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ABSOLUTENESS OF INTUITIONISTIC LOGIC

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PREFACE

The present treatise is a corrected version of the author's Ph.D. dissertation, written at the University of Amsterdam in 1974/75 under the direction of Professor A.S. Troelstra. My research, as well as the production of the dissertation, were generously supported by the Mathematical Center, and as is customary for such dissertations, it now appears as a Mathematical Centre Tract.

The text that follows is divided into two parts. Part A deals with theories whose arithmetical fragment is part of IA^* := Heyting's arithmetic IA extended with transfinite induction over all recursive well-orderings. Such theories (as well as some others closely related to them) are named "regular". It is shown that fairly strong intuitionistic theories, and - in particular - the intuitionistic impredicative theory of types, are regular.

In part B we treat maximality (or "absoluteness") properties of intuitionistic (Heyting's) propositional and predicate logic IL_0 and IL_1 for regular theories. Here, L is said to be *maximal* (or "absolute") for T if

$$\nvdash_L F[P_1,\ldots,P_k] \Rightarrow \nvdash_T F[A_1,\ldots,A_k]$$

for some arithmetical relations A_i of the same arity as P_i $(i=1,\ldots,k)$. The maximality is *uniform* if the A_i's are independent of F. We are also interested in having the substituted relations A_i as low as possible in the arithmetical hierarchy.

Refined versions of the results of part A are incorporated in LEIVANT [A], while the theorems of part B together with several other maximality results are proven in LEIVANT [B]. Nevertheless, the present exposition might still be useful to the interested reader. In contrast to the aforementioned papers, we use here natural deduction systems, and the proofs, especially in part B, illustrate the convenience of using natural deduction to straightforwardly formalize one's intuitive ideas. The sections in part B motivating the proofs are particularly relevant here. The sequential calculi used in our [A] and [B] allow more succinct presentations, but at the cost of concealing to some extent the motivating ideas. Also, our exposition here is more leisurely, so that, in conjunction with the use made of natural deduction, the effect is to reduce the effort required from the reader.

To do justice to the reader, we should place the maximality results proven here amongst other similar results.

Since we prove "absoluteness" of IL, the interest in treating propositional logic IL_0 lies only in reducing the complexity of the substituted sentences (0-ary relations). D.H.J. de JONGH and C.SMORYNSKI [73] have proved that there exist uniform arithmetical substitutions, and also - locally - Σ_1^0 substitutions for IL_0 and T = IA. Theorem I of part B below improves this by making the Σ_1^0 substitutions depend only on the number of propositional letters in the schema F. However, by a uniformization lemma proved in LEIVANT [B] §1.2, already the local Σ_1^0-absoluteness implies completely uniform absoluteness with Σ_1^0 substitutions. From this, using metamathematical properties of IL_0, one easily derives uniform absoluteness also with (binary) disjunctions of Π_1^0 sentences as substitutions (idem, §1.6). Similar statements are also true when IA is replaced by any regular T, and also for T extended with either Church's Thesis CT_0 or the Independence-of-Premiss Principle IP_0 (cf. TROELSTRA [73] for their statement). When Markov's Principle M is added, Σ_1^0 absoluteness fails (since $\vdash_{IA+M} \neg\neg A \to A$ for any Σ_1^0 sentence A), but IL_0 is uniformly Σ_2^0 absolute for T + IP_0 + M (cf. idem).

Turning to Intuitionistic Predicate Logic IL_1, we should start by mentioning a proof of DE JONGH [73] of a relativized version of absoluteness. Theorem II of part B below states the uniform Π_2^0 absoluteness of IL_1 for any regular theory T. IL_1 is also uniformly Σ_2^0 absoluteness for such T (LEIVANT [B] §2.4) but not even locally Σ_1^0 absolute for IA (LEIVANT [76]; this was also proved in §B.6 of the original version of the dissertation). Nevertheless, Σ_1^0-absoluteness does hold for certain fragments of IL_1 (LEIVANT [B] thm.2.VII).

All theories mentioned above, for which IL_1 is proved absolute, are r.e., and the "regular" ones are in IA^*. Allowing for more complex substitutions, one obtains in one stroke maximality of IL_1, for all regular theories, with the substitutions independent also of the theory; namely - one proves the uniform maximality of IL_1 for IA^* (LEIVANT [B] thm.2.VIII). The complexity of the substitutions may be somewhat reduced for IL_0.

For logic with equality, we have the Π_2^0 (and Σ_2^0) uniform absoluteness, for regular theories, of IL_1 extended with the following axioms:

$$\forall x \forall y (x=y \vee \neg x=y)$$

$$\exists x_1 \exists x_2 \ldots \exists x_n [\bigwedge_{0<i<j\leq n} \neg x_i = x_j] \qquad n = 2,3,\ldots$$

(idem §2.6).

As noted in section 2 of the introduction below, there is no straightforward connection between classical and intuitionistic absoluteness. For Classical Predicate Logic CL_1, local Δ_2^0 absoluteness (for sound arithmetic theories) is an immediate consequence of the refinement obtained by HILBERT and BERNAYS [39] to Gödel's Completeness Theorem. The uniformization technique of LEIVANT [B] mentioned above may be applied here, to yield uniform Σ_2^0 (and Π_2^0) absoluteness of CL_1 (idem, thm.2II).

INTRODUCTION

(For unexplained terminology see the preliminary section P below.)

Int. 1. The concept of absoluteness and the main results

Our central aim in this treatise is to prove that the formal systems of intuitionistic propositional and predicate logics (L_0 and L_1 resp.) are "schematically complete" for intuitionistic (Heyting's) arithmetic A, as well as for certain extensions of A. Let us first describe these results as cases of a general type of problems.

Let L be a system of logic, and let M be a system of mathematics based on the language of L; i.e., the language of M contains all the logical constants of the language of L as well as constants or definable objects for each type of parameter of that language. (Examples: (1) L is first order predicate logic and M is ZF set theory; (2) L is second order logic and M is second order arithmetic; (3) L is first order logic without first-order parameters (but with first-order "bound" variables, of course), and M is second order logic.)

Let C be a class of defined constants in the language of M. A schema F in the language of L is *C-absolute for M* if for each instance F^* of F which comes from F by substituting constants of C for parameters of the corresponding type,

(1) $\vdash_M F^*$.

L is said to be *C-absolute for M* if

(2) $L = \{ F \mid F \text{ is } C\text{-absolute for } M \}$,

i.e., if

(3) $\vdash_L F \iff (\forall C\text{-inst. } F^* \text{ of } F) \vdash_M F^*$

(Several other alternatives have been proposed to name the above property of L: "L is maximal (schematically complete, saturated) w.r.t. M", and "M is faithful to L".)

When M is based on L, the implication from left to right in (3) is trivial, the interesting part being of course the converse direction. This has been occasionally expressed in a contrapositive-like form:

(4) $$\not\vdash_{L} F \quad \Rightarrow \quad (\ \exists C\text{-inst. } F^{*} \text{ of } F\ )\ \not\vdash_{M} F^{*}.$$

In our treatment below we prove cases of (4), which is intuitionistically independent of (3). Assuming however Markov's principle for prim. rec. predicates,

$$M_{PR} \qquad \neg\neg\exists x\, A(x) \ \rightarrow\ \exists x\, A(x) \qquad (A \text{ arithmetical, quantifier free}).$$

(4) clearly implies (3) (by contraposition).

Actually the results proved below give the instance F^{*} of F for (4) constructively, independently of the premiss, and quite uniformly. Namely, for a large class of "regular" theories T, which includes A (cf. A.1):

<u>THEOREM I</u>. *Given a regular theory T, L_0 is Σ^0_1-absolute for T. For any given schema* F *of L_0 the substitutions depend only on the number of propositional variables in* F.

<u>THEOREM II</u>. *For T as above, L_1 is Π^0_2-absolute for T, with substitutions which depend on T only.*

These theorems are stated in more detail in B.0 below.

D.H.J. de Jongh had proved already in 1969 the absoluteness of L_0 for intuitionistic arithmetic A (and extensions $A^{\prec}$ of A with transfinite induction over some prim. rec. well ordering $\prec$). SMORYNSKI [72] shows that the meta-substitutions may be taken to be Σ^0_1 (though depending on each schema), and H. FRIEDMAN [73] proves that by allowing the meta-substitutions to be Π^0_2 one gets uniform absoluteness. This last result is a corollary of our theorem II.

All the results just mentioned were obtained by classical methods. Uniform absoluteness is however formalized as a Π^0_2 statement, since it has roughly the form

$$(\forall \text{ schema } F)\ [\ \forall x\ \neg\underline{Pr}_{L}(x,\ulcorner F\urcorner) \ \rightarrow\ \forall x\ \neg\underline{Pr}_{T}(x,\ulcorner fF\urcorner)\ ]$$

where $\underline{Pr}_{L}$ and $\underline{Pr}_{T}$ are prim. rec. proof predicates for L and T resp., and where f is a fixed prim. rec. function. For Π_2^0 sentences, however, provability in classical arithmetic implies provability in intuitionistic arithmetic (cf. TROELSTRA [73]).

So the main novelty of theorem I is the "locally-uniform" Σ_1^0 substitution. We nevertheless present this result in some detail, for two reasons. Firstly, it may be used as an expository introduction to the proof of theorem II; secondly, the method employed might turn out to be helpful in solving a number of other problems concerning the relation between L_0 and A.

As to predicate logic, de Jongh has proved (unpublished) the (local) absoluteness of L_1 for A, but where in each formula all quantifiers are restricted to a fixed unary predicate. This restriction allows a model theoretic treatment using Kripke models with a constant universe, and a special notion of "forced realizability" which utilizes results from the theory of Turing degrees.

Int. 2. Absoluteness in relation with some well-known results

For classical first order logic L_1^C absoluteness is an immediate corollary of HILBERT-BERNAYS [39]'s proof of Gödel's completeness theorem, where one has:

(5) $\nvdash_{L_1^C} \mathrm{F} \Rightarrow$ there is a Δ_2^0 instance F^* of F s.t. $\neg\mathrm{F}^*$.

Hence, if $\nvdash_{L_1^C} \mathrm{F}$ and $\vdash_M \mathrm{F}^*$ then M is not (classically) sound for Δ_2^0 sentences, and thus L_1^C is Δ_2^0-absolute for any theory M (in a language which extends the language of Peano's arithmetic) which is sound for Δ_2^0 sentences.

The same situation occurs not only for L_1^C, but even for classical simple type theory L_ω^C.

It seems here the right place to note that absoluteness results for classical systems are hardly related to absoluteness of (the corresponding) intuitionistic systems. Given (4) for classical L^C, M^C, nothing is said even about the propositional rule of excluded third, $p \vee \neg p$ (which is intuitionistically invalid and unprovable). Conversely, if (4) is given for intuitionistic systems L^I and M^I, then $\nvdash_{M^I} \mathrm{F}$ does not necessarily imply $\nvdash_{M^C} \mathrm{F}$ for the classical completion M^C of M^I. Hence the easy proof of absoluteness for L_1^C is of no help in solving the problem for L_1, while the uniform result for L_1 does not imply the uniform Π_2^0 absoluteness of L_1^C.

Another blind alley is to try to imitate the *method of the proof* of (5) in the treatment of the intuitionistic case. Of course, there is a completeness result for L_1 relative to Kripke's semantics which is analogous to (5), i.e.:

(6) $\quad \not\vdash_{L_1} F \quad \Rightarrow \quad$ there is a Δ^0_2 Kripke model K in which F is not valid

(cf. e.g. THOMASON [68]). Here, however, K is not necessarily a Kripke model for any numerical instrance F* of F since the models standing at each node are not necessarily models of arithmetic. Therefore every use of Kripke's semantics must refer here directly to Kripke's models for *arithmetic*, as done in de Jongh's and Smorynski's proofs mentioned above, but this is a totally different method.

Let us finally compare absoluteness with a conservative extension result. Let $\bar{A}$ denote arithmetic A with predicate variables and with an axiom schema of arithmetical comprehension:

$$\mathrm{ACA} \qquad \exists X\, \forall x\, [\ Ax \leftrightarrow Xx\ ]$$

(A in the language of A). Then $\bar{A}$ is a conservative extension of A (cf. TROELSTRA [73] 1.9.8), and L_1 is trivially contained in $\bar{A}$. $\bar{A}$ is also conservative over L_1 (compare MAEHARA [58] thm.1), i.e.,

$$L_1 \not\vdash A[P_1,\dots,P_n] \quad \Rightarrow \quad \bar{A} \not\vdash A[P_1,\dots,P_n]$$

for any schema $A[P_1,\dots,P_n]$ of L_1. Absoluteness of L_1 for A reads on the other hand

$$L_1 \not\vdash A[P_1,\dots,P_n] \quad \Rightarrow \quad A \not\vdash A[B_1,\dots,B_n]$$

for every $B_1,\dots,B_n$ in the language of A! Notice that the absoluteness of L_1 for A implies the absoluteness of L_1 for $\bar{A}$; this can be easily derived from the fact that $\bar{A}$ is conservative over A.

Int. 3. Infinitary derivations and the subformula property.

The central method of proof used below in establishing absoluteness of L_0 and L_1 is an analysis of infinitary derivations, i.e. (roughly), of derivations with the "ω-rule".

SCHÜTTE [51] seems to have been the first one to notice the usefulness of systems of infinitary derivations for the metamathematical study of arithmetic. He was chiefly interested in extending to arithmetic one of the main advantages of Gentzen's systems for logic, namely - their subformula property. We say that a proof-figure π *satisfies the subformula property* if every formula which occurs in π is a subformula of the formula derived by π. A calculus $\mathbb{C}$ of proof-figures *satisfies the subformula property* if the subsystem $\mathbb{C}_0$ of $\mathbb{C}$ containing only those proof-figures of $\mathbb{C}$ which satisfy the subformula property is complete for $\mathbb{C}$, i.e., if for every proof-figure π of $\mathbb{C}$ deriving a formula (or a sequent) σ there is a proof π_0 of $\mathbb{C}$ which derives σ and satisfies the subformula property. Gentzen proved that his sequential systems for first order classical and intuitionistic logic satisfy the subformula property (by "cut elimination"; cf. GENTZEN [35]), and PRAWITZ [65] proved that the same holds for GENTZEN's [35] system of natural deduction (by "normalization").

Although cut elimination and normalization for the corresponding calculi for arithmetic can also be carried out with some important metamathematical consequences (such as consistency and the "existential definability" property), the subformula property for these calculi is *not* implied.

Call a calculus $\mathbb{C}$ of proof-figures for arithmetic *good*, if there is a predicate Inf(x,y) such that the following hold, for each sequence of formulas $F_1,\ldots,F_k,G$ $(k \geq 0)$.

(1) If $\frac{F_1 \ldots F_k}{G}$ is an instance of an inference rule of $\mathbb{C}$, then
$\vdash_{\mathbb{C}} \mathrm{Inf}(\langle \ulcorner F_1 \urcorner,\ldots,\ulcorner F_k \urcorner\rangle, \ulcorner G \urcorner)$,

(2) $\vdash_{\mathbb{C}} \mathrm{Inf}(\langle \ulcorner F_1 \urcorner,\ldots,\ulcorner F_k \urcorner\rangle, \ulcorner G \urcorner) \rightarrow .\ F_1 \ \& \ldots \& \ F_k \rightarrow G.$

Thus, e.g., if $\mathbb{C}$ generates HA and has all sentences of HA as axioms, $\mathbb{C}$ is not good. But all standard calculi for arithmetic are good.

THEOREM. *If an r.e. good calculus $\mathbb{C}$ of finatary proof figures is complete for Heyting's arithmetic* A *and satisfies the elementary derivability conditions (cf.* TN1 *below, or e.g.* SMORYNSKI [75]*) then $\mathbb{C}$ proves that $\mathbb{C}$ does not satisfy the subformula property.*

PROOF. Let $\mathbb{C}$ be a calculus as above; the subformula property of $\mathbb{C}$ is formally expressible as an arithmetical (actually a Π^0_2) sentence, $\underline{\mathrm{Sp}}_{\mathbb{C}}$ say. Since the proof figures of $\mathbb{C}$ are finite, we can prove in A by induction on the length of proof figures that $\underline{\mathrm{Sp}}_{\mathbb{C}}$ implies the local reflection principle for $\mathbb{C}$; i.e.,

(7) $$\mathrm{Sp}_{\mathbb{C}} \vdash_{A} \exists p\ \underline{\mathrm{Prf}}_{\mathbb{C}}(p, \ulcorner F \urcorner) \to F$$

for each specific arithmetical F (cf. TN1). Taking in (7) in particular $F :\equiv \neg \underline{\mathrm{Sp}}_{\mathbb{C}}$ we get (since $\mathbb{C}$ is complete for A)

$$\underline{\mathrm{Sp}}_{\mathbb{C}} \vdash_{\mathbb{C}} \exists p\ \underline{\mathrm{Prf}}_{\mathbb{C}}(p, \ulcorner \neg \underline{\mathrm{Sp}}_{\mathbb{C}} \urcorner) \to \neg \underline{\mathrm{Sp}}_{\mathbb{C}}$$

and so by propositional logic

$$\vdash_{\mathbb{C}} \exists p\ \underline{\mathrm{Prf}}_{\mathbb{C}}(p, \ulcorner \neg \underline{\mathrm{Sp}}_{\mathbb{C}} \urcorner) \to \neg \underline{\mathrm{Sp}}_{\mathbb{C}}.$$

But this implies by the theorem of LÖB [55]

$$\vdash_{\mathbb{C}} \neg \underline{\mathrm{Sp}}_{\mathbb{C}}$$

since $\mathbb{C}$ is assumed to satisfy the elementary derivability conditions. QED.

Schütte's idea was to restore the subformula property for the systems of arithmetic by giving up the finiteness condition. The reason that cut-elimination for arithmetic does not imply the subformula property is the presence of the induction rule; hence this rule, which for a natural deduction system may be given by

$$\frac{\begin{array}{cc} & [A(a)] \\ \Gamma & \Delta(a) \\ A(\bar{0}) & A(a+1) \end{array}}{\forall x A(x)}$$

(using the notations of GENTZEN [35], PRAWITZ [65]), is replaced by an instance of an infinitary $\forall$-introduction rule (ω-rule):

$$\frac{\begin{array}{cccc} & & \Gamma & \\ & & A(\bar{0}) & \\ & \Gamma & \Delta(\bar{0}) & \\ & A(\bar{0}) & A(\bar{1}) & \dots\dots \\ \Gamma & \Delta(\bar{0}) & \Delta(\bar{1}) & \\ A(\bar{0}) & A(\bar{1}) & A(\bar{2}) & \end{array}}{\forall x A(x)}$$

(compare PRAWITZ [71]). Obviously, this infinitary $\forall$-introduction rule may take over the role of the finitary ∀I. Similarly the [∃E] inference rule

$$\begin{array}{cc}
 & [A(a)] \\
\Gamma & \Delta(a) \\
\exists x A(x) & B \\
\hline
 & B
\end{array}$$

may be replaced by a corresponding infinitary rule

$$\begin{array}{cccc}
 & [A(\bar{0})] & [A(\bar{1})] & \\
\Gamma & \Delta(\bar{0}) & \Delta(\bar{1}) & \cdots\cdot \\
\exists x A(x) & B & B & \\
\hline
 & & B &
\end{array}$$

By iterating these translations each finitary derivation Δ is mapped into a well-founded infinitary derivation Δ^{∞} having the same derived formula and the same open assumptions as Δ. (For a formal definition of the infinitary derivations see A.1).

The mapping above may be described as one which replaces (hereditarily) each expression (i.e., formula or derivation) $\varepsilon(\vec{p})$ which "depends" on a list $\vec{p}$ of parameters, and where the parameters range *implicitly* over the natural numbers, by an *explicit* enumeration $\{\varepsilon(\vec{n})\}_{\vec{n}}$ of the closed expressions which correspond to a substitution $[\vec{n}/\vec{p}]$ of numerals for those parameters.

For the system of infinitary proof figures obtained in this manner, a normalization theorem can be proved (cf. e.g. A.3 below), and in this case the subformula property does follow. The method leads also to a number of interesting applications (cf. e.g., KREISEL-LEVY [68] remark on p.126, PARIKH [73], PARSONS [60], LEIVANT [A]). The general pattern of these applications consists in embedding the (finitary) formal system to be investigated into a ("semi-formal") system of infinitary proof figures, which then allows a smoother proof theoretic analysis.

It is precisely this method which is used in the proof of absoluteness in part B below. We treat those theories whose arithmetical fragment can be embedded as above in a system of infinitary proofs of arithmetic. These proofs are subsequently transformed ("normalized") to ones which satisfy a number of structural properties, the most important of which is the subformula property described above.

Int. 4. Provably correct derivations; regular theories

An infinitary derivation of the kind described in Int. 3 may be viewed formally as an assignment of sequents (or their Gödel codes) to certain nodes of the universal spread; the assignment may be made total by attaching 0 to the rest of the nodes (A.1.1 below). But while for a calculus $\mathbb{C}$ of finitary proof figures (based on an r.e. set of inference rules) we may formally define a *prim. rec.* proof predicate $\underline{\mathrm{Prf}}_{\mathbb{C}}(p,\ulcorner F\urcorner)$, this obviously cannot be done for the arithmetization of infinitary proofs. If $\underline{\mathrm{Prf}}^{\infty}(\phi,\ulcorner F\urcorner)$ should be a formal proof predicate for the proofs described in Int. 3 above, then we should have (in elementary analysis V_0 plus $\mathbf{AC}_{00}$ as defined in section P below)

$$(8) \qquad F \quad \rightarrow \quad \exists\phi\ \underline{\mathrm{Prf}}^{\infty}(\phi,\ulcorner F\urcorner)$$

for every *prenex* arithmetical F (compare A.2.2.1). So the system of infinitary proof figures is classically complete, and $\underline{\mathrm{Prf}}^{\infty}$ cannot even be arithmetical.

The completeness of the infinitary systems for classical truth expressed by (8) illustrates the potential generality of the analysis of infinitary proof figures as a technique in meta-arithmetic: whatever classically sound theory T is given, an embedding of its arithmetical fragment into infinitary proofs is guaranteed. On the other hand one may wish to utilize the recursive enumerability of the embedded theory T, as we do in part B below, and so one has to restrict the class of infinitary derivations into which T is embedded. There are several simple methods for doing this, all having more or less equal merits. We find it particularly convenient to restrict the image of the embedding by requiring that each infinitary proof figure in it is *proved* to be a correct proof in a given (r.e.) theory T_1 (in a language extending the language of analysis V_0). I.e., one considers the derivations which are shown in T_1 to be well-founded and to respect the inference rules. An enumeration of these derivations can easily be extracted from an enumeration of T_1, and so the class of infinitary derivations considered is r.e. in T_1.

To sum up, we wish to exploit two properties of a given theory T: firstly, that the arithmetic fragment $A[T]$ of T, i.e., the arithmetic sentences provable in T, is r.e.; and secondly, that $A[T]$ can be investigated through an analysis of the structure of infinitary derivations. Both conditions are indeed satisfied if

(9) $$A[T] \subseteq A^{\infty}[T_1]$$

for some r.e. T_1, where $A^{\infty}[T_1]$ is (roughly, cf. A.1.2) the system of infinitary derivations proved in T_1 to be correct and normal, and where the inclusion refers to the derived sentences. When this is the case, we say that T is *regular* (A.1.2). For the proof of theorem II in B.4 below, the infinitary derivations examined have to be recursive, so for that proof (9) is strengthened to

(10) $$A[T] \subseteq A^{\infty}_{\mathrm{rec}}[T_1]$$

where $A^{\infty}_{\mathrm{rec}}[T_1]$ are (roughly) the recursive derivations of $A^{\infty}[T_1]$. When T satisfies (10) (and another minor condition, cf. A.1.2) we say that T is *strongly regular*.

Our feeling now is that regularity (as well as strong regularity) are conditions which are quite general and natural. There are a number of arguments supporting this feeling.

[a] By (8) above we have for any theory $T \supseteq V_0 + \mathbf{AC}_{00}$

(11) $$A_{\mathrm{p}}[T] \subseteq A^{\infty}[T]$$

where $A_{\mathrm{p}}[T]$ is the fragment of prenex formulae of $A[T]$. As $A_{\mathrm{p}}[T]$ is classically complete for $A[T]$, (11) implies that *any* classical r.e. theory T satisfies

(12) $$A[T] \subseteq A_{\mathrm{p}}[\,T + V_0 + \mathbf{AC}_{00}\,] \subseteq A^{\infty}[\,T + V_0 + \mathbf{AC}_{00}\,],$$

(where the first inclusion is trivial) and so any such T which is consistent with $V_0 + \mathbf{AC}_{00}$ is regular.

[b] Obviously, regularity and strong regularity are preserved under restriction. It is therefore quite satisfactory to know that some strong theory, in which a large part of current intuitionistic mathematics can be formalized, is (strongly) regular. We indeed show in A.4 below that intuitionistic type theory L_{ω} is strongly regular.

[c] If T is (strongly) regular and sound, then so is T extended with any schema of transfinite induction over some prim.rec. well-ordering (cf. A.2.4 for a precise statement, a proof and a discussion of its significance).

[d] The class of regular theories is closed under the operation of adding self-consistency (A.2.2.4), and so this class is closed under transfinite progressions along Σ^0_1 paths in Kleene's O.

Int. 5. The method of proof

The proofs of theorems I and II are both composed of two parts.

(i) A reduction of the problem, using proof-theoretic methods. For theorem I we show, roughly, that given a regular theory T and a schema F of L_0, if F^* is a Σ^0_1 meta-substitution of F then

$$(13) \qquad \nvdash_{L_0} F \quad \text{and} \quad \vdash_T F^* \quad \Rightarrow \quad \vdash_T U^*$$

where U is a *specific* schema and where U^* comes from U by the same meta-substitution (B.2.0). The proof theoretic reduction of theorem II is similar, with L_1 in place of L_0, with T strongly regular and with Π^0_2 meta-substitutions. In the proof of theorem I the schema U is fixed for all schemata with a certain bound on the number of propositional letters used, while in the proof of theorem II U is fixed altogether.

(ii) A solution of the reduced problem. We find instances U^* of the corresponding U and of the kind required, for which $\vdash_T U^*$ is impossible.

Step (ii) uses the recursive enumerability of T, and is (in both proofs) a generalization of Gödel's first incompleteness theorem (B.1,B.5). On the other hand the proof of step (i) in each case utilizes the embedding of $A[T]$ in the set of normal infinitary derivations. (Here we define "normality" in a somewhat broader sense which renders the arguments a bit simpler).

The idea of the proof-theoretic reduction is the following. Let $T = A^\infty[T_1]$. Then (13) is a consequence of the provability in T_1 of

$$(14) \qquad (\nvdash_{L_0} F) \ \& \ \underline{\mathrm{Prf}}^\infty(\phi,\ulcorner F^*\urcorner) \ \rightarrow \ \exists\psi\ \underline{\mathrm{Prf}}^\infty(\psi,\ulcorner U^*\urcorner).$$

Assuming the premiss of (14), one analyses the structure of ϕ and, using $\nvdash_{L_0} F$, shows how to "extract" a derivation ψ (for U^*) from ϕ.

The precise nature of this "extraction" will be clear from the heuristic discussions (B.2.1, B.4.2) and from the technical details of the proofs (B.2.2-6, B.4.3-11). There is however a difference between the proofs of the two theorems which should be noted outright. In contrast to L_0, L_1 is not decidable, and as a consequence one has to weaken the proof-theoretic reduction of theorem II to

$$(15) \qquad (\not\vdash_{L_1} \mathrm{F}) \ \& \ \underline{\mathrm{Prf}}^{\infty}_{\mathrm{rec}}(\mathrm{d}, \ulcorner \mathrm{F}^{*} \urcorner) \rightarrow \neg\neg\exists\psi \ \underline{\mathrm{Prf}}^{\infty}(\psi, \ulcorner \mathrm{U}^{*} \urcorner)$$

where $\underline{\mathrm{Prf}}^{\infty}_{\mathrm{rec}}$ is the proof predicate for recursive infinitary derivations. Compared to (14), the premiss here is strengthened and the conclusion is weakened. Furthermore, (15) is not proved in the r.e. theory T_1, but in a certain Σ^0_2-enumerated extension of it (cf. B.3).

Int. 6. Normalization of infinitary derivations; regularity of the theory of types

In Int. 3 above we have quoted Schütte's result stating that every infinitary derivation (of arithmetic) can be brought into a "normal" ("cut free") form which satisfies the subformula property; this in turn is used in our proof of absoluteness as indicated in Int. 5 (the additional structural requirements we are using are inessential to the proof of normalization). The traditional proofs of normalization of infinitary derivations (SCHÜTTE [51][60], FEFERMAN [68], TAIT [68], MARTIN-LÖF [68]) all use ordinal assignments, following GENTZEN's [36][38] consistency proofs. This evolution is quite evident: ordinals can be assigned to well-founded infinitary trees in a natural way, so extending Gentzen's idea was the first thing which came to mind while passing from finite to infinitary proof figures.

In part A below we present however a new proof of normalization which does not use the technique of ordinal assignments. We do so simply to permit a certain generalization which will be explained below, and for which the technique of ordinal assignment is not so adequate.

Cut elimination for (a sequential calculus for) the classical theory of types L^c_ω is known since TAKAHASHI [67] (for the theory of species L_2 proofs were discovered independently also by PRAWITZ [68] and TAIT [66]). From the work of GIRARD [71][72] (as expounded in detail in MARTIN-LÖF [73]) we also know an effective procedure which transforms each proof of L_ω into a normal one; and like for Gentzen's systems for first order logic L_1, we get for L_ω (and ipso facto for L_2) normal proofs which do satisfy the subformula property. However, when a normal proof π of L_2 proves a formula F in which a second order quantifier occurs, then the subformula property of π is of limited interest: suppose e.g. that $\exists$X G[X] is a subformula of F, then so is G[H] for every formula H including e.g. F itself. This is an evident drawback if one refers to the interpretation of arithmetic in L_2, as given by PRAWITZ [65], since under this interpretation first order sentences of arithmetic are *always* mapped to second order formulae.

Consequently, a system of type theory which does satisfy the subformula property for arithmetical sentences must be built up firstly by extending the language to include the language of arithmetic, and secondly by expanding the first order parametric expressions into explicit infinitary proof figures (as in Int. 3 for first order arithmetic). I.e., a system L_ω^∞ is adopted for the union of the languages of A and of L_ω, whose inference rules are those of the infinitary system for arithmetic, plus the rules of L_ω for higher order quantification.

We are now ready to justify our abandoning the technique of ordinal assignment. We wish to prove a normalization theorem for L_ω^∞, because then we may conclude that L_ω is regular: L_ω is embedded in L_ω^∞ in an obvious manner, and every normal derivation in L_ω of an arithmetical sentence must actually be a purely arithmetical derivation, because it must satisfy the subformula property. So, if T is a theory in which these facts are provable, then

$$A[L_\omega] \subseteq A^\infty[T]$$

(cf. A.4.9), and so L_ω is regular.

It is easily seen however (cf. TN 3) that if the normalization theorem for L_ω^∞ was to be proved by assigning an ordinal notation to each proof figure, then notations should be available for all "provable ordinals" of L_ω. Such notations are unfortunately not known at present.

There remains the possibility of assigning ordinals (in place of ordinal notations) to the proof figures, as done e.g. by SCARPELLINI [71] p.156; the proof of normalization is then carried out in some formal set theory (ZF say). But then it seems unrealistic to expect either an optimal result, or a proof within L_ω of a normalization theorem for the systems obtained by restricting L_ω^∞ to languages with a bound on formula-complexity. The method described in part A below does have, on the other hand, the properties just mentioned, in analogy to the well-known normalization proofs for arithmetic.

In proving the normalization theorem for L_ω^∞ (A.4) we combine the ideas of PRAWITZ's [71] "validity" argument, the work of GIRARD [71][72] and the "geometrical" treatment of infinitary proof figures of LEIVANT [A]. For another application of the normalization theorem for L_ω^∞ see TN 4.

TECHNICAL NOTES TO THE INTRODUCTION

TN 1. The elementary derivability conditions for an r.e. system T and a provability predicate $\underline{\mathrm{Pr}}_T$ for it are

$\underline{\mathrm{D1}}$. $\qquad \vdash_T F \;\Rightarrow\; \vdash_T \underline{\mathrm{Pr}}_T(\ulcorner F \urcorner)$

$\underline{\mathrm{D2}}$. $\qquad \vdash_T \underline{\mathrm{Pr}}_T(\ulcorner F \urcorner) \rightarrow \underline{\mathrm{Pr}}_T(\ulcorner \underline{\mathrm{Pr}}_T(\ulcorner F \urcorner) \urcorner)$

$\underline{\mathrm{D3}}$. $\qquad \vdash_T \underline{\mathrm{Pr}}_T(\ulcorner F \rightarrow G \urcorner)\ \&\ \underline{\mathrm{Pr}}_T(\ulcorner F \urcorner) \rightarrow \underline{\mathrm{Pr}}_T(\ulcorner G \urcorner)$.

