we list some of the possibilities most natural for separation postulates. They are numbered with a letter N, from "normality", since the strongest conditions N6, N8 could be considered as normality postulates.

N1. $\wedge p\ \wedge q(p \# q \rightarrow \vee R(p \underline{\in} R \;\&\; q\ \omega\ R))$.
N2. $\wedge p\ \wedge q(p \# q \rightarrow \vee R\ \vee S(p \underline{\in} R \;\&\; q \underline{\in} S \;\&\; \varphi\, RS = 0))$.
N3. $\wedge p\ \wedge Q(p\ \omega\ Q \rightarrow \vee R(p \underline{\in} R \;\&\; \varphi\, RQ = 0))$.
N4. $\wedge p\ \wedge Q(p\ \omega\ Q \rightarrow \vee R(p\ \omega\ R \;\&\; Q \Subset R))$.
N5. $\wedge p\ \wedge Q(p\ \omega\ Q \rightarrow \vee R\ \vee S(p \underline{\in} R \;\&\; Q \Subset S \;\&\; \varphi\, RS = 0))$.
N6. $\wedge P\ \wedge Q(\varphi\, PQ = 0 \rightarrow \vee P'\ \vee Q'(\varphi\, P'Q' = 0 \;\&\; P \Subset P' \;\&\; Q \Subset Q'))$.
N7. $\wedge p\ \wedge Q(p \underline{\in} Q \rightarrow \vee R(p \underline{\in} R \Subset Q))$.
N8($\mathfrak{B}$). $\wedge P \in \mathfrak{B}\ \wedge Q \in \mathfrak{B}\ (P \Subset Q \rightarrow \vee R \in \mathfrak{B}\ (P \Subset R \Subset Q))$.
N8 = N8($\mathfrak{P}$).

N6 and N8($\mathfrak{B}$) will play the most important role.
The following implications are trivial: N6 $\rightarrow$ N5 $\rightarrow$ (N4 & N3) $\rightarrow$ N2 $\rightarrow$ N1; (N4 $\vee$ N3) $\rightarrow$ N1; N8 $\rightarrow$ N7.

2.2. Definition. Let Π be the species of point generators of an I-space Γ. A subspecies $\Pi_1 \subset \Pi$ is called a spread representation of Γ, if the following conditions are fulfilled:
a) There exists a spread with a defining pair $\langle \theta, \vartheta \rangle$, and with Π_1 as the species of spread elements. $\langle \theta, \vartheta \rangle$ is called the defining pair of the representation.
b) ϑ^* is a mapping of θ into $\mathfrak{P}$ such that
$\vartheta\langle i_1, \ldots, i_k \rangle = \langle \vartheta^*\langle i_1 \rangle, \vartheta^*\langle i_1, i_2 \rangle, \ldots, \vartheta^*\langle i_1, \ldots, i_k \rangle\rangle$; we put $\bar{\vartheta}\langle i_1, \ldots, i_k \rangle = \vartheta^*\langle i_1 \rangle \vartheta^*\langle i_1, i_2 \rangle \ldots \vartheta^*\langle i_1, \ldots, i_k \rangle$.

If $\vartheta^*\langle i_1,\ldots,i_k\rangle = \gamma i_k$, Π_1 is called a normal representation.
c) $\wedge\langle P_n\rangle_n \in \Pi\ \vee\langle Q_n\rangle_n \in \Pi_1(\langle P_n\rangle_n \simeq \langle Q_n\rangle_n)$.

2.3. Remarks. a) A normal representation has a property which is very convenient in formulation:

$$\langle P_1,\ldots,P_k\rangle \in \vartheta\,\theta \leftrightarrow \langle \gamma i_1,\ldots,\gamma i_k\rangle \in \theta,$$

hence $\langle P_1,\ldots,P_k\rangle \in \vartheta\,\theta$ is a decidable property.
b) A normal representation is entirely determined by Π_1; a spread representation in general, strictly spoken, not (different pairs $\langle\theta,\vartheta\rangle$ may produce the same species Π_1), but since in our applications no confusion is to be expected, we shall neglect this subtlety in the sequel.
c) A finitary spread representation may always be supposed to be normal, since $\vartheta\theta$ contains only finitely many sequences of a given length.

2.4. Definition. A spread representation Π_1 of an I-space $\langle\varphi,\Pi\rangle$ is called perfect if the defining pair $\langle\theta,\vartheta\rangle$ of the representation satisfies the following condition:
$\langle i_1,\ldots,i_k\rangle \in \theta\ \&\ \langle Q_n\rangle_n \in \Pi\ \&\ \langle Q_n\rangle_n \in \bar{\vartheta}\langle i_1,\ldots,i_k\rangle \rightarrow \vee\langle j_n\rangle_n\ \vee\langle R_n\rangle_n \in \Pi_1\ \wedge m(\vartheta^*\langle j_1,\ldots,j_m\rangle = R_m\ \&\ \langle j_1,\ldots,j_k\rangle = \langle i_1,\ldots,i_k\rangle\ \&\ \langle Q_n\rangle_n \simeq \langle R_n\rangle_n)$.
If Π_1 is normal we obtain a simpler formulation:
$\langle P_1,\ldots,P_k\rangle \in \vartheta\,\theta\ \&\ \langle Q_n\rangle_n \in \Pi\ \&\ \langle Q_n\rangle_n \in P_1\ldots P_k \rightarrow \vee\langle R_n\rangle_n \in \Pi_1(\langle R_n\rangle_n \simeq \langle Q_n\rangle_n\ \&\ \langle P_1,\ldots,P_k\rangle = \langle R_1,\ldots,R_k\rangle)$.

2.5. Definition. We say that a subspecies Π_1 of Π possesses the inclusion property if
$\wedge\langle P_n\rangle_n \in \Pi_1\ \vee\langle Q_n\rangle_n \in \Pi_1(\langle P_n\rangle_n \simeq \langle Q_n\rangle_n\ \&\ \wedge n(P_1\ldots P_n \Subset Q_1\ldots Q_n))$.

2.6. Definition. A spread representation Π_1 of an I-space $\langle\varphi,\Pi\rangle$ with a defining pair $\langle\theta,\vartheta\rangle$ is called a strong inclusion representation (in short $\Subset$-representation) if the following assertion is true:
$\langle i_1,\ldots,i_k\rangle \in \theta \rightarrow \bar{\vartheta}\langle i_1,\ldots,i_k\rangle \Subset \bar{\vartheta}\langle i_1,\ldots,i_{k-1}\rangle$, for all $\langle i_1,\ldots,i_k\rangle \in \theta$, $k > 1$.

2.7. Definition. Let $\Gamma = \langle\varphi,\Pi\rangle$ be an I-space. An I-space Δ is called an inessential extension of Γ, if $\mathfrak{A}(\Gamma) = \langle A_n\rangle_n$, $\mathfrak{A}(\Delta) = \langle A_n\rangle_n \cup \langle B_n\rangle_n$, $\langle A_n\rangle_n$, $\langle B_n\rangle_n$ disjoint sequences of different elements and if
a) $\wedge n\ \vee P \in \mathfrak{P}(\Gamma)\ (B_n \sim_\Delta P)$,
b) $\langle R_n\rangle_n \in \Pi(\Delta) \leftrightarrow \vee\langle R'_n\rangle_n \in \Pi(\Gamma)\ \wedge n(R_n \sim_\Delta R'_n)$.
Δ is homeomorphic to Γ, as is trivially seen; it can also be deduced from 1.32.

2.8. Lemma. To an I-space Γ with a spread representation Π_1 always an inessential extension Δ can be found with a normal representation Π_2 such that

a) Π_1 is perfect iff Π_2 is perfect.

b) Π_1 possesses the inclusion property iff Π_2 possesses the inclusion property.

c) Π_1 is a $\Subset$-representation iff Π_2 is a $\Subset$-representation.

Proof. Let Π_1 be given by a defining pair $\langle \theta, \vartheta \rangle$, and let $\langle \sigma_i \rangle_i$ be an enumeration of θ without repetitions. We put $\mathfrak{A}(\Delta) = \langle A_n \rangle_n \cup \langle B_n \rangle_n$, $\langle A_n \rangle_n = \mathfrak{A}(\Gamma)$, $\wedge n(B_n \sim \vartheta^* \sigma_n)$.
Now we construct Π_2 with a defining pair $\langle \theta', \vartheta' \rangle$ such that $\langle j_1, \ldots, j_k \rangle \in \theta' \leftrightarrow \langle \gamma_\Delta j_1, \ldots, \gamma_\Delta j_k \rangle = \langle B_{i_1}, \ldots, B_{i_k} \rangle$ & $\sigma_{i_1} = \langle l_1 \rangle$ & $\sigma_{i_2} = \langle l_1, l_2 \rangle$ & ... & $\sigma_{i_k} = \langle l_1, \ldots, l_k \rangle \in \theta$;
$\vartheta' \langle j_1, \ldots, j_k \rangle = \langle \gamma_\Delta j_1, \ldots, \gamma_\Delta j_k \rangle$.
(b), (c) are trivial.
(a) is proved by a verification somewhat lengthy but very straightforward; hence the proof is omitted.

2.9. Remark. In many cases, if we want to prove a topological property (e.g. metrizability) for a space Γ, using the existence of certain representations, we can use a normal representation instead, with properties analogous to the properties of the original representation, for an inessential extension of Γ.

Conversely, the existence of a spread representation in general for a special type of spaces (or for a special example) is in most cases more easily demonstrated than the existence of a normal representation; lemma 2.8 makes an easy transition possible.

2.10. Some of the most natural representation postulates are:

R1. There exists a spread representation $\Pi_1 \subset \Pi$.
R2. There exists a perfect representation $\Pi_1 \subset \Pi$.
R3. There exists a perfect representation $\Pi_1 \subset \Pi$ which possesses the inclusion property.
R4. There exists a $\Subset$-representation $\Pi_1 \subset \Pi$.
R5. $\wedge \langle P_n \rangle_n \in \Pi \ \vee \langle Q_n \rangle_n \in \Pi \ (\langle P_n \rangle_n \simeq \langle Q_n \rangle_n$ & $\wedge n(Q_{n+1} \Subset Q_n))$.

The following implications are trivial: R3 $\rightarrow$ R2 $\rightarrow$ R1; R4 $\rightarrow$ R5.
The postulates R5, R3, R4 will prove of special importance.
In this paragraph we want to develop the properties of spaces which satisfy some representation and normality postulates. Because of their complication the postulates R1-R4 cannot be called elegant; therefore most theorems about spaces satisfying one or more of these postulates must be considered as tools destined for application to more naturally defined spaces.

2.11. Definition. An abstract I-space in which R5 holds is called an abstract IR-space. Any space homeomorphic to an abstract IR-space is called an IR-space.
IR-spaces are classically equivalent to regular spaces with a countable basis. An IR-basis is defined in the same manner as an I-basis (3.1.26).
Remark. A subspace [P] of an IR-space is again an IR-space, for if $\langle Q_n\rangle_n \in P$ & $\wedge n(Q_{n+1} \Subset Q_n)$, then also $\wedge n(PQ_{n+1} \Subset_P PQ_n)$ (1.29).

2.12. Theorem. Let Γ, Δ be two I-spaces such that the conditions (a),(b) of 1.32 are fulfilled, and let a postulate "Ax" hold in Γ. If "Ax" is one of the postulates N1-8, R1-5, "Ax" is also valid for Δ. Especially this is true for Δ_1, defined by $\langle \varphi, \Pi^*(\Gamma)\rangle$.
The proof is trivial in all cases.

2.13. Theorem. In an IR-space the following assertions hold for all p, R, V, W.
a) N3.
b) N7.
c) $\{p\} \Subset V \leftrightarrow p \underline{\in} V$.
d) $\text{Int}^*V = \text{Int } V$.
e) $\vee \langle P_n\rangle_n \in p\ \vee m(P_1 \dots P_m \Subset R) \rightarrow p \underline{\in} R$.
f) $V \Subset'' W \leftrightarrow V \subset \text{Int } W$.
Proof. (a) Let $\langle P_n\rangle_n\ \omega\ R$, $\langle P_n\rangle_n \simeq \langle Q_n\rangle_n$ & $\wedge n(Q_{n+1} \Subset Q_n)$. $\langle Q_n\rangle_n\ \omega\ R$, so there exists a ν such that $\varphi\ Q_1 \dots Q_\nu R = 0$. $Q_1 \dots Q_{\nu+2} \Subset Q_1 \dots Q_{\nu+1} \Subset Q_1 \dots Q_\nu$.
Since $\langle P_n\rangle_n \in Q_1 \dots Q_{\nu+2}$ there exists a μ such that $P_1 \dots P_\mu \subset Q_1 \dots Q_{\nu+1}$; so $P_1 \dots P_\mu \Subset Q_1 \dots Q_\nu$. Hence $\langle P_n\rangle_n^* \underline{\in} Q_1 \dots Q_\nu$ & $\varphi\ Q_1 \dots Q_\nu R = 0$.
(b) There exists a $\langle Q_n\rangle_n \in p$ such that $\wedge n(Q_{n+1} \Subset Q_n)$. For a certain ν $Q_1 \dots Q_\nu \Subset R$. $Q_1 \dots Q_{\nu+2} \Subset Q_1 \dots Q_{\nu+1} \Subset Q_1 \dots Q_\nu$; it follows that if $\langle P_n\rangle_n \in p$, a μ can be found such that $P_1 \dots P_\mu \subset Q_1 \dots Q_{\nu+1}$, and hence $P_1 \dots P_\mu \Subset Q_1 \dots Q_\nu$. Thus we may take Q_ν for the S in the assertion N7.
(c) If $\{p\} \Subset V$ there exists a $\langle Q_n\rangle_n \in p$, such that $\wedge n(Q_{n+1} \Subset Q_n)$, and a μ can be found such that $Q_1 \dots Q_\mu \subset V$. It follows that $p \in Q_1 \dots Q_{\mu+1} \Subset Q_1 \dots Q_\mu \subset V$, so $p \underline{\in} V$.
If $p \underline{\in} V$, $\langle P_n\rangle_n^* = p$, a μ and a Q can be found such that $P_1 \dots P_\mu \Subset Q \subset V$. If $\langle R_n\rangle_n \in \Pi$, there exists a ν such that

$$\varphi\ P_1 \dots P_\mu R_1 \dots R_\nu = 0 \vee R_1 \dots R_\nu \subset Q \subset V.$$

Hence $\langle R_n\rangle_n^* \# p \vee \vee n(R_1 \dots R_n \subset V)$.
(d) $p \in \text{Int}^*V \rightarrow p \underline{\in} V$. So an R can be found such that $p \underline{\in} R \subset V$. There exist S_1, S_2, such that $p \underline{\in} S_1 \Subset S_2 \Subset R \subset V$, as follows from (b).

Let $\langle Q_n\rangle_n \in S_1$. For a certain μ, $Q_1 \ldots Q_\mu \subset S_2$, so $Q_1 \ldots Q_\mu \Subset R$. $S_1 \subset \mathrm{Int}^*[R] \subset \mathrm{Int}^*V$. For this reason $p \,\underline{\in}\, \mathrm{Int}^*V$, therefore Int^*V is open.

Int V is the species of interior points of V. If $p \in \mathrm{Int}\ V$, an open species W can be found such that $p \in W \subset V$. So an R can be found such that $p \,\underline{\in}\, R \subset V$, hence $p \in \mathrm{Int}^*V$. Since $\mathrm{Int}^*V \subset \mathrm{Int}\ V$ is trivial, we have proved $\mathrm{Int}^*V = \mathrm{Int}\ V$.

(e) Let $\langle P_n\rangle_n$, $\langle S_n\rangle_n \in p$, $P_1 \ldots P_\mu \Subset R$, $\langle P_n\rangle_n \simeq \langle Q_n\rangle_n$, $\wedge n(Q_{n+1} \Subset Q_n)$. A ν can be found such that $Q_1 \ldots Q_\nu \subset R$. As before, we have $Q_1 \ldots Q_{\nu+2} \Subset Q_1 \ldots Q_{\nu+1} \Subset Q_1 \ldots Q_\nu$. For a certain λ $S_1 \ldots S_\lambda \subset Q_1 \ldots Q_{\nu+1} \Subset Q_1 \ldots Q_\nu \subset R$ (since $\langle S_n\rangle_n \in Q_1 \ldots Q_{\nu+2}$), therefore $p \,\underline{\in}\, R$.

(f) Let $V \Subset'' W$, $p = \langle P_n\rangle_n^* \in V$, $\wedge n(P_{n+1} \Subset P_n)$. A ν can be found such that $P_1 \ldots P_{\nu+1} \Subset P_1 \ldots P_\nu \subset W$; hence $p \,\underline{\in}\, W$, so $V \subset \mathrm{Int}\ W$.

Conversely, let $V \subset \mathrm{Int}\ W$. If $p = \langle P_n\rangle_n^* \in V$, then for a certain ν and a certain R $P_1 \ldots P_\nu \Subset R \subset V$. Hence $[P_1 \ldots P_\nu] \subset W$, so $V \Subset'' W$.

2.14. Corollary to 2.13(c). In an IR-space, the species $\langle \mathrm{Int}\ [\gamma n]\rangle_n$ is a basis for the open sets.
Proof trivial.

2.15. Remark. As a consequence of 2.13, all ways of introducing a topology, mentioned in 1.31, turn out to be equivalent if R5 is satisfied.

2.16. Theorem. The following assertions are true in an IR-space.

a) A point p is a closure point of a species V, iff
$\wedge R(p \,\underline{\in}\, R \rightarrow \vee q(q \in [R] \cap V))$.
A point p is a weak closure point of a species V, iff
$\wedge R(p \,\underline{\in}\, R \rightarrow [R] \cap V \neq \emptyset)$.

b) $[R]$ is a closed pointspecies for every R.

c) $([P_1] \cup [P_2] \cup \ldots \cup [P_\mu])^- = [P_1 + \ldots + P_\mu]$ for every species $\{P_1, \ldots, P_\mu\}$.

Proof. (a) Let p be a closure point of V. $p \,\underline{\in}\, R \rightarrow p \in \mathrm{Int}[R]$. $\vee q(q \in \mathrm{Int}[R] \cap V)$, hence also $\vee q(q \in [R] \cap V)$.
Suppose $\wedge R(p \,\underline{\in}\, R \rightarrow \vee q(q \in [R] \cap V))$. If $p \in W$, W open, then there is a Q such that $p \,\underline{\in}\, Q \subset W$, and from $\vee q(q \in [Q] \cap V)$ it follows that $\vee q(q \in W \cap V)$.
The proof in the case of weak closure points is analogous.
(b) Let $\langle P_n\rangle_n^*$ be a closure point of R, and let $\langle P_n\rangle_n\ \omega\ R$. Then there exists a ν such that $\varphi P_1 \ldots P_\nu R = 0$. By 2.13(a), a Q can be found such that $\langle P_n\rangle_n^* \,\underline{\in}\, Q$ & $\varphi QR = 0$. Also $\varphi QR = 1$, since $\langle P_n\rangle_n$ is a closure point of R. ($\langle R_n\rangle_n \in QR \rightarrow \varphi R_1QR = 1$; $\varphi R_1QR = 1 \rightarrow \varphi QR = 1$). In this way a con-

tradiction is obtained, hence $\neg\langle P_n\rangle_n\ \omega\ R$, so $\langle P_n\rangle_n \in R$.
(c) Let $\langle Q_n\rangle_n \in P_1 + \ldots + P_\mu$, $\langle Q_n\rangle_n^* \underline{\in} R$. For a certain ν, $Q_1\ldots Q_\nu \Subset R$. $\varphi\ Q_1\ldots Q_\nu\ (P_1 + \ldots + P_\mu) = 1 \rightarrow$ $\vee i(\varphi\ Q_1\ldots Q_\nu P_i = 1)$.
Let $\varphi\ Q_1\ldots Q_\nu P_\lambda = 1$;
then $\vee q(q \in [Q_1\ldots Q_\nu] \cap ([P_1] \cup \ldots \cup [P_\mu]))$.
Therefore $\vee q(q \in [R] \cap ([P_1] \cup \ldots \cup [P_\mu]))$, and
$[P_1 + \ldots + P_\mu] \subset ([P_1] \cup \ldots \cup [P_\mu])^-$ (by (a)).
On the other side, $[P_i] \subset [P_1 + \ldots + P_\mu]$ for $1 \leqslant i \leqslant \mu$.
So $[P_1] \cup \ldots \cup [P_\mu] \subset [P_1 + \ldots + P_\mu]$. By (b), we obtain
(1.2.17) $([P_1] \cup \ldots \cup [P_\mu])^- \subset [P_1 + \ldots + P_\mu]$.

2.17. Theorem. In every IR-space the following implications hold:
a) $V \Subset W \rightarrow V^- \Subset W$; b) $V \Subset W \rightarrow V^- \Subset \text{Int } W$.
Proof. (a) Let $p = \langle P_n\rangle_n^* = \langle Q_n\rangle_n^*$, $\wedge n(Q_{n+1} \Subset Q_n)$. There is a ν such that $[Q_1\ldots Q_\nu] \cap V = \emptyset \vee Q_1\ldots Q_\nu \subset W$. Hence $[Q_1\ldots Q_{\nu+1}] \cap V^- = \emptyset \vee Q_1\ldots Q_{\nu+1} \subset W$.
A μ can be found such that $P_1\ldots P_\mu \subset Q_1\ldots Q_{\nu+1}$, therefore $[P_1\ldots P_\mu] \cap V^- = \emptyset \vee P_1\ldots P_\mu \subset W$; thus we have shown that $V^- \Subset W$.
(b) $V \Subset W \rightarrow V^- \Subset W$ (a); $V^- \Subset W \rightarrow V^- \Subset'' W$ (1.20(b)); hence (2.13(f)) $V^- \subset \text{Int } W$.

2.18. Theorem. In an IR-space we can characterize the notions "continuous mapping" and "weakly located subspecies" in the following manner:
a) A mapping δ from an IR-space Γ_1 into an IR-space Γ_2 is a continuous mapping iff for all p, S:
$p \in \Pi^o(\Gamma_1)\ \&\ \delta p \underline{\in}_{\Gamma_2} S\ \&\ S \in \mathfrak{P}(\Gamma_2) \rightarrow$
$\vee R \in \mathfrak{P}(\Gamma_1)\ (p \underline{\in}_{\Gamma_1} R\ \&\ \delta[R]_{\Gamma_2} \subset [S]_{\Gamma_2})$.
b) A subspecies V of Π^o, Π^o the species of points of an IR-space, is weakly located iff
$\wedge p\ \wedge R(p \underline{\in} R \rightarrow (\vee q(q \in [R] \cap V) \vee \vee S(p \underline{\in} S \subset R\ \&$
$[S] \cap V = \emptyset)))$.
Proof. Trivial.

2.19. Theorem. If Γ, Δ are IR-spaces, and ξ is a homeomorphism from Γ onto Δ, then $V \Subset_\Gamma W \leftrightarrow \xi V \Subset_\Delta \xi W$. Likewise for $\Subset'$, $\Subset''$.
Proof. For $\Subset'$, $\Subset''$ the result is a trivial consequence of 2.13(f), 1.18. Let $p = \langle P_n\rangle_n^* \in \Pi^o(\Delta)$, and suppose $V \Subset_\Gamma W$. $\xi^{-1}p = \langle Q_n\rangle_n^* \in \Pi^o(\Gamma)$, $\wedge n(Q_{n+1} \Subset_\Gamma Q_n)$. For a certain ν $[Q_1\ldots Q_\nu]_\Gamma \cap V = \emptyset \vee Q_1\ldots Q_\nu \subset W$.
$\xi^{-1}p \underline{\in}_\Gamma Q_1\ldots Q_\nu$, hence there is an $R \in \mathfrak{P}(\Delta)$ such that $p \underline{\in}_\Gamma R \subset \xi[Q_1\ldots Q_\nu]_\Gamma$ (2.18(a)). For a certain μ we obtain $P_1\ldots P_\mu \Subset_\Delta R \subset \xi[Q_1\ldots Q_\nu]_\Gamma$. Therefore $[P_1\ldots P_\mu] \cap$

$\xi V = \emptyset \vee P_1 \ldots P_\mu \subset \xi W$. This proves $\xi V \Subset_\Delta \xi W$. The implication in the reverse direction is proved likewise.

2.20. Theorem. If Γ is an IR-space in which R1 holds, then R4 holds in Γ.
Proof. Let Π_1 be a spread representation for Γ; we may suppose Π_1 to be normal, with a defining pair $\langle \theta, \vartheta \rangle$.
To every point generator $\langle P_n \rangle_n \in \Pi_1$ a point generator $\langle Q_n \rangle_n$ can be found such that $\wedge n(Q_{n+1} \Subset Q_n)$.
Therefore there exists a sequence of mappings $\langle \psi_n \rangle_n$ from Π_1 into $\mathfrak{P}(\Gamma)$, such that $\wedge n \wedge \langle P_m \rangle_m \in \Pi_1$ $(\psi_{n+1} \langle P_m \rangle_m \Subset \psi_n \langle P_m \rangle_m)$, and $\wedge \langle P_m \rangle_m$ $(\langle \psi_n \langle P_m \rangle_m \rangle_n \simeq \langle P_m \rangle_m)$.
As a consequence of Brouwers principle, there is also a sequence of mappings $\langle \eta_n \rangle_n$ from Π_1 into $\vartheta\theta$, such that for every $\langle P_m \rangle_m \in \Pi_1$ $\psi_n \langle P_m \rangle_m$ can be calculated from an initial segment $\langle P_1, \ldots, P_t \rangle = \eta_n \langle P_m \rangle_m$.
We may suppose that for all $\langle P_m \rangle_m \in \Pi_1$ and for every n $\eta_n \langle P_m \rangle_m$ is an initial segment of $\eta_{n+1} \langle P_m \rangle_m$.
We define mappings ψ'_n from $\eta_n \Pi_1$ into $\vartheta\theta$ by

$$\psi'_n \eta_n \langle P_m \rangle_m = \psi_n \langle P_m \rangle_m .$$

The species of segments $\eta_{n+1} \langle P'_m \rangle_m$ such that $\eta_n \langle P_m \rangle_m$ is an initial segment of $\eta_{n+1} \langle P'_m \rangle_m$ ($\langle P_m \rangle_m$ fixed) can be enumerated (as a consequence of the enumeration principle).
Let $\eta_1 \Pi_1$ be enumerated as $\langle X_i \rangle_i$. If a certain element of $\eta_n \Pi_1$, say $\eta_n \langle P_m \rangle_m$ is denoted by $X_{i_1, \ldots, i_n}$, the species of $\eta_{n+1} \langle P'_m \rangle_m$ such that $\eta_n \langle P_m \rangle_m$ is an initial segment of $\eta_{n+1} \langle P'_m \rangle_m$ can be enumerated as $\langle X_{i_1, \ldots, i_n, k} \rangle_k$. Hence we obtain inductively a sequence $X_{i_1, \ldots, i_n}$ for every finite sequence $\langle i_1, \ldots, i_n \rangle$.
Now a $\Subset$-representation Π_2 with a defining pair $\langle \theta', \vartheta' \rangle$ can be constructed for Γ. θ' is the species of all finite sequences of natural numbers, and we put $\vartheta'^* \langle i_1, \ldots, i_n \rangle = \psi'_n X_{i_1, \ldots, i_n}$.

2.21. Theorem. Let Γ be an IR-space in which R3 holds, and let $\{V_i : i \in I\}$, $I \subset \underset{\cdot}{N}$, be a covering of Γ. Then $\{\mathrm{Int}\ V_i : i \in I\}$ is also a covering of Γ.
Proof. Suppose p to be an arbitrary point of Γ, and let Π_1 be a normal perfect representation of Γ with the inclusion property. There exist $\langle P'_n \rangle_n$, $\langle P_n \rangle_n \in \Pi_1$, such that $p = \langle P'_n \rangle^*_n = \langle P_n \rangle^*_n$, $\wedge n(P'_1 \ldots P'_n \Subset P_1 \ldots P_n)$. A function ψ associates with every element $\langle Q_n \rangle_n \in \Pi_1$ a natural number m such that $\langle Q_n \rangle^*_n \in V_m$. m is known from an initial segment of finite length, $\langle Q_1, \ldots, Q_k \rangle$. Let $\psi \langle P_n \rangle_n = \mu$, μ known from $P_1, \ldots, P_\nu$. Since the representation is perfect, we

may reason as follows.
If $\langle R_n\rangle_n \in P_1 \dots P_\nu$, there is a $\langle P''_n\rangle_n \in \Pi_1$ with $P_i = P''_i$ for $1 \leqslant i \leqslant \nu$, $\langle P''_n\rangle_n \simeq \langle R_n\rangle_n$. $\psi\langle P''_n\rangle_n = \mu$, so $\langle P''_n\rangle^*_n = \langle R_n\rangle^*_n \in V_\mu$.
Therefore $P_1 \dots P_\nu \subset V_\mu$. $P'_1 \dots P'_\nu \Subset P_1 \dots P_\nu$, hence $p \underline{\in} P_1 \dots P_\nu \subset V_\mu$, so $p \in \text{Int } V_\mu$.

