INVESTIGATIONS ON THE INTUITIONISTIC PROPOSITIONAL CALCULUS

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A thesis submitted to the Graduate School of the University of Wisconsin in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

by
Dick Herman Jacobus de Jongh

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To Professors: KLEENE
ROBBIN
ACZEL

This thesis having been approved in respect to form and mechanical execution is referred to you for judgment upon its substantial merit.

Robert M. Bock
Dean

Approved as satisfying in substance the doctoral thesis requirement of the University of Wisconsin.

Stephen C. Kleene
Major Professor
Joel W. Robbin
Peter Aczel

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DICK HERMAN JACOBUS de JONGH

A thesis submitted in partial fulfillment of the
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I am indebted to E.W. Beth, who introduced me to logic as a science, and who first established the problems with which this thesis is concerned.

I am also indebted to A. Heyting, under whose tutelage I learned the principles of intuitionism.

Finally, I owe much to the patient endeavor of S.C. Kleene, who proposed one of the major problems of this thesis, and with whose help this thesis reached maturity.

Dedication

to the memory of my teachers

D.K. de Jongh and E.W. Beth

Chapter I.

Introduction

The first investigations in the field of semantics of intuitionistic logic were by Beth with the aid of his semantic tableaux [2],[ ]. His completeness results for the predicate calculus were improved by Dyson and Kreisel [7]. Later Kripke [14] built on this work, and with slightly changed semantic tableaux and a different interpretation reached more easily completeness results closely related to the earlier ones. Independently Beth in his last publications [3],[ ] and de Jongh [8] worked on these methods. Later Aczel [1] gave a Henkin-type completeness proof built on the same principles.

We will start out with a description of a completeness theorem for the propositional calculus which is slightly different from but obviously equivalent to Kripke's.

In our propositional calculus we will study formulas $U,V,W,\ldots,U_1,U_2,\ldots,V_1,V_2,\ldots,W_1,W_2,\ldots$, built from atomic formulas $A_1,A_2,\ldots$, by means of the connectives $\&, v, \supset$ and $\neg$. For a system of axiom schemas for the intuitionistic propositional calculus Pp, and for the classical propositional calculus PC, see e.g. [11].

Def. A *P.O.G.-set* is a partially ordered set with one maximal element (the greatest element).

We will write F for the set of all formulas.

Def. An *I-valuation* is a quadruple $<P,\leq,p_0,w>$, where P is a P.O.G.-set with partial ordering $\leq$ and maximal element p_0, and w is a function with domain $P\times F$ and range the set $\{0,1\}$ such that for all $p\varepsilon P$:

(i) For an atomic formula A_1,$w(p,A_1)=1$ iff, for all $p'\leq p$, $w(p',A_1)=1$.

(ii) $w(p,U\&V)=1$ iff $w(p,U)=1$ and $w(p,V)=1$.

(iii) $w(p,U\vee V)=1$ iff $w(p,U)=1$ or $w(p,V)=1$.

(iv) $w(p,U\supset V)=1$ iff, for all $p'\leq p$, $w(p',U)=0$ or $w(p',V)=1$.

(v) $w(p,\neg U)=1$ iff, for all $p'\leq p$, $w(p',U)=0$.

Remark. (i) is assumed only for atomic formulas, but can be proven thence for all formulas by using (ii)-(v).

The following completeness theorem [14] can be proven by using semantic tableaux or from the theory of pseudo-Boolean algebras (see Chapter IV).

Th.1.1. For all formulas U, U is a theorem of Pp (1) iff, for all I-valuations for all P.O.G.-sets$<P,\leq,p_0>$$w(p_0,U)=1$, or (2) iff, for all I-valuations on finite P.O.G.-sets, $w(p_0,U)=1$.

In [14] Kripke also describes an intuitive interpretation of I-valuations. The P.O.G.-set is taken as a set of possible situations (stages), where the partial ordering plays the role of time, i.e. if $q \leq p$, then q is supposed to be a possible future situation as seen from p. Time is seen as discreet, one can move from an earlier stage to any possible later one, but one can stay at any stage for an unlimited amount of time. The stages can be represented by certain sets of axioms, certain methods of derivation or in a special application of ours in Chapter VI, by the computability of certain functions.

We can extend the concept of I-valuation to the predicate calculus. There an I-valuation is a sextuple $\langle P,\leq,p_0,d,D,w\rangle$, where $P,\leq,p_0$ and w have the same meaning as before, D is a non-empty set (interpreted as a set of individuals) and d is a function from P into $\mathcal{P}(D)$ (the power set of D) such that, if $p' \leq p$, the $d(p) \subseteq d(p')$. The domain of w is now the set of couples $\langle p,U\rangle$, where U is a formula without free variables built from atoms A_i, $A_j(\ )$, $A_k(\ ,\ )$, ... $(i,j,k=1,\ldots\infty)$, individual constants from d(p), variables and quantifiers. And w has to fulfill the additional properties:

(vi) $w(p,\forall xU(x))=1$ iff for all $p' \leq p$ and $u \in d(p')$ $w(p',U(u))=1$.

(vii) $w(p,\exists xU(x))=1$ iff for some $u \in d(p)$ $w(p,U(u))=1$.

Chapter II.

Connectives and Operators.

In this chapter we will try to develop a general concept of connective for Pp, with respect to the semantics described in Chapter I. Our work-model is of course, PC. There we have the following well-known situation.

A valuation for a set of formulas of PC is a function from this set into the set $\{0,1\}$. Then for any connective a for PC with respect to these semantics we want to have a procedure that enables us to extend a valuation for n formulas $U_1,..,U_n$ to a valuation for the formulas $U_1,....,U_n,a(U_1,...,U_n)$. The solution here is to represent any n-ary connective a by a function (n-ary I-operator) from $\{0,1\}^n$ into $\{0,1\}$. Vice-versa we can for any such operator introduce a connective that it represents. It can then be proven that all the connectives produced in this way can be defined in a natural way from the standard connectives $\&, v, \supset$ and $\neg$, i.e., for any n-ary operator a there is a formula $U(A_1,...,A_n)$, containing as connectives only $\&, v, \supset$ and $\neg$, such that for all formulas V containg as connectives $\&, v, \supset, \neg$ and a, V and V* have the same valuation for any valuation of their atoms, where V* is obtained from V by replacing all well-formed parts of V of the form $a(W_1,\ldots,W_n)$ by $U(W_1,\ldots,W_n)$.

To be able to proceed in a similar way for Pp, we have to be able to talk about "P.O.G.-sets of n-tuples of 0's and 1's". Since there are no easy unique representations for P.O.G.-sets as there are for totally ordered sets ($\{1,\ldots,n\}$), we go about this in the following way.

We take a countably infinite A, and we define B as the set of all finite P.O.G.-sets with elements in A. Now we define:

Def. An *I-function* f is a function with domain a P.O.G.-set $P\varepsilon B$ and range the set $\{0,1\}$ with the property: for all $p',p\varepsilon P$, if $p'\leq p$ and $f(p)=1$, then $f(p')=1$.

Def. An *I^n-function* f is a function with domain a P.O.G.-set $P\varepsilon B$ and range the set $\{0,1\}^n$ with the property that the function f^m defined on P by $f(p)=(f(p))(m)$ (the m-th element of the sequence f(p)) is an I-function for all m ($1\leq m\leq n$).

So for any I^n-function f there is an n-tuple of I-functions $(f^1,\ldots,f^n)$ with the same domain, and vice versa; sometimes we will write $(f^1,\ldots,f^n)$ for f. We will write D_f for the domain of f, m_f for the maximum element of D_f. For the partial ordering of D_f we will often write $\leq_f$; if it is obvious which I^n-function is meant we will just write $\leq$. We write F^n for the set of all I^n-functions, F for F^1.

We restrict ourselves to finite P.O.G.-sets, since all the important properties of the standard connectives can be described with finite P.O.G.-sets, and there is no problem in using intuitionistic methods. We will use these finite P.O.G.-sets in the Chapters II,III and the first part of Chapter V; in the last part of V we will study I^n-functions with infinite domains, but there we will not be able to use intuitionistic methods. When in the Chapters II,III and the first part of V we use reasonings that are based on the law of the excluded middle, the properties in question are always decidable.

We do not want the difference between two isomorphic P.O.G.-sets to play a role in the theory; for that reason we define an equivalenc relation on F^n.

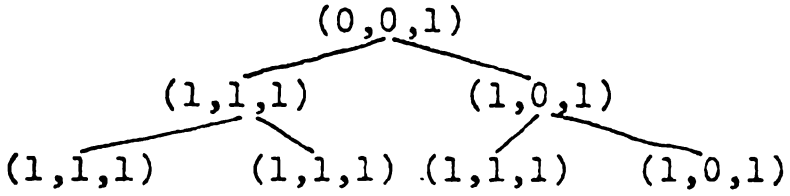
Def. Two I^n-functions f and g are *congruent by* ϕ iff ϕ is an isomorphism from D_f onto D_g such that $f(p)=g(\phi(p))$ for all $p \varepsilon D_f$.

Def. f is *congruent to* g (in symbols $f \equiv g$) iff f is congruent to g by ϕ for some ϕ.

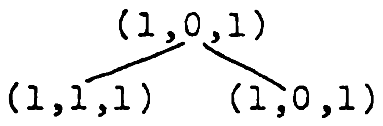
It is obvious that $\equiv$ is an equivalence relation. Further it is clear that for any other countably infinite set A we would get an exactly similar set of congruence classes.

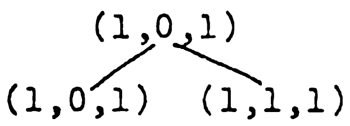
To each I-valuation on a sequence of atomic formulas $A_1,\ldots,A_n$ there corresponds naturally a congruence class of I^n-functions. Also, if we take the P.O.G.-sets for the I-valuations from B, then there is a 1-1 correspondence between the I-valuations for $A_1,\ldots,A_n$ and the I^n-functions.

To illustrate the following discussions we will use pictures of I^n-functions. The following is an example of a picture of an I^3-function:



(Similar pictures are common in lattice-theory, see e.g. [4].) Note that such pictures represent I^n-functions only up to congruence, and moreover that two different pictures can represent congruent I^n-functions, e.g.:





Now we are ready to define n-ary operators in such a way that, if we have an I-valuation for formulas $U_1,\ldots,U_n$, we can extend it to an I-valuation for $U_1,\ldots,U_n$, *a*$(U_1,\ldots,U_n)$, if *a* is the connective that represents the operator a, keeping in mind that we want the result to be independent of which particular P.O.G.-set we choose from a class of isomorphic ones for the I-valuation.

Def. An n-ary *I-operator* a is a function from F^n into F with the properties:

(i) $D_{a(f)}=D_f$ for all $f\varepsilon F^n$,

(ii) If $f\equiv g$ by ϕ, then $a(f)\equiv a(g)$ by ϕ.

Here the similarity with the case for PC changes, since the set of I-operators does not even come close to the set of I-operators that represent connectives definable from the standard connectives $\&, v, \supset$ and $\neg$. We now look to our intuitive interpretation for help in restricting this class of I-operators. To begin with we will want the valuation of *a*$(U_1,\ldots,U_n)$ on $p\varepsilon P$ to be dependent only on the valuations of $U_1,\ldots,U_n$ for $p'\leq p$ in P, since a connective should in one way or other be a restriction on the possible future valuations of $U_1,\ldots,U_n$. To describe this property formally we will need some more definitions.

If $P\varepsilon B$ and $p\varepsilon P$, we write $P(p)$ for the set $\{p'\varepsilon P:p'\leq p\}$, $P[p]$ for the set $\{p'\varepsilon P:p'<p\}$. If $f\varepsilon F^n$ and $p\varepsilon D_f$, we write f_p for the restriction of f to $D_f(p)$. f_p is obviously again an I^n-function.

Obvious properties are:

$D_{f_p}=D_f(p)$. $f_{m_f}=f$. $(f_p)_p=f_p$. For all $p'\leqq p$, $(f_p)_{p'}=f_{p'}$. If $g\equiv f_p$ for some $p\in D_f$, then we call g a *sub-I^n-function* of f, and we write $g\leq f$. It is obvious that $\leq$ is transitive and reflexive (is a pseudo-ordering), and that $f\equiv g$ implies $f\leq g$, as well as that $f\leq g$ and $g\leq f$ together imply $f\equiv g$.

Examples.

If

$$f=\begin{tikzcd} & (1,0,1) \arrow[dl, dash] \arrow[dr, dash] & \\ (1,1,1) \arrow[dr, dash] & & (1,0,1) \arrow[dl, dash] \\ & (1,1,1) & \end{tikzcd},\quad g=\begin{tikzcd} (1,0,1) \arrow[d, dash] \\ (1,1,1) \end{tikzcd}\quad \text{with } D_f=\begin{tikzcd} & p_1 \arrow[dl, dash] \arrow[dr, dash] & \\ p_2 \arrow[dr, dash] & & p_3 \arrow[dl, dash] \\ & p_4 & \end{tikzcd}$$

and

$$D_g=\begin{tikzcd} p_3 \arrow[d, dash] \\ p_4 \end{tikzcd},$$

then $g=f_{p_3}$. If

$$h=\begin{tikzcd} (1,0,1) \arrow[d, dash] \\ (1,1,1) \end{tikzcd}\quad \text{with } D_h=\begin{tikzcd} p_5 \arrow[d, dash] \\ p_6 \end{tikzcd},$$

then $h\equiv g$ and so $h\leq f$.

We are now ready to define a smaller class of I-operators

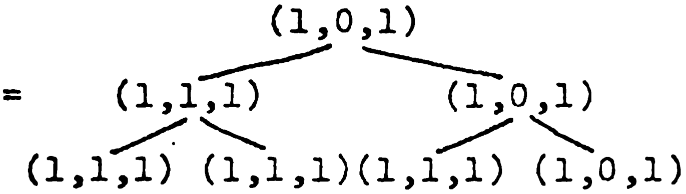
Def. An *ordered* I-operator is an n-ary I-operator with the property:

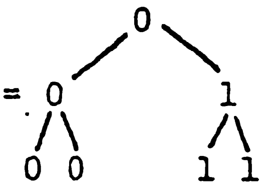
(iii) for all $f\in F^n$, if $p\in D_f$, then $(a(f))(p)=(a(f_p))(p)$.

For ordered I-operators we can derive the following stronger statements.

If $p'\leqq_f p$, then $(a(f_p))(p')=(a((f_p)_{p'}))(p')=(a(f_{p'}))(p')=(a(f))(p')$. This implies $a(f_p)=(a(f))_p$; and, if $q\in D_g$, and $f_p\equiv g_q$ by ϕ, then $(a(f))_p\equiv(a(g))_q$ by ϕ; and at last as a particular case of this, if $g \leq f$, then for some $p \varepsilon D_f$, $a(g) \equiv (a(f))_p$.

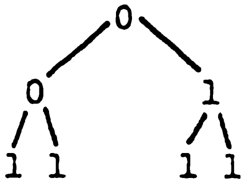
As an example, if b is an I-operator that maps

j= 

on j'= 

,

then b is certainly not an ordered I-operator, since the congruence class of I^3-functions represented by the picture (1,1,1) has three of its elements as sub-I^3-functions of j, which by an ordered I-operator should be mapped onto three congruent I-functions, and such is not the case here. If the I-operator c maps j onto j"=



, then c could be an ordered I-operator. The class of ordered I-operators still comes out to be too extensive. But before we restrict this class even more we will discuss a very important property of ordered I-operators.

<u>Def</u>. The *characteristic set* C_a of an ordered I-operator a is the set of all $f \varepsilon F^n$ such that $(a(f))(m_f)=1$.

<u>Th.2.1</u>. A subset G of F^n is the characteristic set of some ordered I-operator a, which is then unique,

iff G has the property:

(*) for all $f,g\varepsilon F^n$, if $f\varepsilon G$ and $g\leq f$, then $g\varepsilon G$.

Proof. $\Longrightarrow$ If $f\varepsilon C_a$, then $(a(f))(m_f)=1$, so for all $p\varepsilon D_f$ $(a(f))(p)=1$. Now, if $g\leq f$, then according to the properties of ordered I-operators $(a(g))(m_g)=(a(f)(p)$ for some $p\varepsilon D_f$, so $(a(g))(m_g)=1$ and $g\varepsilon C_a$.

The uniqueness part of the theorem is obvious, since, if a is an ordered I-operator, $f\varepsilon F^n$ and $p\varepsilon D_f$, then $(a(f))(p)=(a(f_p))(m_{f_p})$, which is determined by the fact whether $f_p\varepsilon C_a$ or $f_p\notin C_a$.

$\Longleftarrow$ Suppose G has the property (*). Then define the I-operator a in the following way: for any $f\varepsilon F^n$ and $p\varepsilon D_f$, $(a(f))(p)=1$ iff $f_p\varepsilon G$ (of course take $D_{a(f)}=D_f$). To prove that a is indeed an ordered I-operator with $C_a=G$, we have to show:

(i) For all $f\varepsilon F^n$ and $p\varepsilon D_f$, if $(a(f))(p)=1$ and $p'\leqq_f p$, then $(a(f))(p')=1$.

(ii) For all f, $g\varepsilon F^n$, if $f\equiv g$ by ϕ, then $a(f)\equiv a(g)$ by ϕ.

(iii) For all $p\varepsilon D_f$, $(a(f))(p)=(a(f_p)(p)$.

(iv) For all $f\varepsilon F^n$, $f\varepsilon G$ iff $(a(f))(m_f)=1$.

Proofs:

(i) If $(a(f)(p)=1$, then, by the way we defined a, $f_p\varepsilon G$. If now $p'\leqq_f p$, then $f_{p'}\leq f_p$, so by (*) $f_{p'}\varepsilon G$, and $(a(f))(p')=1$.

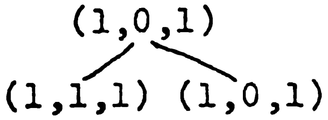
(ii) If $f\equiv g$ by ϕ, then, for all $p\varepsilon D_f$, $f_p\equiv g_{\phi(p)}$ by the restriction of ϕ to $D_f(p)$, and so $f_p\varepsilon G$ iff $g_{\phi(p)}\varepsilon G$.

This in its turn implies $(a(f))(p)=(a(g))(\phi(p))$ for all $p \varepsilon D_f$, so that indeed $a(f) \equiv a(g)$ by ϕ.

