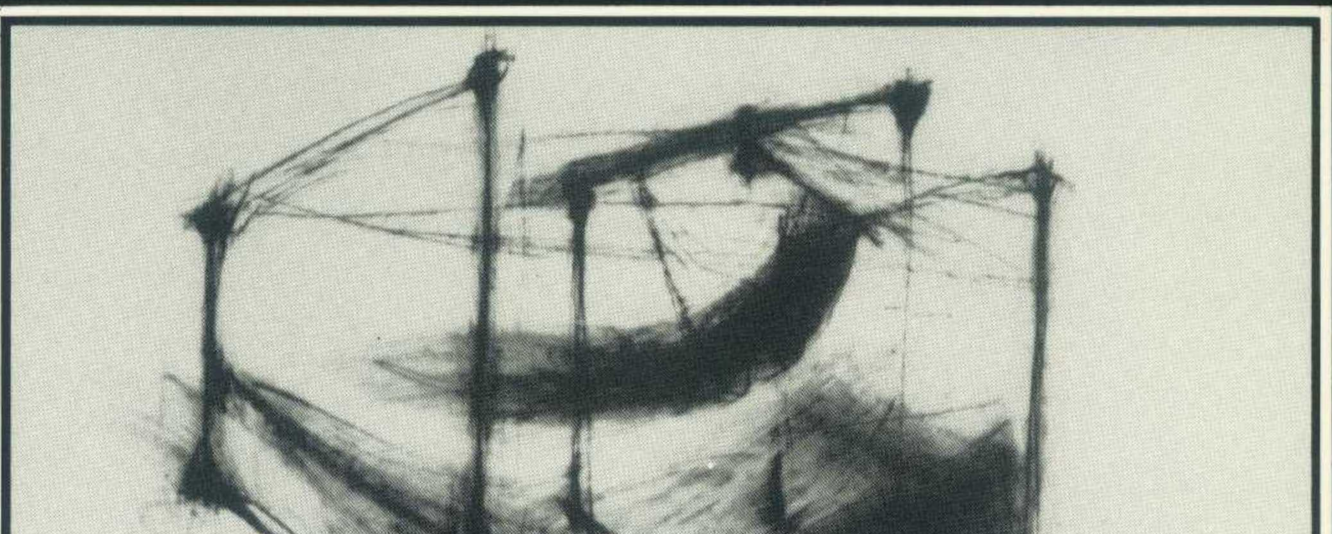
MANY-DIMENSIONAL MODAL LOGIC



YDE VENEMA

foar myn heit
en foar myn mem

CONTENTS.

CHAPTER 1. DIMENSIONS IN MODAL LOGIC. 1

CHAPTER 2. RULES FOR THE UNDEFINABLE. 5

2.1. Introduction 6
2.2. Sahlqvist theorems 9
2.3. Sahlqvist tense formulas 11
2.4. The D-operator 16
2.5. The Main Proof 19
2.6. Uni-directional complications 26
2.7. The SD-theorem 31
2.8. The SNΞ-theorem 36
2.9. Conclusions, remarks and questions 38

CHAPTER 3. THE SQUARE UNIVERSE. 43

3.1. Introduction 44
3.2. Two-dimensional cylindric logic 48
3.3. A modal logic of binary relations 54
3.4. A two-dimensional temporal logic 70
3.5. Two-dimensional algebras 74
3.6. Conclusions, remarks and questions 78

CHAPTER 4. QUANTIFIERS AND CUBES. 81

4.1. Algebraizing restricted first order logic 82
4.2. Cylindric modal logic 88
4.3. Characterizing n-cubes 93
4.4. Axiomatizing n-cubes 97
4.5. Harvest 99
4.6. Using the D_n-irreflexivity rule 102
4.7. Conclusions, remarks and questions 108

CHAPTER 5. PERIODS IN PLANES. 113

5.1. Introduction 114
5.2. The system HS 119
5.3. Intervals as computation paths 129
5.4. A modal operator for chopping intervals 133
5.5. Conclusions, remarks and questions 137

Chapter 6. Conclusions. 139

Appendix A. Modal Similarity Types. 143

Appendix B. Consequences of Derivation Systems. 157

Appendix C. Index. 161

Appendix D. Samenvatting. 166

Bibliography. 169

Dankwoord/Acknowledgements. 178

CHAPTER 1.

DIMENSIONS IN MODAL LOGIC.

Outline.

The following pages form the introduction to the dissertation.

We start with an informal definition of what we understand by the term "many-dimensional modal logic"; we announce the kind of questions that will be raised, and we discuss the main themes running through the dissertation.

MANY-DIMENSIONAL MODAL LOGIC.

The *subject* of this dissertation is many-dimensional modal logic, as stated by the title. Treating a branch of modal logic, we assume some familiarity with the tree itself, in particular with its Kripke semantics: whenever we introduce a formalism of (not necessarily monadic) modal operators, we will assume that implicitly, Kripke frames for this language are defined, in the form of pairs (W, I) with W a universe of possible worlds and I a function associating an accessiblity relation with each operator.
We mentioned the Kripke semantics for modal languages here, because we need it to clim up the particular branch that we will study and give an informal definition of many-dimensional modal logic: we call a modal formalism *α-dimensional* (for some ordinal α), if in the intended semantics, the universe of a frame consists of (a subset of) α-tupels over a more basic set. This means that we have *two kinds* of semantics for such logics: the wide class of Kripke frames, and the narrower, intended class, where frames will be called *cubes*, or *squares* in the two-dimensional case.

Many-dimensional modal logics have been studied in the literature, for many different reasons and from many different backgrounds, ranging from linguistics via philosophy and mathematics to computer science; in section 3.1 we give an overview, for the two-dimensional case. As these formalisms have a lot in common from a technical point of view, one aim of this dissertation is to study many-dimensional modal logics in a systematic way. This uniform approach shows the following questions to be the central ones:

expressiveness: in particular, what is the relation between many-dimensional modal logic and classical logic?
definability: (how) can we characterize the cubes among the Kripke frames?
completeness: (how) can we axiomatize the formulas valid in the cubes?

The reader will recognize these questions as the traditional ones in modal logic. In the context of many-dimensional modal logic however, they have a quite characteristic flavour. Apart from this, through the branch of many-dimensional modal logic more vessels are running, vessels that have a scope reaching far beyond their technical interest. In fact, the results in this dissertation could have been perceived from at least three different perspectives, each of which would have lead to a different title:

EXTENDED CORRESPONDENCE THEORY

Correspondence theory studies the relation between modal formalisms and classical first order logic as languages for Kripke frames and models. The main references to correspondence theory are van Benthem [13, 14].
The two kinds of semantics for many-dimensional modal logics give rise to two kinds of correspondence theory:
Concerning the 'intended semantics', we will concentrate on the models. As the modal propositional variables are assigned a subset of tupels over the base set, α-dimensional modal languages can be compared with first order signatures having arbitrarily many α-adic relation symbols (note that in one-dimensional modal logic, only the accessibility

relations may have rank > 1). This increase in expressive power, which can go as far as first order logic itself, is in fact the main reason for the introduction of many-dimensional modal logics.

For the correspondence theory of the 'unstructured' Kripke semantics, we will mainly look at the frames. There is nothing new under the sun here: it is well-known that in general, on the frame level modal formalisms correspond to an extension of the first order calculus where universal quantifications are allowed over subsets of the universe. In many cases the second order correspondent of a formula can be replaced by a first order equivalent. Sahlqvist's theorem gives a wide set of formulas having this nice property (and others), and will an inportant instrument to us.

MODAL DERIVATION RULES.

In all many-dimensional modal formalism going beyond a certain degree of simplicity, it turned out that the task of giving a transparent axiomatization of cube validity was by no means simple. For a two-dimensional temporal logic, Gabbay [31] introduced a new kind of derivation rule, the so-called irreflexivity rule. Adding this rule to an orthodox, finite derivation system, he obtained a relatively simple completeness result. This idea was followed by many authors working on modal formalisms with a more-dimensional flavour.
In isolation, such a rule often leaves the impression to act like a deus ex machina. There is an obvious generalization however, which is already treated by Gabbay [31]. Another aim of this dissertation is to make this general concept more explicit and to prove an abstract result about rules like the irreflexivity rule.

A MODAL PERSPECTIVE ON ALGEBRAIC LOGIC

An important area in the theory of modal logic is formed by the duality theory between Relational Kripke Frames and Boolean Algebras with Operators. For an overview of duality theory, which started with Jónsson and Tarski [60] and thus actually precedes Kripke semantics, we refer to Goldblatt [43].)
Duality theory constitutes a bridge between modal logic and algebraic logic. The latter can be seen as a research program aiming at extending the succesfull treatment of propositional logic in *Boolean* Algebras, to more complex logical formalisms. An excellent overviewing introduction to algebraic logic is Németi [89].
Now the usual direction taken in duality theory is to apply (universal) algebra to (modal) logic, the outstanding example being Goldblatt and Thomason's characterization of modal definability. Our direction is the converse: we will take a possible world point of view on (more-dimensional) modal algebras. In this way we will introduce two novelties to algebraic logic: Sahlqvist's theory and modal derivation rules.
There are many more-dimensional modal formalisms; we have chosen to treat those, in which we encounter the most intensively studied logical algebras, viz. relation algebras and cylindric algebras, as modal algebras.
Viewed from a different angle, this is an essay on algebraic logic.

THE TEMPORAL DIMENSION.

There is one aspect of many-dimensional modal logic that we have not mentioned yet, namely the connection with temporal logic. Our standard references to the logic of time are van Benthem [12] and Gabbay [35].
In fact the most extensive line of papers in many-dimensional modal logic sprang from the wish to devise a logical formalism for the concept of time that is more adequate than the usual one-dimensional temporal languages.
From a technical point of view, the temporal aspect can be incorporated in the general framework without too many problems. However, there are some questions that are particularly interesting in the context of temporal logic; we mention *expressive completeness* (a finite set of modal operators having, in some sense, the same expressive power as first order logic).

Besides that, the modal logic of time *periods* has a special relation with many-dimensional modal logic — actually, at the foundation of this dissertation laid the following observation: if we identify an interval with the pair consisting of its beginning and its end point, then the universe of intervals is a subset of the Cartesian square, viz. the north-western halfplane (all pairs having a first coordinate smaller than the second one). In this perspective, the modal logic of intervals meets our definition of many-dimensional modal logic.

SURVEY.

The first chapter after this introduction is devoted to a discussion on modal derivation rules, as our results in this area underly all other material.
Various two-dimensional logics are treated in the third chapter, which is the central one of the dissertation.
In chapter 4 we unfold our modal perspective on cylindric algebras and related notions.
The fifth chapter deals with the modal logic of intervals.
In the conclusions we will briefly return to the issues raised in this introduction.
The dissertation contains two appendices: in the first one we summarize the background knowledge presupposed for reading this work; in the second one we defend our perspective on the consequence relation ($\Sigma \models \phi$).
Furthermore, the booklet contains an index, a bibliography, a summary in Dutch and acknowledgements.
For a more detailed outline we refer to the title pages of the various chapters.

CHAPTER 2.

RULES FOR THE UNDEFINABLE.

Outline.

In this chapter we prove a meta-theorem on completeness, for modal axiom systems having unorthodox derivation rules styled after Gabbay's Irreflexivity Rule.

In the introduction we sketch the problem, defining the notions of a non-ξ rule and the class of frames that it characterizes. Section 2 gives some preliminary facts on the Sahlqvist theorem and its algebraic meaning. In section 3 we define the formulas that our meta-theorem admits as axioms, and an alternative version of canonical structures; we prove a persistence result relating these definitions. For a nice formulation of our result, we need the difference operator: this matter is dealt with in section 4. The sections 5, 7 and 8 are devoted to the proof of the main theorem: section 5 treats the simplest case, where all operators are diamonds and the only non-ξ rule is D-irreflexivity. Later we extend the result to similarity types with polyadic operators (section 7) and axiom systems with arbitrary many non-ξ rules (section 8). In section 6 we discuss why tense diamonds behave better than uni-directional ones. In the last section 9, we draw our conclusions and mention some questions for further research.

2.1 Introduction.

Let us for the moment consider the simplest tense similarity type with two operators F and P. It is well-known that the logic K^t4, being the extension of the basic tense logic K^t with the axiom (4): $FFp \rightarrow Fp$, completely axiomatizes the class Fr^t_4 of transitive tense frames, i.e. frames $\mathfrak{F} = (W, R_F)$ where R_F is transitive (and $R_P = (R_F)^{-1}$). Adding the axiom (T): $Gp \rightarrow p$ then gives a complete axiomatization of the class Fr^t_{4T} of *reflexive* transitive tense frames.
Now suppose we want to axiomatize the class Fr^t_{i4} of *irreflexive* transitive tense frames. There is no modal or tense formula corresponding to irreflexivity in the same manner as (4) and (T) correspond to resp. transitivity and reflexivity. So for Fr^t_{i4} it is (in principle) less clear how to find an axiomatization than for Fr^t_4 or Fr^t_{4T}. The usual procedure, establishing the completeness of K^t4 itself[1] for Fr^t_{i4}, consists of starting with some model $\mathfrak{M}$ for a consistent set of formulas Σ and then transforming $\mathfrak{M}$ into an *irreflexive* model $\mathfrak{M}'$ for Σ.
A different road was taken by Gabbay in [31], where he suggested to add (to a similar logic) a special derivation rule, which he baptized the *irreflexivity rule*. This rule can be formulated as follows:

$$(IR) \qquad \vdash \neg(Gp \rightarrow p) \rightarrow \phi \;\Rightarrow\; \vdash \phi, \text{ if } p \notin \phi.$$

Gabbay's completeness proof then consists of constructing a tranisitive irreflexive model right away, without passing models that may be bad in the sense that they have reflexive points.
This idea was followed by many authors who wanted to give axiomatizations for classes of frames defined by conditions which are not directly expressible in the intensional language. Examples include Burgess [23], Zanardo [140] for branching-time temporal logics, Kuhn [68] and Venema [131, 130, 132, 133] for many-dimensional modal logics (of intervals), and Gabbay and Hodkinson [36], de Rijke [104], Roorda [107]. There is an independent Bulgarian line of papers: Passy-Tinchev [95], Gargov and Goranko [39], Gargov, Passy and Tinchev [41] where similar rules are used in a context of enriched modal formalisms. Finally, in the first order temporal logic of program verifications there is a related concept called 'clock rule' (cf. Sain [117], Andréka, Németi and Sain [10] and the references therein). Gabbay [35] contains a lot of new material concerning the irreflexivity and related rules, for example giving a general procedure to find axiomatizations for *any* first order definable temporal connective, over the class of linear orders.

So the question naturally arises whether anything general can be said about logics having rules like the irreflexivity rule. Let us first have a closer look at IR; we suggest to concentrate on the 'converse' statement, i.e.

If ϕ is consistent and does not use p,
then $\phi \wedge \neg(Gp \rightarrow p)$ is consistent.

[1]In this sense the example is not representative: For K^t4, the irreflexivity rule is *conservative* (cf. section 7).

In other words, to a consistent formula ϕ we may always add a conjunct of the form $\neg(Gp \to p)$ *witnessing* irreflexivity.

More general, for an arbitrary similarity type, we set

Definition 2.1.1.
Let ξ be a modal formula in the proposition letters $p_0, \ldots, p_{n-1}$. For a class K of frames, set $\mathsf{K}_{-\xi}$ as the class of *non-ξ frames* in K, i.e. the frames $\mathfrak{F} = (W, I)$ in K such that for no world w in W, $\mathfrak{F}, w \models \xi$. For Ξ a set of formulas, $\mathsf{K}_{-\Xi}$ is the intersection of the $\mathsf{K}_{-\xi}$, $\xi \in \Xi$. ⊞

Recall that for a formula ϕ, Fr_ϕ is defined as the class of frames where ϕ is valid. Note that in general, the following three classes of frames, all defined using the negation of ξ, are *distinct*:

(i) $\mathsf{Fr}_{\neg\xi}$ (i.e. the class of frames with $F \models \neg\xi$),
(ii) $\overline{\mathsf{Fr}_\xi}$ (i.e. the complement of Fr_ξ),
(iii) $\mathsf{Fr}_{-\xi}$.

For, $\mathfrak{F}$ is in $\mathsf{Fr}_{\neg\xi}$ iff *for all* valuations V and *all* worlds w, $\mathfrak{F}, V, w \models \neg\xi$; $\mathfrak{F}$ is in the second class iff *there are* a valuation V and a world w with $\mathfrak{F}, V, w \models \neg\xi$, and $\mathfrak{F} \in \mathsf{Fr}_{-\xi}$ means that *for every* world w *there is* a valuation V with $\mathfrak{F}, V, w \models \neg\xi$.
This means, so to speak, that $-\xi$ 'corresponds' to the second order formula

$$\forall x_1 \exists P_0 \ldots P_n \neg\xi^1(x_0),$$

where $\xi^1(x_0)$ is the local model correspondent of ξ, every monadic predicate P_i being the first order counterpart of the propositional variable p_i in ξ. Thus we are studying classes of frames that are definable in a version of second order logic where we have a restricted possibility to use existential quantification over monadic predicates.

As an example, consider the formula $\xi = Gp \to Pp$ which is locally equivalent on the frame level to $\exists y(Rxy \wedge R^{-1}xy)$. So $\mathsf{Fr}_{-\xi}$ is the class of frames $\mathfrak{F}$ with $\mathfrak{F} \models \forall x \forall y(Rxy \to \neg R^{-1}xy)$ i.e. the class of *asymmetric* frames, while $\overline{\mathsf{Fr}_\xi}$ is the class of frames with $\mathfrak{F} \models \exists x \forall y(Rxy \to \neg R^{-1}xy)$. The negation $Gp \wedge H\neg p$ of ξ can be shown to be globally equivalent to the formula $\neg\exists x \exists y Rxy$, so $\mathsf{Fr}_{\neg\xi}$ finally is the class of frames with empty R. As another example, one can show $\mathsf{Fr}_{-(Gp \to FFp)}$ to be the class of *intransitive* frames. In these two examples the second order definition of $\mathsf{Fr}_{-\xi}$ can be replaced by a first order one, but this need not always be the case.

Now suppose we want to axiomatize the logic $\Theta(\mathsf{Fr}_{-\xi})$ consisting of all formulas valid in $\mathsf{Fr}_{-\xi}$. Let ϕ be a $\Theta(\mathsf{Fr}_{-\xi})$-consistent formula, then there is a model $\mathfrak{M} = (\mathfrak{F}, V)$ such that $\mathfrak{F}$ is in $\mathsf{Fr}_{-\xi}$ and with a world w in $\mathfrak{M}$ where $\mathfrak{M}, w \models \phi$. Let $p_0, \ldots, p_{n-1}$ be *new* propositional variables, in the sense that they are not elements of $Dom(V)$. As $\mathfrak{F}, w \not\models \xi$, there is a valuation V' such that $\mathfrak{F}, V', w \models \neg\xi(p_0, \ldots, p_{n-1})$. Now let V'' be defined by

$$\begin{array}{rcll} V''(q) & = & V(q) & \text{if } q \in Dom(V) \\ V''(p_i) & = & V'(p_i) & \text{for } i = 0, \ldots, n-1. \end{array}$$

then clearly we have $(\mathfrak{F}, V''), w \models \phi \wedge \neg\xi$.

This means that

$$\phi \wedge \neg\xi(p_0, \ldots, p_{n-1}) \text{ is } \Theta(\mathsf{Fr}_{-\xi})\text{-consistent if } \phi \text{ is } \Theta(\mathsf{Fr}_{-\xi})\text{-consistent and none of the } p_i \text{ occurs in } \phi.$$

Taking the converse again of the above proposition, we have a formulation of the $\neg\xi$-consistency rule. (This rule is called the $I\xi$-rule in Gabbay [35].)

Definition 2.1.2.
Let $\xi(p_0, \ldots, p_{n-1})$ be a modal formula. The $\neg\xi$- *consistency rule*, or shorter: the *non-ξ rule* is the following derivation rule:

$$(N\xi R) \qquad \vdash \neg\xi(p_0, \ldots, p_{n-1}) \rightarrow \phi \ \Rightarrow \vdash \phi, \ \text{ if } \vec{p} \notin \phi.$$

If Λ is a derivation system and ξ a formula (Ξ a set of formulas), then $\Lambda(-\xi)$ ($\Lambda(-\Xi)$) denotes the system Λ extended with the non-ξ rule (all non-ξ rules, $\xi \in \Xi$). ⊞

The paragraph above definition 2.1.2 can be seen as a proof of the *soundness* of $N\xi R$ with respect to $\mathsf{Fr}_{-\xi}$: if $\mathsf{Fr}_{-\xi} \models \neg\xi(p_0, \ldots, p_{n-1}) \rightarrow \phi$ and no p_i occurs in ϕ, then $\mathsf{Fr}_{-\xi} \models \phi$.
The aim however is of course to try and show *completeness* for non-ξ rules; this will be the main subject of this chapter. We should note at this moment that in general we do not have an isolated $N\xi R$ added to a minimal (tense) logic, but a situation in which we add possibly more than one $N\xi R$ to a logic having other axioms besides the basics.
So the general question, raised by Gabbay [31, 35] is the following: we have a similarity type S, an S-logic Λ which is (strongly) sound and complete with respect to a class of frames K, and a set of formulas Ξ. The question now is the following

Is $\Lambda(-\Xi)$ strongly complete with respect to $\mathsf{K}_{-\Xi}$?

Gabbay proves a generalized irreflexivity lemma stating that a $\Lambda(-\xi)$-consistent set Σ of formulas has a model $\mathfrak{M}$ with $\mathfrak{M} \models \Theta(\mathsf{Fr}_{\Lambda,-\Xi})$. Unfortunately, this is not enough to prove completeness, for we have to find a model $\mathfrak{M}$ such that the underlying *frame* is in $\mathsf{Fr}_{-\Xi}$.
In general this seems to be difficult and maybe even impossible to establish. Therefor we concentrate on logics with a special, nice kind of axioms, viz. so-called Sahlqvist formulas. For some of these logics we can get a positive answer to the above question. The answer we obtain is partial because our proof method will turn out to be highly sensitive to the similarity type of the logic. In particular, and maybe surprisingly, there is a striking difference in our approach between *tense* similarity types (i.e. where the language has a 'converse' operator for each of its diamonds) and uni-directional ones (where no operator has a converse).
Furthermore, we feel our proofs become more perspicuous if we add a special operator, the so-called *difference operator*, to the language. We would like to stress the point, that in many applications (in fact for *all* logics discussed here), this will turn out to be only an apparent extension of the language because the operator is *definable* in the old language, at least over the class of frames that we want to axiomatize.

2.2 Sahlqvist Theorems.

It is well-known that on the level of frames every formula ϕ locally and globally has a second order equivalent ϕ^2. In many important cases however, it turns out that this formula ϕ^2 has a much simpler first order equivalent (in the corresponding frame language L_S). Well-known examples include reflexivity for $p \to \Diamond p$ and the Church-Rosser property for $\Diamond\Box p \to \Box\Diamond p$. A general theorem in this direction was found by Sahlqvist (cf. [111]). The *correspondence* part of Sahlqvist's theorem gives a decidable set of modal S-formulas having a local equivalent in L_S. In [14], van Benthem provides a quite perspicuous algorithm to find this first order correspondent ϕ^s of a Sahlqvist formula ϕ. (At the end of section 3, we will give our version of this *substitution method.*) The second, *completeness* part of the Sahlqvist theorem states that adding a set Σ of Sahlqvist axioms to the minimal S-logic K_S, we obtain a complete axiomatization for the class of frames Fr_Σ. An accessible version of the proof of this part can be found in Sambin-Vaccaro [118], from which we took some terminology. The correspondence and completeness part of Sahlqvist's theorem are closely connected; in Kracht [66] they are studied in a unifying framework.

Definition 2.2.1
A *strongly positive formula* is a conjunction of formulas $\Box_1 \dots \Box_m p_i$ $(m \geq 0)$. A formula is *positive* (*negative*) if every propositional variable occurs under an *even (odd)* number of negation symbols. A modal formula is *untied* if it is obtained from strongly positive formulas and negative ones by applying only $\wedge$ and arbitrary existential modal operators. Formulas of the form $\phi \to \psi$ with ϕ a tense formula and ψ a positive one, are called *Sahlqvist* formulas[2]. ⊞

Theorem 2.2.2. SAHLQVIST.
Let σ be a Sahlqvist formula. Then

(i) σ is canonical: $\mathfrak{F}^c_{K_S\sigma} \models \sigma$.
(ii) $K_S\sigma$ is strongly sound and complete with respect to Fr_σ.
(iii) There is an effectively obtainable L_S-formula $\sigma^s(x_0)$ such that for all $\mathfrak{F}$ in Fr, w in $\mathfrak{F}$:

$$\mathfrak{F}, w \models \sigma \iff \mathfrak{F} \models \sigma^s[x_0 \mapsto w].$$

Proof.
For **(i)** we refer to Sambin-Vaccaro [118]; **(ii)** is immediate by **(i)**. The last part **(iii)** will be proved in the next section, where we will also give the algorithm to find $\sigma^s(x_0)$. ⊞

A typical example of a formula which is *not* Sahlqvist, is $\Box\Diamond p \to \Diamond\Box p$. A typical example of a Sahlqvist formula is $\Diamond\Box p \to \Box\Diamond p$; its first order correspondent is $\forall y_0 y_1((Rxy_0 \wedge Rxy_1) \to \exists z(Ry_0z \wedge Ry_1z))$.

The remainder of this section, which is the result of joint work with Maarten de Rijke, is

[2] In fact, we may even consider the wider set of formulas obtained from (basic) Sahlqvist formulas by applying *duals* of existential modal operators.

not needed for understanding the rest of Chapter 2.

Although the Sahlqvist theorem is a very nice and important instrument in proving modal completeness, it seems not to belong to the standard luggage of modal logicians. It will then hardly come as a surprise that the result is virtually unknown in algebraic logic. Quite deplorably so, because it has very interesting consequences for the theory of Boolean Algebras with operators, consequences which are almost trivial to obtain, by a suitable arrangement of known results.

Definition 2.2.3.
Let S be a modal similarity type. All notions defined in 2.2.1 apply to (algebraic) S-terms as well as to (modal) S-formulas, cf. Appendix A11. A *Sahlqvist equation* is an equation of the form[3] $r \leq t$ where r is an untied term and t a positive one. A *Sahlqvist variety* is a variety axiomatized by Sahlqvist equations. The *Sahlqvist correspondent* of a Sahlqvist equation η: $r \leq t$ is given as the L_S-formula $\forall x_0 \sigma^s$ where σ is the *underlying modal formula* $(r \rightarrow t)$ of η. ⊞

These equations are abundant in algebraic logic: *all* axioms of cylindric and relation algebras are Sahlqvist equations (or have such equivalents). For example, we can rewrite the RA-axiom $x^{\smile}; -(x;y) \leq -y$ as $x^{\smile}; -(x;y) \wedge y \leq 0$[4]. The Sahlqvist correspondent of the Sahlqvist equation η is the universal closure of the Sahlqvist correspondent of the underlying modal formula of η: the correspondent of the 'Church-Rosser equation' $\diamond - \diamond - x \leq -\diamond - \diamond x$ is $\forall x y_0 y_1((Rxy_0 \wedge Rxy_1) \rightarrow \exists z(Ry_0 z \wedge Ry_1 z))$.

Our main result about Sahlqvist equations is that they are preserved under taking *canonical embedding algebras*. This theorem forms a considerable strengthening of a result by Henkin, Monk and Tarski in [53]: they call a term *positive in the wider sense* if no subterm beginning with the negation symbol contains a variable. A *positive equation in the wider sense* is of the form $r = t$ with r and t positive terms. In section 2.7 of [53] the authors prove that positive equations are preserved under taking canonical embedding algebras. It is straightforward to verify that splitting up a positive equation $r = t$ into $r \leq t$ and $t \leq r$, we obtain two Sahlqvist equations. On the other hand, there are many Sahlqvist equations that are not positive, for example the 'Church-Rosser equation' mentioned above.
The following result is immediate by 2.2.2 and the definitions:

Theorem 2.2.4.
Let S be a modal similarity type, $\mathfrak{F}$ a frame and η a Sahlqvist equation. Then

$$\mathfrak{Cm}\mathfrak{F} \models \eta \iff \mathfrak{F} \models \eta^s.$$

Theorem 2.2.5.
Let $\mathfrak{A}$ be a Boolean Algebra with Operators and η a Sahlqvist equation. Then

$$\mathfrak{A} \models \eta \iff \mathfrak{Em}\mathfrak{A} \models \eta.$$

[3]An equivalent definition, which is perhaps more perspicuous from the algebraic point of view, is: a *Sahlqvist equation* is of the form $r = 0$, where r is an *untied term*.

[4]This important observation was made by Johan van Benthem, cf. section 3.3.5.

Proof.
Let η be valid in $\mathfrak{A}$; assume η is of the form $\phi \leq \psi$. Set $\mathfrak{A}_\eta$ as the free algebra (of suitable cardinality) over the variety V_η. Let $K_S\eta$ be the extension of the minimal S-logic K_S with the axiom $\phi \to \psi$, and set $\mathfrak{F}_\eta$ as the canonical frame of this logic, then $\mathfrak{F}_\eta = \mathfrak{Cs}\mathfrak{A} - \eta$ by A39.
Now by canonicity of Sahlqvist formulas, cf. Theorem 2.2.2, $\mathfrak{F}_\eta \models \phi \to \psi$. This implies $\mathfrak{Cm}\mathfrak{F}_\eta \models \eta$, and as $\mathfrak{Cm}\mathfrak{F}_\eta = \mathfrak{Em}\mathfrak{A}_\eta$, the theorem follows by the observation (cf. Goldblatt [43], 3.2.5(6)) that $\mathfrak{Em}\mathfrak{A}$ is a homomorphic image of $\mathfrak{Em}\mathfrak{A}_\eta$. ⊞

Maybe the nicest aspect of the above theorem is that it frees us from giving tedious algebraic derivations for Sahlqvist equations, allowing us to focus on reasoning in *atom structures*. The following example of this feature will be used in chapter 4:

Corollary 2.2.6.
Let V be a Sahlqvist variety and η_1, η_2 two Sahlqvist equations. Then

$$\mathsf{AtV} \models \eta_1^s \leftrightarrow \eta_2^s \iff \mathsf{V} \models \eta_1 \leftrightarrow \eta_2.$$

Proof.
($\Rightarrow$) Let $\mathfrak{A}$ be an algebra in V with $\mathfrak{A} \models \eta_i$. By Theorem 2.2.5, η_i holds in $\mathfrak{Em}\mathfrak{A} = \mathfrak{Cm}\mathfrak{Cs}\mathfrak{A}$. So by Theorem 2.2.4 η_i^s is valid in the canonical structure $\mathfrak{Cs}\mathfrak{A}$. By assumption then, η_j^s is valid in $\mathfrak{Cs}\mathfrak{A}$ as well. But then again $\mathfrak{Em}\mathfrak{A} \models \eta_j$, so η_j holds in $\mathfrak{A}$ as $\mathfrak{A}$ is a subalgebra of $\mathfrak{Em}\mathfrak{A}$.
($\Leftarrow$) Let $\mathfrak{F}$ be a frame in AtV with $\mathfrak{F} \models \eta_i^s$. Then $\mathfrak{Cm}\mathfrak{F} \models \eta_i \Rightarrow \mathfrak{Cm}\mathfrak{F} \models \eta_j \Rightarrow \mathfrak{F} \models \eta_j^s$. ⊞

2.3 Sahlqvist tense formulas.

In the previous section we saw that a Sahlqvist formula is *canonical*: if it holds in a canonical model, then it is valid on *all* models on the underlying canonical frame. In this chapter we develop and use non-standard notions of canonical structures, for which we have to adapt the proof of the Sahlqvist theorem. In fact we will show that van Benthem's substitution method (which deals with Kripke frames) also works for the following class of *general* frames:

Definition 2.3.1.
A general frame $\mathfrak{G} = (\mathfrak{F}, A)$ is *discrete* if for all worlds w in $\mathfrak{F}$, $\{w\} \in A$. ⊞

A crucial distinction will be made among the *diamonds* of the similarity type, between the uni-directional ones and those of which the converse diamond also belongs to S, cf. Appendix A.40.

Definition 2.3.2.
A *Sahlqvist tense formula*, or shortly: an *St-formula* is a Sahlqvist formula satisfying the extra constraint that all boxes occurring in strongly positive formulas are *tense* boxes. ⊞

As an example of a Sahlqvist formula which is not an St-formula, we can take our Church-Rosser formula $\Diamond\Box p \rightarrow \Box\Diamond p$ (at least, if $\Diamond$ is not a tense diamond). Our 'tense axiom' $p \rightarrow \Box^{-1}\Diamond p$ itself is an St-formula. Note that in a tense similarity type, there is no distinction between Sahlqvist formulas and St-formulas.

The theorem that we need is the following:

Theorem 2.3.3.
Let $\mathfrak{G} = (\mathfrak{F}, A)$ be a discrete general tense frame and σ a Sahlqvist tense formula such that $\mathfrak{G} \models \sigma$. Then $\mathfrak{F} \models \sigma$.

The remainder of this section is devoted to prove Theorem 2.3.3; as a side result, we can give an easy formulation of the algorithm producing the first order correspondent of a Sahlqvist formula.

The definition of Sahlqvist formulas is a syntactic one, but in fact the important constraint on the consequent is a semantic one, viz. monotonicity:

Definition 2.3.4.
Let V and V' be two valuations on a frame $\mathfrak{F}$. V' is *wider than* V, notation: $V \leq V'$, if for all atoms p, $V(p) \subseteq V'(p)$. A modal formula ϕ is *monotone* if for all $\mathfrak{F}, V, V'$ and w:

$$\mathfrak{F}, V, w \models \phi \text{ and } V \leq V' \text{ imply } \mathfrak{F}, V', w \models \phi$$ ⊞

We also need related concepts for the first order model-language.

Definition 2.3.5.
Let Q be the set of propositional variables of the language. Recall that $L_{S,Q}$ denotes the first order language with S-accessibility predicates and a monadic predicate P_i for every propositional variable $p_i \in Q$. The *sign* of an occurrence of a predicate T in a formula ϕ is defined by induction to ϕ: T occurs positively in the atomic formula $Tx_0 \ldots x_{n-1}$. If T occurs positively (negatively) in ϕ, then it occurs negatively (positively) in $\neg\phi$, and positively (negatively) in $\phi \vee \psi$ and $\exists x\phi$. An $L_{S,Q}$-formula is *positive (negative)* if all occurrences of Q-predicates are positive (negative).
An $L_{S,Q}$-formula $\phi(x_1, \ldots, x_n)$ is *monotone* if for all valuations V, V' and all n-tupels $w_1, \ldots, w_n$:

$$\mathfrak{F}, V \models \phi[w_1, \ldots, w_n] \text{ and } V \leq V' \text{ imply } \mathfrak{F}, V' \models \phi[w_1, \ldots, w_n].$$ ⊞

Note that in the above definition it does not matter how the *accessibility* predicates occur in a formula. There is a lot to be said about the above concepts, but we confine ourselves to the following facts, of which the proof is standard:

Lemma 2.3.6.
(i) If ϕ is a positive (negative) formula, then so is ϕ^1.
(ii) Negations of positive (negative) formulas are equivalent to negative (positive) ones.
(iii) Positive formulas are monotone.

From here until 2.3.13 we fix the St-formula σ and the general frame $\mathfrak{G} = (\mathfrak{F}, A)$, $\mathfrak{F} = (W, R_\nabla)_{\nabla \in S}$. To establish the validity of σ in $\mathfrak{F}$, we must prove that for every valuation V, the model $\mathfrak{F}, V \models \sigma$. So let us start with defining a set of valuations for which we already know that $\mathfrak{F}, V \models \sigma$ (the proof is standard):

Definition 2.3.7.
A valuation V is *admissible* if $V(p) \in A$ for all atoms p.

Lemma 2.3.8.
For all admissible valuations V, $\mathfrak{F}, V \models \sigma$. ⊞

We now proceed to define a second kind of valuations, intuitively those forming the *minimal* valuations needed to make the strongly positive formulas, (these being the 'real' antecedent of the Sahlqvist formula σ,) true in a world of W.

Definition 2.3.9.
First we define *basic rudimentary formulas*, or short, br-formulas: a basic rudimentary formula of *length* 0 is of the form $\beta(x, y) \equiv x = y$. If $\beta(x, x_n)$ is a basic rudimentary formula of length n and $R_\Diamond$ is the accessibility symbol of a tense diamond, then $\exists x_n(\beta(x, x_n) \wedge R_\Diamond x_n y)$ is a basic rudimentary formula of length $n + 1$.
A *rudimentary* formula, or short, an r-formula, is of the form

$$\rho(x_1, \ldots, x_n, y) \equiv \bigvee_{1 \leq i \leq n} \beta_i(x_i, y),$$

where every β_i is a disjunction of basic rudimentary formulas in x_i and y.
A subset X of W is *rudimentary in* $w_1, \ldots, w_n \in W$ if for some rudimentary formula $\rho(x_1, \ldots, x_n, y)$, $X = \{v \in W \mid \mathfrak{F} \models \rho(w_1, \ldots, w_n, v)\}$.
A valuation V is *rudimentary* if for all atoms p, $V(p)$ is rudimentary. ⊞

Note that, intuitively, a basic rudimentary formula $\beta(x, y)$ of length n describes the existence and form of an path from x to y following tense accessibility relations. A rudimentary formula $\rho(x_1, \ldots, x_n, y)$ describes the position of y with respect to $x_1, \ldots, x_n$ in the frame, in terms of 'tense paths' leading from x_i to y, for every x_i.

Lemma 2.3.10.
Rudimentary valuations on discrete general tense frames are admissible.

Proof.
It is sufficient to prove that for every r-formula $\rho(x_1, \ldots, x_n, y)$, the sets $X_{\rho,\vec{w}} = \{v \in W \mid \mathfrak{F} \models \rho(w_1, \ldots, w_n, v)\}$ are in A for all n-tupels $\vec{w} = (w_1, \ldots, w_n)$ of worlds in W. Because A is closed under finite unions, we can do with showing the above for *basic* rudimentary

formulas. By induction to the length k of a basic formula $\beta(x,y)$ we prove the following claim:

$$\text{For every } w \in W,\ X_{\beta,w} \in A.$$

For $k = 0$, we have $X_{\beta,w} = \{w\}$ in A by the discreteness of $\mathfrak{G}$.
For $k = m+1$, let $\beta(x,y)$ be of the form $\exists x_n(\beta'(x,x_n) \wedge R_\Diamond x_n y)$ where $\Diamond$ is a tense diamond.
Now $X_{\beta,w} = \{v \in W \mid \mathfrak{F} \models \beta(w,v)\}$ is the set of worlds v such that there is a $u \in W$ with $\mathfrak{F} \models \beta'(w,u)$ and $\mathfrak{F} \models R_\Diamond uv$.
So $X_{\beta,w}$ contains precisely the worlds having an $R_\Diamond$-*predecessor* in $X_{\beta',w}$, or

$$X_{\beta,w} = \{v \in W \mid v \text{ has an } R_\Diamond^{-1}\text{-successor in } X_{\beta',w}\}.$$

By the induction hypothesis, $X_{\beta',w}$ is in A, and by the fact that we are in a *tense* frame, $(R_\Diamond)^{-1}$ is the accessibility relation of $\Diamond^{-1}$. So $X_{\beta,w} = m_{\Diamond^{-1}}(X_{\beta,w}) \in A$, cf. Appendix A.17. ⊞

Note that in the above proof it is essential to have *tense* operators in *tense* frames.

Lemma 2.3.11.
Let ψ be an untied formula. Then its first order model-equivalent $\psi^1(x_0)$ is equivalent to

$$\exists x_1 \dots x_n\Big(\pi \wedge \bigwedge_{i<k} \forall y(\rho_i(\vec{x},y) \rightarrow P_i y) \wedge \bigwedge_{j<m} N_j(u_j)\Big).$$

where the x_i's are distinct variables different from x_0, all the variables u_i are among $x_0,\dots,x_n$, π is a conjunction of atomic $L_S(x_0,\dots,x_n)$-formulas (i.e. atomic accessibility formulas of the form $R_\nabla(x_{i_0},\dots,x_{i_{\ell(\nabla)}})$ with ∇ an arbitrary S-operator and every variable in $\{x_0,\dots,x_n\}$), the ρ_i's are suitable rudimentary formulas, and the N_j's are negative.

Proof.
By a straightforward induction to the complexity of untied formulas, cf. Sambin and Vaccaro [118]. ⊞

Lemma 2.3.12.
Let $\sigma = \psi_1 \rightarrow \psi_2$ be a Sahlqvist formula. Then $\sigma^1(x_0)$ is equivalent to

$$\forall x_1 \dots x_n\Big(\Big(\pi \wedge \bigwedge_{i<k} \forall y(\rho_i(\vec{x},y) \rightarrow P_i y)\Big) \rightarrow \gamma_2(x_0,\dots,x_n)\Big).$$

where the antecedent is as in the previous lemma and the consequent γ_2 is some positive formula.

Proof.
Let $N(x_0,\dots,x_n)$ be the formula $\bigwedge_{j<m} N_j(u_j)$, then N is negative. By lemma 2.3.11, the local model correspondent $\sigma^1(x_0)$ of σ is equivalent to

$$\forall x_1 \dots x_n\Big(\Big(\pi \wedge \bigwedge_{i<k} \forall y(\rho_i(\vec{x},y) \rightarrow P_i y) \wedge N\Big) \rightarrow \psi_2^1(x_0)\Big).$$

So, by moving the negative N from the antecedent to the consequent, we obtain

$$\forall x_1 \ldots x_n\Big(\big(\pi \wedge \bigwedge_{i<k} \forall y(\rho_i(\vec{x}, y) \rightarrow P_i y)\big) \rightarrow \big(\neg N \vee \psi_2^1(x_0)\big)\Big).$$

where the antecedent is already as desired, and the consequent is positive as it is a disjunction of two positive formulas (cf. 2.3.6). ⊞

Proof of Theorem 2.3.3.
Let σ be of the form $\psi_1 \rightarrow \psi_2$, where ψ_1 is untied and ψ_2 is positive. We use the notation of the previous lemmas and set

$$\gamma_1(x_0, \ldots, x_n) \equiv \pi \wedge \bigwedge_{i<k} \forall y(\rho_i(\vec{x}, y) \rightarrow P_i y)$$

Obviously, $\sigma^1(x_0)$ is equivalent to $\forall x_1 \ldots x_n(\gamma_1 \rightarrow \gamma_2)$, where γ_2 is positive and hence monotone.
So by the fact that $\mathfrak{G} = (\mathfrak{F}, A) \models \sigma$ we get

$$\text{for all } \textit{admissible} \text{ valuations } V,\ \mathfrak{F}, V \models \forall x_0 \ldots x_n(\gamma_1 \rightarrow \gamma_2). \tag{1}$$

Our aim is to show that this implies $\mathfrak{F} \models \sigma$, or equivalently

$$\text{for } \textit{all} \text{ valuations } V,\ \mathfrak{F}, V \models \forall x_0 \ldots x_n(\gamma_1 \rightarrow \gamma_2). \tag{†}$$

So let a valuation V be given, together with worlds $w_0, w_1, \ldots, w_n \in W$ for which we have

$$\mathfrak{F}, V \models \gamma_1(w_0, w_1, \ldots, w_n). \tag{2}$$

Now let V^- be the rudimentary valuation that precisely 'fits' in γ_1, i.e. $V^-(p_i) = \{v \in W \mid \mathfrak{F} \models \rho_i(\vec{w}, v)\}$, then

$$\mathfrak{F}, V^- \models \gamma_1(w_0, w_1, \ldots, w_n). \tag{3}$$

V^- is admissible by lemma 2.3.10, so (1) and (3) give

$$\mathfrak{F}, V^- \models \gamma_2(w_0, w_1, \ldots, w_n). \tag{4}$$

But by (2) and definition of V^-, we have $V^- \leq V$. Together with the fact that γ_2 is monotone, this yields

$$\mathfrak{F}, V \models \gamma_2(w_0, w_1, \ldots, w_n), \tag{5}$$

which ensures (†). ⊞

As a matter of fact, from this proof it is only a minor step to give the algorithm producing the correspondent $\sigma^s(x_0)$ of an arbitrary (i.e. not necessarily tense) Sahlqvist formula:

Definition 2.3.13.
For a Sahlqvist formula σ, let $\sigma^s(x_0)$ be the L_S-formula

$$\forall x_1 \ldots x_n(\pi \rightarrow (\gamma_2(x_0, \ldots, x_n)[\rho_i(\vec{x}, u)/P_i u]))$$

(i.e. we substitute, everywhere in γ_2, $\rho_i(\vec{x}, u)$ for the atomic formula $P_i u$.) ⊞

Theorem 2.3.14 (SAHLQVIST CORRESPONDENCE).
Let σ be an arbitrary Sahlqvist formula, w a world in a frame $\mathfrak{F}$. Then

$$\mathfrak{F}, w_0 \models \sigma \iff \mathfrak{F} \models \sigma^s[x_0 \mapsto w_0].$$

Proof.
($\Rightarrow$) Let $w_1, \ldots, w_n$ be such that $\mathfrak{F} \models \pi[w_0, \ldots, w_n]$. This implies that, with V^- the valuation such that

$$V^-(p_i) = \{v \in W \mid \mathfrak{F} \models \rho_i(\vec{w}, v)\},$$

we have

$$\mathfrak{F}, V^- \models \pi \wedge \forall y(\rho_i(\vec{x}, y) \rightarrow P_i y)[w_0, \ldots, w_n].$$

So by the assumption $\mathfrak{F}, w_0 \models \sigma$, lemma 2.3.12 gives $\mathfrak{F}, V^- \models \gamma_2(x_0, \ldots, x_n)$. By definition of V^- we immediately obtain

$$\mathfrak{F} \models (\gamma_2(x_0, \ldots, x_n)[\rho_i(\vec{x}, u)/P_i u])[w_0, \ldots, w_n],$$

which is what we desired.

($\Leftarrow$) Here we can copy the proof of Theorem 2.3.13, after making the observation that now

$$\mathfrak{F}, V^-, w_0 \models \sigma$$

by definition of σ^s and the assumption $\mathfrak{F} \models \sigma^s[w_0]$. ⊞

2.4 The D-operator.

An important rôle in this dissertation is played by the so-called *difference* operator D. This operator is special in having the *inequality* relation as its intended accessibility relation:

Definition 2.4.1.
Let S be a similarity type containing the monadic operator D. An S-frame $\mathfrak{F} = (W, R_\nabla)_{\nabla \in S}$ is called *(D-)standard* if

$$R_D = \{(s,t) \in {}^2W \mid s \neq t\}.$$

As abbreviations we use $\underline{D}\phi \equiv \neg D \neg \phi$, $O\phi \equiv \phi \wedge \underline{D}\neg\phi$, $E\phi \equiv \phi \vee D\phi$.
For K a class of S-frames, we denote the class of standard frames in K by $\mathsf{K}^{\neq}$.

When referring to standard frames, we will suppress mentioning the inequality relation R_D. Thus we may identify standard frames for S with the frames for the similarity type obtained by dropping D from S. In the sequel we will frequently omit the adjective 'standard' when referring to the intended semantics, explicitly using the term 'non-standard' for the frames with $R_D \neq \{(s,t) \in {}^2W \mid s \neq t\}$. Note that in a standard model we have

$$\begin{array}{lll} \mathfrak{M}, w \models D\phi & \text{iff} & \text{there is a } v \neq w \text{ with } \mathfrak{M}, v \models \phi, \\ \mathfrak{M}, w \models O\phi & \text{iff} & w \text{ is the } only \text{ world with } \mathfrak{M}, w \models \phi, \\ \mathfrak{M}, w \models E\phi & \text{iff} & \text{there is a world } v \text{ with } \mathfrak{M}, v \models \phi. \end{array}$$

In many examples the D-operator is *definable* in the poorer language; for example, over the class LI of irreflexive linear orderings we have

$$\mathsf{LI} \models D\phi \leftrightarrow (F\phi \vee P\phi).$$

All of the similarity types studied in this dissertation have the property that over the class of frames to be axiomatized, the D-operator is definable.

The D-operator was introduced independently by various authors, including, in (probably) chronological order: Sain [113, 116], Koymans [65] and Gargov-Passy-Tinchev [41]. A nice feature of this new operator, and the main reason for its introduction, is the fact that it greatly increases the expressive power of the language. For example, *irreflexivity* is easily seen to be characterized by the formula $\Diamond p \rightarrow Dp$. Maarten de Rijke proved many results on the expressiveness and completeness of modal and tense logics having a D-operator, cf. [104]. We only need the following:

Definition 2.4.2.
Let S be a similarity type containing D. For Λ an S-logic, ΛD denotes the logic Λ extended with the following axioms:
$(D1)$ $p \rightarrow \underline{D}Dp$
$(D2)$ $DDp \rightarrow (p \vee Dp)$
$(D3_\nabla)$ $\nabla(p_1, \ldots, p_n) \rightarrow \bigwedge Ep_i$.
ΛD^+ is the logic ΛD extended with the *irreflexivity rule for D*:

(IR_D) $\quad \vdash Op \rightarrow \phi \ \Rightarrow \vdash \phi$, if $p \not\in \phi$.

Instead of $K_{\{D\}}$ (the minimal D-logic), we write K_D, instead of $K_D D$: KD. ⊞

Theorem 2.4.3.
For any similarity type S, $K_S D$ and $K_S D^+$ are both strongly sound and complete with respect to the class of standard S-frames.

Proof.
Cf. de Rijke [104]. ⊞

As a corollary of this completeness theorem some nice semantic properties of the operators are also *provable*:

Lemma 2.4.4.

(i) $\vdash KD^{(+)} \vdash E(Op \wedge \phi) \wedge E(Op \wedge \neg\phi) \rightarrow \bot$.

(ii) If ∇ is an S-operator, then
$\vdash K_S D^{(+)} \vdash (\nabla(\ldots, Op \wedge \phi, \ldots) \wedge \nabla(\ldots, Op \wedge \neg\phi, \ldots)) \rightarrow \bot$.
(iii) If ∇ is an S-operator, then
$\vdash K_S D^{(+)} \vdash \bigwedge_i \nabla(\ldots, Op \wedge \phi_i, \ldots) \rightarrow \nabla(\ldots, Op \wedge \bigwedge_i \phi_i, \ldots)$.

Proof.
By showing that the above schemes of formulas are semantically *valid* in standard S-frames, and then using the completeness theorem for $KD^{(+)}$. ⊞

Combining the notions of Sahlqvist (tense) formulas and the D-operator, we seem to have two options. Because of the general result on Sahlqvist correspondence, we know that every Sahlqvist formula σ has a local correspondent $\sigma^{s\prime}(x_0)$ in the language L_S where R_D is the symbol for the accessibility relation of D. However, we are almost exclusively interested in the way this equivalence works out for the *standard* S-frames; this means that we will only consider interpretations where R_D is the inequality relation. It is then very natural to let this preference be reflected in the syntax, by a slight abuse of notation:

Definition 2.4.5.
Let S be a similarity type and σ a Sahlqvist formula. If S does not contain the D-operator, $\sigma^s(x_0)$ denotes the ordinary first order Sahlqvist equivalent of σ given in Definition 2.3.13. If S does contain D, $\sigma^{s\prime}(x_0)$ denotes this ordinary first order equivalent, $\sigma^s(x_0)$ is $\sigma^{s\prime}(x_0)$ with every occurrence of R_D replaced by $\neq$. ⊞

As an example, the Sahlqvist correspondent of $\Diamond p \rightarrow Dp$ is not $\forall x_1(Rx_0x_1 \rightarrow R_Dx_0x_1)$, but $\forall x_1(Rx_0x_1 \rightarrow x_0 \neq x_1)$, (or even better: $\neg Rx_0x_0$.) With this notation we have equivalence of σ and σ^s for the standard frames:

Theorem 2.4.6.
Let σ be a Sahlqvist formula, w a world in a standard frame $\mathfrak{F}$. Then

$$\mathfrak{F}, w \models \sigma \iff \mathfrak{F} \models \sigma^s(w_0).$$

Proof.
Straightforward by Theorem 2.3.14 and the definitions of σ^s and standard frames. ⊞

However, by restricting our attention to standard frames we loose the automatic completeness of Sahlqvist's theorem: where we do have, for a set of Sahlqvist axioms Σ,

$$K_S D\Sigma \text{ is strongly sound and complete w.r.t. } \mathsf{Fr}_\Sigma,$$

we are not (yet) sure whether

$$K_S D^+\Sigma \text{ is strongly sound and complete w.r.t. } \mathsf{Fr}^{\neq}_\Sigma.$$

In the next section we will prove the above statement, for Sahlqvist *tense* axioms.

2.5 The main proof.

This subsection contains the main idea of the proof on the Sahlqvist theorem in a context with modal derivation rules. To keep notation as simple as possible, we consider a tense similarity type S having besides the difference operator D only one pair $\{F, P\}$ of tense operators. We let $\Diamond$ range over the monadic modal operators, $\Box$ is the dual of $\Diamond$, and $\Diamond^{-1}$ is the converse of $\Diamond$, i.e. $F^{-1} = P$, $P^{-1} = F$ and $D^{-1} = D$. Note that for this similarity type there is no distinction between ordinary Sahlqvist formulas and Sahlqvist tense formulas. We intend to prove the following theorem, keeping some generalizations and corollaries for later subsections.

Theorem 2.5.1. SD-THEOREM (monadic operators).
Let S be a tense similarity type with three diamonds F, P and D, and let σ be a Sahlqvist formula. Then $K^tD^+\sigma$ is strongly sound and complete with respect to $\mathsf{Fr}_{\sigma}^{t,\neq}$.

Recall that $K^tD^+\sigma$ has the following axioms:

(CT)	all classical tautologies
(DB)	$\Box(p \to q) \to (\Box p \to \Box q)$
(CV)	$\phi \to HFp$
$(D1)$	$p \to \underline{D}Dp$
$(D2)$	$DDp \to (p \vee Dp)$
$(D3)$	$\Diamond p \to p \vee Dp$
(σ)	σ

Its derivation rules are

(MP)	Modus Ponens
(UG)	Universal Generalization
(SUB)	Substitution

and the irreflexivity rule for D:

$$(IR_D) \qquad \vdash Op \to \phi \;\Rightarrow\; \vdash \phi, \text{ if } p \notin \phi.$$

Note that the above theorem is not an automatic corollary of the ordinary Sahlqvist theorem, because of the special interpretation for the accessibility relation of D that we have in mind, namely the inequality relation, and the fact that the axiom system has the unorthodox derivation rule IR_D. The difference with the ordinary Sahlqvist case shows itself in the fact that the logic $K^tD^+\sigma$ is *not* canonical:
Consider the set $\{\phi \to D\phi \mid \phi \text{ a formula}\}$. This set is consistent, so it must be contained in a maximal consistent set Δ which is a world in the canonical frame. Clearly however, Δ is R_D-reflexive, so inequality is *not* the canonical D-accessibility relation. In other words: the canonical frame is not standard.

So it turns out that the canonical frame is bad because it contains R_D-reflexive worlds. A naive approach to this problem is to simply throw them out of the canonical universe.

This is not sufficient however: consider the set

$$\{p_0 \wedge \underline{D}\neg p_0\} \cup \{F\top\} \cup \{G(\phi \to \underline{D}\phi) \mid \phi \text{ a formula}\}.$$

It is consistent, so it has a MC extension $\Delta \in W^c$. Δ itself is not R_D-reflexive, but all of its R_F-successors are. So Δ, having at least one R_F-successor, is an unwelcome inhabitant of the canonical frame too.

Now instead of successively throwing bad MCSs out of the canonical frame, we feel it is better to follow a more constructive path, defining a canonical-like model consisting only of good MCSs. To give this notion of a 'good' MCS, we need some auxiliary definitions. The first one is meant to provide us with a unique representation

$$\phi_0 \wedge \Diamond_1(\phi_1 \wedge \ldots \Diamond_{n-1}(\phi_{n-1} \wedge \Diamond_n\phi_n)),$$

for every formula ϕ.

Definition 2.5.2: Diamond Forms.

For notational elegance, instead of $\vee$ we take $\wedge$ as our basic boolean connective, and we add the *dummy diamond* $\odot$ to the set of monadic operators. This operator has the following interpretation:

$$\mathfrak{M}, w \models \odot\phi \iff \mathfrak{M}, w \models \phi.$$

Formula paths and their *lengths* are defined by induction:

(0) If ϕ is a formula, $\langle\phi\rangle$ is a formula path of length 0.

(1) For a formula ϕ, $\Diamond \in \{F, P, D, \odot\}$ and t a formula path of length n, $\langle(\phi, \Diamond), t\rangle$ is a formula path of length $n + 1$.

For t a formula path, the formula $\Phi\mu(t)$ is defined as

(0) $\Phi\mu(\langle\phi\rangle) = \phi$

(1) $\Phi\mu(\langle(\psi, \Diamond), t\rangle) = \psi \wedge \Diamond\Phi\mu(t)$

Notions like 'consistency' apply to formula paths as if they were formulas.

For ϕ a formula, its *path representation* $Pr(\phi)$ is the following formula path:

$$\begin{array}{lll}
(\text{at}) & Pr(p) & = \langle p\rangle \\
(\neg) & Pr(\neg\psi) & = \langle\neg\psi\rangle \\
(\wedge) & Pr(\psi \wedge \chi) & = \begin{cases} \langle(\psi, \Diamond), Pr(\chi')\rangle & \text{if } \chi \equiv \Diamond\chi', \Diamond \in \{F, P, D\} \\ \langle(\psi, \odot), Pr(\chi)\rangle & \text{otherwise} \end{cases} \\
(\Diamond) & Pr(\Diamond\psi) & = \langle(\top, \Diamond), Pr(\psi)\rangle.
\end{array}$$

The *diamond form* $N(\phi)$ of a formula ϕ is a representation of ϕ as $\Phi\mu(Pr(\phi))$, viz.

$$\phi_0 \wedge \Diamond_1(\phi_1 \wedge \ldots \Diamond_{n-1}(\phi_{n-1} \wedge \Diamond_n\phi_n))$$

Let t be a formula path, ζ a formula and m a natural number. By a nested induction to m and t we define $W^p(\zeta, m, t)$ as the following formula path:

$$\begin{array}{lll}
W^p(\zeta, 0, \langle\phi\rangle) & = & \langle\zeta \wedge \phi\rangle \\
W^p(\zeta, 0, \langle(\psi, \Diamond), t\rangle & = & \langle(\zeta \wedge \psi, \Diamond), t\rangle \\
W^p(\zeta, m+1, \langle\phi\rangle) & = & \langle\phi\rangle \\
W^p(\zeta, m+1, \langle(\psi, \Diamond), t\rangle) & = & \langle(\psi, \Diamond), W^p(\zeta, m, t)\rangle.
\end{array}$$

For ζ and ϕ formulas and m a natural number, we set[5]

$$W(\zeta, m, \phi) = \Phi\mu(W^p(\zeta, m, Pr(\phi))). \qquad \boxplus$$

The intuitive meaning of $W(\zeta, m, \phi)$ is the following: let ϕ have a diamond form

$$\phi_0 \wedge \Diamond_1(\phi_1 \wedge \ldots \Diamond_{n-1}(\phi_{n-1} \wedge \Diamond_n\phi_n)),$$

then $W(\zeta, m, \phi)$ is ϕ with ζ added as a *witness* at level m, viz.

$$\phi_0 \wedge \Diamond_1(\phi_1 \wedge \ldots \Diamond_m(\zeta \wedge \phi_m \wedge \Diamond_{m+1}(\phi_{m+1} \wedge \ldots \Diamond_{n-1}(\phi_{n-1} \wedge \Diamond_n\phi_n)\ldots))\ldots),$$

if $m \leq n$. Otherwise $W(\zeta, m, \phi) = \phi$.

As an example, the diamond form of

$$\phi = \Diamond q \wedge (q \wedge \Diamond\Diamond r)$$
is
$$\Diamond q \wedge \odot(q \wedge \Diamond(\top \wedge \Diamond r))$$

so

$$\begin{aligned}
W(\zeta, 0, \phi) &= \zeta \wedge \Diamond q \wedge \odot(q \wedge \Diamond(\top \wedge \Diamond r)) \\
W(\zeta, 1, \phi) &= \Diamond q \wedge \odot(\zeta \wedge q \wedge \Diamond(\top \wedge \Diamond r)) \\
W(\zeta, 2, \phi) &= \Diamond q \wedge \odot(q \wedge \Diamond(\zeta \wedge \top \wedge \Diamond r)) \\
W(\zeta, 3, \phi) &= \Diamond q \wedge \odot(q \wedge \Diamond(\top \wedge \Diamond(\zeta \wedge r))) \\
W(\zeta, 4, \phi) &= \Diamond q \wedge \odot(q \wedge \Diamond(\top \wedge \Diamond r)).
\end{aligned}$$

Definition 2.5.3.
A set of formulas Σ is *distinguishing*, or a *d-theory* if
(i) it is maximal consistent and
(ii) for every ϕ in Σ and natural number m, there is a propositional variable p with $W(Op, m, \phi)$ in Σ. $\boxplus$

Note that as d-theories are MCSs, the canonical accessibility relations R^c_F, R^c_P and R^c_D for F, P and D have the ordinary meaning:

$$R^c_\Diamond \Sigma\Delta \text{ iff for all } \phi \in \Delta,\ \Diamond\phi \in \Sigma$$

We want to take the d-theories as the possible worlds in our version of the canonical model. A minimal constraint which a canonical-ish model must meet is that every consistent set of formulas is somehow to be found as (part of) a possible world. In our setting this means that every consistent set must have a distinguishing extension.
First we need a lemma of a rather technical nature:

Lemma 2.5.4.
If p does not occur in ϕ or η, then $\vdash W(Op, m, \phi) \rightarrow \eta \Rightarrow \vdash \phi \rightarrow \eta$.