The local reflection principle is proved in $\mathbb{C}$ by a straightforward induction on the length of derivations as follows. Each inference step of $\mathbb{C}$ is of the form

$$\frac{\langle\ \sigma_i\ \rangle_{i<n}}{\tau}\ [\rho]$$

where $\sigma_i (i<n)$, τ are formulae or sequents whose validity is equivalent to certain sentences $F_i (i<n)$, G (resp.). Assume now that a proof figure Δ of $\mathbb{C}$ is given, with

$$\Delta \;\equiv\; \frac{\langle\ \begin{array}{c}\Gamma_i\\ \vdots\\ \sigma_i\end{array}\ \rangle_{i<n}}{\tau}\ .$$

Δ is finite, and so there is a (restricted) truth definition $\underline{\mathrm{Tr}}$ in A which applies to all formulae occurring in Δ (cf. e.g. TROELSTRA [73] 1.5.4). By ind. hyp. we have $\underline{\mathrm{Tr}}(\ulcorner F_i \urcorner)$ $(i<n)$, and since $\bigwedge_i F_i \rightarrow G$ is simply a rule of $\mathbb{C}$, we thus get $\underline{\mathrm{Tr}}(\ulcorner G \urcorner)$.

The predicate $\underline{\mathrm{Tr}}$ above depends on Δ, but if Δ is known to satisfy the subformula property, then $\underline{\mathrm{Tr}}$ depends only on the derived formula of Δ. So we actually have, in A (and hence in $\mathbb{C}$)

$$(1) \qquad \exists p\,[\ \underline{\mathrm{Prf}}_{\mathbb{C}}(p, \ulcorner F \urcorner)\ \ \&\ \ \text{"p satisfies the subformula property"}\ ] \rightarrow \rightarrow \underline{\mathrm{Tr}}(\ulcorner F \urcorner)$$

for each sentence F. But

$$\underline{\mathrm{Sp}}_{\mathbb{C}} :\equiv\ \exists p\ \underline{\mathrm{Prf}}_{\mathbb{C}}(p, \ulcorner F \urcorner) \rightarrow \exists p\,[\ \underline{\mathrm{Prf}}_{\mathbb{C}}(p, \ulcorner F \urcorner)\ \&\ \text{"p satisfies the subformula property"}\ ],$$

so (1) implies

$$\underline{\mathrm{Sp}}_{\mathbb{C}} \vdash \exists p\ \underline{\mathrm{Prf}}(p, \ulcorner F \urcorner) \to F$$

for each sentence F.

TN 2. KREISEL [65] proves that no r.e. system $\mathbb{C}$ of finitary proof figures built up from derived rules of A and which is complete for A can be *proved in* A to satisfy the subformula property. Our statement is stronger since the subformula property is simply false (not only unprovable) provided $\mathbb{C}$ is sound for Σ^0_2 sentences (i.e., for $\neg\underline{\mathrm{Sp}}_{\mathbb{C}}$).

The reference to the reflection principle made in Kreisel's proof mentioned above is redundant, since the result quoted is obvious already from Gödel's second incompleteness theorem: one proves trivially in A that no derivation of $\bar{0} = \bar{1}$ may satisfy the subformula property, and so if

$$\vdash_{\mathbb{C}} \underline{\mathrm{Sp}}_{\mathbb{C}}$$

then $\mathbb{C}$ proves its own consistency.

TN 3. Suppose that we can prove in L_ω for a certain prim. rec. well-ordering $\prec$

$$\mathrm{TI}^{\prec} \ :\equiv\ \forall x\, [\, \forall y{\prec}x\ P(y) \to P(x)\,] \to \forall x\ P(x)$$

where P is a predicate-parameter. We then have (trivially) a proof $\{d^{\prec}\}$ of $L^{\infty}_{\omega,\mathrm{rec}}$ for $\mathrm{TI}^{\prec}$.

Suppose that $\{d^{\prec}\}$ is normalized into $\{d^{\prec}_N\}$; an anlysis of $\{d^{\prec}_N\}$ using the subformula property, shows that $\{d^{\prec}_N\}$ must have a specific structure for which the Brouwer-Kleene well-ordering $\prec'$ associated with $\{d^{\prec}_N\}$ is equivalent (in Y_0) to $\prec$ itself (i.e., $\mathrm{TI}^{\prec}$ and $\mathrm{TI}^{\prec'}$ are equivalent in Y_0 for each predicate P in the language of Y_0). Taking various $\prec$ we find that the ordinal of $\{d^{\prec}_N\}$ may be any "provable ordinal" of L_ω, i.e., any ordinal over which t.i. is provable in L_ω.

TN 4. A memo of G. Kreisel from 1973 proposes another application of the normalization of L^{∞}_{ω}. Kreisel's aim there is to answer a question of M.J. Beeson about a possible intuitionistic analogue to the KREISEL-SHOENFIED-WANG [60] completeness result (which reads: Peano's arithmetic extended with transfinite induction over all prim. rec. well-orderings is complete for classically true sentences). As a partial answer to that question, Kreisel's memo sketches a possible proof that Heyting's arithmetic A extended with t.i. over all prim. rec. well-orderings is complete at least for $A[L_2]$ (i.e., the arithmetical fragment of the theory of species). A system similar to $L^{\infty}_{2,\mathrm{rec}}$ (:= the recursive derivations in L^{∞}_{2}) is presented, and it is assumed that the normalization of that system can be proved. But if {d} is a normal proof of $L^{\infty}_{2,\mathrm{rec}}$ which derives an arithmetical sentence F, then {d} is actually a proof of $A^{\infty}_{\mathrm{rec}}$ by the subformula property. By t.i. over the Brouwer-Kleene well-ordering corresponding to {d}, and using a restricted truth definition for the subformula of F one gets that F is true.

The missing normalization step is proved in A.4 below. The proof remains however incomplete, since one uses not only t.i. over the proof tree {d}, but also the fact that {d} describes a correct derivation; this last assumption is a Π^0_1 sentence which is not necessarily provable in A. However, this hiatus may be filled up as follows.

Given a quantifier free unary predicate E, define

$$\begin{aligned} x \prec_E y \;\; :\equiv \;\; & x < y \;\;\&\;\; \forall z \le y\; E(z) \\ \vee \;\; & y < x \;\;\&\;\; \exists z \le y\; \neg E(z) \end{aligned}$$

$\prec_E$ is of course prim. rec., and if $\forall x\, E(x)$ then $\prec_E$ is simply $<$, and so it is certainly well founded. Let

$$A^E(x) \;:\equiv\; \exists s\; \forall z \prec_E x\;\; z \epsilon s$$

where

$z \epsilon s$:≡ z is an element of the finite set of natural numbers encoded by s (via the binary encodement say).

It is easily seen that, in A,

$$\forall y \prec_E x\; A^E(y) \;\rightarrow\; A^E(x)$$

and so by t.i. over $\prec_E$

(1) $\forall x\, A^E(x)$.

But in A one proves outright

(2) $\neg E(x) \rightarrow \neg A^E(x)$

Since E is decidable, we get from (1) and (2)

$$\forall x\, E(x).$$

So, if $\forall x\, E(x)$ is true, then $\prec_E$ is well-founded and $\forall x\, E(x)$ is provable by t.i. over $\prec_E$. This completes now Kreisel's sketch: given a derivation π of L_ω which proves an arithmetical sentence F, one maps trivially π into an infinitary derivation {d} of $L^\infty_{\omega,rec}$ for F. By the normalization theorem of A.4 below, d is mapped into a *normal* derivation {e} of $L^\infty_{\omega,rec}$ for F which is, by the subformula property, a purely arithmetical derivation. Now one looks at A extended with t.i. over $\prec_E$ and over $\prec_e$, where $\prec_E$ is defined as above if $\forall x\, E(x)$ expresses the local correctness of the derivation {e}, and where $\prec_e$ is the Brouwer-Kleene well-ordering associated with the proof-tree {e}. In the extended theory we can now conclude as above that F is true. We thus have:

<u>THEOREM</u>: *L_ω is conservative over A extended with t.i. over prim. rec. well-founded orderings.*

PRELIMINARIES

P.1. SYNTACTICAL NOTIONS

P.1.1. The propositional constants we use are &, v, → and ⊥ (for absurdity); negation is definable in terms of → and ⊥:

$$F \;:\equiv\; F \to \bot$$

We find it convenient to distinguish syntactically between "free" and "bound" variables. The label "variable" is reserved to "bound" variables, while the "free" variables we call *parameters*.

P.1.2. We often have to distinguish between different occurrences of the same syntactic object σ (usually a formula, sometimes a term or a parameter). An accurate definition of "an occurrence of σ in τ" may be found e.g. in STEEN [72] p.13. We shall write $\underline{\sigma}$ (underlined) when referring to an occurrence of σ; usually the specific occurrence refered to will be either obvious from the context or irrelevant to it.

In a formula $\underline{F} \to \underline{G}$, $\underline{F}$ as well as all its sub- (occurrences of) formulae are said to be *negatively* bound by the shown occurrence of →. $\underline{B}$ is said to be a *negative subformula* of A if the number of implications negatively binding $\underline{B}$ in A is odd; if this number is even then $\underline{B}$ is a *positive subformula* of A. (compare PRAWITZ [65] p.43).

P.1.3. We shall usually use *natural deduction* calculi for generating formal theories. In these calculi there are for each logical constant κ an introduction rule [κI] and an elimination rule [κE]. The natural deduction calculi were invented by G. GENTZEN [35]; good introductions to them may also be found in PRAWITZ [65], [71]. We shall freely use the terminology of these works for dealing with natural deductions.

We also adopt the following convention: if Δ is a natural deduction deriving a formula F (or a sequent s) we write $\overset{\Delta}{F}$ (resp., $\overset{\Delta}{s}$) in place of Δ when we wish to express this fact explicitly. On the other hand $\frac{\Delta}{G}$ (with a separating horizontal line) stands for the deduction which extends $\Delta \equiv \overset{\Delta}{F}$ by deriving G from F.

P.2. FORMAL SYSTEMS

P.2.1. *Intuitionistic propositional and first-order predicate logics (L_0 and L_1 respectively)*

The language of L_0 is built up from the propositional parameters ("letters") $p_0, p_1, \ldots$ and from the propositional constants &, v, $\rightarrow$ and $\perp$. The language of L_1 is built up as usual from predicate parameters $P_i^n (i, n \geq 0, P_i^n$ is n-place), first-order parameters $a_0, a_1, \ldots$ and first-order variables $x_0, x_1, \ldots$, the propositional constants and the first-order quantifiers $\forall, \exists$.

The theories L_0 and L_1 are generated by the corresponding natural deduction calculi (GENTZEN [35], PRAWITZ [65]). A rough picture of these calculi may be obtained by looking at A.1.1. below.

P.2.2. *Second-order logic L_2 (the theory of species)*

The language of L_2 contains, in addition to the second order parameters P_i^n of L_1 also second order variables $X_i^n (i, n \geq 0)$ and corresponding second order quantifiers $\forall^{(n)}, \exists^{(n)}$.

The theory L_2 is now generated by a natural deduction calculus as in version I of PRAWITZ [65] p.65, i.e., without referring to λ-abstraction (compare A.4.1 below).

P.2.3. *The theory of types L_ω*

The simple types are generated inductively by starting with 0 as a basic type (the type of "first-order objects"), and passing from a sequence $\tau_1, \ldots, \tau_n$ of types $(n \geq 0)$ to a new type $(\tau_1, \ldots, \tau_n)$, the type of properties of tuples $\langle T_1, \ldots, T_n \rangle$ of terms of types $\tau_1, \ldots, \tau_n$ respectively. In particular ( ) is the type of propositions.

The language of L_ω is built up now similarly to the language of L_2, but with variables and predicates P_i^τ, X_i^τ $(i \geq 0)$ for each type τ and with corresponding quantifiers $\forall^\tau$, $\exists^\tau$.

The intuitionistic (simple) type theory L_ω is generated once again by a natural deduction system in an obvious manner (for details see MARTIN-LÖF [73]).

The theories L_k (k = 0,1,2,3...) may now be defined to be L_ω restricted to the types whose definition is of length $\leq$ k.

P.2.4. *Intuitionistic (Heyting's) arithmetic*

Here we have in addition to the first order variables and parameters of L_1 also a first-order constant 0 and function symbols f_i^n $(i\geq 0, n\geq 1)$. Each f_i^n denotes a function from $\mathbb{N}^n$ to $\mathbb{N}$, and we may take f_0^1 to denote the successor function. The first-order terms are now built up in a standard manner. If a term contains occurrences of ("free") variables we shall say that it is a *pseudo-term*; otherwise it is a *pure term*.

The language of A contains only a single second order predicate = which is binary; we write of course $t = s$ for $=(t,s)$.

A is now generated by a natural deduction calculus which includes, in addition to the inference rules of L_1:

(i) inference rules expressing Peano's third and fourth axioms;

(ii) an inference rule expressing the principle of induction;

(iii) all defining equations for prim.rec. functions, where each f_i^n is interpreted as the i'th n-place prim.rec. fnction.

For details cf. PRAWITZ [71].

P.2.5. L_1A: L_1 *extended to the language of* A.

The language of L_1A is the union of the languages of L_1 and of A, i.e., we extend the language of A with predicate letters P_i^n $(i,n\geq 0)$.

L_1A is now generated by a calculus of natural deductions based on the rules of L_1.

Note that the language of L_1A is more restricted than the language of HAS_0 (Heyting's arithmetic with species variables) of TROELSTRA [73] 1.9.3, where quantification over second order variables is also allowed.

P.2.6. *Elementary analysis* V_0

The language of V_0 is the extension of the language of A obtained by allowing function parameters g_i^n $(i,n\geq 0)$, function variables ϕ_i^n $(i,n\geq 0)$ and function quantifiers $\forall\phi_i^n$, $\exists\phi_i^n$ $(i\geq 0)$.

The natural deduction calculus for V_0 is obtained by joining to the inference rules of A inference rules for function quantification; e.g., the rule of $\forall$-elimination for function-variables:

$$\frac{\forall\phi_i^n\ \forall[\phi]}{A[h^n]}$$

where h^n is either a function constant f_j^n $(j\geq 0)$ or a function parameter g_j^n $(j\geq 0)$.

Note that we do not have in $\mathcal{V}_0$ any comprehension rule, and consequently the function parameters and variables may be interpreted to range over prim.rec. functions. $\mathcal{V}_0$ is therefore a conservative extension of $\mathcal{A}$ (cf. HOWARD-KREISEL [66], where $\mathcal{V}_0$ is denoted by $\mathbb{H}$).

P.2.7. *Classical theories*

For each intuitionistic theory $\mathcal{T}$ one obtains the classical completion $\mathcal{T}^c$ of $\mathcal{T}$ by joining to $\mathcal{T}$ either the axiom schema of double negation,

$$\neg\neg A \to A,$$

or the axiom schema of excluded third,

$$A \vee \neg A.$$

P.3. ARITHMETIZATION OF METAMATHEMATICAL NOTIONS

P.3.1. Finite sets of numbers $\{n_0,\dots,n_k\}$ may be encoded by

$$\{n_0,\dots,n_k\} \mapsto \sum_{i\leq k} 2^{n_i}$$

which is a one-to-one prim.rec. function.

The set-theoretical relations $\in$, $\subset$ etc. are then encoded by prim.rec. relations for which we use the same notation ($\in$, $\subset$ etc.).

P.3.2. Let $\langle\!\langle\ \rangle\!\rangle$ stand for the coding of finite sequences given in TROELSTRA [73] 1.3.9. We take

$$\langle n_0,\dots,n_k\rangle = \langle\!\langle n_0,\dots,n_k\rangle\!\rangle + 1.$$

The prim.rec. functions for projection $(n)_i$, cancatenation $u*v$ and length $\underline{\mathrm{lth}}(n)$ corresponding to the coding $\langle\rangle$ are then defined (as for $\langle\!\langle\rangle\!\rangle$) in an obvious manner.

We shall use the following properties of $\langle\rangle$:

(1) $\langle\rangle$ is onto the positive integers.

So an algorithm may produce the code of a node in the universal spread which satisfies a certain property, and may yield 0 when no such node exists.

(2) $n_i < \langle n_0,\dots,n_k\rangle \qquad (i\leq k)$

(3) $\langle n_0,\dots,n_k\rangle < \langle n_0,\dots,n_k,n_{k+1},\dots,n_{k+m}\rangle$

(4) $u < v \Rightarrow u * \langle m\rangle < v * \langle m\rangle$ and $\langle m\rangle * u < \langle m\rangle * v$.

We let $u \prec v$ stand for the prim.rec. relation "u is (a code of) a proper initial segment of (the sequence encoded by) v", and we let tail be a prim. rec. function which satisfies

$$\underline{\text{tail}}(\langle n_0,\ldots,n_k\rangle) = \langle n_1,\ldots,n_k\rangle$$

P.3.3. We shall frequently use the notations of KLEENE [52][69] for dealing with general recursive functions: the standard prim.rec. predicates T, T^{ϕ}, the result-extracting function U,

$\{n\}$ (resp. $\{n\}^{\phi}$) for the partial recursive (resp. recursive in ϕ) function with index n ,

$!\{n\}(x)$ for $\exists y\, T(n,x,y)$

$!!\{n\}$ for $\forall x\, !\{n\}(x)$ (i.e., $\{n\}$ is a total function)

$t \simeq s$ for "t and s are both well-defined and equal, or they are both undefined".

P.3.4. We shall implicitly assume throughout this treatise that some standard Gödel coding of syntactical objects is given.

For arithmetization of proofs we shall use:

$\underline{\text{Der}}_{\mathcal{T}}(x)$ for "x encodes a derivation of (a standard calculus generating) the theory $\mathcal{T}$";

$\underline{\text{Prf}}_{\mathcal{T}}(x,y)$ for "x encodes a proof of $\mathcal{T}$ for the formula (sentence, sequent) encoded by y";

$\underline{\text{Pr}}_{\mathcal{T}}(y)$ for $\exists x\, \text{Prt}_{\mathcal{T}}(x,y)$.

P.4. MATHEMATICAL SCHEMATA.

P.4.1. The axiom-schema of choice from numbers to numbers AC_{00} reads:

$$AC_{00}: \quad \forall x \exists y\, A(x,y) \rightarrow \exists \phi \forall x\, A(x,\phi(x))$$

P.4.2. The schema of transfinite induction $TI^{\prec}$ over a (fixed) binary relation $\prec$

$$TI^{\prec}: \quad \forall x\, [\, \forall y \prec x\, A(y) \rightarrow A(x)\,] \rightarrow \forall x\, A(x)$$

P.4.3. The axiom-schema of bar induction, or "induction over well founded trees"

$$\mathrm{BI} \;:\quad \forall\phi\ [\ \underline{\mathrm{WF}}(\phi) \rightarrow \underline{\mathrm{Ind}}\,[A,\phi]\ ]$$

where

$$\mathrm{WF}(\phi) \;:\equiv\; \forall\chi\exists x\ \phi(\bar{\chi}(x)) = 0$$

$$\underline{\mathrm{Ind}}[A,\phi] \;:\equiv\; \forall u\ \{\ \forall n\ J[A,\phi,u*\langle n\rangle] \rightarrow J[A,\phi,u]\ \} \rightarrow \forall u\ J[A,\phi,u]$$

$$J[A,\phi,u] :\equiv \forall v \prec u\ \phi(v) \neq 0 \rightarrow A(u)$$

($\prec$ is the initial-segment relation between codes of finite sequences).

Here the function ϕ is thought of as representing a tree, namely, the set of nodes u in the universal spread which satisfy

$$\forall v \prec u\ \phi(v) \neq 0$$

When ϕ is known to satisfy

$$\phi(u) = 0 \rightarrow \phi(u*\langle n\rangle) = 0$$

then we may replace the J above by

$$J_1[A,\phi,u] \;:\equiv\; \phi(u) \neq 0 \rightarrow A(u).$$

BI is easily seen to be derivable in $V_0^c + AC_{00}$. It is also not difficult to verify that our schema BI is a special case of the schema of bar induction for monotonic predicates BI_M of HOWARD-KREISEL [66] p.326 as well as of the schema of bar induction for decidable predicates BI_D on p.336 there.

PART A. Regular theories and normalization of infinitary derivations.

A.1. DEFINITION OF REGULAR THEORIES

This chapter is a self-contained introduction to part B. The reader who wishes to do so may skip A.2 - A.4.

1.1. DESCRIPTION OF A^∞

By a *sentence* we mean a closed formula of A. A *sequent* is a syntactical object of the form $\underline{a} \Rightarrow F$ where $\underline{a}$ is a finite set of sentences and F is a sentence. $\underline{a}$ is the *precedent*, or *antecedent* of the sequent, and F is the *succedent*, or the *conclusion*.

We use here the absurdity symbol $\bot$ though it is definable as $\bar{0} = \bar{1}$ (GENTZEN [33] §6) because one of our aims is to get a formal separation between logic and arithmetic.

Propositional rules of A^∞:

$$[\mathrm{T}] \qquad \underline{a} \Rightarrow F \qquad \text{where } F \in \underline{a}$$

$$[\&\mathrm{I}] \quad \frac{\underline{a} \Rightarrow F_0 \qquad \underline{a} \Rightarrow F_1}{\underline{a} \Rightarrow F_0 \& F_1} \; ; \qquad [\&\mathrm{E}_i] \quad \frac{\underline{a} \Rightarrow F_0 \& F_1}{\underline{a} \Rightarrow F_i} \qquad (i=0,1)$$

$$[\to\mathrm{I}] \quad \frac{\underline{a}, F \Rightarrow G}{\underline{a} \Rightarrow F \to G} \; ; \qquad [\to\mathrm{E}] \quad \frac{\underline{a} \Rightarrow F \to G \qquad \underline{a} \Rightarrow F}{\underline{a} \Rightarrow G}$$

(where $\underline{a}, F$ stands for $\underline{a} \cup \{F\}$)

$$[\mathrm{vI}_i] \quad \frac{\underline{a} \Rightarrow F_i}{\underline{a} \Rightarrow F_0 \mathrm{v} F_1} \quad (i=0,1) \; ; \qquad [\mathrm{vE}] \quad \frac{\underline{a} \Rightarrow F_0 \mathrm{v} F_1 \qquad \underline{a}, F_0 \Rightarrow G \qquad \underline{a}, F_1 \Rightarrow G}{\underline{a} \Rightarrow G}$$

$$[\bot] \quad \frac{\underline{a} \Rightarrow \bot}{\underline{a} \Rightarrow F}$$

Quantification and arithmetical rules of A^{∞}:

[TE] $\underline{a} \Rightarrow E$ where E is a true equation when every function symbol f^{i}_{j} is interpreted as the j'th i-place prim. rec. function.

[FE] $\dfrac{\underline{a} \Rightarrow E}{\underline{a} \Rightarrow \bot}$ where E is a false equation.

[∀I] $\dfrac{\{\underline{a} \Rightarrow F(\bar{n})\}_{n>\omega}}{\underline{a} \Rightarrow \forall x F(x)}$

[∀E] $\dfrac{\underline{a} \Rightarrow \forall x F(x)}{\underline{a} \Rightarrow F(t)}$ (t a term); [∃I] $\dfrac{\underline{a} \Rightarrow F(t)}{\underline{a} \Rightarrow \exists x F(x)}$

[∃E] $\dfrac{\underline{a} \Rightarrow \exists x F(x) \qquad \{\underline{a}, F(\bar{n}) \Rightarrow G\}_{n<\omega}}{\underline{a} \Rightarrow G}$

A function ϕ is a derivation of A^{∞} (notation: $\underline{\mathrm{Der}}^{\infty}(\phi)$) if

(1) $\{\, n \mid \phi n \neq 0 \,\}$ is a tree of (codes of) finite sequents under the obvious partial ordering:

$$\phi u = 0 \longrightarrow \phi(u{*}\langle n\rangle) = 0,$$
$$\phi(u{*}\langle n\rangle) = 0 \longrightarrow \phi(u{*}\langle n+1\rangle) = 0;$$

(where * denotes concatenation of sequent numbers).

(2) For every u (= the code of a node in the universal spread) $(\phi u)_0$ is the code of one of the inference rules ρ above (under some fixed encodement), while $(\phi u)_1$ and $(\phi(u{*}\langle n\rangle))_1$ $(n<\omega)$ are codes of sequents which relate as the conclusion and the premiss sequents of ρ (and when no n'th premiss is required, $(\phi(u{*}\langle n\rangle))_1 = 0$).

(3) ϕ is well-founded: $\forall X \exists x \cdot \phi(\bar{X}(x)) = 0$.

<u>EXAMPLE 1</u>. The ("informal") derivation

$$\dfrac{[T] \quad \{A\} \Rightarrow A \qquad\qquad [TE] \quad \{A\} \Rightarrow \bar{0} = \bar{0}}{[\&I] \quad \{A\} \Rightarrow A \;\&\; \bar{0} = \bar{0}}$$

is formalized[*] by the function ϕ defined by

$$\begin{aligned}
\phi\langle\,\rangle &:= \langle \ulcorner \&\mathrm{I} \urcorner, \ulcorner \{A\} \Rightarrow A \;\&\; \bar{0} = \bar{0} \urcorner \rangle \\
\phi\langle 0 \rangle &:= \langle \ulcorner \mathrm{T} \urcorner, \ulcorner \{A\} \Rightarrow A \urcorner \rangle \\
\phi\langle 1 \rangle &:= \langle \ulcorner \mathrm{TE} \urcorner, \ulcorner \{A\} \Rightarrow \bar{0} = \bar{0} \urcorner \rangle \\
\phi u &:= 0 \quad \text{for every } u \notin \{\langle\,\rangle, \langle 0 \rangle, \langle 1 \rangle\}.
\end{aligned}$$

<u>EXAMPLE 2</u>. The derivation

$$\frac{\{\ [\mathrm{TE}] \quad \emptyset \Rightarrow f_k(\bar{n})=0\ \}_{n<\omega}}{[\forall\mathrm{I}] \quad \emptyset \Rightarrow \forall x\ f_k(x)=0}$$

is formalized by the ϕ defined by

$$\begin{aligned}
\phi\langle\rangle &:= \langle \ulcorner \forall\mathrm{I} \urcorner, \ulcorner \emptyset \Rightarrow \forall x f_k(x)=0 \urcorner \rangle \\
\phi\langle n \rangle &:= \langle \ulcorner \mathrm{TE} \urcorner, \ulcorner \emptyset \Rightarrow f_k(\bar{n})=0 \urcorner \rangle \qquad \text{for } n<\omega \\
\phi u &:= 0 \quad \text{if } \underline{\mathrm{lth}}(u) > 1.
\end{aligned}$$

A number d is a *recursive derivation of* A^{∞} (notation: $\underline{\mathrm{Der}}^{\infty}_{\mathrm{rec}}(d)$) if $\{d\}$ is a total function (i.e., $\forall x \exists y T(d,x,y)$) and clauses (1)-(3) above hold when ϕ and $=$ are replaced by $\{d\}$ and $\simeq$ respectively.

$$\begin{aligned}
\underline{\mathrm{Prf}}^{\infty}(\phi,s) &:\equiv \underline{\mathrm{Der}}^{\infty}(\phi) \;\&\; (\phi\langle\rangle)_1 = s \\
\underline{\mathrm{Prf}}^{\infty}(\phi,\ulcorner F \urcorner) &:\equiv \underline{\mathrm{Prf}}^{\infty}(\phi,\ulcorner \Rightarrow F \urcorner)
\end{aligned}$$

(The formal ambiguity of $\underline{\mathrm{Prf}}^{\infty}$ will never cause any trouble.)
A derivation ϕ is *normal* (notation: $\underline{\mathrm{NDer}}^{\infty}(\phi)$) if:

(1) No major (i.e., leftmost) premiss of an elimination rule in ϕ is derived by an instance of an introduction rule;

(2) No major premiss of an elimination rule nor a premiss of an instance of [∃I] or [FE] is derived by an instance of [vE], [∃E] or [⊥].

(The reference to [∃I] and [FE] in (2) is made for technical reasons: it simplifies a bit the proofs in part B, since it implies that equations may stand only at top nodes of the normal derivations treated there.)
Predicates like $\underline{\mathrm{NDer}}^{\infty}_{\mathrm{rec}}(d)$, $\underline{\mathrm{NPrf}}^{\infty}_{\mathrm{rec}}(d,\ulcorner F \urcorner)$, $\underline{\mathrm{Prf}}^{\infty}_{\mathrm{rec}}(d,s)$ etc. are defined now in an obvious manner.

The central property of normal derivations is the *subformula property*: every formula occurring in a normal derivation is a subformula of the derived sequent. Another property of normal derivations which we use is the disjunction instantiation property: If $\underline{\mathrm{Prf}}^{\infty}(\phi, \ulcorner F \vee G \urcorner)$ then $\underline{\mathrm{Prf}}^{\infty}(\phi^{\langle 0 \rangle}, \ulcorner F \urcorner)$ or $\underline{\mathrm{Prf}}^{\infty}(\phi^{\langle 0 \rangle}, \ulcorner G \urcorner)$, where $\phi^{\langle 0 \rangle}$ is the "main" subderivation of ϕ $(\phi^{\langle 0 \rangle}(u) := \phi(\langle 0 \rangle * u))$ (see A.3.8/9).

REMARK. The use of sequents in the formulation of A^{∞} above should not mislead the reader to view A^{∞} as a sequential calculus. Sequents are used here only as a convenience in describing a natural-deduction system. In sequential calculi the precedent and the succedent of a sequent play a symmetric role, and there are two introduction rules for each logical constant, while here (as in all natural deduction systems) there is an introduction and an elimination rule for each constant, both operating on the succedent.

1.2. REGULAR AND STRONGLY REGULAR THEORIES

Let T be a theory in the language of analysis. Write

$$A^{\infty}[T] \quad :\equiv \quad \{F \mid T \vdash \neg\neg\exists\phi\ \underline{\mathrm{NPrf}}^{\infty}(\phi, \ulcorner F \urcorner)\}$$

$$A^{\infty}_{\mathrm{rec}}[T] :\equiv \quad \{F \mid \exists d\ [T \vdash \underline{\mathrm{NPrf}}^{\infty}_{\mathrm{rec}}(d, \ulcorner F \urcorner)]\}$$

or, otherwise stated,

$$\underline{\mathrm{Pr}}_{A^{\infty}[T]}(\ulcorner F \urcorner) \quad :\equiv \underline{\mathrm{Pr}}_{T}\ulcorner \neg\neg\exists\phi\ \underline{\mathrm{NPrf}}^{\infty}(\phi, \ulcorner F \urcorner)\urcorner$$

$$\underline{\mathrm{Pr}}_{A^{\infty}_{\mathrm{rec}}[T]}(\ulcorner F \urcorner) :\equiv \exists d\, \underline{\mathrm{Pr}}_{T}\ulcorner \underline{\mathrm{NPrf}}^{\infty}_{\mathrm{rec}}(d, \ulcorner F \urcorner)\urcorner.$$

In A.3 below we prove (in $V_0 + L_{\omega} + \mathrm{BI}$ say) that each derivation ϕ of A^{∞} can be mapped into a derivation ϕ_N which is normal, recursive in ϕ and proves the same sequent as ϕ. Hence, if $T \supseteq V_0 + L_{\omega} + \mathrm{BI}$, we can replace in all the definition above $\underline{\mathrm{NPrf}}$ by $\underline{\mathrm{Prf}}$.

We could formally strenghten the absoluteness results proved in part B below by modifying the definitions of $\underline{\mathrm{Pr}}_{A^{\infty}[T]}$, $\underline{\mathrm{Pr}}_{A^{\infty}_{\mathrm{rec}}[T]}$ above in yet another way, namely - by inserting double negations wherever they make sense. We do not see however any natural applications of these refinements.

An r.e. set A^* of sentences of arithmetic closed under Modus Ponens is a *regular number theory* when for some consistent r.e. theory T in the language of analysis (cf. P), $A^* \subseteq A^\infty[T]$. A theory T^* in a language extending the language of arithmetic is *regular* if $A[T^*]$, i.e., the arithmetical fragment of T^*, is a regular number theory. When referring to A^*, T^* as above, we shall assume that $T \supseteq V_0 + \mathrm{BI}$. This assumption does not of course affect the generality of the discussion, since anyhow

$$A[T^*] \subseteq A^\infty[T] \subseteq A^\infty[T + V_0 + \mathrm{BI}]$$

and we may replace a given theory T by $T + V_0 + \mathrm{BI}$. On the other hand, this convention renders the set of infinitary derivations of $A^\infty[T]$ closed under operations which are proved in $V_0 + \mathrm{BI}$ to preserve the correctness of proofs.