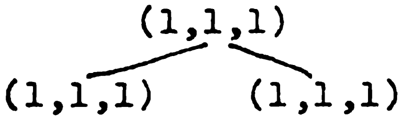
(iii) $(a(f_p))(p)=1$ iff $(f_p)_p \varepsilon G$, and $(a(f))(p)=1$ iff $f_p \varepsilon G$. But $(f_p)_p=f_p$. So $(a(f_p))(p)=(a(f))(p)$.

(iv) $(a(f))(m_f)=1$ iff $f_{m_f} \varepsilon G$. but $f_{m_f}=f$. so $f \varepsilon G$ iff $(a(f))(m_f)=1$.

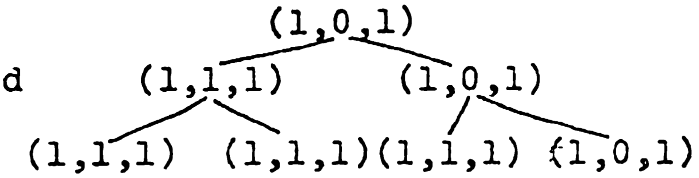
In our last example, if c is an ordered I-operator, then (1,0,1), (1,1,1) and



and εC_c



and



$\not\varepsilon C_c$.

Next come some definitions needed to restrict the class of ordered I-operators even more.

<u>Def</u>. The partially ordered set $<Q, \leqq_Q>$ is an α-*reduction* of the partially ordered set $<P, \leqq>$ w.r. (with respect) to r, r' iff $r, r' \varepsilon P$, $P[r']=P(r)$ and $<Q \leqq_Q> = <P-\{r'\}, \leqq>$.

It is obvious that $\leqq_Q$ has to be a partial ordering again, since the restriction of any partial ordering is again a partial ordering.

<u>Def</u>. The partially ordered set $<Q, \leqq_Q>$ is a β-*reduction* w.r. to r, r' of the partially ordered set $<P, \leqq>$ iff $r, r' \varepsilon P$, $r \neq r'$, $P[r]=P[r']$, $Q=P-\{r'\}$, and, for all $p, p' \varepsilon Q$, $p' \leqq_Q p$ iff $(p' \leqq p$ or $(p'=r$ and $r' \leqq p))$.

$\leq_Q$ will always be a partial ordering, if r and r' fulfill the required properties in P. The reflexive and symmetric properties are immediately clear. For the transitive property, assume that $p'' \leq_Q p'$ and $p' \leq_Q p$. Then there are four possibilities:

(1) $p'' \leq p'$ and $p' \leq p$. Then $p'' \leq p$ and so $p'' \leq_Q p$.

(2) $p'' \leq p'$ and $p' = r$ and $r' \leq p$. Then either $p'' = p'$, so $p'' = r$ and $r' \leq p$, so $p'' \leq_Q p$; or $p'' < p' = r$, so (by $P[r] = P[r']$) $p'' < r'$ and $p'' \leq_Q p$.

(3) $p'' = r$ and $r' \not\leq p'$; and $p' \leq p$. Then $r' \leq p$, so $p'' \leq_Q p$.

(4) $p'' = r$ and $r' \leq p'$, $p' = r$ and $r' \leq p$. This is impossible since $r' \not\leq r$.

Def. The partially ordered set Q is a *reduction* (w.r. to r,r') of the partially ordered set P iff Q is an α- or β-reduction of P (w.r. to r,r').

Def. If $f, g \in F^n$, then g is a reduction (α-reduction, β-reduction) of f w.r. to r,r' iff D_g is a reduction (α-reduction, β-reduction) of D_f w.r. to r,r', $f(r) = f(r')$ and for all $p \in D_g$, $f(p) = g(p)$.

Intuitively, what we do in a reduction of an I^n-function is to identify two elements of its domain. Our intuitive interpretation sees in a β-reduction two points with the same valuation under f, all possible future states being the same for the two points.

Obviously there are intuitively no qualms about identifying two such points. In an α-reduction one of the two points is a future state as seen from the other, but, if we move from the "earlier" point to the "later", we do not change essentially, since we keep the same valuation under f, and we do not loose any future possibilities, since the "earlier" point has only one immediate predecessor. So, intuitively we can just as well identify these two points (or leave the "earlier" one of them out). We now want to construct a class of operators, the members of which do not differentiate in their treatment of an I^n-function and its reductions. For this purpose we will need some more definitions and a theorem.

Def. An I^n-function is *irreducible* iff, there is no I^n-function g that is a reduction of f.

Def. An I^n-function g is a *normal form* of the I^n-function f iff, g is irreducible and there is a sequence of I^n-functions $f_0,\dots,f_k$ $(k\geq 1)$ such that $f_0=f$, $f_k=g$, and for all i $(2\leq i\leq k)$ f_i is a reduction of f_{i-1}.

We now want to prove that the normal form of an I^n-function is unique up to congruence, and that congruent I^n-functions have congruent normal forms. This will be easier, when we use the strongly isotone functions studied in [9].

Def. A function ϕ from $\langle P,\leq_1\rangle$ onto $\langle Q,\leq_2\rangle$ is *strongly isotone* iff,

(i) for all $p', p \varepsilon P$, if $p' \leqq_1 p$, then $\phi(p') \leqq_2 \phi(p)$ (if fulfills (i), then we call ϕ *isotone*),

(ii) for all $p', p \varepsilon P$, if $\phi(p') \leqq_2 \phi(p)$, then for some $p'' \leqq_1 p$, $\phi(p'') = \phi(p')$.

Def. A function from a partially ordered set P onto a partially ordered set Q is a *reduction- (α-reduction-, β-reduction) function* iff, for some $r, r' \varepsilon P$, Q is a reduction (α-reduction, β-reduction) of P w.r. to r, r', $\phi(r') = r$, and for all $p \varepsilon Q$, $\phi(p) = p$.

The next theorem establishes a connection between strongly isotone functions and reductions. This theorem was implicit in [9]. It follows from Th.4.5 and Th.4.6 of that article almost immediately, and a proof can also be destilled from Th.4.7 of [9]. However, we will give a simple direct proof here.

Th.2.2. For any two finite partially ordered sets P and Q the following two statements are equivalent:

(1) there is a strongly isotone function from P onto Q,

(2) there is a sequence of partially ordered sets $P_1, \ldots, P_k$ $(k \geqq 1)$ such that $P = P_1$, $Q = P_k$ and for all i $(2 \leqq i \leqq k)$, P_{i-1} is isomorphic to, or a reduction of P_i.

Proof. (2)$\Rightarrow$(1). We have to prove, (a) any isomorphism is strongly isotone, (b) any reduction-function is strongly isotone, (c) a composition of strongly isotone functions is strongly isotone. The proof for a is trivial.

(b) Assume ϕ is a reduction-function from $\langle P,\leq_1\rangle$ onto $\langle Q,\leq_2\rangle$ (w.r. to r,r') and assume $p'\leq_1 p$. Then, if $p'\neq r'$ and $p\neq r'$, then $\phi(p')=p'$ and $\phi(p)=p$, so $\phi(p')\leq_2\phi(p)$. If $p=r'$, then $p'\leq_1 r'$, and either $p'=r'$, so $\phi(p')=\phi(p)$, or $p'\leq_1 r$, so $\phi(p')\leq_2\phi(r)=\phi(r')=\phi(p)$. If $p'=r'$, then $p=\phi(p)$ and $r'\leq_1 p$, so according to the definition of reduction $\phi(p')=\phi(r')=r\leq_2 p=\phi(p)$. To prove (ii), assume $\phi(p')\leq_2\phi(p)$. Then either $\phi(p')\leq_1 p$, or in a β-reduction $\phi(p')=r$ and $r'\leq_1 p$, so $\phi(r')=\phi(p')$, or in an α-reduction $p'=r'$ and $p=r$, so $\phi(p)=\phi(p')=r$.

(c) That (i) carries over in composition, is trivial. To show the same for (ii), assume ϕ from P onto Q and ψ from Q onto $\langle R,\leq_3\rangle$ are strongly isotone, and assume $\psi(\phi(q))\leq_3\psi(\phi(p))$. Then for some $r\varepsilon Q$, $r\leq_2\phi(p)$ and $\psi(r)=\psi(\phi(q))$. Now $r=\phi(s)$ for some $s\varepsilon P$, so for some $q'\leq_1 p$, $\psi(\phi(q'))=\psi(r)=\psi(\phi(q))$.

(1)$\Longrightarrow$(2). Let us write P_ϕ for the set of elements $p\varepsilon P$ such that $\phi(p)=\phi(q)$ for some $q\neq p$. First we will prove that ,if $P_\phi=\emptyset$, then ϕ is an isomorphism from P onto Q. In that case, if $q\leq_1 p$, then obviously $\phi(q)\leq_2\phi(p)$; if $\phi(q)\leq_2\phi(p)$, then for some $r\leq_1 p$, $\phi(q)=\phi(r)$, so $q=r$ and $q\leq_1 p$, since P_ϕ is empty. Now assume P_ϕ contains at least two elements. We will prove that in that case there is a partially ordered set R, a reduction-function ψ from P onto R, and a strongly isotone function χ from R onto Q such that $\chi\psi=\phi$. Assume p is a minimal element of P_ϕ, and assume $\phi(q)=\phi(p)$, $r\leq_1 q$, and $r\in P_\phi$. Then $\phi(r)\leq_2\phi(q)=\phi(p)$. So, for some $s\leq_1 p$, $\phi(s)=\phi(r)$. Since p is minimal in P_ϕ, s=p and $\phi(r)=\phi(p)$. This implies that the set $S=\{q\in P\colon \phi(q)=\phi(p)\}$ is M-closed in P_ϕ. There are now two possibilities. I. S has a minimal element $q\neq p$. Assume $r<p$. Then $\phi(r)\leq_2\phi(p)=\phi(q)$. So, for some $s\leq_1 q$, $\phi(r)=\phi(s)$. Since p is minimal in P_ϕ, s=r. So $P[p]\subseteq P[q]$. By symmetric considerations $P[q]\subseteq P[q]$. So $P[p]=P[q]$. Then take for $\langle R,\leq_3\rangle$ the β-reduction w.r. to p,q. II. p is the only minimal element of S. Obviously there has to be an element $q\in S$ that is an immediate successor of p. Assume $r\leq q$. Then $\phi(r)\leq_2\phi(q)=\phi(p)$. So for some $s\leq_1 p$, $\phi(r)=\phi(s)$. This means that either r=p, or r=s and so $r\leq_1 p$. So we have proved that $P[q]=P(p)$. Then take for $\langle R,\leq_3\rangle$ the α-reduction of P w.r. to p,q. In both cases we define ψ as the reduction-function from P onto R. Then we define χ by, for all $r\in R$, $\chi(r)=\phi(r)$. This function is clearly onto. Now assume $s\leq_3 r$. Then either $s\leq_1 r$, so $\chi(s)\leq_2\chi(r)$, or (if R is an α-reduction of P) s=q and r=p, so $\chi(s)=\chi(q)=\chi(p)=\chi(r)$, or (if R is a β-reduction of P) s=p and $q\leq_1 r$, so $\chi(s)=\chi(p)=\chi(q)\leq_2\chi(r)$. This means that χ is isotone.

Now assume $\chi(s)\leq_2\chi(r)$. Then, for some $t\leq_1 r$, $\phi(t)=\phi(s)$. This implies that $\psi(t)\leq_3 r$ and $\chi(\psi(t))=\phi(\psi(t))=\phi(t)=\phi(s)=\chi(s)$. So χ is a strongly isotone function from R onto Q. R contains less elements than P, and, if we take $R=P_1$ and repeat the process for P_1 and Q, etc., then we get a sequence $P_0,\ldots,P_k$ as required.

Corollary 1. If $f,g\varepsilon F^n$, then the following statements are equivalent.

(1) There is a sequence $f_1,\ldots,f_k$ $(k\geq 1)$ such that $f=f_1$, $g=f_k$ and, for all i $(2\leq i\leq k)$, f_i is a reduction of f_{i-1}.

(2) There is a sequence $f_1,\ldots,f_k$ $(k\geq 1)$ such that $f=f_1$, $g=f_k$ and, for all i $(2\leq i\leq k)$ f_i is congruent to a reduction of f_{i-1}.

(3) There is a strongly isotone function ϕ from D_f onto D_g such that, for all $p\varepsilon D_f$, $g(\phi(p))=f(p)$.

In these cases we will call g a *reduced form* of f (by ϕ).

Corollary 2. $f\varepsilon F^n$ is irreducible, iff all reduced forms of f are congruent to f.

Next we prove three lemmas on the way to the uniqueness theorem.

Lemma 2.1. If g is a reduced form of f by ϕ , then for all $p\epsilon D_f$ $g_{\phi(p)}$ is a reduced form of f_p.

Proof. It is immediately clear that $\phi(D_f(p))= D_g(\phi(p))$. Since the restriction of a strongly isotone function is strongly isotone, $g_{\phi(p)}$ is then a reduced form of f_p by the restriction of ϕ to $D_f(p)$.

Lemma 2.2. If g is a normal form of f by ϕ, then for all $p\epsilon D_f$, $g_{\phi(p)}$ is a normal form of f_p.

Proof. According to lemma 2.1 $g_{\phi(p)}$ is a reduced form of f_p. Assume h is a reduced form of $g_{\phi(p)}$ by ψ $(D_h \cap D_g = \emptyset)$. Then we define an I^n-function k by $D_k= (D_g-D_g(\phi(p)))\cup D_h$, and for all $r\epsilon D_g-D_g(\phi(p))$, $k(r)=f(r)$, and for all $r\epsilon D_h$, $k(r)=h(r)$, and for all r', $r\epsilon D_k$ $r' \leqq_k r$ iff $r' \leqq_g r$ or $r' \leqq_h r$, or $r'\epsilon D_h$ and $r \geqq_g \phi(p)$. Then k is a reduced form of g by χ defined by, for all $q\epsilon D_g$, $\chi(q)=q$ if $q\epsilon D_g-D_g(\phi(p))$, $\chi(q)=\psi(q)$ if $q\epsilon D_g(\phi(p))$. Then it is clear that χ is strongly isotone. This means that h is congruent to $g_{\phi(p)}$, and so $g_{\phi(p)}$ is a normal form of f_p.

Corollary. If f is irreducible and $g \leq f$, then g is irreducible.

Lemma 2.3. If $f\epsilon F^n$, $r,r'\epsilon D_f$, $r\neq r'$, and $f_r \equiv f_{r'}$, then f is not irreducible.

Proof. Assume f is irreducible, $r,r'\epsilon D_f$, $r\neq r'$ and $f_r \equiv f_{r'}$ by ϕ. Without losing generality, we can assume that r is a minimal element with this property, i.e., there are no $s,s'\varepsilon D_f$ $(s<r)$ such that $s\neq s'$ and $f_s\equiv f_{s'}$. Now assume $p\varepsilon D_f(r)$, $p\neq r$; then $p=\phi(p)$, since, if $p\neq\phi(p)$, then $f_p\equiv f_{\phi(p)}$ by ϕ would be contrary to our assumption that r is minimal. This is also true for any $p\varepsilon D_f(r')$, $p\neq r'$, since ϕ is onto. So actually we have $D_f[r]=D_f[r']$. But then the conditions for a β-reduction are fulfilled, and f cannot be irreducible.

After one more definition we are ready to prove our uniqueness theorem. Another proof of Th 2.3 will follow from Th. 5.14.

Def. The *depth* of an I^n-function f is the maximal length of the chains w.r. to $<_f$ in D_f.

Th.2.3. If $f\equiv h$, g is a normal form of f, and k is a normal form of h, then $g\equiv k$.

Proof. We proceed with induction on the depth of f. (The depth of h is of course equal to the depth of f.)

(1) The depth of f is 1. Then D_f and D_h consist of only one point, f and h are irreducible, and so the result is immediate.

(2) Assume the depth of f is m, and the theorem holds for I^n-functions with depth $<m$. According to lemma 2.2, there is a function ϕ from D_f onto D_g such that, for all $p\varepsilon D_f$, $g_{\phi(p)}$ is congruent to a normal form

of f_p, and a function ψ from D_f onto D_k such that for all $p\varepsilon D_f$, $k_{\psi(p)}$ is congruent to a normal form of f_p.

We will now show that for all $p',p\varepsilon D_f$, $\phi(p')\leq_g\phi(p)$ iff $\psi(p')\leq_k\psi(p)$.

In the first place, consider $p',p<_f m_f$. If $\phi(p')\leq_g\phi(p)$, then $g_{\phi(p')}\leq g_{\phi(p)}$. By the induction hypothesis $k_{\psi(p')}\equiv g_{\phi(p')}$ and $k_{\psi(p)}\equiv g_{\phi(p)}$. So $k_{\psi(p')}\leq k_{\psi(p)}$. That means that there is an $r\leq_k\psi(p)$ such that $k_{\psi(p')}\equiv k_r$. But then the irreducibility of k implies by lemma 2.3 that $\psi(p')=r$, and so $\psi(p')\leq_k\psi(p)$. Symmetric considerations give us the implication in the other direction. So indeed, (*) for all $p',p<_f m_f$, $\phi(p')\leq_g\phi(p)$ iff $\psi(p')\leq_k\psi(p)$. Now we consider two possible cases.

(1). $\phi(D_f-\{m_f\})$ has a greatest element $r_1=\phi(p_1)$. Then by (*) $\psi(D_f-\{m_f\})$ also has a greatest element, namely $\psi(p_1)$. We consider two subcases. I. $f(m_f)=f(p_1)$. Then $\phi(m_f)=\phi(p_1)$; otherwise there would exist an α-reduction of g w.r. to $m_g=\phi(m_f),\phi(p_1)$, and g is irreducible. For the same reasons, $\psi(m_f)=\psi(p_1)$. II. $f(m_f)\neq f(p_1)$. Then for obvious reasons, $\phi(m_f)\neq\phi(p_1)$ and $\phi(p)<_g\phi(m_f)$ for all $p<_f m_f$. Also $\psi(p)<_k\psi(m_f)$ for all $p<_f m_f$.

(2). $\phi(D_f-\{m_f\})$ has more than one maximal element. Then again, for all $p<m_f$, $\phi(p)<_g\phi(m_f)$. Otherwise D_g would have more than one maximal element. If $\phi(D_f-\{m_f\})$ has more than one maximal element, then so has $(D_f-\{m_f\})$, so also $\psi(p)<_k\psi(m_f)$ for all $p<_f m_f$.