Proof.

[5]To be precise, we define a function from S-formulas to formulas in the extended similarity type with the dummy operator.

By induction to m.
If $m = 0$, $W(Op, m, \phi)$ is equivalent to $Op \wedge \phi$, so $\vdash W(Op, m, \phi) \rightarrow \eta$ implies $\vdash Op \rightarrow (\phi \rightarrow \eta)$, whence by an application of IR_D we obtain $\vdash \phi \rightarrow \eta$.
If $m = k + 1$, distinguish two cases:
If ϕ is an atom or a negation, then $W(Op, m, \phi) = \phi$, so the claim is immediate.
In the other case we have $Pr(\phi) = \langle(\psi, \diamond), Pr(\chi)\rangle$ for some ψ, χ (where $\diamond \in \{F, P, D, \odot\}$), so $W(Op, k+1, \phi) = \psi \wedge \diamond W(Op, k, \chi)$. The claim is now proved as follows:

$$\begin{array}{rlr} & \vdash (\psi \wedge \diamond W(Op, k, \chi)) \rightarrow \eta & \text{(assumption)} \\ \Rightarrow & \vdash \diamond W(Op, k, \chi)) \rightarrow (\psi \rightarrow \eta) & \text{(propositional logic)} \\ \Rightarrow & \vdash W(Op, k, \chi) \rightarrow \Box^{-1}(\psi \rightarrow \eta) & \text{(tense logic)} \\ \Rightarrow & \vdash \chi \rightarrow \Box^{-1}(\psi \rightarrow \eta) & \text{(induction hypothesis)} \\ \Rightarrow & \vdash \diamond\chi \rightarrow (\psi \rightarrow \eta) & \text{(tense logic)} \\ \Rightarrow & \vdash (\psi \wedge \diamond\chi) \rightarrow \eta & \text{(propositional logic),} \end{array}$$

and we are finished, as an easy proof shows that $\vdash \phi \leftrightarrow (\psi \wedge \diamond\chi)$. ⊞

The following propositions form our version of Gabbay's generalized irreflexivity lemma (cf. [35]):

Lemma 2.5.5.
Let Σ be a consistent set in which the variable p does not occur, and $\phi \in \Sigma$. Then $\Sigma \cup \{W(Op, m, \phi)\}$ is consistent for all m.

Proof.
Suppose otherwise, then $\vdash W(Op, m, \phi) \rightarrow \neg\psi$ for some $m \in \omega$ and $\psi \in \Sigma$. By Lemma 2.5.4 this would imply $\vdash \phi \rightarrow \neg\psi$, contradicting the consistency of Σ. ⊞

Lemma 2.5.6.
If Σ is a consistent set, then there is a distinguishing Σ' containing Σ.

Proof.
Let Q be the set of propositional variables in Σ, assume that Q is countable[6] and let $p_0, p_1, \ldots$ be mutually distinct propositional variables not in Q; set, for $0 \leq \xi \leq \omega$, $Q_\xi = Q \cup \{p_i \mid i < \xi\}$.
For a set Δ of formulas in Q_ω, let $PV(\Delta)$ be the set of propositional variables appearing in (formulas of) Δ. A theory Δ is called an *approximation* if Δ is consistent, $\Sigma \subseteq \Delta$ and $PV(\Delta) = Q_n$ for some $n < \omega$. In this case p_{n+1} is called the *new* variable for Δ and denoted by p_Δ.
Now let Δ be an approximation and (ϕ, m) a *potential shortcoming*, i.e. ϕ is a formula in Q_ω and $m \in \omega$. The pair (ϕ, m) is called a *shortcoming* of Δ if $\phi \in \Delta$ while no witness $W(Op, m, \phi)$ is in Δ. Assume that we have a wellordering $\mathcal{W}$ of the set $\Phi(M(Q_\omega)) \times \omega$ of potential shortcomings. If Δ has shortcomings, let (ϕ_Δ, m_Δ) be the *first* (in $\mathcal{W}$) of Δ's shortcomings. Now set

$$\Delta^+ = \begin{cases} \Delta & \text{if } \Delta \text{ has no shortcomings} \\ \Delta \cup \{W(Op_\Delta, m_\Delta, \phi_\Delta)\} & \text{otherwise} \end{cases}$$

[6]This restriction can easily be lifted.

We claim that if Δ is an approximation, then so is Δ^+:
Δ^+ is consistent by lemma 2.5.5; the other conditions are straightforward.
We now define the following sequence of theories $\Sigma_0, \Sigma_1, \ldots$:

$$\begin{aligned} \Sigma_0 &= \Sigma \\ \Sigma_{2n+1} &= \begin{cases} \Sigma_{2n} \cup \{\phi_n\} & \text{if } \Sigma_{2n+1} \cup \{\phi_n\} \text{ is consistent} \\ \Sigma_{2n} \cup \{\neg\phi_n\} & \text{otherwise} \end{cases} \\ \Sigma_{2n+2} &= \begin{cases} (\Sigma_{2n+1})^+ & \text{if } \Sigma_{2n+1} \text{ has shortcomings} \\ \Sigma_{2n+1} & \text{otherwise} \end{cases} \end{aligned}$$

and set $\Sigma' = \bigcup_{n<\omega} \Sigma_n$.
It is then straightforward to prove the following:
(0) $(\Sigma_n)_{n<\omega}$ is an increasing sequence.
(1) Every Σ_n is an approximation.
(2) For every Q_ω-formula ϕ, either ϕ or $\neg\phi$ is in Σ'.
(3) For every Q_ω-formula ϕ and $m \in \omega$, there is a witness $W(Op, m, \phi)$ in Σ'.

This gives all the desired properties of Σ'. ⊞

The fact that any consistent set is contained in a d-theory, means that in a certain sense there are *enough* distinguishing sets. Note however, that we needed to extend the language to prove lemma 2.5.6. This could mean that problems might arise if we want to show that every d-theory Γ containing a formula $\Diamond\phi$ has a distinguishing $\Diamond$-successor Δ with $\phi \in \Delta$. For, in context of ordinary maximal consistent sets, this proposition is proved by showing that the set

$$\{\phi\} \cup \{\psi \mid \Box\psi \in \Gamma\}$$

has a maximal consistent extension. We might do the same here, but then we have to show that this set has a distinguishing extension *in the same proposition letters*. We choose a different proof, using the fact that because the language has the O-operator, the distinguishing Γ contains a complete description of Δ:

Lemma 2.5.7.
If Γ is a d-theory and $\Diamond\phi \in \Gamma$, then there is a d-theory Δ with $\phi \in \Delta$ and $R^c_\Diamond \Gamma\Delta$.

Proof.
As $\Diamond\phi$ is in Γ, so is $\Diamond(\phi \wedge Op)$ for some atom p. Let Δ be the set $\{\psi \mid \Diamond(Op \wedge \psi) \in \Gamma\}$. Δ is consistent, for assume otherwise, then there are $\psi_1, \ldots, \psi_n$ in Δ with every $\Diamond(Op \wedge \psi_i)$ in Γ and

$$\vdash (\bigwedge_i \psi_i) \rightarrow \bot$$

By lemma 2.4.4 we have

$$\vdash \bigwedge_i (\Diamond(Op \wedge \psi_i)) \rightarrow \Diamond(Op \wedge \bigwedge_i \psi_i)$$

So $\Diamond(Op \wedge \bigwedge_i \psi_i)$ and hence $\Diamond\bot$ is in Γ, contradicting its consistency.
As $\Diamond Op \in \Gamma$, for every ψ either $\Diamond(Op \wedge \psi)$ or $\Diamond(Op \wedge \neg\psi)$ is in Γ, so clearly Δ is maximal.

The fact that $R^c_\diamond \Gamma\Delta$ is immediate by definition of Δ.
To prove that Δ is distinguishing, let $\psi \in \Delta$, and $m \in \omega$. We have to show that for some q, $W(Oq, m, \psi)$ is in Δ:
By definition of Δ, $\diamond(Op \wedge \psi) \in \Gamma$. As Γ is distinguishing, there is a q with

$$W(Oq, m+2, \diamond(Op \wedge \psi))$$

in Γ. But a simple calculation shows this formula to be equivalent to

$$\top \wedge \diamond(Op \wedge W(Oq, m, \psi)),$$

whence $W(Oq, m, \psi) \in \Delta$. ⊞

These two lemmas are sufficient to establish that there are *enough* d-theories. There is still one difference with the ordinary case which we need to discuss: suppose we would take the set of *all* distinguishing sets to form the universe of our canonical model. Then there would be *too many* worlds, for consider two D-theories Δ, Δ' with $p \wedge \underline{D}\neg p \in \Delta$, $p \wedge \underline{D}p \in \Delta'$. If both were to be in our 'canonical' model, the underlying frame would be non-standard, for Δ' is not an R_D-successor of Δ, while clearly $\Delta \neq \Delta'$. This inspires the following definition:

Definition 2.5.8.
Two distinguishing theories Γ and Δ are *connected*, notation: $\Gamma \sim_D \Delta$, if either $\Gamma = \Delta$ or $R^c_D \Gamma\Delta$. A *set* of d-theories is called *connected* if all pairs of its members are.

Lemma 2.5.9.
$\sim_D$ is an equivalence relation.

Proof.
Reflexivity of $\sim_D$ is immediate.
For symmetry, let $\Gamma \sim_D \Delta$. If $\Gamma = \Delta$, we are finished. If not, we have $R^c_D \Gamma\Delta$. Now R^c_D is a symmetric relation (this is an immediate consequence of having the Sahlqvist axiom $D1$ in the logic). So we have $R^c_D \Delta\Gamma$, implying $\Delta \sim_D \Gamma$.
For transitivity of $\sim_D$, it suffices to show that R^c_D is *pseudo-transitive*:

$$\forall x \forall y \forall z((xRy \wedge yRz) \rightarrow (x = z \vee xRz))$$

But this is immediate by the fact that pseudo-transitivity is the Sahlqvist correspondent of axiom $D3$, and the completeness part of Sahlqvist's theorem. ⊞

Definition 2.5.10: d-canonical structures.
A *d(istinguishing)*-canonical frame is of the form $\mathfrak{F}^d = (W^d, R^d_F, R^d_P, R^d_D)$ where W^d is a connected set of distinguishing theories, and the R^d's are the R^c's restricted to W^d.
Define also *d-canonical models* $\mathfrak{M}^d = (\mathfrak{F}^d, V^d)$ and *d-canonical general frames* $\mathfrak{G}^d = (\mathfrak{F}^d, A^d)$, where V^d is V^c restricted to W^d and A is given by $X \in A^d$ iff $X = V^d(\phi)$ for some ϕ. ⊞

In the sequel we will have a particular d-canonical model, frame, etc. in mind, viz. the

one consisting of all worlds connected to a fixed d-theory Σ. Therefor, we will frequently speak about *the* d-canonical model, frame, etc.
We need several nice properties of the d-canonical model. The easiest to establish is the truth lemma, via the fact that the d-canonical frame is a tense frame and standard:

Lemma 2.5.11.
Let $\mathfrak{F}^d$ be a d-canonical frame, then
(i) R^d_F and R^d_P are each others converse.
(ii) R^d_D is the inequality relation.

Proof.
(i) is immediate by the fact that $\mathfrak{F}^d$ is a substructure of the canonical frame.
For (ii), the connectedness of $\mathfrak{F}^d$ implies that $\Gamma \neq \Delta \Rightarrow R^d_D\Gamma\Delta$. The fact that every d-theory contains a witness $p \wedge \underline{D}\neg p$ ensures that no element of W^d is R^d_D-reflexive, so R^d_D is contained in the inequality relation. ⊞

Lemma 2.5.12.

$$\mathfrak{M}^d \models \phi\ [w] \text{ iff } \phi \in w.$$

Proof.
By a formula induction, of which we only give the induction step for the modal operators: Let ϕ be of the form $\Diamond\psi$.
First, suppose $\mathfrak{M}^d, w \models \phi$. We show that this implies the existence of a v with $R^d_\Diamond wv$ and $\mathfrak{M}^d, v \models \psi$: for $\Diamond \in \{F, P\}$ this is immediate by lemma 2.5.7, for $\Diamond = D$ we also need lemma 2.5.11, namely the fact that v is an R^d_D-successor of w if $v \neq w$. By the induction hypothesis then, we get: there is a v with $R^d_\Diamond wv$ and $\psi \in v$. So by definition of $R_\Diamond$ we get $\Diamond\psi \in w$.
For the other direction, suppose $\Diamond\psi \in w$. By Lemma 2.5.6 there is a v with $R^d_\Diamond wv$ and $\psi \in v$. By the induction hypothesis $\mathfrak{M}^d, v \models \psi$. Again, for $\Diamond \in \{F, P\}$ this immediately implies $\mathfrak{M}^d, w \models \Diamond\psi$, for $\Diamond = D$ we need lemma 2.5.11 once more (now we use $R_D \subseteq\ \neq$).
In both cases we find the desired $\mathfrak{M}^d, w \models \phi$. ⊞

So it is left to prove that the underlying d-canonical frame is in Fr_σ, or, equivalently, to show that $\mathfrak{F}^d, V \models \sigma$ for all valuations V. This is immediate by the following lemma and Theorem 2.3.3.

Lemma 2.5.13.
Any d-canonical general frame is discrete.

Proof.
Let w be a d-theory or world in a d-canonical general frame $\mathfrak{G}^d = (\mathfrak{F}^d, A^d)$. Let p be the propositional variable such that $Op \in w$, then by the truth lemma w is the *only* d-theory of $\mathfrak{G}^d$ with $Op \in w$. So $\{w\} = V^d(Op) \in A^d$. ⊞

Proof of theorem 2.5.1.
Soundness is immediate.
For completeness, suppose $\Sigma \not\vdash \phi$, then $\Sigma \cup \{\neg\phi\}$ is consistent, so by lemma 2.5.6 there is

a d-theory Σ' with $\Sigma \cup \{\neg\phi\} \subseteq \Sigma'$.
Let $\mathfrak{M}^d = (\mathfrak{F}^d, V^d)$ be the d-canonical model with $\Sigma' \in W^d$. By lemma 2.5.13 and Theorem 2.3.3, $\mathfrak{F}^d \models \sigma$ and by the truth lemma, $\mathfrak{M}^d \models \psi$ for all $\psi \in \Sigma \cup \{\neg\phi\}$.
So we obtained $\Sigma \not\models_{\mathsf{Fr}^{t,\neq}_{\sigma}} \phi$. ⊞

2.6 Uni-directional Complications.

In this section, which is not needed for understanding the sequel, we will see where our proof fails for a monadic similarity type S which is not versatile. It suffices to take the case where we have only one diamond F besides D. We would like to extend the results of the previous section to this case, but there seem to be two problems:

The first of these was already noted by Gabbay [31] and is also discussed in Gargov and Goranko [39].
The point is the following. In the previous section we saw that it is not sufficient to prove completeness by purging the canonical frame of R_D-reflexive points: their predecessors also needed to be thrown out, and the predecessors of those, ad infinitum. In our 'constructive' approach this problem arises in the following way: it is not sufficient to show that $Op \wedge \phi$ is consistent if ϕ is so, we must also prove that $\phi_0 \wedge \Diamond_1(Op \wedge \phi_1)$ is all right if $\phi_0 \wedge \Diamond_1\phi_1$ is, etc. In the tense-logical situation, we can do this by changing our 'perspective' on the formula, namely by moving the ϕ_1-position to the top level: we look at $\phi_1 \wedge \Diamond_1^{-1}\phi_0$ (which is consistent iff $\phi_0 \wedge \Diamond_1\phi_1$ is so), then we insert Op, obtaining $(Op \wedge \phi_1) \wedge \Diamond_1^{-1}\phi_0$. Returning to the old 'perspective' we see that indeed $\phi_0 \wedge \Diamond_1(\phi_1 \wedge Op)$ is consistent if $\phi_0 \wedge \Diamond_1\phi_1$ is consistent. It will be clear that *tense operators* are indespensable instruments for this surgery.

We will now prove that it really goes wrong in the uni-directional case:

Definition 2.6.1.
Let ρ be the formula $G(p \to Dp)$, ρ' the formula $\rho \wedge F\top$. ⊞

Note that ρ is a Sahlqvist formula (cf. the footnote to definition 2.2.1), its equivalent $\rho^{s'}$ is $\forall x \forall y(Rxy \to R_D yy)$. So ρ says: all R-successors are R_D-reflexive.

Recall that $K_F D^+ \rho'$ is the axiom system with the following axioms:

(CT)	all classical tautologies
(DB)	$\Box(p \to q) \to (\Box p \to \Box q)$
$(D1)$	$p \to \underline{D}Dp$
$(D2)$	$DDp \to (p \vee Dp)$
$(D3)$	$\Diamond p \to p \vee Dp$
(ρ')	ρ'

Its derivation rules are MP, UG, SUB and IR_D. If we had an analogon of theorem 2.5.1 for this logic, $K_F D^+ \rho'$ should be inconsistent, for we have

Proposition 2.6.2.
$\mathsf{K}^{\neq}_{\rho'} = \emptyset$.

Proof.
It suffices to show that ρ' only has non-standard frames. Assume $\mathfrak{F} \models \rho'$, where $\mathfrak{F} = (W, R, R_D)$ and w is a world of $\mathfrak{F}$. By $\mathfrak{F}, w \models F\top$, w has a successor v, by $\mathfrak{F} \models \rho^{s\prime}(w)$, v is R_D-reflexive. But then $\mathfrak{F}$ is not standard. ⊞

But, $K_F D^+ \rho'$ is *not* inconsistent, as we can easily show by considering non-standard frames again:

Proposition 2.6.3.
$K_F D^+(\rho') \not\vdash \bot$.

Proof.
We will define a $K_F D^+ \rho'$-consistent set Δ. Consider the following non-standard frame $\mathfrak{F} = (W, R, R_D)$:

$$\begin{array}{lcl} W & = & \{w, v\} \\ R & = & \{(w, v)\} \\ R_D & = & \{(w, v), (v, w), (v, v)\}, \end{array}$$

and set $\Delta = \{\phi \mid \mathfrak{F}, w \models \phi\}$. Clearly then $\bot \notin \Delta$. We show that Δ contains the axioms of $K_F D^+ \rho'$ and is closed under its rules. For the axioms, this is fairly trivial: for instance, ρ' is in Δ as $\mathfrak{F} \models \forall y(Rxy \rightarrow R_D yy)[w]$. Concerning the rules, the only thing worth treating is that Δ is closed under IR_D: but this is immediate by the fact that w itself is R_D-irreflexive. ⊞

This problem is not difficult to mend: a close inspection of the completeness proof in the previous section reveals that the essential property that we need and which versatile logics automatically give us, is the *deep insertion property*

(DIP) $\quad \vdash W(Op, m, \phi) \rightarrow \eta \;\Rightarrow\; \vdash \phi \rightarrow \eta$
for all $m \in \omega$ and p not occurring in ϕ or η.

The idea is now to extend the definition of the irreflexivity rule so as to obtain a logic in which the extension lemma holds again:

Definition 2.6.4.
Define the following set of derivation rules:

(IR_D^*) $\quad \vdash \neg W(Op, m, \psi) \;\Rightarrow\; \vdash \neg\psi$
for all $m \in \omega$ and $p \notin \psi$.

Lemma 2.6.5.
Let Λ be a logic having IR_D^*. Then Λ has DIP.

Proof.
By the following chain of consequences (where we assume that p does not occur in ϕ or in η):

$$\begin{array}{lll} & \vdash W(Op, m, \phi) \to \eta & \text{(assumption)} \\ \Rightarrow & \vdash \neg(\neg\eta \wedge W(Op, m, \phi)) & \text{(proplog)} \\ \Rightarrow & \vdash \neg W(Op, m+1, \neg\eta \wedge \phi) & \text{(evaluation of } W) \\ \Rightarrow & \vdash \neg(\neg\eta \wedge \phi) & (IR_D^*) \\ \Rightarrow & \vdash \phi \to \eta & \text{(proplog)} \end{array}$$

⊞

So for a similarity type where not all diamonds have converses, it is necessary to have the rule IR_D^* instead of IR_D. This was already noted by Gabbay [31] and by Gargov and Goranko [39], from which we derived the above example. It is not yet clear whether this extension is also ***sufficient*** to prove the analogon of the SD-theorem, at least if we want to consider axiom systems with ***arbitrary*** Sahlqvist axioms. For, there is another difference between the tense logical case and the unidirectional one.

This second problem seems to be more serious; assume that, analogous again to the previous section, we have constructed a d-canonical model $\mathfrak{M}^d$ for a MCS Σ. We want to prove $\mathfrak{F}^d \models \sigma$, where σ is the Sahlqvist axiom added to the logic $K_S D^+$. In the tense logical case, we could do this, by using a special kind of valuations which we called ***rudimentary***. We showed that for such a valuation $\mathfrak{F}^d, V \models \sigma$. This path however can only be taken if we have the converse diamond of $\mathfrak{F}$ in the language (cf. the proof of Lemma 2.3.10); in the uni-directional case, rudimentary valuations need not be ***admissible***. It even turns out that not every d-canonical frame validates σ. We consider an example:

Definition 2.6.6.
Let γ be the formula $\sigma = FGp \to GFp$. ⊞

We have already met γ in section 2; its first order equivalent is the *Church-Rosser* formula

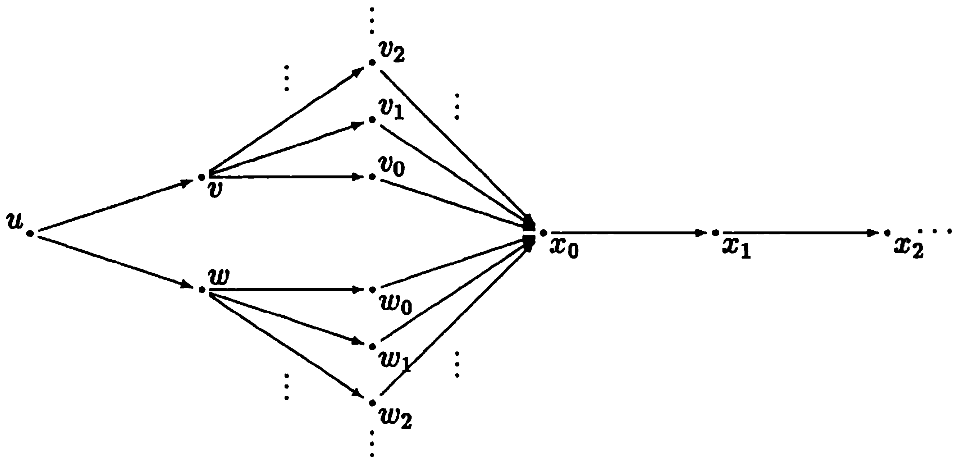
$$\gamma^s(x) = \forall y \forall z (Rxy \wedge Rxz \to \exists t (Ryt \wedge Rzt))$$

We will give a distinguishing theory Δ with $\mathfrak{F}^d \not\models \gamma(\Delta)$ for the d-canonical frame of Δ.

Definition 2.6.7.
Consider the following standard frame $\mathfrak{F} = (W, R)$:
The set of possible worlds is given as $W = \{u, v, w\} \cup \{v_n, w_n, x_n \mid n \in \omega\}$.
The accessibility relation R holds as follows: Ruv, Ruw,Rvv_n and Rww_n, all n, Rv_nx_0 and Rw_nx_0, all n, and Rx_nx_{n+1}, all n, viz. the picture on the next page.
Finally, we define a model $\mathfrak{M}$ on $\mathfrak{F}$. Let the propositional variables of the language be named $p, q, r, p_0, p_1, p_2, \ldots$
The valuation V is defined by

$$\begin{array}{lll} V(p) = \{u\} & V(q) = \{v\} & V(r) = \{w\} \\ V(p_{3n}) = \{v_n\} & V(p_{3n+1}) = \{w_n\} & V(p_{3n+2}) = \{x_n\}. \end{array}$$

⊞



Lemma 2.6.8.
$\mathfrak{M} \models \sigma\gamma$ for all substitutions σ.

Proof.
It is our aim to show that for all formulas ϕ and $t \in U$:

$$\mathfrak{M}, t \models FG\phi \to GF\phi$$

For $t \neq u$ this is immediate by $\mathfrak{F} \models \gamma^s(t)$.
For $t = u$, let $V_\phi = \{n \in \omega \mid \mathfrak{M}, v_n \models \phi\}$ and $W_\phi = \{n \in \omega \mid \mathfrak{M}, w_n \models \phi\}$.
By a straightforward induction to ϕ we can show:

V_ϕ and W_ϕ are either both finite or both cofinite.

Now assume $\mathfrak{M}, u \models FG\phi$; without loss of generality we suppose that $\mathfrak{M}, v \models G\phi$. So V_ϕ contains *all* v_n, but then V_ϕ and W_ϕ are both infinite. This implies $\mathfrak{M}, w \models F\phi$. As we have $\mathfrak{M}, v \models F\phi$ too, we obtain $\mathfrak{M}, u \models GF\phi$. ⊞

Definition 2.6.9.
Let for $t \in W$, Δ_t be the set $\{\phi \mid \mathfrak{M}, t \models \phi\}$. ⊞

Lemma 2.6.10.
For every t in W, Δ_t is distinguishing.

Proof.
By induction to m we will prove:

For all $t \in W$, $\phi \in \Delta_t$, there is a p such that $W(Op, m, \phi) \in \Delta_t$.

For $m = 0$, let $t \in W$. By definition of the valuation V, there is a propositional variable p_t such that $V(p_t) = \{t\}$. So $\mathfrak{M}, t \models Op_t$, giving $W(Op_t, 0, \phi) \in \Delta_t$.
For $m = k + 1$, let $t \in W$ and $\phi \in \Delta_t$. The only interesting case is where ϕ has the form $\psi \wedge \Diamond\chi$.
If $\mathfrak{M}, t \models \psi \wedge \Diamond\chi$, there is a t' with $R_\Diamond tt'$ and $\mathfrak{M}, t' \models \chi$. By the induction hypothesis, there is a p with $\mathfrak{M}, t' \models W(Op, k, \chi)$. But this means

$$\psi \wedge \Diamond W(Op, k, \chi) = W(Op, k + 1, \phi) \in \Delta_t.$$ ⊞

Lemma 2.6.11.
Let $\mathfrak{F}^d$ be the d-canonical frame of Δ_u. Then $\mathfrak{F}^d \not\models \gamma^s(\Delta_u)$.

Proof.
It is straightforward to verify that in $\mathfrak{M}^d$, Δ_v and Δ_w are R_F-successors of Δ_u. Let Σ be a maximal consistent R_F^c-successor of both Δ_v and Δ_w. We can prove that such a Σ cannot be distinguishing, by showing that for each propositional variable s

$$Gs \rightarrow s \in \Sigma.$$

For, if $s \in \{p, q, r\} \cup \{p_{3n+1}, p_{3n+2} \mid n \in \omega\}$, we have $G(Gs \rightarrow s)$ in Δ_v, so by the truth lemma $\mathfrak{M}^d, \Delta_v \models G(Gs \rightarrow s)$, immediately giving the above claim. For $s \in \{p_{3n} \mid n \in \omega\}$ we can prove something similar, now using Δ_w. ⊞

Note that in the situation above, we have an example of a Sahlqvist formula which is not persistent with respect to the class of discrete frames: let $\mathfrak{G} = (\mathfrak{F}, A)$ be the general frame with $\mathfrak{F}$ as defined in 2.6.7 and $X \in A$ if either X or its complement is finite. Then $\mathfrak{G}$ is discrete, $\mathfrak{G} \models \gamma$, while $\mathfrak{F} \not\models \gamma$.

Sahlqvist *tense* formulas however are still persistent for discrete general frames. Note that for a uni-directional similarity type, atoms are the only strongly positive formulas, so the set of St-formulas is rather small. Still, for this restricted set we do have a completeness theorem:

Definition 2.6.12.
Let S be an arbitrary similarity type of constants and diamonds. $K_S D^*$ is the basic S-logic extended with the set of rules IR_D^*. ⊞

Theorem 2.6.13.
Let S be an arbitrary similarity type of constants and diamonds, and Σ a set of Sahlqvist tense formulas. Then

$$K_S D^* \Sigma \text{ is strongly sound and complete for } K_\Sigma^{\neq}.$$

Proof.
An copy of the proof in section 5, using lemma 2.6.5 instead of 2.5.4.

We conjecture that for any *individual* set of Sahlqvist axioms, the completeness like in Theorem 2.6.13 can be shown to hold, but we are doubtful whether there is a uniform proof (analogous to that of Theorem 2.5.1) taking care of all Sahlqvist axiomatizations at once. On the other hand, Goranko [45] announces a general *weak completeness proof*, for arbitrary canonical formulas.

2.7 The SD-theorem.

There are some problems involved, mainly of a technical nature, in extending the completeness proof of the SD-theorem to languages having dyadic operators.

First of all we have to make clear what we mean by a Sahlqvist formula in a dyadic language. In fact, the definition and all the results in section 2 already apply to arbitrary similarity types. The following point is worth some discussion, however: in a similarity type with only diamonds and constants, we allow boxed atoms in the strongly positive formulas. A naive approach to define Sahlqvist triangle formulas would then be to allow duals of dyadic operators too. But de Rijke showed that the formula

$$(p \underline{\triangle} p) \underline{\triangle} p \rightarrow (p \triangle p) \triangle p$$

is *not acceptable* as a Sahlqvist formula, as it does not have a first order equivalent on the frame level. So for triangle similarity types, the atoms and negative formulas are the only admissible building blocks of Sahlqvist antecedents. This implies that for arbitrary similarity types, the difference between Sahlqvist *tense* formulas and ordinary Sahlqvist formulas is caused by the nature of the *diamonds* alone.

On the other hand, there is a difference between *versatile* (cf. Appendix A.40) triangle similarity types and uni-directional ones, analogous to the monadic case: if we consider a language and semantics which are not versatile, one irreflexivity rule is not sufficient, but we have to add infinitely many rules, allowing the building in of witnesses at all depths in a formula. To avoid these technical complications, we have to get familiar with the *versatile* logic of dyadic operators. Let us for the moment consider a similarity type consisting of three dyadic operators $\triangle_0$, $\triangle_1$ and $\triangle_2$. Frames for this similarity type have the form $\mathfrak{F} = (W, R_0, R_1, R_2)$, where R_i is the ternary accessibility relation of $\triangle_i$. Recall that the truth definition of a dyadic operator gives

$$u \models \phi \triangle_i q \iff \text{there are } v, w \text{ with } R_i uvw,\ v \models \phi \text{ and } w \models \psi.$$

In the intended *versatile* semantics, the three R_i's are 'directions' of one ternary relation R; as a standard we take $R = R_0$.

Definition 2.7.1.
A frame $\mathfrak{F} = (W, R_0, R_1, R_2)$ is a *versatile* frame if it satisfies the following conditions, for $i = 0, 1, 2$ (we write $2 + 1 = 0$):

$$(Qi) \qquad R_i uvw \rightarrow R_{i+1} vwu$$

The class of versatile frames is denoted by Fr^v. ⊞

Analogous to the monadic case, Fr^v can be quite easily characterized and axiomatized:

Definition 2.7.2.
Define the following formulas, for $i = 0, 1, 2$:

$$(Vi) \qquad p \wedge \neg(r \triangle_{i+1} p) \triangle_i r \to \bot,$$

and set $V \equiv V1 \wedge V2 \wedge V3$.
Let K_S^v be the versatile S-logic, i.e. the minimal S-logic K_S extended with the axiom V. ⊞

Note that Vi is a Sahlqvist formula: p is strongly positive, $\neg(r \triangle_{i+1} p)$ is negative and r is again strongly positive, so $p \wedge \neg(r \triangle_{i+1} p) \triangle_i r$ is untied, and as $\bot$ is positive, we are finished. This means that we immediately have the following:

Theorem 2.7.3.
For $i = 0, 1, 2$: $\mathfrak{F} \models Qi \iff \mathfrak{F} \models Vi$.

Proof.
The proposition is immediate by the Sahlqvist theorem, but we give a direct proof (taking $i = 0$):
($\Rightarrow$) Suppose that for some model $\mathfrak{M}$ on $\mathfrak{F}$, $\mathfrak{M}, u \models p \wedge \neg(r \triangle_1 p) \triangle_0 r$. By the truth definiton of $\triangle_0$, there are v, w with $R_0 uvw$, $v \models \neg(r \triangle_1 p)$, $w \models r$, while $u \models p$. $\mathfrak{F} \models Q0$ implies $R_1 vwu$, so by the truth definition of $\triangle_1$ we get $v \models r \triangle_1 p$ and find the desired contradiction.
($\Leftarrow$) Let (u, v, w) be in R_0. We want to show $(v, w, u) \in R_1$. Suppose otherwise and consider a valuation V with $V(p) = \{u\}$, $V(r) = \{w\}$. Then $v \models \neg(r \triangle_1 p)$, so $u \models \neg(p \triangle_1 r) \triangle_0 r$. By $\mathfrak{F} \models V_1$ we then have $u \models \neg p$, contradicting $V(p) = \{u\}$. ⊞

Theorem 2.7.4: Soundness and Completeness.
K_S^t is strongly sound and complete with respect to the versatile S-frames.

Proof.
Immediate by the fact that the axioms are Sahlqvist formulas and 2.2.2. ⊞

Corollary 2.7.5.
The following deduction rule is a derived rule of K_S^v:

$$\vdash \neg(p \wedge q \triangle_i r) \iff \vdash \neg(q \wedge r \triangle_{i+1} p).$$

Proof.
By the observation that the rule is *sound* in the class of S-versatile frames. ⊞

Note that intuitively, $\mathfrak{M} \models \neg(p \wedge q \triangle_i r)$ denotes the impossibility of the existence of a triple (u, v, w) in R with $u \models p$, $v \models q$ and $w \models r$.
We can easily generalize this idea to operators of rank $r \neq 2$. For example, for the monadic case we have

$$\vdash \neg(p \wedge \Diamond q) \iff \vdash \neg(q \wedge \Diamond^{-1} p)$$

as a derived rule of the minimal tense logic.

Now we are ready to add monadic tense operators, including the D-operator to the language.

Definition 2.7.6.
Let S be a versatile similarity type having constants, monadic tense operators $\{\Diamond_i, \Diamond_i^{-1} \mid i < \alpha\}$ and dyadic operators $\{\triangle_0^j, \triangle_1^j, \triangle_2^j \mid j < \beta\}$.
The *versatile S-logic* K_S^v is defined as the extension of the minimal S-logic K_S with the tense axiom CV for every diamond pair, and the versatility axiom V for every triple of triangles. ⊞

Theorem 2.7.7. THE SD-THEOREM.
Let S be a versatile similarity type containing D and Σ a set of Sahlqvist formulas. Then

$$K_S^t D^+ \Sigma \text{ is strongly sound and complete for } \mathsf{K}_\Sigma^{t,\neq}.$$

The remainder of this section will be devoted to the proof of this theorem. For notational simplicity, we assume that $S = \{D, F, P, \triangle_0, \triangle_1, \triangle_2\}$ and that Σ is a singleton $\{\sigma\}$. From now on we abbreviate the logic $K_S^t D^+(\sigma, -\xi)$ by Λ. Formulating in this context the notions we defined in the monadic case causes some technical problems. The main idea is exactly the same, however:

Definition 2.7.8.
Formula trees and their *depth* are inductively defined as follows:

(0) Formulas are formula trees of depth 0.
(1) If ψ is a formula, $\Diamond$ is a diamond and t' is a formula tree of depth n, then $\langle(\psi, \Diamond), t'\rangle$ is a formula tree of depth $n + 1$.
(2) If ψ is a formula, $\triangle$ is a triangle and t_0, t_1 are formula trees of depths n_0, n_1, then $\langle(\psi, \triangle), t_0, t_1\rangle$ is a formula tree of depth $1 + \max(n_0, n_1)$.

For t a formula tree, the formula $\Phi\mu(t)$ is given as
(0) $\Phi\mu(\langle\phi\rangle) = \phi$
(1) $\Phi\mu(\langle(\psi, \Diamond), t'\rangle) = \psi \wedge \Diamond\Phi\mu(t')$
(2) $\Phi\mu(\langle(\psi, \triangle), t_0, t_1\rangle) = \psi \wedge \Phi\mu(t_0)\triangle\Phi\mu(t_1)$
For ϕ a formula, its *tree representation* $Tr(\phi)$ is the following formula tree:
(at) $Tr(p) = \langle p\rangle$
($\neg$) $Tr(\neg\phi) = \langle\neg\phi\rangle$
($\wedge$) $Tr(\phi \wedge \psi) = \langle(\phi, \odot), Tr(\psi)\rangle$
($\Diamond$) $Tr(\Diamond\psi) = \langle(\top, \Diamond), Tr(\psi)\rangle$
($\triangle$) $Tr(\psi\triangle\chi) = \langle(\top, \triangle), Tr(\psi), Tr(\chi)\rangle$ ⊞

Analogous to the monadic case, we want to be able to place ξ-witnesses in *every* node of a tree. Different from the monadic case, nodes will now be named by sequences of 0's and 1's (think of going left or right.)

Definition 2.7.9.
Let 2^* be the set of sequences in the alphabet $\{0, 1\}$. Inductively 2^* can be defined by: (i) the empty sequence ϵ is in 2^*, and (ii) if s is in 2^*, then so are $s * 0$ and $s * 1$.
Now let ζ be a formula, s a sequence and t a formula tree. We define $W(\zeta, s, t)$, the *tree*

t witnessing ζ *at node* s, by a nested induction to s and t:

$$\begin{array}{lcl}
W^t(\zeta,\epsilon,\langle\phi\rangle) & = & \langle\zeta\wedge\phi\rangle \\
W^t(\zeta,\epsilon,\langle(\psi,\Diamond),t'\rangle) & = & \langle(\zeta\wedge\psi,\Diamond),t'\rangle \\
W^t(\zeta,\epsilon,\langle(\psi,\vartriangle),t_0,t_1\rangle) & = & \langle(\zeta\wedge\psi,\vartriangle),t_0,t_1\rangle \\
W^t(\zeta,s*i,\langle\phi\rangle) & = & \langle\phi\rangle \\
W^t(\zeta,s*i,\langle(\psi,\Diamond),t'\rangle) & = & \langle(\psi,\Diamond),W^t(\zeta,s,t')\rangle \\
W^t(\zeta,s*0,\langle(\psi,\vartriangle),t_0,t_1\rangle) & = & \langle(\psi,\vartriangle),W^t(\zeta,s,t_0),t_1\rangle \\
W^t(\zeta,s*1,\langle(\psi,\vartriangle),t_0,t_1\rangle) & = & \langle(\psi,\vartriangle),t_0,W^t(\zeta,s,t_1)\rangle
\end{array}$$

For ζ and ϕ formulas and s a sequence, we set

$$W(\zeta,s,\phi)=\Phi\mu(W^t(\zeta,s,Tr(\phi))).$$ ⊞

Definition 2.7.10.
A set of formulas Δ is *distinguishing* if it is maximal consistent, and for every ϕ in Δ and s in 2^*, there is a propositional variable p with $W(Op,s,\phi)\in\Delta$. ⊞

Lemma 2.7.11.
If Op has no letters in common with ϕ and η, then for all sequences s:

$$\vdash W(Op,s,\phi)\to\eta \;\Rightarrow\vdash\phi\to\eta.$$

Proof.
We prove the lemma by induction to the length of s:
If $s=\epsilon$, then $W(Op,s,\phi)=Op\wedge\phi$, so the proposition is immediate by IR_D:

$$\vdash (Op\wedge\phi)\to\eta \;\Rightarrow\vdash Op\to(\phi\to\eta)\;\Rightarrow\vdash\phi\to\eta.$$

If s has a positive length, distinguish the following cases:
(1) ϕ is an atom or a negation. As this implies $W(Op,s,\phi)=\phi$, there is nothing to prove.
(2) If ϕ has the form $\Diamond\psi$ or $\psi\wedge\chi$, we have a situation analogous to the monadic case, so for the proof we refer to 2.5.4.
(3) So the only interesting case is where ϕ has the form $\psi\vartriangle\chi$. Without loss of generality we may assume $s=s'*0$ and $\vartriangle=\vartriangle_1$.
Abbreviate $W(Op,s*0,\phi)$ by ϕ' and $W(Op,s,\psi)$ by ψ', then $\phi'=\psi'\vartriangle\chi$.
The proof now goes as follows:

$$\begin{array}{lll}
 & \vdash\phi'\to\eta & \text{(assumption)} \\
\Rightarrow & \vdash\psi'\vartriangle_1\chi\to\eta & \text{(definition)} \\
\Rightarrow & \vdash(\neg\eta\wedge\psi'\vartriangle_1\chi)\to\bot & \text{(propositional logic)} \\
\Rightarrow & \vdash(\psi'\wedge\chi\vartriangle_2\neg\eta)\to\bot & \text{(Corollary 2.7.5.)} \\
\Rightarrow & \vdash\psi'\to\neg(\chi\vartriangle_2\neg\eta) & \text{(propositional logic)} \\
\Rightarrow & \vdash\psi\to\neg(\chi\vartriangle_2\neg\eta) & \text{(Induction Hypothesis)} \\
\Rightarrow & \vdash(\psi\wedge\chi\vartriangle_2\neg\eta)\to\bot & \text{(propositional logic)} \\
\Rightarrow & \vdash(\neg\eta\wedge\psi\vartriangle_1\chi)\to\bot & \text{(Corollary 2.7.5.)} \\
\Rightarrow & \vdash\psi\vartriangle_1\chi\to\eta & \text{(propositional logic)} \\
\Rightarrow & \vdash\phi\to\eta & \text{(definition)}
\end{array}$$ ⊞

Lemma 2.7.12.
Every consistent set has a distinguishing extension.

Proof.
Analogous to lemma 2.5.6. ⊞

Lemma 2.7.13.
If Γ is distinguishing and $\delta \triangle \pi \in \Gamma$, then there are d-theories Δ and Π with $\delta \in \Delta$, $\pi \in \Pi$ and $R^c_{\triangle}\Gamma\Delta\Pi$.

Proof.
As $\delta \triangle \pi$ is in Γ, we have $(\delta \wedge Od)\triangle(\pi \wedge Op)$ in Γ for some propositional variables d and p. Set

$$\begin{aligned} \Delta &= \{\phi \mid (\phi \wedge Od)\triangle Op \in \Gamma\} \\ \Pi &= \{\psi \mid Od\triangle(\psi \wedge Op) \in \Gamma\} \end{aligned}$$

The argument that Δ and Π are maximal and consistent is just like in 2.5.7. To show that $R^c_{\triangle}\Gamma\Delta\Pi$, let $\phi \in \Delta$ and $\psi \in \Pi$; we have to prove that $\phi\triangle\psi \in \Gamma$.
As $(\phi \wedge Od)\triangle Op$ is in Γ, so is either $(\phi \wedge Od)\triangle(Op \wedge \psi)$ or $(\phi \wedge Od)\triangle(Op \wedge \neg\psi)$. If the latter were the case, then the formula $Od\triangle(Op \wedge \neg\psi)$ would be in Γ too. But this would imply $\neg\psi$ in Π, contradicting the consistency of Π.
The proof that Δ and Π are both distinguishing is again analogous to the monadic case. ⊞

Definition 2.7.14.
Distinguishing canonical structures are defined as in definition 2.5.10. ⊞

Lemma 2.7.15.
Let $\mathfrak{M}^d$ be a d-canonical model, Γ a world in $\mathfrak{M}^d$. Then

$$\mathfrak{M}^d, \Gamma \models \phi \iff \phi \in \Gamma.$$

Proof.
By a formula induction, of which we only give the step for $\phi = \psi\triangle\chi$:
By the truth definition, $\mathfrak{M}^d, \Gamma \models \psi\triangle\chi$ implies that there are Δ, Π with $R^d\Gamma\Delta\Pi$ and $\mathfrak{M}^d, \Delta \models \psi$, $\mathfrak{M}^d, \Pi \models \chi$. By the induction hypothesis, ψ is in Δ and χ is in Π, so by definition of R^d, $\psi\triangle\chi \in \Gamma$.
For the other direction, suppose $\psi\triangle\chi \in \Gamma$. By 2.7.13 there are Δ, Π with $R^c\Gamma\Delta\Pi$ and $\psi \in \Delta$, $\chi \in \Pi$. The induction hypothesis now gives $\mathfrak{M}^d, \Delta \models \psi, \mathfrak{M}^d, \Pi \models \chi$, so by the truth definition, $\mathfrak{M}^d, \Gamma \models \psi\triangle\chi$. ⊞

Lemma 2.7.16.
Let $\mathfrak{F}$ be a d-canonical frame. Then $\mathfrak{F}$ is in $\mathsf{Fr}^{v;\neq}_{\sigma}$.

Proof.
The proof for $\mathfrak{F}$ in $\mathsf{Fr}^{\neq}_{\sigma}$ is as in section 2.5. $\mathfrak{F}$ is versatile by the fact that $\mathfrak{F}$ is a substructure of the canonical versatile frame $\mathfrak{F}^c$ and the fact that the *universal* L_S-formula defining versatile frames is preserved under taking substructures. ⊞

Proof of theorem 2.7.7.
Exactly like the proof of theorem 2.5.1. ⊞

2.8 The SNΞ-theorem.

We are now ready to prove our main completeness theorem for a versatile logic having other non-ξ rules besides IR_D.

Definition 2.8.1.
Let S be a versatile similarity type containing the D-operator, Σ a set of Sahlqvist formulas and Ξ a set of arbitrary formulas. $K_S^v D^+(\Sigma, -\Xi)$ is the logic $K_S^v D^+$ extended with the axioms Σ and the non-ξ rules for all $\xi \in \Xi$. ⊞

Recall that the above definition implies that the *rules* of $K_S^v D^+(\Sigma, -\Xi)$ are MP, UG, SUB, IR_D and $\{N\xi R \mid \xi \in \Xi\}$.
If the similarity type contains only constants and diamonds, then the system has the following *axioms*:

(CT)	all classical tautologies
(DB)	$\Box(p \to q) \to (\Box p \to \Box q)$
(CV)	$\phi \to \Box\Diamond^{-1}p$
$(D1)$	$p \to \underline{D}Dp$
$(D2)$	$DDp \to (p \vee Dp)$
$(D3)$	$\Diamond p \to p \vee Dp$
(Σ)	Σ

If there are also triangles around, then the system has the versatility axiom V too (cf. 2.7.2).

Note that the class $\mathsf{Fr}^{v,\neq}_{(\Sigma,-\Xi)}$ was defined as the class of standard versatile S-frames with

$$\mathfrak{F} \models \sigma \quad \text{for all } \sigma \text{ in } \Sigma$$
$$\mathfrak{F}, w \not\models \xi \quad \text{for all } w \text{ in } \mathfrak{F},\ \xi \text{ in } \Xi$$

If all ξ's have local first order equivalents $\xi^f(x)$on the frame level (for example, if every ξ is a Sahlqvist formula too), then $\mathsf{Fr}^{v,\neq}_{(\Sigma,-\Xi)}$ is elementary, as we have

$$\mathfrak{F} \text{ in } \mathsf{Fr}_{-\xi} \iff \mathfrak{F} \models \forall x \neg \xi^f(x).$$

So, the theory below takes care of many classes of frames, for example the asymmetric or intransitive frames (cf. the characterizations given in the introduction).

Theorem 2.8.2. SNΞ-THEOREM.
Let S, Σ and Ξ be as in definition 2.8.1. Then

$$K_S^v D^+(\Sigma, -\Xi) \text{ is strongly sound and complete for } \mathsf{Fr}^{v,\neq}_{(\Sigma,-\Xi)}.$$

The proof of Theorem 2.8.2 is in fact a straightforward adaptation of the proof in section 2.7. There we started with a MCS Δ and inserted in Δ, for every $s \in 2^*$ and formula $\phi \in \Delta$, formulas $W(Op, s, \phi)$, in order to witness the R_D-irreflexivity of all worlds connected to Δ. Here we will add more formulas (of the form $W(\neg\xi(p_1, \dots, p_n), s, \phi)$), this time in order to ensure that the canonical-like general frame we end with is not only standard (with respect to R_D), but also in $\mathsf{Fr}_{-\Xi}$. So we set

Definition 2.8.3.
A set Δ of S-formulas is *witnessing* if it is distinguishing and satisfies that for all sequences $s \in 2^*$, formulas $\phi \in \Delta$ and $\xi \in \Xi$, there are propositional variables $p_1, \dots, p_n$ with $W(\neg\xi(p_1, \dots, p_n), s, \phi) \in \Delta$.

Lemma 2.8.4.
Every maximal consistent set Δ has a witnessing extension Δ'.

Proof.
An straightforward analogon of 2.7.12. ⊞

Definition 2.8.5.
A *w(itnessing)-canonical frame* is of the form $\mathfrak{F}^w = (W^w, R^w_\nabla)_{\nabla \in S}$ where W^w is a $\sim_D$-connected set of witnessing theories and R^w_∇ is the canonical accessibility relation of ∇, restricted to W^w. *Witnessing models* and *witnessing general frames* are also defined in the obvious way. For a w-theory Δ, *the* w-canonical frame (model, etc.) of Δ is the w-canonical frame with $\Delta \in W^w$. If we want to make the set Ξ explicit, we use the term *w-canonical structure witnessing against* Ξ. ⊞

Lemma 2.8.6: Truth Lemma.
Let $\mathfrak{M}^w$ be a w-canonical model, Δ a world of $\mathfrak{M}^w$. Then

$$\mathfrak{M}^w, \Delta \models \phi \iff \phi \in \Delta.$$

Proof.
In the same manner as in section 7, we prove that for every w-theory Δ and for every diamond $\Diamond$, triangle $\vartriangle$ we have

$$\begin{array}{rcl} \Diamond\phi \in \Delta & \iff & \text{there is a w-theory } \Delta' \text{ with } (\Delta, \Delta') \in R^w_\Diamond \text{ and } \phi \in \Delta', \\ \phi_1 \vartriangle \phi_2 \in \Delta & \iff & \text{there are w-theories } \Delta_1, \Delta_2 \text{ with} \\ & & (\Delta, \Delta_1, \Delta_2) \in R^w_\vartriangle \text{ and } \phi_i \in \Delta_i. \end{array}$$

As we can also show that $\mathfrak{F}^w$ is standard, the truth lemma follows by a straightforward formula induction. ⊞

Lemma 2.8.7.
Let $\mathfrak{G}^w = (\mathfrak{F}^w, A^w)$ be a w-canonical general versatile frame witnessing against Ξ. Then $\mathfrak{F}^w$ is in $\mathsf{Fr}^{v,\neq}_{-\Xi}$.

Proof.
Let Δ be a world of $\mathfrak{F}^w$. As Δ is a w-theory of the logic, we can find for every $\xi \in \Xi$

propositional variables $\vec{p}$ with $\neg\xi(\vec{p}) \in \Delta$. By the truth lemma then $\mathfrak{M}^w, u \models \neg\xi(\vec{p})$ So $\mathfrak{F}^w, \Delta \not\models \xi$, for all $\xi \in \Xi$. The proof that $\mathfrak{F}^w$ is standard and versatile runs just like in section 7. ⊞

Proof of theorem 2.8.2.
Soundness is already proved in the introduction to this chapter. For completeness, let Δ be a $K_S^t D^+(\Sigma, -\Xi)$- consistent set of formulas. By the extension lemma, Δ is contained in a w-theory Δ'. Let $\mathfrak{M}^w$ be the w-canonical model of Δ'. By the truth lemma,

$$\mathfrak{M}^w, \Delta' \models \phi \text{ for all } \phi \in \Delta'.$$

A (by now) standard argument shows that $\mathfrak{F}^w$ is versatile, so by lemma 2.8.7, $\mathfrak{F}^w$ is in $\mathsf{Fr}^{v,\neq}_{-\Xi}$. It is in Fr_Σ by the facts that $\mathfrak{G}^w$ is discrete (every w-theory is distinguishing!) and that $\mathfrak{G}^w \models \Sigma$. So we have satisfied Δ in a model based on a frame in the intended class $\mathsf{Fr}^{v,\neq}_{(\Sigma,-\Xi)}$. ⊞

Just like in section 6, we can prove a poorer version of Theorem 2.8.2 for arbitrary (not versatile) similarity types, but we leave this to the reader.

2.9 Conclusions, Remarks and Questions.

2.9.1 General Conclusions.

This chapter was a study in the semantics and axiomatics of non-ξ rules, styled after Gabbay's (Generalized) Irreflexivity Rule.
On the semantic side, we defined $\mathsf{K}_{-\Xi}$ as the class of frames $\mathfrak{F}$ in K where no $\xi \in \Xi$ holds anywhere, i.e. for no $\xi \in \Xi$ is there a w in $\mathfrak{F}$ with $\mathfrak{F}, w \models \xi$. In general, such a class will not be *definable* by a modal formula. Natural examples are formed by the irreflexive, asymmetric or transitive frames.
The main result of this chapter, the $SN\Xi$-theorem 2.8.2 states that under certain conditions, classes of the form $\mathsf{K}_{-\Xi}$ are *axiomatizable*, by a derivation system having a non-ξ rule for every $\xi \in \Xi$. In the various sections of this chapter we have discussed these conditions.
The most elegant formulation of the $SN\Xi$-theorem is in the case where the similarity type is *versatile* and contains the D-operator. For such a similarity type, our result gives a nice derivation system for every class $\mathsf{K}_{-\Xi}$ where K is a class of D-standard, versatile frames which is characterized by a set of *Sahlqvist axioms*. For poorer similarity types, there are various options, of which we list a few:

(i) If the similarity type is not versatile, we have to add a *schema* of non-ξ rules (cf. sections 6 and 7).

(ii) If not all diamonds are tense, only *Sahlqvist tense* formulas are allowed as axioms (cf. sections 5 and 6).
(iii) If the similarity type S does not contain the D-operator, the theorem does not apply directly.

Fortunately, this does not mean that the full power of the $SN\Xi$-theorem is lost for these poorer similarity types; one only has to work a bit harder for it. To give an example: in many cases, over the class $\mathsf{K}_{-\Xi}$ we can *define* the D-operator in the poorer formalism, so that we can work with this defined D'-operator. Each chapter of this thesis contains a worked out example of this idea.
So, more than a theorem, the $SN\Xi$-concept is a *procedure* to find axiomatizations for non-ξ classes:

(i) Find the proper characterization of the class (maybe in an extended similarity type).
(ii) Apply the $SN\Xi$-theorem, immediately obtaining a strongly sound and complete derivation system.
(iii) Try to simplify this system.

This schedule will be used througout this dissertation.

It would be unfair not to mention the fact that axiomatizations using non-ξ rules have some *disadvantages* too: first of all, such axiomatizations may not have all the nice mathematical properties that orthodox axiomatization have. For example (cf. Goldblatt [43]): define, for a logic Λ, the corresponding algebraic variety V_Λ of Boolean Algebras with Operators as the class of algebras where the set of equations $\{\phi = 1 \mid \Lambda \vdash \phi\}$ is valid. Now for a finite *orthodox* Λ, the complement of V_Λ will be closed under ultraproducts, while this need not be the case for an unorthodox Λ. Second, by the nature of the derivation rule, it may be necessary to add new propositional variables to the language in order to derive a formula ϕ, whence we have *less control* on derivations in these unorthodox systems.

These disadvantages take us to the question, in which cases a non-ξ rule can be *eliminated* from a system.

2.9.2. Conservativity.

An interesting point which has not been discussed yet concerns the question whether non-ξ rules add new theorems to a logic.

Some scattered results are known:
In the introduction we saw an example where a rule is *conservative*: the logic K^t4 already axiomatizes the class of irreflexive transitive tense frames, so adding IR does not produce any new theorem.
On the other hand, adding IR to $K^tL(Gp \rightarrow p)$ makes this logic inconsistent, so here IR is not conservative. In Zanardo [141], Zanardo replaced the irreflexivity rule used in

Burgess [23] to axiomatize a branching-time temporal logic, by (infinitely many) axioms. An similar case is found in cylindric modal logic and the modal logic of relation algebras (cf. Venema [131, 132]), where adding a non-ξ rule to a finite set of axioms creates a finite derivation system for a logic which is known not to be finitely axiomatizable when only the orthodox derivation rules MP, UG and SUB are allowed. A striking difference between a uni-directional similarity type and its tense counterpart concerns the modal logic of the two-dimensional 'domino relation', where an axiomatization of the uni-directional modal logic needs *both* infinitely many axioms *and* a non-ξ rule (cf. Kuhn [68]), while the tense logic allows a finite and orthodox axiomatization (cf. Venema [135]).

The general question

> Are there natural (syntactic/semantic) criteria deciding when a non-ξ rule is conservative over a derivation system?

lies (almost) completely open. We have one minor result: recall that a formula is *closed* if it does not contain propositional variables (only constants).

Definition 2.9.1.
A logic Λ has the *interpolation property* (IP) if $\Lambda \vdash \phi \rightarrow \psi$ implies the existence of an *interpolant* χ in the common language of ϕ and ψ, such that $\Lambda \vdash \phi \rightarrow \chi$ and $\Lambda \vdash \chi \rightarrow \psi$. ⊞

Proposition 2.9.2.
Let Λ be a logic and ξ a formula, such that
(i) Λ has the IP.
(ii) for every closed formula γ, $\Lambda(-\xi) \vdash \gamma$ implies $\Lambda \vdash \gamma$.
Then $N\xi R$ is conservative over Λ.

Proof.
Assume that Λ and ξ satisfy (i) and (ii). Denote derivability in Λ by $\vdash$. To show that NξR is conservative over Λ, we must prove

$$\vdash \neg\xi(\vec{p}) \rightarrow \phi \quad \Rightarrow \vdash \phi, \text{ if no } p_i \text{ occurs in } \phi$$

So assume $\vdash \neg\xi(\vec{p}) \rightarrow \phi$ where $\vec{p} \notin \phi$. By (i) there is an interpolant γ for $\neg\xi(\vec{p})$ and ϕ; γ must be closed, as $\neg(\vec{p})$ and ϕ do not share any variables.
As $\vdash \neg\xi(\vec{p}) \rightarrow \gamma$, one application of $N\xi R$ shows that γ is a $\Lambda(-\xi)$-theorem, so by (ii), $\vdash \gamma$. Now $\vdash \phi$ is immediate by $\vdash \gamma \rightarrow \phi$. ⊞

2.9.3. Questions and Remarks.

We end this chapter with some miscellaneous questions and remarks:

(i) The most obvious question is whether the $SN\Xi$-result can be extended to similarity types not having the D-operator or tense diamonds, and to arbitrary canonical formulas. Independently from our result, Goranko [45] announces a similar meta-theorem on *weak* completeness, for arbitrary canonical formulas. Hodkinson [35] extends our result to a similarity type where diamonds come in pairs too, here having *complementary* accessibility relations ($R_{-\diamond} = (R_\diamond)^c$).

(ii) Call a class *negatively definable* if it is of the form $\mathsf{Fr}_{-\Xi}$. There seems to be an interesting connection between this notion and what Kracht calls *describable properties*, cf. [66]. Is there a *structural characterization* for negatively definable classes, like there is for modally definable classes? It is not difficult to see that negatively definable classes are closed under disjoint unions and generated subframes; any $\mathsf{Fr}_{-\Xi}$ *reflects* p-morphic images, and if it is elementary, ultrafilter extensions too. Do these preservation properties give the desired characterization for (elementary) negatively definable classes?

(iii) Let Λ be the set of formulas $\Theta(\mathsf{Fr}_{(\Sigma,-\Xi)})$, and Fr_Λ the class of frames where Λ is valid. What is the relation between $\mathsf{Fr}_{(\Sigma,-\Xi)}$ and Fr_Λ?

(iv) Consider the tense similarity type with diamonds $\{F, P, D\}$. To axiomatize the irreflexive frames, we now have the choice between the F-irreflexivity *rule* and the *axiom* $Fp \rightarrow Dp$. When and how can rules be replaced by axioms, and vice versa?

(v) An interesting aspect of non-ξ rules is that in some sense they behave like axioms; in the introduction we already saw how they *characterize* the class $\mathsf{K}_{-\xi}$ as the class of frames where $N\xi R$ is *sound*.
Maybe it is better to use the term *anti-axioms*[7], however, according to their behaviour in derivation systems: an orthodox derivation system $MD = (AX, \{MP, UG, SUB\})$ generates a logic, to be precisely, the *smallest* set of formulas containing the axioms AX which is closed under MP, UG and SUB. For the set Λ of formulas generated by the axiom system $MD(-\xi) = (AX, \{MP, UG, SUB, N\xi R\})$, we may add the clause that Λ, if consistent, must also be the *least* set of formulas *not containing* the formula ξ (to be more precisely, not containing ξ in any 'existential position', like in $\diamond(\delta \wedge \xi)$).

(vi) In the following chapters we will see *examples* of applications of the $SN\Xi$-theorem in algebraic logic, but there is also a *general* perspective. Recall that in the theory of Boolean Algebras with Operators one is interested in representing algebras over sets, and defines *canonical extensions* for this aim. Now in fact, our 'constructive' way of defining distinguishing and witnessing theories, leading to the notion of distinguishing resp. witnessing canonical frames, constitutes a *new set representation of free algebras* over sets. In this new way of representing algebras, one seems to have *more control* on the properties of the frame than in the ordinary representation over ultrafilters. Obvious questions are to extend the construction to *arbitrary* algebras, and to investigate its (algebraic) properties.