A theory T^* is *strongly regular* if there is a theory T (as above) s.t.

$$A[T^*] \subseteq A^\infty_{\mathrm{rec}}[T]$$

and where

$$T^+ :\equiv T + \mathrm{AC}^-_{00} + \Pi^C_1$$

is consistent. Here AC^-_{00} is a "negative" intuitionistic version of the axiom of choice from numbers to numbers:

$$\mathrm{AC}^-_{00} \qquad \forall x \neg\neg\exists y\, A(x,y) \longrightarrow \neg\neg\exists\alpha\forall x\, A(x,\alpha x)$$

and Π^0_1 is the set of all true Π^0_1 sentences. Formally, a proof predicate $\underline{\mathrm{Prf}}_{T^+}$ for T^+ may be defined from a proof predicate $\underline{\mathrm{Prf}}_T$ for T by the Π^0_1 predicate

$$\underline{\mathrm{Prf}}_{T^+}(p,\ulcorner F\urcorner) :\equiv \exists x{<}p \text{ "x encodes a conjunction of instances of BI, } \mathrm{AC}^-_{00} \text{ and of true } \Pi^0_1 \text{ sentences" } \;\&\; \underline{\mathrm{Prf}}_T(p,\underline{\mathrm{imp}}(x,\ulcorner F\urcorner))$$

where $\underline{\mathrm{imp}}$ is a prim.rec. function which satisfies

$$\underline{\mathrm{imp}}(\ulcorner G\urcorner,\ulcorner F\urcorner) = \ulcorner G \to F\urcorner.$$

The motivation for the condition on T^+ is of a technical nature, and will be clear from the proof of theorem II in B.

A.2. GENERAL PROPERTIES OF THE CLASS OF REGULAR THEORIES

The aim of this chapter is to show that the class of regular theories is quite large, and that it satisfies some natural closure properties. (compare Int.4).

2.1. ARITHMETIC, THE THEORY OF SPECIES AND TYPE THEORY ARE (STRONGLY) REGULAR

There is an obvious embedding of A in $A^{\infty}_{\mathrm{rec}}$ (cf. Int.3), and so

$$A \subseteq A^{\infty}_{\mathrm{rec}}[V_0 + \mathrm{BI}].$$

Hence A is regular. In A.4 below we also prove that the theory of species L_2, as well as type theory are regular; namely,

$$A[L_2] \subseteq A^{\infty}_{\mathrm{rec}}[L_2 + V_0 + \mathrm{BI}]$$

$$A[L_\omega] \subseteq A^{\infty}_{\mathrm{rec}}[L_\omega + V_0 + \mathrm{BI}]$$

(actually BI is redundant everywhere, and even after dropping it the inclusions are proper. See LEIVANT [A]). So L_2 and L_ω are also regular. Of course, the last assertion implies the first two ones, since $A \subseteq A[L_2] \subseteq A[L_\omega]$, and when $T_1 \subseteq T_2$, then the (strong) regularity of T_2 implies trivially that of T_1.

Assuming that $L_\omega + \mathrm{AC}^{-}_{00} + \Pi^0_1$ is consistent (or that $L^c_\omega + \mathrm{AC}_{00}$ is 1-consistent, cf. KREISEL-LEVY [68] §9) we have

$$(L_\omega)^+ \subseteq L_\omega + AC^-_{00} + \Pi^0_1$$

(the operation $(\ )^+$ is defined in A.1.2) must also be consistent. Hence A, L_2, L_ω are all *strongly* regular.

2.2. ADDING SELF-CONSISTENCY TO A REGULAR THEORY

2.2.1. LEMMA. *There exists a prim.rec. function* p *s.t.*

$$\vdash_A \mathrm{F} \rightarrow \underline{\mathrm{Prf}}^\infty_{\mathrm{rec}}(\mathrm{p}(\ulcorner \mathrm{F} \urcorner), \ulcorner \mathrm{F} \urcorner)$$

for any prenex Π^0_2 *sentence* F.

PROOF. If E is an equation, and $\{\mathrm{p}(\ulcorner \mathrm{E} \urcorner)\}$ describes the singleton derivation

$$[\mathrm{TE}] \quad \Rightarrow \mathrm{E}$$

then, clearly,

$$\mathrm{E} \longrightarrow \underline{\mathrm{Prf}}^\infty_{\mathrm{rec}}(\mathrm{p}(\ulcorner \mathrm{E} \urcorner), \ulcorner \mathrm{E} \urcorner).$$

Next, if F is Σ^0_1, $\mathrm{F} \equiv: \exists x \mathrm{E}x$, then

(1) $$\mathrm{F} \longrightarrow \underline{\mathrm{Prf}}^\infty_{\mathrm{rec}}(\mathrm{p}(\ulcorner \mathrm{F} \urcorner), \ulcorner \mathrm{F} \urcorner)$$

where $\{\mathrm{p}(\ulcorner \mathrm{F} \urcorner)\}$ describes

$$[\mathrm{TE}] \quad \Rightarrow \mathrm{E}(\overline{\mu x.\mathrm{E}x})$$

$$[\exists \mathrm{I}] \quad \Rightarrow \mathrm{F}$$

Finally, if F is Π^0_2, $\mathrm{F} \equiv: \forall x \mathrm{F}_0 x$, then (1) where $\{\mathrm{p}(\ulcorner \mathrm{F} \urcorner)\}$ is defined to be the description of

$$\frac{\langle \phi_n \rangle_{n<\omega}}{[\forall \mathrm{I}] \quad \Rightarrow \mathrm{F}}$$

when ϕ_n is described by $\mathrm{p}(\ulcorner \mathrm{F}_0(\bar{n}) \urcorner)$, i.e. -

$$\{p(\ulcorner F\urcorner)\}\langle\ \rangle \ := \langle\ulcorner\forall I\urcorner,\ulcorner\Rightarrow F\urcorner\rangle$$

$$\{p(\ulcorner F\urcorner)\}(\langle n\rangle * u) := \{p(\ulcorner F_0(\bar{u})\urcorner)\}u.$$

p is now a prim.rec. function by the s.m.n. theorem. □

<u>REMARK</u>. (1) above is of course uniform, i.e., for a Π^0_2 open formula F(x),

$$\forall x\ [\ F(x) \ \rightarrow\ \underline{\mathrm{Prf}}_{\mathrm{rec}}(p(\ulcorner F(\bar{x})\urcorner),\ulcorner F(\bar{x})\urcorner)\ ]$$

(where $\ulcorner F(\bar{x})\urcorner$ is $\underline{\mathrm{sub}}(\ulcorner F(a)\urcorner,\underline{\mathrm{num}}(x))$:= the result of substituting the numeral with value x for the parameter a in F). So, for a Σ^0_3 sentence $\exists x F(x) \equiv: G$

$$G \ \rightarrow\ \exists x\ \underline{\mathrm{Prf}}^{\infty}_{\mathrm{rec}}(p(\ulcorner F(\bar{x})\urcorner),\ulcorner F(\bar{x})\urcorner)$$

$$\rightarrow\ \exists d\ \underline{\mathrm{Prf}}^{\infty}_{\mathrm{rec}}(d,\ulcorner G\urcorner) \quad \text{(trivially).}$$

But here d cannot depend primitive recursivelu on $\ulcorner G\urcorner$. (Already for Σ^0_2 sentences $G \equiv \exists x\ F(x)$ one cannot have d depending recursively on G, since one can extract recursively from d a number p s.t. F(p). A partial resursive ϕ yielding $d = \phi\ulcorner G\urcorner$ would therefore allow one to decide recursively membership in an arbitrary Π^0_1 set.)

<u>2.2.2. LEMMA</u>. *Let* $\mathcal{T}$ *be (strongly) regular,*

$$\mathcal{T} \subseteq \mathcal{A}^{\infty}[\mathcal{T}_1] \qquad (\ \mathcal{T} \subseteq \mathcal{A}^{\infty}_{\mathrm{rec}}[\mathcal{T}_1]\ )$$

say. If F *is a* Σ^0_3 *sentence consistent with* $\mathcal{T}_1$ *(with* $\mathcal{T}_1^+$*) then* $\mathcal{T}$ + {F} *is (strongly) regular.*

<u>PROOF</u>. By the remark at the end of 2.2.1.

$$F \vdash_{\mathcal{A}} \exists d\ \underline{\mathrm{Prf}}^{\infty}_{\mathrm{rec}}(d,\ulcorner F\urcorner)$$

and since $\mathcal{T}_1 \supseteq \mathcal{V}_0 \supset \mathcal{A}$ we thus have

$$\vdash_{\mathcal{T}_1 + \{F\}} \exists d\ \underline{\mathrm{Prf}}^{\infty}_{\mathrm{rec}}(d,\ulcorner F\urcorner),$$

i.e.

$$\vdash_{\mathcal{A}^{\infty}_{\mathrm{rec}}[\mathcal{T}_1+\{F\}]} F$$

so

$$T + \{F\} \;\subseteq\; A^{\infty}_{\mathrm{rec}}[T_1 + \{F\}]$$

$$\subseteq\; A^{\infty}[T_1 + \{F\}].$$

Together with the consistency conditions assumed for $T_1 + \{F\}$ this concludes the proof. □

2.2.3. LEMMA. *If* T *is regular then it is consistent.*

PROOF. We have trivially (in V_0)

$$\forall\phi\ \neg\underline{\mathrm{NPrf}}(\phi,\ulcorner\bot\urcorner)$$

and so, if T_1 is consistent then $A^{\infty}[T_1]$ is also consistent, and so must be every $T \subseteq A^{\infty}[T_1]$. □

2.2.4. PROPOSITION. *Let* T *be (strongly) regular,*

$$T \subseteq A^{\infty}[T_1] \qquad (\, T \subseteq A^{\infty}_{\mathrm{rec}}[T_1]\,)$$

where T_1 (T_1^+) *is sound for negations of* Π^0_1 *sentences. If* $\underline{\mathrm{Con}}_T$ *is a canonical consistency sentence for* T *then* $T + \underline{\mathrm{Con}}_T$ *is (strongly) regular.*

PROOF. T is regular and therefore r.e. By 2.2.3 T is also consistent, and so $\underline{\mathrm{Con}}_T$ is a true Π^0_1 sentence which must therefore be consistent with T_1 (with T_1^+). So by 2.2.2 $T + \underline{\mathrm{Con}}_T$ is (strongly) regular. □

By the same token, if T_1 is sound for negations of Π^0_2 sentences, then T+ (global ω-consistency of T) is regular, since the statement added is Π^0_2. Note that the global consistency of T is equivalent to uniform reflection for Π^0_3 on T + (uniform reflection on T) (cf. SMORYNSKI [77] thm.1.1.).

2.3. THEORIES "GENERATED BY TRANSFINITE INDUCTION" ARE REGULAR

Let $\prec$ be a binary predicate and write $x\prec y$ for $\prec(x,y)$.

$$\underline{\mathrm{Step}}^{\prec}_{x}[A(x)] \;:\equiv\; \forall y\prec x\ A(y) \;\rightarrow\; A(x)$$

$$\underline{\mathrm{TI}}^{\prec}_{x}[A(x)] \;:\equiv\; \forall x\ \underline{\mathrm{Step}}^{\prec}_{x}[A(x)] \;\rightarrow\; \forall x\ A(x).$$

2.3.1. PROPOSITION. *Let T be (strongly) regular, $T \subseteq A^{\infty}[T_1]$ where T_1 is sound for Δ^1_1 sentences. Let $\prec$ be a prim.rec. well-ordering, then*

$$T^{\prec} :\equiv T + \{\ \underline{\mathrm{TI}}_x^{\prec}[A(x)]\ \}_{A \text{ arithmetical}}$$

is (strongly) regular.

PROOF. Let $x \prec y$ be expressed by an equation $f_k(x,y)=0$ (f_k prim.rec.), and define

$$x \prec_z y \quad :\equiv \quad x \prec y \prec z$$

Given an arithmetical formula $A(x)$ define the partial recursive function $\phi_z \equiv \lambda u.\phi(z,u) \equiv \lambda u.\{a\}(z,u)$ as a formal description of the following derivation of A^{∞}.

$$\dfrac{\left\{\begin{array}{c}\Sigma_{n,z}\\ \forall x\ \underline{\mathrm{Step}}_x^{\prec}[A(x)] \Rightarrow \bar{n} \preccurlyeq \bar{z} \ \to\ A(\bar{n})\end{array}\right\}_{n<\omega}}{[\forall I] \qquad \forall x\ \underline{\mathrm{Step}}_x^{\prec}[A(x)] \Rightarrow \forall x \preccurlyeq \bar{z}\ A(x)}$$

where $\Sigma_{n,z}$ is

(i) the derivation (represented by)

$$\begin{array}{lll} & [T]\ \ \bar{n} \prec \bar{z} & [T]\ \ \bar{n} = \bar{z} \\ & [FE]\ \ \bot & [FE]\ \ \bot \\ \bar{n} \prec \bar{z} \vee \bar{n} = \bar{z} & [\bot]\ \ A(\bar{n}) & [\bot]\ \ A(\bar{n}) \\ \hline & [\vee E]\ \ A(\bar{n}) & \\ & [\to I]\ \ \bar{n} \preccurlyeq \bar{z} \ \to\ A(\bar{n}) & \end{array}$$

if $\bar{n} \preccurlyeq \bar{z}$ is false (and where we have skipped the premisses of all sequents);
(ii) the derivation

$$\dfrac{\dfrac{\begin{array}{ll} [T] & \forall x\ \underline{\mathrm{Step}}_x^{\prec}[A(x)] \Rightarrow \forall x\ \underline{\mathrm{Step}}_x^{\prec}[A(x)] \\ [\forall E] & \qquad \ldots \ \Rightarrow \forall y \prec \bar{n}\ A(y) \ \to\ A(\bar{n}) \end{array} \qquad \dfrac{\Delta_n}{\forall x\ \underline{\mathrm{Step}}_x^{\prec}[A(x)] \Rightarrow \forall y \prec \bar{n}\ A(y)}}{[\to E] \qquad \forall x\ \underline{\mathrm{Step}}_x^{\prec}[A(x)] \Rightarrow A(\bar{n})}}{[\to I] \qquad \forall x\ \underline{\mathrm{Step}}_x^{\prec}[A(x)] \Rightarrow \bar{n} \preccurlyeq \bar{z} \ \to\ A(\bar{n})}$$

if $\bar{n}\preceq\bar{z}$ is true, and where Δ_n is described by ϕ_n.

ϕ is defined here in terms of $\ulcorner\phi\urcorner = a$ (via $\{a\}$), and so ϕ is well defined by Kleene's recursion theorem (cf. e.g. KLEENE [52] p.352, thm. XXVII).

We may pick the index a to be that one given (primitive recursively) by the proof of the recursion theorem, and define d(z) by $\{d(z)\} \equiv \lambda u.\{a\}(z,u)$. d is a prim. rec. function by the s.m.n. theorem.

Further, let $\{e\} = \{e^{\prec,A}\}$ describe the derivation

$$\begin{array}{ll}
 & \left\{ \begin{array}{c} \Delta_z \\ \forall x\ \underline{\mathrm{Step}}^{\prec}_{x}[A(x)] \Rightarrow \forall x\preceq z\ A(x) \end{array} \right\}_{z<\omega} \\
\hline
[\forall I] & \forall x\ \underline{\mathrm{Step}}^{\prec}_{x}[A(x)] \Rightarrow \forall z \forall x\preceq z\ A(x) \\
[\to I] & \Rightarrow \forall x\ \underline{\mathrm{Step}}^{\prec}_{x}[A(x)] \to \forall z \forall x\preceq z\ A(x)
\end{array}$$

where Δ_z is described by $\{d(z)\}$. The derived formula of $\{e\}$ is clearly a variant of $\underline{\mathrm{TI}}^{\prec}_{x}[A(x)]$.

Using a suitable instance of $\underline{\mathrm{TI}}^{\prec}_{x}[B(x)]$, namely with

$$B_A(y) :\equiv \underline{\mathrm{Prf}}^{\infty}_{\mathrm{rec}}(e^{\prec y,A}, \ulcorner \underline{\mathrm{TI}}^{\prec y}_{x}[A(x)] \urcorner) ,$$

we find quite directly that $\{e^{\prec,A}\}$ is total, and describes a derivation in $A^{\infty}_{\mathrm{rec}}$ of $\underline{\mathrm{TI}}^{\prec}_{x}[A(x)]$. Hence

$$\begin{aligned} T^{\prec} &\subseteq A^{\infty}[T_1 + (\forall \mathrm{arith}.A)\ \underline{\mathrm{TI}}^{\prec}_{x}[B_A(x)]\ ] \\ &\equiv: A^{\infty}[T_1 + W^{\prec}]. \end{aligned}$$

It is easily verified that $W^{\prec}$ is a set of Δ^1_1 sentences, and since T_1 is assumed to be sound for Δ^1_1 sentences, $T_1 + W^{\prec}$ must be consistent. Hence $T^{\prec}$ is regular. The proof for strong regularity is similar. □

2.3.2. The interest in proposition 2.3.1 springs from the proof theoretic power of the schemata $\underline{\mathrm{TI}}^{\prec}_{x}[A(x)]$ (with $\prec$ a prim.rec. well-ordering, A arithmetical) which are complete for classically true arithmetical sentences (cf. KREISEL, SHOENFIELD & WANG [60] §7 thm.6).

It should be noted that the converse of 2.3.1 is false: not every regular theory is an extension of $\mathcal{A}$ (say) by $\{\underline{\mathrm{TI}}^{\prec}_{x}[A(x)]\}_{A\ \mathrm{arith.}}$ for some $\prec$. The main reason for this is simply that a regular theory is not necessarily sound: let F be a false but $\mathcal{A}$-independent Π^0_2 sentence; then $\mathcal{A} + F$ is regular by 2.2.2.

If we restrict attention to classically *sound* regular theories, and relativize the whole discussion to classical systems, then the converse of 2.3.1 does hold, simply by the Kreisel-Shoenfield-Wang theorem mentioned above. For intuitionistic truth and formal systems we do not however yet have an analogue to that theorem (compare TN4).

A.3. NORMALIZATION IN A^{∞}

3.1. AN INFORMAL DESCRIPTION OF THE "NORMALIZATION STRATEGY"

We prove here that every derivation ϕ of A^{∞} for a sequent $\underline{a} \Rightarrow F$ can be transformed into a derivation ϕ_N for $\underline{a} \Rightarrow F$ which is normal in the sense of 1.1, i.e., no major premiss of an instance of an elimination rule in ϕ_N is derived by an introduction rule, and no major premise of an instance of an elimination rule, of [∃I] or of [FE] is derived by an instance of [vE], [∃E] or [⊥]. Further, ϕ_N is recursive and continuous in ϕ, that is, the value of ϕ_N at any given node u in the universal spread is computed recursively from the value of ϕ at a finite number of nodes.

The transformation of ϕ into ϕ_N uses, as in the treatment of finite natural deductions, "reduction-steps" which eliminate local violations in ϕ of the requirements of normality. For lack of a better name we call these violations *cuts*, in analogy to the traditional nomenclature for sequential calculi.

One unfortunate situation is that a reduction which eliminates one cut may create new ones; this is familiar from the finitary case. Here, in addition, the number of reduction-steps cannot be finite, and their order is important. What is essential to the success of the procedure we shall describe is that for each specific node u we can compute $\phi_N(u)$ by performing only a *finite* number of reductions on ϕ. An insight into this can be obtained by looking at the more general treatment given in LEIVANT [A], where it is also shown that the order of the reductions is relevant only up to obvious requirements.

The properties of reduction sequences proved in LEIVANT [A] are in a way analoguous to the strong normalization property of finitary proofs, i.e. - every reduction sequence starting with a given finitary natural deduction terminates. The proofs use however arguments on the *geometry* of infinitary derivations which are combinatorially tedious. For our purpose here all this is irrelevant, so we confine ourselves to the more modest task of giving *one* method of obtaining ϕ_N from ϕ.

The normalization strategy we use can be roughly described as follows. Suppose that we have computed already $\phi_N(v)$ for every $v < u$. Under our conventions on the coding of sequences this means in particular that $\phi_N(v)$ is given for every $v \prec u$. These values have been computed by constructing a certain reduction sequence

$$(*) \qquad \phi \models^1 \phi_1 \models^1 \dots \models^1 \phi_k$$

(cf. 3.1 below), where for $v < u$ $\phi_N(v) := \phi_k(v)$. To compute $\phi_N(u)$ we show now how to extend (*) by

$$\phi_k \models^1 \phi_{k+1} \dots \models^1 \phi_{k+m}$$

so that $\phi_N(v) := \phi_{k+m}(v)$ for $v \leq u$. Examine the inference rules of ϕ_k at $u, u*\langle 0\rangle, u*\langle 0,0\rangle, \dots$, as long as these are eliminations (or [∃I] or [FE]); since ϕ_k is well-founded, this must come to an end. If no cut occurs immediately above any of the examined nodes, let $m := 0$ (i.e., stop); else - let ϕ_{k+1} be obtained by a reduction at the *uppermost* (i.e., maximal) such cut. The process is repeated as long as cuts are found, and our point (to be proved below) is that this may happen only finitely many times.

It should be noted that in ϕ_k above more than one cut can occur along $u, u*\langle 0\rangle, \dots$; namely, if we have a chain of instances of [vE] and [∃E]. If the inference rule in ϕ_k at u is not an elimination (or [∃I] or [FE]) then the sequence $u, u*\langle 0\rangle, \dots$ is empty, and the condition for $m := 0$ is satisfied trivially.

3.2. SOME CONVENTIONS

We slightly modify the formulation of A^∞ in 1.1, so as to allow a smoother exposition. Let an *indexed formula* be a pair $\langle n,F\rangle$, which we write as ${}^nF$, where n is a natural number and F a formula. A *sequent* is now a syntactical object of the form $\underline{a} \Rightarrow G$, where $\underline{a}$ is a finite set of indexed formulae and G is a formula. The inference rules remain as in 1.1, except for the "discharging" inferences; i.e., the [vE] rule for example takes the form

$$\begin{array}{ll} & \underline{a} \Rightarrow A_1 \vee A_2 \quad \underline{a}, {}^{k}A_1 \Rightarrow B \quad \underline{a}, {}^{k}A_2 \Rightarrow B \\ \hline [\vee E_k] & \qquad\qquad \underline{a} \Rightarrow B \end{array}$$

$[\rightarrow I_k]$ and $[\exists E_k]$ are defined similarly.
W.l.o.g. we make the convention that two occurrences of the same formula which are "discharged" at distinct nodes of a derivation ϕ are given distinct indices. Note that normalization for the modified A^∞ implies normalization for the original formulation: indices can be just ignored in ϕ and ϕ_N. They are indeed useful only as track-keepers through the normalization proof.

There is one more modification we make in A^∞, but this one does weaken the results. We add to the rules of A^∞ the replacement rule

$$\begin{array}{ll} & \underline{a} \Rightarrow F(t) \\ \hline [R] & \underline{a} \Rightarrow F(t') \end{array} \qquad \text{where } t' = t.$$

The reason we make this modification is of course our concern for the reader's time and patience: its presence allows a simplified formulation of the reduction steps (see 3.3.2 below).

The normalization proof for the original version can be found in LEIVANT [A] (where it is shown that the obvious "term replacing" derivations which take the place of [R] can be inserted into the proof of normalization). In the absoluteness proofs in part B we do use however normalization for the original A^∞, without [R], because a separation between logic and arithmetic is utilized there, and this separation is destroyed by using [R] in the reductions.

To shorten the discussion of infinitary proof figures we shall use the following notational conventions. Given a derivation ϕ of A^∞ we shall write $\rho^{\phi,u}$ and $s^{\phi,u}$ for the inference rule and the sequent (respectively) standing at the node u in ϕ, i.e.,

$$\phi(u) = \langle \ulcorner \rho^{\phi,u} \urcorner, \ulcorner s^{\phi,u} \urcorner \rangle,$$

and when $s^{\phi,u} \equiv \underline{b} \Rightarrow G$ then $\underline{a}^{\phi,u} := \underline{b}$, $F^{\phi,u} :\equiv G$. Also, we shall freely use *geometrical* representation of derivations, e.g.,

$$\phi = \frac{\{\ \phi^{\langle m\rangle}\ \}_{m<\omega}}{[\rho^{\phi,\langle\rangle}] \quad \underline{a}^{\phi,\langle\rangle} \Rightarrow F^{\phi,\langle\rangle}}$$

(where $\phi^{u} := \lambda x.\phi(u{*}x)$). $\phi[\underline{a}]$ will denote the result of joining the finite set of indexed formulae $\underline{a}$ to all the premisses of sequents in ϕ, i.e.,

$$\phi[\underline{a}](u) := \langle \ulcorner \rho^{\phi,u} \urcorner, \ulcorner \underline{a}^{\phi,u} \cup \underline{a} \Rightarrow F^{\phi,u} \urcorner \rangle .$$

$$\phi[{}^{k}A] := \phi[\{{}^{k}A\}]$$

When $\phi = \begin{array}{c}\phi\\ \underline{a}, {}^{k}A \Rightarrow B\end{array}$, $\psi = \begin{array}{c}\psi\\ \underline{b} \Rightarrow A\end{array}$ then

$$\begin{array}{c}\psi\\ [{}^{k}A]\\ \phi\\ \underline{a} \Rightarrow B\end{array}$$

is defined as the derivation which comes by replacing (or "grafting on") each top node of $\phi[\underline{b}]$ of the form $\underline{c} \cup \underline{b}, {}^{k}A \Rightarrow A$ (where necessarily $\underline{a} \subseteq \underline{c}$) by $\psi[\underline{c}]$, and dropping $[{}^{k}A]$ from all premisses of the result.

Finally, we shall mark in this chapter by asterisks in the margins those passages which can be omitted when only the negative (i.e., free of v and of ∃) fragment of the language of *A* is treated: the reader will get a more transparent view of the proof by skipping these sections on a first reading. The beginning of a paragraph to be skipped is marked by /*, its end by */, and isolated phrases by *.

3.3. PROPER REDUCTIONS

3.3.1. The *critical inference rules* are all the elimination rules ([&E],[→E],[vE],[∀E] and [∃E]), [∃I] and [FE]. These are the rules which may induce a cut:

$\underline{\mathrm{Cut}}(\phi,u) :\equiv$ "$\rho^{\phi,u}$ is an elimination and $\rho^{\phi,u*\langle 0\rangle}$ an introduction rule, or $\rho^{\phi,u}$ is critical and $\rho^{\phi,u*\langle 0\rangle}$ is [vE], [∃E] or [⊥]".

3.3.2. *Detour reductions*

(i) $\&_i$-reduction:

$$\begin{array}{l} \begin{array}{cc} \phi^{\langle 0,0\rangle} & \phi^{\langle 0,1\rangle} \\ \underline{a} \Rightarrow A_0 & \underline{a} \Rightarrow A_1 \end{array} \\ \hline [\&I] \quad \underline{a} \Rightarrow A_0 \,\&\, A_1 \\ [\&E] \quad \underline{a} \Rightarrow A_i \end{array} \quad \models^1 \quad \begin{array}{c} \phi^{\langle 0,i\rangle} \\ \underline{a} \Rightarrow A_i \end{array} \qquad (i=0,1).$$

(ii) $\rightarrow$-reduction:

$$\begin{array}{lll} & \phi^{\langle 0,0\rangle} & \\ & \underline{a}, {}^{k}A \Rightarrow B & \phi^{\langle 1\rangle} \\ \cline{1-2} [\rightarrow I] & \underline{a} \Rightarrow A \rightarrow B & \underline{a} \Rightarrow A \\ \hline [\rightarrow E] & \qquad \underline{a} \Rightarrow B & \end{array} \quad \models^1 \quad \begin{array}{c} \phi^{\langle 1\rangle} \\ [{}^{k}A] \\ \phi^{\langle 0,0\rangle} \\ \underline{a} \Rightarrow B \end{array}$$

* (iii) v_i-reduction:

$$\begin{array}{llll} & \phi^{\langle 0,0\rangle} & & \\ & \underline{a} \Rightarrow A_i & \phi^{\langle 1\rangle} & \phi^{\langle 2\rangle} \\ [vI] & \underline{a} \Rightarrow A_1 \vee A_2 & \underline{a}, {}^{k}A_1 \Rightarrow B & \underline{a}, {}^{k}A_2 \Rightarrow B \\ \hline [vE] & \qquad \underline{a} \Rightarrow B & & \end{array}$$

$$\models^1 \quad \begin{array}{c} \phi^{\langle 0,0\rangle} \\ [{}^{k}A_i] \\ \phi^{\langle i\rangle} \\ \underline{a} \Rightarrow B \end{array} \qquad (i=1,2).$$

(iv) $\forall$-reduction:

$$\begin{array}{l}
\qquad \phi^{\langle 0,n\rangle} \\
\{\ \underline{a} \Rightarrow A(\bar{n})\ \}_{n<\omega} \\ \hline
[\forall I]\quad \underline{a} \Rightarrow \forall x A(x) \\
[\forall E]\quad \underline{a} \Rightarrow A(t)
\end{array}
\quad \models^{1} \quad
\begin{array}{l}
\qquad \phi^{\langle 0,n_0\rangle} \\
\qquad \underline{a} \Rightarrow A(\bar{n}_0) \\ \hline
[R]\quad \underline{a} \Rightarrow A(t)
\end{array}$$

where n_0 is the value of the term t (recall that we treat only sentences, and so all terms are closed).

* (v) $\exists$-reduction:

$$\begin{array}{ll}
\begin{array}{l}
\qquad \phi^{\langle 0,0\rangle} \\
\qquad \underline{a} \Rightarrow A(t) \\ \hline
[\exists I]\quad \underline{a} \Rightarrow \exists x A(x)
\end{array}
&
\begin{array}{c}
\phi^{\langle n+1\rangle} \\
\{\ \underline{a}, {}^{k}A(\bar{n}) \Rightarrow B\ \}_{n<\omega}
\end{array}
\\ \hline
[\exists E] & \qquad \underline{a} \Rightarrow B
\end{array}
\quad \models^{1} \quad
\begin{array}{c}
\psi \\
[{}^{k}A(\bar{n}_0)] \\
\phi^{\langle n_0+1\rangle} \\
\underline{a} \Rightarrow B
\end{array}$$

where n_0 is the value of t, and

$$\psi := \begin{array}{l}
\qquad \phi^{\langle 0,0\rangle} \\
\qquad \underline{a} \Rightarrow A(t) \\ \hline
[R]\quad \underline{a} \Rightarrow A(\bar{n}_0)
\end{array}$$

/* 3.3.3. *Permutative reductions*

Let ρ be a critical inference rule.

$\vee\rho$-reduction:

$$\cfrac{\cfrac{\overset{\phi^{\langle 0,0\rangle}}{\underline{a} \Rightarrow A_1 \vee A_2} \qquad \{\, \overset{\phi^{\langle 0,j\rangle}}{\underline{a}, {}^{k}A_j \Rightarrow B} \,\}_{j=1,2}}{[\vee E] \qquad \underline{a} \Rightarrow B} \qquad \{\, \phi^{\langle n\rangle} \,\}_{n>0}}{[\rho] \quad \underline{a} \Rightarrow C}$$

$$\models^1 \quad \cfrac{\overset{\phi^{\langle 0,0\rangle}}{\underline{a} \Rightarrow A_1 \vee A_2} \qquad \{\, [\rho] \ \cfrac{\overset{\phi^{\langle 0,j\rangle}}{\underline{a}, {}^{k}A_j \Rightarrow B} \qquad \{\, \phi^{\langle n\rangle}[{}^{k}A_j] \,\}_{n>0}}{\underline{a}, {}^{k}A_j \Rightarrow C} \,\}_{j=1,2}}{[\vee E] \qquad \underline{a} \Rightarrow C}$$

*/ $\exists\rho$-reduction: analogous, with $[\exists E]$ in place of $[\vee E]$.

3.3.4. *Absurdity reductions*

Let ρ be a critical inference other than [FE].

$\bot\rho$-reduction:

$$\cfrac{\begin{array}{c} \phi^{\langle 0,0\rangle} \\ \underline{a} \Rightarrow \bot \\ [\bot] \quad \underline{a} \Rightarrow A \end{array} \qquad \{\, \phi^{\langle n+1\rangle} \,\}_{n<\omega}}{[\rho] \quad \underline{a} \Rightarrow B} \qquad \models^1 \qquad \begin{array}{c} \phi^{\langle 0,0\rangle} \\ \underline{a} \Rightarrow \bot \\ [\bot] \quad \underline{a} \Rightarrow B \end{array}$$

$\bot$-[FE]-reduction:

$$\begin{array}{c} \phi^{\langle 0,0\rangle} \\ [\mathrm{FE}]\ \dfrac{[\bot]\ \dfrac{\underline{a} \Rightarrow \bot}{\underline{a} \Rightarrow E}}{\underline{a} \Rightarrow \bot} \end{array} \quad \models^{1} \quad \begin{array}{c} \phi^{\langle 0,0\rangle} \\ \underline{a} \Rightarrow \bot \end{array}$$

The absurdity reductions are the converse of the expansion reductions of PRAWITZ [71], 3.3.3. While Prawitz's aim is to show that the intuitionistic absurdity rule can be reduced to a Post rule (more generally, that an intuitionistic first order system is conservative over the system of its Post rules, cf. PRAWITZ [71], 3.5.5), our aim is to get normal derivations where, roughly speaking, breaking into the internal structure of formulae is avoided when possible.

