# MODAL CORRESPONDENCE THEORY

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The greatest debt of all - and I am not ashamed of this well-worn phrase - is owed to my parents, however. Although this book will be nothing but a meaningless array of symbols to them, I dedicate it to

*A.K. van Benthem*

*J.M.G. van Benthem - Eggermont*

knowing that they will understand the spirit of this dedication.

## MODAL CORRESPONDENCE THEORY

CONTENTS:

PART I



PART II



The three papers in part II are reprinted here by the kind permission of the American Mathematical Society.

## I.1 INTRODUCTION

This dissertation is about a certain class of formulas of monadic second-order logic with a single binary predicate constant, the modal formulas. These formulas are of the form

$$(\forall P_1)\dots(\forall P_n)\phi(P_1,\dots, P_n, R),$$

where $P_1,\dots, P_n$ are unary predicate variables and R is the binary predicate constant. $\phi(P_1,\dots, P_n, R)$ is a formula of monadic first-order logic based on $P_1,\dots, P_n$ with restricted quantifiers. This can be stated more formally as follows. $\phi(P_1,\dots, P_n, R)$ belongs to the smallest class E of expressions satisfying the following four conditions,

(i) for each individual variable x, $P_1x,\dots, P_nx$ are expressions in E

(ii) if $\alpha$ is an expression in E, then so is $\neg\alpha$

(iii) if $\alpha$ and $\beta$ are expressions in E, then so is $(\alpha \to \beta)$

(iv) if $\alpha$ is an expression in E, then so is $(\forall y)(Rxy \to \alpha)$,

for any two distinct variables x and y.

Finally $\phi$ is required to have exactly one free individual variable.

If $\alpha$ is a modal formula, we write $\overline{\alpha}$ for the universal closure of $\alpha$ taken with respect to its one free individual variable.

The exact connection between this definition of modal formulas and more traditional ones will become clear at the end of this introduction and in chapter I.2.

Modal formulas derive their interest from two sources. In the first place, according to a theorem by S.K. Thomason (cf. [23]) there exists an effective translation $\tau$ from sentences in the language of monadic second-order logic with one binary predicate constant R to modal formulas, and a modal formula $\delta$ such that, for all sentences $\phi$ and sets of sentences $\Gamma$ in this language,

$$\Gamma \models \phi \text{ iff } \{\overline{\tau(\gamma)} \mid \gamma \in \Gamma\} \cup \{\overline{\delta}\} \models \overline{\tau(\phi)}.$$

(Here $\models$ denotes logical consequence. In the limiting case where $\Gamma$ is empty and $\phi$ is universally valid, we write $\models \phi$.) H.C. Doets showed recently that an effective translation $\delta$ exists from second-order sentences to sentences of the form $(\forall R)(\exists P)\psi(R, P)$, where $\psi(R, P)$ is a first-order sentence in the binary predicate variable R and the unary predicate variable P, such that for all second-order sentences $\phi$,

$$\models \phi \text{ iff } \models \delta(\phi).$$

Combining these results it appears that the modal formulas are, in a sense, a reduction class for second-order logic. An effective translation T exists from second-order sentences to modal formulas such that, for all second-order sentences $\phi$,

$$\models \phi \text{ iff } \overline{\delta} \models \overline{T(\phi)}.$$

The $\overline{\delta}$ cannot be omitted here, for the set of universally valid modal formulas is recursive, whereas the set of universally valid second-order sentences is not.

The second source of interest in modal formulas lies in the well-known possible worlds semantics for modal logic. The clauses of S. Kripke's truth definition (cf. [12]) are reflected in our syntactic clauses (i),..., (iv).

From both these points of view the following question seems a natural one. Which modal formulas are first-order definable? More precisely, fixing $L_0$ to be the first-order language with equality containing the binary R mentioned above as its only predicate constant, we ask which modal formulas are logically equivalent to $L_0$-formulas. Taking this relation of logical equivalence between modal formulas and $L_0$-formulas as our object of study we are led to an obvious converse of our first question. Which $L_0$-formulas are modally definable? More precise formulations of these questions will be found in chapters I.2 and I.6.

The above questions are treated in part I which is intended to give a survey of this area of research. Part II consists of three published contributions of our own to the subject. In addition to these we mention Van Benthem [1]. Also all results in part I that are not explicitly attributed to a particular person or the folk literature are new as far as we know.

We now give a short description of part I. In the remainder of this introduction it will be shown how modal formulas as defined here are related to modal formulas defined in a more traditional (and in fact the usual) way. Moreover, a semantic characterization is given of those formulas of monadic first-order logic that have restricted quantifiers.

I.2 contains some standard notions and results to give a first impression of modal formulas. Our question about first-order definable modal formulas is stated in a precise manner. This leads to two different versions, one for modal formulas $\phi$ ("local" correspondence) and one for modal sentences $\overline{\phi}$ ("global" correspondence). Defining M1 as $\{\phi \mid \phi$ is a modal formula logically equivalent to some $L_0$-formula with the same free variable as $\phi\}$ and $\overline{\text{M}}$1 as $\{\phi \mid \phi$ is a modal formula for which $\overline{\phi}$ is logically equivalent to some $L_0$-sentence} we obtain a surprising result: $M1 \subseteq \overline{M}1$, but $M1 \neq \overline{M}1$. (For $\overline{M}1$, cf. Segerberg [18], Thomason [24], and Sahlqvist [16]; for M1, cf. Van Benthem [1].)

I.3 gives an algebraic characterization of $\overline{M}1$. In Goldblatt [7] it is shown that a modal formula is in $\overline{M}1$ iff it is preserved under ultraproducts. This is an instance of the general result that a $\Pi^1_1$-sentence is first-order definable iff it is preserved under ultraproducts. Goldblatt's result is sharpened here to preservation under ultrapowers. It is also proved that a set of modal formulas defines either an $L_0$-elementary class of models, or an $L_0$-$\Delta$-elementary class that is not $L_0$-elementary, or a class that is not $L_0$-$\Sigma\Delta$-elementary. Examples of all three kinds are given.

More syntactic information on first-order definable modal formulas is provided by two methods introduced in I.4. It appears that, whereas $\overline{M}1$ was the most natural class to characterize algebraically, M1 is a more suitable object for study now. The first method yields "positive" results, showing certain formulas to be in M1. It proceeds roughly as follows. Call the $L_0$-formula $\psi$ a <u>substitution-instance</u> of the modal formula $(\forall P_1)\ldots(\forall P_n)\phi$, if $\psi$ is obtained from $\phi$ by substituting $L_0$-formulas for the predicate variables. (But see chapter I.4 for the exact formulations!) Clearly, a modal formula implies each of its substitution-instances. $M1^{sub}$ is the class of modal formulas which are implied by a conjunction of their substitution-instances. It is shown that $M1^{sub} \subseteq M1$ and that $M1^{sub}$ is recursively enumerable. But $M1^{sub} \neq M1$, as appears from an example in I.2. Still, this method leads to a generalization of a theorem by H. Sahlqvist (cf. [16]), which was the most comprehensive result until now.

The second method yields "negative" results, showing certain formulas to be outside of M1. Here the Löwenheim-Skolem theorem is used as follows. We show that the modal formula under consideration holds in an uncountable model, for some assignment to its free variable, but that it does not hold in any countable elementary submodel of a suitable kind. A number of examples obtained in this way show that the generalisation of Sahlqvist's result referred to above is "almost" the best possible result.

A combination of the two methods leads to a complete syntactic classification of the first-order definable modal reduction principles. We do not define this notion here, but the definition is in chapter I.4. (Cf. II.2 and Fitch [6].) Many of the better-known axioms used in modal logic are modal reduction principles.

I.5 deals with particular cases where R satisfies some fixed property. To give an example: which modal formulas are first-order definable, given that R is transitive? One of the results is that all modal reduction principles are first-order definable in this case.

I.6 is concerned with the dual question about modally definable $L_0$-formulas. In Kaplan [11] the more general question was asked which classes of models are defined by (sets of) modal formulas. This question was answered in Goldblatt & Thomason [8], using algebraic techniques. For classes of models definable by an $L_0$-sentence their result assumes a very elegant form. This is all we need here, and we give a new proof of the relevant result, which avoids their use of so-called "modal algebras".

In addition a number of preservation results are proved for various model-theoretic notions occurring in Goldblatt & Thomason's theorem. This has the following consequence for modally definable $L_0$-sentences. These are all equivalent to $L_0$-sentences of the form $(\forall x)\phi$, where $\phi$ is an $L_0$-formula with the one free variable x constructed using atomic formulas, $\bot$ (a sign standing for a contradiction, the so-called falsum), conjunction, disjunction and restricted quantifiers.

Let us now mention some of the main questions we left open. To begin with, is M1 recursively enumerable, and what about $\bar{M}1$? We doubt if M1 and $\bar{M}1$ are even arithmetical, in view of our result (cf. p. 30) that these classes are not provably arithmetical in ZF. Take $\bar{P}1$ to be the class of $L_0$-sentences defined by a modal formula in the global sense. Is $\bar{P}1$ recursively enumerable, and is $\bar{P}1$ recursive in $\bar{M}1$? Finally, consider the $M1^{sub}$ of chapter I.4. It is recursively enumerable, but is it recursive?

Other important questions arise when we consider the notion of completeness, which is not treated in this dissertation. (Proving completeness theorems has been the main activity in modal logic for quite some time.) Consider the class $\bar{C}1$ of modal formulas which are complete with respect to some first-order property of R expressed by an $L_0$-sentence. It is easy to see that $\bar{C}1$ is arithmetical. K. Fine proved that $\bar{C}1$ is not contained in $\bar{M}1$ (cf. [5]) and S.K. Thomason proved that $\bar{M}$ 1 is not contained in $\bar{C}1$ (cf. [22]). On the other hand, the modal formulas described in theorem 4.13 are in $\bar{M}1 \cap \bar{C}1$ (cf. Sahlqvist [16]) and it is an open question if $M1^{sub} \subseteq \bar{M}1 \cap \bar{C}1$. What, then, is the exact relation of $\bar{M}1$ to $\bar{C}1$, and can $\bar{M}1 \cap \bar{C}1$ be characterized in some model-theoretic fashion?

We conclude this introduction with two results about modal formulas. $L_1$ is the first-order language with an infinite set of unary predicate constants and one binary predicate constant R. A modal formula as defined above is a formula of the form $(\forall P_1)\ldots(\forall P_n)\phi(P_1,\ldots P_n, R)$, where $\phi$ is an m-formula as defined below.

1.1 Definition

An m-formula is a member of the smallest class X of $L_1$-formulas satisfying

(i) for each unary predicate constant P and each individual variable x, $Px \in X$

(ii) if $\alpha \in X$, then $\neg\alpha \in X$

(iii) if $\alpha \in X$ and $\beta \in X$, then $(\alpha \to \beta) \in X$

(iv) if $\alpha \in X$, then $(\forall y)(Rxy \to \alpha) \in X$, provided that x and y are distinct individual variables

In chapter I.2 the traditional $\Box, \Diamond$ -notation is used for modal formulas. These are then translated into formulas of the form $(\forall P_1)\ldots(\forall P_n)\phi(P_1,\ldots, P_n, R)$, where $\phi$ is an $L_1$-formula of an even more special kind:

1.2 Definition

An M-formula is a member of the smallest class X of $L_1$-formulas satisfying

(i) for each unary predicate constant P and each individual variable x, $Px \in X$

(ii) if $\alpha \in X$, then $\neg\alpha \in X$

(iii) if $\alpha$ and $\beta$ have the same free variables and are both in X, then $(\alpha \to \beta) \in X$

(iv) if $\alpha \in X$ and y is the free variable of $\alpha$, then $(\forall y)(Rxy \to \alpha) \in X$, provided that x is distinct from y.

m-formulas have at least one free variable, M-formulas have exactly one.

1.3 Lemma

Any m-formula $\alpha$ is equivalent to a Boolean combination of M-formulas, each with their free variable among those of $\alpha$.

Proof: The assertion is proved by induction on the complexity of m-formulas. In order to simplify the proof the clauses (iii) and (iv) of the above definitions are temporarily replaced by analogous clauses for conjunction ($\wedge$), disjunction ($\vee$) and restricted existential quantification ($(\exists y)(Rxy \wedge\ )$). As we are only trying to prove an equivalence this change is harmless.

The cases $\alpha = Px$, $\alpha = \neg\beta$, $\alpha = \beta \wedge \gamma$ and $\alpha = \beta \vee \gamma$ are trivial. It remains to consider $\alpha = (\exists y)(Rxy \wedge \beta)$. By the induction hypothesis $\beta$ is equivalent to a Boolean combination of M-formulas each with their free variable among those of $\beta$. By the theorem on distributive normal forms $\beta$ is then equivalent to a formula of the form $\sum_{i=1}^{n} \prod_{j=1}^{n_i} \beta_{ij}$, where $\beta_{ij}$ is an M-formula.

(As for the notation, we stipulate that $\sum_{i=1}^{n} \phi_i =_{def} (\phi_1 \vee \ldots \vee \phi_n)$ and $\prod_{i=1}^{n} \phi_i =_{def} (\phi_1 \wedge \ldots \wedge \phi_n)$.)

By standard logic, $(\exists y)(Rxy \wedge \sum_{i=1}^{n} \prod_{j=1}^{n_i} \beta_{ij})$ is equivalent to $\sum_{i=1}^{n} (\exists y)(Rxy \wedge \prod_{j=1}^{n_i} \beta_{ij})$. So it suffices to consider the members of this disjunction. If none of the $\beta_{ij}$'s have a free variable y then $(\exists y)(Rxy \wedge \prod_{j=1}^{n_i} \beta_{ij})$ is equivalent to $(\exists y)(Rxy \wedge (Py \vee \neg Py)) \wedge \prod_{j=1}^{n_i} \beta_{ij}$, for an arbitrary unary predicate constant P. This is a Boolean combination of M-formulas of the required kind. Otherwise, let $\beta_i^1$ be the conjunction of those $\beta_{ij}$'s with y as their free variable and let $\beta_i^2$ be the conjunction of the remainder. Then $(\exists y)(Rxy \wedge \prod_{j=1}^{n_i} \beta_{ij})$ is equivalent to $(\exists y)(Rxy \wedge \beta_i^1) \wedge \beta_i^2$, again a Boolean combination of M-formulas of the required kind.

QED.

1.4 Corollary

Any m-formula with one free variable is equivalent to an M-formula.

Proof: A Boolean combination of M-formulas with the same free variable is itself an M-formula. QED.

Before stating the next result we mention a few notational conventions. $L_1$-models will be denoted by M or N, possibly with subscripts or superscripts. When we want to be explicit we write M = <W, R, V>, where W is the domain of M, R is the interpretation of the predicate constant R (a harmless autonomy occurs here) and V(P) is the set of those members of W for which $P^M$ holds. The sign $\models$, which was used already to denote logical consequence and universal validity, will denote truth in a model when occurring in a context $M \models \phi$. Other model-theoretic notions will be used as well, following the conventions of Chang & Keisler [2]. Two possibly lesser-known notations are used. FV($\alpha$) is the set of individual variables occurring free in $\alpha$, and $[t_1/x_1,\ldots, t_n/x_n]\phi$ is the result of simultaneously substituting $t_1$ for $x_1,\ldots, t_n$ for $x_n$ in $\phi$. More information about terminology is to be found in chapter I.2.

1.5 Definition

$M_1 = \langle W_1, R_1, V_1\rangle$ is a generated submodel of $M_2 = \langle W_2, R_2, V_2\rangle$ ($M_1 \subsetneq M_2$) if $M_1$ is a submodel of $M_2$ and, for all $w \in W_1$ and $v \in W_2$ such that $R_2wv$ holds, $v \in W_1$.

1.6 Definition

$\phi$, with the free variables $x_1,\ldots, x_n$, is invariant for generated submodels if, for all models $M_1$ and $M_2$ such that $M_1 \subsetneq M_2$ and all

$w_1, \ldots, w_n \in W_1$, $M_1 \models \phi\,[w_1, \ldots, w_n]$ iff $M_2 \models \phi\,[w_1, \ldots, w_n]$.

1.7 Definition

C is a p-relation between $M_1 = \langle W_1, R_1, V_1 \rangle$ and $M_2 = \langle W_2, R_2, V_2 \rangle$ if the following four conditions are satisfied,

(i) the domain of C is $W_1$ and the range of C is $W_2$

(ii) for each $w \in W_1$ and $v \in W_2$ such that Cwv, and each unary predicate constant P, $w \in V_1(P)$ iff $v \in V_2(P)$

(iii) for each $w, w' \in W_1$ and $v \in W_2$ such that $R_1ww'$ and Cwv there exists a $v' \in W_2$ with $R_2vv'$ and $Cw'v'$

(iv) for each $v, v' \in W_2$ and $w \in W_1$ such that $R_2vv'$ and Cwv there exists a $w' \in W_1$ with $R_1ww'$ and $Cw'v'$.

1.8 Definition

$\phi$, with the free variables $x_1, \ldots, x_n$, is invariant for p-relations if, for all models $M_1$ and $M_2$, all p-relations C between $M_1$ and $M_2$, and all $w_1, \ldots, w_n \in W_1$, $w'_1, \ldots, w'_n \in W_2$ such that $Cw_1w'_1, \ldots, Cw_nw'_n$, $M_1 \models \phi\,[w_1, \ldots, w_n]$ iff $M_2 \models \phi\,[w'_1, \ldots, w'_n]$.

These concepts are of interest only for formulas with free variables. An $L_1$-sentence invariant for generated submodels is either universally valid or a contradiction, as is easily seen using the methods of chapter I.2.

1.9 Theorem

An $L_1$-formula $\phi$ containing at least one free variable is equivalent to an m-formula iff it is invariant for generated submodels and p-relations.

Proof: One direction is easy. Each m-formula is invariant for generated submodels and p-relations, as a simple induction shows.

On the other hand, let $\phi$ have this property and let $FV(\phi) = \{x_1,\ldots, x_n\}$. Define $m(\phi) = \{\psi \mid \psi$ is an m-formula, $\phi \models \psi$, $FV(\psi) \subseteq FV(\phi)\}$. We will show that $m(\phi) \models \phi$. By the compactness theorem, this implies $\psi \models \phi$, for some $\psi \in m(\phi)$, whence clearly $\models \phi \leftrightarrow \psi$. Since the proof uses a construction which recurs at various places in I.6, it will be given in quite some detail.

Let $M_1 \models m(\phi)[w_1,\ldots, w_n]$. Introduce individual constants $\underline{w}_1,\ldots, \underline{w}_n$. The notation $\underline{w}$ is consistently used to introduce a unique individual constant for an object w. Adding $\underline{w}_1,\ldots, \underline{w}_n$ to $L_1$ gives a language $L_{11}$. $M_1$ is then expanded to an $L_{11}$-model $M_{11}$ by interpreting $\underline{w}_1$ as $w_1,\ldots, \underline{w}_n$ as $w_n$. Let $\phi^* = [\underline{w}_1/x_1,\ldots, \underline{w}_n/x_n]\,\phi$.

Define $m(L_{11})$ to be the class of those sentences (!) of $L_{11}$ that are obtained by starting with atomic formulas of the forms Px or Pc and applying $\neg$, $\rightarrow$, $(\forall y)(Rxy \rightarrow$ or $(\forall y)(Rcy \rightarrow$ , where x and y are distinct individual variables and c is an arbitrary individual constant of $L_{11}$. (m-formulas always had at least one free variable, but this relaxation of the definition generates sentences as well.)

Each finite subset of $\{\phi^*\} \cup \{\psi \mid \psi \in m(L_{11})$ and $M_{11} \models \psi\}$ has a model. For suppose otherwise. Then, for some $\psi_1,\ldots, \psi_k$ as described, $\phi^* \models \neg(\psi_1 \wedge \ldots \wedge \psi_k)$, but, since $M_1 \models m(\phi)[w_1,\ldots, w_n]$, it follows that $M_{11} \models \neg(\psi_1 \wedge \ldots \wedge \psi_k)$, contradicting $M_{11} \models \psi_1 \wedge \ldots \wedge \psi_k$. So there exists a model $N_{11}$ for the whole set. $N_{11}$ is an $L_{11}$-model satisfying the following two conditions,

(i) $N_{11} \models \phi^*$

(ii) $N_{11}$-$m(L_{11})$-$M_{11}$,

where (ii) is short for "for each $\phi \in m(L_{11})$, $N_{11} \models \phi$ iff $M_{11} \models \phi$".

For each c and w such that c is an individual constant in $L_{11}$, w is an element of the domain of $N_{11}$, and $N_{11} \models Rcx[w]$, add a new constant $k_{cw}$ to $L_{11}$ to obtain $L_2$. Then expand $N_{11}$ to an $L_2$-model $N_2$ by interpreting each $k_{cw}$ as w. $m(L_2)$ is defined in the obvious way.

Each finite subset of $\{\psi \mid \psi \in m(L_2)$ and $N_2 \models \psi\} \cup \{Rck_{cw} \mid N_2 \models Rck_{cw}\}$ has a model which is an expansion of $M_{11}$. To prove this, consider $\psi_1, \ldots, \psi_k$ as described, together with $Rc_1k_{c_1w_1}, \ldots, Rc_lk_{c_lw_l}$. Add $Rck_{cw}$ for each $k_{cw}$ occurring in $\psi_1 \wedge \ldots \wedge \psi_k$ which is not among $k_{c_1w_1}, \ldots, k_{c_lw_l}$, say for $k_{c'_1w'_1}, \ldots, k_{c'_sw'_s}$. Then take distinct variables $x_1, \ldots, x_l$, $y_1, \ldots, y_s$ not occurring in $\psi_1 \wedge \ldots \wedge \psi_k$ and substitute them for $k_{c_1w_1}, \ldots, k_{c_lw_l}, k_{c'_1w'_1}, \ldots, k_{c'_sw'_s}$ respectively to obtain $(\psi_1 \wedge \ldots \wedge \psi_k)'$. Then $N_{11} \models (\exists x_1)(Rc_1x_1 \wedge \ldots \wedge (\exists x_l)(Rc_lx_l \wedge (\exists y_1)(Rc'_1y_1 \wedge \ldots \wedge (\exists y_s)(Rc'_sy_s \wedge (\psi_1 \wedge \ldots \wedge \psi_k)'\ldots)$. This sentence is in $m(L_{11})$ and therefore it also holds in $M_{11}$, since $N_{11}$-$m(L_{11})$-$M_{11}$. It is now clear how $M_{11}$ can be expanded to a model for $\{\psi_1, \ldots, \psi_k, Rc_1k_{c_1w_1}, \ldots, Rc_lk_{c_lw_l}\}$.

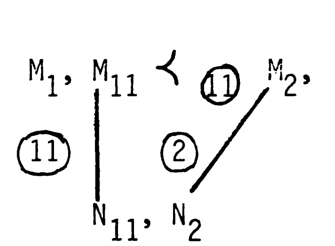
Using a well-known model-theoretic argument it follows that the above set has a model $M_2$ satisfying the following conditions,

(i) $M_{11} \prec_{L_{11}} M_2$ (i.e., $M_{11}$ is an $L_{11}$-elementary submodel of $M_2$)

(ii) $N_2$-$m(L_2)$-$M_2$,

where (ii) has the obvious meaning. This situation may be pictured as:

models:




languages: $L_1$, $L_{11}$, $L_2$, $L_2$

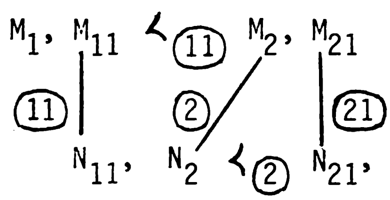
This construction is repeated, but now starting from $M_2$. For each c and w such that c is a constant in $L_2$, w is an element in the domain of $M_2$ and $M_2 \models Rcx[w]$, add a new constant $k_{cw}$ to $L_2$ to obtain $L_{21}$. $M_2$ is then expanded to an $L_{21}$-model $M_{21}$ by interpreting $k_{cw}$ as w. Using an argument similar to the one given above one sees that each finite subset of $\{\psi \mid \psi \in m(L_{21})$ and $M_{21} \models \psi\} \cup \{Rck_{cw} \mid k_{cw} \in L_{21}-L_2$ and $M_{21} \models Rck_{cw}\}$ has a model which is an expansion of $N_2$. Therefore this set has a model $N_{21}$ satisfying the following two conditions,

(i) $N_2 \prec_{L_2} N_{21}$

(ii) $N_{21}$-$m(L_{21})$-$M_{21}$.

In the picture this leads to:

models:



languages: $L_1$, $L_{11}$, $L_2$, $L_2$, $L_{21}$

Iterating this construction yields two elementary chains $M_1, M_2, \ldots$ and $N_{11}, N_{21}, \ldots$ with limits M and N, respectively. The required conclusion follows from the assumption on $\phi$ and the fundamental theorem on elementary chains. Since $N_{11} \models \phi^*$, $N \models \phi^*$. The submodel $N_c$ of N generated by the constants in $\bigcup_n L_n$ is a generated submodel of N and therefore $N_c \models \phi^*$, by the invariance of $\phi$ for generated submodels. The following defines a p-relation C between $N_c$ and the generated submodel $M_c$ of M generated by the constants of $\bigcup_n L_n$. Define Cwv to hold if, for some constant $c \in \bigcup_n L_n$, $w = c^N$ and $v = c^N$. The construction of the chains guarantees that C satisfies the four properties required. By the invariance of $\phi$ for p-relations, $M_C \models \phi^*$, and, using the invariance of $\phi$ for generated submodels once more, $M \models \phi^*$. This implies that $M_{11} \models \phi^*$, so $M_1 \models \phi[w_1, \ldots, w_n]$. QED.

The use of constants $k_{cw}$, rather than $\underline{w}$, in this proof serves to avoid the following complication. Let $c_1$ and $c_2$ be constants of $L_{11}$ and let $N_2 \models Rc_1x[w]$ and $N_2 \models Rc_2x[w]$. $\{Rc_1\underline{w}, Rc_2\underline{w}\}$ need not have a model which is an expansion of $M_{11}$. The method used only leads to the $L_{11}$-sentence $(\exists x_1)(Rc_1x_1 \wedge Rc_2x_1)$, but this is not a sentence in $m(L_{11})$ and therefore need not be true in $M_{11}$. Using $k_{c_1w}$ and $k_{c_2w}$ leads to the $m(L_{11})$-sentence $(\exists x_1)(Rc_1x_1 \wedge (\exists x_2)Rc_2x_2)$, in which the information about $c_1$ and $c_2$ having a common R-successor is lost.

## I.2 PRELIMINARY NOTIONS AND RESULTS

The usual set-theoretic and model-theoretic notation will be used in the metalanguage, including the abbreviations $\forall$ (for all), $\exists$ (there exists), $\Rightarrow$ (if...then...), $\Leftrightarrow$ (if and only if), & (and) and $\sim$ (not). In the formal languages we have $\forall$, $\exists$, $\rightarrow$, $\leftrightarrow$, $\wedge$ and $\neg$, as well as $\vee$ (or). The terminology will be standard, unless explicit exceptions are made. (E.g., the term "model" will be used in a special way, to be explained shortly.) We presuppose the standard results of classical logic, as contained in Enderton [3], Shoenfield [19], or Chang & Keisler [2].

We shall be concerned with the following formal languages:

$L_m$, the language of modal propositional logic, has an infinite set of proposition letters, the Boolean operators $\neg$, $\rightarrow$, $\wedge$, $\vee$, $\leftrightarrow$ (the last three being considered to be defined in terms of the first two in the usual way) and the unary modal operators $\Box$ and $\Diamond$ ($\Diamond$ being considered to be defined as $\neg\Box\neg$.)

$L_0$ is the first-order language with identity and one other, binary predicate constant R.

$L_1$ is the first-order language with R and identity, and an infinite set of unary predicate constants. A fixed 1-1 correspondence is assumed to exist between the proposition letters of $L_m$ and the unary predicate constants of $L_1$.

$L_2$ is the second-order language with R and identity, and an infinite set of unary predicate variables. Again, a fixed 1-1 correspondence is assumed between the proposition letters of $L_m$ and the unary predicate variables of $L_2$.

We write p, q, r,...; $p_1$, $p_2$,... for proposition letters of $L_m$; P, Q, R,...; $P_1$, $P_2$,... for unary predicate constants of $L_1$ as well as for unary predicate variables of $L_2$, P is supposed to correspond to p, $P_1$ to $p_1$, etc. $\alpha$, $\beta$,..., $\phi$, $\psi$,..., possibly with subscripts, denote formulas; and $\Gamma$, $\Delta$, $\Sigma$,..., possibly with subscripts, denote sets of formulas. Sometimes superscripts are used in order to emphasize that a formula is a formula of a certain language; thus $\phi^m$ denotes an $L_m$-formula and $\psi^1$ an $L_1$-formula. Finally, the signs $\bot$ (<u>falsum</u>) and T (<u>verum</u>) are used as abbreviations for an arbitrary contradiction or universally valid formula, respectively.

Formulas of $L_m$ may be regarded as abbreviations of certain formulas of either $L_1$ or $L_2$, via the "translation" ST(-) defined below.

2.1 Definition

Let x be a fixed variable, and let P be the unary predicate constant in $L_1$ corresponding to the proposition letter p. ST($\phi$) is defined inductively for $L_m$-formulas $\phi$ by:

(i) $ST(p) = Px$

(ii) $ST(\neg\alpha) = \neg ST(\alpha)$

(iii) $ST(\alpha \to \beta) = ST(\alpha) \to ST(\beta)$

(iv) $ST(\Box\alpha) = (\forall y)(Rxy \to [y/x]\ ST(\alpha))$, where y does not occur in $ST(\alpha)$.

For a set $\Gamma$ of $L_m$-formulas $ST(\Gamma) = \{ST(\gamma) \mid \gamma \in \Gamma\}$.

It may be seen that the ST-counterparts of $L_m$-formulas are essentially just those M-formulas of $L_1$ (definition 1.2) with x as their free variable, and that their universal closures with respect to the unary predicate symbols occurring in them are essentially the modal formulas of $L_2$ as described in I.1. From now on the term "modal formula" will be applied to $L_m$-formulas, their ST-counterparts in $L_1$ and the universal closures of the latter in $L_2$. The context will always make it clear which meaning is intended.

A structure for $L_0$ (or $L_2$) consists of a non-empty set W and a binary relation R on W; F = <W, R> is called a frame. (Likewise, we write $F_1 = \langle W_1, R_1 \rangle$, etc.) A structure for $L_1$ may conveniently be considered as a triple M = <W, R, V> or a pair M = <F, V>, where F = <W, R> is a frame and V assigns to each unary predicate constant P of $L_1$ a subset V(P) of W. (Likewise, we write $M_1 = \langle W_1, R_1, V_1 \rangle = \langle F_1, V_1 \rangle$, etc.) Structures for $L_1$ are called models, and V is called a valuation on F. (In current model-theoretic terminology structures for any language L are called "L-models", but we will use the more neutral "L-structure", reserving the term "model" for $L_1$-structures.)

The basic truth definitions for $L_m$-formulas, due essentially to S. Kripke, can now be given.

2.2 Definition

If $\phi$ is an $L_m$-formula with the proposition letters $p_1, \ldots, p_n$ (corresponding to the unary predicate symbols $P_1, \ldots, P_n$) and M = <F, V> = <W, R, V> is a model with $w \in W$, then

(i) $M \models \phi\,[w] \Leftrightarrow M \models ST(\phi)\;\; w$

(ii) $M \models \phi \qquad \Leftrightarrow M \models (\forall x) ST(\phi)$

(iii) $F \models \phi\,[w] \Leftrightarrow F \models (\forall P_1)\ldots(\forall P_n)ST(\phi)\,[w]$

(iv) $F \models \phi \Leftrightarrow F \models (\forall x)(\forall P_1)\ldots(\forall P_n)ST(\phi)$

For a set $\Gamma$ of $L_m$-formulas, $M \models \Gamma\,[w]$ holds iff, for all $\gamma \in \Gamma$, $M \models \gamma\,[w]$, and similarly for $M \models \Gamma$, $F \models \Gamma\,[w]$ and $F \models \Gamma$.

Many of the fundamental properties of the truth definition for $L_m$-formulas follow immediately from definition 2.2; the following is an example.

2.3 Lemma

If f is an isomorphism from $F_1$ onto $F_2$, then, for all $L_m$-formulas $\phi$ and all $w \in W_1$,

$$F_1 \models \phi\,[w] \Leftrightarrow F_2 \models \phi\,[f(w)].$$

Proof: Here, and henceforth, a proof by simple induction on the complexity of formulas will be omitted.

The next definitions and lemmas up to and including 2.18 are (our versions of) standard results from the folk literature.

2.4 Definition

$F_1$ is a generated subframe of $F_2$ ($F_1 \subsetneq_{\rightarrow} F_2$; cf. definition 1.5) if $F_1$ is a subframe of $F_2$ and, for all $w \in W_1$ and $v \in W_2$, if $R_2wv$, then $v \in W_1$. If $F_1 \subsetneq_{\rightarrow} F_2$ and V is a valuation on $F_2$, then $\underline{V_1}$ is the valuation on $F_1$ defined by $V_1(p) = V(p) \cap W_1$.

The notion "generated subframe" is closely related to the better-known notion "end extension". (Cf. Chang & Keisler [2].)

2.5 Lemma (Generation Lemma, Segerberg [17], Feferman [4])

If $F_1$ is a subframe of $F_2$, then $F_1 \subsetneq_{\rightarrow} F_2$ if and only if, for each valuation V on $F_2$, each $w \in W_1$ and each $L_m$-formula $\phi$,

$<F_2, V> \models \phi [w] \Leftrightarrow <F_1, V_1> \models \phi [w]$.

2.6 Corollary

If $F_1 \subsetneq_{\rightarrow} F_2$, then, for all $w \in W_1$ and all $L_m$-formulas $\phi$,

$F_2 \models \phi [w] \Leftrightarrow F_1 \models \phi [w]$

$F_2 \models \phi \quad \Rightarrow F_1 \models \phi$

2.7 Definition

If F is a frame and $w \in W$, then TC(F, w) is the smallest $F_1 = <W_1, R_1> \subsetneq_{\rightarrow} F$ with $w \in W_1$; i.e., $W_1 = \cap \{X \subseteq W \mid w \in X \;\&\; (\forall x \in W)(\forall y \in W)((x \in X \;\&\; Rxy) \Rightarrow y \in X)\} = \{u \in W \mid$ a sequence $v_1, \ldots, v_n$ exists with $w = v_1$, $u = v_n$ and $Rv_iv_{i+1}$ for all i $(1 \leqq i \leqq n-1)\}$.

Clearly, $F \models \phi [w]$ iff $TC(F, w) \models \phi [w]$.

2.8 Definition

Let $\{F_i \mid i \in I\}$ be a set of frames. Set $F_i' = <W_i', R_i'>$, where $W_i' = \{<i, w> \mid w \in W_i\}$ and $R_i' = \{<<i, w>, <i, v>> \mid <w, v> \in R_i\}$. Then the disjoint union $\bigoplus \{F_i \mid i \in I\}$ of the set $\{F_i \mid i \in I\}$ is the frame $<\bigcup_{i \in I} W_i', \bigcup_{i \in I} R_i'>$.

For each $i \in I$, $F_i$ is isomorphic to the generated subframe $F_i'$ of $\bigoplus \{F_i \mid i \in I\}$, whence the following corollary.

2.9 Corollary

For each $i \in I$, $w \in W_i$ and $L_m$-formula $\phi$,

$F_i \models \phi\ [w] \Leftrightarrow \bigoplus \{F_i \mid i \in I\} \models \phi\ [\langle i, w\rangle]$;

hence $\bigoplus \{F_i \mid i \in I\} \models \phi$ iff, for all $i \in I$, $F_i \models \phi$.

Corollary 2.9 implies that $(\forall x)(\forall y)Rxy$ is not equivalent to a modal formula - it is not preserved under disjoint unions.

2.10 Definition

A function f from $F_1$ onto $F_2$ is a p-morphism if

$(\forall w \in W_1)(\forall v \in W_1)(R_1wv \Rightarrow R_2f(w)f(v))$ and

$(\forall w \in W_1)(\forall v \in W_2)(R_2f(w)v \Rightarrow (\exists u \in W_1)(Rwu \ \&\ f(u) = v))$.

If V is a valuation on $F_2$, then $f^{-1}(V)$ is the valuation on $F_1$ defined by $f^{-1}(V)(p) \quad = \quad \{w \in W_1 \mid f(w) \in V(p)\}$.

The concept of a "p-morphism" was first defined by K. Segerberg in "Decidability of S4.1", Theoria 34 (1968), pp. 7-20. An earlier, similar notion ("strongly isotone function") is in D.H.J. de Jongh & A.S. Troelstra: "On the connection of partially ordered sets with some pseudo-Boolean algebras", Indagationes Mathematicae 28:3 (1966), pp. 317-329.

2.11 Lemma (p-morphism theorem, Segerberg [17])

A function f from $F_1$ onto $F_2$ is a p-morphism iff, for all $w \in W_1$, all valuations V on $F_2$ and all $L_m$-formulas $\phi$,

$\langle F_2, V\rangle \models \phi\ [f(w)] \Leftrightarrow \langle F_1, f^{-1}(V)\rangle \models \phi\ [w]$.

2.12 Corollary

If f is a p-morphism from $F_1$ onto $F_2$, then, for all $w \in W_1$ and all $L_m$-formulas $\phi$,

$$F_1 \models \phi[w] \Rightarrow F_2 \models \phi[f(w)]$$

$$F_1 \models \phi \Rightarrow F_2 \models \phi.$$

2.13 Definition

$U = <\{0\}, \emptyset>$

$I = <\{0\}, \{<0, 0>\}>.$

2.14 Corollary (cf. Makinson [15])

For all $L_m$-formulas $\phi$ and all frames F, if $F \models \phi$, then $U \models \phi$ or $I \models \phi$.

Proof: If $F \models (\exists x)(\forall y)\neg Rxy$, then, for any $w \in W$ with $(\forall y \in W)\sim Rwy$, $<\{w\}, \emptyset> \underset{\rightarrow}{\subseteq} F$. Therefore, by corollary 2.6, $<\{w\}, \emptyset> \models \phi$ and, by lemma 2.3, $U \models \phi$.

If $F \models (\forall x)(\exists y)Rxy$, then f defined by $f(w) = 0$ for all $w \in W$, is a p-morphism from F onto I, and so, by corollary 2.12, $I \models \phi$. QED.

Corollary 2.14 implies that $(\forall x)(\exists y)(Rxy \wedge \neg Ryx)$ is not equivalent to a modal formula - it does not hold in U or I. (But it is preserved under generated subframes and disjoint unions.)

2.15 Lemma (tree lemma)

Any modal formula which is not universally valid has a counter-example on a finite irreflexive intransitive tree.

Proof: The notion of "tree" is taken for granted here. If the modal formula $\phi$ is not universally valid, there exists a frame F and $w \in W$ such that $\sim F \models \phi [w]$ and, by 2.7, $\sim TC(F, w) \models \phi [w]$. An irreflexive and intransitive tree T is defined from $TC(F, w) = \langle W_1, R_1 \rangle$ by taking the finite sequences $\langle w_1, \ldots, w_n \rangle$ of elements $w_1, \ldots, w_n$ of $W_1$ satisfying $R_1 w_i w_{i+1}$ for all i ($1 \leqq i \leqq n-1$), as its nodes, and the set of pairs $\langle\langle w_1, \ldots, w_n \rangle, \langle w_1, \ldots, w_n, w_{n+1} \rangle\rangle$ (for which $R_1 w_n w_{n+1}$ holds) as its ordering relation. f defined by $f(\langle w_1, \ldots, w_n \rangle) = w_n$ is a p-morphism from T onto TC(F, w), so, by corollary 2.12, $\sim T \models \phi [\langle w \rangle]$.