[7] This explains our notation '$-\xi$'

CHAPTER 3.

THE SQUARE UNIVERSE.

Outline.

Giving a detailed treatment of two-dimensional modal logics, this chapter forms the pivot of the thesis.

In the introduction we give an overview of the literature on two-dimensional modal logics; we also provide a uniform, technical framework for the topic. Section 2 deals with a rather simple formalism, two-dimensional cylindric modal logic. In section 3 we give an extensive account of the modal counterpart $\mathcal{CC}\mathcal{S}$ of relation algebras. In section 4 we add a temporal component to the formalism $\mathcal{CC}\mathcal{S}$, obtaining a two-dimensional temporal logic. The last but one section shows what results in algebraic logic our modal approach yields. We finish in section 6 with conclusions, and questions for further research.

3.1 Introduction.

3.1.1. Why.

There are several reasons for developing a framework of modal logic in which the possible worlds are *pairs* of elements of the model instead of the points themselves, and in the literature one can find similar ideas arising in this direction, from different backgrounds, sometimes in quite different disciplines.

First, in *tense logic* there is a research line inspired by *linguistic* motivations. In ordinary tense logic as developed by Prior (cf. [100]), the truth of a formula is only dependent of the form of the formula itself and the time when its truth is evaluated. However, temporal discourse has both a *referential* and a *deictic* side: the truth of a proposition may change not only with the point of *reference*, but also with the point of *utterance* by the speaker. In the seventies, the development of formal linguistics and the strive to provide a logical foundation for it, led people like Gabbay, Guenthner, Kamp, Segerberg and Åqvist to investigate two-dimensional temporal logics taking care of these phenomena, cf. for example [30, 33, 62, 119, 142]. So these papers all have a temporal aspect in common, e.g. models always having some ordering relation. For an overview we refer to Gabbay [35], in section 4 we will give some more details on two-dimensional temporal logic.
In order to provide a modal framework where the absolute (logical) and the relative (e.g. physical) necessity can be distinguished, both Humberstone [55] and van Fraassen [29] suggest a two-dimensional approach. Studying Humberstone's system, Kuhn gives a technical account in [67] of the two-dimensional modal logic of the domino relation ($(u,v)R(x,y)$ iff $v = x$).
More abstract is the approach of V. Shekhtman, whose motivation to study many-dimensional modal logics seems to come mainly from pure logic. In [120], his aim is to axiomatize the 'Cartesian product' of two modal logics. For example, the intended frames of the Cartesian product of $S4$ and $S5$ consists of a set of possible worlds of the form $U_0 \times U_1$, with a reflexive transitive relation on U_0 and an equivalence relation U_1. An earlier article in this direction is Davis [26].
Related frameworks with a two-dimensional *flavour* are branching-time temporal logics (cf. Zanardo [140]), combinations of modal and temporal logics (cf. Thomason [128]), and in computer science, formalizations of the behaviour of distributed systems (cf. Spaan [124]. In these three areas, the worlds in the intended frames can be seen as pairs consisting of a timepoint, together with respectively a branch of time, a state of affairs and a computation sequence.
In chapter 5 we will give a detailed account of how temporal logics of *intervals* fit in the two-dimensional picture.
Many of the systems described above may be perceived as following a general trend in modal logic, namely of bridging the gap between the old-fashioned intensional framework, which is simple, elegant and has nice computational properties, and classical first order logic which is expressive and perhaps still more familiar. Examples of other extensions

of the classical modal formalism are: Blackburn [19] and Gargov and Goranko [39] add 'nominals' or 'names' to the language, these being atomic propositions holding at unique possible worlds. Orłowska [92], Humberstone [56] and Goranko [44] enrich the system with new operators, e.g. one for having the complement of the accessibility relation of the original ◇ as its accessibility relation. The D-operator discussed in the previous chapter can be seen as another example.
Let us have a closer look at this relation between modal logics and classical logic. It is well-known (cf. van Benthem [14]), that (on the model level) modal logic corresponds to a fragment of first order logic, and that there exists standard translations from modal logics to first order formulas. The co-domain of these translations is formed by a set of first order formulas having one free variable, in a language with a fixed set of accessibility predicates and arbitrarily many monadic predicate variables. One-dimensional extensions of the simple modal logic aim at reaching a larger set of first order formulas, but still these formulas have only monadic predicates (besides the accessibility predicates) and one free variable. In the above-mentioned two-dimensional extensions, these parameters are shifted too: the co-domain of the translation[1] may now contain formulas in *two* free variables and with *dyadic* predicate variables.
These characteristics of two-dimensional modal logics are shared by the *algebraic* theory of binary relations (cf. Németi [89] for an introductory overview), which forms our last source motivating the study of two-dimensional modal logics. The analogies between algebras of binary relations and two-dimensional modal logics are striking but (as far as we know) they have never been made this explicit. We hope to do so in this chapter — in fact the idea can be put into one slogan:

> Algebras of binary relations are the modal algebras of two-dimensional modal logics.

This idea forms our guideline in choosing the particular examples of two-dimensional logics that we will study.

3.1.2. How.

In this subsection we will acquaint the reader with the technicalities of our approach to many-dimensional modal logics. We will first give the general setting, then we mention two examples known from the literature. So, to start with the general idea, we need:

Definition 3.1.1.
Consider the similarity type S_2 with a modal constant δ, the following monadic operators: ⬙, ⟐, ⬙′, ⟐′, ⬖, ⬗, ⊗, ⊖ and ⦶, and the dyadic operator $\circ$. ⊞

By Appendix A.4,5 we have a definition of a semantics for S_2 and its subtypes. The *intended* semantics for S_2 however has a two-dimensional character, the set of possible

[1] Note that for two-dimensional logics, this translation (cf. section 3.3.3 for details) is *not* the same as the standard correspondence map as defined for the similarity type.

worlds consisting of a Carthesian square[2] and the interpretation map having a fixed and uniform definition for this kind of universes.

Definition 3.1.2.
A *two-dimensional frame* or a *square* is a frame $\mathfrak{F} = (W, I)$ where $W = U \times U$ for some set U, and the definition of I is given as follows:

$$
\begin{array}{lcl}
I(\delta) & = & \{(u,v) \mid u = v\} \\
I(\diamondminus) & = & \{((u,v),(x,y)) \mid v = y\} \\
I(\diamondbar) & = & \{((u,v),(x,y)) \mid u = x\} \\
I(\diamondminus') & = & \{((u,v),(x,y)) \mid u \neq x, v = y\} \\
I(\diamondbar') & = & \{((u,v),(x,y)) \mid u = x, v \neq y\} \\
I(\diamondleftblack) & = & \{((u,v),(x,y)) \mid u = y\} \\
I(\diamondtopblack) & = & \{((u,v),(x,y)) \mid v = x\} \\
I(\otimes) & = & \{((u,v),(x,y)) \mid u = y, v = x\} \\
I(\obar) & = & \{((u,v),(x,y)) \mid u = x = y\} \\
I(\ominus) & = & \{((u,v),(x,y)) \mid v = x = y\} \\
I(\circ) & = & \{((u,v),(w,x),(y,z)) \mid u = w, v = z, x = y\}.
\end{array}
$$

A *two-dimensional* or *square* model is a model based on a two-dimensional frame. ⊞

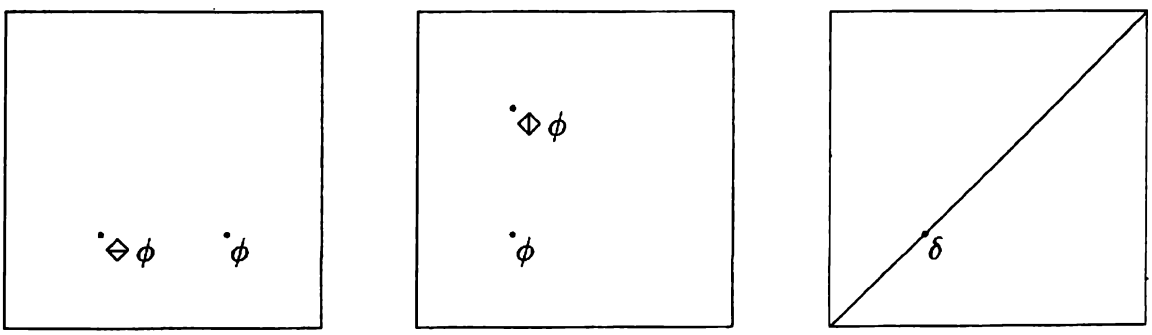
We are interested in the following questions for subtypes S of S_2:

(1) Can we distinguish the squares among the (abstract) S-frames? And if so, in which language (M_S or L_S) and how exactly?
(2) Can we give a derivation system generating the S-formulas valid in the class of two-dimensional frames?
(3) Which fragment of first order logic does the set of S-formulas capture?

A nice thing about two-dimensional modal logic is that structures can be represented geometrically, in a very intuitive way. Let $\mathfrak{M}$ be a square model, and (x, y) a world in $\mathfrak{M}$, then

$$
\begin{array}{lcl}
\mathfrak{M}, x, y \models \diamondminus \phi & \iff & \text{there is a z in } U \text{ with } \mathfrak{M}, z, y \models \phi \\
\mathfrak{M}, x, y \models \diamondbar \phi & \iff & \text{there is a z in } U \text{ with } \mathfrak{M}, x, z \models \phi \\
\mathfrak{M}, x, y \models \delta & \iff & x = y
\end{array}
$$

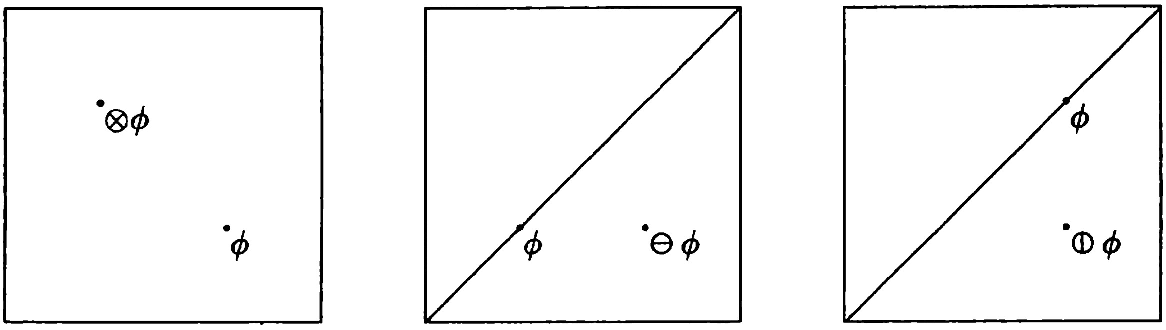
viz.



[2]We do not consider the case where the universe is of the form $U_0 \times U_1$ with possibly *different* base sets U_0 and U_1. For this wider case an analogous theory can be developed.

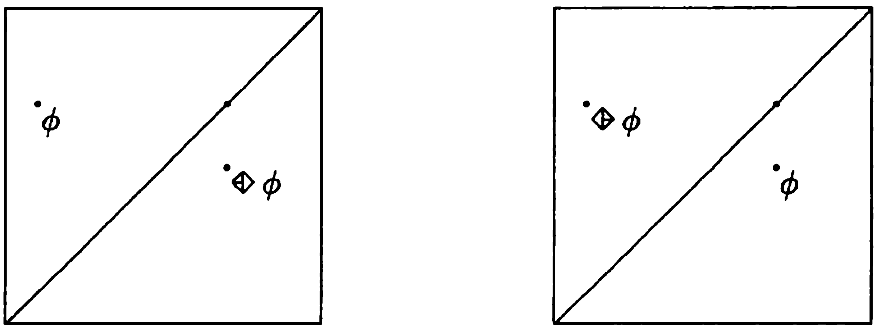
$$\mathfrak{M}, x, y \models \otimes\phi \iff \mathfrak{M}, y, x \models \phi.$$
$$\mathfrak{M}, x, y \models \ominus\phi \iff \mathfrak{M}, y, y \models \phi$$
$$\mathfrak{M}, x, y \models \obar\phi \iff \mathfrak{M}, x, x \models \phi.$$

viz.



$$\mathfrak{M}, x, y \models \diamondminus\phi \iff \text{there is a z in } U \text{ with } \mathfrak{M}, z, x \models \phi.$$
$$\mathfrak{M}, x, y \models \diamondbar\phi \iff \text{there is a z in } U \text{ with } \mathfrak{M}, y, z \models \phi$$

viz.



We do not show the pictures of the irreflexive versions $\diamondminus', \diamondbar'$ of $\diamondminus, \diamondbar$; for the picture of the dyadic operator $\circ$, we refer to section 3.3.2.

In the literature some subsystems of D_2 have been studied explicitly: we mention the similarity type SEG, with operators $\{\diamondminus, \diamondbar, \otimes, \ominus, \obar\}$. It was first studied by Segerberg in [119], where he gave a finite axiomatization of the SEG-formulas valid in the class of two-dimensional frames. He also showed that this validity problem is decidable, and he introduced the intuitive symbols $\diamondminus$, $\diamondbar$, etc. Kuhn treated the $\{\diamondbar\}$-fragment KU in a paper [68], providing an complete derivation system with infinitely many derivation rules and a non-ξ rule. In Venema [135] it was shown, that adding the inverse operator $\diamondbar$ to KU, one can give a finite and orthodox axiomatization.

Finally, a remark about conventions: as the interpretation of the operators in D_2 is uniformly defined for all squares, we usually neglect mentioning the interpretation when referring to a two-dimensional frame. Also, par abus de notation, we will not write $\mathfrak{M} = (U \times U, V)$ but $\mathfrak{M} = (U, V)$ to denote a two-dimensional model. The purpose of this is that now models can be seen as structures for classical first order logic with dyadic predicates: every modal propositional variable, corresponding to a dyadic predicate in the first order logic, is indeed interpreted as a binary relation over U.

3.2 Two-dimensional cylindric modal logic.

3.2.1. Two-dimensional cylindric modal logic.

In this section we will focus our attention on a particular two-dimensional modal logic:

Definition 3.2.1.
Consider the similarity type of *cylindric modal logic of dimension 2* $CML_2 = \{\diamondminus, \diamondbar, \delta\}$, for which we introduce some auxiliary terminology. A CML_2-frame is called a *2-frame* and usually represented as a quadruple $F = (W, H, V, D)$ where H, V and D are the accessibility relations of respectively $\diamondminus$, $\diamondbar$ and δ. A *2-model* is a model based on a 2-frame. *Two-dimensional frames* (*squares*) have been defined in 3.1.2. The class of CML_2-squares is denoted by C_2.
For a 2-formula ϕ, its *mirror image* ϕ^m is obtained by replacing all occurrences in ϕ of $\diamondminus$ by $\diamondbar$ and vice versa. ⊞

The symbols H, V and D are mnemonics for the horizontal resp. vertical accessibility relation and the diagonal, cf. the pictures in the introduction. Note that in two-dimensional models, the *projection* operators $\ominus, \obar$, and the *domino* operators $\diamondplus, \diamondplus$ can be *defined* as abbreviations in CML_2. This is not the case for the *irreflexive* versions $\diamondminus', \diamondbar'$ of $\diamondminus, \diamondbar$, but then, strangely enough, we do have a definition of the D-operator (cf. section 2.4) in our language:

Definition 3.2.2.
We use the following abbreviations:

$$
\begin{array}{rclrcl}
\diamondplus\,\phi & = & \diamondminus\diamondbar\,\phi & Z_H\phi & = & \obar\,\diamondminus(\neg\delta \wedge \diamondbar\,\phi) \\
\ominus\,\phi & = & \diamondminus(\delta \wedge \phi) & Z_V\phi & = & \ominus\,\diamondbar(\neg\delta \wedge \diamondminus\,\phi) \\
\obar\,\phi & = & \diamondbar(\delta \wedge \phi) & D_2\phi & = & Z_H\phi \vee Z_V\phi.
\end{array}
$$

Proposition 3.2.3.
Let $\mathfrak{M} = (U, V)$ be a two-dimensional model, $u, v \in U$. Then

(i)	$\mathfrak{M}, u, v \models \diamondplus\,\phi$	$\iff$	there are x, y in U with $\mathfrak{M}, x, y \models \phi$,
(ii)	$\mathfrak{M}, u, v \models \ominus\,\phi$	$\iff$	$\mathfrak{M}, v, v \models \phi$,
(iii)	$\mathfrak{M}, u, v \models \obar\,\phi$	$\iff$	$\mathfrak{M}, u, u \models \phi$,
(iv)	$\mathfrak{M}, u, v \models D_2\phi$	$\iff$	there is a $(u', v') \neq (u, v)$ with $\mathfrak{M}, u', v' \models \phi$.

Proof.
We only treat **(iv)**. First we show

$$
\begin{array}{rl}
\mathfrak{M}, u, v \models & Z_H\phi \\
\iff & \mathfrak{M}, u, v \models \obar\,\diamondminus(\neg\delta \wedge \diamondbar\,\phi) \\
\iff & \mathfrak{M}, u, u \models \diamondminus(\neg\delta \wedge \diamondbar\,\phi) \\
\iff & \text{there is a } u' \text{ with } \mathfrak{M}, u', u \models \neg\delta \wedge \diamondbar\,\phi \\
\iff & \text{there is a } u' \neq u \text{ with } \mathfrak{M}, u', u \models \diamondbar\,\phi \\
\iff & \text{there are } u', w \text{ with } u' \neq u \text{ and } \mathfrak{M}, u', w \models \phi
\end{array}
$$

So $u, v \models Z_H\phi$ iff there is a point with a *different first* coördinate where ϕ holds. Likewise, $u, v \models Z_V\phi$ iff there is a point with a *different second* coördinate where ϕ holds. As two pairs of points are different iff (at least) one of their coördinates is different, this implies that D_2 indeed has the inequality relation as its accessibility relation, in two-dimensional frames. ⊞

We chose to study the fragment CML_2 of S_2 in order to have the important two-dimensional *cylindric algebras* (cf. Henkin-Monk-Tarski [53]) as (a subclass of) our modal algebras; this connection however will not be studied before section 3.5.

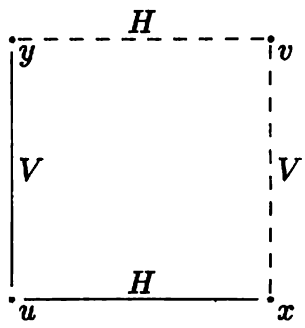
As was already mentioned in the introduction, we are interested in the connection between 2-frames and two-dimensional frames. It is not very hard to give a characterization of two-dimensional frames in the first order language L_{CML_2}, and as the analogous task for an extension of CML will be undertaken in the next section, we omit it here.
The axiomatization problem is much more interesting, and the remainder of this section will be devoted to it. We propose the following axioms:

Definition 3.2.4.
Consider the following CML_2-formulas:
$(CH1)$ $p \to \diamondminus p$
$(CH2)$ $p \to \boxminus \diamondminus p$
$(CH3)$ $\diamondminus \diamondminus p \to \diamondminus p$
$(CH4)$ $\diamondminus \diamondvert p \to \diamondvert \diamondminus p$
$(CH5)$ $\diamondminus \delta$
$(CH6)$ $\diamondminus(\delta \wedge p) \to \boxminus(\delta \to p)$
$(CH7)$ $(\delta \wedge \diamondminus(\neg p \wedge \diamondvert p)) \to \diamondvert(\neg\delta \wedge \diamondminus p)$
The mirror image of (CHi) is denoted by (CVi), $(CHVi)$ denotes $(CHi \wedge CVi)$. ⊞

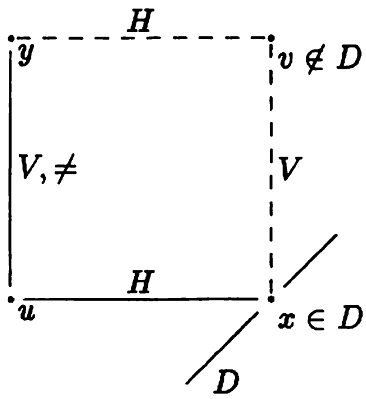
All these formulas are in Sahlqvist form. It is convenient to have names for their Sahlqvist correspondents:

Definition 3.2.5.
Define the following L_{CML_2}-formulas:
$(NH1)$ Hxx
$(NH2)$ $\forall y(Hxy \to Hyx)$
$(NH3)$ $\forall y \forall z((Hxy \wedge Hyz) \to Hxz)$
$(NH4)$ $\forall uy((Hxu \wedge Vuy) \to \exists v(Vxv \wedge Hvy))$
$(NH5)$ $\exists y(Hxy \wedge Dy)$
$(NH6)$ $\forall yy'((Hxy \wedge Hxy' \wedge Dy \wedge Dy') \to y = y')$
$(NH7)$ $\forall uy((Dx \wedge Hxu \wedge Vuy \wedge u \neq y) \to \exists v(Vxv \wedge \neg Dv \wedge Hvy))$.
The mirror image of (NHi) is denoted by (NVi), $NHVi = NHi \wedge NVi$. ⊞

So, $NH1$, $NH2$ and $NH3$ express that H is respectively reflexive, symmetric and transitive; together they state that H is an equivalence relation. $NH5$ and $NH6$ then mean that in every H-equivalence class there is exactly one element on the diagonal D ($NH5$ for existence and $NH6$ for unicity). The meaning of $NH4$ and $NH7$ is best made clear by the following pictures:



NH4



NH7

For these formulas, the correspondence part of Sahlqvist's theorem means the following:

Theorem 3.2.6.
Let $\mathfrak{F}$ be a 2-frame, then for $i = 1, \ldots, 7$, $Z \in \{H, V\}$:

$$\mathfrak{F}, w \models CZi \iff \mathfrak{F} \models NZi[x \mapsto w].$$

Proof.
The claim is immediate by the fact that $(CZi)^s = (NZi)$ and theorem 2.2.2. We give some details of the computation of the Sahlqvist equivalent $(CH7)^s$ of $CH7$, as defined in 2.3.13. Clearly we have $\mathfrak{F}, w \models CH7$ iff

$$\begin{aligned}\mathfrak{F}, w \models \forall P \quad & \big[(Dx_0 \wedge \exists x_1(Hx_0x_1 \wedge \neg Px_1 \wedge \exists x_2(Vx_1x_2 \wedge Px_2)) \\ & \rightarrow \exists z_1(Vx_0z_1 \wedge \neg Dz_1 \wedge \exists z_2(Hz_1z_2 \wedge Pz_2))\big],\end{aligned}$$

which is equivalent to

$$\begin{aligned}\mathfrak{F}, w \models \forall P \forall x_1 \forall x_2 \quad & \big[(Dx_0 \wedge Hx_0x_1 \wedge \neg Px_1 \wedge Vx_1x_2 \wedge Px_2 \\ & \rightarrow \exists z_1(Vx_0z_1 \wedge \neg Dz_1 \wedge \exists z_2(Hz_1z_2 \wedge Pz_2))\big].\end{aligned}$$

Then the previous statements are equivalent to

$$\begin{aligned}\mathfrak{F}, w \models \forall P \forall x_1 \forall x_2 \quad & \big[(Dx_0 \wedge Hx_0x_1 \wedge Vx_1x_2 \wedge \forall y(x_2 = y \rightarrow Py)) \\ & \rightarrow (Px_1 \vee \exists z_1(Vx_0z_1 \wedge \neg Dz_1 \wedge \exists z_2(Hz_1z_2 \wedge Pz_2)))\big].\end{aligned}$$

So by definition 2.3.13 we obtain $(CH7)^s$ by substituting $v = x_2$ for Pv everywhere in the above formula (the 'minimal' substitution making the antecedent true). This gives

$$\begin{aligned}\mathfrak{F}, w \models \forall x_1 \forall x_2 \quad & \big[(Dx_0 \wedge Hx_0x_1 \wedge Vx_1x_2) \\ & \rightarrow (x_1 = x_2 \vee \exists z_1(Vx_0z_1 \wedge \neg Dz_1 \wedge \exists z_2(Hz_1z_2 \wedge z_2 = x_2)))\big],\end{aligned}$$

or, equivalently

$$\begin{aligned}\mathfrak{F}, w \models \forall x_1 \forall x_2 \quad & \big[(Dx_0 \wedge Hx_0x_1 \wedge Vx_1x_2 \wedge x_1 \neq x_2) \\ & \rightarrow \exists z_1(Vx_0z_1 \wedge \neg Dz_1 \wedge Hz_1x_2))\big],\end{aligned}$$

which is NH7. ⊞

In fact, $(CH7)$ is the modal counterpart of (a simplified version of) Henkin's equation which plays an important rôle in the theory of cylindric algebras. For details we refer to subsection 3.5.2.

3.2.2. Two-dimensional cylindric completeness.

In this section we prove a completeness result for CML_2: we will give a strongly sound and complete axiom system for the set of 2-formulas that are valid in the class C_2 of squares.

Definition 3.2.7.
Let A_2 be the basic axiom system K_{CML_2} for CML_2 extended with the axioms $CHV1$, $\ldots$, $CHV7$. ⊞

Theorem 3.2.8.
A_2 is strongly sound and complete with respect to C_2.

The proof of theorem 3.2.8 will consist of two parts: First we show that A_2 is strongly sound and complete for the class of so-called *hypercylindric frames*, and then we show that these hypercylindric frames and the two-dimensional ones have the same modal theory.

Definition 3.2.9.
A 2-frame F is *hypercylindric* if $F \models CHV1 \wedge \ldots \wedge CHV7$. The class of hypercylindric 2-frames is denoted by HCF_2. ⊞

Theorem 3.2.10.
A_2 is strongly sound and complete with respect to HCF_2.

Proof.
An immediate consequence of the completeness part 2.2.2(ii) of Sahlqvist's theorem. ⊞

Note that for CML_2 the notion of a *zigzag morphism* boils down to the following: let F and F' be two 2-frames, then a map $f : W \mapsto W'$ is a zigzag morphism iff it satisfies the following properties:
(1) f is a homomorphism
(2) Du if $D'fu$
(3) If $H'fuv'$ then there is a $v \in W$ such that Huv and $fv = v'$
(4) If $V'fuv'$ then there is a $v \in W$ such that Vuv and $fv = v'$
We will call a map f satisfying the first two conditions a *potential zigzag morphism.*

The following lemma states that every hypercylindric frame is a zigzagmorphic image of a disjoint union of two-dimensional frames. This immediately implies that $\Theta(\mathsf{C}_2) = \Theta(\mathsf{HCF}_2)$.

Theorem 3.2.11.

$$\mathsf{HCF}_2 = \mathbf{H_f P_f C_2}.$$

Proof.
Clearly every square is hypercylindric, so $\mathbf{H_f P_f C_2} \subseteq \mathsf{HCF}_2$.
For the other direction, observe that $H|V$ is an equivalence relation, in fact the accessibility relation of the $S5$-diamond $\diamondsuit$. Call a frame *connected* if this relation $H|V$ is total, *nice* if it is connected and hypercylindric. It is an easy observation that every hypercylindric frame is a disjoint union of nice frames, so it suffices to show that

every nice frame is a zigzagmorphic image of a square.

So, let $\mathfrak{F} = (W, H, V, D)$ be a nice frame. We will define a chain of potential zigzag morphisms $(f_\xi)_{\xi<\lambda}$ (where λ is the maximum of $|W|$ and ω), such that the union f_λ of this chain is the desired zigzag. Every map f_ξ should be seen as an approximation of f_λ. Look at the set of *potential defects* $P = \lambda \times \lambda \times W \times \{\ominus, \diamondsuit\}$. Call the quadruple $(\beta, \gamma, v, \Diamond) \in P$ a *defect* of a homomorphism $f : \xi \times \xi \mapsto W$ (where $\xi < \lambda$), if it defies one of the zigzag conditions (3) or (4), e.g. for (4): $\Diamond = \diamondsuit$, $(\beta, \gamma) \in \xi \times \xi$ and $Vf(\beta, \gamma)v$ while there is no $\gamma' \in \xi$ such that $f(\beta, \gamma') = v$; f is called *perfect* if it has no defects. Assume that P is well-ordered, then we may speak of the *first* defect $D(f)$ of an imperfect potential zigzag morphism $f : \xi \times \xi \mapsto W$. By the following lemma such a map has an extension f' lacking the defect $D(f)$:

Claim.
Let $f : \xi \times \xi \mapsto W$ be a potential zigzag morphism, $(\beta, \gamma, v, \Diamond)$ a defect of f. Then there is a potential zigzagmorphism $f' \supset f$, $f' : (\xi + 1) \times (\xi + 1) \mapsto W$ such that $(\beta, \gamma, v, \Diamond)$ is not a defect of f'.

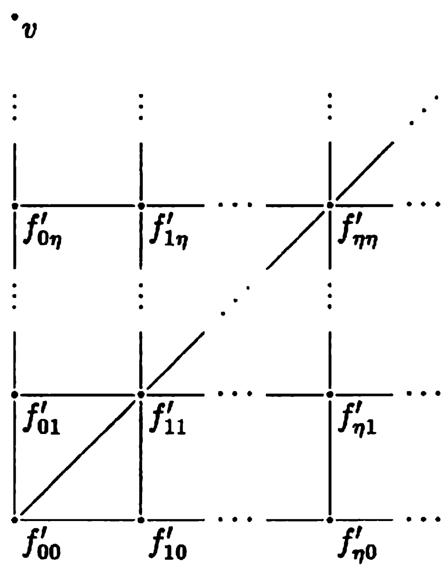
Proof.
Without loss of generality we assume that $\beta = \gamma = 0$ and $\Diamond = \diamondsuit$.
We first set

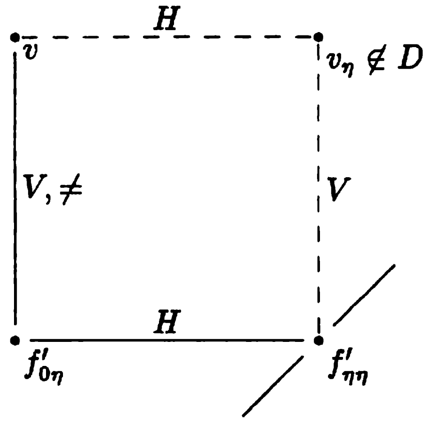
$$\begin{aligned} f'(\zeta, \eta) &= f(\zeta, \eta) \text{ for } \zeta, \eta < \xi, \\ f'(0, \xi) &= v, \end{aligned}$$

viz. the left picture on the next page (where we denote $f'(\zeta, \theta)$ by $f'_{\zeta\theta}$).
Next we are concerned with the $f'(\eta, \xi)$, $\eta < \xi$. By assumption we have $v \neq f(0, \eta)$, and as f is a potential zigzag morphism we get a situation as showed in the right picture. By $\mathfrak{F} \models NV7(f'(\eta, \eta))$, $\mathfrak{F}$ has a $v_\eta \notin D$ with $Hv_\eta v$ and $Vv_\eta f'(\eta, \eta)$. We define

$$f'(\eta, \xi) = v_\eta.$$

and set $f'(\xi, \xi)$ as the unique diagonal H-successor of any/all of the $f'(\eta, \xi)$.
It is straightforward to verify that with this definition the part of f' defined up till now satisfies both conditions (1) and (2).





For the definition of $f'(\xi, \eta)$ $(\eta < \xi)$, we use the same trick as above to ensure $f'(\xi, \eta) \notin D$: as $f'(\xi, \xi)$ is in D and $f'(\eta, \xi)$ is not, they cannot be identical. So $f'(\eta, \xi)$ can be defined as any non-diagonal H-successor of $f'(\eta, \eta)$ which is a V-successor of $f'(\xi, \xi)$ (such a $f'(\xi, \eta)$ exists by $NH7$). ⊞

We now define the chain of maps as follows:

$$\begin{array}{lcll} f_0 & = & \{((0,0),u)\} & \text{for some } u \text{ on the diagonal of } F. \\ f_{\xi+1} & = & \begin{cases} f_\xi & \text{if } f_\xi \text{ is perfect.} \\ (f_\xi)' & \text{otherwise} \end{cases} & \\ f_\theta & = & \bigcup_{\xi<\theta} f_\xi & \text{if } \theta \leq \lambda \text{ is a limit ordinal} \end{array}$$

It is now straightforward to verify that f_λ has the desired properties: first of all it is a potential zigzag morphism as all the maps in the chain are. Suppose that f_λ is not a zigzagmorphism, then there are quadruples in P witnessing this shortcoming. Let $\pi = (\beta, \gamma, v, \Diamond)$ be the first of these in the well-ordering of P, suppose its ordinal number is η. Take $\theta = max(\beta + 1, \gamma + 1)$, then π is a defect of f_θ. It need not be its first one, but there can be at most η problems before π that are more urgent. So π must be the first defect of $f_{\theta+\eta}$, whence it can not be a defect of $f_{\theta+\eta+1}$. But this gives a contradiction, since $f_{\theta+\eta+1} \subseteq f_\lambda$. So f_λ is a zigzag.
Finally, the proof that f_λ is surjective is straightforward by the connectedness of $\mathfrak{F}$. ⊞

Proof of theorem 3.2.8.
Immediate by 3.2.10 and 3.2.11.

⊞

3.3 A Modal Logic of Binary Relations.

In this rather large section we treat a second two-dimensional logic, in more detail. Instead of giving an overview, we let the titles of the subsections speak for themselves.

3.3.1. (Representable) Relation Algebras.

In the algebraic theory of binary relations (cf. Németi [89] for an overview), one studies operations on the set of binary relations.

Definition 3.3.1.
Let U be some unspecified set. $Re(U)$ is defined as the set of all binary relations on U, i.e. $Re(U) = \{R \mid R \subseteq U \times U\}$.
The *composition* $R|S$ of two relations R and S is defined by

$$R|S = \{(s,t) \in U \times U \mid \exists u((s,u) \in R \wedge (u,t) \in S)\},$$

the *converse* of a relation R is

$$R^{-1} = \{(s,t) \in U \times U \mid (t,s) \in R\}$$

and finally the *diagonal* is the relation

$$Id = \{(s,t) \in U \times U \mid s = t\}. \qquad \boxplus$$

Although Tarski [126] was not the first one to suggest an *algebraic* treatment to the subject (cf. Maddux [80] for a historical introduction), his approach set the standard:

Definition 3.3.2.
A *relation type algebra* is defined as a Boolean Algebra with the following operators: a binary ;, a unary $\breve{}$ and a constant 1'.
The class FRA of *full relation algebras* consists of those relation type algebras that are isomorphic to an algebra of the form

$$\mathfrak{Re}(U) = (Re(U), \cup, ^c, |, ^{-1}, Id).$$

The class RRA of *representable relation algebras* is defined as the variety generated by FRA, i.e. RRA = V(FRA). $\boxplus$

The question naturally arises as to study the representable relation algebras. By applying some elementary universal algebra (cf. Burris-Sankappanavar [24]), we obtain RRA = HSP(FRA) and by Birkhoff's theorem we have $\mathfrak{A}$ is in RRA iff all the equations holding in FRA are true in $\mathfrak{A}$. Tarski proved that every representable relation algebra is a subalgebra of a direct product of full relation algebras, i.e. RRA = SP(FRA). Some reflection then shows that every RRA can be embedded in an algebra of the form

$(P(E), \cup, ^c, |, ^{-1}, Id)$ where E is an equivalence relation over some set U.

In order to enumerate the equations holding in the variety RRA, Tarski proposed the following axiomatization:

Definition 3.3.3.
A *relation algebra* is a relation type algebra $\mathfrak{A} = (A, +, -, ;, \breve{}, 1')$ in which the following axioms are valid:

$(RA0)$ Axioms stating that $(A, +, -)$ is a Boolean Algebra
$(RA1)$ $(x + y); z = x; z + y; z$
$(RA2)$ $(x + y)\breve{} = x\breve{} + y\breve{}$
$(RA4)$ $(x; y); z = x; (y; z)$
$(RA5)$ $x; 1' = x$
$(RA6)$ $(x\breve{})\breve{} = x$
$(RA7)$ $(x; y)\breve{} = y\breve{}; x\breve{}$
$(RA8)$ $x\breve{}; -(x; y) \leq -y$.

The class of relation algebras is denoted by RA. ⊞

For an introduction to the theory of relation algebras and their arithmetic, we refer to Jónsson [58, 59], or to Tarski-Givant [127], where the formalism $\mathcal{L}^{\times}$ can be seen as an alphabetic variant for the arithmetic of relation algebras that are generated by one element.

It soon turned out that the RA-axioms do not exhaustively generate all valid principles governing binary relations. There are RAs that are not representable, as was first shown by Lyndon [73]; perhaps the simplest, finite example was provided by McKenzie and can be found in [127]. The question whether finitely many equations might be added to the RA-axioms was answered negatively by Monk in [81], while in [83] he showed that it is not even sufficient to add infinitely many axioms in only a finite number of relation variable symbols. Recent further strengthenings of this negative result have been found by Haiman [47] and by Andréka [5].
On the other hand, explicit *infinite* axiomatizations are known: cf. Lyndon [73] or McKenzie [74]. Unfortunately, these axiomatizations are intuitively not very appealing. Wadge gave another way of recursively enumerating *Equ*(RRA) using a Gentzen-type deduction method, cf. Wadge [137] or Maddux [77]. Here variables referring to elements of the domain are introduced again in the proofs, thus violating the paradigm of algebraic logic *not* having such variables.
Natural *sufficient* but not necessary conditions for representability can be found in Maddux [75].

The study of relation algebras is not restricted to algebraic logic: in de Roever [106], they are used in a computer framework science framework, for proving program correctness. Recently, van Benthem [15] found interesting applications for relation algebras in a general theory of information processing. He shows various connections with linguistics and inference systems. These patterns are also present in Roorda [107], who treats a *modal* logic closely related to the formalism presented in the next section.

3.3.2. A Modal Logic of Binary Relations.

In this section we give the modal system in which the modal algebras have the type of relation algebras. Therefore this logic must have the following signature:

Definition 3.3.4.
$\mathcal{CC\delta}$ is the modal similarity type $\{\circ, \otimes, \delta\}$ with $\circ$ a dyadic, $\otimes$ a monadic modal operator, and δ a modal constant. ⊞

Just like in the case of CML_2, we have two kinds of semantics: the intended models have a cartesian square as the set of possible worlds; they form a subclass of the class of $\mathcal{CC\delta}$-models provided by the general definition of a semantics for a modal similarity type.

Definition 3.3.5.
A *$\mathcal{CC\delta}$-frame* is a quadruple $\mathfrak{F} = (W, C, R, I)$ with $C \subseteq {}^3W$, $R \subseteq {}^2W$ and $I \subseteq W$. A $\mathcal{CC\delta}$-frame is *two-dimensional*, or a *square*, if W is of the form $W = U \times U$ for some set U, and

$$\begin{array}{rcl} C &=& \{((u,v),(w,x),(y,z)) \in {}^3(U \times U) \mid u = w \wedge x = y \wedge v = z\} \\ R &=& \{((u,v),(x,y)) \in {}^2(U \times U) \mid u = y \wedge v = x\} \\ I &=& \{(u,v) \in U \times U \mid u = v\}. \end{array}$$

The class of two-dimensional $\mathcal{CC\delta}$-frames (squares) is denoted by SQ. ⊞

So, in a two-dimensional model $\mathfrak{M}$ we have

$$\begin{array}{lcl} \mathfrak{M}, u, v \models \delta & \iff & u = v \\ \mathfrak{M}, u, v \models \otimes\phi & \iff & \mathfrak{M}, v, u \models \phi \\ \mathfrak{M}, u, v \models \phi \circ \psi & \iff & \text{there is a } w \text{ with } \mathfrak{M}, u, w \models \phi \text{ and } \mathfrak{M}, w, v \models \psi, \end{array}$$

viz.

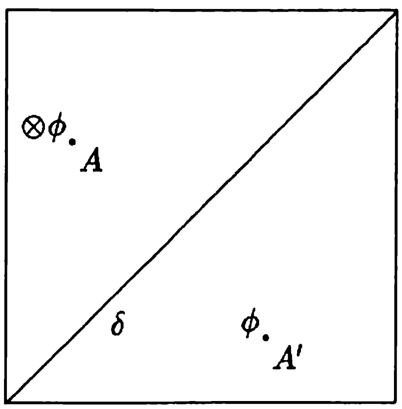


fig. 1: δ and $\otimes$

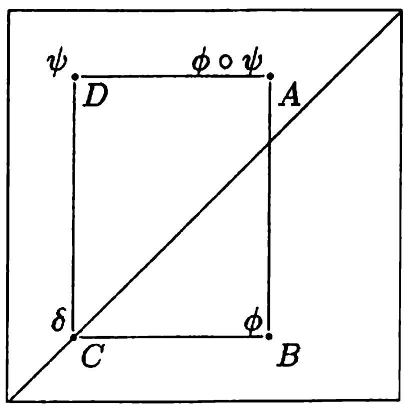


fig. 2: $\circ$

As we have already seen, the constant δ is true precisely on the diagonal, to verify if $A \models \otimes\phi$ we look at the image A' of A after reflecting in the diagonal. The formula $\phi \circ \psi$ holds at a point A if we can build a rectangle $ABCD$ with the following properties:

$B \models \phi$, $D \models \psi$ and C lies on diagonal, where B is the vertex on the same *vertical* line as A and D on the same *horizontal* as A.

The connection with RRAs is given by the following

Proposition 3.3.6.
$\mathfrak{F}$ is a square iff $\mathfrak{Cm}\mathfrak{F}$ is (isomorphic to) a full relation algebra. ⊞

Note that the cylindric operators $\ominus$ and $\oplus$, their irreflexive companions $\ominus', \oplus'$ and the D-operator can be seen as *abbreviated operators* of $\mathcal{CC\delta}$:

Definition 3.3.7.
Define

$$\begin{aligned}
\ominus\phi &= \top \circ \phi \\
\oplus\phi &= \phi \circ \top \\
\ominus'\phi &= \neg\delta \circ \phi \\
\oplus'\phi &= \phi \circ \neg\delta. \\
D'\phi &= \top \circ \phi \circ \neg\delta \vee \neg\delta \circ \phi \circ \top.
\end{aligned}$$

It is a straightforward exercise to verify that in the squares, these defined operators indeed have the right semantics. We only state

Proposition 3.3.8.

$$\mathsf{SQ} \models D\phi \leftrightarrow D'\phi.$$

⊞

3.3.3. $\mathcal{CC\delta}$ and Classical Logic.

In the introduction we have already mentioned that two-dimensional models for $\mathcal{CC\delta}$ can also be seen as ordinary structures for a first order language with dyadic predicates, and that in fact, this identification was precisely the reason to study (the algebraic version of) systems like $\mathcal{CC\delta}$. In this section we will see how far the expressive power of $\mathcal{CC\delta}$ takes us in classical logic. In section 3.4 we give some examples of how this language can express properties of binary relations like transitivity or irreflexivity.

Definition 3.3.9.
Let L be an ordinary signature of first order logic, N a set of L-formulas, $k \leq \omega$ an ordinal and X a set of variables in L. We set

$$\begin{aligned}
N^2 &= \{\phi \in N \mid \phi \text{ contains only dyadic predicates } \} \\
N(X) &= \{\phi \in N \mid \text{ all free variables of } \phi \text{ are in } X \} \\
N_k &= \{\phi \in N \mid \text{ all variables of } \phi \text{ are in } \{x_0, \ldots, x_{k-1}\} \}.
\end{aligned}$$

⊞

We will show that $\mathcal{CC\delta}$ has the same expressive power as $L_3^2(x_0, x_1)$, a fragment of L which we will call "the three variable fragment of first order logic", by a slight abus de langue. We hasten to remark that this claim is an immediate consequence of the fact that the corresponding relation *algebraic* system has the same property. This matter is also

treated in detail in Tarski-Givant [127]. We prove this correspondence directly in order to give a clear picture of what is going on.
Note that, as we have two kinds of semantics for $\mathcal{CC\delta}$, namely the squares and the wider class of more general $\mathcal{CC\delta}$-models, we also have *two* kinds of correspondence maps. The one below is directed to the squares:

Definition 3.3.10.
Let $(\cdot)^\circ$ be the following translation from $\mathcal{CC\delta}$-formulas to $L_3^2(x_0, x_1)$:

$$\begin{aligned} p_l^\circ &= P_l x_0 x_1 \\ (\phi \wedge \psi)^\circ &= \phi^\circ \wedge \psi^\circ \\ (\neg\phi)^\circ &= \neg\phi^\circ \\ (\otimes\phi)^\circ &= \phi^\circ(x_0/x_1, x_1/x_0) \\ (\phi \circ \psi)^\circ &= \exists x_2(\phi^\circ(x_2/x_1) \wedge \psi^\circ(x_2/x_0)). \end{aligned}$$ ⊞

Note that in the above definition we tacitly assumed that the substitution of variables can be performed in L_3^2. This is not very difficult but rather tedious to establish, so we refer to Gabbay [32] or Tarski-Givant [127]. The following proposition states that every $\mathcal{CC\delta}$-formula has an equivalent in $L_3^2(x_0, x_1)$:

Proposition 3.3.11.
Let $\mathfrak{M} = (U, V)$ be a two-dimensional model. Then

$$\mathfrak{M}, u_0, u_1 \models \phi \iff \mathfrak{M} \models \phi^\circ[x_i \mapsto u_i].$$

Proof.
By a trivial formula induction. ⊞

To show that conversely, every $L_3^2(x_0, x_1)$-formula has an equivalent in $\mathcal{CC\delta}$, is a bit harder. Maybe the easiest proof uses a second subset of L as an intermediate system:

Defininition 3.3.12.
In this definition we assume $\{i, j, k\} = \{0, 1, 2\}$. By induction we define the sets $L^+(x_i, x_j)$: All atomic formulas in $L^2(x_i, x_j)$ are (atomic) formulas of $L^+(x_i, x_j)$. $L^+(x_i, x_j)$ is closed under Boolean formula-building, and finally, if ϕ is in $L^+(x_i, x_j)$ and ψ is in $L^+(x_j, x_k)$, then $\exists x_j(\phi \wedge \psi)$ is in $L^+(x_i, x_k)$. ⊞

$L^+(x_i, x_j)$ is designed to be the *exact* first order counterpart of $\mathcal{CC\delta}$:

Definition 3.3.13.
Let $(\cdot)^{ij}$ be the following translation from $L^+(x_i, x_j)$ to $\mathcal{CC\delta}$:

$$\begin{aligned} (P_l x_i x_j)^{ij} &= p_l & \qquad (x_i = x_j)^{ij} &= \delta \\ (P_l x_j x_i)^{ij} &= \otimes p_l & (x_j = x_i)^{ij} &= \delta \\ (P_l x_i x_i)^{ij} &= \oplus(p_l \wedge \delta) & (x_i = x_i)^{ij} &= \top \\ (P_l x_j x_j)^{ij} &= \ominus(p_l \wedge \delta) & (x_j = x_j)^{ij} &= \top \end{aligned}$$

$$\begin{aligned} (\neg\phi)^{ij} &= \neg\phi^{ij} \\ (\phi \wedge \psi)^{ij} &= \phi^{ij} \wedge \psi^{ij} \\ \exists x_k(\phi(x_i, x_k) \wedge \psi(x_k, x_j))^{ij} &= \phi^{ik} \circ \psi^{kj}. \end{aligned}$$

Proposition 3.3.14.
Let $\mathfrak{M}$ be a two-dimensional model, ϕ in $L^+(x_i, x_j)$. Then

$$\mathfrak{M} \models \phi[x_i \mapsto u_i, x_j \mapsto u_j] \iff \mathfrak{M}, u_i, u_j \models \phi^{ij}.$$

Proof.
By a trivial formula-induction. ⊞

So we are finished if we can prove that every $L_3^2(x_0, x_1)$- formula has an equivalent in $L^+(x_0, x_1)$. We need something stronger:

Proposition 3.3.15.
Every formula in L_3^2 is equivalent to a Boolean combination of L^+-formulas.

Proof.
By induction to the complexity of L_3^2-formulas. We only treat the induction step involving the existential quantifier.
Let $\phi \in L_3^2$ be of the form $\phi = \exists x_2 \psi$. As ϕ is in L_3^2, so is ψ. By induction hypothesis then, ψ is equivalent to a Boolean combination of $L^+(x_i, x_j)$-formulas. Assume that this combination is in disjunctive normal form, and distribute the $\exists x_2$ over the disjuncts. This shows that in fact we may assume that ψ is equivalent to a *conjunction* $\psi_{01} \wedge \psi_{02} \wedge \psi_{12}$ with $\psi_{ij} \in L^+(x_i, x_j)$. Clearly then ϕ is equivalent to $\psi_{01} \wedge \exists x_2(\psi_{02} \wedge \psi_{12})$ which is a Boolean combination of two $L^+(x_0, x_1)$-formulas. ⊞

Proposition 3.3.16.
Let ϕ be in $L_3^2(x_0, x_1)$. Then ϕ has an equivalent $\phi^\bullet$ in *CCS* such that for all two-dimensional models:

$$\mathfrak{M} \models \phi[x_i \mapsto u_i] \iff \mathfrak{M}, u_0, u_1 \models \phi^\bullet.$$

Proof.
Let ϕ^+ be ϕ's $L^+(x_0, x_1)$-equivalent, which exists by the previous proposition. Then take $\phi^\bullet = (\phi^+)^{01}$. The claim then follows by proposition 3.3.14. ⊞

3.3.4. *CCS* and Relation Algebras.

In this subsection the exact relation between *CCS* and RAs will be discussed. By a result of Maddux [76], it is known that the algebraic equations defining RA have first order equivalents in the frame language. Our aim is to show that there are *Sahlqvist formulas* characterizing AtRA. Particularly in this area, we are deeply indebted to Johan van Benthem for his guidance into correspondence theory. The observation that the RA-axioms are Sahlqvist forms was made and worked out during conversations with him, and is also reported on in van Benthem [16].
The first order *language* of the Sahlqvist correspondents is slightly different from our L_{CCS} having C, R and I as accessibility predicates, however. Because of the intended interpretation of $(\cdot)^{-1}$ as taking the *unique* converse of a relation, the *relation* symbol R is replaced by a *function* symbol f. This inspires the following definition:

Definition 3.3.17.
An *arrowframe* is a quadruple $\mathfrak{F} = (W, C, f, I)$ with $C \subseteq {}^3W$, $f \in {}^WW$ and $I \subseteq W$. Elements of W are called *arrows*. An *arrowmodel* is a $\mathcal{CC}\delta$-model based on an arrowframe. L_A is the first order language for arrowframes, having relation symbols C (ternary), I (unary) and a function symbol f (unary). ⊞

Of course, the arrowframes form a subclass of the $\mathcal{CC}\delta$-frames, viz. those $\mathcal{CC}\delta$ frames where R is *functional.* For the truth relation in arrowmodels we obtain

$$\mathfrak{M}, w \models \otimes\phi \iff \mathfrak{M}, fw \models \phi.$$

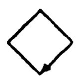
Note that in all squares R is functional.
From now on we will concentrate on arrowframes rather than on $\mathcal{CC}\delta$-frames, and for example use the first order language L_A instead of $L_{\mathcal{CC}\delta}$. This change of language is not essential, because the class of arrowframes is quite easy characterizable by a Sahlqvist formula:

Proposition 3.3.18.
Let $\mathfrak{F} = (W, C, R, I)$ be a $\mathcal{CC}\delta$-frame, then

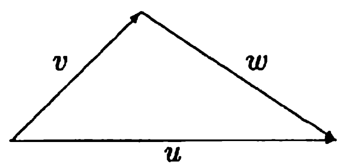
$$\begin{array}{lcl} \mathfrak{F} \models \otimes p \to \neg \otimes \neg p & \iff & \mathfrak{F} \models \forall x \forall y \forall z (Rxy \wedge Rxz \to y = z) \\ \mathfrak{F} \models \neg \otimes \neg p \to \otimes p & \iff & \mathfrak{F} \models \forall x \exists y Rxy \\ \mathfrak{F} \models \otimes p \leftrightarrow \neg \otimes \neg p & \iff & R \text{ is functional.} \end{array}$$

Proof.
These are standard Sahlqvist equivalences, cf. 2.2.2. ⊞

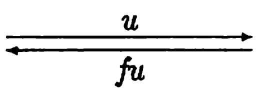
Arrowframes are known from the literature, cf. Maddux [76]. (The nice name 'arrowframe' is due to Johan van Benthem.) Informally, we will use the following pictures to represent arrows and their relations:

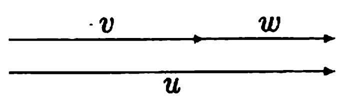


Iu



or



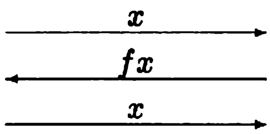


u, fu $Cuvw$

We now give the modal version of the RA-axioms:

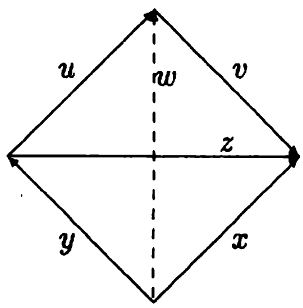
Definition 3.3.19.
Set the following (pairs of) $CC\delta$- and L_A-formulas

(CC0) $\otimes p \leftrightarrow \neg \otimes \neg p$
(CC1) $p \to \otimes\otimes p$
(AR1) $ffx = x$

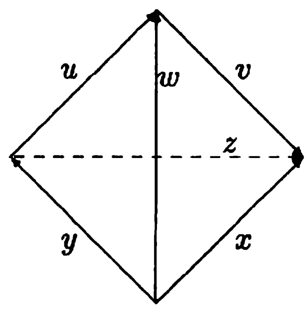


AR1

(CC2) $p \circ (q \circ r) \to (p \circ q) \circ r$
(CC3) $(p \circ q) \circ r \to p \circ (q \circ r)$
(AR2) $\forall yzuv((Cxyz \wedge Czuv) \to \exists w(Cxwv \wedge Cwyu))$
(AR3) $\forall ywuv((Cxwv \wedge Cwyu) \to \exists z(Cxyz \wedge Czuv))$

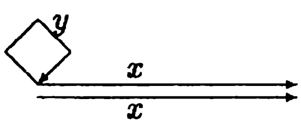


AR2

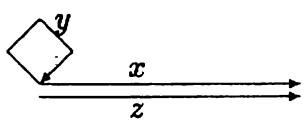


AR3

(CC4) $p \to \delta \circ p$
(CC5) $\delta \circ p \to p$
(AR4) $\exists y(Iy \wedge Cxyx)$
(AR5) $\forall yz(Cxyz \wedge Iy) \to x = z)$



AR4

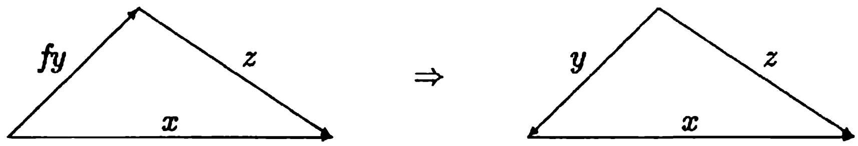


AR5

(CC6) $\otimes(p \circ q) \to (\otimes q \circ \otimes p)$
(CC7) $(\otimes q \circ \otimes p) \to \otimes(p \circ q)$
(AR6) $\forall yz(Cfxyz \to Cxfzfy)$
(AR7) $\forall yz(Cxfzfy \to Cfxyz)$

*AR*6 and *AR*7

$(CC8)$ $\otimes p \circ \neg(p \circ q) \wedge q \rightarrow \bot$
$(AR8)$ $\forall yz(Cxfyz \rightarrow Czyx)$



*RA*8

(A $CC\delta$-frame or) an arrowframe is called *relational* if ($CC0$ and) $CC1, \ldots, CC8$ hold in it. The class of relational frames is denoted by AR. ⊞

The relational frames are the *atom structures* of the relation algebras; we state the converse proposition, which is immediate by Appendix A.19:

Theorem 3.3.20.
RA = CmAR. ⊞

Warning 3.3.21: It is tempting to see these pictures as graphs, and reason accordingly. But as not all relational frames are squares, such reasoning could lead to wrong conclusions: one should be careful not to use intuitions about graphs that are not explicitly justified by the axioms. However, it might follow from theorem 5 of Maddux [77] that in subgraphs containing not more than *four* points, all graph-based intuitions are *sound.* (By lack of space, we can not go into details.)

As we had already announced, the CC-axioms are Sahlqvist formulas; so we have both a correspondence and a completeness result.

Definition 3.3.22.
Let AR be the minimal axiom system $K_{CC\delta}$, extended with $CC0 \ldots CC8$ as axioms. ⊞

Theorem 3.3.23.
For $i = 1, \ldots, 8$: $\mathfrak{F} \models CCi \iff \mathfrak{F} \models ARi$.

Theorem 3.3.24.
AR is strongly sound and complete with respect to AR.

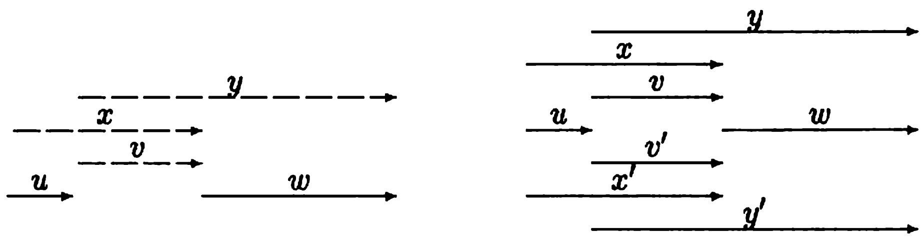
Proofs.
Immediate by the Sahlqvist form of the axioms and theorem 2.2.2. ⊞

3.3.5. Characterizing squares.

In this section we set out to characterize the class SQ of two-dimensional arrow- resp. $\mathcal{CCS}$-frames. The simplest way to do so is in the first order language L_A.

Definition 3.3.25.
Define the following L_A-formulas:
(ARU) $\forall xuu'vv'ww'yy'((Cxuv \wedge Cx'uv' \wedge Cyvw \wedge Cy'v'w) \rightarrow v = v')$
(ARE) $\forall uw\exists vxy(Cxuv \wedge Cyvw)$.
⊞

$\mathfrak{F} \models ARE$ means that for every arrow pair u and w there is a *connecting* arrow v, while $\mathfrak{F} \models ARU$ means that such a connecting arrow must be unique, viz.



ARE *ARU*

Theorem 3.3.26.
Let $\mathfrak{F}$ be an arrowframe. Then

$$\mathfrak{F} \text{ in SQ} \iff \mathfrak{F} \models AR1, \ldots, AR8, ARU \;\&\; ARE.$$

Proof.
The direction from left to right is straightforward, so we only prove the other side: suppose $\mathfrak{F} = (W, C, f, I)$ is a relational arrow frame satisfying ARU and ARE. We will show that $\mathfrak{F}$ is isomorphic to the square based on $I \subseteq W$. To this end we give an isomorphism $g : W \mapsto I \times I$. Before doing so, we prove an extra fact about F, namely that the mirror images of $AR4$ and $AR5$ hold in it:

Claim 1: $\mathfrak{F} \models \forall x\exists y(Iy \wedge Cxxy)$ and $\mathfrak{F} \models \forall xyz(Cxzy \wedge Iy \rightarrow z = y)$.
Proof. It belongs to the standard arithmetic of relation algebras that $x; 1' = x$ holds in an RA. This implies that if $\mathfrak{F}$ is in AR, we have $\mathfrak{F} \models p \circ \delta \leftrightarrow p$. Taking the Sahlqvist correspondents of the formulas $p \circ \delta \rightarrow p$ and $p \rightarrow p \circ \delta$, we immediately obtain claim 1.

We can now define a *unique* 'left point'- arrow l_u and 'right point'-arrow r_u of an arrow u:

Claim 2: $\mathfrak{F} \models \forall u\exists! l(Culu \wedge Il)$ and $\mathfrak{F} \models \forall u\exists! r(Cuur \wedge Ir)$.
Proof: Existence is given for l by $(AR4)$, and for r by claim 1.
Uniqueness we only prove for l: suppose there are l, l' with $Culu \wedge Cul'u \wedge Il \wedge Il'$. By $AR2$ there is an m with $Cml'l$. It is straightforward to verify that by Il and Il' this implies $l = m = l'$.

So we are justified in defining the 'left point' l_u and 'right point' r_u of an arrow u as the arrows satisfying the first resp. last condition of the second claim. We can now define the isomorphism $g : W \mapsto {}^2I$ by

$$gu = (l_u, r_u)$$

Claim 3: g is surjective.
Proof: Let l, r be in I. By ARE there are v, x, y with $Cxlv$ and $Cyvr$.
$Cxlv$ and Il imply $x = v$ by $AR5$, $Cyvr$ and Ir imply $y = v$ by claim 1.
So we have $Cvlv$ and $Cvvr$, implying $gv = (l, r)$.

Claim 4: g is injective.
Proof: Suppose $gu = gu'$. Set $l = l_u = l_{u'}$, $r = r_u = r_{u'}$, then both u and u' connect $l_u = l_{u'}$ and $r_u = r_{u'}$, so by ARU we have $u = u'$.