In all these cases it is clear that for all $p',p\epsilon D_f$, $\phi(p')\leqq_g\phi(p)$ iff $\psi(p')\leqq_k\psi(p)$. This implies that for all $p',p\epsilon D_f$, $\phi(p')=\phi(p)$ iff $\psi(p')=\psi(p)$; also that $f(p)=g(\phi(p))=k(\psi(p))$ for all $p\epsilon D_f$. If we now define χ from D_g onto D_k by, $\chi(q)=r$ iff for some $p\epsilon D_f$, $\phi(p)=q$ and $\psi(p)=r$, then χ is uniquely defined and $g\equiv k$ by χ.

Def. Two I^n-functions f and g are *equivalent* (in symbols $f\approx g$) iff they have congruent normal forms.

With the help of Th.2.3 it is easy to see that $\approx$ is an equivalence relation, and that congruent I^n-functions are equivalent.

Def. A *normal* n-ary I-operator a is an ordered I-operator such that, if $f\epsilon C_a$ and $g\approx f$, then $g\epsilon C_a$.

Def. The *normalized characteristic* set C^*_a of a normal n-ary I-operator a is the set of all irreducible I^n-functions in C_a.

Def. A *finite (infinite)* normal I-operator is a normal I-operator with a normalized characteristic set consisting of a finite (infinite) number of congruence classes.

As an example, the ordered I-operator c of the example given on page 12 is not a normal I-operator, because $\begin{array}{c}(1,1,1)\\ / \quad \backslash \\ (1,1,1)\;(1,1,1)\end{array}$ ≈ (1,1,1), since $\begin{array}{c}(1,1,1)\\ | \\ (1,1,1)\end{array}$ is a β-reduction of $\begin{array}{c}(1,1,1)\\ / \quad \backslash \\ (1,1,1)(1,1,1)\end{array}$ and (1,1,1) is an α-reduction of $\begin{array}{c}(1,1,1)\\ | \\ (1,1,1)\end{array}$.

Chapter III.

Definability of I-operators.

In this chapter we will investigate the relationship between the connectives $\&, \vee, \supset, \neg$, and the set of normal I-operators. We will show that these standard connectives are represented by normal I-operators, and that the set of normal I-operators definable in these I-operators is not the whole set of normal I-operators, but contains all the finite normal I-operators.

Of course we first have to exhibit the I-operators that represent the standard connectives, and define what we mean by definability in the set of I-operators.

Def. For all positive integers n, and all i $(1\leq i\leq n)$, the n-ary I-operator u_i^n is defined thus: for all $f\varepsilon F^n$, $u_i^n(f)=f^i$.

Def. The n-ary I-operator a is the *composition* of the n-ary I-operators $b_1,\ldots,b_m$ by the m-ary I-operator c iff, for all $f\varepsilon F^n$, $a(f)=c(b_1(f),\ldots,b_m(f))$.

Def. An I-operator a is *definable* from a set S of I-operators iff, there is a sequence $a_1,\ldots,a_n$ such that $a_n=a$ and for each m $(1\leq m\leq n)$, either $a_m\varepsilon S$, or $a_m=u_i^j$ for some i,j, or a_m is the composition of $a_{k_1},\ldots,a_{k_s}$ by by $a_{k_{s+1}}$, where for all t ($1\leq t\leq s+1$), $1\leq k_t\leq m$.

Def. The *closure* of a set of I-operators is the set of all I-operators definable from S. A set of I-operators is *closed* iff, it is equal to its closure.

The set of all I-operators is obviously closed.

Th.3.1 The set of ordered I-operators is closed.

Proof. (a). u_1^n is ordered, since $(u_1^n(f_p))(p)=((f_p)^1)(p)=f^1(p)=(u_1^n(f))(p)$. (b). If a is the composition of $b_1,\ldots,b_m$ by c, and $b_1,\ldots,b_m$ and c are ordered, then a is an ordered I-operator, since $(a(f_p))(p)=(c(b_1(f_p),\ldots,b_m(f_p)))(p)=(c((b_1(f))_p,\ldots,(b_m(f))_p))(p)=(c((b_1(f),\ldots,b_m(f))_p))(p)=(a(f))(p)$.

Th.3.2. The set of normal I-operators is closed.

Proof. Because of Th.3.1 we only have to prove (a) and (b) as follows. (a). If $f\varepsilon C_{u_i^n}$ and $g\approx f$, then $g\varepsilon C_{u_i^n}$, for all i ($1\leq i\leq n$). For all i ($1\leq i\leq n$), $f\varepsilon C_{u_i^n}$ iff $f^i(m_f)=1$. If g is a reduction of f, then it is easy to see that $f(m_f)=g(m_g)$, so for all i ($1\leq i\leq n$), $f^i(m_f)=g^i(m_g)$ and $f\varepsilon C_{u_i^n}$ iff $g\varepsilon C_{u_i^n}$. Hence, by induction over the sequences of reductions from f and g to their normal forms, $f\approx g$ implies $f\varepsilon C_{u_i^n}$ iff $g\varepsilon C_{u_i^n}$.

(b). If a is the composition of $b_1,\ldots,b_m$ by c, and $b_1,\dots,b_m$ and c are normal, then a is normal, i.e., if $f \varepsilon C_a$ and $g \simeq f$, then $g \varepsilon C_a$. Again it will be sufficient to prove that, if g is a reduction of f, then $g \varepsilon C_a$ iff $f \varepsilon C_a$. Assume g is a reduction of f w.r. to r,r'. Then for all i $(1 \leq i \leq m)$, $(b_i(f))(r)=(b_i(f))(r')=(b_i(g))(r)$, and for all $s \leq r$, $(b_i(f))(s)=(b_i(g))(s)$; also $D_{b_i(f)}=D_f$ and $D_{b_i(g)}=D_g$, complete with their partial orderings. So, for all i $(1 \leq i \leq m)$, $b_i(g)$ is a reduction of $b_i(f)$ w.r. to r,r'. But then also the I^m-function $(b_1(g),\dots,b_m(g))$ is a reduction of $(b_1(f),\dots,b_m(f))$ w.r. to r,r'. Then, since we assumed c to be normal $f \varepsilon C_a$ iff $g \varepsilon C_a$.

We present the I-operators representing the connectives $\&, \vee, \supset$ and $\neg$ in accordance with the results in Chapter I. In each case we will use the same symbol for the connective and the representing I-operator. For all $f \varepsilon F^2$, $p \varepsilon D_f$:

$(f^1 \& f^2)(p)=1$ iff, $f^1(p)=1$ and $f^2(p)=1$,

$(f^1 \vee f^2)(p)=1$ iff, $f^1(p)=1$ or $f^2(p)=1$,

$(f^1 \supset f^2)(p)=1$ iff, for all $p' \leq_f p$, $f^1(p')=0$ or $f^2(p')=1$.

For all $f \varepsilon F$, $p \varepsilon D_f$:

$(\neg f)(p)=1$ iff, for all $p' \leq p$, $f(p')=0$.

Lemma 3.1. For all $f \varepsilon F^2$, $p \varepsilon D_f$:

(a) $(f^1 \& f^2)(p)=1$ iff, for all $p' \leq_f p$, $f^1(p')=1$ and $f^2(p')=1$, (b) $(f^1 \vee f^2)(p)=1$ iff, for all $p' \leq_f p$, $f^1(p')=1$

or $f^2(p')=1$.

Proof. (a). $\Rightarrow (f^1\&f^2)(p)=1$ implies $f^1(p)=1$ and $f^2(p)=1$, by the definition of &. Then by the definition of I-function, for all $p'\leq_f p$, $f^1(p')=1$ and $f^2(p')=1$.
$\Leftarrow$ Trivial.

(b). $\Rightarrow (f^1 v f^2)(p)=1$ implies $f^1(p)=1$ or $f^2(p)=1$. If $f^1(p)=1$, then for all $p'\leq_f p$, $f^1(p')=1$, so $(f^1 v f^2)(p)=1$. Similarly, if $f^2(p)=1$. $\Leftarrow$ Trivial.

The definitions of the I-operators representing the standard connectives and lemma 3.1 immediately suggest the following definition.

Def. An I-operator a is *pseudo-classical* iff, there exists an operator a_c of PC (considered as a function from $\{0,1\}^n$ into $\{0,1\}$), with the property: for all $f\varepsilon F^n$ and $p\varepsilon D_f$, $(a(f))(p)=1$ iff, for all $p'\leq_f p$, $a_c(f(p'))=1$.

Th.3.3. & and $\neg$ are finite normal I-operators.

Proof. If $f\varepsilon C_\&$, then for all $p\leq D_f$, $f^1(p)=1$ and $f^2(p)=1$. Now define an I^2-function g with a domain consisting of a single element, and $g^1(m_g)=g^2(m_g)=1$. Then the function ϕ from D_f onto D_g defined by, for all $p\varepsilon D_f$, $\phi(p)=m_g$, is strongly isotone. So $g\propto f$, and $C^*_\&=\{f\colon f\leq g\}$.
The proof for $\neg$ is similar.

Th.3.4. $\&, v, \supset$ and $\neg$ are pseudo-classical I-operators.

Proof. If $\&_c, v_c, \supset_c$ and $\neg_c$ are respectively the symbols for the conjunction, disjunction, implication, and negation of PC, then, for all $f \varepsilon F^2$ and $p \varepsilon D_f$:

$(f^1 \& f^2)(p)=1$ iff for all $p' \leqq_f p$ $(f^1(p'))\&_c (f^2(p'))=1$.

$(f^1 v f^2)(p)=1$ iff for all $p' \leqq_f p$ $(f^1(p'))v_c (f^2(p'))=1$.

$(f^1 \supset f^2)(p)=1$ iff for all $p' \leqq_f p$ $(f^1(p'))\supset_c (f^2(p'))=1$.

For all $f \varepsilon F$ and $p \varepsilon D_f$, $(\neg f)(p)=1$ iff for all $p' \leqq_f p \neg_c(f(p'))=1$.

Def. An I-operator is a *standard* I-operator, if a is definable from &, v, $\supset$, $\neg$.

We define iterated conjunction and disjunction in the natural way by induction, thus. If $f \varepsilon F^n$, then $\overset{n}{\bigcup}(f)=(\overset{n-1}{\bigcup}(u^n_1(f),\dots,u^n_{n-1}(f)))vu^n_n(f)$, and $\overset{n}{\bigcap}(f)=$ $(\overset{n-1}{\bigcap}(u^n_1(f),\dots,u^n_{n-1}(f)))\&u^n_n(f)$. It is obvious then that $\bigcup$ and $\bigcap$ naturally represent the connectives $\bigcup$ and $\bigcap$ designating iterated conjunction and disjunction. Further $\bigcup$ is definable from v (actually from {v}); $\bigcap$ is definable from &. $\bigcup$ and $\bigcap$ are pseudo-classical I-operators corresponding to the classical iterated conjunction and disjunction $\bigcup_c$ and $\bigcap_c$; i.e. $(\overset{n}{\bigcup}(f))(p)=1$ iff for all $p' \leqq_f p$ $\overset{n}{\bigcup}_c(f(p'))=1$; $(\overset{n}{\bigcap}(f))(p)=1$ iff for all $p' \leqq_f p$ $\overset{n}{\bigcap}_c(f(p'))=1$. Also $(\overset{n}{\bigcup}(f))(p)=1$ iff $\bigcup_{i=1}^{n}{}_c(f^i(p))=1$, and $(\overset{n}{\bigcap}(f))(p)=1$ iff $\bigcap_{i=1}^{n}{}_c(f^i(p))=1$. We will also write $\bigcup_{i=1}^{n}(f^i)$ for $\overset{n}{\bigcup}(f)$, $\bigcap_{i=1}^{n}(f^i)$ for $\overset{n}{\bigcap}(f)$, and $\bigcup_{j \varepsilon J}(f^j)$ for $\overset{k}{\bigcup}(u^n_{j_1}(f),\dots,u^n_{j_k}(f))$ if $J=\{j_1,\dots,j_k\}(1 \leqq j_1,\dots,j_k \leqq n)$, etc. We will also write $a=\bigcup_{i=1}^{k} a_i$ iff a is defined by:

for all $f \varepsilon F^n$, $a(f)=\bigcup_{i=1}^{k}(a_i(f))$, etc.

Th.3.5. All pseudo-classical I-operators are standard.

Proof. Assume a is a pseudo-classical I-operator, correspoding to a_c of PC, i.e. $(a(f))(p)=1$ iff, for all $p' \leqq_f p$, $a_c(f(p'))=1$. Bring a_c into the conjunctive normal form. Then for all $t \varepsilon \{0,1\}^n$, $a_c(t)=\bigcap_{i=1}^{k}{}_c(a_c^i(t))$ $(k \geq 1)$, where for each i $(1 \leq i \leq k)$, $a_c^i(t^1,\ldots,t^n)=(\bigcup_{j \varepsilon J_i}{}_c(\neg_c t^j))v_c(\bigcup_{m \varepsilon M_i}{}_c t^m)$ for some $J_i, M_i \subseteq (1,\ldots,n)$. For each i $(1 \leq i \leq k)$ there are three possible cases. (a). $J_i \neq \emptyset$ and $M_i \neq \emptyset$. Then $a_c^i(t)=(\bigcap_{j \varepsilon J_i}{}_c t^j) \supset_c (\bigcup_{m \varepsilon M_i}{}_c t^m)$. (b). $J_i=\emptyset$. Then $a_c^i(t)=\bigcup_{m \varepsilon M_i}{}_c(t^m)$. (c). $M_i=\emptyset$. Then $a_c^i(t)=\neg_c(\bigcap_{j \varepsilon J_i}{}_c t^j)$. Now, if $a^1,\ldots,a^k$ are the pseudo-classical I-operators corresponding to $a_c^1,\ldots,a_c^k$, then $a=\bigcap_{i=1}^{k}a^i$, where the a^i in the respective cases are, (a) $a^i(f)=(\bigcap_{j \varepsilon J_i}f^j) \supset (\bigcup_{m \varepsilon M_i}f^m)$, (b) $a^i(f)=\bigcup_{m \varepsilon M_i}(f^m)$, (c) $a^i(f)=\neg(\bigcap_{j \varepsilon J_i}(f^j))$. This is easy to check with the help of the definitions of the I-operators $\supset, \neg, \bigcup$ and $\bigcap$.

Th.3.6. All standard I-operators are normal.

Proof. Since we have already shown that the normal I-operators form a closed set (Th.3.2), it is by Th.3.4 sufficient to show that all pseudo-classical I-operators are normal. It is immediately clear from the definition of pseudo-classical I-operator that they are ordered. So assume a is a pseudo-classical I-operator corresponding to

a_c of PC, $f \in C_a$ and $g \approx f$. Then it is sufficient to show that $g \in C_a$. From $f \in C_a$ it follows that $a_c(f(p))=1$ for all $p \in D_f$. But $g \approx f$ implies that for all $p \in D_f$ there is a $q \in D_g$ such that $f(p)=g(q)$. So also for all $q \in D_g$, $a_c(g(q))=1$, which means that $g \in C_a$.

As the main result in this Chapter we will prove that all finite normal I-operators are standard. This can be done indirectly by applying some results from [9]. (see Th.5.16 (cf this thesis)), but we will give a direct proof here that gives us actually a definition for the normal I-operator. For this purpose we will need the following lemmas.

Lemma 3.2. (a). If $C_a=\bigcup_{i=1}^{n} C_{a_i}$, then $a=\bigcup_{i=1}^{n} a_i$. (b). If $C_a=\bigcap_{i=1}^{n} C_{a_i}$, then $a=\bigcap_{i=1}^{n} a_i$. If a is normal, then (c) if $C^*_a=\bigcup_{i=1}^{n} C^*_{a_i}$, then $a=\bigcup_{i=1}^{n} a_i$, (d) if $C^*_a=\bigcap_{i=1}^{n} C^*_{a_i}$, then $a=\bigcap_{i=1}^{n} a_i$.

Proof. (a). Assume $C_a=\bigcup_{i=1}^{n} C_{a_i}$, then $(a(f))(p)=1$ iff, $f_p \in \bigcup_{i=1}^{n} C_{a_i}$, so $(a(f))(p)=1$ iff, $\bigcup_{i=1C}^{n}((a_i(f))(p)=1$. So, indeed $a=\bigcup_{i=1}^{n} a_i$. (b) is proved in exactly the same way as (a). (c). If $C^*_a=\bigcup_{i=1}^{n} C^*_{a_i}$, then $C_a=\{f\colon (Eg)(g \approx f \text{ and } g \in C^*_a)\}=\bigcup_{i=1}^{n}\{f\colon (Eg)(g \approx f \text{ and } g \in C^*_{a_i})\}=\bigcup_{i=1}^{n} C_{a_i}$. So, according to (a), $a=\bigcup_{i=1}^{n} a_i$. (d) is proved in the same way as (c).

Lemma 3.3. Let f and g be irreducible I^n-functions, and let $q_1,\ldots,q_k$ be the direct predecessors of m_g in D_g w.r. to $<_g$. Then $f \leq g$ iff, for all $p \in D_f$, either there exists an i ($1\leq i\leq k$) such that $f_p\leq g_{q_i}$, or $f(p)=g(m_g)$ and for each i ($1\leq i\leq k$), there exists a $p'\leq_f p$ such that $f_{p'}\equiv g_{q_i}$.

Proof. ⇒ For k=0 the lemma is trivial. So assume k>0. There are two possible cases. (a) $f\equiv g_q$ by ϕ for some $q<_g m_g$. Then for some i ($1\leq i\leq k$), $q\leq_g q_i$. So for all $p\in D_f$, $\phi(p)\leq_g q_i$, and, since $f_p\equiv g_{\phi(p)}$, $f_p\leq g_{q_i}$. (b) $f\equiv g$ by ϕ. Then for each $p\in D_f$ there are two possible subcases. I. $p<_f m_f$. Then again for some i, $\phi(p)\leq_g q_i$ and $f_p\leq g_{q_i}$. II. $p=m_f$. Then for each i ($1\leq i\leq k$), $\phi^{-1}(q_i)\leq_f p$ and $f_{\phi^{-1}(q_i)}\equiv g_{q_i}$.