The following general lemma now implies that $\phi$ has a counterexample on a finite subtree of T. QED.

2.16 Lemma

Let F be an irreflexive intransitive tree, V a valuation on F and $w \in W$, and let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$ be $L_m$-formulas such that $M = \langle F, V \rangle \models \alpha_i [w]$, for all i ($1 \leqq i \leqq n$), and $\sim M \models \beta_j [w]$; for all j ($1 \leqq j \leqq m$). Then there exists a finite submodel M' of M with w in its domain such that $M' \models \alpha_i [w]$, for all i ($1 \leqq i \leqq n$), and $\sim M' \models \beta_j [w]$, for all j ($1 \leqq j \leqq m$).

Proof: The lemma is proved by induction on the number of occurrences of Boolean and modal operators in $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$. The only non-trivial case is when each of the $\alpha_i$ and $\beta_j$ is either a proposition letter or a formula of the form $\Box\gamma$. (In the other cases Boolean reductions may be used.) Let $\Box\gamma_1, \ldots, \Box\gamma_r$ be the formulas of the second kind occurring among $\beta_1, \ldots, \beta_m$. Choose $w_1, \ldots, w_r \in W$ such that

$\sim M \models \gamma_i [w_i]$ and $Rww_i$ for all i $(1 \leq i \leq r)$. By the induction hypothesis, finite submodels $M'_1, \ldots, M'_r$ of M exist, containing $w_1, \ldots, w_r$ respectively, such that $\sim M'_i \models \gamma_i [w_i]$ and, for all $\Box\theta$ occurring among $\alpha_1, \ldots, \alpha_n$, $M'_i \models \theta [w_i]$, for all i $(1 \leq i \leq r)$. The submodel of M obtained from the union of $M'_1, \ldots, M'_r$ by adding w is the required M'. QED.

2.17 Definition

If $M_1 = \langle W_1, R_1, V_1\rangle$ and $M_2 = \langle W_2, R_2, V_2\rangle$ are models, and $\Gamma$ is a set of modal formulas closed under the formation of subformulas, then a function g from $M_1$ onto $M_2$ is an f-morphism with respect to $\Gamma$ if the following three conditions hold,

$(\forall w \in W_1)(\forall v \in W_1)(R_1wv \Rightarrow R_2g(w)g(v))$

$(\forall w \in W_1)(M_1 \models p [w] \Leftrightarrow M_2 \models p [g(w)])$ for all proposition letters p

$(\forall w \in W_1)(\forall \phi)(\Box\phi \in \Gamma \Rightarrow (M_1 \models \Box\phi [w] \Rightarrow M_2 \models \Box\phi [g(w)]))$.

2.18 Lemma (filtration lemma; cf. Segerberg [ 17 ])

If g is an f-morphism with respect to $\Gamma$ from $M_1$ onto $M_2$, then, for all $w \in W_1$ and $\phi \in \Gamma$,

$M_1 \models \phi [w] \Leftrightarrow M_2 \models \phi [g(w)]$.

The standard example is the following. Let $M = \langle W, R, V\rangle$ be a model and $\Gamma$ a set of modal formulas closed under the formation of subformulas. For any $w \in W$, define $g_\Gamma(w) = \{\phi \in \Gamma \mid M \models \phi [w]\}$, $R_\Gamma g_\Gamma(w) g_\Gamma(v)$ iff each $\Box\phi \in \Gamma \cap g_\Gamma(w)$ is in $g_\Gamma(v)$, and $V_\Gamma(p) = \{g_\Gamma(w) \mid p \in g_\Gamma(w)\}$ for all proposition letters $p \in \Gamma$. $g_\Gamma$ is an f-morphism from M onto the standard $\Gamma$-filtration of M: $\langle\{g_\Gamma(w) \mid w \in W\}, R_\Gamma, V_\Gamma\rangle$. Clearly, if $\Gamma$ is finite, then so is the standard $\Gamma$-filtration of M.

If R is transitive, then the Lemmon $\Gamma$-filtration $\langle\{g^L_\Gamma(w) \mid w \in W\}$, $R^L_\Gamma$, $V^L_\Gamma\rangle$ of M is defined as above, but for the definition of its relation - we now set $R^L_\Gamma g^L_\Gamma(w) g^L_\Gamma(v)$ iff, for all $\Box\phi \in \Gamma \cap g^L_\Gamma(w)$, both $\Box\phi$ and $\phi$ are in $g^L_\Gamma(v)$. Again $g^L_\Gamma$ is an f-morphism, and $R^L_\Gamma$ is transitive.

$(\forall x)(\exists y)(Rxy \wedge Ryy)$ is preserved under generated subframes, disjoint unions and p-morphic images, although it is not equivalent to a modal formula. This can be shown using lemma 6.14, but it may be of interest to note here that the following simple argument using the concept of a Lemmon-filtration suffices.

Suppose that $(\forall x)(\exists y)(Rxy \wedge Ryy)$ is equivalent to the modal formula $\phi$. Since $\phi$ does not hold on $\langle \mathbb{N}, <\rangle$ (the natural numbers with the "smaller than" ordering) there exists a valuation $V'$ on $\langle \mathbb{N}, <\rangle$ and an $n \in \mathbb{N}$ such that $\sim\langle \mathbb{N}, <, V'\rangle \models \phi[n]$. For $\Gamma = \neg\phi$ + its subformulas take the Lemmon $\Gamma$-filtration of $\langle \mathbb{N}, <, V'\rangle$. Since $\Gamma$ is finite, this is a finite model $\langle W, R, V\rangle$ with a transitive R. $\langle W, R, V\rangle$ also satisfies $(\forall x)(\exists y)Rxy$, for $\langle \mathbb{N}, <\rangle \models (\forall x)(\exists y)Rxy$ and any f-morphism is an $L_0$-homomorphism. $\sim\langle W, R, V\rangle \models \phi[g^L_\Gamma(n)]$, but this is a contradiction. For in any finite transitive frame satisfying $(\forall x)(\exists y)Rxy$, $(\forall x)(\exists y)(Rxy \wedge Ryy)$ holds, so $\langle W, R\rangle \models \phi$ and therefore $\langle W, R, V\rangle \models \phi[g^L_\Gamma(n)]$. QED.

This concludes the exposition of standard results. We have arrived at the main definitions.

2.19 Definition

If $\phi^m$ is a modal formula and $\phi^o$ a formula of $L_0$ with one free variable, then (recall that $F = \langle W, R\rangle$)

$$E(\phi^m, \phi^o) \Leftrightarrow (\forall F)(\forall w \in W)(F \models \phi^m[w] \Leftrightarrow F \models \phi^o[w]).$$

(This is the so-called local correspondence.)

If $\phi^m$ is a modal formula and $\phi^o$ an $L_0$-sentence, then

$\overline{E}(\phi^m, \phi^o) \Leftrightarrow (\forall F)(F \models \phi^m \Leftrightarrow F \models \phi^o)$.

(This is the so-called global correspondence.)

$M1 = \{\phi^m \mid \text{for some } L_0\text{-formula } \phi^o, E(\phi^m, \phi^o)\}$.

$\overline{M}1 = \{\phi^m \mid \text{for some } L_0\text{-sentence } \phi^o, \overline{E}(\phi^m, \phi^o)\}$.

If $E(\phi^m, \phi^o)$ and $\phi^o$ has the free variable x, then $\overline{E}(\phi^m, (\forall x)\phi^o)$, whence $M1 \subseteq \overline{M}1$. The converse does not hold, however, as the next lemma shows. (The local notion is stronger than the global one here, in contrast to usual mathematical practice.)

As remarked in the introduction, $\overline{M}1$ has an elegant semantic characterization (cf. chapter I.3), while M1 is a more natural object for syntactic studies (cf. chapter I.4). If frames are considered to be the basic semantic structures, then $\overline{M}1$ would be, in a sense, the more fundamental class. It may be of interest to observe, however, that in Kripke's original semantics frames were considered together with a distinguished element of their domain ("the actual world"). If couples $\langle F, w\rangle$ with $w \in W$ are considered to be the basic semantic structures, then M1 is the more fundamental class.

Before stating the next lemma we explain one more notational convention.

$\{x_i, y_j \mid i \in I, j \in J\}$ is short for $\{x_i \mid i \in I\} \cup \{y_j \mid j \in J\}$, and similarly for longer sequences $x_i, y_j, z_k, \ldots$ and sequences in which double or triple subscripts are used.

2.20 Lemma

$\overline{E}(\Box\Diamond\Box\ \Box p \to \Diamond\Diamond\Box\Diamond p, (\forall x)(\exists y)Rxy)$.

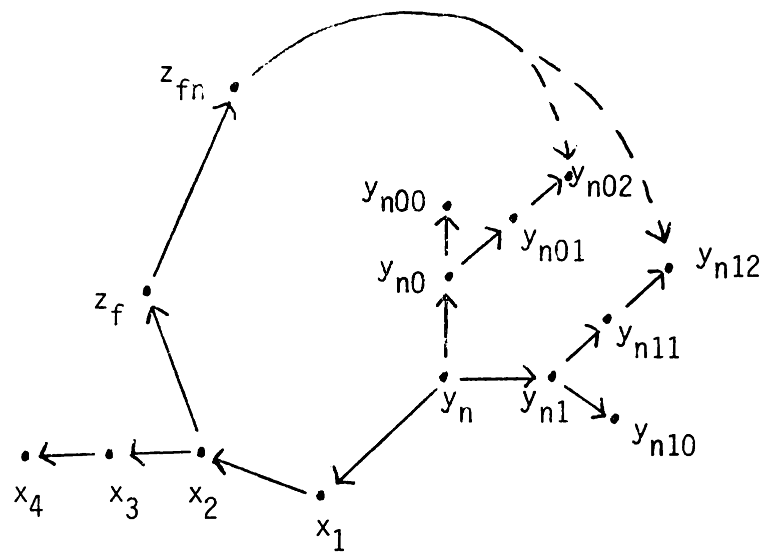
$\Box\Diamond\Box\Box p \to \Diamond\Diamond\Box\Diamond p \notin M1$.

Proof: If $F \models (\forall x)(\exists y)Rxy$, then, for any modal formula $\phi$, $F \models \Box\phi \to \Diamond\phi$. This implies that $F \models \Box\Diamond\Box\Box p \to \Diamond\Diamond\Box\Diamond p$.
If $\sim F \models (\forall x)(\exists y)Rxy$, then, for some $w \in W$, $F \models \neg(\exists y)Rxy\,[w]$. It suffices to observe that, for such a w and all modal formulas $\phi$, $F \models \Box\phi\,[w]$ and $F \models \neg\Diamond\phi\,[w]$. This proves the first assertion.

The second assertion is proved as follows. A frame $F = \langle W, R\rangle$ and a $v \in W$ are given such that, for the modal formula $\phi^m$ in question, $F \models \phi^m\,[w]$. W is uncountable, but it is shown that, for no countable elementary subframe F' of F containing a certain countable subset of W, $F' \models \phi^m\,[w]$. From the Löwenheim-Skolem theorem it follows that $\phi^m$ is not equivalent to an $L_0$-formula.

$W = \{x_1, x_2, x_3, x_4\} \cup \{y_n, y_{ni}, y_{nij} \mid n \in \mathbb{N}, i \in \{0, 1\}, j \in \{0, 1, 2\}\} \cup \{z_f, z_{fn} \mid f: \mathbb{N} \to \{0, 1\}, n \in \mathbb{N}\}$.

$R = \{\langle x_1, x_2\rangle, \langle x_2, x_3\rangle, \langle x_3, x_4\rangle\} \cup \{\langle x_2, z_f\rangle, \langle z_f, z_{fn}\rangle, \langle z_{fn}, y_{nf(n)2}\rangle \mid f: \mathbb{N} \to \{0, 1\}, n \in \mathbb{N}\} \cup \{\langle x_1, y_n\rangle, \langle y_n, y_{ni}\rangle, \langle y_{ni}, y_{nij}\rangle, \langle y_{ni1}, y_{ni2}\rangle \mid n \in \mathbb{N}, i \in \{0, 1\}, j \in \{0, 1\}\}$.




Let V be any valuation on F satisfying $\langle F, V\rangle \models \Box\Diamond\Box\Box p\ [x_1]$. We show that $\langle F, V\rangle \models \Diamond\Diamond\Box\Diamond p\ [x_1]$, thereby establishing that $F \models \phi^m\ [x_1]$. Since $\langle F, V\rangle \models \Box\Diamond\Box\Box p\ [x_1]$, $\langle F, V\rangle \models \Diamond\Box\Box p\ [y_n]$ for all $n \in$ IN. So, for all $n \in$ IN, either $\langle F, V\rangle \models \Box\Box p\ [y_{n0}]$, in which case $\langle F, V\rangle \models p\ [y_{n02}]$, or $\langle F, V\rangle \models \Box\Box p\ [y_{n1}]$, in which case $\langle F, V\rangle \models p\ [y_{n12}]$. Let $f$: IN $\rightarrow \{0, 1\}$ satisfy $\langle F, V\rangle \models p\ [y_{nf(n)2}]$ for all $n \in$ IN. Then $\langle F, V\rangle \models \Diamond p\ [z_{fn}]$ for all $n \in$ IN, so $\langle F, V\rangle \models \Box\Diamond p\ [z_f]$, and therefore $\langle F, V\rangle \models \Diamond\Diamond\Box\Diamond p\ [x_1]$.

Let F' be any countable elementary subframe of F with a domain containing $\{x_1, x_2, x_3, x_4\} \cup \{y_n, y_{ni}, y_{nij} \mid n \in \text{IN}, i \in \{0, 1\}, j \in \{0, 1, 2\}\}$. Take any $z_f \in W\text{-}W'$ and put $V(p) = \{y_{nf(n)2} \mid n \in \text{IN}\}$. Then $\langle F', V\rangle \models \Box\Diamond\Box\Box p\ [x_1]$, because $\langle F', V\rangle \models \Box p\ [x_4]$, $\langle F', V\rangle \models \Box\Box p\ [x_3]$, $\langle F', V\rangle \models \Diamond\Box\Box p\ [x_2]$, and $\langle F', V\rangle \models p\ [y_{nf(n)2}]$, $\langle F', V\rangle \models \Box p\ [y_{nf(n)1}]$, $\langle F', V\rangle \models \Box p\ [y_{nf(n)0}]$, $\langle F', V\rangle \models \Box\Box p\ [y_{nf(n)}]$, $\langle F', V\rangle \models \Diamond\Box\Box p\ [y_n]$. Also $\sim\langle F', V\rangle \models \Diamond\Diamond\Box\Diamond p\ [x_1]$, for $\sim\langle F', V\rangle \models \Diamond p\ [x_4]$, $\sim\langle F', V\rangle \models \Box\Diamond p\ [x_3]$, and, for all $i, n \in$ IN, $\sim\langle F', V\rangle \models \Diamond p\ [y_{ni0}]$, $\sim\langle F', V\rangle \models \Box\Diamond p\ [y_{ni}]$, and finally $\sim\langle F', V\rangle \models \Box\Diamond p\ [z_g]$ for any $z_g \in W'$. To see this, note that $z_g \neq z_f$, so $g \neq f$ and, for at least one $n \in \mathbb{N}$, $g(n) \neq f(n)$. For such an n, $\sim\langle F', V\rangle \models \Diamond p [z_{gn}]$, since $\sim\langle F', V\rangle \models p [y_{ng(n)2}]$, and therefore $\sim\langle F', V\rangle \models \Box\Diamond p [z_g]$. It follows that $\sim F' \models \Box\Diamond\Box\Box p \to \Diamond\Diamond\Box\Diamond p [x_1]$.

QED.

The last result of this chapter shows that set-theoretic principles from outside ZF may be necessary for proving equivalences of the form $E(\phi^m, \phi^o)$. As will be shown in corollary 2.22, it follows from this that E is not provably arithmetical in ZF. In chapter I.4 the result is used in the proof that $M_1^{sub} \neq M1$.

In the remainder of this chapter $\phi^m$ will stand for the modal formula $(\Box p \to \Box\Box p) \wedge \Box(\Box p \to \Box\Box p) \wedge (\Box\Diamond p \to \Diamond\Box p)$, and $\phi^o$ for the formula $(\forall y)(Rxy \to (\forall z)(Ryz \to Rxz)) \wedge (\forall y)(Rxy \to (\forall u)(Ryu \to (\forall v)(Ruv \to Ryv))) \wedge (\exists y)(Rxy \wedge (\forall z)(Ryz \to z = y))$. Note that $F \models (\forall y)(Rxy \to (\forall z)(Ryz \to Rxz)) [w]$ does not imply transitivity of R even on TC(F, w). $F = \langle \mathbb{N}, \{\langle 0, n\rangle, \langle n, n+1\rangle \mid n \in \mathbb{N}\}\rangle$ and $w = 0$ provide a counterexample: $\langle 1, 2\rangle \in R$ and $\langle 2, 3\rangle \in R$, but $\langle 1, 3\rangle \notin R$. But in conjunction with $F \models (\forall y)(Rxy \to (\forall u)(Ryu \to (\forall v)(Ruv \to Ryv))) [w]$ this formula guarantees that R is transitive on TC(F, w).

2.21 <u>Lemma</u>

(AC) $E(\phi^m, \phi^o)$

$ZF \vdash E(\phi^m, \phi^o) \to AC^{uo}$,

where $AC^{uo}$ is the axiom of choice for unordered pairs.

<u>Proof</u>: It is provable without the axiom of choice that $E(\Box p \to \Box\Box p, (\forall y)(Rxy \to (\forall z)(Ryz \to Rxz)))$ and $E(\Box(\Box p \to \Box\Box p), (\forall y)(Rxy \to (\forall u)(Ryu \to (\forall v)(Ruv \to Ryv))))$, using the methods developed in chapter I.4.

The following result follows from theorem 2 of II.2. On the <u>transitive</u> frames $E(\Box\Diamond p \to \Diamond\Box p, (\exists y)(Rxy \wedge (\forall z)(Ryz \to z = y)))$ holds.

These facts, combined with the preceding remarks, prove that $E(\phi^m, \phi^o)$. If $F \models \phi^m [w]$, then $F \models (\forall y)(Rxy \to (\forall z)(Ryz \to Rxz)) [w]$ and $F \models (\forall y)(Rxy \to (\forall u)(Ryu \to (\forall v)(Ruv \to Ryv))) [w]$, hence R is transitive on TC(F, w). By corollary 2.6, $TC(F, w) \models \phi^m [w]$ and, by the result from II.2, $TC(F, w) \models (\exists y)(Rxy \wedge (\forall z)(Ryz \to z = y)) [w]$, which implies $F \models (\exists y)(Rxy \wedge (\forall z)(Ryz \to z = y)) [w]$. It follows that $F \models \phi^o [w]$. If, on the other hand, $F \models \phi^o [w]$, then $F \models \Box p \to \Box\Box p [w]$ and $F \models \Box(\Box p \to \Box\Box p) [w]$, and again TC(F, w) is a transitive frame satisfying $TC(F, w) \models \Box\Diamond p \to \Diamond\Box p [w]$. Another application of 2.6 yields $F \models \phi^m [w]$.

The proof of theorem 2 in II.2 depends on the axiom of choice. Our second assertion is a kind of weak converse. Note that $\sim ZF \vdash AC^{u0}$, as is proved in Jech [ 10 ].

Let $\{A_i \mid i \in I\}$ be a set of disjoint unordered pairs. An application of $E(\phi^m, \phi^o)$ yields a set of representatives for $\{A_i \mid i \in I\}$. Take some w outside $\bigcup_{i\in I} A_i$, and let $R = \{\langle x, y\rangle \mid (x = w \;\&\; y \in \bigcup_{i\in I} A_i)$ or, for some $i \in I$, $x \in A_i \;\&\; y \in A_i\}$ and $F = \langle \bigcup_{i\in I} A_i \cup \{w\}, R\rangle$.
$F \models (\forall y)(Rxy \to (\forall z)(Ryz \to Rxz)) [w]$ and $F \models (\forall y)(Rxy \to (\forall u)(Ryu \to (\forall v)(Ruv \to Ryv))) [w]$, so $F \models (\Box p \to \Box\Box p) \wedge \Box(\Box p \to \Box\Box p) [w]$.
Since $\sim F \models (\exists y)(Rxy \wedge (\forall z)(Ryz \to z = y)) [w]$, $\sim F \models \phi^m [w]$, and this can only be the case because $\sim F \models \Box\Diamond p \to \Diamond\Box p [w]$. If V is any valuation on F for which $\langle F, V\rangle \models \Box\Diamond p [w]$ and $\sim\langle F, V\rangle \models \Diamond\Box p [w]$ (i.e., $\langle F, V\rangle \models \Box\Diamond\neg p [w]$), then $V(p) - \{w\}$ is the required set, having exactly one member in common with each $A_i$. QED.

2.22 Corollary

E is not provably arithmetical in ZF.

Proof: $ZF + AC \vdash E(\phi^m, \phi^o)$ and $ZF \vdash E(\phi^m, \phi^o) \rightarrow AC^{uo}$.
The latter implies, by Jech's result, that $\sim ZF \vdash E(\phi^m, \phi^o)$.
But then E cannot be provably arithmetical in ZF, since ZF+AC is conservative over ZF with respect to arithmetical statements. (If $\phi$ is arithmetical, i.e., all quantifiers in $\phi$ are relativized to $\omega$, and $ZF + AC \vdash \phi$, then, since $ZF \vdash (ZF)^L$ and $ZF \vdash (AC)^L$, $ZF \vdash \phi^L$, where L defines the constructible universe. Now $\omega$ is absolute and, therefore, $ZF \vdash \phi$.)

QED.

A similar argument shows that M1 and $\bar{M}1$ are not provably arithmetical in ZF. E.g., for the case of $\bar{M}1$, we use the fact that (*) $ZF \vdash \phi^m \in \bar{M}1 \rightarrow COF$, where COF is the principle that any linear ordering without a last element has a cofinal subset whose complement is also cofinal. For countable orderings, this principle is provable in ZF, but its general form is not. (Cf. Jech [ 10 ], p. 96.) (*) is easily proved using the Löwenheim-Skolem theorem for single formulas (which is provable in ZF): if COF has a counterexample F, then $F \models \phi^m$; but $\phi^m$ holds on no countable linear ordering without a last element.

## I.3 AN ALGEBRAIC CHARACTERIZATION OF $\overline{M}1$

This chapter begins with the results of II.3, continues with a few results about preservation of second-order sentences under ultraproducts and ends up with a few topics in modal model theory.

3.1 Lemma (R.I. Goldblatt)

If $\{F_i \mid i \in I\}$ is a set of frames and U an ultrafilter on I, then the ultraproduct $\prod_U F_i$ is isomorphic to a generated subframe of the ultrapower $\prod_U \oplus \{F_i \mid i \in I\}$.

This lemma was stated by Goldblatt in a private communication to the author.

3.2 Definition

$\underline{FR(\phi)} = \{F \mid F \models \phi\}$

$\underline{FR(\Gamma)} = \bigcap_{\phi\in\Gamma} FR(\phi)$

3.3 Definition

A class of frames is

elementary, if it equals $FR(\phi)$, for some $L_0$-sentence $\phi$

$\Delta$-elementary, if it is an intersection of elementary classes

$\Sigma$-elementary, if it is a union of elementary classes

$\Sigma\Delta$-elementary, if it is a union of $\Delta$-elementary classes.

This hierarchy does not extend beyond ΣΔ-elementary classes: it collapses, since a class of frames is ΣΔ-elementary iff it is closed under $L_0$-elementary equivalence.

3.4 Theorem

A ΣΔ-elementary class of frames closed under disjoint unions and generated subframes is closed under ultraproducts and is, therefore, Δ-elementary.

A Σ-elementary class of frames closed under disjoint unions and generated subframes is elementary.

Proof: A ΣΔ-elementary class of frames is closed under elementary equivalence and, therefore, closed under ultrapowers and isomorphic images. So, if it is also closed under disjoint unions and generated subframes, lemma 3.1 implies that it is closed under ultraproducts. A class of frames closed under elementary equivalence and ultraproducts is Δ-elementary.

A Σ-elementary class is ΣΔ-elementary. So, if it is closed under disjoint unions and generated subframes, it is Δ-elementary. A class of frames which is both Σ-elementary and Δ-elementary is elementary.

QED.

3.5 Corollary

If Γ is a set of modal formulas, then

FR(Γ) is ΣΔ-elementary ⇒ FR(Γ) is Δ-elementary

FR(Γ) is Σ-elementary ⇒ FR(Γ) is elementary.

If $\phi$ is a modal formula, then

$FR(\phi)$ is $\Sigma\Delta$-elementary $\Rightarrow$ $FR(\phi)$ is elementary.

Proof: Modal formulas are preserved under disjoint unions and generated subframes, by 2.9 and 2.6. Moreover, if $FR(\phi)$ is $\Delta$-elementary, it is elementary. This follows from the observations on universal second-order sentences to be made below. QED.

Standard compactness arguments show that, for all second-order sentences $\phi$ of the form $(\forall X_1)\ldots(\forall X_k)\psi$, where $X_1,\ldots, X_k$ are predicate variables and $\psi$ is a first-order sentence, the following two equivalences hold:

$FR(\phi)$ is $\Sigma\Delta$-elementary $\Leftrightarrow$ $FR(\phi)$ is $\Sigma$-elementary

$FR(\phi)$ is $\Delta$-elementary $\Leftrightarrow$ $FR(\phi)$ is elementary.

Also, $FR(\phi)$ is elementary $\Leftrightarrow$ $\phi$ is preserved under ultraproducts, which follows from the fact that existential second-order sentences are preserved under ultraproducts. For, clearly both $FR(\phi)$ and its complement are closed under isomorphic images, so, if they are both closed under ultraproducts, they will be elementary, by Keisler's characterization of elementary classes.

Reformulating the above results the following characterization of $\overline{M}1$ is obtained.

3.6 Theorem

For any modal formula $\phi$ the following three statements are equivalent:

FR($\phi$) is elementary (i.e, $\phi \in \overline{M}1$)

FR($\phi$) is closed under ($L_0$-) elementary equivalence

FR($\phi$) is closed under ultrapowers.

For M1 the following similar, but less elegant, characterization may be proved using the same methods.

3.7 Theorem

A modal formula $\phi$ is in M1 iff, for all frames F and sets I such that $w_i \in W$, for each $i \in I$, and all ultrafilters U on I,

$(\forall i \in I)F \models \phi\,[w_i] \Rightarrow \prod_U F \models \phi\,[(\langle w_i\rangle_{i\in I})_U]$.

This theorem is used in the only proof we have been able to find for

3.8 Lemma

If $\Box\phi \in M1$, then $\phi \in M1$.

Proof: If $\phi \notin M1$, then, by theorem 3.7, there are $F = \langle W, R\rangle$, I, $\{w_i \mid i \in I\}$ and U with, for each $i \in I$, $F \models \phi\,[w_i]$, but $\sim\prod_U F \models \phi\,[(\langle w_i\rangle_{i\in I})_U]$. Take some v outside the domain of $\prod_U F$, and let $F_i$ be the frame $\langle W \cup \{v\}, R \cup \{\langle v, w_i\rangle\}\rangle$. Since $F \models \phi\,[w_i]$, $F_i \models \phi\,[w_i]$ and $F_i \models \Box\phi\,[v]$.
We show that $\sim\prod_U F_i \models \Box\phi\,[(\langle v\rangle_{i\in I})_U]$, thereby proving that $\Box\phi \notin M1$.

For each $i \in I$, $F_i \models (\forall x)(Rx_1x \leftrightarrow x = x_2)\,[v, w_i]$ and, therefore, by the theorem of Łoś, $\prod_U F_i \models (\forall x)(Rx_1x \leftrightarrow x = x_2)\,[(\langle v\rangle_{i\in I})_U, (\langle w_i\rangle_{i\in I})_U]$. So, $(\langle v\rangle_{i\in I})_U$ has exactly one R-successor in $\prod_U F_i$, viz.

$(\langle w_i\rangle_{i\in I})_U$. Clearly, $F \subsetneq F_i$, and therefore $\prod_U F \subsetneq \prod_U F_i$. This is an instance of the following general fact used in the proof of lemma 3.1:

If $F_i \subsetneq F_i'$ for all $i \in I$, and U is an ultrafilter on I, then $\prod_U F_i \subsetneq \prod_U F_i'$.

(The proof of this is straightforward.) Now let V be any valuation on $\prod_U F$ such that $\langle\prod_U F, V\rangle \models \neg\phi\,[\,(\langle w_i\rangle_{i\in I})_U\,]$. V is also a valuation on $\prod_U F_i$, and, by lemma 2.5, $\langle\prod_U F_i, V\rangle \models \neg\phi\,[\,(\langle w_i\rangle_{i\in I})_U\,]$. This implies that $\langle\prod_U F_i, V\rangle \models \neg\Box\phi\,[\,(\langle v\rangle_{i\in I})_U\,]$. QED.

The converse of lemma 3.8 is a part of lemma 4.2.

In order to put theorem 3.6 into perspective we mention a few results without proof. Second-order sentences of the form $(\forall P_1)\ldots(\forall P_n)(\forall x_1)\ldots(\forall x_k)\phi$, where $\phi$ is constructed using atomic formulas of the form $P_ix_j$ for each i, j ($1 \leqq i \leqq n$, $1 \leqq j \leqq k$), $L_0$-formulas with free variables among $x_1,\ldots, x_k$, and Boolean operators, are preserved under ultraproducts. Sentences of the form $(\forall P_1)\ldots(\forall P_n)(\exists x_1)\ldots(\exists x_k)\phi$, with $\phi$ as in the preceding sentence, are preserved under ultrapowers. But not every sentence of this last form is preserved under ultraproducts, as is shown by the following sentence $\psi$ defining the finite irreflexive linear orderings. Let $\chi = \chi(R, =)$ express that R is a discrete linear ordering with a first and a last element. Then take $\psi = (\forall P)(\chi \wedge ((\forall x)(\forall y)((Px \wedge \neg Py) \rightarrow Rxy) \rightarrow ((\exists z)(\neg(\exists y)(Ryz \wedge \neg Pz) \vee (\exists z)(\neg(\exists y)Rzy \wedge Pz) \vee (\exists z)(Pz \wedge (\exists u)(\neg Pu \wedge \neg(\exists v)(Rzv \wedge Rvu))))))$. Using the rules for obtaining a prenex normal form $\psi$ is easily brought into the form $(\forall P)(\exists x_1)\ldots(\exists x_6)\phi$, where $\phi$ is as above.

The limitations of these results are shown by the following sentence $\alpha$, defining the natural numbers with $<$, which is clearly not preserved under ultrapowers. Let $\beta = \beta(R, =)$ axiomatize the $L_0$-theory of the natural numbers with $<$. Then take $\alpha = (\forall P)(\beta \wedge ((\exists x)(\neg(\exists y)Ryx \wedge Px) \rightarrow ((\forall x)(\forall y)((Rxy \wedge \neg(\exists z)(Rxz \wedge Rzy)) \rightarrow (Px \rightarrow Py)) \rightarrow (\forall x)Px)))$. Again using the rules for obtaining a prenex normal form $\alpha$ is easily brought into either of the forms $(\forall P)(\forall x_1)(\forall x_2)(\exists x_3)(\exists x_4)\phi$ or $(\forall P)(\exists x_1)(\exists x_2)(\forall x_3)(\forall x_4)\phi$, where $\phi$ is as above. So allowing any other combination of first-order quantifiers than the two mentioned above leads to essentially second-order sentences.

Treating modal formulas as second-order formulas in the way we do here makes it interesting to study modal model theory as a first step towards the model theory for second-order logic, where results are so regrettably scarce. A few topics will be mentioned here.

In II.1 an uncountable frame F is presented, such that $F \models \Box\Diamond p \rightarrow \Diamond\Box p$, but, for no countable elementary subframe F' of F, $F' \models \Box\Diamond p \rightarrow \Diamond\Box p$. This may be interpreted as a failure of the Löwenheim-Skolem property for modal formulas. But, defining more purely modal notions like those in definition 3.9 below, we get the following problem.

3.9 <u>Definition</u>

If F is a frame and M a model, then the <u>modal theory</u> of F ($Th_m(F)$) is $\{\phi \mid \phi$ is a modal formula and $F \models \phi\}$, and the <u>modal theory</u> of M ($Th_m(M)$) is $\{\phi \mid \phi$ is a modal formula and $M \models \phi\}$.

Is there, for any frame, a countable frame with the same modal theory? For models the answer is affirmative, as follows trivially from the Löwenheim-Skolem theorem. For frames the answer is negative, as is shown by S.K. Thomason in "Reduction of tense logic to modal logic. I", the Journal of Symbolic Logic 39:3 (1974), pp. 549-551.

In the statement of Thomason's result in the introduction the consequence relation $\models$ for modal formulas was not defined explicitly in modal terminology. If this is done, as follows,

3.10 <u>Definition</u>

If $\Gamma$ is a set of modal formulas and $\phi$ is a modal formula, then $\Gamma \models \phi \Leftrightarrow (\forall F)(F \models \Gamma \Rightarrow F \models \phi)$.

it becomes a matter of interest to determine the smallest cardinality <u>m</u> for which the following holds,

For all sets $\Gamma$ of modal formulas and all modal formulas $\phi$, if $\sim\Gamma \models \phi$, then, for some frame F of cardinality smaller than <u>m</u>, $F \models \Gamma$ and $\sim F \models \phi$.

Obviously, such an <u>m</u> exists, as a Hanf-type argument shows.

There is a peculiar mixture of first and second-order elements in the behaviour of modal formulas promising an attractive area for investigation. An example of this concludes the present chapter.

It follows from the tree lemma (2.15) that any modal formula which is not universally valid has a counterexample on a finite frame. So, the set of universally valid modal formulas is recursive, in view of Post's theorem and the fact that the set of universally valid modal formulas is recursively enumerable. (This follows from the usual modal completeness theorems, or from the completeness theorem for $L_1$ via the ST-translation of II.2.) On the other hand, the relation $\models$ is highly complex, even in the form $\phi \models \psi$, where $\phi$ and $\psi$ are modal formulas. $\{\psi \mid \psi$ is a modal formula and $\delta \models \psi\}$, where $\delta$ is the particular modal formula used by Thomason in his translation (cf. the introduction), is not recursively enumerable, since it is a reduction class for all universally valid second-order sentences. The difference between universal validity and logical consequence for modal formulas is also illustrated by the following result, for which some auxiliary notation is needed.

3.11 Definition

$\Box^0\phi = \phi$; $\Box^{n+1}\phi = \Box\Box^n\phi$.

$\Diamond^0\phi = \phi$; $\Diamond^{n+1}\phi = \Diamond\Diamond^n\phi$.

3.12 Lemma

For all finite F, if $F \models ((\Box p \wedge \neg\Box^2 p) \rightarrow \Diamond(\Box^2 p \wedge \neg\Box^3 p)) \wedge (\Box p \rightarrow p)$, then $F \models \Box p \rightarrow \Box^2 p$.

It is not the case that $((\Box p \wedge \neg\Box^2 p) \rightarrow \Diamond(\Box^2 p \wedge \neg\Box^3 p)) \wedge (\Box p \rightarrow p) \models \Box p \rightarrow \Box^2 p$.

Proof: Let F be a finite frame such that (*) $F \models ((\Box p \wedge \neg\Box^2 p) \rightarrow \Diamond(\Box^2 p \wedge \neg\Box^3 p)) \wedge (\Box p \rightarrow p)$. Suppose that $\sim F \models \Box p \rightarrow \Box^2 p$: we shall derive a contradiction. For some valuation V on F and some $w_1 \in W$, $\langle F, V\rangle \models \Box p\,[w_1]$ and $\sim\langle F, V\rangle \models \Box^2 p\,[w_1]$.

By (*), $\langle F, V\rangle \models \Diamond(\Box^2 p \wedge \neg\Box^3 p)[w_1]$, so $w_2 \in W$ exists such that $Rw_1w_2$ and $\langle F, V\rangle \models \Box^2 p \wedge \neg\Box^3 p[w_2]$. Obviously, $w_2 \neq w_1$.

Let $w_1,\ldots, w_n$ be elements of W such that $Rw_iw_{i+1}$, for each i ($1 \leqq i \leqq n-1$) and $w_i \neq w_j$, for each i, j ($1 \leqq i \neq j \leqq n$) and $\langle F, V\rangle \models \Box^i p \wedge \neg\Box^{i+1} p[w_i]$, for each i ($1 \leqq i \leqq n$), hold. This sequence can be extended to a sequence $w_1,\ldots, w_n, w_{n+1}$ with the same properties, using the general principle

For all modal formulas $\phi$ and $\psi$ and all frames F, if $F \models \phi$, then $F \models [\psi/p]\phi$ for all proposition letters p.

This principle follows from a simple observation. If V is a valuation on F and V' is like V but for its p-value, which is $\{w \in W \mid \langle F, V\rangle \models \psi[w]\}$, then $\langle F, V\rangle \models [\psi/p]\phi[w] \Leftrightarrow \langle F, V'\rangle \models \phi[w]$. (If this were an elementary text book we would formulate the principle as the so-called "substitution lemma".)

The $w_{n+1}$ referred to above is found by noting that (*) and the above principle imply that $F \models (\Box^n p \wedge \neg\Box^{n+1} p) \to \Diamond(\Box^{n+1} p \wedge \neg\Box^{n+2} p)$. (Substitute $\Box^{n-1}p$ for p.) Therefore, $\langle F, V\rangle \models \Diamond(\Box^{n+1} p \wedge \neg\Box^{n+2} p)[w_n]$, so a $w_{n+1}$ exists with $\langle F, V\rangle \models \Box^{n+1} p \wedge \neg\Box^{n+2} p[w_{n+1}]$. For each $i \leqq n$, $w_{n+1} \neq w_i$, because $\Box^{n+1}p \to \Box^i p$ holds on F. (Use the fact that $F \models \Box p \to p$, and apply the above principle several times.)

This construction shows that F is infinite, which is our contradiction.

The second assertion of the lemma is proved by an example taken from Makinson [ 14 ]. Consider the frame $\langle \mathbb{N}, R\rangle$, with $R = \{\langle m, n\rangle \mid m \in \mathbb{N}, n \in \mathbb{N}, m \leqq n \text{ or } m = n+1\}$. R is not transitive, and therefore $\Box p \to \Box^2 p$ does not hold on this frame, but it is easy to check that $((\Box p \wedge \neg\Box^2 p) \to \Diamond(\Box^2 p \wedge \neg\Box^3 p)) \wedge (\Box p \to p)$ holds on it. QED.

## I.4 SYNTACTIC RESULTS ON M1

The first five lemmas of this chapter list some simple properties of E and M1.

### 4.1 Lemma

For all modal formulas $\phi$ and $\psi$ and all $L_0$-formulas $\alpha$ and $\beta$,

$E(\phi, \alpha)$ & $E(\psi, \beta) \Rightarrow E(\phi \wedge \psi, \alpha \wedge \beta)$

$E(\phi, \alpha)$ & $E(\psi, \beta) \Rightarrow E(\phi \vee \psi, \alpha \vee \beta)$, provided that $\phi$ and $\psi$ have no proposition letters in common

$E(\phi, \alpha) \Leftrightarrow E([\neg p/p]\phi, \alpha)$, for all proposition letters p.

Proof: For all modal formulas $\phi$ and $\psi$, $F \models \phi \wedge \psi [w]$ iff $F \models \phi [w]$ and $F \models \psi [w]$. If $\phi$ and $\psi$ have no proposition letters in common, then $F \models \phi \vee \psi [w]$ iff $F \models \phi [w]$ or $F \models \psi [w]$. This is easily provable using the fact that, if $V_1$ and $V_2$ agree on the proposition letters occurring in $\phi$, then $\langle F, V_1\rangle \models \phi [w]$ iff $\langle F, V_2\rangle \models \phi [w]$. Finally, $F \models \phi [w]$ iff $F \models [\neg p/p]\phi [w]$, for all proposition letters p. QED.