Claim 5: g is a homomorphism.
Proof: Let $u \in I$. By $Cuuu$ we find $u = l_u = r_u$, so $gu = (u, u) \in I$.
For f we have to show $fgu = gfu$. Now $fgu = (r_u, l_u)$. For gfu, we find $l_{fu} = r_u$, as $fl_u \in I$ and Cul_uu implies $Cfufufl_u$. Similarly, $r_{fu} = l_u$. So, indeed, $gfu = fgu$.
For C, suppose $Cwuv$. We have to show $l_w = l_u$, $r_w = r_v$ and $r_u = l_v$. We only treat $r_u = l_v$: using $AR2$ and $AR3$ we can easily show that both $x = r_u$ and $x = l_v$ satisfy $Cuux$ and $Cvxv$, so by ARU we have $r_u = l_v$.

Claim 6: g is an anti-homomorphism.
Proof: For I: if $l_u = r_u$, we have both u and l_u connecting fu and u, so $u = l_u$ whence $u \in I$.
The part for f is already proved in the previous claim.
For C, suppose $Cgwgugv$, then $l_w = l_u$, $r_w = r_v$ and $r_u = l_v$. Define $m = r_u (= l_v)$.
$Cuum$ and $Cvmv$ imply the existence of an x with $Cxuv$. We will prove that $x = w$. $Cxuv$ implies $l_x = l_u$ (e.g. by the previous claim, where we proved g to be a homomorphism) and $r_x = r_v$. But then $l_x = l_w$ and $r_x = r_w$, so we obtain $x = w$ by the injectivity of g. $Cwuv$ is then immediate. ⊞

Unfortunately, SQ is not characterizable in *CCδ*. Of course, this is obvious as FRA (= CmSQ) is not a variety: it is not closed under products or subalgebras. However, after adding the D-operator (cf. section 2.4), we can define SQ in the new language, by *Sahlqvist* formulas:

Definition 3.3.27.
CCD is the similarity type *CCδ* augmented with the difference operator D. ⊞

Note that the *CCδ*-frames can be identified with the class of *standard* *CCD*-frames. (We refer to definition 2.4.1 and the remarks below it for notions concerning the D-operator, like the definition of O and E, or the identification of *CCδ*-frames with standard *CCD*-frames.)

Definition 3.3.28.
Define the following *CCD*-formulas:

(CCU) $(O\neg p \circ q \circ O\neg r \wedge E(\neg p \circ \neg q \circ \neg r)) \rightarrow \bot$
(CCE) $p \wedge Eq \rightarrow E(p \circ \top \circ q)$.

Proposition 3.3.29.
Let $F = (W, C, f, I)$ be a (D-standard) relational arrowframe. Then

$$\mathfrak{F} \models CCU \iff \mathfrak{F} \models ARU$$
$$\mathfrak{F} \models CCE \iff \mathfrak{F} \models ARE.$$

Proof.
The proposition can be proved by the Sahlqvist form of CCU and CCE, but in this case a direct proof is much more perspicuous; we only treat $CCU \Rightarrow ARU$:
Suppose $\mathfrak{F} \not\models ARU$, then there are arrows u, w, $v \neq v'$, x, x', y and y' as in the picture above theorem 3.3.26. Define a valuation V on $\mathfrak{F}$ with $V(p) = W - \{u\}$, $V(r) = W - \{w\}$ and $V(q) = \{v\}$. Recall that (in a standard frame) $O\phi$ holds at a world iff this world is the *only one* where ϕ holds. Now let z be an arrow with $Czuy$ and $Czxw$; z' is defined likewise. It is straightforward to prove that under this valuation, $z \models O\neg p \circ q \circ O\neg r$ and $z' \models \neg p \circ \neg q \circ \neg r$. This latter fact implies $z \models E(\neg p \circ \neg q \circ \neg r)$, showing $\mathfrak{M}, z \not\models CCU$.
⊞

We now have our desired characterization of the squares: they are the D-standard relational arrowframes where CCU and CCE are valid:

Theorem 3.3.30.
$\mathsf{SQ} = \mathsf{AR}^{\neq}_{CCE \wedge CCU}$.

Proof
By 3.3.23 and 3.3.26. ⊞

We have already seen that on the class of squares, the D-operator is *definable.* Theorem 3.3.30 seems to contradict the fact that SQ is not characterizable in $\mathcal{CC6}$ — why not take the D'-versions of CCE and CCU? The point is that for the D-operator as a *primitive* we have *stipulated* that inequality be its accessibility relation, for the *defined* D'-operator this will not hold for all frames in AtRRA, only for the squares.

3.3.6. Axiomatizing Squares.

As we have found a characterization of the class of two-dimensional frames in terms of Sahlqvist $\mathcal{CCD}$-formulas, we can obtain a completeness result which is an almost immediate consequence of the SD-theorem. Formally however, we have to extend the language once again, as in the formulation of this theorem one needs an *versatile* similarity type:

Definition 3.3.31.
Let $\mathcal{CCD}^t$ be the similarity type $\mathcal{CCD}$ extended with two dyadic modal operators $\circ_1$ and $\circ_2$ besides $\circ$. ⊞

The set $\{\circ, \circ_1, \circ_2\}$ should be seen as a triple of versatile dyadic operators, cf. definition

2.7.1. Note that we can identify $CC\delta$-structures (with CCD-structures and thus) with versatile CCD^t-structures: in versatile CCD^t-frames, the accessibility relations of $\circ_1$ and $\circ_2$ are given by $R_{\circ_1}vwu \leftrightarrow Cuvw \leftrightarrow R_{\circ_2}wuv$.

Definition 3.3.32.
Let $ACCD^t$ be the axiom system $K^t_{CCD^t}D^+$ (cf. 2.7.2 and 2.4.2), extended with the axioms $CC0, \ldots, CC8$, CCE and CCU and the following:
(XD) $Dp \leftrightarrow (\neg\delta \circ p \circ \top) \vee (\top \circ p \circ \neg\delta)$.

Theorem 3.3.33.
$ACCD^t$ is strongly sound and complete with respect to the class of squares.

Proof.
Immediate by the SD-theorem 2.7.7 and 3.3.30. (In fact, the axiom XD is superfluous; note that it is *sound.*) ⊞

We will now show that we can in fact formulate a much simpler sound and complete axiom system which does not need to go beyond the borders of the old language $CC\delta$.

Definition 3.3.34.
Let AR^+ be the extension of AR with the irreflexivity rule for D':
$(IR_{D'})$ $\vdash (p \wedge \neg D'p) \rightarrow \phi \Rightarrow \vdash \phi$, if $p \notin \phi$. ⊞

In other words, AR^+ has as its axioms: all propositional tautologies, distribution for $\circ$ and $\otimes$, and $CC0 \ldots CC8$. Its rules are MP, UG, SUB and $IR_{D'}$.
We will prove that $ACCD^t$ is a *conservative extension* of AR^+, using induction to the length of $ACCD^t$-derivations. Loading the induction hypothesis makes the proof more perspicuous; we need the following definition:

Definition 3.3.35.
Let $(\cdot)^o$ be the following translation from CCD^t-formulas to $CC\delta$-formulas:

$$\begin{array}{rcl} p^o &=& p \\ \delta^o &=& \delta \\ (\neg\phi)^o &=& \neg\phi^o \\ (\phi \wedge \psi)^o &=& \phi^o \wedge \psi^o \\ (\otimes\phi)^o &=& \otimes\phi^o \end{array} \qquad \begin{array}{rcl} (\phi \circ \psi)^o &=& \phi^o \circ \psi^o \\ (\phi \circ_1 \psi)^o &=& \psi^o \circ \otimes\phi^o \\ (\phi \circ_2 \psi)^o &=& \otimes\psi^o \circ \phi^o \\ (D\phi)^o &=& D'\phi^o \end{array}$$

Proposition 3.3.36.
For a CCD^t-formula ϕ:

$$ACCD^t \vdash \phi \iff AR^+ \vdash \phi^o.$$

Proof.
The direction $\Leftarrow$ is easy, as all AR^+-axioms are $ACCD^t$-axioms and the irreflexivity rule for D' is easily seen to be a derived rule of $ACCD^t$. (Here the axiom XD comes handy.)
The proof for the other direction is by induction to the derivation of ϕ in $ACCD^t$.
For the basis step, in the next section we will show that

$(*)$ $AR \vdash \alpha^o$ for all axioms α of $ACCD^t$.

If ϕ is derived from earlier theorems by applying one of the orthodox derivation rules, ϕ^o is derived from the translations of these earlier $ACCD^t$-theorems by applying the same rule in AR^+.
So the only case left is where the last step in the $ACCD^t$-derivation of ϕ used the D-irreflexivity rule: we have $ACCD^t \vdash (p \wedge \neg Dp) \rightarrow \phi$, where p does not occur in ϕ. By the induction hypothesis, $AR^+ \vdash (p \wedge \neg D'p) \rightarrow \phi^o$, so an application of $(IR_{D'})$ gives $AR^+ \vdash \phi^o$. ⊞

Theorem 3.3.37. SOUNDNESS AND COMPLETENESS.

$$\Sigma \vdash_{AR^+} \phi \iff \Sigma \models_{SQ} \phi.$$

Proof.
Soundness is immediate. For completeness, let $\Sigma \models_{SQ} \phi$, then $\Sigma \vdash_{ACCD^t} \phi$ by 3.3.33. By definition this means that there are $\sigma_1, \ldots, \sigma_n$ in Σ such that $ACCD^t \vdash (\sigma_1 \wedge \ldots \wedge \sigma_n) \rightarrow \phi$. By 3.3.36 we obtain $AR^+ \vdash ((\sigma_1 \wedge \ldots \wedge \sigma_n) \rightarrow \phi)^o$, but as ϕ and the σ_i are $CC\delta$-formulas, we have $((\sigma_1 \wedge \ldots \wedge \sigma_n) \rightarrow \phi)^o = (\sigma_1 \wedge \ldots \wedge \sigma_n) \rightarrow \phi$. So $(\sigma_1 \wedge \ldots \sigma_n) \rightarrow \phi$ itself is an AR^+-theorem. This gives $\Sigma \vdash_{AR^+} \phi$. ⊞

3.3.7. Some Arrow-Arithmetic.

The previous section had an open end: for some $CC\delta$-formulas we have to show that they are derivable in AR^+.

Proposition 3.3.38.
Let $(\cdot)^o$ be as in 3.3.35. For every $ACCD^t$-axiom α, $AR^+ \vdash \alpha^o$.

Proof.
In fact, the D'-irreflexivity rule is needed for none of these derivations. We will frequently use the completeness theorem 3.2.24, giving *semantic* proofs about relational arrowframes instead of formal derivations. We let O' and E' denote the obvious abbreviations, i.e. $O'\phi = \phi \wedge \neg D'\phi$, $E'\phi = \phi \vee D'\phi$.

Claim 1: $AR \vdash V^o$
Proof. Recall that $V \equiv V0 \wedge V1 \wedge V2$ is the *tense* axiom associated with the operator triple $\{\circ, \circ_1, \circ_2\}$ of CCD^t.
We only treat

$$V_2 = p \wedge \neg(r \circ_2 p) \circ_1 r \rightarrow \bot.$$

An evaluation shows that $V2^o$ is

$$p \wedge r \circ \otimes\neg(\otimes p \circ r) \rightarrow \bot,$$

which by $CC0$, $CC6$ and $CC1$ is equivalent to

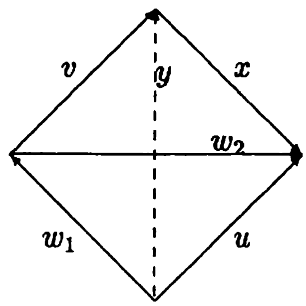
$$p \wedge r \circ \neg(\otimes r \circ p) \rightarrow \bot$$

and then by $CC1$ to

$$p \wedge \otimes\otimes r \circ \neg(\otimes r \circ p) \rightarrow \bot.$$

So by one application of SUB to the axiom $CC8$ it follows that this formula is an AR-theorem.

Claim 2: $AR \vdash D1^o$.
Proof. Recall that $D1$ is the axiom $p \rightarrow \underline{D}Dp$, so it is sufficient to show that AR $\models p \rightarrow \neg D'\neg D'p$.
So suppose that $\mathfrak{M}$ is a model based on a relational arrowframe, and that $\mathfrak{M}, u \models p \wedge D'\neg D'p$. We will derive a contradiction from this.
By $u \models D'\neg D'p$ and the definition of D', there are w_1, w_2 with Cuw_1w_2 and either $w_1 \notin I$, $w_2 \models \neg D'p \circ \top$, or $w_2 \notin I$ and $w_1 \models \top \circ \neg D'p$.
Without loss of generality we assume the first, so there are v, x with Cw_2vx and $v \models \neg D'p$.
By $AR2$ there is a y with Cyw_1v and $Cuyx$, viz.



$Cuyx$ implies $Cyufx$, so $u \models p$ gives $y \models p \circ \top$.
Cyw_1v implies $Cvfw_1y$, and as $fw_1 \notin I$ (by $w_1 \notin I$), we get $v \models \neg\delta \circ (p \circ \top)$, contradicting $v \models \neg D'p$. ⊞

To show that $AR \vdash D2^o$, $AR \vdash D3^o$ (cf. 2.4.2 for definitions), we first prove an auxiliary result. Recall that $\oplus\phi$ is the formula $\top \circ \phi \circ \top$.

Claim 3: $AR \vdash \oplus p \rightarrow p \vee D'p$.
Proof. All of the following formulas are AR-theorems:

$$\begin{array}{ll}
\oplus p \rightarrow & (\delta \vee \neg\delta) \circ p \circ \top \\
\oplus p \rightarrow & (\neg\delta \circ p \circ \top) \vee (\delta \circ p \circ \top) \\
\oplus p \rightarrow & D'p \vee (\delta \circ p \circ (\delta \vee \neg\delta)) \\
\oplus p \rightarrow & D'p \vee ((\delta \circ p \circ \neg\delta) \vee (\delta \circ p \circ \delta)) \\
\oplus p \rightarrow & D'p \vee ((\top \circ p \circ \neg\delta) \vee p) \\
\oplus p \rightarrow & D'p \vee D'p \vee p.
\end{array}$$

Claim 4: $AR \vdash D2^o$, $AR \vdash D3^o$.
Proof. It is fairly easy to establish that the following formulas are AR-theorems:

$$D'D'p \rightarrow \oplus p, \qquad \otimes p \rightarrow \oplus p, \qquad p \circ q \rightarrow \oplus p, \qquad q \circ p \rightarrow \oplus p.$$

The derivation of $D2^o$ and $D3^o$ in AR is then easy to find, by claim 3.

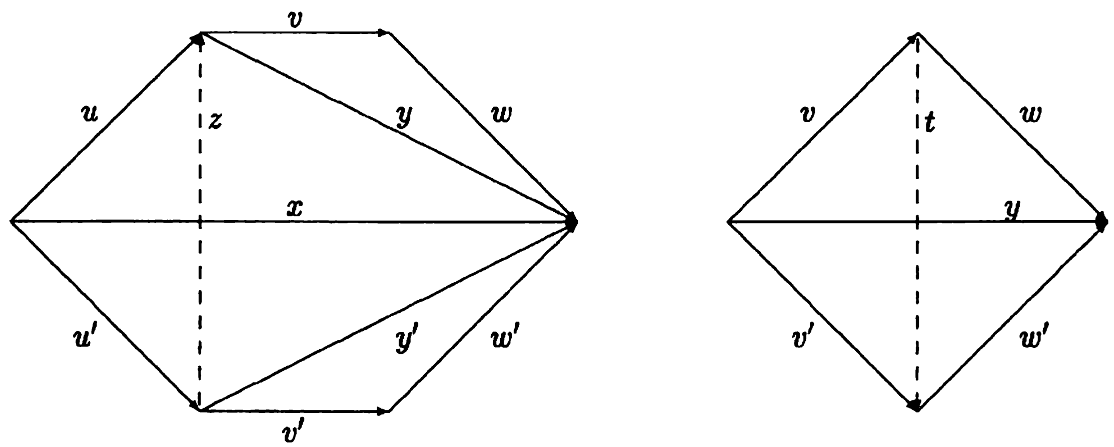
Claim 5: $AR \vdash CCE^o$.

Proof. We will show that $\mathsf{AR} \models p \wedge E'q \rightarrow E'(p \circ \top \circ q)$.
Let u be a world with $u \models p \wedge E'q$. By definition of E', we have $u \models q$ or $u \models \top \circ q \circ \neg\delta$ or $u \models \neg\delta \circ q \circ \top$, of which we only treat the last case. Analogous to the proof of claim 2, using the same terminology (and the same picture), we find $w_1 \models \neg\delta$, $v \models q$ and $u \models p$. This gives $y \models p \circ \top \circ q$.
If $x \in I$, we have $u = y$, so $u \models p \circ \top \circ q$.
If $x \notin I$ we have $u \models (p \circ \top \circ q) \circ \neg\delta \Rightarrow u \models \top \circ (p \circ \top \circ q) \circ \neg\delta \Rightarrow u \models D'(p \circ \top \circ q)$.
So we obtain $u \models p \vee D'p$, QED.

Claim 6: $AR \vdash CCU^o$.
Proof.
We will show that

$$\mathsf{AR} \models (O'\neg p \circ q \circ O'\neg r) \wedge E'(\neg p \circ \neg q \circ \neg r) \rightarrow \bot.$$

Suppose otherwise, that in a relational arrowmodel, $x \models (O'\neg p \circ q \circ O'\neg r) \wedge E'(\neg p \circ \neg q \circ \neg r)$. It is a tedious, but not too difficult exercise to show that this implies $x \models \neg p \circ \neg q \circ \neg r$. So there are $u, y, v, w, u', y', v', w'$ such that $u \models O'\neg p$, $v \models q$, $w \models O\neg r$, $u' \models \neg p$, $v' \models \neg q$, $w' \models \neg r$, where these arrows are situated as depicted as in the left figure:



First we show that $u = u'$: by $Cxuy$ and $Cxu'y'$ there is a z with $Cu'xu$ and $Cy'zy$. Suppose that $z \notin I$, then $u' \models \neg p \Rightarrow u \models \neg p \circ \neg\delta \Rightarrow u \models \top \circ \neg p \circ \neg\delta \Rightarrow u \models D'\neg p$, which would contradict $u \models O'\neg p$. So z is in I. But then $u = u'$ by $AR5$ and $y = y'$ by claim 1 in 2.3.26.
So we get a picture as in the right figure. (Note that now we have a reduced 'graph' of four 'points', cf. 3.3.21).
There is a t with $Cv'vt$ and $Cwtw'$. In the same way as for z, we can show that $t \in I$. But then $Cv'vt$ implies $v = v'$, contradicting $v \models q$ and $v' \models \neg q$. ⊞

3.4 A Two-Dimensional Temporal Logic.

In the introduction to this chapter we mentioned extended tense logics as one of the main examples of modal formalisms with a two-dimensional semantics. In this section we will develop such a two-dimensional temporal logic as a simple extension of $\mathcal{CC\delta}$, compare it with some of the existing two-dimensional tense systems and prove some results concerning expressiveness and completeness.

The main idea behind our system $\mathcal{CC\lambda}$ is very simple: as the ordering relation of temporal structures is a *binary* relation, in a two-dimensional modal logic we can introduce a modal *constant* referring to this relation.

The technical framework of this section is closely connected to that in chapter 5. Many results carry over, in both directions. We have tried to keep the overlap as small as possible. For more information on the technical side of this congruence we refer to [131].

Definition 3.4.1.

$\mathcal{CC\lambda}$ is the similarity type $\mathcal{CC\delta}$ extended with a modal constant λ. A *two-dimensional frame* for $\mathcal{CC\lambda}$ is a pair $\mathfrak{T} = (T, <)$ with $<$ a binary relation on T. A *two-dimensional model* is a $\mathcal{CC\lambda}$-model of which the frame is two-dimensional.

The $\mathcal{CC\delta}$-operators obtain their usual interpretation in two-dimensional models for $\mathcal{CC\lambda}$, for λ we have

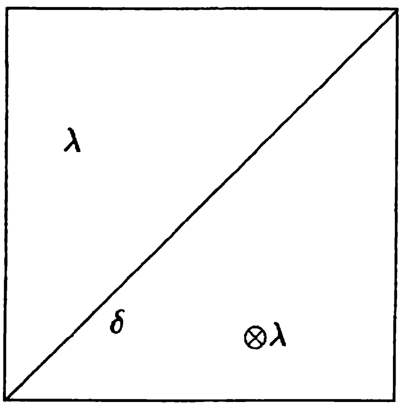
$$\mathfrak{M}, s, t \models \lambda \iff s < t.$$

As abbreviations we define the *compass-operators* by

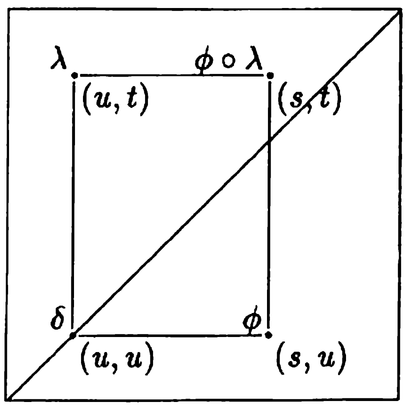
$$\Diamond_N \phi = \phi \circ \otimes\lambda$$
$$\Diamond_W \phi = \otimes\lambda \circ \phi \qquad\qquad \Diamond_E \phi = \lambda \circ \phi$$
$$\Diamond_S \phi = \phi \circ \lambda.$$

Note that in a two-dimensional model $\mathfrak{M}$, these compass-operators get their natural interpretation, e.g.

$$\begin{array}{ll} \mathfrak{M}, s, t \models \Diamond_S \phi & \iff \\ \iff & \text{there is a } u \text{ with } s, u \models \phi \text{ and } u, t \models \lambda \\ \iff & \text{there is a } u \text{ with } s, u \models \phi \text{ and } u < t \\ \iff & \text{there is a point } (s, u) \textit{ south} \text{ of } (s, t) \text{ with } s, u \models \phi, \end{array}$$



λ, δ and $\otimes\lambda$



south.

A nice consequence of having an explicit referent to the (ordering) relation in the object language, is that it becomes very easy to *characterize* properties of $<$:

Definition 3.4.2.
Consider the following $CC\lambda$-formulas:

(TR)	$\lambda \circ \lambda \to \lambda$	(transitivity)
(IR)	$\lambda \to \neg\delta$	(irreflexivity)
(TO)	$\lambda \vee \delta \vee \otimes\lambda$	(totality)
(LN)	$TR \wedge IR \wedge TO$	(linearity)
(DI)	$\lambda \circ \lambda \to \lambda \circ (\lambda \wedge \neg(\lambda \circ \lambda)) \wedge (\lambda \wedge \neg(\lambda \circ \lambda)) \circ \lambda$	(discreteness)
(DE)	$\lambda \to \lambda \circ \lambda$	(denseness)
(W)	$\oplus p \to \oplus(p \wedge \boxminus \boxbar \neg p)$	(well-orderings)
(FP)	$\Diamond \top$	(first point)
(LP)	$\Diamond \top$	(last point)

⊞

Proposition 3.4.3.
Let $\mathfrak{T} = (T, <)$ be a two-dimensional frame. Then

(i)	$\mathfrak{T} \models TR$	$\iff$	$<$ is transitive.
(ii)	$\mathfrak{T} \models IR$	$\iff$	$<$ is irreflexive.
(iii)	$\mathfrak{T} \models TO$	$\iff$	$<$ is total.
(iv)	$\mathfrak{T} \models LN$	$\iff$	$<$ is linear.

Now suppose $<$ is linear. Then

(v)	$\mathfrak{T} \models DI$	$\iff$	$<$ is discrete.
(vi)	$\mathfrak{T} \models DE$	$\iff$	$<$ is dense.
(vii)	$\mathfrak{T} \models W$	$\iff$	$<$ is well-ordered.
(viii)	$\mathfrak{T} \models FP$	$\iff$	T has a first point.
(ix)	$\mathfrak{T} \models LP$	$\iff$	T has a last point.

Proof.
As an example, we prove **(v)**. Let $\mathfrak{T}$ be linear.
First, assume that $\mathfrak{T}$ is discrete, and that $\mathfrak{M}$ is a model on $\mathfrak{T}$ with $\mathfrak{M}, s, t \models \lambda \circ \lambda$. Clearly then t is a successor of s, but not the immediate one. So let u be the immediate successor of s. By linearity of $<$ we have $s < u < t$, and as u is the immediate successor of s: $s, u \models \lambda \wedge \neg(\lambda \circ \lambda)$. So $s, t \models (\lambda \wedge \neg(\lambda \circ \lambda)) \circ \lambda$.
We treat the other conjunct of the consequent in DI likewise, here considering the immediate predecessor of t.
Now, assume that $\mathfrak{T} \models DI$ and let $s < t$. We have to find an immediate successor for s. If t is the immediate successor of s, we are finished. Otherwise, $s, t \models \lambda \circ \lambda$ (in every model on $\mathfrak{T}$), so $s, t \models (\lambda \wedge \neg(\lambda \circ \lambda)) \circ \lambda$ by assumption. By the truth definition, there is a u with $s, u \models \lambda \wedge \neg(\lambda \circ \lambda)$ and $u, t \models \lambda$. It is then straightforward to verify that this u is the immediate successor of s. ⊞

Compared to the existing two-dimensional tense logics, we feel that $CC\lambda$ has the advantage of being both quite expressive and perspicuous. In fact, concerning the first point, all of the systems known to us can be seen as subsystems of $CC\lambda$. For example, the system studied by Åqvist in [142] uses a set of operators that can be defined as the following subtype of $CC\lambda$:

$$\{bf = \lambda, id = \delta, af = \otimes\lambda, \langle P\rangle = \diamondsuit, \langle F\rangle = \diamondsuit, \langle O\rangle = \ominus, \langle X\rangle = \otimes\}.$$

As a second example, one of the systems discussed by Gabbay in [35] has two modal operators, F and P, with F having the following semantics:

$$\mathfrak{M}, s, t \models F\phi \iff \begin{array}{ll} \text{Either} & s = t \text{ and for some } t' > t,\ \mathfrak{M}, s, t' \models \phi \\ \text{or} & s < t \text{ and } \mathfrak{M}, t, t \models \phi \\ \text{or} & s > t \text{ and for some } s < u < t,\ \mathfrak{M}, u, u \models \phi. \end{array}$$

It is a straightforward exercise to show that $F\phi$ can be defined in $CC\lambda$ as

$$(\delta \rightarrow \diamondsuit\,\phi) \wedge (\lambda \rightarrow \ominus\,\phi) \wedge (\otimes\lambda \rightarrow \diamondsuit\,\diamondsuit(\delta \wedge \phi)).$$

Of course, for practical purposes such operators may be necessary: Gabbay's motivation for the introduction of F is that it exactly captures the future perfect tense in English. However, we feel that it is better to use a formalism where the *basic* operators have a more perspicuous semantics, provided that this clarity does not stand in the way of the system's expressive power.

We will now pin down this expressive power of $CC\lambda$ precisely:

Definition 3.4.4.
Let $L^{<}$ denote the set of first order formulas in an ordinary signature with one fixed dyadic predicate symbol $<$ (which is, of course, to be interpreted as the ordering relation in two-dimensional models). ⊞

Recall from the previous section that $L_3^{<2}(x_0, x_1)$ is the set of first order formulas, in a language with binary predicates, one of which is $<$, using only the variables x_0, x_1 and x_2, of which x_0 and x_1 are free. The results in section 3.3 immediately give:

Proposition 3.4.5.
Over the class of two-dimensional models, every $CC\lambda$-formula has a $L_3^{2<}(x_0, x_1)$-equivalent, and vice versa. ⊞

Suppose now that we *restrict* the valuations on two-dimensional models in such a way that atomic propositions correspond to *monadic* predicates instead of dyadic ones. This is not an unusual or unnatural restriction in two-dimensional temporal logics; in fact, both of the systems mentioned above satisfy this constraint.

Definition 3.4.6.
Let $\mathfrak{T} = (T, <)$ be a two-dimensional frame. A $CC\lambda$-valuation $V : Q \mapsto P(T \times T)$ is called *flat* if

$$\text{for all } s, t, t' \in T,\ q \in Q,\ \ (s,t) \in V(q) \iff (s, t') \in V(q).$$

A two-dimensional model $\mathfrak{M} = (T, V)$ is *flat* if V is so. ⊞

So a model is flat if the truth of the *atomic* formulas only depends on the first coördinate of the evaluation point. This means that in fact, flat models are not structures for *dyadic* predicates, but for *monadic* ones. For more information on the subject of flat versus ordinary two-dimensional tense logic we refer to Gabbay [35] (where these types of logic are called *weak* resp. *strong*). To connect the notion of flatness with first order logic, we need

Definition 3.4.7.
$L^{1<}$ is the set of first order formulas in a language with one binary predicate $<$ and arbitrary monadic predicates. $\mathsf{MOD}(L^{1<})$ denotes the class of structures for $L^{1<}$ (in the ordinary sense of first order model theory). ⊞

Clearly then $\mathsf{MOD}(L^{1<})$ can be identified with the class of flat $CC\lambda$-models. An almost immediate consequence of 3.4.5 is

Proposition 3.4.8.
Over the class of flat two-dimensional models, every $CC\lambda$-formula has an equivalent in $L_3^{1<}(x_0, x_1)$, and vice versa.

Proof.
By slightly adapting the definitions of the translations already given for ordinary $CC\delta$ in section 3.3.3.
The basic $CC\lambda$-proposition p now can be translated as Px_0.
For the other direction, we may proceed as if an atom Px_0 were of the form Px_0x_0 and then continue as in 3.3.16. ⊞

A fortiori, every $CC\lambda$-formula has an equivalent in $L^{1<}(x_0, x_1)$. We will now prove the converse of this fact, establishing an *expressive completeness* result, in the style of Kamp's famous result stating that over the class of continous linear orderings, every formula in $L^{1<}(x_0)$ has an equivalent in the one-dimensional formalism with the operators S ('Since') and U ('Until') (cf. Kamp [61], or Gabbay [35] for a more accessible proof).

Theorem 3.4.9.
Over the class of flat models based on a linear frame, every $L^{1<}(x_0, x_1)$-formula has an equivalent in $CC\lambda$, and vice versa.

Proof.
Let ϕ be a formula in $L^{1<}(x_0, x_1)$. By results of Gabbay [32], resp. Immerman and Kozen [57], $L^{1<}$ has *Henkin-dimension* three, resp. the *three-variable property* over the class of linear orderings, both implying that ϕ has an equivalent in $L_3^{1<}(x_0, x_1)$. By proposition 3.4.8 then, ϕ has a $CC\lambda$-equivalent. ⊞

Note that the restriction to *flat* $CC\lambda$/*monadic* predicates is essential here, as it is shown in Venema [136] that no *finite* system of two-dimensional temporal operators can be as expressive as $L_3^{2<}(x_0, x_1)$.
Our notion of expressive completeness is not the only one possible, and $CC\lambda$ is not the only two-dimensional expressively complete system. We refer to Gabbay [35] for more details.

We now turn to completeness matters, the last topic of this section. We will show that $\mathcal{CC}\lambda$ allows very simple axiomatizations, simple at least on top of the completeness theorem for $\mathcal{CC}\delta$.

Definition 3.4.10.
Let AL^+, ADE^+ and AD^+I be the axiom system AR^+ of $\mathcal{CC}\delta$, extended with the following axioms:

AL^+ : $AR^+ + LN$
ADE^+ : $AR^+ + LN + DI$
ADI^+ : $AR^+ + LN + DE$.

Theorem 3.4.11.
AL^+, ADI^+ and ADE^+ are strongly sound and complete with respect to respectively the classes of linear orderings, discrete linear orderings and dense linear orderings.

Proof.
We only show completeness. For AL^+, let Δ be an AL^+-consistent set of formulas. Considering λ as a propositional variable in $\mathcal{CC}\delta$, we obtain by $AL^+ \supset AR^+$ and the completeness theorem 3.3.37 for $\mathcal{CC}\delta$, that Δ is satisfiable in a two-dimensional model $\mathfrak{M} = (U, V)$. Define $< := V(\lambda)$, then by 3.4.2(iv) and the fact that $\mathfrak{M} \models LN$, $<$ is a linear ordering. Clearly then Δ is satisfiable in a linear two-dimensional model.
The proofs for ADE^+ and ADI^+ are analogous, using 3.4.2(iv) and 3.4.2(v). ⊞

3.5 Two-Dimensional Algebras.

3.5.1. Two-dimensional Cylindric Algebras.

CML_2 is not the only 'simple' subtype of S_2; the reason why we picked it out is that the Boolean CML_2-algebras are well-known in algebraic logic under the name cylindric type algebras (of dimension two). We change some of our notation, in order to keep in tune with the algebraic standard:

Definition 3.5.1.
In this section we write c_0, c_1 and d_{01} for the operators $\ominus$, $\oplus$ and δ, and T_0 (or: $\sim_0$), T_1 (or: $\sim_1$) resp. E_{01} for the accessibility relations H, V resp. D. ⊞

Definition 3.5.2.
Boolean CML_2-algebras are called *cylindric type algebras*; cylindric type algebras in the variety generated by CmC_2 are called *Representable Cylindric Algebras of dimension 2*; this variety itself is denoted by RCA_2. ⊞

For a discussion of these notions we refer to the next chapter where (representable) cylin-

dric algebras of arbitrary dimension are treated; the standard textbook is Henkin-Monk-Tarski [53].
The set $\mathit{Equ}(\mathsf{RCA}_2)$ is finitely axiomatizable, and a finite, explicit set of equational axioms has been known for a long time, cf. [53], pp. 79–84. It is interesting to note that the method used there to prove the representation theorem (originating with Andréka and Németi), is virtually the same as the two-dimensional bulldozing technique Segerberg used to prove his completeness result for the two-dimensional similarity type *SEG*, cf. Segerberg [119].

By Appendix A.19, our completeness result 3.2.8 has an immediate algebraic counterpart in the form of a *new* finite axiom system for $\mathit{Equ}(\mathsf{RCA}_2)$:

Definition 3.5.3.
Define the following CML_2-equations (where $\{i,j\} = \{0,1\}$):

$(CE1_i)$ $x \leq c_i x$
$(CE2_i)$ $x \leq -c_i - c_i x$
$(CE3_i)$ $c_i c_i x \leq c_i x$
$(CE4_i)$ $c_i c_j x \leq c_j c_i x$
$(CE5_i)$ $c_i d_{01} = 1$
$(CE6_i)$ $c_i(d_{01} \cdot x) \leq -c_i(d_{01} \cdot -x)$
$(CE7_i)$ $d_{01} \cdot c_i(-x \cdot c_j x) \leq c_j(-d_{01} \cdot c_i x)$.

$CE = \{CEl_{i(j)} \mid l = 1, \ldots, 7,\ \{i,j\} = \{0,1\}\}$. ⊞

Theorem 3.5.4.
CE axiomatizes RCA_2.

Proof.
Immediate by A.19 and 3.2.8. ⊞

3.5.2. Simplifying Henkin's equation.

As we have mentioned before, it has been known for a long time that the class RCA_2 can be finitely axiomatized. This standard axiomatization is slightly different from ours:

Definition 3.5.5.
Consider the following CML_2-equations:

$(CE2\frac{1}{2}{}_i)$ $c_i(x \cdot c_i y) = c_i x \cdot c_i y$
(H_{ij}) $c_i(x \cdot -y \cdot c_j(x \cdot y) \leq c_j(-d_{01} \cdot c_0 x))$.

A cylindric type algebra $\mathfrak{A}$ is a *Cylindric Algebra* if $CE1$, $CE2\frac{1}{2}$, $CE4$, $CE5$ and $CE6$ hold in it. The variety of Cylindric Algebras is denoted by CA_2. ⊞

Among algebraic logicians, H is known as *Henkin's equation*. Traditionally, it is the equations $CE1$, $CE2\frac{1}{2}$, $CE4$, $CE5$, $CE6$ and H that are used to axiomatize RCA_2. It is immediate that this system is equivalent to the set CE, as both axiomatize the same variety RCA_2. We will prove the equivalence directly however, because this proof is a nice illustration of how easy Sahlqvist's theorem can make life, enabling us to reason in the frames instead of giving algebraic derivations. First:

Proposition 3.5.6.
Let $\mathfrak{A}$ be a CA_2-type algebra with $\mathfrak{A} \models CE1$. Then

$$\mathfrak{A} \models CE2\tfrac{1}{2} \iff \mathfrak{A} \models CE2 \wedge CE3.$$

Proof.
By 2.2.6, the Sahlqvist form of the equations gives us the advantage it is sufficient to prove that the *first order* Sahlqvist correspondents to be equivalent. As we have $(CE2\frac{1}{2})^s =$

$$(\dagger) \qquad \forall x \forall y((T_i ux \wedge T_i xy) \leftrightarrow (T_i ux \wedge T_i uy)),$$

we have to prove that for T_i*-reflexive* frames $\mathfrak{F}$:

$$\mathfrak{F} \models (\dagger) \iff T_i \text{ is transitive and symmetric.}$$

But this is almost immediate by the definitions. ⊞

Corollary 3.5.7.
A cylindric type algebra $\mathfrak{A}$ is a cylindric algebra iff $\mathfrak{A} \models CE1, \ldots, CE6$. ⊞

Now we will prove that in the variety of cylindric algebras, Henkin's equation is equivalent to $CE7$:

Proposition 3.5.8.
Let $\mathfrak{A}$ be in CA_2. Then

$$\mathfrak{A} \models H \iff \mathfrak{A} \models CE7.$$

Proof.
By a similar argument as before, it suffices to prove that on the class of cylindric *frames*, the Sahlqvist correspondents of H and $CE7$ are equivalent. Now $CE7^s = \forall x NHV7(x)$ and H^s has the form

$$(H^s) \qquad \forall u \forall v \forall w((u \sim_0 v \sim_1 w \wedge v \neq w) \to \exists x(\neg Dx \wedge u \sim_1 x \wedge (x \sim_0 v \vee x \sim_0 w))).$$

So we have to show that for a cylindric frame $\mathfrak{F}$

$$\mathfrak{F} \models H^s \iff \mathfrak{F} \models \forall x NHV7(x).$$

The following pictures show the meaning of H^s and $NHV7$ for cylindric frames:

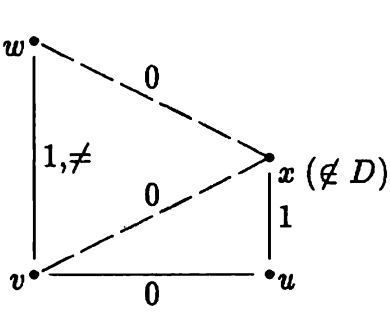


fig. 1: H^s.

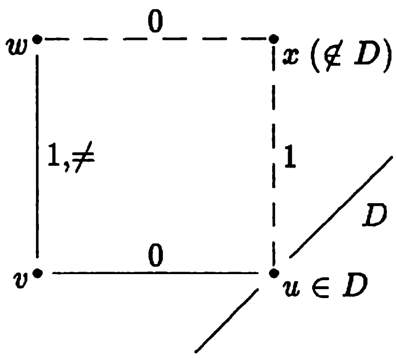


fig. 2: $N7$.

(⇐)
Assume that $\mathfrak{F} \models NHV7$. To prove that $\mathfrak{F} \models H^s$, let u, v and w be worlds in $\mathfrak{F}$ with $u \sim_0 v \sim_1 w$ and $v \neq w$. We have to find an x with $x \notin D$, $u \sim_1 x$ such that x is in the 0-equivalence class with v or with w.
Distinguish the following cases:
Case 1: $u \in D$. Then $\mathfrak{F} \models NHV7(u)$ immediately gives us the desired x, with $x \sim_0 w$.
Case 2: $u \notin D$. Then u itself is the desired x, as $u \sim_0 v$ and $u \sim_1 u$.

(⇒)
For the other direction, we assume that $\mathfrak{F} \models H^s$, we consider arbitrary worlds u, v and w in $\mathfrak{F}$ with $u \notin D$, $u \sim_0 v \sim_1 w$ and $v \neq w$, and set ourselves the task to find an x with $x \notin D$ and $u \sim_1 x \sim_0 w$, viz. figure 2.
By $\mathfrak{F} \models H^s$, there is a $y \notin D$ with $u \sim_1 y$ and $y \sim_0 v$ or $y \sim_0 w$. Distinguish

Case 1: $y \sim_0 w$.
This means we are finished immediately: take $x = y$.

Case 2: $y \sim_0 v$.
By $\mathfrak{F} \models NHV4$, there is a z in $\mathfrak{F}$ with $u \sim_1 z \sim_0 w$, viz. figure 3.

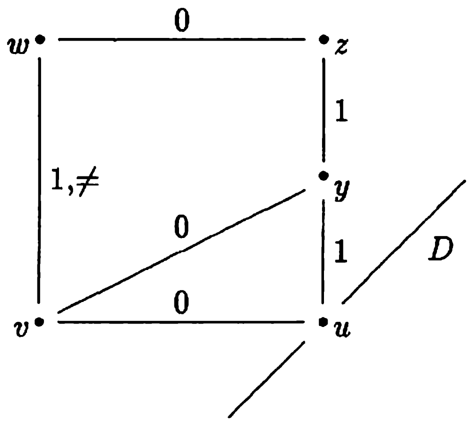


fig. 3.

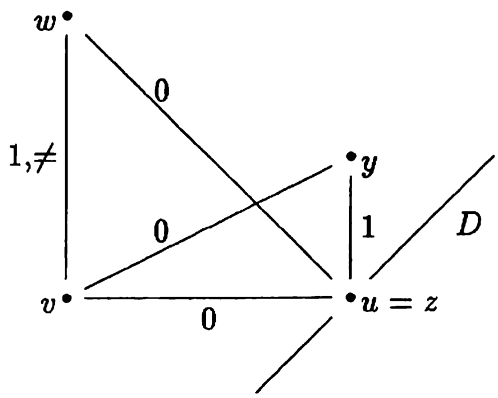


fig. 4.

Distinguish

Case 2.1: $z \notin D$. Again we are finished: take $x = z$.

Case 2.2: $z \in D$. This implies $z = u$ by $\mathfrak{F} \models NHV6$, so we have the situation as in figure 4.
We now have $w \sim_0 z = u \sim_0 v \sim_0 y$, so $y \sim_0 w$ after all, and we are back in case 1: take $x = y$. ⊞

3.5.3. Relation Algebras.

The completeness theorem for $\mathcal{CC\delta}$ has as an immediate corollary a finite derivation system generating $\mathit{Equ}(\mathsf{RRA})$:

Definition 3.5.9.
Define the following derived $\mathcal{CC\delta}$ term:

$$d'(x) \equiv 0'; x; 1 + 1; x; 0'.$$ ⊞

Clearly $d'(x)$ is the algebraic version of the defined D'-operator.

Definition 3.5.10.
Let Ψ_2 be the smallest set of $\mathcal{CC\delta}$-equations containing $RA0, \ldots, RA8$ which is closed under ordinary algebraic deduction and under the following closure operation:

$$y \cdot -d'(y) \leq t(x_0, \ldots x_{n-1}) \;/\; t(x_0, \ldots, x_{n-1}) = 1$$
provided y does not occur among the $\vec{x}$.

Theorem 3.5.11.

$$\Psi_2 = \mathit{Equ}(\mathsf{RRA})$$

Proof.
Immediate by the fact that the derivation system generating Ψ_2 is the algebraic version (cf. Appendix A.33) of AR^+, the completeness theorem 3.3.37 and 3.3.6. ⊞

3.6 Conclusions, Remarks and Questions.

3.6.1. General Conclusions.

In this chapter we have given a setup for a uniform, systematic study of two-dimensional modal logics (section 1). Three examples have been worked out in detail: CML_2 (section 2), $\mathcal{CC\delta}$ (section 3) and $\mathcal{CC\lambda}$ (section 4). CML_2 and $\mathcal{CC\delta}$ form the modal counterpart of two-dimensional cylindric resp. relation algebras. In these two formalisms we have put some modal machinery in action: for CML_2 we used Sahlqvist's theorem to find a complete axiomatization of the formulas valid in the squares; to obtain an analogous result for *CCδ* we needed the SD-theorem of the first chapter as well. In section 4 we added to *CCδ* a constant referring to the ordering relation in a time structure and obtained a two-dimensional *temporal* logic, *CCλ*. For this similarity type we found, besides a rather easy completeness proof, a result on expressive completeness, in the style of Kamp's theorem. The modal approach to the algebraic framework turned out to be rather fruitful: in section 5 we saw how the modal completeness results for CML_2 and *CCδ* yielded finite derivation systems for the equations valid in the classes of representable cylindric resp. relation algebras[3] We also proved that in cylindric algebras, Henkin's equation could be simplified to $CE7$ (cf. definition 3.5.3), a possible candidate as the *shortest equation* in *Equ*(RCA_2) – *Equ*(CA_2) (cf. Problem 10 in [7]).

3.6.2. Questions and Remarks.

Lots of topics have been left untouched. To mention a few:

(1) There are more two-dimensional modal operators possible besides the ones we have mentioned; for example, operators corresponding to the algebraic Q-operators defined by Jónsson [59]. The corresponding modal similarity type would contain *all* two-dimensional, first order definable, *additive* modal operators[4]. Andréka-Németi [9] and Venema [136] study the expressive power of this similarity type; in both papers it is proved that over the class of squares, the system with Q-operators is still less expressive than first order logic $L^2(x_0, x_1)$.
Other examples are non-additive modal operators, like two-dimensional versions of S and U ('Since' and 'Until'), or modal versions of operators that are not first order definable, like the *Kleene star.*

(2) We have confined ourselves to studying *squares*; it is also interesting to study *rectangle* models where the set of possible worlds W is of the form $W = U_0 \times U_1$ with U_0 and U_1 possibly *distinct*. This is the approach taken by e.g. Shekhtman [120].

(3) There are more connections between relation algebras and modal logics. In Orlowska [92, 93] similarity types have some additional algebraic structure, namely that of a *relation algebra.* The idea is that if $\Diamond_a$ and $\Diamond_b$ are diamonds, then so are $\Diamond_{a^{-1}}$, $\Diamond_{-a}$, $\Diamond_{a;b}$, etc. In the intended frames one should find $R_{\Diamond_{-a}} = -R_{\Diamond_a}$, etc. This formalism can be *interpreted* in *CCδ*, cf. [131]. Note that, extending this relation algebra with the Kleene star, we would obtain an enriched *dynamic logic.*

(4) An intrigueing question (at least to the author): is AtRRA elementary?

Finally, many of the remarks made in the conclusions of chapter 4 apply to the modal version *CCδ* of (representable) relation algebras as well.

---

[3] For a more detailed discussion of the relation algebraic case, the reader is referred to the conclusions of the next chapter, where an analogous result on higher-dimensional cylindric algebras is treated.

[4] cf. also 5.5.2(ii).

CHAPTER 4.

QUANTIFIERS AND CUBES.

Outline.

Our aim in this chapter is to treat (a restricted version of) classical first order logic as if it were a modal formalism.

In the introduction we define restricted first order logic, its algebraization in cylindric algebras, and some related notions. Section 2 treats the modal perspective on these matters. In section 3 we will see how to characterize the intended frames (the cubes) in the modal language. Section 4 treats the complete axiomatization for cube validity yielded by this characterization. As corollaries to this result, in section 5 we find several completeness results defined in the introduction. In section 6 an example is given of a derivation where a non-ξ rule is actually used. We finish with giving our conclusions and making some remarks; in particular we will show that in fact, ordinary first order logic has a modal counterpart too.

4.1 Algebraizing Restricted First Order Logic.

In chapter 3 we saw how two-dimensional modal logics form an extension of ordinary modal logics in capturing a larger fragment of first-order logic. The aim that we set ourselves in this chapter is then a natural one:

> to devise and study a many-dimensional modal formalism which is equally expressive as first order logic itself.

The obvious approach is to try and mimick the language of the first order predicate calculus, *by looking at quantifiers as if they were modal operators.* Indeed, quite some authors have observed the resemblance between quantifiers and modal ($S5$-)operators, some references are listed in Kuhn [67].
In the implementation of this idea we meet technical obstacles, however, one of which concerns the following. If our many-dimensional modal logics are to be the analogons of the two-dimensional ones, we want the modal version of a formula $\phi(x_0, \ldots, x_{n-1})$ to be evaluated at n-tupels of elements of the model. Now in unrestricted first order logic, formulas may have arbitrary many free variables. This means that we have to think about the precise nature of our possible worlds.
One possibility is to allow all finite sequences as possible worlds in a model; this is the approach taken by Kuhn in [67]. A consequence of this choice is that his modal language *PREDBOX* is *sorted*, having for example distinct propositional variables for every n (corresponding to n-adic predicate symbols of first order logic). The most important result of [67] is a completeness theorem for *PREDBOX*.
The alternative approach we concentrate on in this chapter is motivated mainly by the desire to study a modal logic which bears a close resemblance to intensively studied *algebras* of relations. For an overview of the algebraic approach towards relations we refer to Németi [89]. Just like in the two-dimensional case, our guideline will be that the algebras of polyadic relations are to be (a subclass of) the modal algebras of our system. Now it happens to be that the most important algebraic theory of relations concerns the *cylindric* algebras (cf. Henkin-Monk-Tarski [53]). Concentrating on cylindric algebras forces us to study modal similarity types where the dimension of the models in the intended semantics is a fixed ordinal α. This implies that again we do not reach all formulas of ordinary first order logic, but (a fragment of) L^α_α: the set of first order formulas that use $\leq \alpha$ variables in a language where all predicate symbols are α-ary.

It is the following version of first order logic of which cylindric algebras form the direct counterpart:

Definition 4.1.1.
L^r_α, the language of *restricted first order logic*, is defined as follows: its *alphabet* consists of the set of variables $\{v_i \mid i < \alpha\}$, it has got a countable set Q of *relation symbols* $(R_0, R_1, \ldots)$, identity $(=)$, the Boolean connectives $\neg, \vee$ and the quantifiers $\exists v_i$. *Formulas* of L^r_α are defined as usual, with the restriction that all *atomic formulas* are of the form

$v_i = v_j$ or $R_l(v_0 \ldots v_i \ldots)_{i<\alpha}$. As abbreviations we will freely use $\wedge, \rightarrow, \forall v_i$, etc. ⊞

For $\alpha < \omega$, we get a logic with *finitely many variables*. Such logics have been studied in the literature, for purely logical reasons (Henkin [52], Henkin-Monk-Tarski [53], Tarski-Givant [127], Sain [115], Monk [84]) or because of their relation with temporal logics in computer science (Gabbay [32],Immerman-Kozen [57]). For $\alpha \geq \omega$ the logic is sometimes called the *finitary logic of infinitary relations*, cf. Sain [112].
In fact, the variables in atomic relational formulas do not provide any information. In the sequel we will leave them out, writing R_l for $R_l(v_0 \ldots v_i \ldots)_{i<\alpha}$, so we have $L^r_\alpha \subseteq L^r_\beta$ if $\alpha \leq \beta$.
Besides possibly having a deviant number of variables, the language L^r_α differs in two respects from ordinary first order logic: first, because all relations in L^r_α have the same rank α, and second, because atomic formulas like $Rv_1v_1v_1$ or $Rv_1v_0v_2$ are not allowed. In section 7 we will discuss how to overcome these restrictions; for a detailed discussion of the connection between logic and cylindric algebras we refer to section 4.3 of Henkin-Monk-Tarski [53].
The *model theory* of L^r_α is the same as for ordinary first order logic:

Definition 4.1.2.
A *structure* of L^r_α is a pair $\mathfrak{M} = (U, V)$ such that U is a universe and V is an interpretation function mapping every R_i to a subset of ${}^\alpha U$. Truth of a formula in a model is defined as usual: let u be in ${}^\alpha U$, then

$$\begin{array}{lll} \mathfrak{M} \models v_i = v_j[u] & \text{if} & u_i = u_j, \\ \mathfrak{M} \models R_l[u] & \text{if} & u \in V(R_l), \\ \mathfrak{M} \models \exists v_i \phi[u] & \text{if} & \mathfrak{M} \models \phi[u(u'_i/u_i)] \text{ for some } u'_i \in U, \\ \text{etc.} & & \end{array}$$

An α-formula ϕ is *(α-)valid* in $\mathfrak{M}$ (notation: $\mathfrak{M} \models \phi$) if $\mathfrak{M} \models \phi[u]$ for all $u \in {}^\alpha U$. An α-formula is *(α)-valid* (notation: $\models_\alpha \phi$) if it is valid in every structure of L^r_α. ⊞

Note that we now have several versions of validity for α-formulas, as we may see them as β-formulas as well, for every $\beta \geq \alpha$. Fortunately, these notions of validity coincide:

Proposition 4.1.3.
Let $\alpha < \beta$, $\phi \in L^r_\alpha$. Then $\models_\alpha \phi \iff \models_\beta \phi$.

Proof.
(This proof is an adaptation of a proof by Simon, cf. [121].)
First we show that, for any two ordinals α, β, one can see a model $\mathfrak{M}_\alpha$ for L^r_α as a model $\mathfrak{M}_{\beta/\alpha}$ for L^r_β: if $\mathfrak{M}_\alpha = (U, V_\alpha)$, set $\mathfrak{M}_{\beta/\alpha} = (W, V_{\beta/\alpha})$ with

$$V_{\beta/\alpha}(R_l) = \{(u_0, \ldots, u_i, \ldots)_{i<\beta} \in {}^\beta U \mid (u_0, \ldots, u_i, \ldots)_{i<\alpha} \in V_\alpha(R_l)\}.$$

Now let ϕ be a formula in $L^r_{\min(\alpha,\beta)}$. By induction to the complexity of ϕ, it is straightforward to show that

$$\mathfrak{M}_\alpha \models \phi[(u_0, \ldots, u_i, \ldots)_{i<\alpha}] \text{ iff } \mathfrak{M}_{\beta/\alpha} \models \phi[(u_0, \ldots, u_i, \ldots)_{i<\beta}].$$

Now we can easily prove the proposition: for the direction from right to left, let ϕ be the α-formula and suppose that $\not\models_\alpha \phi$. Then there is an α-structure $\mathfrak{M}_\alpha$ with a sequence $u \in {}^\alpha U$ such that $\mathfrak{M} \models \neg\phi[u]$. By the above claim we have that, for $(u_\alpha, \ldots, u_j, \ldots)_{\alpha \leq j < \beta}$ an arbitrary sequence, $\mathfrak{M}_{\beta/\alpha} \models \neg\phi[(u_0, \ldots, u_i, \ldots)_{i<\beta}]$. So $\not\models_\beta \phi$.
The direction from left to right is similar. ⊞

Now we turn to the algebraization of the above logics (for the general idea of algebraizations we refer to Blok-Pigozzi [20]). We start with cylindric set algebras of dimension α. These are for L^r_α what Boolean set algebras are for propositional logic.

Definition 4.1.4.
Let U be some unspecified set, α an ordinal and $i < \alpha$. The *i-th cylindrification on $Sb({}^\alpha U)$* is the following operation C_i on $Sb({}^\alpha U)$: for $X \subseteq {}^\alpha U$

$$C_i(X) = \{u \in {}^\alpha U \mid u[u'_i/u_i] \in X \text{ for some } u'_i \in U\}.$$

The *i, j-diagonal in ${}^\alpha U$* is the set

$$D_{ij} = \{u \in {}^\alpha U \mid u_i = u_j\}.$$

The *α-dimensional full cylindric set algebra on U* is the structure

$$\mathfrak{Cs}_\alpha(U) = (Sb({}^\alpha U), \cup, -, C_i, D_{ij})_{i,j<\alpha}.$$ ⊞

The idea is now to abstract away from the set-represented background:

Definition 4.1.5.
A *cylindric type algebra of dimension α* is an algebra $\mathfrak{A} = (A, +, -, c_i, d_{ij})_{i,j<\alpha}$ with $(A, +, -)$ a Boolean Algebra, d_{ij} a constant and c_i a normal, additive unary operator, for all $i, j < \alpha$.
We define the following classes of algebras: FCS_α is the class of full α-dimensional cylindric set algebras over some set U. The class RCA_α of *representable cylindric algebras of dimension α* is the variety generated by FCS_α.
The algebraic language used to describe these algebras is denoted by $\mathcal{L}_\alpha$. ⊞

This framework forms the basis of the *algebraization* of L^r_α:

Definition 4.1.6.
Let ϕ be an α-formula. The *corresponding $\mathcal{L}_\alpha$-term* ϕ^t of ϕ is defined by

$$\begin{array}{rcl} (R_l)^t & = & x_l \\ (v_i = v_j)^t & = & d_{ij} \\ (\neg\phi)^t & = & -\phi^t \\ (\phi \vee \psi)^t & = & \phi^t + \psi^t \\ (\exists v_i \phi)^t & = & c_i \phi^t. \end{array}$$ ⊞

The exact connection between valid L^r_α-formulas and equations holding in RCA_α (or, equivalently, in FCS_α) is the following:

Proposition 4.1.7.
Let ϕ be an α-formula, then

$$\models_\alpha \phi \iff \mathsf{RCA}_\alpha \models \phi^t = 1.$$

Proof.
The basic idea of this proof is that there is a 1–1-correspondence between structures for L^r_α and (countably generated) α-dimensional set algebras. For, let $\mathfrak{M} = (U, V)$ be an L^r_α-structure. Define, for any $\phi \in L^r_\alpha$,

$$\phi^{\mathfrak{M}} = \{u \in {}^\alpha U \mid \mathfrak{M} \models \phi[u]\}.$$

It is immediately seen that $\{\phi^{\mathfrak{M}} \mid \phi \in L^r_\alpha(Q)\}$ forms, together with the obvious operations, a subalgebra of $\mathfrak{Cs}_\alpha(U)$ which is generated by $\{R^{\mathfrak{M}}_l \mid R_l \in Q\}$. If we now give the following interpretation in $\mathfrak{Cs}_\alpha(U)$ to the variables of $\mathcal{L}_\alpha$:

$$h(x_l) = V(R_l),$$

then one can prove by a straightforward induction that 'algebra and structure coincide', i.e. for all $\phi \in L^r_\alpha$

$$h(\phi^t) = \phi^{\mathfrak{M}}.$$

This correspondence immediately gives

$$\not\models_\alpha \phi \Rightarrow \mathsf{RCA}_\alpha \not\models \phi^t = 1.$$

The other direction of this theorem is proved in an analogous way, by turning cylindric set algebras into L^r_α-structures. ⊞

By proposition 4.1.7, finding complete derivation systems for $\models_\alpha$ and for RCA_α are really two sides of the same coin. In the monograph "Cylindric Algebras" of Henkin, Monk and Tarski, the following axiomatization is suggested, as a first approximation of representabe cylindric algebras.

Definition 4.1.8.
Consider the following $\mathcal{L}_\alpha$-equations[1]:

$(C1_i)$	$c_i 0 = 0$
$(C2_i)$	$x \leq c_i x$
$(C3_i)$	$c_i(x \cdot c_i y) \leq c_i x \cdot c_i y$
$(C4_{ij}$	$c_i c_j x \leq c_j c_i x$
$(C5_i)$	$d_{ii} = 1$
$(C6_{ij})$	$c_i(d_{ij} \cdot x) \cdot c_i(d_{ij} \cdot -x)) = 0$
$(C7_{ijk})$	$d_{ij} = c_k(d_{ik} \cdot d_{kj})$.

Finally, for finite α we set $C1 \equiv \bigwedge_i C1_i$, etc., taking $C4 \equiv \bigwedge_{i,j} C4_{ij}$, $C6 \equiv \bigwedge_{i \neq j} C6_{ij}$ and

[1]The names of the (hypercylindric) axioms in this chapter and chapter 3 do *not* correspond. We list the normality axiom ($C1$), in order to keep in tune with the standard cylindric algebra terminology. (Recall definitions A.29, A.30 for our standard.) .

$C7 \equiv \bigwedge_{i,j,k} C7_{ijk}$. If $\alpha \geq \omega$, we let $C1, \ldots, C7$ be the corresponding equation *schemata*. An α-cylindric type algebra $\mathfrak{A}$ is a *cylindric algebra of dimension* α (short: a CA_α), if $\mathfrak{A} \models C0, \ldots, C7$. The class of these algebras is denoted by CA_α. ⊞

Analogous to the situation for relation algebras, cf. 3.3.1, $Equ(\mathsf{RCA}_\alpha)$ turned out to be very hard to axiomatize, at least for the cases $\alpha > 2$. For $\alpha = 2$ we refer to section 3.5 of this dissertation. Certainly $C0, \ldots, C7$ are not sufficient in axiomatizing RCA_α. A fortiori, though $Equ(\mathsf{RCA}_\alpha)$ is known to be recursively enumerable, it was shown by Monk in [83] that for $\alpha > 2$, no finite schema of equations can generate $Equ(\mathsf{RCA}_\alpha)$, if one allows only the ordinary derivation rules; in the same article he gave a complete system with infinitely many axioms. Recently, Andréka [5] gave a very strong generalization of the negative result by Monk. Roughly speaking, she proved that if Σ is a set of equations axiomatizing the class RCA_α, $\alpha > 2$, then for all natural numbers n, and all ordinals $i < \alpha$, there are infinitely many equations $\eta \in \Sigma$ such that η contains more than n operation symbols, more than n variables and a diagonal constant with index i. On the other hand, in [8] Andréka and Németi defined a finite schema of axioms and rules generating $Equ(\mathsf{RCA}_\alpha)$, but this system has an axiom which is not in equational form.

In this chapter we will look at cylindric algebras from the modal point of view. Actually,

our modal language can be seen as an *alphabetical variant* of L^r_α!

In other words, we may look at restricted first order logic *as if it were* a modal logic, in the sense that we may *read* the existential quantification $\exists v_i$ as a *diamond* $\Diamond_i$.