⇐ Consider the set P of all $p\in D_f$ such that for some i ($1\leq i\leq k$), $f_p\leq g_{q_i}$. For each $p\in P$ there is exactly one $q\in D_g$ such that $f_p\equiv g_q$ (lemma 2.3). This defines a function ϕ from P into D_g. ϕ is an isomorphism, since $p'\leq_f p$ implies $f_{p'}\leq f_p$, so $g_{\phi(p')}\leq g_{(p)}$, and so $\phi(p')\leq_g \phi(p)$, and inversely. Also, if $p\in P$ and $p'\leq_f p$, then $p'\in P$. If $q\in D_f-P$, then $f(q)=g(m_g)$ and for each i ($1\leq i\leq k$) there exists a $p'\leq_f q$ such that $f_{p'}\equiv g_{q_i}$. So for each $q\in D_f-P$ and $p\in \bar{P}$, $q\leq_f p$. Now there are two possible cases. (1) D_f-P has more than one minimal element. Assume s and s' are minimal and $s\neq s'$. Then there is a β-reduction of f w.r. to s,s' contrary to the fact that f is irreducible ($D_f[s]=D_f[s']=P$, $f(s)=f(s')=g(m_g)$). (2) D_f-P has exactly one minimal element s. Assume s' is a direct successor of s w.r. to $<_f$. Then s is the only immediate predecessor of s' w.r. to $<_f$. For if $t\leq_f s'$, then either $t\varepsilon P$, so $t<_f s$, or $t\varepsilon D_f-P$, so $s\leq_f t<_f s'$, and so s=t. This means that there is an α-reduction of f w.r. to s,s', contrary to the hypothesis that f is irreducible. So D_f-P contains only one element m_f, and we can extend ϕ to Df by defining $\phi(m_f)=m_g$. Then $f\equiv g$ by ϕ, so $f\leq g$.

Lemma 3.4. Let $f,g\varepsilon F^n$, and let $q_1,\ldots,q_k$ be the direct predecessors of m_g w.r. to $<_g$. Then $f\equiv g$ iff $f\leq g$ and for no i $(1\leq i\leq k)$ $f\leq g_{q_i}$.

Proof. Trivial.

Lemma 3.5. If f and g are irreducible I^n-functions, and $q_1,\ldots,q_k$ are the direct predecessors of m_g w.r. to $<_g$, and for each i $(1\leq i\leq k)$, $q_{i1},\ldots,q_{ik_i}$ $(k_i\geq 0)$ are the direct predecessors of q_i w.r. to $<_g$, then $f\leq g$ iff, for all $p\varepsilon D_f$, either there exists an i $(1\leq i\leq k)$ such that $f_p\leq g_{q_i}$, or $f(p)=g(m_g)$ and for each i $(1\leq i\leq k)$, there is a $p'\leq_f p$ such that $f_{p'}\leq g_{q_i}$, but for no j $(1\leq j\leq k_i)$, $f_{p'}\leq g_{q_{ij}}$.

Proof. Immediate from Lemmas 3.3 and 3.4.

Th.3.7. All finite normal I-operators are standard.

Proof. A finite normal I-operator a has a normalized characteristic set $C_a^*=\{f:\ f\leq g_i\}$ for some sequence $(g_1,\ldots,g_k)$ $(k\geq 0)$, where for each i,j $(1\leq i,j\leq k,\ i\neq j)$ not $g_i\leq g_j$, since there are only a finite number of maximal congruence classes in C^*_a (w.r. to the relation $\leq$). If $k=0$, then C^*_a is empty and $a=u^n_1\&\neg u^n_1$. If $k\geq 1$, then assume for all i ($1\leq i\leq k$), a_i is the normal I-operator with $C^*_{a_i}=\{f: f\leq g_i\}$. Then $C^*_a=\bigcup_{i=1}^{k} C^*_{a_i}$, and by Lemma 3.2 $a=\bigcup_{i=1}^{k} a_i$, so a is definable in the a_i. That means in the proof we can restrict ourselves to the case that $k=1$, i.e. we can assume $C^*_a=\{f: f\leq g\}$ for some irreducible $g\in F^n$.

Without loss of generality we can assume $g^i(m_g)=0$ for $i=1,\dots,m$ ($1\leq m\leq n$) and $g^i(m_g)=1$ for $i=m+1,\dots,n$. Now let b be the m-ary normal I-operator with $C^*_b=\{f\in F^m: f\leq(g^1,\dots,g^m)\}$. $(g^1,\dots,g^m)$ obviously is an irreducible I^m-function, so b is again a finite normal I-operator. Now define the normal n-ary I-operator b_1 by $b_1(f)=b(u^n_1(f),\dots,u^u_m(f))$. Then $C^*_a=C^*_{b_1}\cap\bigcap_{i=m+1}^{n} C^*_{u^n_1}$, so by Lemma 3.2 $a=b_1\&\bigcap_{i=m+1}^{n} u^n_1$, which means that a is definable in b. So, in the proof we can restrict ourselves to the case that $g^i(m_g)=0$ for all i ($1\leq i\leq n$).

Since all standard I-operators are normal, we only have to investigate the behavior of a with respect to the irreducible I^n-functions; if we find a standard I-operator that agrees with a there, then a has to be standard.

We will now proceed to prove the theorem by induction on the depth of g.

(a) The depth of g is 1. Then $D_g=\{m_g\}$, and $D_f=\{m_f\}$ for all $f\le g$. Hence, for all irreducible $f\varepsilon F^n$ and $p\varepsilon D_f$, $(a(f))(p)=1$ iff $f_p\varepsilon C^*_a$, i.e. $(a(f))(p)=1$ iff $f_p\le g$, so (since $D_g=\{m_g\}$) $(a(f))(p)=1$ iff, for all $p'\varepsilon D_{f_p}$, $f(p')=g(m_g)=(0,\dots,0)$, and so $(a(f))(p)=1$ iff, for all $p'\le_f p$, $\bigcap_{i=1}^{n}{}_c\neg_c f^i(p')=1$. So $a(f)=\bigcap_{i=1}^{n}(\neg f^i)=\bigcap_{i=1}^{n}(\neg u_i^n(f))$.

(b) The depth of g is m>1, and we assume that the theorem holds for all I^n-functions with depth<m. Assume further that m_g has k direct predecessors ($k\ge1$) w.r. to $<_g$, and that for each i ($1\le i\le k$) q_i has k_i direct predecessors $q_{i1},\dots,q_{ik_i}$ ($k_i\ge0$). Also assume that, for each i and j ($1\le i\le k$, $1\le j\le k_i$), a_i and a_{ij} are the normal I-operators with respective normalized characteristic sets $C^*_{a_i}=\{f\colon f\le g_{q_i}\}$ and $C^*_{a_{ij}}=\{f\colon f\le g_{q_{ij}}\}$. Then by the induction hypothesis we can assume that a_i and a_{ij} are standard I-operators, so we just have to prove that a is definable in the a_i and a_{ij}. Now, for any irreducible $f\varepsilon F^n$, $p\varepsilon D_f$, $(a(f))(p)=1$ iff $f_p\le g$; i.e.(Lemma 3.5) $(a(f))(p)=1$ iff, for all $p'\le_f p$, either there exists an i ($1\le i\le k$) such that $f_{p'}\le g_{q_i}$, or $f(p')=g(m_g)=(0,\dots,0)$ and for each i ($1\le i\le k$) there is a $p''\le_f p'$ such that $f_{p''}\le g_{q_i}$ but for no j ($1\le j\le k_i$) $f_{p''}\le g_{q_{ij}}$. From this it follows that $(a(f))(p)=1$ iff, for all $p'\le_f p$, either there is an i $(1\leq i\leq k)$ such that $f_{p'}\leq g_{q_i}$, or for all i $(1\leq i\leq n)$, $f^i(p')=0$ and for each i $(1\leq i\leq k)$ it is not true that (1) for all $p''\leqq_f p'$ not $f_{p''}\leq g_{q_i}$ or for some j $(1\leq j\leq k_i)$, $f_{p''}\leq g_{q_{ij}}$. But (1) is equivalent, if $k_i\neq 0$, to (2) for some $p''\leqq_f p'$, $(a_i(f))(p'')\supset_c\bigcup_{j=1}^{k_i}{}_c((a_{ij}(f))(p''))=1$ and, if $k_i=0$, to (2') for all $p''\leqq_f p'$, $\neg_c((a_i(f))(p'))=1$, which in turn are equivalent respectively to (3) $((a_i\supset\bigcup_{j=1}^{k_i}a_{ij})(f))(p')=1$, (3') $((\neg a_i)(f))(p')=1$. Then $(a(f))(p)=1$ iff, for all $p'\leqq_f p$, $((\bigcup_{i=1}^{k}{}_c(a_i(f))(p'))\vee_c (\neg_c ((\bigcup_{i=1}^{n}{}_c(u_i^n(f))(p'))\&_c\neg_c\bigcup_{i=1}^{k}{}_c(((a_i\supset\bigcup_{j=1}^{k_i}a_{ij})(f))(p'))))=1$ (with $a_i\supset\bigcup_{j=1}^{k_i}a_{ij}$ replaced by $\neg a_i$ if $k_i=0$). But then $(a(f))(p)=1$ iff, for all $p'\leqq_f p$, $(((\bigcup_{i=1}^{n}u_i^n)\vee\bigcup_{i=1}^{k}(a_i\supset\bigcup_{j=1}^{k_i}a_{ij}))(f))(p')\supset_c((\bigcup_{i=1}^{k}a_i)(f))(p')=1$. But this means that $a=((\bigcup_{i=1}^{n}u_i^n)\vee\bigcup_{i=1}^{k}(a_i\supset\bigcup_{j=1}^{k_i}a_{ij}))\supset\bigcup_{i=1}^{k}a_i$, with $a_i\supset\bigcup_{j=1}^{k_i}a_{ij}$ replaced by a_i if $k_i=0$.

<u>Corollary</u> to the proof of Th.3.7. For the special case that for each m $(1\leq m\leq n)$ there is an i $(1\leq i\leq k)$ such that $g^m(q_i)=0$, the formula obtained for a in the proof can be slightly simplified. Namely, in that case, from for each i $(1\leq i\leq k)$ there is a $p''\leqq_f p'$ such that $f_{p''}\leq g_{q_i}$, it follows by the definition of I-function that $f^m(p')=0$ for all m $(1\leq m\leq n)$. So we can leave out the requirement that $f(p')=g(m_g)$, and we end up with $a=\bigcup_{i=1}^{k}(a_i\supset\bigcup_{j=1}^{k_i}a_{ij})\supset\bigcup_{i=1}^{k}a_i$ (with $a_i\supset\bigcup_{j=1}^{k}a_{ij}$ replaced by $\neg a_i$ if $k_i=0$).

We will continue our investigations on I-operators in Chapter V. In Chapter IV we will apply Th.3.7. to another problem.

Chapter IV.

A Characterization of the Intuitionistic Propositional Calculus

In this chapter we will find a characterization of Pp from above, i.e. we will describe a property of Pp that no consistent propositional calculus stronger than Pp possesses. By a propositional calculus stronger than Pp we understand one in which all formulas provable in Pp are provable, and some others as well, and which is closed under substitution and modus ponens. (Closure under substitution is, of course, assured if no particular axioms are postulated, but only axiom schemata.) By a formula we understand a formula built up from atoms $A_1, A_2, \ldots$ with the connectives &, v, $\supset$ and $\neg$.

Lukasiewicz [15] proposed the conjecture that Pp can be characterized from above by the property: for any formulas U,V, if $\vdash_{Pp} UvV$, then $\vdash_{Pp} U$ or $\vdash_{Pp} V$. This conjecture was disproved by Kreisel and Putnam [13], who showed that Pp + the axiom schema $(\neg U \supset VvW) \supset (\neg U \supset V) v (\neg U \supset W)$ has the same property.

In [12] Kleene proved a stronger property of Pp, and he subsequently proposed to the author the conjecture that this property characterizes Pp from above. First, one defines a notion $K|_{T}U$ for any sequence K of formulas, any formula U, and any propositional calculus T, from the notion $\vdash_{T}$ of provability in T. Kleene states the definition in [12] in particular for the case T is Pp (see [12] §4), and he proves (among other things) that, for each U,V,W, if $U|_{Pp}U$ and $\vdash_{Pp}U\supset V\vee W$, then $\vdash_{Pp}U\supset V$ or $\vdash_{Pp}U\supset W$. Kleene's conjecture, which we will confirm in this chapter is: if T is a propositional calculus at least as strong as Pp, possessing the property

(*) for each U,V,W if $U|_{T}U$ and $\vdash_{T}U\supset V\vee W$ then $\vdash_{T}U\supset V$ or $\vdash_{T}U\supset W$,

then T is Pp. Also we will give another characterization of Pp from above by replacing (*) by

(**) for each U,V if $U|_{T}U$, $\vdash_{T}U\supset V$ and $\vdash_{T}V\supset U$ then $V|_{T}V$.

Before we will be able to do this, we will have to discuss pseudo-Boolean algebras and their connection with I-valuations. The duals of pseudo-Boolean algebras (Brouwerian algebras) and their connection with intuitionistic logic were first discussed by McKinsey and Tarski [16], [17]. A *pseudo-Boolean algebra* is an abstract algebra with three binary operations $\cup$, $\cap$, $\Rightarrow$(relative pseudo-complement), one unary operation —— (pseudo-complement), and two constants $\mathbf{1}$ and $\mathbf{0}$.. We will use as variables over elements of these algebras $\alpha,\beta,\gamma,\alpha_1,\alpha_2,\ldots,\beta_1,\beta_2,\ldots$. Terms are then defined in the usual way and if $U(\alpha_1,\ldots,\alpha_n)$ and $V(\alpha_1,\ldots,\alpha_n)$ are terms, then $U(\alpha_1,\ldots,\alpha_n)=V(\alpha_1,\ldots,\alpha_n)$ is an equation. We say that this equation is valid in an algebra *A*, if for all $\alpha_1,\ldots,\alpha_n \in A$, $U(\alpha_1,\ldots,\alpha_n)=V(\alpha_1,\ldots,\alpha_n)$. An equation is valid in a class of algebras, if it is valid in all algebras of the class. A set of equations defines the class of algebras in which all the equations are valid. The class of pseudo-Boolean algebras can be defined by a set of equations (see e.g. [20]).

A formula $U(\&,v,\supset,\neg,A_1,\ldots,A_n)$ is said to be valid in pseudo-Boolean algebra *A*, iff the equation $U^*(\cap,\cup,\Rightarrow,-,\alpha_1,\ldots,\alpha_n)= 1$ is valid in *A*, where U^* is formed from $\alpha_1,\ldots,\alpha_n$ by means of $\cap,\cup,\Rightarrow$ and $-$, in exactly the same way as U from $A_1,\ldots,A_n$ by means of $\&,v,\supset$ and $\neg$. McKinsey and Tarski [20] proved the following theorem.

Th.4.1. The propositional formula U is derivable in Pp (a) iff U is valid in every pseudo-Boolean algebra, and (b) iff U is valid in every finite pseudo-Boolean algebra.

Def. For any propositional calculus T stronger than Pp we say a pseudo-Boolean algebra *A* is a *T-pseudo-Boolean algebra* iff, for each formula U such that $\vdash_T U$, U is valid in *A*.

Th.4.2. For every propositional calculus T stronger than Pp, $\vdash_T U$ iff, U is valid in each T-pseudo-Boolean algebra.

Proof. $\Rightarrow$Trivial. $\Leftarrow$ Follows from a particular case of a theorem of Birkhoff [5] that says that in a class of algebras defined by equations an equation is valid iff it is derivable from the defining equations by means of the following four rules. (i) $U=U$. (ii) If $U*V$ then $V=U$. (iii) If $U=V$ and $V=W$, then $U=W$. (iv) If $U=V$ and $W=X$, and U' results from U by replacing some occurrences of W by X, then $U'=V$. It is easy to see that the rules (i),...,(iv) can be simulated in the logic, if we realize that from $U=V$ we can derive $U\to V=1$ and $V\to U=1$ and conversely.

Another special case of a theorem of Birkhoff [5] we will use is:

Th.4.3. The class of all T-pseudo-Boolean algebras is closed under the formation of sub-algebras, homomorphisms and direct products.

Now we are ready to look at the relationship between pseudo-Boolean algebras and I-valuations (or I-functions). The following definitions are from [9] (see also [19]).

If a partially ordered set $\langle V,\leq\rangle$ is a complete lattice, then $\alpha\varepsilon V$ is called *join-irreducible* iff $\alpha>\bigcup\{\beta:\beta<\alpha\}$. The set of all join-irreducible elements of V will be denoted by V^0.

Def. A lattice A is called *join-representable*, iff A is complete and completely distributive, and every $\alpha\varepsilon A$ can be written as $\alpha=\bigcup\{\beta:\beta\leq\alpha \text{ and } \beta\varepsilon A^0\}$.

Def. A subset F of $<P,\leq>$ is called *M-closed* iff for all $p,q\varepsilon P$, $p\varepsilon F$ and $q\leq p$ implies $q\varepsilon F$.

The set of all M-closed subsets of a partially ordered set P will be denoted by $\overline{P}$. $\overline{P}$ is then complete and completely distributive.

Th.4.4. ([9],[19]). Every join-representable lattice A is isomorphic to $\overline{A^0}$.

Th.4.5. (e.g. [4]). A complete and completely distributive lattice is a pseudo-Boolean algebra, if we define $\alpha\rightarrow\beta=\bigcup\{\gamma:\alpha\cap\gamma\leq\beta\}$.

Every finite distributive lattice is complete and completely distributive, and join-representable (e.g. [4]). So this theorem implies that every finite distributive lattice is a pseudo-Boolean algebra $\overline{P}$ for some partially ordered set P. Since for every partially ordered set P, $\overline{P}$ is a distributive lattice, there is therefore a 1-1 correspondence between finite pseudo-Boolean algebras and finite partially ordered sets.

Def. If P is a partially ordered set, then P is *T-admissible* iff $\overline{P}$ is a T-pseudo-Boolean algebra.

Th.4.6. (essentially in [9]). If P is a P.O.G.-set, then there is the following correspondence between any I-valuation $<P,w>$ and the pseudo-Boolean algebra

$\bar{P}$: for all formulas U and V, if $F_1=\{p: w(p,U)=1\}$ and $F_2=\{p: w(p,V)=1\}$, then (i) $F_1 \cap F_2=\{p: w(p,U\&V)=1\}$, (ii) $F_1 \cup F_2=\{p: w(p,U \vee V)=1\}$, (iii) $F_1 \Rightarrow F_2=\{p: w(p,U \supset V)=1\}$, (iv) $-F_1=\{p: w(p,\neg U)=1\}$.