### 4.2 Lemma

For all modal formulas $\phi$ and $\psi$,

(i) $\phi \in M1$ & $\psi \in M1 \Rightarrow \phi \wedge \psi \in M1$

(ii) $\phi \in M1$ & $\psi \in M1 \Rightarrow \phi \vee \psi \in M1$, provided that $\phi$ and $\psi$ have no proposition letters in common

(iii) $\phi \in M1 \Leftrightarrow [\neg p/p]\phi \in M1$, for all proposition letters p

(iv) $\phi \in M1 \Leftrightarrow \Box\phi \in M1$.

Proof: (i), (ii) and (iii) follow from lemma 4.1. One direction of (iv) is lemma 3.8, the other is proved as follows. If $\phi \in M1$, then, for some $L_0$-formula $\psi$, $E(\phi, \psi)$, where $\psi$ has one free variable, say x. For any variable y not occurring in $\psi$, $E(\Box\phi, (\forall y)(Rxy \rightarrow [y/x]\psi))$ and so $\Box\phi \in M1$. This is so, because, for all frames F and $w \in W$, $F \models \Box\phi[w]$ iff $(\forall v \in W)(Rwv \Rightarrow F \models \phi[v])$. QED.

4.3 Lemma

The following implications do not hold for all modal formulas $\phi$ and $\psi$,

(i) $\phi \in M1 \Rightarrow \neg\phi \in M1$

(ii) $\phi \in M1 \Rightarrow \Diamond\phi \in M1$

(iii) $\phi \in M1\ \&\ \psi \in M1 \Rightarrow (\phi \rightarrow \psi) \in M1$

(iv) $\phi \in M1 \Rightarrow [\neg p/q]\phi \in M1$

(v) $\phi \wedge \psi \in M1 \Rightarrow \phi \in M1\ \&\ \psi \in M1$.

Proof: In II.1 the modal formula $\Box\Diamond p \rightarrow \Diamond\Box p$ is shown to be outside M1. This formula is equivalent to $\neg(\Box\Diamond p \wedge \Box\Diamond\neg p)$ and to $\Diamond(\Diamond p \rightarrow \Box p)$. On the other hand the following formulas are in M1: $\Box\Diamond p$, $\Box\Diamond\neg p$, $\Diamond\Box p$ and $\Diamond p \rightarrow \Box p$, with $L_0$-equivalents $\neg(\exists y)Rxy$, $\neg(\exists y)Rxy$, $(\exists y)(Rxy \wedge \neg(\exists z)Ryz)$ and $(\forall y)(Rxy \rightarrow (\forall z)(Rxz \rightarrow z = y))$, respectively. By this, (i), (ii) and (iii) are obvious.

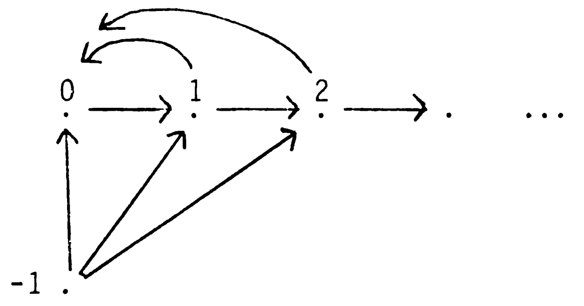
For (iv) consider $\phi = (\Diamond p \wedge \Diamond q) \rightarrow \Diamond(p \wedge \Diamond q)$. $\phi \in M1$, because $E(\phi, (\forall y)(Rxy \rightarrow (\forall z)(Rxz \rightarrow Ryz)))$. We will show that $[\neg p/q]\phi =$

$(\Diamond p \wedge \Diamond \neg p) \to \Diamond(p \wedge \Diamond \neg p)$ is not in M1.

Let $F = \langle W, R\rangle$ be the frame with

$W = \{-1, 0, 1, 2, \ldots\}$

$R = \{\langle -1, i\rangle, \langle i, i+1\rangle, \langle i+1, 0\rangle \mid i \in \mathbb{N}\}$.




$F \models (\Diamond p \wedge \Diamond \neg p) \to \Diamond(p \wedge \Diamond \neg p)\ [-1]$. To see this, let V be a valuation on F such that $\langle F, V\rangle \models \Diamond p \wedge \Diamond \neg p\ [-1]$. For some i, j $\in \mathbb{N}$, $\langle F, V\rangle \models p\ [i]$ and $\langle F, V\rangle \models \neg p\ [j]$. Either $\langle F, V\rangle \models \neg p\ [0]$, in which case $\langle F, V\rangle \models p \wedge \Diamond \neg p\ [i]$ and $\langle F, V\rangle \models \Diamond(p \wedge \Diamond \neg p)\ [-1]$, or $\langle F, V\rangle \models p\ [0]$, in which case $\langle F, V\rangle \models p \wedge \Diamond \neg p\ [k]$, where k is the greatest number smaller than j such that $\langle F, V\rangle \models p\ [k]$.

If our formula were in M1 it would have to hold at -1 in proper elementary extensions of F. Let F' be any proper elementary extension of F (in which $\mathbb{N}$ gets a "tail"), and set $V(p) = \mathbb{N}$.
Then $\langle F', V\rangle \models \Diamond p \wedge \Diamond \neg p\ [-1]$, but $\sim\langle F', V\rangle \models \Diamond(p \wedge \Diamond \neg p)\ [-1]$, for the only R- successors of $n \in \mathbb{N}$ remain 0 and n+1.

(v) follows from the example in lemma 2.21. $\Box p \to \Box\Box p \in$ M1, so $\Box(\Box p \to \Box\Box p) \in$ M1, by lemma 4.2(iv) above, and $(\Box p \to \Box\Box p) \wedge \Box(\Box p \to \Box\Box p) \in$ M1. Also, $(\Box p \to \Box\Box p) \wedge \Box(\Box p \to \Box\Box p) \wedge (\Box\Diamond p \to \Diamond\Box p) \in$ M1, as was shown in the proof of lemma 2.21.

But, as noted above, $\Box\Diamond p \to \Diamond\Box p \notin$ M1. QED.

Recall our use of $\perp$ and T as signs for formulas which are everywhere false and everywhere true, respectively. This notation greatly simplifies the statement of the subsequent results in this chapter.

4.4 Definition

A closed formula is a modal formula containing only occurrences of $\perp$, T, Boolean operators and modal operators.

4.5 Definition

A modal formula $\phi$ is monotone in the proposition letter p if, for all models $M = \langle W, R, V\rangle$, all $w \in W$ and all valuations V' such that $V'(p) \supseteq V(p)$,

$M \models \phi [w] \Rightarrow \langle W, R, V'\rangle \models \phi [w]$.

4.6 Definition

A modal formula $\phi$ is positive if it is constructed using only $\perp$, T, proposition letters, $\wedge$, $\vee$, $\Box$ and $\Diamond$ .

Any positive formula is monotone in all its proposition letters. We have a proof of the converse which is too complicated to be trusted, so we omit it here.

4.7 Lemma

Any closed formula is in M1.

If a modal formula $\phi$ is monotone in p, then $\phi \in M1$ iff $[\perp/p]\phi \in M1$.

Proof: Treating $\perp$ and T as primitives we add the clauses $ST(\perp)$ =

$(\forall x)\neg(Rxx \rightarrow Rxx)$ and $ST(T) = (\forall x)(Rxx \rightarrow Rxx)$ to definition 2.1. Then $ST(\phi)$ will be an $L_0$-formula for any closed modal formula $\phi$.

The second assertion is proved by observing that, for any modal formula $\phi$ monotone in p, and any frame F and $w \in W$, $F \models \phi[w]$ iff $F \models [\bot/p]\phi[w]$. From left to right this is obvious, and from right to left it follows from the fact that $\{w \in W \mid F \models \bot[w]\} = \emptyset$ and $\phi$'s being monotone in p. QED.

4.8 Definition

The degree $d(\phi)$ of a modal formula $\phi$ is defined inductively according to the clauses

$d(\bot) = d(T) = 0$

$d(p) = 0$ for a proposition letter p

$d(\neg\alpha) = d(\alpha)$

$d(\alpha \rightarrow \beta) = \max(d(\alpha), d(\beta))$

$d(\Box\alpha) = d(\alpha) + 1$

Restricting the modal formulas to those in which no iterations of the modal operators occur, as described in Lewis [13], trivializes the problem of characterizing M1. This follows from the next lemma.

4.9 Lemma

If a modal formula $\phi$ has degree $\leq 1$, then $\phi \in M1$.

Proof: Case 1: $d(\phi) = 0$. Then no modal operators occur in $\phi$, it is a propositional formula, and there are two possibilities. Either $\phi$ is a tautology, in which case $E(\phi, Rxx \rightarrow Rxx)$, or $\phi$ is not a tautology, and

$E(\phi, \neg(Rxx \rightarrow Rxx))$, since a falsifying valuation exists.

Case 2: $d(\phi) = 1$.

The term "rewriting" will mean the following in this proof: "taking equivalents using the universally valid formulas $\neg\Diamond\alpha \leftrightarrow \Box\neg\alpha$, $\neg\Box\alpha \leftrightarrow \Diamond\neg\alpha$, $\neg\neg\alpha \leftrightarrow \alpha$, $\Diamond(\alpha \vee \beta) \leftrightarrow (\Diamond\alpha \vee \Diamond\beta)$, $\Box(\alpha \wedge \beta) \leftrightarrow (\Box\alpha \wedge \Box\beta)$, $((\alpha \vee \beta) \rightarrow \gamma) \leftrightarrow ((\alpha \rightarrow \gamma) \wedge (\beta \rightarrow \gamma))$ for all $\alpha$, $\beta$ and $\gamma$, as well as other propositional tautologies, like the De Morgan and distributive laws."

Replace each occurrence of $\bot$ in $\phi$ by $(p \wedge \neg p)$ and each occurrence of $T$ by $(p \vee \neg p)$, where p is any proposition letter. Then rewrite $\phi$ as a conjunction of disjunctions $\prod_{i=1}^{n} \sum_{j=1}^{n_i} \phi_{ij}$, where each $\phi_{ij}$ is either a (negation of a) proposition letter, or a (negation of a) formula of the form $\Diamond\psi_{ij}$, where $\psi_{ij}$ is a conjunction of (negations of) proposition letters. It will be shown that any $\sum_{j=1}^{n_i} \phi_{ij}$ is in M1, whence, by lemma 4.2, $\phi$ is in M1.

Only the most complex case for $\sum_{j=1}^{n_i} \phi_{ij}$ will be treated, degenerate cases being obvious. Let $\phi_i = \sum_{j=1}^{n_i} \phi_{ij}$ be $\alpha_1 \vee \ldots \vee \alpha_k \vee \neg\Diamond\beta_1 \vee \ldots \vee \neg\Diamond\beta_l \vee \Diamond\gamma_1 \vee \ldots \vee \Diamond\gamma_m$, with $\alpha_1, \ldots, \alpha_k$ (negations of) proposition letters. First a few trivial cases have to be excluded. If, for some proposition letter p, p and $\neg p$ occur among $\alpha_1, \ldots, \alpha_k$, then $\phi_i$ is universally valid, so, clearly, it is in M1. If p and $\neg p$ occur as conjuncts in some $\beta_i$, then $\beta_i$ is equivalent to $\bot$, $\Diamond\beta_i$ is equivalent to $\Diamond\bot$, which is equivalent to $\bot$, and so $\neg\Diamond\beta_i$ is equivalent to T and $\phi_i$ is again universally valid. If p and $\neg p$ occur in some $\gamma_j$, then $\Diamond\gamma_j$ may be replaced by $\bot$, for similar reasons, and therefore, dropped from the disjunction. Rewrite $\phi_i$ as $\phi_i^1 = (\neg(\alpha_1 \vee \ldots \vee \alpha_k) \wedge \Diamond\beta_1 \wedge \ldots \wedge \Diamond\beta_l) \rightarrow \Diamond\gamma_1 \vee \ldots \vee \Diamond\gamma_m$. $\neg(\alpha_1 \vee \ldots \vee \alpha_k)$ may be rewritten as a conjunction like the $\beta$'s and $\gamma$'s. It may be assumed that no proposition letter occurs

twice in any of the conjunctions. If $p_1, \ldots, p_n$ are the proposition letters occurring in $\phi_i^1$, then let $P_1, \ldots, P_{2^n}$ be the list of conjunctions specifying, for each $p_i$, whether or not it "obtains". (Compare the well-known "state descriptions".) Rewrite $\phi_i^1$ as a conjunction $\prod_{j=1}^{s} \phi_{ij}^1$, where each $\phi_{ij}^1$ is of the form $(P_{k_1} \wedge \Diamond P_{l_1} \wedge \ldots \wedge \Diamond P_{l_m}) \rightarrow \Diamond P_{m_1} \vee \ldots \vee \Diamond P_{m_r}$.
(This rewriting also uses the fact that any modal formula of the form $\Diamond\alpha \leftrightarrow (\Diamond(\alpha \wedge p) \vee \Diamond(\alpha \wedge \neg p))$ is universally valid.)

Using lemma 4.2 again, it suffices to find $L_0$-equivalents for formulas of this form. Assume that no repetitions occur among $P_{l_1}, \ldots, P_{l_m}$ or among $P_{m_1}, \ldots, P_{m_r}$. The following possibilities arise.

Case 1: Some $l_i$ is among the $m_j$'s. Then $\phi_{ij}^1$ is universally valid and, therefore, trivially in M1.

Case 2: No $l_i$ is among the $m_j$'s.

Subcase 2.1: $k_1$ is among the $m_j$'s. In this case $k_1$ is not among the $l_i$'s, and our formula is equivalent to

$$(\forall y_1)(Rxy_1 \rightarrow \ldots \rightarrow (\forall y_m)(Rxy_m \rightarrow (\prod_{1 \leq i \neq j \leq m} (x \neq y_i \wedge y_i \neq y_j) \rightarrow Rxx)\ldots).$$

Subcase 2.2: $k_1$ is not among the $m_j$'s.

Subcase 2.2.1: $k_1$ is not among the $l_i$'s. Then $\phi_{ij}^1$ is equivalent to

$$\neg(\exists y_1)(Rxy_1 \wedge \ldots \wedge (\exists y_m)(Rxy_m \wedge \prod_{1 \leq i \neq j \leq m} (x \neq y_i \wedge y_i \neq y_j))\ldots).$$

Subcase 2.2.2: $k_1$ is among the $l_i$'s. $\phi_{ij}^1$ is now equivalent to

$$\neg(\exists y_1)(Rxy_1 \wedge \ldots \wedge (\exists y_m)(Rxy_m \wedge \prod_{1 \leq i \neq j \leq m} y_i \neq y_j)\ldots).$$

The proof that these equivalences hold is too tedious to be given here. Going through an example will convince the reader. QED.

Lemma 4.9 may also be proved using the characterization of M1 obtained in theorem 3.7. The idea is to use the fact that, if the formula in question can be falsified in the ultraproduct, this is due to the existence of enough R-successors of $(<w_i>_{i\in I})_U$. But this fact can be transferred to F itself, by the theorem of Łoś.

Theorems 4.11 and 4.13 will now be proved using the method of substitutions described briefly in the introduction. After that the class $M1^{sub}$ of formulas for which this method works is introduced, and shown to be a proper subset of M1. This method is very useful in the actual practice of "reading off" $L_0$-equivalents for modal formulas. A few examples will be given, but for more applications the reader should consult Van Benthem [ 1 ] .

4.10 Definition

$R^0xy$ stands for $x = y$

$R^{n+1}xy$ stands for $(\exists z_{n+1})(R^n x z_{n+1} \wedge R z_{n+1} y)$.

It is more convenient to consider $R^1xy$ not as $(\exists z_1)(x = z_1 \wedge Rz_1y)$, but as Rxy.

Recall the notation $\Box^i$, $\Diamond^i$ of definition 3.11.

4.11 Theorem

If the modal formula $\psi$ is positive and the modal formula $\phi$ is constructed using $\Box^i p$ for proposition letters p and $i \in \mathbb{N}$, $\perp$, T, V, $\wedge$ and $\Diamond$, then $\phi \to \psi \in M1$.

Proof: We first reduce the assertion to be proved to the case without mention of "v". Use the equivalences mentioned in the proof of lemma 4.9 in order to rewrite $\phi$ as a disjunction of formulas constructed using formulas of the form $\Box^i p$, $\bot$, T ,$\wedge$ and $\Diamond$. Then rewrite $\phi \to \psi$ as a conjunction of implications, each of which has one of these disjuncts as its antecedent formula.

Lemma 4.7 helps in removing the proposition letters which occur in $\phi \to \psi$, but not both in $\phi$ and in $\psi$. (In a sense these do not contribute anything vital to the formula.) Let p be such a proposition letter. If it occurs in $\psi$, then $\phi \to \psi$ is monotone in p, and $\bot$ may be substituted for it. If it occurs in $\phi$, then T may be substituted for it. For, by lemma 4.2, $[\neg p/p](\phi \to \psi)$ may be considered instead of $\phi \to \psi$, and this formula is monotone in p. Substituting $\bot$ for p in $[\neg p/p](\phi \to \psi)$ has the same effect as substituting T for p in $(\phi \to \psi)$.

Consider some $\phi \to \psi$ obtained through these manipulations. Write $ST(\phi \to \psi)$ in such a way that no two quantifiers have the same bound variable. In this way, there corresponds, to each occurrence of $\Box$ and $\Diamond$ in $\phi \to \psi$, a unique bound variable in $ST(\phi \to \psi)$. From $ST(\phi \to \psi)$ $L_0$-formulas $CV(p, \phi)$ will be extracted for each proposition letter p, which, on substitution in a slightly modified form of $ST(\phi \to \psi)$, will yield the required $L_0$-equivalent.

Consider $ST(\phi)$ occurring as the antecedent formula in $ST(\phi \to \psi)$. Move all existential quantifiers corresponding to occurrences of $\Diamond$ in $\phi$ to the front. This is possible by the operations that bring formulas into a prenex normal form, because only occurrences of $\wedge$ have to be "crossed". This yields $(\exists y_1)\ldots(\exists y_k)\phi'$, so $ST(\phi \to \psi)$ may now be written as $(\forall y_1)\ldots(\forall y_k)(\phi' \to ST(\psi))$.

Fix a variable u not occurring in $ST(\phi \to \psi)$. Let $\bar{p}$ be an occurrence of p in $\phi$. $v(\bar{p})$ is the bound variable $y_i$ in $ST(\phi)$ corresponding to the innermost occurrence of $\Diamond$ in $\phi$ the scope of which contains $\bar{p}$, or, if no such occurrence of $\Diamond$ exists, $v(\bar{p}) = x$. For the greatest number j such that $\bar{p}$ occurs within a subformula of $\phi$ of the form $\Box^j p$ put $CV(\bar{p}, \phi) = R^j v(\bar{p})u$. $CV(p, \phi)$ is defined as $\sum_{\substack{\text{all occurrences} \\ \bar{p} \text{ of } p \text{ in } \phi}} CV(\bar{p}, \phi)$.

Finally take alphabetic variants, if necessary, to ensure that the $CV(p, \phi)$'s and $(\forall y_1)\ldots(\forall y_k)(\phi' \rightarrow ST(\phi))$ have no bound variables in common.

The $L_0$-equivalent $s(\phi \rightarrow \psi)$ of $\phi \rightarrow \psi$ is obtained by substituting, for each proposition letter p and corresponding unary predicate constant P, and each individual variable z, $[z/u]CV(p, \phi)$ for Pz in $(\forall y_1)\ldots(\forall y_k)(\phi' \rightarrow ST(\phi))$.

A number of examples illustrating the above procedure will follow this proof, the remainder of which consists in showing that, for all frames F and all $w \in W$, $F \models \phi \rightarrow \psi\ [w]$ iff $F \models s(\phi \rightarrow \psi)\ [w]$.

One direction is immediate. If $F \models \phi \rightarrow \psi\ [w]$, then, for the proposition letters $p_1, \ldots, p_n$ occurring in $\phi \rightarrow \psi$, $F \models (\forall P_1)\ldots(\forall P_n)ST(\phi \rightarrow \psi)\ [w]$, and so $F \models (\forall P_1)\ldots(\forall P_n)(\forall y_1)\ldots(\forall y_k)(\phi' \rightarrow ST(\psi))\ [w]$, or $F \models (\forall y_1)\ldots(\forall y_k)(\forall P_1)\ldots(\forall P_n)(\phi' \rightarrow ST(\psi))\ [w]$. $s(\phi \rightarrow \psi)$ is an instantiation of this formula, so $F \models s(\phi \rightarrow \psi)\ [w]$. (Compare the remark preceding theorem 4.16.)

For the converse, suppose that, for some valuation V, $\langle F, V\rangle \models \phi\ [w]$. $\langle F,V\rangle \models \psi[w]$ is to be proved. Clearly, $\langle F,V\rangle \models (\exists y_1)\ldots(\exists y_k)\phi'[w]$, and so, for some $w_1, \ldots, w_k \in W$, $\langle F, V\rangle \models \phi'\ [w, w_1, \ldots, w_k]$, where $w_i$ is assigned to $y_i$ for each i ($1 \leq i \leq k$). The valuation V' is defined, for each proposition letter p, by $V'(p) = \{v \in W \mid F \models CV(p, \phi)\ [w, w_1, \ldots, w_k, v]$, where v is assigned to u $\}$. It can now be shown that

$V'(p) \subseteq V(p)$ for all proposition letters p

$\langle F, V'\rangle \models \phi'\ [w, w_1, \ldots, w_k]$.

A detailed proof of this would yield no additional insights, for these two assertions are obvious consequences of the definition of the $CV(p, \phi)$'s.

Substitute the $CV(p, \phi)$'s for the P's in $\phi'$: this gives $\phi''$. Since $\langle F, V' \rangle \models \phi' [w, w_1, \ldots, w_k]$, $F \models \phi'' [w, w_1, \ldots, w_k]$.
From $F \models s(\phi \to \psi) [w]$ it then follows that $F \models \psi' [w, w_1, \ldots, w_k]$, where $\psi'$ is obtained from $ST(\psi)$ by the same substitution. This amounts to $\langle F, V' \rangle \models ST(\psi) [w]$ and, therefore, using the facts that $V'(p) \subseteq V(p)$ for all proposition letters p, and that $\psi$ is monotone in all its proposition letters, it is seen that $\langle F, V \rangle \models ST(\psi) [w]$, i.e., $\langle F, V \rangle \models \psi [w]$. QED.

The following seven examples are well-known modal axioms. The modal logic involved is mentioned between parentheses in each case.

(1) $\Box p \to p$ (T)

ST: $(\forall y)(Rxy \to Py) \to Px$

$CV(p, \Box p)$: $Rxu$

s: $(\forall y)(Rxy \to Rxy) \to Rxx$, or, after simplification,

$Rxx$.

(2) $\Box p \rightarrow \Box\Box p$ (S4)

ST: $(\forall y)(Rxy \rightarrow Py) \rightarrow (\forall z)(Rxz \rightarrow (\forall v)(Rzv \rightarrow Rxv))$

CV(p, $\Box p$): Rxu

s: after a similar simplification, $(\forall z)(Rxz \rightarrow (\forall v)(Rzv \rightarrow Rxv))$.

(3) $p \rightarrow \Box\Diamond p$ (B)

ST: $Px \rightarrow (\forall y)(Rxy \rightarrow (\exists z)(Ryz \wedge Pz))$

CV(p, p): $x = u$

s: $x = x \rightarrow (\forall y)(Rxy \rightarrow (\exists z)(Ryz \wedge x = z))$, or, after simplification,

$(\forall y)(Rxy \rightarrow Ryx)$.

(4) $\Diamond\Box p \rightarrow \Box p$ (S5)

ST: $(\exists y)(Rxy \wedge (\forall z)(Ryz \rightarrow Pz)) \rightarrow (\forall v)(Rxv \rightarrow Pv)$

CV(p, $\Diamond\Box p$): Ryu

s: $(\forall y)((Rxy \wedge (\forall z)(Ryz \rightarrow Ryz)) \rightarrow (\forall v)(Rxv \rightarrow Ryv))$, or,

after simplification,

$(\forall y)(Rxy \rightarrow (\forall v)(Rxv \rightarrow Ryv))$.

(5) $\Diamond\Box p \rightarrow \Box\Diamond p$ (S4.2) is treated similarly, yielding after simplification,

$(\forall y)(Rxy \rightarrow (\forall z)(Rxz \rightarrow (\exists v)(Rzv \wedge Ryv)))$.

(6) $(\Diamond\Box p \wedge p) \rightarrow \Box p$ (S4.4)

ST: $((\exists y)(Rxy \wedge (\forall z)(Ryz \rightarrow Pz)) \wedge Px) \rightarrow (\forall v)(Rxv \rightarrow Pv)$

CV(p, $\Diamond\Box p \wedge p$): $Ryu \vee x = u$

s: $(\forall y)((Rxy \wedge (\forall z)(Ryz \rightarrow (Ryz \vee x = z)) \wedge (Ryx \vee x = x)) \rightarrow (\forall v)(Rxv \rightarrow$

$(Ryv \vee x = v)))$, or, simplified, $(\forall y)(Rxy \rightarrow (\forall v)(Rxv \rightarrow (Ryv \vee x = v)))$.

(7) $\Box(\Box p \rightarrow q) \vee \Box(\Box q \rightarrow p)$ (S4.3)

This formula has to be rewritten first to $\Diamond(\Box p \wedge \neg q) \rightarrow \Box(\Diamond \neg q \vee p)$, and then, using lemma 4.2, to $\Diamond(\Box p \wedge q) \rightarrow \Box(\Diamond q \vee p)$.

ST: $(\exists y)(Rxy \wedge (\forall z)(Ryz \rightarrow Pz) \wedge Qy) \rightarrow (\forall s)(Rxs \rightarrow ((\exists t)(Rst \wedge Qt) \vee Ps))$.

CV$(p, \Diamond(\Box p \wedge q))$: Ryu

CV$(q, \Diamond(\Box p \wedge q))$: $y = u$

s: $(\forall y)((Rxy \wedge (\forall z)(Ryz \rightarrow Ryz) \wedge y = y) \rightarrow (\forall s)(Rxs \rightarrow ((\exists t)(Rst \wedge y = t) \vee Rys)))$, or, simplified,

$(\forall y)(Rxy \rightarrow (\forall s)(Rxs \rightarrow (Rsy \vee Rys)))$.

A similar procedure yields for $\Box((\Box p \wedge p) \rightarrow q) \vee \Box(\Box q \rightarrow p)$
$(\forall y)(Rxy \rightarrow (\forall s)(Rxs \rightarrow (Rsy \vee Rys \vee s = y)))$.

4.12 Definition

Positive and negative occurrences of a proposition letter p in a modal formula are defined inductively according to the clauses

(i) p occurs positively in p

(ii) p does not occur in $\bot$ or T

(iii) a positive (negative) occurrence of p in $\alpha$ is a negative (positive) occurrence of p in $\neg\alpha$.

(iv) a positive (negative) occurrence of p in $\alpha$ is a negative (positive) occurrence of p in $\alpha \rightarrow \beta$, but a positive (negative) occurrence of p in $\beta \rightarrow \alpha$.

(v) a positive (negative) occurrence of p in $\alpha$ is a positive (negative) occurrence of p in $\Box\alpha$.

From this definition the following derived rule may be obtained,

(vi) a positive (negative) occurrence of p in $\alpha$ is a positive (negative) occurrence of p in $\alpha \wedge \beta$, $\beta \wedge \alpha$, $\alpha \vee \beta$, $\alpha \vee \beta$ and $\Diamond\alpha$.

The next theorem is slightly more general than 4.11. (Cf. Sahlqvist [16].)

4.13 Theorem

If a modal formula $\phi$ is constructed using proposition letters and their negations, $\bot$, T, $\wedge$, $\vee$, $\Box$ and $\Diamond$, and $\phi$ satisfies, for all proposition letters p occurring in it, either

no positive occurrence of p is in a subformula of $\phi$ of one of the forms $\alpha \wedge \beta$ or $\Box\alpha$ within the scope of some $\Diamond$,

or

no negative occurrence of p is in a subformula of $\phi$ of one of the forms $\alpha \wedge \beta$ or $\Box\alpha$ within the scope of some $\Diamond$,

then $\phi \in$ M1.

Proof: If some proposition letter p occurs only positively in $\phi$, then $\phi$ is monotone in p, and, by lemma 4.7, we may consider $[\bot/p]\ \phi$ instead. If a proposition letter p occurs only negatively in $\phi$, then it occurs only positively in $[\neg p/p]\phi$, a formula which may be considered instead of $\phi$, by lemma 4.2. Then we substitute $\bot$ for p. By using lemma 4.2 once more, and contracting double negations, we make every remaining proposition letter satisfy the second condition of the theorem.

Rewrite the negation of the formula just obtained as a formula $\psi$ constructed using (negations of) proposition letters, $\bot$, T, $\wedge$, $\vee$, $\Box$ and $\Diamond$, by the interchange laws $\neg\Diamond\alpha \leftrightarrow \Box\neg\alpha$, $\neg\Box\alpha \leftrightarrow \Diamond\neg\alpha$, de Morgan laws and, again, double negation. Now no positive occurrence of a proposition letter in $\psi$ remains in a subformula of $\psi$ of the form $\alpha \vee \beta$ or $\Diamond\alpha$ in the scope of some $\Box$.

A subformula $\Box\alpha$ of $\psi$ is equivalent to a conjunction of formulas of the form $\Box^i p$ and n-formulas, i.e., formulas in which no proposition letter occurs positively. This is proved by induction on $\alpha$. The cases $\alpha = p, \neg p, \bot, \top$ and $\alpha = \alpha_1 \wedge \alpha_2$ are trivial. If $\alpha = \alpha_1 \vee \alpha_2$ or $\alpha = \Diamond\beta$, then no proposition letter occurs positively in it, since $\Box\alpha$ satisfies the same condition as $\psi$. Finally, if $\alpha = \Box\beta$, use the induction hypothesis and the law $\Box(\gamma \wedge \delta) \leftrightarrow (\Box\gamma \wedge \Box\delta)$. Transform $\psi$ into $\psi'$ by replacing occurrences of $\Box\alpha$ which do not lie within the scope of another $\Box$ by equivalents of the kind described here.

A second induction establishes that each subformula $\alpha$ of $\psi'$ is equivalent to a disjunction of formulas constructed using formulas of the form $\Box^i p$, n-formulas, $\wedge$ and $\Diamond$. The cases $\alpha = p, \neg p, \bot, \top$ and $\alpha = \alpha_1 \vee \alpha_2$ are trivial. If $\alpha = \Diamond\beta$, then use the law $\Diamond(\gamma \vee \delta) \leftrightarrow (\Diamond\gamma \vee \Diamond\delta)$, and if $\alpha = \alpha_1 \wedge \alpha_2$, use the propositional distributive laws. Finally, if $\alpha = \Box\beta$, then, by the above, it is either an n-formula, or of the form $\Box^i p$. Applying this result to $\psi'$ itself a disjunction $\psi'' = \psi_1 \vee \ldots \vee \psi_n$ is obtained, with the $\psi_i$'s constructed as indicated. $\psi''$ is obtained by rewriting $\neg\phi$, so $\phi \leftrightarrow \neg\psi'' \leftrightarrow (\neg\psi_1 \wedge \ldots \wedge \neg\psi_n)$. In view of lemma 4.2, it suffices to consider the $\neg\psi_i$'s.

$ST(\psi_i)$ can be written in the form $(\exists y_1)\ldots(\exists y_k)\psi_i'$, as in the proof of theorem 4.11, but now only with respect to those occurrences of $\Diamond$ with a positive occurrence of a proposition letter in their scope. For each proposition letter p $CV(p, \psi_i)$ may be defined as before, and then substituted in $(\forall y_1)\ldots(\forall y_k)\neg\psi_i'$. This yields the required equivalent $s(\neg\psi_i)$, as may be shown in almost the same way as in the previous proof.

Again it is obvious that $F \models \neg\psi_i [w]$ implies $F \models s(\neg\psi_i) [w]$. For the converse, suppose that $\sim F \models \neg\psi_i [w]$. Then, for some valuation

V on F, $\langle F, V\rangle \models \psi_i\,[w]$ and so $\langle F, V\rangle \models \psi_i'\,[w, w_1, \ldots, w_k]$ for some $w_1, \ldots, w_k \in W$. Defining V' using the $CV(p, \psi_i)$'s as before yields

$\langle F, V'\rangle \models \psi_i'\,[w, w_1, \ldots, w_k]$

$V'(p) \subseteq V(p)$ for all proposition letters p.

(The second assertion is now needed in proving that n-formulas remain true in the transition from V to V'.)

From this it follows that $F \models \psi_i''\,[w, w_1, \ldots, w_k]$, where $\psi_i''$ is $\psi_i'$ with the $CV(p, \psi_i)$'s substituted for the P's. But $s(\neg\psi_i) = (\forall y_1)\ldots(\forall y_k)\neg\psi_i''$ and, therefore, $\sim F \models s(\neg\psi_i)\,[w]$. QED.

$\Diamond(p \wedge \Box\Diamond\neg p) \rightarrow (\Diamond\Box p \vee \Box\Box\neg p)$ is a formula which can be treated using theorem 4.13, but not using theorem 4.11. It will be obvious from previous arguments that any modal formula is equivalent to one constructed using proposition letters and their negations, $\bot$, T, $\wedge$, $\vee$, $\Box$ and $\Diamond$. Applying the relevant laws here yields

$\Box(\neg p \vee \Diamond\Box p) \vee \Diamond\Box p \vee \Box\Box\neg p$,

satisfying the second condition of the theorem. Rewriting its negation yields $\Diamond(p \wedge \Box\Diamond\neg p) \wedge \Box\Diamond\neg p \wedge \Diamond\Diamond p$, which is already a $\psi_i$. (The only n-formula occurring in it is $\Box\Diamond\neg p$.)

$ST(\psi_i) = (\exists y)(Rxy \wedge Py \wedge (\forall z)(Ryz \rightarrow (\exists v)(Rzv \wedge \neg Pv))) \wedge (\forall w)(Rxw \rightarrow (\exists s)(Rws \wedge \neg Ps)) \wedge (\exists t)(Rxt \wedge (\exists r)(Rtr \wedge Pr))$.

$CV(p, \psi_i) = (y = u \vee r = u)$.

$ST(\neg\psi_i)$ becomes, after simplification,

$(\forall y)(Rxy \rightarrow (\forall t)(Rxt \rightarrow (\forall r)(Rtr \rightarrow ((\forall z)(Ryz \rightarrow (\exists v)(Rzv \wedge v \neq y \wedge v \neq r)) \rightarrow (\exists w)(Rxw \wedge (\forall s)(Rws \rightarrow (s = y \vee s = r)))))))$.

The idea in the previous proofs has been to consider the modal formula $\phi = (\forall P_1)\ldots(\forall P_n)\psi(P_1,\ldots, P_n, R)$, rewrite it, with parameters $y_1,\ldots, y_k$ in front, to get $(\forall P_1)\ldots(\forall P_n)(\forall y_1)\ldots(\forall y_k)\psi'$, and then to find $L_0$-formulas $\alpha_1,\ldots, \alpha_n$ with free variables among $x, y_1,\ldots, y_k$ to be substituted for $P_1,\ldots P_n$. This yields an $L_0$-formula $s(\phi)$ equivalent to $\phi$. Here the direction from $\phi$ to $s(\phi)$ takes care of itself (a universal instantiation has taken place), but the converse requires proof. Assuming that $\langle F, V\rangle \models \neg\phi\,[w]$, it is shown that already $\langle F, V'\rangle \models \neg\phi\,[w]$, where V' is a valuation defined by the $\alpha_i$'s.Pushing the $\alpha_i$'s from the valuation into $\neg\phi$ yields a counterexample to $s(\phi)$.

From this point of view those modal formulas $\phi$ are of interest for which $\langle F, V\rangle \models \phi\,[w]$ implies $\langle F, V_1\rangle \models \phi\,[w]$ or ... or $\langle F, V_m\rangle \models \phi\,[w]$, where $V_1,\ldots, V_m$ are $L_0$-definable valuations. Most formulas in M1 with which we are acquainted fall into this category, also those not covered by theorem 4.13 (like the ones mentioned in the third and fourth clause of theorem 4.19). Further investigation of this had led to slight extensions of theorem 4.13 with liberalized conditions on the occurrences of proposition letters, but these are not stated here, because the gain in generality is offset by an enormous cost in technical complications.

The two definitions below describe the class $M1^{sub}$ of modal formulas amenable to treatment by the method of substitutions.

4.14 Definition

If $\zeta$ is an $L_1$-formula of the form $(\forall y_1)\ldots(\forall y_k)\eta$, where $\eta = \eta(x, y_1,\ldots, y_k, P_1,\ldots, P_n)$ and $x, y_1,\ldots, y_k$ do not occur as bound variables in $\eta$, then $\chi$ is called a substitution instance of $\zeta$ if there are $L_0$-formulas $\alpha_1, \ldots, \alpha_n$ and a variable s not occurring in $\zeta$ satisfying the following three conditions for each i ($1 \leq i \leq n$),

each free variable of $\alpha_i$ is among $x, y_1, \ldots, y_k, s$

no bound variable of $\alpha_i$ occurs in $\zeta$

$\chi = [\alpha_1/P_1, \ldots, \alpha_n/P_n]\zeta$, i.e., $\zeta$ with subformulas of the form $P_i z$ replaced by $[z/s]\alpha_i$.

4.15 Definition

If $\phi$ is the modal formula $(\forall P_1)\ldots(\forall P_n)\psi(P_1, \ldots, P_n, R)$, then $S(\phi) = \{\chi \mid \chi$ is a substitution instance of an $L_1$-formula $\zeta$ logically equivalent to $\psi\}$.

$M1^{sub} = \{\phi \mid \phi$ is a modal formula and $S(\phi) \models \phi\}$.

If $\chi \in S(\phi)$, then $\phi$ implies $\chi$. For, suppose that $\phi = (\forall P_1)\ldots(\forall P_n)\psi$, $\zeta \leftrightarrow \psi$ and $\chi$ is a substitution instance of $\zeta = (\forall y_1)\ldots(\forall y_k)\eta$. Then $(\forall P_1)\ldots(\forall P_n)\psi$ implies $(\forall P_1)\ldots(\forall P_n)\zeta = (\forall P_1)\ldots(\forall P_n)(\forall y_1)\ldots(\forall y_k)\eta$ and, since this formula is equivalent to $(\forall y_1)\ldots(\forall y_k)(\forall P_1)\ldots(\forall P_n)\eta$, it implies $\chi$.

4.16 Theorem

$M1^{sub} \subseteq M1$.

$M1^{sub}$ is recursively enumerable.

$M1^{sub} \neq M1$.

Proof: If $S(\phi) \models \phi$, then, by the compactness theorem, for some finite conjunction $\chi$ of formulas in $S(\phi)$, $\chi \models \phi$, and therefore $E(\phi, \chi)$, using the above remark.