Before turning over to this modal perspective, we discuss two interesting notions that are related to restricted first order logic and cylindric algebras, namely *type-free* or *typeless* logic and *schema validity* in first order logic.

Typeless logic arises out of abstracting away from the ranks of relation symbols in ordinary first order logic. Typeless logic is studied in e.g. Henkin-Monk-Tarski [53], Andréka-Gergely-Németi [6], Simon [121, 122].

Definition 4.1.10.
The language L_{tf} of *type free* or *typeless* logic is the same as L^r_ω. A *type* for L_{tf} is a map $\rho : \omega \mapsto \omega$ assigning to each relation symbol R_l a finite *rank* $\rho(l)$.
Now let ϕ be a type-free formula. Define the ρ-instatiation ϕ^ρ as the first order formula obtained from ϕ by replacing all atomic (type-free) subformulas R_l by the ρ-typed $R_l(v_0 \ldots v_{\rho(l)-1})$. ⊞

This idea of giving types to L^r_ω-formulas lies behind the *model theory* of typeless logic too:

Definition 4.1.11.
A *model* or *structure* for L_{tf} is a pair $\mathfrak{M} = (U, V)$ such that there exists a type ρ with the property that $\mathfrak{M}$ is a structure for the restricted first order logic of similarity type ρ (or equivalently, V is a funcion mapping every relation symbol R_l to a $\rho(l) - 1$-ary relation on U).

A typeless formula ϕ is *valid in* $\mathfrak{M}$, notation: $\mathfrak{M} \models_{tf} \phi$, if $\mathfrak{M}$ is of type ρ and ϕ^ρ is valid in $\mathfrak{M}$ in the usual classical sense. ϕ is *type-free valid*, notation: $\models_{tf} \phi$, if ϕ is valid in all models for L_{tf}. ⊞

So an ω-formula is type-free valid if it is valid in any model, no matter how we type the relation symbols of ϕ. A simple example: the typed instance $Pv_0v_1 \to \forall v_2 Pv_0v_1$ of the typeless formula $P \to \forall v_2 P$ is valid, but its colleague $Pv_0v_1v_2 \to \forall v_2 Pv_0v_1v_2$ is not, so $P \to \forall v_2 P$ is *not* a type-free valid formula.
Note that as $L_{tf} = L^r_\omega$, we have defined yet another kind of validity for the ω-formulas (besides $\models_\alpha$, $\alpha \geq \omega$). But again, type-free validity is equivalent to ω-validity:

Proposition 4.1.12.
Let ϕ be an ω-formula. Then $\models_{tf} \iff \models_\omega \phi$.

Proof.
Analogous to the proof of 4.2.3, we can convert typeless models into ω-models and the other way round, now giving each predicate an individual treatment according to the type it has, resp. should get. ⊞

A completeness proof for typeless logic is given by Simon [121, 122]. We will come back to his result in the conclusions to this chapter.

The second concept we (briefly) mention is that of *schema validity*, cf. Németi [88], Rybakov [110]. Formula schemas are used widely in logic, e.g. in *axiomatizations* of first order logics: an example of such a formula schema is $\phi \to \exists v_i \phi$. Formally we set:

Definition 4.1.13.
Let Q_{fm} be a set of formula variables (i.e. variables ranging over formulas), and assume that we have a set $\{v_i \mid i \in \omega\}$ of individual variables. *Formula schemas* are defined by induction:

(i) ϕ is a schema if $\phi \in Q_{fm}$,
(ii) $v_i = v_j$ is a schema if $i, j < \omega$,
(iii) $\exists v_i \sigma$, $\neg\sigma$, $\sigma \vee \xi$ are schemas if $i \in \omega$ and σ, ξ are schemas.

An *instance* of a schema σ is a first order formula we obtain by substituting first order formulas for the formula variables in σ. A formula schema σ is *valid* if every instance of it is valid as a first order formula. ⊞

By Proposition 0.3 in Németi [88], schema validity is yet another (alphabetical) variant of ω-validity or typeless validity: if we replace the formula variables by predicate symbols, a schema is valid iff the resulting ω-formula is ω-valid (or typeless valid). We will not go into details.

4.2 Cylindric Modal Logic.

Fix an ordinal α with $2 \leq \alpha$.

Looking at the definition of restricted first order logic, we immediately observe that its semantics is already of a modal nature; an atomic formula $v_i = v_j$ can be seen as a modal constant, the quantifier $\exists v_i$ behaves like a diamond, etc. In the following three sections we will take this modal viewpoint seriously, studying an alphabetical variant of L^r_α in which the modal features of the system are more clearly reflected:

Definition 4.2.1.
CML_α is the modal similarity type having constants δ_{ij} for $i, j < \alpha$ and diamonds $\Diamond_i$ for $i < \alpha$. For a set of propositional variables Q, the formulas of the language $M(CML_\alpha, Q)$ are called *cylindric modal formulas in Q of dimension α*, or shortly, *α-formulas in Q*. We abbreviate $\Box_i\phi \equiv \neg\Diamond_i\neg\phi$. ⊞

Note that we now have *three* kinds of notation for the same similarity type:

first order	algebraic	modal
L^r_α	$\mathcal{L}_\alpha$	CML_α
$v_i = v_j$	d_{ij}	δ_{ij}
$\neg$	$-$	$\neg$
$\vee$	$+$	$\vee$
$\exists v_i$	c_i	$\Diamond_i$

Definition 4.2.2.
The first order language L_{CML_α} corresponding to CML_α has monadic predicates E_{ij} and dyadic predicates T_i, $i, j < \alpha$. CML_α-frames are called *α-frames* and formally denoted as $\mathfrak{F} = (W, I)$, where W is the universe and I the function interpreting the E_{ij} and T_i as unary resp. binary relations on W. ⊞

We will be sloppy about the difference between the syntax of L_{CML_α} and its semantics, using the same symbols T_i, E_{ij} to denote both predicate symbols and their interpretation in an α-frame: a frame is thus denoted as $F = (W, T_i, E_{ij})_{i,j<\alpha}$.

Definition 4.2.3.
Let U be some set. By the *α-cubic frame* or the *α-cube* based on U we understand the Kripke frame $(W, T_i, E_{ij})_{i,j<\alpha}$, where $W = {}^\alpha U$, T_iuv iff $u_j = v_j$ for all $j \neq i$ (i.e. u and v may differ only in their i-th coördinate), and $u \in E_{ij}$ iff $u_i = u_j$. The class of α-cubes is denoted by C_α. An α-formula ϕ is *α-valid*, notation: $\models_\alpha \phi$ if $\mathsf{C}_\alpha \models \phi$. ⊞

The name 'cubes' for the intended frames is taken from a paper [99] by Prijatelj who studies related structures modelling natural language phenomena. Note that by this definition, the α-cubic *models* can be identified with the structures for L^r_α, so the notation $\models_\alpha$ is unambiguous for modal and restricted first order formulas.

Obviously, the *full* cylindric set algebras are the complex algebras of the cubes:

Proposition 4.2.4.
$\mathsf{FCS}_\alpha = \mathbf{Cm}(\mathsf{C}_\alpha)$. ⊞

As was mentioned in the introduction, Cylindric Algebras were developed as an approximation to Representable Cylindric Algebras. We will now develop and discuss the modal counterpart of Cylindric Algebras:

Definition 4.2.5.
Consider the following α-formulas:

$(CM1_i)$ $p \to \Diamond_i p$
$(CM2_i)$ $p \to \Box_i \Diamond_i p$
$(CM3_i)$ $\Diamond_i \Diamond_i p \to \Diamond_i p$
$(CM4_{ij}$ $\Diamond_i \Diamond_j p \to \Diamond_j \Diamond_i p$
$(CM5_i)$ δ_{ii}
$(CM6_{ij})$ $\Diamond_i(\delta_{ij} \wedge p) \to \Box_i(\delta_{ij} \to p))$
$(CM7_{ijk})$ $\delta_{ij} \leftrightarrow \Diamond_k(\delta_{ik} \wedge \delta_{kj})$,

and the L_{CML_α}-formulas

$(N1_i)$ $T_i xx$
$(N2_i)$ $\forall y(T_i xy \to T_i yx)$
$(N3_i)$ $\forall y \forall z((T_i xy \wedge T_i yz) \to T_i xz)$
$(N4_{ij})$ $\forall y(\exists z(T_i xz \wedge T_j zy) \to \exists u(T_j xu \wedge T_i uy))$
$(N5_i)$ $E_{ii} x$
$(N6_{ij})$ $\forall y \forall z((T_i xy \wedge E_{ij} y \wedge T_i xz \wedge E_{ij} z) \to y = z)$
$(N7_{ijk})$ $E_{ij} x \leftrightarrow \exists y(T_k xy \wedge E_{ik} y \wedge E_{kj} y)$.

For $CM1, \ldots, CM7, N1, \ldots, N7$ we have the obvious definition (analogous to 4.1.8). An α-frame $\mathfrak{F}$ is called *cylindric* if $\mathfrak{F} \models CM1 \ldots CM7$. The class of cylindric frames is denoted by CF_α. ⊞

By the Sahlqvist correspondence theorem 2.2.2, the above are equivalent pairs of formulas, cf. section 3.2.1 for more details.

Proposition 4.2.6.
Let $\mathfrak{F}$ be an α-frame, u a world in $\mathfrak{F}$. Then for $l = 1, \ldots, 7$ and $i, j, k < \alpha$:

$$\mathfrak{F}, u \models CMl_{i(j(k))} \iff \mathfrak{F} \models Nl_{i(j(k))}(u).$$ ⊞

Just like for the two-dimensional case, the triple of modal axioms $CM1$, $CM2$ and $CM3$ is equivalent to the algebraic pair $C2$ and $C3$, and, analogous to 3.5.7, we obtain the cylindric algebras as the complex algebras of cylindric frames:

Proposition 4.2.7.
$\mathsf{CA}_\alpha = \mathbf{Cm}(\mathsf{CF}_\alpha)$. ⊞

In section 3.2, where we discussed cylindric modal logic of dimension 2, we found a complete axiomatization A_2 for C_2 by adding only one axiom to the set corresponding to the CA_2-axioms. Here the situation is much more complex, as we should have expected by the fact that RCA_α is not finitely axiomatizable for $\alpha > 2$. Still, the extra 'simplified Henkin'-axiom of A_2 plays a significant rôle in the higher-dimensional case too:

Definition 4.2.8.
Define the following formulas in CML_α resp. L_{CML_α}, resp. the following equation in $\mathcal{L}_\alpha$.

$(CM8_{ij})$ $(\delta_{ij} \wedge \Diamond_i(\neg p \wedge \Diamond_j p)) \to \Diamond_j(\neg\delta_{ij} \wedge \Diamond_i p)$
$(N8_{ij})$ $\forall y \forall z((E_{ij}x \wedge T_i xy \wedge T_j yz \wedge y \neq z) \to \exists u(\neg E_{ij}u \wedge T_j xu \wedge T_i uz))$
$(C8_{ij})$ $d_{ij} \cdot c_i(-x \cdot c_j x) \leq c_j(-d_{ij} \cdot c_i x)$

Call an α-frame F *hypercylindric* if F is cylindric and $F \models C8$; a cylindric algebra $\mathfrak{A}$ is *hypercylindric* if $\mathfrak{A} \models CA8$. The classes of hypercylindric α-frames and hypercylindric algebras of dimension α are denoted by HCF_α resp. HCA_α. ⊞

Recall that in section 3.5.2 we proved that over the class of cylindric algebras, $C8_{ij}$ is equivalent to *Henkin's equation* $c_i(x \cdot y \cdot c_j(x \cdot -y)) \leq c_j(c_i x \cdot -d_{ij})$.
Clearly we have

Lemma 4.2.9.
$\mathsf{HCA}_\alpha = \mathbf{Cm}(\mathsf{HCF}_\alpha)$.

Proof.
Immediate by proposition 4.2.7 and the form of $CM8/C8$.. ⊞

Definition 4.2.10.
Let A_α be the derivation system having as its axioms

(CT) all propositional tautologies
(DB) $\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$
(CM) $CM1, \ldots, CM8$.

and MP, UG and SUB as its derivation rules. ⊞

Theorem 4.2.11.
A_α is strongly sound and complete with respect to HCF_α.

Proof.
Sahlqvist completeness, cf. Theorem 2.2.2. ⊞

As an exercise in the arithmetic and semantics of 'hypercylindric' modal logic, we show how to approximate the D-operator in CML. In the remainder of this section we assume that α is *finite* and write n instead. First some preliminaries:

Definition 4.2.12.
Let $\Gamma = \{i_1, \ldots, i_m\}$ be a subset of $\{0, \ldots, n-1\}$, with $i_1 < i_2 < \ldots < i_m$. Set

$$\begin{aligned}
\Diamond_\Gamma \phi &\equiv \Diamond_{i_1} \cdots \Diamond_{i_m} \phi \\
\oplus^n_\Gamma \phi &\equiv \Diamond_{\{0,\ldots,n-1\}\setminus\Gamma} \phi \\
\oplus^n_i \phi &\equiv \oplus_{\{i\}} \phi \\
\oplus^n \phi &\equiv \Diamond_{\{0,\ldots,n-1\}} \phi \\
\bigcirc_{ij} \phi &\equiv \Diamond_i(\delta_{ij} \wedge \phi).
\end{aligned}$$

We also use the boxes $\Box_\Gamma$, $\boxplus_\Gamma$, $\boxplus_i$ and $\boxplus$. In L_{CML_n}, we abbreviate

$$\begin{array}{lll} xT_i \mid T_j y & \equiv & \exists z(T_i xz \wedge T_j zy) \\ xH_i^n y & \equiv & xT_0 \mid \ldots \mid T_{i-1} \mid T_{i+1} \mid \ldots \mid T_{n-1} y. \end{array}$$

⊞

Convention 4.2.13.

(i) By the axioms $CM1$, ..., $CM4$ it is clear that in a (hyper)cylindric frame, any operator $\Diamond_\Gamma$ will behave like an $S5$-operator; this fact will be used without further notice.

(ii) We will usually denote the accessibility relation of $\Diamond_i$ by $\sim_i$.

(iii) If no confusion arises concerning the dimension n, we will suppress the superscript in $\Diamond\!\!\!\!+$-operators.

(iv) In the sequel we will frequently pretend that CML has diamonds $\oplus_i$ with H_i as their accessibility relation.

(v) From the axioms $CM5$, $CM6$ and $CM7$ we can deduce that $\Box_i(\delta_{ij} \to p) \to \Diamond_i(\delta_{ij} \wedge p)$ is a theorem of A_n. Hence for every world u in a hypercylindric frame there is *exactly* one world v with $u \sim_i v$ and $E_{ij}v$, if $i \neq j$. For this v we have that $u \models \bigcirc_{ij}\,\phi$ iff $v \models \phi$. We will pretend that the language L_{CML_n} has a *function* symbol f_{ij} acting as the accessibility function of $\bigcirc_{ij}$, analogous to the situation for $\otimes$ in $\mathcal{CCS}$, cf. 3.3.18.

In the two-dimensional case we could define the D-operator by observing that in a two-dimensional model, two worlds are different iff at least one of their coördinates is different. We used operators Z_H and Z_V referring to points with different first resp. second coordinates. This inspires the following definition:

Definition 4.2.14.
Let $S(i)$ be the successor of i, modulo n (i.e. $S(n-1) = 0$). Set

$$\begin{array}{lll} Z_i\phi & \equiv & \bigcirc_{S(i),i}\,\Diamond_i(\neg\delta_{S(i),i} \wedge \oplus_i\,\phi) \\ D_n\phi & \equiv & \bigvee_{i<n} Z_i\phi \\ \underline{D}_n\phi & \equiv & \neg D_n\neg\phi. \end{array}$$

Proposition 4.2.15.
Let $\mathfrak{M}$ be a cubic model. Then

(i) $\mathfrak{M}, u \models Z_i\phi$ iff there is a v with $v_i \neq u_i$ and $\mathfrak{M}, v \models \phi$.

(ii) $\mathfrak{M}, u \models D_n\phi$ iff there is a v with $v \neq u$ and $\mathfrak{M}, v \models \phi$.

Proof.
The first part (i) is given by (take $i = 0$):

$$\begin{array}{ll} & (u_0, u_1, u_2, \ldots, u_{n-1}) \models Z_0\phi \\ \Longleftrightarrow & (u_0, u_1, u_2, \ldots, u_{n-1}) \models \bigcirc_{10}\,\Diamond_0(\neg\delta_{10} \wedge \oplus_0\,\phi) \\ \Longleftrightarrow & (u_0, u_0, u_2, \ldots, u_{n-1}) \models \Diamond_0(\neg\delta_{10} \wedge \oplus_0\,\phi) \\ \Longleftrightarrow & (v_0, u_0, u_2, \ldots, u_{n-1}) \models \neg\delta_{10} \wedge \oplus_0\,\phi, \text{ for some } v_0 \\ \Longleftrightarrow & (v_0, u_0, u_2, \ldots, u_{n-1}) \models \oplus_0\,\phi, \text{ for some } v_0 \neq u_0 \\ \Longleftrightarrow & \text{there is a } v \text{ with } v_0 \neq u_0 \text{ such that } v \models \phi. \end{array}$$

(ii) is an immediate consequence of (i). ⊞

Note that we need a *finite* dimension n to define the D_n-operator.
We will now show that the axiom systen A_n proves some nice behaviour of the D_n-operator, viz. the D_n-versions of the D-axioms (cf. 2.4.2). Readers disinterested in technicalities are advised to skip this proof, as it is not needed for reading or understanding the sequel.

Proposition 4.2.16.

(i) $A_n \vdash D_n(p \vee q) \rightarrow (D_n p \vee D_n q)$
(ii) $A_n \vdash D_n \underline{D}_n p \rightarrow p$
(iii) $A_n \vdash D_n D_n p \rightarrow (p \vee D_n p)$
(iv) $A_n \vdash \Diamond_i p \rightarrow (p \vee D_n p)$.

Proof.
By theorem 4.2.15, to show $\vdash_n \phi$ we may equally well prove $\mathsf{HCF}_n \models \phi$. We omit the (simple) proof of (i). For the proof of the other items, we first need an extra $\vdash_n$-theorem. Assume $k \notin \{i, j\}$, then

$$\mathsf{HCF}_n \models \Diamond_k(\delta_{ij} \wedge \Diamond_i(\neg\delta_{ij} \wedge \Diamond_j \phi)) \leftrightarrow (\delta_{ij} \wedge \Diamond_i(\neg\delta_{ij} \wedge \Diamond_j \Diamond_k \phi)),$$

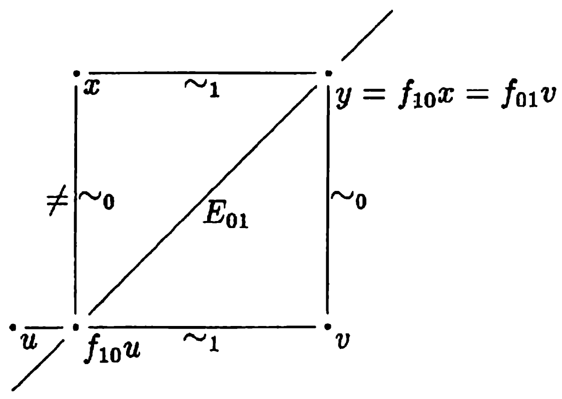
which is more or less straightforward to show. (The idea is that step by step we can pull resp. push the $\Diamond_k$-diamond through the part of the formula where nothing refers to the k-th dimension.) Actually, we need the dual

$$(*) \qquad \vdash_n \Box_k(\delta_{ij} \rightarrow \Box_i(\neg\delta_{ij} \rightarrow \Box_j \phi)) \leftrightarrow (\delta_{ij} \rightarrow \Box_i(\neg\delta_{ij} \rightarrow \Box_j \Box_k \phi)).$$

Now we start proving (ii):
Let $\mathfrak{F}$ be a hypercylindric frame, $\mathfrak{M}$ a model on $\mathfrak{F}$, u in $\mathfrak{F}$ such that $\mathfrak{M}, u \models D_n \underline{D}_n p$. Without loss of generality we may assume $\mathfrak{M}, u \models Z_0 \underline{D}_n p$.
This implies

$$f_{10}(u) \models \Diamond_0(\neg\delta_{10} \wedge \ddot{\Phi}_0 \underline{D}_n p)$$

so there is an x with $f_{10}(u) \sim_0 x$ and $x \models \neg\delta_{10} \wedge \Phi_0 \underline{D}_n p$. By $x \models \neg\delta_{10}$ we have $f_{10}(u) \neq x$.
Let $y = f_{01}(x)$.
As $\mathfrak{F} \models N8(y)$, there is a v with $\neg E_{01} v$ and $f_{10}(u) \sim_1 v \sim_0 y$, viz.



Now we have:

$$
\begin{array}{lll}
 & x \models \oplus_0 \neg D_n \neg p & \\
\Rightarrow & x \models \oplus_0 \neg Z_0 \neg p & (\text{def. } D_n) \\
\Rightarrow & x \models \oplus_0 \neg \Diamond_1(\delta_{10} \wedge \Diamond_0(\neg\delta_{10} \wedge \oplus_0 \neg p)) & (\text{def. } Z_0) \\
\Rightarrow & x \models \oplus_0 \Box_1(\delta_{10} \rightarrow \Box_0(\neg\delta_{10} \rightarrow \boxplus_0 p)) & (\text{modlog}) \\
\Rightarrow & x \models \Diamond_1 \ldots \Diamond_{n-1}\Box_1(\delta_{10} \rightarrow \Box_0(\neg\delta_{10} \rightarrow \Box_{n-1} \ldots \Box_2\Box_1 p)) & (\text{def. } \oplus_0, \boxplus_0) \\
\Rightarrow & x \models \Diamond_1 \ldots \Diamond_{n-1}\Box_{n-1} \ldots \Box_2\Box_1(\delta_{10} \rightarrow \Box_0(\neg\delta_{10} \rightarrow \Box_1 p)) & (*) \\
\Rightarrow & x \models \Diamond_1\Box_1(\delta_{10} \rightarrow \Box_0(\neg\delta_{10} \rightarrow \Box_1 p)) & (N2) \\
\Rightarrow & x \models \Box_1(\delta_{10} \rightarrow \Box_0(\neg\delta_{10} \rightarrow \Box_1 p)) & (\Diamond_1 \text{ is } S5) \\
\Rightarrow & y \models \Box_0(\neg\delta_{10} \rightarrow \Box_1 p)) & (x \sim_1 y \in E_{01}) \\
\Rightarrow & v \models \Box_1 p & (y \sim_0 v \notin E_{01}) \\
\Rightarrow & u \models p & (v \sim_1 u)
\end{array}
$$

This proves $\mathsf{HCF}_n \models D_n\underline{D}_n p \rightarrow p$, so we are finished with **(ii)**.

Before turning to **(iii)**, we establish the following *claim:*

$$(\dagger) \qquad \mathsf{HCF}_n \models \oplus p \rightarrow p \vee D_n p.$$

Let u be a world of a model on a hypercylindric frame $\mathfrak{F}$ such that $u \models \oplus p$. We have to show $u \models p \vee D_n p$.
Let w be the p-world which can be reached from u via a path following the $\sim_i$. If $w = u$, $u \models p$ is immediate.
Otherwise, there is an $i < n$ and a $v \in \mathfrak{F}$ with $u \neq v$ and $u \sim_i vH_i w$. Without loss of generality we assume $i = 0$. By $\mathfrak{F} \models N8$, there is a t with $t \notin E_{01}$, $f_{10}u \sim_0 t \sim_1 v$. It is then straightforward to show that $v \models \oplus p$ implies $f_{10}(u) \models \Diamond_0(\neg\delta_{01} \wedge \Diamond_1 \oplus_0 p)$, giving $u \models Z_0 p$. So $u \models D_n p$.
This proves the *claim*.

The remaining proofs are now simple. For **(iii)**, it is straightforward to show that $\vdash_n D_n\phi \rightarrow \oplus\phi$, so $\vdash_n D_n D_n p \rightarrow \oplus p$, whence (iii) follows by ($\dagger$). We obtain **(iv)** by $\vdash_n \Diamond_i p \rightarrow \oplus p$ and ($\dagger$). $\boxplus$

4.3 Characterizing n-cubes.

In this section and the following we assume α is finite, and write n instead.

We want to give a characterization of the cubes within the class of all n-frames, starting from the class of hypercylindric frames. Obviously, C_n is not modally definable. We will show however, that it does allow an 'SNΞ-characterization', i.e. we can find a formula β (short for: 'bad') such that the cubes are the non-β hypercylindric frames, cf. definition 2.1.1.

Our intuition is to generalize the following property of 3-cubes to the n-dimensional case: if u and v are two *different* points on the same line, then the planes through u and through v, orthogonal to this line, are *disjoint*.

Definition 4.3.1.
Let u be a point in an n-frame $\mathfrak{F} = (W, I)$. For $i < n$, the *i-hyperplane through u* is defined as $H_i^u = \{v \in W \mid uH_iv\}$. ⊞

For n-cubes this definition coincides with ordinary mathematical usage: let u be a point in an n-cube $\mathfrak{F} = ({}^nU, I)$, then $H_i^u = \{v \in {}^nU \mid v_i = u_i\}$.
In the sequel, we concentrate on 0-hyperplanes. Any cube $\mathfrak{F}$ has the following property:

$$\text{for all } u, v \text{ in } \mathfrak{F}\text{: if } u \neq v \text{ and } uT_0v, \text{ then } H_0^u \cap H_0^v = \emptyset.$$

Using an L_{CML_n}-formula, we can express this as $\mathfrak{F} \models DH'$, where

$$DH' \equiv \forall x \forall y((T_0xy \wedge x \neq y) \rightarrow \neg H_0xy).$$

However, we want a description which is close to the *modal* language. Unfortunately, we cannot express the fact that y with $y \sim_0 x$ differs from x *unless x is on a diagonal E_{0i}*. In that case we can express

$$E_{0i}x \wedge \exists y(T_0xy \wedge \neg E_{0i}y)$$

in the modal language. For a point x not on any diagonal, we use a trick: the idea is to bring back the '0-badness' of x to the '0-badness' of $f_{10}(x)$. What this means precisely is stated in the next definition and proposition.

Definition 4.3.2.
Define the following formulas in L_{CML_n}:

$$\begin{aligned} BAD(x) &\equiv \exists y(\neg E_{0i}y \wedge T_0f_{10}(x)y \wedge H_0xy) \\ DH(x) &\equiv \neg BAD(x) \\ DH &\equiv \forall x DH(x). \end{aligned}$$

A world u in a frame $\mathfrak{F}$ is *bad* if $\mathfrak{F} \models BAD(u)$. An n-frame $\mathfrak{F}$ is said to *have disjoint hyperplanes* if $\mathfrak{F}$ is hypercylindric and satisfies $\mathfrak{F} \models DH$. ⊞

The following lemma ensures us that the epitheton 'having disjoint hyperplanes' is attributed to the right class of frames:

Lemma 4.3.3.
Let $\mathfrak{F}$ be a hypercylindric frame. Then

$$\mathfrak{F} \models DH \iff \mathfrak{F} \models DH'.$$

Proof.
($\Rightarrow$) Assume $\mathfrak{F} \not\models DH'$ because it has worlds u and v with uT_0v, uH_0v and $u \neq v$. As $\mathfrak{F} \models N8(f_{10}u)$, there is a y with $f_{10}u \sim_0 y \sim_1 v$ and $y \notin E_{01}$. Clearly then yH_0v, so uH_0y. This means that u is bad.

($\Leftarrow$) Let $\mathfrak{F} \not\models DH$ because u is bad. If y is the world with $y \notin E_{01}$ and $f_{10}u \sim_0 y H_0 u$, then clearly $f_{10}u$ and y are points with $y \notin E_{01}$, $f_{10}u \sim_0 y$ and $H_0^{f_{10}u} \cap H_0^y \neq \emptyset$. ⊞

We now give the CML_n-characterization of badness:

Definition 4.3.4.
Recall that the Z_0-operator is introduced in 4.2.14. Define the following n-formula:

$$\beta(p) \equiv p \rightarrow Z_0 p.$$

Lemma 4.3.5.
Let $\mathfrak{F}$ be a hypercylindric n-frame, u a world in $\mathfrak{F}$. Then

$$\mathfrak{F}, u \not\models \beta \iff \mathfrak{F} \models DH(u).$$

Proof.
By the fact that $BAD(x_0)$ is the Sahlqvist correspondent of β, cf. 2.2.2. ⊞

We can now give the desired characterization, be it not precisely of the cubes: the following theorem states that the frames with disjoint hyperplanes are precisely the *disjoint unions* of the cubic frames.

Theorem 4.3.6.
$\mathsf{HCF}_{n,-\beta} = \mathbf{P_f}\mathsf{C}_n$.

Proof.
It is clear that all n-cubes have disjoint hyperplanes, so $\mathbf{P_f}\mathsf{C}_n \subseteq \mathsf{HCF}_{n,-\beta}$.
To prove the other inclusion is quite tedious: first, let $\mathfrak{F}$ be a hypercylindric frame. As $\Diamond$ is an $S5$-diamond in $\mathfrak{F}$, $\mathfrak{F}$ is a disjoint union of *connected* generated subframes, i.e. parts where the accessibility relation $T_0|H_0$ is total. Call a frame *nice* if it is a connected n-frame with disjoint hyperplanes. Clearly then it suffices to show that

(∗) every nice n-frame is isomorphic to a cube.

To get an intuition about how the proof of (∗) runs, let us first consider an n-cube $\mathfrak{C}$ over the set U. An i-hyperplane in $\mathfrak{C}$ is completely determined by the i-th coördinate of its members, i.e. by an element of U. So the idea will be to represent an arbitrary nice frame over its hyperplanes. The problem is that if $i \neq j$, the set of i- resp j- hyperplanes are different, while there can only be one base set U. The solution to this problem is to relate these disjoint sets of hyperplanes: in a cubic frame again, we can find the second coördinate of a world $u = (u_0, u_1, \ldots, u_{n-1})$ not only by considering H_1^u, but also by looking at the 0-hyperplane of $(u_1, u_1, \ldots, u_{n-1}) = f_{01}(u)$.
Now we can turn to technicalities:
Let $\mathfrak{F} = (W, \sim_i, E_{ij})_{i,j<n}$ be a nice n-frame; set

$$U \text{ is the set of the 0-hyperplanes of } \mathfrak{F}.$$

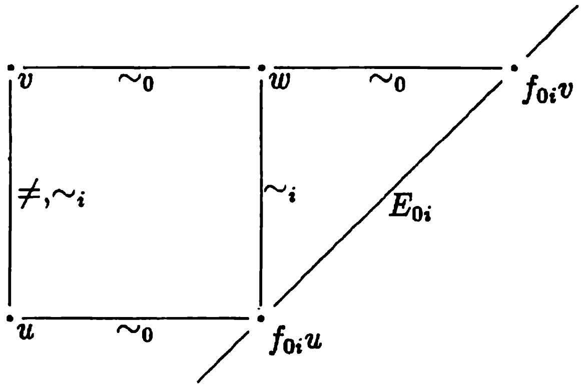
Define the map $h : W \mapsto {}^nU$ by

$$h(u) = (H_0^u, H_0^{f_{01}u}, \ldots, H_0^{f_{0,n-1}u}).$$

In this proof we will write $H(v)$ for H_0^v.

Claim 1. h is injective.

Proof. Let $u \neq u'$. As $\mathfrak{F}$ is connected, there must be $u_0, \ldots, u_n$ such that $u = u_0 \sim_0 u_1 \ldots u_{n-1} \sim_{n-1} u_n = u'$. As $u_0 \neq u_n$ there must be a *first* i with $u_i \neq u_{i+1}$.
If $i = 0$, we have $H(u) \neq H(u_1) = H(u')$, so we are finished.
If $i > 0$, defining $v = u_{i+1}$, we have $u \sim_i v$, $u \neq v$. By $N8$ then there is a w with $v \sim_0 w \sim_i f_{0i}u$ and $w \notin E_{0i}$, viz.



We have $w \sim_0 f_{0i}v$ and $w \neq f_{0i}v$. So by disjoint hyperplanes, $H(f_{0i}v) \neq H(w) = H(f_{0i}u)$. But $v \sim_{i+1}| \ldots |\sim_{n-1} u'$ implies $f_{0i}v \sim_{i+1}| \ldots |\sim_{n-1} f_{0i}u'$, so $H(f_{0i}v) = H(f_{0i}u')$. This gives $H(f_{0i}u) \neq H(f_{0i}u')$, so $h(u) \neq h(u')$. *This proves claim 1.*

Claim 2. h is a homomorphism.

Proof.
For the diagonals: $u \in E_{ij} \Rightarrow f_{0i}u = f_{0j}u \Rightarrow h_i(u) = h_j(u) \Rightarrow h(u) \in D_{ij}$.
For the cylinders: suppose $u \sim_i u'$.
If $i = 0$, then for $j \neq 0$ we have $f_{0j}u = f_{0j}u'$, so $h_j(u) = h_j(u')$.
If $i \neq 0$, then uH_0u', so $h_0(u) = h_0(u')$, and for $j \notin \{0, i\}$ we can show $f_{0j}u \sim_i f_{0j}u'$, implying $H(f_{0j}u) = H(f_{0j}u')$.
So always we obtain $h_j(u) = h_j(u')$ for $j \neq i$, giving $h(u) \sim_i h(u')$.
This proves claim 2.

Claim 3. h is an antihomomorphism.

Proof.
For the diagonal, suppose $h_i(u) = h_j(u)$. This gives $f_{0i}uH_0f_{0j}u$, so by $f_{0i}u \sim_0 f_{0j}u$ and disjoint hyperplanes we get $f_{0i}u = f_{0j}u$. So $f_{0i}u \in E_{ij}$; but then u must be on the ij-diagonal too.
For the cylinders, suppose $h(u) \sim_i h(u')$, or $h_j(u) = h_j(u')$ for $j \neq i$.
We only treat the case $i = 0$: by connectedness there is a v with $u \sim_0 vH_0u'$. By $u \sim_0 v$ and claim 2 we have $h_j(u) = h_j(v)$ for $j \neq 0$, so $h_j(v) = h_j(u')$ for $j \neq 0$. But we also have $h_0(v) = h_0(u')$, as vH_0u'. So $h(v) = h(u')$ and thus by injectivity, $v = u'$. Now $u \sim_0 u'$ is immediate by definition of v. *This proves claim 3.*

Claim 4. h is surjective.

Proof.
By induction to the number k of coördinates differing from the first one, we prove that every $a \in {}^nU$ is the image of a world in $\mathfrak{F}$ under h:
For $k = 0$, let G be the 0-hyperplane in $\mathfrak{F}$ such that $a = (G, \ldots, G)$. As G is not empty, there is a u in G. It is now straightforward to verify that $a = h(f_{n,n-1} \cdots f_{21}f_{10}(u))$.
For $k > 0$, assume that $a = (G, G_1, G_2, \ldots, G_k, G, \ldots, G)$ (without loss of generality).
By the induction hypothesis, $a' = (G, G, G_2, \ldots, G_k, G, \ldots, G)$ is the h-image of some u' in $\mathfrak{F}$. By claim 3, $u' \in E_{01}$. By connectedness, there is a $u_1 \in G$ with $u' \sim_0 u_1$. $\mathfrak{F} \models N4$ implies the existence of a u with $u' \sim_1 u \sim_0 f_{10}u_1$. It is straightforward to verify that $h_i(u) = a_i$ for all i:
For $i \neq 1$: $u \sim_1 u'$ gives $h(u) \sim_1 h(u')$ by claim 2, so $h_i(u) = h_i(u') = a'_i = a_i$.
For $i = 1$: $f_{01}u = f_{10}u_1$ by definition of u, so $h_1(u) = H(f_{01}u) = H(f_{10}u_1) = G_1 = a_1$.
So $h(u) = a$. *This proves claim 4.* ⊞

We would like to thank Hajnal Andréka, István Németi and Ildikó Sain for bringing theorem 3.2.5 of Henkin-Monk-Tarski [53] to our attention. Our result above is closely related to this theorem, in fact it can be seen as as its modal, frame-based version: i-hyperplanes in frames corresponding to i-thin elements of algebras, etc.

4.4 Axiomatizing n-cubes.

Having our characterization of n-cubes in terms of CML_n-formulas and their negations, the road to find an axiom system for C_n is more or less straightforward, if we take the $SN\Xi$-theorem as our guide. Analogous to the $CC\delta$-case, we first give an axiomatization in an extended language and then simplify the system, turning back to the old language.

Definition 4.4.1.
CML_nD is the similarity type of CML_n, extended with the difference operator D. Let $ACML_nD$ be the basic derivation system $K_{CML_nD}D^+$ extended with the axioms $CM1$, $\ldots$, $CM8$, and with the non-β rule:

$$(N\beta R) \qquad \vdash \neg\beta(p) \to \phi \Rightarrow \vdash \phi, \text{ if } p \notin \beta.$$

⊞

In other words, $ACML_nD^+$ has as its axioms: all propositional tautologies (CT), distribution for all diamonds (DB), the difference axioms ($D1$), ($D2$) and ($D3$), and the hypercylindric modal axioms (CM). Its derivation rules are: Modus Ponens (MP), Universal Generalization (UG), Substitution (SUB), the irreflexivity rule for D (IR_D) and the non-β rule ($N\beta R$).
Note that $ACML_nD^+$ is a *tense* logic, as axiom $CM2$ ensures that every $\diamond_i$ is its *own* converse. So completeness is immediate:

Theorem 4.4.2.
$ACML_nD^+$ is strongly sound and complete with respect to C_n.

Proof.
By the $SN\Xi$-theorem 2.8.2, $ACML_nD^+$ is strongly sound and complete with respect to the class of n-frames F satisfying (i) $\mathfrak{F} \models CM1, \ldots, CM8$ and (ii) for all u, $\mathfrak{F}, u \not\models \beta$, in other words: the class $\mathsf{HCF}_{-\beta}$ of frames with disjoint hyperplanes. The theorem is then immediate by theorem 4.3.6 and the fact that for *any* class K of frames, the semantic consequence relations over K and $\mathbf{P_f}\mathsf{K}$ coincide. ⊞

In order to simplify $ACML_nD^+$, recall that D_n is our CML_n-version of the difference operator.

Definition 4.4.3.
Let A_n^+ be the derivation system with *axioms* CT, DB and CM. Its *rules* are MP, UG, SUB and the irreflexivity rule for D_n:

$$(IR_{D_n}) \qquad \vdash (p \wedge \neg D_n p) \rightarrow \phi \;\Rightarrow \vdash \phi, \text{ if } p \not\in \phi.$$

Theoremhood of ϕ in A_n^+ is denoted by $A_n^+ \vdash \phi$ or by $\vdash_n^+ \phi$. ⊞

We will show completeness for A_n^+ by proving that $ACML_nD^+$ is conservative over it. Just as in the case of CCD and $CC\delta$ (cf. section 3.3.6), we will use a translation map:

Definition 4.4.4.
Let Q be a set of propositional variables. The translation $(\cdot)^o$ mapping CML_nD-formulas in Q to CML_n-formulas in Q is defined as the unique extension of the identity map on Q which is a homomorphism with respect to the CML_n-operators and satisfies $(D\phi)^o = D_n\phi^o$. ⊞

Now we can prove the conservativity of $ACML_nD^+$ over A_n^+:

Lemma 4.4.5.
Let ϕ be a CML_nD-formula. Then
$ACML_nD^+ \vdash \phi \Rightarrow A_n^+ \vdash \phi^o$.

Proof.
By induction to the derivation of ϕ in $ACML_nD^+$.
If ϕ is an axiom of $ACML_nD^+$, then $A_n^+ \vdash \phi^o$ either because ϕ^o is an A_n^+-axiom itself, or by lemma 4.2.16.
Otherwise, ϕ is obtained by a derivation rule from theorems to which the induction hypothesis applies. If this rule was one of MP, UG, SUB or (IR_D), we can use the same rule to obtain a derivation for ϕ^o in A_n^+.
We concentrate on the case where the non-β rule was the last one applied: assume that $ACML_nD^+ \vdash \phi$ because of $ACML_nD^+ \vdash \neg\beta(p) \rightarrow \phi$, $p \not\in \phi$.

By the induction hypothesis we get $A_n^+ \vdash (\neg\beta \rightarrow \phi)$, so

$$\begin{array}{llr} & \vdash \neg\beta \rightarrow \phi^o & \text{(definition } (\cdot)^o) \\ \Rightarrow & \vdash \neg\phi^o \rightarrow \beta & \text{(proplog)} \\ \Rightarrow & \vdash \neg\phi^o \rightarrow (p \rightarrow Z_0 p) & \text{(definition of } \beta) \\ \Rightarrow & \vdash \neg\phi^o \rightarrow (p \rightarrow D_n p) & \text{(definition of } D_n) \\ \Rightarrow & \vdash (p \wedge \neg D_n p) \rightarrow \phi^o & \text{(proplog).} \end{array}$$

Finally, one application of the D_n-irreflexivity rule gives $A_n^+ \vdash \phi^o$, as desired. ⊞

And so we can finish with our completeness result:

Theorem 4.4.6. SOUNDNESS & COMPLETENESS

$$\Sigma \vdash_n^+ \phi \iff \Sigma \models_n \phi.$$

Proof.
By theorem 4.4.2 and lemma 4.4.5, analogous to 3.3.37. ⊞

4.5 Harvest.

In this section we show how the completeness proof 4.4.6 for modal n-formulas has several nice corollaries. We will show how it yields (and in some cases, *is*) a completeness proof for: α-valid L_α^r/CML_α-formulas, type-free valid formulas, valid schemas of formulas and the equations holding in RCA_α.

First, we consider infinite-dimensional cylindric modal logic. Recall that CML_α and L_α^r are alphabetical variants. This means that we can use proposition 4.1.3 to give completeness proofs for α-valid modal α-formulas where $\alpha \geq \omega$.
We start with the case where $\alpha = \omega$: let ϕ be an ω-formula. As ϕ uses only finitely many symbols, there is a finite n such that ϕ is an n-formula. By 4.1.3, ϕ is ω-valid iff ϕ is n-valid. So by the completeness theorems for n-validity, an ω-formula ϕ is ω-valid iff there is an n such that $\vdash_n^+ \phi$. This motivates us to add a *schema* of D_n-irreflexivity rules to ω-derivation systems. In order to give a uniform definition for all dimensions, we will then also add the rules IR_{D_n} to the systems A_m^+ with $n < m < \omega$. This is harmless, as the rule IR_{D_n} is *sound* for α-validity if $n < \alpha$.
For an arbitrary infinite ordinal α the situation is essentially the same as for ω, yet technically more complicated:

Definition 4.5.1.
Let $\alpha > \omega$ be an ordinal, $P_\omega(\alpha)$ the set of finite subsets of α. Let G_α be a family $\{g_I^\alpha \mid I \in P_\omega(\alpha)\}$ of bijections $g_I^\alpha : I \mapsto |I|$. (Assume, for ordinals α, β, that $g_I^\alpha = g_I^\beta$ for all I, so that we may suppress the superscripts.)

Now let ϕ be a CML_α-formula. The *index set* $I(\phi)$ of ϕ is defined as the set of indices occurring in one of the modal operators of ϕ. Replacing every operator δ_{ij} resp. c_i by $\delta_{g_{I(\phi)}(i),g_{I(\phi)}(j)}$ resp. $\diamond_{g_{I(\phi)}(i)}$, we obtain a formula in $CML_{|I(\phi)|}$. This formula is called the *finite-dimensional version of* ϕ, notation: $\phi\downarrow$.
For restricted formulas and $\mathcal{L}_\alpha$-terms and -equations we use the analogous terminology.
⊞

Definition 4.5.2.
Let A'_α be the extension of the CML_α-axiom system A_α (defined in 4.2.10) with the *schema* of rules

$$IR_\alpha = \{IR_{D_n} \mid \omega > n \leq \alpha\}.$$

For $\alpha \leq \omega$, A^+_α is the system A'_α.
For $\alpha > \omega$, A^+_α is the system A'_α extended with the *gravity rule*

$$(G) \qquad \vdash \phi\downarrow \Rightarrow \vdash \phi.$$

Theorem 4.5.3.
Let α be an arbitrary ordinal, Σ a set of CML_α-formulas and ϕ a CML_α-formula. Then

$$\Sigma \vdash^+_\alpha \phi \iff \Sigma \models_\alpha \phi.$$

Proof.
We only treat weak completeness, which is immediate for finite α by 4.4.6.
For infinite α, suppose $\models_\alpha \phi$.
If $\alpha = \omega$, this means that $\models_n \phi$ for some $n \in \omega$. By weak completeness for the finite case we have $\vdash^+_n \phi$, so $\vdash^+_\omega$, as every $\vdash^+_n$-derivation is also a $\vdash^+_\omega$-derivation.
For $\alpha > \omega$, clearly $\models_\alpha \phi\downarrow$, for reasons of symmetry. By 4.4.6 we obtain $\models_n \phi\downarrow$ where n is the size of the index set of ϕ. By completeness, $\vdash^+_n \phi\downarrow$, so $\vdash^+_\alpha \phi\downarrow$ and hence $\vdash^+_\alpha \phi$, by an application of the gravity rule. ⊞

An immediate consequence of the above theorem is that we can see A^+_α as a complete derivation system for α-validity of L^r_α-formulas, and A^+_ω for typeless validity. Perhaps the following presentation of these systems is more perspicous:

Definition 4.5.4.
Let A^r_α be the following derivation system for L^r_α having the following axioms:

(CT)	all propositional tautologies
(DB)	$\forall v_i(P \rightarrow R) \leftrightarrow (\forall v_i P \rightarrow \forall v_i R)$
$(CR1)$	$P \rightarrow \exists v_i P$
$(CR2)$	$P \rightarrow \forall v_i \exists v_i P$
$(CR3)$	$\exists v_i \exists v_i P \rightarrow \exists v_i P$
$(CR4)$	$\exists v_i \exists v_j P \rightarrow \exists v_j \exists v_i P$
$(CR5)$	$v_i = v_i$
$(CR6)$	$\exists v_i(v_i = v_j \wedge P) \rightarrow \forall v_i(v_i = v_j \rightarrow P))$
$(CR7)$	$v_i = v_j \leftrightarrow \exists v_k(v_i = v_k \wedge v_k = v_j)$
$(CR8)$	$(v_i = v_j \wedge \exists v_i(\neg P \wedge \exists v_j P)) \rightarrow \exists v_j(v_i \neq v_j \wedge \exists v_i P)$.

The derivation rules of A^r_α are MP, SUB and UG (here: ϕ / $\forall v_i\phi$).
Rewriting the definition of D_n in L^r_n, we obtain

$$D^r_n(\phi) = \bigvee_i \exists v_{S(i)}\big(v_{S(i)} = v_i \wedge \exists v_i(v_i \neq v_{S(i)} \wedge \exists v_0 \dots v_{i-1}v_{i+1} \dots v_{n-1}\phi)\big),$$

where $S(i)$ is the successor of i modulo n, as defined in 4.2.14.
The derivation system A^{r+}_α is defined as the extension of A^r_α with
(i) the schema $IR^r_\alpha = \{IR^r_{D_n} \mid \omega > n \leq \alpha\}$ of rules, where:

$(IR^r_{D_n})$ $\quad\quad \vdash (P \wedge \neg D_n(P)) \rightarrow \phi \Rightarrow \vdash \phi$, if $P \notin \phi$.

(ii) the gravitation rule, in case $\alpha > \omega$:

(G^r) $\quad\quad \vdash \phi\downarrow \Rightarrow \vdash \phi$. ⊞

The following theorems are immediate by the fact that A^{r+}_α is the derivation system A^+_α, rewritten in L^r_α, and by Theorem 4.5.3.

Theorem 4.5.5: COMPLETENESS FOR RESTRICTED LOGIC.
Let ϕ be an L^r_α-formula. Then

$$\vdash^{r+}_\alpha \phi \iff \models_\alpha \phi.$$ ⊞

Theorem 4.5.6: COMPLETENESS FOR TYPELESS LOGIC.
Let ϕ be a typeless formula. Then

$$\vdash^{r+}_\omega \phi \iff \models_{tf} \phi.$$ ⊞

The theorem above indicates a possible solution to Problem 4.16 of Henkin-Monk-Tarski [53], as A^{r+}_ω is a proof calculus for type-free valid formulas which involves only type-free valid formulas. By turning over to a suitable alphabetical variant, the above theorem is also a completeness result for *valid schemas of first order formulas*, cf. the remarks below 4.1.13.
Note that in fact, we did not need cylindric algebras to obtain these results. But of course, a finite derivation system for RCA_α is important in its own right. To formulate it, an algebraic version of the D_n-operator is needed:

Definition 4.5.7.
Define the following $\mathcal{L}_n$-term ($S(i)$ is as in 4.2.14):

$$d_n(x) = \bigvee_{i<n} c_{S(i)}\big(d_{S(i),i} \cdot c_i(-d_{S(i),i} \cdot c_0 \dots c_{i-1}c_{i+1} \dots c_{n-1}x)\big).$$

Definition 4.5.8.
Let α be an arbitrary ordinal.
Recall that $C1 \dots C8$ is an axiomatization of the hypercylindric algebras (cf. 4.1.8 and 4.2.8). Let Σ_α be the smallest set of $\mathcal{L}_\alpha$-equations containing $C0, \dots, C8$, which is closed under

(i) ordinary algebraic deduction
(ii) the following closure operations, for any n with $\omega > n \leq \alpha$:

$$y \cdot -d_n(y) \leq t(x_0, \ldots, x_{n-1}) \ / \ t(x_0, \ldots, x_{n-1}) = 1$$
if y does not occur among the $\vec{x}$.

(iii) the gravitation rule, if $\alpha > \omega$:

$$\eta\downarrow \ / \ \eta.$$

Theorem 4.5.9.

$$\Sigma_\alpha = Equ(\mathsf{RCA}_\alpha).$$

Proof.
Clearly Σ_α is the algebraic version of A^+_α (cf. Appendix A.33), so the theorem is immediate by 4.5.3 and 4.2.4. ⊞

4.6 Using the D_n-irreflexivity rule.

As the set of equations characterizing RCA_α is known not to be finitely axiomatizable using the ordinary derivation rules, the same applies to the set of valid α-formulas. So the rule IR_{D_n} really gives us new theorems, as we do obtain a complete system by adding it to the finite axiom system A_α. So there must be a formula ϕ with

(i) $\mathsf{C}_\alpha \models \phi$
(ii) $\not\vdash_\alpha \phi$
(iii) $\vdash^+_\alpha \phi$.

For the related cylindric algebraic question, I. Sain [11] was the first one to find such a derivation of an equation using the new rule. Here we present our own example, in the modal formalism CML_3. This example shows that the completeness proof for $\vdash^+_\alpha$ is more than just an abstract proof for the existence of a derivation system: one can actually work in it.
The material in this section contains many essential contributions by Hajnal Andréka, Ildikó Sain and István Németi.

Definition 4.6.1.
Consider the following CML_3-formulas:

$$\begin{array}{rcl}
\gamma' &=& (\Diamond_1\Diamond_2 r \wedge \bigcirc_{01}\Diamond_1\Diamond_2 r \wedge \bigcirc_{02}\Diamond_1\Diamond_2 r) \rightarrow (\delta_{01} \vee \delta_{02} \vee \delta_{12}) \\
\gamma &=& \boxplus\gamma' \\
\psi_i &=& \boxplus(r_i \rightarrow r) \wedge \boxplus(\Diamond_0 r \rightarrow \Diamond_0 r_i) \\
\psi' &=& \boxplus((r_0 \rightarrow \neg r_1) \wedge (r_1 \rightarrow \neg r_2) \wedge (r_2 \rightarrow \neg r_0)) \\
\psi &=& r_0 \wedge \psi_0 \wedge \psi_1 \wedge \psi_2 \wedge \psi' \\
\rho &=& \gamma \wedge \psi
\end{array}$$

To give some intuitions: let R denote $V(r_i)$, etc. In a 3-cubic model, γ expresses that the *domain* of R, (i.e. the set $\{s \in U \mid (s,t,u) \in R$ for some $t, u \in U\})$ has at most *two* elements; ψ says that we can split R in three disjoints parts R_0, R_1, R_2 in such a way that the domain of R contains at least *three* elements. So ρ is not satisfiable in a cube:

Proposition 4.6.2.
$\mathsf{C}_3 \models \neg\rho$.

Proof.
Let $\mathfrak{M} = (U,V)$ be a 3-cubic model, and suppose $\mathfrak{M},(s_0,t,u) \models \psi$. Then there are $s_1, s_2 \in U$ such that for $i = 0,1,2$: $\mathfrak{M},(s_i,t,u) \models r_i \wedge r$, and s_0, s_1 and s_2 are mutually distinct. (This means that the domain of R contains at least three elements.) So for the triple $s = (s_0, s_1, s_2)$ we have $s \models \neg(\delta_{01} \vee \delta_{02} \vee \delta_{12})$.
We also have

$s \models \Diamond_1\Diamond_2 r$,

$s \models \bigcirc_{01} \Diamond_1\Diamond_2 r$ (as $(s_1, s_1, s_2) \models \Diamond_1\Diamond_2 r$)

$s \models \bigcirc_{02} \Diamond_1\Diamond_2 r$ (as $(s_2, s_1, s_2) \models \Diamond_1\Diamond_2 r$)

So $s \models \neg\gamma'$ whence $\mathfrak{M},(s_0,t,u) \models \neg\gamma$. ⊞

Our second aim is to show that $\neg\rho$ is not derivable without the IR_{D_n}-rule:

Proposition 4.6.3.
$A_3 \not\vdash \neg\rho$.

Proof.
By our completeness result 4.2.11 it suffices to show that ρ is satisfiable in some hypercylindric frame.

Let A, B and C be the (mutually disjoint) sets $\{a_0, a_1\}$, $\{b_0, b_1\}$ and $\{c_0, c_1\}$ and set $U = A \cup B \cup C$, $\mathfrak{A}''$ the full cylindric set algebra $\mathfrak{Cs}_3(U)$. Let $s \subseteq {}^3U$ be the set $A \times B \times C$ and $\mathfrak{A}'$ the subalgebra of $\mathfrak{A}''$ which is generated by s. In Andréka [5] it is shown that such an s is an atom of $\mathfrak{A}'$. $\mathfrak{A}'$ is finite, so it has an atom structure $\mathfrak{F}' = (W', \sim_i', E'_{ij})$. Clearly $\mathfrak{A}'$ is representable and hence hypercylindric. Note that s itself is a possible world of $\mathfrak{F}'$. The frame we are after is obtained by *splitting* s into three parts. (The notion of splitting atomic cylindric algebras is well-known, cf. Henkin-Monk-Tarski [53].)
Set $\mathfrak{F} = (W, \sim_i, E_{ij})$, where

$$W = (W' - \{s\}) \cup \{s_0, s_1, s_2\}$$

for some new elements s_0,s_1 and s_2; the relations on W are defined as follows:
First let $f : W \mapsto W'$ be a function such that

$$f(t) = \begin{cases} s & \text{if } t = s_i \\ t & \text{otherwise.} \end{cases}$$

Then we set

$$\begin{aligned} E_{ij} &= \{t \mid f(t) \in E'_{ij}\} \\ \sim_i &= \{(t,u) \mid (f(t), f(u)) \in \sim_i'\}. \end{aligned}$$

Claim 1. f is a zigzagmorphism.
Proof. This is almost immediate. (Note that $\mathfrak{F}' \models \neg E_{ij}s$ as $\mathfrak{A}'' \models s \cap D_{ij} = \emptyset$.)

Claim 2. $\mathfrak{F}$ is hypercylindric.
Proof. We leave it to the reader to verify that, with the above definitions, $\mathfrak{F}$ is cylindric. (In fact, this is a well-known aspect of 'splitting' in cylindric algebras, cf. [53].)
We do want to show that $\mathfrak{F} \models N8$: let u, v, w be in W such that $u \in E_{01}$, $u \sim_0 v \sim_1 w$ and $v \neq w$. We want to show the existence of an x with $u \sim_1 x \sim_0 w$ and $x \notin E_{01}$.
Now the only non-trivial case is where $\{v, w\} \subseteq \{s_0, s_1, s_2\}$. Without loss of generality we may assume $v = s_0, w = s_1$.
We now need a special feature of the original algebra $\mathfrak{A}'$ that we started with. Consider the sets

$$\begin{array}{rcll} q & = & \{(b_m, b_m, c_k) \mid \{m, k\} \subseteq \{0,1\}\} & \subseteq B \times B \times C \\ q' & = & \{(b_m, b_{1-m}, c_k) \mid \{m, k\} \subseteq \{0,1\}\} & \subseteq B \times B \times C. \end{array}$$

We want to show that q is the element u of $\mathfrak{F}$ we are discussing, and q' the element x we are looking for.

Claim 2.1. q and q' are atoms of $\mathfrak{A}'$.
Proof. Note that

$$\begin{array}{rcl} B \times B \times C & = & U \times B \times C \cap B \times U \times C \\ & = & C_0 s \cap C_1(D_{01} \cap C_0 s), \end{array}$$

so this element is generated by s in the original full set algebra $\mathfrak{A}''$, and thus an element of $\mathfrak{A}'$. But then so are $q = D_{01} \cap B \times B \times C$ and $q' = (B \times B \times C) - D_{01}$. We omit the proof that q and q' are atomic (cf. Andréka [5].) This proves *claim 2.1.*

Claim 2.2. $\mathfrak{F} \models q \sim_0 s_i \sim_0 q', q \sim_1 s_i \sim_1 q'$.
Proof. By definition of $\sim_0$ and $\sim_1$, it is sufficient to show that $q \sim_0' s \sim_0' q'$ and $q \sim_1' s \sim_1' q'$ in $\mathfrak{F}'$. By definition of $\mathfrak{F}'$, this is equivalent to showing that $C_0 q = C_0 s = C_0 q'$ and $C_1 q = C_1 s = C_1 q'$ hold in $\mathfrak{A}''$. But this is immediate by the definitions. This proves *Claim 2.2.*

Claim 2.3. $\mathfrak{F} \models E_{01}q, \neg E_{01}q'$.
Proof. In $\mathfrak{A}''$, and therefore in $\mathfrak{A}'$, $q \subseteq D_{01}$, so $\mathfrak{F}' \models E_{01}q$. But then $\mathfrak{F} \models E_{01}q$ too, by definition of E_{01}. A similar proof goes for $\mathfrak{F} \models \neg E_{01}q'$.
This proves *claim 2.3.*

Now as $u \sim_0 q$ (by $u \sim_0 v = s_0 \sim_0 q$) and both u and q are in E_{01}, they must be identical by $N6$. By $\neg E_{01}q'$, $w = s_1 \sim_0 q$ and $u = q \sim_1 q'$ then, q' indeed has the desired properties. This proves *claim 2.*

So now, if we can define a model $\mathfrak{M}$ on $\mathfrak{F}$ such that ρ is true somewhere, we are finished. Set

$$\begin{array}{rcl} V(r) & = & \{s_0, s_1, s_2\} \\ V(r_i) & = & \{s_i\}. \end{array}$$

and $\mathfrak{M} = (\mathfrak{F}, V)$.

Claim 3. $\mathfrak{F}, V, s_0 \models \rho$.

Proof. It will be clear that $\mathfrak{M}, s_0 \models \psi$.
To show that $\mathfrak{M} \models \gamma'$, consider the cubic model $\mathfrak{M}'' = (\mathfrak{F}'', V'')$ with $\mathfrak{F}'' = \mathfrak{Cs}(U)(= \mathfrak{AtA}'')$ and $V''(r) = s$. Clearly $\mathfrak{M}'' \models \gamma'$ as the domain of s contains only two elements (a_0 and a_1).
By constructing the obvious zigzagmorphism $g : \mathfrak{F}'' \mapsto \mathfrak{F}'$, and defining a model $\mathfrak{M}'$ on $\mathfrak{F}'$ with $V'(r) = \{s\}$, it is standard modal theory (cf. van Benthem [14]) to show that $\mathfrak{M}' \models \gamma'$.
By the same procedure, now with respect to $f : \mathfrak{F} \mapsto \mathfrak{F}'$, we prove that $\mathfrak{M} \models \gamma'$. So $\mathfrak{M}, s_o \models \boxplus \gamma'$. This proves *claim 3.* ⊞

We conclude this section by showing that $\neg\rho$ is derivable if we have the new derivation rule at our disposal:

Proposition 4.6.4.
$A_3^+ \vdash \neg\rho$.

Proof.
We will not give the actual derivation, which would hardly give any insights; our task will be to prove

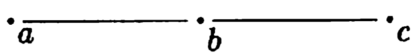
$$(*) \qquad \mathsf{HCF}_3 \models (p \wedge \neg D_3 p) \rightarrow \neg\rho.$$

After establishing this, we reason as follows:
By 4.2.11, $A_3 \vdash (p \wedge \neg D_3 p) \rightarrow \neg\rho$, so by one application of the D_3-irreflexivity rule: $A_3^+ \vdash \neg\rho$.

To prove (*), let $\mathfrak{F}$ be hypercylindric, V and a such that $\mathfrak{F}, V, a \models p \wedge \rho$. We will show that $\mathfrak{F}, V, a \models D_3 p$.