2.22. Remark to 2.21. Let Γ be an I-space in which R3 holds, and let $\{V_i : i \in I\}$, $I \subset \underline{N}$, be a covering of Γ. Then $\wedge p \ \vee P \ \vee m(p \underline{\in} P \subset V_m)$.
(this follows from the proof of 2.21.)

2.23. Theorem. Let Γ be an I-space in which R1 and the conclusion of theorem 2.21 holds. Then every mapping δ of Γ into a separable metric space Δ with metric ρ is a continuous mapping.
Proof. Let $\langle p_i\rangle_i$ be a basic pointspecies for Δ. To every n and every $q \in \Gamma$ a p_i can be found such that $\rho(\delta q, p_i) < 2^{-n}$. There is a function ψ_ν and a spread representation $\Pi_1 \subset \Pi = \Pi(\Gamma)$ such that for a $\langle P_n\rangle_n \in \Pi_1$, $\psi_\nu\langle P_n\rangle_n$ is a natural number m for which $\rho(\delta\langle P_n\rangle^*_n, p_m) < 2^{-\nu}$.
$I_\nu = \{i : \vee\langle P_n\rangle_n \in \Pi_1(\psi_\nu\langle P_n\rangle_n = i)\}$. We put:

$$V_{i,\nu} = \{q : q \in \Pi^o(\Gamma) \ \& \ \rho(\delta q, p_i) < 2^{-\nu}\}$$

for every $i \in I_\nu$. $\{V_{i,\nu} : i \in I_\nu\}$ is a covering, therefore $\{\text{Int } V_{i,\nu} : i \in I_\nu\}$ is a covering too.
If $q \in \Gamma$, $q \in \text{Int } V_{\mu,\nu}$ we obtain:

$$\wedge r \in \text{Int } V_{\mu,\nu} \ (\rho(\delta q, \delta r) < 2^{-\nu+1}).$$

therefore δ is continuous.

2.24. Theorem. In an I-space Γ in which R3 holds, we are able to prove:

$$V \Subset' W \leftrightarrow V \Subset W.$$

Proof. Let $V \Subset' W$, $\langle P''_n\rangle_n \in \Pi$, and let Π_1 be a normal perfect representation of Γ with the inclusion property. There exist $\langle P'_n\rangle_n$, $\langle P_n\rangle_n \in \Pi_1$ such that $\langle P''_n\rangle_n \simeq \langle P'_n\rangle_n \simeq \langle P_n\rangle_n$, $\wedge n(P'_1 \dots P'_n \Subset P_1 \dots P_n)$.
A mapping ψ from Π_1 into $\{0, 1\}$ is defined, such that

$$\psi\langle S_n\rangle_n = 0 \rightarrow \langle S_n\rangle^*_n \notin V, \ \psi\langle S_n\rangle_n = 1 \rightarrow \langle S_n\rangle^*_n \in W.$$

For every $\langle S_n\rangle_n \in \Pi_1$, $\psi\langle S_n\rangle_n$ is determined by an initial segment of finite length; suppose $\psi\langle P_n\rangle_n$ to be determined by $\langle P_1, \dots, P_\mu\rangle$.
Since Π_1 is perfect, to every point generator $\langle T_n\rangle_n \in P_1 \dots P_\mu$ a $\langle T'_n\rangle_n \in \Pi_1$, $\langle T_n\rangle_n \simeq \langle T'_n\rangle_n$, can be found, such that $T'_i = P_i$ for $1 \leqslant i \leqslant \mu$. We put

$$\Pi_2 = \{\langle S_n\rangle_n : \wedge i(1 \leqslant i \leqslant \mu \rightarrow S_i = P_i) \ \& \ \langle S_n\rangle_n \in \Pi_1\}.$$

We remark that $\psi\Pi_2 = 0 \vee \psi\Pi_2 = 1$. In the first case $V \cap [P_1 \ldots P_\mu] = \emptyset$, in the second case $P_1 \ldots P_\mu \subset W$. $P'_1 \ldots P'_\mu \Subset P_1 \ldots P_\mu$. On that account, there exists a λ such that $\varphi P''_1 \ldots P''_\lambda P'_1 \ldots P'_\mu = 0 \vee P''_1 \ldots P''_\lambda \subset P_1 \ldots P_\mu$. The first is impossible. We conclude that $[P''_1 \ldots P''_\lambda] \cap V = \emptyset \vee P''_1 \ldots P''_\lambda \subset W$, and our theorem is proved.

2.25. Theorem. If Γ is an IR-space in which R3 holds, then for every representation Π_1 of Γ:
$\bigwedge \langle P_n \rangle_n \in \Pi_1 \bigvee m([P_1 \ldots P_m] \cap V = \emptyset \vee P_1 \ldots P_m \subset W) \leftrightarrow V \Subset W$.
Proof. The implication from the right to the left is trivial. Let $p \in \Pi^o$. There is a $\langle P_n \rangle_n \in \Pi_1$ such that $p = \langle P_n \rangle_n^*$, and we see that the left condition implies (by application to $\langle P_n \rangle_n$) $p \notin V \vee p \in W$, hence $V \Subset' W$. Then also $V \Subset W$ (2.24).

2.26. Theorem. If Γ is an I-space, and Π_1 a $\Subset$-representation for Γ, then
$\bigwedge n \langle P_n \rangle_n \in \Pi_1 \bigvee m(\varphi P_1 \ldots P_m Q = 0 \vee P_1 \ldots P_m \subset R) \leftrightarrow Q \Subset R$.
Proof. Trivial.

2.27. Lemma. Suppose Γ to be an IR-space in which N8($\mathfrak{B}$) holds, $\mathfrak{B} \subset \mathfrak{P}(\Gamma)$. Let $T_0, T_1 \in \mathfrak{B}$, $T_1 \Subset T_0$. Then there is a continuous mapping f from Π^o into $\underline{R}$, such that for any p:
$p \in T_1 \rightarrow f(p) = 1$; $p \notin T_0 \rightarrow f(p) = 0$; $0 \nmid f(p) \nmid 1$.
Proof. We construct a species of lattice elements $T_\alpha \in \mathfrak{B}$, $\alpha = m2^{-n}$, $n = 0,1,2,3,\ldots$, $m = 0,1,\ldots,2^n$, such that

$$\alpha > \beta \leftrightarrow T_\alpha \Subset T_\beta.$$

This construction can be carried out inductively. For suppose that all $T_{k2^{-\nu}}$, $0 \leqslant k \leqslant 2^{-\nu}$ already have been constructed, in agreement with the conditions mentioned before. We construct $T_{(2\mu+1)2^{-\nu-1}} \in \mathfrak{B}$, by applying N8($\mathfrak{B}$) to

$T_{(2\mu+2)2^{-\nu-1}} \Subset T_{2\mu\, 2^{-\nu-1}}$.
Thus we obtain:

$T_{(2\mu+2)2^{-\nu-1}} \Subset T_{(2\mu+1)2^{-\nu-1}} \Subset T_{2\mu\, 2^{-\nu-1}}$.
Let $\langle P_n \rangle_n \in \Pi$. We define:

$\psi^P(n,k) = \psi(n,k) = \sup\{m2^{-n} : \varphi P_1 \ldots P_k T_{m2^{-n}} = 1 \vee m = 0\}$.

We have

$$k > k' \rightarrow \psi(n,k) \leqslant \psi(n,k') \text{ for all } n,k,k' \quad (1)$$

For every n there exist t(m,n), m = 1,...,2^n, such that

$\varphi P_1 \ldots P_{t(m,n)} T_{m2^{-n}} = 0 \vee P_1 \ldots P_{t(m,n)} \subset T_{m2^{-n}-2^{-n-1}} \in T_{(m-1)2^{-n}}$

$t^P(n) = t(n)$ is a monotonously increasing function which satisfies:

$t(n) \geqslant \sup\{t(m,n): 1 \leqslant m \leqslant 2^n\}$.

For example, we may take:

$t(n) = \sup\{t(m,n),\ t(n-1) + 1: 1 \leqslant m \leqslant 2^n\}$.

Therefore for arbitrary but fixed ν, μ:

$$\varphi P_1 \ldots P_{t(\nu)} T_{\mu 2^{-\nu}} = 0 \vee P_1 \ldots P_{t(\nu)} \in T_{(\mu-1)2^{-\nu}},\quad 1 \leqslant \mu \leqslant 2^{-\nu}. \tag{2}$$

If $k, k' \geqslant t(\nu)$, $k \geqslant k'$, there are two possibilities, $\psi(\nu, k') = 0$ or $\psi(\nu, k') > 0$.

$$\psi(\nu, k') = 0 \rightarrow \psi(\nu, k) = 0 \text{ (by (1))} \tag{3}$$
$$\psi(\nu, k') > 0 \rightarrow P_1 \ldots P_{k'} \in T_{\psi(\nu,k')-2^{-\nu}} \tag{4}$$

since from (2) it follows that:

$\varphi P_1 \ldots P_{k'} T_{\psi(\nu,k')} = 0 \vee P_1 \ldots P_{k'} \in T_{\psi(\nu,k')-2^{-\nu}}$.

(4), combined with $k \geqslant k'$ leads to:

$$\psi(\nu, k') > 0 \rightarrow P_1 \ldots P_k \in T_{\psi(\nu,k')-2^{-\nu}} \tag{5}$$

We conclude:

$$\psi(\nu, k') > 0 \rightarrow \psi(\nu, k) \geqslant \psi(\nu, k') - 2^{-\nu} \tag{6}$$

(1), (3) and (6) together learn us that

$|\psi(\nu, k) - \psi(\nu, k')| \leqslant 2^{-\nu}$.

Therefore:

$$\wedge n \wedge k \wedge k'(k', k \geqslant t(n) \rightarrow |\psi(n, k') - \psi(n, k)| \leqslant 2^{-n}) \tag{7}$$

If $\psi(\nu, \lambda) = 1$, then for all $n \geqslant \nu$ $\psi(n, \lambda) = 1$.
If $\psi(\nu, \lambda) < 1$, $\varphi P_1 \ldots P_\lambda T_{\psi(\nu,\lambda) + 2^{-\nu}} = 0$.
Combining both cases, we obtain:

$$\wedge n \wedge n' \wedge k(n \geqslant n' \rightarrow \psi(n, k) \leqslant \psi(n', k) \leqslant \psi(n, k) + 2^{-n}) \tag{8}$$

From (7), (8) we are able to deduce that $\lim_{n \to \infty} \psi(t(n))$ exists.
For let $n \leqslant n'$. Then $|\psi(n, t(n)) - \psi(n', t(n'))| \leqslant$
$$|\psi(n, t(n)) - \psi(n, t(n'))| + |\psi(n, t(n')) - \psi(n', t(n'))| \leqslant 2^{-n} + 2^{-n} = 2^{-n+1}. \tag{9}$$

Moreover, the value of this limit is independent of the particular function t(n) chosen. For let t'(n) be another monotonously increasing function which satisfies

$\wedge n \wedge k' \wedge k(k', k \geqslant t'(n) \rightarrow |\psi(n,k') - \psi(n,k)| < 2^{-n}),$

then either $t(n) \geqslant t'(n)$ or $t'(n) \geqslant t(n)$, hence

$|\psi(n, t(n)) - \psi(n, t'(n))| \leqslant 2^{-n}.$

On that account we are justified in defining a function F on Π by: $F\langle P_n\rangle_n = \lim_{n \to \infty} \psi(n, t(n))$. Next we prove:

$$\wedge \langle Q_n\rangle_n \in \Pi \wedge \langle R_n\rangle_n \in \Pi(\langle Q_n\rangle_n \simeq \langle R_n\rangle_n \rightarrow F\langle Q_n\rangle_n = F\langle R_n\rangle_n) \quad (10)$$

We suppose $\Pi(\Gamma) = \Pi^*(\Gamma)$; this may be done without losing generality, since we have proved 2.12.
Thus, together with $\langle Q_n\rangle_n$, $\langle R_n\rangle_n$, $\langle S_n\rangle_n$ with $S_n = R_nQ_n$ is also a member of $\Pi(\Gamma)$. We prove (10) by demonstrating

$F\langle Q_n\rangle_n = F\langle S_n\rangle_n$, $F\langle R_n\rangle_n = F\langle S_n\rangle_n$.

First we define functions ψ^Q, ψ^R, ψ^S, t^Q, t^R analogous to the functions ψ^P, t^P in the foregoing part of the proof. We obtain immediately from the definition:

$$\psi^S(n,k) \leqslant \psi^Q(n,k) \text{ for all } n,k. \quad (11)$$

From a careful consideration of (2) it will be clear that a function t^S, analogous to t^P, may be taken to be equal to t^R or equal to t^Q.
Take a fixed number ν. Then

$$\psi^Q(\nu, t^Q(\nu)) = 0 \rightarrow \psi^S(\nu, t^Q(\nu)) = 0 \quad (12)$$

We have further

$$\psi^Q(\nu, t^Q(\nu)) > 0 \rightarrow Q_1 \ldots Q_{t^Q(\nu)} \Subset T_{\psi^Q(\nu, t^Q(\nu)) - 2^{-\nu}}$$
$$\rightarrow Q_1R_1Q_2R_2 \ldots Q_{t^Q(\nu)} R_{t^Q(\nu)} \Subset T_{\psi^Q(\nu, t^Q(\nu)) - 2^{-\nu}} \quad (13)$$

In both cases, (12) and (13), we obtain

$$\psi^Q(\nu, t^Q(\nu)) - 2^{-\nu} \leqslant \psi^S(\nu, t^Q(\nu)) \quad (14)$$

Combining (11), (14) we draw the conclusion that

$\wedge n(|\psi^Q(n, t^Q(n)) - \psi^S(n, t^Q(n))| \leqslant 2^{-n})$

Hence $F\langle S_n\rangle_n = F\langle Q_n\rangle_n$.
Likewise $F\langle S_n\rangle_n = F\langle R_n\rangle_n$. We are now justified in defining f by: $f(\langle P_n\rangle_n^*) = F\langle P_n\rangle_n$.
It remains to be shown that f satisfies the conditions mentioned in the lemma.
Let $\langle P_n\rangle_n \in T_\alpha$, then $\wedge m(\varphi P_1 \ldots P_m T_\alpha = 1)$; so $\wedge n \wedge k(\psi(n,k) > \alpha)$; hence $F\langle P_n\rangle_n \nless \alpha$. We have proved:

$$\wedge p(p \in T_\alpha \rightarrow f(p) \nless \alpha). \quad (15)$$

Let $\neg\langle P_n\rangle_n \in T_\alpha$. It follows from (2) that

$$\varphi P_1 \ldots P_{t(\nu)} T_{\alpha+2^{-\nu}} = 0 \vee P_1 \ldots P_{t(\nu)} \Subset T_\alpha.$$

The second possibility is excluded, therefore

$$\varphi P_1 \ldots P_{t(\nu)} T_{\alpha+2^{-\nu}} = 0.$$

Hence $\psi(\nu, t(\nu)) < \alpha + 2^{-\nu}$.
From $\wedge n(\psi(n, t(n)) < \alpha + 2^{-n})$ we see that $F\langle P_n\rangle_n \not> \alpha$.
We have thus proved:

$$\wedge p(p \notin T_\alpha \rightarrow f(p) \not> \alpha). \tag{16}$$

From (15) we deduce: $p \in T_1 \rightarrow f(p) = 1$, and from (16) we obtain: $p \notin T_0 \rightarrow f(p) = 0$.
Finally f has to be proved continuous.
Let $q = \langle Q_n\rangle_n$ be an arbitrary point of Γ. We shall prove

$$\wedge q \wedge \varepsilon \vee R \wedge r \in R(|f(r) - f(q)| < \varepsilon \;\&\; q \underline{\in} R) \tag{17}$$

Let $\langle Q_n\rangle_n^* = \langle P_n\rangle_n^*$, $\wedge n(P_{n+1} \Subset P_n)$, and let $\psi = \psi^P$, $t = t^P$ be the functions defined before. Take a fixed ν; $\langle R_n\rangle_n$ arbitrary.

$$\begin{aligned}&\psi(\nu, t(\nu)) = 0 \rightarrow \varphi P_1 \ldots P_{t(\nu)} T_{2^{-\nu}} = 0;\\ &\psi(\nu, t(\nu)) = 0 \;\&\; \langle R_n\rangle_n \in P_1 \ldots P_{t(\nu)} \rightarrow \langle R_n\rangle_n \notin T_{2^{-\nu}}\end{aligned} \tag{18}$$

$$\begin{aligned}&\psi(\nu, t(\nu)) = 1 \rightarrow P_1 \ldots P_{t(\nu)} \Subset T_{1-2^{-\nu}};\\ &\psi(\nu, t(\nu)) = 1 \;\&\; \langle R_n\rangle_n \in P_1 \ldots P_{t(\nu)} \rightarrow \langle R_n\rangle_n \in T_{1-2^{-\nu}}\end{aligned} \tag{19}$$

$$\begin{aligned}0 < \psi(\nu, t(\nu)) < 1 \rightarrow &\varphi P_1 \ldots P_{t(\nu)} T_{\psi(\nu,t(\nu))+2^{-\nu}} = 0\\ &\& \; P_1 \ldots P_{t(\nu)} \Subset T_{\psi(\nu,t(\nu))-2^{-\nu}};\end{aligned}$$

$$\begin{aligned}&0 < \psi(\nu, t(\nu)) < 1 \;\&\; \langle R_n\rangle_n \in P_1 \ldots P_{t(\nu)} \rightarrow\\ &\quad \neg\langle R_n\rangle_n \in T_{\psi(\nu,t(\nu))+2^{-\nu}} \;\&\; \langle R_n\rangle_n \in T_{\psi(\nu,t(\nu))-2^{-\nu}}\end{aligned} \tag{20}$$

From (15),(16),(18),(19),(20) we see that:

$$\left\{\begin{aligned}&\wedge m \wedge \langle R_n\rangle_n \in \Pi\,(\langle R_n\rangle_n \in P_1 \ldots P_{t(m)} \rightarrow\\ &\qquad |F\langle R_n\rangle_n - \psi(m, t(m))| \not> 2^{-m})\\ &q \underline{\in} P_1 \ldots P_{t(\nu)}\,.\end{aligned}\right. \tag{21}$$

Combining (21) and (9) we see that

$$\wedge\langle R_n\rangle_n \in \Pi\,(\langle R_n\rangle_n \in P_1 \ldots P_{t(\nu)} \rightarrow |F\langle P_n\rangle_n - F\langle R_n\rangle_n| < 4.2^{-\nu})$$

since $|F\langle R_n\rangle_n - F\langle P_n\rangle_n| \not> |F\langle P_n\rangle_n - \psi(\nu, t(\nu))| + |F\langle R_n\rangle_n - \psi(\nu, t(\nu))| \not> 2^{-\nu+1} + 2^{-\nu} < 4.2^{-\nu}$.
If we take ν so large that $4.2^{-\nu} < \varepsilon$, we have proved (17).

2.28. Lemma. Let Γ be an I-space in which R4 holds; let Π_1 be a $\Subset$-representation for Γ with a defining pair $\langle \theta, \mathfrak{d}\rangle$.

$$\mathfrak{C} = \{\bar{\mathfrak{d}}\langle i_1, \ldots, i_k\rangle : \langle i_1, \ldots, i_k\rangle \in \theta\}.$$

Suppose $\mathfrak{C} \subset \mathfrak{B}$, N8($\mathfrak{B}$) holds for Γ. Then Γ is metrizable.
Proof. We enumerate all pairs

$\langle \mathfrak{I}\langle i_1,\ldots,i_{k+1}\rangle, \mathfrak{I}\langle i_1,\ldots,i_k\rangle\rangle$ such that $\langle i_1,\ldots,i_k,i_{k+1}\rangle \in \theta$, in an enumeration $\langle\langle Q_i, Q_i^!\rangle\rangle_i$. To every pair $\langle Q_i, Q_i^!\rangle$ of the enumeration a continuous function f_i can be constructed, according to lemma 2. 27, such that for any p, $p \in Q_i \rightarrow f_i(p) = 1$, $p \notin Q_i^! \rightarrow f_i(p) = 0$.
We define

$$\rho(p,q) = \sum_{i=1}^{\infty} 2^{-i} | f_i(p) - f_i(q) | .$$

We must show that ρ represents a metrization of Γ.
To achieve this we must prove for every p, q, r:

$$\rho(p,q) = \rho(q,p) \tag{1}$$
$$\rho(p,q) \nless 0 \ \& \ (p \# q \leftrightarrow \rho(p,q) > 0) \tag{2}$$
$$\rho(p,q) \ngtr \rho(p,r) + \rho(r,q) \tag{3}$$
$$\wedge\varepsilon \vee T(p \underline{\in} T \ \& \ T \subset U_\varepsilon(p)) \tag{4}$$
$$\wedge R(p \underline{\in} R \rightarrow \vee\varepsilon(U_\varepsilon(p) \subset R)) \tag{5}$$

(1) and the first part of (2) are trivial. The second part of (2) is demonstrated as follows. Let $r \# s$, $r = \langle R_n\rangle_n^*$, $s = \langle S_n\rangle_n^*$, $\langle R_n\rangle_n, \langle S_n\rangle_n \in \Pi_1$. For a certain ν, $\varphi R \ldots R_\nu S_1 \ldots S_\nu = 0$.
There is a μ such that $\langle R_1 \ldots R_{\nu+1}, R_1 \ldots R_\nu\rangle = \langle Q_\mu, Q'_\mu\rangle$.

$$r \in R_1 \ldots R_{\nu+1} \rightarrow f_\mu(r) = 1; \quad s \notin R_1 \ldots R_\nu \rightarrow f_\mu(s) = 0.$$

Hence $\rho(r,s) \nless 2^{-\mu}$, and $\rho(r,s) > 0$.
(3) follows from

$$\wedge i(| f_i(p) - f_i(q)| \ngtr | f_i(p) - f_i(r)| + | f_i(q) - f_i(r)|).$$

Proof of (4). Let p be an arbitrary point. Choose a natural number ν, such that $\sum_{i=\nu+1}^{\infty} 2^{-i} = 2^{-\nu} < \varepsilon\, 2^{-1}$. $f_1,\ldots,f_\nu$ are continuous functions, so there exist $T_1,\ldots T_\nu$ such that

$$\wedge q(q \in T_i \rightarrow | f_i(q) - f_i(p) | < \frac{2^{i-1}\varepsilon}{\nu}), \text{ for } p \underline{\in} T_i,\ 1 \leqslant i \leqslant \nu \tag{6}$$

If $T = T_1 \ldots T_\nu$, we deduce from (6) for any $q \in T$:

$$\rho(p,q) = \sum_{i=1}^{\infty} | f_i(p) - f_i(q)| 2^{-i} < \varepsilon 2^{-1} + \sum_{i=1}^{\nu} 2^{-i} | f_i(p) - f_i(q)|$$
$$< \varepsilon 2^{-1} + \varepsilon 2^{-1} = \varepsilon.$$

Proof of (5). Let $p \underline{\in} R$, p, R arbitrary, $\langle P_n\rangle_n \in p$, $\langle P_n\rangle_n \in \Pi_1$. A ν can be found such that $P_1 \ldots P_\nu \Subset R$, and there exists a μ such that $\langle P_1 \ldots P_\nu P_{\nu+1}, P_1 \ldots P_\nu\rangle = \langle Q_\mu, Q'_\mu\rangle$. By definition of f_μ, $f_\mu(p) = 1$. Let q be an arbitrary point.

$$\begin{aligned} q \in U_{2^{-\mu}}(p) &\rightarrow \rho(p,q) < 2^{-\mu} \\ &\rightarrow | f_\mu(p) - f_\mu(q)| 2^{-\mu} < 2^{-\mu} \\ &\rightarrow | f_\mu(p) - f_\mu(q)| < 1 \\ &\rightarrow f_\mu(q) > 0 \\ &\rightarrow \neg\neg q \in P_1 \ldots P_\nu \\ &\rightarrow q \in P_1 \ldots P_\nu \Subset R \\ &\rightarrow q \underline{\in} R. \end{aligned}$$

2.29. Corollary to 2.28. If Γ is a space which satisfies the requirements of 2.28, then Γ can be embedded topologically in the hilbert cube by a mapping g:

$$g(p) = \langle f_i(p) \rangle_i.$$

2.30. Theorem. If R1 holds in a metrizable IR-space $\Gamma = \langle V_0, \mathfrak{T} \rangle$, and ρ is a metric on V_0 such that $\langle V_0, \mathfrak{T}(\rho) \rangle = \langle V_0, \mathfrak{T} \rangle$, then $\langle V_0, \rho \rangle$ has a point representation.
Proof. By I5, a point p(Q) can be associated with every lattice element Q such that $\varphi Q = 1$. Let $\langle \theta, \vartheta \rangle$ be the defining pair of a spread Π_1 which represents Γ.

If $\langle Q_n \rangle_n \in \Pi_1$, then $\langle p(Q_1 \ldots Q_n) \rangle_n$ converges with respect to ρ to $\langle Q_n \rangle_n^* = q$; for since $q \in U_\varepsilon(q)$, it follows that for a certain R $q \underline{\in} R \subset U_\varepsilon(q)$; then for a certain ν $Q_1 \ldots Q_\nu \Subset R$, so $p(Q_1 \ldots Q_\nu) \in R$. $\{ p(Q) : Q \in \bar{\vartheta}\, \theta \}$ is a basic pointspecies for Γ.

3. CIN- and PIN-spaces.

3.1. In this paragraph we treat some special cases of IR-spaces, in which Π, the species of point generators, can be eliminated as an undefined object. Thus, in a sense, we obtain a "pointless" topology. The expression "topology without points" was first coined in MENGER 1940. From the various theories discussed there, the theory of MOORE 1935 somewhat resembles the approach of the CIN-spaces; the theory of WALD 1932 on the other hand is more related to the PIN-spaces. One of the main differences between our approach and these theories is that the notion of strong inclusion is not a primitive one in our system.

3.2. Definition. An abstract CIN-space is defined as an abstract I-space, such that the following postulates are fulfilled.

C1. There exists a sequence of species of lattice elements, $\langle \mathfrak{A}_i \rangle_i$, $\mathfrak{A}_i = \langle A_{i,j} \rangle_j$, such that $\mathfrak{A} \subset \mathfrak{A}_1$, $\wedge i(\mathfrak{A}_{i+1} \subset \mathfrak{A}_i)$, $\wedge n \wedge m \wedge i \wedge j(\varphi A_{n,i} A_{m,j} = 1 \rightarrow A_{n,i} A_{m,j} \in \mathfrak{A}_n)$.

C2. $\varphi A_{1,i(1)} \ldots A_{n,i(n)} = 1 \rightarrow \vee k(\varphi A_{1,i(1)} \ldots A_{n,i(n)} A_{n+1,k} = 1)$.

C3. $\wedge i \wedge j \vee k(A_{i,j} \Subset A_{i,k})$, $\wedge i \wedge j(\varphi A_{i,j} = 1)$.

C4. $\wedge \langle P_n \rangle_n (\langle P_n \rangle_n \in \Pi \leftrightarrow \wedge n \vee j(P_n = A_{n,j} \;\&\; \varphi P_1 \ldots P_n = 1))$.

N6.

N9. $\wedge i \wedge j \wedge k(A_{i,j} \Subset A_{i,k} \rightarrow \vee l(A_{i,j} \Subset A_{i,l} \Subset A_{i,k}))$.

A CIN-space is a topological space homeomorphic to an abstract CIN-space. The species $\langle A_{i,j} \rangle_{i,j}$ is called a CIN-covering system. A CIN-basis is defined in the same manner as an I-basis in 3.1.26. The letter "C" is derived from "covering".

3.3. Theorem. In a CIN-space
a) postulate I5 is derivable from the other postulates, and
b) $N8(\mathfrak{A}_1)$ holds.
Proof. (a). Let $\varphi P = 1$. $P = Q_1 + \ldots + Q_\nu$, Q_i $(1 \leqslant i \leqslant \nu)$ a meet of elements of $\mathfrak{A}$. Since $\mathfrak{A} \subset \mathfrak{A}_1$, there is a Q_μ, $1 \leqslant \mu \leqslant \nu$, $Q_\mu \in \mathfrak{A}_1$, $\varphi Q_\mu = 1$. Let $Q_\mu = A_{1,i(1)}$. By means of repeated application of C2 we prove inductively the existence of a sequence $\langle A_{n,i(n)}\rangle_n$ such that $\wedge n(\varphi A_{1,i(1)} \ldots A_{n,i(n)}=1)$. By C4, $\langle A_{n,i(n)}\rangle_n \in \Pi$, hence $\langle A_{n,i(n)}\rangle_n^* \in P$.
(b) Immediate by N9.