The second assertion is proved by inspection of the definition of $M1^{sub}$. $\phi \in M1^{sub}$ iff $S(\phi) \models \phi$ iff, for some $\chi_1, \ldots, \chi_m \in S(\phi)$, $\chi_1 \wedge \ldots \wedge \chi_m \models \phi$. The two predicates used in the third equivalent are recursively enumerable: $\chi \models \phi$ iff $\chi \models (\forall P_1)\ldots(\forall P_n)\psi$ (for some $L_1$-formula $\psi$) iff $\chi \models \psi$ (since $\chi$ is an $L_0$-formula, i.e., without occurrences of unary predicate symbols); and logical consequence in $L_1$ is a recursively enumerable notion. Moreover, $S(\phi)$ is a recursively enumerable set. $\chi \in S(\phi)$ iff there are formulas $\alpha_1, \ldots, \alpha_n$ as described in definition 4.14 and a formula $\zeta$ with $\models \psi \leftrightarrow \zeta$ (a recursively enumerable predicate again) such that $\chi = [\alpha_1/P_1, \ldots, \alpha_n/P_n]\zeta$.

The example treated in lemma 2.21 can be used to show that $M1^{sub} \neq M1$. Let $\phi^m = (\Box p \rightarrow \Box\Box p) \wedge \Box(\Box p \rightarrow \Box\Box p) \wedge (\Box\Diamond p \rightarrow \Diamond\Box p)$ and $\phi^o = (\forall y)(Rxy \rightarrow (\forall z)(Ryz \rightarrow Rxz)) \wedge (\forall y)(Rxy \rightarrow (\forall u)(Ryu \rightarrow (\forall v)(Ruv \rightarrow Ryv))) \wedge (\exists y)(Rxy \wedge (\forall z)(Ryz \rightarrow y = z))$.

By theorem 4.11,

$E(\Box p \rightarrow \Box\Box p, (\forall y)(Rxy \rightarrow (\forall z)(Ryz \rightarrow Rxz)))$

and, therefore, by lemma 4.2(iv),

$E(\Box(\Box p \rightarrow \Box\Box p), (\forall y)(Rxy \rightarrow (\forall u)(Ryu \rightarrow (\forall v)(Ruv \rightarrow Ryv))))$.

These equivalences do not depend on the axiom of choice. In the proof of lemma 2.21 it was shown that $E(\phi^m, \phi^o)$, using the last two equivalences and theorem 2 of II.2, which depends on the axiom of choice; so $\phi^m \in M1$. It was also shown that $ZF \vdash E(\phi^m, \phi^o) \rightarrow AC^{uo}$, where $AC^{uo}$ is the axiom of choice for unordered pairs. Closer inspection of the proof reveals that "$\phi^o \models \phi^m$" is provable without the axiom of choice, and that in fact (1) $ZF \vdash$ "$\phi^m \models \phi^o$" $\rightarrow AC^{uo}$.

Suppose that $\phi^m \in M1^{sub}$. Then, for some $\chi_1, \ldots, \chi_m \in S(\phi)$, $\phi^m \models \chi = \chi_1 \wedge \ldots \wedge \chi_m$ and $\chi \models \phi^m$. The argument given above shows easily that

(2) ZF $\vdash$ "$\phi^m \models \chi$".

Since $\chi \models \phi^m$ iff $\chi \models \phi_1^m$, where $\phi_1^m$ is $\phi^m$ without its second-order quantifiers, it is also clear that

(3) ZF $\vdash$ "$\chi \models \phi^m$".

It follows, by the above, that

(4) ZF + AC $\vdash$ "$\chi \models \phi^o$".

But then, by the argument used in the proof of corollary 2.22,

(5) ZF $\vdash$ "$\chi \models \phi^o$" (since logical consequence in $L_0$ is arithmetical).

(2) and (5) imply that ZF $\vdash$ "$\phi^m \models \phi^o$", and this yields, in combination with (1), ZF $\vdash$ $AC^{\omega}$, contradicting the result in Jech [10] that $\sim$ZF $\vdash$ $AC^{\omega}$. So the original supposition is false: $\phi^m \notin M1^{sub}$. QED.

The next theorem shows that the various conditions on the occurrences of proposition letters in the statements of theorems 4.11 and 4.13 are necessary. As soon as combinations $\Box(\ldots\Diamond\ldots)$ or $\Box(\ldots \vee \ldots)$ are allowed in the antecedent formula, or proposition letters occur negatively in the consequent formula, the resulting implication may be outside of M1. This is shown by the first four formulas. The fifth has been added, because it is of an unusual type not found in ordinary modal logics.

4.17 Theorem

(i) $\Box\Diamond p \rightarrow \Diamond\Box p \notin \bar{M}1$.

(ii) $\Box(p \vee q) \rightarrow \Diamond(\Box p \vee \Box q) \notin M1$.

(iii) $\Box(\Box p \vee p) \rightarrow \Diamond(\Box p \wedge p) \notin M1$.

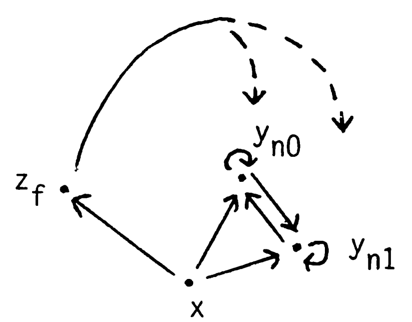
(iv) $\Diamond p \rightarrow \Diamond(p \wedge \Box\neg p) \notin \bar{M}1$.

(v) $\Diamond\Box(\Box p \rightarrow p) \notin M1$.

Proof: Of these cases (iv) is proved in a conventional way, the others are proved using a method introduced in II.1. A frame F is given with an uncountable domain W and a $w \in W$ such that $F \models \phi\,[w]$ for the modal formula $\phi$ in question. It is then shown that, for no countable elementary subframe F' of F with a domain containing w and a countable set of other elements of W (to be specified in each case), $F' \models \phi\,[w]$. It follows from the Löwenheim-Skolem theorem that $\phi \notin M1$. If it can be shown that $F \models \phi$, then it even follows that $\phi \notin \bar{M}1$.

(i) : cf. II.1.

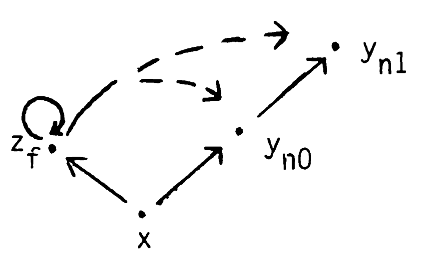
(ii) : Take $W = \{x, y_{n0}, y_{n1}, z_f \mid n \in \mathbb{N}, f: \mathbb{N} \to \{0, 1\}\}$, and $R = \{\langle x, y_{ni}\rangle, \langle y_{ni}, y_{nj}\rangle, \langle x, z_f\rangle, \langle z_f, y_{nf(n)}\rangle \mid n \in \mathbb{N};\ i, j \in \{0, 1\};$ $f: \mathbb{N} \to \{0, 1\}\}$.




$F \models \Box(p \vee q) \to \Diamond(\Box p \vee \Box q)\,[w]$, which may be seen as follows. Let $\langle F, V\rangle \models \Box(p \vee q)\,[w]$. Then, either for some $n \in \mathbb{N}$, $\langle F, V\rangle \models p\,[y_{ni}]$ for each $i \in \{0, 1\}$, in which case $\langle F, V\rangle \models \Box p\,[y_{n0}]$ and so $\langle F, V\rangle \models \Diamond(\Box p \vee \Box q)\,[x]$, or, for each $n \in \mathbb{N}$ there is an $i \in \{0, 1\}$ such that $\langle F, V\rangle \models q\,[y_{ni}]$. In this last case take $f: \mathbb{N} \to \{0, 1\}$ such that $\langle F, V\rangle \models q\,[y_{nf(n)}]$ for all $n \in \mathbb{N}$. Then $\langle F, V\rangle \models \Box q\,[z_f]$ and so, in this case too, $\langle F, V\rangle \models \Diamond(\Box p \vee \Box q)\,[x]$.

We shall now show that, if F' is a countable elementary subframe of F with a domain containing x and $y_{ni}$ for each $n \in IN$ and $i \in \{0, 1\}$, then $\sim F' \models \Box(p \vee q) \rightarrow \Diamond(\Box p \vee \Box q)$ [x ]. Let $z_g \in W-W'$, and set V(p) = $\{y_{ng(n)} \mid n \in IN\}$ and V(q) = $\{y_{n(1-g(n))} \mid n \in IN\} \cup \{z_f \mid z_f \in W'\}$. Then $\langle F', V\rangle \models \Box(p \vee q)$ [x ], but, as will be shown presently, $\sim\langle F', V\rangle \models \Diamond(\Box p \vee \Box q)$ [x] . Clearly, $\sim\langle F', V\rangle \models \Box p$ $[y_{ni}]$ and $\sim\langle F', V\rangle \models \Box q$ $[y_{ni}]$ for each $n \in IN$ and $i \in \{0, 1\}$. Any f with $z_f \in W'$ differs from g for at least one $n \in IN$, so, for no $z_f \in W'$, $\langle F', V\rangle \models \Box p$ $[z_f]$ . Since F' is an elementary subframe of F, any f with $z_f \in W'$ differs from 1-g for at least one $n \in IN$. (If $z_{1-g}$ were in W', then $z_g$ would be, since it is $L_0$-expressible that each $z_f$ has a "complementary" element $z_{1-f}$.)
Therefore, for no $z_f \in W'$, $\langle F', V\rangle \models \Box q$ $[z_f]$ .

(iii) Take $W = \{x, y_{ni}, z_f \mid n \in IN; i \in \{0, 1\}; f: IN \rightarrow \{0, 1\}\}$ and $R = \{\langle x, y_{n0}\rangle, \langle y_{n0}, y_{n1}\rangle, \langle x, z_f\rangle, \langle z_f, z_f\rangle, \langle z_f, y_{nf(n)}\rangle \mid n \in IN; f: IN \rightarrow \{0, 1\}\}$.



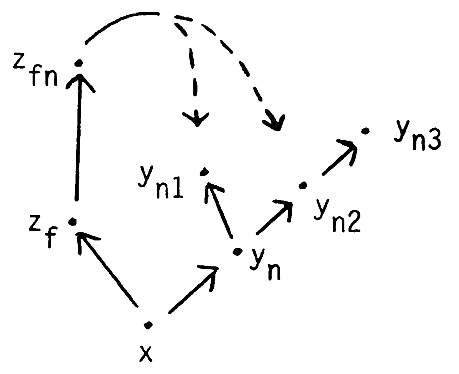

We will show that $F \models \Box(\Box p \vee p) \rightarrow \Diamond(\Box p \wedge p)$ [ x ] .
Let $\langle F, V\rangle \models \Box(\Box p \vee p)$ [ x ]. Then an $f: IN \rightarrow \{0, 1\}$ exists such that, for each $n \in IN$, $\langle F, V\rangle \models p$ $[y_{nf(n)}]$. Also $\langle F, V\rangle \models p$ $[z_g]$ for all $z_g \in W$, and therefore $\langle F, V\rangle \models \Box p \wedge p$ $[z_f]$, so $\langle F, V\rangle \models \Diamond(\Box p \wedge p)$ [ x ].

If F' is a countable elementary subframe of F, with a domain containing x and $y_{ni}$ for each $n \in$ IN and $i \in \{0, 1\}$, then $\sim F' \models \Box(\Box p \vee p) \to \Diamond(\Box p \wedge p)[x]$, as we will show now. Let $z_g \in W\text{-}W'$, and set $V(p) = \{y_{ng(n)} \mid n \in \text{IN}\} \cup \{z_f \mid z_f \in W'\}$. Clearly, $\langle F', V\rangle \models \Box(\Box p \vee p)[x]$, and it is also easy to see that $\sim\langle F', V\rangle \models \Box p \wedge p[y_{n0}]$ for each $n \in$ IN, and $\sim\langle F', V\rangle \models \Box p \wedge p[z_f]$ for each $z_f \in W'$.

The last formula shows how tricky this subject is. For the formula $\Box(\Box p \vee p) \to \Diamond\Box p$, which seems to violate the conditions of theorem 4.11 in exactly the same way as $\Box(\Box p \vee p) \to \Diamond(\Box p \wedge p)$, is in M1! For all frames F and $w \in W$, $F \models \Box(\Box p \vee p) \to \Diamond\Box p[w] \Leftrightarrow F \models \Box p \to \Diamond\Box p[w] \Leftrightarrow F \models (\exists y)(Rxy \wedge (\forall z)(Ryz \to Rxz))[w]$.

(iv) A better known equivalent of $\Diamond p \to \Diamond(p \wedge \Box\neg p)$ is <u>Löb's formula</u> $\Box(\Box p \to p) \to \Box p$. (This "induction principle" reflects a form of Löb's theorem for arithmetic. Cf. Solovay [21].)
A straightforward argument shows that, for all frames F and $w \in W$, $F \models \Box(\Box p \to p) \to \Box p[w] \Leftrightarrow F \models (\forall y)(Rxy \to (\forall z)(Ryz \to Rxz))[w]$ & $\sim(\exists f\colon \text{IN} \to W)(f(0) = w \;\&\; (\forall n \in \text{IN})Rf(n)f(n+1))$. (Cf. Van Benthem [1].)
Of course, well-foundedness is not first-order definable, so Löb's formula is not in $\bar{M}1$.

(v) Take $W = \{x, y_n, y_{ni}, z_f, z_{fn} \mid n \in \text{IN};\ i \in \{1, 2, 3\};\ f\colon \text{IN} \to \{1, 2\}\}$ and $R = \{\langle x, y_n\rangle, \langle y_n, y_{ni}\rangle, \langle y_{n2}, y_{n3}\rangle, \langle x, z_f\rangle, \langle z_f, z_{fn}\rangle, \langle z_{fn}, y_{nf(n)}\rangle \mid n \in \text{IN};\ i \in \{1, 2\};\ f\colon \text{IN} \to \{1, 2\}\}$.




We will show that $F \models \Diamond\Box(\Box p \to p)$ $[x]$. Suppose that $\langle F, V\rangle \models \Box\Diamond(\Box p \wedge \neg p)$ $[x]$: a contradiction follows. Take $f\colon \mathbb{N} \to \{1, 2\}$ such that, for all $n \in \mathbb{N}$, $\langle F, V\rangle \models \neg p$ $[y_{nf(n)}]$.
Then $\langle F, V\rangle \models \Box\Diamond\neg p$ $[z_f]$; but also $\langle F, V\rangle \models \Diamond(\Box p \wedge \neg p)$ $[z_f]$, which is a contradiction.

If F' is a countable elementary subframe of F with a domain containing x, $y_n$ and $y_{ni}$, for each $n \in \mathbb{N}$ and $i \in \{1, 2, 3\}$, then $\sim F' \models \Diamond\Box(\Box p \to p)$ $[x]$, by the following argument. Let $z_g \in W - W'$. Note that no $z_{gn} \in W'$. Set $V(p) = \{y_{n3} \mid n \in \mathbb{N}\} \cup \{y_{nh(n)} \mid n \in \mathbb{N};\ h(n) = 1$ if $g(n) = 2$, $h(n) = 2$ if $g(n) = 1\}$. $\langle F', V\rangle \models \Box\Diamond(\Box p \wedge \neg p)$ $[x]$, as is easy to check, so $\sim\langle F', V\rangle \models \Diamond\Box(\Box p \to p)$ $[x]$. QED.

4.18 Definition

A modal reduction principle is a modal formula of the form $\vec{M}p \to \vec{N}p$, where $\vec{M}$ and $\vec{N}$ are (possibly empty) sequences of modal operators $\Box$ and $\Diamond$.

Many axioms used in modal logic are modal reduction principles as the examples after theorem 4.11 show.

A combination of the method of substitutions and the Löwenheim-Skolem type argument of the above proof leads to the following result,

4.19 Theorem

A modal reduction principle $\vec{M}p \to \vec{N}p$ is in M1 iff it has one of the following forms:

(i) $\Diamond^i \Box^j p \to \vec{N}p$, for some $i, j \in \mathbb{N}$ and arbitrary $\vec{N}$

(ii) $\vec{M}p \to \Box^i \Diamond^j p$, for some $i, j \in \mathbb{N}$ and arbitrary $\vec{M}$

(iii) $\Box^i \vec{M}_1 p \to \vec{N}_2 \vec{M}_1 p$, for some $i \in \mathbb{N}$ such that length $(\vec{N}_2) = i$ and arbitrary $\vec{M}_1$

(iv) $\vec{M}_2 \vec{M}_1 p \to \Diamond^i \vec{M}_1 p$, for some $i \in \mathbb{N}$ such that length $(\vec{M}_2) = i$ and arbitrary $\vec{M}_1$.

Proof: It is easy to prove that modal reduction principles of these forms are in M1. (i) and (ii) follow from theorem 4.11 and (iii) and (iv) are equivalent to closed formulas, which are in M1 by lemma 4.7. The proof of the converse is quite complicated: the reader is referred to II.2. QED.

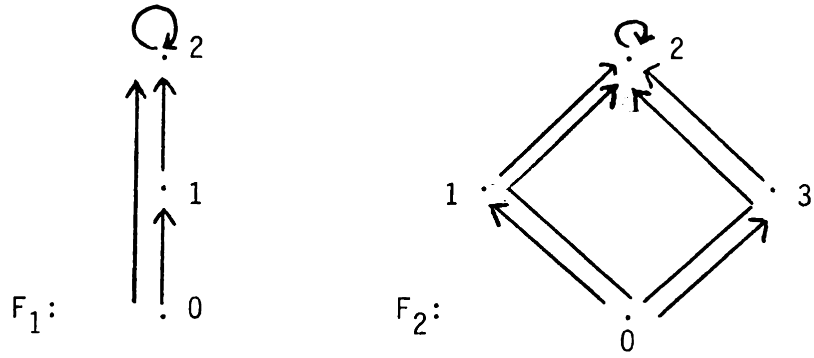
This theorem settles a problem of Fitch [6], as far as M1 is concerned.

It remains to be seen if modal reduction principles, or indeed modal formulas with one proposition letter, are in any sense typical for modal formulas in general.

4.20 Lemma

The modal formula $\Box((\Box p \wedge p) \to q) \vee \Box(\Box q \to p)$ (CF) is not equivalent to any modal formula with only one proposition letter.

Proof: Consider the frames $F_1$ and $F_2$ with $W_1 = \{0, 1, 2\}$, $R_1 = \{<0, 1>, <0, 2>, <1, 2>, <2, 2>\}$, $W_2 = \{0, 1, 2, 3\}$ and $R_2 = \{<0, 1>, <0, 2>, <0, 3>, <1, 2>, <3, 2>, <2, 2>\}$.




By theorem 4.11, $E(CF, (\forall y)(Rxy \rightarrow (\forall z)(Rxz \rightarrow (Ryz \vee Rzy \vee z = y))))$, so CF holds in $F_1$ but not in $F_2$. But we will show that for any modal formula $\phi$ with only one proposition letter, if $F_1 \models \phi$, then $F_2 \models \phi$, from which the lemma follows.

$f_1 = \{<0, 0>, <1, 1>, <3, 1>, <2, 2>\}$,

$f_2 = \{<0, 0>, <1, 1>, <2, 2>, <3, 2>\}$ and

$f_3 = \{<0, 0>, <1, 2>, <2, 2>, <3, 1>\}$

are p-morphisms from $F_2$ onto $F_1$. Let V be any valuation on $F_2$. Consider the p-values only, for some proposition letter p. If 1 and 3 are both in, or both outside, V(p), then, by lemma 2.11, for all modal formulas $\phi$ whose only proposition letter is p, and i = 0, 1, 2, 3, $<F_2, V> \models \phi[i] \Leftrightarrow <F_1, V^1> \models \phi[f_1(i)]$, where $V^1(p) = V(p) - \{3\}$. If one of 1 and 3 is in V(p) and the other is not, then one of them is in V(p) iff 2 is. Say this is 1 (the other case is clearly symmetric), then, by lemma 2.11, for all modal formulas $\phi$ whose only proposition letter is p, and i = 0, 1, 2, 3, $<F_2, V> \models \phi[i] \Leftrightarrow <F_1, V^2> \models \phi[f_3(i)]$, where $V^2(p) = V(p) - \{1, 3\}$ if $1 \in V(p)$ and $V^2(p) = (V(p) - \{3\}) \cup \{1\}$ if $1 \notin V(p)$. So, if a modal formula containing only one proposition letter can be falsified in $F_2$, it can be falsified in $F_1$, which proves the claim made above. QED.

## I.5 RELATIVE CORRESPONDENCE

In lemma 2.20 the modal formula $\Box\Diamond\Box\Box p \rightarrow \Diamond\Diamond\Box\Diamond p$ was considered, which is not in M1, but holds on all frames satisfying $(\forall x)(\exists y)Rxy$. A similar example is provided by theorem 4.17. $\Box(\Box p \vee p) \rightarrow \Diamond(\Box p \wedge p)$ is not in M1, but it holds on all reflexive frames. (If $F = \langle W, R\rangle$ is a frame with a reflexive R and V is a valuation on F satisfying $\langle F, V\rangle \models \Box(\Box p \vee p)\ [w]$, then either, for some $v \in W$ with Rwv, $\langle F, V\rangle \models \Box p\ [v]$, in which case $\langle F, V\rangle \models \Box p \wedge p\ [v]$ and $\langle F, V\rangle \models \Diamond(\Box p \wedge p)\ [w]$, or, for all $v \in W$ with Rwv, $\langle F, V\rangle \models p\ [v]$, in which case $\langle F, V\rangle \models \Box p \wedge p\ [w]$, so again $\langle F, V\rangle \models \Diamond(\Box p \wedge p)\ [w]$.)

These examples indicate that certain restrictive conditions on the binary relation R will change the behaviour of E and M1 considerably. In this chapter the main restrictive condition to be studied is transitivity, but the first result is about a stronger restriction which makes all modal formulas first-order definable.

Lemma 4.9 says that any modal formula with degree $\leq 1$ is in M1. In II.2 it is shown that in $FR(\Diamond\Box p \leftrightarrow \Box p) \cap FR(\Box\Box p \leftrightarrow \Box p)$ each modal formula is equivalent to one of degree $\leq 1$. Theorem 4.11 enables us to prove the following,

$E(\Diamond\Box p \rightarrow \Box p,\ (\forall y)(Rxy \rightarrow (\forall z)(Rxz \rightarrow Ryz)))$

$E(\Box p \rightarrow \Diamond\Box p,\ (\exists y)(Rxy \wedge (\forall z)(Ryz \rightarrow Rxz)))$

$E(\Box\Box p \rightarrow \Box p,\ (\forall y)(Rxy \rightarrow (\exists z)(Rxz \wedge Rzy)))$

$E(\Box p \rightarrow \Box\Box p,\ (\forall y)(Rxy \rightarrow (\forall z)(Ryz \rightarrow Rxz)))$

It is clear that the second of these relational conditions may be contracted to $(\exists y)Rxy$, by virtue of the fourth. Let $\psi$ be their conjunction; we have proved:

5.1 Lemma

On $FR((\forall x)\psi)$ each modal formula has a (local) $L_0$-equivalent.

This result implies that each modal formula is first-order definable on the basis of S5; but it is even slightly stronger in that not each frame satisfying $(\forall x)\psi$ has a relation which is an equivalence relation. E.g., $\langle\{0, 1\}, \{\langle 0, 1\rangle, \langle 1, 1\rangle\}\rangle \in FR((\forall x)\psi)$.

The following two results on modal reduction principles are from II.2, where their (long) proofs are found.

5.2 Theorem

On $FR((\forall x)(\exists y)Rxy)$ the modal reduction principles with (local) $L_0$-equivalents are exactly those of the forms

$\Diamond^i\Box^j p \rightarrow \vec{M}p$ or

$\vec{M}p \rightarrow \Box^i\Diamond^j p$, where $i, j \in \mathbb{N}$ and $\vec{M}$ is a sequence of modal operators

$\vec{M}p \rightarrow \vec{N}p$, where $\vec{M}$ and $\vec{N}$ are sequences of modal operators of the same length, such that, for all $i \in \mathbb{N}$, if $(\vec{M})_i = \Diamond$, then $(\vec{N})_i = \Diamond$.

5.3 Theorem

On the transitive frames all modal reduction principles have (local) $L_0$-equivalents.

Not all modal formulas have $L_0$-equivalents on the transitive frames. E.g., $\Box(\Box p \rightarrow p) \rightarrow \Box p$ is still equivalent to well-foundedness of the converse relation of the binary relation R (cf. theorem 4.17). Another example is the formula $\Box(\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow p$, which has no $L_0$-equivalent even on the frames with a transitive, reflexive and connected relation. (Cf. Van Benthem [ 1 ].) This formula is of some interest because of its connection with intuitionistic logic: it axiomatizes the strongest modal logic for which Gödel's embedding of intuitionistic logic into modal logic works. We do not formulate this more precisely, because it would lead us away from our main theme, but note that a correspondence theory for intuitionistic formulas would provide an example of the situation discussed in this chapter. (The class of frames would be restricted to the transitive and reflexive ones and other conditions might have to be added.)

The difference between M1 and $\bar{M}1$ virtually disappears modulo transitivity, as is apparent from the next two lemmas.

If $\phi$ is a modal formula and $\psi$ an $L_0$-sentence such that $\bar{E}(\phi, \psi)$ holds, then $\psi$ may be taken to be of the form $(\forall x)\chi$, where $\chi$ is an $L_0$-formula with only restricted quantifiers. This will follow from theorem 6.21, but for the case of transitive frames a more direct proof is given here.

5.4 Definition

For $L_0$-formulas $\phi$ with no bound occurrences of the variable x, $\bar{R}_x(\phi)$ is defined inductively according to the clauses

$\bar{R}_x(\alpha) = \alpha$ for atomic formulas $\alpha$

$\bar{R}_x(\neg\alpha) = \neg\bar{R}_x(\alpha)$

$\bar{R}_x(\alpha \to \beta) = \bar{R}_x(\alpha) \to \bar{R}(\beta)$

$\bar{R}_x((\forall y)\alpha) = \bar{R}_x([x/y]\alpha) \wedge (\forall y)(Rxy \to \bar{R}_x(\alpha))$.

5.5 Lemma

If $F$ is a transitive frame $\langle W, R\rangle$, $w \in W$ and $w_1,\ldots, w_m \in TC(F, w)$, then, for any $L_0$-formula $\phi = \phi(x, y_1,\ldots, y_m)$ with no bound occurrences of $x$,

$TC(F, w) \models \phi[w, w_1,\ldots, w_m] \Leftrightarrow F \models \bar{R}_x(\phi)[w, w_1,\ldots, w_m]$.

Proof: Use induction on the complexity of $\phi$, noting that the domain of $TC(F, w)$ is $\{w\} \cup \{v \in W \mid Rwv\}$. QED.

5.6 Corollary

If $\phi$ is a modal formula and $\psi$ an $L_0$-sentence in which the variable $x$ does not occur such that $\bar{E}(\phi, \psi)$, then $\bar{E}(\phi, (\forall x)\bar{R}_x(\psi))$.

Proof: If $F \models \phi$, then, by corollary 2.6, $(\forall w \in W)(TC(F, w) \models \phi)$, so $(\forall w \in W)(TC(F, w) \models \psi)$. From lemma 5.5 it then follows that $(\forall w \in W)(F \models \bar{R}_x(\psi)[w])$, i.e. $F \models (\forall x)\bar{R}_x(\psi)$.

If $F \models (\forall x)\bar{R}_x(\psi)$, then $(\forall w \in W)(F \models \bar{R}_x(\psi)[w])$, so, by lemma 5.5, $(\forall w \in W)(TC(F, w) \models \psi)$ and, therefore, $(\forall w \in W)(TC(F, w) \models \phi)$ and $(\forall w \in W)(TC(F, w) \models \phi[w])$. By corollary 2.6, $(\forall w \in W)F \models \phi[w]$, i.e., $F \models \phi$. QED.

5.7 Lemma

If $\psi$ is an $L_0$-sentence of the form $(\forall x)\chi$, where $\chi$ contains only restricted quantifiers, and $\phi$ is a modal formula, then on the class of transitive frames the following equivalence holds for any variable y not occurring in $\psi$,

$\bar{E}(\phi, \psi) \Leftrightarrow E(\phi \wedge \Box\phi, \chi \wedge (\forall y)(Rxy \rightarrow [y/x]\chi))$.

<u>Proof</u>: $\Rightarrow$: If $F \models \phi \wedge \Box\phi\ [w]$, then $TC(F, w) \models \phi$ and, therefore, $TC(F, w) \models \psi$, so $TC(F, w) \models \chi\ [w]$ and $(\forall v \in W)(Rwv \Rightarrow TC(F, w) \models \chi\ [v])$. Since $L_0$-formulas with only restricted quantifiers are invariant for generated subframes, $F \models \chi\ [w]$ and $(\forall v \in W)(Rwv \Rightarrow F \models \chi\ [v])$.

If $F \models \chi \wedge (\forall y)(Rxy \rightarrow [y/x]\chi)[w]$, then $TC(F, w) \models \chi \wedge (\forall y)(Rxy \rightarrow [y/x]\chi)\ [w]$ (this formula is restricted), so $TC(F, w) \models (\forall x)\chi$. It follows that $TC(F, w) \models \phi$, so $TC(F, w) \models \phi \wedge \Box\phi$ and $TC(F, w) \models \phi \wedge \Box\phi\ [w]$, from which, again by 2.6, $F \models \phi \wedge \Box\phi\ [w]$.

$\Leftarrow$: If $F \models \phi$, then $F \models \phi \wedge \Box\phi$, so $(\forall w \in W)(F \models \phi \wedge \Box\phi\ [w])$ and, trivially, $(\forall w \in W)(F \models \chi\ [w])$, i.e., $F \models \psi$.

If $F \models \psi$, then $(\forall w \in W)(F \models \chi \wedge (\forall y)(Rxy \rightarrow [y/x]\chi)\ [w])$, so $(\forall w \in W)(F \models \phi \wedge \Box\phi\ [w])$ whence, trivially, $F \models \phi$. QED.

5.8 <u>Corollary</u>

If $\phi$ is a modal formula, then on the transitive frames, $\phi \in \bar{M}1$ iff $\phi \wedge \Box\phi \in M1$.

<u>Proof</u>: The direction from left to right follows from lemmas 5.6 and 5.7. If $\phi \wedge \Box\phi \in M1$, say $E(\phi \wedge \Box\phi, \psi)$, where $\psi$ has the one free variable x, then $\bar{E}(\phi, (\forall x)\psi)$. QED.

The following list of questions ends this chapter.

(1) Is $\phi \in \bar{M}1 \Leftrightarrow \phi \in M1$ valid for all modal formulas $\phi$ on the transitive

frames?

A class of finite frames closed under isomorphic images is Σ-elementary. (Use the $L_0$-sentences describing the members up to isomorphism.)

(2) Does every modal formula have a first-order equivalent on the finite frames?

The subject of intuitionism was mentioned in this chapter. Now intuitionistic formulas behave better than modal formulas in some ways. Let us restrict attention to transitive and reflexive frames F, and valuations V on them satisfying, for any proposition letter p, $(\forall w \in W)(\forall v \in W)(Rwv \Rightarrow (w \in V(p) \Rightarrow v \in V(p)))$. Then results like the following hold (cf. Smorynski [20]):

For all frames F, valuations V and intuitionistic formulas $\phi$, $(\forall w \in W)(\forall v \in W)(Rwv \Rightarrow (\langle F, V\rangle \models \phi[w] \Rightarrow \langle F, V\rangle \models \phi[v]))$.

For all frames F, valuations V and intuitionistic formulas $\phi$, if, for some $w \in W$, $\langle F, V\rangle \models \phi[w]$, then a finite submodel M of $\langle F, V\rangle$ exists such that $M \models \phi[w]$.

The first result does not hold for modal formulas in general. (E.g., negations of proposition letters need not be preserved under R-successors.) Inspection of Smorynski's proof shows that the second result does hold for all modal formulas, given these frames and this kind of valuation. The result does not hold for arbitrary valuations, however. E.g., if V on F = $\langle \mathbb{N}, \leqq\rangle$ is given by $V(p) = \{0, 2, 4, \ldots\}$, then $\langle F, V\rangle \models \Box\Diamond p \wedge \Box\Diamond\neg p\,[0]$, but this modal formula holds at 0 in no finite submodel of $\langle F, V\rangle$. The result does not hold for arbitrary transitive frames either. E.g., if V on F = $\langle \mathbb{N}, <\rangle$ satisfies the above condition, then $\langle F, V\rangle \models \Box\Diamond\top\,[0]$, but this modal formula holds at 0 in no finite submodel of $\langle F, V\rangle$.

Because of these results we formulate as a final question

(3) Does every intuitionistic formula have a first-order equivalent?

We have no doubt that this question is known to people working on intuitionistic logic or intermediate logics.

## I.6 MODAL DEFINABILITY

This chapter is concerned with the question which is complementary to the one of chapter I.2, viz. which $L_0$-formulas are modally definable?

6.1 Definition

$\underline{P1}$ = {α | α is an $L_0$-formula with one free variable such that, for some modal formula φ, E(φ, α)}.

$\underline{\bar{P}1}$ = {α | α is an $L_0$-sentence such that, for some modal formula φ, $\bar{E}$(φ, α)}.

The first results of this chapter are about P1, but the main emphasis will be on $\bar{P}1$, for which an algebraic characterization is "almost" available.

6.2 Lemma

If α and β are $L_0$-formulas with one and the same free variable x, then

(i) if α ∈ P1 and β ∈ P1, then α ∧ β ∈ P1

(ii) if α ∈ P1 and β ∈ P1, then α ∨ β ∈ P1

(iii) if α ∈ P1, then (∀y)(Rxy → [y/x]α) ∈ P1, provided that y does not occur in α.

Proof: (i) follows from lemma 4.1, and so does (ii). (If $E(\phi, \alpha)$ and $E(\psi, \beta)$ for modal formulas $\phi$ and $\psi$, then change the proposition letters in $\phi$ and $\psi$ so that none occur in both $\phi$ and $\psi$. This amounts to a change of bound variables in an $L_2$-formula. After such a change lemma 4.1 is directly applicable.) (iii) follows from lemma 4.2(iv). QED.

6.3 Lemma

P1 is not closed under $\neg$.

P1 is not closed under restricted existential quantification.

Proof: $Rxx \in P1$, because of $E(\Box p \to p, Rxx)$, but $\neg Rxx \notin P1$. For, $\langle IN, \rangle \models \neg Rxx\,[0]$ and f defined by $f(n) = 0$ for all $n \in IN$, is a p-morphism from $\langle IN, \rangle$ onto $I = \langle\{0\}, \{\langle 0, 0\rangle\}\rangle$, but $\sim I \models \neg Rxx\,[0]$, and corollary 2.12 can be applied.

An argument similar to that proving $(\forall x)(\exists y)(Rxy \wedge Ryy)$ to be outside of $\bar{P}1$ (cf. the example after lemma 2.18) shows that $(\exists y)(Rxy \wedge Ryy) \notin P1$, from which the second assertion follows. QED.

An algebraic characterization result for $L_0$-formulas modally definable in the local sense could be extracted from the proof of theorem 6.15, but, since $\bar{P}1$ is our main object of interest in this chapter, this is omitted. Instead, a preservation result is given for the main semantic notions of chapter I.2. (Cf. the Lyndon homomorphism theorem in Chang & Keisler [2], or the main result of Feferman [4].) In the statement and the proof of this as well as later results of this chapter $\bot$ and T will be abbreviations for $(\forall x)\neg(Rxx \to Rxx)$ and $(\forall x)(Rxx \to Rxx)$, respectively. Formal languages L will be used consisting of $L_0$ with added individual constants.

6.4 Definition

If L is a first-order language containing the binary predicate constant R, then the restricted positive formulas of L are the L-formulas belonging to the smallest class RF1(L) containing $\bot$ and all atomic formulas of the forms $Rt_1t_2$ and $t_1 = t_2$, where $t_1$ and $t_2$ are variables or individual constants, which is closed under $\wedge$, $\vee$, restricted universal quantification of the form $(\forall y)(Rty \rightarrow$ and restricted existential quantification of the form $(\exists y)(Rty \wedge$ , where t is a constant or a variable distinct from $y$.

Formulas of $RF1(L_0)$ contain at least one free variable. As soon as individual constants are present this need no longer be the case.

The following definitions and results up to and including theorem 6.7 are stated for $L_0$-formulas with one free variable, but are easily extended to the case of an arbitrary number of free variables.

6.5 Definition

An $L_0$-formula $\phi$ with one free variable is invariant for generated subframes if, for all frames $F_1$ (= $\langle W_1, R_1 \rangle$) and $F_2$ such that $F_1 \subsetneq F_2$ and all $w \in W_1$, $F_1 \models \phi[w] \Leftrightarrow F_2 \models \phi[w]$.

6.6 Definition

An $L_0$-formula $\phi$ with one free variable is preserved under p-morphisms if, for all frames $F_1$ (= $\langle W_1, R_1 \rangle$) and $F_2$, all p-morphisms f from $F_1$ onto $F_2$ and all $w \in W_1$, $F_1 \models \phi[w] \Rightarrow F_2 \models \phi[f(w)]$.

6.7 Theorem

An $L_0$-formula with one free variable is invariant for generated subframes and preserved under p-morphisms iff it is equivalent to a restricted positive $L_0$-formula with the same free variable.

Proof: Any restricted positive formula $\phi$ of $L_0$ with the free variables $x_1,\ldots, x_k$ is invariant for generated subframes. Any restricted positive formula $\phi$ of $L_0$ with the free variables $x_1,\ldots, x_k$ is preserved under p-morphisms. Both of these results are proved by a simple induction on the complexity of $\phi$.

Now let the $L_0$-formula $\phi$ with the one free variable x be invariant for generated subframes and preserved under p-morphisms. An argument rather analogous to the one used in the proof of theorem 1.9 shows that $\phi$ is equivalent to a restricted positive formula with the one free variable x:

Let $1(\phi) = \{\psi \mid \psi \in RF1(L_0),\ \psi$ has the one free variable x, and $\phi \models \psi\}$. It will be shown that $1(\phi) \models \phi$, from which the conclusion follows by the compactness theorem. Let $F_1^1 \models 1(\phi)\,[w]$. After adding an individual constant $\underline{w}$ to $L_0$ to obtain $L_1$ $F_1$ is expanded to an $L_1$-structure $F_1$ by interpreting $\underline{w}$ as w. In the remainder of this chapter "$L_1$" will be used to denote this language or a similar one: the notational convention of chapter I.2 regarding the use of "$L_1$" is hereby dropped.

Each finite subset of $\{[\underline{w}/x]\phi\} \cup \{\neg\psi \mid \psi$ is a sentence in $RF1(L_1)$ and $F_1 \models \neg\psi\}$ has a model. Otherwise, $[\underline{w}/x]\phi \models \neg(\neg\psi_1 \wedge \ldots \wedge \neg\psi_m)$ for some $\psi_1,\ldots, \psi_m$ as described, so $[\underline{w}/x]\phi \models \psi_1 \vee \ldots \vee \psi_m$, contradicting the fact that $RF1(L_1)$ is closed under $\vee$ and $F_1 \models \neg(\psi_1 \vee \ldots \vee \psi_m)$. It follows that the above set has a model, say $G_1$. (From now on the capital letter G, possibly with subscripts and/or superscripts, will also denote frames.) This yields the following situation:

frames: $F_1^1$, $F_1$

$$\uparrow \text{(1)}$$

$G_1$,

languages: $L_0$, $L_1$

where $G_1 \models [\underline{w}/x]\phi$

and $G_1 - 1(L_1) - F_1$, where "$G - 1(L) - F$" abbreviates "for all sentences $\phi$ in RF1(L), if $G \models \phi$, then $F \models \phi$".

Elementary chains $F_1, F_2, \ldots$ and $G_1, G_2, \ldots$ will now be constructed using the following general method. Let a language $L_n$ and $L_n$-structures $F_n$ and $G_n$ be given such that $G_n - 1(L_n) - F_n$. For each c and w, where c is an individual constant in $L_n$, w is in the domain of $G_n$ and $G_n \models Rcx[w]$, add a new constant $\underline{w}$ to $L_n$ to obtain $L_n^1$. Expand $G_n$ to an $L_n^1$-structure $G_n^1$ by interpreting each $\underline{w}$ as w.