3.3.5. If $\phi^u \models^1 \psi$ and ϕ_1 comes from ϕ by replacing ϕ^u by ψ (i.e.,

$$\phi_1 v := \begin{cases} \phi v & \text{if } v \not\succcurlyeq u \\ \psi w & \text{if } v = u * w \end{cases} \qquad)$$

then $\phi \models^1 \phi_1$, and we also write more specifically $\phi \models^1_u \phi_1$.

$$\phi \models^n \phi_1 \ :\equiv\ \exists \psi\ [\ \phi = \psi^{\langle 0\rangle}\ \&\ \phi_1 = \psi^{\langle n\rangle}\ \&$$
$$\forall i{<}n\ \psi^{\langle i\rangle} \models^1 \psi^{\langle i+1\rangle}\ ]$$

$$\phi \models \phi_1 \ :\equiv\ \exists n\ \ \phi \models^n \phi_1\ .$$

3.4. DEFINITION OF THE NORMALIZATION STRATEGY; THE NORMALIZATION PREDICATE

3.4.1. Write $\langle n\rangle^i$ for $\underbrace{\langle n,\ldots,n\rangle}_{i\ \text{times}}$.

$$\underline{\mathrm{Infl}}(\phi,u,v)\ :\equiv\ \exists i{<}u\ [\ u = v * \langle 0\rangle^i\ \ \&\ \ \forall w_{v \preccurlyeq w \precneqq u}\ \text{"}\rho^{\phi,w}\text{ is critical"}\ ]$$

$$\equiv\ \text{"u influences v in }\phi\text{".}$$

3.4.2. Define

$$(1) \qquad \underline{\mathrm{Clear}}(\phi,u) \;:\equiv\; \forall w\,[\,\underline{\mathrm{Infl}}(\phi,w,u) \to \neg\underline{\mathrm{Cut}}(\phi,w)\,]$$

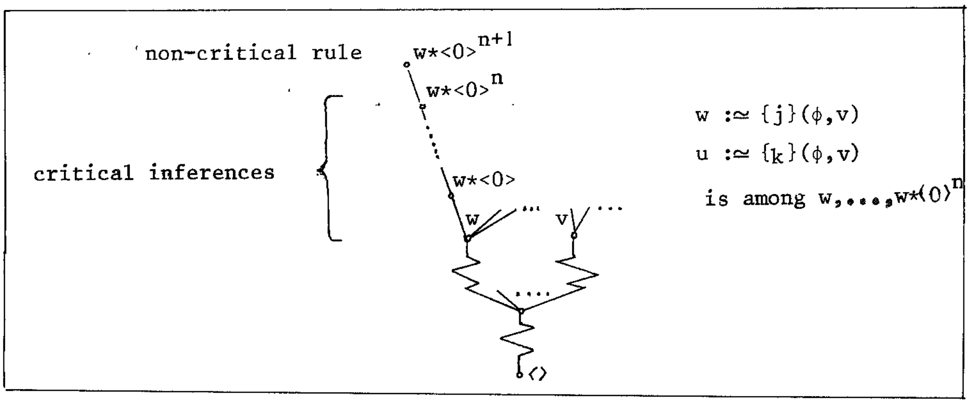
$$(2) \qquad \{j\}(\phi,v) \;:=\; \mu u{\le}v\,.\,\exists w[\underline{\mathrm{Infl}}(\phi,w,u)\;\&\;\underline{\mathrm{Cut}}(\phi,w)].$$

I.e., u is a "clear node" in the derivation ϕ if no cut occurs at a node which "influences" u in ϕ. The function $\{j\}$ picks out the first node up to the argument v which is not clear in ϕ. Here the natural ordering $\le$ is taken in the definition in order to simplify the definition of the normalization procedure defined below. Under our conventions $u \preccurlyeq v$ implies $u \le v$, but replacing $\le$ in the definition of $\{j\}$ by $\preccurlyeq$ would destroy the linear order of the reduction steps we have in mind, and implied by 3.4.4 below.

We further define

$$(3) \qquad \begin{array}{lll} \{k\}(\phi,v) & :\simeq 0 & \text{if } \underline{\mathrm{Clear}}(\phi,v), \\ & :\simeq \underline{\max}[\,u \mid \underline{\mathrm{Infl}}(\phi,u,\{j\}(\phi,v))\;\&\;\underline{\mathrm{Cut}}(\phi,u)\,] & \text{otherwise.} \end{array}$$

I.e., for the node $\{j\}(\phi,v) =: w$ defined above $\{k\}$ picks the maximal node y which influences w and where a cut occurs.



3.4.3. We define the derivation $\{r\}(\phi,u) \equiv \lambda v.\{r_0\}(\phi,u,v)$ by

$$(4) \qquad \begin{array}{ll} \phi \vdash^{1}_{u} \{r\}(\phi,u) & \text{if } \underline{\mathrm{Cut}}(\phi,u) \\ \{r\}(\phi,u) :\simeq \phi & \text{otherwise.} \end{array}$$

Using this notation we now define the normalization functional

$$\phi \mapsto \lambda v.\Psi(\phi,v)$$

by defining $\Psi \equiv \{n\}$ through

$$\begin{aligned}
(5) \qquad \Psi(\phi,v) \ &:\simeq \{n\}(\{r\}(\phi,\{k\}(\phi,v)),v) && \text{if } \{k\}(\phi,v) \not\simeq 0, \\
&:\simeq \phi v && \text{otherwise.}
\end{aligned}$$

It is seen outright that (5) is a correct Herbrand-Gödel definition of Ψ as a partial recursive functional (compare e.g. PETER[67] p.195 or KLEENE[52] sec.54). The reader less familiar with the Herbrand-Gödel definition may prefer to note that Ψ is well-defined by Kleene's recursion theorem (KLEENE[52] p.352, thm. XXVII) and that the index n is given primitive recursively (from the definition (5)) by the proof of that theorem. The intuitive meaning of $\Psi \equiv \{n\}$ is this: if

$$\exists w \leq v \ \neg \underline{\mathrm{Clear}}(\phi,w)$$

then

$$\begin{aligned}
w \ :\simeq \{j\}(\phi,v) &\simeq \mu w \leq v.\neg \underline{\mathrm{Clear}}(\phi,w) \\
&\not\simeq 0
\end{aligned}$$

and so

$$\{k\}(\phi,v) \simeq: u \neq 0$$

as in the illustration above. Letting

$$\phi \models^1_u \phi_1$$

we take

$$\Psi(\phi,v) \ :\simeq \Psi(\phi_1,v).$$

If

$$\neg\exists w \leq v\ \neg\underline{\mathrm{Clear}}(\phi, w)$$

then $\{j\}(\phi,v) \simeq 0$ and so $\{k\}(\phi,v) \simeq 0$ and

$$\Psi(\phi,v) :\simeq \phi v.$$

In other words, $\Psi(\phi,v)$ is obtained by a series of reductions

$$\phi =: \phi_0 \models^{1}_{u_0} \phi_1 \models^{1}_{u_1} \cdots \models^{1}_{u_t} \phi_{t+1} \cdots$$

where

$$u_t :\simeq \{k\}(\phi_t, v) \neq 0.$$

If and when for some t we get

$$\forall w \leq v\ \underline{\mathrm{Clear}}(\phi_t, v)$$

we stop and set

$$\Psi(\phi,v) :\simeq \phi_{t+1}(v).$$

Note that $\Psi(\phi,v) \equiv \{n\}(\phi,v)$ may be defined by different reduction sequences for various values of v, and so it is not evident prima facie that $\{n\}(\phi,v)$ is at all a derivation.

$$\underline{\mathrm{Nmble}}(\phi) :\equiv\ !!\{n\}^{\phi}\ \&\ \underline{\mathrm{WF}}(\{n\}^{\phi})$$

$$\equiv \forall v \exists x\ T^{\phi}(n,v,x)\ \&\ \forall \chi \exists y\ \{n\}(\phi, \bar{\chi}(y)) \simeq 0$$

$$\equiv \text{"}\phi \text{ is normalizable"}.$$

We shall also refer below to the function

$$\{\ell\}(\phi,v) :\simeq \begin{cases} \{\ell\}(\{r\}(\phi,\{k\}(\phi,v)),v) + 1 & \text{if } \exists w{\le}v\ \neg\text{Clear}(\phi,w) \\ 0 & \text{otherwise.} \end{cases}$$

which, like {n} above, may be formally defined by a use of the recursion theorem. Intuitively $\{\ell\}^{\phi}$ measures the length of computation of $\{n\}^{\phi}$.

3.4.4. When for some v $\{k\}(\phi,v) \simeq: u \neq 0$ and $\phi \models^{1}_{u} \phi_1$, we write $\phi \models^{1}_{*} \phi_1$. If $v_1 \le v_2$ and

$$w :\simeq \{j\}(\phi,v_1) :\simeq \mu z{\le}v_1 . \neg\underline{\text{Clear}}(\phi,z)$$
$$\neq 0$$

then

$$\{j\}(\phi,v_2) \simeq \{j\}(\phi,v_1)$$

trivially, and so

$$\{k\}(\phi,v_2) \simeq \{k\}(\phi,v_1).$$

Hence $\phi \models^{1}_{*} \phi_1$ for at most one ϕ_1.

We further define $\models^{t}_{*}$ to be the t-time iteration of $\models^{1}_{*}$. So $\{n\}(\phi,v)$ is (when converging) $\phi_1(v)$, where ϕ_1 is uniquely determined by

$$\phi \models^{\{\ell\}(\phi,v)}_{*} \phi_1$$

3.4.5. LEMMA. *Let* $\underline{\text{Der}}^{\infty}(\phi)$.

(a) $\underline{\text{Nmble}}(\phi) \leftrightarrow \forall\psi\ [\ \phi \models^{1}_{*} \psi \to \underline{\text{Nmble}}(\psi)\ ]$

(b) $\phi \models^{1} \psi \to s^{\phi,\langle\rangle} = s^{\psi,\langle\rangle}$

(c) $\phi \models^{1} \psi \to \{n\}^{\phi} \cong \{n\}^{\psi}$.

PROOF. Obvious from the definitions. □

3.4.6. LEMMA. *Let* $\underline{\text{Der}}^{\infty}(\phi)$. *If* $\phi \models^{1}_{w} \psi$ *and* $w \nprec u$ *then* $\psi u = \phi u$ and $\forall m\ s^{\psi,u*<m>} = s^{\phi,u*<m>}$.

PROOF. Immediate. □

3.4.7. LEMMA. *Let* $\underline{\text{Der}}^{\infty}(\phi)$. *If*

(1) $\underline{\text{Clear}}(\phi,v)$,

(2) $\phi \models^{1}_{w} \psi$ *and*

(3) $w \not< v$

then $\underline{\text{Clear}}(\psi,v)$.

PROOF. Assume (1) - (3), and towards proving $\underline{\text{Clear}}(\psi,v)$ assume

(4) $\underline{\text{Infl}}(\psi,u,v)$, $u = v*\langle 0\rangle^{n}$.

If $w \leqslant u*\langle 0\rangle$ then we must have by (3) and (4)

(5) $w = v*\langle 0\rangle^{k}$ for some k, $0 \leq k \leq n+1$.

We know from 3.4.6 that (2) implies

(6) $\forall y \not\gtrless w\ \psi(y) = \phi(y)$

and by (4)

(7) $\forall i \leq n$ "$\rho^{\psi,v*\langle 0\rangle^{i}}$ is critical".

From (5), (6) and (7)

(8) $\forall i < k$ "$\rho^{\phi,v*\langle 0\rangle^{i}}$ is critical"

while (2) implies that $\rho^{\phi,w}$ must also be critical, from which by (8) and (5)

(9) $\underline{\text{Infl}}(\phi,w,v)$

and so by (1) $\neg\underline{\text{Cut}}(\phi,w)$ contradicting (2). Hence

(10) $w \not\leqslant u*\langle 0\rangle$.

But now we get from (6)

$$(11)\qquad \begin{cases} \rho^{\psi,u} = \rho^{\phi,u} \\[1em] \rho^{\psi,u*<0>} = \rho^{\phi,u*<0>} \end{cases}$$

and from (4), (10) and (6)

$$(12)\qquad \underline{\mathrm{Infl}}(\phi,u,v).$$

(1) and (12) imply $\neg\underline{\mathrm{Cut}}(\phi,u)$, from which by (11) $\neg\underline{\mathrm{Cut}}(\psi,u)$, as required. □

3.4.8. PROPOSITION. *If* $\underline{\mathrm{Prf}}^{\infty}(\phi,s)$ *and* $\underline{\mathrm{Nmble}}(\phi)$ *then* $\underline{\mathrm{NPrf}}^{\infty}(\{n\}^{\phi},s)$.

PROOF. Assume the premiss; then by the definition of the predicate Nmble we have

$$(1)\qquad !!\{n\}^{\phi} \quad \& \quad \underline{\mathrm{WF}}(\{n\}^{\phi}).$$

It remains to prove that $\{n\}^{\phi}$ is locally correct and cut free.

Fix a node v. By the argument of 3.4.4 $!\{n\}(\phi,v)$ implies that

$$(2)\qquad \phi \models_{*}^{\{\ell\}(\phi,v)} \psi_v$$

for a certain derivation ψ_v for which

$$(3)\qquad \forall w \le v\ \underline{\mathrm{Clear}}(\psi_v,w)$$

$$(4)\qquad \{n\}^{\phi}(v) :\simeq \psi_v(v).$$

Likewise we have for each m a derivation $\psi_{v*<m>}$ s.t.

$$(5)\qquad \phi \models_{*}^{\{\ell\}(\phi,v*<m>)} \psi_{v*<m>}$$

$$(6)\qquad \forall w \le v*<m>\ \underline{\mathrm{Clear}}(\psi_{v*<m>},w)$$

$$(7)\qquad \{n\}^{\phi}(v*<m>) :\simeq \psi_{v*<m>}(v*<m>).$$

By 3.4.4 reduction sequence (2) is necessarily a subsequence of (5) (for each m). Fixing m, we thus have for certain $\chi_0,\dots,\chi_t$, $w_1,\dots,w_t$, $t \ge 0$,

(8) $$\psi_v = \chi_0 \models^{1}_{w_1} \chi_1 \models^{1}_{w_2} \cdots \models^{1}_{w_t} \chi_t = \psi_{v*\langle m\rangle}.$$

We prove by induction on t that

(9) $$\forall w \leq v \ \underline{\mathrm{Clear}}(\chi_t, w)$$

(10) $$\chi_t(v) = \psi_v(v)$$

(11) $$s^{\chi_t, v*\langle m\rangle} = s^{\psi_v, v*\langle m\rangle} \qquad \text{for the given m.}$$

For t = 0 (9) - (11) are trivial (cf. (3)). Assuming (9) - (11) for t, we have by (9)

(12) $$w_{t+1} \not\leq v, \quad w_{t+1} \not\preccurlyeq v$$

and so by 3.4.7

(13) $$\forall w \leq v \ \underline{\mathrm{Clear}}(\chi_{t+1}, w)$$

while by 3.4.6 (12) implies

$$\chi_{t+1}(v) = \chi_t(v) = \psi_v(v)$$

and

$$s^{\chi_{t+1}, v*\langle m\rangle} = s^{\chi_t, v*\langle m\rangle} = s^{\psi_v, v*\langle m\rangle}$$

This completes the induction. We thus have from (8)

(14) $$\psi_{v*\langle m\rangle}(v) = \psi_v(v) \qquad \text{(by (10))}$$

$$\simeq \{n\}^{\phi}(v) \qquad \text{(by (4))}$$

(15) $$s^{\psi_{v*\langle m\rangle}, v*\langle m\rangle} = s^{\psi_v, v*\langle m\rangle} \qquad \text{(by (11))}$$

$$= s^{\{n\}^{\phi}, v*\langle m\rangle} \qquad \text{(by (7))}$$

Hence $s^{\{n\}^{\phi}, v}$ and $s^{\{n\}^{\phi}, v*\langle m\rangle}$ relate according to the inference rule $\rho^{\psi_v, v} = \rho^{\{n\}^{\phi}, v}$ and thus $\{n\}^{\phi}$ is a locally correct derivation. Further, we have

$$(16) \qquad \rho^{\{n\}^{\phi},v} = \rho^{\psi_v,v} \qquad \text{(by (4))}$$

$$= \rho^{\psi_{v*<0>},v} \qquad \text{(by (14))}$$

while

$$(17) \qquad \rho^{\{n\}^{\phi},v*<0>} = \rho^{\psi_{v*<0>},v*<0>} \qquad \text{(by (7))}$$

But (6) implies that $\neg\underline{\mathrm{Cut}}(\psi_{v*<0>},v)$ and so (16) and (17) imply $\neg\underline{\mathrm{Cut}}(\phi^{\{n\}},v)$, since $\underline{\mathrm{Cut}}(\chi,u)$ depends on $\rho^{\chi,u}$ and $\rho^{\chi,u*<0>}$ only. Hence $\{n\}^{\phi}$ is cut-free. □

3.5. GENERALIZED REDUCTIONS; STABILITY

The concepts defined in this section are analogues of the ones defined for the finitary natural deduction system for $\mathcal{A}$ in LEIVANT [74], i.e., they are based on the ideas of PRAWITZ [71]'s "validity argument".

3.5.1. The measure of complexity μ on the sentences of the language of $\mathcal{A}$ is defined by recursion on their length as follows.

$$\mu(E) := 0 \qquad \text{if } E \text{ is an equation}$$

$$\mu(A \,\&\, B) := \mu(A \vee B) := \underline{\max}[\mu(A), \mu(B)]$$

$$\mu(\forall x A(x)) := \mu(\exists x A(x)) := \mu(A(\bar{0}))$$

$$\mu(A \to B) := \underline{\max}[\mu(A) + 1, \mu(B)].$$

We also write, for a derivation ϕ of $\mathcal{A}^{\infty}$,

$$|\phi| :\equiv F^{\phi,\langle\rangle}$$

i.e. - $|\phi|$ is the derived sentence of ϕ.

3.5.2. We define now simultaneously by (metamathematical) recursion (on n) two predicates:

$\underline{St}_n(\phi)$ (for "ϕ is stable and $\mu(|\phi|) \le n$")

and

$\phi \overset{n}{\Vdash} \psi$ (for ϕ s.t. $\mu(|\phi|) \le n$).

The metamathematical recursion yields an explicit definition of $\underline{St}_n$ and $\overset{n}{\Vdash}$ in terms of $\underline{St}_m$ and $\overset{m}{\Vdash}$ with $m<n$. When ϕ is recursive, $\phi x \simeq \{d\}x$ say, then these predicates are arithmetical (compare LEIVANT [74] §7), and given a (hyperarithmetical) truth definition for the full language of A, one can define (arithmetically in this truth definition) predicates $\underline{St}$ and $\Vdash$ s.t.

$$(1) \qquad \left.\begin{array}{l} \underline{St}(\{d\}) \longleftrightarrow \underline{St}_n(\{d\}) \\ \{d\} \Vdash \{e\} \longleftrightarrow \{d\} \overset{n}{\Vdash} \{e\} \end{array}\right\} \quad \text{where } n := \mu(|\{d\}|).$$

But there are no *arithmetical* predicates $\underline{St}$, $\Vdash$ satisfying (1). Likewise, a truth definition for the full language of V_0 provides uniform predicates $\underline{St}$ and $\Vdash$ for arbitrary derivations ϕ.

3.5.3. Assume now $\underline{St}_m$, $\overset{m}{\Vdash}$ to be defined for every $m<n$, and let $\mu(|\phi|) \le n$

(i) If $\phi \models^1_u \psi$ where $u :\simeq \{k\}(\phi,<>) \ne 0$ then $\phi \overset{n}{\Vdash}{}^1 \psi$. ($\{k\}$ is defined in 3.4.3. Note that we may have $\chi_1 \models^1_* \chi_2$ while $\{k\}(\chi_1,<>) = 0$).

(ii)
$$\phi = \dfrac{\begin{array}{c}\phi^{\langle 0\rangle}\\ \underline{a} \Rightarrow A_0\end{array} \quad \begin{array}{c}\phi^{\langle 1\rangle}\\ \underline{a} \Rightarrow A_1\end{array}}{[\&I] \ \ \underline{a} \Rightarrow A_0 \& A_1} \quad \overset{n}{\Vdash}{}^1 \quad \begin{array}{c}\phi^{\langle i\rangle}\\ \underline{a} \Rightarrow A_i\end{array} \qquad (i=0,1)$$

(iii)
$$\dfrac{\begin{array}{c}\phi^{\langle 0\rangle}\\ \underline{a}, {}^kA \Rightarrow B\end{array}}{[\to I] \ \ \underline{a} \Rightarrow A \to B} \quad \overset{n}{\Vdash}{}^1 \quad \begin{array}{c}\psi\\ [{}^kA]\\ \phi^{\langle 0\rangle}\\ \underline{a} \Rightarrow B\end{array}$$

whenever $\psi = \begin{array}{c}\psi\\ \underline{a} \Rightarrow A\end{array}$ and $\underline{St}_{\mu(A)}(\psi)$.

* (iv)
$$\dfrac{\begin{array}{c}\phi^{\langle 0\rangle}\\ \underline{a} \Rightarrow A_i\end{array}}{[\vee I] \ \ \underline{a} \Rightarrow A_0 \vee A_1} \quad \overset{n}{\Vdash}{}^1 \quad \begin{array}{c}\phi^{\langle 0\rangle}\\ \underline{a} \Rightarrow A_i\end{array} \qquad (i=0,1)$$

(v)

$$\begin{matrix}\phi^{\langle 0\rangle}\\ \dfrac{\{\ \underline{a}\Rightarrow A(\bar{m})\ \}_{m<\omega}}{[\forall I]\ \ \underline{a}\Rightarrow \forall x A(x)}\end{matrix} \quad \overset{n}{\Vdash}^{1} \quad \begin{matrix}\phi^{\langle m\rangle}\\ \underline{a}\Rightarrow A(\bar{m})\end{matrix} \qquad (m<\omega)$$

/* (vi)

$$\begin{matrix}\phi^{\langle 0\rangle}\\ \dfrac{\underline{a}\Rightarrow A(t)}{[\exists I]\ \ \underline{a}\Rightarrow \exists x A(x)}\end{matrix} \quad \overset{n}{\Vdash}^{1} \quad \begin{matrix}\phi^{\langle 0\rangle}\\ \underline{a}\Rightarrow A(t)\end{matrix}$$

(vii)

$$\dfrac{\overset{\phi^{\langle 0\rangle}}{\underline{a}\Rightarrow A_1 \vee A_2} \qquad \overset{\phi^{\langle j\rangle}}{\{\ \underline{a},{}^{k}A_j\Rightarrow B\ \}_{j=1,2}}}{[\vee E]\qquad \underline{a}\Rightarrow B} \quad \overset{n}{\Vdash}^{1} \quad \begin{matrix}\phi^{\langle j\rangle}\\ \underline{a},{}^{k}A_j\Rightarrow B\end{matrix} \qquad (j=1,2)$$

(viii)

$$\dfrac{\overset{\phi^{\langle 0\rangle}}{\underline{a}\Rightarrow \exists x A(x)} \qquad \overset{\phi^{\langle m+1\rangle}}{\{\ \underline{a},{}^{k}A(\bar{m})\Rightarrow B\ \}_{m<\omega}}}{[\exists E]\qquad \underline{a}\Rightarrow B} \quad \overset{n}{\Vdash}^{1} \quad \begin{matrix}\phi^{\langle m+1\rangle}\\ \underline{a},{}^{k}A(m)\Rightarrow B\end{matrix} \qquad (m<\omega)$$

*/

(ix)

$$\begin{matrix}\phi^{\langle 0\rangle}\\ \dfrac{\underline{a}\Rightarrow \bot}{[\bot]\ \ \underline{a}\Rightarrow A}\end{matrix} \quad \overset{n}{\Vdash}^{1} \quad \begin{matrix}\phi^{\langle 0\rangle}\\ \underline{a}\Rightarrow \bot\end{matrix}$$

Notice that the reduction is always at $\langle\rangle$ in clauses (ii)-(ix).

(x) $\overset{n}{\Vdash}$ is the transitive closure of $\overset{n}{\Vdash}^{1}$:

$$\phi \overset{n}{\Vdash}^{k} \phi_1 \ :\equiv\ \exists\psi\ [\ \phi = \psi^{\langle 0\rangle}\ \ \&\ \ \forall i<k\ \psi^{\langle i\rangle} \overset{n}{\Vdash}^{1} \psi^{\langle i+1\rangle}\ \ \&\ \ \psi^{\langle k\rangle} = \phi_1\ ]$$

$$\phi \overset{n}{\Vdash} \phi_1 \ :\equiv\ \exists k\ \phi \overset{n}{\Vdash}^{k} \phi_1$$

(so in particular $\phi \overset{n}{\Vdash}^{0} \phi$ and $\phi \Vdash \phi$).

(xi) $\underline{\mathrm{St}_n}(\phi)\ :\equiv\ \forall\phi_1\ [\ \phi \overset{n}{\Vdash} \phi_1 \ \rightarrow\ \underline{\mathrm{Nmble}}(\phi_1)\ ]$.

We refer to reduction steps (ii)-(ix) as *improper reductions*. Reductions (vii) and (viii) we label more specifically as *simplifications* (because of their similarity to PRAWITZ [71]'s "immediate simplification" reductions).

In the discussion below we write $\underline{\mathrm{St}}(\phi)$ for $\underline{\mathrm{St}_n}(\phi)$ and $\phi \Vdash \psi$ for $\phi \overset{n}{\Vdash} \psi$ where $n := \mu(|\phi|)$.

3.5.4. LEMMA.

(a) $\underline{St}(\phi) \to \underline{Nmble}(\phi)$

(b) $\phi \models \phi_1$ *and* $\underline{St}(\phi) \to St(\phi_1)$

(c) $\underline{St}(\phi)$ *iff* $[\underline{Nmble}(\phi)$ *and* $[\phi \Vdash \psi$ *by improper reductions* $\to \underline{Nmble}(\psi)]]$.

PROOF. (a) Immediate, since $\phi \overset{0}{\models} \phi$. (b) and (c) are immediate from the definitions. □

3.5.5. $\phi = \genfrac{}{}{0pt}{}{\phi}{\underline{a}, {}^kA \Rightarrow B}$ is *stable at* ${}^kA$ if whenever

$$\psi = \genfrac{}{}{0pt}{}{\psi}{\underline{b} \Rightarrow A}, \quad \underline{St}(\psi)$$

then $\underset{\phi}{\overset{\psi}{[{}^kA]}}$ is stable.

$\phi = \genfrac{}{}{0pt}{}{\phi}{\underline{a} \Rightarrow B}$ is *strongly stable* (notation: $\underline{SSt}(\phi)$) if when $\underline{a} = \{{}^{n_i}A_i\}_i$, and ψ_{i,n_i} are stable derivations with $|\psi_{i,n_i}| = A_i$, then

$$\underset{\phi}{\left\{ \overset{\psi_{i,n_i}}{[{}^{n_i}A_i]} \right\}_{{}^{n_i}A_i \in \underline{a}}} =: \theta$$

is stable, where θ is defined analogously to the definition at the end of 3.2 for the substitution of a single derivation ψ. We write here $\phi \mapsto \theta$.

3.5.6. LEMMA. $\underline{SSt}(\phi) \longrightarrow \underline{St}(\phi)$.

PROOF. Immediate, since the singleton derivations

$$\psi_{i,n_i} := [\ [T]\ \underline{a} \Rightarrow A_i\ ]$$

are trivially stable. □

3.6. THE STABILITY THEOREM; TREATMENT OF THE NON-CRITICAL INFERENCE RULES

3.6.1. Our aim is to prove the following

THEOREM (in $\mathcal{V}_0 + \mathrm{BI}$)

(i) $\underline{\mathrm{Der}}^{\infty}(\phi) \rightarrow \underline{\mathrm{SSt}}(\phi)$

(ii) $\underline{\mathrm{Der}}^{\infty}(\phi) \rightarrow \underline{\mathrm{St}}(\phi)$

(iii) $\underline{\mathrm{Der}}^{\infty}(\phi) \rightarrow \underline{\mathrm{Nmble}}(\phi)$.

Here (i) implies (ii) by 3.5.6, and (ii) implies (iii) by 3.5.4. The proof of (i) proceeds by BI on the proof-tree ϕ. I.e., one proves

$$(1) \qquad \forall m\ \underline{\mathrm{SSt}}(\phi^{u*\langle m\rangle}) \rightarrow \underline{\mathrm{SSt}}(\phi^{u})$$

and so by BI $\underline{\mathrm{SSt}}(\phi^{\langle\rangle})$ and $\phi^{\langle\rangle} = \phi$.
For the top nodes of ϕ, i.e. - where $\rho^{\phi,u}$ is [T] or [TE], the premiss of (1) is satisfied trivially, and the conclusion is immediate from the definitions. So our main concern is to prove (1) for the other cases for $\rho^{\phi,u}$. The cases of non-critical inference rules are treated in 3.6.3 below, the proof for the critical rules being postponed to 3.7.

3.6.2. LEMMA. *Assume* $\underline{\mathrm{Der}}^{\infty}(\phi)$ *and* $\underline{\mathrm{Clear}}(\phi,\langle\rangle)$. *Then*

(i) $\underline{\mathrm{Nmble}}(\phi) \longleftrightarrow \forall m\ \underline{\mathrm{Nmble}}(\phi^{\langle m\rangle})$

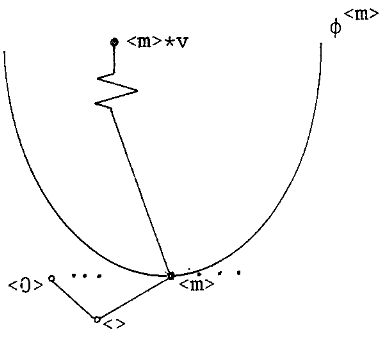
(ii) *If* $\underline{\mathrm{Nmble}}(\phi)$, *then*

$$\{n\}^{\phi^{\langle m\rangle}} = (\{n\}^{\phi})^{\langle m\rangle}$$

PROOF. [a] Assume $\underline{\mathrm{Nmble}}(\phi)$. For each m and v we prove

$$(1) \qquad \{n\}(\phi^{\langle m\rangle},v) \simeq \{n\}(\phi,\langle m\rangle * v),$$

i.e., (ii). But then $!!\{n\}^{\phi}$ implies that $!!\{n\}^{\phi^{\langle m\rangle}}$ and $\underline{\mathrm{WF}}(\{n\}^{\phi})$ implies $\underline{\mathrm{WF}}(\{n\}^{\phi^{\langle m\rangle}})$, so $\underline{\mathrm{Nmble}}(\phi^{\langle m\rangle})$ for every m.



The proof proceeds by induction on $\{\ell\}(\phi,\langle m\rangle * v)$, i.e., we prove by induction $\forall\phi\forall i\ P(\phi,i)$, where

$$P(\phi,i) :\equiv \{\ell\}(\phi,\langle m\rangle * v) \simeq i \ \rightarrow\ \{n\}(\phi^{\langle m\rangle},v) \simeq \{n\}(\phi,\langle m\rangle * v).$$

Since, by $\underline{\text{Nmble}}(\phi)$, $!\{\ell\}(\phi,\langle m\rangle * v)$, we get (1).

Basis. If $\{\ell\}(\phi,\langle m\rangle * v) \simeq 0$ then

(2) $\underline{\text{Clear}}(\phi,\langle m\rangle * v)$,

and so trivially

(3) $\underline{\text{Clear}}(\phi^{\langle m\rangle},v)$.

Hence

$$\begin{aligned} \{n\}(\phi^{\langle m\rangle},v) &:\simeq \phi^{\langle m\rangle}(v) && \text{by (3)} \\ &= \phi(\langle m\rangle * v) \\ &\simeq: \{n\}(\phi,\langle m\rangle * v) && \text{by (2).} \end{aligned}$$

Ind. Step. Let $\{\ell\}(\phi,\langle m\rangle * v) > 0$, $\phi \Vdash^{1}_{u} \phi_1$ for some u, where $\underline{\text{Infl}}(\phi,u,w)$ for some $w \le \langle m\rangle * v$.