Proof. F_1 and F_2 are M-closed, so $F_1 \cap F_2$, $F_1 \cup F_2$, $F_1 \Rightarrow F_2$ and $-F_1$ are well-defined.
(i) $F_1 \cap F_2=\{p: w(p,U)=1\}$and $w(p,V)=1$ $=\{p: w(p,U\&V)=1\}$.
(ii) $F_1 \cup F_2=\{p: w(p,U)=1\}$or $w(p,V)=1$ $=\{p: w(p,U \vee V)=1\}$.
(iii) $p \varepsilon F_1 \Rightarrow F_2$ iff $P(p) \subseteq F_1 \Rightarrow F_2$. $P(p) \subseteq F_1 \Rightarrow F_2$ iff $P(p) \cap F_1 \subseteq F_2$. $P(p) \cap F_1 \subseteq F_2$ iff for all $p' \leq p$ if $w(p,U)=1$ then $w(p,V)=1$, so ultimately $p \varepsilon F_1 \Rightarrow F_2$ iff $w(p,U \supset V)=1$, and indeed $F_1 \Rightarrow F_2=$ $\{p: w(p,U \supset V)=1\}$. (iv) Similar to (iii).

Lemma 4.1. For any P.O.G.-set P with maximum element p_0 and any formula $U(A_1,\ldots,A_n)$, $w(p_0,U)=1$ for all I-valuations $<P,p_0,w>$ iff $U^*(\alpha_1,\ldots,\alpha_n)=P$ for all $\alpha_1,\ldots,\alpha_n \varepsilon \bar{P}$.

Proof. Immediate by induction on the result of Th.4.6.

Th.4.7. If P is a P.O.G.-set with maximum element p_0, then P is T-admissible iff, for all I-valuations $<P,w>$, and all formulas U such that $\vdash_T U$, $w(p_0,U)=1$.

Proof. Immediate from the lemma, and the definition of T-admissible.

The main theorem of this chapter is a little bit stronger than we need to establish the results predicted above. Probably the double negation of this theorem is valid intuitionistically. We have not checked this.

Th.4.8. If T is a consistent propositional calculus stronger than Pp, then for each integer $r\geq 2$ there is a formula $U\supset V_1 v \ldots v V_s$ $(s\geq r)$ such that $U|_{T}U$ and $\vdash_T U\supset V_1 v\ldots v V_s$, but not $\vdash_T U\supset V_{i_1} v \ldots v V_{i_k}$ for any proper subsequence $(i_1,\ldots,i_k)$ $(k\geq 1)$ of $(1,\ldots,s)$.

Proof. We will first construct a finite P.O.G.-set $\langle P',\leq_0,p_0\rangle$, having $p_1,\ldots,p_k$ $(k\geq 1)$ as the immediate predecessors of p_0 , such that P' is not T-admissible, but for all i $(1\leq i\leq k)$ $P'(p_i)$ is T-admissible. For this purpose we start with a formula X such that $\vdash_T X$, but not $\vdash_{Pp} X$. There is an I-valuation $\langle P'',p_0',w\rangle$ such that $w(p_0',X)=0$. If P" has the desired properties, then take P'=P". Otherwise there is a $p\varepsilon P''$ such that P"(p) has the desired properties. For, if q is a minimal element of P", then $\overline{P''(q)}$ is a two-element Boolean algebra. So in that case P"(q) is certainly T-admissable, since T is assumed to be consistent and therefore does not contain theorems that are not provable in PC. We take then P'=P"(p) for some P"(p) with the desired properties. Since P' is not T-admissable and therefore $\overline{P'}$ is not a Boolean algebra, P' contains more than one element and $k\geq 1$.

Once we have constructed P', we construct another P.O.G.-set P. If $k \geq r$, then we take $P=P'$, and in the proof we will take $s=k$. If $k<r$, then we take a natural number j such that $jk \geq r$, and we construct $j-1$ partially ordered sets P_i $(1 \leq i \leq j-1)$ (disjoint from P and from each other) that are isomorphic images of $P'-\{p_0\}$ by ϕ_i and have partial orderings $\leq_i$. Then we take $P= P' \cup P_1 \cup \ldots \cup P_{j-1}$ and the partial ordering on P as for all $p', p \in P$, $p' \leq p$ iff $p' \leq_0 p$ or $p' \leq_i p$ for some i $(1 \leq i \leq j-1)$ or $p=p_0$. It is clear then that P is a P.O.G.-set with maximum element p_0. Also that for all $p<p_0$, $P(p)$ is T-admissible, since either $P(p)=P'(p)$, or $P(p) \equiv P'(\phi_i^{-1}(p))$ for some i $(1 \leq i \leq j-1)$. But P is not T-admissible, as we prove in the following way. There is a formula Y and an I-valuation $<P',w>$ such that $w(p_0,Y)=0$ while $\vdash_T Y$. We now define an I-valuation $<P,w'>$ in the following way: for any atomic formula A_i, if $p \in P'$, then $w'(p,A_i)=w(p,A_i)$, if $p \in P_j$, then $w'(p,A_i)=w(\phi_j^{-1}(p,A_i))$. Then we can prove that $w'(p_0,U)= w(p_0,U)$ for all formulas U, by induction on the length of U. Of course, for all $p \in P_i$ and all U, $w'(p,U)=w(\phi_i^{-1}(p),U)$. The statement to be proved is true for atomic formulas. If it is true for U and V, then it follows immediately for $U \& V$, $U v V$, $U > V$ and $\neg V$ by applying the definition of I-valuation. So $w'(p_0,Y)=0$ and P is not T-admissible. We will take $s=jk$ in the proof in this case.

Assume P contains n+1 elements. Then we will construct an irreducible I^n-function g with $D_g=P$ such that for all m $(1\leq m\leq n)$ $g^m(p)=0$ for some $p<p_0$. Assume $p_0,\ldots,p_n$ is an enumeration of the elements of P, and assume $p_1,\ldots,p_s$ are the direct predecessors of p_0 $(s\geq r\geq 2)$. Then we define $g^m(p_i)=1$ iff $p_i\leq p_m$ for all m $(1\leq m\leq n)$ and all i $(1\leq i\leq n)$ and $g(p_0)=(0,\ldots,0)$. Then obviously g is an I^n-function. g is irreducible, since for all $p,p'\varepsilon P$, if $p\neq p'$, then $g(p)\neq g(p')$. Also for all j $(1\leq j\leq n)$ $g^j(p)=0$ for some $p<p_0$, namely if $2\leq j\leq n$ then take $p=p_1$ and if $j=1$ then take $p=p_2$. All this implies that we can apply the corollary of Th.3.7 to the I-operator with normalized characteristic set $\{f\colon f\leq g\}$.

Assume $a,a_1,\ldots,a_s$ are the standard I-operators with normalized characteristic sets $\{f\varepsilon F^n\colon f\leq g\}$ and $\{f\varepsilon F^n\colon f\leq g_{p_i}\}$ $(1\leq i\leq s)$, and assume that $U,V_1,\ldots,V_s$ are the formulas corresponding to $a,a_1,\ldots,a_s$ formed from the atoms $A_1,\ldots,A_n$. Now we will prove:

(a) $\vdash_T U\supset V_1 v\ldots vV_s$, (b) not $\vdash_T U\supset V_{i_1} v\ldots vV_{i_k}$ for any proper subsequence $(i_1,\ldots,i_k)$ of $(1,\ldots,s)$, (c) $U|_T U$.

(a) The crucial point of the proof is that the class of T-pseudo-Boolean algebras does not contain a pseudo-Boolean algebra on which $U\supset V_1 v\ldots vV_s$ is not valid, a "counter-example" to this formula. More precisely, for any pseudo-Boolean algebra A on which $U \supset V_1 \vee \ldots \vee V_s$ is not valid, the pseudo-Boolean algebra $\bar{F}$ is isomorphic to a subalgebra of a homomorphism of A, and so Th.4.3 implies that, since $\bar{F}$ is not a T-pseudo-Boolean algebra, A cannot be one, and therefore $\vdash_T U \supset V_1 \vee \ldots \vee V_s$. We will now prove this assertion. Assume $U \supset V_1 \vee \ldots \vee V_s$ is not valid on A. Then there are elements $\alpha_1, \ldots, \alpha_n \varepsilon A$ such that $U^* \Rightarrow V^* \cup \ldots \cup V^*(\alpha_1, \ldots, \alpha_n) \neq \mathbb{1}$. Now we take the relativization A_{U^*} of A with respect to U^* (see e.g. [20]), i.e. the sublattice of elements of A $\leq U^*$. With an appropriate relative pseudo-complement defined this is a pseudo-Boolean algebra and a homomorphic image of A. The $\mathbb{1}$-element of A_{U^*} is U^* of A, and the homomrphism ϕ can be written in A as $\phi(\alpha) = \alpha \& U^*$. Now we write $\phi(\alpha_i) = \beta_i$ for all i $(1 \leq i \leq n)$; then $V_1^* \cup \ldots \cup V_s^*(\beta_1, \ldots, \beta_n) < U^*(\beta_1, \ldots, \beta_n) = \mathbb{1}$ in A_{U^*}.

Now we take the sub-algebra B of A_{U^*} generated by $\beta_1, \ldots, \beta_n$. We claim that B is isomorphic to $\bar{F}$. Take any element $W^*(\beta_1, \ldots, \beta_n)$ of B; $W^*(\beta_1, \ldots, \beta_n) = (U^* \cap W^*)(\beta_1, \ldots, \beta_n)$. Now assume b is the standard I-operator corresponding to $(U \& W)(A_1, \ldots, A_n)$. Then b is a normal I-operator with $C_b^* \leq C_a^*$. So b is also a finite normal I-operator. This means that any element of B corresponds to a finite normal I-operator b with $C_b^* \leq C_a^*$. Now assume W_1^* and W_2^* are two elements of B corresponding to the I-operators b_1 and b_2. It is easy to see that, if $C_{b_1}^* \leq C_{b_2}^*$, then $\vdash_{Pp}(W_1 \supset W_2)(A_1, \ldots, A_n)$, so $W_1^* \leq W_2^*$ in B. On the other hand, assume that

$W^*_1 \leq W^*_2$. We will prove that then $C^*_{b_1} \subseteq C^*_{b_2}$. For that purpose we assume that $C^*_{b_1} \not\subseteq C^*_{b_2}$, and deduce that $\vdash_{Pp}((W_1 \supset W_2) \supset (U \supset (V_1 v \ldots v V_s)))(A_1, \ldots, A_n)$. Assume this formula is not a theorem of Pp. Then there is an I-valuation $\langle Q, \leq_\Delta, q_0, w \rangle$ such that $w(q_0, ((W_1 \supset W_2) \supset (U \supset (V_1 v \ldots v V_s)))) = 0$. Then, for some $q \varepsilon Q$, $w(q, W_1 \supset W_2) = 1$ and $w(q, U \supset (V_1 v \ldots v V_s)) = 0$. Then again there is an $r \leq_\Delta q$ such that $w(r, W_1 \supset W_2) = w(r, U) = 1$ and $w(r, V_1 v \ldots v V_s) = 0$, so $w(r, V_i) = 0$ for all i $(1 \leq i \leq s)$. Now take an irreducible I^n-function h corresponding to the restriction of the I-valuation to Q(r), i.e. an irreducible I^n-function equivalent to the one corresponding to this restriction. Then $((b_1 \supset b_2)(h))(m_h) = 1$ and $(a(h))(m_h) = 1$, so $h \varepsilon C^*_a$, and $(a_i(h))(m_h) = 0$, so $h \not\varepsilon C^*_{a_i}$ for any i $(1 \leq i \leq s)$. But this implies that $h \equiv g$. But if $C^*_{b_1} \subseteq C^*_{b_2}$, then there is a $p \varepsilon P$ such that $g_p \varepsilon C^*_{b_1}$ and $g_p \not\varepsilon C^*_{b_2}$, so $(b_1(g))(p) = 1$ and $(b_2(g))(p) = 0$, and $((b_1 \supset b_2)(g))(m_g) = 0$, and we have a contradiction. So we have now proved that $\vdash_{Pp}(W_1 \supset W_2) \supset (U \supset (V_1 v \ldots v V_s))$. This implies that $W^*_1 \Rightarrow W^*_2 \leq U^* \Rightarrow (V^*_1 \cup \ldots \cup V^*_s)$ in B and A_{U^*}. But we had assumed that $W^*_1 \leq W^*_2$, so $W^*_1 \Rightarrow W^*_2 = \mathbb{1}$ and $U^* \Rightarrow (V^*_1 \cup \ldots \cup V^*_s) = \mathbb{1}$ in A_{U^*}. But this again gives us a contradiction, and we have proved $C^*_{b_1} \subseteq C^*_{b_2}$.

The result we have reached now is that B is isomorphic to the lattice of all normal I-operators b with normalized characteristic sets $C^*_b \subseteq C^*_a$. This lattice is isomorphic to $\overline{P}$, since for every M-closed subset R of

P there is exactly one such I-operator, namely the I-operator with normalized charcteristic set $\{h\varepsilon F^n$: h irreducible and $h \leq g_p$ for some $p \varepsilon R\}$. This concludes the proof that $U \supset V_1 \vee \ldots \vee V_s$ is valid on every T-pseudo-Boolean algebra, and so $\vdash_T U \supset V_1 \vee \ldots \vee V_s$.

(b) To prove that not $\vdash_T U \supset V_{i_1} \vee \ldots \vee V_{i_k}$, assume $t \not\varepsilon (i_1, \ldots, i_k)$ $(1 \leq t \leq s)$. Then $g_{p_t} \varepsilon C^*_a$, so $(a(g))(p_t)=1$; but for all j $(1 \leq j \leq k)$ $f_{p_t} \not\varepsilon C^*_{a_{i_j}}$, so $(a_{i_j}(f))(p_t)=0$ for all j $(1 \leq j \leq k)$. All this implies that $((a \supset a_{i_1} \vee \ldots \vee a_{i_k})(g))(p_t)=0$, so $U \supset V_{i_1} \vee \ldots \vee V_{i_k}$ is not valid on $\overline{P(p_t)}$; and, since $\overline{P(p_t)}$ is a T-pseudo-Boolean algebra, not $\vdash_T U \supset V_{i_1} \vee \ldots \vee V_{i_k}$.

(c) Here we rely on the corollary to the proof of Th.3.7. Assume that for all i $(1 \leq i \leq s)$ p_i has t_i $(t_i \geq 0)$ immediate predecessors $p_{i1}, \ldots, p_{it_i}$ in P, and assume that for all i,j $(1 \leq i \leq s, 1 \leq j \leq t_i)$ a_{ij} is the I-operator with normalized characteristic set $C^*_{a_{ij}} = \{h \varepsilon F^n$: $h \leq g_{p_{ij}}\}$ and that $V_{ij}(A_1, \ldots, A_n)$ are the corresponding formulas. Then according to the corollary,

$$a=((a_1 \supset (a_{11} \vee \ldots \vee a_{1t_1}) \vee \ldots \vee (a_{s1} \vee \ldots \vee a_{st_s}))) \supset (a_1 \vee \ldots \vee a_s)$$

(in case $t_i=0$, use $\neg a_i$ instead of $a_i \supset (a_{i1} \vee \ldots \vee a_{it_i})$ for each i $(1 \leq i \leq s)$). To show that $U |_T U$ it is sufficient to show that not $U |_T (V_1 \supset V_{11} \vee \ldots \vee V_{1t_1}) \vee \ldots \vee (V_s \supset V_{s1} \vee \ldots \vee V_{st_s})$. To prove that it is sufficient to show that for no i $(1 \leq i \leq s)$ $U \vdash_T V_i$ $V_{i1} \vee \ldots \vee V_{it_i}$ for any i $(1 \leq i \leq s)$. But this is immediate from the facts that $f_{p_i} \varepsilon C^*_a$, $f_{p_i} \varepsilon C^*_{a_i}$, but for no j $(1 \leq j \leq t_i)$ $f_{p_i} \varepsilon C^*_{a_{ij}}$ (or in the case $t_i=0$, $f_{p_i} \not\varepsilon C^*_{\neg a_i}$).

For the proof of our second characterization of Pp from above we will need another definition and a theorem.

Def. a is a *connected* I-operator iff a is ordered and for all $f,g \varepsilon C_a$ there exists an $h \varepsilon C_a$ such that $f \leq h$ and $g \leq h$.

It is obvious that a normal I-operator a is connected iff for all $f,g \varepsilon C^*_a$ there is an $h \varepsilon C^*_a$ such that $f \leq h$ and $g \leq h$. Also that a finite normal I-operator is connected iff there exists an I^n-function g such that $C^*_a = \{f \varepsilon F^n: f \leq g\}$.

Th.4.9. For any formula U, $U|_{Pp}U$ iff the standard I-operator corresponding to U is connected.

Proof. $\Longrightarrow$ If $U|_{Pp}U$, then according to [12], for any formulas V,W, if $\vdash_{Pp} U \supset V \vee W$, then $\vdash_{Pp} U \supset V$ or $\vdash_{Pp} U \supset W$. Now assume the normal I-operator c corresponds to U. Assume $f,g \varepsilon C^*_c$, and assume further that $p_1,\ldots,p_r$ are the immediate predecessors of m_f w.r. to $<_f$, and that $q_1,\ldots,q_s$ are the immediate predecessors of m_g w.r. to $<_g$ $(r \geq 0, s \geq 0)$, and for all i $(1 \leq i \leq r)$ a_i is the normal I-operator with normalized characteristic set $\{k \varepsilon F^n: k \leq f_{p_i}\}$, and for all j $(1 \leq j \leq s)$ b_j is the normal I-operator with normalized characteristic set $\{k \varepsilon F^n: k \leq g_{q_j}\}$, and a and b are the normal I-operators with normalized characteristic sets $\{k \varepsilon F^n: k \leq f\}$ and $\{k \varepsilon F^n: k \leq g\}$. Also assume that $V_0,V_1,\ldots,V_r,W_0,W_1,\ldots,W_s$

are the corresponding formulas.