3.4. Remark. a) Elements of a CIN-covering system with different indices are not necessarily different.
b) Π can be eliminated completely from the postulates, if we combine I4 and C4 to

C5. $\wedge \langle P_n\rangle_n (\wedge n \ \vee j (P_n = A_{n,j}) \ \& \ \wedge n(\varphi P_1 \ldots P_n = 1) \rightarrow \langle P_n\rangle_n \in \Sigma)$.

and afterwards define Π by:

$$\langle P_n\rangle_n \in \Pi \leftrightarrow \wedge n \ \vee j(P_n = A_{n,j}) \ \& \ \langle P_n\rangle_n \in \Sigma.$$

c) The family of CIN-spaces coincides classically with the family of separable complete metric spaces.
The proof follows from FROLÍK 1962, theorem 3.1 (proved in FROLÍK 1960 2.8, 2.14) and the observations
1) Every CIN-space is completely regular, since it is metrizable.
2) $\langle\langle \text{Int } A_{i,n}\rangle_n\rangle_i = \langle \mathfrak{A}_i\rangle_i$ satisfies the conditions of FROLÍK 1962, theorem 3.

3.5. Theorem. In a CIN-space R3 and R4 hold.
Proof. We show that Π is a normal perfect representation with a defining pair $\langle \theta, \vartheta\rangle$,
$\langle P_1, \ldots, P_n\rangle \in \vartheta \ \theta \leftrightarrow \wedge t(1 \leqslant t \leqslant n \rightarrow P_t \in \mathfrak{A}_t) \ \& \ \varphi P_1 \ldots P_n = 1$.
C2 guarantees us that θ is in fact a spread law. We prove Π to be perfect as follows. Let $\langle A_{n,i(n)}\rangle_n \in \Pi$, $\langle A_{n,i(n)}\rangle_n \in A_{1,j(1)} \ldots A_{\nu,j(\nu)}$. We define $\langle A_{n,k(n)}\rangle_n$ by:

$$1 \leqslant n \leqslant \nu \rightarrow k(n) = j(n)$$
$$n > \nu \rightarrow k(n) = i(n)$$

$\langle A_{n,k(n)}\rangle_n \in \Pi$ by C4. $\langle A_{n,k(n)}\rangle_n \simeq \langle A_{n,i(n)}\rangle_n$, hence our representation is perfect.

We can construct to every $\langle A_{n,i(n)}\rangle_n \in \Pi$ an $\langle A_{n,j(n)}\rangle_n \in \Pi$, such that $\langle A_{n,i(n)}\rangle_n \simeq \langle A_{n,j(n)}\rangle_n$, and

$$\wedge n(A_{1,i(1)} \ldots A_{n,i(n)} \Subset A_{1,j(1)} \ldots A_{n,j(n)}),$$
$$\wedge n(A_{1,j(1)} \ldots A_{n+1,j(n+1)} \Subset A_{1,j(1)} \ldots A_{n,j(n)})$$

This construction is carried out by induction. By C3, there

is an $A_{1,j(1)}$ such that $A_{1,i(1)} \Subset A_{1,j(1)}$. To $A_{2,i(2)}$ we can find an $A_{2,k}$ such that $A_{2,i(2)} \Subset A_{2,k}$. Thus we obtain $A_{1,i(1)} A_{2,i(2)} \Subset A_{1,j(1)} A_{2,k}$. By N9, there is an $A_{2,j(2)}$ such that $A_{1,i(1)} A_{2,i(2)} \Subset A_{2,j(2)} \Subset A_{1,j(1)} A_{2,k}$. Hence $A_{1,i(1)} A_{2,i(2)} \Subset A_{1,j(1)} A_{2,j(2)} \Subset A_{1,j(1)}$.
Suppose $A_{1,j(1)}, \dots, A_{\nu,j(\nu)}$ to be already constructed. To $A_{\nu+1,i(\nu+1)}$ an $A_{\nu+1,k}$ with $A_{\nu+1,i(\nu+1)} \Subset A_{\nu+1,k}$ can be found. It follows that
$A_{1,i(1)} \dots A_{\nu+1,i(\nu+1)} \Subset A_{1,j(1)} \dots A_{\nu,j(\nu)} A_{\nu+1,k}$.
We construct (by an application of N9) an $A_{\nu+1,j(\nu+1)}$ such that
$A_{1,i(1)} \dots A_{\nu+1,i(\nu+1)} \Subset A_{\nu+1,j(\nu+1)} \Subset A_{1,j(1)} \dots A_{\nu,j(\nu)} A_{\nu+1,k}$.
We conclude that
$A_{1,i(1)} \dots A_{\nu+1,i(\nu+1)} \Subset A_{1,j(1)} \dots A_{\nu+1,j(\nu+1)} \Subset A_{1,j(1)} \dots A_{\nu,j(\nu)}$.
As a consequence, Π possesses the inclusion property.
Further, if we replace every $\langle A_{n,i(n)}\rangle_n \in \Pi$ by the corresponding $\langle A_{n,j(n)}\rangle_n$, constructed as indicated before, we obtain a $\Subset$-representation.

3.6. Corollary to 3.5.
a) A CIN-space is metrizable.
b) In a CIN-space for all V, W: $V \Subset' W \rightarrow V \Subset W$.
c) Every mapping of a CIN-space into a separable metric space is continuous.
d) Every one-to-one mapping of a CIN-space Γ onto a CIN-space Γ' is a homeomorphism between Γ and Γ'.
Proof. (a) follows from 2.28, (b) from 2.25, (c) from 2.23, and (d) is an immediate consequence of (c), 2.30, (a), 2.2.5.

3.7. Theorem. If Γ is a CIN-space, V a closed weakly located pointspecies of Γ, then $V \Subset'' W \rightarrow V \Subset W$ for any pointspecies W. If V is weakly located, then $V \Subset W \leftrightarrow V^- \subset \mathrm{Int}\, W$.
Proof. Let $\langle P_n\rangle_n \in \Pi$, $\langle P_n\rangle_n$ arbitrary. We can find a $\langle Q_n\rangle_n$, such that $\langle Q_n\rangle_n \simeq \langle P_n\rangle_n$, $\wedge n(Q_{n+1} \Subset Q_n)$, $\langle Q_n\rangle_n \in \Pi$.
Since V is weakly located, we have

$$\wedge n(\vee q(q \in [Q_n] \cap V) \vee \vee m(m > n \;\&\; [Q_m] \cap V = \emptyset)) \qquad (1)$$

(For if there is an R such that $\langle P_n\rangle_n^* \in R \subset Q_n$, $[R] \cap V = \emptyset$, there is also a $Q_m \Subset R \subset Q_n$). We can select from $\langle Q_n\rangle_n$ a subsequence $\langle Q'_n\rangle_n$, $\wedge n(Q'_{n+1} \Subset Q'_n)$, such that for a certain sequence of points $\langle q_n\rangle_n$, $q_n = \langle S^n_m\rangle^*_m$, the following assertion holds:

$$\wedge n([Q'_n] \cap V = \emptyset \vee q_n \in [Q'_n] \cap V) \qquad (2)$$

For if $Q'_n = Q_k$, we take $Q'_{n+1} = Q_{k+1}$ if we know that $\vee q(q \in [Q_{k+1}] \cap V)$, and $Q'_{n+1} = Q_m, m > n+1$, if we know that $[Q_m] \cap V = \emptyset$, depending on the decision which can be made according to (1).

If $[Q_1'] \cap V = \emptyset$, we remark that $\langle P_n\rangle_n^* \in Q_1'$, so there is a ν such that $P_1 \ldots P_\nu \Subset Q_1'$; $[P_1 \ldots P_\nu] \cap V = \emptyset$.
If $q_1 \in [Q_1'] \cap V$, we construct a finitary spread Π_1 with a defining pair $\langle \theta, \vartheta\rangle$, such that $\vartheta\langle i_1, \ldots, i_k\rangle = \langle \gamma i_1, \ldots, \gamma i_k\rangle$ as follows.

$$\langle R_1, \ldots, R_n\rangle \in \vartheta\,\theta \leftrightarrow \vee k(1 \leqslant k \leqslant n \,\&\, \langle R_1, \ldots, R_k\rangle = \langle Q_1', \ldots, Q_k'\rangle \,\&\, q_k \in [Q_k'] \cap V \,\&\, \wedge t(k+t \leqslant n \rightarrow R_{k+t} = Q_k' S_1^k \ldots S_{k+t}^k)) \qquad (3)$$

Every spread element is a point generator and represents a point of V, and satisfies $\wedge\langle T_n\rangle_n (\langle T_n\rangle_n \in \Pi_1 \rightarrow \wedge m(T_{m+1} \subset T_m))$. Because of $V \Subset'' W$, we have

$$\wedge\langle T_n\rangle_n \in \Pi_1 \; \vee i(T_i \subset W).$$

Π_1 is finitary, therefore a natural number ν must exist (1.1.12) such that $T_\nu \subset W$ for every $\langle T_n\rangle_n \in \Pi_1$. We have

$$\langle Q_1', \ldots, Q_\nu'\rangle \in \vartheta\,\theta \rightarrow Q_\nu' \subset W.$$
$$\langle Q_1', \ldots, Q_\nu'\rangle \notin \vartheta\,\theta \rightarrow [Q_\nu'] \cap V = \emptyset.$$

A natural number μ can be found such that $P_1 \ldots P_\mu \subset Q_\nu'$, hence in the first case $P_1 \ldots P_\nu \subset W$, in the second case $[P_1 \ldots P_\nu] \cap V = \emptyset$. Q.e.d. The second part follows easily with 3.2.17.

3.8. Definition. An abstract PIN-space is an abstract I-space such that N6, N8 hold, and for which Π can be described by the following postulate P (from "point"):

$$\text{P.} \quad \Sigma = \Pi.$$

A PIN-space is a topological space which is homeomorphic to an abstract PIN-space. A PIN-basis is defined in the same manner as an I-basis (3.1.26).
Remark. A PIN-space is therefore a space which satisfies I1-3, N6, N8. Π is defined in terms of φ.

3.9. Theorem. Every PIN-space is a CIN-space.
Proof. Let $\langle P_n\rangle_n$ be a fixed enumeration of the lattice elements of a PIN-space Γ, such that $\wedge n(n > 1 \rightarrow \varphi P_n = 1)$, $P_1 = A_0$. From this enumeration we construct an enumeration $\langle\!\langle Q_i, Q_i'\rangle\!\rangle_i$ of all pairs $\langle Q_i, Q_i'\rangle$ with $Q_i, Q_i' \in \mathfrak{P}(\Gamma)$, $\varphi Q_i Q_i' = 0$, $Q_1 = A_0 \vee Q_1' = A_0$, $\wedge i(\varphi Q_i = 1 \vee \varphi Q_i' = 1)$, if necessary with repetitions to grant a denumerably infinite sequence.
We define:
$\mathfrak{A}_i = \{P\colon \varphi P = 1 \,\&\, \wedge j(j \leqslant i \rightarrow \varphi PQ_j = 0 \vee \varphi PQ_j' = 0)\}$.
We see that $\mathfrak{A} \subseteq \mathfrak{A}_1$, since $\wedge n(\varphi A_n Q_n = 0 \vee \varphi A_n Q_1' = 0)$; $\wedge i(\mathfrak{A}_{i+1} \subset \mathfrak{A}_i)$; $\wedge i \wedge P \wedge Q(P \in \mathfrak{A}_i \,\&\, \varphi PQ = 1 \rightarrow PQ \in \mathfrak{A}_i)$. C1 is therefore satisfied.
If $\varphi P_1 \ldots P_n = 1$, $\wedge i(1 \leqslant i \leqslant n \rightarrow P_i \in \mathfrak{A}_i)$, then there is a $P_{n+1} \in \mathfrak{A}_{n+1}$ such that $\varphi P_1 \ldots P_n P_{n+1} = 1$, for if

$\varphi P_1 \ldots P_n Q_{n+1} = 1$, we take $P_{n+1} = P_1 \ldots P_n Q_{n+1}$, and if $\varphi P_1 \ldots P_n Q_{n+1} = 0$, we take $P_{n+1} = P_1 \ldots P_n$. So we have proved C2.
C3 follows from N6, for if $P \in \mathfrak{A}_i$, $\wedge j(j \leqslant i \rightarrow \varphi PQ_j = 0 \vee \varphi PQ_j' = 0)$, there exist according to N6, R_j for every $j \leqslant i$, such that $P \Subset R_j$ & $(\varphi R_j Q_j = 0 \vee \varphi R_j Q_j' = 0)$; and if we take $R = R_1 \ldots R_i$ then $P \Subset R$ & $R \in \mathfrak{A}_i$. N9 is an immediate consequence of N8. To obtain a CIN-space, we must afterwards restrict Π to Π' consisting of all point generators $\langle R_n \rangle_n$ such that

$$\wedge m(m \leqslant n \rightarrow \varphi R_n Q_m = 0 \vee \varphi R_n Q_m' = 0)$$

We remark that the two spaces $\langle \varphi, \Pi' \rangle$ and $\langle \varphi, \Pi \rangle$ satisfy the conditions of 1.32; on that account they define the same topology, and their notions of strong inclusion coincide.

3.10. Definition. A point p of an abstract I-space Γ is called decidable, if

$$\wedge P(p \in P \vee p \;\omega\; P).$$

3.11. Every PIN-space possesses an enumerable set of decidable points, dense in the space.
Proof. We show that every $Q \in \mathfrak{P}$ with $\varphi Q = 1$ contains a decidable point. To a certain Q with $\varphi Q = 1$ we can find an enumeration $\langle P_n \rangle_n$ of all lattice elements P such that $\varphi P = 1$, with $P_1 = Q$.
We define $\langle R_n \rangle_n \in \Pi$ as follows.
$R_1 = P_1$; if $R_1, \ldots, R_\nu$ already have been defined, we define $R_{\nu+1} = P_{\nu+1}$ if $\varphi R_1 \ldots R_\nu P_{\nu+1} = 1$, $R_{\nu+1} = R_\nu$ if $\varphi R_1 \ldots R_\nu P_{\nu+1} = 0$.
If $\varphi ST = 0$, $\varphi S = 1$, $\varphi T = 1$, there exist μ, ν such that $S = P_\nu$, $T = P_\mu$. Suppose $\mu < \nu$. Then $\varphi R_1 \ldots R_{\mu-1} P_\mu = 0 \vee \varphi R_1 \ldots R_\mu P_\mu = 1$. If $\varphi R_1 \ldots R_\mu P_\mu = 1$, $R_\mu = P_\mu$, and thus $\varphi R_1 \ldots R_\mu R_\nu = 0$.
Hence if $\lambda = \sup\{\mu, \nu\}$, then $\varphi R_1 \ldots R_\lambda S = 0 \vee \varphi R_1 \ldots R_\lambda T = 0$. So $\langle R_n \rangle_n \in \Pi$.
$\langle R_n \rangle_n$ is decidable, for if $\varphi S = 1$, $S = P_\nu$ for a certain ν, then either $R_\nu = P_\nu$, and in this case $\langle R_n \rangle_n \in P_\nu$, or $\varphi R_1 \ldots R_{\nu-1} P_\nu = 0$, hence $\langle R_n \rangle_n \;\omega\; P_\nu$.
If we associate with every Q with $\varphi Q = 1$ an enumeration as indicated in the beginning of the proof, we obtain an enumerable set of decidable points. This species is dense in the space, for if $q \underline{\in} R$, $q = \langle Q_n \rangle_n^*$, $\wedge n(Q_{n+1} \Subset Q_n)$, there is a ν such that $q \underline{\in} Q_\nu \Subset R$; and hence there is a decidable $p \in Q_\nu \subset \mathrm{Int}[R]$.

4. Topological products.

4.1. Definition. Let Γ_i, $i = 1,2,\ldots$ be a finite or a denumerably infinite sequence of I-spaces. $\mathfrak{A}(\Gamma_i) = \mathfrak{A}_i$, $\mathfrak{P}(\Gamma_i) = \mathfrak{P}_i$, $\varphi_{\Gamma_i} = \varphi_i$. Arbitrary elements of $\mathfrak{P}_i$ are denoted by capitals with upper index i (and indexed below if necessary): P^i, Q^i, R^i, S^i, T^i. Now we define a product space Γ as follows.

$\mathfrak{A}(\Gamma) = \mathfrak{A} = \{\langle P^i \rangle_i : \vee n \wedge m(\varphi_m P^m = 1 \;\&\; (m > n \rightarrow P^m = A^m_\infty))\}$.

$\mathfrak{A}_\pi = \{\langle P^i \rangle_i : \vee n \wedge m(m > n \rightarrow P^m = A^m_\infty)\}$.

We define functions π_j from $\mathfrak{A}_\pi$ into $\mathfrak{A}_j$:

$\langle P^i \rangle_i \in \mathfrak{A}_\pi \rightarrow \pi_j \langle P^i \rangle_i = P^j$.

$\mathfrak{P} = \mathfrak{P}(\Gamma)$ is the free distributive lattice with the elements of $\mathfrak{A}$ as generators, with a zero-element A_0, an all-element A_∞, and operators $+$, $\cdot$. Arbitrary elements of $\mathfrak{P}$ are denoted by capitals P, Q, R, S, T, with indexes below if necessary.

A defining function $\varphi = \varphi_\Gamma$ is declared on $\mathfrak{A}_\pi$ by:

$\varphi P_1 \ldots P_n = 1 \leftrightarrow \wedge i(\varphi_i \pi_i P_1 \ldots \pi_i P_n = 1)$, $(P_1, \ldots, P_n \in \mathfrak{A}_\pi)$.

We put $\varphi A_0 = 0$. φ satisfies I1, I2 with respect to $\mathfrak{A}$ and can therefore be extended to $\mathfrak{P}$. Elements of $\mathfrak{A}_\pi$ will sometimes, somewhat less formal, be written as sequences $(P^1, P^2, \ldots)$ instead of being written as $\langle P^i \rangle_i$. If we define $\sim\; = \;\sim_\Gamma$ with respect to elements of $\mathfrak{P}$ and $\mathfrak{A}_\pi$ as in 1.6, we remark that:

a) Every finite meet of elements of $\mathfrak{A}$ is equivalent to an element of $\mathfrak{A}_\pi$: $\langle P^i_1 \rangle_i \langle P^i_2 \rangle_i \ldots \langle P^i_n \rangle_i \sim \langle P^i_1 \ldots P^i_n \rangle_i$. Hence we treat an element of $\mathfrak{P}$ in the sequel always as a join of elements of $\mathfrak{A}_\pi$.

b) $A_0 \sim \langle A^i_0 \rangle_i$; $A_\infty \sim \langle A^i_\infty \rangle_i$.

c) $(P^1, \ldots, P^{i-1}, Q^i, P^{i+1}, \ldots) + (P^1, \ldots, P^{i-1}, R^i, P^{i+1}, \ldots)$
$\sim (P^1, \ldots, P^{i-1}, Q^i + R^i, P^{i+1}, \ldots)$

Finally we define:

$\Pi(\Gamma) = \Pi = \{\langle P_n \rangle_n : \wedge n(P_n \in \mathfrak{A}_\pi) \;\&\; \wedge i(\langle \pi_i P_n \rangle_n \in \Pi^*(\Gamma_i))\}$.

π_j can be extended to Π, Π^o by stipulating:

$$\langle P_n \rangle_n \in \Pi \rightarrow \pi_j \langle P_n \rangle_n = \langle \pi_j P_n \rangle_n$$
$$\pi_j \langle P_n \rangle^*_n = \langle \pi_j P_n \rangle^*_n .$$

In the sequel we formulate a number of theorems for the product of a finite or denumerably infinite sequence of I-spaces, but in the proof we restrict ourselves to the denumerably infinite case, since the finite case is proved easily by omitting some details of the proof for the infinite case.

4.2. Theorem. Let Γ_i, $i = 1,2,\ldots$ be a finite or a denumerably infinite sequence of I-spaces. The product Γ of this sequence if again an I-space.

Proof. I1, I2 are already valid by definition 4.1. I3 follows from remark (a) in 4.1. Since

$\varphi(P_1 + \ldots + P_\mu)(Q_1 + \ldots + Q_\nu) = 0 \leftrightarrow$
$\Lambda i \, \Lambda j(1 \leqslant i \leqslant \mu \; \& \; 1 \leqslant j \leqslant \nu \rightarrow \varphi P_i Q_j = 0)$,
the requirement for an element of Π to satisfy the splitting condition for all pairs of lattice elements P, Q such that $\varphi PQ = 0$, is equivalent with the validity of the splitting condition with respect to all pairs $P, Q \in \mathfrak{A}$ such that $\varphi PQ = 0$. If $\varphi \langle P^i \rangle_i \langle Q^i \rangle_i = 0$, there is a ν such that $\varphi_\nu P^\nu Q^\nu = 0$. If $\langle R_n \rangle_n \in \Pi$, then $\langle R_n^\nu \rangle_n \in \Pi^*(\Gamma_\nu)$, if $\pi_\nu R_n = R_n^\nu$. There is a μ such that

$$\varphi_\nu R_1^\nu \ldots R_\mu^\nu P^\nu = 0 \vee \varphi_\nu R_1^\nu \ldots R_\mu^\nu Q^\nu = 0.$$

Hence $\varphi R_1 \ldots R_\mu \langle P^i \rangle_i = 0 \vee \varphi R_1 \ldots R_\mu \langle Q^i \rangle_i = 0$.
I5 is satisfied, since if $\varphi P = 1$, we can find a $\langle R_n^i \rangle_n \in \pi_i P$ for every i. Then if $\langle R_n \rangle_n$ is defined by $\pi_i R_n = R_{n-i+1}^i$ for $n \geqslant i$, $\pi_i R_n = A_\infty^i$ for $n < i$, it follows that $\langle R_n \rangle_n \in P$.

4.3. Remark. Since the symbols for elements of the species $\mathfrak{P}_i, \mathfrak{P}$ are taken from disjoint species, in most cases we can use one symbol for the notions $\Subset$, $\subset$, $\underline{\in}$, $\in$, $\#$, $\simeq$, $*$, without ambiguity.

4.4. Lemma. Γ is the product of a finite or denumerably infinite sequence of Γ_i. With notations as in 4.1, 4.3, we have:

a) $P \subset Q \leftrightarrow \Lambda i(\pi_i P \subset \pi_i Q)$ for all $P, Q \in \mathfrak{A}$.
b) For all $\langle P_n \rangle_n, \langle Q_n \rangle_n \in \Pi$: $\langle P_n \rangle_n \simeq \langle Q_n \rangle_n \leftrightarrow$
   $\Lambda m(\langle \pi_m P_n \rangle_n \simeq \langle \pi_m Q_n \rangle_n)$.
c) For all $\langle P_n \rangle_n, \langle Q_n \rangle_n \in \Pi$,
   $\langle P_n \rangle_n \# \langle Q_n \rangle_n \leftrightarrow \mathrm{V}m(\langle \pi_m P_n \rangle_n \# \langle \pi_m Q_n \rangle_n)$.
d) For all $\langle P_n \rangle_n \in \Pi$, $Q \in \mathfrak{A}$: $\langle P_n \rangle_n \in Q \leftrightarrow \Lambda i(\langle \pi_i P_n \rangle_n \in \pi_i Q)$.
e) For all $P, Q \in \mathfrak{A}$: $P \Subset Q \leftrightarrow \Lambda i(\pi_i P \Subset \pi_i Q)$.
f) For all $\langle P_n \rangle_n \in \Pi$, and all $Q \in \mathfrak{A}$:
   $\langle P_n \rangle_n^* \underline{\in} Q \leftrightarrow \Lambda i(\langle \pi_i P_n \rangle_n^* \underline{\in} \pi_i Q)$.

Proof. (a) Let $\pi_i P = P^i$, $\pi_i Q = Q^i$, $R = \langle R^i \rangle_i \in \mathfrak{A}$. Suppose $\Lambda i(P^i \subset Q^i)$. $\varphi PR = 1 \rightarrow \Lambda i(\varphi_i P^i R^i = 1)$
$\rightarrow \Lambda i(\varphi_i Q^i R^i = 1)$
$\rightarrow \varphi QR = 1$.
Conversely, suppose $P \subset Q$, $\varphi P^\mu R^\mu = 1$. It follows that
$\varphi P(A_\infty^1, \ldots, A_\infty^{\mu-1}, R^\mu, A_\infty^{\mu+1}, \ldots) = 1$,
hence $\varphi Q(A_\infty^1, \ldots, A_\infty^{\mu-1}, R^\mu, A_\infty^{\mu+1}, \ldots) = 1$,
This implies in turn $\varphi_\mu Q^\mu R^\mu = 1$; so $P^\mu \subset Q^\mu$.
(b), (c), (d) are trivial.
(e) We suppose (P, Q as before) $\Lambda i(P^i \Subset Q^i)$. Let $\langle R_n \rangle_n \in \Pi$, $\pi_i R_n = R_n^i$. Then $\langle R_n^i \rangle_n \in \Pi^*(\Gamma_i)$.
There exists a ν such that $\Lambda i(i > \nu \rightarrow P^i = Q^i = A_\infty^i)$. Thus

$$\Lambda i \, \Lambda n(i > \nu \rightarrow R_1^i \ldots R_n^i \subset Q^i). \qquad (1)$$

There exist $\mu_1, \mu_2, \ldots, \mu_\nu$ such that

$$\wedge i(1 \leqslant i \leqslant \nu \rightarrow \varphi_i R_1^i \ldots R_{\mu_i}^i P^i = 0 \vee R_1^i \ldots R_{\mu_i}^i \subset Q^i).$$

Take $\mu = \sup\{\mu_i : 1 \leqslant i \leqslant \nu\}$. Then

$$\bigvee_{1 \leqslant i \leqslant \nu} (\varphi_i R_1^i \ldots R_\mu^i P^i = 0) \vee \bigwedge_{1 \leqslant i \leqslant \nu} (R_1^i \ldots R_\mu^i \subset Q^i).$$

By (a), combined with (1) we obtain

$$\varphi R_1 \ldots R_\mu P = 0 \vee R_1 \ldots R_\mu \subset Q.$$

Conversely, we suppose: $P \Subset Q$. Let $\langle R_n^\mu \rangle_n \in \Pi(\Gamma_\mu)$. We construct $\langle R_n \rangle_n \in \Pi$ in the following manner. Let $\langle R_n^i \rangle_n \in \Pi(\Gamma_i)$, $\langle R_n^i \rangle_n \in P^i$ for every $i \neq \mu$. Now we put:

$\wedge n \wedge i((i \leqslant \mu \rightarrow \pi_i R_n = R_n^i) \ \& \ (i > \mu \ \& \ n \geqslant i \rightarrow \pi_i R_n = R_{n-i+1}^i) \ \& \ (i > \mu \ \& \ n < i \rightarrow \pi_i R_n = A_\infty^i))$.

A ν can be found such that $\varphi R_1 \ldots R_\nu P = 0 \vee R_1 \ldots R_\nu \subset Q$.

$$\wedge i(i \neq \mu \rightarrow \varphi_i R_1^i \ldots R_\nu^i P^i = 1).$$

Therefore

$$\varphi_\mu R_1^\mu \ldots R_\nu^\mu P^\mu = 1 \rightarrow R_1 \ldots R_\nu \subset Q.$$
$$\rightarrow R_1^\mu \ldots R_\nu^\mu \subset Q^\mu.$$

So we have proved for an arbitrary μ: $P^\mu \Subset Q^\mu$.

(f) is easily derived from (e) as follows:

If $\langle R_n \rangle_n^* \underline{\in} P$, there is a μ such that $R_1 \ldots R_\mu \Subset P$. We conclude $\wedge n(\pi_n R_1 \ldots \pi_n R_\mu \Subset \pi_n P)$ (by (e)), therefore $\wedge m(\langle \pi_m R_n \rangle_n^* \underline{\in} \pi_m P)$. Conversely, if $\wedge m(\langle \pi_m R_n \rangle_n^* \underline{\in} \pi_m P)$, there are $\nu, \mu_1, \mu_2, \ldots, \mu_\nu$ such that $\wedge i(i > \nu \rightarrow \pi_i P = A_\infty^i)$, $\wedge i(1 \leqslant i \leqslant \nu \rightarrow \pi_i R_1 \ldots \pi_i R_{\mu_i} \Subset \pi_i P)$.

If $\mu = \sup\{\mu_i : 1 \leqslant i \leqslant \nu\}$, then $R_1 \ldots R_\mu \Subset P$, hence $\langle R_n \rangle_n^* \underline{\in} P$.

4.5. Theorem. If in every Γ_i of a finite or denumerably infinite sequence of I-spaces "Ax" holds, "Ax" one of the postulates N1-6, R5, then "Ax" holds in Γ (the product of the Γ_i) too.

If in every Γ_i N8 holds, then N8($\mathfrak{A}$) ($\mathfrak{A}$ defined as in 4.1) holds in Γ.

Proof. (a) Ax = N6. Let $P = \langle P^i \rangle_i \in \mathfrak{A}$, $Q = \langle Q^i \rangle_i \in \mathfrak{A}$, $\varphi PQ = 0$. There is a μ such that $\varphi_\mu P^\mu Q^\mu = 0$. By the validity of N6 for every Γ_i, there are P_1^μ, Q_1^μ, such that $P^\mu \Subset P_1^\mu$, $Q^\mu \Subset Q_1^\mu$, $\varphi P_1^\mu Q_1^\mu = 0$. Then $P \Subset (A_\infty^1, \ldots, A_\infty^{\mu-1}, P_1^\mu, A_\infty^{\mu+1}, \ldots) = P_1$, $Q \Subset (A_\infty^1, \ldots, A_\infty^{\mu-1}, Q_1^\mu, A_\infty^{\mu+1}, \ldots) = Q_1$, while $\varphi P_1 Q_1 = 0$.