$\underline{\text{Clear}}(\phi,\langle\rangle)$ (which we are assuming from the start) implies $\underline{\text{Clear}}(\phi_1,\langle\rangle)$ by 3.4.7, and also $u \neq \langle\rangle$, which in turn implies that

$$(4) \qquad \begin{cases} \phi_1^{\langle q\rangle} = \phi^{\langle q\rangle} & \text{for } q \neq (u)_0 \\ \phi^{\langle (u)_0\rangle} \models^{1}_{*} \phi_1^{\langle (u)_0\rangle} \end{cases}$$

So ϕ_1 satisfies the lemma's conditions, while $\{\ell\}(\phi_1, \langle m\rangle * v) = \{\ell\}(\phi, \langle m\rangle * v) \dot{-} 1$. Hence

$$\begin{aligned} \{n\}(\phi^{\langle m\rangle}, v) &\simeq \{n\}(\phi_1^{\langle m\rangle}, v) && \text{by (4), in any case (cf. 3.4.4)} \\ &\simeq \{n\}(\phi_1, \langle m\rangle * v) && \text{by ind.hyp.} \\ &\simeq \{n\}(\phi, \langle m\rangle * v) && \text{since by 3.4.4 } \{n\}^{\phi} = \{n\}^{\phi_1} \end{aligned}$$

[b] Assume $\forall m\ \underline{\mathrm{Nmble}}(\phi^{\langle m\rangle})$. First, $\underline{\mathrm{Clear}}(\phi, \langle\rangle)$ implies $\{n\}(\phi, \langle\rangle) \simeq \phi\langle\rangle$; so to prove $\underline{\mathrm{Nmble}}(\phi)$ it suffices to prove

$$\{n\}(\phi, \langle m\rangle * v) \simeq \{n\}(\phi^{\langle m\rangle}, v).$$

We cannot here use induction on $\{\ell\}(\phi, \langle m\rangle * v)$, because $!\{\ell\}(\phi, \langle m\rangle * v)$ is precisely what we have to prove. So fixing m and v, we proceed by induction on

$$\{p\}(\phi, m, v) :\simeq \sum_{w \le \langle m\rangle * v} \{\ell\}(\phi^{\langle (w)_0\rangle}, \underline{\mathrm{tail}}(w))$$

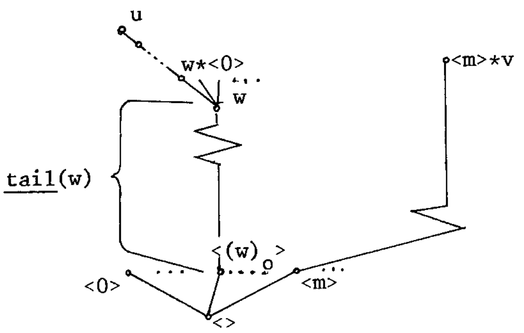
where $\underline{\mathrm{tail}}(w) := \langle (w)_1, \ldots, (w)_{\underline{\mathrm{lth}}(w) \dot{-} 1}\rangle$.

Basis. $\{p\}(\phi, m, v) \simeq 0$. Then for every $w \le \langle m\rangle * v$ $\{\ell\}(\phi^{\langle (w)_0\rangle}, \underline{\mathrm{tail}}(w)) \simeq 0$, hence $\underline{\mathrm{Clear}}(\phi^{\langle (w)_0\rangle}, \underline{\mathrm{tail}}(w))$ and so $\underline{\mathrm{Clear}}(\phi, w)$. So we have

$$\begin{aligned} \{n\}(\phi, \langle m\rangle * v) &:\simeq \phi(\langle m\rangle * v) \\ &= \phi^{\langle m\rangle}(v) \\ &\simeq: \{n\}(\phi^{\langle m\rangle}, v). \end{aligned}$$

Ind. Step. Let $\{p\}(\phi, m, v) > 0$. Then there is at least one $w \le \langle m\rangle * v$ s.t. $\neg\underline{\mathrm{Clear}}(\phi^{\langle (w)_0\rangle}, \underline{\mathrm{tail}}(w))$ and so $\neg\underline{\mathrm{Clear}}(\phi, w)$. Hence, $\phi \models^{1}_{*} \phi_1$, where the reduction occurs at some node u which influences w, and

$$(5) \qquad \begin{cases} \phi_1^{\langle q\rangle} = \phi^{\langle q\rangle} & \text{for } q \ne (u)_0 = (w)_0 \\ \phi^{\langle (w)_0\rangle} \models^{1}_{*} \phi_1^{\langle (w)_0\rangle} & \text{at } \underline{\mathrm{tail}}(u) \end{cases}$$



Since clearly $\{k\}(\phi^{\langle (w)_0\rangle}, \underline{\mathrm{tail}}(w)) \simeq \underline{\mathrm{tail}}(u)$ we have $\{\ell\}(\phi_1^{\langle (w)_0\rangle}, \underline{\mathrm{tail}}(w)) \simeq \{\ell\}(\phi^{\langle (w)_0\rangle}, \underline{\mathrm{tail}}(w)) \dot{-} 1$ and so

$$\{p\}(\phi_1, m, v) < \{p\}(\phi, m, v)$$

while (as in [a]) we conclude by 3.4.7 from the assumed $\underline{\mathrm{Clear}}(\phi, \langle\rangle)$ that

$$\underline{\mathrm{Clear}}(\phi_1, \langle\rangle).$$

Hence

$$\begin{aligned} \{n\}(\phi, \langle m\rangle * v) &\simeq \{n\}(\phi_1, \langle m\rangle * v) && \text{since by 3.4.5 } \{n\}^{\phi} = \{n\}^{\phi_1} \\ &\simeq \{n\}(\phi_1^{\langle m\rangle}, v) && \text{by ind.hyp.} \\ &\simeq \{n\}(\phi^{\langle m\rangle}, v) && \text{by (5) and 3.4.5.} \quad \square \end{aligned}$$

<u>3.6.3. LEMMA</u> (in V_0). *Let* $\mathrm{Der}^{\infty}(\phi)$.

(i) *If* $\rho^{\phi, \langle\rangle}$ *is* [→I]

$$\phi = \cfrac{\overset{\phi^{\langle 0\rangle}}{\underline{a}, {}^{k}A \Rightarrow B}}{\underline{a} \Rightarrow A \to B}$$

say, and $\phi^{\langle 0\rangle}$ *is stable at* ${}^{k}A$ *then* $\underline{\mathrm{St}}(\phi)$.

(ii) *If* $\rho^{\phi,\langle\rangle}$ *is a non-critical inference other than* [→I], *and* $\forall m\ \underline{St}(\phi^{\langle m\rangle})$ *then* $\underline{St}(\phi)$.

<u>PROOF</u>. Let $\phi \Vdash^{n} \psi$; we have to show $\underline{Nmble}(\psi)$.

<u>Case [a]</u>. $n = 0$, $\psi = \phi$. In any case we assume here $\forall m\ \underline{St}(\phi^{<m>})$ and so by 3.5.4 $\forall m\ \underline{Nmble}(\phi^{<m>})$. By 3.6.2 this implies $\underline{Nmble}(\phi)$.

<u>Case [b]</u>. $n = m+1$, $\phi \models^{1} \phi_1 \Vdash^{m} \psi$. Here $\rho^{\phi,<>}$ is not critical, so $\underline{Clear}(\phi,<>)$ and hence $\{k\}(\phi,<>) \simeq 0$; thus necessarily $\phi \Vdash^{1} \phi_1$ by an improper reduction. When $\rho^{\phi,<>}$ is [R] such a reduction is impossible. We are then left with the following cases.

If $\rho^{\phi,\langle\rangle}$ is [→I] as in (i) above, then

$$\phi \Vdash \overset{\psi}{\underset{\phi^{\langle 0\rangle}}{[{}^{k}A]}} = \phi_1 \qquad \text{,for some stable } \psi,$$

so $\underline{St}(\phi_1)$ since $\phi^{\langle 0\rangle}$ is stable at ${}^{k}A$. Hence $\underline{Nmble}(\psi)$.

If $\rho^{\phi,\langle\rangle}$ is [&I], [∀I], [vI] or [⊥I] the proof is similar. □

/* 3.7.1. ϕ is *stable at* ${}^{k}A$ *under* $\overset{\psi}{\underline{a} \Rightarrow A \vee B}$ if whenever

$$\psi \Vdash \begin{array}{c} \xi \\ \underline{b} \Rightarrow A \\ \hline [\vee I]\quad \underline{b} \Rightarrow A \vee B \end{array}$$

then $\overset{\xi}{\underset{\phi}{[{}^{k}A]}}$ is stable. (A symmetric definition for $\psi = \overset{\psi}{B \vee A}$.)

ϕ is *stable at* ${}^{k}A(\bar{n})$ *under* $\overset{\psi}{\underline{a} \Rightarrow \exists x A(x)}$ if whenever

$$\psi \Vdash \begin{array}{c} \xi \\ \underline{b} \Rightarrow A(t) \\ \hline [\exists I]\quad \underline{b} \Rightarrow \exists x A(x) \end{array}, \qquad t = \bar{n}$$

and

$$\theta := \begin{array}{c} \xi \\ \underline{b} \Rightarrow A(t) \\ \hline [R]\quad \underline{b} \Rightarrow A(\bar{n}) \end{array}$$

*/ then $[{}^{k}A(\bar{n})]^{\theta}_{\phi}$ is stable.

3.7.2. <u>MAIN LEMMA</u>. *Let* ϕ *be a derivation of* A^{∞} *which satisfies one of* (i)-(iii) *below; then* $\underline{\mathrm{St}}(\phi)$.

(i) $\rho^{\phi,\langle\rangle}$ *is a critical inference rule other than* [vE] *and* [∃E] *(i.e.* - [&E], [→E], [∀E], [∃I] *or* [FE]*) and* $\bigwedge\limits_{i=0,1} \underline{\mathrm{St}}(\phi^{\langle i\rangle})$.

/* (ii) $\rho^{\phi,\langle\rangle}$ *is* $[\mathrm{vE}_k]$, $F^{\phi,\langle 0\rangle} \equiv: A_1 \vee A_2$, $\bigwedge\limits_{i=0,1,2} \underline{\mathrm{Nmble}}(\phi^{\langle i\rangle})$, *and* $\phi^{\langle j\rangle}$ *is stable at* ${}^{k}A_j$ *under* $\phi^{\langle 0\rangle}$ $(j=1,2)$.

(iii) $\rho^{\phi,\langle\rangle}$ *is* $[\exists\mathrm{E}_k]$, $F^{\phi,\langle 0\rangle} \equiv: \exists x A(x)$, *and for each* $m < \omega$ $\underline{\mathrm{Nmble}}(\phi^{\langle m\rangle})$

*/ *and* $\phi^{\langle m+1\rangle}$ *is stable at* ${}^{k}A(\bar{m})$ *under* $\phi^{\langle 0\rangle}$.

<u>PROOF</u>. (I) Method: the proof proceeds by a primary BI on the tree $\{n\}^{\phi^{\langle 0\rangle}}$. I.e., formally speaking we prove

$$(1) \qquad \forall m\, Q(\phi, u * \langle m\rangle) \rightarrow Q(\phi,u)$$

where

$$Q(\phi,u) :\equiv \forall\psi\, [\, \underline{\mathrm{Der}}^{\infty}(\psi) \;\&\; \underline{\mathrm{Nmble}}(\psi^{\langle 0\rangle}) \;\&\; \text{"}\{n\}^{\psi^{\langle 0\rangle}} \text{ is a subtree of } (\{n\}^{\phi^{\langle 0\rangle}})^{u}\text{"} \rightarrow \text{"the lemma holds for } \psi\text{"}\,].$$

Assuming $\underline{\mathrm{WF}}(\{n\}^{\phi^{\langle 0\rangle}})$, as we do by the lemma's assumptions, we get from (1) by BI $Q(\phi,\langle\rangle)$, which trivially implies that the lemma holds for ϕ.

Further, we shall use a secondary (ordinary) induction on $\{\mathcal{L}\}(\phi^{\langle 0\rangle},\langle\rangle)$, i.e., we prove $\forall m\, R(\phi,m)$ where

$$R(\phi,m) :\equiv \forall\psi\, [\, \underline{\mathrm{Der}}^{\infty}(\psi) \;\&\; \text{"}\{n\}^{\psi^{\langle 0\rangle}} \text{ is a subtree of } \{n\}^{\phi^{\langle 0\rangle}}\text{"} \;\&\; \{\mathcal{L}\}(\psi^{\langle 0\rangle},\langle\rangle) \simeq m \rightarrow \text{"the lemma holds for } \psi\text{"}\,].$$

Now assuming $!\{\mathcal{L}\}(\phi^{\langle 0\rangle},\langle\rangle)$ as we do by the lemma's assumptions, we get trivially from $R(\phi,\{\mathcal{L}\}(\phi^{\langle 0\rangle},\langle\rangle))$ that the lemma holds for ϕ.

(II) A preliminary observation: If ϕ satisfies one of (i)-(iii), and $\phi \models^1_* \phi_1$ by a reduction at some $\langle 0\rangle * u \simeq \{k\}(\phi, \langle\rangle)$, then ϕ_1 satisfies the same condition as ϕ does. The proof is immediate from the definitions.

(III) Let ϕ satisfy one of (i)-(iii), $\phi \Vdash^1_{u_1} \phi_1 \Vdash^1_{u_2} \phi_2 \cdots \Vdash^1_{u_n} \phi_n$. We have to prove $\underline{\text{Nmble}}(\phi_n)$.

If $n = 0$ and $\underline{\text{Clear}}(\phi, \langle\rangle)$, then $\underline{\text{Nmble}}(\phi)$ by 3.6.2(i). If $\neg\underline{\text{Clear}}(\phi, \langle\rangle)$, $\{k\}(\phi, \langle\rangle) \simeq \langle\rangle$, then $\phi \models^1_{\langle\rangle} \psi$ for some ψ, and we shall see in (IV) below that then $\text{St}(\psi)$; hence by 3.4.4/5 $\underline{\text{Nmble}}(\phi)$. Finally, if $\{k\}(\phi, \langle\rangle) \simeq \langle 0\rangle * u$ then $\phi \models^1_* \psi$ with ψ satisfying the lemma's conditions by (II) above, and $\phi^{\langle 0\rangle} \models^1_u \psi^{\langle 0\rangle}$ where $u \simeq \{k\}(\phi^{\langle 0\rangle}, \langle\rangle)$. So $\{n\}^{\phi^{\langle 0\rangle}} = \{n\}^{\psi^{\langle 0\rangle}}$ while $\{\ell\}(\phi^{\langle 0\rangle}, \langle\rangle) \simeq \{\ell\}(\psi^{\langle 0\rangle}, \langle\rangle) + 1$. Hence by the secondary ind.hyp. applied to ψ, $\text{St}(\psi)$ and so (3.4.4/5) $\underline{\text{Nmble}}(\phi)$.

Next, if $n > 0$, it obviously suffices to prove $\underline{\text{St}}(\phi_1)$. If $\phi \models^1_* \phi_1$ we have $\underline{\text{St}}(\phi_1)$ as above.

If $\phi \Vdash^1 \phi_1$ by an improper reduction then $\rho^{\phi, \langle\rangle}$ must be one of [∃I], [∨E], [∃E]. For the first one, $\phi_1 = \phi^{\langle 0\rangle}$ which is assumed stable (case (i) of the lemma); for the last two, $\phi_1 = \phi^{\langle m+1\rangle}$ for some $m < \omega$, which is assumed stable (cases (ii), (iii)). So $\underline{\text{St}}(\phi_1)$ in any case.

(IV) It remains to prove that whenever $\phi \models^1_{\langle\rangle} \psi$ and $\langle\rangle \simeq \{k\}(\phi, \langle\rangle)$ then $\underline{\text{St}}(\psi)$. We inspect cases for the type of reduction, i.e. - for $\rho^{\phi, \langle\rangle}$ and $\rho^{\phi, \langle 0\rangle}$.

<u>case (a)</u>. $\rho^{\phi, \langle\rangle}$ is [→E],

<u>subcase (aα)</u>. $\rho^{\phi, \langle 0\rangle}$ is [→I],

$$\phi = \cfrac{\dfrac{\overset{\displaystyle\phi^{\langle 0,0\rangle}}{\underline{a}, {}^{k}A \Rightarrow B}}{\underline{a} \Rightarrow A \to B} \qquad \overset{\displaystyle\phi^{\langle 1\rangle}}{\underline{a} \Rightarrow A}}{a \Rightarrow B}$$

$$\models^1_{\langle\rangle} \quad \begin{array}{c} \phi^{\langle 1\rangle} \\ [{}^{k}A] \\ \phi^{\langle 0,0\rangle} \\ \\ \underline{a} \Rightarrow B \end{array} \quad =: \psi.$$

Then, since $\underline{\mathrm{St}}(\phi^{\langle 1\rangle})$,

$$\phi^{\langle 0\rangle} \Vdash^{1}_{\langle\rangle} \psi$$

by an improper reduction; and as $\underline{\mathrm{St}}(\phi^{\langle 0\rangle})$ by assumption, $\underline{\mathrm{St}}(\psi)$.

/* <u>subcase (aβ)</u>. $\rho^{\phi,\langle 0\rangle}$ is $[\vee E]$;

$$\phi = \dfrac{\dfrac{\begin{array}{c}\phi^{\langle 0,0\rangle}\\ \underline{a}\Rightarrow A_1 \vee A_2\end{array} \qquad \left\{\begin{array}{c}\phi^{\langle 0,j\rangle}\\ \underline{a},{}^{k}A_j \Rightarrow B\to C\end{array}\right\}_{j=1,2}}{[\vee E] \qquad \underline{a}\Rightarrow B\to C} \qquad \begin{array}{c}\phi^{\langle 1\rangle}\\ \underline{a}\Rightarrow B\end{array}}{[\to E] \qquad \underline{a}\Rightarrow C}$$

$$\models^{1}_{\langle\rangle} \quad \dfrac{\begin{array}{c}\phi^{\langle 0,0\rangle}\\ \underline{a}\Rightarrow A_1 \vee A_2\end{array} \qquad \left\{\dfrac{\begin{array}{c}\phi^{\langle 0,j\rangle}\\ \underline{a},{}^{k}A_j \Rightarrow B\to C\end{array} \qquad \begin{array}{c}\phi^{\langle 1\rangle}[{}^{k}A_j]\\ \underline{a},{}^{k}A_j \Rightarrow B\end{array}}{[\to E] \qquad \underline{a},{}^{k}A_j\Rightarrow C}\right\}_{j=1,2}}{[\vee E] \qquad \underline{a}\Rightarrow C}$$

$$=: \quad \psi$$

We assume $\{k\}(\phi,\langle\rangle) \simeq \langle\rangle$, and so $\underline{\mathrm{Clear}}(\phi^{\langle 0\rangle},\langle\rangle)$. Hence, by 3.6.2, $\underline{\mathrm{St}}(\phi^{\langle 0\rangle})$ implies $\underline{\mathrm{Nmble}}(\phi^{\langle 0,0\rangle})$, and $\{n\}^{\psi^{\langle 0\rangle}} = \{n\}^{\phi^{\langle 0,0\rangle}}$ is a proper subtree of $\{n\}^{\phi^{\langle 0\rangle}}$. So ψ has a "lower" BI measure than ϕ, and to conclude that $\underline{\mathrm{St}}(\psi)$ it remains to check that ψ satisfies case (ii) of the lemma.

We have found that $\underline{\mathrm{Nmble}}(\psi^{\langle 0\rangle})$. We check next that $\underline{\mathrm{Nmble}}(\psi^{\langle j\rangle})$ $(j=1,2)$, also by BI. $\psi^{\langle j,1\rangle} := \phi^{\langle 1\rangle}[{}^{k}A_j]$ is stable by assumption, since $\phi^{\langle 1\rangle}[{}^{k}A_j]$ and $\phi^{\langle 1\rangle}$ behave in the same way for all properties concerning reductions. Finally,

$$\phi^{\langle 0\rangle} \Vdash \phi^{\langle 0,j\rangle} = \psi^{\langle j,0\rangle}$$

by a simplification reduction; so $\underline{\mathrm{St}}(\phi^{\langle 0\rangle})$ implies $\underline{\mathrm{St}}(\psi^{\langle j,0\rangle})$. Hence $\psi^{\langle j\rangle}$ satisfies case (i) of the lemma. Since $\{n\}^{\psi^{\langle j,0\rangle}} = \{n\}^{\phi^{\langle 0,j\rangle}}$ is a proper subtree of $\{n\}^{\phi^{\langle 0\rangle}}$, we get by BI hyp. $\mathrm{St}(\psi^{\langle j\rangle})$ and so $\mathrm{Nmble}(\psi^{\langle j\rangle})$ $(j=1,2)$.

It remains to verify that $\psi^{\langle j\rangle}$ is stable at ${}^{k}A_j$ under $\psi^{\langle 0\rangle}$. So assume

$$(1) \qquad \psi^{\langle 0\rangle} = \phi^{\langle 0,0\rangle} \Vdash^{n} \begin{array}{c} \xi \\ \underline{a} \Rightarrow A_j \\ \hline \underline{a} \Rightarrow A_1 \vee A_2 \end{array}$$

If $n = 0$, then $\mathrm{Cut}(\phi^{\langle 0\rangle},\langle\rangle)$, so $\{k\}(\phi,\langle\rangle) \not\simeq \langle\rangle$, contradicting the assumption of (IV). Else, and

$$\phi^{\langle 0,0\rangle} \models^{1}_{u} \theta \,, \quad u \simeq \{k\}(\phi^{\langle 0,0\rangle},\langle\rangle)$$

then

$$\{k\}(\phi,\langle\rangle) \simeq \langle 0,0\rangle * u \not\simeq \langle\rangle,$$

again a contradiction.

Finally, if $\phi^{\langle 0,0\rangle} \Vdash^{1} \theta$ by an improper reduction, then this must be a simplification, because reduction (1) preserves the derived formula. So $\rho^{\phi,\langle 0,0\rangle}$ is [vE] or [∃E], and as $\rho^{\phi,\langle 0\rangle}$ here is [vE], we thus have $\mathrm{Cut}(\phi,\langle 0\rangle)$, contradiction $\{k\}(\phi,\langle\rangle) \simeq \langle\rangle$ once again. Hence (1) is simply impossible under the inspected conditions.

*/ subcase (aγ) $\rho^{\phi,\langle 0\rangle}$ is [∃E]. Similar to (aβ).

subcase (aδ) $\rho^{\phi,\langle 0\rangle}$ is [⊥].

$$\phi = \begin{array}{c} \begin{array}{cc} \begin{array}{c} \phi^{\langle 0,0\rangle} \\ \underline{a} \Rightarrow \bot \\ \hline [\bot] \quad \underline{a} \Rightarrow A \to B \end{array} & \begin{array}{c} \phi^{\langle 1\rangle} \\ \underline{a} \Rightarrow A \end{array} \end{array} \\ \hline [\to E] \qquad a \Rightarrow B \end{array}$$

$$\models^{1}_{\langle\rangle} \quad \begin{array}{c} \phi^{\langle 0,0\rangle} \\ \underline{a} \Rightarrow \bot \\ \hline [\bot] \quad \underline{a} \Rightarrow B \end{array} \quad =: \psi$$

Here $\underline{St}(\phi^{\langle 0\rangle})$ is assumed, while $\phi^{\langle 0\rangle} \Vdash \phi^{\langle 0,0\rangle}$ by an improper reduction, so $\underline{St}(\phi^{\langle 0,0\rangle})$ outright from the definition of stability.

<u>case (b)</u>. $\rho^{\phi,\langle\rangle}$ is [&E] - similar to case (a).

<u>case (c)</u>. $\rho^{\phi,\langle\rangle}$ is [$\forall$E].

<u>subcase (cα)</u>. $\rho^{\phi,\langle 0\rangle}$ is [$\forall$I].

$$\phi = \cfrac{\cfrac{\left\{\begin{array}{c}\phi^{\langle 0,m\rangle}\\ \underline{a} \Rightarrow A(\bar{m})\end{array}\right\}_{m<\omega}}{[\forall I]\quad \underline{a} \Rightarrow \forall x A(x)}}{[\forall E]\quad \underline{a} \Rightarrow A(t)} \quad \models^{1} \quad \cfrac{\begin{array}{c}\phi^{\langle 0,m_0\rangle}\\ \underline{a} \Rightarrow A(\bar{m}_0)\end{array}}{[R]\quad \underline{a} \Rightarrow A(t)} \quad =:\ \psi$$

where m_0 is the value of t. Here $\underline{St}(\phi^{\langle 0\rangle})$ by assumption, and $\phi^{\langle 0\rangle} \Vdash \phi^{\langle 0,m_0\rangle} = \psi^{\langle 0\rangle}$. So $\underline{St}(\psi^{\langle 0\rangle})$ and by 3.6.3(ii) $\underline{St}(\psi)$.

Other subcases of (c) are treated like the analogous subcases of (a).

<u>case (d)</u>. $\rho^{\phi,\langle\rangle}$ is [FE] or [$\exists$I]; the proof is as for (aβ) - (aδ).

/* <u>case (e)</u>. $\rho^{\phi,\langle\rangle}$ is [vE].

<u>subcase (eα)</u>. $\rho^{\phi,\langle\rangle}$ is [vI].

$$\phi = \cfrac{\cfrac{\begin{array}{c}\phi^{\langle 0,0\rangle}\\ \underline{a} \Rightarrow A_i\end{array}}{[\text{vI}]\quad \underline{a} \Rightarrow A_1 \vee A_2} \qquad \left\{\begin{array}{c}\phi^{\langle j\rangle}\\ \underline{a}, {}^{k}A_j \Rightarrow B\end{array}\right\}_{j=1,2}}{[\text{vE}_k] \qquad\qquad \underline{a} \Rightarrow B}$$

$$\models^{1}_{\langle\rangle} \quad \begin{array}{c}\phi^{\langle 0,0\rangle}\\ {[{}^{k}A_i]}\\ \phi^{\langle i\rangle}\\ \underline{a} \Rightarrow B\end{array} \quad =:\ \psi \qquad\qquad (i=1,2).$$

Then $\underline{St}(\psi)$ outright from the statement of condition (ii).

subcase (eβ). $\rho^{\phi,\langle 0\rangle}$ is [vE].

$$\phi = \dfrac{\dfrac{\overset{\phi^{\langle 0,0\rangle}}{\underline{a} \Rightarrow B_1 \vee B_2} \qquad \left\{\overset{\phi^{\langle 0,i\rangle}}{\underline{a}, {}^{\ell}B_i \Rightarrow A_1 \vee A_2}\right\}_{i=1,2}}{[\vee E_\ell] \qquad \underline{a} \Rightarrow A_1 \vee A_2} \qquad \left\{\overset{\phi^{\langle j\rangle}}{\underline{a}, {}^{k}A_j \Rightarrow C}\right\}_{j=1,2}}{[\vee E_k] \qquad \underline{a} \Rightarrow C}$$

$$\models^{1}_{\langle\rangle} \dfrac{\overset{\phi^{\langle 0,0\rangle}}{\underline{a} \Rightarrow B_1 \vee B_2} \qquad \left\{\dfrac{\overset{\phi^{\langle 0,i\rangle}}{\underline{a}, {}^{\ell}B_i \Rightarrow A_1 \vee A_2} \qquad \left\{\overset{\phi^{\langle j\rangle}[{}^{\ell}B_i]}{\underline{a}, {}^{\ell}B_i, {}^{k}A_j \Rightarrow C}\right\}_{j=1,2}}{[\vee E_k] \qquad \underline{a}, {}^{\ell}B_i \Rightarrow C}\right\}_{i=1,2}}{[\vee E_\ell] \qquad \underline{a} \Rightarrow C}$$

$$=: \psi$$

and recall that we assume $\{k\}(\phi,\langle\rangle) \simeq \langle\rangle$. We find that the BI measure of ψ is lower than that of ϕ as in (aβ), and that Nmble($\psi^{\langle 0\rangle}$), Nmble($\psi^{\langle i,j\rangle}$) (i=1,2; j=0,1,2) and that the induction measure of $\psi^{\langle i\rangle}$ is lower than that of ϕ (i=1,2). To conclude St($\psi^{\langle i\rangle}$) and so Nmble($\psi^{\langle i\rangle}$) (i=1,2) as in (aβ) we have to check here that $\psi^{\langle i\rangle}$ satisfies case (ii) of the lemma, i.e., that in addition to the above $\psi^{\langle i,j\rangle}$ is stable at ${}^{k}A_j$ under $\psi^{\langle i,0\rangle}$ (i=1,2; j=1,2). I.e., we have to verify that when

$$(1) \qquad \psi^{\langle i,0\rangle} = \phi^{\langle 0,i\rangle} \Vdash \dfrac{\overset{\xi}{\underline{a}, {}^{\ell}B_i \Rightarrow A_j}}{\underline{a}, {}^{\ell}B_i \Rightarrow A_1 \vee A_2} =: \theta$$

then

$$\begin{array}{l} \xi \\ [{}^{k}A_j] \\ \quad \phi^{\langle j\rangle}[{}^{\ell}B_i] \end{array}$$

is stable (i,j=1,2). But $\phi^{\langle 0\rangle} \Vdash \phi^{\langle 0,i\rangle}$ by a simplification, so with (1) we have $\phi^{\langle 0\rangle} \Vdash \theta$, and so, since ϕ satisfies case (ii) of the lemma, $\begin{array}{l} \xi \\ [{}^{k}A_j] \\ \phi^{\langle j\rangle}[{}^{\ell}B_i] \end{array}$ is stable as required. Now we can apply BI hyp. to $\psi^{\langle i\rangle}$, and find that $\underline{St}(\psi^{\langle i\rangle})$ $(i=1,2)$.

We conclude from this $\underline{St}(\psi)$ as in (aβ) (i.e., as in (aβ) the stability of $\psi^{\langle i\rangle}$ at ${}^{\ell}B_i$ under $\psi^{\langle 0\rangle}$ is satisfied vacuously because $\{k\}(\phi,\langle\rangle) \simeq \langle\rangle$).

<u>subcases (eγ),(eδ).</u> $\rho^{\phi,\langle 0\rangle}$ is [∃E] or [⊥]. Analogous to (eβ) (compare (aδ)).

<u>case (f).</u> $\rho^{\phi,\langle\rangle}$ is [∃E] - like case (e), mutatis mutandis (compare also

*/ case (c) for the use of [R]). □

<u>3.7.3. LEMMA</u> (in $\mathcal{V}_0$+BI). *Let* $\underline{Der}^{\infty}(\phi)$; *if* $\forall m\ \underline{SSt}(\phi^{\langle m\rangle})$ *then* $\underline{SSt}(\phi)$.

<u>PROOF</u>. [a] If ϕ is a singleton derivation (i.e., $\rho^{\phi,\langle\rangle}$ is [T] or [TE]) then $\underline{SSt}(\phi)$ outright from the definition.

[b] If $\rho^{\phi,\langle\rangle}$ is [→I],

$$\phi = \frac{\overset{\phi^{\langle 0\rangle}}{\underline{a}, {}^{k}B \Rightarrow C}}{\underline{a} \Rightarrow B \to C} \longmapsto \phi_1 = \underset{\phi}{\left\{ \begin{matrix} \psi_{i,n_i} \\ [{}^{n_i}A_i] \end{matrix} \right\}_{{}^{n_i}A_i \in \underline{a}}}$$

then by our convention on indexing ${}^{k}B \notin \underline{a}$, so $\phi_1^{\langle 0\rangle}$ is stable at ${}^{k}B$, and by 3.6.3(i) ϕ_1 is stable.

[c] If $\rho^{\phi,\langle\rangle}$ is a non-critical inference other than [→I], then $\phi \mapsto \phi_1$ implies outright $\phi^{\langle m\rangle} \mapsto \phi_1^{\langle m\rangle}$, so $\forall m\ \underline{SSt}(\phi^{\langle m\rangle})$ implies $\forall m\ \underline{St}(\phi_1^{\langle m\rangle})$ and by 3.6.3(ii) $\underline{St}(\phi_1)$.

[d] If $\rho^{\phi,\langle\rangle}$ is a critical inference other than [vE], [∃E] we obtain that $\underline{SSt}(\phi)$ as in [c], using 3.7.2(i) in place of 3.6.3(ii).

[e] $\rho^{\phi,\langle\rangle}$ is [vE] or [∃E]; $\phi \mapsto \phi_1$ implies $\phi^{\langle 0\rangle} \mapsto \phi_1^{\langle 0\rangle}$, so if $\phi_1^{\langle 0\rangle} \Vdash \psi$ then we get from $\underline{SSt}(\phi^{\langle 0\rangle})$ that $\underline{St}(\psi)$. Consequently, we find as in [b] above that ϕ_1 satisfies the conditions of 3.7.2(ii) (respectively, 3.7.2(iii)), and so $\underline{St}(\phi_1)$. □

This concludes the proof of theorem 3.6.1.

3.8. THE SUBFORMULA PROPERTY

For simplicity, we refer to A^{∞} as given in 1.1, i.e., without indexing and without the replacement rule [R].

Ad hoc definitions:

F sub of G $:\equiv$ F is a subformula of G or $F \equiv \bot$.

F sub of $\underline{a}$ $:\equiv$ F sub of some $G \in \underline{a}$.

$\underline{a}$ sub of $\underline{b}$ $:\equiv$ every $F \in \underline{a}$ is sub of $\underline{b}$.

$s_1 = \underline{a}\Rightarrow F$ sub of $s_2 = \underline{b}\Rightarrow G$ $:\equiv$ $\underline{a} \cup \{F\}$ sub of $\underline{b} \cup \{G\}$.

<u>THEOREM</u> (subformula property). *Let* <u>NDer</u>$^{\infty}(\phi)$, *then for every* u $s^{\phi,u}$ *sub of* $s^{\phi,\langle\rangle}$.

(Note that the theorem refers also to equations!)