We claim that each finite subset of $\Delta = \{\psi \mid \psi$ is a sentence in $RF1(L_n^1)$ and $G_n^1 \models \psi\}$ has a model which is an expansion of $F_n$. For, let $\psi_1, \ldots, \psi_k \in \Delta$, containing the constants $\underline{w}_1, \ldots, \underline{w}_l$ from $L_n^1$-$L_n$. There are constants $c_1, \ldots, c_l$ of $L_n$ such that

$G_n^1 \models Rc_1x_1 \wedge \ldots \wedge Rc_lx_l \wedge [x_1/\underline{w}_1, \ldots, x_l/\underline{w}_l](\psi_1 \wedge \ldots \wedge \psi_k)[w_1, \ldots, w_l]$,

where $x_1, \ldots, x_l$ are variables not occurring in $(\psi_1 \wedge \ldots \wedge \psi_k)$. Therefore, $G_n \models (\exists x_1)(Rc_1x_1 \wedge \ldots \wedge (\exists x_l)(Rc_lx_l \wedge [x_1/\underline{w}_1, \ldots, x_l/\underline{w}_l](\psi_1 \wedge \ldots \wedge \psi_k))\ldots)$ and so this $RF1(L_n)$-sentence (!) holds in $F_n^1$. From this the claim easily follows, and a standard model-theoretic argument will even establish there is a model $F^1_n$ for $\Delta$ such that

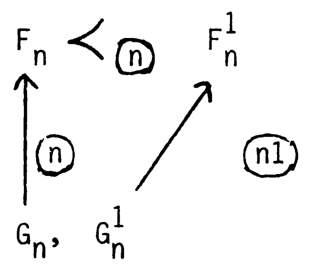
$F^1_n$ is an $L^1_n$-structure

$F_n \prec_{L_n} F^1_n$ (i.e., $F_n$ is an $L_n$-elementary substructure of $F^1_n$)

$G^1_n - 1(L^1_n) - F^1_n$.

Picture this as:

frames:




languages: $L_n$, $L^1_n$, $L^1_n$

For each c and w, where c is an individual constant in $L^1_n$, w is in the domain of $F^1_n$ and $F^1_n \models Rcx\ [w]$, add a new constant $k_{cw}$ to $L^1_n$ to obtain $L_{n+1}$. Expand $F^1_n$ to an $L_{n+1}$-structure $F_{n+1}$ by interpreting each $k_{cw}$ as w.

Each finite subset of $\Gamma = \{ \psi \mid \psi$ is a sentence of $RF1(L_{n+1})$ and $F_{n+1} \models \neg\psi\} \cup \{Rck_{cw} \mid k_{cw}$ is a constant in $L_{n+1}-L^1_n$ such that $F_{n+1} \models Rck_{cw}\}$ has a model which is an expansion of $G^1_n$. To see this, let $\neg\psi_1,\ldots, \neg\psi_k \in \Gamma$ and consider $Rc_1k_{c_1w_1},\ldots, Rc_1k_{c_1w_1}$. (If $\neg\psi_1,\ldots, \neg\psi_k$ contain other constants from $L_{n+1}-L^1_n$ besides $k_{c_1w_1},\ldots, k_{c_1w_1}$, then add the relevant $Rck_{cw}$'s. So one may as well suppose that $k_{c_1w_1},\ldots, k_{c_1w_1}$ are all the constants from $L_{n+1}-L^1_n$ occurring in $\neg\psi_1,\ldots, \neg\psi_k$.) If $\{\neg\psi_1,\ldots, \neg\psi_k, Rc_1k_{c_1w_1},\ldots, Rc_1k_{c_1w_1}\}$ is not satisfiable in an expansion of $G^1_n$, then, for any sequence of variables $x_1,\ldots, x_1$ not occurring in $\neg\psi_1,\ldots, \neg\psi_k$,

$G_n^1 \models (\forall x)(Rc_1x_1 \rightarrow \ldots (\forall x_l)(Rc_lx_l \rightarrow [x_1/k_{c_1w_1}, \ldots, x_l/R_{c_l}k_{c_lw_l}](\psi_1 \vee \ldots \vee \psi_k))\ldots)$.

Moreover, since this RFl($L_n^1$)-sentence (!) holds in $G_n^1$, it also holds in $F_n^1$, as $G_n^1$ - 1($L_n^1$) - $F_n^1$. This contradicts the fact that $F_n^1 \models Rc_1x_1 \wedge \ldots \wedge Rc_lx_l \wedge [x_1/k_{c_1w_1}, \ldots, x_l/k_{c_lw_l}](\neg\psi_1 \wedge \ldots \wedge \neg\psi_k)[w_1, \ldots, w_l]$.

Two remarks should be made at this point. As the reader will no doubt have noticed, there was a slight inexactness in the construction of $F_n^1$. Constants $\underline{w}_1, \ldots, \underline{w}_l$ were considered, occurring in $(\psi_1 \wedge \ldots \wedge \psi_k)$, and $c_1, \ldots, c_l$ such that $G_n^1 \models (Rc_1x_1 \wedge \ldots \wedge Rc_lx_l \wedge [x_1/\underline{w}_1, \ldots, x_l/\underline{w}_l]\ (\psi_1 \wedge \ldots \wedge \psi_k)[w_1, \ldots, w_l]$. It was then concluded that $G_n \models (\exists x_1)(Rc_1x_1 \wedge \ldots \wedge (\exists x_l)(Rc_lx_l \wedge [x_1/\underline{w}_1, \ldots, x_l/\underline{w}_l]\ (\psi_1 \wedge \ldots \wedge \psi_k))\ldots)$. But suppose that, e.g., $w_1$ and $w_2$ are the same element, i.e., $\underline{w}_1 = \underline{w}_2$, but $c_1$ and $c_2$ are different. (In other words, $(c_1)^{G_n^1}$ and $(c_2)^{G_n^1}$ have the R-successor $w_1$ in common.) Then the above sentence should start with $(\exists x_1)(Rc_1x_1 \wedge Rc_2x_1 \wedge \ldots$ . Here this inexactness is harmless, since the new sentence is in RFl($L_n$) as well. But with $F_{n+1}$ this would be serious. For $\{Rc_1\underline{w}, Rc_2\underline{w}, \neg\psi(\underline{w})\}$ the same construction would lead to $(\forall y_1)(Rc_1y_1 \rightarrow (Rc_2y_1 \rightarrow \psi(y_1)))$ which is <u>not</u> in RFl($L_n^1$).
The $k_{cw}$-complication serves to avoid this in a similar way as explained after the proof of theorem 1.9.

The second remark concerns $\perp$. If no $\neg\psi_1, \ldots, \neg\psi_k$ are present in the previous argument, then $(\forall x_1)(Rc_1x_1 \rightarrow \ldots (\forall x_l)(Rc_lx_l \rightarrow \perp)\ldots)$ is to be considered. Here is, where we need $\perp$ essentially. (In fact, what is needed is the existence of at least one sentence $\psi$ in RFl($L_{n+1}$) such that

$F_{n+1} \models \neg\psi . \perp$ is such a sentence, and in some cases it may be the only one, e.g., if $F_{n+1} = \langle \{0\}, \{\langle 0, 0 \rangle\} \rangle$.)

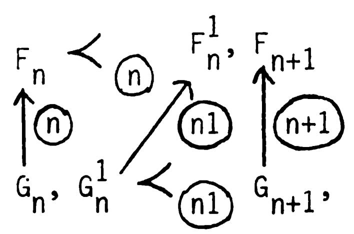
Again a standard model-theoretic argument establishes the existence of an $L_{n+1}$-structure $G_{n+1}$ satisfying

$$G_n^1 \prec_{L_n^1} G_{n+1}$$

$$G_{n+1} - 1(L_{n+1}) - F_{n+1}.$$

Picture this as:

frames:




languages: $L_n$, $L_n^1$, $L_n^1$, $L_{n+1}$,

It will be clear now how the two elementary chains $F_1, F_2, \ldots$ and $G_1, G_2, \ldots$ are constructed, together with the languages $L_1, L_2, \ldots$ Several applications of the fundamental theorem on elementary chains, in combination with the initial assumptions on $\phi$, will yield the required conclusion. $[\underline{w}/x]\phi$ holds in the limit G of the chain $G_1, G_2, \ldots$ By the invariance of $\phi$ for generated subframes, $TC(G, \underline{w}^G) \models [\underline{w}/x]\phi$. This generated subframe of G is exactly the substructure of G with a domain consisting of the $c^G$'s, where c is a constant in $\bigcup_n L_n$. For $w = c^G$ in the domain of $TC(G, \underline{w}^G)$ put $f(w) = c^F$, where F is the limit of the chain $F_1, F_2, \ldots$ We claim that f is a p-morphism from $TC(G, \underline{w}^G)$ onto $TC(F, \underline{w}^F)$.

That f is well-defined follows from the fact that if $c_1^G = c_2^G$, then, for a suitably large n, $c_1 \in L_n$ and $c_2 \in L_n$, $G_n \models c_1 = c_2$ and so, since $G_n - 1(L_n) - F_n$, $F_n \models c_1 = c_2$ and, therefore, $F \models c_1 = c_2$. That f is onto follows from the observation that TC(F, $\underline{w}^F$) consists exactly of the interpretations of the $\bigcup_n L_n$-constants in F. $Rc_1^G c_2^G$ implies $Rc_1^F c_2^F$, by an argument similar to the one showing f to be well-defined. This proves the first condition in the definition of a p-morphism. For the second one, if $Rc_1^F v$ in TC(F, $\underline{w}^F$), then $v = c_2^F$ for some $\bigcup_n L_n$-constant $c_2$ (one of the $k_{cw}$'s will serve), so $v = f(c_2^G)$.

$\phi$ is preserved under p-morphisms and, therefore, TC(F,$\underline{w}^F$)$\models [\underline{w}/x]\phi$. It follows from this, by the invariance of $\phi$ for generated subframes, that $F \models [\underline{w}/x]\phi$, and so $F_1 \models [\underline{w}/x]\phi$, i.e., $F_1^1 \models \phi[w]$. QED.

6.8 Corollary

Each formula in P1 is equivalent to a restricted positive formula with the same free variable.

Proof: Each formula in P1 is invariant for generated subframes and preserved under p-morphisms, because its defining modal formula is. (Cf. corollaries 2.6 and 2.12.) QED.

The final result on P1 is a constructive one, showing how modal definitions may be obtained for certain $L_0$-formulas.

6.9 Definition

A $\bar{\forall}$-formula is an $L_0$-formula with one free variable, which is of the form $U\psi$, where U is a (possibly empty) sequence of restricted universal quantifiers and $\psi$ is an $L_0$-formula in which only atomic formulas, $\wedge$ and $\vee$ occur.

Many relational conditions occurring in the literature are of this form, e.g., reflexivity, transitivity and symmetry, but also the oft-mentioned property of having no more than a given number of R-incomparable R-successors at any given point.

6.10 Lemma

Each $\bar{\forall}$-formula is in P1, and its modal definition can be obtained constructively from it.

Proof: Let $\phi$ be a $\bar{\forall}$-formula $U\psi$. Using the propositional distributive laws, write $\psi$ as a conjunction $\prod_{i=1}^{n} \psi_i$ of disjunctions $\psi_i$ of atomic formulas.

Since $\phi$ is equivalent to $\prod_{i=1}^{n} U\psi_i$, it suffices to consider the conjuncts $U\psi_i$, by lemma 6.2. Rewrite $U\psi_i$ to a formula of the form "$\neg$-sequence of restricted existential quantifiers-conjunction of negated atomic formulas". Remove repetitions from this conjunction, and also drop one of each pair $\neg x = y$, $\neg y = x$ in it. Take a different bound variable for each quantifier.

A tree $T_y$ is constructed inductively for each variable y occurring in $\psi_i$. If no restricted quantifiers of the form $(\exists z)(Ryz \wedge$ occur in $\psi_i$, then $T_y$ consists of a single node y. If not, then $T_y$ is constructed from $T_{z_1}, \ldots, T_{z_m}$, where $z_1, \ldots, z_m$ are the variables z such that $(\exists z)(Ryz \wedge$ occurs in $\psi_i$, by joining their topnodes to a new topnode y.

For each node y in the tree $T_x$, where x is the one free variable of

$\psi_i$, a modal formula (y) is defined inductively as the conjunction of

$\Diamond$(z), for each immediate descendant z of y,

$\Box p_{yz}$, for each $\neg$Ryz occurring in the propositional matrix of $\psi_i$,

$\neg p_{zy}$, for each $\neg$Rzy occurring in the propositional matrix of $\psi_i$,

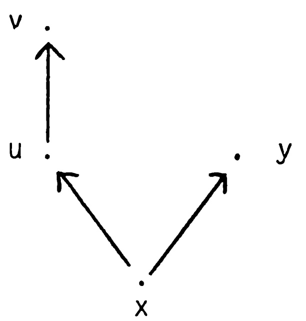
$q_{yz}$, for each $\neg$y=z occurring in the propositional matrix of $\psi_i$,

$\neg q_{zy}$, for each $\neg$z=y occurring in the propositional matrix of $\psi_i$

(or T, if the conjunction is empty).

$\neg$(x) is the modal formula defining $U\psi_i$. This is easily shown by noting that, for all frames F = <W, R> and each $w \in W$, $F \models \neg U\psi_i\,[w]$ iff, for some valuation V on F, $\langle F, V\rangle \models (x)\,[w]$. QED.

$(\forall y)(Rxy \rightarrow (\forall u)(Rxu \rightarrow (\forall v)(Ruv \rightarrow Ryv)))$ will serve as an example. Rewriting it as $\neg(\exists y)(Rxy \wedge (\exists u)(Rxu \wedge (\exists v)(Ruv \wedge \neg Ryv)))$ yields the tree $T_x$:




(y) = $\Box p_{yv}$

(v) = $\neg p_{yv}$

(u) = $\Diamond \neg p_{yv}$

(x) = $\Diamond \Box p_{yv} \wedge \Diamond\Diamond \neg p_{yv}$.

$\neg$(x) is equivalent to $\Diamond \Box p_{yv} \rightarrow \Box \Box p_{yv}$.

The second part of this chapter is devoted to $\bar{P}1$ and to $L_0$-sentences in general.

6.11 Lemma

$\bar{P}1$ is closed under conjunctions, but not under disjunctions or negations.

(Note that there is no natural formulation for clauses involving restricted quantification in $\bar{P}1$. Compare the difficulty in explaining $F \models \Box\phi$ in terms of $F \models \phi$: $F \models \Box\phi$ iff $(\forall w \in W)\ F \models \Box\phi\,[w]$ iff $(\forall w \in W)(\forall v \in W)(Rwv \Rightarrow F \models \phi\,[v])$, but this does not help.)

Proof of lemma 6.11: If $\alpha$ and $\beta$ are $L_0$-sentences in $\bar{P}1$, then, for some modal formulas $\phi$ and $\psi$, $\bar{E}(\phi, \alpha)$ and $\bar{E}(\psi, \beta)$. Then also $\bar{E}(\phi \wedge \psi, \alpha \wedge \beta)$, for $F \models \phi \wedge \psi$ iff $F \models \phi$ and $F \models \psi$. The corresponding result for disjunction does not hold, even if $\phi$ and $\psi$ have no proposition letters in common. $(\forall x)Rxx \in \bar{P}1$ $(\bar{E}(\Box p \rightarrow p, (\forall x)Rxx))$ and $(\forall x)(\forall y)(Rxy \rightarrow Ryx) \in \bar{P}1$ $(\bar{E}(\Diamond\Box p \rightarrow p, (\forall x)(\forall y)(Rxy \rightarrow Ryx))$, but $(\forall x)Rxx \vee (\forall x)(\forall y)(Rxy \rightarrow Ryx) \notin \bar{P}1$, for this sentence is not preserved under disjoint unions. E.g., it holds in both $\langle\{0\}, \emptyset\rangle$ and $\langle\{0, 1\}, \{\langle 0, 0\rangle, \langle 0, 1\rangle, \langle 1, 1\rangle\}\rangle$, but not in their disjoint union. Finally, $(\forall x)Rxx \in \bar{P}1$, but $\neg(\forall x)Rxx \notin \bar{P}1$, since it is not preserved under generated subframes. E.g., it holds in $\langle\{0, 1\}, \{\langle 0,0\rangle\}\rangle$, but not in $\langle\{0\}, \{\langle 0, 0\rangle\}\rangle$. QED.

A very general result is found in Goldblatt & Thomason [9], which gives an algebraic characterization of the classes of frames definable by a set of modal formulas (i.e., as $FR(\Gamma)$ for a set $\Gamma$ of modal formulas).

If only $\Sigma\Delta$-elementary classes of frames are considered their result assumes a particularly elegant form as stated below. This last result is proved here in a non-algebraic fashion. This does not yield a complete characterization for $\bar{P}1$, though, since the $L_0$-sentences characterized by it are exactly those which are definable by a class of modal formulas, rather than by a single one. Such difficulties were not encountered in chapter I.3, because modal formulas defined by a class of $L_0$-sentences are definable by a single $L_0$-sentence already, as a simple argument showed. The analogous question for the present case is still open. Theorem 20.10 in Goldblatt [8] does characterize $\bar{P}1$ algebraically, but the additional notion involved ("completed ultraproduct") is not as (elegant and) natural as the ones occurring in theorem 6.15 below.

6.12 Definition

If $F = \langle W, R\rangle$ is a frame, then the ultrafilter extension $F^*$ of F is the frame $\langle W^*, R^*\rangle$ with the set $W^*$ of all ultrafilters on W as its domain and the relation $R^*U_1U_2$ between ultrafilters $U_1$ and $U_2$ defined by $(\forall X \subseteq W)(X \in U_2 \Rightarrow \{w \in W \mid (\exists v \in W)(Rwv \ \& \ v \in X)\} \in U_1)$.

6.13 Definition

If $M = \langle W, R, V\rangle$ is a model and $\phi$ a modal formula, then $V(\phi) = \{w \in W \mid M \models \phi[w]\}$.

Recall definition 3.9: for a model M, $Th_m(M) = \{\phi \mid \phi$ is a modal formula such that $M \models \phi\}$, and $Th_m(F)$ is defined similarly. The obvious extension to a class $\mathcal{K}$ of frames is:
$Th_m(\mathcal{K}) = \bigcap_{F \in \mathcal{K}} Th_m(F)$.

6.14 <u>Lemma</u> (R.I. Goldblatt & S.K. Thomason)

If $F^*$ is the ultrafilter extension of F, then $Th_m(F^*) \subseteq Th_m(F)$.

<u>Proof</u>: This is shown by an argument much like the standard completeness proofs in modal logic. Suppose that, for some valuation V on F and some modal formula $\phi$, $\langle F, V\rangle \models \neg\phi\ [w]$, where $w \in W$. It will be proved that, for some valuation $V^*$ on $F^*$ and some ultrafilter U, $\langle F^*, V^*\rangle \models \neg\phi\ [U]$.

Define $V^*$ by $V^*(p) = \{U \mid U$ is an ultrafilter on W and $V(p) \in U\}$. It follows that, for all modal formulas $\phi$, $V^*(\phi) = \{U \mid V(\phi) \in U\}$. This is proved by induction on the complexity of $\phi$, where the cases $\phi$ is a proposition letter, $\phi = \neg\psi$ and $\phi = \psi \rightarrow \chi$ are trivial. Consider the case $\phi = \Diamond\psi$. If $U \in V^*(\Diamond\psi)$, then, for some U' with $R^*UU'$, $U' \in V^*(\psi)$, so, by the induction hypothesis, $V(\psi) \in U'$ and, therefore, by the definition of $R^*$, $\{w \in W \mid (\exists v \in W)(Rwv \ \&\ v \in V(\psi))\} \in U$. $V(\Diamond\psi)$ is exactly this set, so it belongs to U. The converse is the only serious step. Let $V(\Diamond\psi) \in U$, i.e., $\{w \in W \mid (\exists v \in W)(Rwv \ \&\ v \in V(\psi))\} \in U$. It is to be shown that, for for some U' with $R^*UU'$, $V(\psi) \in U'$. Such a U' is found by noting that $\{X \subseteq W \mid \{w \in W \mid (\forall v \in W)(Rwv \Rightarrow v \in X)\} \in U\} \cup \{V(\psi)\}$ has the finite intersection property, and then applying the basic theorem on ultrafilters to this set, yielding a U' with $V(\psi) \in U'$ and $R^*UU'$. That the finite intersection property holds is shown as follows. Suppose that, for $X_1, \ldots, X_k$ as described, $X_1 \cap \ldots \cap X_k \cap V(\psi) = \emptyset$, i.e., $X_1 \cap \ldots \cap X_k \subseteq W - V(\psi)$. Then $\{w \in W \mid (\forall v \in W)(Rwv \Rightarrow v \in X_1 \cap \ldots \cap X_k)\} = \bigcap_{i=1}^{k} \{w \in W \mid (\forall v \in W)(Rwv \Rightarrow v \in X_i)\} \subseteq \{w \in W \mid (\forall v \in W)(Rwv \Rightarrow v \notin V(\psi))\}$. But the first set is in U and therefore the second would be, contradicting the assumption that $\{w \in W \mid (\exists v \in W)(Rwv \ \&\ v \in V(\psi))\} \in U$.

So, starting with $\langle F, V\rangle \models \neg\phi\ [w]$, i.e., with $w \in V(\neg\phi)$, $\{V(\neg\phi)\}$

is extended to an ultrafilter $U$, and then, by the above, $U \in V^*(\neg\phi)$, so $\langle F^*, V^*\rangle \models \neg\phi\,[U]$. QED.

6.15 <u>Theorem</u> (R.I. Goldblatt & S.K. Thomason)

A class of frames closed under elementary equivalence is of the form $FR(\Gamma)$ for a set $\Gamma$ of modal formulas iff it is closed under generated subframes, disjoint unions, p-morphisms and its complement is closed under ultrafilter extensions.

<u>Proof</u>: The original proof used algebraic notions, which made it possible to apply Birkhoff's theorem on equational classes of algebras. Here the argument is purely modal.

A class of frames of the form $FR(\Gamma)$ for a set $\Gamma$ of modal formulas satisfies the four closure properties mentioned above because of corollaries 2.6, 2.9, 2.12 and lemma 6.14, respectively.

Now let $\mathcal{K}$ be a class of frames closed under elementary equivalence, generated subframes, disjoint unions and p-morphisms, while its complement is closed under ultrafilter extensions. The first three closure properties imply that $\mathcal{K}$ is $\Delta$-elementary, by theorem 3.4. So, for some set $\Sigma$ of $L_0$-sentences, $\mathcal{K} = FR(\Sigma)$.

For an arbitrary frame $F$ with $F \models Th_m(\mathcal{K})$ it will be shown that $F \in \mathcal{K}$, and, therefore, since, quite trivially, each $F \in \mathcal{K}$ satisfies $Th_m(\mathcal{K})$, $\mathcal{K} = FR(Th_m(\mathcal{K}))$, which proves the above assertion.

For each $X \subseteq W$ take a proposition letter $p_X$ and set $V(p_X) = X$ to obtain a model $M(F) = \langle F, V\rangle$. For each modal formula $\phi$ such that $\phi \notin Th_m(M(F))$, a frame $F_\phi$, $w_\phi \in W_\phi$ and a valuation $V_\phi$ on $F_\phi$ exist satisfying $\langle F_\phi, V_\phi\rangle \models Th_m(M(F))\,[w_\phi]$, but $\sim\langle F_\phi, V_\phi\rangle \models \phi\,[w_\phi]$.

This is so, because otherwise, for some $\phi \notin Th_m(M(F))$, $\Sigma \cup ST(Th_m(M(F))) \models ST(\phi)$, whence, by compactness, $\Sigma \cup \{ST(\psi)\} \models ST(\phi)$ for some $\psi \in Th_m(M(F))$. It follows that $\Sigma \models ST(\psi) \rightarrow ST(\phi)$, so $\psi \rightarrow \phi \in Th_m(\mathcal{K})$, $F \models \psi \rightarrow \phi$, $M(F) \models \psi \rightarrow \phi$ , and, since $M(F) \models \psi$, $M(F) \models \phi$, contradicting the original assumption about $\phi$. By confining attention to $TC(F_\phi, w_\phi)$ (a frame in $\mathcal{K}$, because $\mathcal{K}$ is closed under generated subframes) and noting that, for all modal formulas $\alpha$ in $Th_m(M(F))$, $\Box\alpha \in Th_m(M(F))$, it may be supposed without loss of generality that $\langle F_\phi, V_\phi\rangle \models Th_m(M(F))$ and $\sim\langle F_\phi, V_\phi\rangle \models \phi$ (use lemma 2.5). The disjoint union of $\{\langle F_\phi, V_\phi\rangle \mid \phi \notin Th_m(M(F))\}$ is a model $M_1$ ($= \langle F_1, V_1\rangle$) such that $F_1 \in \mathcal{K}$ ($\mathcal{K}$ is closed under disjoint unions of _frames_ and it is obvious how a disjoint union of _models_ is defined in a completely analogous fashion) and $Th_m(M_1) = Th_m(M(F))$.

Starting from this frame $F_1 \in \mathcal{K}$ with a valuation $V_1$ such that the resulting model has the same modal theory as $M(F)$, a series of further models is constructed:

6.16 _Definition_ (Fine [ 5 ])

A model $M = \langle W, R, V\rangle$ is _1-saturated_ if, for all sets $\Gamma$ of modal formulas such that for each finite subset $\Gamma_0$ of $\Gamma$ a $w \in W$ exists with $M \models \Gamma_0 [w]$, there is a $w \in W$ with $M \models \Gamma [w]$.

A model $M = \langle W, R, V\rangle$ is _2-saturated_ if, for all sets $\Gamma$ of modal formulas and all $w \in W$ such that for each finite subset $\Gamma_0$ of $\Gamma$ a $v \in W$ exists with $Rwv$ and $M \models \Gamma_0 [v]$, there is a $v \in W$ with $Rwv$ and $M \models \Gamma [v]$.

Familiar model-theoretic arguments will give a 1- and 2-saturated elementary extension for $M_1$, say $M_2$ (= $<F_2, V_2>$). (Note that the continuum hypothesis is not needed in this case, because $M_2$ need not be saturated in the full model-theoretic sense of the term.) $F_2 \in \mathcal{K}$, because $\mathcal{K}$ is closed under elementary equivalence, and $Th_m(M_2) = Th_m(M_1)$, since $M_2$ may be taken to be an $L_1$ (in the sense of chapter I.2) -elementary extension of $M_1$.

The following defines a p-morphism h from $F_2$ onto $F^*$. For $w \in W_2$, let $h(w) = \{V(\phi) \mid \phi$ is a modal formula such that $M_2 \models \phi\,[w]\}$, where V was the valuation of M(F). It will be shown that

(i) $h(w) \in W^*$

(ii) h is onto

(iii) $(\forall w \in W_2)(\forall v \in W_2)(R_2wv \Rightarrow R^*h(w)h(v))$

(iv) $(\forall w \in W_2)(\forall v \in W^*)(R^*h(w)v \Rightarrow (\exists u \in W_2)(R_2wu \;\&\; h(u) = v))$.

(i): Clearly, each $V(\phi)$ is a subset of W. h(w) is a filter on W, for, if $V(\phi_1)$ and $V(\phi_2) \in h(w)$, then so is $V(\phi_1) \cap V(\phi_2)$ (= $V(\phi_1 \wedge \phi_2)$), and if $V(\phi) \in h(w)$ and $V(\phi) \subseteq Y$, then $M(F) \models \phi \to p_Y$, so $M_2 \models \phi \to p_Y$ ($Th_m(M_2) = Th_m(M_1) = Th_m(M(F))$) and $M_2 \models \phi \to p_Y\,[w]$. Then $V(p_Y) = Y \in h(w)$. h(w) is also an ultrafilter, because for each $Y \subseteq W$, either $M_2 \models p_Y\,[w]$, or $M_2 \models p_{W-Y}\,[w]$, since $\neg p_Y \leftrightarrow p_{W-Y} \in Th_m(M(F))$. So, either $V(p_Y) = Y \in h(w)$, or $V(p_{W-Y}) = W-Y \in h(w)$.

(ii): Let U be an ultrafilter on W and consider $\Gamma = \{p_X \mid X \in U\}$. For each finite subset $\Gamma_0$ of $\Gamma$, $M_2 \models \Gamma_0\,[w]$ for some $w \in W_2$, because otherwise $M_2 \models \neg\Pi\Gamma_0$, so $M(F) \models \neg\Pi\Gamma_0$, contradicting the finite intersection property for U. By 1-saturatedness a $w \in W_2$ exists such that $M_2 \models \Gamma\,[w]$, and clearly $h(w) = U$.

(iii): If w and $v \in W_2$ with $R_2wv$, and $X \in h(v)$, then $M_2 \models \phi\,[v]$ for some modal formula $\phi$ such that $V(\phi) = X$. It follows that $M_2 \models \Diamond\phi\,[w]$, so $V(\Diamond\phi) = \{w \in W \mid (\exists v \in W)(Rwv \ \& \ v \in V(\phi) \ (= X))\} \in h(w)$. By definition 6.12, this shows that $R^*h(w)h(v)$.

(iv): If, for some $w \in W_2$ and $U \in W^*$, $R^*h(w)U$, then consider $\Delta = \{\phi \mid \phi$ is a modal formula such that $V(\phi) \in U\}$. If $\Delta_0$ is a finite subset of $\Delta$, then $V(\Pi\Delta_0) = \bigcap_{\delta \in \Delta_0} V(\delta) \in U$, so $\{w \in W \mid (\exists v \in W)(Rwv \ \& \ v \in V(\Pi\Delta_0))\} \in h(w)$, by the definition of $R^*$. This set is $V(\Diamond\Pi\Delta_0)$, so $M_2 \models \Diamond\Pi\Delta_0 [w]$. By 2-saturatedness, a $v \in W_2$ exists such that $R_2wv$ and $M_2 \models \Gamma [v]$. Clearly, $h(v) = U$.

Since $\mathcal{K}$ is closed under p-morphisms, we have proved that $F^* \in \mathcal{K}$, so, since the complement of $\mathcal{K}$ is closed under ultrafilter extensions, $F \in \mathcal{K}$. QED.

As an example consider purely existential $L_0$-sentences. These are preserved under ultrafilter extensions, because it is easy to see that f, defined for each $w \in W$ by $f(w) = \{X \subseteq W \mid w \in X\}$, is an isomorphism between F and a subframe of $F^*$. (e.g., $R^*f(w)f(v)$ iff $(\forall X \subseteq W)(v \in X \Rightarrow w \in \{u \in W \mid (\exists s \in W)(Rus \ \& \ s \in X)\})$ iff Rwv.) We shall return to this subject at the end of this chapter. Now let $\phi$ be an $L_0$-sentence of the form $(\forall x)\psi$, where $\psi$ is a $\bar{\forall}$-formula (as described in definition 6.9). It is easy to see that $FR(\phi)$ satisfies the conditions of theorem 6.15, so $\phi$ is modally definable. This proves a version of lemma 6.10, but for the global correspondence only, and a little less constructive.

We conclude with a series of preservation results for the semantic notions of theorem 6.15.

6.17 Definition

If L is a first-order language obtained from $L_0$ by adding a (possibly empty) set of individual constants, then

RF2(L) is the class of L-formulas constructed using atomic formulas with variables and/or constants, $\bot$, $\wedge$, $\vee$, $\forall$, $\exists$ and restricted universal quantifiers of the form $(\forall y)(Rty \rightarrow$ , where t is an individual constant or a variable distinct from y,

RF3(L) is the class of L-formulas constructed using atomic formulas with variables and/or constants, negations of such formulas, $\wedge$, $\vee$, $\forall$ and restricted existential quantifiers of the form $(\exists y)(Rty \wedge$ , where t is an individual constant or a variable distinct from y,

RF4(L) is the class of L-formulas constructed using atomic formulas with variables and/or constants, negations of such formulas, $\wedge$, $\vee$, $\exists$ and restricted quantifiers of the forms $(\forall y)(Rty \rightarrow$ and $(\forall y)(Ryt \rightarrow$ , where t is an individual constant or a variable distinct from y.

The task of finding more appropriate names for these classes is left to the imaginative reader.

For convenience, we restate definitions 6.5 and 6.6 for the case of $L_0$-sentences and add a new notion.

6.18 Definition

An $L_0$-sentence $\phi$ is preserved under p-morphisms if, for all frames $F_1$ and $F_2$, and all p-morphisms f from $F_1$ onto $F_2$, $F_1 \models \phi$ only if $F_2 \models \phi$.

An $L_0$-sentence $\phi$ is preserved under generated subframes if, for all frames $F_1$ and $F_2$ such that $F_2 \subsetneq F_1$, $F_1 \models \phi$ only if $F_2 \models \phi$.

An $L_0$-sentence $\phi$ is preserved under disjoint unions if, for all sets of frames $\{F_i \mid i \in I\}$ with $F_i \models \phi$ for all $i \in I$, $\bigoplus\{F_i \mid i \in I\} \models \phi$.

6.19 Theorem (R.I. Goldblatt)

An $L_0$-sentence is preserved under p-morphisms iff it is equivalent to a sentence in $RF2(L_0)$.

Proof: This proof will be similar to that of theorem 6.7, and therefore details will be omitted, wherever possible. The same holds for the remaining proofs in this chapter.

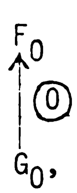
Formulas in $RF2(L_0)$ are preserved under p-morphisms, as an easy induction on the complexity of formulas shows. More precisely, if $\phi$ is a formula in $RF2(L_0)$ with the free variables $x_1,\ldots, x_k$, and f is a p-morphism from $F_1$ onto $F_2$, then, for all $w_1,\ldots, w_k \in W_1$, $F_1 \models \phi [w_1,\ldots, w_k] \Rightarrow F_2 \models \phi [f(w_1),\ldots, f(w_k)]$.

Let $\phi$ be preserved under p-morphisms. Define $2(\phi) = \{\psi \mid \psi$ is a sentence in $RF2(L_0)$ and $\phi \models \psi\}$. We will show that $2(\phi) \models \phi$, and again the conclusion follows by compactness, for $RF2(L_0)$ is closed under $\wedge$.

Starting with $F_0$ such that $F_0 \models 2(\phi)$, elementary chains $F_0$, $F_1$, $F_2,\ldots$ and $G_0$, $G_1$, $G_2,\ldots$ are constructed. The only salient points are the construction principle and the final reasoning.

First, each finite subset of $\{\phi\} \cup \{\neg\psi \mid \psi$ is a sentence in $RF2(L_0)$ and $F_0 \models \neg\psi\}$ has a model, as a by now familiar argument shows. Let $G_0$ be a model for this set. The starting point for the construction is

frames:



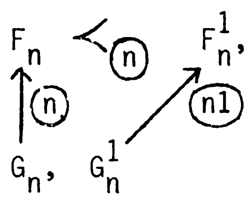
languages: $L_0$,

where $F_0$ and $G_0$ are $L_0$-structures and

$G_0$ - 2($L_0$) - $F_0$, i.e., for any sentence $\phi$ in RF2($L_0$), if $G_0 \models \phi$, then $F_0 \models \phi$. (The general notion G - 2(L) - F is defined in the same way.)

Now let $F_n$, $G_n$ and $L_n$ be given such that $F_n$ and $G_n$ are $L_n$-structures satisfying $G_n$ - 2($L_n$) - $F_n$. Add new constants $\underline{w}$, for each w in the domain of $G_n$, to obtain the language $L^1_n$. Expand $G_n$ to an $L^1_n$-structure $G^1_n$ by interpreting each $\underline{w}$ as w. Then each finite subset of $\Gamma = \{\psi \mid \psi$ is a sentence in RF2($L^1_n$) and $G^1_n \models \psi\}$ has a model which is an expansion of $F_n$. (To see this, let $\psi_1, \ldots, \psi_k \in \Gamma$ contain $\underline{w}_1, \ldots, \underline{w}_l$ from $L^1_n$-$L_n$. Then, for any $x_1, \ldots, x_l$ not occurring in $(\psi_1 \wedge \ldots \wedge \psi_k)$, $G_n \models (\exists x_1) \ldots (\exists x_l) [x_1/\underline{w}_1, \ldots, x_l/\underline{w}_l] (\psi_1 \wedge \ldots \wedge \psi_k)$, so this sentence, being in RF2($L_n$), holds in $F_n$.) It follows that $\Gamma$ has a model $F^1_n$, which is an $L_n$-elementary extension of $F_n$, with $G^1_n$ - 2($L^1_n$) - $F^1_n$. The situation is now:

frames:




languages: $L_n$, $L^1_n$, $L^1_n$

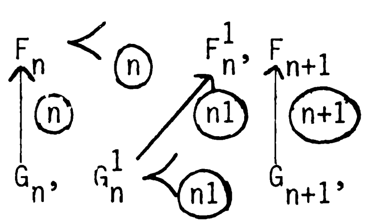
For each c and w, where c is a constant in $L^1_n$, w is in the domain of $F^1_n$ and $F^1_n \models$ Rcx [ w ], add a new constant $k_{cw}$ to $L^1_n$. Also add a new constant $\underline{w}$ for each w in the domain of $F^1_n$. These additions yield a language $L_{n+1}$, and $F^1_n$ is expanded to an $L_{n+1}$-structure $F_{n+1}$ by interpreting each $k_{cw}$ as w and each $\underline{w}$ as w. Each finite subset of $\Delta = \{\neg\psi \mid \psi$ is a sentence in $RF2(L_{n+1})$ and $F_{n+1} \models \neg\psi\} \cup \{Rck_{cw} \mid k_{cw} \in L_{n+1}-L_n^1$ and $F_{n+1} \models Rck_{cw}\}$ has a model which is an expansion of $G_n^1$. For, consider $\neg\psi_1, \ldots, \neg\psi_k \in \Delta$, as well as $Rc_1k_{c_1w_1}, \ldots, Rc_lk_{c_lw_l}$. By adding $Rck_{cw}$'s it may be supposed that all constants of the form $k_{cw}$ belonging to $L_{n+1}-L_n^1$ which occur in $\neg\psi_1, \ldots, \neg\psi_k$ are among $k_{c_1w_1}, \ldots, k_{c_lw_l}$. Let $\neg\psi_1, \ldots, \neg\psi_k$ also contain $\underline{w}_1, \ldots, \underline{w}_m$ from $L_{n+1}-L_n^1$. If $\neg\psi_1 \wedge \ldots \wedge \neg\psi_k \wedge Rc_1k_{c_1w_1} \wedge \ldots \wedge Rc_lk_{c_lw_l}$ were not satisfiable in an expansion of $G_n^1$, then, for some variables $x_1, \ldots, x_m, y_1, \ldots, y_l$ not occurring in this formula,

$G_n^1 \models (\forall x_1)\ldots(\forall x_m)(\forall y_1)(Rc_1y_1 \to \ldots (\forall y_l)(Rc_ly_l \to$
$[x_1/\underline{w}_1, \ldots, x_m/\underline{w}_m, y_1/k_{c_1w_1}, \ldots, y_l/k_{c_lw_l}](\psi_1 \vee \ldots \vee \psi_k))\ldots)$.

But then this $RF2(L_n^1)$-sentence (!) would be true in $F_n^1$, which contradicts the definition of $\Delta$. It follows that $\Delta$ has a model $G_{n+1}$ which is an $L_n^1$-elementary extension of $G_n^1$ satisfying $G_{n+1} - 2(L_{n+1}) - F_{n+1}$.