(b) Ax = N1-5 is treated quite analogously.

(c) Suppose N8 is valid in every Γ_i. If $P_i = \langle P^i \rangle_i \in \mathfrak{A}$, $Q = \langle Q^i \rangle_i \in \mathfrak{A}$, then $P \Subset Q \leftrightarrow \wedge i(P^i \Subset Q^i)$. If we construct for every i an R^i such that $P^i \Subset R^i \Subset Q^i$, it follows that if $R = \langle R^i \rangle_i$, $P \Subset R \Subset Q$.

Hence N8($\mathfrak{A}$) holds in Γ.

(d) Ax = R5. Let $\langle P_n \rangle_n \in \Pi(\Gamma)$, $P_n = \langle P_n^i \rangle_i$. $\langle P_m^i \rangle_m \in \Pi^*(\Gamma_i)$.

For every i a $\langle Q^i_n\rangle_n \in \Pi(\Gamma_i)$ can be found such that $\wedge n(Q^i_{n+1} \Subset Q^i_n)$, $\langle Q^i_n\rangle \simeq \langle P^i_n\rangle$. If we define $\langle S_n\rangle_n$ by $\pi_j S_n = Q^j_{n-j+1}$ for $n \geqslant j$, $\pi_j S_n = A^j_\infty$ for $n < j$, then $\langle S_n\rangle_n \simeq \langle Q_n\rangle_n$ (4.4(b)) and $\wedge n(S_{n+1} \Subset S_n)$ (4.4(e)).

4.6. Theorem. The product Γ of a finite or denumerably infinite sequence of IR-spaces Γ_i is homeomorphic to the topological product of the Γ_i (and may therefore be written as $\prod_{i=1}^{\infty} \Gamma_i$).

Proof. The proof can be given by simple verification, using the fact that Γ is again an IR-space (a consequence of the previous lemma).

4.7. Theorem. The topological product of a finite or denumerably infinite sequence of CIN-spaces Γ_i is homeomorphic to a CIN-space Γ'.

Proof. Let Γ be the product of the Γ_i. $\Gamma = \langle \varphi, \Pi\rangle$; now we construct a CIN-space $\Gamma' = \langle \varphi, \Pi^+\rangle$ which is homeomorphic to Γ. Let $\mathfrak{A}_j(\Gamma_i) = \langle A^i_{j,k}\rangle_k$. $\mathfrak{A}_1 = \mathfrak{A}$. $\mathfrak{A}_{i+1}$ consists of all sequences of the following form:

$(A^1_{i,j(1)}, A^2_{i-1,j(2)}, \ldots, A^i_{1,j(i)}, P^{i+1}, \ldots, P^{i+p}, A^{i+p+1}_\infty, A^{i+p+2}_\infty, \ldots)$

with $\varphi_{i+k} P^{i+k} = 1$ for $1 \leqslant k \leqslant p$. $p = 0$ or $p > 0$.

By this definition C1 is automatically satisfied. C2 is trivial if we realize that, if $\mathfrak{A}_i = \langle A_{i,j}\rangle_j$, then $A_{1,i(1)} \ldots A_{n,i(n)} \in \mathfrak{A}_n$. C3 is proved as follows.

Let $A_{i,j} = (A^1_{i-1,j(1)}, \ldots, A^{i-1}_{1,j(i-1)}, P^i, \ldots, P^{i+p}, A^{i+p+1}_\infty, \ldots)$

We construct $A_{i,k}$ as follows.

For every $t, 1 \leqslant t \leqslant i-1$, we choose an $A^{i-t}_{t,k(i-t)}$ such that $A^{i-t}_{t,j(i-t)} \Subset A^{i-t}_{t,k(i-t)}$.

If $A_{i,k} = (A^1_{i-1,k(1)}, \ldots, A^{i-1}_{1,k(i-1)}, A^i_\infty, A^{i+1}_\infty, \ldots)$ then $A_{i,j} \Subset A_{i,k}$.

N9 is proved quite analogously.

N6 follows from 4.5.

We define Π^+:

$\langle P_n\rangle_n \in \Pi^+ \leftrightarrow \wedge i(\langle \pi_i P_{n+i}\rangle_n \in \Pi(\Gamma_i)\ \&\ \langle \pi_i P_n\rangle_n \in \Pi^*(\Gamma_i))$.

Let $\langle P_n\rangle_n \in \Pi$. $\pi_i P_n = P^i_n$. There exist $\langle Q^i_n\rangle_n \in \Pi(\Gamma_i)$ with $\langle Q^i_n\rangle_n \simeq \langle P^i_n\rangle_n$, and a function $m(k,i)$ such that

$\wedge k \wedge i(P^i_1 \ldots P^i_{m(k,i)} \subset Q^i_1 \ldots Q^i_k)$

We put $m(k) = \sup\{m(i,k) : 1 \leqslant i \leqslant k\}$. Hence

$P^i_1 \ldots P^i_{m(k)} \subset Q^i_1 \ldots Q^i_k$, $1 \leqslant i \leqslant k$. (*)

We construct $\langle R_n\rangle_n \in \Pi^+$, $\pi_i R_n = R^i_n$, such that

$\wedge i \wedge n(R^i_{n+i-1} = Q^i_n)\ \&\ \wedge n(n < i \rightarrow R^i_n = A^i_\infty)$.

We see from (*):

$P_1^i \ldots P_{m(k)}^i \subset R_1^i \ldots R_k^i, \; 1 \leqslant i \leqslant k.$

Hence $P_1 \ldots P_{m(k)} \subset R_1 \ldots R_k$.
If we restrict the pointgenerators to Π^+, the resulting space is again an I-space; by 1.32 the spaces $\langle \varphi, \Pi \rangle$ and $\langle \varphi, \Pi^+ \rangle$ are homeomorphic, and their relations of strong inclusion coincide.

5. Examples.

5.1. In this paragraph we want to treat various examples of topological spaces.

Let Γ be a certain metrizable space, with a metric ρ, and let ψ be a standard mapping from a lattice $\mathfrak{P}$ onto a located system (closed with respect to $\cap$, $\cup$) of closed pointspecies of Γ, and let φ be defined from ψ. Then I1-2 are automatically satisfied. Let Π be a species of sequences of elements of $\mathfrak{P}$. We suppose

$$\{\mathrm{Int}\,\psi P : P \in \mathfrak{P}\} \text{ is a basis for } \Gamma. \tag{1}$$

$$\psi P \cap \psi Q = \emptyset \rightarrow \vee \varepsilon (U_\varepsilon(\psi P) \cap U_\varepsilon(\psi Q) = \emptyset) \tag{2}$$

$$\langle P_n \rangle_n \in \Pi \rightarrow \text{diameter } \psi P_1 \ldots P_n \text{ converges to zero, } \wedge n(\psi P_1 \ldots P_n \neq \emptyset) \tag{3}$$

In order to prove I3 it is sufficient to prove that

$$P \subset Q \leftrightarrow \psi P \subset \psi Q \tag{4}$$

The implication from the left to the right is proved thus. Let $P \subset Q$, $r \in \psi P$, $U_\varepsilon(r) \cap \psi Q = \emptyset$ for a certain $\varepsilon > 0$. Since (1) holds, there is an R such that $\psi R \cap \psi Q = \emptyset$, $r \in \psi R$. Then $\varphi PR = 1$, $\varphi QR = 0$. $P \subset Q \;\&\; \varphi PR = 1 \rightarrow \varphi QR = 1$ can be proved from I1-2 only; $\varphi QR = 0$ therefore contradicts $P \subset Q$, hence $U_\varepsilon(r) \cap \psi Q \neq \emptyset$.
Since ψQ is located, either $\vee p(p \in U_\varepsilon(r) \cap \psi Q)$, or for a $\delta < \varepsilon$ $U_\delta(r) \cap \psi Q = \emptyset$. Since we can prove $U_\delta(r) \cap \psi Q \neq \emptyset$ the latter possibility is excluded, therefore $\vee p(p \in U_\varepsilon(r) \cap \psi Q)$. This holds for every ε, hence $r \in (\psi Q)^- = \psi Q$. This proves (4), since the implication in the reverse direction is trivial. (The proof can be simplified if we suppose ψQ to be metrically located.)
(2) and (3) imply the validity of I4. I5 can be satisfied by taking a sufficiently big species for Π.

Now we suppose that we have proved $\Delta = \langle \varphi, \Pi \rangle$ to be an I-space.
If we define ξ by:

$$\langle P_n \rangle_n \in \Pi \rightarrow \xi \langle P_n \rangle_n^* \in \bigcap_{n=1}^{\infty} \psi P_n \tag{5}$$

then ξ is a bi-unique mapping from Δ onto Γ.
If Δ is an IR-space, ξ is continuous. For let $U_\varepsilon(p) \subset V$, V a pointspecies of Γ. If $\langle P_n\rangle_n \in \Pi$, we conclude from (3) that for a certain ν $p \notin \psi P_1 \ldots P_\nu \vee P_1 \ldots P_\nu \subset V$, so $\xi^{-1}p \Subset \xi^{-1}V$, hence ξ is continuous.
Finally we remark that

$$U(\varepsilon, \psi P) \cap U(\varepsilon, (\psi Q)^c) = \emptyset \rightarrow P \Subset Q$$
$$U(\varepsilon, \psi P) \subset \psi Q \rightarrow P \Subset Q. \tag{6}$$

5.2. Theorem. $\underset{\sim}{Q}$ is an IR-space; $\Subset$ and $\Subset'$ are different relations in $\underset{\sim}{Q}$.
Proof. We put $[r, r']_Q = \{r'' : r'' \in \underset{\sim}{Q} \;\&\; r \leqslant r'' \leqslant r'\}$. $\{[r, r']_Q : r \leqslant r' \;\&\; r, r' \in \underset{\sim}{Q}\}$ is a located system. After closure with respect to $\underline{U}$ we can define a standard mapping ψ and a mapping φ defined from ψ. Π is defined by:

$$\langle P_n\rangle_n \in \Pi \leftrightarrow \vee r \; \vee r' \; \vee r'' \; (r, r', r'' \in \underset{\sim}{Q} \;\&\; r' \leqslant r \leqslant r'' \;\&\; \wedge n(\psi P_n = [r', r'']_Q \cap [r-2^{-n}, r+2^{-n}]_Q).$$

I1-5 are now valid according to 5.1. R5 is also trivially fulfilled. There remains to be proved that ξ, defined as in 5.1 (5) is a homeomorphism.
If $\langle P_n\rangle_n^* \in V$, there is a ν such that $P_1 \ldots P_\nu \Subset V$. Let $\psi P_1 \ldots P_\nu = [r, r']_Q$, $\psi Q_n = [r-2^{-n}, r+2^{-n}]_Q$, $\psi R_n = [r'-2^{-n}, r'+2^{-n}]_Q$ for every n; then $\langle Q_n\rangle_n$, $\langle R_n\rangle_n \in \Pi$, and there is a μ such that $R_1 \ldots R_\mu \subset V$, $Q_1 \ldots Q_\mu \subset V$. Therefore $[r-2^{-\mu}, r'+2^{-\mu}]_Q \subset V$. Hence ξV is a neighbourhood of $\xi\langle P_n\rangle_n^*$. ξ is proved to be continuous after the argument of 5.1.

Thus we have constructed an abstract IR-space Δ, homeomorphic to Γ. In Δ the following equivalence holds for any P, Q:

$$P \Subset' Q \leftrightarrow P \subset Q.$$

This can be seen as follows: $A_i \Subset' A_j \leftrightarrow A_i \subset A_j$ for any i, j, since for all $r, r', r'' \in \underset{\sim}{Q}$ $r \in [r', r'']_Q \vee r \notin [r', r'']_Q$. If $P \sim P_1 + \ldots + P_\nu$, $P_1, \ldots, P_\nu \in \mathfrak{A}$, $i \neq j \rightarrow \varphi P_i P_j = 0$, then $P_1 + \ldots + P_\nu \Subset' P_1 + \ldots + P_\nu$, hence also $P \Subset' P$; we conclude that $P \subset Q \rightarrow P \Subset' Q$. This proves $\Subset$ and $\Subset'$ to be different.

5.3. Corollaries. a) $\underset{\sim}{Q}$ does not possess a perfect representation with inclusion property. (Cf. 2.25)
b) $\underset{\sim}{Q}$ is an IR-space which is not a CIN-space. (3.5, 2.25)

5.4. We want to prove that $\underline{H}$ is a CIN-space; we begin with some notations and definitions. ρ is the metric in $\underline{H}$ as defined in 2.14. $\underline{H}^{(n)}$ denotes the following subspace of $\underline{H}$:

$$\{x : x \in \underline{H} \;\&\; \wedge j(j > n \rightarrow x_j = 0)\}.$$

$\underline{H}^{(n)}$ is homeomorphic to $\underline{R}^n$.
Let $\langle p_i \rangle_i$ be an enumeration of the rational points of $\underline{H}$, and $\langle r_i \rangle_i$ an enumeration of the rational numbers.
We define:

$$B_{i,j} = \{x : \rho(x, p_i) < r_j\}.$$

Arbitrary intersections of a finite number of $B_{i,j}$ are marked by $B, B', \ldots$.
A $B_{i,j}$ can be described as a species $\{x : f(x) \not> 0\}$ with f, defined on $\underline{H}$, given by:

$f(x) = \sum_{j=1}^{\infty} x_j^2 + \sum_{j=1}^{n} b_j x_j + a$, b_j, a rational numbers, while n is such that $p_i \in \underline{H}^{(n)}$.

5.5. Lemma. It can be decided whether an intersection B of a finite sequence $B_{i_1,j_1}, \ldots, B_{i_\nu,j_\nu}$ is secured or not.
Proof. Without loss of generality we may suppose:

$$\neg \bigvee_{1 \leqslant n \leqslant \nu} \bigvee_{1 \leqslant m \leqslant \nu} (B_{i_n,j_n} \subset B_{i_m,j_m} \ \& \ n \neq m)$$

Let $p_{i_k} \in \underline{H}^{(\sigma)}$, for $1 \leqslant k \leqslant \nu$.
We remark that B is secured iff $B \cap \underline{H}^{(\sigma)}$ is secured.
For let $B_{i_k,j_k} = \{x : f_k(x) \not> 0\}$ for $1 \leqslant k \leqslant \nu$.
If $q = (q_1, q_2, \ldots, q_\sigma, q_{\sigma+1}, \ldots)$, and we put $q' = (q_1, \ldots, q_\sigma, 0, 0, \ldots)$ then $f_k(q) \not> 0 \rightarrow f_k(q') \not> 0$ for $1 \leqslant k \leqslant \nu$.
So if $q \in B$, then $q' \in B \cap \underline{H}^{(\sigma)}$.

$$f_k(x) = \sum_{i=1}^{\infty} x_i^2 + \sum_{i=1}^{\sigma} b_{k,i} x_i + a_k = 0, \quad 1 \leqslant k \leqslant \nu,$$

a_k, $b_{k,i}$ rational numbers.
$B \cap \underline{H}^{(\sigma)}$ is secured iff the following system of equations

$$\overline{f}_k(x) = \sum_{i=1}^{\sigma} x_i^2 + \sum_{i=1}^{\sigma} b_{k,i} x_i + a_k = 0, \quad 1 \leqslant k \leqslant \nu \qquad (*)$$

has a real solution.
If we subtract $\overline{f}_1(x) = 0$ from the other equations, we obtain ν-1 linear equations in σ unknowns. If there is no solution for them, then B is empty. If there are λ independent solutions, such that for example $x_1, \ldots, x_\lambda$ can be chosen independently, we obtain a quadratic expression in λ unknowns, by substituting the general solution in $\overline{f}_1(x) = 0$. If the hyperconic in $\underline{R}^\lambda$, represented by this equation has no real part, then $B \cap \underline{H}^{(\sigma)}$ is empty, and then B is also empty. The manipulation of these equations does not present any difficulties from an intuitionistic point of view, since all coefficients are rational numbers.

5.6. Lemma. Let B, B' be constructed as non-empty intersections of a finite number of $B_{i,j}$, taken from the sequence $B_{i_1,j_1}, \ldots, B_{i_\nu,j_\nu}$, and suppose $p_{i_k} \in \underline{H}^{(\sigma)}$ for $1 \leqslant k \leqslant \nu$.

Then $\rho(B, B') = \rho(B \cap \underline{H}^{(\sigma)}, B' \cap \underline{H}^{(\sigma)})$.
Proof. $\rho(B, B')$, if defined, certainly satisfies:
$\rho(B, B') \not> \rho(B \cap \underline{H}^{(\sigma)}, B' \cap \underline{H}^{(\sigma)})$.
Now let us suppose $q \in B$, $q' \in B'$, $q = \langle q_i \rangle_i$, $q' = \langle q_i' \rangle_i$.
We put $q'' = (q_1, \ldots, q_\sigma, 0, 0, 0, \ldots)$; $q''' = (q_1', \ldots, q_\sigma', 0, 0, \ldots)$.
We suppose B, B' to be defined by:

$$\left.\begin{array}{l} B = \{x : f_i(x) \not> 0,\ 1 \leqslant i \leqslant \lambda\} \\ B' = \{x : f_i(x) \not> 0,\ \lambda + 1 \leqslant i \leqslant \nu\}. \end{array}\right\} \qquad (1)$$

If $q \in B$, then also $q'' \in B$; if $q' \in B'$, then also $q''' \in B'$, as follows immediately from (1). Further $\rho(q'', q''') \not> \rho(q, q')$.
Hence $\rho(B \cap \underline{H}^{(\sigma)}, B' \cap \underline{H}^{(\sigma)}) \not> \rho(B, B')$.

5.7. Lemma. Let B, B' be defined as in 5.6. If B'^c denotes the complement of B', we have:
$\rho(B, B'^c) = \rho(B \cap \underline{H}^{(\sigma+1)}, B'^c \cap \underline{H}^{(\sigma+1)})$.
Proof. As before, we conclude: if $\rho(B, B'^c)$ is defined, then
$\rho(B, B'^c) \not> \rho(B \cap \underline{H}^{(\sigma+1)}, B'^c \cap \underline{H}^{(\sigma+1)})$.
Now let $q \in B$, $q' \in B'^c$. $\sum_{i=\sigma+1}^{\infty} q_i^2 = s^2$, $\sum_{i=\sigma+1}^{\infty} q_i'^2 = t^2$, $s, t \not< 0$; $q = \langle q_i \rangle_i$, $q' = \langle q_i' \rangle_i$.
We put $q'' = (q_1, \ldots, q_\sigma, s, 0, 0, \ldots)$; $q''' = (q_1', \ldots, q_\sigma', t, 0, 0, \ldots)$.
If we consider (1) in the proof of 5.6, we see that $q'' \in B$, $q''' \in B'^c$.

$$\rho(q'', q''') = \sum_{i=1}^{\sigma} (q_i - q_i')^2 + (s-t)^2; \ \rho(q, q') = \sum_{i=1}^{\infty} (q_i - q_i')^2.$$

$$\rho(q'', q''') - \rho(q, q') =$$

$$s^2 + t^2 - 2st - \sum_{i=\sigma+1}^{\infty} q_i^2 - \sum_{i=\sigma+1}^{\infty} q_i'^2 + 2 \sum_{i=\sigma+1}^{\infty} q_i q_i' =$$

$$-2 \left(\sum_{i=\sigma+1}^{\infty} q_i^2\right)^{2^{-1}} \left(\sum_{i=\sigma+1}^{\infty} q_i'^2\right)^{2^{-1}} + 2 \sum_{i=\sigma+1}^{\infty} q_i q_i'.$$

It follows from the Cauchy-Schwarz inequality, that
$\rho(q, q') - \rho(q'', q''') \not< 0$. Therefore
$\rho(B \cap \underline{H}^{(\sigma+1)}, B'^c \cap \underline{H}^{(\sigma+1)}) \not> \rho(B, B'^c)$.

5.8. Theorem. $\underline{H}$ is a CIN-space.
Proof. We construct an abstract I-space Δ homeomorphic to $\underline{H}$, using a standard mapping ψ defined on a located system obtained from the $B_{i,j}$ by closure with respect to $\cap$, $\underline{\cup}$. φ is defined from ψ. The $B_{i,j}$ with their intersections constitute a located system as a consequence of 5.5, 5.6, 5.7.
$\mathfrak{A}_k$ is the species of all finite meets $A_{n_1} \ldots A_{n_s}$, $\varphi A_{n_1} \ldots A_{n_s} = 1$, such that for a certain t, $1 \leqslant t \leqslant s$, $A_{n_t} = B_{i,j}$ with $r_j > k^{-1}$.
$\Pi(\Delta)$ is given by the following definition:

$$\langle P_n \rangle_n \in \Pi(\Delta) \leftrightarrow \bigwedge n (P_n \in \mathfrak{A}_n \ \& \ \varphi P_1 \ldots P_n = 1) \qquad (4)$$

Now I1-5 are satisfied.

From lemma 5.7 and 5.1 (6) we draw the conclusion that for any $P, Q \in \mathfrak{A}_1$:

$$P \Subset Q \leftrightarrow \rho(\psi P, (\psi Q)^c) > 0. \qquad (5)$$

For let $\psi P = B$, $\psi Q = B'$ and let $\underline{H}^{(\sigma+1)}$ be defined as in lemma 5.7.
Then $P \Subset Q$ implies: $\wedge \langle R_n \rangle_n \in \Pi(\Delta)\ \vee m(\underline{H}^{(\sigma+1)} \cap \psi R_1 \cap \ldots \cap \psi R_m \cap B = \emptyset \vee \psi R_1 \cap \ldots \cap \psi R_m \cap \underline{H}^{(\sigma+1)} \subset B' \cap \underline{H}^{(\sigma+1)})$.
$\{\psi P \cap \underline{H}^{(\sigma+1)} : P \in \mathfrak{P}\}$ is a located system in $\underline{H}^{(\sigma+1)}$.
Anticipating properties of LDFTK- and DFTK-spaces, treated in chapter IV (see especially 4.1.30) we obtain

$\rho(\psi P \cap \underline{H}^{(\sigma+1)}, (\psi Q)^c \cap \underline{H}^{(\sigma+1)}) > 0$

hence we conclude to (5).
C1, C4 are valid as a consequence of our definitions. C2 is easily verified. C3 is proved thus: let $A_{n_1} \ldots A_{n_\nu} \in \mathfrak{A}_\mu$. So $\sigma, i, j, \in \underline{N}$ can be found, such that $\psi A_{n_\sigma} = B_{i,j}$ & $r_i < \mu^{-1}$. There exists an r_k such that $r_i < r_k < \mu^{-1}$; therefore $A_{n_1} \ldots A_{n_\nu} \Subset A_m$ if $\psi A_m = B_{k,j}$.
N6 and N9 follow from lemma 5.6, 5.7 and the fact that the proportions of a finite intersection of $B_{i,j}$ are continuous functions of the r_i. Consider for example $B = B_{i1,j1} \cap \ldots \cap B_{i_\mu, j_\mu}$, $B' = B_{i_{\mu+1}, j_{\mu+1}} \cap \ldots \cap B_{i_\nu, j_\nu}$. If $B \cap B' \neq \emptyset$, then $\rho(B, B') > 0$. We can find k_n, $1 \leqslant n \leqslant \nu$, such that for every n, $1 \leqslant n \leqslant \nu$, $\psi^{-1} B_{i_n, j_n} \Subset \psi^{-1} B_{i_n, k_n}$ (by taking r_{k_n} slightly greater than r_{j_n}), while $B'' \cap B''' = \emptyset$ for $B'' = B_{i1,k1} \cap \ldots \cap B_{i_\mu, k_\mu}$, $B''' = B_{i_{\mu+1}, k_{\mu+1}} \cap \ldots \cap B_{i_\nu, k_\nu}$.
Likewise in the case of N9.

Thus we have proved Δ to be a CIN-space.
ξ is defined as in 5.1. ξ is continuous according to 5.1; ξ^{-1} is easily proved to be continuous using (5).

5.9. Theorem. $\underline{F}$ is a CIN-space.
Proof. We construct an abstract CIN-space Δ which will be proved homeomorphic to $\underline{F}$. $\mathfrak{A} = \mathfrak{A}(\Delta)$, $\Pi = \Pi(\Delta)$, $\varphi = \varphi_\Delta$ etc. The metric of $\underline{F}$ will be denoted by ρ; we recall that the species of rational polygonal functions on $[0,1]$ was denoted by $\underline{F}^\circ$. We define for $f, g \in \underline{F}$:

$$\left.\begin{array}{l} f < g \leftrightarrow \wedge x(x \in [0,1] \rightarrow f(x) < g(x)) \\ f > g \leftrightarrow g < f \\ f \leqslant g \leftrightarrow \wedge x(x \in [0,1] \rightarrow f(x) \not> g(x)) \\ f \geqslant g \leftrightarrow g \leqslant f \\ f \circ g \leftrightarrow \neg f \leqslant g \ \& \ \neg f \geqslant g. \end{array}\right\} \qquad (1)$$

sup(f, g), inf(f, g) are defined by:
$(\sup(f,g))(r) = \sup\{f(r), g(r)\}$, $(\inf(f,g))(r) = \inf\{f(r), g(r)\}$.
We remark that

$$\left.\begin{array}{l}\bigwedge f\ \bigwedge g(f, g \in \underline{F}^{o}\ \&\ f \leqslant g \rightarrow f < g \vee \neg\, f < g) \\ \bigwedge f\ \bigwedge g(f, g \in \underline{F}^{o} \rightarrow f \leqslant g \vee g \leqslant f \vee f \circ g)\end{array}\right\} \qquad (2)$$

We define $[f, g] = \{f' : f' \in \underline{F}\ \&\ f \leqslant f' \leqslant g\}$.
The species $[f, g]$ with $f, g \in \underline{F}^{o}$, $f \leqslant g$ constitute a species F^* which is denumerably infinite. We remark that for arbitrary $[f_1, g_1], [f_2, g_2] \in F^*$:

$$[f_1, g_1] \cap [f_2, g_2] = \emptyset \vee [f_1, g_1] \cap [f_2, g_2] = [\sup(f_1, f_2), \inf(g_1, g_2)] \in F^* \qquad (3)$$

$$[f_1, g_1] \subset [f_2, g_2] \leftrightarrow f_2 \leqslant f_1\ \&\ g_1 \leqslant g_2. \qquad (4)$$

Let ψ be a standard mapping defined for $\mathfrak{P}$, such that $\mathfrak{A} = \langle A_n \rangle_n$ is mapped onto F^*. φ is defined from ψ.

We remark that for any species $\{n_1, \ldots, n_k\}$ such that $\varphi A_{n_1} \ldots A_{n_k} = 1$

$$\bigvee i(A_{n_1} \ldots A_{n_k} \sim A_i) \qquad (5)$$

$\mathfrak{A}_k$ is defined as the species of all finite meets $A_{n_1} \ldots A_{n_m}$ with $\varphi A_{n_1} \ldots A_{n_m} = 1$, such that, if $\varphi A_{n_1} \ldots A_{n_m} = [f, g]$, then $\rho(f, g) < k^{-1}$.
We define Π according to C4:

$$\langle P_n \rangle_n \in \Pi \leftrightarrow \bigwedge n(\varphi P_1 \ldots P_n = 1\ \&\ P_n \in \mathfrak{A}_n) \qquad (6)$$

I1-5 are now easily proved if we use the reasoning in 5.1. Thus we have defined an I-space Δ. We want to show Δ to be a CIN-space. C1, C2 do not present any difficulties.