<u>PROOF</u> (in $\mathcal{V}_0$; BI is not used). The theorem is an immediate consequence (by ordinary induction on the codes of nodes) of

(i) if $u = v*\langle m\rangle$ is a major premiss of an elimination rule, then $F^{\phi,u}$ sub of $\underline{a}^{\phi,u}$.

(ii) Otherwise, then $S^{\phi,u}$ sub $S^{\phi,v}$.

(ii) is clear by inspection of cases. If $\rho^{\phi,u}$ is [FE],

$$\begin{array}{lll} \textcircled{u}\mapsto & & \underline{a} \Rightarrow E \\ \cline{3-3} \textcircled{v}\mapsto & [FE] & \underline{a} \Rightarrow \bot \end{array}$$

say, then $\rho^{\phi,u}$ cannot be an introduction, since E is an equation, and cannot be $[\bot]$ or an elimination – by our definition of normality. So necessarily $\rho^{\phi,u*\langle 0\rangle}$ is [T], and so E sub $\underline{a}$ and (ii) is satisfied.

(i) is proved by induction on the length of the branch $u, u*\langle 0\rangle, u*\langle 0,0\rangle, \ldots$ in ϕ, which by <u>WF</u>(ϕ) must be finite. Since ϕ is normal, if $\rho^{\phi,u}$ is an elimination, then $\rho^{\phi,u*\langle 0\rangle}$ is either an elimination or [T], so by ind.hyp. (i) holds for $u*\langle 0\rangle$, i.e.,

$$F^{\phi,u*\langle 0\rangle} \text{ sub of } \underline{a}^{\phi,u*\langle 0\rangle} = \underline{a}^{\phi,u} \quad . \tag{1}$$

(a) If $\rho^{\phi,u}$ is [&E] or [∀E] then $\underline{a}^{\phi,u} = \underline{a}^{\phi,u*\langle 0\rangle}$ and $F^{\phi,u}$ sub of $F^{\phi,u*\langle 0\rangle}$, so by (1) we have (i) for u.

(b) If $\rho^{\phi,u}$ is [→E], then $\underline{a}^{\phi,u} = \underline{a}^{\phi,u*\langle 0\rangle} = \underline{a}^{\phi,u*\langle 1\rangle}$ while $F^{\phi,u}$ and $F^{\phi,u*\langle 1\rangle}$ sub of $F^{\phi,u*\langle 0\rangle}$, so by (1) we are done.

(c) If $\rho^{\phi,u}$ is [vE] or [∃E], then $s^{\phi,u*\langle m+1\rangle}$ sub of $s^{\phi,u*\langle 0\rangle}$, and $s^{\phi,u}$ sub of $s^{\phi,u*\langle 1\rangle}$; so by (1) u satisfies (i). □

3.9. THE DISJUNCTION INSTANTIATION PROPERTY

PROPOSITION. *If* $\vdash_{A^*} A_1 \vee A_2$ *(where* A^* *is* A^∞, $A^\infty[T]$ *or* $A^\infty_{\mathrm{rec}}[T]$ *for any* $T \supseteq A$*) then either* $\vdash_{A^*} A_1$ *or* $\vdash_{A^*} A_2$.

PROOF. By the normalization theorem (3.4.8, 3.6.1) if $\underline{\mathrm{Pr}}^\infty(\ulcorner A_1 \vee A_2 \urcorner)$ then for some ϕ $\underline{\mathrm{NPrf}}^\infty(\phi, \ulcorner \Rightarrow A_1 \vee A_2 \urcorner)$. Inspection on the cases for $\rho^{\phi,\langle\rangle}$ shows that the only possible case for this inference is [vI]. So $\underline{\mathrm{Prf}}^\infty(\phi^{\langle 0\rangle}, \ulcorner \Rightarrow A_i \urcorner)$ for i=1 or 2. □

A.4. NORMALIZATION IN L_2^∞; L_ω IS REGULAR

4.0. As explained in Int. 6, the chief raison d'être of the departure of the normalization proof in A.3 from the traditional method of ordinal assignments is our wish to smoothly extend the proof to higher order systems: the "abstract" argument used in the proof may be adapted to the infinitary system L_ω^∞ which combines A^∞ with the language and the inference rules of simple type theory L_ω. For the reader familiar with GIRARD [71],[72] it is probably clear by now how to combine Girard's proof of normalization of L_ω with the argument of A.3 so as to get a normalization proof for L_ω^∞. The more sceptic reader might like some details, and as a compromise we give below a somewhat detailed indication of the proof for the system L_2^∞, which combines A^∞ with the language and rules of the theory of species L_2. This should make it clear that the notions of A.3, though applying to infinitary proof figures, may be combined without further ado with Girard's proof for the corresponding finitary systems. A detailed normalization proof for L_ω^∞ may then be easily supplied by the patient reader.

The main consequence of the proof is that type theory L_ω (and ipso facto also the theory of species L_2) are regular theories, namely

$$A[L_\omega] \subseteq A^\infty_{\mathrm{rec}}[L_\omega + \mathrm{BI}]$$

(for refinements cf. LEIVANT [A]). Assuming that $L_\omega + \mathrm{BI} + \mathrm{AC}_{00}^-$ is consistent (cf. A.1) we have that L_ω is also strongly regular. This last assumption is an immediate consequence of the consistency of $L_\omega^C + \mathrm{AC}_{00}$.

As indicated in TN4, another corollary of the normalization of L_ω^∞ is that L_ω is conservative over A extended with the schema of transfinite induction over each well-founded p.r. ordering.

4.1. DESCRIPTION OF L_2^∞

The language of L_2^∞ is the language of A extended with variables and parameters for species of n-tuples, $\{X_i^n\}_{i<\omega}$, $\{P_i^n\}_{i<\omega}$, $n<\omega$, and corresponding universal quantification $\forall X_i^n$.

When V^n is a variable or a parameter, F is a formula, G is a pseudo-formula (i.e. - some first order variables possibly occur unbounded in G) and $\vec{x}$ is an n-tuple of first-order variables, we write $F[G[\vec{x}]/V]$ for the (pseudo-) formula which comes from F by substituting (simultaneously) $G[\vec{t}/\vec{x}]$ for every occurrence $P(\vec{t})$ in F. Usually no confusion occurs if we skip $\vec{x}$, and so we shall do below.

The inference rules of L_2 are those of A^∞, with the addition of the second order quantification rules:

$$[\forall^2 I] \qquad \frac{\underline{a} \Rightarrow A}{\underline{a} \Rightarrow \forall X\ A[X/P]}$$

where P does not occur in $\underline{a}$ (and X,P are of the same type).

$$[\forall^2 E] \qquad \frac{\underline{a} \Rightarrow \forall X\ A}{\underline{a} \Rightarrow A[G/X]}$$

Derivations and recursive derivations are defined now as for A^∞. We shall use the notations $\underline{\mathrm{Der}}^\infty$, $\underline{\mathrm{Der}}^\infty_{\mathrm{rec}}$ etc. in this chapter for derivations of L_2^∞.

Without loss of generality we assume that each derivation satisfies the convention on parameters of PRAWITZ [71] 1.2.4, i.e. - that no parameter which occurs in the derived sequent of a derivation ϕ is the proper parameter of any $[\forall^2 I]$-inference, and that the same parameter is not the proper parameter of two distinct $[\forall^2 I]$-inferences. Note that any given derivation ϕ may be made to conform to this convention by replacing any occurrence of a parameter p by p_{j+u} where u is the code of the node for which that occurrence acts as the proper parameter when there is such a node (else u := 0), and where j is the largest index of the parameters which occur in the derived sequent of ϕ.

4.2. PROPER REDUCTIONS; NORMALIZATION

The *critical* inference rules of L_2^∞ are those of A^∞ plus $[\forall^2 E]$. Cuts are defined accordingly as in 3.3.1.

Reductions $\models^1$, $\models^n$, $\models$ etc. on derivations are defined as in 3.3, with the addition of permutative- and absurdity-reductions with $[\forall^2 E]$ as the lower (critical) inference rule, and of

$$\forall^2\text{-reductions:} \qquad \cfrac{\cfrac{\begin{array}{c}\phi^{\langle 0,0\rangle}\\ \underline{a} \Rightarrow F[P/X]\end{array}}{[\forall^2 I]\quad \underline{a} \Rightarrow \forall X\ F}}{[\forall^2 E]\quad \underline{a} \Rightarrow F[G/X]} \qquad \models^1 \qquad \begin{array}{c}\phi^{\langle 0,0\rangle}[G/P]\\ \underline{a} \Rightarrow F[G/X]\end{array}$$

Other functions and predicates defined in 3.3 are now adapted to L_2^∞ by taking into account the above modifications.

4.3. BASES, BASING FUNCTIONS

A *basis* is a set $\mathcal{B}$ of derivations (of L_2^∞) which satisfies:

$$\phi \in \mathcal{B} \text{ and } \phi \models^1_* \phi_1 \quad \Rightarrow \quad \phi_1 \in \mathcal{B}.$$

A *basing function* is a finite function β from the second order parameters of the language to bases. We write $\beta = \{\langle P_i, \mathcal{B}_i\rangle\}_{i=1,\dots,k}$ also as

$$\begin{bmatrix}\mathcal{B}_1\\ P_1\end{bmatrix} \cdots \begin{bmatrix}\mathcal{B}_k\\ P_k\end{bmatrix}.$$

This notation makes it easy to denote an extension of a given basing function. (A basing function does not range over *occurrences* of parameters: our convention on parameters makes this unnecessary.)

Given a formula F and a basing function β, the basing function $(\beta \restriction F)$ is defined to be the restriction of β to parameters occurring in F.

4.4. GENERALIZED REDUCTIONS; STABILITY

The measure μ on formulae of L_2^{∞} is defined as in 3.5.1, with the additional clause:

$$\mu(\forall XA) := \mu(A).$$

We refer to triplets $\langle\phi,\beta,F\rangle$, where ϕ is a derivation, β a basing function and F a formula, s.t. $|\phi| :\equiv F^{\phi,\langle\rangle}$ is a substitution instance F^{*} of F, and where $\beta = (\beta\restriction F)$ (F may be thought of as the "skeleton" of ϕ).

We define by metamathematical recursion the predicates $\Vdash^{n}$ and $\underline{\mathrm{St}}_n$ over the triplets satisfying $\mu(F) \leq n$ $(n\geq 0)$. The nature of this recursion is the same as in 3.5.2, and we shall omit the index n

(i) If $\phi \models^{1}_{u} \phi_1$ where $u :\simeq \{k\}(\phi,\langle\rangle) \neq 0$ then

$$\langle\phi,\beta,F\rangle \Vdash^{1} \langle\phi_1,\beta,F\rangle.$$

(ii) If $\underline{\mathrm{St}}(\psi,\beta,F)$, $|\psi| = F^{*}$ then

$$\left\langle\frac{\phi^{\langle 0\rangle}}{[\to I_m]\quad \underline{a} \Rightarrow F^{*} \to G^{*}}, \beta,\ F \to G\right\rangle$$

$$\Vdash^{1} \left\langle \overset{\psi}{\underset{\phi^{\langle 0\rangle}}{[^{m}F^{*}]}}, \beta, G\right\rangle.$$

(iii) $$\left\langle\frac{\phi^{\langle 0\rangle}[P]}{[\forall^{2}I]\quad \underline{a} \Rightarrow \forall X\ F^{*}}, \beta,\ \forall X\ F\right\rangle$$

$$\Vdash^{1} \langle\phi^{\langle 0\rangle}[G/P], \beta[^{\mathcal{B}}_{P}], F[P/X])\rangle$$

where G is any formula, P is the proper parameter of the main inference of ϕ and $\mathcal{B}$ is any basis.

(iv)
$$\left\langle \frac{\phi^{\langle 0\rangle}}{[\&I]\quad \underline{a} \Rightarrow \qquad \& \ F^*_1}, \beta,\ F_0 \ \&\ F_1 \right\rangle$$

$$\Vdash^1 \quad \langle \phi^{\langle i\rangle}, \beta, F^*_i \rangle \qquad\qquad (i=0,1).$$

Other improper reductions are adapted from the corresponding reductions in 3.5.3 in the same manner.

$\Vdash$ is the transitive closure of $\Vdash^1$.

$$\underline{St}(\phi,\beta,F) \ :\equiv\ \forall\phi_1,\beta_1,F_1\ \{\ \langle\phi,\beta,F\rangle \Vdash \langle\phi_1,\beta_1,F_1\rangle \ \longrightarrow\ M(\phi_1,\beta_1,F_1)\ \}$$

where $M(\phi_1,\beta_1,F_1)$ is the formal predicate expressing

$$\mathrm{Nmble}(\phi_1)\ \&\ [\ F_1 \equiv P(\vec{t}) \longrightarrow \phi_1 \in \beta_1(P)\ ].$$

Here, if $P \notin \underline{\mathrm{Domain}}(\beta)$ we let $\phi_1 \in \beta_1(P)$ be definitionally true.

ϕ is *strongly stable* (notation: $\underline{SSt}(\phi)$) if for every derivation ϕ^* which comes from ϕ by substituting (pseudo-) formulae for parameters, and for every basis function β, if

$$\phi^{*+} \ =\ \begin{array}{c} \psi_i \\ \{\ [\ {}^{n_i}A_i]\ \}_{i\in I} \\ \phi^* \end{array} \qquad \text{(like in 3.5.5)}$$

where $\underline{St}(\psi_i,(\beta\upharpoonright A_i),A_i)$, $\ {}^{n_i}A_i \in \underline{a}^{\phi,\langle\rangle}$ (i∈I) then $\underline{St}(\phi^{*+},(\beta\upharpoonright|\phi|),|\phi|)$.

4.5. LEMMA. *Let* $\underline{\mathrm{Der}}^{\infty}(\phi)$.

(i) *If* $\underline{SSt}(\phi)$ *then* $\underline{St}(\phi,(\beta\upharpoonright|\phi|),|\phi|)$ *for any basing function* β.

(ii) *If* $\underline{St}(\phi,\beta,F)$ *for some* β,F, *then* $\underline{\mathrm{Nmble}}(\phi)$.

PROOF. (i) Let, in the definition of $\underline{SSt}(\phi)$ above, $\phi^* := \phi$ and $\phi^{*+} := \phi^*$ by taking

$$\psi_i \ :=\ [\ [T]\ \underline{a}^{\phi^*,\langle\rangle} \Rightarrow A_i\ ]$$

(singleton derivations).

(ii) Take $\Vdash$ of length 0 in the definition of $\underline{St}(\phi,\beta,F)$. □

4.6.1. LEMMA. *Given a basing function* β *and a formula* F, *let*

$$[\![\beta,F]\!] := \{\ \phi\ \mid\ \text{"}|\phi| \text{ is a substitution instance of } F\text{"}$$
$$\&\ \underline{\mathrm{St}}(\phi,(\beta{\restriction}F),F)\ \}.$$

Then ⟦β,F⟧ *is a basis.*

PROOF. Immediate from the definitions. □

4.6.2. SUBSTITUTION LEMMA. *Let* P *not occur in* G.

$$\underline{\mathrm{St}}(\phi,\beta,F[G/P]) \longleftrightarrow \underline{\mathrm{St}}(\phi,\beta\left[\substack{[\![\beta,G]\!]\\ P}\right],F)$$

(or, put differently,

$$[\![\beta,F[G/P]]\!] = [\![\beta\left[\substack{[\![\beta,F]\!]\\ P}\right],F]\!]\ \textit{).}$$

PROOF. By induction on $\mu(F)$.

I. Assume

(1) $\underline{\mathrm{St}}(\phi,\beta^*,F)$ where $\beta^* := \beta\left[\substack{[\![\beta,G]\!]\\ P}\right]$.

If $F \equiv P$, $F[G/P] \equiv G$, then (1) implies $\phi \in \beta^*(P) = [\![\beta,G]\!]$, and so $\underline{\mathrm{St}}(\phi,\beta,F[G/P])$ outright.

If $F \not\equiv P$, let

$$\langle\phi,\beta,F[G/P]\rangle \Vdash^{k} \langle\psi,\gamma,H\rangle.$$

We prove by (a second) induction on k that

$$M(\psi,\gamma,H) :\equiv \underline{\mathrm{Nmble}}(\psi)\ \ \&\ \ [\ H \equiv Q(\vec{t}) \rightarrow \psi \in \gamma(Q)\ ].$$

Basis. $k = 0$, $\psi = \phi$, and so $\underline{\mathrm{Nmble}}(\psi)$ by (1) and 4.5. Further, if $H \equiv F[G/P] \equiv Q(\vec{t})$ then $F \not\equiv P$ implies $F \equiv Q(t)$, and so

$$\begin{aligned} \psi = \phi &\in \beta^*(Q) && \text{by (1)}\\ &=: \beta(Q) = \gamma(Q) && \text{since } Q \not\equiv P \end{aligned}$$

Ind. step. Let

$$\begin{aligned}\langle\phi,\beta,F[G/P]\rangle &\Vdash^{1} \langle\phi_1,\beta_1,F_1[G/P]\rangle\\ &\Vdash^{k} \langle\psi,\gamma,H\rangle\end{aligned}$$

and inspect cases for the first reduction. The only non-trivial cases are the following

[a] [$\rightarrow$I]-reduction; $F \equiv A \rightarrow B$,

$$\phi_1 = \begin{array}{c}\chi\\ {[^{k}A[G/P]]}\\ \phi^{\langle 0\rangle}\end{array}, \qquad \beta_1 = \beta, \qquad F_1 \equiv B$$

where $\underline{\mathrm{St}}(\chi,\beta,A[G/P])$. But $\mu(A) < \mu(F)$, and so by the first ind.hyp. $\underline{\mathrm{St}}(\chi,\beta^{*},A)$. Hence

$$\langle\phi,\beta^{*},F\rangle \Vdash^{1} \langle\phi_1,\beta^{*},F_1\rangle\,,$$

and so by (1) $\underline{\mathrm{St}}(\phi_1,\beta^{*},F_1)$. We may therefore apply the second ind.hyp. to $\langle\phi_1,\beta_1,F_1[G/P]\rangle$ and conclude $M(\psi,\gamma,H)$.

[b] [$\forall^{2}$I]-reduction; $F \equiv \forall X\ A$

$$\phi_1 = \phi^{\langle 0\rangle}[D/Q], \qquad \beta_1 = \beta[\genfrac{}{}{0pt}{}{\mathcal{B}}{Q}], \qquad F_1 \equiv A[Q/X].$$

So

$$\langle\phi,\beta^{*},F\rangle \Vdash^{1} \langle\phi_1,\beta^{*}[\genfrac{}{}{0pt}{}{\mathcal{B}}{Q}],F_1\rangle,$$

and so by (1)

$$\underline{\mathrm{St}}(\phi_1,\beta[\genfrac{}{}{0pt}{}{[\![\beta,G]\!]}{P}][\genfrac{}{}{0pt}{}{\mathcal{B}}{Q}],F_1). \tag{2}$$

In order to apply ind.hyp. to $\langle\phi_1,\beta_1,F_1[G/P]\rangle$ we have to know however that (1) applies, i.e., that

$$\underline{\mathrm{St}}(\phi_1,\beta_1^{*},F) \quad \text{where} \quad \beta_1^{*} = \beta[\genfrac{}{}{0pt}{}{[\![\beta_1,G]\!]}{P}][\genfrac{}{}{0pt}{}{\mathcal{B}}{Q}]. \tag{3}$$

But $(\beta_1 \vdash G) = (\beta \vdash G)$ because P does not occur in G, and so

$$[\![\beta_1, G]\!] = [\![\beta, G]\!] \;;$$

hence (2) implies (3), as required.

II. Assume

$$(4) \qquad \underline{St}(\phi,\beta,F[G/P])$$

and let

$$\langle\phi,\beta^*,F\rangle \Vdash^k \langle\psi,\gamma,H\rangle .$$

As in I above we prove that $M(\psi,\gamma,H)$ by induction on k. The induction step is symmetric to that in I, while for the induction basis we note that when $F \equiv P$, $F[G/P] \equiv G$, then

$$\phi \in \beta^*(P) := [\![\beta, G]\!]$$

by (4). □

The reader might note, in connection to the proof above, that GIRARD's [72] proof of the substitution lemma is not quite accurate: the application of the ind.hyp. given at bottom p.II.1.6 should yield $[\underline{\alpha},\gamma/\underline{A},G]\underline{\sigma}$ within the l.h.s., in place of $[\underline{\alpha},\underline{A}]\underline{\sigma}$.

4.7. LEMMA. *Let* $\langle\phi,\beta,F\rangle$ *be a triplet as above, and* $\rho^{\phi,\langle\rangle}$ *be a non-critical inference. If* $\forall m\ \underline{SSt}(\phi^{\langle m\rangle})$ *then* $\underline{SSt}(\phi)$.

PROOF. The proof is totally analogous to that of 3.6.3. To take as an example the only essentially new case, let $\rho^{\phi,\langle\rangle}$ be $[\forall^2 I]$,

$$\phi = \cfrac{\overset{\phi^{\langle 0\rangle}[P]}{\underline{a} \Rightarrow F[P/X]}}{\underline{a} \Rightarrow \forall X\ F} \quad , \quad \underline{a} = \{ {}^{n_i}A_i \}_i$$

and let ϕ^{*+} come from ϕ as in 4.4, i.e

$$\phi^{*+} = \begin{array}{c} \psi_i \\ \{[\,{}^{n_i}A_i\,]\}_i . \\ \phi^* \end{array}$$

In analogy to the proof of 3.6.3, we have to show that $\underline{\mathrm{St}}(\xi,\beta,F[P/X])$ for any basing function β, and where

$$\xi := \begin{array}{c} (\phi^{*+})^{\langle 0\rangle}[G] \\ \underline{a} \Rightarrow F[G/X] \end{array}$$

for some formula G. P cannot occur in $\underline{a}$ and therefore, it does not occur in any ψ_i (by the convention on parameters). Hence ξ may be obtained from $\phi^{\langle 0\rangle *}$ by first substituting G for P and then substituting the derivations ψ_i. Since $\underline{\mathrm{SSt}}(\phi^{\langle 0\rangle})$ is assumed we thus get $\underline{\mathrm{St}}(\xi,\beta,F[P/X])$ as required. □

4.8. LEMMA. *Let* $\langle\phi,\beta,F\rangle$ *be a triplet as above,* $\rho^{\phi,\langle\rangle}$ *be a critical node. If* $\forall m\ \underline{\mathrm{SSt}}(\phi^{\langle m\rangle})$ *then* $\underline{\mathrm{SSt}}(\phi)$.

PROOF. Here again the proof is analogous to the proof in 3.7.2-3 for the first order case. Since $\rho^{\phi,\langle\rangle}$ is not $[\forall^2 I]$,

$$\langle\phi,\beta,F\rangle \Vdash^1 \langle\phi_1,\beta_1,F_1\rangle$$

must imply that $\beta_1 = \beta$, $F_1 \equiv F$; hence the proof in 3.7.2 for derivations is trivially adapted to triplets. The only exception is the case $\rho^{\phi,\langle\rangle} = [\forall^2 E]$, where we have to show (in step (IV)) that if

$$\langle\phi,\beta,F[G/X]\rangle \Vdash^1_{\langle\rangle} \langle\phi_1,\beta,F[G/X]\rangle$$

where

$$\phi = \begin{array}{c} \left.\begin{array}{c} \xi[P] \\ \underline{a} \Rightarrow F^*[P/X] \\ \hline \underline{a} \Rightarrow \forall X\ F^* \end{array}\right\}\eta , \\ \hline \underline{a} \Rightarrow F^*[G^*/X] \end{array} \qquad \phi_1 = \begin{array}{c} \xi \\ \underline{a} \Rightarrow F^*[G^*/X] \end{array}$$

then $\underline{St}(\phi_1,\beta,F[G/X])$. We have, by an improper reduction,

$$\langle\eta,\beta,\forall XF\rangle \mid\!\models^1 \langle\phi_1,\beta[^{\mathcal{B}}_{P}],F[P/X]\rangle$$

for any basis $\mathcal{B}$. Since $\underline{St}(\eta,\beta,\forall XF)$ is assumed here, we thus get

$$(1) \qquad \underline{St}(\phi_1,\beta[^{\mathcal{B}}_{P}],F[P/X]).$$

But picking up in particular $\mathcal{B} := [\![\beta,G]\!]$ we get from (1) by the substitution lemma

$$\underline{St}(\phi_1,\beta,F[G/X])$$

as required. □

4.9. THEOREM. *Every derivation of L_2^∞ is normalizable.*

PROOF. Assume $\underline{Der}^\infty(\phi)$. As in 3.6.1, we get from 4.7, 4.8 by BI $\underline{SSt}(\phi)$ and so by 4.5 Nmble(ϕ). □

4.10. THE REGULARITY OF L_2, L_ω

Since there is a truth definition for L_2 in L_3 (by Tarski's method, compare e.g. TARSKI [36]), the proof of normalization of L_2 above is easily seen to be formalizable in L_3 + BI, and so

$$(1) \qquad A[L_2] \subseteq A[L^\infty_{2,\mathrm{rec}}] \subseteq A^\infty_{\mathrm{rec}}[L_3 + \mathrm{BI}]$$

where $L^\infty_{2,\mathrm{rec}}$ is the system of recursive derivations of L_2^∞. The first inclusion is an immediate corollary of the obvious embedding of L_2 in $L^\infty_{2,\mathrm{rec}}$. When a derivation $\{d\}$ of $L^\infty_{2,\mathrm{rec}}$ proves an arithmetical sentence, then the normal form $\{n\}^{\{d\}}$ of $\{d\}$ is a derivation in $L^\infty_{2,\mathrm{rec}}$ which satisfies the subformula property, and therefore must actually be a derivation of A^∞_{rec}. The local correctness and wellfoundedness of $\{n\}^{\{d\}}$ are proved in the theory in which the normalization of $\{d\}$ is proved, hence the second inclusion in (1).

Actually the embedding of L_2 in $L^\infty_{2,\mathrm{rec}}$ mentioned above assigns to each particular (finitary) proof π of L_2 a proof ϕ of L_2^∞ where there is a bound n on complexity of formulae. Thus for ϕ the normalization proof uses the predicates $\|\!\models^{m}$, $\underline{\mathrm{St}}_m$ only for $m \leq n$, and using a truth definition in Π^1_k-analysis for a suitably large $k = k(n)$ this proof is formalizable in $L^\infty_2 + \mathrm{BI}$. Hence (1) is refined to

$$A[L_2] \leq A^\infty_{\mathrm{rec}}\,[L_2 + \mathrm{BI}].$$

Analogously we have

$$A[L_\omega] \leq A^\infty_{\mathrm{rec}}[L_\omega + \mathrm{BI}].$$

PART B. Absoluteness theorems.

B.0. STATEMENT OF THE RESULTS

When $F[p_1,\dots,p_k]$ is a scheme of L_0 with (at most) the k propositional letters shown, and when $A_1,\dots,A_k$ are arithmetical sentences, write $F[A_1,\dots,A_k]$ for the sentence which comes from $F[p_1,\dots,p_k]$ by substituting A_i for every occurrence of p_i $(i=1,\dots,k)$. When $F[P_1^{n_1},\dots,P_k^{n_k}]$ is a scheme of L_1 with (at most) the k predicate letters shown, where $P_i^{n_i}$ is n_i-place, and $A_i^{n_i}$ is an arithmetical formula with n_i free variables $(i=1,\dots,k)$, write $F[A_1,\dots,A_k]$ for the formula which comes from $F[P_1,\dots,P_k]$ by replacing every atomic subformula $P_i(x_1,\dots,x_{n_i})$ by $A_i(x_1,\dots,x_{n_i})$.

Regular and strongly regular number theories are defined in A.1 above.

THEOREM I (Locally uniform Σ_1^0 absoluteness of L_0).
Let A^* *be a regular number theory. For every* $k < \omega$ *there are* Σ_1^0 *sentences* $A_1,\dots,A_k$ *s.t.*

$$L_0 \nvdash F[p_1,\dots,p_k] \quad \Rightarrow \quad A^* \nvdash F[A_1,\dots,A_k].$$

Or more precisely: there is a quantifier-free (q.f.) formula $E_0(x)$ *s.t.*

$$\forall k \forall x_{\underline{L_0\text{-Fml}}(x)}\left[\neg\underline{\mathrm{Pr}}_{L_0}(x) \ \ \& \ \ \nu(x)\le k \ \rightarrow \ \neg\underline{\mathrm{Pr}}_{A^*}(\underline{\mathrm{sub}}^k_{\Sigma_1^0}(x,\ulcorner E_0\urcorner))\right]$$

is provable in A + "A^* *is regular", where*

$\underline{L_0\text{-Fml}}(x)$:≡ "x *is the g.n. of a schema in the language of* L_0";

$\nu(\ulcorner F\urcorner)$:= "*the number of propositional letters occurring in* F", *and* $\underline{\mathrm{sub}}^{k}_{\Sigma^0_1}$ *is a prim. rec. function which satisfies*

$$\underline{\mathrm{sub}}^{k}_{\Sigma^0_1}(\ulcorner F[p_1,\ldots,p_k]\urcorner, \ulcorner E_0\urcorner) = \ulcorner F[\exists x E_0\langle k,x\rangle,\ldots,\exists x E_0\langle k,x\rangle]\urcorner.$$

THEOREM II (Globally uniform Π^0_2 absoluteness for L_1).
Let A^* *be a strongly regular number theory. There are* Π^0_2 *predicates* $\{A^j_i\}_{i,j<\omega}$ *s.t.*

$$L_1 \nvdash F[P_{i_1}^{n_i},\ldots,P_{i_k}^{n_k}] \Rightarrow A^* \nvdash F[A_{i_1}^{n_1},\ldots,A_{i_k}^{n_k}].$$

Or more precisely: there is a q.f. formula $E_1(x)$ *s.t.*

$$\forall x_{\underline{L_1\text{-Fml}}(x)}\left[\neg\underline{\mathrm{Pr}}_{L_1}(x) \rightarrow \neg\underline{\mathrm{Pr}}_{A^*}(\underline{\mathrm{sub}}_{\Pi^0_2}(x,\ulcorner E_1\urcorner))\right]$$

where $\underline{\mathrm{sub}}_{\Pi^0_2}$ *is a prim. rec. function which satisfies*

$$\underline{\mathrm{sub}}_{\Pi^0_2}(\ulcorner F[P_1^{n_1},\ldots,P_k^{n_k}]\urcorner, \ulcorner E_1\urcorner) = \ulcorner F[Q_1^{n_1},\ldots,Q_k^{n_k}]\urcorner.$$

where

$$Q_i^{n_i}(\vec{z}) :\equiv \forall x\exists y\, E_1\langle x,y,i,n_i,\langle\vec{z}\rangle\rangle.$$

B.1. RECURSION-THEORETIC SOLUTION OF A REDUCED FORM OF THOEREM I

1.0. We wish to find Σ_1^0 sentences $A_1,\dots,A_k$ s.t.

$$(*) \qquad \nvdash_{L_0} F[p_1,\dots,p_k] \quad \Rightarrow \quad \nvdash_{A^*} F[A_1,\dots,A_k].$$

If the theories L_0 and A^* are replaced by their classical completions then $A_1,\dots,A_k$ may be defined by truth-tables arguments using recursion-theoretic methods only, as in KRIPKE [63] and in MYHILL [72]. The complications for the intuitionistic case are the result of the presence of implications in the schema F, or more precisely - of negative nestings of implications. It is in such cases that the intuitionistic interpretation of the logical constants is expressed in an impredicative manner ("for every construction... there exists a construction...").