The situation is now:

frames:




languages: $L_n$, $L_n^1$, $L_n^1$, $L_{n+1}$,

Once the elementary chains are constructed, the limits F and G are taken. Since $G_0 \models \phi$, $G \models \phi$. f defined as before is a p-morphism from G onto F this time. (Each element of F and G is the interpretation of some constant of $\bigcup_n L_n$. Note how the second clause in the definition of a p-morphism holds because of the $k_{cw}$'s used in the construction.) By the assumption on $\phi$, $F \models \phi$ and, therefore, $F_0 \models \phi$. QED.

6.20 Theorem (R.I. Goldblatt)

An $L_0$-sentence is preserved under generated subframes iff it is equivalent to a sentence in RF3($L_0$).

Proof: The argument is similar to the preceding one. The three main differences are:

In the construction of $F^1_n$ constants $\underline{w}$ are added for each w in the domain of $G_n$ such that $G_n \models Rcx[w]$ for some $L_n$-constant c. This is because only restricted existential quantifiers are available now, so it is impossible to take a $\underline{w}$ for each w in the domain of $G_n$.

In the construction of $G_{n+1}$ it suffices to take constants $\underline{w}$ for each w in the domain of $F^1_n$. One then considers the set $\Delta = \{\neg\psi \mid \psi$ is a sentence in RF3($L_{n+1}$) and $F_{n+1} \models \neg\psi\} \cup \{Rc\underline{w} \mid c$ is a constant in $L^1_n$ and $F^1_n \models Rcx[w]\}$ For $\neg\psi_1,\ldots,\neg\psi_k \in \Delta$ and $Rc_1\underline{w}_1,\ldots,Rc_l\underline{w}_l \in \Delta$, where $\neg\psi_1,\ldots,\neg\psi_k$ contain no additional $(L_{n+1}-L^1_n)$-constants (this may be ensured by adding $Rc\underline{w}$'s) the argument goes as follows. Suppose that $\neg\psi_1 \wedge \ldots \wedge \neg\psi_k \wedge Rc_1\underline{w}_1 \wedge \ldots \wedge Rc_l\underline{w}_l$ is not satisfiable in an expansion of $G^1_n$. Let $x_1,\ldots,x_l$ be variables not occurring in this formula. Then, for $r \leqq l$, $G^1_n \models (\forall x_1)\ldots(\forall x_r)(\neg Rc_1x_1 \vee \ldots \vee \neg Rc_rx_r \vee \alpha \vee [x_1/\underline{w}_1,\ldots,x_r/\underline{w}_r](\psi_1 \vee \ldots \vee \psi_m))$, where in this RF3($L^1_n$)-sentence r may be smaller than l, because of the following. In case $w_1 = w_2$ and $Rc_1\underline{w}_1$ and $Rc_2\underline{w}_2$ occur, just take one $x_1$, write $\neg Rc_1x_1$ in front and put $\neg Rc_2x_1$ in the disjunction $\alpha$. Etc.

The final reasoning concerns the limits G and F of the chains obtained in this fashion. $G_0 \models \phi$ and, therefore, $G \models \phi$. The substructure $G^1$ of G with the interpretations of the $\bigcup_n L_n$-constants in G as its domain is a generated subframe of G, by the manner of choosing constants in the construction of $F_n^1$. So $G^1 \models \phi$. The function f defined as before is an isomorphism now from $G^1$ onto F. ($RF3(L_n)$ contains negations of atomic formulas as well, so f becomes a 1-1 p-morphism, i.e. an isomorphism.)
So $F \models \phi$ and $F_0 \models \phi$. QED.

In order to deal with disjoint unions it becomes necessary to complicate these proofs by constructing systems of elementary chains simultaneously. This method proves theorems 6.23 and 6.28, but it has not led to a proof of the expected result:

An $L_0$-sentence is preserved under disjoint unions iff it is equivalent to a sentence of the form $(\forall x)\phi$, where $\phi$ is in $RF4(L_0)$.
(Note that a double universal quantifier cannot be allowed. E.g., $(\forall x)(\exists y)Rxy$ is preserved under disjoint unions, as is $(\forall x)(\exists y)Ryx$, but $(\forall x)(\forall y)Rxy$ is not.)

The next theorem is the main result about $\bar{P}1$, comparable to theorem 6.7 for P1.

6.21 Theorem

An $L_0$-sentence is preserved under p-morphisms, generated subframes and disjoint unions iff it is equivalent to a sentence of the form $(\forall x)\phi$, where $\phi$ is in $RF1(L_0)$.

Proof: One direction follows directly from previous observations. For the other, suppose that the $L_0$-sentence $\phi$ is preserved under p-morphisms, generated subframes and disjoint unions. It will be shown that $I(\phi) = \{(\forall x)\psi \mid \psi$ is a formula in $RF1(L_0)$ with the one free variable x and $\phi \models (\forall x)\psi\} \models \phi$ Then, by the compactness theorem and the law $(\forall x)(\alpha \wedge \beta) \leftrightarrow (\forall x)\alpha \wedge (\forall x)\beta$, the conclusion follows.

Let $F_0^1 \models I(\phi)$. Take constants $\underline{w}$ for each w in the domain of $F_0^1$. For each $\underline{w}$, $L_{\underline{w}} =_{def} L_0 \cup \{\underline{w}\}$. Expand $F_0^1$ to $F_0$ by interpreting each $\underline{w}$ as w. For each $\underline{w}$, $F_0$ is an $L_{\underline{w}}$-structure. Each finite subset of $\Gamma_{\underline{w}} = \{\phi\} \cup \{\neg\psi \mid \psi$ is a sentence in $RF1(L_{\underline{w}})$ and $F_0 \models \neg\psi\}$ has a model. If not, then, for $\neg\psi_1, \ldots, \neg\psi_k \in \Gamma_{\underline{w}}$, $\{\phi, \neg\psi_1, \ldots, \neg\psi_k\}$ has no model, i.e. $\phi \models \psi_1 \vee \ldots \vee \psi_k$, and so $\phi \models (\forall x)[x/\underline{w}](\psi_1 \vee \ldots \vee \psi_k)$. (Minor troubles with bound variables may always be avoided by taking suitable alphabetic variants, so they will not be mentioned.) But then $F_0^1 \models (\forall x)[x/\underline{w}](\psi_1 \vee \ldots \vee \psi_k)$, contradicting $F_0^1 \models [x/\underline{w}](\neg\psi_1 \wedge \ldots \wedge \neg\psi_k)[w]$. So $\Gamma_{\underline{w}}$ has a model $G_{\underline{w}}$. Defining $L_0(G_{\underline{w}})$ as $L_{\underline{w}}$ and $\underline{G}_0$ as $\{G_{\underline{w}} \mid \underline{w}$ and $G_{\underline{w}}$ as described above$\}$ the following situation is reached:

For each $G \in \underline{G}_0$, $G\text{-}1(L_0(G))\text{-}F_0$, where G-1(L)-F was defined in the proof of theorem 6.7

For each $G \in \underline{G}_0$, $F_0$ is an $L_0(G)$-structure

For different G's $\in \underline{G}_0$, the languages $L_0(G)$ have disjoint sets of individual constants.

Again elementary chains will be constructed, according to the following principle.

Let $\underline{G}_n$, $F_n$ and, for each $G \in \underline{G}_n$, $L_n(G)$ be given such that, for each $G \in \underline{G}_n$, $F_n$ is an $L_n(G)$-structure and $G\text{-}1(L_n(G))\text{-}F_n$, while different

languages $L_n(G)$ have disjoint sets of individual constants. Consider any $G \in \underline{G}_n$ and add, for each w in the domain of G such that, for some constant c in $L_n(G)$, $G \models Rcx\ [w]$, a new constant $\underline{w}$ to obtain $L_n^1(G)$. G is then expanded to an $L_n^1(G)$-structure $G^1$ by interpreting each $\underline{w}$ as w. Each finite subset of $\Delta_n(G) = \{\psi \mid \psi$ is a sentence of $RF1(L_n^1(G))$ such that $G^1 \models \psi\}$ has a model which is an expansion of $F_n$. To prove this, let $\psi_1,\ldots, \psi_k \in \Delta_n(G)$ contain the $(L_n^1(G)-L_n(G))$-constants $\underline{w}_1,\ldots, \underline{w}_l$ such that $G \models Rc_i x_i\ [w_i]$ for each i ($1 \leq i \leq l$), where $c_1,\ldots, c_l$ are $L_n(G)$-constants. For variables $x_1,\ldots, x_l$ not occurring in $\psi_1,\ldots, \psi_k$,
$G \models (\exists x_1)(Rc_1x_1 \wedge \ldots \wedge (\exists x_l)(Rc_lx_l \wedge [x_1/\underline{w}_1,\ldots, x_l/\underline{w}_l]\ (\psi_1 \wedge \ldots \wedge \psi_k))\ldots)$. This is a sentence in $RF1(L_n(G))$, so it holds in $F_n$, since $G\text{-}1(L_n(G))\text{-}F_n$.

A similar argument establishes that each finite subset of $\bigcup_{G\in\underline{G}_n} \Delta_n(G)$ has a model which is an expansion of $F_n$.
(The above argument can be given for finitely many G's at the same time, because the languages $L_n(G)$ involved have disjoint sets of individual constants.) So $\bigcup_{G\in\underline{G}_n} \Delta_n(G)$ has a model $F_n^1$ satisfying for each $G \in \underline{G}_n$,

$F_n^1$ is an $L_n^1(G)$-structure

$F_n \prec_{L_n(G)} F_n^1$

$G^1\text{-}1(L_n^1(G))\text{-}F_n^1$.

Now for the other direction:

Consider any $L_n^1(G)$. Add, for each c and w such that c is a constant in $L_n^1(G)$, w is in the domain of $F_n^1$ and $F_n^1 \models Rcx\ [w]$, a new constant $k_{cw}$ to obtain $L_n^2(G)$. Note that different $L_n^2(G)$'s get disjoint sets of individual constants. $F_n^1$ is expanded to $F_n^2$ by interpreting each $k_{cw}$ from each $L_n^2(G)$ as w. In this way $F_n^2$ becomes an $L_n^2(G)$-structure for each $G \in \underline{G}_n$.

Each finite subset of $\Sigma_n(G) = \{\neg\psi \mid \psi$ is a sentence in $RF1(L_n^2(G))$ such that $F_n^2 \models \neg\psi\} \cup \{Rck_{cw} \mid k_{cw} \in L_n^2(G)-L_n^1(G)\}$ has a model which is an expansion of $G^1$. The argument showing this is the same as in previous proofs. So $\Sigma_n(G)$ has a model $G^2$ satisfying

$$G^1 \prec_{L_n^1(G)} G^2$$

and

$$G^2\text{-}1(L_n^2(G))\text{-}F_n^2.$$

Next, take new constants $\underline{w}$ for elements w in the domain of $F_n^2$ not named by any constant in any $L_n^2(G)$. Expand $F_n^2$ to $F_{n+1}$ by interpreting each $\underline{w}$ as w. Since, by our construction, $F_{n+1} \models \bar{I}(\phi)$, the procedure followed in the construction of $\underline{G}_0$ may be repeated with respect to these constants to obtain models $G_{\underline{w}}$ with corresponding languages $L_{n+1}(G_{\underline{w}}) = L_0 \cup \{\underline{w}\}$.

Defining $\underline{G}_{n+1}$ as $\{G^2 \mid G \in \underline{G}_n\} \cup \{G_{\underline{w}} \mid G_{\underline{w}}$ constructed in the preceding paragraph$\}$ and $L_{n+1}(G^2)$ as $L_n^2(G)$, the original situation applies again. For each $G \in \underline{G}_{n+1}$, $F_{n+1}$ is an $L_{n+1}(G)$-structure and $G\text{-}1(L_{n+1}(G))\text{-}F_{n+1}$, while different $L_{n+1}(G)$'s have disjoint sets of individual constants.

This construction yields a set of elementary chains, each beginning with a member G of some $\underline{G}_n$, as well as the chain $F_0, F_1, F_2, \ldots$ Call the limit of the last chain F and that of a chain starting with G, C(G). Then $C(G) \models \phi$, since $G \models \phi$. As in previous proofs, define a p-morphism $f_G$ from the generated subframe of C(G) consisting of the interpretations of the constants in the language of C(G), onto a generated subframe of F. (It should be clear from the construction that this is possible.) If C is the disjoint union of the C(G)'s, then $C \models \phi$, for $\phi$ is preserved under disjoint unions. The union of the p-morphisms $f_G$ is a p-morphism from a generated subframe C' of C onto F. $\phi$ is preserved under generated subframes, so

$C' \models \phi$, and $\phi$ is preserved under p-morphisms, so $F \models \phi$. It follows that $F_0 \models \phi$ and $F_0^1 \models \phi$. QED.

In the preservation result involving disjoint unions restricted quantifiers of the form $(\forall y)(Ryt \rightarrow$ were mentioned. This motivates the formulation of a number of similar results for tense-logical formulas, i.e., modal formulas with restricted quantifiers of this kind as well.

6.22 Definition

If $F = \langle W, R\rangle$ is a frame and $w \in W$, then $\overline{TC}(F, w)$ is the smallest subframe $\langle W_1, R_1\rangle$ of F with a domain satisfying $w \in W_1$ and $(\forall w \in W_1)(\forall v \in W)((Rwv \vee Rvw) \Rightarrow v \in W_1)$.

It is easy to see that any frame F is (isomorphic to) a disjoint union of subframes of the form $\overline{TC}(F, w)$, called the components of F.

6.23 Definition

An $L_0$-sentence $\phi$ is invariant for disjoint unions if, for all sets $\{F_i \mid i \in I\}$ of frames, $\bigoplus \{F_i \mid i \in I\} \models \phi$ iff $(\forall i \in I)F_i \models \phi$.

6.24 Definition

A $\bar{p}$-morphism from a frame $F_1$ onto a frame $F_2$ is a p-morphism from $F_1$ onto $F_2$ satisfying the additional property $(\forall w \in W_1)(\forall v \in W_2)(R_2vf(w) \Rightarrow (\exists u \in W_1)(Ruw \,\&\, f(u) = v))$.

6.25 Definition

An $L_0$-sentence $\phi$ is preserved under $\bar{p}$-morphisms if, for all $\bar{p}$-morphisms f from a frame $F_1$ onto a frame $F_2$, $F_1 \models \phi$ only if $F_2 \models \phi$.

6.26 Definition

$\vec{\forall}$ stands for a restricted universal quantifier of the form $(\forall y)(Rty \rightarrow$.

$\overleftarrow{\forall}$ stands for a restricted universal quantifier of the form $(\forall y)(Ryt \rightarrow$.

$\vec{\exists}$ stands for a restricted existential quantifier of the form $(\exists y)(Rty \wedge$.

$\overleftarrow{\exists}$ stands for a restricted existential quantifier of the form $(\exists y)(Ryt \wedge$.

Reviewing our previous results concerning $L_0$-sentences now using this notation yields:

| (preserved under) | (syntactic form) |
|---|---|
| p-morphisms | atomic formulas, $\bot$, $\wedge$, $\vee$, $\forall$, $\exists$, $\vec{\forall}$ |
| generated subframes | (negations of) atomic formulas, $\wedge$, $\vee$, $\forall$, $\vec{\exists}$ |
| ? | $\forall x$: (negations of) atomic formulas, $\wedge$, $\vee$, $\exists$, $\vec{\forall}$, $\overleftarrow{\forall}$. |
| p-morphisms and generated subframes and disjoint unions | $\forall x$: atomic formulas, $\bot$, $\wedge$, $\vee$, $\vec{\exists}$, $\vec{\forall}$. |

We will now add to these:

| | |
|---|---|
| $\bar{p}$-morphisms | atomic formulas, $\bot$, $\wedge$, $\vee$, $\forall$, $\exists$, $\overrightarrow{\forall}$, $\overleftarrow{\forall}$. |
| (invariant for) disjoint unions | $\forall x$: atomic formulas, $\neg$, $\wedge$, $\vee$, $\overrightarrow{\forall}$, $\overleftarrow{\forall}$, $\overrightarrow{\exists}$, $\overleftarrow{\exists}$. |
| $\bar{p}$-morphisms and (invariant for) disjoint unions | $\forall x$: atomic formulas, $\bot$, $\wedge$, $\vee$, $\overrightarrow{\forall}$, $\overleftarrow{\forall}$, $\overrightarrow{\exists}$, $\overleftarrow{\exists}$. |

More precisely,

6.27 <u>Definition</u>

If L is a first-order language obtained from $L_0$ by adding a (possibly empty) set of individual constants, then

<u>RF5(L)</u> is the set of formulas constructed from atomic formulas using $\neg$, $\wedge$, $\vee$, $\overrightarrow{\forall}$, $\overleftarrow{\forall}$, $\overrightarrow{\exists}$ and $\overleftarrow{\exists}$,

<u>RF6(L)</u> is the set of formulas constructed from atomic formulas and $\bot$, using $\wedge$, $\vee$, $\forall$, $\exists$, $\overrightarrow{\forall}$ and $\overleftarrow{\forall}$,

<u>RF7(L)</u> is the set of formulas constructed from atomic formulas and $\bot$, using $\wedge$, $\vee$, $\overrightarrow{\forall}$, $\overleftarrow{\forall}$, $\overrightarrow{\exists}$ and $\overleftarrow{\exists}$.

6.28 <u>Theorem</u>

An $L_0$-sentence is invariant for disjoint unions iff it is equivalent to a sentence of the form $(\forall x)\phi$, where $\phi$ is in RF5($L_0$)

An $L_0$-sentence is preserved under $\bar{p}$-morphisms iff it is equivalent to a sentence in RF6($L_0$)

An $L_0$-sentence is invariant for disjoint unions and preserved under $\bar{p}$-morphisms iff it is equivalent to a sentence of the form $(\forall x)\phi$, where $\phi$ is in RF7($L_0$).

Proof: Only a sketch of a proof will be given, and that for the first assertion only. The second one is proved almost like theorem 6.19 and the third follows by a combination of the arguments for the first two.

One direction is easy, so consider the other one, and let $\phi$ be an $L_0$-sentence invariant for disjoint unions. Let $\bar{5}(\phi) = \{(\forall x)\psi \mid \psi \in RF5(L_0)$ with the one free variable x and $\phi \models (\forall x)\psi\}$. We shall show that $\bar{5}(\phi) \models \phi$, which yields the required conclusion.

Let $F_0^1 \models \bar{5}(\phi)$. Write $F_0^1$ as a disjoint union of its components in some way. E.g., $F_0^1 = \bigoplus \{F_{0w}^1 \mid w \in I\}$, where each $F_{0w}^1$ is of the form $\overline{TC}(F_0^1, w)$ for a w in the domain of $F_0^1$. Consider any $F_{0w}^1$. Add a constant $\underline{w}$ to $L_0$ to obtain $L_{\underline{w}}$ and expand $F_{0w}^1$ to $F_{0w}$ by interpreting $\underline{w}$ as w. Doing this for all $w \in I$ yields $F_0 = \bigoplus \{F_{0w} \mid w \in I\}$. Each finite subset of $\{\psi \mid \psi$ is a sentence in $RF5(L_{\underline{w}})$ such that $F_0 \models \psi\} \cup \{\phi\}$ has a model, and so the whole set has a model $G_{\underline{w}}$. Defining $\underline{G}_0$ as the set of all $G_{\underline{w}}$'s obtained in this way and $L_0(G_{\underline{w}})$ as $L_{\underline{w}}$, the following situation arises:

For each $G \in \underline{G}_0$, $G$-$5(L_0(G))$-$F_0$ (where G-5(L)-F has the by now familiar meaning),

For different G's in $\underline{G}_0$, the languages $L_0(G)$ have disjoint sets of individual constants,

All constants from $L_0(G)$ are interpreted in one component of $F_0$, in which no interpretations of constants from different languages $L_0(G')$ occur.

The general construction starts from this situation, but now with subscripts n instead of 0.

For each $G \in \underline{G}_n$, add constants $\underline{w}$ to $L_n(G)$ for those w's in the domain of G which satisfy $G \models Rcx \vee Rxc\,[w]$ for some $L_n(G)$-constant c. This yields $L_n^1(G)$ and G is expanded to an $L_n^1(G)$-structure $G^1$ by interpreting each $\underline{w}$ as w. (Take different $\underline{w}$'s for elements from different G's, so as to keep the languages disjoint.) Each finite subset of $\bigcup_{G \in \underline{G}_n} \{\psi \mid \psi$ is a sentence in $RF5(L_n^1(G))$ such that $G^1 \models \psi\}$ has a model which is an expansion of $F_n$, so the whole set has a model $F_n^1$ satisfying, for each $G \in \underline{G}_n$,

$$F_n \prec_{L_n(G)} F_n^1$$

$$G_n^1\text{-}5(L_n^1(G))\text{-}F_n^1$$

all constants of $L_n^1(G)$ are interpreted in one component of $F_n^1$, viz. that where those of $L_n(G)$ were interpreted.

For the other direction, take constants $\underline{w}$ for those elements w in the domain of $F_n^1$ which satisfy $F_n^1 \models Rcx \vee Rxc\,[w]$ for some $L_n^1(G)$-constant c, to obtain $L_n^2(G)$. Also take, for each component of $F_n^1$ in which no interpretation of any constant occurs as yet, an element w in that component and a corresponding constant $\underline{w}$ to obtain new languages $L_{\underline{w}} = L_0 \cup \{\underline{w}\}$. Expand $F_n^1$ to $F_{n+1}$ by interpreting each $\underline{w}$ as w. Repeat the procedure of the beginning of this proof with respect to the last-mentioned $\underline{w}$'s. For the first-mentioned, consider $\{\psi \mid \psi$ is a sentence in $RF5(L_n^2(G))$ such that $F_{n+1} \models \psi\}$. Each finite subset of this set has a model which is an expansion of $G^1$, and therefore the whole set has a model $G^2$ satisfying

$$G^1 \prec_{L_n^1(G)} G^2$$

$$G^2\text{-}5(L_{n+1}(G^2))\text{-}F_{n+1}, \text{ where } L_{n+1}(G^2) =_{def} L_n^2(G).$$

Finally, define $\underline{G}_{n+1}$ as the union of $\{G^2 \mid G \in \underline{G}_n\}$ and the set of $G_{\underline{w}}$'s obtained for the new languages $L_{\underline{w}}$.

This procedure yields elementary chains starting from G's in some $\underline{G}_n$, with chain limits C(G). The constants interpreted in C(G) form a component C'(G) of it. From the construction it can be seen that the disjoint union $G^*$ of these components is isomorphic to the limit F of the chain $F_0$, $F_1$, $F_2$,... Now G $\models \phi$, for each G in each $\underline{G}_n$, and so C(G) $\models \phi$. C'(G) $\models \phi$, by the invariance of $\phi$ for disjoint unions, and, for the same reason, $G^* \models \phi$. It follows that F $\models \phi$ and $F_0^1 \models \phi$. QED.

Tense-logical formulas are invariant for disjoint unions and preserved under $\bar{p}$-morphisms, so theorem 6.28 is applicable to $L_0$-sentences defined by tense-logical formulas.

No preservation result has been given for disjoint unions, so an obvious open question remains. The same question is open for ultrafilter extensions. This chapter ends with the few results we have on this subject.

First recall that a frame F is isomorphic to a subframe of its ultrafilter extension $F^*$. The reason is that for $w^* = \{X \subseteq W \mid w \in X\}$ and $v^* = \{X \subseteq W \mid v \in X\}$, where w and v $\in$ W, $R^*w^*v^*$ iff Rwv and $w^* = v^*$ iff w = v. (The second assertion is trivial and the first follows easily using the definition of $R^*$.) So, for all practical purposes, we may consider F as a subframe of $F^*$. This implies that existential $L_0$-sentences are preserved under ultrafilter extensions, but such a result is hardly exciting. A little more information is provided by lemma 6.30 below.

6.29 <u>Definition</u>

For a fixed variable u, the <u>r(u)-formulas</u> are the $L_0$-formulas obtained by starting with atomic formulas of the forms Rux, Rxu, u = x and x = u, where x is a variable different from u, and applying $\neg$, $\wedge$ and two kinds of restricted existential quantification, forming $(\exists y)(Ruy \wedge [y/u]\phi)$ or $(\exists y)(Ryu \wedge [y/u]\phi)$ from $\phi$, if $y$ does not occur in $\phi$.

E.g., for a variable $x$ different from $u$ and $i \in \mathbb{N}$, the formula $R^i ux$ is an $r(u)$-formula (cf. definition 4.10).

6.30 Lemma

If $\phi = \phi(u, x_1, \ldots, x_k)$ is an $r(u)$-formula, then, for any frame $F = \langle W, R\rangle$, any $w_1, \ldots, w_k \in W$ and $U \in W^*$,
$F^* \models \phi\,[U, w_1^*, \ldots, w_k^*] \Leftrightarrow \{v \in W \mid F \models \phi\,[v, w_1, \ldots, w_k]\} \in U$.

Proof: We argue by induction on the complexity of $\phi$.

$\phi$ is $Rux$: $F^* \models Rux\,[U, w_1^*]$ iff $R^*Uw_1^*$ iff $\{v \in W \mid Rvw_1\} \in U$ (by an easy deduction) iff $\{v \in W \mid F \models Rux\,[v, w_1]\} \in U$.

$\phi$ is $Rxu$: this is proved analogously, using the fact that $R^*w_1^*U$ iff $\{v \in W \mid Rw_1v\} \in U$.

$\phi$ is $u = x$: $F^* \models u = x\,[U, w_1^*]$ iff $U = w_1^*$ iff $\{w\} \in U$ iff $\{v \in W \mid F \models u = x\,[v, w_1]\} \in U$.

$\phi$ is $x = u$: this is proved analogously.

$\phi$ is $\neg\psi$ or $\phi_1 \wedge \phi_2$: these cases follow by standard arguments, using the characteristic properties of ultrafilters.

$\phi$ is $(\exists y)(Ruy \wedge [y/u]\phi)$: $F^* \models (\exists y)(Ruy \wedge [y/u]\phi)\,[U, w_1^*, \ldots, w_k^*]$ iff, for some $V \in W^*$, $R^*UV$ and $F^* \models \phi\,[V, w_1^*, \ldots, w_k^*]$ iff (by the induction hypothesis), for some $V \in W^*$, $R^*UV$ and $\{v \in W \mid F \models \phi\,[v, w_1, \ldots, w_k]\} \in V$. Now apply the following general principle:

If $\phi = \phi(y, y_1, \ldots, y_k)$, then, for any $w_1, \ldots, w_k \in W$ and any $U \in W^*$,

$\{v \in W \mid (\exists z \in W)(Rvz \;\&\; F \models \phi\,[z, w_1,\ldots, w_k\,])\} \in U \Leftrightarrow$ for some $V \in W^*$, $R^*UV$ and $\{v \in W \mid F \models \phi\,[v, w_1,\ldots, w_k\,]\} \in V$.

(The standard deduction leading to this principle is omitted. Use the fundamental theorem on ultrafilters.)

The list of equivalences continues with

$\{v \in W \mid (\exists z \in W)(Rvz \;\&\; F \models \phi\,[z, w_1,\ldots, w_k\,])\} \in U$, i.e.,

$\{v \in W \mid F \models (\exists y)(Ruy \wedge [y/u]\phi)\,[v, w_1,\ldots, w_k\,]\} \in U$.

$\phi$ is $(\exists y)(Ryu \wedge [y/u]\phi)$: this is proved analogously, but now using the principle:

If $\phi = \phi(y, y_1,\ldots, y_k)$, then, for any $w_1,\ldots, w_k \in W$ and any $U \in W^*$, $\{v \in W \mid (\exists z \in W)(Rzv \;\&\; F \models \phi\,[z, w_1,\ldots, w_k\,])\} \in U \Leftrightarrow$ for some $V \in W^*$, $R^*VU$ and $\{v \in W \mid F \models \phi\,[v, w_1,\ldots, w_k\,]\} \in V$. QED.

6.31 <u>Corollary</u>

If $\phi = \phi(u, x_1,\ldots, x_k)$ is an r(u)-formula, then, for any frame $F = \langle W, R\rangle$ and any $w, w_1,\ldots, w_k \in W$,

$F^* \models \phi\,[w^*, w_1^*,\ldots, w_k^*\,] \Leftrightarrow F \models \phi\,[w, w_1,\ldots, w_k]$ .

<u>Proof</u>: By lemma 6.30, $F^* \models \phi\,[w^*, w_1^*,\ldots, w_k^*\,]$ iff

$\{v \in W \mid F \models \phi\,[v, w_1,\ldots, w_k\,]\} \in w^*$ iff

$w \in \{v \in W \mid F \models \phi\,[v, w_1,\ldots, w_k\,]\}$ iff $F \models \phi\,[w, w_1,\ldots, w_k\,]$. QED.

The corollary implies that any sentence obtained from an r(u)-formula by existential quantification is preserved under ultrafilter extensions, which extends our result about existential formulas. Yet this result does not exhaust the class of sentences preserved under ultrafilter extensions.

E.g., $(\forall x)Rxx$ and $(\forall x)(\forall y)Rxy$ have this property as well (although $(\forall x)\neg Rxx$ does not). Let us treat the first and the third formula. If $F \models (\forall x)Rxx$, $U \in W^*$ and X is any set in U, then $\{w \in W \mid (\exists v \in W)(Rwv \,\&\, v \in X)\} \in U$, for it contains X; and so $R^*UU$. But although $\langle IN,< \rangle \models (\forall x)\neg Rxx$, $\sim\langle IN,< \rangle^* \models (\forall x)\neg Rxx$. For any free ultrafilter U on IN and $X \in U$, $\{w \in IN \mid (\exists v \in IN)(w < v \,\&\, v \in X)\} = IN$, since X is infinite, and, since $IN \in U$, $R^*UU$.

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## II.1:A NOTE ON MODAL FORMULAE AND RELATIONAL PROPERTIES

Consider modal propositional formulae, constructed using proposition-letters, connectives and the modal operators $\Box$ and $\Diamond$. The semantic structures are frames, i.e., pairs $\langle W, R\rangle$ with $R \subseteq W^2$. Let F, V be variables ranging respectively over frames and functions from the set of proposition-letters into the powerset of W. Then the relation

$$w \models \alpha(\text{in } \langle F, V\rangle),\ w \in W,$$

may be defined, for arbitrary formulae $\alpha$, following the Kripke truth-definition. From this we may further define

$$F \models \alpha\,[w] \Leftrightarrow (\forall V)(w \models \alpha(\text{in } \langle F, V\rangle)),$$
$$F \models \alpha \Leftrightarrow (\forall w)_{w \in W}\,(F \models \alpha\,[w]).$$

Now, to every modal formula $\alpha$ there corresponds some property $P_\alpha$ of R. A particular example is obtained by considering the well-known translation of modal formulae into formulae of monadic second-order logic with a single binary first-order predicate. For these particular $P_\alpha$ we have

$$F \models \alpha\,[w] \Leftrightarrow F \models P_\alpha\,[w]$$

for all F and $w \in W$. These formulae $P_\alpha$ are, however, rather intractable and more convenient ones can often be found. An especially interesting case occurs when $P_\alpha$ may be taken to be some first-order formula. For example, it can be seen that

$$F \models (\Box p \rightarrow \Box\Box p)\,[w] \Leftrightarrow F \models (\forall y)(Rxy \rightarrow (\forall z)(Ryz \rightarrow Rxz))\,[w]$$

for all F and $w \in W$. It is customary to talk about a related correspondence, namely when for all F we have

$$F \models \alpha \Leftrightarrow F \models P_\alpha.$$

Note that this correspondence holds whenever the first one above holds. The main purpose of this note is to prove

Theorem 1

There is no first-order formula $\phi$ such that $F \models \phi \Leftrightarrow F \models \Box\Diamond p \rightarrow \Diamond\Box p$ for all F.

Proof:

Suppose such a formula $\phi$ does exist: we shall deduce a contradiction. Consider the frame $F = \langle W, R\rangle$ where

$W = \{q\} \cup \{q_n \mid n \in \omega\} \cup \{q_{n,i} \mid n \in \omega,\ i \in 2 = \{0, 1\}\} \cup \{r_f \mid f \in 2^\omega\}$,

$R = \{\langle q, q_n\rangle \mid n \in \omega\} \cup \{\langle q, r_f\rangle \mid f \in 2^\omega\} \cup \{\langle q_n, q_{n,i}\rangle \mid n \in \omega,\ i \in 2\}$
$\quad \cup \{\langle q_{n,i}, q_{n,i}\rangle \mid n \in \omega,\ i \in 2\} \cup \bigcup_{f\in 2^\omega} \{\langle r_f, q_{n,f(n)}\rangle \mid n \in \omega\}$.

Lemma A

$F \models \Box\Diamond p \rightarrow \Diamond\Box p$.

Proof:

It is easy to see that $F \models (\Box\Diamond p \rightarrow \Diamond\Box p)\,[w]$ for all $w \in W - \{q\}$; this hinges on the fact that for all $n \in \omega$, $i \in 2$:

$$q_{n,i} \models p \Leftrightarrow q_{n,i} \models \Diamond p \Leftrightarrow q_{n,i} \models \Box p.$$

Now, suppose that $q \models \Box\Diamond p$ (in $\langle F, V\rangle$) for some V. Then there is an $f \in 2^\omega$ such that $q_{n,f(n)} \models p$ for all $n \in \omega$. But then $r_f \models \Box p$ and so $q \models \Diamond\Box p$. QED.

It follows immediately from lemma A that $F \models \phi$. Hence, by the Löwenheim-Skolem theorem there is a countable elementary substructure $F' = \langle W', R'\rangle$ of F such that $q \in W'$ and $q_n, q_{n,0}, q_{n,1} \in W'$ for all $n \in \omega$.

Lemma B

$F' \not\models (\Box\Diamond p \rightarrow \Diamond\Box p)\,[q]$.

Proof:

Since W is uncountable and W' is countable, we can pick an element $r_g$ of W - W'. Define

$$V(p) = \{q_{n,g(n)} \mid n \in \omega\}.$$

First, we claim that $q \models \Box\Diamond p$ (in <F', V>). It is easy to see that $q_n \models \Diamond p$. In order to show that $r_f \models \Diamond p$, proceed as follows. For any $f \in 2^\omega$, define $\sim f(n) = 1 - f(n)$ for all $n \in \omega$. Then if $r_f \in W'$ it follows that $r_{\sim f} \in W'$. (This may be seen by exhibiting a first-order formula which forces it to be true (in F and so in F'). For example, let $A_1(x)$ express: Rqx and x has exactly two R-successors; $A_2(x)$ express: Rqx and not $A_1(x)$. Then take

$(\forall x)(A_2(x) \rightarrow (\exists y)(A_2(y) \ \& \ (\forall z)(A_1(z) \rightarrow (\forall u)((Rzu \ \& \ Rxu) \rightarrow \neg Ryu))))$.)

Hence, if $r_f \in W'$ then $f \neq \sim g$ because $\sim\sim g = g$ and $r_g \notin W'$. Therefore, $f(k) = g(k)$ for some k and so $r_f \models \Diamond p$ because $Rr_f q_{k,f(k)}$. This completes the proof of our first claim.

Secondly, we claim that $q \not\models \Diamond\Box p$ (in <F', V>). For, $Rq_n q_{n,\sim g(n)}$ and so $q_n \models \Diamond \neg p$, for all $n \in \omega$. Also, if $r_f \in W'$ then $f \neq g$ and so $f(k) \neq g(k)$ for some $k \in \omega$. Since $Rr_f q_{k,f(k)}$ we deduce that $r_f \models \Diamond \neg p$. This completes the proof of the second claim and hence the lemma. QED.

Finally, it follows immediately from the second lemma that $F' \not\models \phi$. This contradiction proves the theorem. QED.

In order to place the main theorem above in perspective, we conclude the paper with a positive result which we state without proof. Let $\Box^0 p = \Diamond^0 p = p$ and $\Box^{n+1} p = \Box\Box^n p$, $\Diamond^{n+1} p = \Diamond\Diamond^n p$ for all $n \in \omega$.

Theorem 2

For every modal formula $\phi$ of the form

$$\Diamond^k \Box^l p \rightarrow M_1 \ldots M_n p, \quad k, l, n \in \omega,$$

where $M_1, \ldots, M_n$ are modal operators (i.e., $\Box$ or $\Diamond$), there exists a first-order formula $\phi^*$ (in R and =) such that

$$F \models \phi^* [w] \Leftrightarrow F \models \phi [w]$$

for all F and $w \in W$.

(Our convention is that if $n = 0$ then $<M_1,\ldots,M_n>$ is empty.)
Although we shall not prove this result we shall describe how to obtain $\phi^*$ from $\phi$. To do this, we need some more notation.
Let $Q(\Diamond) = \exists$, $Q(\Box) = \forall$, $C(\Diamond) = \&$, $C(\Box) = \rightarrow$. If $u = <u_1,\ldots, u_n>$ is an n-tuple of variables, define $Q(M_1\ldots M_n, u, v)$ to be

(i) empty if $<M_1,\ldots, M_n>$ is empty,

(ii) $(Q(M_1)u_1)(Rvu_1C(M_1)$ if $n = 1$ and

(iii) $(Q(M_1)u_1)(Rvu_1C(M_1)\ \ldots(Q(M_n)u_n)(Ru_{n-1}u_nC(M_n)$ if $n > 1$.

Also define $R^0xy$ = '$x = y$'; $R^1xy = Rxy$; $R^2xy = (\exists z)(Rxz\ \&\ Rzy)$, etc.
Now let $y = <y_1,\ldots, y_k>$; $z = <z_1,\ldots, z_n>$.
Let v be x if $k = 0$ and $y_k$ otherwise; let w be x if $<M_1,\ldots, M_n>$ is empty and $z_n$ otherwise. Finally, define

$$\phi^* = Q(\Box^k, y, x)Q\ (\ M_1\ldots M_n, z, x)R^1vw)\ldots).$$

Remark

Following a suggestion from the referee we have discovered that a more general result than theorem 2 is contained in a paper by H. Sahlqvist: 'Completeness and correspondence in the first and second order semantics for modal logic', which is to appear in the *Proceedings of the Third Scandinavian Logic Symposium, Uppsala 1973*, North-Holland, Amsterdam.

Remarks (added in proof, July 1974)

1. In the proof of theorem 1 it is actually sufficient to take any countably infinite elementary substructure of F. For every such structure will contain q and infinitely many $q_n$'s. A counterexample for $\Box\Diamond p \rightarrow \Diamond\Box p$ can be found as before. So we have shown the Löwenheim-Skolem theorem fails for modal logic in the following sense: There exists an uncountable frame with no countably infinite elementary subframe satisfying the same modal formulas. Of course we were using a hybrid formulation since 'elementary substructure' was taken in its predicate-logical sense. If we try to define more purely modal notions, however, the situation becomes rather trivial. E.g.,

$$(F_1 = <W_1, R_1>, F_2 = <W_2, R_2>).$$

Define $F_1 \preceq_m F_2$ by (i) $W_1 \subseteq W_2$; (ii) $R_1 = R_2 \cap (W_1 \times W_1)$; (iii) for all modal $\phi$, $w \in W_1$, valuations $V(V_1 = V \upharpoonright W_1)$: $\langle F_1, V_1\rangle \models \phi[w]$ iff $\langle F_2, V\rangle \models \phi[w]$. It turns out that $F_1 \preceq_m F_2$ iff (i) $F_1 \subseteq F_2$ and (ii) for all $w \in W_1$, $v \in W_2$, $R_2wv$: $v \in W_1$.
It is obvious now how the Löwenheim-Skolem property fails with respect to this notion of elementary substructure.