Let $\psi A_i = [f_1, f_1']$, $\psi A_j = [f_2, f_2']$. Then it is not difficult to prove

$$A_i \Subset A_j \leftrightarrow f_2 < f_1 \leqslant f_1' < f_2'. \qquad (7)$$

(To prove the implication from the left to the right, consider $\langle P_n \rangle_n \in \Pi$ such that $\psi P_1 \ldots P_n = [f_1 - (3n)^{-1}, f_1 + (3n)^{-1}]$.)
Keeping in mind (5), we see that for any meets $A_{n_1} \ldots A_{n_t}$, $A_{m_1} \ldots A_{m_s}$ such that $\psi A_{n_1} \ldots A_{n_t} = [f_1, f_1']$, $\psi A_{m_1} \ldots A_{m_s} = [f_2, f_2']$:

$$A_{n_1} \ldots A_{n_t} \Subset A_{m_1} \ldots A_{m_s} \leftrightarrow f_2 < f_1 \leqslant f_1' < f_2'. \qquad (8)$$

Now C3 is verified by remarking that for any $[f, g] \in F^*$
$\rho(f, g) < k^{-1} \rightarrow \bigvee n(\rho(f - n^{-1}, g + n^{-1}) < k^{-1})$.
If $P = P_1 + \ldots + P_\nu$, $Q = Q_1 + \ldots + Q_\mu$, $\varphi PQ = 0$, $P_i \Subset P_{i,j}$, $Q_j \Subset Q_{i,j}$, $\varphi P_{i,j} Q_{i,j} = 0$ for $1 \leqslant i \leqslant \nu$, $1 \leqslant j \leqslant \mu$, then

$$P \Subset \sum_{i=1}^{\nu} \prod_{j=1}^{\mu} P_{i,j} = P',\ Q \Subset \sum_{j=1}^{\mu} \prod_{i=1}^{\nu} Q_{i,j} = Q',\ \varphi P'Q' = 0. \qquad (9)$$

It follows from (9) that it suffices to verify N6 in the case $\psi P = [f_1, f_1']$, $\psi Q = [f_2, f_2']$.
$\psi P \cap \psi Q = \emptyset \rightarrow \neg \sup(f_1, f_2) \leqslant \inf(f_1', f_2')$. Let for an $r \in \underline{Q}$ $\sup\{f_1(r), f_2(r)\} - \inf\{f_1'(r), f_2'(r)\} > 2\lambda^{-1}$. Then $[f_1 - \lambda^{-1}, f' + \lambda^{-1}] \cap$

$[f_2-\lambda^{-1}, f'_2+\lambda^{-1}] = \emptyset$. Thus we obtain $P \in P'$, $Q \in Q'$, $\varphi P'Q' = 0$, if $\psi P' = [f_1-\lambda^{-1}, f'_1+\lambda^{-1}]$, $\psi Q' = [f_2-\lambda^{-1}, f'_2+\lambda^{-1}]$.
N9 follows immediately from (8) and:

$$f_2 < f_1 \leqslant f'_1 < f'_2 \rightarrow f_2 < 2^{-1}(f_1+f_2) < f_1 \leqslant f'_1 < 2^{-1}(f'_1+f'_2) < f'_2.$$

Δ is therefore a CIN-space. ξ is defined according to 5.1, and ξ, ξ^{-1} are readily proved to be continuous.

5.10. Theorem. If θ is a spread law which satisfies the following condition:
(*) If $\sigma \in \theta$, then either σ has only one immediate descendant or σ has at least two different descendants of equal length.
Then $\underline{D}(\theta)$ is a PIN-space.
Proof. We construct a PIN-space Δ, defined by $\langle \varphi, \Pi \rangle$, homeomorphic to $\underline{D}(\theta)$. If $\tau = \langle i_1, \ldots, i_k \rangle \in \theta$, we put:

$$V_\tau = \{\alpha: \alpha \in \underline{D}(\theta) \;\&\; \bar{\alpha}(k) = \langle i_1, \ldots, i_k \rangle\}.$$

The species $\{V_\tau\}_{\tau \in \theta, \tau \neq \emptyset}$ can be enumerated. We remark that

$$\wedge\tau \wedge\sigma(\tau, \sigma \in \theta \rightarrow V_\tau \subset V_\sigma \vee V_\sigma \subset V_\tau \vee V_\tau \cap V_\sigma = \emptyset). \quad (1)$$

f is a bi-unique mapping of $\underline{N}$ onto θ.
ψ is a standard mapping with respect to the located system $\{V_\tau: \tau \in \theta\}$; ψ is defined on $\mathfrak{P}(\Delta)$, such that if $\mathfrak{A}(\Delta) = \langle A_n \rangle_n$ then $\psi A_n = V_{f(n)}$.
φ is defined from ψ; $\Pi(\Delta) = \Sigma(\Delta)$. I1-3 are proved as indicated in 5.1. To prove I4 we show that if $\alpha \in \underline{D}(\theta)$, then $\langle A_{f^{-1}\bar{\alpha}(n)} \rangle_n \in \Pi(\Delta)$.
We remark that

$$\varphi A_{n_1} \ldots A_{n_t} = 1 \rightarrow \vee m(A_{n_1} \ldots A_{n_t} \sim_\Delta A_m) \quad (2)$$

It is sufficient to prove that $\langle A_{f^{-1}(\bar{\alpha}(n))} \rangle_n$ separates all pairs A_s, A_t such that $\varphi A_s A_t = 0$. This follows from (2). Take an arbitrary pair A_σ, A_τ with $\varphi A_\sigma A_\tau = 0$. Then $V_{f(\sigma)} \cap V_{f(\tau)} = \emptyset$; let $f(\sigma) = \langle i_1, \ldots, i_\mu \rangle$, $f(\tau) = \langle j_1, \ldots, j_\nu \rangle$, $\mu \leqslant \nu$. It follows that $\langle i_1, \ldots, i_\mu \rangle \neq \langle j_1, \ldots, j_\mu \rangle$.
Then also $\bar{\alpha}(\mu) \neq \langle i_1, \ldots, i_\mu \rangle \vee \bar{\alpha}(\mu) \neq \langle j_1, \ldots, j_\mu \rangle$, hence $V_{\bar{\alpha}(\mu)} \cap V_{f(\sigma)} = \emptyset \vee V_{\bar{\alpha}(\mu)} \cap V_{f(\tau)} = \emptyset$, and therefore

$$\varphi A_{f^{-1}(\bar{\alpha}(\mu))} A_\sigma = 0 \vee \varphi A_{f^{-1}(\bar{\alpha}(\mu))} A_\tau = 0.$$

I5 is immediate.
Next we prove:
To every $\langle P_n \rangle_n \in \Pi$ a monotonously increasing sequence $\langle n_i \rangle_i$ can be found such that for some $\alpha \in \underline{D}(\theta)$ $\wedge i(\psi P_1 \ldots P_{n_i} \subset V_{\bar{\alpha}(i)})$ holds.
Suppose already proved for a certain ν: $\psi P_1 \ldots P_{n_\nu} \subset V_\sigma$.
We must prove the existence of a descendant τ of σ and a number $n_{\nu+1} \in \underline{N}$, $n_{\nu+1} > n_\nu$, such that $P_1 \ldots P_{n_{\nu+1}} \subset V_\tau$.
There are the following possibilities.
a) σ has only one immediate descendant σ_1. We may take $\tau = \sigma_1$, $n_{\nu+1} = n_\nu + 1$.

b) σ has at least two different descendants of equal length, σ_1, σ_2; $V_{\sigma_1} \cap V_{\sigma_2} = \emptyset$.
Hence there exists a μ such that $\psi P_1 \ldots P_\mu \cap V_{\sigma_1} = \emptyset \vee \psi P_1 \ldots P_\mu \cap V_{\sigma_2} = \emptyset$. Let $\psi P_1 \ldots P_\mu = V_{\tau_1} \cup \ldots \cup V_{\tau_\lambda}$, $V_{\tau_i} \cap V_{\tau_j} = \emptyset$ for $i \neq j$. There exists a $\mu_1 \geqslant \mu$ such that: $\Lambda i\, \Lambda j(i \neq j \ \&\ 1 \leqslant i,j \leqslant \lambda \rightarrow V_{\tau_i} \cap \psi P_1 \ldots P_{\mu_1} = \emptyset \vee V_{\tau_j} \cap \psi P_1 \ldots P_{\mu_1} = \emptyset)$.
On this account a V_{τ_η}, $1 \leqslant \eta \leqslant \lambda$ must exist, such that $\psi P_1 \ldots P_{\mu_1} \subset V_{\tau_\eta}$. Now we can take $n_{\nu+1} = \mu_1$, τ the immediate descendant of σ such that $V_{\tau_\eta} \subset V_\tau \subset V_\sigma$. Now we are able to prove:

$$\Lambda i\, \Lambda j(A_i \Subset A_j \leftrightarrow A_i \subset A_j)$$

The implication from the left to the right is trivial. Suppose for certain ν, μ $A_\nu \subset A_\mu$. Let $\langle P_n \rangle_n \in \Pi$. $\langle n_i \rangle_i$ is a strictly monotonously increasing sequence such that $\psi P_1 \ldots P_{n_i} \subset V_{\bar{\alpha}(i)}$ for a certain $\alpha \in \underline{D}(\theta)$. Let $f(\nu)$ be a sequence of length λ. Then either $\bar{\alpha}(\lambda) = f(\nu)$, or $\bar{\alpha}(\lambda) \neq f(\nu)$. In the first case $P_1 \ldots P_{n_\lambda} \subset A_\nu$; in the second case $\varphi P_1 \ldots P_{n_\lambda} A_\nu = 0$. Therefore $A_\nu \Subset A_\mu$, hence also $A_\nu \subset A_\mu$.

Let $V_{\sigma_1} \cap \ldots \cap V_{\sigma_\mu} \subset V_{\tau_1} \cap \ldots \cap V_{\tau_\nu}$; $i \neq j \rightarrow V_{\sigma_i} \cap V_{\sigma_j} = \emptyset$, $V_{\tau_i} \cap V_{\iota} = \emptyset$.
Then $\Lambda i \vee j(1 \leqslant i \leqslant \mu \ \&\ 1 \leqslant j \leqslant \nu \rightarrow V_{\sigma_i} \subset V_{\tau_j})$.
Hence if $\psi P = V_{\sigma_1} \cup \ldots \cup V_{\sigma_\mu}$, $\psi Q = V_{\tau_1} \cup \ldots \cup V_{\tau_\nu}$, it follows that $P \Subset Q$. Thus we have proved

$$\Lambda P\, \Lambda Q(P \Subset Q \leftrightarrow P \subset Q).$$

Now N6, N8 are trivial.
ξ is defined as in 5.1; ξ, ξ^{-1} are proved to be continuous without difficulty, as in previous examples.

5.11. Remark. The topological product of a denumerable infinity of copies of $\underline{N}$ can be considered as a space $\underline{D}(\theta)$, θ consisting of all finite sequences of natural numbers.

The positively irrational numbers > 0 (i.e. positive real numbers which lie apart from every rational number) are exactly those positive real numbers which possess a unique development in a non-terminating continued fraction. (See BROUWER 1920, p. 959, DIJKMAN 1952, p. 52).

Therefore, according to a well-known argument (e.g. KURATOWSKI 1958, § 14 V), they are homeomorphic to the topological product of a denumerable infinity of copies of $\underline{N}$.

5.12. Remark. The construction of a located system depends in the examples treated in an essential way on the individual properties of the spaces considered.
We showed without great difficulty that any CIN-space is

metrizable, but theorems in the reverse direction are not so easy to prove. (compare FREUDENTHAL 1936, 7.16, and 4.2.3 in this thesis).
This is a consequence of the difficulties encountered in proving the existence of suitable located systems for a whole class of metric spaces.

CHAPTER IV

LDFTK-SPACES

1. DFTK-spaces.

1.1. In this paragraph the reference "FREUDENTHAL 1936" will be shortened to "FR".

DFTK-spaces were introduced in FR. In this paragraph we want to prove every DFTK-space to be a PIN-space. Further a number of theorems and lemmas which will be useful later on will be proved.

To start with, we resume a number of definitions and theorems from FR with only inessential changes in terminology. The notion of a DFTK-space will be slightly widened; every DFTK-space in our sense is homeomorphic to a DFTK-space in the sense of FR.

1.2. Definitions. $\mathfrak{A} = \langle A_n \rangle_n$, $\mathfrak{P}$ is constructed from $\mathfrak{A}$ as usual. A function φ is defined on $\mathfrak{P}$ as in 3.1.2; φ satisfies I1, I2. We introduce the following postulates for φ:

F. $\varphi A_{n_1} \ldots A_{n_k} = 1 \longrightarrow \wedge n \vee m(m > n \;\&\; \varphi A_{n_1} \ldots A_{n_k} A_m = 1)$.

T. $\varphi A_{n_1} \ldots A_{n_s} = 1 \;\&\; \varphi A_{m_1} \ldots A_{m_t} = 1 \;\&$
$\varphi A_{n_1} \ldots A_{n_s} A_{m_1} \ldots A_{m_t} = 0 \longrightarrow \vee n \wedge m(m \geqslant n \longrightarrow$
$\varphi A_{n_1} \ldots A_{n_s} A_m = 0 \vee \varphi A_{m_1} \ldots A_{m_t} A_m = 0)$.

K. $\wedge k \vee l \wedge n(\varphi(A_k + A_{k+1} + \ldots + A_{k+l})A_n = 1)$.

If r is a function from $\underline{N}$ into $\underline{N}$ such that $\wedge k \wedge n(\varphi(A_k + A_{k+1} + \ldots + A_{k+r(k)})A_n = 1)$, then r is called a K-function.

1.3. Theorem. (FR 2.2) If φ is a function on the lattice $\mathfrak{P}$ which satisfies I1, I2, F, T, K, then also the following assertions are valid:

a) $\varphi P = 1 \longrightarrow \wedge n \vee m(m > n \;\&\; \varphi P A_m = 1)$.

b) $\varphi P = 1 \;\&\; \varphi Q = 1 \;\&\; \varphi PQ = 0 \longrightarrow \vee n \wedge m(m \geqslant n \longrightarrow \varphi A_m P = 0 \;\underline{\vee}\; \varphi A_m Q = 0)$.

c) $\varphi P = 1 \;\&$ r is a K-function $\longrightarrow \wedge k(\varphi P(A_k + \ldots + A_{k+r(k)}) = 1)$.

1.4. Definition. $\subset$, $\sim$ are defined as in 3.1.6. (FR 2.6).
$P \in^* Q \longleftrightarrow \vee n \wedge m(m \geqslant n \;\&\; \varphi A_m P = 1 \longrightarrow A_m \subset Q)$. (FR 2.6).

1.5. Theorem. φ satisfies I1-2, F, T, K. Then

a) I3 holds for $\subset$ (FR 2.11).

b) Theorem 3.1.7 is valid for $\subset$, $\sim$.

c) $P \in^* Q$ & $P \in^* R \longrightarrow P \in^* QR$ (FR 2.12); $P \in^* Q$ & $R \in^* Q \longrightarrow P+R \in^* Q$ (FR 2.12); $P \in^* Q$ & $Q \subset R \longrightarrow P \in^* R$, $P \subset Q$ & $Q \in^* R \longrightarrow P \in^* R$. (FR 2.8); $P \in^* Q \longrightarrow P \subset Q$ (FR 2.10).

Proof. 3.1.7 is proved on assumption of I1-3 only, hence (b) is a consequence of (a).

1.6. Definition. A species of lattice elements $\{P_1, \ldots, P_\nu\}$ is called an LQ-covering (lattice quasi-covering) if $\Lambda m(\varphi A_m(P_1 + \ldots + P_\nu) = 1)$. $\{P_1, \ldots, P_\nu\}$ is called an L-covering (lattice covering) if $\vee n \Lambda m \vee k(m \geqslant n \longrightarrow A_m \in^* P_k$ & $1 \leqslant k \leqslant \nu)$. (cf. FR 3.2).

1.7. Definition. A_m is of degree n means $m \geqslant n$. $A_{m_1} \ldots A_{m_t}$ is of degree n means: for some i $(1 \leqslant i \leqslant t)$ A_{m_i} is of degree n. An LQ-covering is of degree n if every element of the covering is of degree n.

Remark: the degree of an LQ-covering is defined only for LQ-coverings whose elements are meets of elements of $\mathfrak{A}$. (cf. FR 3.1, 3.2).

1.8. Definition. P' is an L-neighbourhood of degree n of P, if $P \in^* P'$, $P' = P_1 + \ldots + P_k$, $\varphi PP_m = 1$ for $1 \leqslant m \leqslant k$, and P_m of degree n for $1 \leqslant m \leqslant k$.

A piece of degree n is an L-neighbourhood of degree n of a meet of elements of $\mathfrak{A}$ of degree n. (Fr 3.3).

1.9. Theorem. φ satisfies I1-2, F, T, K. Then

a) If $\{P_1, \ldots, P_n\}$ is an LQ-covering, then for any P: $P \in^* P^* = \Sigma\{P_i : 1 \leqslant i \leqslant n$ & $\varphi P_i P = 1\}$ (FR 3.6).

b) $\varphi PQ = 0 \longrightarrow \vee P^* \vee Q^*(P \in^* P^*$ & $Q \in^* Q^*$ & $\varphi P^* Q^* = 0)$ (FR 3.8).

c) $P \in^* Q \longrightarrow \vee R(P \in^* R \in^* Q)$ (FR 3.9).

d) $P \in^* Q \longrightarrow \vee k \Lambda n(n \geqslant k$ & $\varphi A_n P = 1 \longrightarrow A_n \in^* Q)$ (FR 3.10).

e) If $P \in^* Q$, a $\nu \in \underline{N}$ can be found such that for every piece S of degree $n \geqslant \nu$ $\varphi PS = 1 \longrightarrow S \in^* Q$.

f) If $\varphi PQ = 0$, a $\nu \in \underline{N}$ can be found such that for every piece S of degree $n \geqslant \nu$ $\varphi PS = 0 \vee \varphi QS = 0$.

g) If $\{P_1, \ldots, P_n\}$ is an LQ-covering, and $P_i \subset P_i'$ for $1 \leqslant i \leqslant n$, then $\{P_1', \ldots, P_n'\}$ is an LQ-covering. Likewise for L-coverings.

Proof. (e), (f) could be proved by means of FR 3.11-15, but since proofs are omitted there, it is simpler to give a straightforward demonstration.

(e). From (d) it follows that for every lattice element R of degree $n \geqslant \nu$ we have $\varphi PR = 1 \longrightarrow R \in^* Q$, since R can be written as A_mR', $m \geqslant n \geqslant \nu$, hence $\varphi PR = 1 \rightarrow \varphi PA_m = 1$; $\varphi PA_m = 1 \rightarrow A_m \Subset^* Q$ (d); $A_m \Subset^* Q \rightarrow R'A_m \Subset^* Q$ (1.5(b), (c)). If $P_1 \Subset^* P_4$, we are able to construct P_2, P_3 such that $P_1 \Subset^* P_2 \Subset^* P_3 \Subset^* P_4$.

There exist ν_1, ν_2, ν_3 such that for any lattice element R of degree $n \geqslant \nu_i$ $\varphi P_iR = 1 \rightarrow R \Subset^* P_{i+1}$ for i = 1, 2, 3. We put $\nu = \sup \{\nu_1, \nu_2, \nu_3\}$. Let S be a piece of degree $n \geqslant \nu$, $S = T_1 + \ldots + T_\mu$, S a neighbourhood of T; $T, T_1, \ldots, T_\mu$ lattice elements of degree n. Suppose $\varphi P_1S = 1$. For a certain λ, $1 \leqslant \lambda \leqslant \mu$, $\varphi P_1T_\lambda = 1$. Hence $T_\lambda \Subset^* P_2$, therefore $T_\lambda \subset P_2$ (1.5(c)). If $1 \leqslant i \leqslant \mu$, $\varphi T_\lambda T = 1$, $\varphi TT_i = 1$. Hence it follows that $\varphi P_2T = 1$, consequently $T \Subset^* P_3$, so $T \subset P_3$; we conclude that $\varphi P_3T_i = 1$, therefore $T_i \Subset^* P_4$.
This holds for every i, $1 \leqslant i \leqslant \mu$, hence $S \Subset^* P_4$ by 1.5(c).
(f) Let $\varphi PQ = 0$. We construct P^*, Q^* such that $P \Subset^* P^*$, $Q \Subset^* Q^*$, $\varphi P^*Q^* = 0$ (using (c)).
There is a ν such that for a piece S of degree $n \geqslant \nu$:

$$\varphi SP = 1 \rightarrow S \Subset^* P^*; \ \varphi SQ = 1 \rightarrow S \Subset^* Q^*.$$

$$\begin{aligned} \varphi SP = 1 \ \& \ \varphi SQ = 1 &\rightarrow S \Subset^* P^* \\ &\rightarrow S \subset P^* \\ S \subset P^* \ \& \ \varphi SQ = 1 &\rightarrow \varphi P^*Q = 1 \\ &\rightarrow \varphi P^*Q^* = 1\text{: contradiction.} \end{aligned}$$

Hence $\varphi SP = 0 \vee \varphi SQ = 0$.
(g) is trivial.

1.10. Definition. A centered system $\langle P_n \rangle_n$, $\forall m(P_m \neq A_\infty)$, is called a DFTK-point generator, if to every P_n a piece S_n of degree m_n can be found, such that $P_n \subset S_n$, and where $\langle m_n \rangle_n$ is a sequence increasing beyond all bounds. $\langle P_n \rangle_n \in Q$ is defined just as in 3.1.10 (see also FR 4.1, 4.2).

1.11. Lemma. A DFTK-point generator satisfies the splitting condition with respect to every pair T_1, T_2 such that $\varphi T_1T_2 = 0$.
Proof. Suppose for every piece of degree $n \geqslant \nu$ (1.9(f))

$$\varphi T_1S = 0 \vee \varphi T_2S = 0.$$

Let $\langle R_n \rangle_n$ be a DFTK-point generator. There is a μ such that $R_\mu \subset S$, S a piece of degree $n \geqslant \nu$. Then $\varphi T_1S = 0 \vee \varphi T_2S = 0$. Then also $\varphi R_\mu T_1 = 0 \vee \varphi R_\mu T_2 = 0$, hence certainly $\varphi R_1 \ldots R_\mu T_1 = 0 \vee \varphi R_1 \ldots R_\mu T_2 = 0$.

1.12. Lemma. If $\varphi P = 1$ there exists a DFTK-point generator $\langle Q_n \rangle_n$, $\langle Q_n \rangle_n \in P$.
Proof. $P = P_1 + \ldots + P_\nu$, $P_1, \ldots, P_\nu$ meets of elements of $\mathfrak{A}$; $\langle Q_n \rangle_n \in P_\lambda$ for $1 \leqslant \lambda \leqslant \nu$ implies $\langle Q_n \rangle_n \in P$. Hence we may restrict ourselves to a P which is a meet of elements of $\mathfrak{A}$. Applying postulate F we prove inductively the existence

of a centered sequence $\langle Q_n\rangle_n$, Q_n of degree n, $\wedge n(\varphi Q_1 \ldots Q_n P = 1)$. (FR 2.3).
$\langle Q_n\rangle_n$ is a DFTK-point generator, since $Q_n \subset S_n$, where $S_n = \Sigma\{A_i : \varphi A_i Q_n = 1 \ \& \ n \leqslant i \leqslant n + r(n)\}$ (r is a K-function, and we use 1.9(a), 1.5(c)).
S_n is a piece of degree n.

1.13. Definition. Between DFTK-point generators a relation $\simeq$ and a relation # can be introduced, as in 3.1.11. # is a pre-apartness relation, according to the proof of 3.1.12, using 1.11. The equivalence class with respect to $\simeq$ which contains $\langle P_n\rangle_n$ will be denote by $\langle P_n\rangle_n^*$.

1.14. Theorem. If φ is a function on $\mathfrak{P}$ which satisfies I1-2, F, T, K, then φ and the corresponding species of DFTK-point generators Π define an I-space $\langle \varphi, \Pi\rangle$.
Proof. Immediate by 1.2, 1.5(a), 1.11, 1.12.

1.15. Definition. The I-space defined in 1.14 is called an abstract DFTK-space; every topological space homeomorphic to an abstract DFTK-space is called a DFTK-space.

1.16. Lemma. If $\langle P_n\rangle_n \in \Sigma$, then $\langle P_1 \ldots P_n\rangle_n$ is a DFTK-point generator.
Proof. Let $\langle P_n\rangle_n \in \Sigma$; $\{A_\nu, A_{\nu+1}, \ldots, A_{\nu+r(\nu)}\}$ (r a K-function) is an LQ-covering.

A number μ can be found such that for every pair A_i, A_j, $\nu \leqslant i \leqslant j \leqslant \nu + r(\nu)$, $\varphi A_i A_j = 0$, the following assertion is true:

$$\varphi A_i P_1 \ldots P_\mu = 0 \vee \varphi A_j P_1 \ldots P_\mu = 0. \tag{1}$$

There exists a certain A_λ, $\nu \leqslant \lambda \leqslant \nu + r(\nu)$, such that $\varphi A_\lambda P_1 \ldots P_\mu = 1$ (using 1.3(c)).
We define $Q_\nu = \Sigma\{A_i : \nu \leqslant i \leqslant \nu + r(\nu) \ \& \ \varphi A_i A_\lambda = 1\}$.
We remark that as a consequence of (1)

$$\nu \leqslant i \leqslant \nu + r(\nu) \ \& \ \varphi A_i P_1 \ldots P_\mu = 1 \longrightarrow \varphi A_i A_\lambda = 1. \tag{2}$$

Therefore, if $Q'_\nu = \Sigma\{A_i : \nu \leqslant i \leqslant \nu + r(\nu) \ \& \ \varphi A_i P_1 \ldots P_\mu = 1\}$, then $P_1 \ldots P_\mu \Subset^* Q'_\nu$ (by 1.9(a)), and $Q'_\nu \subset Q_\nu$ (by 1.5(b), (2)). Hence (by 1.5(c)) $P_1 \ldots P_\mu \subset Q_\nu$; Q_ν is a piece of degree ν, since $A_\lambda \Subset^* Q_\nu$ (by 1.9(a)). This proves our assertion.

1.17. Theorem. (FR 5.3). Every DFTK-space possesses a finitary perfect representation, with a defining pair $\langle \theta, \vartheta\rangle$; for every k, the finite species $\mathfrak{C}_k = \{\gamma i_k : \langle i_1, \ldots, i_k\rangle \in \theta\}$ is an L-covering by pieces of degree k, which are joins of elements of $\mathfrak{A}$.

1.18. Definition. If a DFTK-space can be represented by a spread Π_1 such that $\wedge\langle P_n\rangle_n \in \Pi_1 \wedge n(P_{n+1} \Subset^* P_n)$, then Π_1 is called a $\Subset^*$-representation for the space. (cf. FR 5.5)

1.19. Theorem. Every DFTK-space possesses a $\Subset^*$-representation. (FR 5.6, without proof).
Proof. Let Π_0 be a normal finitary perfect representation, with a defining pair $\langle\theta, \vartheta\rangle$, for a DFTK-space Γ, according to 1.17. The species $\mathfrak{C}_k$ are defined as in 1.17. We suppose $\mathfrak{C}_n = \{P_1^n, \ldots, P_{k(n)}^n\}$. Every P_t^n is a piece of degree n.

$$Q_t = \{A_i : 1 \leq i \leq 1 + r(1) \;\&\; \varphi P_t^1 A_i = 1\}.$$

(r is a K-function).
We conclude (1.9(a)) that $P_t^1 \Subset^* Q_t$.
Now we construct, by induction, to every $\langle P_{s_1}^1, \ldots, P_{s_k}^k\rangle \in \vartheta\,\theta$ a lattice element $Q_{s_1,\ldots,s_k}$ such that $P_{s_1}^1 \ldots P_{s_k}^k \Subset^* Q_{s_1,\ldots,s_k} \Subset^* Q_{s_1,\ldots,s_{k-1}}$, in the following manner.
We suppose the $Q_{s_1,\ldots,s_n}$ to be already constructed for all $n \leq \nu$; let $t_1, \ldots, t_\nu$ be a sequence such that $t_1 < t_2 < \ldots < t_\nu$ and such that for every k, $1 \leq k \leq \nu$ the following assertion holds:

$$t \geq t_k \;\&\; \langle P_{s_1}^1, \ldots, P_{s_k}^k\rangle \in \vartheta\,\theta \;\&\; \varphi A_t P_{s_1}^1 \ldots P_{s_k}^k = 1$$
$$\rightarrow A_t \Subset^* Q_{s_1,\ldots,s_k}.$$

The existence of this sequence follows from our induction hypothesis and 1.9(d).
Let $\langle P_{s_1}^1, \ldots, P_{s_{\nu+1}}^{\nu+1}\rangle \in \vartheta\,\theta$. We define

$$Q_{s_1,\ldots,s_{\nu+1}} = \{A_i : t_\nu \leq i \leq t_\nu + r(t_\nu) \;\&\; \varphi A_i P_{s_1}^1 \ldots P_{s_{\nu+1}}^{\nu+1} = 1\}$$

hence

$$P_{s_1}^1 \ldots P_{s_{\nu+1}}^{\nu+1} \Subset^* Q_{s_1,\ldots,s_{\nu+1}} \Subset^* Q_{s_1,\ldots,s_\nu}.$$

A $t_{\nu+1} > t_\nu$ can be found such that (1) holds for $k = \nu+1$.
Next we want to prove that for any $\langle P_{s_i}^i\rangle_i \in \Pi_0$ the corresponding sequence $\langle Q_{s_1,\ldots,s_i}\rangle_i$ is a splitting system.
Let $\varphi ST = 0$, $\varphi S = 1$, $\varphi T = 1$. For a certain ν (by 1.11)

$$\varphi P_{s_1}^1 \ldots P_{s_\nu}^\nu S = 0 \vee \varphi P_{s_1}^1 \ldots P_{s_\nu}^\nu T = 0.$$

A μ can be found such that for all $m \geq \mu$ (1.3(b))

$$\varphi P_{s_1}^1 \ldots P_{s_\nu}^\nu S = 0 \rightarrow \varphi A_m S = 0 \vee \varphi A_m P_{s_1}^1 \ldots P_{s_\nu}^\nu = 0$$
$$\varphi P_{s_1}^1 \ldots P_{s_\nu}^\nu T = 0 \rightarrow \varphi A_m T = 0 \vee \varphi A_m P_{s_1}^1 \ldots P_{s_\nu}^\nu = 0.$$

Hence for all $n \geq \nu$, $m \geq \mu$

$$\varphi P_{s_1}^1 \ldots P_{s_n}^n S = 0 \rightarrow \varphi A_m S = 0 \vee \varphi A_m P_{s_1}^1 \ldots P_{s_n}^n = 0$$
$$\varphi P_{s_1}^1 \ldots P_{s_n}^n T = 0 \rightarrow \varphi A_m T = 0 \vee \varphi A_m P_{s_1}^1 \ldots P_{s_n}^n = 0.$$

Therefore for all $n \geqslant \nu$, $m \geqslant \mu$

$$\varphi A_m S = 0 \vee \varphi A_m T = 0 \vee \varphi A_m P^1_{s_1} \ldots P^n_{s_n} = 0.$$

If we choose $t_\lambda \geqslant \mu$, $\lambda \geqslant \nu$, we see that

$\Lambda i(t_\lambda \leqslant i \leqslant r(t_\lambda) + t_\lambda \;\&\; \varphi A_i P^1_{s_1} \ldots P^n_{s_n} = 1 \rightarrow \varphi A_i S = 0) \vee$

$\Lambda i(t_\lambda \leqslant i \leqslant r(t_\lambda) + t_\lambda \;\&\; \varphi A_i P^1_{s_1} \ldots P^n_{s_n} = 1 \rightarrow \varphi A_i T = 0)$.