As in A.3., let us count the negative nestings of implications by a measure μ, i.e.,

$$\begin{aligned}
&\mu\ulcorner F\urcorner := 0 \text{ for atomic } F,\\
&\mu\ulcorner F\&G\urcorner := \mu\ulcorner F\vee G\urcorner := \underline{\max}[\mu\ulcorner F\urcorner,\mu\ulcorner G\urcorner],\\
&\mu\ulcorner F\to G\urcorner := \underline{\max}[\mu\ulcorner F\urcorner+1,\mu\ulcorner G\urcorner]; \text{ and for the full language of } L_1,\\
&\mu\ulcorner \forall xF\urcorner := \mu\ulcorner \exists xF\urcorner := \mu\ulcorner F\urcorner
\end{aligned}$$

We shall see that for schemata F s.t. $\mu\ulcorner F\urcorner \leq 1$ the classical recursion-theoretic methods work. The complexity involved in the growth of the μ-measure is further illustrated by the fact (cf. LEIVANT [74]) that the consistency of A_k is provable in A_{k+1} for every k, where

$$A_k :\equiv A \text{ restricted to formulae } F \text{ s.t. } \mu\ulcorner F\urcorner \leq k.$$

1.1. STATEMENT OF THE REDUCED SOLUTION

We define a sequence U_k of propositional schemata, where $U_k \equiv U_k[p_1,\dots,p_k]$ and $\mu\ulcorner U_k\urcorner \leq 1$ as follows.

$$
\begin{aligned}
U_0 \quad &:\equiv \bot \\
U_1[p] &:\equiv p \vee \neg p.
\end{aligned}
$$

Assuming U_k to be defined, let

$$
\begin{aligned}
U_k^i[p_1,\dots,p_{k+1}] \quad &:\equiv U_k[p_1,\dots,p_{i-1},p_{i+1},\dots,p_{k+1}] \\
U_{k+1}[p_1,\dots,p_{k+1}] &:\equiv \bigvee_{i=1,\dots,k+1} [p_i \rightarrow U_k^i].
\end{aligned}
$$

We shall solve in this section (*) for the schemata U_k, i.e.,

<u>PROPOSITION</u>. *We can uniformly construct* Σ_1^0 *sentences* $A_1^k,\dots,A_k^k$ *s.t.*

$$\nvdash_{A^*} U_k[A_1^k,\dots,A_k^k] \qquad (k<\omega).$$

Here A^* *may be taken to be any consistent r.e. extension of* A *which satisfies disjunction instantiation (the so-called "disjuction property"), i.e.,*

$$\vdash_{A^*} A \vee B \quad \Rightarrow \quad [\vdash_{A^*} A \text{ or } \vdash_{A^*} B].$$

1.2. Actually proposition 1.1 gives a solution of (*) for *all* schemata F s.t. $\mu\ulcorner F\urcorner \leq 1$, on account of the following

<u>PROPOSITION</u>. *For any schema* F *of* L_0 *s.t.* $\mu\ulcorner F\urcorner \leq 1$,

$$\nvdash_{L_0} F[p_1,\dots,p_k] \quad \Rightarrow \quad \vdash_{L_0} F \rightarrow U_k.$$

<u>SKETCH OF PROOF</u>. Use a primary induction on k (= the number of propositional letters occurring in F), secondary induction on the length of F, and ternary induction on the length of the left main subformula of F. □

<u>1.3. LEMMA</u> (propositional logic. Compare KLEENE [52] §33).
[a1] *If* <u>G</u> *is a positive occurrence of a subformula of* F*, then*

$$E \vdash_{L_0} G \to H \quad \Rightarrow \quad E \vdash_{L_0} F \to F[H/\underline{G}]$$

(where $F[H/\underline{G}]$ *comes from* F *by replacing the occurrence* $\underline{G}$ *by* H*).*

[a2] *If* $\underline{G}$ *is a negative occurrence in* F*, then*

$$E \vdash_{L_0} G \to H \quad \Rightarrow \quad E \vdash_{L_0} F[H/\underline{G}] \to F.$$

[b] *Let* F^q *be the propositional schema which comes from* F *by replacing (simultaneously) every occurrence of some (fixed) propositional letter* p *in* F *by* $p\vee q$*, where* q *is a fixed propositional letter. Then*

$$\neg q \vdash_{L_0} F^q \to F.$$

<u>PROOF</u>. [a]: Straightforward by induction on the length of F (simultaneously for [a1] and [a2]).

[b]: Since $q \vdash_{L_0} p\vee q$, we get by repeated application of [a2] (*) $\vdash_{L_0} F^{q^-} \to F$, where F^{q^-} comes from F by replacing only negative occurrences $\underline{p}$ in F by $p\vee q$. But $\neg q \vdash_{L_0} p\vee q \to p$, so we get by iterated application of [a1]: (**) $\neg q \vdash_{L_0} F^q \to F^{q^-}$. (*) and (**) yield [b]. □

1.4. SIMPLIFIED DEFINITION OF EFFECTIVELY INSEPARABLE R.E. SETS

It is just to smoothen the exposition that we use the following

<u>LEMMA</u>. *Two disjoint r.e. sets* A,B *are effectively inseparable iff there is a (total) recursive function* f *s.t.*

$$\left.\begin{array}{l} W_i \cap A = \emptyset \\ W_j \cap B = \emptyset \end{array}\right\} \Rightarrow f(i,j) \notin W_i \cup W_j$$

<u>PROOF</u>.

I. The "if" direction is trivial, since the function f satisfies more than what is required from a function of effective inseparability (cf. e.g. ROGERS [67] p.94).

II. Let, on the other hand f_1 be a (partial) recursive function for the effective inseparability of A and B, and let i,j satisfy

(1) $A \cap W_i = \emptyset, \quad B \cap W_j = \emptyset.$

By the reduction principle (cf. ROGERS [67], p.72) there are functions g,h s.t.

(2) $W_{g(i)} \subseteq W_i; \quad W_{h(j)} \subseteq W_j,$

(3) $W_{g(i)} \cup W_{h(j)} = W_i \cup W_j$ and

(4) $W_{g(i)} \cap W_{h(j)} = \emptyset.$

Take now

(5) $W_{g'(i)} := W_{g(i)} \cup B; \quad W_{h'(j)} := W_{h(j)} \cup A.$

Then

(6) $W_{g'(i)} \supseteq B; \quad W_{h'(j)} \supseteq A$

while by (4), (2), (1) and the assumed $A \cap B = \emptyset$,

(7)
$$W_{g'(i)} \cap W_{h'(j)} = [W_{g(i)} \cap W_{h(j)}] \cup [W_{g(i)} \cap A] \cup \\ \cup [W_{h(j)} \cap B] \cup [A \cap B] \quad = \emptyset.$$

For the f defined by

$$f(i,j) := f_1(g'(i),h'(j))$$

we have now, by (6) (7) and the choice of f_1 that $f(i,j) \notin W_i \cup W_j$ as required. It is easily seen, in addition, that f may be extended to a total function. □

1.5. DEFINITION OF THE DESIRED Σ_1^0 SENTENCES

The following construction generalizes the method of MYHILL [72]. Let A,B be r.e. sets, effectively inseparable (in the sense of 1.4) through the function f, and let A^* be any consistent r.e. extension of A which satisfies the disjunction instantiation property. Following SHEPHERDSON [60] we may define (explicitly) a Σ_1^0 formula $F(a) \equiv \exists x F_0(x,a)$ s.t.

(1) $$A = \{m \mid\ \vdash_{A^*} F(\bar{m})\}; \quad B = \{m \mid\ \vdash_{A^*} \neg F(\bar{m})\}$$

(To see that this holds also intuitionistically, either inspect Shepherdson's proof, or observe that the equations above are formalizable as Π_2^0 statements and recall that for such sentences derivability in classical arithmetic implies derivability in intuitionistic arithmetic.)

We construct now, by recursion on k, an infinite sequence $\{A_i^k\}_{i<\omega}$ s.t.

(2) $$\nvdash_{A^*} U_k[A^k_{i_1},\dots,A^k_{i_k}] \text{ for every distinct } i_1,\dots,i_k.$$

Basis: By the assumed properties of A^* there is a Σ_1^0 Rosser sentence R for A^*; set $A_i^1 :\equiv R$ for every i.

Recursion step: Assume A_i^k, $i < \omega$ to be defined and to satisfy (2). We define a sequence of Σ_1^0 sentences $\{G_j^k\}_{j<\omega}$ s.t. no finite boolean combination of the G_j^k's implies in A^* $U_k[A^k_{i_1},\dots,A^k_{i_k}]$ for some distinct $i_1,\dots,i_k$. (By a boolean combination we mean here a set $\{H_j\}_j$ where H_j is either G_j^k or $\neg G_j^k$.)

Sub-basis: Let

(3) $$W_{g(1,k)} := \{m \mid \exists \text{ distinct } i_1,\dots,i_k \text{ for which } F(\bar{m}) \vdash_{A^*} U_k[A^k_{i_1},\dots,A^k_{i_k}]\}$$

(4) $$W_{h(1,k)} := \{m \mid \exists \text{ distinct } i_1,\dots,i_k \text{ for which } \neg F(\bar{m}) \vdash_{A^*} U_k[A^k_{i_1},\dots,A^k_{i_k}]\}$$

Now $W_{g(1,k)} \cap A = \emptyset$ and $W_{h(1,k)} \cap B = \emptyset$ by (1) and (2). Hence

$$f(g(1,k),h(1,k)) \notin W_{g(1,k)} \cup W_{h(1,k)}.$$

Define

$$G_1^k :\equiv F(f(g(1,k),h(1,k)));$$

then

(5) $$\left.\begin{matrix} G_1^k \\ \neg G_1^k \end{matrix}\right\} \nvdash_{A^*} U_k[A^k_{i_1},\dots,A^k_{i_k}] \qquad \text{as required.}$$

Sub-recursion step: Assume that $G^k_1,\dots,G^k_l$ are defined, and satisfy

(6) $G^* \not\vdash_{A^*} U_k[A^k_{i_1},\dots,A^k_{i_k}]$ for every boolean combination G^* of $G^k_1,\dots,G^k_l$ and every distinct $i_1,\dots,i_k$.

Define

$$W_{g(l+1,k)} := \{m \mid \exists \text{ distinct } i_1,\dots,i_k \text{ s.t. } F(\bar{m}),\ G^* \vdash_{A^*} U_k[A^k_{i_1},\dots,A^k_{i_k}] \text{ for some boolean combination } G^* \text{ of } G^k_1,\dots,G^k_l\}$$

$$W_{h(l+1,k)} := \{m \mid \dots \neg F(\bar{m}),\ G^* \vdash_{A^*} U_k[A^k_{i_1},\dots,A^k_{i_k}] \dots\}.$$

As in the treatment of the sub-basis we have here

$$W_{g(l+1,k)} \cap A = \emptyset;\quad W_{h(l+1,k)} \cap B = \emptyset.$$

So, defining

$$G^k_{l+1} :\equiv F(f(g(l+1,k),h(l+1,k))),$$

we have

$$G^* \not\vdash_{A^*} U_k[A^k_{i_1},\dots,A^k_{i_k}] \text{ for every boolean combination } G^* \text{ of } G^k_1,\dots,G^k_{l+1}.$$

Main recursion step continued: Define now A^{k+1}_i to be (the purely Σ^0_1 equivalent of) $A^k_i \vee G^k_i$. To conclude the proof, assume

$$\vdash_{A^*} U_{k+1}[A^{k+1}_{i_1},\dots,A^{k+1}_{i_{k+1}}] \text{ for some distinct } i_1,\dots,i_{k+1}.$$

By the disjunction instantiation property of A^* we get, w.l.o.g.,

$$\vdash_{A^*} A^{k+1}_{i_1} \rightarrow U_k[A^{k+1}_{i_2},\dots,A^{k+1}_{i_{k+1}}].$$

But recalling the definition of A^{k+1}_j, this implies

$$G^k_{i_1} \vdash_{\mathcal{A}^*} U_k[A^k_{i_2} \vee G^k_{i_2}, \ldots, A^k_{i_{k+1}} \vee G^k_{i_{k+1}}]$$

which by 1.3 [b] implies

$$G^k_{i_1}, \neg G^k_{i_2}, \ldots, \neg G^k_{i_{k+1}} \vdash_{\mathcal{A}^*} U_k[A^k_{i_2}, \ldots, A^k_{i_{k+1}}] \text{ ,}$$

contradicting the construction of the sequence G^k_j. Hence

$$\nvdash_{\mathcal{A}^*} U_{k+1}[A^{k+1}_{i_1}, \ldots, A^{k+1}_{i_{k+1}}]$$

as required. □

Note, finally, that the above construction can be rendered totally uniform. That is, every A^k_i can be presented as $\exists x B(f'(i,k),x)$ for a suitable total recursive function f'. This formula does not belong, strictly speaking, to the formalism of $\mathcal{A}$. But it is equivalent to the following formula of prim. rec. arithmetic:

$$\exists z\ T(e,\langle i,k\rangle,(z)_0)\ \&\ F_0(U((z)_0),(z)_1),$$

where e is the g.n. of the function f', T and U are Kleene's computation-predicate and result-extrating function respectively. We have thus proved theorem I for schemata F s.t. $\mu\ulcorner F\urcorner \leq 1$. □

B.2. PROOF-THEORETIC REDUCTION OF THEOREM I

2.0. Here we prove, for a regular number theory $A^* \subseteq A^\infty[T]$,

<u>PROPOSITION</u>. *If* $\nvdash_{L_0} F[p_1,\dots,p_k]$ *and* $\vdash_{A^*} F[A_1,\dots,A_k]$, *then*

$$\vdash_{A^\infty[T]} U_k[A_1,\dots,A_k]$$

for any Σ^0_1 *sentences* $A_1,\dots,A_k$.

To simplify notations we shall actually prove the proposition for $A^* \subseteq A^\infty_{\mathrm{rec}}[T]$ (in place of $A^\infty[T]$). A proof of the general version stated above is obtained trivially mutatis mutandis. Combined with the solution given in section 2 for the schemata U_k, this implies theorem I, since $A^\infty_{\mathrm{rec}}[T]$ as well as $A^\infty[T]$ are r.e. and satisfy the disjunction instantiation property.

The proposition is proved as follows. In 2.1 - 2.7 below we prove (for some prim.rec. f)

$$(1)\quad \vdash_{Y_0+\mathrm{BI}} \neg\underline{\mathrm{Pr}}_{L_0}(\ulcorner F\urcorner) \;\&\; \underline{\mathrm{Nprf}}^\infty_{\mathrm{rec}}(d,\ulcorner F[A_1,\dots,A_k]\urcorner)$$
$$\to \underline{\mathrm{Nprf}}^\infty_{\mathrm{rec}}(fd,\ulcorner U_k[A_1,\dots,A_k]\urcorner).$$

So, for a theory $T \supseteq Y_0 + \mathrm{BI}$ and a proof-predicate $\underline{\mathrm{Pr}}_T$ for it which is proved in A to be closed under Modus Ponens,

$$(2)\quad \vdash_A \underline{\mathrm{Pr}}_T\ulcorner\neg\underline{\mathrm{Pr}}_{L_0}(\ulcorner F\urcorner)\urcorner \;\&\; \underline{\mathrm{Pr}}_T\ulcorner\underline{\mathrm{NPrf}}^\infty_{\mathrm{rec}}(d,\ulcorner F[A_1,\dots,A_k]\urcorner)\urcorner$$
$$\to \underline{\mathrm{Pr}}_T\ulcorner\underline{\mathrm{NPrf}}^\infty_{\mathrm{rec}}(fd,\ulcorner U_k[A_1,\dots,A_k]\urcorner)\urcorner.$$

But $\underline{\mathrm{Pr}}_{L_0}$ is a prim.rec. predicate, so (2) implies

$$(3)\quad \neg\underline{\mathrm{Pr}}_{L_0}(\ulcorner F\urcorner)\ \&\ \underline{\mathrm{Pr}}_{A^*}(\ulcorner F[A_1,\dots,A_k]\urcorner) \rightarrow \underline{\mathrm{Pr}}_{A^\infty_{\mathrm{rec}}[T]}(\ulcorner U_k[A_1,\dots,A_k]\urcorner)^\urcorner$$

$$\text{for any } A^* \subseteq A^\infty_{\mathrm{rec}}[T].$$

(3) is proved in any extension of A where $A^* \subseteq A^\infty_{\mathrm{rec}}[T]$ is proved.

2.1. HEURISTICAL CONSIDERATION LEADING TO THE REDUCTION

2.1.1. Assume the premiss of 2.0(1). It means that a normal derivation d of F in A^∞_{rec} is given where some quantification or arithmetical rule must occur, because $\neg\underline{\mathrm{Pr}}_{L_0}\ulcorner F\urcorner$. We "climb up" in the proof-tree d in search for such an occurrence, starting at the root $\langle\ \rangle$.

To allow a smoother semi-formal exposition, let us write - for a node u - $\rho^{d,u}$ for the inference rule encoded by $(\{d\}u)_0$, and

$$s^{d,u} \equiv \underline{a}^{d,u} \Rightarrow F^{d,u}$$

for the sequent coded by $(\{d\}u)_1$.

At every stage of our search in d we arrive at some node u where the sentence $F^{d,u}$ is a Σ^0_1 substitution of a schema of L_0, and where $\neg\underline{\mathrm{Pr}}_{L_0}(\ulcorner s^{d,u}\urcorner)$, i.e. $\underline{a}^{d,u} \Rightarrow F^{d,u}$ cannot be proven using the rules of L_0 only.

Suppose now that a node u is "selected" at a given stage of the search. If $\rho^{d,u}$ is a propositional rule, then at least one of the premisses $u{*}\langle n\rangle$, $n \le 2$ of u in d must satisfy $\neg\underline{\mathrm{Pr}}_{L_0}(\ulcorner s^{d,u*\langle n\rangle}\urcorner)$, because $\neg\underline{\mathrm{Pr}}_{L_0}(\ulcorner s^{d,u}\urcorner)$ since u is "selected". We "climb up" to the leftmost of these premisses.

$\rho^{d,u}$ cannot be [∀I] or [∀E], by the subformula property of d, because ∀ does not occur in $F[A_1,\dots,A_k]$.

If $\rho^{d,u}$ is [∃E], and $\neg\underline{\mathrm{Pr}}_{L_0}(\ulcorner s^{d,u*\langle 0\rangle}\urcorner)$ (i.e., the major premiss is not provable using propositional rules only), then we climb up to $u{*}\langle 0\rangle$. Else, we proceed simultaneously to all minor premisses $u{*}\langle n+1\rangle$, $n \le \omega$. The major premiss $F^{d,u*\langle 0\rangle} \equiv: \exists z Cz$ must be a Σ^0_1 sentence, by the subformula property. So for every n

$$s^{d,u*\langle n\rangle} \equiv \underline{a}^{d,u}, C\bar{n} \Rightarrow F^{d,u}$$

where $C\bar{n}$ is an equation, and $F^{d,u}$ is a Σ^0_1 substitution of a propositional schema. It is easy to see (2.3 below) that if $\underline{\mathrm{Pr}}_{L_0}(\ulcorner \underline{a}^{d,u}, C\bar{n} \Rightarrow F^{d,u}\urcorner)$ for some n, then $\underline{\mathrm{Pr}}_{L_0}(\ulcorner \underline{a}^{d,u} \Rightarrow F^{d,u}\urcorner)$, which contradicts our assumption that the node u is selected. It follows that all nodes $u*\langle n+1\rangle$ corresponding to the minor premises of $\rho^{d,u}$ satisfy our conditions on "selected" nodes.

Now since d is a well founded tree, any successive selection of nodes as above must terminate. Such a "search" cannot terminate at a top-node of the derivation d, because
(i) if $\rho^{d,u}$ = [TE] then $F^{d,u}$ is an equation, and so u is not selected;
(ii) if $\rho^{d,u}$ = [T] then $\underline{\mathrm{Pr}}_{L_0}(\ulcorner s^{d,u}\urcorner)$.
Hence the search determined by any successive choice of minor (or major) premisses of instances of [∃E] must stop at some node u s.t. $\rho^{d,u}$ is either [∃I] or [FE].

2.1.2. Let us now consider how this information on the "search" described above may be used to construct a proof in A^∞_{rec} for $U_k[A_1,\dots,A_k]$. To start with, take the simplest case, where k = 1, $F \equiv F[\exists x Ex]$, and let u be some terminating node of the search.

Case 1. $\rho^{d,u}$ = [∃I]

$$\begin{array}{lll} & \rho^{d,u*\langle 0\rangle} & \underline{a} \Rightarrow Et \\[1em] \text{the node } \textcircled{u}\!\longrightarrow & [\exists I] & \underline{a} \Rightarrow \exists x Ex \end{array}$$

Obviously, the inference rule $\rho^{d,u*\langle 0\rangle}$ cannot be an introduction rule. If $\rho^{d,u*\langle 0\rangle}$ is [→E], then we have the configuration

$$\text{the node } \textcircled{u*0}\!\longrightarrow \quad \frac{\underline{a} \Rightarrow G \rightarrow Et \qquad \underline{a} \Rightarrow G}{\underline{a} \Rightarrow Et}$$

But no subformula of $F[A_1,\dots,A_k]$ has the form G→Et where Et is an equation. So $\rho^{d,u*\langle 0\rangle}$ cannot be [→E], and the cases [&E] and [∀E] are ruled out likewise. $\rho^{d,u*\langle 0\rangle}$ cannot be one of [⊥], [vE], [∃E], by our definition of normality. We are thus left with the case that $u*\langle 0\rangle$ is a top node of d, and $\rho^{d,u*\langle 0\rangle}$ is [TE] or [T]. In the first case we may construct

$$\begin{array}{ll} [\mathrm{TE}] & \Rightarrow Et \\ [\exists \mathrm{I}] & \Rightarrow \exists xEx \\ [\mathrm{vI}_0] & \Rightarrow \exists xEx \vee \neg\exists xEx \end{array}$$

So we have obtained a derivation for $U_1[\exists xEx]$.

On the other hand, the case $\rho^{d,u*\langle 0\rangle} = [\mathrm{T}]$ is ruled out as follows. Assume that $\rho^{d,u*\langle 0\rangle} = [\mathrm{T}]$. Then $Et \in \underline{a}$, and since d derives a sequent $\Rightarrow F$ with an empty precedent, Et must be "discharged" in d somewhere below the node u. Again by the subformula property of d, this discharge cannot be at an instance of $[\rightarrow \mathrm{I}]$ or of $[\mathrm{vE}]$, and so it must be at an instance of $[\exists \mathrm{E}]$, and we should have the following configuration (where $t = \bar{n}$).

$$\begin{array}{cccc} & & \underline{a} \Rightarrow E\bar{n} & \\ & \textcircled{u}\rightarrow & \underline{a} \Rightarrow \exists xEx & \\ & \cdots & & \cdots \\ \underline{b} \Rightarrow \exists xEx & & \underline{b}, E\bar{n} \Rightarrow B & \\ \hline & \textcircled{v}\rightarrow & [\exists \mathrm{E}] \quad \underline{b} \Rightarrow B & \end{array}$$

Here the two indicated occurrences of Σ_1^0 formulae must be identical for the case considered. Since the node u is selected, so must be v, but not $v*\langle 0\rangle$. This means that $\neg \mathrm{Pr}_{L_0}(\ulcorner \underline{a} \Rightarrow \exists xEx \urcorner)$, but $\underline{\mathrm{Pr}}_{L_0}(\ulcorner \underline{b} \Rightarrow \exists xEx \urcorner)$. From the configuration just shown we must have, however, $\underline{b} \subseteq \underline{a}$, and this is a contradiction.

Case 2. $\rho^{d,u} = [\mathrm{FE}]$,
$$\begin{array}{ll} & \underline{a} \Rightarrow E \\ [\mathrm{FE}] & \underline{a} \Rightarrow \bot \end{array}$$
say.

As in case 1, we find that $u*\langle 0\rangle$ must be a top node of d, and since E here is a false equation, we are left with the case that $\rho^{d,u*\langle 0\rangle}$ is [T]; so we must find in d the following configuration:

$$\begin{array}{ccccc} & & [\mathrm{T}] \quad \underline{a} \Rightarrow E & & \\ & \textcircled{u}\rightarrow & \underline{a} \Rightarrow \bot & & \\ & \cdots & \Sigma_n & \Sigma_{n+1} & \cdots \\ \underline{b} \Rightarrow \exists xEx & & \underline{b}, E\bar{n} \Rightarrow B & & \\ \hline & \textcircled{v}\rightarrow & [\exists \mathrm{E}] \quad \underline{b} \Rightarrow B & & \end{array}$$

and we may assume w.l.g. (by the well-foundedness of d) that the configuration of the type shown does not repeat itself within any of the subderivations Σ_m. Since u is selected, so must be v, and hence $v*\langle m+1\rangle$ for every $m < \omega$. Each search in a subderivation Σ_m must come to an end at some node u_m, and the argument of case 1 (about ruling out $\rho^{d,u*\langle 0\rangle} = [T]$) shows that since $v*\langle 0\rangle$ is not selected, ρ^{d,u_m} is not $[\exists I]$, and must therefore be [FE]. Hence we can extract from the configuration above the derivation:

$$\begin{array}{ll}
\multicolumn{2}{c}{[T]\quad \exists xEx \Rightarrow \exists xEx \qquad \left\{\begin{array}{ll}[T] & \exists xEx, E\bar{n} \Rightarrow E\bar{n}\\ [FE] & \exists xEx, E\bar{n} \Rightarrow \bot\end{array}\right\}_{n<\omega}}\\
\hline
[\exists E] & \exists xEx \Rightarrow \bot\\
[\to I] & \quad\Rightarrow \neg\exists xEx\\
[vI_1] & \quad\Rightarrow \exists xEx \text{ v } \neg\exists xEx
\end{array}$$

and again we find a derivation in A^∞_{rec} for $U_1[\exists xEx]$. This concludes our observation on the case that $k = 1$, $F = F[\exists xEx]$.

2.1.3. Consider now the case $k = 2$, i.e., $F \equiv F[\exists xE_0x, \exists xE_1x]$. Here the following configuration may occur

$$\frac{\underline{a} \Rightarrow \exists xE_0x \qquad \{\Sigma_n\}_{n<\omega}}{\textcircled{u}\!\longrightarrow\ [\exists E]\quad \underline{a} \Rightarrow B}$$

where the node u is selected, and the search continues to the minor subderivations Σ_n (i.e. $-\ \underline{Pr}_{L_0}(\ulcorner \underline{a} \Rightarrow \exists xE_0x\urcorner)$). But now, from our argument for the case $k = 1$ it is clear that, for the node u_m at which the search in the minor subderivation Σ_m terminates $F^{u_m} \nVdash \exists xE_0x$ $(m<\omega)$. So we may apply the argument for the case $k = 1$ to each of the minor subderivations separately, and extract from each of these a derivation Σ^*_m for $\exists xE_1x \text{ v } \neg\exists xE_1x$. Since the method of doing this is uniform, we can actually collect the derivations Σ^*_m to yield the following derivation of A^∞_{rec}.

$$\dfrac{[T]\quad \exists xE_0x \Rightarrow \exists xE_0x \qquad \left\{\begin{array}{c}\Sigma^*_m\\ \exists xE_0x \Rightarrow \exists xE_1x \vee \neg\exists xE_1x\end{array}\right\}_{m<\omega}}{\begin{array}{ll}{[\exists E]} & \exists xE_0x \Rightarrow \exists xE_1x \vee \neg\exists xE_1x\\ {[\to I]} & \quad\;\; \Rightarrow \exists xE_0x \to \exists xE_1x \vee \neg\exists xE_1x\\ {[\vee I_0]} & \quad\;\; \Rightarrow U_2[\exists xE_0x, \exists xE_1x]\end{array}}$$

Iterating this process with some technical symmetrization arguments, we obtain 2.0(1).

2.2. NOTATIONS

$\underline{\text{Subordinated}}\ (d,u,v) :\equiv \exists w,n<v\ \big[\ v=w*\langle 0\rangle\ \ \&\ \ w*\langle n+1\rangle \prec u\ \ \&\ \ \rho^{d,w} = [\exists E]\big]$

$\equiv$ "v is a major premiss node of an instance of $[\exists E]$ in d, and u is a node in one of the minor subderivations of this instance".

Here $\prec$ stands for the initial-segment relation (between sequent-numbers).

$\underline{\text{Selected}}\ (d,u) :\equiv$ "$F^{d,u}$ is not an equation" $\&\ \neg\underline{\text{Pr}}_{L_0}(\ulcorner s^{d,u}\urcorner)\ \&$
$\forall w\prec u\ \big[\ \underline{\text{Subordinated}}(d,u,w*\langle 0\rangle) \to \underline{\text{Pr}}_{L_0}(\ulcorner s^{d,w*\langle 0\rangle}\urcorner)\ \big].$

When $\underline{\text{NPrf}}^{\infty}_{\text{rec}}(d,\ulcorner F[A_1,\ldots,A_k]\urcorner)$ $(A_1,\ldots,A_k\ \Sigma^0_1$ sentences) write

$\underline{b}^{d,u} := \{F^{d,v} \mid \underline{\text{Subordinated}}(d,u,v)\}$

$a_0^{d,u} := \{E \in \underline{a}^{d,u} \mid E \text{ an equation}\}$

$U^{d,u} :\equiv U_m[A_{i_1},\ldots,A_{i_m}]$ where $\{A_{i_1},\ldots,A_{i_m}\} := \{A_1,\ldots,A_k\} \setminus \underline{b}^{d,u}$

(set-theoretic difference)

2.3. LEMMA. *Let* $A_1,\dots,A_k$ *be* Σ^0_1 *sentences, let* $\underline{a} \Rightarrow G$ *be formed of subformulae of* $F[A_1,\dots,A_k]$ *only, where* G *is not an equation, and let* E *be an equation. Then*

$$\underline{Pr}_{\mathcal{L}_0}(\ulcorner \underline{a},E \Rightarrow G\urcorner) \;\Rightarrow\; \underline{Pr}_{\mathcal{L}_0}(\ulcorner \underline{a} \Rightarrow G\urcorner) .$$

PROOF. Let Π be a normal proof for $\underline{a},E\Rightarrow G$ which uses propositional inference-rules only, and let Π^* come from Π be eliminating E from all precedents of sequents in Π. Check by inspection on cases for inference rules that Π^* is a correct derivation. (Note that by normality no formula of the form $E \to H$ may occur in Π). □

2.4. LEMMA. (in A) *Assume* $\underline{NPrf}^{\infty}_{rec}(d,\ulcorner F[A_1,\dots,A_k]\urcorner)$;

(a) $\underline{Selected}(d,u) \longrightarrow \left[\, \rho^{d,u} = [\exists I] \;\vee\; \rho^{d,u} = [FE] \;\vee\; \exists n{\le}2\ \underline{Selected}(d,u{*}\langle n\rangle) \right]$

(b) $\underline{Selected}(d,u) \;\&\; \rho^{d,u} = [\exists E] \;\&\; \underline{Pr}_{\mathcal{L}_0}(\ulcorner s^{d,u*\langle 0\rangle}\urcorner)$

$\longrightarrow \forall n{>}0\ \underline{Selected}(d,u{*}\langle n\rangle)$.

PROOF. Assume $\rho^{d,u} \neq [\exists I],[FE]$ and the premiss of (a), and consider cases for $\rho^{d,u}$. $\rho^{d,u}$ cannot be [T] or [TE], because $\underline{Selected}(d,u)$. $\rho^{d,u}$ is not $[\forall I]$ or $[\forall E]$ by the subformula property of d. If $\rho^{d,u}$ is a propositional inference-rule, the proof is immediate. We are left with the case that $\rho^{d,u}$ is $[\exists E]$. If $\neg\underline{Pr}_{\mathcal{L}_0}(\ulcorner s^{d,u*\langle 0\rangle}\urcorner)$ then we are done (for part (a)). Else, then $\underline{Selected}(d,u{*}\langle n\rangle)$ for every n > 0 by 2.3. □

2.5. ASSIGNMENT OF DERIVATIONS TO THE SELECTED NODES.

Assume $\underline{NPrf}^{\infty}_{rec}(d,\ulcorner F[A_1,\dots,A_k]\urcorner)$ as above. We define a function $\{a(d,u)\}$ recursive in $\{d\}$ and u by the conditions given below (compare the definition of $\{n\}$ in A.3.4.3). By the s.m.n.-theorem a(d,u) is then a prim.rec. function. $\{a(d.u)\}$ is intended to be the formal description of a derivation of A^∞ for $\underline{a}_0^{d,u} \cup \underline{b}^{d,u} \Rightarrow U^{d,u}$.

(i) If $\neg\underline{Selected}(d,u)$, then $\{a(d,u)\} \equiv 0$.

(ii) Else, and $\rho^{d,u} = [\exists I]$, then $\{a(d,u)\}$ describes the finite derivation

$$[\rho^{d,u*\langle 0\rangle}] \quad \underline{a}_0^{d,u} \cup \underline{b}^{d,u} \Rightarrow F^{d,u*\langle 0\rangle}$$

$$[\exists I] \quad \underline{a}_0^{d,u} \cup \underline{b}^{d,u} \Rightarrow F^{d,u}$$

$$\left.\begin{array}{ll}\text{instances of} & [\rightarrow I] \\ \text{and of} & [\vee I]\end{array}\right\} \qquad \vdots$$

$$\underline{a}_0^{d,u} \cup \underline{b}^{d,u} \Rightarrow U^{d,u}$$

Note that, by the argument of 2.1.2, $F^{d,u} \notin \underline{b}^{d,u}$ and that $\rho^{d,u*\langle 0\rangle}$ is either [T] or [TE]. So the figure above is indeed a derivation.

(iii) Else, and $\rho^{d,u}$ = [FE]. Let $\{a(d,u)\}$ describe formally

$$[T] \quad \underline{a}_0^{d,u} \cup \underline{b}^{d,u} \Rightarrow F^{d,u*\langle 0\rangle}$$

$$[FE] \quad \underline{a}_0^{d,u} \cup \underline{b}^{d,u} \Rightarrow \bot$$

$$[\bot] \quad \underline{a}_0^{d,u} \cup \underline{b}^{d,u} \Rightarrow U^{d,u}$$

(iv) Else, and $\rho^{d,u}$ is a propositional inference rule. Let $u*\langle n\rangle$ be the leftmost premiss of u in d s.t. <u>Selected</u>$(d,u*\langle n\rangle)$ (cf. 2.4.(a)), and let $\{a(d,u)\} := \{a(d,u*\langle n\rangle)\}$.