2. Yet another modification of the proof of theorem 1 enables us to prove that $\Box\Diamond p \rightarrow \Diamond\Box p$ has no first-order equivalent on *countable* frames. For this one needs a set S of first-order formulas describing a point like q with R-successors of two kinds. (Those of the first kind have exactly two R-successors, those of the second kind share exactly one R-successor with every point of the first kind; also some additional requirements should be included.) Add formulas requiring the existence of n different points of the first kind for every n. Also add the purported first-order equivalent. It is clear that S is finitely satisfiable, so it should be satisfiable in a countably infinite domain. But a contradiction can be obtained through a counterexample like before.
On the other hand it is easy to see that for all *transitive* frames F, $w \in W$: $F \models \Box\Diamond p \rightarrow \Diamond\Box p\,[w] \leftrightarrow F \models (\exists y)(Rxy \wedge (\forall z)(Ryz \rightarrow z = y))\,[w]$.

## II.2: MODAL REDUCTION PRINCIPLES

### 1. INTRODUCTION

Modal reduction principles (MRPs) are modal formulas of the following form: $\vec{M}p \to \vec{N}p$, where $\vec{M}$, $\vec{N}$ are (possibly empty) sequences of modal operators (i.e. $\Box$ or $\Diamond$ ). Short-hand notation: $\vec{M}$, $\vec{N}$. We study a certain semantic correspondence between modal formulas and relational properties and obtain two main results.

(1) On transitive semantic structures every MRP corresponds to a first-order relational property.

(2) For the general case a syntactic criterion exists for distinguishing modal formulas with corresponding first-order properties from the others.

The first reference to a problem like this we found in [ 3 ], where it is shown that MRPs of the form $\Diamond^i$, $\vec{N}$ or $\vec{M}$, $\Box^i$ have corresponding first-order properties. ($\Diamond^0$ = –: the empty sequence. $\Diamond^1 = \Diamond$. $\Diamond^{i+1} = \Diamond\Diamond^i$. $\Box^i$: similarly.) An extension to the case $\Diamond^i\Box^j$, $\vec{N}$ was given in [ 1 ], as well as a proof that $\Box\Diamond$, $\Diamond\Box$ has no corresponding first-order property. The methods of the latter paper are used extensively here.

## 2. PRELIMINARIES

Two formal languages are used: one for modal propositional logic, the other for predicate-logic. Primitive signs for the first: $\neg$ (not), $\rightarrow$ (if...then), $\perp$ (falsum), $\Box$ (necessarily) - and $\wedge$ (and), $\vee$ (or), $\Diamond$ (possibly), T (verum) etc. are defined in the usual manner; for the second: $\neg$, $\rightarrow$, $\perp$, $\forall$ (for all) - and $\exists$ etc. defined in the usual way. Lower case Greek letters are used for formulas. The modal semantic structures are *frames*: ordered couples <W, R> with R a binary relation on W. (These may also be regarded as semantic structures for a predicate-logic with a single binary predicate-letter R.) Notation for frames: F (= <W, R>). Ordered couples <F, V> with F a frame and V a *valuation*, i.e. a function from the set of proposition-letters into the power-set of W, are called *models*. Notation for models: M (= <W, R, V>). The well-known Kripke truth-definition defines the notion $M \models \alpha [w]$, for a model M, $w \in W$, $\alpha$ a modal formula. We define a second notion: $F \models \alpha [w]$ by means of: for all V: $\langle F, V\rangle \models \alpha [w]$. $F \models \alpha$ is defined by: for all $w \in W$: $F \models \alpha [w]$.

The correspondence we consider is the following:

For all F, w: $F \models \phi_m [w] \Leftrightarrow F \models \phi_r [w]$, where $\phi_m$ is a modal formula, $\phi_r$ any formula (but mostly first-order) expressing a relational property of R. ($\phi_r$ has exactly one free variable.) Results about this correspondence are given in [ 2 ]. We state a few for future reference.

To every $\phi_m$ there corresponds a first-order relational property in the following sense. Let $ST(\phi_m)$ be the standard first-order translation of $\phi_m$.

(ST(p) = Px, P a one-place predicate-letter

$ST(\perp) = Px \wedge \neg Px$

$ST(\neg\alpha) = \neg ST(\alpha)$

$ST(\alpha \rightarrow \beta) = ST(\alpha) \rightarrow ST(\beta)$

$ST(\Box\alpha) = (\forall y)(Rxy \rightarrow [y/x]\ ST(\alpha))$, where y does not occur in $ST(\alpha)$.

The only free variable in $ST(\alpha)$ is x.)

Let M be a model, $M^r$ the predicate-logical structure corresponding to M in the obvious way. Then: $M \models \phi_m [w] \Leftrightarrow M^r \models ST(\phi_m) [w]$.

This gives us, for a $\phi_m$ with proposition-letters $p_1, \ldots, p_n$, $(M = \langle F, V \rangle)$: $F \models \phi_m [w] \Leftrightarrow F \models (\forall P_1) \ldots (\forall P_n)\ ST(\phi_m)\ [w]$.
The $\phi_r$ obtained in this way is second-order. Very often a first-order $\phi_r$ exists, however. The main (and open) problem is to characterize the class M1 of modal formulas with corresponding first-order properties. Not much is known about M1. [2] contains, amongst others, some closure-conditions (M1 is closed under $\wedge$, $\Box$, not under $\neg$, $\vee$, $\rightarrow$, $\Diamond$), but the main result seems to be essentially the following theorem (based directly on a theorem of Sahlqvist's. Cf. [4]).

Theorem 1

Every modal formula $\phi_m$ of the form $\alpha \rightarrow \beta$, with:

(1) $\alpha$ is constructed from $\bot$, T, p's and $\neg$p's using $\wedge$, $\vee$, $\Diamond$, $\Box$.

(2) no unnegated p occurs in $\alpha$ inside the scope of some $\Diamond$ which is itself inside the scope of some $\Box$.

(3) no unnegated p occurs in $\alpha$ in a subformula $\gamma \vee \delta$ which is inside the scope of some $\Box$.

(4) $\alpha(\beta)$ is monotone or antitone in its proposition-letters that do not occur in $\beta(\alpha)$.

(5) $\beta$ is monotone in all its proposition-letters that occur in $\alpha$ as well.

has a first-order corresponding $\phi_r$, obtainable from it in a constructive manner.

The limits of this theorem are given by the following three formulas: $\Box\Diamond p \rightarrow \Diamond\Box p$; $\Box(\Box p \vee p) \rightarrow \Diamond(\Box p \wedge p)$; $\Box(p \vee q) \rightarrow (\Diamond\Box p \vee \Diamond\Box q)$. They do not have corresponding first-order properties.

For our special formulas, the MRPs, we obtain a full solution of the characterization problem. We need a special case of theorem 1 for that. All MRPs of the form $\Diamond^i \Box^j, \vec{N}$ have corresponding first-order properties. This implies the same fact for those of the form $\vec{M}, \Box^i \Diamond^j$ by virtue of the inversion-principle IP: $F \models M_1 \ldots M_k\ p \rightarrow N_1 \ldots N_m\ p\ [w] \Leftrightarrow F \models \overline{N}_1 \ldots \overline{N}_m\ p \rightarrow \overline{M}_1 \ldots \overline{M}_k\ p\ [w]$, where $\overline{\Box} = \Diamond$ and $\overline{\Diamond} = \Box$.

## 3. MRPs ON TRANSITIVE FRAMES

We restrict attention to frames with transitive R. On these frames the following formulas hold:

(i) $\Box\Diamond\Diamond p \leftrightarrow \Box\Diamond p$ (and so, by IP, $\Diamond\Box\Box p \leftrightarrow \Diamond\Box p$)

(ii) $\Box\Diamond\Box\Diamond p \leftrightarrow \Box\Diamond p$ (and so, by IP, $\Diamond\Box\Diamond\Box p \leftrightarrow \Diamond\Box p$).

Proof: E.g. (ii): Let $\langle F, V\rangle \models \Box\Diamond\Box\Diamond p\,[w]$. Consider any y with Rwy. $\langle F, V\rangle \models \Diamond\Box\Diamond p\,[y]$. So there is a z with Ryz and $\langle F, V\rangle \models \Box\Diamond p\,[z]$. R transitive: Rwz, so $\langle F, V\rangle \models \Diamond\Box\Diamond p\,[z]$. Therefore there is a u with Rzu and so $\langle F, V\rangle \models \Diamond p\,[u]$, and finally a v with Ruv and $\langle F, V\rangle \models p\,[v]$. R transitive: Ryv, so $\langle F, V\rangle \models \Diamond p\,[y]$. So $\langle F, V\rangle \models \Box\Diamond p\,[w]$.
Conversely, let $\langle F, V\rangle \models \Box\Diamond p\,[w]$. Consider any y with Rwy. $\langle F, V\rangle \models \Diamond p\,[y]$ and, by the transitivity of R, $\langle F, V\rangle \models \Box\Diamond p\,[y]$. So there is a z with Ryz and, again by transitivity, $\langle F, V\rangle \models \Box\Diamond p\,[z]$. In other words: $\langle F, V\rangle \models \Box\Diamond\Box\Diamond p\,[w]$. QED.

Remark: Although our methods are purely semantic, facts like the above could be proved syntactically as well, using the minimal modal system with $\Box p \rightarrow \Box\Box p$ added.

The above allows us to restrict attention to sequences of modal operators of the following types: $\Diamond^i$, $\Diamond^i\Box$, $\Diamond^i\Box\Diamond$, $\Box^i$, $\Box^i\Diamond$, $\Box^i\Diamond\Box$.
The only relevant MRPs then (excluding those that have first-order equivalents by virtue of section 2) are:

1. $\Diamond^k\Box\Diamond$, $\Diamond^l\Box$
2. $\Diamond^k\Box\Diamond$, $\Diamond^l\Box\Diamond$
3. $\Diamond^k\Box\Diamond$, $\Box^l\Diamond\Box$
4. $\Box^k\Diamond$, $\Diamond^l\Box$
5. $\Box^k\Diamond$, $\Diamond^l\Box\Diamond$
   $\Box^k\Diamond$, $\Box^l\Diamond\Box$ (= type 1, by IP)
   $\Box^k\Diamond\Box$, $\Diamond^l\Box$ (= type 5, by IP)
6. $\Box^k\Diamond\Box$, $\Diamond^l\Box\Diamond$
   $\Box^k\Diamond\Box$, $\Box^l\Diamond\Box$ (= type 2, by IP)

Theorem 2

On transitive frames all MRPs have first-order equivalents.

Proof: We will exhibit first-order equivalents for each of the six types in the above list. Two preliminary results:

(a): If $\phi_m$ has no proposition-letters (so only $\perp$, T and operators may occur) then $F \models \phi_m [w] \Leftrightarrow F \models ST(\phi_m) [w]$. For these $\phi_m$'s $ST(\phi_m)$ is a first-order formula in R. So giving, for some $\phi_m$, an equivalent formula of this kind is as good as giving a first-order corresponding property.

(b):

(AC) Lemma 1

Let R be a transitive binary relation on a set X. If for all $x \in X$ there exists a $y \in X$ with $y \neq x$ and Rxy then two disjoint cofinal subsets of X exist. (Y cofinal in X means: for all $x \in X$ there exists a $y \in Y$ such that Rxy.)

Proof: Enumerate X as $\{x_0, \ldots, x_\gamma, \ldots\}$ using some initial ordinal number. Consider the set C of pairs <Y, Z> with (i) $Y \cap Z = \emptyset$, (ii) $Y, Z \subseteq X$, (iii) $\forall y \in Y\ \exists z \in Z$: Ryz; $\forall z \in Z\ \exists y \in Y$: Rzy, (iv) $\forall y \in Y\ \exists y' \in Y$: Ryy'; $\forall z \in Z\ \exists z' \in Z$: Rzz'. C is not empty: $\langle \emptyset, \emptyset \rangle \in C$. We apply Zorn's lemma to the binary relation $\leqq$ given by $\langle Y_1, Z_1 \rangle \leqq \langle Y_2, Z_2 \rangle$ iff $Y_1 \subseteq Y_2$; $Z_1 \subseteq Z_2$. Clearly every chain is bounded. So we are ready if we can show that for a $\leqq$-maximal <Y, Z>: $Y \cup Z = X$. Suppose <Y, Z> maximal, but $x \in X$, $x \notin Y$, $x \notin Z$.

(1) If Rxy for some $y \in Y$ we would have: $\langle Y \cup \{x\}, Z \rangle \in C$ (transitivity of R),

(2) Rxz for some $z \in Z$: similarly.

In both cases contradiction with the maximality of <Y, Z>.

(3) If these cases do not apply we construct $Y_1$, $Z_1$ such that $\langle Y \cup Y_1, Z \cup Z_1 \rangle \in C$. Put x in $Y_1$. Take the first $y_\gamma$ in X with $Rxx_\gamma$, $x \neq x$ . Put it in $Z_1$. ($x_\gamma \notin Y \cup Z$!) Repeat this. In the course of this process it may happen that e.g. u is put in $Z_1$, but all v with Ruv, $v \neq u$ have been put in $Y_1 \cup Z_1$ already at some earlier stage. We may

break off then. (For suppose $x_\gamma$ is the first in X with $Rux_\gamma$, $u \neq x_\gamma$. Assume $x_\gamma \in Z_1$. Since $x_\gamma \neq u$ the process did not stop there and we have $Rx_\gamma v$, with $v \in Y_1$. By transitivity Ruv.) $u \in Y_2$: similarly. If this does not happen the process may be stopped after $\omega$ steps. QED.

One more definition: $R^0xy$: $x = y$; $R^1xy$: $Rxy$; $R^2xy$: $(\exists z)(Rxz \wedge Rzy)$; etc.

We now list the first-order properties $\phi_r$ corresponding to the MRPs in our previous list. We may assume $k \geqq 1$, $l \geqq 1$.

| $\phi_m$ of type: | $\phi_r$: |
|---|---|
| 1. | $(\forall y)\ [\ (R^kxy \wedge (\forall z)(Ryz \rightarrow (\exists u)Rzu)) \rightarrow (\exists v)(R^lxv \wedge (\forall w)(Rvw \rightarrow v = w \wedge Ryv))\ ]$ |
| 2. | $k \geqq l$: T. $k < l$: $\Box^k\Diamond T \vee \Diamond^l \Box \bot$ |
| 3. | $(\forall y)\ [\ (R^kxy \wedge (\forall z)(Ryz \rightarrow (\exists u)Rzu)) \rightarrow (\forall v)(R^lxv \rightarrow (\exists w)(Rvw \wedge (\forall s)(Rws \rightarrow s = w \wedge Ryw)))]$ |
| 4. | $\Diamond^k \Box \bot \vee (\exists y)(R^lxy \wedge (\forall z)(Ryz \rightarrow z = y))$ |
| 5. | $\Diamond^k \Box \bot \vee \Diamond^l \Box \Diamond T$ |
| 6. | $\Diamond^k \Box \bot \vee \Diamond^l \Box \Diamond T$ |

We check the cases 6, 4 and 3.

6: Let $F \models \Box^k\Diamond\Box p \rightarrow \Diamond^l \Box\Diamond p\ [w]$. Take $V(p) = W$. Then either not $\langle F, V\rangle \models \Box^k \Diamond \Box p\ [w]$ (and so $F \models \Diamond\Box^k \bot\ [w]$) or $\langle F, V\rangle \models \Diamond^l\Box\Diamond p\ [w]$ (so $F \models \Diamond^l\Box\Diamond T\ [w]$.)

Conversely, suppose $F \models \Diamond^k \Box \bot\ [w]$. Then trivially $F \models \Box^k\Diamond\Box p \rightarrow \Diamond^l\Box\Diamond p\ [w]$. If $F \models \Diamond^l\Box\Diamond T\ [w]$ we reason as follows: we have a y with $R^lwy$ such that $F \models \Box\Diamond T\ [y]$. Suppose $\langle F, V\rangle \models \Box^k\Diamond\Box p\ [w]$. We will show that $\langle F, V\rangle \models \Diamond^l\Box\Diamond p\ [w]$. Consider any z with Ryz. $\langle F, V\rangle \models \Diamond T\ [z]$. Because of transitivity: $\langle F, V\rangle \models \Box\Diamond T\ [z]$. This implies $\langle F, V\rangle \models \Diamond^i T\ [z]$, for every $i \geqq 1$.

If $k \geqq l+1$: $\langle F, V\rangle \models \Box^{k-l}\Diamond\Box p\ [y]$. But also if $k < l+1$: $\langle F, V\rangle \models \Box\Diamond\Box p\ [y]$. ($\Box p \rightarrow \Box\Box p$.) So we have in any case $\langle F, V\rangle \models \Box^i\Diamond\Box p\ [z]$, where $i \geqq 1$. By the above we get: $\langle F, V\rangle \models \Diamond^i\Diamond\Box p\ [z]$, which reduces to: $\langle F, V\rangle \models \Diamond\Box p\ [z]$. Using $\langle F, V\rangle \models \Box\Diamond T\ [z]$ once more we get $\langle F, V\rangle \models \Diamond\Diamond p\ [z]$ and so finally $\langle F, V\rangle \models \Diamond p\ [z]$.

4: It is easy to check that if $\phi_r$ holds so does $\phi_m$. For the converse suppose not $F \models \phi_r [w]$. So $F \models \Box^k \Diamond T [w]$ and $F \models (\forall y)(R^1xy \rightarrow (\exists z)(Ryz \wedge z \neq y)) [w]$. We look for a V with $\langle F, V\rangle \models \Box^k \Diamond p [w]$ and $\langle F, V\rangle \models \Box^1 \Diamond \neg p [w]$. First we apply the lemma with $X = \{z \mid R^1xz\}$. Let Y be one of the cofinal sets obtained. $V(p) =_{def} Y \cup \{z \mid R^{k+1}xz \text{ and } z \notin X\}$.

3: Again $\phi_r$ obviously implies $\phi_m$. Now suppose not $F \models \phi_r [w]$. Then there are y, v with: $R^kwy$, $F \models \Box\Diamond T [y]$, $R^1wv$, and $F \models (\forall w)(Rzw \rightarrow (\exists s)(Rws \wedge (s \neq w \vee \neg Ryw))) [v]$. Clearly $F \models \Box\Diamond T [v]$. A V is required such that $\langle F, V\rangle \models \Diamond^k \Box \Diamond p [w]$, $\langle F, V\rangle \models \Diamond^1 \Box \Diamond \neg p [w]$. In fact we want: $\langle F, V\rangle \models \Box\Diamond p [y]$, $\langle F, V\rangle \models \Box\Diamond \neg p [v]$. Apply the lemma with $X = \{z \mid Rvz \text{ and } Ryz\}$. Let Y be one of the cofinal sets obtained. $V(p) =_{def} Y \cup \{z \mid R^2yz \text{ and } z \notin X\}$. QED.

Remarks:

(1) Not all modal formulas are equivalent to a first-order property on transitive frames. E.g. LF $(\Box(\Box p \rightarrow p) \rightarrow \Box p)$ expresses well-foundedness of the converse relation of R. More precisely: if R transitive then $F \models LF [w] \Leftrightarrow$ there is no $f \in W^\omega$ such that $f(0) = w$; $Rf(i)f(i+1)$, all $i \in \omega$. (In fact LF implies transitivity itself.)

(2) Theorem 2 shows that all MRPs have corresponding first-order properties on the basis of most well-known modal logics. For the characteristic axiom of S4: $\Box p \rightarrow \Box\Box p$ is equivalent to: $(\forall y)(Rxy \rightarrow (\forall z)(Ryz \rightarrow Rxz))$.

## 4. MRPs ON FRAMES WITH SUCCESSORS

We now consider frames with successors, i.e. $F \models (\forall x)(\exists y)Rxy$.

Definition: a *$\leq$-formula* is a MRP of the form $M_1 \ldots M_k,\ N_1 \ldots N_k$ with: $M_i = \Diamond$ implies $N_i = \Diamond$, for all i. We are going to prove

**Theorem 3**

On frames with successors the MRPs with corresponding first-order properties are exactly those of the types (1) $\leq$-formulas

(2) $\Diamond^i \Box^j, \vec{N}$

(3) $\vec{M}, \Box^i \Diamond^j$

Proof:

The formulas of the kinds described have corresponding first-order properties. For $\leq$-formulas are universally valid on frames with successors and the others are even in M1 (Cf. section 2). So we have to show that MRPs $\vec{M}, \vec{N}$ that are not $\leq$-formulas and contain an occurrence of $\Box\Diamond$ in $\vec{M}$, and one of $\Diamond\Box$ in $\vec{N}$ have no first-order equivalent on frames with successors. In order to do this we need the following

**Lemma 2**

MRPs of the following types have no first-order equivalent on frames with successors:

(1) $\Box\Diamond\vec{M},\ \Diamond\Box\vec{N}$.

(2) $\Box\Diamond\vec{M},\ \Diamond\Diamond\vec{N}$; $\Diamond\Box$ occurs in $\Diamond\Diamond\vec{N}$; $\vec{M}, \vec{N}$ is not a $\leq$-formula.

(3) $\Box\Diamond\vec{M},\ \Box\Diamond\vec{N}$; $\Diamond\Box$ occurs in $\Box\Diamond\vec{N}$; $\vec{M}, \vec{N}$ is not a $\leq$-formula.

(4) $\Box\Diamond\Diamond^i\Box^j,\ \Box\Box\vec{N}$; $\Diamond\Box$ occurs in $\vec{N}$.

(5) $\Diamond\vec{M},\ \Box\vec{N}$; $\Box\Diamond$ occurs in $\vec{M}$, $\Diamond\Box$ occurs in $\vec{N}$; $\vec{M}, \vec{N}$ a $\leq$-formula.

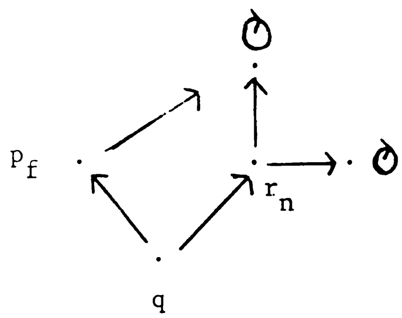
Proof:

For the proof of this lemma it is essential to know the method of proof in [ 1 ]. We will state the main steps in proving (1), but the remainder of the proof will be as short as possible.

(1): Consider the frame $F = \langle W, R\rangle$, given by

$W = \{q\} \cup \{r_n, r_{n.1}, r_{n.2} \mid n \in \omega\} \cup \{p_f \mid f \in \{1, 2\}^\omega\}$.

$R = \{\langle q, r_n\rangle, \langle r_n, r_{n.1}\rangle, \langle r_n, r_{n.2}\rangle, \langle r_{n.1}, r_{n.1}\rangle, \langle r_{n.2}, r_{n.2}\rangle \mid n \in \omega\} \cup \{\langle q, p_f\rangle \mid f \in \{1, 2\}^\omega\} \cup \bigcup_{f \in \{1,2\}^\omega} \{\langle p_f, r_{n,f(n)}\rangle \mid n \in \omega\}$.




a) $F \models \Box\Diamond\vec{M},\ \Diamond\Box\vec{N}\ [q]$.

W is uncountable. Take a countable elementary substructure F' of F with domain containing q, $r_n$, $r_{n.1}$, $r_{n.2}$, all $n \in \omega$.

b) $F' \not\models \Box\Diamond\vec{M},\ \Diamond\Box\vec{N}\ [q]$.

(Use a f belonging to a $p_f$ that is not in the domain of F' in order to find a suitable counter-example.)

If $\Box\Diamond\vec{M},\ \Diamond\Box\vec{N}$ were equivalent to a first-order formula it would have to hold in F' at q.

From now on we will just give the frames needed for applying this method of proof and some comment.

(2): Let $\Box\Diamond\vec{M},\ \Diamond\Diamond\vec{N}$ be $\Box\Diamond M_1 \ldots M_k,\ \Diamond\Diamond N_1 \ldots N_l$. (k= 0 means: $\vec{M}$ is empty.) For some i: $N_i = \Box$(i) . For some j: $M_j = \Diamond$, $N_j = \Box$ or $k \neq 1$ (ii). We associate trees $T_n$ with $\vec{M}$ as follows. Start with $r_n$, $r_{n.1}$, $r_{n.2}$. Let $Rr_n r_{n.1}$, $Rr_n r_{n.2}$. If $M_1 = \Box$, add $r_{n.1.1}$, $r_{n.2.1}$ and let $Rr_{n.1} r_{n.1.1}$, $Rr_{n.2} r_{n.2.1}$. If $M_1 = \Diamond$, add $r_{n.1.1}$, $r_{n.1.2}$, $r_{n.2.1}$, $r_{n.2.2}$ and let $Rr_{n.1} r_{n.1.1}$, $Rr_{n.1} r_{n.1.2}$, $Rr_{n.2} r_{n.2.1}$, $Rr_{n.2} r_{n.2.2}$. Etc. Let $\{e_1^n, \ldots, e_N^n\}$ be the set of end-points. $N \geq 2$. Now add a point $s_n$ and let $Res_n$, for every end-point e. In this way we have obtained a frame $F_n$ where $\Diamond\vec{M}p$ is true at $r_n$, if p is true at some end-point e. Also if p is true in only one end-point e $\Diamond N_1 \ldots N_l p$ will not be true in $r_n$, because of (ii). Also take a copy of this structure, say F, with minimal point r. Let i be the first-number for which $N_i = \Box$. We now describe the frame. (Identify $F_n$ with the set of its points for convenience.)

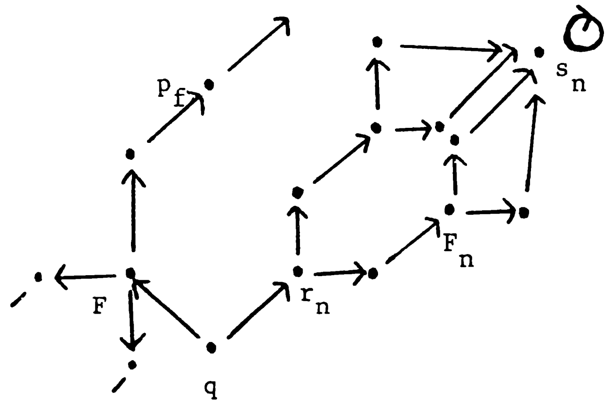
$$W = \{q, p_1, \ldots, p_i\} \cup \{r_n \mid n \in \omega\} \cup F \cup \bigcup_{n \in \omega} F_n \cup \{p_f, p^n_{f.i+1}, \ldots, p^n_{f.1} \mid f \in \{1, \ldots, N\}^\omega, n \in \omega\}.$$

(If i = 1 only $p_f$'s.) Identify r and $p_1$.

$R = \{\langle q, r_n\rangle \mid n \in \omega\} \cup$ the structure of the $F_n$'s and F as explained $\cup$
$\{\langle q, p_1\rangle, \langle p_1, p_2\rangle, \ldots, \langle p_{i-1}, p_i\rangle\} \cup \{\langle p_i, p_f\rangle \mid f \in \{1,\ldots, N\}^{\omega}\} \cup$
$\{\langle p_f, p^n_{f.i+1}\rangle, \ldots, \langle p^n_{f.l-1}, p^n_{f.l}\rangle \mid f \in \{1,\ldots, N\}^{\omega}, n \in \omega\} \cup$
$\bigcup_{f\in\{1,\ldots,N\}^{\omega}} \{\langle p^n_{f.l}, e^n_{f(n)}\rangle \mid n \in \omega\}$.

(F is needed for making $\Box\Diamond\vec{M}$ true at q in the countable elementary subframe.)

E.g. $\Box\Diamond\Box\Diamond$ , $\Diamond\Diamond\Box\Box$

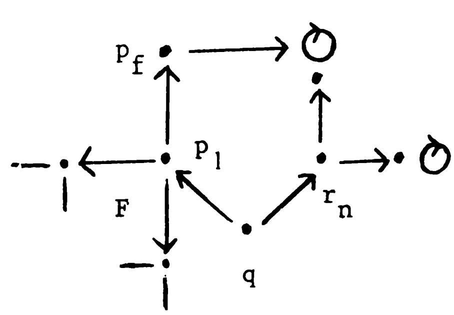



(3): Let $\Box\Diamond\vec{M}$, $\Box\Diamond\vec{N}$ be $\Box\Diamond M_1\ldots M_k$, $\Box\Diamond N_1\ldots N_l$. (k = 0 means: $\vec{M}$ empty.) Let i be the first number for which $N_i = \Box$.

$W = \{q, p_1,\ldots, p_i\} \cup \{r_n, r_{n.1}, r_{n.2} \mid n \in \omega\} \cup F$ (as constructed above)
$\cup \{p_f \mid f \in \{1, 2\}^{\omega}\}$. Identify $p_1$ and r.

$R = \{\langle q, r_n\rangle, \langle r_n, r_{n.1}\rangle, \langle r_n, r_{n.2}\rangle, \langle r_{n.1}, r_{n.1}\rangle, \langle r_{n.2}, r_{n.2}\rangle \mid n \in \omega\}$
$\cup \{\langle q, p_1\rangle,\ldots, \langle p_{i-1}, p_i\rangle\} \cup \{\langle p_i, p_f\rangle \mid f \in \{1, 2\}^{\omega}\} \cup$ the structure of F $\cup \bigcup_{f\in\{1,2\}^{\omega}} \{\langle p_f, r_{n,f(n)}\rangle \mid n \in \omega\}$.

E.g. $\Box\Diamond\Diamond\Box$, $\Box\Diamond\Box\Diamond$.



Suppose $\vec{M}, \vec{N}$ is a $\leq$-formula, but $\vec{M} \neq \vec{N}$. (We need this case in the next section.) Let $M_j$ be the first $\Box$ in $\vec{M}$ for which $N_j = \Diamond$. A minor modification in the given frame suffices. Instead of F use $\{u_1, \dots, u_j\}$ with $\langle p_1, u_1\rangle, \dots, \langle u_{j-1}, u_j\rangle \in R$. (This new frame has a point without a successor: $u_j$.)

(4): Let $\vec{N}$ be $\Box^k \Diamond^l \Diamond \Box N_1 \dots N_m$. ($m = 0$ means: $\vec{N}$ is empty.) So we have: $\Box \Diamond \Diamond^i \Box^j, \sqsubseteq \Box \Box^k \Diamond^l \Diamond \Box N_1 \dots N_m$.

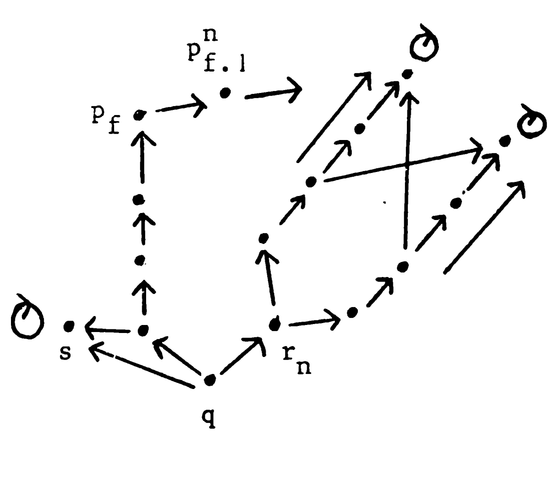
Case 1: $k < i$.

For every $n \in \omega$ we construct $U_n$ as follows: take $r_n, r^1_{n.1}, \dots, r^1_{n.i+1}, r^2_{n.1}, \dots, r^2_{n.i+1}$ and let R on $U_n$ be: $\{\langle r_n, r^1_{n.1}\rangle, \langle r^1_{n.1}, r^1_{n.2}\rangle, \dots, \langle r^1_{n.i}, r^1_{n.i+1}\rangle, \langle r^1_{n.i+1}, r^1_{n.i+1}\rangle, \langle r^1_{n.l+k}, r^1_{n.i+1}\rangle, \langle r^1_{n.l+k}, r^2_{n.i+1}\rangle, \langle r_n, r^2_{n.1}\rangle, \langle r^2_{n.1}, r^2_{n.2}\rangle, \dots, \langle r^2_{n.i}, r^2_{n.i+1}\rangle, \langle r^2_{n.i+1}, r^2_{n.i+1}\rangle, \langle r^2_{n.l+k}, r^2_{n.i+1}\rangle, \langle r^2_{n.l+k}, r^1_{n.i+1}\rangle\}$.

$W = \{q, p_1, \dots, p_{k+l+2}\} \cup \bigcup_{n\in\omega} U_n \cup \{p_f, p^n_{f.1}, \dots, p^n_{f.m} \mid f \in \{1, 2\}^\omega, n \in \omega\}$ (if $m = 0$ only $p_f$'s appear) $\cup \{s\}$.

$R = \{\langle q, r_n\rangle \mid n \in \omega\} \cup$ the structure of the $U_n$'s as described $\cup$ $\{\langle q, p_1\rangle, \langle p_1, p_2\rangle, \dots, \langle p_{k+l+1}, p_{k+l+2}\rangle\} \cup \{\langle p_{k+l+2}, p_f \mid f \in \{1, 2\}^\omega\}$ $\cup \{\langle p_f, p^n_{f.1}\rangle, \dots, \langle p^n_{f.m-1}, p^n_{f.m}\rangle \mid f \in \{1, 2\}^\omega, n \in \omega\} \cup$ $\bigcup_{f\in\{1,2\}^\omega} \{\langle p^n_{f.m}, r^{f(n)}_{n.i+1}\rangle \mid n \in \omega\} \cup \{\langle p_1, s\rangle, \langle s, s\rangle, \langle q, s\rangle\}$.

E.g. $\Box\Diamond\Diamond\Diamond\Diamond\Box, \sqsubseteq \Box \Box\Diamond\Box\Diamond$



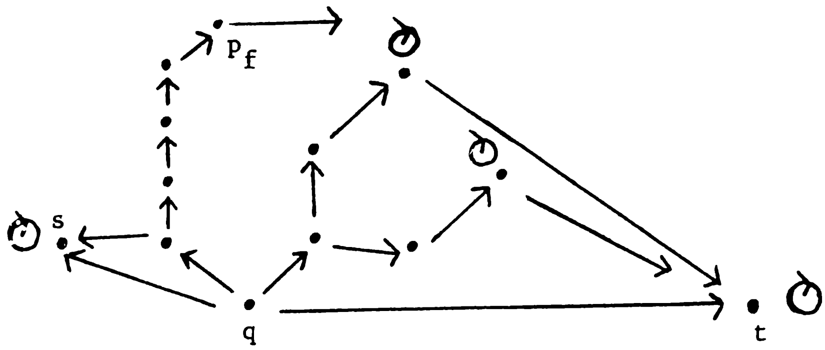
Case 2: $k \geqq i$.

$U_n$ is like before, but in the R-structure the ordered couples with $r^1_{n.1+k}$, $r^2_{n.1+k}$ must be dropped.

W = like before, but with a new element t added.

R = like before, but for the simpler structure of the $U_n$'s and the addition: $\{\langle q, t\rangle, \langle t, t\rangle, \langle r^1_{n.i+1}, t\rangle, \langle r^2_{n.i+1}, t\rangle \mid n \in \omega\}$.

E.g. $\Box\Diamond\Diamond\Box$, $\Box\Box\Box\Box\Diamond\Box$



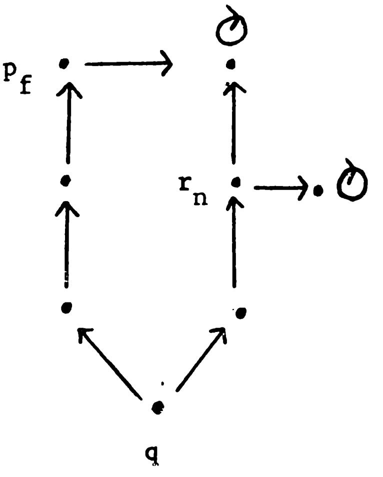

(5): Let $\Diamond\vec{M}$, $\Box\vec{N}$ be $\Diamond M_1, \ldots, M_k$, $\Box N_1 \ldots N_k$. $\vec{M}$, $\vec{N}$ a $\leqq$-formula. Let i be the first number such that $M_i = \Box$, $M_{i+1} = \Diamond$. Let j be the first number such that $N_j = \Diamond$, $N_{j+1} = \Box$. Suppose $i \leqq j$. (If this is not the case we use the inversion-principle IP and treat $\Diamond\bar{N}_1 \ldots \bar{N}_k$, $\Box\bar{M}_1 \ldots \bar{M}_1$ instead. $\bar{N}_1 \ldots \bar{N}_k$, $\bar{M}_i \ldots \bar{M}_k$ is a $\leqq$-formula!)

$W = \{q, r^1, \ldots, r^i, p_1, \ldots, p_j\} \cup \{r_n, r_{n.1}, r_{n.2} \mid n \in \omega\} \cup \{p_f \mid f \in \{1,2\}^\omega\}$.

$R = \{\langle q, r^1\rangle, \ldots, \langle r^{i-1}, r^i\rangle, \langle q, p_1\rangle, \ldots, \langle p_{j-1}, p_j\rangle\} \cup \{\langle r^i, r_n\rangle, \langle r_n, r_{n.1}\rangle, \langle r_n, r_{n.2}\rangle, \langle r_{n.1}, r_{n.1}\rangle, \langle r_{n.2}, r_{n.2}\rangle \mid n \in \omega\} \cup \{\langle p_j, p_f\rangle \mid f \in \{1, 2\}^\omega\} \cup \bigcup_{f \in \{1,2\}^\omega} \{\langle p_f, r_{n.f(n)}\rangle \mid n \in \omega\}$.

(The hard part here is to prove that the MRP considered holds in this frame. Use the fact that $\vec{M}$, $\vec{N}$ is a $\leqq$-formula and note that $N_{i+1} = \Diamond$, $M_{j+1} = \Box$.)

E.g. $\Diamond\Box\Diamond\Box$, $\Box\Box\Diamond\Box$




QED.

We can finish the proof of theorem 3 now. Start with $\vec{M}$, $\vec{N}$: not a $\leqq$-formula, $\Box \Diamond$ occurs in $\vec{M}$, $\Diamond \Box$ in $\vec{N}$. Let $\vec{M} = M_1 \ldots M_k$, $\vec{N} = N_1 \ldots N_l$. Let i be the smallest number for which $M_i = \Box$, $M_{i+1} = \Diamond$ or $N_i = \Diamond$, $N_{i+1} = \Box$ (★).

Case 1: $M_i \ldots M_k$, $N_i \ldots N_l$ is a $\leqq$-formula. Then for some $n < i$: $M_n = \Diamond$, $N_n = \Box$. Take the largest such n and consider $M_n \ldots M_k$, $N_n \ldots N_k$. ($l = k$!) We use lemma 2(5) now as follows. Take points $q_1, \ldots, q_{n-1}$, q with $Rq_1q_2, \ldots, Rq_{n-1}q$. Add these to the frame constructed in the proof of lemma 2(5). This gives a frame $F_0$ in which: $F_0 \models \vec{M}, \vec{N} [q_1] \Leftrightarrow F_0 \models M_n \ldots M_k, N_n \ldots N_k [q] \Leftrightarrow F \models M_n \ldots M_k$, $N_n \ldots N_k [q]$. Call this procedure 'fixing the first n-1 modalities'.

Case 2: $M_i \ldots M_k$, $N_i \ldots N_l$ is not a $\leqq$-formula.
(a): Suppose $M_i \ldots M_k = \Box \Diamond M_{i+2} \ldots M_k$. If $N_i N_{i+1} = \Diamond \Box, \Diamond\Diamond, \Box\Diamond, \Box\Box$ – while $M_i \ldots M_k$ is of the form described in lemma 2(4) – we may use the corresponding clauses of the lemma, fixing the first i-1 modalities. If $N_i N_{i+1} = \Box\Box$, but $M_i \ldots M_k$ not of the required form, we move on to the right. Let $i_1$ be the smallest number > i for which the situation (★) occurs and repeat the procedure.
(b): If $N_i \ldots N_l = \Diamond \Box N_{i+2} \ldots N_l$ and we are not in case (a), we act just like before, using the dual form of lemma 2 which we did not state. (It amounts to inverting $M_i \ldots M_k$, $N_i \ldots N_l$ to $\overline{N}_i \ldots \overline{N}_l$, $\overline{M}_i \ldots \overline{M}_k$.) QED.