We conclude that

$$\varphi Q_{s_1, \ldots, s_{\lambda+1}} S = 0 \vee \varphi Q_{s_1, \ldots, s_{\lambda+1}} T = 0.$$

Therefore it will be clear how a $\Subset^*$-representation can be constructed from the $Q_{s_1, \ldots, s_k}$.

1.20. Theorem. In a DFTK-space $P \Subset Q \leftrightarrow P \Subset^* Q$.
Proof. $P \Subset^* Q \rightarrow P \Subset Q$ is proved thus. Let $\langle R_n \rangle_n \in \Sigma$. Then $\langle R_1 \ldots R_n \rangle_n$ is a DFTK-point generator by 1.16. Let $\langle S_n \rangle_n$ be a DFTK-point generator such that S_n is a piece of degree n, $\Lambda n \; \mathrm{V} m(R_1 \ldots R_m \subset S_n)$. ($\langle S_n \rangle_n$ exists as a consequence of the definition of a DFTK-point generator). A ν can be found such that for $n \geqslant \nu$ $\varphi P S_n = 1 \rightarrow S_n \subset Q$. Further a μ can be found such that $R_1 \ldots R_\mu \subset S_\nu$, hence

$$\begin{aligned}\varphi P R_1 \ldots R_\mu = 1 &\rightarrow \varphi P S_\nu = 1\\ &\rightarrow S_\nu \subset Q\\ &\rightarrow R_1 \ldots R_\mu \subset Q, \text{ hence } P \Subset Q.\end{aligned}$$

Conversely, let $P \Subset Q$. Let Π_1 be a $\Subset^*$-representation with a defining pair $\langle \theta, \mathfrak{J} \rangle$ as described in 1.19. There exists a function ψ from Π_1 into $\underline{N}$ such that

$$\langle R_n \rangle_n \in \Pi_1 \;\&\; \psi \langle R_n \rangle_n = m \;\&\; \varphi R_1 \ldots R_m P = 1 \rightarrow R_1 \ldots R_m \subset Q.$$

m is known from a finite initial segment $\langle R_1, \ldots, R_s \rangle$; we may always suppose $s \geqslant m$. Since Π_1 is finitary, there exists a $\nu \in \underline{N}$ such that $\psi \langle R_n \rangle_n$ is known from $\langle R_1, \ldots, R_\nu \rangle$ for any $\langle R_n \rangle_n \in \Pi_1$ (using the fan theorem), while $\psi \langle R_n \rangle_n \leqslant \nu$. Now we remark (using 1.9(a), (g))

$$P \Subset^* \Sigma \{ \mathfrak{J} \langle i_1, \ldots, i_\nu \rangle : \langle i_1, \ldots, i_\nu \rangle \in \theta \;\&\; \varphi P \mathfrak{J} \langle i_1, \ldots, i_\nu \rangle = 1 \} \subset Q.$$

Hence by 1.5(c): $P \Subset^* Q$.

1.21. Theorem. Every DFTK-space is a PIN-space.
Proof. This follows from 1.9(b), (c), 1.11 and 1.16; for if Π denotes the set of DFTK-point generators, and φ is the function which satisfies I1-2, F, T, K, then $\langle \varphi, \Pi \rangle$, $\langle \varphi, \Sigma \rangle$ define homeomorphic spaces, since the conditions of 3.1.30 are satisfied.

1.22. Lemma. If $\{Q_1, \ldots, Q_n\}$ is an L-covering of a DFTK-space, an L-covering $\{Q'_1, \ldots, Q'_n\}$ can be found such that $Q'_i \Subset Q_i$ for $1 \leqslant i \leqslant n$.
Proof. There exist a ν and a function ψ such that for all $n \geqslant \nu$

$$\psi A_n = k \longrightarrow A_n \Subset Q_k \ \& \ 1 \leqslant k \leqslant n.$$

We put $Q''_k = \{A_i : \nu \leqslant i \leqslant r(\nu) + \nu \ \& \ \psi A_i = k\}$ (r is a K-function.) Then by 1.5(c) $Q''_k \Subset Q_k$ for $1 \leqslant k \leqslant n$. We construct Q'_k (1.9(c)) such that $Q''_k \Subset Q'_k \Subset Q_k$ for $1 \leqslant k \leqslant n$. $\{Q'_1, \ldots, Q'_n\}$ is an L-covering since there exists a μ such that for all $m \geqslant \mu$, $1 \leqslant i \leqslant n$, $\varphi Q''_i A_n = 1 \longrightarrow A_n \Subset Q'_i$ (1.9(d)).

1.23. Lemma. If $\{P^i_1, \ldots, P^i_{f(i)}\}$ is an L-covering for $1 \leqslant i \leqslant n$, then $\{P^1_{j_1} P^2_{j_2} \ldots P^n_{j_n} : \wedge k(1 \leqslant k \leqslant n \longrightarrow 1 \leqslant j_k \leqslant f(k)\}$ is also an L-covering.
Proof. There exist $\nu_1, \nu_2, \ldots, \nu_n$ such that we have

$$m \geqslant \nu_i \ \& \ 1 \leqslant i \leqslant n \longrightarrow \vee k(A_m \Subset P^i_k \ \& \ 1 \leqslant k \leqslant f(i)).$$

Hence if $\nu = \sup\{\nu_1, \ldots, \nu_n\}$, then for $m \geqslant \nu$ there are $j_1, \ldots, j_n$ such that $A_m \Subset P^k_{j_k}$ for $1 \leqslant k \leqslant n$. Therefore (1.5(c)) $A_m \Subset P^1_{j_1} \ldots P^n_{j_n}$.

1.24. Lemma. If $\{V_1, \ldots, V_n\}$ is a covering of a DFTK-space, then an L-covering $\{R_1, \ldots, R_n\}$ can be found such that $R_i \subset V_i$ for $1 \leqslant i \leqslant n$.
Proof. We suppose Π_0 to be the finitary perfect representation with a defining pair $\langle \theta, \mathfrak{d} \rangle$, and $\mathfrak{C}_n$ the species, introduced in 1.17, $\mathfrak{C}_n = \{P^n_1, \ldots, P^n_{k(n)}\}$. $\{P^n_1, \ldots, P^n_{k(n)}\}$ is an L-covering for every n. A function ψ of Π_0 into $\{1, \ldots, n\}$ exists, such that

$$\langle P^n_{s_n} \rangle_n \in \Pi_0 \ \& \ \psi \langle P^n_{s_n} \rangle_n = m \longrightarrow \langle P^n_{s_n} \rangle^*_n \in V_m.$$

m is known from an initial segment of length t, $\langle P^1_{s_1}, \ldots, P^t_{s_t} \rangle$. Since the representation is perfect, $P^1_{s_1} \ldots P^t_{s_t} \subset V_m$.
Π_0 is finitary, therefore a ν can be found such that $\psi \langle P^n_{s_n} \rangle_n$ is known from $\langle P^1_{s_1}, \ldots, P^\nu_{s_\nu} \rangle$, for every $\langle P^n_{s_n} \rangle_n \in \Pi_0$. Thus a function ψ' can be found such that

$$\psi' \langle P^1_{s_1}, \ldots, P^\nu_{s_\nu} \rangle = m \longrightarrow P^1_{s_1} \ldots P^\nu_{s_\nu} \subset V_m.$$

The species $\{\bar{\mathfrak{d}}\langle i_1, \ldots, i_\nu \rangle : \langle i_1, \ldots, i_\nu \rangle \in \theta\}$ (ν fixed) is an L-covering (1.23).
Hence if we put

$$R_m = \Sigma\{\bar{\mathfrak{d}}\langle i_1, \ldots, i_\nu \rangle : \langle i_1, \ldots, i_\nu \rangle \in \theta \ \& \ \psi' \langle \gamma i_1, \ldots, \gamma i_\nu \rangle = m\}$$

then $\{R_1, \ldots, R_m\}$ is an L-covering, and $R_i \subset V_i$ for $1 \leqslant i \leqslant n$.

1.25. Theorem. $\{Q_1, \ldots, Q_n\}$ is an L-covering of a DFTK-space iff $\{Q_1, \ldots, Q_n\}$ is a covering.
Proof. Let $\{Q_1', \ldots, Q_n'\}$, $Q_i' \Subset Q_i$ for $1 \leqslant i \leqslant n$, be an L-covering constructed according to 1.22.
There is a $\nu \in \underline{N}$ such that for all pieces S of degree $m \geqslant \nu$ the following assertion is valid: $\varphi Q_i' S = 1 \longrightarrow S \Subset Q_i$ (for $1 \leqslant i \leqslant n$) (1.9(e)). To every DFTK-point generator $\langle P_n \rangle_n$ a piece S of degree ν, and a μ can be found such that $P_\mu \subset S$. On this account there exists a λ such that $P_\mu \subset Q_\lambda$, therefore $\langle P_n \rangle_n \in Q_\lambda$. If $\{Q_1, \ldots, Q_n\}$ is a covering, there exists (1.24) an L-covering $\{R_1, \ldots, R_n\}$, $R_i \subset Q_i$ for $1 \leqslant i \leqslant n$, hence $\{Q_1, \ldots, Q_n\}$ is also an L-covering (1.9(g)).

1.26. Theorem. If $\{V_1, \ldots, V_n\}$ is a covering of a DFTK-space, then there exists a covering $\{P_1, \ldots, P_n\}$ such that $[P_i] \Subset V_i$ for $1 \leqslant i \leqslant n$.
Proof. By 1.22, 1.24, 1.25.
Remark. This theorem could also have been obtained as a consequence of 3.2.21, but then we should have to prove 1.25 separately.

1.27. Theorem. Every DFTK-space is an LC-space and conversely (FR 7.17).
Remark. From now on we shall use the existence of an adequate metric for a DFTK-space without further comment in our proofs.

1.28. Theorem. Let $\Gamma = \langle \varphi, \Pi \rangle$ be a DFTK-space, $\varepsilon > 0$, $\mathfrak{A}(\Gamma) = \langle A_n \rangle_n$. Then
a) diameter $[A_n]$ converges to zero with increasing n (FR 6.5).
b) The diameter of a piece of degree n is smaller than ε for almost all n.
Proof. (b) is an immediate consequence of (a).

1.29. Definition. A DFTK-basis is defined quite analogously to an I-basis.
Remark. Let $\langle V_n \rangle_n$ be a located system of non-empty species of an LC-space $\langle V_o, \mathfrak{T} \rangle$. Let $\mathfrak{P}$ be defined as usual from $\mathfrak{A} = \langle A_n \rangle_n$, and let ψ be a standard mapping defined on $\mathfrak{P}$ such that $\psi A_n = V_n$, and suppose φ to be defined from ψ. Then $\langle V_n \rangle_n$ is a DFTK-basis for $\langle V_o, \mathfrak{T} \rangle$ iff φ satisfies I1-2, F, T, K, and $\bigcap_{n=1}^{\infty} \psi P_n$ contains exactly one point for every $\langle P_n \rangle_n \in \Sigma$.

1.30. Lemma. Let Γ be a DFTK-space, and let V be a

locatedpointspecies of Γ. $V \Subset_\Gamma W \leftrightarrow \forall\varepsilon(\varepsilon > 0 \ \& \ U(\varepsilon, V) \subset W)$.
Proof. In 3.3.7 was proved: $V \Subset_\Gamma W \leftrightarrow V^- \subset \text{Int } W$, for located V. If for some positive ε, $U(\varepsilon, V) \subset W$, then $V^- \subset \text{Int } W$.
Suppose $V^- \subset \text{Int } W$. V^- is an LC-space (2.3.9). Hence to every $p \in V^-$ a $\delta > 0$ can be found such that $U(2\delta, p) \subset W$. The species of the $U(\delta, p)$ is an open covering of V^-; as a consequence there exists (2.3.4(d)) a quasi-finite subcovering $\{U_1, \ldots, U_n\}$, $U_i = U(\delta_i, p_i)$ for $1 \leq i \leq n$. $\delta = \inf\{\delta_i : 1 \leq i \leq n\}$.
Hence $U(\delta, V) \subset U(\delta, \bigcup_{i=1}^n U_i) \subset \bigcup_{i=1}^n U(\delta_i + \delta, p_i) \subset \bigcup_{i=1}^n U(2\delta_i, p_i) \subset W$.

1.31. Lemma. Let V, W be located and relatively located pointspecies of a DFTK-space Γ. Then
a) $V \Subset W \rightarrow \forall P \ \forall P'(V \subset P \Subset P' \subset W)$.
b) $V \cap W = \emptyset \rightarrow \forall P \ \forall Q(V \subset P \ \& \ W \subset Q \ \& \ \varphi PQ = 0)$.
Proof. (a) $V \Subset W \rightarrow V \Subset' W$. This implies that $\{W, V^c\}$ covers Γ. Hence by 1.25, 1.24, there exist P', Q' such that $P' \subset W$, $Q' \subset V^c$; $\{P', Q'\}$ covers Γ. By 1.26, there exist P, Q such that $P \Subset P'$, $Q \Subset Q'$, $\{P, Q\}$ covers Γ.
$p \in V \rightarrow p \notin V^c$; $p \notin V^c \rightarrow p \notin Q$; $p \notin Q \rightarrow p \in P$.
Hence $V \subset P \Subset P' \subset W$.
(b) It follows from the fact that V, W are relatively located, and from 2.3.10(c) that for a certain $\delta > 0$, $U(\delta, V) \cap U(\delta, W) = \emptyset$. We construct located pointspecies V', W' such that $U(4^{-1}\delta, V) \subset V' \subset U(2^{-1}\delta, V)$, $U(4^{-1}\delta, W) \subset W' \subset U(2^{-1}\delta, W)$ (2.3.11(a)); $U(2^{-1}\delta, V') \cap U(2^{-1}\delta, W') = \emptyset$. $V \Subset V'$ & $W \Subset W'$ (1.30).
Applying (a) we construct P, Q such that $V \subset P \subset V'$, $W \subset Q \subset W'$, $\varphi PQ = 0$.

1.32. Lemma. Let Γ be a DFTK-space and $\langle V_n \rangle_n$ a DFTK-basis for Γ. $\langle W_n \rangle_n$ is a sequence of pointspecies such that $\langle V_n \rangle_n \cup \langle W_n \rangle_n$ is a located system. Then $\langle V_n \rangle_n \cup \langle W_n \rangle_n$ is a PIN-basis for Γ.
Proof. We suppose $\wedge n(V_n \neq \emptyset)$, $\wedge n(W_n \neq \emptyset)$. (This can be done according to the definition of a PIN-basis.)
We define an I-space $\Delta = \langle \varphi, \Pi \rangle$ by $\mathfrak{A}(\Delta) = \langle A_n \rangle_n \cup \langle B_n \rangle_n$, $\Pi(\Delta) = \Sigma(\Delta)$. ψ is defined on $\mathfrak{P}(\Delta)$ as a standard mapping which satisfies $\psi A_n = V_n$, $\psi B_n = W_n$; $\varphi_\Delta = \varphi$ is defined from ψ.
Let Γ_1 be defined by $\mathfrak{A}(\Gamma_1) = \langle A_n \rangle_n$, $\varphi_{\Gamma_1} = \varphi_\Delta \mid \mathfrak{P}(\Gamma_1)$, $\Pi(\Gamma_1) = \Sigma(\Gamma_1)$. Γ_1 is an abstract DFTK-space, homeomorphic to Γ. We put $\Gamma_2 = \langle \varphi_\Delta, \Pi(\Gamma_1) \rangle$.
We prove successively that Γ_1 is homeomorphic to Γ_2, and that Γ_2 is homeomorphic to Δ.
In Γ_2, I1, 2, 5 are satisfied. I3, 4 can be proved for Γ_2 by the methods described in 3.5.1.

$\Pi^o(\Gamma_1) = \Pi^o(\Gamma_2)$; on this account $V \Subset_{\Gamma_1} W \longleftrightarrow V \Subset_{\Gamma_2} W$. Therefore Γ_2 has to be an IR-space, since Γ_1 is an IR-space. We obtain (3.2.13(c)) $p \underline{\in}_{\Gamma_2} V \longleftrightarrow \{p\} \Subset_{\Gamma_2} V \longleftrightarrow \{p\} \Subset_{\Gamma_1} V \longleftrightarrow p \underline{\in}_{\Gamma_1} V$, and this implies in turn that Γ_1, Γ_2 are homeomorphic.

Let $\langle R_n \rangle_n \in \Sigma(\Delta)$, and let $\langle\langle Q_i, Q_i^! \rangle\rangle_i$ be an enumeration of all pairs $\langle Q_i, Q_i^! \rangle$ such that $\varphi Q_i Q_i^! = 0$.
There exists a sequence $\langle n_i \rangle_i$, $\Lambda i(n_{i+1} > n_i)$ such that $\Lambda i\ \Lambda j(1 \leqslant j \leqslant i \longrightarrow \varphi R_1 \ldots R_{n_i} Q_j = 0 \vee \varphi R_1 \ldots R_{n_i} Q_j^! = 0)$.
As a consequence of 1.31(b) we are able to construct $P_i \in \mathfrak{P}(\Gamma_1)$ such that $\Lambda i(R_1 \ldots R_{n_i} \subset_\Delta P_i \ \&\ \Lambda j(1 \leqslant j \leqslant i \longrightarrow \varphi P_i Q_j = 0 \vee \varphi P_i Q_j^! = 0))$. Hence $\Lambda k \vee m(R_1 \ldots R_m \subset_\Delta P_1 \ldots P_k)$. We conclude that (3.1.32) Γ_2, Δ are homeomorphic, and $\Subset_{\Gamma_2} = \Subset_\Delta$.

As a consequence of 1.31, in Δ N6, N8 also hold, as will be proved now.
Suppose Δ to be homeomorphic to Γ_1 by a homeomorphism ξ. If $P \Subset_\Delta Q$, then $\xi P \Subset_{\Gamma_1} \xi Q$, ξP, ξQ located in Γ_1. Then P', Q', R' can be found such that $\xi P \subset P' \Subset_{\Gamma_1} R' \Subset_{\Gamma_1} Q' \subset \xi Q$, hence $P \subset_\Delta P' \Subset_\Delta R' \Subset_\Delta Q' \subset Q$. N8 is proved likewise.

1.33. Remark. If Γ, Δ are defined as in 1.32, the following theorems remain valid for Δ (as a consequence of 1.32, 1.31) if we interprete P, Q etc. as to belong to $\mathfrak{P}(\Delta)$, but $\langle A_n \rangle_n = \mathfrak{A}(\Gamma_1)$: 1.3, 1.9, 1.17, 1.19, 1.20, 1.22, 1.24, 1.25, 1.26.

2. LDFTK-spaces.

2.1. Definition. A metric locally DFTK-space is a metric space with a point representation, such that to every point a closed neighbourhood can be found, which is an LC-space (equivalently DFTK-space) in the relative topology.

2.2. Definition. An abstract PIN-space which satisfies the following two postulates
L1. $\Lambda P \vee \langle Q_n \rangle_n (P = A_\infty \vee \langle [Q_n] \rangle_n$ is a DFTK-basis for $[P])$.
L2. $\Lambda P \vee Q(P \neq A_\infty \longrightarrow P \Subset Q \ \&\ Q \neq A_\infty)$.
is called an abstract locally DFTK-space (abstract LFDTK-space). A space which is homeomorphic to an abstract LDFTK-space is called a locally DFTK-space (LDFTK-space).

2.3. Theorem. If $\langle V_o, \rho \rangle$ is a metric locally DFTK-space, then $\langle V_o, \mathfrak{T}(\rho) \rangle$ is an LDFTK-space and conversely.
Proof. Let $\langle V_o, \rho \rangle$ be a metric locally DFTK-space. We

put $\langle V_o, \mathfrak{T}(\rho)\rangle = \Gamma$ To every point $p \in V_o$ a real number $\varepsilon > 0$ and $B, C \subset V_o$ can be found, such that $F(p, \varepsilon, B, C)$ holds, where $F(p, \varepsilon, B, C)$ is defined by

$$F(p, \varepsilon, B, C) \leftrightarrow U(2^{-2}\varepsilon, p) \subset B \subset U(2^{-1}\varepsilon, p) \subset U(\varepsilon, p) \subset C$$
and B, C LC-spaces in the relative topology. (1)

We remark that

$$\{\text{Int } B : \vee p \ \vee \varepsilon \ \vee C(p \in V_o \ \& \ \varepsilon > 0 \ \& \ F(p, \varepsilon, B, C)\}$$

is an open covering of Γ.

Using the intuitionistic analogue of Lindelöf's theorem (2.2.6) we obtain sequences $\langle B_n\rangle_n, \langle C_n\rangle_n, \langle \varepsilon_n\rangle_n, \langle p_n\rangle_n$, such that $\langle B_n\rangle_n$ is a covering of Γ, and $\wedge n(F(p_n, \varepsilon_n, B_n, C_n))$.

Next we construct sequences $\langle B_{n,m}\rangle_m$ for every n, such that $B_{n,m}$ is an LC-space in the relative topology for all n, m, and such that (using 2.3.11(a))

$$B_n = B_{n,1} \ \& \ \wedge m(U(2^{-k-2}\varepsilon_n, B_{n,m}) \subset B_{n,m+1} \subset U(2^{-k-1}\varepsilon_n, B_{n,m})) \qquad (2)$$

We see that $\wedge n \wedge m(B_{n,m} \subset C_n)$. We re-enumerate $\langle B_{n,m}\rangle_{n,m}$ as $\langle D_n\rangle_n$ by putting $D_{t(n,m)} = B_{n,m}$ where t is a bi-unique mapping from $\underline{N} \times \underline{N}$ onto $\underline{N}$, and r, s are mappings such that $rt(n,m) = n$, $st(n,m) = m$. We put

$$E_1 = D_1, E_{n+1} = E_n \ \underline{\cup} \ D_n.$$

We see that $\wedge n(E_n \subset E_{n+1})$; E_n is an LC-space in the relative topology, therefore $\langle E_n\rangle_n$ is a located system.
Let f be a mapping from $\underline{N}$ into $\underline{N}$ defined by

$$f(n) = \sup\{t(r(k), s(k)+1) : 1 \leq k \leq n\}.$$

It follows that a sequence $\langle \delta_n\rangle_n$ can be found such that

$$U(\delta_n, E_n) \subset E_{f(n)}$$

since $D_{t(n,m)} \subset U(\varepsilon, D_{t(n,m)}) \subset B_{n,m+1}$ for a certain ε. If we put

$$g(1) = 1, g(n+1) = fg(n), \ F_n = E_{g(n)}$$

then $\langle F_n\rangle_n$ is a located system of LC-spaces. Defining η_n as $\delta_{g(n)}$ we obtain

$$U(\eta_n, F_n) \subset F_{n+1}. \qquad (3)$$

Now we construct a DFTK-basis $\langle G_{2,n}\rangle_n$ for F_2, such that $\langle G_{2,n}\rangle_n \cup \{F_1\}$ is a located system in F_2. (This is possible by 2.3.12; to see this we remark that if $\langle U_i\rangle_i$ in 2.3.12 is a DFTK-basis, then $\langle W_i\rangle_i$ is also a DFTK-basis, as a consequence of 1.29, remark.)

If we put $H_{1,n} = G_{2,n} \cap F_1$, then $\langle H_{1,n}\rangle_n$ is a located system, and a DFTK-basis for F_1.
For let $\mathfrak{C}$ be the system obtained by closing $\langle G_{2,n}\rangle_n$ with respect to $\cap$, and let $V, W \in \mathfrak{C}$. If $U(\delta, V \cap F_1) \cap U(\delta, W \cap F_1) \cap F_2 \subset U(\varepsilon, V \cap W \cap F_1)$, and if we put $\delta' = \inf(\eta_1, \delta)$, we obtain $U(\delta', V \cap F_1) \cap U(\delta', W \cap F_1) \subset U(\delta', V \cap F_1) \cap U(\delta', W \cap F_1) \cap F_2 \subset U(\varepsilon, V \cap W \cap F_1)$. Therefore $\langle H_{1,n}\rangle_n$ is a located system.
Let $\langle G_{3,n}\rangle_n$ be a DFTK-basis for F_3, such that $\langle H_{1,n}\rangle_n \cup \langle G_{3,n}\rangle_n \cup \{F_2\}$ is a located system in F_3. We put $H_{2,n} = G_{3,n} \cap F_2$. $\langle H_{1,n}\rangle_n \cup \langle H_{2,n}\rangle_n$ is a located system, $\langle H_{2,n}\rangle_n$ is a DFTK-basis for F_2.
We proceed by induction. Let us suppose that we have already proved that $\langle H_{1,n}\rangle_n \cup \langle H_{2,n}\rangle_n \cup \ldots \cup \langle H_{k-1,n}\rangle_n$ is a located system, and $\langle H_{i,n}\rangle_n$ is a DFTK-basis for F_i, $1 \leqslant i \leqslant k-1$.

We construct a DFTK-basis $\langle G_{k+1,n}\rangle_n$ for F_{n+1}, such that $\langle H_{1,n}\rangle_n \cup \ldots \cup \langle H_{k-1,n}\rangle_n \cup \langle G_{k+1,n}\rangle_n \cup \{F_k\}$ is a located system in F_{n+1} (2.3.12).

If we put $G_{k+1,n} \cap F_k = H_{k,n}$, then $\langle H_{k,n}\rangle_n$ is a DFTK-basis for F_n, and $\langle H_{1,n}\rangle_n \cup \ldots \cup \langle H_{k,n}\rangle_n$ is a located system.
For let $\mathfrak{C}$ be the system obtained by closure of $\langle H_{1,n}\rangle_n \cup \ldots \cup \langle H_{k-1,n}\rangle_n \cup \langle G_{k+1,n}\rangle_n$ with respect to $\cap$, and let $V, W \in \mathfrak{C}$. If $U(\delta, V \cap F_k) \cap U(\delta, W \cap F_k) \cap F_{k+1} \subset U(\varepsilon, V \cap W \cap F_k)$, and if we put $\delta' = \inf(\eta_k, \delta)$, then

$$U(\delta', V \cap F_k) \cap U(\delta', W \cap F_k) = U(\delta', V \cap F_k) \cap U(\delta', W \cap F_k) \subset F_{k+1} \subset U(\varepsilon, V \cap W \cap F_k).$$

In this way we obtain a system $\langle H_{n,m}\rangle_{n,m}$ which will be proved to be a PIN-basis for a PIN-space Δ which satisfies L1-2.

We put $K_{\tau(n,m)} = H_{n,m}$, $\nu = \inf\{n\colon K_n \neq \emptyset\}$. $\langle L_n\rangle_n$ is defined by stipulating $L_i = L_\nu = K_\nu$ for $1 \leqslant i \leqslant \nu$; $n > \nu \rightarrow L_n = K_n$ if $K_n \neq \emptyset$, $L_n = L_\nu$ if $K_n = \emptyset$.

We construct Δ by means of a standard mapping ψ such that $\psi A_n = L_n$, $\mathfrak{A}(\Delta) = \langle A_n\rangle_n$. $\varphi_\Delta = \varphi$ is defined from ψ. As a consequence of 1.29, 1.32 $\langle \psi P \cap L_n\rangle_n$ is a PIN-basis for every $P \neq A_\infty$. If $\psi P \subset F_n$, then $\langle \psi P \cap H_{n,m}\rangle_m$ is a PIN-basis for ψP.
I1-2 are automatically satisfied. We prove $P \subset Q \leftrightarrow \psi P \subset \psi Q$ in the usual way, as described in 3.5.1. This proves I3.
Next we prove for a separating system $\langle T_n\rangle_n$: $\bigcap_{n=1}^{\infty} \psi R_n$ contains exactly one point, and the diameter of $\psi R_1 \ldots R_k$ tends to zero with increasing k.
There exist a ν and a μ, $\nu, \mu \in \underline{N}$, such that $\psi R_1 \ldots R_\nu \subset F_\mu$.