(i) As $a \models \psi$, there are b, c in F with

$a \sim_0 b \sim_0 c$
$a \models r_0 \wedge \neg r_1 \wedge \neg r_2$
$b \models \neg r_0 \wedge r_1 \wedge \neg r_2$
$c \models \neg r_0 \wedge \neg r_1 \wedge r_2$



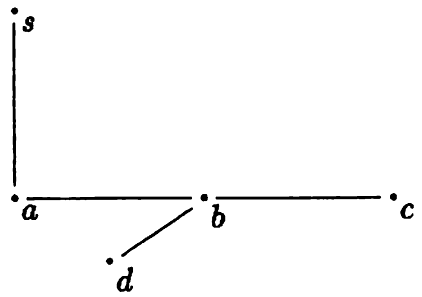
(ii) Actually, all we need to remember is:

a, b and c are distinct
$a \sim_0 b \sim_0 c$
$a \models r, b \models r$ and $c \models r$.

We will show that $\mathfrak{M} \models \gamma$ causes the *hyperplanes* through a and b to coincide, implying that a and b are *bad* points (cf. definition 4.3.2).

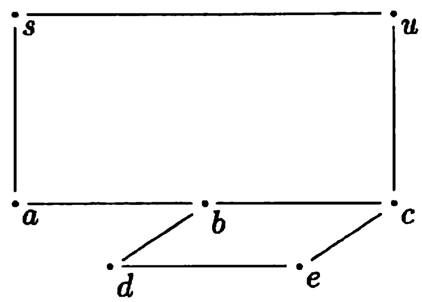
(iii) Let $s = f_{10}a$, $d = f_{20}b$, i.e.

$E_{01}s,\ a\sim_1 s$
$E_{02}d,\ b\sim_2 d$



(iv) By $\mathfrak{F} \models N8$, there are u, e with

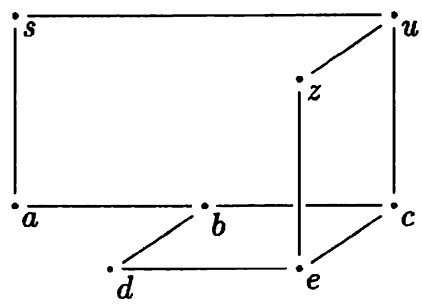
$\neg E_{01}u,\ s\sim_0 u\sim_1 c$
$\neg E_{02}e,\ d\sim_0 e\sim_2 c$



(v) By $\mathfrak{F} \models N4$, there is a z with

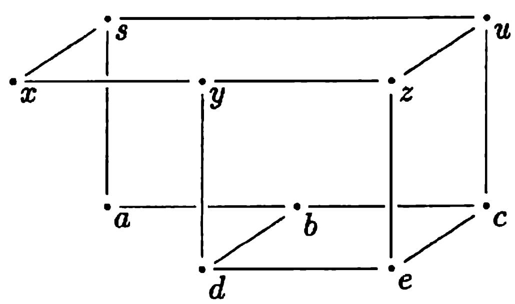
$e\sim_1 z\sim_2 u.$

This also gives:
$\neg E_{02}z,\ \neg E_{01}z.$



(vi) By $N4$ again, there are x, y with

$d\sim_1 y\sim_0 z$
$s\sim_2 x\sim_0 z$



In fact, we will prove that $H_0^a \cap H_0^b \neq \emptyset$ by showing $x = y$.

(vii) We have that

$$\left.\begin{array}{lclcl} a\models r & \Rightarrow & s\models \delta_{01}\wedge\Diamond_1 r & \Rightarrow & x\models \delta_{01}\wedge\Diamond_2\Diamond_1 r \\ b\models r & \Rightarrow & d\models \delta_{02}\wedge\Diamond_2 r & \Rightarrow & y\models \delta_{02}\wedge\Diamond_1\Diamond_2 r \\ c\models r & \Rightarrow & e\models \Diamond_2 r & \Rightarrow & z\models \Diamond_1\Diamond_2 r \end{array}\right\} \Rightarrow$$

$$
\begin{aligned}
& z \models \bigcirc_{01} \Diamond_1 \Diamond_2 r \wedge \bigcirc_{02} \Diamond_1 \Diamond_2 r \wedge \Diamond_1 \Diamond_2 r \\
\Rightarrow\ & z \models \delta_{01} \vee \delta_{02} \vee \delta_{12} && \text{(as } z \models \gamma'\text{)} \\
\Rightarrow\ & z \models \delta_{12} && \text{(cf. (v))}
\end{aligned}
$$

(viii) Now

$E_{12}z$ and $z \sim_0 y$ give $E_{12}y$, so with $E_{02}y$ we get $E_{01}y$.
We already had $E_{01}x$, so by $x \sim_0 y$ and $\mathfrak{F} \models N6$ we obtain $x = y$.

(ix) This gives

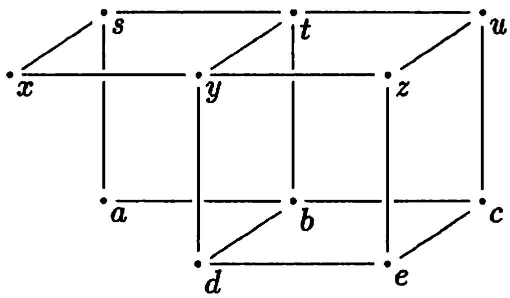
$a \sim_1 s \sim_2 x = y \sim_1 d \sim_2 b$,
so a and b really are in the same 0-hyperplane:
aH_0b.

Recall that we assumed $a \models p$. We will prove that this implies $a \models Z_0p$, and hence $a \models D_3p$.

(x) By $\mathfrak{F} \models N8(s)$, there is a t with

$s \sim_0 t \sim_1 b$, $\neg E_{01}t$.

Note that aH_0t
by aH_0bH_0t.



(xi) We can now prove $a \models D_3p$:

$$
\begin{aligned}
& a \models p \\
\Rightarrow\ & t \models \neg\delta_{01} \wedge \Diamond_1 \Diamond_2 p \\
\Rightarrow\ & s \models \delta_{01} \wedge \Diamond_0(\neg\delta_{01} \wedge \Diamond_1 \Diamond_2 p) \\
\Rightarrow\ & a \models \bigcirc_{10} \Diamond_0(\neg\delta_{01} \wedge \Diamond_1 \Diamond_2 p) \\
\Rightarrow\ & a \models Z_0 p \\
\Rightarrow\ & a \models D_3 p.
\end{aligned}
$$

So we have proved indeed that $HCF_3 \models (\rho \wedge p) \to D_3p$. ⊞

4.7 Conclusions, Remarks and Questions.

4.7.1. General Conclusions.

In this chapter we have given a detailed account of how (a restricted version of) classical first order logic can be seen as a modal logic (section 1).
We defined a similarity type in which the Kripke structures have the type of atom stucures of cylindric algebras, and the intended frames or cubes are the atom strucures of full cylindric set algebras (section 2).
The class of (disjoint unions of) cubes allows a nice characterization in terms of Sahlqvist formulas and a non-ξ rule (section 3).
So, by applying our $SN\Xi$-theorem, we found a finite axiomatization of the modal formulas valid in the cubes (section 4).
In section 5 we saw that this completeness theorem has several nice corollaries:

(i) a completeness theorem for restricted first order logic (both the finitary and the infinitary versions).
(ii) a proof calculus for type-free valid formulas, indicating a possible solution to Problem 4.16 of Henkin-Monk-Tarski [53].
(iii) a finite derivation system generating $\mathit{Equ}(\mathsf{RCA}_\alpha)$, for all α.

In section 6 we showed that our completeness results are more than abstract proof of the existence of certain derivation systems: the new rule can actually be used to find derivations.

It was generally assumed that the equations valid in representable cylindric algebras were *not* finitely axiomazible. In our opinion, the main contribution of this chapter lies in showing the validity of this assumption to depend on the *derivation rules* one allows in an axiomatization. By adding rules that are non-orthodox from the traditional algebraic viewpoint, existing finite (and hence incomplete) axiomatizations can be turned into (finite) complete derivation systems.
In this connection (but not only in this connection) it is interesting to note the following: independently of our result, András Simon found a proof calculus for typeless validity (and thus, for the related notions), in which another kind of unorthodox derivation rule appears (cf. Simon [121, 122]). Simon's methods seem to be complementary with ours in that he concentrates on finite dimensional while we on finite dimensional cylindric algebras. Cf. also 4.7.4(ii).

We feel that, compared to the algebraic approach, the main advantage of the *modal* paradigm is, that by concentrating on the frames/atom structures of a similarity type, the *local* aspects of truth (i.e. at each world individually) can be emphasized, rather than the *global* ones (i.e. at the model/algebra as a whole). The notion of a non-ξ rule, which plays such an important rôle in our axiomatizations, is quite natural within the modal framework where the focus is on individual possible worlds, while it is less so in the algebraic way of thinking. For details we refer to the second appendix.

4.7.2. Unrestricted First Order Logic.

The reader might have objections to our claim that we have treated predicate logic as if it were a modal formalism, as the classical logic L^r_α we have 'modalized' seems to be rather odd. This oddity is only apparent however, as all of the restrictions imposed on first order formulas in L^r_α can be overcome. As these transpositions are wellknown from the theory of algebraic logic (cf. Németi [89] or Henkin-Monk-Tarski [53]), we will be rather brief.
The simplest constraint to be removed is the fixed rank of predicate symbols: without any modifications one may assign every predicate variable in L^r_α/ atomic proposition in CML_α a rank $\beta < \alpha$. In the axiom system however, one has to do some bookkeeping concerning *free* and *bound* occurrences of variables, and add an axiom schema '$\phi \to \forall v_i \phi$, for v_i not occurring free in ϕ'. (These technical complications form a good reason to consider only restricted logics with a fixed rank for all predicates.)
We now turn to the question of the *fixed order* of the variables in atomic formulas.
First, atoms like $Rv_0v_1v_0v_3$, having an equivalent $\exists v_2(v_2 = v_0 \wedge Rv_0v_1v_2v_3)$, never pose any problems.
For the case where a *simultaneous* substitution of variables is involved, consider the example $Rv_1v_0v_2$ (with R a ternary predicate symbol). If we have more dimensions at our disposal, we can use the extra variables as buffers and consider the formula

$$\exists v_3 \exists v_4 (v_3 = v_0 \wedge v_4 = v_1 \wedge \exists v_0 \exists v_1 (v_0 = v_4 \wedge v_1 = v_3 \wedge Rv_0v_1v_2))$$

which is equivalent to $Rv_1v_0v_2$.
Bringing the above three ideas together, one can show that indeed every ordinary first order ($L_{\omega\omega}$-)formula has an equivalent in L^r_α/CML_α.

If we do not have these extra buffer variables, for example if we study logic with finitely many variables, we are in trouble, at least if we want to stay in the similarity type of cylindric algebras. However, *polyadic algebras* (introduced by Halmos [48]) have operations for *switching dimensions*. They do not have diagonals, so if we want to have a algebraic counterpart of the identities $v_i = v_j$ too, we have to study *polyadic equality algebras* (cf. Nëmeti [89] or Henkin-Monk-Tarski [53]): these have the combined expressive power of cylindric and polyadic algebras. From the modal perspective, to obtain the full power of L_n (unrestricted first order logic with n variables), we have to add operators $\otimes_{ij}$ ($i, j < n$) to CML_n, with the following semantics in cubes (we give an example for $n = 4$):

$$\mathfrak{M}, (u_0, u_1, u_2, u_3) \models \otimes_{23}\phi \iff \mathfrak{M}, (u_0, u_1, u_3, u_2) \models \phi.$$

It is not very difficult to axiomatize cubic validity for the resulting similarity type. In fact, adding the *parametric* axioms

$$(p \wedge \neg D'p) \to (\otimes_{ij} q \leftrightarrow \bigcirc_{ij} \Diamond_j (q \wedge \bigcirc_{ji} \Diamond_i p))$$

on top of A_n^+, is sufficient (cf. the proof of theorem 5.4.7, where we work out an example of a completeness proof for a derivation system having parametrical axioms.)

4.7.3. Interpolation and Amalgamation.

In section 2.9.2 we saw how the question whether non-ξ rules are *conservative* over a given logic, has to do with the interpolation property of the logic. The latter property is connected to the amalgamation property of the corresponding algebraic variety (cf. Pigozzi [98]).

Definition 4.7.1.
A class K of algebras is said to have the *amalgamation property* (*AP*), if for any $\mathfrak{A}$, $\mathfrak{B}_1$, $\mathfrak{B}_2$ in K with $\mathfrak{A} \subseteq \mathfrak{B}_i$, there are a $\mathfrak{C}$ in K and embeddings $f_i : \mathfrak{B}_i \rightarrowtail \mathfrak{C}$ such that $f_1 \restriction \mathfrak{A} = f_2 \restriction \mathfrak{A}$.
⊞

Theorem 4.7.2.
Let S be a similarity type, Λ an S-logic. Then Λ has the interpolation property iff $\mathbf{CmFr}_\Lambda$ has the amalgamation property. ⊞

For a proof of this theorem we refer to Pigozzi [98]. The amalgamation property of cylindric and related algebras has been studied quite intensively, for example by Németi, Pigozzi and Sain, cf. Sain [115] for an overview. A maybe unexpected corollary of our completeness theorem for cylindric modal logic is that no finitely based variety between HCA and RCA has the amalgamation property. This result is in itself not spectacular, but the proof method is a novelty:

Theorem 4.7.3.
Let K be a finitely based variety with $\mathsf{HCF}_\alpha \subseteq \mathsf{K} \subseteq \mathsf{RCA}_\alpha$. Then K does not have *AP*.

Proof.
Suppose otherwise and set Λ as the finite, orthodox axiom system axiomatizing **At**K, then Λ has *IP* by 4.7.2. Let Λ' be the logic Λ extended with the set of *closed* α-valid CML_α-formulas as axioms, and Λ^+ the extension of Λ' with the D_n-irreflexivity rules.
We will prove that Λ' and Λ^+ have the properties (i) and (ii) of 2.9.2:
For (i), it is not hard to prove that Λ' has *IP* too. Now Λ^+ proves only α-valid formulas, so Λ^+ is consistent, and by definition of Λ', Λ^+ proves all *closed* α-valid formulas that are Λ'-theorems; this gives (ii).
By theorem 2.9.2 then, IR_{D_n} is conservative over Λ', but as Λ' is an extension of A_α, this means that Λ' is a sound and complete axiomatization of the α-cubes. Turning back to the algebras, we would obtain an (orthodox!) axiomatization for RCA_n with only finitely many equations using variables. This contradicts the result by Andréka [5] mentioned below definition 4.1.8. ⊞

A similar result can be proved for the finitely based varieties between RA and RRA (cf. chapter 4).

4.7.4. Questions.

(i) Just like for two-dimensional logics, we can study various α-dimensional similarity types – we have already encountered the operator $\otimes_{ij}$ in subsection 4.7.2. Consider the following problem: is there a similarity type of (first order definable/ permutation invariant/...) operators, so that cube validity becomes axiomatizable by an *orthodox* system having *finitely* many axioms (or axiom schemas in the infinite-dimensional case)?
This problem is (part of) the modal counterpart of the so-called *finitization quest* in algebraic logic, cf. Biró [18], Maddux [79], Németi [89], Sain [114]. The upshot of these papers seems to be that for *first order definable* extensions of *CML* (or *CC*δ), the answer to the above question is negative. For similarity types *no* having constants referring to identity of the corresponding first order logic, there are positive solutions, witness Sain [114].
Is there a connection with the fact that in our derivation systems we *need* the diagonal constants to define the *CML*/*CC*δ-versions of the D-operator which play such an important rôle in our story?

(ii) We have already mentioned Simon's independent completeness result for typeless logic (cf. Simon [121, 122]), and the fact that both his and our derivation system use unorthodox derivation rules (in fact, in the Appendix B we show why this is *necessarily* so). But, his and our systems share another characteristic: both rules are of the form

$$(R) \qquad \vdash \phi \Rightarrow \vdash \psi, \text{ provided } C.$$

Here C is some *constraint* on the syntax of ϕ and ψ, in Simon's rule concerning the cylindric operators appearing in ϕ and ψ, in our D_n-irreflexivity rule concerning the propositional variables.
Is there some intrinsic complexity in the notions to be axiomatized, *forcing* the unorthodox derivation rules to have such constraints? Is this question related to the previous one?

CHAPTER 5.

PERIODS IN PLANES.

Outline.

In this chapter we show how one can treat temporal logics of periods in the style of two-dimensional modal logic, looking at intervals as points in a plane.

We start with a general introduction to period-based temporal logics, followed by a more technical part motivating the treatment of interval logic as two-dimensional modal logic (section 1). The main part of the chapter is formed by section 2, where we discuss a particular system called HS. We treat expressiveness, definability and completeness. In section 3 we make a connection between interval logics and logics of computation, viewing intervals as sequences of computation states. This section contains an example of a derivation system where we can eliminate the irreflexivity rule. Before finishing off with our conclusions (section 5), we treat a proper extension of HS with a modal operator for chopping intervals.

5.1 Introduction.

5.1.1. Time in Periods.

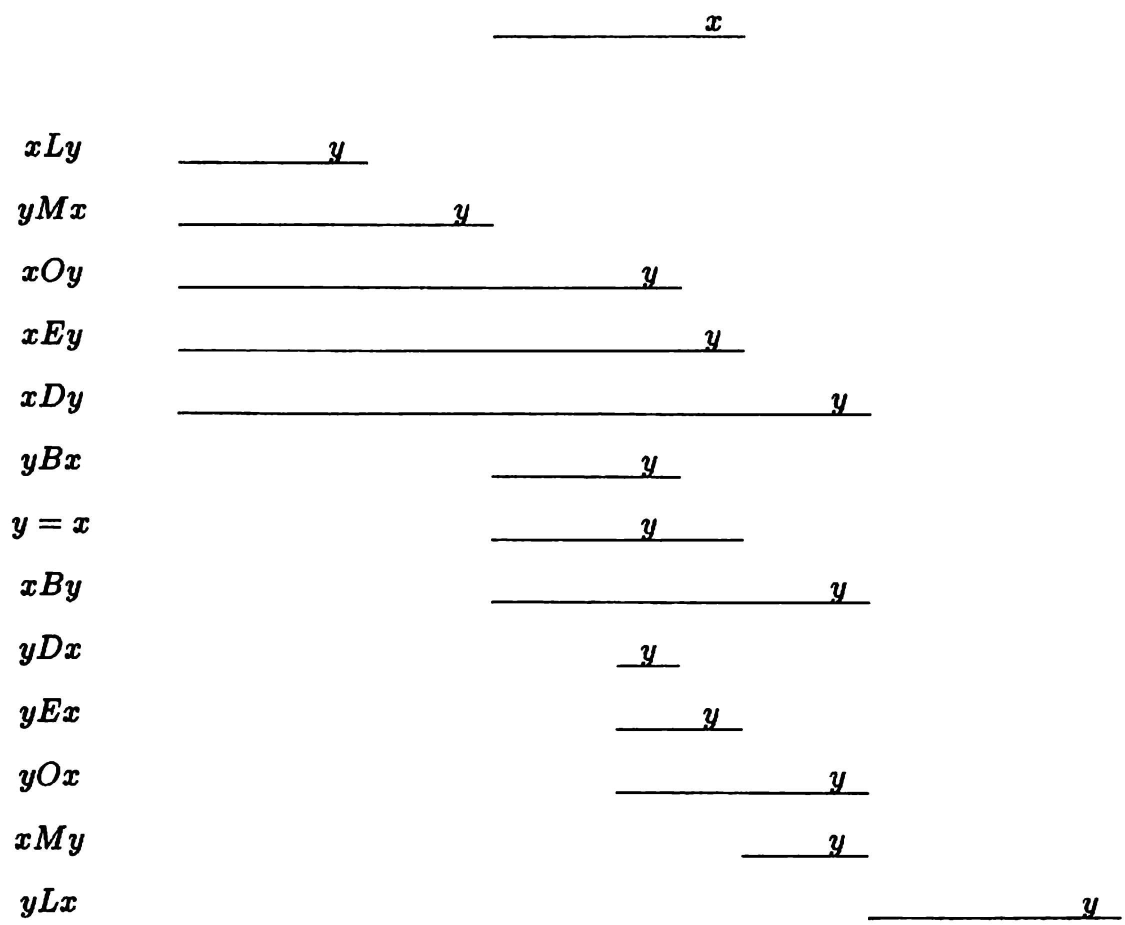
When developing a logical formalism for the concept of time, one has to make (at least) three fundamental choices: one has to choose a temporal *ontology*, select a *formal language*, and fix the nature of the *semantics*, i.e. the interplay between ontology and language. Let us consider these possibilities in more detail.

First of all, one has to select a suitable *ontology*. Questions that need answering here are: What are to be the primary entities of the structure representing time? Which basic relations do we impose on them? What constraints should these relations meet in a 'proper' representation of time?
In temporal logic, the dominant approach is to represent time as a series of *points*, together with a binary relation which should be a (linear?) ordering. Probably, the motivation for this choice originates with the fact that most people have a picture of such a flow of moments in mind when thinking of the concept 'time'. It is debatable however how intuitive this picture is. Does it not spring from years of (naive or scientific) reasoning and training?
And indeed, in this century it has been argued that from a psychological or philosophical point of view, it is much more natural to consider an ontology where *periods* of time are the basic objects, and points are more abstract entities, *constructed* from these periods. There is also a fairly important research line where a structure representing time itself is considered to be too abstract a notion, and *events*, being objects representing 'something happening in (space)time' are the building blocks.
Of course there is not one sole approach which is best for each and every purpose; in general it is much more interesting to see which ontology suits which aims best, and how different ontologies relate to each other. For a thorough treatment of these matters we refer to van Benthem [12].
Our focus point here will be a formalism where uninterrupted periods of time are, when not the basic objects, at least the central ones. (The issue "periods as basic objects" versus "periods as constructed entities" will be discussed in the next section.)
Informally, a period i will be represented by a horizontal line:

$$\underline{\qquad\qquad\qquad i\;}$$

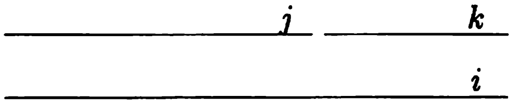
So let us assume now that we have a set of periods of which we want to make a structure representing time. The question is which relations we need connecting the periods. There is not such an obvious candidate as the binary ordering relation for point structures, and the literature has seen many different approaches.
First, we have to consider the question whether we want to allow *periods of no duration* in the structure, and a unary relation telling us whether a period is of this form. One might argue that such periods are nothing but time*points* in disguise, but then again this may very well turn into an advantage! Besides that, it may come handy to have such

durationless point-like periods to represent instantaneous events like the exploding of a bomb.
Looking at binary relations between periods, let us for the moment confine ourselves to structures representing a 'linear' stretch of time — of course we are using an intuitive notion of linearity as we have no means yet to define it formally. It is well-known that for linear time, there are thirteen possible positions of two periods with respect to each other. (In a formalized setting, Allen [2] and van Benthem [12] give simple proofs for this.) Fixing the position of x, we show the thirteen different habitats of y in the following picture:



The letters L, M, O, E, D and B are mnemonics for resp. *is later than*, *meets*, *overlaps*, *ends*, *during* and *begins*.
So there are numerous ways to define our basic relations. A not unusual choice is to take D and L as basic (for example, Humberstone [54], Röper [108] and van Benthem [12] take this approach), but one finds also M (cf. Allen-Hayes [4]), or B and E (cf. Halpern-Shoham [50]).
Another possibility is a ternary relation like $Aijk$, informally: i is the sum of j and k, viz:

Aijk



This relation is the accessibility relation of the *versatile* logic studied in Venema [133]. One may wonder whether this choice is an important one. Are not all relations interdefinable? The answer to this question brings us to the second field, after the ontology, in which choices are to be made concerning the formalization of the notion of time, namely the *formal language* one wants to use to describe the structures.
The main choice here, both for point-based and period-based structures, seems to lie between *first order logic* and some kind of *modal* formalism. (An interesting alternative is developed by Ladkin and Maddux [69] who develop a calculus for intervals in the setting of *relation algebras*...) Returning to the 'interdefinability question' formulated above, the point is that while the definition of one relation in terms of others can usually be written quite easily in first order logic, this description may go beyond the expressive power of the modal formalism. As usual, this limited expressiveness of modal formalisms can also turn into an advantage, for reasons of elegance and complexity. Again, our aim here is not so much to give a detailed discussion of the merits of both approaches, as to give a *technical* treatment of the subject, here concentrating on the link between the theory of many-dimensional modal logics and the modal logics of intervals.
Where in point-based logic the literature is far too extensive to summarize here, publications on modal logics of periods are far more scarce: van Benthem [12], Galton [37], Humberstone [54], Nishimura [91], Röper [108] and Richards c.s. [103] all have a linguistic/philosophical motivation to study the modal logic of intervals. There are also computer scientists interested in the field, e.g. from the area of or artificial intelligence (Halpern and Shoham [50]), and there is a close connection between interval-based temporal logics and the logics developed in the field of program verification like *process logic* ([49, 109, 51]). In Moszkowski [87] an interval-based formalism is used itself as a programming language. Treatments of period-based temporal logics from more logical/mathematical perspectives are given in Burgess [23], Reynolds [101], Venema [130, 133] and White [138].

Finally, there is one more area in which one has to select one out of several options, namely in the *semantics* of the formalism, or the interplay between language and ontology. It is quite common in papers on modal interval-logic to impose *constraints* on the semantics like *homogeneity*: A proposition is true at a period iff it is true at all of its subperiods. Although it is clear that in many applications it may be useful to have conditions on the formalism of this kind, we believe it is better first to develop a framework which is as general as possible, and then to *add* suitable conditions on the semantics wherever it is needed.

To sum up: we will study *modal* logics of *periods*, with a *most general* semantics.

5.1.2. Intervals as Two-Dimensional Points.

In the previous subsection, we already hinted at the idea that taking periods as 'central' objects does not force us to leave points out of the picture. On these pages we will go deeper into the relation between point-based and period-based structures.

As we have already mentioned, some of the reasoning in favour of periods is of a fundamental, metaphysical nature, when it is argued that moments 'do not really exist', and periods are more natural and concrete objects. But there is also a more pragmatic lobby: in a formal approach to linguistics or artificial intelligence it can be quite *useful* to have entities representing stretches of time in the models, for example to provide interpretations for linguistic expressions like 'yesterday', or propositions like 'the robot performed its task'. While we do support this second argument, the validity of the first one is in our eyes too dependent on the particular context to treat it in general, and a discussion about it is definitely beyond the scope of this dissertation. Besides that, we feel that it is at least *interesting* to study temporal structures where periods are entities constructed out of points.

Turning to technical implementations, again we have more than one option available: periods can be *convex sets* or *intervals* (or *open sets*, or ...). From now on we confine ourselves to structures where time is supposed to be linear; this restriction allows us a more streamlined presentation — it can be lifted without too many difficulties.

Definition 5.1.1.
Let $\mathfrak{T} = (T, <)$ be a *temporal order*, i.e. $<$ is a total linear ordering on T. A subset C of T is called *convex* if

$$\forall stu(s \in C \wedge u \in C \wedge s < t < u \rightarrow t \in C).$$

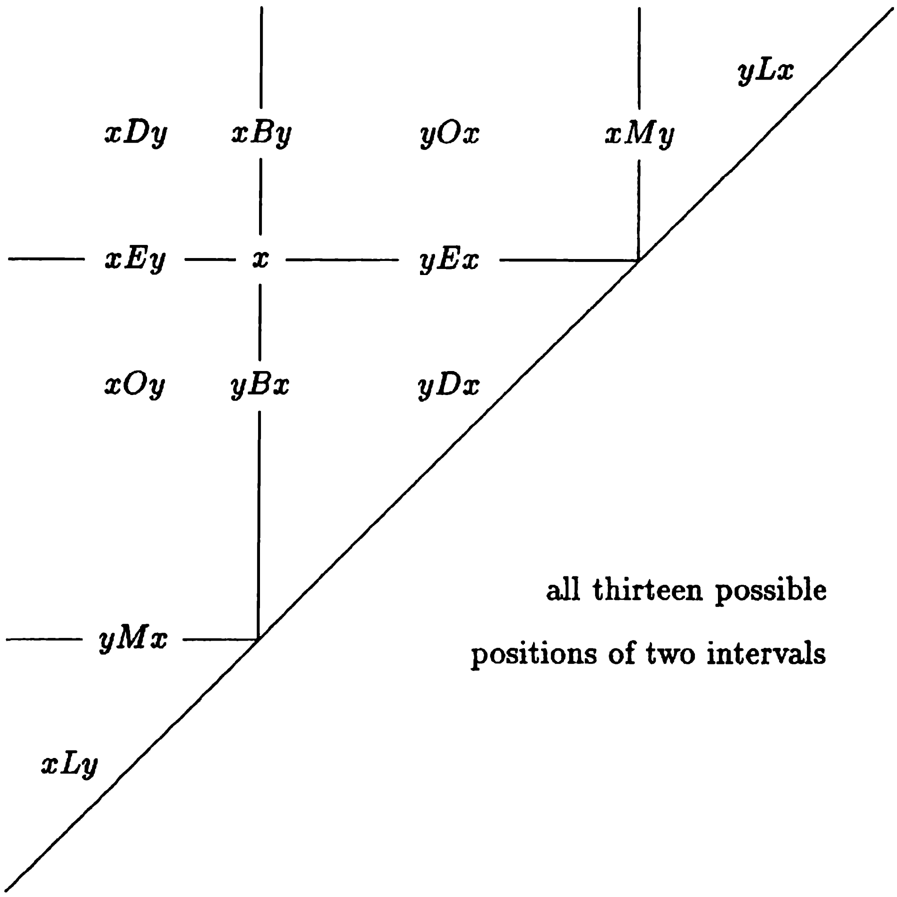
The set of convex set in $\mathfrak{T}$ is denoted by $CONV(\mathfrak{T})$.
An *interval* of $\mathfrak{T}$ is a pair (s,t) with $s \leq t$, usually denoted with *straight* brackets, $[s,t]$. We set

$$INT(\mathfrak{T}) = \{[s,t] \mid s \leq t\}. \qquad \boxplus$$

If we identify an interval $[s,t]$ with the closed subset $\{x \in T \mid s \leq x \leq t\}$ (in general, we will *not* do so), it will be clear that all intervals are convex, but the converse does not hold: the open set of rationals $\{q \in Q \mid \sqrt{2} < q < \pi\}$ will not be an interval, for example.

Our attention will be directed towards interval-structures. Admittedly, one can imagine situations where arbitrary convex sets are needed. In such a case, the following trick may offer consolation: take a linear order $\mathfrak{T} = (T, <)$ and let $\overline{\mathfrak{T}}$ be the Dedekind-completion of $\mathfrak{T}$. By a simple topological argument, every convex set in $\mathfrak{T}$ can be represented by a subset in $\overline{\mathfrak{T}}$ of the following kind: $\langle s,t\rangle$, $\langle s,t]$, $[s,t\rangle$ or $[s,t]$, with $s,t \in \overline{T}$. Now $INT(\overline{\mathfrak{T}})$ only contains the *closed* intervals $[s,t]$, but at least we have reduced the second-order logic of T to some first order interval-like structure. Clearly this is a matter for further research. On the other hand, we have several reasons for preferring intervals to convex sets.
First of all, in a language having variables referring to points, convex sets immediately

take us to the *second order* predicate calculus of the point structures, where intervals can be referred to in first order logic. (Note that although we will study the *modal* logic of periods, this issue is relevant in the correspondence theory.) Related to this matter is the following point: we are interested in the connection between structures where periods are the basic entities, and point structures where they are constructed objects. So it will be relevant to construct a point ordering $\mathfrak{T}(\mathfrak{P})$ associated with a given period structure $\mathfrak{P}$. This construction, although mathematically interesting, can be quite complicated if one wants $\mathfrak{P}$ to be isomorphic to the convex-set structure based on $\mathfrak{T}(\mathfrak{P})$, cf. van Benthem [12] or Thomason [129], where the authors use second-order notions like *filters*. For intervals however, this construction is quite simple, cf. Allen-Hayes [4] or theorem 5.2.11 here.
The main reason for taking intervals instead of convex sets leads us to the connection between the modal logic of intervals and many-dimensional modal logic: if periods are intervals, i.e. *pairs* of time points, we have a nice two-dimensional picture of $INT(\mathfrak{T})$, where each interval finds its own spot in the plane, and we can give an geometrical interpretation of the relations between intervals, analogous to the situation in two-dimensional modal logic. For example, let $\mathfrak{T} = (T, <)$ be a *linear* temporal order. $INT(\mathfrak{T})$ can be depicted as the North-Western halfplane, the intervals of no duration can be identified precisely with the points on the diagonal, and the thirteen possible positions of an interval $[u, v]$ with respect to a given interval $[s, t]$ are shown in the following figure:



For example, a *beginning* interval $[s, u]$ of $[s, t]$ (i.e. $u < t$), is situated right *south* of $[s, t]$, an interval $[u, s]$ *meeting* $[s, t]$ lies on the horizontal line of which the diagonal point $[s, s]$ lies south of $[s, t]$, etc.

5.2 The system HS.

5.2.1. The System HS: Introduction.

In this section we will undertake a detailed treatment of the interval-based modal logic devised by Halpern and Shoham in [50]. Their original system has three pairs of tense operators, with respectively B, E and M as intended accessibility relations. We propose a slight simplification of their similarity type:

Defininition 5.2.1.
HS is the similarity type having two pairs of diamonds: $\diamondT, \diamondB$ and $\diamondR, \diamondL$. Besides the boxes ($\boxT$,$\boxB$, $\boxR$ and $\boxL$), we use the following abbreviations:

$$\begin{array}{rcl}
\ominus\,\phi & = & (\boxR\perp\wedge\,\phi)\vee\diamondR(\boxR\perp\wedge\,\phi) \\
\obar\,\phi & = & (\boxB\perp\wedge\,\phi)\vee\diamondB(\boxB\perp\wedge\,\phi) \\
\diamondH'\,\phi & = & \diamondL\,\phi\vee\diamondR\,\phi \\
\diamondH\,\phi & = & \phi\vee\diamondH'\,\phi \\
\diamondV'\,\phi & = & \diamondT\,\phi\vee\diamondB\,\phi. \\
\diamondV\,\phi & = & \phi\vee\diamondV'\,\phi. \\
\diamondplus\,\phi & = & \diamondH\,\diamondV\,\phi\vee\diamondV\,\diamondH\,\phi.
\end{array}$$

⊞

Frames for this similarity type will have four binary accessibility relations. We are only interested in the *tense* frames, however:

Definition 5.2.2.
An *HS-frame* is a triple $\mathfrak{F} = (I, B, E)$ with I a set of possible worlds called *intervals* and B and E binary relations on I. The relations B, B^{-1}, E, E^{-1} are the *accessibility relations* of resp. $\diamondT$, $\diamondB$, $\diamondL$ and $\diamondR$.
An HS-frame is *two-dimensional* if $I = INT(\mathfrak{T})$ for some (linear!) irreflexive temporal order $\mathfrak{T}$, and B and E are the beginning- resp. end interval relations, i.e.

$$\begin{array}{rcl}
[s,t]B[u,v] & \Longleftrightarrow & s = u \text{ and } t < v \\
{[s,t]}E[u,v] & \Longleftrightarrow & t = v \text{ and } u < s.
\end{array}$$

In this case we call $\mathfrak{F} = (INT(\mathfrak{T}), B, E)$ the frame *based on* $\mathfrak{T}$, notation: $\mathfrak{F} = \mathfrak{I}(\mathfrak{T})$. Frequently however, we will identify $\mathfrak{F}$ with $\mathfrak{T}$ and write $\mathfrak{F} = (T, <)$ etc. ⊞

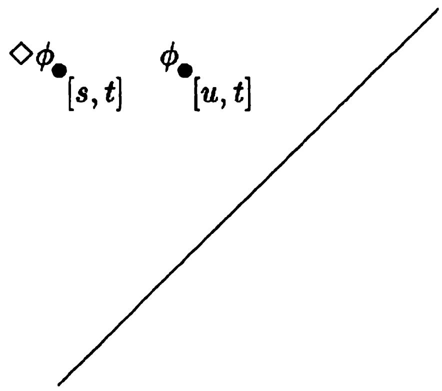
Note that we could also have defined a two-dimensional *non-linear* semantics for HS. The problem is that for a non-linear temporal ordering $(T, <)$, the set $T \times T$ does not allow such nice pictures as for linear orderings; one has to resort to Riemann-like surfaces.
The operators now have a two-fold intuitive meaning. In terms of intervals, we get for example

$$\begin{array}{rcl}
i \models \diamondB\,\phi & \Longleftrightarrow & \text{there is an interval } j \text{ starting } i \text{ with } j \models \phi \\
i \models \diamondL\,\phi & \Longleftrightarrow & i \text{ ends an interval } j \text{ with } j \models \phi \\
i \models \boxB\perp & \Longleftrightarrow & i \text{ is a `point' interval} \\
i \models \obar\,\phi & \Longleftrightarrow & \phi \text{ holds in the starting `point' of } i.
\end{array}$$

In the two-dimensional pictures, we have a *compass-interpretation* like for the $\mathcal{CC}\lambda$-operators, cf. section 3.4. Note however that we do not have all pairs (s, t) at our disposal, only the 'directed' ones $[s, t]$ in the North-Western halfplane. For example, we get

$$[s,t] \models \Diamond \phi \iff \text{there is a } u \text{ with } s < u \leq t \text{ and } [u,t] \models \phi.$$

viz.



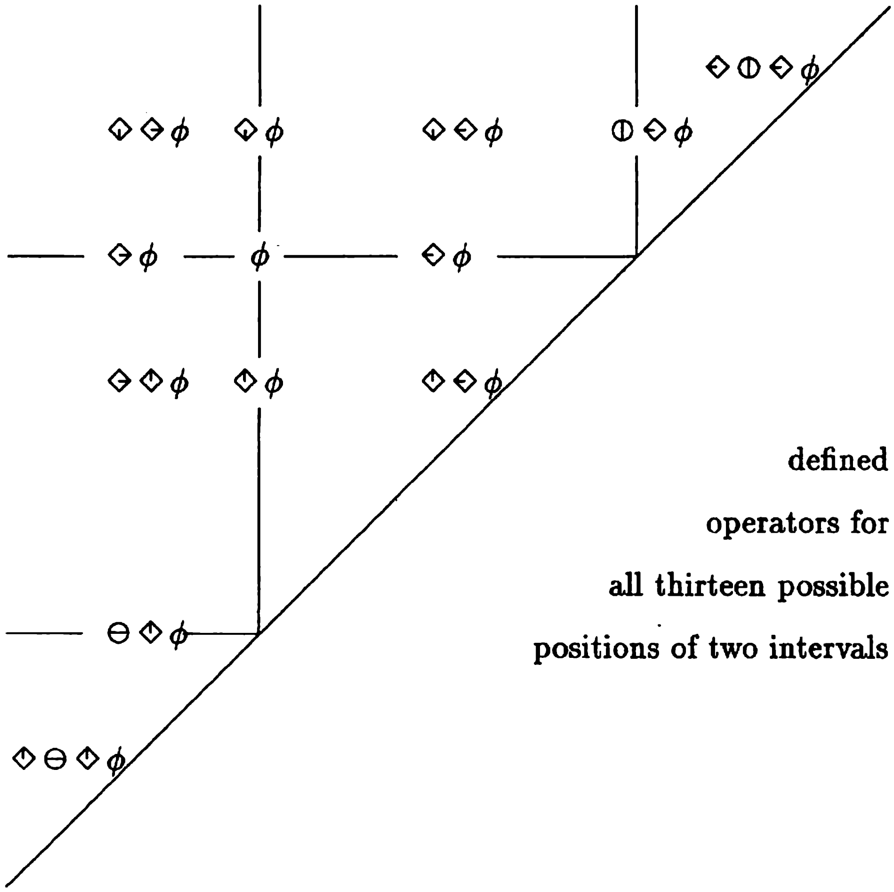
A nice property of the system is that for each of the thirteen possible interval relations, an operator can be defined in HS having this relation as its accessibility relation. For example:

$$[s,t] \models \oplus \ominus \phi \text{ iff there is an interval } [u,s] \textit{ meeting } [s,t] \text{ where } \phi \text{ holds.}$$

(This particular definition shows how we could get rid of the operators in the original system of Halpern and Shoham that referred to the 'meeting' relation.)
For the other positions, the appropriate definition can be found in the figure on the next page:

By using the formula $\Box \bot$ one also differentiate between stretched intervals and those of no duration.

Note that it is very easy in this formalism to model temporal phenomena studied in for instance linguistics, by imposing constraints on the semantics. For example, the principle of *homogeneity* (if a proposition holds at an interval then it holds at all subintervals) is expressed by the formula $\phi \rightarrow \Box \phi \wedge \boxminus \phi \wedge \Box \boxminus \phi$. Or, looking at the diagonal worlds as time points, we can treat the *progressive tense* as follows: 'John is sleeping' holds at $[t, t]$ iff $\Diamond \Diamond$ 'John sleeps' holds at $[t, t]$.



defined operators for all thirteen possible positions of two intervals

5.2.2. The system HS: expressiveness.

For both kinds of semantics for HS, HS-frames and two-dimensional frames, we can look at the expressive power of HS. In the next section we concentrate on the interval-based HS-frames, here we will look at the capacity of HS to distinguish classes of linear orders. First we quote Halpern and Shoham [50]; HS gives us very simple characterizations of some natural classes of two-dimensional frames. Some examples are:

Definition 5.2.3.
Consider the following HS-formulas:

(FP)	$\boxminus \bot \vee \Diamond \boxminus \bot$
(NLP)	$\Diamond \top$
$(length1)$	$\Diamond \top \wedge \boxdot \boxdot \bot$
(DE)	$\neg length1$
(DI)	$\boxdot \bot \vee length1 \vee (\Diamond\, length1 \wedge \Diamond\, length1)$

Proposition 5.2.4.
Let $\mathfrak{T} = (T, <)$ be a temporal order. Then

(i) $\mathfrak{T} \models FP \iff T$ has a first point.

(ii) $\mathfrak{T} \models NLP \iff <$ is unbounded to the right.

(iii) $\mathfrak{T} \models DE \iff$ < is dense.
(iv) $\mathfrak{T} \models DI \iff$ < discrete.

Proof.
Analogous to 3.4.2. ⊞

We now turn to a comparison between the expressive power of HS and ordinary (one-dimensional) point based temporal logics. By the latter we mean the ordinary tense logical systems having operators like F,P, S and U, where the semantics of the operators have a first order definition. By a standard correspondence-theoretic argument, the upper bound in expressiveness for these point based formalisms is the universal monadic second order logic of linear orderings.

Definition 5.2.5.
Recall that $L^{1<}(x_0)$ is the set of first order formulas having one free variable x_0, in a language with arbitrary many monadic predicates and one binary <. Let $UM^<$ be the set of second order formulas of the form $\forall P_0 \ldots \forall P_{n-1} \forall x_0 \phi$, with $\phi \in L^{1<}(x_0)$ and $P_0, \ldots, P_{n-1}$ the monadic predicates occurring in ϕ. ⊞

The following lemma expresses that HS is at least as expressive as $UM^<$:

Lemma 5.2.6.
Any two linear frames with the same HS-theory are $UM^<$-equivalent.

Proof.
By a result of J. Stavi (cf. [35] for a long awaited proof in print), we know that the set of temporal connectives $S'U' = \{S, U, S', U'\}$ is expressively complete with respect to the class of linear orders, where these operators are defined as follows (read $\mathfrak{T}, t \models p$ for Pt):

$$\mathfrak{T}, t \models U(p,q) \iff \exists y > t(Py \wedge \forall u(t < u < y \rightarrow Qu))$$

$$\begin{array}{lll} \mathfrak{T}, t \models U'(p,q) \iff & & \\ & (a) & \exists v > t\, \forall u(t < u < v \rightarrow Qu) \\ \wedge & (b) & \forall v > t\, (\forall u(t < u < v \rightarrow Qu) \rightarrow \\ & & (Qv \wedge \exists w > v \forall u(v < u < w \rightarrow Qu))) \\ \wedge & (c) & \exists y > t(\neg Qy \wedge Py \wedge \\ & & \forall v((t < v < y \wedge \exists u(t < u < v \wedge \neg Qu)) \rightarrow Pv)). \end{array}$$

S and S' are defined likewise, with respect to the past. Intuitively, $U'(p,q)$ holds at t if there is a point $v \in T$ and a gap g between t and v, such that (i) p holds everywhere between t and g, (ii) q holds everywhere between g and s, and (iii) $\neg p$ is true arbitrarily soon after the gap.

These operators can easily be 'circumscribed' in HS, for example $U'(p,q)$ by:

$$\begin{array}{lll} & & \Box \bot \\ \wedge & (a) & \Diamond \underline{POINT}(q) \\ \wedge & (b) & \Box(\underline{POINT}(q) \rightarrow \Diamond \underline{POINT}(q)) \\ \wedge & (c) & \Diamond(\ominus(\neg q \wedge p) \wedge \Box(\neg \underline{POINT}(q) \rightarrow \ominus p)). \end{array}$$

where $\underline{POINT}(q)$ is a formula meaning 'q holds at every point inside the interval'; take, for example, $\underline{POINT}(q) = \boxdot\boxminus(\boxdot\bot \rightarrow q)$.
Now suppose $\mathfrak{T}$ and $\mathfrak{T}'$ are two linear orders which are not $UM^<$-equivalent; by the expressive completeness of $S'U'$, this implies that $\mathfrak{T}$ and $\mathfrak{T}'$ are not $S'U'$-equivalent. But then there is also an HS-formula separating them. ⊞

Lemma 5.2.7.
The ordinals ω^ω and $\omega^\omega + \omega^\omega$ are $UM^<$-equivalent, but they do not have the same HS-theory.

Proof.
We use a result by Büchi and Siefkes, [21] p. 91: every ordinal α has a unique representation $\alpha = \omega^\omega \cdot \nu + \omega^{q-1} \cdot k_{q-1} + \cdots + \omega^0 \cdot k_0$, where k and q are finite ordinals. Call $\omega^\omega \cdot \nu$ the *ω-head* and $\omega^{q-1} \cdot k_{q-1} + \cdots \omega^0 \cdot k_0$ the *ω-tail* of α. The authors prove that two countable ordinals α and β are $UM^<$-equivalent iff either (i) $\alpha = \beta < \omega^\omega$ or (ii) $\omega^\omega \leq \alpha, \beta$ and α and β have the same ω-tail. So the ordinals $\omega^\omega$and $\omega^\omega + \omega^\omega$ are $UM^<$-equivalent.
It remains to be proved that there is an HS-formula valid in one of the frames and not in the other. Call an ordering *iso-choppable* if it can be decomposed into a head and a tail that are isomorphic. Then clearly $\omega^\omega + \omega^\omega$ is iso-choppable, ω^ω is not.
Formally, a frame $\mathfrak{T}$ is iso-choppable iff the following holds: there is a bijection f from a head P of $\mathfrak{T}$ to the corresponding tail Q $(= T - P)$ that is order-preserving, i.e. $s > t$ implies $f(s) > f(t)$.
Now consider the following HS-formula ϕ:

	$\boxplus(p \rightarrow \boxdot\bot) \wedge \boxplus(q \rightarrow \boxdot\bot)$	(T is the disjunct union of the point-sets $P \subseteq V(p)$ and $Q \subseteq V(q)$)
$\wedge$	$\boxplus(\boxdot\bot \rightarrow (p \leftrightarrow \neg q))$	
$\wedge$	$\diamondplus(\boxdot\bot \wedge p) \wedge \diamondplus(\boxdot\bot \wedge q)$	$(P, Q \neq \emptyset)$
$\wedge$	$\boxplus(f \rightarrow (\obar p \wedge \ominus q))$	$(f \subseteq P \times Q)$
$\wedge$	$\boxplus(p \rightarrow \dot{\Diamond}(f \wedge \boxdot\neg f \wedge \dot{\Box}\neg f))$	(f is a function)
$\wedge$	$\boxplus(q \rightarrow \Diamond f)$	(f is surjective)
$\wedge$	$\boxplus(f \rightarrow \boxminus\neg f)$	(f is injective)
$\wedge$	$\boxplus(f \rightarrow \boxdot\boxminus\neg f)$	(f is order-preserving)

Then clearly $\mathfrak{T}$ is iso-choppable iff ϕ is satisfiable in $\mathfrak{T}$. So $\omega^\omega \models \neg\phi$, while $\omega^\omega + \omega^\omega \not\models \neg\phi$, whence these frames have a different HS-theory. ⊞

So indeed, for any first order definable point logic PL, on the class of linear frames HS-equivalence is a strictly finer sieve than PL-equivalence:

Theorem 5.2.8.
Let PL be an intensional point logic with first order definable operators. Then

(i) Any two HS-equivalent linear frames have the same PL-theory.
(ii) ω^ω and $\omega^\omega + \omega^\omega$ are PL-equivalent, but not HS-equivalent

Proof.
Immediate by the previous lemmas and the fact that any PL-formula has a $UM^<$-equivalent on the frame level. ⊞

5.2.3. The System HS: (In)Completeness.

Concerning the complexity of the validity problem for several classes of structures, Halpern and Shoham [50] prove, by constructing HS-formulas encoding the computation of a Turing machine, the first two of the following facts; the third one is then immediate.

Fact 5.2.8.

(i) The validity problem for any class of temporal structures containing a frame having an infinitely ascending sequence of time points is r.e.-hard.

(ii) If every frame in such a class is Dedekind-complete, the validity problem is Π^1_1-hard.

(iii) The HS-theories of the: natural numbers, the integers and the reals, are not recursively axiomatizable.

More positive is the following news:

Proposition 5.2.9.
For the following (classes of) frames, the HS-theory is recursively enumerable: the linear orders, the dense and the discrete linear orders and the ordering of the rationals.

Proof (van Benthem).
In the same way as for $CC\delta$ or $CC\lambda$, we can show that on the frame level, every HS-formula has an equivalent of the form $\forall P_0 \ldots P_{n-1} \forall x_0 \forall x_1 \phi$, where ϕ has two free variables x_0 and x_1, and all its predicate symbols are dyadic and in $\{<, P_0, \ldots, P_{n-1}\}$. The proposition is then immediate by the fact that the universal second order theory of the mentioned (classes of) frames is recursively enumerable. ⊞

The problem is of course to give *explicit* derivation systems. We will apply the technique developed in the second chapter, first looking for a proper characterization of the two-dimensional frames among the HS-frames.

Definition 5.2.10.
Consider the following L_{HS}-formulas (R ranges over B, E):

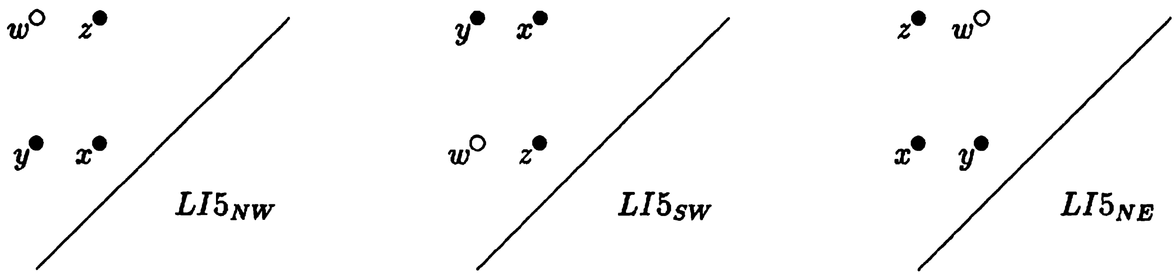
($LI1_R$) $\forall xyz(xRy \wedge yRz \rightarrow xRz)$
($LI2a_R$) $\forall xyz(xRy \wedge xRz \rightarrow yRz \vee y = z \vee zRy)$
($LI2b_R$) $\forall xyz(yRx \wedge zRx \rightarrow zRy \vee y = z \vee yRz)$
($LI3_R$) $\forall x(\neg\exists z\, zRx \vee \exists y(yRx \wedge \neg\exists z\, zRy))$
($LI4_R$) $\forall x(\neg\exists z\, zBx \leftrightarrow \neg\exists z\, zEx)$
($LI5_{NW}$) $\forall xyz(xBy \wedge xEz \rightarrow \exists w(yEw \wedge zBw))$
($LI5_{SW}$) $\forall xyz(xBy \wedge zEx \rightarrow \exists w(wEy \wedge zBw))$
($LI5_{NE}$) $\forall xyz(yBx \wedge xEz \rightarrow \exists w(yEw \wedge wBz))$
($LI6$) $\forall x \neg\exists y(xBy \wedge xEy)$.

$LI1 \ldots LI6$ denote the obvious conjunctions (e.g. $LI1 = LI1_B \wedge LI1_R$). ⊞

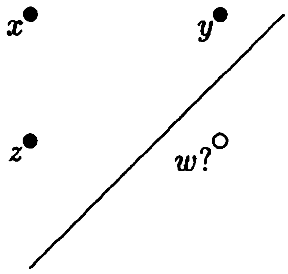
The intuitive meaning of these formulas is the following:

$LI1_R$: R is transitive.
$LI2a_R$: R is not branching to the right.
$LI2b_R$: R is not branching to the left.
$LI3_R$: if an interval is stretched, then there is a 'point'-interval R-below it.
$LI4_R$: 'having no proper beginning' means 'having no proper end'
$LI6$: B and E are disjoint.

The $LI5$-axioms express some kind of compass-related Church-Rosser property, best made clear in the following pictures:



Note that LI_{SE} (with the obvious definition) does not hold everywhere in two-dimensional frames, viz.



Theorem 5.2.11.
Let $\mathfrak{F}$ be an HS-frame. Then

$$\mathfrak{F} \text{ is (isomorphic to) a linear two-dimensional frame} \iff \mathfrak{F} \models LI1 \ldots LI6.$$

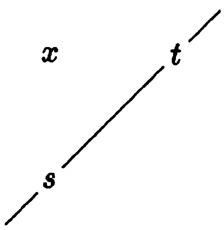
Proof.
We only prove the direction from right to left.
Let $\mathfrak{F} = (I, B, E)$ be an HS-frame where $LI1 \ldots L16$ are valid. Define T as the set of point-intervals in I:

$$T = \{x \in I \mid \neg\exists z\; zBx\},$$

and set a relation $<$ on T by

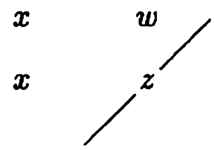
$$s < t \iff \exists x\,(sBx \wedge tEx),$$

viz.



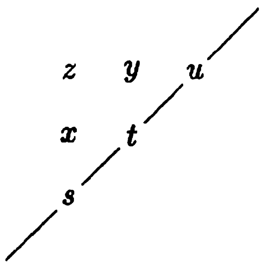
We will show that $\mathfrak{F} \simeq \mathfrak{I}(T,<)$.
First we need

Claim 1: B and E are irreflexive.
Proof. Suppose that xBx. Let z be the point-interval with zEx. By $LI5_{NE}$, there is a w with zBw and wEx, viz.

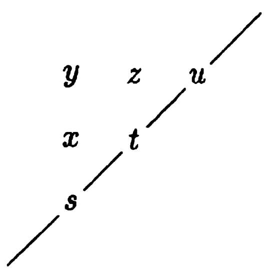


By wEx and the linearity axioms we have either wEz or $w = z$ or zEw. The first alternative is impossible as z is a point-interval, the third by the disjointness of B and E. But $w = z$ is also impossible, as zBw would imply zBz, again contradicting the fact that $z \in T$.

Claim 2: $(T,<)$ is a linear order.
Proof. For transitivity, suppose $s < t < u$. Then there are x and y with sBx, tEx, tBy and uEy. By $LI5$ there is a z with xBz, yEz, viz



By transitivity of B and E we have sBz, uEz. So $s < u$.
For irreflexivity, suppose $t < t$. Then there would be an x with tBx and tEx, contradicting $LI6$.
Finally we prove linearity: suppose $s < t$, $s < u$. By definition there are x, y with sBx, tEx, sBy and uEy. By $LI2$ either xBy or $x = y$ or yBx.
In the case $x = y$, $LI2$ gives that either tEu or $t = u$ or uEt. The cases where tEu or uEt are ruled out by definition of T. So $t = u$.
In the case xBy, by $LI5_{NE}$ there is a z with tBz and zEy, viz.



Now it is easy to prove that uEz. This gives $t < u$.
The case where yBx is similar, so we have proved that $<$ is linear to the right. Left-linearity is similar.

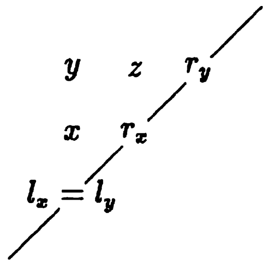
Claim 3: $x \notin T \to (\exists! l \in T : lBx \wedge \exists! r \in T : rEx)$.
Proof. Straightforward, using $LI3$, $LI2$ and $LI7$.

So for every stretched interval $x \notin T$ there is a unique *starting point* l_x and a unique *end point* r_x. Now we can define the isomorphism $g : I \mapsto INT(T, <)$:

$$g(x) = \begin{cases} [x, x] & \text{if } x \in T \\ [l_x, r_x] & \text{otherwise} \end{cases}$$

Claim 4: g is surjective.
Proof. Let $[s, t]$ be in $INT(T, <)$. If $s = t$, $[s, t] = g(s)$. Otherwise, $s < t$ gives an x with sBx and tEx, so $[s, t] = g(x)$.

Claim 5: g preserves B and E.
Proof. Suppose xBy. We only treat the case where $x \notin T$. Clearly $l_x = l_y$. By $LI5_{NE}$ there is a z with r_xBz, zEy, viz.



By r_xBz, $z \notin T$. So r_yEz. By definition of $<$ then, we have $r_x < r_y$.
So $g(x) = [l_x, r_x]B[l_x, r_y] = g(y)$.

Claim 6: g is injective.
Proof. Suppose $g(x) = g(y) = [s, t]$. If $s = t$, by irreflexivity of $<$ we obtain $x = s = t$ and $y = s = t$, so $x = y$. If $s < t$, then by $l_x = l_y$ we have either xBy or yBx or $x = y$. The first two possibilities are ruled out by the previous claim, as they would imply $r_x \neq r_y$. So $x = y$.

Claim 7: g anti-preserves B and E.
Proof. Suppose $g(x)Bg(y)$, then $l_x = l_y$ and $r_x < r_y$. By definition of $<$ there is a z with r_xBz, r_yEz. $\mathfrak{F}$ satisfies $LI5_{NW}$, so there is a y' with zEy' and xBy'. By $g(y') = g(y)$ we then find $y = y'$. So xBy. ⊞

Having our characterization of two-dimensional frames, we can now turn this result into an axiomatization of the set of HS-formulas valid in two-dimensional frames. First we discuss the axioms we need, and the formula to be used in a non-ξ rule.

Definition 5.2.12.
Consider the following HS-formulas ($\Diamond$ ranges over ⟐, ⟐):

$(MI1_\diamond)$	$\Diamond\Diamond \rightarrow \Diamond p$
$(MI2a_\diamond)$	$\Diamond^{-1}\Diamond p \rightarrow (\Diamond p \vee p \vee \Diamond^{-1}p)$
$(MI2b_\diamond)$	$\Diamond\Diamond^{-1}p \rightarrow (\Diamond^{-1}p \vee p \vee \Diamond p)$
$(MI3_\diamond)$	$\Diamond^{-1}\top \rightarrow \Diamond^{-1}\Box^{-1}\bot)$
$(MI4)$	$\Box_S \bot \leftrightarrow \Box_E \bot$
$(MI5_{NW})$	$\Diamond_W \Diamond_N p \rightarrow \Diamond_N \Diamond_W p$
$(MI5_{NE})$	$\Diamond_E \Diamond_N p \rightarrow \Diamond_N \Diamond_E p$
$(MI5_{SE})$	$\Diamond_W \Diamond_S p \rightarrow \Diamond_S \Diamond_W p$
(κ)	$\Box_S p \rightarrow \Diamond_W p$

For $MI1 \ldots MI5$ again we have the appropriate definitions. ⊞

These formulas are the ('negative') correspondents of the formulas characterizing the two-dimensional frames:

Proposition 5.2.13.
Let $\mathfrak{F}$ be an HS-frame. Then
(i) For $l = 1, \ldots, 5$: $\mathfrak{F} \models MIl \iff \mathfrak{F} \models LIl$.
(ii) $\mathfrak{F}$ in $\mathsf{Fr}_{-\kappa} \iff \mathfrak{F} \models LI6$.

Proof.
By the Sahlqvist form of the formulas, cf. 2.2.2. ⊞

We are now ready to define our axioms system. Just like for most of the completeness proofs given in this dissertation, we could start by extending the similarity type HS with the D-operator, then give the axioms and rules and finally reduce the system again. As we consider this procedure to be standard by now, we immediately turn to the simpler system. First of all, we need a *defined* D-operator:

Definition 5.2.14.
Abbreviate

$$D'\phi = \Diamond_W' \Diamond_N \phi \vee \Diamond_N' \Diamond_W \phi$$ ⊞

Clearly, in two-dimensional frames, D' plays the rôle of the difference operator.

Definition 5.2.15.
Let AHS be the minimal tense HS-logic K^t_{HS} extended with the axioms $MIL1, \ldots, MIL5$. AHS^+ is the axiom system AHS extended with the D'-irreflexivity rule. ⊞

Theorem 5.2.16. SOUNDNESS AND COMPLETENESS.
AHS^+ is strongly sound adn complete with respect to the class of linear irreflexive frames.