Then we have not $\vdash_{Pp} U\supset(V_0\supset V_1 v\ldots vV_r)$ and not $\vdash_{Pp} U\supset(W_0\supset W_1 v\ldots vW_s)$, since $((c\supset(a\supset a_1 v\ldots va_r))(f))(m_f)=0$ and $((c\supset(b\supset b_1 v\ldots vb_s))(g))(m_g)=0$. This implies that also not $\vdash_{Pp} U\supset((V_0\supset V_1 v\ldots vV_r)v(W_0\supset W_1 v\ldots vW_s))$. Then for some irreducible $h\varepsilon F^n$, $((c\supset((a\supset a_1 v\ldots va_r)v(b\supset b_1 v\ldots vb_s))$ $(h))(m_h)=0$. Now for some $p\varepsilon D_h$, $(c(h))(p)=1$, $((a\supset a_1 v\ldots va_r)$ $(h))(p)=0$ and $((b\supset b_1 v\ldots vb_s)(h))(p)=0$. Since h is irreducible, h_p is irreducible, and there are irreducible h' and h" such the $h'\lesssim h_p$, $h''\lesssim h_p$, $(a(h'))(m_{h'})=1$. $(a(h''))(m_{h''})=1$, $((a_1 v\ldots va_r)(h'))(m_{h'})=0$ and $((b_1 v\ldots vb_s)(h''))(m_{h''})=0$. So $h'\varepsilon C^*_a$, $h''\varepsilon C^*_b$, for no i $(1\leqq i\leqq k)$, $h'\varepsilon C^*_{a_i}$, for no i $(1\leqq i\leqq s)$, $h'\varepsilon C^*_{b_i}$. This implies that $h'\equiv f$ and $h''\equiv g$. So $f\leq h_p$ and $g\leq h_p$, and we have completed the proof that c is connected.

Assume a corresponds to U, and a is connected. Now assume not $\vdash_{Pp} U\supset V$ and not $\vdash_{Pp} U\supset W$, with the I-operators b and c corresponding to V and W. Then there is an $f\varepsilon C^*_a$, $f\not\varepsilon C^*_b$ and a $g\varepsilon C^*_a$, $g\not\varepsilon C^*_c$. Since a is connected, there is an I^n-function $h\varepsilon C^*_a$ such that $f\leq h$ and $g\leq h$. This implies $h\not\varepsilon C^*_b$, $h\not\varepsilon C^*_c$ and $h\not\varepsilon C^*_b\vee C^*_c$, so $h\not\varepsilon C^*_{bvc}$. This means that $h\not\varepsilon C^*_{a\supset bvc}$, so not $\vdash_{Pp} U\supset VvW$. Now we have proved that for all V,W, if $\vdash_{Pp} U\supset VvW$, then $\vdash_{Pp} U\supset V$ or $\vdash_{Pp} U\supset W$. Then according to [12], $U|_{Pp}U$.

Th.4.10. If T is a consistent propositional calculus at least as strong as Pp, and
(**) for each U,V, if $U|_T U$, $\vdash_T U \supset V$ and $\vdash_T V \supset U$, then $V|_T V$, then T is Pp.

Proof. To begin with, the property (**) holds for Pp, since if $U|_{Pp} U$, then the operator corresponding to U is connected, and this is of course a property that is invariant under logical equivalence. If T contains a theorem not contained in Pp, then we will again use the formulas $U, V_1, \ldots, V_s$ used in the proof of Th.4.8. For these formulas, $U|_T U$, $\vdash_T U \supset V_1 v \ldots v V_s$ and $\vdash_{Pp} (V_1 v \ldots v V_s) \supset U$, so $\vdash_T (V_1 v \ldots v V_s) \supset U$, but not $V_1 v \ldots v V_s |_T V_1 v \ldots v V_s$, since not $V_1 v \ldots v V_s \vdash_T V_1$. This means that if T is stronger than Pp, then T does not have the property (**) and our theorem is proved.

Chapter V.

More Results about Definability of I-operators. Generalized I-operators.

The first part of this chapter will be devoted to some more results about the definability of I-operators, the most important result being that not all normal I-operators are standard. The last part will be devoted to a generalization of the concept of I^n-function to I^n-functions with infinite domains and to the consequent generalization of the concepts of I-operators, characteristic sets, etc. Here we will reach a completeness theorem, but we will have to use classical methods.

The clearest method to prove that not all normal I-operators are standard uses I^n-functions on trees. As this is also an interesting subject in itself, we will start with an exposition on these I^n-functions.

Def. A *tree* is a P.O.G.-set such that for all $t \varepsilon T$ the set $\{t' \varepsilon T: t \leq t'\}$ is finite and linearly ordered.

Def. If $f \varepsilon F^n$, then f is *tree-irreducible* if D_f is a tree, and for every $g \varepsilon F^n$, if g is a reduced form of f, and D_g is a tree, then $g \equiv f$.

Lemma 5.1. If T is a tree, then, for all $t\varepsilon T$, T(t) is a tree, and T-T(t) is a tree.

Proof. Trivial.

Th.5.1. If f is tree-irreducible, then for all $p\varepsilon D_f$ f_p is tree-irreducible.

Proof. By the lemma $D_f(p)$ is again a tree. If there were a non-isomorphic reduced form g of f_p by ϕ, then we could construct a non-isomorphic reduced form h (by ψ) of f by defining $D_h=(D_f-D_f(p))\cup D_g$, $q'\leq_h q$ iff $q'\leq_f q$ or $q'\leq_g q$ or $\phi^{-1}(q')\leq_f q$ for all $q',q\varepsilon D_h$, .$h(q)=g(q)$ for all $q\varepsilon D_g$, $h(q)=f(q)$ for all $q\varepsilon D_f-D_f(p)$, and $\phi(p')=p'$ for all $p'\varepsilon D_f-D_f(p)$, and $\phi(p')=\psi(p')$ for all $p'\varepsilon D_f(p)$.

Th.5.2. $f\varepsilon F^n$ is tree-irreducible iff D_f is a tree, and (1) f allows no α-reduction, (2) there are no $r,r',t\varepsilon D_f$, such that $r\neq r'$, r' is an immediate predecessor of t, $r\leq_f t$ and $f_{r'}\equiv f_r$.

Proof. $\Longrightarrow$(1) is obvious.

(2) Assume there are $r,r'\varepsilon D_f$ such that r' is an immediate predecessor of t and $r\leq_f t$, and $f_r\equiv f_{r'}$ by ϕ. Then define g as the restriction of f to $D_f-D_f(r')$. By the lemma D_g is a tree. Now define ψ on D_f as follows: $\psi(p)=p$ iff $p\not\varepsilon D_f(r')$, $\psi(p)=\phi(p)$ iff $p\varepsilon D_f(r')$. To prove that ψ is strongly isotone we have to prove the properties (i) and (ii) of the definition of strongly isotone. Property (i) is immediately obvious. To prove (ii), assume $\psi(q)\leq\psi(p)$.

Now there are three possibilities:

I. $\psi(q)\notin D_f(r)$, $\psi(p)\notin D_f(r)$. Then $\psi(q)=q$, $\psi(p)=p$, so $q\leq p$.

II. $\psi(q)\varepsilon D_f(r)$, $\psi(p)\notin D_f(r)$. Then $\psi(p)=p$, so $\psi(q)\leq p$, and $\psi(\psi(q))=\psi(q)$.

III. $\psi(q)\varepsilon D_f(r)$, $\psi(p)\varepsilon D_f(r)$. Then there are again two possibilities: (a) $\psi(p)=p$. Then $\psi(q)\leq\psi(p)=p$, and $\psi(\psi(q))=\psi(q)$. (b) $\psi(p)\neq p$, so $p=\phi(\psi(p))$ and $\phi(\psi(q))\leq p$, and $\psi(\phi(\psi(q)))=\psi(q)$, since on $D_f(r)$ ϕ is the inverse of ψ.

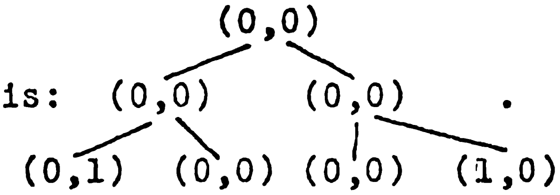
$\Longleftarrow$ This we will prove by induction on the depth of f. For depth 1 the result is trivial. So assume the depth of f is m and the theorem is valid for depth $<m$, and assume (1) and (2). Obviously (1) and (2) also hold for the sub-I^n-functions of f. So, if we assume that $p_1,\ldots,p_k$ $(k\geq 1)$ are the immediate predecessors of m_f w.r. to $<_f$, then the induction hypothesis assures us that $f_{p_1},\ldots,f_{p_k}$ are tree-irreducible. Now there are two possibilities:

I. $k=1$. Now, if g is a reduced form of f by ϕ, then $\phi(D_f(p_1))$ is siomorphic to $D_f(p_1)$, since according to the induction hypothesis f_{p_1} is tree-irreducible. But also $\phi(m_f)\neq\phi(p_1)$, otherwise ϕ would be an α-reduction-function, contrary to (1). This implies that $f\equiv g$ by ϕ, and f is tree-irreducible.

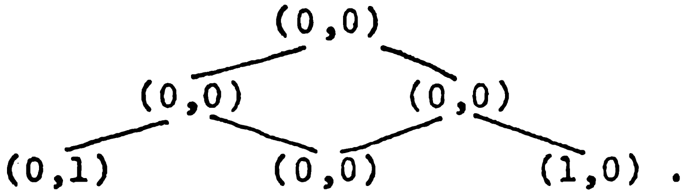
II. $k>1$. Again assume g is a reduced form of f by ϕ. Assume $r,r'\varepsilon D_f$, $r\neq r'$ and $\phi(r)=\phi(r')$. If we assume that $r'=m_f$, thenfor some m $(1\leq m\leq k)$ $r\not\leq p_m$. But then

$\phi(p_m)\leq\phi(m_f)=\phi(r)$, and, since ϕ is strongly isotone, for some $s\leq r$ $\phi(s)=\phi(p_m)$. This means that we can assume that $r\neq m_f$ and $r'\neq m_f$. In that case for exactly one i and exactly one j ($1\leq j,i\leq k$) $r\leq p_i$ and $r'\leq p_j$, since D_f is a tree. Also $i\neq j$, otherwise $g_{\phi(p_i)}$ would be a non-congruent reduced form of f_{p_i} contrary to the induction hypothesis. But $\phi(p_i)\leq\phi(p_j)$ or $\phi(p_j)\leq\phi(p_i)$, otherwise D_g would not be a tree. Assume $\phi(p_j)\leq\phi(p_i)$. Then, since ϕ is strongly isotone, $\phi(p_j)=\phi(q)$ for some $q\leq p_i$. Since f_{p_j} is tree-irreducible, $\phi(D_f(p_j))$ is isomorphic to $D_f(p_j)$, and for the same reason, $\phi(D_f(q))$ is isomorphic to $D_f(q)$. Now, if $s\leq p_j$, then $\phi(s)\leq\phi(q)$. So for some $s'\leq q$ $\phi(s')=\phi(s)$. But the fact that $\phi(D_f(q))$ is isomorphic to $D_f(q)$.then implies that this s' is unique. The same thing holds inversely, so $f_{p_j}\equiv f_q$, and $q\leq m_f$, the immediate successor of p_j, contrary to (2). So for all $r,r'\varepsilon D_f$, $r\neq r'$ implies $\phi(r)\neq\phi(r')$. The properties of strongly isotone functions then imply that ϕ is an isomorphism. So f is a tree-irreducible I^n-function.

An example of an I^2-function that is tree-irreducible, but not irreducible, is: .



It allows a β-reduction to the irreducible I^2-function:



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Th.5.3. If $f \varepsilon F^n$, D_f is a tree, and p_1 is the only immediate predecessor of m_f w.r. to $<_f$, then f is tree-irreducible iff, f_{p_1} is tree-irreducible and $f(m_f) \neq f(p)$.

Proof. Immediate from Th.5.2.

Th.5.4. If $f \varepsilon F^n$, D_f is a tree, and $p_1,\ldots,p_k$ $(k \geq 2)$ are the only immediate predecessors of m_f w.r. to $<_f$, then f is tree-irreducible iff, for all i $(1 \leq i \leq k)$ f_{p_i} is tree-irreducible and for no i,j $(1 \leq i,j \leq k,\ i \neq j)$ $f_{p_i} \leq f_{p_j}$.

Proof. $\Longrightarrow$ Immediate from Th.5.1 and Th.5.2.

$\Longleftarrow$ Assume f not tree-irreducible, and apply Th.5.2. As no α-reduction is possible, there must be $r,r' \varepsilon D_f$ $(r \neq r')$ such that $f_r \equiv f_{r'}$, and $r \leq t$, t being the direct successor of r'. Obviously $r < m_f$ and $r' < m_f$, so for some i,j $(1 \leq i,j \leq k)$ $r \leq p_i$ and $r' \leq p_j$. As p_i is tree-irreducible $i \neq j$. But then $t = m_f$, $r' = p_j$ and $f_{p_j} \leq f_{p_i}$, contrary to hypthesis. So f is tree-irreducible.

We will now prove that in each equivalence-class of I^n-functions there is a tree-irreducible I^n-function, unique up to congruence. The meaning of this theorem is that in our discussions in the Chapters II and III we could have restricted ourselves to I^n-functions on trees instead of P.O.G.-sets. The intuitive interpretation of Chapter I does not give grounds either for or against restricting ourselves to trees. Intuitively that means the choice between excluding or not excluding the possibility that two states incomparable in time both have the same possible future state in common.

Lemma 5.2. If P is a finite P.O.G.-set, and for all $p \varepsilon P$ p has at most one immediate successor, then P is a tree.

Proof. Take any $p \varepsilon P$.. Then define a sequence $p_0, \ldots, p_m$ for some $m \geq 0$ in the following way: $p = p_0$; for all integers i, if p_{i-1} is the maximum element of P, then $i-1=m$. If p_{i-1} is smaller than this maximum element, then p_i is the unique immediate successor of p_{i-1}. The sequence thus obtained is the set $\{p' \varepsilon P: p \leq p'\}$, and so this set is linearly ordered, and P is a tree.

Th.5.5. For any $h \varepsilon F^n$, there is an f such that $h \approx f$ and f is tree-irreducible. This f is unique up to congruence.

Proof. Assume $g\varepsilon F^n$, g irreducible and $g\approx h$. We will construct a tree-irreducible f such that $f\approx g$. If D_g is a tree, our problem is solved. So we assume that D_g is not a tree. By lemma 5.2 there is then an $r\varepsilon D_g$ such that r has more than one immediate successor. Let us assume that r is minimal with respect to this property, and that $s_1,\dots,s_k$ are the immediate successors of r. Then $D_g(r)$ is a tree, and, for all $r'\varepsilon D_g(r)$, if $r'\leq s$, then $s\leqq r$ or $r\leqq s$. Now we take k-1 trees from A, $T_1,\dots,T_{k-1}$, disjoint from D_g and from each other, such that for all i $(1\leqq i\leqq k-1)$; T_i is isomorphic to $D_g(r)$ by ϕ_i. Then we define an I^n-function g' as follows: $D_{g'}=D_g\cup\bigcup_{i=1}^{k-1}T_i$; for all $p\varepsilon D_g$ $g'(p)=g(p)$ and for all $p\varepsilon T_i$ $(1\leqq i\leqq k-1)$ $g'(p)=g(\phi_i(p))$; and for all $p',p\varepsilon D_{g'}$, $p'\leqq_{g'}p$ iff, either $p',p\varepsilon D_g(r)$ and $p'\leqq_g p$, or $p',p\varepsilon D_g-D_g(r)$ and $p'\leqq_g p$, or for some i $(1\leqq i\leqq k-1)$ $p',p\varepsilon T_i$ and $\phi_i(p')\leqq_g\phi_i(p)$, or for some i $(1\leqq i\leqq k-1)$ $p'\varepsilon T_i$ and $s_i\leqq_g p$, or $p'\varepsilon D_g(r)$ and $s_k\leqq_g p$. Then the function ϕ from $D_{g'}$ onto D_g defined by $\phi(p)=p$ for all $p\varepsilon D_g$ and $\phi(p)=\phi_i(p)$ for $p\varepsilon T_i$, is strongly isotone. So g' is equivalent to g. If $D_{g'}$ is not a tree, then we repeat the same procedure for g' etc. . As the number of elements with more than one immediate successor deminishes each time, the process must end. The end product f has then a tree as domain. We will prove by induction on the depth of g (= the depth of f) that f is tree-irreducible. For depth 1 this is trivial. So we now assume that the depth of g is m and that this process applied to any irreducible I^n-function of depth $<m$ delivers a tree-irreducible I^n-function. When we look at the construction of f above, we see that $m_f=m_g$, and that the immediate predecessors $p_1,\ldots,p_k$ of m_g w.r. to $<_g$ are also the immediate predecessors of m_f w.r. to $<_f$. We also see in that construction that the same process was applied to g_{p_i} for all i $(1\leq i\leq k)$ with the function f_{p_i} as outcome for all i $(1\leq i\leq k)$. The induction hypothesis then states that f_{p_i} is tree-irreducible for all i $(1\leq i\leq k)$. Now we study two cases. (1). $k=1$. Then $f(m_f)=g(m_g)\neq g(p_1)=f(p_1)$, and by Th.5.3 f is tree-irreducible. (2). $k>1$. Then for no i,j $(1\leq i,j\leq k, i\neq j)$ $f_{p_i}\leq f_{p_j}$, since that would imply $g_{p_i}\leq g_{p_j}$ (see lemma 2.2), which in its turn would imply that g is not irreducible(by lemma 2.3). Then Th.5.4 implies that f is irreducible.

Now we prove that f is unique, by induction on the depth of f. For depth 1 it is again trivial. If the depth of f is m, the we assume the theorem for I^n-functions with depth $<m$. Assume that $p_1,\ldots,p_k$ are the immediate predecessors of m_f w.r. to $<_f$. Then according to the induction hypothesis $f_{p_1},\ldots,f_{p_k}$ are uniquely determined. It is then very easy to see that f is also uniquely determined.

Lemma 5.3 If f,g are tree-irreducible, $f\approx h_1$, $g\approx h_2$, h_1 and h_2 irreducible and $h_1\leq h_2$, then $f\leq g$.

Proof. Clear from the construction in Th.5.5.

Th.5.6. A normal I-operator a is uniquely characterized set C_a^{**} of all tree-irreducible $f\varepsilon F^n$ in its characteristic set (the *tree-characteristic* set of a), and there is a function from C_a^* onto C_a^{**} that is an isomorphism w.r. $\leq$, and if it maps f onto g then $f\approx g$.

Proof. Immediate from Th.5.5 and lemma 5.3.

Th.5.7. For $n\geq 2$ not all normal n-ary I-operators are standard.