$\psi R_1 \ldots R_\nu$ is a PIN-space with a PIN-basis $\langle \psi R_1 \ldots R_\nu \cap L_n \rangle_n$. Therefore $\langle R_1 \ldots R_\nu R_n \rangle_n$ must fulfil the splitting condition with respect to all pairs $\langle R_1 \ldots R_\nu P,\ R_1 \ldots R_\nu Q \rangle$ such that $\varphi R_1 \ldots R_\nu PQ = 0$; hence $\bigcap_{n=1}^{\infty} \psi R_n$ contains exactly one point of ψR_ν, and the diameter of $\psi R_1 \ldots R_\nu R_n$ tends to zero with increasing n.
Conversely, if $p \in \Gamma$, there is a ν such that $p \in F_\nu$. $\langle F_\nu \cap L_n \rangle_n$ is a PIN-basis for F_ν, therefore a splitting system $\langle R_n \rangle_n$ can be found such that $\bigcap_{n=1}^{\infty} \psi R_n = \{p\}$. If we put $\Sigma(\Delta) = \Pi(\Delta)$, it follows from the preceding considerations that I4, I5 are satisfied. N6, N8 remain to be proved. Let $P, Q \in \mathfrak{P}$. From the construction of the F_n it is seen that a ν can be found such that $\psi P,\ \psi Q \subset F_\nu$. If $\langle S_n \rangle_n$ is a sequence such that $\Lambda n(\psi S_n = F_n)$, it is a consequence of (3) that

$$\Lambda n(S_n \Subset_\Delta S_{n+1})$$

$P, Q \subset S_\nu \Subset_\Delta S_{\nu+1} \Subset_\Delta S_{\nu+2}$. If $\varphi PQ = 0$ we construct P', Q' such that $P \Subset_{S_{\nu+2}} P'$ & $Q \Subset_{S_{\nu+2}} Q'$ & $\varphi P'Q' = 0$. It follows that $P \Subset_{S_{\nu+2}} P'S_{\nu+1}$ & $Q \Subset_{S_{\nu+2}} Q'S_{\nu+1}$ & $\varphi P'Q'S_{\nu+1} = 0$. Using lemma 3.1.28 we draw the conclusion that

$$P \Subset_\Delta P'S_{\nu+1} \ \&\ Q \Subset_\Delta Q'S_{\nu+1} \ \&\ \varphi P'Q'S_{\nu+1} = 0.$$

This proves N6.
If $P \Subset_\Delta Q$, we construct an R such that $P \Subset_{S_{\nu+1}} R \Subset_{S_{\nu+1}} Q \Subset_\Delta S_{\nu+1}$. Hence we obtain (3.1.30) $P \Subset_\Delta R \Subset_\Delta Q$, and this proves N8; Δ is therefore a PIN-space.

Finally we must construct a homeomorphism ξ from Δ onto Γ. We put (as in 3.1.26, 3.5.1)

$$\langle P_n \rangle_n \in \Pi(\Delta) \longrightarrow \xi \langle P_n \rangle_n^* \in \bigcap_{n=1}^{\infty} \psi P_n .$$

It is readily verified that ξ is bi-unique. ξ is continuous, since Δ is a PIN-space. ξ^{-1} is continuous, since in Γ every point has a neighbourhood which is an LC-space; as a consequence of 1.27, 1.21, 3.3.9, 3.3.6, ξ^{-1} is continuous on such a neighbourhood, therefore ξ^{-1} is continuous on Γ.
Now we prove this theorem in the reverse direction. An LDFTK-space is a PIN-space, and therefore metrizable; as a consequence of 3.3.9, 3.3.5, and 3.2.30 it has a point representation.

It follows from L1, L2 that to every point a neighbourhood can be found which is an LC-space, for if $\langle P_n \rangle_n \in \Pi$, $\langle P_n \rangle_n \in Q$, $Q \neq A_\infty$, there is an R (L2) such that $Q \Subset R$, $R \neq A_\infty$, so $\langle P_n \rangle_n^* \in \operatorname{Int} R$. R is a neighbourhood which is an LC-space in the relative topology.

2.4. Definition. An LDFTK-basis is defined analogous to an I-basis.

2.5. Example. $\underline{R}^n$ possesses an LDFTK-basis consisting of all species $\{(x_1, \ldots, x_n) : \Lambda i(1 \leq i \leq n \rightarrow x_i \in [k_i 2^{-m}, (k_i+1)2^{-m}]\}$, $|k_i|, m \in \underline{N}$.

3. Covering theorems.

3.1. Theorem. Let $\langle P_n \rangle_n$ be a star-finite covering (cf. 3.1.37) of an LDFTK-space. Then there exists a star-finite covering $\langle P_n^* \rangle_n$ such that $\Lambda n(P_n \Subset P_n^*)$ and $\Lambda i\ \Lambda j(\varphi P_i^* P_j^* = 1 \leftrightarrow \varphi P_i P_j = 1)$.
Proof. Suppose $\langle Q_n \rangle_n$ to be an arbitrary star-finite covering. We shall construct a star-finite covering $\langle Q_n' \rangle_n$ such that for a certain ν, $Q_\nu \Subset Q_\nu'$, $\Lambda n(n \neq \nu \rightarrow Q_n' = Q_n)$, $\Lambda n(\varphi Q_\nu' Q_n = 1 \leftrightarrow \varphi Q_\nu Q_n = 1)$; the construction is described below.

We put $\Sigma\{Q_j : \varphi Q_j Q_\nu = 1\} = Q_\nu''$
$\Sigma\{Q_j : \varphi Q_j Q_\nu'' = 1\} = Q_\nu'''$.

If $\langle R_n \rangle_n \in \Pi$, there exists a Q_μ such that $\langle R_n \rangle_n^* \subseteq Q_\mu$ (3.3.9, 3.3.5, 3.2.21) hence for a certain λ, $R_1 \ldots R_\lambda \Subset Q_\mu$. $\varphi Q_\nu Q_\mu = 0$ implies $\varphi R_1 \ldots R_\lambda Q_\nu = 0$. $\varphi Q_\nu Q_\mu = 1$ implies $Q_\mu \subset Q_\nu'$, so $R_1 \ldots R_\lambda \subset Q_\nu''$. Therefore $Q_\nu \Subset Q_\nu''$; likewise we prove $Q_\nu'' \Subset Q_\nu'''$. Now we take R_ν'' to be

$$R_\nu'' = \Sigma\{Q_j : \varphi Q_j Q_\nu = 1 \ \&\ \varphi Q_j Q_\nu = 0\}.$$

$Q_\nu''' = R_\nu'' + Q_\nu''$; $\varphi R_\nu'' Q_\nu = 0$.

We construct a Q_ν^* such that $Q_\nu \Subset Q_\nu^*$ & $\varphi Q_\nu^* R_\nu'' = 0$ and we put $Q_\nu' = Q_\nu^* Q_\nu''$. It follows that $Q_\nu \Subset Q_\nu'$, $\varphi Q_\nu' R_\nu'' = 0$.

Suppose $\varphi Q_\nu Q_n = 0$ & $\varphi Q_\nu' Q_n = 1$. $\varphi Q_n Q_\nu'' = 0$ contradicts $\varphi Q_\nu' Q_n = 1$; $\varphi Q_n Q_\nu'' = 1$ implies $Q_n \subset R_\nu''$ (since $\varphi Q_\nu Q_n = 0$), hence $\varphi Q_n Q_\nu' = 0$ which contradicts $\varphi Q_\nu' Q_n = 1$.
Therefore $\Lambda n(\varphi Q_\nu' Q_n = 1 \leftrightarrow \varphi Q_\nu Q_n = 1)$.

Now we apply this construction to $\langle P_n \rangle_n$. At the first step $\langle P_n \rangle_n$ is changed into $\langle P_1^*, P_2, P_3, \ldots \rangle$, $P_1 \Subset P_1^*$. We repeat this construction; at the k^{th} step $\langle P_1^*, \ldots, P_{k-1}^*, P_k, P_{k+1}, \ldots \rangle$ is changed into $\langle P_1^*, \ldots, P_k^*, P_{k+1}, P_{k+2}, \ldots \rangle$, $P_k \Subset P_k^*$. By this method a sequence $\langle P_n^* \rangle_n$ is constructed such that $\Lambda n(P_n \Subset P_n^*)$. We must prove

$$\Lambda i\ \Lambda j(\varphi P_i^* P_j^* = 1 \leftrightarrow \varphi P_i P_j = 1) \qquad (1)$$

In our construction $\varphi P_\nu = 0$ implies $\varphi P_\nu^* = 0$. Hence $\varphi P_\nu P_\nu = 1 \leftrightarrow \varphi P_\nu^* P_\nu^* = 1$ is trivial.
Suppose $\nu < \mu$. At the ν^{th} step we constructed from $\langle P_1^*, \ldots, P_{\nu-1}^*, P_\nu, \ldots \rangle$ $\langle P_1^*, \ldots, P_\nu^*, P_{\nu+1}, \ldots \rangle$ such that

$$\varphi P_\mu P_\nu = 1 \longleftrightarrow \varphi P_\mu P^*_\nu = 1. \quad (2)$$

At the μ^{th} step we constructed from $<P^*_1, \ldots, P^*_{\mu-1} . P_\mu, \ldots>$ $<P^*_1, \ldots, P^*_\mu, P_{\mu+1}, \ldots>$ such that

$$\varphi P_\mu P^*_\nu = 1 \longleftrightarrow \varphi P^*_\mu P^*_\nu = 1. \quad (3)$$

It follows from (2), (3) that

$$\varphi P^*_\nu P^*_\mu = 1 \longleftrightarrow \varphi P_\nu P_\mu = 1. \quad (4)$$

This proves (1).

3.2. Theorem. Let $<Q_n>_n$ be a covering of an LDFTK-space Γ. Then there exists a covering $<Q^*_n>_n$ of Γ, such that $\wedge n(Q^*_n \Subset Q_n)$.
Proof. Let us suppose first $\wedge n(Q_n \neq A_\infty)$.
We construct an R_1 such that $Q_1 \Subset R_1$, $R_1 \neq A_\infty$ (L2). R_1 is a DFTK-space, therefore a natural number n(1) can be found such that $\{Q_1, \ldots, Q_{n(1)}\}$ is a covering of R_1. We put $Q_1 = S_1$, $Q_1 + \ldots + Q_{n(1)} = S_2$. Using L2 we construct an R_2 such that $S_2 \Subset R_2$, $R_2 \neq A_\infty$. Then we can find a natural number $n(2) > n(1)$ such that $\{Q_1, \ldots, Q_{n(2)}\}$ covers R_2.

Carrying on inductively we obtain sequences $<R_i>_i, <S_i>_i$, $<n(i)>_i$ such that $n(i) < n(i+1)$, $\{Q_1, \ldots, Q_{n(i)}\}$ a covering of R_i, $R_i \neq A_\infty$, $Q_1 + \ldots + Q_{n(i-1)} = S_i \Subset R_i$ for every i.

Our next step is the construction of a sequence $<R^!_i>_i$ such that $\wedge i(S_i \Subset R^!_i \Subset R_i)$ (N8). 1.26 implies the existence of $Q^!_1, \ldots, Q^!_{n(1)}$ which cover R_1 such that $Q^!_i \Subset_{R_1} Q_i$ for $1 \leqslant i \leqslant n(1)$. Since $Q_1 \Subset R_1$, we obtain $Q^!_1 \Subset Q_1$ (lemma 3.1.30). We put $Q^*_1 = Q^!_1$. Defining $Q''_i = Q^!_i R^!_1$, we see that $Q''_1 \sim Q^*_1$, $Q''_i \Subset Q_i$ for $1 \leqslant i \leqslant n(1)$ (3.1.28), $\{Q''_1, \ldots, Q''_{n(1)}\}$ covers $R^!_1$. $\{Q^*_1, Q_2, \ldots, Q_{n(2)}\}$ covers R_2; this is seen as follows: $p \in R_2 \longrightarrow p \in Q_j$, $1 \leqslant j \leqslant n(2)$. $j = 1 \longrightarrow p \in R_1$. R_1 is covered by $\{Q^*_1, Q_2, \ldots, Q_{n(1)}\}$, hence $p \in Q^*_1 \vee (p \in Q_j \;\&\; 2 \leqslant j \leqslant n(1))$. This proves our assertion.

We conclude to the existence of a covering $\{Q^*_1, Q^{**}_2, \ldots, Q^{**}_{n(1)}, Q^!_{n(1)+1}, \ldots, Q^!_{n(2)}\}$ of R_2, such that for $2 \leqslant i \leqslant n(1)$ $Q^{**}_i \Subset_{R_2} Q_i$, and for $n(1) < j \leqslant n(2)$ $Q^!_j \Subset_{R_2} Q_j$ (1.26).
Since $Q_1 + \ldots + Q_{n(1)} = S_2 \Subset R_2$, it follows that $Q^{**}_i \Subset Q_i$ for $1 < i \leqslant n(1)$. We put $Q^*_i = Q^{**}_i + Q''_i$ for $1 \leqslant i \leqslant n(1)$. $\{Q^*_1, \ldots, Q^*_{n(1)}\}$ covers $R^!_1$. We define $Q''_j = Q^!_j R^!_2$, $n(1) < j \leqslant n(2)$; $Q''_j \Subset Q_j$ for $n(1) < j \leqslant n(2)$ (3.1.30).

We proceed inductively. Suppose already constructed $\{Q^*_1, \ldots, Q^*_{n(i)}, Q^!_{n(i)+1}, \ldots, Q^!_{n(i+1)}\}$ such that $Q_k \Subset Q^*_k$ for $1 \leqslant k \leqslant n(i)$, $\{Q^*_1, \ldots, Q^*_{n(k)}\}$ covers $R^!_k$ for $k \leqslant i$, $\{Q^*_1, \ldots, Q^*_{n(i)}, Q^!_{n(i)+1}, \ldots, Q^!_{n(i+1)}\}$ covers R_{i+1}, $Q^!_j \Subset_{R_{i+1}} Q_j$ for $n(i) < j \leqslant n(i+1)$.

Then $\{Q_1^*, \ldots, Q_{n(i)}^*, Q_{n(i)+1}, \ldots, Q_{n(i+2)}\}$ covers R_{i+2}, since $p \in R_{i+2} \rightarrow (p \in Q_j \;\&\; j > n(i)) \vee (p \in Q_j \;\&\; j \leqslant n(i))$; $p \in Q_j \;\&\; j \leqslant n(i) \rightarrow p \in R_{i+1}$; $p \in R_{i+1} \rightarrow (p \in Q_j^* \;\&\; 1 \leqslant j \leqslant n(i)) \vee (p \in Q_j \;\&\; n(i) < j \leqslant n(i+1))$.

This enables us to construct a covering $\{Q_1^*, \ldots, Q_{n(i)}^*, Q_{n(i)+1}^{**}, \ldots, Q_{n(i+1)}^{**}, Q'_{n(i+1)+1}, \ldots, Q'_{n(i+2)}\}$ of R_{i+2} such that $Q_j^{**} \Subset_{R_{i+2}} Q_j$ for $n(i) < j \leqslant n(i+1)$, $Q'_j \Subset_{R_{i+2}} Q_j$ for $n(i+1) < j \leqslant n(i+2)$.

$S_{i+2} = Q_1 + \ldots + Q_{n(i+1)} \Subset R_{i+2}$, hence $Q_j^{**} \Subset Q_j$ for $n(i) < j \leqslant n(i+1)$.

We put $Q''_j = Q'_j R'_{i+1}$ for $n(i) < j \leqslant n(i+1)$. Then $Q''_j \Subset Q_j$ for $n(i) < j \leqslant n(i+1)$. If we put $Q_j^* = Q_j^{**} + Q''_j$, $n(i) < j \leqslant n(i+1)$, we obtain a covering $\{Q_1^*, \ldots, Q_{n(i+1)}^*, Q'_{n(i+1)+1}, \ldots, Q'_{n(i+2)}\}$ of R_{i+2}, such that $Q_1^*, \ldots, Q_{n(i+1)}^*$ cover R_{i+1}, $Q'_j \Subset_{R_{i+2}} Q_j$ for $n(i+1) < j \leqslant n(i+2)$.

There remains to be proved that $<Q_n^*>_n$ is a covering. $p \in Q_j \;\&\; j \leqslant n(k) \rightarrow p \in Q_1 + \ldots + Q_{n(k)} \Subset R'_{k+1}$. If $p \in R'_{k+1}$ it follows that $p \in Q_\mu^*$ for a certain μ, $1 \leqslant \mu \leqslant n(k+1)$.

Finally, we remove the restriction $\wedge n(Q_n \neq A_\infty)$. In the construction described before, $\{Q_1^*, \ldots, Q_{n(i)}^*\}$ could be constructed from $\{Q_1, \ldots, Q_{n(i+1)}\}$.

If $Q_j = A_\infty$ for a j such that $n(i+1) < j \leqslant n(i+2)$ we put $Q_j^* = A_\infty$, $Q_k^* = A_0$ for $n(i) < k \;\&\; k \neq j$. If there is no such j, $n(i+1) < j \leqslant n(i+2)$, we proceed with our construction as before.

3.3. Theorem. Let $<P_n>_n$ be a covering of an LDFTK-space Γ. Then there exists a star-finite refinement $<Q_n>_n$ of $<P_n>_n$. If $\wedge n(P_n \neq A_\infty)$ we may suppose $\wedge n(Q_n \Subset P_n)$.

Proof. We suppose first $\wedge n(P_n \neq A_\infty)$.

We make use of theorem 3.2. There exists a covering $<P''_n>_n$ such that $\wedge n(P''_n \Subset P_n)$. Now we shall construct sequences $<Q''_n>_n, <Q'_n>_n, <Q_n>_n$, such that $\wedge n(Q''_n \Subset Q'_n \Subset Q_n \;\&\; Q''_n \subset P''_n \;\&\; Q_n \Subset P_n)$. The construction of these sequences is carried out by means of induction, but the regularity of the construction will only become apparent after two steps.

We begin with the construction of a sequence $<n(i)>_i$, $n(1) = 1$, $\wedge n(n(i) < n(i+1))$, such that $\{P''_1, \ldots, P''_{n(i+1)}\}$ is a covering of $P_1 + \ldots + P_{n(i)}$ for every i.

We put $P''_i = Q''_i$ for $1 \leqslant i \leqslant n(2)$, and we shall construct Q'_i, Q_i $1 \leqslant i \leqslant n(2)$ such that $Q''_i \Subset Q'_i \Subset Q_i \Subset P_i$. Then we construct Q''_j, $n(2) < j \leqslant n(3)$, such that $\varphi Q_1 Q''_j = 0$, $Q'_1 + \ldots + Q'_{n(2)} + Q''_j \sim Q'_1 + \ldots + Q'_{n(2)} + P''_j$, $Q''_j \subset P''_j$. (The details of this construction will be described afterwards).

We remark: $P''_1 + \ldots + P''_{n(3)} \sim Q'_1 + \ldots + Q'_{n(2)} + P''_{n(2)+1} + \ldots + P''_{n(3)} \sim Q'_1 + \ldots + Q'_{n(2)} + Q''_{n(2)+1} + \ldots + Q''_{n(3)}$ (using

$Q'_1 + \ldots + Q'_{n(2)} \subset Q_1 + \ldots + Q_{n(2)} \subset P_1 + \ldots + P_{n(2)} \subset P''_1 + \ldots + P''_{n(3)}$).
If we construct Q'_j, Q_j for $n(2) < j \leqslant n(3)$, such that $Q''_j \Subset Q'_j \Subset Q_j$, $\varphi Q_j Q_1 = 0$, $Q_j \Subset P_j$, then $\{Q_1, \ldots, Q_{n(3)}\}$ is a covering of $P''_1 + \ldots + P''_{n(3)}$ (by 3.1.38)

Suppose now that Q''_j, Q'_j, Q_j have already been constructed for $n(1) \leqslant j \leqslant n(k)$, such that
a) $Q''_j \Subset Q'_j \Subset Q_j$, $Q''_j \subset P''_j$, $Q_j \Subset P_j$ for $1 \leqslant j \leqslant n(k)$.
b) $\{(Q_1, \ldots, Q_{n(i)}\}$ is a covering of $P''_1 + \ldots + P''_{n(i)}$ for $i \leqslant k$.
c) $\varphi(Q_1 + \ldots + Q_{n(i)})(Q_{n(i+1)+1} + \ldots + Q_{n(i+2)}) = 0$ for $1 \leqslant i \leqslant k-2$.
d) $(Q'_1 + \ldots + Q'_{n(i)}) + Q''_j \sim (Q'_1 + \ldots + Q'_{n(i)}) + P''_j$ for $n(i) < j \leqslant n(i+1)$, $i < k$.
e) $P''_1 + \ldots + P''_{n(i)} \subset Q'_1 + \ldots + Q'_{n(i)}$ for $1 \leqslant i \leqslant k$.
It follows that $Q_1 + \ldots + Q_{n(i-1)} \Subset P_1 + \ldots + P_{n(i-1)} \subset P''_1 + \ldots + P''_{n(i)} \subset Q'_1 + \ldots + Q'_{n(i)}$.
We construct Q''_j, Q'_j, Q_j for $n(k) < j \leqslant n(k+1)$ as follows. We begin with constructing the Q''_j such that
$\varphi(Q_1 + \ldots + Q_{n(k-1)})Q''_j = 0$,
$(Q'_1 + \ldots + Q'_{n(k)}) + Q''_j \sim (Q'_1 + \ldots + Q'_{n(k)}) + P''_j, Q''_j \Subset P''_j$;
the details of this construction will be given afterwards.

We construct the Q'_j, Q_j such that $Q''_j \Subset Q'_j \Subset Q_j \Subset P_j$, $\varphi(Q_1 + \ldots + Q_{n(k-1)})Q_j = 0$. We remark that

$Q'_1 + \ldots + Q'_{n(k)} \subset Q_1 + \ldots + Q_{n(k)} \subset P_1 + \ldots + P_{n(k)} \subset P''_1 + \ldots + P''_{n(k+1)}$,

hence $P''_1 + \ldots + P''_{n(k+1)} \sim P''_1 + \ldots + P''_{n(k+1)} + Q'_1 + \ldots + Q'_{n(k)} \sim P''_1 + \ldots + P''_{n(k)} + Q'_1 + \ldots + Q'_{n(k)} + P''_{n(k)+1} + \ldots + P''_{n(k+1)} \sim Q'_1 + \ldots + Q'_{n(k)} + P''_{n(k)+1} + \ldots + P''_{n(k+1)} \sim Q'_1 + \ldots + Q'_{n(k)} + Q''_{n(k)+1} + \ldots + Q''_{n(k+1)}$.
Therefore (3.1.38) $\{Q_1, \ldots, Q_{n(k+1)}\}$ is a covering of $P''_1 + \ldots + P''_{n(k+1)}$.
We remark: $P''_1 + \ldots + P''_{n(k+1)} \subset Q'_1 + \ldots + Q'_{n(k+1)}$.
So the conditions (a)-(e) are also satisfied for k+1 instead of k.

Finally we describe the construction of the Q''_j, $n(k) < j \leqslant n(k+1)$ in detail. $Q'_1 + \ldots + Q'_{n(k)} + P''_j$ is a DFTK-space with a DFTK-basis $\langle [T_n] \rangle_n$. There is a μ such that

$$m \geqslant \mu \ \& \ \varphi(Q_1 + \ldots + Q_{n(k-1)})T_m = 1 \longrightarrow T_m \subset Q'_1 + \ldots + Q'_{n(k)}.$$

We put ($n(k) < j \leqslant n(k+1)$, $k > 1$)

$$Q''_j = \Sigma\{T_m P''_j : \varphi T_m(Q_1 + \ldots + Q_{n(k-1)}) = 0 \ \& \ \mu \leqslant m \leqslant \mu + r(\mu)\}$$
$$R''_j = \Sigma\{T_m P''_j : \varphi T_m(Q_1 + \ldots + Q_{n(k-1)}) = 1 \ \& \ \mu \leqslant m \leqslant \mu + r(\mu)\}$$

where r is a K-function as introduced in 1.2.

$Q_j'' + R_j'' \sim P_j''$; $R_j'' \subset (Q_1' + \ldots + Q_{n(k)}')P_j''$;
$\varphi Q_j''(Q_1 + \ldots + Q_{n(k-1)}) = 0$.
$Q_1' + \ldots + Q_{n(k)}' + Q_j'' \sim Q_1' + \ldots + Q_{n(k)}' + Q_j'' + R_j'' \sim$
$Q_1' + \ldots + Q_{n(k)}' + P_j''$.

From (a), (b), (c) we see that $\langle Q_n\rangle_n$ is a star-finite refinement of $\langle P_n\rangle_n$.

If we do not know if $\wedge n(P_n \neq A_\infty)$ holds, we can apply our construction to $\langle R_n\rangle_n$, obtained as follows:

$\neg \vee m(m \leqslant n \ \& \ P_m = A_\infty) \longrightarrow R_n = P_n$.
$\vee m(m \leqslant n \ \& \ P_m = A_\infty) \longrightarrow (R_n = A_{n-k+1} \ \& \ k = \inf\{m : P_m = A_\infty\})$.

3.4. Corollary to 3.3, 3.2. Let $\langle Q_n\rangle_n$ be a covering of an LDFTK-space Γ. Then there exists a star-finite covering $\langle P_n\rangle_n$ of Γ such that $\wedge n \vee m(P_n \in Q_m)$.

3.5. Theorem. Let Γ be an LDFTK-space.

a) Every open or enumerable covering of Γ possesses a refinement consisting of an enumerable sequence of lattice elements.
b) Every open or enumerable covering possesses a star-finite enumerable refinement consisting of lattice elements.

Proof. (a). We use the enumeration principle. Γ has a perfect representation Π_1. To every $\langle R_n\rangle_n \in \Pi_1$ a natural number m can be assigned by a function ψ such that $\psi\langle R_n\rangle_n = m$ implies: $\langle R_n\rangle_n^* \subseteq \gamma m$ and $[\gamma m]$ is contained in an element of the covering (cf. 2.2.6). $\psi\Pi_1$ can be enumerated, and this proves (a).
(b) is an immediate consequence of (a) and 3.3.

4. Located pointspecies and completeness.

4.1. Theorem. Let Γ be an LDFTK-space. Γ can be metrized by a metric ρ such that

a) Every located non-empty pointspecies of Γ is metrically located with respect to ρ.
b) $\langle \Pi^o(\Gamma), \rho\rangle$ is metrically complete.

Proof. Let $\langle P_n\rangle_n$ be a star-finite covering of Γ, $\wedge n(P_n \neq A_\infty)$, and suppose $\langle Q_n\rangle_n$ to be constructed according to 3.1, so $\wedge n(P_n \in Q_n \ \& \ Q_n \neq A_\infty)$, $\wedge i \wedge j(\varphi P_i P_j = 1 \longleftrightarrow \varphi Q_i Q_j = 1)$. Let $\langle\langle R_i, R_i'\rangle\rangle_{i=2}^{\infty}$ be an enumeration of all pairs $\langle R_i, R_i'\rangle$ such that $\varphi R_i R_i' = 0$.
The species $\{P_n : \varphi P_n = 1\}$ can be enumerated as $\langle P_n'\rangle_n$, with repetitions if necessary. There is a mapping g from $\underline{N}$ into $\underline{N}$ such that $P_n' = P_{g(n)}$. $\langle Q_n'\rangle_n$ is defined by $Q_n' = Q_{g(n)}$.