Proof.
Soundness is straightforward.
For completeness, we should give a proof that the D'-versions of the D-axioms (cf. definition 2.4.2) are theorems of AHS^+ and that the non-κ rule is a *derived* rule of AHS^+. The theorem is then immediate by the $SN\Xi$-theorem 2.8.2.
The part concerning the axioms we omit; we concentrate on showing the conservativity of $N\kappa R$ over AHS^+:

Assume $AHS^+ \vdash \neg(\boxdot p \rightarrow \Diamond p) \rightarrow \phi$. We have to show $AHS^+ \vdash \phi$. Well:

(1)	$\vdash \neg(\boxdot p \rightarrow \Diamond p) \rightarrow \phi$	(assumption)
(2)	$\vdash \neg\phi \rightarrow (\boxdot p \rightarrow \Diamond p)$	(1, proplog)
(3)	$\vdash \neg\phi \rightarrow (\boxdot \diamondsuit q \rightarrow \Diamond \diamondsuit q)$	(2, SUB, take a $q \not\in \phi$)
(4)	$\vdash q \rightarrow \boxdot \diamondsuit q$	(axiom CV)
(5)	$\vdash \neg\phi \rightarrow (q \rightarrow \Diamond \diamondsuit q)$	(3, 4, modlog)
(6)	$\vdash \neg\phi \rightarrow (q \rightarrow D'q)$	(5, definition D')
(7)	$\vdash (q \wedge \neg D'q) \rightarrow \phi$	(6, proplog)
(8)	$\vdash \phi$	(7, D'-irreflexivity).

⊞

It is now a simple matter to obtain completeness results for the classes of linear frames mentioned in proposition 5.2.4, simply by adding the suitable formulas as axioms to the logic AHS^+, in the same manner as we proved completeness for *dense* and *discrete* $CC\lambda$-logic, cf. section 3.4.

5.3 Intervals as Computation Paths.

In this section we concentrate on the ordering $(\omega, <)$, for two reasons: first of all because we can see the natural numbers as a sequence of computation states. Temporal logics over this frame can thus be used in the theory of verification of program correctness. There is quite an extensive literature on this subject, we refer to Stirling [125] for an overview. The second reason to pick out this ordering is that we have an interesting example here where we can *eliminate* a non-ξ rule from an axiomatization. The idea to look for an *orthodox* axiomatization here sprang from reading Maddux [78], where the related case of relation algebras is treated.

To start with the semantics, ordinary modal logics of computation are one-dimensional, the truth of a formula being evaluated at a single state. It is obvious however, that computation *paths* should also play an important rôle as objects of reasoning. For this reason, formalisms like *process logics* (cf. Harel-Kozen-Parikh [51]) have been developed. The connection with intervals lies in the fact that an interval $[s, t]$ over $(\omega, <)$ can be identified with the set of states $\{u \in T \mid s \leq u \leq t\}$ and thus represents a finite[1] computation path.
For reasons of complexity, it is undesirable to allow arbitrary valuations — we have already seen in 5.2.9, that the validity problem for HS-formulas in $(\omega, <)$ must be Π^1_1-hard. Now the *nature* of a *state* is that it is a complete gathering of 'all information obtained up till

[1]To model *infinite paths* as well, one could have a look at the ordinal $\omega + 1$. This is a matter for further research. The same applies to possible extensions to *branching time* models.

now', so a restriction to one-dimensional *valuations* is quite natural:

Definition 5.3.1.
A model $\mathfrak{T}, V$ is *flat* if V is a *flat* valuation, i.e. if V only depends on the first coördinate: $V(p_i) = INT(\mathfrak{T}) \cap (S_i \times T)$ for some $S_i \subseteq T$. A formula ϕ is *flatly valid* in a class K, notation: $\mathsf{K} \models_\flat \phi$, if ϕ is valid in all flat models based on frames in K. ⊞

Our aim in the remainder of this section will be to axiomatize flat validity over $(\omega, <)$. In a trivial way, flat validity can be reduced to ordinary validity:

Definition 5.3.2.
For an HS-formula ϕ, we set

$$\phi^\flat = \bigwedge_{p \in \phi} \boxplus(p \leftrightarrow \boxbar\, p) \rightarrow \phi.$$

Proposition 5.3.3.
Let $\mathfrak{T}$ be a frame, then $\mathfrak{T} \models_\flat \phi \iff \mathfrak{T} \models \phi^\flat$. ⊞

In section 5.2.2 we saw that HS is in some sense expressively complete. We need this fact in a crucial step of our completeness proof: we will give a 'completeness by completeness'-proof, inspired by Gabbay-Hodkinson [36].

Proposition 5.3.4.
For every first order formula in $L^{1<}(x_0)$, there is an HS-formula ϕ° such that for every flat model $\mathfrak{M} = (\mathfrak{T}, V)$ and $t \in T$:

$$\mathfrak{M} \models \phi[t] \iff \mathfrak{M}, [t,t] \models \phi^\circ.$$

Proof.
By the procedure described in the proof of lemma 5.2.6. ⊞

We now turn to axiomatics. From here until theorem 5.3.11, our results are *not* dependent on the assumption of flatness. First we consider the well-ordered frames:

Definition 5.3.5.
Abbreviate

$$\begin{aligned} SW(\phi) &= \phi \wedge \boxdot \neg\phi \wedge \boxminus \boxbar \neg\phi \\ W &= \diamondplus\, p \rightarrow \diamondplus\, SW(p). \end{aligned}$$

WO denotes the class of well-orderings. ⊞

Intuitively, $SW(\phi)$ holds at an interval w iff w is the first ϕ-world in the lexicographic ordering of the intervals, or, spatially, iff w is the most southern of the most western worlds where ϕ holds.

Proposition 5.3.6.
Let $\mathfrak{T} = (T, <)$ be a linear ordering, then

(i) $\mathfrak{T}$ in WO $\iff$ $\mathfrak{T} \models W$

(ii) $\mathfrak{T} \simeq (\omega, <)$ $\iff$ $\mathfrak{T} \models W \wedge DI$.

Proof.
(i) For ($\Rightarrow$), assume that $\mathfrak{T}$ is a well-ordering and $\mathfrak{T}, [s,t] \models \diamondplus p$. Let P_0 be the set $\{s \in T \mid [s,t] \models p, \text{ for some } t\}$.
As $P_0 \neq \emptyset$, it has a smallest element s_0. Let P_{s_01} be the set $\{t \in T \mid [s_0, t] \models p\}$. As $P_{s_01} \neq \emptyset$, it has a smallest element t_0. It is straightforward to verify that $[s_0, t_0] \models p \wedge \boxdot \neg \wedge \boxminus \boxbar \neg p$.
For ($\Leftarrow$), let P be a non-empty subset of T. By considering a valuation V with $V(p) = \{[s,t] \mid s \in P\}$, it is an easy exercise to verify that there is a southernmost westernmost interval $[s,s]$ where p is true, and that this implies that s is the smallest element of P.
(ii) is easy by (i), as $(\omega, <)$ is the only discrete well-ordering. ⊞

Definition 5.3.7.
AHSW is the axiom system *AHS* extended with the axiom *W*, *AHSN* has *DI* too. We abbreviate $\Omega \vdash \phi$ for $AHSN \vdash \phi^{\flat}$. ⊞

Note that *AHSW* and *AHSN* are *orthodox*: they do not use any non-ξ rule. Our aim is to prove comleteness of Ω for flat validity over the natural numbers. First we show:

Proposition 5.3.8.
$IR_{D'}$ is conservative over *AHSW* (and hence, over *AHSN*).

Proof.
Abbreviate $O'\phi = \phi \wedge \neg D'\phi$. Assume that $\vdash O'p \rightarrow \phi$; we want to prove $\vdash \phi$.
Let us first give the intuitive idea behind the proof: recall that the operator O is the 'only here' operator, i.e. $O\phi$ holds at a world w if w is the *only* world where ϕ holds. So let us read $IR_{D'}$ as: if we can derive ϕ under the assumption that there is a unique world where p holds, then we can derive ϕ. The essential observation of the proof is now that in the *axiom W*, the consequent $\diamondplus SW(p)$ *also* gives us a *unique* world where p holds, namely the southernmost westernmost p-world (cf. the proof of 5.3.6).
Now we can give the technical details. First, we state

$$(*) \qquad AHS \vdash SW(p) \rightarrow \neg D'SW(p).$$

One can prove $(*)$ by showing that $\vdash SW(p) \wedge D'SW(p) \rightarrow \bot$. This proof is quite tedious: it involves spelling out the definition of the D'-operator into a long disjunctive formula, and then showing that none of these disjuncts is consistent with $SW(p)$. This is a straightforward affair, so we do not go into details.
Now we proceed with the following derivation:

1.	$(p \wedge \neg D'p) \rightarrow \phi$	(assumption)
2.	$\neg\phi \rightarrow (p \rightarrow D'p)$	(1, proplog)
3.	$\neg\phi \rightarrow (SW(\neg\phi) \rightarrow D'SW(\neg\phi))$	(2, SUB)
4.	$\neg\phi \rightarrow \neg SW(\neg\phi)$	(3, $*$)
5.	$SW(\neg\phi) \rightarrow \neg\phi$	(def $SW(\neg\phi)$)
6.	$\neg SW(\neg\phi)$	(4,5, proplog)
7.	$\diamondplus \neg\phi \rightarrow \diamondplus SW(\neg\phi)$	(W)
8.	$\boxplus \phi$	(6,7, modlog)
9.	ϕ	(8, $\diamondplus$ is $S5$)

Unfortunately, proposition 5.3.8 does not yield a completeness theorem for the well-ordered frames — recall that by 5.2.8, $\Theta_{HS}(\mathsf{WO})$ cannot be recursively enumerable. There is a subtlety involved here.

Definition 5.3.9.
Let N be a set of first order formulas. A structure $\mathfrak{M} = (\mathfrak{T}, V)$ is *N-definably well-ordered* if for every $N(x)$-formula $\phi(x)$, the set $\phi^{\mathfrak{M}} = \{t \in T \mid \mathfrak{M} \models \phi[x \mapsto t]\}$ has a first element (if it is not empty). A two-dimensional HS-frame $(\mathfrak{T}, V)$ is *HS-definably well-ordered* if for every HS-formula ϕ, the set $\{t \in T \mid \mathfrak{M}, [t,t] \models \diamond \phi\}$ has a first element (if it is not empty). ⊞

The point is that $AHSW$ is *not* complete with respect to WO, but with respect to the HS-definably well-ordered models[2].

Lemma 5.3.10.
Let Δ be an $AHSW$-consistent set, then Δ is satisfiable in an HS-definably well-ordered model.

Proof.
Let Δ be $AHSW$-consistent, then by 5.3.8 Δ is $AHSW^{+}$-consistent too (where $AHSW^{+}$ is the extension of $AHSW$ with the D'-irreflexivity rule).
By the completeness result for HS, the set

$$\Delta' = \Delta \cup \{W(\phi) \mid \phi \text{ an } HS\text{-formula}\}$$

is satisfiable in a linear model $\mathfrak{M} = (\mathfrak{T}, V)$. It is an easy proof that by $\mathfrak{M} \models W$, $\mathfrak{M}$ is HS-definably well-ordered. ⊞

Fortunately, in the case of *flatness* (corresponding to *monadic* predicates in the first order language), we can apply a nice theorem of Doets [27]:

Theorem 5.3.11 (Doets).
Let $\mathfrak{M}$ be an $L^{1<}$-definably well-ordered structure. Then for every $n < \omega$ there is a well-ordered $L^{1<}$-structure $\mathfrak{M}'$ such that $\mathfrak{M} \equiv_n \mathfrak{M}'$ (i.e. for all $L^{1<}$-sentences of quantifier depth $\leq n$, $\mathfrak{M} \models \phi \iff \mathfrak{M}' \models \phi$).

Proof.
We refer to Doets [27], corollary 4.4. ⊞

Putting things together, we can prove our completeness theorem for Ω:

Theorem 5.3.12.
$\Omega \vdash \phi \iff (\omega, <) \models_{\flat} \phi$.

Proof.
Soundness is immediate.
For completeness, let ϕ be Ω-consistent, (i.e. $\phi^{\flat}$ is $AHSN$-consistent), so $\phi^{\flat} \wedge \boxplus DI$ is

[2]This subtlety is overlooked in [78]; the theorems 12 and 13 there are incorrect as stated.

$AHSW$-consistent. Abbreviate $\psi = \phi^\flat \wedge \boxplus DI$.
By 5.2.16, ψ is satisfiable in an HS-definably well-ordered model $\mathfrak{M}'$. By 5.3.4, $\mathfrak{M}'$ is $L^{1<}$-definably well-ordered.
Any HS-formula has an $L^{2<}(x_0, x_1)$-equivalent on the model level (dyadic predicates), so the *flat* formula ψ has an $L^{1<}(x_0, x_1)$-equivalent ψ° (monadic predicates). Let k be the quantifier depth of ψ°. By 5.3.11, there is a well-ordered model $\mathfrak{M} = (\mathfrak{T}, V)$ such that $\mathfrak{M}' \equiv_{k+2} \mathfrak{M}$.
As ψ is satisfiable in $\mathfrak{M}'$, $\mathfrak{M} \models \exists x_0 \exists x_1 \psi^\circ$ by definition of $\equiv_{k+2}$,
Clearly then, ψ is satisfiable in $\mathfrak{M}$, i.e. there are some $s \leq t$ in $\mathfrak{M}$ with $\mathfrak{M}, [s,t] \models \phi^\flat \wedge \boxplus DI$.
As $\mathfrak{T}$ is a well-ordering, $\mathfrak{M} \models \boxplus DI$ implies $\mathfrak{T} \simeq (\omega, <)$ by 5.3.6(ii), so we have found a model $\mathfrak{M}$ on $(\omega, <)$ where $\phi^\flat$ is satisfiable. By definition of $\phi^\flat$, $\mathfrak{M}$ is then a *flat* model for ϕ. ⊞

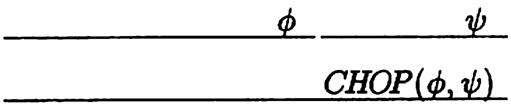
5.4 A modal operator for chopping intervals.

In the previous sections we saw that HS is quite an expressive formalism, being stronger than any point-based system and having a defined operator for each of the thirteen binary interval relations. However, there are limits to its expressiveness, witness the following example.

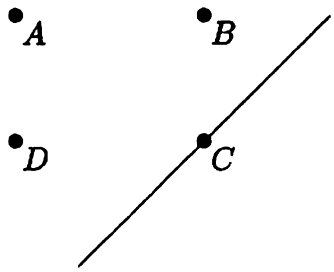
Definition 5.4.1.
Let *CHOP*(ϕ, ψ) be the dyadic operator having the following interpretation in two-dimensional models:

$$\mathfrak{M}, [s,t] \models \mathit{CHOP}(\phi, \psi) \iff \text{there is a u with } s \leq u \leq t \text{ such that } \mathfrak{M}, [s,u] \models \phi \text{ and } \mathfrak{M}, [u,t] \models \psi.$$ ⊞

So, informally, *CHOP*(ϕ, ψ) holds at an interval if it can be chopped into two pieces where ϕ resp. ψ hold, viz.



In the two-dimensional picture, *CHOP*(ϕ, ψ) holds in a point A if we can construct a rectangle $ABCD$ such that ψ holds in B which is equal to or lies east of A, C is situated on the diagonal and ϕ holds at D, equal to or south of A, viz.



So in some sense, *CHOP* can be seen as the interval version of the composition operator $\circ$ in $CC\lambda$. The *CHOP*-operator seems to be a very natural one; one may see it, just like Von Wright's *ANDNEXT*-operator (cf. van Benthem [12]) as a formalization of the temporal connective in sentences like "He came home *and* went to bed". It is also interesting from the point of view of computer science, viz. for the temporal logics used in proving *program correctness*. If we take the approach of the previous section and interpret formulas not in intervals but in computation *paths*, i.e. sequences of computation states, then $CHOP(\phi, \psi)$ holds in those sequences which are the *concatenation* of two sequences where ϕ resp. ψ hold. Some results on this approach can be found in Rosner-Pnueli [109] and in Moszkowski [87].

To show that the *CHOP*-operator is not definable in in the system HS, it is sufficient to prove

Proposition 5.4.2.
The formula $CHOP(p,p)$ has no equivalent in HS.

Proof.
Consider the frame $\mathfrak{R} = (R, <)$, the ordering of the real numbers. We will show that unlike HS-formulas, *CHOP* can 'change slopes', of which we give the following example: Take a valuation V with $V(p) = \{[x,y] \mid x+y \in Z\}$, cf. fig. 1. Then for all HS-valuations, $V(\phi)$ can be depicted in fig. 2: there all only lines with slopes -1, 0 and ∞.

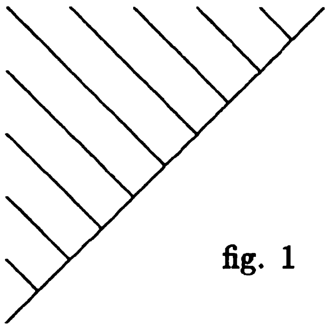


fig. 1

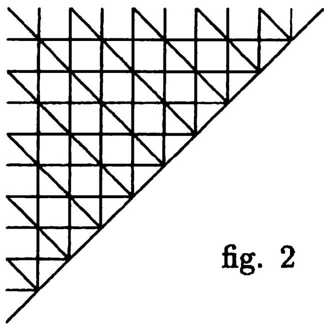


fig. 2

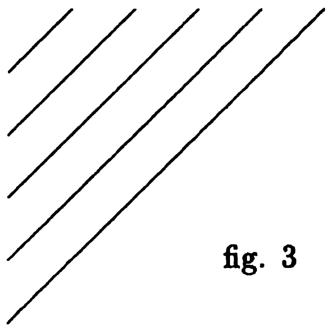
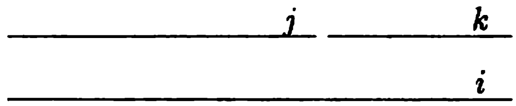


fig. 3

Yet, turning to HSC, one can immediately prove that $V(CHOP(p,p)) = \{[x,y] \mid x - y \in Z\}$, cf. fig 3. So these lines, having a slope of $+1$, differ from every line in fig. 2. This sketches the proof why no HS-formula can be equivalent to $CHOP(p,p)$. ⊞

To prove completeness for a logic having this *CHOP*-operator, there are two ways open to us: the first one is taken in Venema [133], where we considered the system CDT, an extension of HS having, besides *CHOP*, two operators D and T such that *CHOP*, D and T form a set of dyadic *versatile* operators. The system also has a *constant* π being true at the point-intervals. The ternary accessibility relation asociated with the binary operators is the 'and'-relation A already mentioned in the first section: $Aijk$ holds if i is the 'sum' of the adjacent intervals j and k, viz.

$Aijk$



So we could follow the by now familiar procedure of giving a proper characterization of two-dimensional CDT-frames first in the corresponding first order language L_{CDT}, and then turning this characterization into an $SN\Xi$-axiomatization.
We choose a different method here, inspired by Gabbay and Hodkinson [36] who give an axiomatization of irreflexive Since and Until-logic using an axiomatization of the F, P-fragment. This proof uses the notion of a *parametrical definition.*

Definition 5.4.3.
Let HSC be the similarity type HSC extended with the binary operator *CHOP*. ⊞

We do not need the notion of an HSC-model, we simply interpret HSC in (two-dimensional) HS-frames.

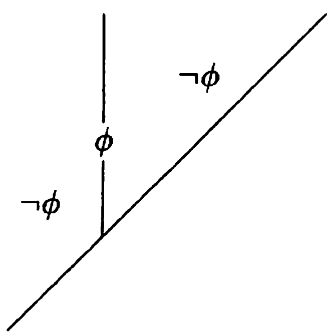
Definition 5.4.4.
Define the following formulas:

$$\begin{aligned} VER(\phi) &= \boxbar\,\phi \wedge \boxminus'\,\boxbar\,\neg\phi \\ Df_{CHOP}(\phi,\psi,\chi) &= (\psi \wedge \ominus\,\chi) \vee (\obar\,\psi \wedge \chi) \vee \diamondminus(\chi \wedge \obar(\diamondminus(\psi \wedge \phi))). \\ X(\phi,\psi,\chi) &= VER(\phi) \rightarrow (CHOP(\psi,\chi) \leftrightarrow Df_{CHOP}(\phi,\psi,\chi)). \end{aligned}$$

The formula ϕ is called the *parameter* of $X(\phi,\psi,\chi)$. ⊞

Proposition 5.4.5.
$X(\phi,\psi,\chi)$ is valid on the class of two-dimensional models.

Proof.
Let $[s,t]$ be an interval with $[s,t] \models VER(\phi)$. This means that the vertical line through $[s,t]$ contains *precisely* those points where ϕ holds, viz.



It is then straightforward to verify that

$$[s,t] \models CHOP(\psi,\chi) \rightarrow Df_{CHOP}(\phi,\psi,\chi).$$

For the other direction, if $[s,t] \models \psi \wedge \ominus\chi$ we have $[s,t] \models \psi$ and $[t,t] \models \chi$, so indeed $[s,t] \models CHOP(\phi,\psi)$.
The case where $[s,t] \models \oplus\psi \wedge \chi$ goes likewise.
If $[s,t] \models \diamond(\chi \wedge \oplus(\diamond(\psi \wedge \phi)))$, there is a $u > s$ with $[u,t] \models \chi$ and $[u,u] \models \diamond(\psi \wedge \phi)$. So there is a $s' < u$ with $[s',u] \models \psi \wedge \phi$. But $[s',u] \models \phi$ implies $s' = s$. This gives $[s,u] \models \psi$ and $[u,t] \models \chi$, so again we obtain $[s,t] \models CHOP(\phi,\psi)$. ⊞

Definition 5.4.6.
Let $AHSC$ be the axiom system AHS extended with the axiom $X(p,q,r)$.

Theorem 5.4.7.
$AHSC$ is a strongly sound and complete derivation system for the set of HSC-formulas that are valid in linear two-dimensional frames.

Proof.
Assume that Σ is a consistent set of HSC-formulas. We will define a two-dimensional model where Σ is satisfied.
We start with copying the first steps of the completeness proof for AHS. In constructing a witnessing extension of Σ (cf. section 2.8), we only take care of the diamonds, treating (outermost) subformulas of the form $CHOP(\phi,\psi)$ as variables in an extended language. In this way, we can show the existence of a pair $(\mathfrak{F},\Lambda)$ with $\mathfrak{F} = (I,B,E)$ an HS-frame and Λ a map assigning a maximal $AHSC$-consistent set of formulas to every $i \in I$ such that:

(i) $\mathfrak{F}$ satisfies $LI1 \ldots LI6$,
(ii) $\diamond\phi \in \Lambda(i) \iff$ there is a j with jBi and $\phi \in \Lambda(j)$, and likewise for the other diamonds.
(iii) For every $i \in I$ there is a propositional variable p_i such that i is the *only* interval with $p_i \in \Lambda(i)$.
(iv) There is an $i \in I$ with $\Sigma \subseteq \Lambda(i)$.

By (i) we may and will consider $\mathfrak{F}$ as a two-dimensional frame based on a linear ordering $\mathfrak{T} = (T,<)$.
Now we define the usual model $\mathfrak{M}$ on $\mathfrak{F}$, by setting

$$V(p) = \{i \in I \mid p \in \Lambda(i)\},$$

and we set out to prove the truth lemma

$$(TC) \qquad \text{For all } HSC\text{-formulas } \phi \text{ and intervals } i \quad \mathfrak{M}, i \models \phi \iff \phi \in \Lambda(i).$$

The claim is proved by an induction to the length of a maximal nesting of $CHOP$-operators in ϕ.
In the base step, we are dealing with $CHOP$-less formulas, for which the proof of TC is

standard.
For the induction step, we only treat the case where $\phi = CHOP(\psi,\chi)$:
Let i be an interval in $\mathfrak{F}$, we will prove TC for the formula $CHOP(\psi,\chi)$.
By (iii) and the definition of VER, we have $\mathfrak{M}, i \models VER(\Diamond p_i)$, so by the fact that $VER(\Diamond p_i)$ is a $CHOP$-less formula, $VER(\Diamond p_i) \in \Lambda(i)$.
As $X(p,q,r)$ is an axiom of $AHSC$, this gives

$$CHOP(\psi,\chi) \in \Lambda(i) \iff Df_{CHOP}(\Diamond p_i, \psi, \chi) \in \Lambda(i).$$

And by the fact that $\mathfrak{M}$ is two-dimensional, we have

$$\mathfrak{M}, i \models CHOP(\psi,\chi) \iff \mathfrak{M}, i \models Df_{CHOP}(\Diamond p_i, \psi, \chi).$$

It is now quite easy to prove the induction step

$$CHOP(\psi,\chi) \in \Lambda(i) \iff \mathfrak{M}, i \models CHOP(\psi,\chi),$$

as we can reduce this proposition to

$$Df_{CHOP}(\Diamond p_i, \psi, \chi) \in \Lambda(i) \iff \mathfrak{M}, i \models Df_{CHOP}(\Diamond p_i, \psi, \chi),$$

which holds by the induction hypothesis. ⊞

5.5 Conclusions, Remarks and Questions.

5.5.1 General conclusions.

In the introduction to this chapter we motivated the study of modal logics of time periods, and we showed how interval-based formalisms can be treated in the style of two-dimensional modal logics. The main part of the chapter concentrated on a system HS devised by Halpern and Shoham. We saw how this formalism is stronger than ordinary point-based logics in distinguishing ordinals, and we gave an $SN\Xi$-completeness proof for the formulas valid in proper point-based interval structures (section 2). For many applications it is natural to impose restrictions to the valuations of interval models. In the case of flat validity, we have seen that a nice consequence of such a restriction can be that validity is axiomatizable by an orthodox derivation system (section 3). In section 4 we treated an extension of HS with a $CHOP$-operator. For this system HSC we coould give a relatively simple axiomatization by extending the HS-axiom system with a parametrical axiom.
As a general conclusion we would say that treating interval logics in a two-dimensional framework is a fruitful approach, also from the perspective of applications.

5.5.2 Questions and Remarks.

(i) At several places in this chapter we have mentioned the close connection between interval logics and system like $CC\lambda$ discussed in section 3.4. For an explicit treatment of this relation we refer to Venema [131]

(ii) Just like for ordinary two-dimensional modal logics, there are many similarity types for interval logics. A very interesting example is Nishimura [91], who studies a formalism having *all* first order definable additive operators. This system can thus be seen as the interval-version of Jónsson's clone of Q-operators (cf. 3.6.2.(1)). It should be a straightforward matter to transform Nishimura's *completeness* theorem into an axiomatization of the representable Q-relation algebras.

(iii) The 'completenes-by-completeness' method, used in section 4 to obtain an *orthodox* axiomatization of the formulas that are flatly valid in $(\omega, <)$, was introduced by Gabbay and Hodkinson [36], who use it for one-dimensional systems *with* non-ξ rules, to axiomatize the ordering $\mathfrak{R}$ of the reals. In Venema [134] the method is used to axiomatize the S, U-logic over $(\omega, <)$ in an *orthodox* system; Reynolds [102] shows the same for the much harder case of the reals. It is interesting to find out how far this method reaches.

CHAPTER 6.

CONCLUSIONS.

Outline

Rather than giving a detailed summary of our results, we briefly discuss some of the main themes of this dissertation.

We have tried to give a general, uniform perspective on many-dimensional modal logics. The uniformity of our approach is mainly implicit, for example in terminology ('squares', 'cubes'), notation ($\Diamond, \otimes_{ij}$) or the style of our proofs (e.g. from characterization to axiomatization). We showed many existing algebraic and modal systems to fit neatly in this framework.
The style of this dissertation has been rather technical, and we have obtained several results on completeness, definability and expressiveness. We refer to the concluding sections of the various chapters for a summary of these technical contributions and questions for further research. Here, we would like to sketch some of the main threads running through this dissertation — it turned out that the topic has more dimensions than we first expected.

SAHLQVIST THEORY AND DUALITY.

A pleasant feature of the systems that we have studied is that *almost all* formulas involved turned out to be Sahlqvist formulas. This enabled us to apply the duality theory between Relational Kripke Frames and Boolean Algebras with Operators, in a very straightforward and eclectic manner: we could choose to work in frames, algebras, first order logic, whatever suited us best. As nice examples of this strategy we mention the easy simplification of Henkin's equation in cylindric algebras (cf. section 3.5) and the first order characterization of the cubic frames for cylindric modal logic (cf. section 4.3).

NEGATIVE DEFINABILITY AND RULES AS ANTI-AXIOMS.

For all but the most simple many-dimensional systems, we have seen that the class P of *proper* frames (squares, cubes, point-based interval structures), allows a nice characterization, of the following form:

> There is a set of positive, and a set of negative characteristics of worlds, both expressible by Sahlqvist formulas, Σ resp. Ξ.
> A frame is proper iff each world has all positive, and no negative characteristics.

For each of these classes we have found a strongly sound and complete derivation system, allowing the following presentation:

> Where Σ is the set of *axioms* of the system, Ξ is the set of *anti-axioms*, i.e. formulas which we strongly want to avoid as theorems. The formal implementation of this intuition consists of the system having, besides the orthodox derivation rules, a non-ξ rule for every $\xi \in \Xi$:

$$\text{if } \vdash \phi \rightarrow \xi(\vec{p}) \text{ and } \vec{p} \not\in \phi, \text{ then } \vdash \neg\phi.$$

All many-dimensional completeness proofs are applications of our general $SN\Xi$-theorem:

> For a sufficiently rich similarity type, a *characterization* in terms of a positive Σ and a negative Ξ, gives rise to an *axiomatization* with Σ as axioms and Ξ as anti-axioms.

MODAL LOGIC AND ALGEBRAIC LOGIC.

The system that we have studied in most detail (CML, $\mathcal{CCS}$) formed the modal counterpart of well-known algebras studied in algebraic logic (cylindric resp. relation algebras). For both formalisms, it was known that any orthodox axiomatization would have an infinite and quite complex set of axioms. We showed, that by adding one relatively transparent derivation rule to a simple, finite axiom system, a finite 'anti-axiomatization' can be obtained. As this kind of rule is deeply rooted in the possible world semantics of modal logic, we feel that a modal perspective on algebraic set representation theory, really can give new insights in algebraic logic.

SEMANTIC CONSTRAINTS AND APPLICATIONS.

Although applications of many-dimensional modal logics have received very little attention in this dissertation, we would like to stress the point that the main motivation for introducing two-dimensional temporal logics and interval logics did not come from logic proper but from linguistics and computer science. In this light, it is both convenient and interesting that by imposing restrictions on the semantics of two-dimensional modal logics of time, restrictions that are inspired by considerations of applications, the resulting system has very nice logical properties too. As examples we mention the expressive completeness result (3.4.9) for $\mathcal{CC}\lambda$, and the orthodox axiomatization of flat validity over the natural numbers (5.3.12).

APPENDIX A.

MODAL SIMILARITY TYPES.

Outline

This appendix is a summary of the background knowledge presupposed for reading this dissertation.

A1. Introduction.

This dissertation treats many different modal-like formalisms, as well as their companions in the theory of Boolean Algebra with Operators. Instead of introducing notions like zigzagmorphism or embedding algebras for every formalism separately, we felt it might be useful for the reader to have a systematic overview. For the abstract notion ranging over all formalisms we use the term *modal similarity type*.
This appendix intends to give a listing of all the notions and facts that we assume as background knowledge for reading this dissertation.
We aimed at a systematic, uniform presentation of the *concepts* involved, not at a complete covering of even the major results in the field.
The material was more or less obtained by amalgamating results from or listed in the following literature, to which we refer for more details, background information, etc.

- J.F.A.K. van Benthem, *Modal Logic and Clasical Logic* [14],
  for correspondence theory and general background in the theory of modal logic.
- S. Burris and H.P. Sankappanavar, *A Course in Universal Algebra* [24],
  for universal algebra.
- R. Goldblatt, *Varieties of Complex Algebras* [43],
  for duality theory and modal model theory.
- L. Henkin, J.D. Monk and Tarski, *Cylindric Algebras* [53],
  the standard reference to algebraic logic.
- I. Németi, *Algebraizations of Quantifier Logics: an Introductory Overview* [89],
  for indeed, an introductory overview to algebraic logic.

A2. Similarity types.

Definition A1: Similarity types.
A *modal similarity type* is a pair $S = (O, \rho)$ with O a set of *modal operators*, and $\rho : O \mapsto \omega$ a map assigning to each operator of O a finite *rank* or *adity*. Modal operators of rank 0 are called *constants*, monadic operators: *diamonds*, and dyadic ones: *triangles*. ⊞

We usually assume the rank of operators known and make no distinction between S and O. As variables ranging over operators we use $\nabla, \nabla_1, \ldots$. If the operators are zero-adic or constants, we use $\delta, \lambda, \pi, \sigma, \ldots$, for monadic symbols we use $\Diamond, \Diamond_1, F, P, D, \ldots$, and for dyadics we take $\vartriangle, \vartriangle_1, \circ, \ldots$.
In the following, we assume familiarity with the Boolean connectives and constants; as basics we take $\neg$ and $\vee$.

Definition A2. Modal languages.
A *modal language* is a pair $M = (S, Q)$, where S is a similarity type and Q is a set of

propositional variables. When no confusion arises we write $M(S)$, $M(Q)$ or M. The set $\Phi(M)$ of *formulas in* M is inductively defined as follows:

(0) The modal and boolean constants and the propositional variables are the *atomic* formulas in M.
(1) If ϕ and ψ are formulas in M, then so are $\neg\phi$ and $\phi \vee \psi$.
(2) If $\phi_1, \ldots, \phi_n$ are formulas in M and ∇ is a modal operator of rank n, then $\nabla(\phi_1, \ldots, \phi_n)$ is a formula in M.

We assume familiarity with the notion of a *formula algebra*; the formula algebra of the language M is denoted by $\mathfrak{Fm}_M$. If the variable p does not occur in ϕ, we write $p \notin \phi$. A formula is *closed* if no variables occur in it, only constants.
For an operator ∇, we abbreviate

$$\underline{\nabla}(\phi_1, \ldots, \phi_n) = \neg\nabla(\neg\phi_1, \ldots, \neg\phi_n)$$

and call $\underline{\nabla}$ the *dual* of ∇. Duals of diamonds are called *boxes*: $\Box\phi = \neg\Diamond\neg\phi$.
To increase readability, we will suppress brackets. We list the operators by decreasing priority: (i) monadic operators ($\neg$, $\Diamond$,$\Box$), (ii) polyadic modal operators, (iii) $\{\wedge, \vee\}$, (iv) $\{\rightarrow, \leftrightarrow\}$.

Definition A3: Classical Languages.
Let $M = (S, Q)$ be a modal language, with $S = \{\nabla_i \mid i < \xi\}$, $Q = \{p_j \mid j < \zeta\}$. The *correspondence map* ℓ assigns an *accessibility* relation symbol $\ell(\nabla_i)$ of adity $\rho(\nabla_i) + 1$ to each operator ∇_i of S and a monadic relation symbol P_j to each propositional variable p_j in Q.
The *corresponding (classical) frame language* L_S has as its predicate symbols the set $\{\ell(\nabla) \mid \nabla \in O\}$. The *corresponding (classical) model language* L_M is L_S extended with all monadic symbols P_j, $j < \zeta$. ⊞

Unless otherwise stated, all definitions in this appendix are understood with respect to a fixed modal similarity type S, c.q. a fixed modal language $M = (S, Q)$.

A3. Frames, models and correspondence.

Definition A4: Frames.
An S*-frame* is a pair $\mathfrak{F} = (W, I)$, which is a structure for L_S in the sense of ordinary first order model theory, i.e. W is a set called the *universe* and I is presented as an interpretation function associating an $n+1$-ary *accessibility relation* with each S-operator of rank n. Elements of W are called *possible worlds.* If $S = \{\nabla_i \mid i < \xi\}$ we may present a frame as $\mathfrak{F} = (W, R_i)_{i<\xi}$ or $\mathfrak{F} = (W, R_\nabla)_{\nabla \in S}$. ⊞

Definition A5: Models
An M*-model* is a pair $\mathfrak{M} = (W, I')$, which is a structure for L_M in the sense of ordinary first order model theory. We usually present a model $\mathfrak{M}$ as a pair $\mathfrak{M} = (\mathfrak{F}, V)$ with $\mathfrak{F} = (W, I)$ an S-frame and V a *valuation*, i.e. a function mapping proposition letters

in Q to subsets of W. This presentation can be brought in accordance with the formal definition by setting $I' = I \cup V$.
V can be extended to a map assigning sets of possible worlds to *all* M-formulas, by the following inductive definition:

$$\begin{array}{lcl} V(\phi \vee \psi) & = & V(\phi) \cup V(\psi) \\ V(\neg\phi) & = & W - V(\phi) \\ V(\nabla(\phi_1, \ldots, \phi_n)) & = & \{w_0 \mid \text{ there are } w_1, \ldots, w_n \text{ in } W \text{ with } R_\nabla(w_0, \ldots, w_n) \\ & & \quad \text{and for all } 0 < i < n\text{: } w_i \in V(\phi_i)\}. \end{array}$$

Definition A6: Truth and validity.
Using the terminology of the previous definition, we can define the notion of *truth*: a formula ϕ is *true* at w in $\mathfrak{M}$, notation: $\mathfrak{M}, w \models \phi$, if $w \in V(\phi)$. The formula ϕ *is true in/holds in* $\mathfrak{M}$, notation: $\mathfrak{M} \models \phi$, if $\mathfrak{M}, w \models \phi$ for all w in $\mathfrak{M}$. ϕ *is valid in* a frame $\mathfrak{F}$ ($\mathfrak{F} \models \phi$) if $(\mathfrak{F}, V) \models \phi$ for all valuations V; ϕ is *valid* in a class K of frames if $\mathfrak{F} \models \phi$ for all $\mathfrak{F}$ in K.
For K a class of models or frames, let $\Theta_S(\mathsf{K})$ be the set of S-formulas holding in K. For Σ a set of formulas, let Fr_Σ be the class of frames in which Σ holds. For a formula ϕ, we write Fr_ϕ instead of $\mathsf{Fr}_{\{\phi\}}$.
A formula ϕ is a *semantic consequence*[1] of a set of formulas Σ *over* a class of frames K, notation: $\Sigma \models_{\mathsf{K}} \phi$ if for every model $\mathfrak{M}$ based on a frame in K, and every world w in $\mathfrak{M}$, $\mathfrak{M}, w \models \phi$ if $\mathfrak{M}, w \models \sigma$ for all $\sigma \in \Sigma$.
A set of formulas Σ *characterizes* a class of frames K if $\mathsf{K} = \mathsf{Fr}_\Sigma$. ⊞

Definition A7: Correspondents.
Let $M = (S, Q)$ be a modal language. By induction to the complexity of formulas in M we define, for every modal formula ϕ in M its classical *local model correspondent* $\phi^1(x_0)$ in L_M:

$$\begin{array}{rcl} (p_i)^1 & = & P_i x_0 \ \ (\text{where } P_i = \ell(p_i)) \\ (\neg\phi)^1 & = & \neg\phi^1 \\ (\phi \vee \psi)^1 & = & \psi^1 \vee \psi^1 \\ (\nabla(\phi_1, \ldots, \phi_n))^1 & = & \exists x_1 \ldots x_n (R_\nabla(x_0, x_1, \ldots, x_n) \wedge \bigwedge_{0<i\leq n} \phi^1(x_i/x_0)). \end{array}$$

The *(classical) local frame correspondent* is defined as the second order formula

$$\phi^2(x_0) \equiv \tilde{\forall} P_1 \ldots \tilde{\forall} P_m \phi^1(x_0),$$

where the second order quantification takes place over these predicates $P_i = \ell(p_i)$ with p_i occurring in ϕ.
The *global* correspondents are defined by a universal first order quantification over the appropriate local correspondent, so the *global model correspondent* is $\forall x_0 \phi^1(x_0)$ and the *global frame correspondent* is $\forall x_0 \phi^2(x_0)$. ⊞

Modal formulas and their classical correspondents are equivalent on the appropriate level:

[1]In Appendix B we discuss the 'global' alternative to this definition, and we give a motivation for choosing our 'local' paradigm.

Theorem A8. Correspondence.
For all models $\mathfrak{M}$, frames $\mathfrak{F}$ and worlds w in $\mathfrak{M}$ resp. $\mathfrak{F}$, and formulas ϕ:

(i)	$\mathfrak{M}, w \models \phi$	$\iff$	$\mathfrak{M} \models \phi^1[x_0 \mapsto w]$
(ii)	$\mathfrak{M} \models \phi$	$\iff$	$\mathfrak{M} \models \forall x_0 \phi^1$
(iii)	$\mathfrak{F}, w \models \phi$	$\iff$	$\mathfrak{F} \models \phi^2[x_0 \mapsto w]$
(iv)	$\mathfrak{F} \models \phi$	$\iff$	$\mathfrak{F} \models \forall x_0 \phi^2$.

Definition A9: Structural Operations on frames.
Let $\mathfrak{F} = (W, I)$ be an S-frame and ∇ an S-operator. A subset $W' \subseteq W$ is *S-hereditary* if for all $\nabla \in O$, $w \in W'$ and $(w, w_1, \ldots, w_n) \in I(\nabla)$ imply $w_i \in W'$, $1 \leq i \leq n$. Let $\mathfrak{F} = (W, I)$ and $\mathfrak{F}' = (W', I')$ be two S-frames, then $\mathfrak{F}'$ is a *generated subframe* of $\mathfrak{F}$ if W' is a ∇-hereditary subset of W, and $I'(\nabla)$ is $I(\nabla)$ restricted to W', for all ∇ in S.
Let $\mathfrak{F} = (W, I)$ and $\mathfrak{F}' = (W', I')$ be two S-frames, then $f : W \mapsto W'$ is a *homomorphism* if

$$(w_0, \ldots, w_n) \in I(\nabla) \Rightarrow (fw_0, \ldots, fw_n) \in I'(\nabla)$$

for all $w_0, \ldots, w_{n-1}$ and ∇. A map $f : W \mapsto W'$ is a *zigzagmorphism*[2] if f is a homomorphism which satisfies, for each operator ∇, the ∇-*zigzagcondition*

$$(ZZ_\nabla) \qquad \begin{array}{l} (fw_0, w'_1, \ldots, w'_n) \in I'(\nabla) \Rightarrow \text{ there are } w_1, \ldots, w_n \text{ such that} \\ (w, w_0, \ldots, w_n) \in I(\nabla) \text{ and } fw_i = w'_i \text{ for all } i. \end{array}$$

Let $\{\mathfrak{F}_i \mid i \in J\}$ be a family of pairwise disjoint S-frames, i.e. $W_i \cap W_j = 0$ if $i \neq j$. The *disjoint union* of the $\mathfrak{F}_i$'s is the frame $\Sigma_{i \in J} \mathfrak{F}_i = (W, I)$ given by $W = \bigcup_{i \in J} W_i$, $I(\nabla) = \bigcup_{i \in J} I_i(\nabla)$.
For a class K of frames, we define $\mathbf{S}_\mathfrak{f}\mathsf{K}$, $\mathbf{H}_\mathfrak{f}\mathsf{K}$ and $\mathbf{P}_\mathfrak{f}\mathsf{K}$ as the classes containing resp. the generated subframes, zigzagmorphic images and disjoint unions of frames in K.

A4. Boolean S-Algebras.

Definition A10: Boolean S-Algebras.
We assume familiarity with the notion of a *Boolean Algebra* (short: BA). As basic operations of a BA we take *addition* (+) and *complementation* (−). *Multiplication* (·) and the constants 0 and 1 can then be seen as derived operations.
Now let $\mathfrak{A} = (A, +, -)$ be a Boolean Algebra.
An n-ary operation $f : A^n \mapsto A$ is called *normal in the i-th coordinate* $(0 < i \leq n)$ if

$$f(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_n) = 0.$$

An n-ary operation $f : A \mapsto A$ is called *additive in the i-th coordinate* $(0 < i \leq n)$ if

$$f(a_1, \ldots, a_{i-1}, a_i + a'_i, a_{i+1}, \ldots, a_n) = \\ = f(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n) + f(a_1, \ldots, a_{i-1}, a'_i, a_{i+1}, \ldots, a_n).$$

Such an operator is *normal (additive)* if it is normal (additive) in all its coordinates.

[2]This notion is also known under the names *p-morphism* and *bounded morphism*.

Now let $S = (O, \rho)$ be a modal similarity type. A *Boolean Algebra with S-operators*, or *Boolean S-Algebra*, is an algebra $\mathfrak{A} = (A, +, -, f)$ where f is a map interpreting every $\nabla \in O$ as a normal, additive operation f_∇ of rank $\rho(\nabla)$ on the Boolean Algebra $(A, +, -)$. Such an algebra is also denoted as $\mathfrak{A} = (A, +, -, f_\nabla)_{\nabla \in S}$.
The class of Boolean S-Algebras is denoted by BAO_S. Boolean S-algebras are sometimes called the *modal* algebras of the similarity type. ⊞

Boolean S-algebras form an alternative semantics for modal languages:

Definition A11. Terms.
Let S be a similarity type, X a set of objects called *variables*. In an algebraic context, formulas of the language $M(S, X)$ may be called *S-terms in X*, or shortly: *terms*. An *S-equation* is a pair (s, t) of S-terms, usually denoted as $s = t$. In an algebraic context, we usually write $+, \cdot, -$ for $\vee, \wedge, \neg$. We abbreviate $s \leq t$ for $-s + t = 1$. ⊞

Definition A12. Algebraic Semantics.
An *assignment* of the variables X in an algebra $\mathfrak{A}$ is a map $h : X \mapsto A$. Such a map can be uniquely extended to a homomorphism $\mathfrak{Fm}_{M(S,X)} \mapsto \mathfrak{A}$ associating an element of A with *every* term. This extension is also denoted by h.
An equation $s = t$ is *valid* in an algebra $\mathfrak{A}$ if for *every* assignment $h : X \mapsto A$, $h(s) = h(t)$.
An equation η is *valid in a class* K of algebras if for all $\mathfrak{A}$ in K, η is valid in $\mathfrak{A}$.
All kinds of validity are denoted by $\models$.
The set of equations valid in K is denoted by $\mathit{Equ}(\mathsf{K})$, the class of algebras where a set of equations E holds, by (V_E). A class K of algebras is *equational* if $\mathsf{K} = \mathsf{V}_E$ for some set of equations E. ⊞

Definition A13.
Let η be the equation $r = s$. The *normal term* of η is defined as $r \cdot s + -r \cdot -s$. The *normal form* of η is set as $r \cdot s + -r \cdot -s = 1$. For a set Σ of equations we define Σ^{nf} as the set of normal forms of the equations in Σ. ⊞

Convention A14.
We will be quite sloppy about the difference between equations and their normal forms. For example, we will use $\mathit{Equ}^{nf}(\mathsf{K})$ as the set of all equations holding in K. This sloppiness is justified by the facts that in the context of Boolean Algebras, an equation η is equivalent to its normal form, the sets Σ and Σ^{nf} characterize the very same class of algebras, etc.

Definition A15. Structural algebraic operations.
Let $\mathfrak{A} = (A, +, -, f)$ and $\mathfrak{A}' = (A', +', -', f')$ be two Boolean S-algebras.
$\mathfrak{A}'$ is a *subalgebra* of $\mathfrak{A}$ if $A' \subseteq A$ and the operations $+', -'$ and f'_∇ of $\mathfrak{A}$ are precisely the $+, -$ and f_∇ of $\mathfrak{A}$, restricted to A'.
A *homomorphism* from $\mathfrak{A}$ to $\mathfrak{A}'$ is a map $\beta : A \mapsto A'$ such that

$$\begin{aligned} \beta(a+b) &= \beta(a) +' \beta(b) \\ \beta(-a) &= -'\beta_a) \\ \beta(f_\nabla(a_1, \ldots, a_n) &= f'_\nabla(\beta a_1, \ldots, \beta a_n). \end{aligned}$$

If the homomorphism $\beta : \mathfrak{A} \mapsto \mathfrak{A}'$ is surjective, we call $\mathfrak{A}'$ a *homomorphic image* of $\mathfrak{A}$.

Let $(\mathfrak{A}_i)_{i\in I}$ be an indexed family of Boolean S-Algebras. The *direct product* $\prod_{i\in I}\mathfrak{A}_i$ is the Boolean S-Algebra $\mathfrak{A} = (A, +, -, f)$ with universe $A = \prod_{i\in I} A_i$ such that

$$\begin{array}{rcl} (a+b)(i) & = & a(i)+b(i) \\ (-a)(i) & = & -(a(i)) \\ (f_\nabla(a_1,\ldots,a_n))(i) & = & (f_i)_\nabla(a_1(i),\ldots,a_n(i)). \end{array}$$

For a class of algebras K, we denote by $\mathbf{S}\mathsf{K}$, $\mathbf{H}\mathsf{K}$ and $\mathbf{P}\mathsf{K}$ the classes of respectively all subalgebras, homomorphic images and products of algebras in K.
$\mathsf{V}\mathsf{K}$ is the least class containing K which is closed under $\mathbf{S}$, $\mathbf{H}$ and $\mathbf{P}$.
A class K is a *variety* if $\mathsf{K} = \mathsf{V}\mathsf{K}$.

Theorem A16.
(i) $\mathsf{V}\mathsf{K} = \mathbf{HSP}\mathsf{K}$.
(ii) V is a variety iff V is an equational class.

A5. Frames and Algebras.

We now have two kinds of semantics for our modalities: relational Kripke structures and Boolean algebras with operators. A framework unifying these approaches is that of general frames, which can be seen as both Kripke frames and complex algebras:

Definition A17: General Frames and Complex Algebras.
Let $S = (O, \rho)$ be a similarity type, ∇ an n-adic operator in S, $\mathfrak{F} = (W, I)$ an S-frame. We define the n-ary operation m_∇ on the powerset $P(W)$ of W by

$$m_\nabla(X_1,\ldots,X_n) = \{w \mid \exists w_1 \ldots \exists w_n(\bigwedge_{0<i\leq n} w_i \in X_i \wedge R_\nabla(w, w_1,\ldots,w_n))\}$$

A *general S-frame* is a pair $\mathfrak{G} = (\mathfrak{F}, A)$ where $\mathfrak{F} = (W, I)$ is an S-frame and $A \subseteq P(W)$ is closed under Boolean operations and under the operations m_∇ for all ∇ in S.
Let $\mathfrak{G}$ be a general S-frame $(\mathfrak{F}, A)$. The *complex algebra* $\mathfrak{Cm}\mathfrak{G}$ of $\mathfrak{G}$ is given as $\mathfrak{A} = (A, \cup, ^c, m_\nabla)_{\nabla\in S}$. The *complex algebra* of a Kripke frame $\mathfrak{F} = (W, I)$ is the complex algebra of the general frame $\mathfrak{G} = (\mathfrak{F}, P(W))$.
For K a class of (general) frames, $\mathbf{Cm}\mathsf{K}$ denotes the class of all complex algebras of frames in K. ⊞

We now turn to a comparison of the set of formulas holding in a class of (general) frames with the set of equations valid in the corresponding class of complex algebras.

Definition A18: Translations.
Let $Q = \{q_i \mid i < \zeta\}$ and $X = \{x_i \mid i < \zeta\}$ be sets of propositional modal resp. algebraic variables. We assume the existence of a bijection identifying q_i with x_i. Thus we are allowed to identify the sets of modal formulas $M(S, Q)$ with the set of algebraic terms $M(S, X)$.
Let ϕ be a modal formula. Its *corresponding algebraic equation* ϕ^α is given as $\phi = 1$.

Let η be an algebraic equation. Seen as a modal formula, its normal term (cf. A13) is called the *corresponding modal formula* of η, notation: η^μ.
For sets Σ, E of formulas resp. equations, Σ^α and E^μ have their usual meaning. ⊞

Theorem A19.
Let $\mathfrak{F}$ be a frame, ϕ an S-formula, K a class of frames. Then

$$\begin{array}{ccc} \mathfrak{F} \models \phi & \iff & \mathfrak{Cm}\mathfrak{F} \models \phi = 1 \\ \Theta(\mathsf{K}) & = & (\mathit{Equ}(\mathbf{Cm}\mathsf{K}))^\mu. \end{array}$$

Definition A20. Atom structures.
We assume familiarity with the notions of *atoms* in Boolean Algebra and *atomic* BAs. Now let $\mathfrak{A} = (A, +, -, f_\nabla)_{\nabla \in S}$ be an atomic Boolean S-algebra. The set of atoms in $\mathfrak{A}$ is denoted by $At\mathfrak{A}$, the *atom structure* $\mathfrak{At}\mathfrak{A}$ is the S-frame $(At\mathfrak{A}, R_\nabla)_{\nabla \in S}$ where R_∇ is given by

$$R_\nabla(a_0, a_1, \ldots, a_n) \iff a_0 \leq f_\nabla(a_1, \ldots, a_n).$$

For a class K of algebras, we let $\mathbf{At}\mathsf{K}$ denote the class of atom structures of atomic algebras in K.

Theorem A21.
$\mathfrak{F} \simeq \mathfrak{At}\mathfrak{A} \iff \mathfrak{A} \simeq \mathfrak{Cm}\mathfrak{F}$.

Definition A22. Canonical structures and embedding algebras.
Let $\mathfrak{A}$ be a Boolean S-algebra.
A subset F of A is a *filter* of $\mathfrak{A}$ if (i) $1 \in F$, (ii) $a, b \in F \Rightarrow a \cdot b \in F$ and (iii) $a \in F$ & $b \geq a \Rightarrow b \in F$. An *ultrafilter* of $\mathfrak{A}$ is a filter U satisfying (iv) $a \notin U \Leftrightarrow -a \in U$.
The *canonical structure* of $\mathfrak{A}$ is the frame $\mathfrak{Cs}\mathfrak{A} = (W, R_\nabla)_{\nabla \in S}$ where W is the set of ultrafilters of $\mathfrak{A}$ and R_∇ is given by

$$R_\nabla(U_0, \ldots, U_n) \iff f_\nabla(a_1, \ldots, a_n) \in U_0 \text{ for all } a_1 \in U_1, \ldots, a_n \in U_n.$$

The *embedding algebra* $\mathfrak{Em}\mathfrak{A}$ of $\mathfrak{A}$ is the complex algebra of canonical extension of $\mathfrak{A}$: $\mathfrak{Em}\mathfrak{A} = \mathfrak{Cm}\mathfrak{Cs}\mathfrak{A}$.

A6. Modal Logics.

Definition A23: Substitutions.
A *substitution* is a function $\sigma : Q \mapsto \Phi(M)$. A substitution σ can be uniquely extended to a homomorphism $\sigma : \mathfrak{Fm}_M \mapsto \mathfrak{Fm}_M$ by setting

$$\begin{array}{lcl} \sigma(\neg\phi) & = & \neg\sigma(\phi) \\ \sigma(\phi \wedge \psi) & = & \sigma(\phi) \wedge \sigma(\psi) \\ \sigma(\nabla(\phi_1, \ldots, \phi_n)) & = & \nabla(\sigma\phi_1, \ldots, \sigma\phi_n). \end{array}$$

Let σ be a substitution such that $\sigma p_i = \phi$, $\sigma p_j = p_j$ if $p_j \neq p_i$. In this case, we denote $\sigma\psi$ by $\psi[\phi/p_i]$. ⊞

In this thesis we identify logics with derivation systems.

Definition A24: Derivation Systems.
A *derivation system* is a pair $MD = (MA, MR)$ with MA a set of formulas called *axioms* and MR a set of derivation rules, a notion for which we only give a semi-formal definition. A *derivation rule* is usually given in the form '$R : \Delta/\phi$, provided C', or, if Δ is a singleton $\{\psi\}$:

$$(R) \qquad \vdash \psi \Rightarrow \vdash \phi, \text{ provided } C.$$

where ϕ and ψ are schemas of formulas and Δ is a set of such schemas, and C a *constraint* on R.
A set Σ of formulas is said to be *closed under R* if any instantiation of ϕ is in Σ whenever the corresponding instantiation of Δ is contained in Σ and the constraint C is met.
A derivation rule is called *orthodox* if it is one of the following three, *Modus Ponens*, *Universal Generalization* or *Substitution*:

(MP) If $\phi \in \Lambda$ and $\phi \to \psi \in \Lambda$ then $\psi \in \Lambda$.
(UG) If $\phi \in \Lambda$ and ∇ is an n-adic operator in M, then $\underline{\nabla}(\phi_1, \ldots, \phi_{i-1}, \phi, \phi_{i+1}, \ldots, \phi_n)$ is in Λ.
(SUB) If $\phi \in \Lambda$ and σ is a substitution then $\sigma\phi \in \Lambda$.

⊞

Definition A25: Logics.
A *(normal) modal logic* in a language M is a subset Λ of $\Phi(M)$ such that
(i) Λ contains the following axioms, the *classical tautologies* and *distribution*:

(CT) all classical tautologies
(DB) $\underline{\nabla}(p_1, \ldots, p_{i-1}, p \to p', p_{i+1}, \ldots, p_n) \leftrightarrow \underline{\nabla}(p_1, \ldots, p_{i-1}, p, p_{i+1}, \ldots, p_n) \to \underline{\nabla}(p_1, \ldots, p_{i-1}, p', p_{i+1}, \ldots, p_n)$

(ii) Λ is closed under the orthodox derivation rules.
A derivation system is called *orthodox* if it contains no derivation rules besides the orthodox ones.
Let MA be a set of axioms and MD a set of derivation rules; the logic $\Lambda(MA, MD)$ is the least set of formulas in M containing MA which is closed under the derivation rules in MD.
For a formula σ we let $\Lambda\sigma$ denote the derivation system Λ extended with σ as an axiom. For a set Σ of formulas we have an analogous convention. ⊞

Definition A26: Derivations.
A *derivation* in Λ is a finite sequence $\phi_0, \ldots, \phi_n$ such that every ϕ_i is either an axiom[3] or obtainable from $\phi_0, \ldots, \phi_{i-1}$ by a derivation rule. A *theorem* of Λ is any formula that can appear as the last item of a derivation. Theoremhood of a formula ϕ in a logic Λ is denoted by $\vdash_\Lambda \phi$. A formula ϕ is *derivable* in a logic Λ from a set of formulas Σ, notation: $\Sigma \vdash_\Lambda \phi$, if there are $\sigma_1, \ldots, \sigma_n$ in Σ with $\vdash (\sigma_1 \wedge \ldots \wedge \sigma_n) \to \phi$.
A formula ϕ is *consistent* if its negation $\neg\phi$ is not a theorem. A set of formulas is *consistent* if the conjunction of any finite subset is consistent and *maximal consistent* if it is

[3] Cf. Appendix B for a motivation of this definition

consistent while it has no consistent proper extension (in the same language). We usually abbbreviate 'maximal consistent set' by 'MCS'.

Definition A26: Properties of logics.
Let Λ be a logic, K a class of frames. Λ is called *sound* with respect to K if $\Lambda \subset \Theta(\mathsf{K})$, and *complete* if $\Theta(\mathsf{K}) \subset \Lambda$. Λ is *strongly sound* if $\Sigma \vdash_\Lambda \phi \Rightarrow \Sigma \models_{\mathsf{K}} \phi$, *strongly complete* if $\Sigma \models_{\mathsf{K}} \phi \Rightarrow \Sigma \vdash_\Lambda \phi$. for all sets of formulas Σ and formulas ϕ.
If Λ is (a derivation system (A, D) which is) sound and complete for a class K of frames, we call Λ an axiomatization for K.

Definition A27: Minimal modal logics.
The *minimal* or *basic* logic K_S of a similarity type S is a defined as having *only* (CT) and (DB) as its axioms, *only* (MP), (UG) and (SUB) as its derivation rules.

Theorem A28.
K_S is strongly sound and complete with respect to Fr_S.

A7. Algebraic derivations.

Definition A29. Algebraic derivation systems.
We assume familiarity with the notion of a *subterm (subformula)* of a given term (formula). Let X be a set of variables, S a similarity type.
An *algebraic derivation system* over X is a pair $AD = (AA, AR)$ consisting of a set AA of *axioms*, i.e. equations over X, and a set AR of *derivation rules* (of which notion we will not give a formal definition, but cf. A24). For a derivation system $AD = (AA, AR)$, we define $Equ(AD)$, the set of equations *generated* by AD, as the smallest set of equations over X that contains AA and is closed under every rule in AR.
If Σ is a set of equations over X, then the *orthodox S-derivation system $D_S(\Sigma)$ over Σ* is the derivation system defined by the following set Σ^+ of axioms:

(i) $s = s$ for all $s \in M(X)$.
(ii) Axioms governing the Boolean part of the algebras.
(iii) N and A, stating that the S-operators are normal resp. additive.
(iv) Σ.

and the following set R_S of rules:

(i) $s = t \;/\; t = s$.
(ii) $r = s,\ s = t \;/\; r = t$.
(iii) Replacement: $s = t \;/\; r[s/x] = r[t/x]$.
(iv) Substitution : $s = t \;/\; s[r/x] = t[r/x]$.