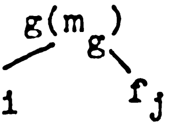
Proof. We will construct a sequence of I^2-functions $\{u_{ij}\}_{\substack{i=1,\dots,\infty\\ j=1,2,3}}$ as follows by induction on i (it is obvious how to do this in an exact way, but very tiresome, so we will do it with the help of pictures) $u_{11}=(1,1)$, $u_{12}=(1,0)$, $u_{13}=(0,1)$, $u_{i+1}=\begin{array}{c}(0,0)\\ \swarrow\ \ \searrow\\ u_{i1}\ \ \ u_{i2}\end{array}$, $u_{i+2}=\begin{array}{c}(0,0)\\ \swarrow\ \ \searrow\\ u_{i1}\ \ \ u_{i3}\end{array}$, , $u_{i+3}=\begin{array}{c}(0,0)\\ \swarrow\ \ \searrow\\ u_{i2}\ \ \ u_{i3}\end{array}$, for all $i\geq 1$. Then we can prove by induction on i (a) for all i,j ($i\geq 1, 1\leq j\leq 3$) u_{ij} is tree-irreducible, and (b) for all $i\leq 1$, if $j\neq k$ ($1\leq j,k\leq 3$) $u_{ij}\nleq u_{ik}$. For i=1 it is trivial and if (b) is true for i=k, then by Th.5.4 (a) is true for i=k+1, and (b) follows immediately for i=k+1.

Now we construct a sequence $\{v_i\}$ of I^2-functions by induction on i from the sequence $\{u_{ij}\}$. $v_i=\begin{array}{c}(0,0)\\ \swarrow\ \ \downarrow\ \ \searrow\\ u_{i1}\ \ u_{i2}\ \ u_{i3}\end{array}$, for all $i\geq 1$.

Again it is obvious that for all i ($1\leq i<\infty$) v_i is tree-irreducible, but also for all i,j ($i\geq 1, j\geq 1$) if $i\neq j$ then $v_i\not\leq v_j$. This is obvious if i>j, and if i<j, then $v_i\leq v_j$ would imply $v_i\equiv u_{km}$, for some k and m ($1\leq k\leq j, 1\leq m\leq 3$), and this is impossible since $D_{u_{km}}$ is a binary tree, and m_{vi} has three immediate predecessors. Now we can for any set of natural numbers $M\subseteq N$ (N being the set of all natural numbers) define an operator a_M, by its tree-characteristic set, $f\in C^{**}_{a_M}$ iff $f\leq v_i$ for some $i\in M$. Now it is obvious from the fact that $i\neq j$ then $v_i\not\leq v_j$ that, if $M\neq L$, then $C^{**}_{a_M}\neq C^{**}_{a_L}$. This implies that there are non-denumerably many normal I-operators, and so they cannot all be standard.

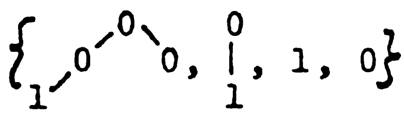
Note that not even all primitive recursive normal I-operators are standard, since the logic is decidable. This is of course a negative theorem, and we will now prove that no I-operator like a_M with M infinite can be a standard I-operator.

Def. A set $\{f_1,\ldots f_m\}\subseteq F^n$ is *independent*, if for no i,j ($1\leq i,j\leq m, i\neq j$) $f_i\leq f_j$.

Def. A normal n-ary I-operator a is *weakly connected of degree m*, if for any independent sequence $f_1,\ldots,f_{m+1}\in C^{**}_a$ there exists a $g\in C^{**}_a$ of the form 

for some i,j ($1\leq i, j\leq m, i\neq j$). (I.e. if there exists a $g\in C^{*}_a$, such that m_g has two direct predecessors q_1 and q_2 in D_g, and $g_{q_1} \equiv f_1$ and $g_{q_2} \equiv f_j$).

Note that, if a is weakly connected of degree 1, then a is connected. Not even for standard I-operators though are these concepts equivalent; e.g. the unary standard I-operator with tree-characteristic set

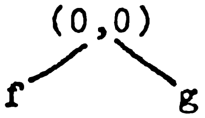


is connected like all finite normal I-operators of which the tree-characteristic (or normalized characteristic) set has only one maximal element, but for the I-functions 0 and 1 we cannot find an I-function g as required by the definition. On the other hand this I-operator is trivially weakly connected of degree 2, since there are no subsets of more than two elements of its tree-characteristic set that are independent.

<u>Def</u>. A normal I-operator is *weakly connected*, if it is weakly connected of degree m for some m. A normal I-operator is *disconnected*, if it is not weakly connected.

From the remarks just made it is easy to conclude that all finite normal I-operators are weakly connected. Examples of disconnected I-operators are the binary I-operators a_M defined in the proof of Th.5.7, in the cases that M is infinite. We will prove that all standard I-operators are weakly connected, but that not all weakly connected normal I-operators are standard. We wil prove the last statement first.

Th.5.8. Not all binary weakly connected normal I-operators are standard.

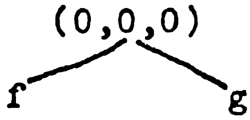
Proof: We consider the I-operators a_M for subsets M of N constructed in the proof of Th.5.7. We define a binary operation C on the set of all independent couples from $(F^2)^2$ by, $C(f,g)=$ 

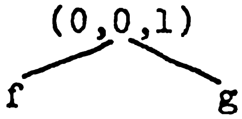
. Then we take for all $M \leq N$ the closure S_M of $C^{**}_{a_M}$ w.r. to C. The set S_M has all the properties required for a tree-characteristic set. Now define for all $M \leq N$ b_M as the normal I-operator with tree-characteristic set S_M. For all $M \leq N$ b_M is weakly connected of degree 1, and for all $M,L \leq N$, if $M \neq L$, then $b_M \neq b_L$. This implies that there are nondenumerably many binary weakly connected normal I-operators, and not all of these can be standard.

The proof of this theorem shows that the sharpest characterization we have as yet been able to give, is by no means sharp enough. It seems that we need a more restrictive concept in the spirit of weakly connected. The set of n-ary weakly connected I-operators is not even a closed set, at least for $n \geq 3$. (for binary I-operators we have not been able to prove this, for unary I-operators the whole situation is special, as we will see later).

Th.5.9. The set of ternary weakly connected normal I-operators is not closed.

Proof. Since & is a finite normal I-operator (Th.3.3), & is weakly connected. In fact & is weakly connected of degree 1. We will now construct two ternary normal I-operators that are weakly connected of degree 1 of which the conjunction is disconnected. Define a_N as in the proof of Th.5.7. Then define the ternary I-operator a' by, for all $f \varepsilon F^3$, $f \varepsilon C^{**}_{a'}$ iff $(f^1,f^2) \varepsilon C^{**}_{a_N}$ and for all $p \varepsilon D_f$, $f^3(p)=1$. And we define the binary operations C and C' on the set of independent couples from $(F^3)^2$ by

C(f,g)= 

and C'(f,g)= 

. Then we take the closures S and S' of $C^{**}_{a'}$ w.r. to C and C'. S and S' again have all the properties required for tree-characteristic sets and $S \cap S' = C^{**}_{a'}$. This means that for the normal I-operators b' and c' defined by $C^{**}_{b'}=S$ and $C^{**}_{c'}=S'$, b'&c'=a'. But is clear that b' and c' are weakly connected, while a' is not weakly connected. This means that the set of ternary weakly connected normal I-operators is not closed.

As &,b' and c' are connected and a' is not, this proof also shows that the set of all connected normal I-operators is not closed. The proof shows too that we cannot prove that all standard I-operators are weakly connected by a simple induction over the number of occurrences of the symbols &, v, $\supset$ and $\neg$ in the definition of the I-operator, since, if a and b are weakly connected, a&b is not necessarily so.

Th.5.10. All standard I-operators are weakly connected.

Proof. By induction over the length of the definition of the I-operator a. If a has length 1, then $a=u_i^n$ for some i. u_i^n is weakly connected of degree 1, since, if $f_1, f_2 \varepsilon C^{**}_{u_i^n}$ and f_1, f_2 independent, then the g defined by, $g^j(m_g)=0$ for $j \neq i$ $(1 \leq j \leq n)$ and g=

$$\begin{tikzcd}[column sep=small] & g(m_g) \arrow[dl, dash] \arrow[dr, dash] & \\ f_1 & & f_2 \end{tikzcd}$$

has the required properties.

Now we assume the theorem is valid for all standard I-operators with length $\leq k$, and we assume a has length k+1. We will treat a as $a \Rightarrow \neg(u_1^n \supset u_1^n)$. This means that we have to look at the cases $a=\neg(u_1^n \supset u_1^n)$, a=bvc, $a=b \supset c$ and a=b&c.

(i). $a=\neg(u_1^n \supset u_1^n)$. In that case $C_a^{**}=\emptyset$, and a is trivially weakly connected of degree 1.

(ii). a=bvc. By the induction hypothesis b and c are weakly connected, assume of degrees k and m. We will prove that then a is weakly connected of degree k+m. Assume $f_1, \ldots, f_{k+m+1}$ independent and in C_a^{**}, then, since

$C^{**}_a=C^{**}_b \cup C^{**}_c$, either if necessary after renumbering $f_1,\ldots,f_{k+1}\varepsilon C^{**}_b$, or $f_1,\ldots,f_{m+1}\varepsilon C^{**}_c$. In both cases we find a g as required.

(iii) a=b⊃c. By the induction hypothesis c is weakly connected, assume of degree m. We will prove that a is weakly connected of degree m. Assume $f_1,\ldots,f_{m+1}$ independent and in $C^{**}_{b\supset c}$, then there are two possibilities:

(1) for all i ($1\leq i\leq m+1$) $f_i\varepsilon C^{**}_c$; then the g we find in C^{**}_c is also an element of $C^{**}_{b\supset c}$,

(2) for some i ($1\leq i\leq m+1$) $f_i\not\varepsilon C^{*}_c$; then take $j\neq i$ ($1\leq j\leq m+1$) and define $g=\underset{f_i \qquad\quad f_j}{(0,0,\ldots,0)}$. Now $f_i\not\varepsilon C^{**}_c$ implies $f_i\not\varepsilon C^{**}_b$, since $f_i\varepsilon C^{**}_{b\ c}$. But then also $g\not\varepsilon C^{**}_b$, $g\not\varepsilon C^{**}_c$. This implies that for all $p\varepsilon D_g$, if $g_p\varepsilon C^{**}_b$, then $g_p\varepsilon C^{**}_c$, so $g\varepsilon C^{**}_{b\ c}$.

(iv) a=b&c. There are three subcases:

(1) $b=u^n_i$, $c=u^n_j$ for some $i,j\leq n$. Then a is weakly connected of degree 1. The proof is similar to the one for case (i).

(2) $b=b_1vb_2$, or $c=c_1vc_2$. Then since $(b_1vb_2)\&c=(b_1\&c)v(b_2\&c)$ and $b\&(c_1vc_2)=(b\&c_1)v(b\&c_2)$ and $b_1\&c$, $b_2\&c$, $b\&c_1$ and $b\&c_2$ are weakly connected by the induction hypothesis, we can apply (ii) again.

(3) Since we can write $(u^n_1\supset u^n_1)\supset u^n_i$ for u^n_i, the only case left to investigate is $a=\bigcap_{i=1}^{n}(a_i\supset b_i)$. We will give the proof for the case that m=2, it is easily seen that the

proof for the general case is similar. By the induction hypothesis b_1,b_2 and $b_1\&b_2$ are weakly connected, assume of respective degrees p,q,r. Assume $s=\mathrm{Max}(p,q,r)$. Then we will prove that a is weakly connected of degree $3s+1$.

Let us assume $f_1,\dots,f_{3s+1}$ is an independent sequence in C^{**}_a. Then (after renumbering if necessary) there are four possible cases.

I. $f_1,\dots,f_{s+1}\varepsilon C^{**}_{b_1}$, $f_1,\dots,f_{s+1}\not\varepsilon C^{**}_{b_2}$. Then $f_1,\dots,f_{s+1}\varepsilon C^{**}_{b_1\&b_2}$, so we can find a g with the required properties in $C^{**}_{b_1\&b_2}$. Then $g\varepsilon C^{**}_a$, since $C^{**}_{b_1\&b_2}\subseteq C^{**}_a$.

II. $f_1,\dots,f_{s+1}\varepsilon C_{b_1}$, $f_1,\dots,f_{s+1}\not\varepsilon C^{**}_{b_2}$. Then for some i,j $(1\leq i,j\leq s+1)$ $g=\underset{f_i\qquad f_j}{g(m_g)}\varepsilon C^{**}_{b_1}$, so $g\varepsilon C^{**}_{a_1\supset b_1}$. Now, since $f_i\not\varepsilon C^{**}_{b_2}$, $f_i\not\varepsilon C^{**}_{a_2}$, also $g\not\varepsilon C^{**}_{b_2}$ and $g\not\varepsilon C^{**}_{a_2}$.

III. $f_1,\dots,f_{s+1}\not\varepsilon C^{**}_{b_1}$, $f_1,\dots,f_{s+1}\varepsilon C^{**}_{b_2}$. Proof similar to case II.

IV. $f_1\not\varepsilon C^{**}_{b_1}$, $f_1\not\varepsilon C^{**}_{b_2}$. Then take $g=\underset{f_1\qquad\qquad f_2}{(0,0,\dots,0)}$ and again $g\varepsilon C^{**}_a$.

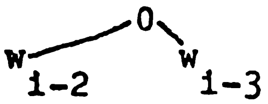
The unary normal I-operators take a very special place in the set of all I-operators. We are able to prove that all unary normal I-operators are standard, since the only infinite normal I-operator is $u^1_1\supset u^1_1$. This we will prove now in the following theorem.

Th.5.11. All unary normal I-operators are standard.

Proof. We define a sequence of the irreducible I-functions $w_i (i=1,\ldots,)$ with the help of pictures in the following way: $w_0=$ 1 , $w_1=$ 0 , $w_2=$

, for all $i\geq 3$

$w_i=$



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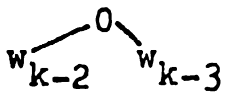
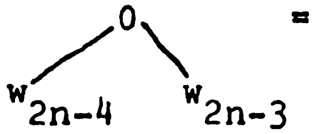
We will prove for all $i\geq 0$ that w_i is tree-irreducible and that for all $j\geq 0$, $w_j\leq w_i$ iff $j=i$ or $j\leq i-2$, by induction on i. The statement is clearly true for $i=0,1,2$. Let us assume $k\geq 3$ and the statement is valid for all $i<k$.

$w_k=$

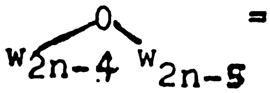


. According to the induction hypothesis w_{k-2} and w_{k-3} are tree-irreducible and $w_{k-3}\not\leq w_{k-2}$. If $j>i$, then clearly $w_j\not\leq w_i$, so $w_{k-2}\not\leq w_{k-3}$. Then according to Th.5.2 w_k is tree-irreducible. Now assume $j\leq k-2$, then there are three possible cases: (1) $j\leq k-4$, then $w_j\leq w_{k-2}\leq w_k$, by the definition of w_k. (2) $j=k-3$, then $w_j\leq w_k$ by the definition of w_k. (3) $j=k-2$, then $w_j\leq w_k$ by the definition of w_k. To conclude, not $w_{k-1}\leq w_k$, since not $w_{k-1}\leq w_{k-2}$, $w_{k-1}\leq w_{k-3}$, and not $w_{k-1}=w_k$.

Next we will prove that this sequence is complete in the sense that all I-functions are equivalent to w_i for some natural numbers. According to the Th.5.5 it is sufficient to prove that all tree-irreducible I-functions are congruent to w_i for some i. We will prove this for tree-irreducible I-functions f by induction on the depth of f.

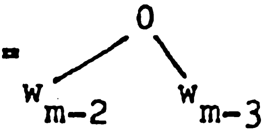
If f has depth 1, then f is clearly congruent to either w_0 or w_1. It is also clear that for all $i \geq 0$, w_{2i-2} and w_{2i-1} have depth i. Now assume f has depth $n>1$, and assume the statement we want to prove is valid for all I-functions with depth $<n$. Assume further m_f has immediate predecessors $p_1,\ldots,p_k$ $(k \geq 1)$. Then for all i $(1 \leq i \leq k)$ f_{p_i} has depth $<n$, so according to the induction hypothesis $f_{p_i} \leq w_j$ for some j $(0 \leq j \leq 2n-1)$. There are now two possible cases: (1) k=1. Then $f(m_f)=0$, $f(p_1)=1$, otherwise there would exist an α-reduction of f w.r.t. to p_1, m_f. Then $f_{p_1} \equiv 1 = w_0$, since, if not $f_{p_1} \equiv 1$, then f_{p_i} is not tree-irreducible and neither is f (Th.5.1). So $f = w_2$. (2) $k \geq 2$. Since for all i,j if $j \leq i-2$ then $w_j \leq w_i$, according to Th.5.4 k=2. That the depth of f is n implies that f≡ 