We consider an inessential extension Δ of Γ, defined as

follows. $\mathfrak{A}(\Delta) = \mathfrak{A}(\Gamma) \cup \langle B_n\rangle_n \cup \langle C_n\rangle_n$, $\mathfrak{A}(\Gamma) = \langle A_n\rangle_n$; $\langle A_n\rangle_n, \langle B_n\rangle_n, \langle C_n\rangle_n$ are disjoint sequences of different elements. $\varphi_\Gamma = \varphi_\Delta | \mathfrak{P}(\Gamma)$.
We put $B_n \sim_\Delta P_n'$, $C_n \sim_\Delta Q_n'$. In the sequel we omit subscripts Δ systematically. Γ is extended to Δ in order to be able to construct a normal perfect representation Π_1 for Δ in accordance with definition 3.2.2. Π_1 is defined by a pair $\langle\theta, \vartheta\rangle$. $\langle\emptyset\rangle \in \theta$, $\langle B_n\rangle \in \vartheta\,\theta$ for every n. B_n is a DFTK-space; therefore a finitary normal perfect representation for B_n can be constructed according to 1.17.
If we define for a fixed ν $\langle\theta_\nu, \vartheta_\nu\rangle$ by

$$\theta_\nu = \{\langle\nu, i_1, \ldots, i_k\rangle : \langle\nu, i_1, \ldots, i_k\rangle \in \theta\}, \quad \vartheta_\nu = \vartheta|\theta_\nu,$$

we can suppose $\langle\theta_\nu, \vartheta_\nu\rangle$ to define a finitary normal perfect representation for $\gamma\nu$.
In this way a perfect representation $\langle\theta, \vartheta\rangle$ is obtained for Δ. We construct a mapping f into $\mathfrak{P}$, defined on $\vartheta\theta - \{\emptyset\}$. $f\langle B_\nu\rangle = C_\nu$. If $\langle P_1, P_2\rangle \in \vartheta\,\theta$, then $f\langle P_1, P_2\rangle$ is chosen such that

$$P_1 P_2 \Subset f\langle P_1, P_2\rangle \Subset f\langle P_1\rangle,$$
$$\varphi P_1 P_2 R_2 = 0 \longrightarrow \varphi f\langle P_1, P_2\rangle R_2 = 0,$$
$$\varphi P_1 P_2 R_2' = 0 \longrightarrow \varphi f\langle P_1, P_2\rangle R_2' = 0.$$

Suppose $f\langle P_1\rangle, f\langle P_1, P_2\rangle, \ldots, f\langle P_1, \ldots, P_n\rangle$ to be defined already, and let $\langle P_1, \ldots, P_{n+1}\rangle \in \vartheta\,\theta$. We construct $f\langle P_1, \ldots, P_{n+1}\rangle$ such that $P_1 \ldots P_{n+1} \Subset f\langle P_1, \ldots, P_{n+1}\rangle \Subset f\langle P_1, \ldots, P_n\rangle$, and

$$\varphi P_1 \ldots P_{n+1} R_{n+1} = 0 \longrightarrow \varphi f\langle P_1, \ldots, P_{n+1}\rangle R_{n+1} = 0,$$
$$\varphi P_1 \ldots P_{n+1} R_{n+1}' = 0 \longrightarrow \varphi f\langle P_1, \ldots, P_{n+1}\rangle R_{n+1}' = 0.$$

Then we define a normal $\Subset$-representation for Δ with a defining pair $\langle\theta', \vartheta'\rangle$, $\langle\emptyset\rangle \in \theta'$, putting

$$\langle Q_1 \ldots Q_n\rangle \in \vartheta'\theta' \leftrightarrow \langle Q_1, \ldots, Q_n\rangle = \langle f\langle P_1\rangle, f\langle P_1, P_2\rangle, \ldots, f\langle P_1, \ldots, P_n\rangle\rangle \;\&\; \langle P_1, \ldots, P_n\rangle \in \vartheta\theta.$$

We enumerate all pairs $\langle\langle k_1, \ldots, k_{n+1}\rangle, \langle k_1, \ldots, k_n\rangle\rangle$ for which $\langle k_1, \ldots, k_{n+1}\rangle \in \theta'$ in a sequence $\langle\langle\sigma_i, \sigma_i'\rangle\rangle_i$ without repetitions, and we put $\vartheta'^*\sigma_i = S_i$, $\vartheta'^*\sigma_i' = S_i'$.
We define a function w from $\underline{N}$ into $\{0, 1\}$:
$w(i) = 1 \longleftrightarrow \sigma_i = \langle k_1, k_2\rangle \in \theta'$ for certain k_1, k_2 and $h(k_1) = 1$, where h is an auxiliary function defined by

$$h(k) = 1 \text{ iff } \gamma k = C_1 \;\&\; g(l) \notin \{g(1), g(2), \ldots, g(l-1)\},$$
$$h(k) = 0 \text{ in all other cases.}$$

To every pair $\langle S_i, S_i'\rangle$ a continuous function $f_i(p)$ can be defined (3.2.27) such that

$$p \in S_i \longrightarrow f_i(p) = 1,\; p \notin S_i' \longrightarrow f_i(p) = 0,\; 0 \not> f(p) \not> 1.$$

Δ (hence also Γ) can be metrized by

$\rho'(p, q) = \sum_{i=1}^{\infty} \{|f_i(p) - f_i(q)| 2^{-i}(1 - w(i)) + |f_i(p) - f_i(q)| w(i)\}(*)$.

We must prove ρ' to be an adequate metric for Δ. This proof closely parallels the proof of the corresponding fact in 3.2.28. Here too we have to prove (1)-(5). (1)-(3) do not present any difficulties. Since the identical mapping of Π^o onto itself can be considered as a mapping of Δ onto $<\Pi^o, \rho'>$, this mapping is continuous by 3.2.22, and this proves (4). The proof of (5) in 3.2.28 remains valid (with small adaptations) in this case; only we have to consider the possibilities $w(\mu) = 1$ and $w(\mu) = 0$ separately.

Let $f_i(p) \# 0$, $f_j(p) \# 0$, $w(i) = w(j) = 1$, $i \neq j$.
$f_i(p) \# 0 \rightarrow p \in S'_i = C_\nu$ for a certain ν.
$f_j(p) \# 0 \rightarrow p \in S'_j = C_\mu$ for a certain μ.
Hence $\varphi C_\nu C_\mu = 1$. $C_\mu \sim Q'_\mu = Q_{g(\mu)}$, $C_\nu \sim Q'_\nu = Q_{g(\nu)}$.
$w(i) = 1 \rightarrow g(\nu) \notin \{g(1), \ldots, g(\nu-1)\}$,
$w(j) = 1 \rightarrow g(\mu) \notin \{g(1), \ldots, g(\mu-1)\}$, therefore $g(\nu) \neq g(\mu)$.
$\{Q_i : \varphi Q_i Q_j = 1\}$ is a finite species for every fixed j. (3.1.37). Therefore the species $\{i : f_i(p) \# 0 \ \& \ w(i) = 1\}$ is contained in a finite species.
If $|f_i(p) - f_i(q)| w(i) \# 0$, then $w(i) = 1 \ \& \ (f_i(p) \# 0 \vee f_i(q) \# 0)$.
Hence $|f_i(p) - f_i(q)| w(i) = 0$ for almost all $i \in \underline{N}$.
This actually proves that the right hand side of (*) converges.

Finally we put $\rho(p, q) = \inf\{\rho'(p, q), 1\}$. ρ satisfies our requirements, as will be proved now.

Suppose V to be a non-empty located pointspecies of Δ, and let $p \in \Pi^o(\Delta)$, p arbitrary.
A $P_{g(n)}$ can be found such that $p \in P_{g(n)}$. We take the least number m such that $P_{g(n)} = P_{g(m)}$, and call it μ. It follows that if $\gamma' C_\mu = \nu$, then $h(\nu) = 1$.

A pair $<B_\mu, P> \in \vartheta\, \theta$ can be found (since Π_1 is a perfect representation) such that $B_\mu \sim P_{g(n)}$, $p \in B_\mu P$. Hence $p \in f<B_\mu, P> \Subset C_\mu$.

$\langle f<B_\mu, P>, f<B_\mu\rangle\rangle = <S_j, S'_j>$ for a certain j. We conclude that $w(j) = 1$, $p \in S_j$, $S'_j = C_\mu$. Therefore $f_j(p) = 1$. $\rho(p, q) < 1 \rightarrow \rho'(p, q) < 1$; this implies in turn $|f_j(p) - f_j(q)| w(j) < 1$. Hence $f_j(q) > 0$, so $q \in C_\mu$. C_μ is a DFTK-space.

Now we can duplicate the reasoning of 2.3.7 very closely. As a result we see that either C_μ does not contain a point of V (in this case we may take $\rho(p, V) = 1$) or there is a finite sequence $\langle q_1, \ldots, q_\lambda \rangle \subset V$ such that if $q \in V \cap C_\mu$, there must be q_i such that $\rho(q_i, q) < 2^{-\nu}$. Hence $\rho(p, V)$ is approximated within $2^{-\nu}$ by $\inf\{1, \rho(q_i, p): 1 \leqslant i \leqslant \lambda\}$.
This proves the existence of the distance function.

Finally we prove (b). Let $<r_n>_n$ be a fundamental sequence with respect to ρ. To every $\mu \geqslant 1$ a ν can be found such that

$$\wedge i \wedge j(i, j \geqslant \nu \to \rho(r_i, r_j) < 2^{-\mu-1}).$$

We take p to be r_ν in our previous considerations, and we construct a λ such that $\rho(r_\nu, q) < 1 \to q \in C_\lambda$.
Therefore $U_1(r_\nu) \subset C_\lambda$, so $\wedge i(i \geqslant \nu \to r_i \in C_\lambda)$.
We conclude that $<r_n>_{n=\nu}^{\infty}$ is a fundamental sequence in C_λ, and converges therefore to a point $r \in C_\lambda$.

4.2. Corollary to 4.1. A metric locally DFTK-space is a complete separable metric space such that to every point a neighbourhood can be found, which is an LC-space in the relative topology; conversely, a space which satisfies these requirements is a metric locally DFTK-space.
Proof. The corollary is an immediate consequence of 4.1 and 2.3.

4.3. Remark. By an argument quite similar to the proof of 2.3.7, and the reasoning in 4.1, we see that the usual metric of $\underline{R}^n$ satisfies condition (a) of 4.1.

5. Topological products.

5.1. In this paragraph we shall demonstrate that the topological product of a denumerably infinite sequence of LDFTK-spaces which satisfy certain requirements, is a PIN-space. The most interesting consequence of this theorem is that $\underline{R}^\infty$ is a PIN-space.

5.2. Lemma. If Γ is the product of a finite or denumerably infinite sequence of PIN-spaces $<\Gamma_1, \Gamma_2, \ldots>$, each of which contains at least two points which lie apart, we can assert, using the notation of 3.4.1, that $\Gamma' = <\varphi, \Sigma>$ and $\Gamma = <\varphi, \Pi>$ are homeomorphic.
Proof. It is immediate that Γ' is also an I-space. Let $<P_n>_n \in \Sigma(\Gamma) = \Pi(\Gamma')$. Now we shall construct a $<Q_n>_n \in \Pi(\Gamma)$ such that $\wedge n(P_1 \ldots P_n \subset Q_n)$. Let $P_1 \ldots P_n = P_{n,1} + \ldots + P_{n,k(n)}$; $P_{n,1}, \ldots, P_{n,k(n)} \in \mathfrak{A}$.
If $P_{n,i} = <P^m_{n,i}>_m$ for every n, i, then we can put

$$Q^m_n = P^m_{n,1} + \ldots + P^m_{n,k(n)},\quad Q_n = <Q^m_n>_m.$$

$<Q_n>_n$ satisfies our requirements. To see this we consider: $R = <A^1_\infty, \ldots, A^{\mu-1}_\infty, R^\mu, A^{\mu+1}_\infty, \ldots>$, $S = <A^1_\infty, \ldots, A^{\mu-1}_\infty, S^\mu, A^{\mu+1}_\infty, \ldots>$, $\varphi_\mu R^\mu S^\mu = 0$ (hence $\varphi RS = 0$).
There exists a ν such that $\varphi P_1 \ldots P_\nu R = 0 \vee \varphi P_1 \ldots P_\nu S = 0$; suppose e.g. $\varphi P_1 \ldots P_\nu R = 0$. $P_1 \ldots P_\nu = P_{\nu,1} + \ldots + P_{\nu,k(\nu)}$.
$\varphi(P_{\nu,1} + \ldots + P_{\nu,k(\nu)})R = 0 \leftrightarrow \wedge i(1 \leqslant i \leqslant k(\nu) \to \varphi P_{\nu,i}R = 0) \leftrightarrow$

$\wedge i(1 \leqslant i \leqslant k(\nu) \longrightarrow \varphi P^{\mu}_{\nu,i} R^{\mu} = 0) \longleftrightarrow \varphi(P^{\mu}_{\nu,i} + \ldots + P^{\mu}_{\nu,k(\nu)})R^{\mu} = 0 \leftrightarrow \varphi Q^{\mu}_{\nu} R^{\mu} = 0.$

This proves $\langle Q^{\mu}_{n} \rangle_n$ to be a point of Γ_{μ} (since Γ_{μ} contains at least two different points). Hence $\langle Q_n \rangle_n$ is by definition a point of Γ. Finally we can apply 3.1.32, therefore $\langle \varphi, \Sigma \rangle$, $\langle \varphi, \Pi \rangle$ are homeomorphic.

5.3. Remark to 5.2. If the Γ_i are DFTK-spaces, then the condition that every space contains at least two different points can be omitted, since this condition is used only to ascertain that $\langle Q_n \rangle_n$ really belongs to Σ, i.e. $Q_{\mu} \not\approx A_{\infty}$ for a certain μ. If the Γ_i are DFTK-spaces, A^i_{∞} can always be replaced by a $P^i \not\approx A^i_{\infty}$, $P^i \sim A^i_{\infty}$, and the construction of $\langle Q_n \rangle_n$ can be modified correspondingly.

5.4. Lemma. Let $\langle \Gamma_1, \ldots, \Gamma_n \rangle$ be a finite sequence of DFTK-spaces. We adopt the notation of 3.4.1. The product Γ of $\langle \Gamma_1, \ldots, \Gamma_n \rangle$ is a DFTK-space with a DFTK-basis $\langle [B_n] \rangle_n$, where $\langle B_n \rangle_n \subset \mathfrak{P}$.
Proof. Let $r_1, \ldots, r_n$ be K-functions for $\Gamma_1, \ldots, \Gamma_n$ respectively. We put $\mathfrak{B}^k = \{ \langle A^1_{m_1}, \ldots, A^n_{m_n} \rangle \ \& \ \wedge i(1 \leqslant i \leqslant n \longrightarrow k \leqslant m_i \leqslant k + r_i(k)) \}$.
We suppose $\mathfrak{B}$ to be an enumeration of $\bigcup\limits_{n=1}^{\infty} \mathfrak{B}^n$ such that all elements of $\mathfrak{B}^k$ precede all elements of $\mathfrak{B}^{k+1} - \mathfrak{B}^k$. $\mathfrak{B} = \langle B_n \rangle_n$. The verification of D, F, T, K and the proof that $\langle [B_n] \rangle_n$ induces the topology of Γ is straightforward; we only have to remark: if $\mathfrak{P}^*$ is the distributive lattice constructed from $\mathfrak{B}$, then $\wedge P \ \vee Q \in \mathfrak{P}^* \ (P \sim Q)$, and to apply remark 5.3.

5.5. Lemma. We use again the notation of 3.4.1. Let Γ be the product of a finite sequence $\langle \Gamma_1, \ldots, \Gamma_n \rangle$ of LDFTK-spaces. Let $P = \langle P^1, \ldots, P^n \rangle$, $P^i \not\approx A^i_{\infty}$ for $1 \leqslant i \leqslant n$. Then if $P \Subset Q$, an R can be found such that $P \Subset R \Subset Q$.
Proof. Let $Q = Q_1 + \ldots + Q_m$, $Q_i \in \mathfrak{A}$ for $1 \leqslant i \leqslant m$. For every Q^i_j such that $Q^i_j = A^i_{\infty}$, we take a $T^i_j \not\approx A^i_{\infty}$, $P^i \Subset T^i_j$. We define T_j by

$$\wedge i(1 \leqslant i \leqslant n \longrightarrow (Q^i_j = A^i_{\infty} \longrightarrow \pi_i T_j = T^i_j) \ \& \ (Q^i_j \not\approx A^i_{\infty} \longrightarrow \pi_i T_j = A^i_{\infty})).$$

We remark that $P \Subset T_j$ $(1 \leqslant j \leqslant m)$, hence $P \Subset T_1 \ldots T_m = T$. Hence also $P \Subset TQ$.
If we put $T = \langle T^1, \ldots, T^n \rangle$, we see that $T^j Q^j_i \not\approx A^j_{\infty}$ for $1 \leqslant j \leqslant n$, $1 \leqslant i \leqslant m$. Therefore $S^j_i, S^j_i \not\approx A^j_{\infty}$, $T^j Q^j_i \Subset S^j_i$ can be found. Putting $S = \langle \sum\limits_{i=1}^{m} S^1_i, \ldots, \sum\limits_{i=1}^{m} S^n_i \rangle$, we see that $P \Subset TQ \Subset S$. It follows from 5.4 that S is a DFTK-space with a DFTK-basis $\langle [B_n] \rangle_n$, $\langle B_n \rangle_n \subset \mathfrak{P}$. On this account

an R can be constructed such that $P \in_S R \in_S TQ \in S$, hence $P \in R \in QT \subset Q$. (3.1.30).

5.6. Theorem. Let $\langle \Gamma_n \rangle_n$ be a sequence of LDFTK-spaces (each of which contains at least two points which lie apart). Using the notations of 3.4.1 we postulate:
$\wedge P^i(P^i \sim A^i_\infty \vee \vee Q^i(\varphi_i Q^i P^i = 0 \ \& \ \varphi_i Q^i = 1))$ for every i. Then the product of $\langle \Gamma_n \rangle_n$ is a PIN-space.
Proof. $\Gamma' = \langle \varphi, \Sigma \rangle$; $\Gamma = \langle \varphi, \Pi \rangle$. By lemma 5.2, Γ, Γ' are homeomorphic. N6 holds in Γ as a consequence of 3.4.5, hence N6 also holds in Γ'. We begin by proving the following assertion.

$$\left.\begin{array}{l} P = \langle P^i \rangle_i \ \& \ P^\mu = A^\mu_\infty \ \& \ (R^i_j = \langle R^i_j \rangle_i, j=1,\ldots,m) \ \& \\ P \in R_1 + \ldots + R_m \ \& \ R = \Sigma\{R_j : R^\mu_j \sim A^\mu_\infty \ \& \ 1 \leqslant j \leqslant m\} \ \& \\ Q^\mu = \Sigma\{R^\mu_j : \neg R^\mu_j \sim A^\mu_\infty \ \& \ 1 \leqslant j \leqslant m\} \ \& \ \neg Q^\mu \sim A^\mu_\infty \rightarrow \end{array}\right\} \quad (1)$$
$$P \in R.$$

If $R \sim R_1 + \ldots + R_m$, (1) is trivial. In all other cases we may suppose $\varphi_\mu Q^\mu = 1$. Let $\langle P_n \rangle_n \epsilon \Pi(\Gamma)$.
$\neg Q^\mu \sim A^\mu_\infty$, hence there exists an S^μ, $\varphi_\mu Q^\mu S^\mu = 0$, $\varphi_\mu S^\mu = 1$.
Let $\langle S^\mu_n \rangle_n \epsilon \Pi(\Gamma_\mu)$, $\wedge n(S^\mu_n \subset S^\mu)$.
Then we construct a point generator $\langle T_n \rangle_n \epsilon \Pi(\Gamma)$, such that for all n $(i \neq \mu \rightarrow T^i_n = P^i_n) \ \& \ T^\mu_n = S^\mu_n$.
Then a ν can be calculated such that

$$\varphi T_1 \ldots T_\nu P = 0 \vee T_1 \ldots T_\nu \subset R_1 + \ldots + R_m.$$

Since $\varphi_\mu S^\mu_1 \ldots S^\mu_\nu Q^\mu = 0$, it follows that $T_1 \ldots T_\nu \subset R \vee \varphi T_1 \ldots T_\nu P = 0$, hence also: $\varphi P_1 \ldots P_\nu P = 1 \rightarrow P_1 \ldots P_\nu \subset R$.
Using this construction repeatedly, we are able to construct from $R_1, \ldots, R_m$ an $R' = R'_1 + \ldots + R'_k$, such that $P \in R' \subset R_1 + \ldots + R_m$, $P^n_\infty = A^n_\infty \rightarrow R'^n_j = A^n_\infty$ for $1 \leqslant j \leqslant k$.

Now we turn to the task of proving N8 for Γ. Since $P' + P'' \in R \leftrightarrow P' \in R \ \& \ P'' \in R$, we may restrict ourselves to the case $P \epsilon \mathfrak{A}$.
If $P \in R_1 + \ldots + R_m$, we can apply the reduction to R', described before.
We define $I = \{i : P^i \neq A^i_\infty\} = \langle i_1, \ldots, i_n \rangle$ ordered according to increasing magnitude. $\Gamma'' = \prod_{i \epsilon I} \Gamma_i$.

To simplify our descriptions we suppose $\langle i_1, \ldots, i_n \rangle = \langle 1, 2, \ldots, n \rangle$. A bi-unique mapping ψ from a species $\mathfrak{B} \subset \mathfrak{P}$ onto $\mathfrak{A}(\Gamma'')$ is defined by $S \epsilon \mathfrak{B} \rightarrow \psi S = \langle \pi_1 S, \ldots, \pi_n S \rangle$, where $\mathfrak{B} = \{P : P \epsilon \mathfrak{A} \ \& \ \wedge i(i > n \rightarrow \pi_i P = A^i_\infty)\}$.
ψ can be extended to joins of elements of $\mathfrak{B}$ such that $\psi(S_1 + S_2) = \psi S_1 + \psi S_2$. We remark (3.4.4(e))

$$S_1, S_2 \epsilon \mathfrak{B} \ \& \ S_1 \in S_2 \leftrightarrow \psi S_1 \in_{\Gamma''} \psi S_2.$$

We have already proved $P, R'_1, \ldots, R'_k$ to belong to $\mathfrak{B}$, hence

$$\psi P \Subset_{\Gamma''} \psi(R'_1 + \ldots + R'_k) = \psi R'.$$

As a consequence of 5. 5 we are able to construct an $R'' \in \mathfrak{P}(\Gamma'')$ (R'' a join of elements of $\mathfrak{A}(\Gamma'')$) such that

$$\psi P \Subset_{\Gamma''} R'' \Subset_{\Gamma''} \psi R'$$

hence $P \Subset \psi^{-1}R'' \Subset R'$.
Therefore N8 holds in Γ, hence in Γ'. This proves our theorem.

5. 7. The most interesting application of the previous theorem is furnished by $\underline{R}^\infty$; $\underline{R}$ satisfies the requirements of 5. 6, so $\underline{R}^\infty$ is a PIN-space.

SAMENVATTING

In hoofdstuk I wordt het begrip topologische ruimte gedefiniëerd en worden vele begrippen en stellingen uit de klassieke topologie, die in de intuitionistische theorie zonder of met geringe wijzigingen kunnen worden overgenomen, opgesomd, veelal zonder bewijs.

In de vierde paragraaf worden de begrippen "metrisch gelocaliseerde puntsoort", "relatief gelocaliseerde puntsoorten", "gelocaliseerde puntsoort" en "gelocaliseerd systeem" ingevoerd.

In hoofdstuk II wordt het begrip metrische ruimte voor het separabele geval besproken; in het bijzonder wordt een intuitionistisch equivalent van de stelling van Lindelöf afgeleid. In de laatste paragraaf van dit hoofdstuk worden de gelocaliseerd compacte ruimten (de "katalogisiert kompakte Raume" uit BROUWER 1926, of de "located compact topological spaces" uit BROUWER 1954) besproken, enkele bekende eigenschappen van deze ruimten opgesomd en enige nieuwe bewezen, die als hulpmiddel optreden in hoofdstuk IV. De behandeling is in hoofdstuk II echter geheel "metrisch".

In hoofdstuk III wordt begonnen met de opbouw van een axiomatische theorie. In §1 worden de I-ruimten geintroduceerd. In §2 worden de zgn. scheidings- en representatie-postulaten en hun consequenties behandeld; de IR-ruimten (analoog aan de klassieke reguliere ruimten met aftelbare basis) worden ingevoerd. §3 bevat de definities van PIN- en CIN-ruimten. Een aantal belangrijke stellingen voor CIN-ruimten (zie 3. 3. 6) gelden als gevolg van de resultaten in §2. CIN-ruimten zijn, klassiek gesproken, volledig metrizeerbare separabele ruimten. Het topologisch product wordt in §4 behandeld, en een aantal belangrijke voorbeelden in §5.

Hoofdstuk IV is gewijd aan de LDFTK-ruimten (analoog aan locaal compacte, separabele metrizeerbare ruimten). In §1 wordt de verbinding tussen de theorie van FREUDENTHAL 1936 (DFTK-ruimten, analoog aan compacta) en de theorie van hoofdstuk III gelegd. §2 bevat een bewijs van de equivalentie van een metrische en een zuiver topologische karakterisering van LDFTK-ruimten. §3 bevat een aantal stellingen over overdekkingen; met behulp van deze stellingen wordt in paragraaf 4 het bestaan van een metriek voor een LDFTK-ruimte bewezen, ten opzichte waarvan elke niet lege gelocaliseerde puntsoort ook metrisch gelocaliseerd is. §5 behandelt het topologisch product van aftelbaar oneindig veel LDFTK-ruimten. Zo blijkt, dat $\underline{R}^{\infty}$ een PIN-ruimte is.

STELLINGEN

I

Het begrip "$\subset$-Ueberdeckung", door Freudenthal ingevoerd, is geen topologisch begrip.

H. Freudenthal, Zum intuitionistischen Raumbegriff, Compositio Math. 4 (1936) blz. 83.

II

Als A een begrensde gelocaliseerde puntsoort in de euclidische n-dimensionale ruimte is, en het complement van A is eveneens gelocaliseerd, dan heeft A een gelocaliseerde rand.

III

Laat F, F' twee lineair recurrente rijen zijn, met resp. $\psi(x)=0$, $\psi'(x)=0$ als karakteristieke vergelijkingen van minimale graad. Dan is $\psi'(x)\ \psi(x)=0$ een karakteristieke vergelijking voor het Cauchy-product F" van F en F'. Is a m-, resp. n-voudige wortel van $\psi(x)=0$, resp. $\psi'(x)=0$ $(m, n > 0)$ dan is a (m+n)-voudige wortel van een karakteristieke vergelijking van minimale graad voor F".

IV

De uitspraak van Rasiowa en Sikorski:
"It is difficult for mathematicians to understand exactly the ideas of intuitionists since the degree of precision in the formulation of intuitionistic ideas is far from the degree of precision to which mathematicians are accustomed in their daily work" doet het intuitionisme geen recht wedervaren.

H. Rasiowa and R. Sikorski, The mathematics of metamathematics, Warszawa 1963, blz. 378.

V

Het voorbeeld dat G. Kreisel geeft om aan te tonen dat voor toepassingen een constructief bewijs niet relevant is, een constructief resultaat wel, is geen goede illustratie van zijn bewering.

G. Kreisel, Interpretation of analysis by means of constructive functionals of finite types, in: Constructivity in mathematics, Amsterdam 1959, blz. 101.

VI

Laat R een commutatieve ring met eenheidselement zijn, en laat $R' \subset R[x_1, \ldots, x_n]$ bestaan uit polynomen, invariant t.o.v. even permutaties van de variabelen. Dan vormen de elementair-symmetrische functies tezamen met $\Sigma x_1^0 x_2^1 x_3^2 \ldots x_n^{n-1}$ (sommatie over alle termen die door een even permutatie van de indices uit $x_1^0 x_2^1 \ldots x_n^{n-1}$ ontstaan) een integriteitsbasis voor R'.

E. Noether, Kőrper und Systeme rationaler Funktionen, Math. Annalen 76 (1915), blz. 183.

VII

Laat R een commutatieve ring met eenheidsselement zijn, en laat $R' \subset R[x_1, \ldots, x_n]$ eveneens een ring met eenheidselement zijn. We definiëren een minimale homogene integriteitsbasis als een integriteitsbasis, bestaande uit homogene polynomen, die niet verkleind kan worden. Het aantal polynomen van elke graad in een minimale homogene integriteitsbasis voor R' ligt eenduidig vast.

E. Noether, Kőrper und Systeme rationaler Funktionen, Math. Annalen 76 (1915), blz. 183.

VIII

De vier polynomen die Masuda aangeeft voor een algebraisch onafhankelijke basis voor L_4 over het grondlichaam k vormen in tegenstelling tot zijn bewering geen integriteitsbasis voor $L_4 \cap k[x_1, \ldots, x_4]$.

K. Masuda, On a problem of Chevalley, Nagoya Math. J. 8 (1955), blz. 63.

IX

Het probleem door G. Birkhoff opgeworpen in de zin:
"It would seem worthwhile to construct propositional calculi based on non-distributive lattices of truth-values" is een schijnprobleem.

G. Birkhoff, Lattice theory, Providence, Rh. I., 1948, blz. 197.

X

Bij de behandeling van de lineaire algebra verdient een meer specifieke term zoals bijv. "lineaire afbeelding" de voorkeur boven het algemene "morfisme".

XI

Als we definiëren: "Een DFTK-ruimte heeft dimensie $\leqslant n$, als er voor elke ε een eindige ε-overdekking met orde n+1 bestaat", dan geldt intuitionistisch de volgende stelling: Een DFTK-ruimte met dimensie $\leqslant n$ kan homeomorf in de euclidische (2n+1)-dimensionale ruimte ingebed worden.

XII

Intuitionistisch geldt: zijn de elementen van een eindige of aftelbaar oneindige overdekking van een volledige metrische ruimte paarsgewijs disjunct, dan zijn ze gesloten.

XIII

Voor DFTK-ruimten zijn de volgende eigenschappen equivalent.

(A) Elke eindige overdekking door soorten die elk minstens een punt bevatten kan in een keten gerangschikt worden.
(B) De enige afsplitsbare deelsoorten van de ruimte zijn de lege soort en de gehele ruimte.

XIV

Het verdient aanbeveling in nog sterkere mate dan thans het geval is, voor de doctoraalstudie wiskunde tentamens te vervangen door zelfstandig literatuuronderzoek, het maken van kleine scripties, het houden van korte voordrachten en het oplossen van eenvoudige research-problemen.

XV

"Floravervalsing" in botanische reservaten is niet altijd verwerpelijk.

A.S. Troelstra, 15 juni 1966.