Remark:
There is a second notion of correspondence: For all F: $F \models \phi_m \Leftrightarrow F \models \phi_r$. ($F \models \phi_m \Leftrightarrow_{def}$ For all $w \in W$: $F \models \phi_m [w]$.) Our results do not imply that formulas without first-order equivalents in our sense of the word have no first-order equivalents in this weaker sense. (Compare the remark at the end of section 5.)

The method of [ 1 ] as used in the preceding proofs allows one to prove the non-existence of first-order equivalents in the weaker sense, provided that $F \models \phi_m$ in the frame F given. Although this is true for some of the frames given it does not hold for all of them. Therefore the problem which formulas have no first-order equivalents in the second sense remains open.

## 5. MRPs ON ARBITRARY FRAMES

In view of the preceding results it now suffices to establish the behaviour of $\leqq$-formulas on arbitrary frames in order to solve the general problem mentioned in the introduction.

### Theorem 4

The $\leqq$-formulas with first-order corresponding properties are exactly those of the types

(1) $\vec{M}, \Box^i \Diamond^j$

(2) $\Diamond^i \Box^j, \vec{N}$

(3) $\Box^i \vec{M}, \overrightarrow{NM}$, where length $(\vec{N}) = i$.

(4) $\overrightarrow{NM}, \Diamond^i \vec{M}$, where length $(\vec{N}) = i$.

Proof: Formulas of type (3) are equivalent to $\neg \Box^i \vec{M}\, T \vee \vec{N}\, T$. For both $F \models \neg \Box^i \vec{M}\, T\, [w]$ and $F \models \vec{N}\, T\, [w]$ imply $F \models \Box^i \vec{M}, \overrightarrow{NM}\, [w]$. Conversely, let $F \models \Box^i \vec{M}\, T \wedge \neg \vec{N}\, T\, [w]$, and $V(p) = W$. Then <F, V> $\models \Box^i \vec{M} p\, [w]$ but not <F, V> $\models \overrightarrow{NM} p\, [w]$.

For the negative part we need:

### Lemma 3

$\leqq$-formulas $\vec{M}, \vec{N}$ with an occurrence of $\Box \Diamond$ in $\vec{M}$ and one of $\Diamond \Box$ in $\vec{N}$ have no corresponding first-order property if they are of one of the following types:

(1) $\Diamond \vec{O} \Box \Box \vec{Q}, \Diamond \vec{P} \Box \Diamond \vec{R}$, where length $(\vec{O})$ = length $(\vec{P})$.

(2) $\Box \vec{O} \Box \Box \vec{Q} \Diamond \Box \vec{S}, \Diamond \vec{P} \Box \Diamond \vec{R} \Diamond \Diamond \vec{T}$, where length $(\vec{O})$ = length $(\vec{P})$, length $(\vec{Q})$ = length $(\vec{R})$.

(3) $\Box \vec{O} \Box \Diamond^i \Box \vec{Q}, \Diamond \vec{P} \Box \Diamond^i \Diamond \vec{R}$, where length $(\vec{O})$ = length $(\vec{P})$, $i \neq 0$.

Remark: Using IP it turns out that the same holds for the types:

(1)' $\Box \vec{O} \Diamond \Box \vec{Q}, \Box \vec{P} \Diamond \Diamond \vec{R}$, where length $(\vec{O})$ = length $(\vec{P})$ and $\Diamond \Box$ occurs on the right-hand side.

(2)' $\Box \vec{O} \Diamond \Box \vec{Q} \Box \Box \vec{S}, \Diamond \vec{P} \Diamond \Diamond \vec{R} \Box \Diamond \vec{T}$, where length $(\vec{O})$ = length $(\vec{P})$, length $(\vec{Q})$ = length $(\vec{R})$.

(3)' $\Box \vec{O} \Diamond \Box^i \Box \vec{Q}, \Diamond \vec{P} \Diamond \Box^i \Diamond \vec{R}$, where length $(\vec{O})$ = length $(\vec{P})$, $i \neq 0$.

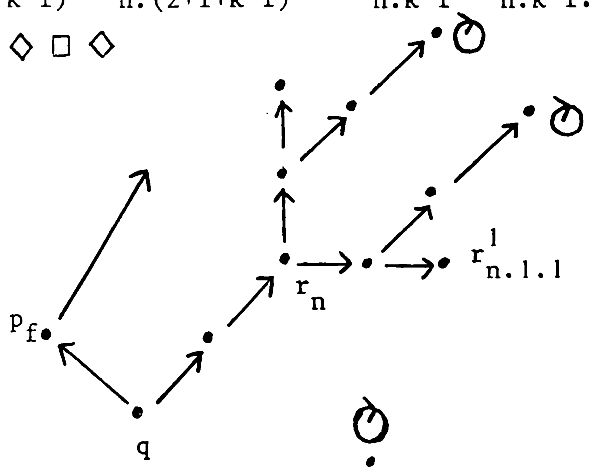
Proof: Again we only give the frames, the method being that of [1].
$\vec{O} = O_1 \ldots O_k$, $\vec{P} = P_1 \ldots P_k$, $\vec{Q} = Q_1 \ldots Q_l$, $\vec{R} = R_1 \ldots R_l$, $\vec{S} = S_1 \ldots S_m$, $\vec{T} = T_1 \ldots T_m$.

(1) Let i be the first number such that $O_i = \Box$, $O_{i+1} = \Diamond$ (a) or, if there is no such number, the first such that $Q_i = \Diamond$ (b). Let j be the first number such that $P_j = \Box$, $j = k + 1$ if no such number exists. We only treat case (a). The solution for case (b) is essentially the same (but easier).

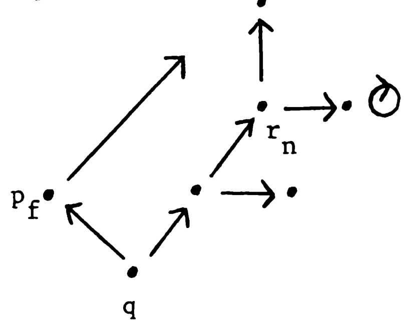
$W = \{q, r^1, \ldots, r^i, p_1, \ldots, p_{j-1}\}$ (if $j = 1$: no $p_i$'s) $\cup \{r_n, r^1_{n.1}, \ldots, r^1_{n.(2+1+k-i)}, r^2_{n.1}, \ldots, r^2_{n.(2+1+k-i)}, r^1_{n.k-i.1}, r^2_{n.k-i.1}\} \cup \{p_f \mid f \in \{1, 2\}^\omega\}$.

$R = \{\langle q, r^1\rangle, \ldots, \langle r^{i-1}, r^i\rangle, \langle q, p_1\rangle, \ldots, \langle p_{j-2}, p_{j-1}\rangle\} \cup$
$\{\langle p_{j-1}, p_f\rangle \mid f \in \{1, 2\}^\omega\}$ (if $j = 1$: take q) $\cup$
$\bigcup_{f \in \{1, 2\}^\omega} \{\langle p_f, r^{f(n)}_{n.(2+1+k-i)}\rangle \mid n \in \omega\} \cup \{\langle p_{j-1}, r_n\rangle, \langle r_n, r^1_{n.1}\rangle,$
$\langle r^1_{n.1}, r^1_{n.2}\rangle, \ldots, \langle r^1_{n.(1+1+k-i)}, r^1_{n.(2+1+k-i)}\rangle, \langle r^1_{n.(2+1+k-i)}, r^1_{n.(2+1+k-i)}\rangle,$
$\langle r^1_{n.k-i}, r^1_{n.k-i.1}\rangle, \langle r_n, r^2_{n.1}\rangle, \langle r^2_{n.1}, r^2_{n.2}\rangle, \ldots, \langle r^2_{n.(1+1+k-i)},$
$r^2_{n.(2+1+k-i)}\rangle, \langle r^2_{n.(2+1+k-i)}, r^2_{n.(2+1+k-i)}\rangle, \langle r^2_{n.k-i}, r^2_{n.k-i.1}\rangle \mid n \in \omega\}$.

E.g. (i) $\Diamond \Box \Diamond \Box \Box$, $\Diamond \Box \Diamond \Box \Diamond$

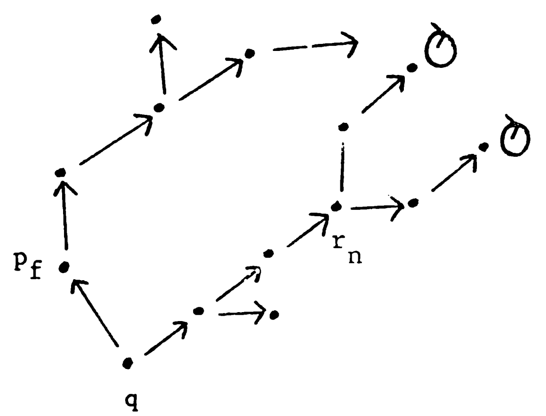


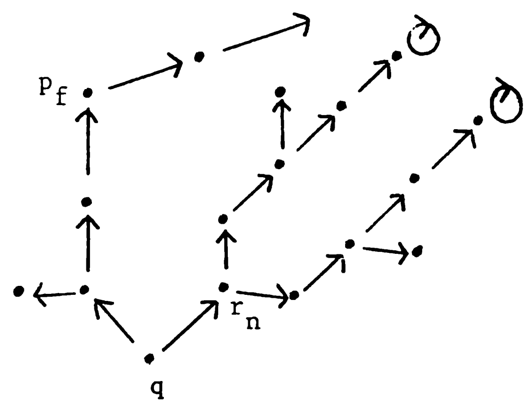

(ii) $\Diamond \Box \Box \Diamond$, $\Diamond \Box \Diamond \Diamond$



(2) We do not prove this case since the idea involved is essentially that of (1). The same tric of adding points without successors, but now at more places, works here as well. (Compare (3).)

E.g. (i) □□□◇□, ◇□ ◇◇◇ (ii) □◇ □□□ ,◇◇◇□◇



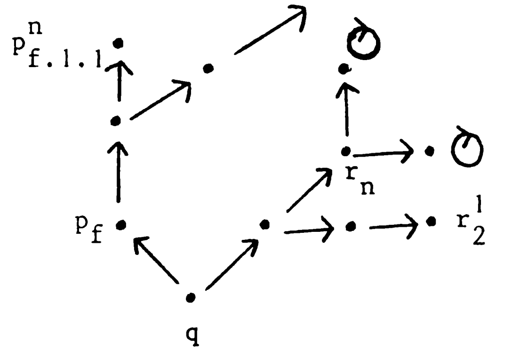




(3) Let j be the first number such that $P_j = \Box$, $j = k + 1$, if no such number exists. h is the number of modalities after the first occurrence of □ on the right-hand side.

$W = \{q, r^1, \ldots, r^{k+1}, r_1^{k+1}, \ldots, r_{i+1}^{k+1}, p_1, \ldots, p_{j-1}\}$ (if $j = 1$: no $p_i$'s) $\cup$ $\{p_f, p_{f.1}^n, \ldots, p_{f.h}^n, p_{h-1-1.1}^n \mid f \in \{1, 2\}^\omega, n \in \omega\} \cup \{r_n, r_{n.1}, r_{n.2} \mid n \in \omega\}$.

$R = \{\langle q, r^1\rangle, \langle r^1, r^2\rangle, \ldots, \langle r^k, r^{k+1}\rangle, \langle r^{k+1}, r_1^{k+1}\rangle, \langle r_1^{k+1}, r_2^{k+2}\rangle, \ldots, \langle r_i^{k+1}, r_{i+1}^{k+1}\rangle, \langle q, p_1\rangle, \langle p_1, p_2\rangle, \ldots, \langle p_{j-2}, p_{j-1}\rangle\} \cup$ $\{\langle r^{k+1}, r_n\rangle, \langle r_n, r_{n.1}\rangle, \langle r_n, r_{n.2}\rangle \mid n \in \omega\} \cup \{\langle p_{j-1}, p_f\rangle \mid f \in \{1, 2\}^\omega\}$ (if $j = 1$: take q) $\cup \{\langle p_f, p_{f.1}^n\rangle, \langle p_{f.1}^n, p_{f.2}^n\rangle, \ldots, \langle p_{f.h-1}^n, p_{f.h}^n\rangle, \langle p_{f.h}^n, r_{n.f(n)}\rangle \mid f \in \{1, 2\}^\omega, n \in \omega\} \cup \{\langle p_{f.h-1-1}^n, p_{f.h-1-1.1}^n\rangle \mid n \in \omega, f \in \{1, 2\}^\omega\}$.

E.g. $\Box\Box\Diamond\Box, \Diamond\Box\Diamond\Diamond$




QED.

Using the lemma in the same way as in the proof of theorem 2 (fixing modalities as needed) we can show that all $\leqq$-formulas of the form $\vec{M}\vec{T}, \vec{N}\vec{U}$ where length $(\vec{M})$ = length $(\vec{N})$ and $\vec{T}, \vec{U}$ is of one of the types described in the lemma, have no first-order equivalents.
The proof may be completed now by considering what $\leqq$-formulas $\vec{M}, \vec{N}$ with $\Box\Diamond$ occurring in $\vec{M}$ and $\Diamond\Box$ in $\vec{N}$, are not excluded by the above and the remark in the proof of lemma 2(3) of section 4 (again fixing irrelevant modalities if needed). It turns out that these can only be of the types mentioned in theorem 4.

Since $\Diamond\Box$ occurs in $\vec{N}$ there is a i with: $M_i = \Box$, $N_i = \Box$.
Since $\Box\Diamond$ occurs in $\vec{M}$ there is a j with: $M_j = \Diamond$, $N_j = \Diamond$.
Take the smallest such i, j and suppose i < j. (If i > j: use IP.)
We will not describe the elimination-process in full detail, but just give the types that are not excluded.

(a) $i \neq 0$. $\Box^i \Box \Box^j \Box\Diamond\vec{M}, \Diamond^i \Box\vec{N} \Box\Diamond\vec{M}$, with length $(\vec{N})$ = j.
(If we try to evade lemma 2(3) by having $\Box\Diamond$ on the right-hand side followed by only $\Diamond$'s it turns out that $\vec{M}$ should contain only $\Diamond$'s in order not to violate lemma 3(3).)

(b) $i \neq 0$. $\Box^i \Box \Box^j \Box\Diamond\Diamond^k, \Diamond^i \Box\vec{N} \Diamond\Diamond\Diamond^k$, length $(\vec{N})$ = j.

(c) $i \neq 0$. $\Box^i \Box \Box^j \Box\Diamond\Diamond^k \Box\vec{M}, \Diamond^i \Box\vec{N}\Diamond\Diamond\Diamond^k\Box\vec{M}$, length $(\vec{N})$ = j.

(No $\Diamond$ may be allowed instead of the last shown $\Box$ on the right-hand side because of lemma 3(2).)

(d) $\Box \Box^j \Box\Diamond\vec{M}, \Box\vec{N} \Box\Diamond\vec{M}$, length $(\vec{N})$ = j.

(Same comment as under (a).)

(e) $\Box\,\Box^j\,\Box\Diamond\Diamond^k$, $\Box\,\vec{N}\,\Diamond\Diamond\Diamond^k$, length $(\vec{N}) = j$.

(f) $\Box\,\Box^j\,\Box\Diamond\Diamond^k\,\Box\,\vec{M}$, $\Box\,\vec{N}\,\Diamond\Diamond\,\Diamond^k\,\Box\,\vec{M}$, length $(\vec{N}) = j$.

(Same comments as under (a).)

(g) $\Box\,\Box^j\,\Box\Diamond\Diamond^k\,\Box\,\vec{M}$, $\Box\,\vec{N}\,\Diamond\Diamond\Diamond^k\Diamond\,\vec{M}$ is excluded, if $\Diamond\Box$ occurs on the right-hand side. (Because of lemma 3(1).)

So the only case that is allowed would be a MRP of the type $\vec{0}$, $\Box^r\,\Diamond^s$. QED.

Corollary:

The MRPs with first-order corresponding properties are exactly those of the types:

(1) $\Diamond^i\,\Box^j$, $\vec{N}$

(2) $\vec{M}$, $\Box^i\,\Diamond^j$

(3) $\Box^i\vec{M}$, $\vec{N}\vec{M}$, where length $(\vec{N}) = i$.

(4) $\vec{N}\vec{M}$, $\Diamond^i\vec{M}$, where length $(\vec{N}) = i$.

Remark: (Cf. the remark at the end of section 4.)

Lemma 3(3) provides us with formulas $\phi_m$ for which no first-order $\phi_r$ exists with: $F \models \phi_m[w] \Leftrightarrow F \models \phi_r[w]$, all F, w, but that do have first-order equivalents in the weaker sense $F \models \phi_m \Leftrightarrow F \models \phi_r$, all F. E.g. $\Box\Box\Diamond\Box$, $\Diamond\Box\Diamond\Diamond$. This formula corresponds to $(\forall x)(\exists y)Rxy$ in the weaker sense. (Of course, if $\phi_m$ is equivalent to $\phi_r$ in the first sense it is equivalent to $(\forall x)\ \phi_r$ in the second sense.)

## 6. SOME USES OF MRPs

(1) Define the *length* $l(M)$ of a modal logic M as the smallest number n such that every sequence of modal operators $\vec{N}$ is equivalent in M to such a sequence of length $\leq n$, if such a number exists; $l(M) = \omega$, otherwise. We have e.g. $l(S5) = 1$, $l(S4.2) = 2$, $l(S4) = 3$. As for transitive frames: $l(\{\Box, \Box\Box\}) = \omega$. For $\Box^k$ is not reducible to a $\vec{M}$ of length $< k$. (Use a linear order of length k.)

MRPs are especially interesting if they serve to establish the length of a logic. Consider S5, with characteristic axioms $\Diamond\Box, \Box$ and $\Box, -$. A more natural way of obtaining a system with length 1 would be by using: $\Diamond\Box, \Box$; $\Box, \Diamond\Box$; $\Box\Box, \Box$; $\Box, \Box\Box$. In [ 2 ] it is shown, using corresponding first-order properties, that this logic can also be axiomatized as $\Diamond\Box, \Box$; $\Box, \Box\Box$; $\Diamond T$. This logic is weaker than S5.

Quite generally we have:

For all $n \in \omega$ there exists a modal logic $M_n$ such that $l(M_n) = n$.

Proof: Let $M_n$ have the characteristic axioms $\Diamond\vec{M}, \vec{M}$; $\vec{M}, \Diamond\vec{M}$; $\Box\vec{M}, \vec{M}$; $\vec{M}, \Box\vec{M}$ for all $\vec{M}$ of length n. Clearly $l(M_n) \leq n$. But not $l(M_n) < n$. For suppose $M_n$ implied $\vec{N}, \vec{O}$, where length $(\vec{N}) = n$, length $(\vec{O}) < n$. Consider the frame $F = \langle W, R\rangle$ with $W = \{1, \ldots, n+1\}$, $R = \{\langle i, i+1\rangle \mid 1 \leq i \leq n\} \cup \{\langle n+1, n+1\rangle\}$. $F \models M_n [1]$, but not $F \models \vec{M}, \vec{N} [1]$ (Let $V(p) = \{n+1\}$.).

$l(M) = l(N)$ does not imply that M and N are deductively equivalent. E.g. $l(\{\Box, \Box\Box; \Box\Box, \Box\}) = 3 = l(S4)$. (Another example was given above.)

(2) Define the *degree* of a modal formula as follows (degree $(\alpha) = d(\alpha)$): $d(p) = 0$; $d(\neg\alpha) = d(\alpha)$; $d(\alpha \to \beta) = \max(d(\alpha), d(\beta))$; $d(\Box\alpha) = d(\Diamond\alpha) = d(\alpha)$, if $\alpha$ is of the form $\Box\beta$ or $\Diamond\beta$; $= d(\alpha)+1$, otherwise.

In S5 there is a theorem about the existence of modal conjunctive normal forms. It states that every formula is reducible to a propositional compound of the types $\Diamond\alpha$, $\Box\alpha$, $\alpha$, where $\alpha$ is a propositional formula. In this case two reductions are performed at once: both length and degree are reduced to 1.

We can separate the two notions and concentrate on a reduction of the degree only. If we look at the form an inductive proof for this kind of assertion would have, we find that we need a principle of the form: $\Box(\Box p \vee \Diamond q \vee r) \leftrightarrow ?$. We study here $\Box(\Box p \vee \Diamond q \vee r) \leftrightarrow \Box\Box p \vee \Box\Diamond q \vee \Box r$. (Cf. [ 2 ], p. 55/6.) One direction of this is trivial so only $\Box(\Box p \vee \Diamond q \vee r) \rightarrow \Box\Box p \vee \Box\Diamond q \vee \Box r$ is relevant. By a general form of IP we may just as well consider $\Diamond\Diamond p \wedge \Diamond\Box q \wedge \Diamond r \rightarrow \Diamond(\Diamond p \wedge \Box q \wedge r)$. This is of the form described in theorem 1. It turns out that its corresponding first-order property is equivalent, after some simplification, to $(\forall y)(Rxy \rightarrow (\forall z)(Rxz \rightarrow (\forall u)(Rzu \rightarrow Ryu)))$. By a translation result from certain predicate-logical formulas to corresponding modal ones ( [ 2 ] also treats the problem, a converse to that of section 2, of determining which relational properties are expressible by means of modal formulas) this is seen to be an equivalent of the MRP $\Diamond\Box, \Box\Box$. So in the logic with $\Diamond\Box p \rightarrow \Box\Box p$ as its single characteristic axiom all formulas are reducible to formulas of degree 1. A general result like the one about length would seem to be provable as well.

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## II.3: MODAL FORMULAS ARE EITHER ELEMENTARY OR NOT ΣΔ-ELEMENTARY

In this paper we prove that for a set L of modal propositional formulas FR(L) (the class of all frames in which every formula of L holds) is elementary, Δ-elementary or not ΣΔ-elementary. For single modal formulas the second of these cases does not occur.

The model theoretic terminology and results used here are from [1]. (The underlying first-order language contains only one, binary, predicate-letter in addition to the identity symbol.) We presuppose familiarity with the usual notions and notations of propositional modal logic. A structure for our first-order language is called a frame. (So a frame is an ordered couple $\langle W, R\rangle$ with domain W and R a binary predicate on W, i.e. a subset of $W \times W$.) A valuation V on F is a function from the set of proposition-letters to the powerset of W. Using the well-known Kripke truth-definition V can be extended to a function from the set of all modal propositional formulas to the power set of W. A modal propositional formula $\phi$ holds in a frame F (= $\langle W, R\rangle$) if for all V on F: $V(\phi) = W$. Notation: $FR(\phi)$ for the class of all frames in which $\phi$ holds. For a set L of modal propositional formulas we define FR(L) as $\bigcap_{\phi \in L} FR(\phi)$. Obviously both FR(L) and cFR(L) (the complement of FR(L)) are closed under isomorphisms.

Using the standard translation which takes modal propositional formulas into formulas of a first-order language containing a single binary predicate-letter and unary predicate-letters (corresponding to the proposition-letters) we see that $FR(\phi)$ is definable by means of a universal second-order formula. This formula contains only unary predicate variables and a single, binary, first-order predicate constant. Consequently, $cFR(\phi)$ is definable using an existential second-order formula.

Let $\{F_i \mid i \in I\}$ be a set of frames. ($F_i = \langle W_i, R_i\rangle$.) The *disjoint union* of this set is $\langle \cup W_i', \cup R_i'\rangle$, where $W_i' =_{def} \{\langle i, w\rangle \mid w \in W_i\}$, $R_i' =_{def} \{\langle\langle i,w\rangle, \langle i, v\rangle\rangle \mid \langle w, v\rangle \in R_i\}$. A frame $F_1$ is a *generated subframe* of $F_2$ if (i) $W_1 \subseteq W_2$, (ii) $R_1 = R_2 \cap (W_1 \times W_1)$, (iii) for all u, $v \in W_2$: $u \in W_1$ & $R_2uv \Rightarrow v \in W_1$. We note that for all L FR(L) is closed under disjoint unions and generated subframes.

Lemma (R.I. Goldblatt)

Let $\{F_i \mid i \in I\}$ be a set of frames with disjoint union F, $G = \prod F_i/U$ an ultraproduct. Then G is isomorphic to a generated subframe of the ultrapower $F^I/U$.

Proof: The map from G to $F^I/U$ defined by $f/U \mapsto f'/U$, where $f'(i) =_{def} \langle i, f(i)\rangle$, is an isomorphism of G onto a generated subframe of $F^I/U$. It is easy to see that it is an isomorphism onto a subframe. Now consider g/U in this subframe with $F^I/U \models Rg/Uh/U$. By Łoś's theorem $\{i \in I \mid F \models Rg(i)h(i)\} \in U$. Since $F_i'$ is a generated subframe of F we see that $\{i \in I \mid h(i) \in W_i'\} \supseteq \{i \in I \mid F \models Rg(i)h(i)\} \cap \{i \in I \mid g(i) \in W_i'\}$. This last set is in U, and so is the first. QED.

Theorem

For all L: FR(L) $\Sigma\Delta$-elementary $\Rightarrow$ FR(L) $\Delta$-elementary. (1)

For all L: FR(L) $\Sigma$-elementary $\Rightarrow$ FR(L) elementary. (2)

For all : FR($\phi$) $\Delta$-elementary $\Rightarrow$ FR($\phi$) elementary. (3)

Proof:

(1) If FR(L) is $\Sigma\Delta$-elementary it is closed under elementary equivalents and therefore under ultrapowers. (By Łoś's theorem an ultrapower of F is elementarily equivalent to F.) But then it is also closed under ultraproducts, because of the lemma and the fact that FR(L) is closed under disjoint unions, generated subframes and isomorphisms. Finally a class closed under elementary equivalents and ultraproducts is $\Delta$-elementary.

(2) If FR(L) is $\Sigma$-elementary it is $\Sigma\Delta$-elementary, and therefore, by the above, $\Delta$-elementary. And a $\Sigma$- and $\Delta$-elementary class is elementary.

(3) (This argument is valid for all universal second order formulas.) Let $\Gamma$ be a set of first-order sentences such that for all F: $F \models \Gamma$ iff $F \models \phi$. Consider $\phi$ with the universal second-order quantifiers dropped as a first-order formula, with the predicate variables regarded as predicate-letters not occurring in $\Gamma$. Call it $\phi^o$. Then $\Gamma \models \phi^o$ and, by compactness, $\Delta \models \phi^o$, for some finite $\Delta \subseteq \Gamma$. Let $\delta$ be the conjunction of $\Delta$. Clearly we have for all F: $F \models \delta$ iff $F \models \phi$. QED.

Corollary

For all $\phi$ the following are equivalent:

(a) $FR(\phi)$ elementary

(b) $FR(\phi)$ closed under elementary equivalents

(c) $FR(\phi)$ closed under ultrapowers

Proof:

(a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c): trivial.

(c) $\Rightarrow$ (a): If $FR(\phi)$ is closed under ultrapowers it is closed under ultraproducts, by a reasoning similar to the above. Also, for every modal formula $\phi$ $cFR(\phi)$ is closed under ultraproducts. (Every class of frames definable by an existential second-order sentence has this property.) Since $FR(\phi)$ and $cFR(\phi)$ are closed under isomorphisms this implies that $FR(\phi)$ is elementary. QED.

Remark: Cf. [3], where it is proved that $FR(\phi)$ is elementary iff it is closed under ultraproducts.

The theorem is the best possible, since all possibilities that are not excluded by it do in fact occur.

Example:

(i) $FR(\Diamond\Box p \rightarrow \Box\Diamond p)$ is elementary. (Cf. [2])

(ii) $FR(\Box\Diamond p \rightarrow \Diamond\Box p)$ is not $\Sigma\Delta$-elementary. (In [2] two elementary equivalent frames are given, of which only one is in this class.)

(iii) Let $\phi_i =_{def} \Diamond^i \Box\bot \rightarrow \Box^{i+1}\bot$. $\bot =_{def} p \wedge \neg p$

$FR(\{\phi_i \mid i \geq 1\})$ is $\Delta$-elementary but not elementary (2).

(Proof: (1).

Definition:

$R^1xy = Rxy$, $R^2xy = (\exists z)(Rxz \wedge Rzy)$, etc.

Let $\psi_i =_{def} (\forall x)((\exists y)(R^ixy \wedge \neg(\exists z)Ryz) \rightarrow \neg(\exists y)R^{i+1}xy)$.

For all F: $F \models \phi_i$ iff $F \models \psi_i$.

(2). Let $F_i =_{def} \langle W_i, R_i\rangle$, $i \geq 1$, where $W_i = \{a, b_1, \ldots, b_i, c_1, \ldots, c_{i+1}\}$

$R_i = \{\langle a, b_1\rangle, \langle a, c_1\rangle, \langle b_j, b_k\rangle, \langle c_l, c_m\rangle \mid k = j+1, m = l+1, 1 \leq j, 1 \leq l, k \leq i, m \leq i+1\}$.

Claim: $F_i \in FR(\phi_j)$, all $j \neq i$.
$F_i \notin FR(\phi_i)$.

If $\{\phi_i \mid i \geqq 1\}$ were elementary we would have a first-order $\psi$ with $\{\psi_i \mid i \geqq 1\} \models \psi$, and $\psi \models \psi_i$, all $i \geqq 1$. Compactness: for some N $\{\psi_1, \ldots, \psi_N\} \models \psi$. Then $\{\psi_1, \ldots, \psi_N\} \models \psi_{N+1}$. But $F_{N+1}$ refutes this, and contradiction. QED. )

Three possibilities are excluded by the theorem:
(1) $FR(\phi)$ $\Delta$-elementary but not elementary.
(2) $FR(\phi)$ $\Sigma$-elementary but not elementary.
(3) $FR(\phi)$ $\Sigma\Delta$-elementary,but not $\Sigma$- (or $\Delta$-) elementary.

An intersection of $FR(\phi)$'s leads to case (1), as part (iii) of the above example shows. For the $\phi_i$'s mentioned there we also have that $\bigcup_{i \geqq 1} FR(\phi_i)$ satisfies (2).

(Proof: Suppose it is elementary. Then for some first-order $\psi$ $\psi_i \models \psi$, all $i \geqq 1$ and $\{\neg\psi_i \mid i \geqq 1\} \models \neg\psi$. Compactness: for some N $\{\neg\psi_1, \ldots, \neg\psi_N\} \models \neg\psi$. So $\{\neg\psi_1, \ldots, \neg\psi_N\} \models \neg\psi_{N+1}$. But this is refuted by the disjoint union of $F_1, \ldots, F_N$. QED. )

We have not been able to find an example of the third kind.

Remark: Our original proof of the statement in the title of this paper was much more complicated. The present proof is due to an idea of R.I. Goldblatt, expressed in the lemma.

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Summary

Modal Correspondence Theory has for its subject the connections between modal formulas and formulas of classical logical systems, both viewed as means of expressing relational properties. Two main questions are treated in this dissertation: which modal formulas are definable in first-order logic and which first-order formulas are definable by means of modal formulas? As for the first, it is shown that a modal formula is first-order definable if and only if it is preserved under ultrapowers. Moreover, two methods are developed, one using first-order substitutions for second-order quantifiers to show constructively that modal formulas satisfying certain syntactic conditions are first-order definable, the other using the Löwenheim-Skolem theorem to show that certain modal formulas are not first-order definable. For the case of modal reduction principles, a class of modal formulas to which most better-known modal axioms belong, these two methods yield a complete syntactic answer to the first question. As for the second question, there is a theorem by R.I. Goldblatt and S.K. Thomason about ΣΔ-elementary classes of relational structures, characterizing the modally definable ones in terms of closure under four algebraic operations. A new proof of this result is given here, as well as a series of preservation results for the algebraic operations it involves. From these results it follows that a first-order formula is modally definable only if it is equivalent to a "restricted positive" formula constructed from atomic formulas and the falsum (a constant denoting a fixed contradiction), using conjunction, disjunction and restricted quantifiers.

Samenvatting

De modale korrespondentietheorie bestudeert het verband tussen modale formules en formules van klassieke logische systemen, beide beschouwd als middel om eigenschappen van relaties uit te drukken. De twee belangrijkste vragen die in dit proefschrift worden behandeld zijn: welke modale formules zijn in de eerste-orde logika definieerbaar en welke eerste-orde formules zijn definieerbaar door middel van modale formules? Met betrekking tot de eerste vraag wordt er aangetoond dat een modale formule juist dan eerste-orde definieerbaar is als hij bewaard blijft onder ultramachten. Bovendien worden er twee methoden ontwikkeld, waarvan de ene (die gebruik maakt van eerste-orde substituties voor universele tweede-orde kwantoren) konstruktief bewijst dat modale formules die aan bepaalde syntaktische kondities voldoen eerste-orde definieerbaar zijn, terwijl de tweede (die berust op de Löwenheim-Skolem stelling) aantoont dat bepaalde modale formules juist niet eerste-orde definieerbaar zijn. Voor het speciale geval van de "modale reduktieprincipes", een klasse van formules waartoe de meeste bekende modale axioma's behoren, geven deze twee methoden samen een volledig, syntaktisch antwoord op de eerste vraag. Met betrekking tot de tweede vraag is er een stelling van R.I. Goldblatt en S.K. Thomason over ΣΔ-elementaire klassen van relationele strukturen, die de modaal definieerbare daaronder karakteriseert met behulp van afgeslotenheid onder een viertal algebraische bewerkingen. Er wordt een nieuw bewijs van dit resultaat gegeven, alsmede een aantal preservatieresultaten voor de vier vermelde algebraische bewerkingen. Uit deze preservatieresultaten valt af te leiden dat elke modaal definieerbare eerste-orde formule logisch equivalent is met een zg. "positieve beperkte" formule, d.w.z. een formule die gekonstrueerd is uit atomaire formules en het falsum (een konstante die een vaste kontradiktie aanduidt), met behulp van konjunktie, disjunktie en beperkte kwantoren.

S T E L L I N G E N

behorend bij het proefschrift

"Modal Correspondence Theory"

van

J.F.A.K. van Benthem

1. In ZF the Boolean prime ideal theorem is equivalent to each of the three following principles,

   (i) Alexander's lemma from topology

   (ii) Any inverse limit of a non-empty set of non-empty finite algebras is itself non-empty

   (iii) If D is a set of finite sets and E is a set such that, for each finite $F \subseteq D$, there is an S satisfying $f \cap S \in E$ for all $f \in F$, then an S exists such that, for all $d \in D$, $d \cap S \in E$.

((i): cf. [ 2 ], (ii): cf. [ 3 ], [ 9 ], (iii): cf. [ 3 ].)

2. In ZF the Hahn-Banach theorem is equivalent to the following theorem of J.L. Kelley's,

If B is a subalgebra of the Boolean algebra A, $\mu_0$ is a measure on B and p is a real-valued function on A satisfying

$p(a) \geq 0$ for all $a \in A$

if $a \leq b$, then $p(a) \leq p(b)$ for all $a, b \in A$

$p(a) + p(b) \geq p(a + b) + p(a \cdot b)$ for all $a, b \in A$

$\mu_0(b) \leqq p(b)$ for all $b \in B$,

then a measure $\mu$ on A exists such that $\mu \upharpoonright B = \mu_0$ and,

for all $a \in A$, $\mu(a) \leqq p(a)$.

(cf. [2], [10], [13].)

3. In ZF Koenig's lemma is equivalent to the axiom of choice for a countable set of finite sets. The remark found in some textbooks that this principle is needed to prove the completeness theorem for single formulas is misleading: completeness and even the Löwenheim-Skolem property for single formulas are provable in ZF.

(cf. [2], [4].)

4. The following generalization of E.W. Beth's definability theorem holds for monadic first-order logic, but not for any first-order logic containing at least one binary predicate constant,
If $\phi = \phi(P, Q_1, \ldots, Q_m)$, where $P, Q_1, \ldots, Q_m$ are predicate constants (P unary) such that any model $\mathfrak{A} = \langle A, P^*, Q_1^*, \ldots, Q_m^* \rangle$ for $\phi$ has at most n different subsets X of A for which $\langle A, X, Q_1^*, \ldots, Q_m^* \rangle \models \phi$, then n formulas $\psi_1 = \psi_1(Q_1, \ldots, Q_m), \ldots, \psi_n = \psi_n(Q_1, \ldots, Q_m)$ exist, each with one free variable x, such that
$\phi \models (\forall x)(Px \leftrightarrow \psi_1) \vee \ldots \vee (\forall x)(Px \leftrightarrow \psi_n)$.

(cf. [3].)

5. Any second-order sentence is logically equivalent (on the class of all general models) to a first-order formula if and only if it is both strongly standard increasing and strongly standard decreasing (i.e., if and only if it is invariant for general models with the same underlying standard model). This answers a question of S. Orey.

(cf. [ 14 ].)

6. Let $\Pi^1_1(R)$ be the class of second-order sentences of the form $(\forall X_1)\ldots(\forall X_n)\phi(X_1,\ldots, X_n, R, =)$, where R is a binary predicate constant and $\phi$ is any first-order sentence in $X_1,\ldots, X_n$, R, =. $\{\phi \in \Pi^1_1(R) \mid$ for some first-order sentence $\psi = \psi(R, =)$, $\phi \leftrightarrow \psi$ holds on all structures $\langle W, R\rangle$ with $W \neq \emptyset$ and $R \subseteq W \times W\}$ is not arithmetical.

If the predicate constant R is omitted, however, yielding $\Pi^1_1$, then $\{\phi \in \Pi^1_1 \mid$ for some first-order sentence $\psi = \psi(=)$, $\phi \leftrightarrow \psi$ holds on all domains$\}$ is arithmetical, in fact $\Sigma^0_2$.

7. P. Lindström's theorem characterizing first-order logic by means of the Löwenheim-Skolem and the compactness properties fails for the case of a first-order logic with only a finite number of predicate constants.

(cf. [ 11 ].)

8. The theorem of C. Åberg to the effect that "there are non-(logical truths) which are logical truths in the sense of some model for ZF" can be proved by the following simple observation. If a formula is a logical truth, then this fact is provable in ZF; but this implication does not hold for

all non-(logical truths). So, for at least one non-(logical truth) $\phi$, ZF + "$\phi$ is a logical truth" is consistent. (Nevertheless, $\phi$ is a logical truth if and only if ZF $\vdash$ "$\phi$ is a logical truth".)

(cf. [ 1 ].)

9. In a correspondence theory for modal predicate logic the sentences $(\forall x)\,\Box Ax \rightarrow \Box(\forall x)Ax$, $\Box(\forall x)Ax \rightarrow (\forall x)\,\Box Ax$ and $(\exists x)\,\Box Ax \rightarrow \Box(\exists x)Ax$ are first-order definable, but $\Box(\exists x)Ax \rightarrow (\exists x)\,\Box Ax$ is not.

(cf. lemma 4.9 of this dissertation.)

10. "Löb's Paradox" of 1955 ('any statement can be proved to be true using only self-reference, induced by a fixed-point construction, and the notion of implication') was also discovered by P.T. Geach around the same time and originates with H.B. Curry in 1942. In fact, this paradox follows immediately from the Liar Paradox when Russell's trick is used to eliminate negation in favour of implication.

(cf. [ 6 ], [ 8 ], [ 12 ].)

11. A short walk to the library will falsify J.R. Danquah's assertion that Bernays' proof of the non-independence of the propositional axioms in Principia Mathematica contains a vicious circle.

(cf. [ 7 ].)

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