$= w_{2n-1}$ or

f≡ 

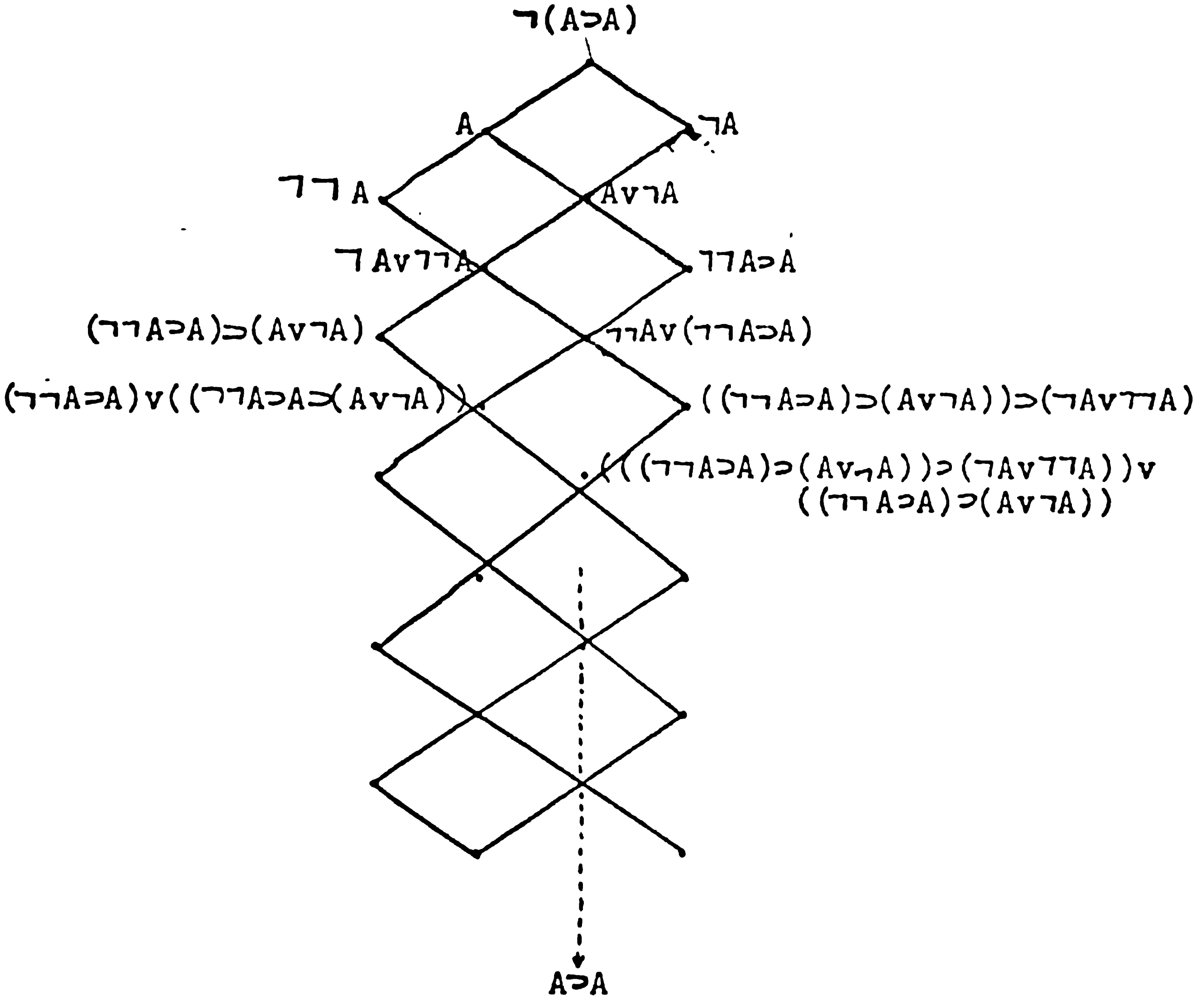
$= w_{2n-2}$

Now we define a sequence c_{ij} $(i=1,\ldots\infty, j=1,2)$ of unary normal I-operators by their tree-characteristic sets. $C^{**}_{c_{01}} = \emptyset$. $C^{**}_{c_{02}} = \{w_i\}_{i=1,\ldots,\infty}$. $C^{**}_{c_{12}} = 0$. For all $i \geq 2$ $C^{**}_{c_{i1}} = \{f \varepsilon F\colon f \leq w_{i-1} \text{ or } f \leq w_{i-2}\} = \{w_0,\ldots,w_{i-1}\}$. $C^{**}_{c_{i2}} = \{f \varepsilon F\colon f \leq w_i\} = \{w_0,\ldots,w_{i-2},w_i\}$. It is clear that this sequence contains all unary normal I-operators. All these I-operators are finite except c_{02}, and $c_{02}=u^1_1\supset u^1_1$, so all unary normal I-operators are standard according to Th.3.7.

But we will give here a simpler way of defining the c_{ij}. We will prove that $c_{01}=\neg(u^1_1\supset u^1_1)$, $c_{02}=u^1_1\supset u^1_1$, $c_{11}=u^1_1$, $c_{12}=\neg$, $c_{21}=u^1_1\vee\neg(u^1_1)$, $c_{22}=\neg\neg$, and for all $i\geq 3$ $c_{i1}=c_{i-1,2}\vee c_{i-2,2}$, $c_{i2}=c_{i-1,2}\supset c_{i-2,1}$. This is evident for c_{01}, c_{02}, c_{11}, c_{12} and c_{21}. f tree-irreducible and $f\in C^*_{\neg\neg}$, iff for no $g\leq f$ $g\in C^*_{\neg}$, i.e. for no $g\leq f$, $g=0$. This is true only for w_0,w_2. And $\{w_0,w_2\}=C^*_{c_{22}}=C^{**}_{c_{22}}$, so indeed $C_{22}=\neg\neg$. Now we prove the last part by induction on i. Assume $m\geq 3$ and assume the definitions are valid for all $i<m$. Then we have to prove $c_{m1}=c_{m-1,2}\vee c_{m-2,2}$, which follows from $C^{**}_{c_{m1}}=C^{**}_{c_{m-1,2}}\cup C^{**}_{c_{m-2,2}}=C^{**}_{c_{m-1,2}\vee_{m-2,2}}$ (reasoning like in lemma 3.1 (c)). And we have to prove $c_{m2}=c_{m-1,2}\supset c_{m-2,1}$. To prove this last statement it is sufficient to establish that $w_m\in C^{**}_{c_{m2}\supset c_{m-1,1}}$, $w_{m-1}\notin C^{**}_{c_{m2}\supset c_{m-1,1}}$, since $C^{**}_{c_{m2}}=\{w_0,\ldots,w_{m-2};w_m\}$. According to the induction hypothesis $w_{m-1}\in C^{**}_{c_{m-1,2}}$, $w_m,w_{m-2}\notin C^{**}_{c_{m-1,2}}$, $w_m,w_{m-1},w_{m-2}\notin C^{**}_{c_{m-2,1}}$. In the first place this implies $w_{m-1}\notin C^{**}_{c_{m2}\supset c_{m-1,1}}$. Further $w_m=$ 

and for all sub-I-functions f of w_{m-2} that are not congruent to w_{m-2}, $g\varepsilon C^{**}_{c_{m-1,2}}$ and $g\varepsilon C^{**}_{c_{m-2,1}}$; and $w_{m-3}\varepsilon C^{**}_{c_{m-1,2}}$ and $w_{m-3}\varepsilon C^{**}_{c_{m-2,1}}$. All this implies that indeed $w_m\varepsilon C^{**}_{c_{m-1,2}\supset c_{m-2,1}}$, since for all $p\varepsilon D_{w_m}$, if $(w_m)_p\varepsilon C^{**}_{c_{m-1,2}}$, then $(w_m)_p\varepsilon C^{**}_{c_{m-2,1}}$.

Now we are able to give a sequence of formulas that comprises all equivalence classes of formulas formed from a single atom A. The formulas here seem to have the shortest length possible. (See for a very similar result [18]).



In the last part of this chapter we will give a short description of how we can generalize our concepts to I^n-functions with infinite domains. We will restrict our attention to countable domains. The generalization to higher cardinalities is easy to make.

Let B' be the set of all P.O.G.-sets from A.

Def. An *II-function* is a function with domain a P.O.G.-set $P \varepsilon B'$ and range the set $\{0,1\}$ with the property: for all $p,p' \varepsilon P$, if $p' \leqq p$ and $f(p)=1$, then $f(p')=1$.

We can now define the concepts of II^n-function, congruence of II^n-functions, II-operator, ordered II-operator, and characteristic set of an II-operator in exactly the same way as in Chapter II. We write D_f for the domain of an II^n-function f, I^n for the set of all II^n-functions, I for I^1. The generalization of the notions of normal form, equivalence and normal I-operator gives some difficulties. If we define normal form in the same way as for I^n-functions, by means of reductions, then not all II^n-functions have a normal form. Th.2.2 is not valid for infinite partially ordered sets. E.g. take $<N,\leqq_1>$ where for all $m,n \varepsilon N$, $m \leqq_1 n$ iff $n=0$, and the set $\{0,1\}$ with the normal ordering. Then there is a strongly isotone function from N onto $\{0,1\}$ (for all n $\phi(n+1)=0$, $\phi(0)=1$), but $\{0,1\}$ cannot be reached from $<N,\leqq_1>$ by reductions. We succeed in defining a normal form with the help of the strongly isotone functions. We can define the concept of reduced form in the same way as in Chapter II. We then give a definition suggested by Th.2.1. Cor.2.

Def. An II^n-function is *irreducible*, if for any $g\varepsilon I^n$, if g is a reduced form of f, then $g\equiv f$.

Def. An II^n-function g is a *normal form* of the II^n-function f, if g is a reduced form of f, and g is irreducible.

We have not succeeded in giving a direct proof of an equivalent of the uniqueness theorem Th.2.3. But we can give an indirect proof based on Th.4.6 of [9].

Th.5.12. (Th.4.6 of [9].) If P and Q are partially ordered sets, then $\overline{Q}$ is a complete subalgebra (i.e. a subalgebra w.r. to $\cup, \cap, \Rightarrow, -, \bigcup$ and $\bigcap$) of $\overline{P}$ iff there exists a strongly isotone mapping from P onto Q. In fact, if ϕ is a strongly isotone function from P onto Q, then the subalgebra A of $\overline{P}$ defined by, $A=\{\alpha\varepsilon\overline{P}$: for all $p,q\varepsilon P$ if $p\varepsilon\alpha$ and $\phi(p)=\phi(q)$ then $q\varepsilon\alpha\}$, is isomorphic to $\overline{Q}$ and forms a complete subalgebra of $\overline{P}$.

Th.5.13. If $f,g\varepsilon I^n$ and g is a reduced form of f, then $\overline{D_g}$ is isomorphic to a complete subalgebra of $\overline{D_f}$ that contains for all i $(1\leq i\leq n)$ the elements $\alpha_i=\{p\varepsilon D_f\colon f^i(p)=1\}$.

Proof. According to Th.5.12, if g is a reduced form of f by ϕ, then $\overline{D_g}$ is isomorphic to the complete subalgebra of $\overline{D_f}$ formed by the set A of elements $\alpha \varepsilon \overline{D_f}$ with the property that for all $p\varepsilon\alpha$, $q\varepsilon D_f$, if $\phi(p)=\phi(q)$, then $q\varepsilon\alpha$. Assume for some i $(1\leq i\leq n)$ $p\varepsilon\alpha_i$ and $\phi(p)=\phi(q)$. Then $f(p)=g(\phi(p))=g(\phi(q))=f(q)$. So, since $f^i(p)=1$, also $f^i(q)=1$, and $q\varepsilon\alpha_i$. So we have proved that for all i $(1\leq i\leq n)\alpha_i\varepsilon A$.

The next theorem besides giving us the necessary apparatus to prove that the normal form is unique up to congruence, also gives us some more insight in the results of Chapters II and III.

Th.5.14. If $f,g\varepsilon I^n$, and g is a normal form of f, then $\overline{D_g}$ is isomorphic to the complete subalgebra of $\overline{D_f}$ generated by the elements $\alpha_i=\{p\varepsilon D_f: f^i(p)=1\}$ $(1\leq i\leq n)$ (i.e. the smallest complete subalgebra containing the α_i) and if ψ is the isomorphism then for all $r\varepsilon D_g$, $g^i(r)=1$ iff $r\varepsilon\psi^{-1}(\alpha_i)$.

Proof. $\overline{D_g}$ is isomorphic to a complete subalgebra A of $\overline{D_f}$. If B is the complete subalgebra of $\overline{D_f}$ generated by the α_i, then $B\subseteq A$. Now, according to Th.5.12 there is a strongly isotone function ϕ from D_g onto B^0. There is an II^n-function h definable on B^0 by, for all $r\varepsilon B^0$, $h(r)=g(s)$, if s is such that $\phi(s)=r$. This is a proper definition, for assume $s,s'\varepsilon D_g,\phi(s)=\phi(s')$, and assume $g^i(s)=1$ for some i $(1\leq i\leq n)$. Then $s\varepsilon\alpha_i$, and by Th.5.12 applied to $\overline{D_g}$ and B, $s'\varepsilon\alpha_i$, so $g^i(s')=1$. By the same reasoning, if $g^i(s')=1$ then $g^i(s)=1$, for all i $(1\leq i\leq n)$. This means that we have proven

$g(s)=g(s')$. So h is properly defined and h is a reduced form of f. As g was assumed to be irreducible it follows that $g\ h$, and $\overline{D_g}$ is isomorphic to B. The last part of the theorem now follows immediately.

<u>Th.5.15</u>. If $f,g,h\varepsilon I^n$, and g and h are normal forms of f, then $g\equiv h$.

<u>Proof</u>. Immediate from the Th.5.14, since, in the first place, both $\overline{D_g}$ and $\overline{D_h}$ are isomorphic to the same subalgebra B of $\overline{D_f}$, so D_g and D_h are isomorphic (Th.4.4), and in the second place, both f and h are determined by the partial ordering of the α_i in B.

Th.5.15 enables us to define the concepts of equivalence, normal II-operator and normalized characteristic set in the same way as in Chapter II. Also, the concept of standard II-operator can be defined in the same way as in Chapter III.

<u>Def</u>. If J has cardinality κ, and $\{a_i\}_{i\varepsilon J}$ is a set of normal II-operators, then $\bigcup_{i\varepsilon J}(a_i)$. $(\bigcap_{i\varepsilon J}(a_i))$ is defined as the II-operator with normalized characteristic set $\bigcup_{i\varepsilon J}(C^*_{a_i})$ $(\bigcap_{i\varepsilon J}(C^*_{a_i}))$. We call these "generalized" II-operators the *κ-disjunction* and *κ-conjunction*.

<u>Def</u>. A *quasi-standard II-operator* is an II-operator the intersection of the sets G of II-operators such that (1) G contains all standard II-operators, (2) J has cardinality $\kappa\leqq 2^{(2^{\aleph_0})}$ and for all $i\varepsilon J$, $a_i\varepsilon G$, implies $\bigcup_{i\varepsilon J}(a_i)\varepsilon G$

and $\bigcap_{i\in J}(a_i)\in G$.

Def. For any cardinal κ the set of *κ-pseudo-Boolean terms* is the intersection of all sets T such that (1) $\alpha,\beta,\gamma,\alpha_1,\alpha_2,\ldots,\beta_1,\beta_2,\ldots\in T$, (2) if U and V are in T, then $U\cup V$, $U\cap V$, $U\Rightarrow V$ and $-U$ are in T, (3) if J has cardinality $\leq\kappa$ and for all $i\in J$, $U_i\in T$, then $\bigcup_{i\in J}U_i\in T$ and $\bigcap_{i\in J}U_i\in T$.

Lemma 5.4. If the pseudo-Boolean algebra *A* has cardinality $2^{\aleph_0}$ and *A* is generated by $\{\alpha_1,\ldots,\alpha_n\}$, then all elements of *A* can be written as $2^{\aleph_0}$ -pseudo-Boolean terms in $\alpha_1,\ldots,\alpha_n$.

Proof. We can define a function from the set of atomic terms onto the set $\{\alpha_1,\ldots,\alpha_n\}$. Then according to the recursion principle for terms (3.2.1 of [10]) there is a homomorphism from the set of $2^{\aleph_0}$ -pseudo-Boolean terms into *A* that is an extension of this function. It is clear that the range of this mapping is a complete subalgebra of *A* containing $\alpha_1,\ldots,\alpha_n$. From this the lemma follows immediately.

Th.5.16. All normal II-operators are quasi-standard.

Proof. If a is a normal II-operator, then a has some normalized characteristic set $\{g_i\}_{i\in J}$. $\{g_i\}_{i\in J}=\bigcup_{i\in J}\{f\in I^n\colon f\leq g_i\}$, so, if for all $i\in J$ a_i are the normal II-operators with $C^*_{a_i}=\{f\in I^n;\ f\leq g_i\}$, then $a=\bigcup_{i\in J}(a_i)$. So we only have to consider the normal II-operators that have a normalized characteristic set with a greatest element. Let us assume then that

$C^*_a=\{f\in I^n: f\leq g\}$ for some irreducible g. Then $\overline{D_g}$ is generated by the $\alpha_i=\{q\in D_g: g^i(q)=1\}$ $(1\leq i\leq n)$. This means that $1_{\overline{D_g}}=D_g=U(\alpha_1,\ldots,\alpha_n)$ for some $2^{\aleph_0}$-pseudo-Boolean term U in $\alpha_1,\ldots,\alpha_n$, according to Lemma 5.4, since $\overline{D_g}$ cannot have more than $2^{\aleph_0}$ elements. Assume that $\{U_i\}_{i\in J}$ is the set of all pseudo-Boolean terms with this property. J then has at most cardinality $2^{(2^{\aleph_0})}$. Assume that, for all $i\in J$, a_i is the normal II-operator corresponding to U_i. (It is obvious that all quasi-standard II-operators are normal by the same reasoning as in Th.3.6.) Then we will prove that $a=\bigcap_{i\in J} a_i$. If we write $V= U_i$, then we have to show that $f\leq g$ iff $V(\beta_1,\ldots,\beta_n)=D_f$ where , for all i, β_i is defined as $\{p\in D_f: f^i(p)=1\}$. First assume $f\leq g$. Without losing generality we can assume that $f=g_q$ for some $q\in D_g$. Then $D_f=D_g(q)$ and $\overline{D_f}$ is a relativization of $\overline{D_g}$ (see Th.3.5 of [9]). Then there is a complete homomorphism from $\overline{D_g}$ onto $\overline{D_f}$ ("complete" meaning a homomorphism also w.r. to the infinite operations) defined by, for all $\overline{D_g}$, $\phi(\alpha)=\alpha\wedge D_f$. This implies, for all i $(1\leq i\leq n)$, that $\phi(\alpha_i)=\beta_i$. And, since ϕ is a complete homomrphism, $U(\beta_1,\ldots,\beta_n)=D_f$. Now assume $f\not\leq g$. Then take an irreducible II-function h such that $f\leq h$ and $g\leq h$. We again assume without losing generality that $f=h_s$ and $g=h_t$, for some $s,t\in D_h$. Let for all i ($1 \leq i \leq n$) $\gamma_i = \{r \varepsilon D_h : h^i(r)=1\}$. Then by lemma 5.4 for some $2^{\aleph_0}$-pseudo-Boolean term W, $W(\gamma_1,\ldots,\gamma_n)=D_g$. But then in $\overline{D_g}$ $W(\alpha_1,\ldots,\alpha_n)=D_g$. So for some $i \varepsilon J$, $W=U_i$. But in $\overline{D_f}$ $W(\beta_1,\ldots,\beta_n) \neq D_f$. So also in $\overline{D_f}$, $V(\beta_1,\ldots,\beta_n) \neq D_f$.

Th.5.17. The unary normal II-operators consist of the standard I-operator, and the II-operator $\bigcup_{i=1}^{\infty} c_{12}$.

Proof. It is easy to check that this system of II-operators is closed under the operations of $\&, \vee, \supset, \neg, \cap, \cup$.

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