⊞

Derivation systems are meant to provide recursive enumerations of the equations that are valid in some variety:

Definition A30.
Let D be a derivation system, K a class of algebras, Σ a set of equations. D is *sound*

for K if $Equ(D) \subseteq Equ(\mathsf{K})$, *complete* if $Equ(\mathsf{K}) \subseteq Equ(D)$. Σ is an *axiomatization* for K if $D_S(\Sigma)$ is a sound and complete derivation system for K. ⊞

Note that the difference between axiomatizations and derivation systems is that the first may only have the 'orthodox' algebraic derivation rules (i) ... (iv).
We now turn to the relation between modal logics and algebraic derivation systems. For *orthodox* modal derivation systems, this relation is well-known:

Theorem A31.
Let $\Lambda = (\Sigma, \{MP, UG, SUB\})$ be an orthodox modal derivation system which is sound and complete with respect to a class of frames K. Then Σ^α is an algebraic axiomatization for **Cm**K. ⊞

For modal axiomatizations having non-orhodox rules, we have to work a little harder:

Definition A32.
Let '$R : \Delta \ / \ \phi$, provided C' be a modal derivation rule. Its *algebraic version* R^A is defined as '$R^A : \Delta^\alpha \ / \ \phi^\alpha$, provided C'.
Let $\Lambda = (\Sigma, \{R_i \mid i \in I\})$ be a modal derivation system . Its *algebraic version* Λ^A is defined as the orthodox algebraic derivation system $D_S(\Sigma^\alpha)$ augmented with the algebraic versions of the *non-orthodox* rules R_i. ⊞

Theorem A33.
Let $\Lambda = (\Sigma, D)$ be a modal derivation system which is sound and complete with respect to a class of frames K. Then the algebraic version Λ^A of Λ is a sound and complete algebraic derivation system for **Cm**K.

Proof sketch.
For notational simplicity, we assume that '$R : \Delta/\phi$, provided C' is the only non-orthodox derivation rule of Λ.
To prove soundness, it suffices to show that $Equ(\mathbf{Cm}\mathsf{K})$ is closed under R^A, because the equations in Σ^α hold in **Cm**K by A19, and the ordinary algebraic axioms and derivation rules raise no problems. So assume that $\Delta^\alpha \subseteq Equ(\mathbf{Cm}\mathsf{K})$, and that the constraint C is met. By A.19, $\Delta \subseteq \Theta(\mathsf{K})$, so by soundness of Λ, $\phi \in \Theta(\mathsf{K})$. But then $\phi^\alpha \in Equ(\mathbf{Cm}\mathsf{K})$ by A.19.
For completeness, assume that the equation η is valid in K. Without loss of generality (cf. A.14) we may assume that η is in normal form $\phi = 1$. By A.19, $\mathsf{K} \models \phi$, so by completeness of Λ, $\vdash_\Lambda \phi$.
So it remains to be proved that the algebraic equations corresponding to Λ-theorems are derivable in Λ^A. This is easily done by induction to the length of the derivation in Λ: for the orthodox modal derivation rules the induction step is standard, for the unorthodox R it is immediate by the definition of R^A. ⊞

A8. Canonical structures.

Definition A34: Canonical Structures.
For Λ a logic in a language M, the Λ-*canonical universe* W^c_Λ is the set of all maximal Λ-consistent sets in M. For ∇ an n-adic modal operator in M, its *canonical accessibility relation* R^c_∇ is defined on W^c by

$$R^c_\nabla(\Delta_0, \dots, \Delta_{n-1}) \iff \text{for all } \phi_1 \in \Delta_1, \dots, \phi_n \in \Delta_n : \nabla(\phi_1, \dots, \phi_n) \in \Delta_0.$$

The Λ-*canonical frame* is given as $\mathfrak{F}^c_\Lambda = (W^c_\Lambda, I^c)$, where I^c is the *canonical interpretation* mapping every operator to its canonical accessibility relation. The *canonical* Λ-*model* is the pair $\mathfrak{M}^c_\Lambda = (\mathfrak{F}^c_\Lambda, V^c)$, where V^c_Λ is the *canonical valuation* assigning to every $p_i \in Q$ the set of MCSs containing p_i, i.e.

$$V^c_\Lambda(p_i) = \{\Delta \in W^c_\Lambda \mid p_i \in \Delta\}.$$

The $\Lambda-$*canonical general frame* is the pair $\mathfrak{G}^c_\Lambda = (\mathfrak{F}^c_\Lambda, A^c_\Lambda)$ where $X \in A^c_\Lambda$ iff $X = V^c_\Lambda(\phi)$ for some $\phi \in \Phi(M)$.
If we want to make the set Q of variables for the language $M = (S, Q)$ explicit, we may write $\mathfrak{F}^c_\Lambda(Q)$, etc.

Theorem A35: Truth Lemma.

$$\mathfrak{M}^c, \Gamma \models \phi \iff \phi \in \Gamma.$$

Proof.
The proof is by induction to the complexity of ϕ. For the atomic case the claim follows by definition. The only non-standard case in the induction step is where $\phi = \nabla(\psi_0, \dots, \psi_{n-1})$, ∇ an n-adic operator. We assume $n = 2$ and write $\phi = \psi \Delta \chi$.
First, suppose $\mathfrak{M}^c, \Gamma \models \psi \Delta \chi$. By the truth definition, there are MCSs Π, Σ with $R^c\Gamma\Pi\Sigma$, $\mathfrak{M}, \Pi \models \psi$ and $\mathfrak{M}, \Sigma \models \chi$. By the Induction Hypothesis, $\psi \in \Pi$ and $\chi \in \Sigma$. By definition of R^c then, $\psi \Delta \chi \in \Gamma$.
For the opposite direction it is sufficient to prove the following claim:

> If Γ is an MCS and $\psi \Delta \chi \in \Gamma$, then there are MCSs Π, Σ with $R^c\Gamma\Pi\Sigma$, $\psi \in \Pi$ and $\chi \in \Sigma$.

To show this, let $\phi_0, \phi_1, \dots$ be an enumeration of the formulas in the language. We will define in a simultaneous, Lindenbaum-like construction, two sequences of sets of formulas $\Pi_0 \subset \Pi_1 \subset \dots$, $\Sigma_0 \subset \Sigma_1 \subset \dots$ such that $\Pi_0 = \{\psi\}$, $\Sigma_0 = \{\chi\}$, all Π_n and Σ_n are finite and consistent, Π_{n+1} is either $\Pi_n \cup \{\phi_n\}$ or $\Pi_n \cup \{\neg\phi_n\}$ and likewise for Σ_n. Furthermore, setting π_n (σ_n) as the conjunction of all formulas in Π_n (Σ_n), we will have $\pi_n \Delta \sigma_n \in \Gamma$ for all n.
The key observation for the induction step of the definition is the following:

$$\begin{array}{ll} & \pi_n \Delta \pi_n \in \Gamma \\ \Rightarrow & \pi_n \wedge (\phi_n \vee \neg\phi_n) \Delta \sigma_n \wedge (\phi_n \vee \neg\phi_n) \in \Gamma \\ \Rightarrow & ((\pi_n \wedge \phi_n) \vee (\pi_n \wedge \neg\phi_n)) \Delta ((\sigma_n \wedge \phi_n) \vee (\sigma_n \wedge \neg\phi_n)) \in \Gamma \\ \Rightarrow & \text{one of} \quad (\pi_n \wedge \phi_n) \Delta (\sigma_n \wedge \phi_n,) \quad (\pi_n \wedge \phi_n) \Delta (\sigma_n \wedge \neg\phi_n), \\ & \qquad (\pi_n \wedge \neg\phi_n) \Delta (\sigma_n \wedge \phi_n), \quad (\pi_n \wedge \neg\phi_n) \Delta (\sigma_n \wedge \neg\phi_n) \quad \text{is in } \Gamma. \end{array}$$

Now for instance in the second case, we take $\Pi_{n+1} = \Pi_n \cup \{\phi_n\}$ and $\Sigma_{n+1} = \Sigma_n \cup \{\neg\phi_n\}$, etc.
It is then straightforward to prove that the Π_n, Σ_n have the properties mentioned above. Let $\Pi = \bigcup_{n<\omega} \Pi_n$, $\Sigma = \bigcup_{n<\omega} \Sigma_n$, then one can easily verify that Π and Σ are MCSs and that $R^c\Gamma\Pi\Sigma$. ⊞

Definition A36: Properties of logics.
A logic Λ is *canonical* if Λ is valid not only on its canonical model (which is always the case, by the truth lemma), but on *every* model based on the canonical frame, i.e. if $\mathfrak{F}^c_\Lambda \models \Lambda$. A formula ϕ is canonical if the logic $K_S\phi$ is canonical.

Theorem A37.
Let Λ be a canonical logic. Then Λ is strongly sound and complete with respect to Fr_Λ.

Definition A38. Free Algebras.
Let Q be a set of variables, K a class of Boolean S-algebras. We assume familiarity with the notion of the *Q-generated free algebra* $\mathfrak{A}_{\mathsf{K}}(Q)$ over K. For a variety axiomatized by a set of equations H, we denote the free algebra by $\mathfrak{A}_H(Q)$.

The canonical frames are the canonical extensions of the free algebras:

Theorem A39.
Let Λ be a modal logic, Q a set of propositional variables. Then

$$\mathfrak{F}^c_\Lambda(Q) = \mathfrak{Cs}\mathfrak{A}_{\Lambda^a}(Q).$$

A9. Versatility.

Nearly always, the frames one has in mind for a modal language, satisfy some extra conditions. An important example is formed by tense logic:

Definition A40. Tense.
Assume that a subset T of the diamonds of S is given as $T = \{F_j, P_j \mid j \in J\}$. Diamonds in this set are called *tense diamonds*, their duals *tense boxes*. We call F_j the *converse* of P_j and the other way round. If $\Diamond$ is a tense diamond, its converse is denoted by $\Diamond^{-1}$. A diamond that is not in T is called *uni-directional*. If all diamonds of a similarity type are in T, we call it a *tense similarity type*.
A frame $(W, R_\nabla)_{\nabla\in S}$ for S is called a *tense frame* if for every $\Diamond \in T$, the accessibility relations of $\Diamond$ and $\Diamond^{-1}$ are each other's converse, i.e. $R_{\Diamond^{-1}} = (R_\Diamond)^{-1}$. For a class K of S-frames, we let K^t denote the class of tense frames in K. ⊞

With emphasis, we want to note that the above definition should be understood as to include the case where a modal operator is its *own* converse.

Definition A41. Tense Logics.
Let S, T be as above. The *minimal tense logic* K^t_S is the minimal S-logic K_S extended

with the following axiom for every $\Diamond \in T$:
$(CV)\ p \to \Box\Diamond^{-1}p$ ⊞

Theorem A42.
K^t_S is strongly sound and complete with respect to the class of all tense frames. ⊞

We want to generalize these concepts to operators of higher rank:

Definition A43: Versatility.
A *versatile* similarity type is a modal similarity type $S = (O, \rho)$ where the set O of operators is given as a (disjoint) union of sets, $O = \bigcup_{j\in J} O_j$, such that $O_j = \{\nabla_{j0}, \ldots, \nabla_{j,n_j}\}$ and all operators in O_j have the same rank $n_j - 1$.
A *versatile frame* for such an S is an S-frame (W, I) where for all $j \in J$, $i \leq n_j$ one has

$$I(\nabla_{ji}) = \{(w_0, w_1, \ldots, w_{n_j}) \mid (w_1, \ldots, w_{n_j}, w_0) \in I(\nabla_{j,i+1}\}$$

For a class K of S-frames, we let K^v denote the class of versatile frames in K. ⊞

We do not exclude the possibility that $O_j = \{\nabla, \ldots, \nabla\}$, i.e. all operators are identical. Once we know that a frame is versatile, it is not necessary to give all of its accessiblity relations. For example, a frame $\mathfrak{F} = (W, R_\Diamond, R_{\Diamond^{-1}})$ can be identified with $\mathfrak{F} = (W, R_\Diamond)$ if $R_{\Diamond^{-1}} = (R_\Diamond)^{-1}$.

Note that the notion 'tense' only applies to diamonds: in a tense similarity type S there is no constraint on the operators of rank > 2. Only if all operators of S are constants or diamonds, do the concepts of 'tense' and 'versatility' coincide, and do we have $\mathsf{K}^t = \mathsf{K}^v$. The notions of tense and versatile operators are known in the theory of Boolean Algebras with Operators under the names of conjugates and residuals.

Appendix B.

Consequences of Derivation Systems.

Outline.

We discuss an alternative for our notion of semantic consequence ($\Sigma \models \phi$) and show that in the context of non-ξ rules, our option behaves nicer.

Recall that we defined a *local* consequence relation for modal formulas by setting

$$\Sigma \models_{\mathsf{K}} \phi \iff \text{for } all \text{ models } \mathfrak{M} = (\mathfrak{F}, V) \text{ with } \mathfrak{F} \text{ in } \mathsf{K} \text{ and } every\ world\ w \text{ in } \mathfrak{M}\text{: } \mathfrak{M}, w \models \Sigma \Rightarrow \mathfrak{M}, w \models \phi.$$

There is a different, *global* paradigm in modal logic, where:

$$\Sigma \models^*_{\mathsf{K}} \phi \iff \text{for } all \text{ models } \mathfrak{M} = (\mathfrak{F}, V) \text{ with } \mathfrak{F} \text{ in } \mathsf{K}\text{: } \mathfrak{M} \models \Sigma \Rightarrow \mathfrak{M} \models \phi.$$

We face an analogous choice in first order logic, if we want to decide what $\Sigma \models \phi$ means, when Σ and ϕ contain *free variables.* (Note that for the formalism L^r_α the question is not only analogous to, but indeed the very same as for CML_α.)

This difference in semantic perspective is reflected in the interpretation of *derivation systems.*
In our approach, $\Sigma \vdash_\Lambda \phi$ holds if there are $\sigma_1, \ldots, \sigma_n \in \Sigma$ with $\vdash_\Lambda (\sigma_1 \wedge \ldots \wedge \sigma_n) \rightarrow \phi$, i.e. derivation rules may only be applied to logical theorems.
In the other line of thinking, $\Sigma \vdash^*_\Lambda \phi$ holds if there is a derivation $\phi_0, \ldots, \phi_n = \phi$ such that every ϕ_i is either an axiom of Λ *or in* Σ, or obtained from an earlier ϕ_j by an application of a derivation rule. In other words: the formulas in Σ are to be used as if they were axioms.
In principle, two choices, both out of two alternatives, would give us four possible pairs consisting of a semantic and an axiomatic notion. Of these, the pairs $\{\models^*, \vdash\}$ and $\{\models, \vdash^*\}$ are ruled out if we want the axiomatic relation to be (strongly) sound and complete with respect to the semantic one: the fact that $p \models^* \Box p$ and $p \not\vdash \Box p$ implies that $\vdash$ cannot be *complete* with respect to $\models^*$, and likewise, the pair $\{\models, \vdash^*\}$ will give problems concerning *soundness*, as $p \not\models \Box p$, yet $p \vdash^* \Box p$.
In this appendix we briefly compare the remaining pairs $\{\models, \vdash\}$ and $\{\models^*, \vdash^*\}$, which we will call 'our' or the 'local' paradigm, respectively the '$*$-style' or 'global' paradigm.

For algebraists, the choice for the $*$-style paradigm seems to be obvious, as equations are always implicitly understood to be universally quantified, and one is interested in an algebra as a whole.

In the *possible world semantics* of modal logic however, we have a strong preference for the *local* paradigm, and we believe that our reasons for this opinion could lead algebraic logicians to think that the local perspective is at least *interesting.* Our motivation for the local variant of semantic consequence and derivation systems is threefold:

First, one can show that $\models$ is *more informative* than $\models^*$. For example (abbreviate $\Box^0\phi = \phi$, $\Box^{n+1}\phi = \Box\Box^n\phi$):

$$\{p_0\} \cup \{\Box^n(p_n \rightarrow (\neg p_{n+1} \wedge \Diamond p_{n+1})) \mid n \in \omega\} \quad \models_{\mathsf{K}} \quad \bot \tag{1}$$

provides us with information about the class K, namely that

$$\text{no } \mathsf{K}\text{-frame contains an infinite sequence } w_0 R w_1 R w_2 \ldots \tag{2}$$

The global version of (1) is vacuously true, so it does not tell us anything. It is not clear to us how to express (2) using $\models^*$, *unless one adds to the language operators enabling a local perspective*, for example the 'only here' operator O. Following an idea by Johan van Benthem, we can show that, letting p be a new variable for Σ, ϕ,

$$\Sigma \models \phi \iff \{EOp\} \cup \{p \to \sigma \mid \sigma \in \Sigma\} \models^* p \to \phi.$$

On the other hand, $\models^*$ can always be reduced to $\models$: let us, in the context of this dissertation, assume[1] that we have an operator $\boxplus$ such that $\boxplus\phi$ holds in a world w iff ϕ holds in *every* world somehow accessible from w. Define $\boxplus\Sigma = \{\boxplus\sigma \mid \sigma \in \Sigma\}$. Then we have

$$\Sigma \models^* \phi \iff \boxplus\Sigma \models \boxplus\phi, \tag{3}$$

as a simple proof shows.

A second, more philosphical reason to prefer $\vdash$ to $\vdash^*$ is that in our opinion, it is an essential characteristic of modal logic that there is not one single notion of validity, not one single logic. This makes for a distinction between logics and theories and it is not clear to us how to represent this distinction in the $*$-style paradigm. We may identify an (orthodox) derivation system Λ with its set of axioms, but there should be a *conceptual difference* between $\Lambda_1 \vdash_{\Lambda_2} \phi$ and $\Lambda_2 \vdash_{\Lambda_1} \phi$. In the $*$-approach however, both $\Lambda_1 \vdash^*_{\Lambda_2} \phi$ and $\Lambda_2 \vdash^*_{\Lambda_1} \phi$ reduce to $\Lambda_1 \cup \Lambda_2 \vdash^*_K \phi$, where K is the minimal modal logic of the similarity type.

Our third and main motivation to focus on the local consequence relation is related to the notion of a non-ξ rule.
Let us consider the simplest case of the irreflexivity rule IR_D for the D-operator:

$$(IR_D) \qquad (p \land \neg Dp) \to \phi \ / \ \phi, \ \text{ if } p \notin \phi.$$

For this rule, problems will rise concerning *soundness* if we adhere to the $*$-paradigm. For, it lies in the *nature* of the D-operator that any standard model $\mathfrak{M}$ can have at most one world where $p \land \neg Dp$ is true. This implies that in any non-trivial standard model $\mathfrak{M}$, $\mathfrak{M} \models \neg(p \land \neg Dp)$, or equivalently, $\mathfrak{M} \models (p \land \neg Dp) \to \bot$. If we want IR_D to be $*$-sound, by the instance $(p \land \neg Dp) \to \bot \ / \ \bot$ of IR_D we are forced to conclude $\mathfrak{M} \models \bot$, which is clearly undesirable.
So at this particular point, the focus on the *local* consequence relation is *essential*:

in the global paradigm non-ξ rules make no sense.

Turning to the example of cylindric modal logic and related notions, we can go even further and claim that no finite $*$-style derivation system can be a sound and complete axiomatization for the cylindric modal formulas valid in the cubes, or the $\mathcal{CCS}$-formulas valid in the squares[2].

[1] This idea stems from Goranko and Passy [46]; using proposition 2.33 in van Benthem [14], one can reduce $\models^*$ to $\models$ in *any* similarity type.

[2] The same claim applies to the equations holding in representable cylindric algebras, typeless valid formulas, etc.

Theorem B.1.
Let $\Lambda = (MA, MD)$ be a derivation system for cylindric modal logic of dimension $n < \omega$, and suppose that Λ is $*$-style sound and complete with respect to cube validity, i.e. $\Sigma \models^*_{\mathsf{C}_n} \phi \iff \Sigma \vdash^*_\Lambda \phi$.
Then either MA or MD is infinite.

Proof.
We will show that unorthodox derivation *rules* can be replaced by *axioms*, in any derivation system which is $*$-style strongly sound and complete with respect to cube validity. Our 'non-finite *derivability* result' then follows by Monks theorem that $\mathit{Equ}(\mathsf{RCA}_n)$ is not finitely *axiomatizable* (and hence, it can be strengthened along the lines of Andréka [5], cf. the remarks below definition 4.1.8).

For a sketch of the proof, suppose that $\Lambda = (MA, MR \cup \{R\})$ is a finite derivation system which is $*$-style sound and complete with respect to the cubes, where $R : \alpha \ / \ \beta$ is an unorthodox derivation rule. (The proof can easily be adapted for rules having constraints.)
Let Λ^A be the derivation system $(MA \cup \{A\}, MR)$, where A is the axiom (schema) $\boxplus\, \alpha \rightarrow \boxplus\, \beta$. We have to show that

$$\Sigma \vdash^*_\Lambda \phi \iff \Sigma \vdash^*_{\Lambda^A} \phi.$$

To prove ($\Leftarrow$), it is sufficient to show that $\boxplus\, \alpha' \rightarrow \boxplus\, \beta'$ is a theorem of Λ, for every instance (α', β') of (α, β). Now as Λ is $*$-style sound, we have $\alpha' \models^* \beta'$, implying $\models^* \boxplus\, \alpha' \rightarrow \boxplus\, \beta'$ by (3). By the supposed completeness of Λ, this implies $\vdash^*_\Lambda \boxplus\, \alpha' \rightarrow \boxplus\, \beta'$.
For ($\Rightarrow$), we have to prove that $\alpha \ / \ \beta$ is a derived rule of Λ^A. This is rather easy, as the following ($*$-style) derivation shows:

(1)	α'	(assumption)
(2)	$\boxplus\, \alpha'$	(1, UG)
(3)	$\boxplus\, \alpha' \rightarrow \boxplus\, \beta'$	(axiom)
(4)	$\boxplus\, \beta'$	(2,3, MP)
(5)	β'	(4, $\boxplus$ is $S5$)

This proves Theorem B.1. $\boxplus$

Conclusion.
The *local* perspective on derivation systems and the semantic consequence relation is *essential* in the idea to use *unorthodox* derivation rules as a means to get round the *non-finite axiomatizability results* in algebraic logic.

Appendix C.

Index.

Outline.

We list the concepts, symbols, etc. that are used in more than one spot of the text, together with the place where they are defined. Numbers refer to text items, not to pages.

$\Diamond_i$ 4.2.1
$\ominus, \oplus$ 3.1.1 (3.1.2), 5.2.1
$\ominus', \oplus'$ 3.1.1 (3.1.2), 5.2.1
$\oplus, \oplus$ 3.1.1 (3.1.2)
$\bigcirc_{ij}$ 4.2.12
$\otimes, \ominus, \oplus$ 3.1.1 (3.1.2)
$\circ$ 3.1.1 (3.1.2)
$\oplus$ 3.2.2, 4.2.12, 5.2.1
$\oplus_i^{(n)}$ 4.2.12
$\Diamond, \Diamond, \Diamond, \Diamond$ 3.4.1, 5.2.1
$\Box, \Box, \Box, \Box$ 3.4.1, 5.2.1
$|$ 3.3.1
$\breve{}$ 3.3.2
; 3.3.2
α-cube 4.2.3
α-formulas 4.2.1
α-frame 4.2.2
α-validity 4.1.2, 4.2.3
β 4.3.4
δ 3.1.1 (3.1.2)
δ_{ij} 4.2.1
$\not\in$ A2
ΛD 2.4.2
ΛD^+ 2.4.2
$\Lambda(\sigma), \Lambda(\Sigma)$ XXXX
$\Lambda(-\xi), \Lambda(-\Xi)$ 2.1.2
Θ A6
κ 5.2.12
ϕ^b 5.3.2
ϕ^s 2.3.13, 2.4.5
$\mathsf{K}_{-\xi}, \mathsf{K}_{-\Xi}$ 2.1.1
Ω 5.3.7
1' 3.3.1
$(\cdot)^1$ A7
$(\cdot)^{-1}$ 3.3.1, A40
$(\cdot)^2$ A7
2-frame, 2-model 3.2.1
$[s, t]$ 5.1.1
$\vdash_\alpha^+$ 4.5.2
$\vdash_n^+$ 4.4.3
$\models_\alpha$ 4.1.2
$\models_{tf}$ 4.1.11
A_α 4.2.10
A_α^+ 4.5.2
A_2 3.2.7
A_n 4.2.10
A_n^+ 4.4.3
$ACCD^t$ 3.3.32
$ACML_nD$ 4.4.10
accessibility relation A4
additive operators A10
admissible valuations 2.3.7
AHS 5.2.15
$AHSC$ 5.3.7
$AHSN$ 5.3.7
$AHSW$ 5.3.7
algebraic derivation system A29
algebraic version of a derivation system A32
AR 3.3.22
AR 3.3.19
AR^+ 3.3.34
$AR1, \ldots, AR8$ 3.3.19
arrow, arrowframe, arrowmodel 3.3.17
ARU, ARE 3.3.25
assignment A12
At, $\mathfrak{At}$ A20
atom structure A20
axiomatization A26, A30

B 5.2.2
BAD, bad 4.3.2
basic logic A27
basic rudimentary formula 2.3.9
Boolean algebra with operators A10
box A2

$C1, \ldots, C7$ 4.1.8
$C8$ 4.2.8
C_α 4.2.3
C_2 3.2.1
CA_α 3.5.5
CA_2 3.5.5
canonical: accessibility relation, frame, model, valuation, universe A34
canonical logic A36
canonical structure A22
$CC\delta$ 3.3.4
$CC\lambda$ 3.4.1
$CC1, \ldots, CC8$ 3.3.19
CCD 3.3.27

CCD^t 3.3.31
CCE, CCU 3.3.82
$CE1, \ldots, CE7$ 3.5.3
CF_α 4.2.5
$CH1, \ldots, CH7$ 3.2.4
characterization A6
$CH1, \ldots, CH7$ 3.2.4
CHOP 5.4.1
$CHV1, \ldots, CHV7$ 3.2.4
closed formula A2
Cm, $\mathfrak{Cm}$ A17
$CM1, \ldots CM7$ 4.2.5
$CM8$ 4.2.8
CML_2 3.2.1
CML_α, CML_n 4.2.1
CML_nD 4.4.1
compass operators 3.4.1
completeness A26, A30
complex algebra A17
conjugates A40
consistent formulas A26
constant A1
constraint A24
converse of a diamond (box) A40
convex set 5.1.1
corresponding algebraic equation A18
corresponding frame language A3
corresponding modal formula A18
corresponding model language A3
$\mathfrak{Cs}$ A22
$\mathfrak{Cs}_\alpha$ 4.1.4
CT A25
cube, cubic: frame, model 4.2.3
CV A41
$CV1, \ldots, CV7$ 3.2.4
cylindric algebra 4.1.8
cylindric algebra (dimension 2) 3.5.5
cylindric frame 4.2.5
cylindric modal logic 4.2.1
cylindric modal logic (dimension 2) 3.2.1
cylindric type algebra 4.1.5
cylindrification 4.1.4

cylindric frame

D 2.4.1
$\underline{D}$ 2.4.1
D_2 3.2.3
D_n, $\underline{D}_n$ 4.2.14
D' 3.3.7, 5.2.14
D-axioms $(D1, D2, D3)$ 2.4.2
D-irreflexivity rule 2.4.2
D-standard frame 2.4.1
DB A27
d-canonical structure 2.5.10
d-theory 2.5.3, 2.7.10
DE 3.4.2, 5.2.3
definably well-ordered 5.3.9
derivability A26
derivation A26
derivation rule A24
derivation system A24
$Df_{CHOP}(\phi, \psi, \chi)$ 5.4.4
DH 4.3.2
DI 3.4.2, 5.2.3
diagonal 4.1.4
diamond A1
direct product A15
discrete general frame 2.3.1
disjoint hyperplanes 4.3.2
disjoint union A8
distinguishing set of formulas 2.5.3, 2.7.10
distribution A25
dual operator A2

E (accessibility relation) 5.2.2
E (operator) 2.4.1
E_{ij} 4.2.2
$\mathfrak{Em}$ A22
embedding algebra A22
Equ A12,A29
equation A11

$\mathfrak{F}^c - \Lambda$ A34
$\mathfrak{F}^d$ 2.5.10
$\mathfrak{F}^w$ 2.8.5
flat model 3.4.6
flat valuation 3.4.6, 5.3.1
flat validity 5.3.1
Fr_Σ A6
FRA 3.3.2
frame A4

FCS_α 4.1.5
full cylindric set algebra 4.1.4
full relation algebra 3.3.2
general frame A17
generated subframe 9A
global frame correspondent A7
global model correspondent A7
$H_i^{(n)}$ 4.2.12
H A15
HCA_α 4.2.8
HCF_α 4.2.8
HCF_2 3.2.7
Henkin's equation 3.5.5
hereditary subset 9A
$\mathbf{H}_\mathfrak{f}$ A9
homomorphism A8, A15
HS 5.2.1
HSC 5.2.1
hypercylindric 2-frame 3.2.9
hypercylindric (α-)algebra 4.2.8
hypercylindric (α-)frame 4.2.8
hyperplane 4.3.1
Id 3.3.1
$\mathfrak{I}(\mathfrak{T})$ 5.2.1
$INT(\mathfrak{T})$ 5.1.1
interpolation property 2.9.1
interval 5.1.1
IR_D^* 2.6.4
IR_D 2.4.2
irreflexivity rule: section 2.1
K A27
K^t A41
K^v 2.7.2, 2.7.6
K^vD^+ 2.8.1, 2.4.2
$L^<$ 3.4.4
$L^{1<}$ 3.4.7
L^2 3.3.9
$\mathcal{L}_\alpha$ 4.1.5
L_A 3.3.17
L_{CCS}, L_{CML}, L_{HS} A3
L_k 3.3.9
L_α^r 4.1.1
L_{tf} 4.1.10
$L(\vec{x})$ 3.3.9
$LI5$ 5.2.10
LN 3..4.2
local frame correspondent A7
local model correspondent A7
M A2
$\mathfrak{M}^c - \Lambda$ A34
$\mathfrak{M}^d$ 2.5.10
$\mathfrak{M}^w$ 2.8.5
many-dimensional modal logic Ch. 1
maximal consistent set A26
MCS A26
$MI1, \ldots, MI5$ 5.2.12
minimal logic A27
minimal tense logic A41
mirror image 3.2.1
modal algebra A10
modal formula A2
modal logic A25
modal similarity type A1
modal language A2
modal operator A1
model A5
Modus ponens A24
MP A24
$N\beta R$ 4.4.1
$N\xi R$ 2.1.1
$N1, \ldots, N7$ 4.2.5
$N8$ 4.2.8
negative formula 2.2.1
$NH1, \ldots, NH7$ 3.25.
$NHV1, \ldots, NHV7$ 3.25.
non-ξ rule 2.1.2
normal operators A10
normal term A13
normal form A13
$NV1, \ldots, NV7$ 3.25.
O (operator) 2.4.1
orthodox derivation rule A24
orthodox derivation system A25, A29
P A15
path representation 2.5.2
$\mathbf{P}_\mathfrak{f}$ A9
positive formula 2.2.1
possible world A4

$Pr(\phi)$ 2.5.2

R_∇ A4
RA 3.3.3
$RA0, \ldots, RA8$ 3.3.3
R^c_∇ A34
RCA_α 4.1.5
RCA_2 3.5.2
relation algebra 3.3.3
relation type algebra 3.3.2
relational arrowframe 3.3.19
$Re(U)$ 3.3.1
$\mathfrak{Re}(U)$ 3.3.1
Replacement A29
representable cylindric algebra 4.1.5
  dimension 2: 3.5.2
representable relation algebra 3.3.2
residual A40
restricted first order logic 4.1.1
RRA 3.3.2
rudimentary: formula, subset,
  valuation 2.3.9

S_2 3.1.2
$(\cdot)^s$ Sahlqvist correspondent 2.2.3, 2.3.13
Sahlqvist equation 2.2.3
Sahlqvist formula 2.2.1
Sahlqvist tense formula 2.3.1
Sahlqvist theorem 2.2.1, 2.3.14, 2.4.6
Sahlqvist variety 2.2.3
schema validity 4.1.13
semantic consequence A6
S A15
SD-theorem 2.5.1, 2.7.7
SNΞ-theorem 2.8.2
$\mathsf{S}_\mathfrak{f}$ A9
soundness A27, A30
SQ 3.3.5
square 3.1.2
standard frame 2.4.1
strong completeness A26
strong soundness A26
strongly positive formula 2.2.1
SUB A24
subalgebra A15
Substitution A24, A29
substitution A23
T_i 4.2.2
tense: box, diamond, frame,
  similarity type A40
tense logic A41
term A11
theorem A26
triangle A1
truth A6
two-dimensional frame 3.1.2
two-dimensional $CC\lambda$-frame 3.4.1
two-dimensional HS-frame 5.2.2
type-free, typeless formulas 4.1.10
type-free validity 4.1.11
UG A24
$UM^<$ 5.2.5
uni-directional diamond A40
Universal Generalization A24
universe A4
untied formula 2.2.1
V (axiom) 2.7.2
V A15
validity A6, A12
valuation A5
variety A16
$VER(\phi)$ 5.4.4
versatile similarity type A43
versatile frame A43, 2.7.1
versatile logic 2.7.2, 2.7.6
w-canonical structures 2.8.5
w-theory 2.8.3
witnessing set of formulas 2.8.3
$W(\psi, n, \phi)$ 2.5.4
$W(\psi, s, \phi)$ 2.7.9
WO 5.3.6
$X(\phi, \psi, \chi)$ 5.4.4
Z_H 3.2.2
Z_i 4.2.14
zigzagmorphism A9
Z_V 3.2.2

APPENDIX D.

SAMENVATTING.

Outline.

We give a summary of the dissertation, in Dutch.

In het inleidende hoofdstuk 1 geven we eerst informeel aan, wat we onder meerdimensionale modale logica's verstaan: een α-dimensionale modale logica, voor een ordinaalgetal α, is een modaal formalisme, waarbij we, binnen de algemene semantiek der Kripke frames, *bedoelde frames* onderscheiden waar het universum bestaat uit α-tupels over een onderliggende verzameling. Dergelijke frames dopen we *kubussen* (*vierkanten* in het tweedimensionale geval).

Hoofdstuk 1 noemt ook de vragen, die in dit proefschrift aan de orde komen: de expressieve kracht van meerdimensionale modale formalismes, de talige karakterizeerbaarheid van de kubussen, en de axiomatizeerbaarheid van de formules die geldig zijn in deze bedoelde frames. Verder introduceren we de belangrijkste onderstromen van de dissertatie: uitgebreide correspondentietheorie, bijzondere modale afleidingsregels en verbanden met algebraïsche en temporele logica.

In het tweede hoofdstuk wordt, betreffende axiomatische volledigheid, een meta-stelling bewezen, die in alle andere hoofdstukken van het proefschrift toegepast wordt. De centrale notie vormt hier de *niet-ξ afleidingsregel*, een generalizatie van Gabbays irreflexiviteitsregel. De bovengenoemde SNΞ-stelling is een aanvulling op de stelling van Sahlqvist, in de zin dat ze voor systemen bestaande uit Sahlqvist axioma's *en* niet-ξ afleidingsregels, automatisch sterke volledigheid geeft voor een klasse van frames die gekarakteriseerd wordt door de axioma's en de afleidingsregels. Een belangrijke randvoorwaarde is, dat het modale similariteitstype versatiel is en de D-operator bevat (of de betreffende operatoren kan definiëren over de te axiomatizeren klasse van frames. In het bewijs van de volledigheidsstelling passen we de traditionele canonieke-framemethode aan; we geven een alternatieve versie van canonieke structuren en bewijzen een nieuw persistentie resultaat voor Sahlqvist formules.

Hoofdstuk 3 is gewijd aan *twee*dimensionale modale logica's. Na een zo volledig mogelijk overzicht van wat de literatuur op dit terrein te bieden heeft, bedden we de meeste bestaande formalismes in één overkoepelend twedimensionaal systeem in.

Daarna worden drie specifieke formalismes in detail behandeld. Eerst twee-dimensionale cylindrische modale logica, waarvoor we een relatief eenvoudig volledigheidsbewijs geven.

De meeste aandacht gaat uit naar $\mathcal{CC\delta}$, een modale logica voor binaire relaties. Voor dit formalisme vinden we een expressief equivalent fragment van de eerste orde logica (het 'drie-variabelen fragment'), we geven een precieze karakterizering van de vierkante $\mathcal{CC\delta}$-frames, en tenslotte vormen we deze karakterizering om tot een volledige axiomatizering voor de vierkanten.

Het derde behandelde formalisme is een uitbreiding van $\mathcal{CC\delta}$ met een temporele component; voor dit systeem kunnen we onder andere een functioneel volledigheidsbewijs geven, over de klasse van lineaire tijdsordes.

Het modale perspectief levert de volgende resultaten op voor algebra's van binaire relaties: een vereenvoudiging van Henkin's gelijkheid in cylindrische algebra's, en een eindig afleidingssysteem voor de equationele theorie der representeerbare relatie-algebra's

Het vierde hoofdstuk laat zien hoe we, geïnspireerd door de overeenkomsten tussen kwantoren en modale operatoren, de eerste orde logica, of een enigzins ingeperkte versie daarvan, kunnen bestuderen als ware het een modaal formalisme, de *cylindrische modale logica.* Deze aanpak is zeer nauw verwant aan de algebraïsche theorie van de predicatenlogica: cylindrische algebras treden op als modale algebras van ons meerdimensionale formalisme. Net als in hoofdstuk 3 karakterizeren we de bedoelde frames in termen van Sahlqvist axioma's en niet-ξ regels, en aldus vinden we, eerst voor het eindig-dimensionale geval, een eindig, sterk correct en volledig afleidingssysteem voor de cylindrische modale formules die geldig zijn in deze kubussen. Uitgaande hiervan kunnen we soortgelijke afleidingsstelsels geven voor onder andere oneindig-dimensionale cylindrische modale logica, typevrij geldige formules, en de equationele theorie van de representeerbare cylindrische algebras.
We geven een voorbeeld van een afleiding voor een formule waarbij een nieuwe niet-ξ afleidingsregel daadwerkelijk gebruikt wordt, en waarvoor dit gebruik ook noodzakelijk is.

In hoofdstuk 5 wordt de modale tijdslogica van intervallen benaderd op de wijze van tweedimensionale modale logica's. We geven eerst een inleiding betreffende tijdslogica's waar periodes in plaats van punten centraal staan, en we motiveren bovengenoemde benadering, die gebaseerd is op het idee, dat we een interval kunnen identificeren met het paar bestaande uit het begin- en het eindpunt van dit interval.
Een specifiek systeem (HS), dat ontworpen is door Halpern en Shoham, wordt daarna onder de loep genomen, tesamen met enkele uitbreidingen en inperkingen. We bewijzen verscheidene resultaten betreffende expressiviteit en volledigheid van deze formalismes.

Hoofdstuk 6 bevat onze conclusies: ten eerste menen we met de SNΞ-stelling uit het eerste hoofdstuk een elegante methode te hebben gevonden, om een deels positieve (de axioma's), deels negatieve (de niet-ξ regels) *karakterizering* van een klasse van frames om te zetten in een *axiomatizering.* Ten tweede zien we deze niet-ξ regels, en ook de gewone Sahlqvist-stelling, als een nuttige bijdrage van het mogelijke-wereldenperspectief uit de modale logica aan de theorie der Boolese algebra's met operatoren. Verder is het opvallend, dat sommige inperkingen op de semantiek van meerdimensionale tijdslogica's, inperkingen die geïnspireerd zijn door mogelijke toepassingen binnen bijvoorbeeld de informatica, systemen opleveren met logisch aantrekkelijke eigenschappen als functionele volledigheid en orthodoxe axiomatizeerbaarheid.

Ten slotte heeft deze dissertatie twee appendices: in appendix A sommen we de noties en resultaten op die we als basiskennis veronderstellen bij de lezer. In appendix B motiveren we onze definitie van de semantische-gevolg relatie: we bewijzen dat alleen in ons 'locale perspectief' de niet-eindige axiomatizeerbaarheidsresultaten uit de algebraïsche logica omzeild kunnen worden door bijzondere afleidingsregels te gebruiken.

Bibliography

[1] S. Abramsky. D. Gabbay and T. Maibaum (eds.), *Handbook of Logic in Computer Science*, Oxford University Press, Oxford, 199?

[2] J. F. Allen, "Maintaining knowledge about temporal intervals", *Artificial Intelligence*, **26** (1983) 832–843.

[3] J. F. Allen and P. J. Hayes, "A commonsense theory of time", *Proceedings of the 9th International Joint Conference on Artificial Intelligience, Los Angeles, California*, Morgan Kauffman, 1985, 528–531.

[4] J. F. Allen and P. J. Hayes, *Moments and Points in an Interval-Based Temporal Logic*, Technical Report 180, University of Rochester, 1987.

[5] H. Andréka, *Complexity of Equations Valid in Algebras of Relations*, Third Doctoral Dissertation, Budapest, 1991.

[6] H. Andréka, T. Gergely and I. Németi, "A simple purely algebraic proof of the completeness of some first-order logics", *Algebra Universalis* **5** (1975) 8–15.

[7] H. Andréka, J.D. Monk and I. Németi (eds.), *Algebraic Logic; Proceedings of the 1988 Budapest Conference on Algebraic Logic*, North-Holland, Amsterdam, 1991.

[8] H. Andréka, I. Németi, "Dimension-complemented and locally finite-dimensional cylindric algebras are elementary equivalent", *Algebra Universalis* **13** (1981) 157–163.

[9] H. Andréka, I. Németi, "On Jónsson's clones of operations on binary operations", in: [7], 431–445.

[10] H. Andréka, I. Németi and I. Sain, "On the strength of temporal proofs", *Theoretical Computer Science* **80** (1991) 125–151.

[11] H. Andréka, I. Németi and I. Sain, personal communication.

[12] J.F.A.K. van Benthem, *The Logic of Time*, Reidel, Dordrecht, 1983.

[13] J.F.A.K. van Benthem, "Correspondence theory", in: [34], 167–247.

[14] J.F.A.K. van Benthem, *Modal Logic and Classical Logic*, Bibliopolis, Naples, 1985.

[15] J.F.A.K. van Benthem, *Language in Action*, North-Holland, Amsterdam, 1991.

[16] J.F.A.K. van Benthem, *Modal Logic and Relational Algebra* (revised version), Institute of Language, Logic and Information, University of Amsterdam, to appear in: [82].

[17] C.H. Bergman, R.D. Maddux and D.L. Pigozzi (eds.), *Algebraic Logic and Universal Algebra in Computer Science*, Lecture Notes in Computer Science 425, Springer Verlag, Berlin, 1990.

[18] B. Biró, "Non-finite-axiomatizability results in algebraic logic", *Journal of Symbolic Logic*, to appear.

[19] P. Blackburn, *Nominal Tense Logic*, PhD Dissertation, University of Edinburgh, Edinburgh, 1989.

[20] W. Blok and D. Pigozzi, "Algebraizable Logics", *Memoirs of the American Mathematical Society*, **77** (1989).

[21] J. Büchi and D. Siefkes, *The Monadic Second Order Theory of All Countable Ordinals*, Springer Lecture Notes in Mathematics **328**, Heidelberg, 1973.

[22] J.P. Burgess, "Decidability for branching time", *Studia Logica*, **39** (1980), 203-218.

[23] J.P. Burgess, "Axioms for tense logics: II. Time periods", *Notre Dame Journal of Formal Logic*, **23** (1982) 375–383.

[24] S. Burris and H.P. Sankappanavar, *A Course in Universal Algebra*, Springer Verlag, Berlin, 1981.

[25] E. Clarke and D. Kozen, eds., *Logics of programs*, Springer Lecture Notes in Computer Science **164** (1984).

[26] C. Davis, "Modal operators, equivalence relations, and projective algebras", *American Journal of Mathematics*, **76** (1954) 747–762.

[27] K. Doets, "Monadic Π^1_1-theories of Π^1_1-properties", *Notre Dame Journal of Formal Logic*, **30** (1989) 224–240.

[28] L. Farinas, "Space as time", *Information Processing Letters*, **17** (1983) 113–115.

[29] B. van Fraassen, "The only necessity is verbal necessity", *Journal of Philosophy*, **74** (1977) 71–85.

[30] D.M. Gabbay, "Two-dimensional propositional tense logic", in: [64], 569–583.

[31] D.M. Gabbay, "An irreflexivity lemma with applications to axiomatizations of conditions on linear frames", in: [85], pp. 67–89

[32] D.M. Gabbay, "Expressive functional completeness in tense logic", in: [85], pp. 91–117.

[33] D.M. Gabbay and F. Guenthner, "A note on systems of n-dimensional tense logics", in: [96], 63–70.

[34] D.M. Gabbay and F. Guenthner, (eds.), *Handbook of Philosophical Logic, vol II*, Reidel, Dordrecht, 1984.

[35] D.M. Gabbay, in collaboration with I. Hodkinson and M. Reynolds, *Temporol Logic: Mathematical Foundations and Computational Aspects*, Oxford University Press, Oxford, 1992, to be published.

[36] D.M. Gabbay and I.M. Hodkinson, "An axiomatization of the temporal logic with Since and Until over the real numbers", *Journal of Logic and Computation* **1** (1990) 229–259.

[37] A. Galton, "The logic of occurrence", in: [38], 169–196.

[38] A. Galton, (ed.), *Temporal Logics and their Applications*, Academic Press, London, 1987.

[39] G. Gargov and V. Goranko, "Modal logic with names", in: [105], 81–103.

[40] G. Gargov and S. Passy, *A Note on Boolean Modal Logic*, in: [97].

[41] G. Gargov, S. Passy and T. Tinchev, "Modal environment for boolean speculations", in: [123], 253–263.

[42] R. Goldblatt, "Metamathematics of modal logic", *Reports on Mathematical Logic*, **6** (1976) 41–77, **7** (1976) 21–52.

[43] R. Goldblatt, "Varieties of complex algebras", *Annals of Pure and Applied Logic*, **44** (1989) 173–242.

[44] V. Goranko, "Modal definability in enriched languages", *Notre Dame Journal of Formal Logic*, **31** (1990) 81–105.

[45] V. Goranko, "Applications of quasi-strucural rules to axiomatizations in modal logic" (abstract), in: [72], p. 119.

[46] V. Goranko, and S. Passy, *Using the Universal Modality: Gains and Questions*, ITLI-Prepublication Series X–90–06, Amsterdam, 1990.

[47] M.D. Haiman, "Arguesian lattices which are not linear", *Bull. Amer. Math. Soc.*, **16,1** (1987), 121–123.

[48] P.R. Halmos, *Algebraic Logic*, Chelsea Publishing Company, New York, 1962.

[49] J. Y. Halpern, Z. Manna and B. Moszkowski, "A hardware semantics based on temporal intervals", *Proceedings 10th Conference on Automata, Languages and Programming, Barcelona*, (1983) 278–291.

[50] J. Y. Halpern and Y. Shoham, "A propositional modal logic of time intervals", in: [70], pp. 279–292.

[51] D. Harel, D. Kozen and R. Parikh, "Process logic: expressiveness, decidability, completeness", *Journal of Computer and System Sciences*, **26** (1983) 222-243.

[52] L. Henkin, *Logical Systems Containing Only a Finite Number of Symbols*, Université de Montréal, 1967.

[53] L. Henkin, J. D. Monk and A. Tarski, *Cylindric Algebras*, North-Holland, Amsterdam, *part I*, 1971, *part II*, 1985.

[54] I.L. Humberstone, "Interval semantics for tense logic: some remarks", *Journal of Philosophical Logic*, **8** (1979) 171–196.

[55] I.L. Humberstone, "Relative necessity revisited", *Reports on Mathematical Logic* **13** (1981) 33–42.

[56] I.L. Humberstone, "Inaccessible worlds", *Notre Dame Journal of Formal Logic*, **24** (1983) 346–352.

[57] N. Immerman and D. Kozen, "Definability with bounded number of bound variables", in: [71], pp. 236–244.

[58] B. Jónsson, "Varieties of relation algebras", *Algebra Universalis*, **15** (1982) 273–298.

[59] B. Jónsson, "The theory of binary relations", in: [7].

[60] B. Jónsson and Alfred Tarski, "Boolean algebras with operators", *American Journal of Mathematics*, **73** (1951) 891–939, **74** (1952) 127–162.

[61] J.A.W. Kamp, *Tense Logic and the Theory of Linear Order*, PhD Dissertation, University of California, Los Angeles, 1968.

[62] Kamp, J.A.W., 'Formal properties of 'now' ", *Theoria*, **37** (1971), 227–273.

[63] S. Kanger (ed.), *Proc. of the Third Scandinavian Logic Symposium Uppsala 1973*, North-Holland, Amsterdam, 1975.

[64] A. Kasher (ed.), *Language in Focus*, Reidel, Dordrecht, 1976.

[65] R. Koymans, *Specifying Message Passing and Time-Critical Systems with Temporal Logic*, PhD Dissertation, Eindhoven University of Technology, 1989.

[66] M. Kracht, "How completeness and correspondence theory got married", in [105], 161–185.

[67] S. Kuhn, "Quantifiers as modal operators", *Studia Logica*, **39** (1980) 145–158.

[68] S. Kuhn, "The domino relation: flattening a two-dimensional logic", *Journal of Philosophical Logic*, **18** (1989) 173–195.

[69] P. Ladkin, *The Logic of Time Representation*, PhD Dissertation, Kestrel Institute, Palo Alto, 1987.

[70] LICS86, *Proceedings of the Symposium on Logic in Computer Science 1986, Cambridge, Massachusettes*, Computer Society Press, Washington, 1986.

[71] LICS87, *Proceedings of the Symposium on Logic in Computer Science 1987, Ithaka, New York*, Computer Society Press, Washington, 1987.

[72] Logic Colloquium '91, *Abstracts of the 9th Internatinal Congress of Logic, Methodology and Philosophy of Sciences, Uppsala, 1991*, Volume I: Logic.

[73] R. C. Lyndon, "The representation of relational algebras", *Annals of mathematics*, **51** (1950) 707–729, **63** (1956) 294–307.

[74] R. MacKenzie, *A General Method for Constructing Elementary Axioms for Classes of Representable Structures*, manuscript, 196?.

[75] R. Maddux, "Some sufficient conditions for the representability of relation algebras", *Algebra Universalis* **8** (1978) 162–172.

[76] R. Maddux, "Some varieties containing relation algebras", *Transactions of the American Mathematical Society*, **272** (1982) 501–526.

[77] R. Maddux, "A sequent calculus for relation algebras", *Annals of Pure and Applied Logic*, **25** (1983) 73–101.

[78] R. Maddux, "Finitary algebraic logic", *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, **35** (1989) 321–332.

[79] R. Maddux, "Nonfinite axiomatizability results for cylindric and relation algebras", *Journal of Symbolic Logic*, **54** (1989) 951–974.

[80] R. Maddux, *The Origin of Relation Algebras in the Development and Axiomatization of the Calculus of Relations*, manuscript, Iowa State University, Ames, 1990.

[81] J.D. Monk, "On representable relation algebras", *Michigan Mathematical Journal*, **11** (1964) 207–210.

[82] *Proceedings Mal′cev Conference on Algebra*, Institute of Mathematics, Siberian Academy of Sciences, Novosibirsk, 199?

[83] J.D. Monk, "Nonfinitizability of classes of representable cylindric algebras", *Journal of Symbolic Logic*, **34** (1969) 331–343.

[84] J.D. Monk, "Provability with finitely many variables", *Proceedings of the American Mathematical Society*, **27** (1971), 353–358.

[85] U. Mönnich (ed.), *Aspects of Philosophical Logic*, Reidel, Dordrecht, 1981.

[86] B. Moszkowski and Z. Manna, "Reasoning in interval temporal logic", in: [25], 371–384.

[87] B. Moszkowski, *Executing Temporal Logic Programs*, Cambridge University Press, Cambridge, 1986.

[88] I. Németi, "On varieties of cylindric algebras with applications to logic", *Annals of Pure and Applied Logic*, **36** (1987), 235–277.

[89] I. Németi, "Algebraizations of quantifier logics: an introductory overview", *Studia Logica*, to appear.

[90] I. Németi, "On cylindric algebraic model theory", in: [17], pp. 37–76.

[91] H. Nishimura, "Interval logics with applications to study of tense and aspect in english", *Publications of the Research Institute for Mathematical Sciences, Kyoto University*, **16** (1980) 417–459.

[92] E. Orłowska, "Relational interpretation of modal logics", in: [7], 443–471.

[93] E. Orłowska, *Modal Relation Algebras*, manuscript, Polish Academy of Sciences, Warsaw, 1991.

[94] R. Parikh (ed.), *Proceedings of the Third Conference on Theoretical Aspects of Reasoning about Knowledge*, Morgan Kauffman, San Mateo, 1990.

[95] S. Passy and T. Tinchev, "PDL with Data Constants", *Information Processing Letters*, **20** (1985) 35–41.

[96] T. Pauli, (ed.), *Philosophical Essays Dedicated to Lennart Åqvist on his Fiftieth Birthday*, 1982.

[97] P. Petkov (ed.), *Mathematical Logic; Proceedings of the Summer School and Conference "Heyting '88"*, Plenum Press, New York and London, 1990.

[98] D. Pigozzi, "Amalgamation, congruence-extension, and interpolation properties in algebras', *Algebra Universalis*, **1** (1972) 269–349.

[99] A. Prijatelj, "Existence Theorems for Models of Dynamic Intensional Logic", *University of Ljubljana, Department of Mathematics, Preprint Series*, **28** (1990), 318.

[100] A. Prior, *Past, Present and Future*, Oxford Clarendon Press, Oxford, 1967.

[101] M. Reynolds, *Axiomatizing Intervals*, manuscript, Department of Computing, Imperial College, London, 1990.

[102] M. Reynolds, *Complete Axiomatization for Until and Since over the Reals — IRR Rule Not Needed*, manuscript, Department of Computing, Imperial College, London, 1991.

[103] B. Richards et alii, *Temporal Representation and Inference*, Academic Press, London, 1989.

[104] M. de Rijke, *The Modal Logic of Inequality*, ITLI-prepublication series LP-90-15, University of Amsterdam, to appear in *Journal of Symbolic Logic.*

[105] M. de Rijke (ed.), *Colloquium on Modal Logic 1991*, Dutch Network for Language, Logic and Information, Amsterdam, 1991.

[106] W.P. de Roever, *Recursive Program Schemes: Semantics and Proof Theory*, PhD Dissertation, Mathematisch Centrum, Amsterdam, 1974.

[107] D. Roorda, *Resource Logics: Proof-theoretical Investigations*, PhD Dissertation, University of Amsterdam, 1991.

[108] P. Röper, "Intervals and tenses", *Journal of Philosophical Logic*, **9** (1980) 451–469.

[109] R. Rosner and A. Pnueli, "A choppy logic", in: [70], pp. 306–313.

[110] V. Rybakov, "Metatheories of the first-order theories", *Abstracts of International Congress of Metamathematics, Kyoto, Japan (August 21–29, 1990)*, p. 2. **8** (1979) 1–45.

[111] H. Sahlqvist, "Completeness and correspondence in the first and second order semantics for modal logic", in: [63], pp. 110–143.

[112] I. Sain, "Finitary logics of infinitary structures are compact", *Abstracts of the American Mathematical Society*, Issue **17/3/3**, *82T–03–197 (April 1982), p. 252.

[113] I. Sain, *Successor Axioms Increase the Program Verification Powers of the "Sometime + Nexttime" Logic*, Preprint No. **23** (1983), Mathematical Institute of the Hungarian Academy of Sciences, Budapest.

[114] I. Sain, *Searching for a Finitizable Algebraization of First Order Logic,* Preprint, Mathematical Institute of the Hungarian Academy of Sciences, Budapest, 1987.

[115] I. Sain, "Beth's and Craig's properties via epimorphisms and amalgamation in algebraic logic", in: [17], 209–226.

[116] I. Sain, "Is "Some-Other-Time" sometimes better than "Sometime" for proving partial correctness of programs?" *Studia Logica*, **47** (1988) 279–301.

[117] I. Sain, "Temporal logics need their clocks", *Theoretical Computer Science*, 1991, to appear.

[118] G. Sambin and V. Vaccaro, "A new proof of Sahlqvist's theorem on modal definability and completeness", *Journal of Symbolic Logic*, **54** (1989) 992–999.

[119] K. Segerberg, "Two-dimensional modal logic", *Journal of Philosophical Logic*, **9** (1980) 77-96.

[120] V.B. Shekhtman, "Two-dimensional modal logic", *Mathematical Notices of the USSR Academy of Sciences*, **23** (1978) 417–424 (translated from the Russian).

[121] A. Simon, *A Complete Calculus for Type-free Logic and Representable Cylindric Algebras*, manuscript, Hungarian Academy of Sciences, Budapest, 1990.

[122] A. Simon, "Finite schema completeness for typeless logic and representable cylindric algebras", in: [7], 665–671.

[123] D. Skordev (ed.), *Mathematical Logic and its Applications*, Plenum Press, New York, 1987.

[124] E. Spaan, "Nexttime is not necessary", in: [94], 241–256.

[125] C. Stirling, *Modal logic and temporal logic*, in: [1].

[126] A. Tarski, "On the calculus of relations", *Journal of Symbolic Logic*, **6** (1941) 73–89.

[127] A. Tarski and S. Givant, *A Formalization of Set theory Without Variables*, AMS Colloquium Publications **41**, 1987.

[128] R.H. Thomason, "Combinations of tense and modality", in: [34], 135–165.

[129] S.K. Thomason, "On constructing instants from events", *Journal of Philosophical Logic*, **13** (1984) 85–96.

[130] Y. Venema, "Expressivenes and completeness of an interval tense logic", *Notre Dame Journal of Formal Logic*, **31** (1990) 529–547.

[131] Y. Venema, *Two-dimensional Modal Logics for Relation Algebras and Temporal Logic of Intervals*, ITLI-prepublication series LP-89-03, University of Amsterdam, Amsterdam, 1989

[132] Y. Venema, *Cylindric Modal Logic*, ITLI-prepublication series ML–91–01, University of Amsterdam, Amsterdam, 1991

[133] Y. Venema, "A modal logic for chopping intervals", *Journal of Logic and Computation*, to appear.

[134] Y. Venema, "Completeness by completeness: Since and Until", in: [105], 279–285.

[135] Y. Venema, "The tense logic of dominoes", *Journal of Philosophical Logic*, to appear.

[136] Y. Venema, "Relational games", in: [7], 695–719.

[137] W. W. Wadge, *A complete natural deduction sustem for the relation calculus*, University of Warwick, Theory of Computation Report No. 5, 1975.

[138] M. J. White, "An 'almost classical' period-based tense logic", *Notre Dame Journal of Formal Logic*, **29** (1988) 438–453.

[139] A. Zanardo, "A finite axiomatization of the set of strongly valid Ockhamist formulas", *Journal of Philosophical Logic*, **14** (1985), 447–468.

[140] A. Zanardo, "A complete deductive system for Since-Until branching-time logic", *Journal of Philosophical Logic*, **20** (1991), 131–148.

[141] A. Zanardo, "Axiomatization of 'Peircean' branching-time logic", *Studia Logica*, to appear.

[142] L. Åqvist, "A conjectured axiomatization of two-dimensional Reichenbachian tense logic", *Journal of Philosophical Logic* 8 (1979) 1–45.

DANKWOORD / ACKNOWLEDGEMENTS

Dit proefschrift is mijn verslag van vier jaar wiskundig onderzoek. Velen hebben me in deze tijd op de een of andere wijze geholpen, inhoudelijk of persoonlijk, structureel danwel éénmalig. De belangrijkste van hen wil ik hier graag mijn oprechte dank betuigen.

Om te beginnen is een goede promotor het halve proefschrift. Ik wil Johan van Benthem dank zeggen voor zijn inspirerende begeleiding, een voor mij perfecte combinatie van het overvloedig aandragen van ideeën en het afstand houden van de uitvoering daarvan. Met name ook ben ik zeer erkentelijk voor het grote vertrouwen dat hij in mij stelde toen ik een tijd lang geen logicus wilde zijn.

Second, this dissertation is a tale of three cities — it ended in a marriage between three partners: (Sahlqvist) correspondence theory from Amsterdam, algebraic logic from Budapest and modal derivation rules from London.

Deeply indebted I am to Andréka Hajnal, Németi István and Sain Ildikó; this dissertation would not have been written without scientific and moral support from Budapest. I thank them for inspiring discussions and long letters on the modal side of algebraic logic.

The results of the first chapter I obtained while visiting London, on Erasmus grant ICP–90–NL–0211. I would like to thank Dov Gabbay for his hospitality and for many stimulating discussions on non-ξ rules and many-dimensional temporal logics.

Many other people have contributed to this dissertation, of whom I thank: Maarten de Rijke, voor het samen werken in modale logica en het kritisch doorlezen van het manuscript, Ian Hodkinson, Robin Hirsch and Mark Reynolds and other people from Imperial College for forming such a stimulating environment for temporal logic research, Yao-Hua Tan en Rineke Verbrugge omdat ze zulke lankmoedige en plezierige kamergenoten waren, and, for various reasons: Patrick Blackburn, Valentin Goranko, Joe Halpern, Steve Kuhn, Peter Ladkin, Roger Maddux, Szabolcs Mikulás, Andreja Prijatelj, Dirk Roorda, Yoav Shoham, András Simon, Dmitri Vakarelov, Alberto Zanardo en de leden van onze vakgroep Logica en Theoretische Informatica.

Ten slotte, het werken aan een proefschrift vereist nogal wat emotionele veerkracht bij de jonge onderzoeker. Zonder steun van familie, vrienden en geliefde had ik de eindstreep niet gehaald. In het bijzonder dank ik mijn vrienden Ditteke, Jacques, Marianne en Marjan omdat ze er altijd voor me waren, mijn huisgenoten Heleen, Paulien en Zilla voor de nestwarmte van Soembawa, and Patxi for the sunshine in my life.

Myn âlden ha my yn alles stipe, ek al wie myn kar net altyd harres. Ik draach dit proefskrift op oan E & T.