(v) Else, and $\rho^{d,u}$ = [∃E];

<u>Subcase A</u>: If $\neg\underline{Pr}_{L_0}(\ulcorner s^{d,u*\langle 0\rangle}\urcorner)$, let $a(d,u) := a(d,u*\langle 0\rangle)$.

<u>Subcase B</u>: Else, and $\exists xEx :\equiv F^{d,u*\langle 0\rangle} \notin \underline{b}^{d,u}$, then let $\{a(d,u)\}$ describe the figure

$$\dfrac{[T]\ \ \underline{a}_0^{d,u} \cup \underline{b}^{d,u}, \exists xEx \Rightarrow \exists xEx \qquad \overset{\Sigma'_n}{\left\{ \underline{a}_0^{d,u*\langle n\rangle} \cup \underline{b}^{d,u}, \exists xEx \Rightarrow U^{d,u*\langle n\rangle} \right\}}_{0<n<\omega}}{[\exists E] \quad \underline{a}_0^{d,u} \cup \underline{b}^{d,u}, \exists xEx \Rightarrow U^{d,u*\langle 1\rangle}}$$

$$[\rightarrow I] \quad \underline{a}_0^{d,u} \cup \underline{b}^{d,u} \Rightarrow \exists xEx \rightarrow U^{d,u*\langle 1\rangle}$$

$$\left.\begin{array}{r}\text{instances of} \\ [\vee I]\end{array}\right\} \qquad \vdots$$

$$\underline{a}_0^{d,u} \cup \underline{b}^{d,u} \Rightarrow U^{d,u}$$

Here, if Σ_n is described by $\{a(d,u{*}\langle n\rangle)\}$, then Σ'_n comes from Σ_n by joining the formula $\exists x Ex$ to all precedents. Note that by the case's conditions

$$\left.\begin{array}{l}
\underline{b}^{d,u*\langle n\rangle} = \underline{b}^{d,u} \cup \{\exists x Ex\} \\
\underline{a}_0^{d,u*\langle n\rangle} = \underline{a}_0^{d,u} \cup \{E\bar{n}\} \\
U^{d,u*\langle n\rangle} \equiv U^{d,u*\langle 1\rangle}
\end{array}\right\} \quad \text{for } n > 0.$$

<u>Subcase C</u>. As subcase B, but $\exists x Ex \in \underline{b}^{d,u}$. Then let $\{a(d,u)\}$ describe

$$\frac{\underline{a}_0^{d,u} \cup \underline{b}^{d,u} \Rightarrow \exists x Ex \qquad \overset{\Sigma_n}{\left\{ \underline{a}_0^{d,u*\langle n\rangle} \cup \underline{b}^{d,u}, \exists x Ex \Rightarrow U^{d,u} \right\}}_{0<n<\omega}}{[\exists E] \quad \underline{a}_0^{d,u} \cup \underline{b}^{d,u} \Rightarrow U^{d,u}}$$

Note that here $U^{d,u*\langle n\rangle} \equiv U^{d,u}$ for every n.

<u>2.6.1. LEMMA</u>.

$$\vdash_{\mathcal{V}_0+\mathrm{BI}} \underline{\mathrm{NPrf}}^{\infty}_{\mathrm{rec}}(d, \ulcorner F[A_1,\ldots,A_k]\urcorner) \;\&\; \underline{\mathrm{Selected}}(d,u) \longrightarrow$$
$$\underline{\mathrm{NPrf}}^{\infty}_{\mathrm{rec}}(a(d,u), \ulcorner \underline{a}_0^{d,u} \cup \underline{b}^{d,u} \Rightarrow U^{d,u}\urcorner).$$

<u>PROOF</u>. Straightforward from the definition of a(d,u) above. □

<u>2.6.2. LEMMA</u>. *For* $F, A_1, \ldots, A_k$ *as above*

$$\underline{\mathrm{Pr}}_{\mathcal{L}_0}(\ulcorner F[A_1,\ldots,A_k]\urcorner) \Rightarrow \underline{\mathrm{Pr}}_{\mathcal{L}_0}(\ulcorner F[p_1,\ldots,p_k]\urcorner)$$

<u>PROOF</u>. Let Δ be a normal proof of $F[A_1,\ldots,A_k]$ which uses propositional inference-rules only. All formulae occurring in Δ are subformulae of $F[A_1,\ldots,A_k]$, and a trivial inspection shows that by replacing $A_1,\ldots,A_k$ throughout the proof by $p_1,\ldots,p_k$ respectively we get a correct derivation of $\mathcal{L}_0$ for $F[p_1,\ldots,p_k]$. □

2.7. PROPOSITION.

$$\underline{\mathrm{NPrf}}_{\mathrm{rec}}(d,\ulcorner F[A_1,\dots,A_k]\urcorner)\ \&\ \neg\underline{\mathrm{Pr}}_{L_0}(\ulcorner F\urcorner) \rightarrow$$
$$\underline{\mathrm{NPrf}}^{\infty}_{\mathrm{rec}}(a(d,\langle\ \rangle),\ulcorner\Rightarrow U_k[A_1,\dots,A_k]\urcorner)\ .$$

PROOF. Use 2.6.1 for u = ⟨⟩ , which by the premiss and 2.6.2 must be a selected node. □

B.3. STRUCTURE OF THE PROOF OF THEOREM II

3.1. PRELIMINARIES

3.1.1. Fix a q.f. formula $E(x) :\equiv f(x)=0$ (where f is a fixed prim.rec. function). We shall use the following notations.

$$E_i^n(z_1,\dots,z_n) :\equiv \forall x \exists y\ E\langle x,y,i,n,\langle z_1,\dots,z_n\rangle\rangle$$

$$E^* :\equiv \forall i,u,z \forall x \exists y\ E\langle x,y,i,u,z\rangle$$

$$B^E[w] :\equiv \forall \langle i,n,z\rangle\ [\ \underline{\mathrm{Ineq}}(\langle i,n,z\rangle,w) \longrightarrow \forall x \exists y\ E\langle x,y,i,n,z\rangle\ ]$$

where $\underline{\mathrm{Ineq}}(a,b)$ is an equation which expresses the inequality $a \neq b$. More intuitively,

$$B^E[\langle j,m,\langle \vec{s}\rangle\rangle] :\equiv \forall \langle i,n,\langle \vec{t}\rangle\rangle_{\langle i,n,\langle \vec{t}\rangle\rangle \neq \langle j,m,\langle \vec{s}\rangle\rangle} E_i^n(\vec{t})$$

We further define the sequent

$$s^E[w] :\equiv B^E[w] \Rightarrow \forall x \exists y\ E\langle x,y,(w)_0,(w)_1,(w)_2\rangle$$

$$\equiv B^E[w] \Rightarrow E_{(w)_0}^{(w)_1}((w)_{2,0},\dots,(w)_{2,(w)_1})$$

i.e.

$$s^E[\langle i,n,\langle \vec{z}\rangle\rangle] \equiv \forall j,m,\vec{w}_{\langle j,m,\vec{w}\rangle \neq \langle i,n,\vec{z}\rangle} E_j^m(\vec{w}) \Rightarrow E_i^n(\vec{z}).$$

The sequents $s^E[w]$ play here the same role as the schemata U_k in the treatment of L_0 above.

3.1.2. Let E be an equation as above. An E-*sentence* is a sentence built up using the formation rules of L_1 only, with E_i^n taken in place of the predicate letters P_i^n $(i,n=0,1,\dots)$. An E-*atom* is an E-sentence of the form $E_i^n(t_1,\dots,t_n)$. We call the indicated occurrences of t_i in the E-atom above $(i=1,\dots,n)$ the *formal occurrences* in $E_i^n(\vec{t})$. Since the order of formally-occurring terms in each E-atom is fixed by the very definition of E_i, it is uniformly decidable whether two E-atoms are instances of the same E_i^n.

Let d be a normal derivation in A_{rec}^{∞} of an E-sentence. By the subformula property of d every formula occurring in d is either an E-sentence, an equation $E(t) = 0$ or a Σ_1^0 sentence $\exists y\, E(\langle p,y,i,n,z\rangle)$. It is easily seen that if we replace every formal occurrence of each term t (in some formula in d) by the numeral $\bar{n}$ s.t. $\bar{n}=t$, we get a correct and normal derivation of the same E-sentence. We call such a normal derivation an E-*derivation*.

Notation: $\underline{\text{E-Der}}(d)$; $\underline{\text{E-Prf}}(d,\ulcorner F\urcorner)$. Since we deal with E-derivations only, we consider only E-atoms of the form $E_i^n(\bar{m}_1,\dots,\bar{m}_n)$. If $F[P_{i_1}^{n_1},\dots,P_{i_q}^{n_q}]$ is a schema of L_1 whose predicate-letters are among those shown, we write F^E for $F[E_{i_1}^{n_1},\dots,E_{i_q}^{n_q}]$. So $\ulcorner F^E\urcorner = \underline{\text{sub}}_{\Pi_2^0}(\ulcorner F\urcorner, \ulcorner E\urcorner)$.

3.1.3. We write $[\exists E^1]$ for an instance of $[\exists E]$ whose major premiss (i.e. the consequent of the leftmost premiss-sequent) has a q.f. matrix. For an instance of $[\exists E]$ which does not satisfy this we write $[\exists E^*]$.

3.2. DERIVATION OF E-SENTENCES IN L_1A

L_1A is L_1 extended to the language of A (cf. P.2.5).

3.2.1. LEMMA. *Let every formula in* $\underline{a}$,F *be either an* E-*sentence, an open* Σ_1^0 *formula or an open equation. Let* $\underline{b}$ *be a set of closed equations. Then*

$$\underline{a} \cup \underline{b} \vdash_{L_1A} F \;\Rightarrow\; \underline{a} \vdash_{L_1A} F .$$

PROOF. Assume $\underline{a} \cup \underline{b} \vdash_{L_1A} F$, and let Δ be a normal derivation of L_1A for $\underline{a} \cup \underline{b} \vdash F$ (cf. PRAWITZ [65]). By induction on the length of Δ, using the subformula property and the definition of E-atoms, one proves easily that every formula occurring in Δ is either an E-sentence or an open Σ_1^0 or q.f. formula. Hence formulae in $\underline{b}$ are actually not used in Δ, and so $\underline{a} \vdash_{L_1A} F$. □

3.2.2. LEMMA. *Let* $\underline{a}$,F *be closed formulae of* L_1. *Then*

$$\underline{a}^E \vdash_{L_1 A} F^E \quad \Rightarrow \quad \underline{a} \vdash_{L_1} F$$

where

$$\underline{a}^E := \{G^E \mid G \in \underline{a}\}.$$

PROOF. The proof of PRAWITZ [65],[71] for the normalization of L_1 applies trivially to L_1A and is easily seen to hold also for our definition of normality (for L_1A only the trivial $\rho\iota$-reductions have to be considered in addition). So let Δ be a normal derivation (in the sense of A.1.1) of L_1A for $\underline{a}^E \vdash F^E$ and let $\underline{G}$ be an occurrence of a formula in Δ, G not an E-sentence. By the subformula property of Δ, G must then have one of the forms

[a] $E\langle u,v,i,n,z\rangle$ or [b] $\exists y\, E\langle u,y,i,n,z\rangle$.

By the normality of Δ $\underline{G}$ must either

(i) be an equation (case [a]) occurring at a top-node of Δ (by 2.1.2)

(ii) occur immediately below a top formula, or

(iii) occur as a premiss of $\exists$E derived by $\forall$E (in case [b]).

Note now that E_i^n is defined so that the order of variables in each E-atom is fixed, so that the two first variables of the matrix are bounded by the $\forall\exists$ quantifiers preceding it. Furthermore, two E-atoms formed from distinct E_i^n are syntactically distinct, and the rule [FE] is not used in L_1A. Hence every occurrence $\underline{G}$ as above must occur in a subderivation of Δ of the form

$$\Pi \;\equiv\; \dfrac{\dfrac{\begin{matrix}\Sigma\\ \forall x\exists yE\langle x,y,i,n,z\rangle\end{matrix}}{\exists yE\langle u,y,i,n,z\rangle}\,\forall E \qquad \begin{matrix}\overset{(1)}{\left[\dfrac{\dfrac{E\langle u,v,i,n,z\rangle}{\exists yE\langle u,y,i,n,z\rangle}\,\exists I}{\forall x\exists yE\langle x,y,i,n,z\rangle}\,\forall I\right]_j}\\ \Gamma\\ H\end{matrix}}{H}\,(1)\ \exists E \quad .$$

Replace the subderivation Π of Δ by

$$\Pi^{*} \quad :\equiv \quad \begin{array}{c} \left[\begin{array}{c} \Sigma \\ \forall x \exists y E\langle x,y,i,n,z\rangle \end{array} \right]_{j} \\ \Gamma \\ H \end{array}$$

Note that Π^{*} is normal. Repeating this operation we get by induction on the number of occurrences of Σ_1^0 formulae in Δ a derivation Δ^{*} where all occurrences are of E-formulae. Replace in Δ^{*} every occurrence $E_i^n(\vec{v})$ of an E-atom (including occurrences as a subformula) by $P_i^n(\vec{v})$, and the result is a correct derivation of L_1 for $\underline{a} \vdash F$. □

3.3. We wish to prove theorem II, which is trivially implied by the following more formal version.

<u>THEOREM II</u> (restated). *For any* $T \supseteq Y_0 + \mathrm{BI}$ *there is a q.f.* E(x) *s.t.*

$$A^{T} \vdash \neg\underline{\mathrm{Pr}}_{L_1}(\ulcorner F \urcorner) \;\rightarrow\; \neg\underline{\mathrm{Pr}}_{A^{\infty}_{\mathrm{rec}}[T]}(\underline{\mathrm{sub}}_{\Pi_2^0}(\ulcorner F \urcorner, \ulcorner E \urcorner))$$

where

$$A^{T} := A + \underline{\mathrm{CMP}}(T) + \underline{\mathrm{Rfn}}_{C_0}(T) + \underline{\mathrm{Con}}(T^{+})$$

$$\underline{\mathrm{CMP}}(T) :\equiv \forall x,y\, [\, \underline{\mathrm{Pr}}_{T}(\underline{\mathrm{imp}}(x,y)) \;\rightarrow\; (\underline{\mathrm{Pr}}_{T}(x) \rightarrow \underline{\mathrm{Pr}}_{T}(y))\,],$$

$$\underline{\mathrm{Rfn}}_{C_0}(T) :\equiv \forall x\, [\, \underline{\mathrm{Pr}}_{T}(x) \;\&\; \text{"x encodes a formula in } C_0\text{"} \;\rightarrow\; \underline{\mathrm{Tr}}_{C_0}(x)\,],$$

and where C_0 *is the class of formulae of the form* $\Pi_2^0 \rightarrow \neg\neg\Sigma_2^0$, *and* $\underline{\mathrm{Tr}}_{C_0}$ *is a truth definition for* C_0.

$$\underline{\mathrm{Con}}(T^{+}) :\equiv \forall x\, [\, \text{"x encodes a conjunction of instances of } AC_{00}^{-}, \text{ of BI, and of true } \Pi_1^0 \text{ sentences"} \;\rightarrow\; \neg\underline{\mathrm{Pr}}_{T^{+}}(\underline{\mathrm{neg}}(x))\,].$$

3.4. THE PROOF-THEORETIC REDUCTION

The proof of theorem II proceeds now as follows. Fix an equation E and a schema F of L_1 as above. In sec. 4.3 below we define a (classically) Π^0_1 predicate $\underline{\mathrm{Crit}}(d,u)$ for which we prove

$$(1)\quad \vdash_{Y_0+\mathrm{BI}} \underline{\mathrm{E\text{-}Prf}}(d,\ulcorner F^E\urcorner) \rightarrow [\ E^* \ \& \ \neg\underline{\mathrm{Pr}}_{L_1 A}(\ulcorner F\urcorner) \ \rightarrow \ \neg\neg\exists u\ \underline{\mathrm{Crit}}(d,u)\ ].$$

Since $T \supseteq Y_0+\mathrm{BI}$ we get from (1)

$$(2)\quad \vdash_{A+\underline{\mathrm{CMP}}(T)} \underline{\mathrm{Pr}}_T\ulcorner\underline{\mathrm{E\text{-}Prf}}(d,\ulcorner F^E\urcorner)\ \longrightarrow\ \mathrm{Pr}_T\ulcorner[E^* \ \&\ \neg\underline{\mathrm{Pr}}_{L_1 A}\ulcorner F\urcorner \rightarrow \neg\neg\exists u\ \underline{\mathrm{Crit}}(d,u)]\urcorner$$

and so

$$(3)\quad \vdash_{A+\underline{\mathrm{CMP}}(T)+\underline{\mathrm{Rfn}}_{C_0}(T)} \underline{\mathrm{Pr}}_T\ulcorner\underline{\mathrm{E\text{-}Prf}}(d,\ulcorner F^E\urcorner) \ \&\ E^* \ \&\ \neg\underline{\mathrm{Pr}}_{L_1 A}\ulcorner F\urcorner \ \rightarrow$$

$$\neg\neg\exists u\ \underline{\mathrm{Crit}}(d,u).$$

On the other hand we prove in 4.7-4.11 below

$$(4)\quad \vdash_{Y_0+\mathrm{BI}+\mathrm{AC}^-_{00}} \underline{\mathrm{E\text{-}Der}}(d)\ \ \&\ \ \underline{\mathrm{Crit}}(d,u)\ \ \&\ \ \underline{\mathrm{Res}}(d,u,x)\ \ \longrightarrow$$

$$\neg\neg\exists\phi\ \underline{\mathrm{NPrf}}^{\infty}(\phi,\ulcorner s^E[x]\urcorner)$$

where

$$\underline{\mathrm{Res}}(d,u,x) :\equiv\ \forall y\ [\ T(d,u,y)\ \longrightarrow$$

$$\text{"if } \underline{\mathrm{succedent}}((Uy)_1) \text{ encodes } E^n_i(\vec{t}) \text{ then } x=\langle i,n,\langle\vec{t}\rangle\rangle\text{"}\ ].$$

Since $T^+ \supseteq Y_0+\mathrm{BI}+\mathrm{AC}^-_{00}$ (cf.A.1.2) and $\underline{\mathrm{CMP}}(T) \rightarrow \mathrm{CMP}(T^+)$ trivially, we get from (4)

$$(5)\quad \vdash_{A+\underline{\mathrm{CMP}}(T)} \underline{\mathrm{Pr}}_T\ulcorner\underline{\mathrm{E\text{-}Der}}(d)\urcorner\ \longrightarrow$$

$$\underline{\mathrm{Pr}}_{T^+}(\ulcorner\underline{\mathrm{Crit}}(d,u)\ \&\ \underline{\mathrm{Res}}(d,u,x)\ \rightarrow\ \neg\neg\underline{\mathrm{NPr}}(\ulcorner s^E[x]\urcorner)\urcorner).$$

But $\underline{\mathrm{Crit}}(d,u)$ and $\underline{\mathrm{Res}}(d,u,x)$ are Π^0_1 and $\mathcal{T}^+$ is complete for true Π^0_1 sentences, so

$$(6)\quad \vdash_{A+\underline{\mathrm{CMP}}(\mathcal{T})} \underline{\mathrm{Pr}}_{\mathcal{T}} \ulcorner \underline{\mathrm{E\text{-}Der}}(d) \urcorner \;\&\; \underline{\mathrm{Crit}}(d,u) \;\&\; \underline{\mathrm{Res}}(d,u,x) \;\to\; \underline{\mathrm{Pr}}_{\mathcal{T}^+} \ulcorner \neg\neg \underline{\mathrm{NPr}}^{\infty} \ulcorner s^{E}[x] \urcorner\urcorner .$$

We have however trivially

$$\vdash_{A} \text{"}\{d\}\text{ is total"} \;\to\; \exists x\, \underline{\mathrm{Res}}(d,u,x)$$

and so

$$\vdash_{A+\underline{\mathrm{CMP}}(\mathcal{T})+\underline{\mathrm{Rfn}}_{\mathcal{C}_0}(\mathcal{T})} \underline{\mathrm{Pr}}_{\mathcal{T}} \ulcorner \underline{\mathrm{E\text{-}Der}}(d) \urcorner \;\to\; \exists x\, \underline{\mathrm{Res}}(d,u,x).$$

Hence we get from (6)

$$(7)\quad \vdash_{A+\underline{\mathrm{Rfn}}_{\mathcal{C}_0}(\mathcal{T})} \underline{\mathrm{Pr}}_{\mathcal{T}} \ulcorner \underline{\mathrm{E\text{-}Der}}(d) \urcorner \;\&\; \exists u\, \underline{\mathrm{Crit}}(d,u) \;\to\; \exists x\, \underline{\mathrm{Pr}}_{\mathcal{T}^+} \ulcorner \neg\neg \underline{\mathrm{NPr}}^{\infty} \ulcorner s^{E}[x] \urcorner\urcorner .$$

Combining (3) and (7) yields

$$(8)\quad \vdash_{A+\underline{\mathrm{CMP}}(\mathcal{T})+\underline{\mathrm{Rfn}}_{\mathcal{C}_0}(\mathcal{T})} \underline{\mathrm{Pr}}_{\mathcal{T}} \ulcorner \underline{\mathrm{E\text{-}Prf}}(d,\ulcorner F \urcorner) \urcorner \;\&\; E^{*} \;\&\; \neg\underline{\mathrm{Pr}}_{\mathcal{L}_1 A} \ulcorner F \urcorner \;\to\; \neg\neg\exists x\, \underline{\mathrm{Pr}}_{\mathcal{T}^+} \ulcorner \neg\neg \underline{\mathrm{NPr}}^{\infty} \ulcorner s^{E}[x] \urcorner\urcorner .$$

But from 3.2.2 we have

$$\vdash_{A} \neg\underline{\mathrm{Pr}}_{\mathcal{L}_1} \ulcorner F \urcorner \;\to\; \neg\underline{\mathrm{Pr}}_{\mathcal{L}_1 A} \ulcorner F^{E} \urcorner \qquad (F \text{ a schema of } \mathcal{L}_1)$$

so

$$(9)\quad \vdash_{A+\underline{\mathrm{CMP}}(\mathcal{T})+\underline{\mathrm{Rfn}}_{\mathcal{C}_0}(\mathcal{T})} \neg\underline{\mathrm{Pr}}_{\mathcal{L}_1} \ulcorner F \urcorner \;\&\; \underline{\mathrm{Pr}}_{A^{\infty}_{\mathrm{rec}}[\mathcal{T}]} \ulcorner F^{E} \urcorner \;\&\; E^{*} \;\to\; \neg\neg\exists x\, \underline{\mathrm{Pr}}_{A^{\infty}[\mathcal{T}^+]} \ulcorner s^{E}[x] \urcorner .$$

This completes the proof theoretic-reduction. Note that for any predicate $\underline{Crit}$ (not necessarily Π^0_1) for which (1) and (4) hold, we could prove a statement (7^+) similar to (7), but with $\underline{Pr}_{T^+}\ulcorner \exists x\, \neg\neg\underline{NPr}^{\infty}\ulcorner s^E[x]\urcorner\urcorner$ as the succedent; there is however no way to pull the existential quantifier out of the provability predicate here.

3.5. SOLUTION OF THE REDUCED PROBLEM.

In this part of the proof of theorem II, given in B.5 below, we prove for every Σ^0_2 theory S the existence of a q.f. $E(x)$ s.t.

(10) $$\vdash_{A+\underline{Con}(S)+\underline{Comp}_{\Sigma^0_2}(S)} \forall x\, \neg\underline{Pr}_S\ulcorner s^E[x]\urcorner \quad \& \quad \neg\neg E^*$$

where $\underline{Pr}_S$ is a fixed Σ^0_2 provability predicate for S, and where

(11) $$\underline{Comp}_{\Sigma^0_2}(S) :\equiv \forall x\, [\, \underline{Tr}_{\Sigma^0_2}(x) \rightarrow \underline{Pr}_S(x)\,].$$

i.e., S is complete for Σ^0_2 sentences. (Here $\underline{Tr}_{\Sigma^0_2}(x)$ is a (canonical) truth definition for Σ^0_2 sentences).

We wish to apply (10) to $S \equiv A^\infty[T^+]$, where T and T^+ are as in A.1.2. First, note

(12) $$\vdash_{V_0} \neg\underline{NPr}^\infty(\ulcorner\bot\urcorner),$$

so

(13) $$\vdash_A \underline{Con}(T^+) \rightarrow \underline{Con}(A^\infty[T^+]).$$

Also, for Σ^0_2 sentences F we have directly (compare A.2.2.1)

(14) $$\vdash_{V_0} F \rightarrow \underline{NPr}^\infty\ulcorner F\urcorner$$

and since $T^+ \supseteq V_0$, and quite trivially $\underline{CMP}(T) \rightarrow \underline{CMP}(T^+)$, this implies

(15) $$\vdash_{A+\underline{CMP}(T)} \underline{Pr}_{T^+}\ulcorner F\urcorner \rightarrow \underline{Pr}_{A^\infty[T^+]}\ulcorner F\urcorner.$$

By the definition of $\underline{\mathrm{Pr}}_{T^+}$ however

$$\vdash_A \mathrm{F} \rightarrow \underline{\mathrm{Pr}}_{T^+} \ulcorner \mathrm{F} \urcorner \qquad \text{for every } \Pi^0_1 \text{ F},$$

and so

$$(16) \qquad \vdash_A \mathrm{F} \rightarrow \underline{\mathrm{Pr}}_{T^+} \ulcorner \mathrm{F} \urcorner \qquad \text{for every } \Sigma^0_2 \text{ F}.$$

Hence we get from (15) and (16)

$$(17) \qquad \vdash_{A+\underline{\mathrm{CMP}}(T)} \mathrm{F} \rightarrow \underline{\mathrm{Pr}}_{A^\infty[T^+]} \ulcorner \mathrm{F} \urcorner \qquad \text{for every } \Sigma^0_2 \text{ formula F}.$$

Now observe that steps (15)-(17) can be uniformly formalized (within A), i.e., (11) holds for $S \equiv A^\infty[T^+]$, as wanted.

From (10) for $S \equiv A^\infty[T^+]$, (9) and (13) we now have by predicate logic

$$(18) \qquad \vdash_{A^T} \neg\underline{\mathrm{Pr}}_{L_1} \ulcorner \mathrm{F} \urcorner \rightarrow \neg\underline{\mathrm{Pr}}_{A^\infty_{\mathrm{rec}}[T]} \ulcorner \mathrm{F}^{\mathrm{E}} \urcorner$$

for some fixed quantifier-free E(x).

We proceed now to prove (1) and (4) (the proof-theoretic reduction) and (1) (the recursion-theoretic solution) which together imply as we have just seen theorem II.

B.4. THE PROOF-THEORETIC REDUCTION FOR THEOREM II

4.1. LEMMA. *Let the numeral* $\bar{n}$ *not occur in* $\underline{a}$, F, ∃xGx.

(i) *If* (1) $\underline{a}, G\bar{n} \vdash_{L_1A} F$ *then*

(2) $\underline{a}, Gv \vdash_{L_1A} F$ *where* v *is a parameter which does not occur in* $\underline{a}$, $G\bar{n}$, F.

(ii) *If* $\underline{a} \vdash_{L_1A} G\bar{n}$ *then* $\underline{a} \vdash_{L_1A} Gv$ *(for* v *as above).*

PROOF. Given a normal derivation of L_1A for (1) replace every occurrence of $\bar{n}$ by v and observe, by inspection on cases for the inference rules, that the result is a correct derivation. The proof of (ii) is similar. □

4.2. SEMI FORMAL HEURISTIC OUTLINE OF THE REDUCTION

4.2.1. *Preliminary notations.*

$R_1(d,u) \;:\equiv\; \neg\underline{Pr}_{L_1A}\ulcorner s^{d,u}\urcorner$.

$R_2(d,u) \;:\equiv\;$ "all equations in $\underline{a}^{d,u}$ are true".

$R_3(d,u) \;:\equiv\;$ "$F^{d,u}$ is an E-sentence".

$R_4(d,u) \;:\equiv\;$ "$F^{d,u}$ is an E-atom, and $\rho^{d,u}$ is [∀I]".

$R_5(d,u) \;:\equiv\;$ "$F^{d,u}$ is a Σ^0_1-sentence".

Note that each $R_j(d,u)$ may be formally defined as a Π^0_1 predicate. Example:

$R_3(d,u) \;:\equiv\; \forall y\, [\; T(d,u,y) \longrightarrow$ "$\underline{\mathrm{succedent}}((Uy)_1)$ is the g.n. of an E-sentence"].

$\underline{\mathrm{Start}}(d,u) \;:\equiv\; \bigwedge_{i=1,2,3} R_i(d,u)$.

$\underline{\mathrm{Crit}}_1(d,u) \;:\equiv\; \bigwedge_{i=1,2,4} R_i(d,u)$.

4.2.2. *Locating an arithmetical inference in* E*-derivations (the predicate* $\underline{\text{Crit}}$*).*

We want to define a predicate $\underline{\text{Crit}}$ and to prove for it 3.4(1),(4). The idea is that when $\underline{\text{E-Der}}(d)$ and $\underline{\text{Crit}}(d,u)$ ("u is a critical node in the proof-tree described by d") then the subderivation d^u of d (where $\{d^u\} := \lambda x.\{d\}(u*x)$) has sufficiently nice properties so as to enable the extraction from it of a derivation for $s^E[w]$ for some w.

As a first attempt to define such a predicate we try as in the proof of theorem I to look, when $\underline{\text{E-Prf}}(d,\ulcorner F\urcorner)$ and $\neg\underline{\text{Pr}}_{L_1A}(\ulcorner F\urcorner)$, for a "genuine" use of an arithmetical inference in d. A starting node for such a search upwards may be any node v of d s.t. $\underline{\text{Start}}(d,v)$. When $\underline{\text{Start}}(d,v)$ we can weakly find (i.e., $\neg\neg\exists$) a node $v*\langle n\rangle$ s.t. $\underline{\text{Start}}(d,v*\langle n\rangle)$, using lemma 4.1 when $\rho^{d,v}$ is [∀I] or $[\exists I^*]$, and the truth of E^* and 3.2.1 when $\rho^{d,v}$ is $[\exists E^1]$ (lemma 4.4 below). Thus the search up in d may continue. The only cases where this process stops are when $R_4(d,v)$ or when $\rho^{d,v}$ is [FE]. In the last case, the definition of normality of A.1.1 implies (as in 2.1.2) that a false equation occurs in $\underline{a}^{d,v}$, contradicting $R_2(d,v)$. Thus, by the well-foundedness of the proof-tree d, we find a node $u \succ v$ s.t. $\underline{\text{Crit}}_1(d,u)$.

When $\underline{\text{Crit}}_1(d,u)$, we can actually find in each subderivation $d^{u*\langle m\rangle}$ an inference of the form

$$(*)\qquad \begin{array}{ll} & G \\ \boxed{u*w} \rightarrow & [\exists I]\ \ F^{d,u*w} \end{array}$$

(G is a true equation and $F^{d,u*w} \equiv F^{d,u*\langle m\rangle}$). So these can be collected to yield a derivation of the form:

$$\begin{array}{c} \langle \Sigma_m \rangle_{m<\omega} \\ \hline [\forall I] \quad B[\langle i,n,\langle \vec{t}\rangle\rangle] \Rightarrow E_i^n(\vec{t}) \end{array}$$

where $F^{d,u} \equiv: E_i^n(\vec{t})$, and each Σ_m is (schematically) of the form (*).

Unfortunately, the crude statement that the situation above occurs is not Π_1^0, essentially because there is no bound on the length of the w corresponding to each $m<\omega$. A certain refinement of the argument is therefore necessary.

4.2.3. *Heuristic for the disjunction-free fragment*

Assume, again, $\underline{\text{E-Der}}(d)$ and $\underline{\text{Crit}}_1(d,u)$. The subderivation d^u of d then takes the form

$$(1)\qquad \frac{\left\{\begin{array}{c}\Sigma_m\\ \underline{a} \Rightarrow \exists y E\langle \bar{m},y,i,n,\langle\vec{t}\rangle\rangle\end{array}\right\}_{m<\omega}}{[\forall I]\quad \underline{a} \Rightarrow \forall x\exists y E\langle x,y,i,n,\langle\vec{t}\rangle\rangle}$$

where each Σ_m is formally described by $d^{u*\langle m\rangle}$.

From each Σ_m we wish to extract a derivation in A^∞ of

$$(2)\qquad B[\langle i,n,\langle\vec{t}\rangle\rangle] \Rightarrow \exists y E\langle \bar{m},y,i,n,\langle\vec{t}\rangle\rangle.$$

Fix some m, and let us analyse the structure of Σ_m.

We assume first that d is a derivation for a disjunction-free E-sentence; this implies by the subformula property that disjunction does not occur in the derivation d, and in particular, in the subderivation Σ_m we are looking at.

In addition we may assume

$$(3)\qquad \forall w \succ u*\langle m\rangle\ \neg\underline{\text{Start}}(d,w).$$

Because if $\underline{\text{Start}}(d,w)$, $w \succ u$ then we could start our initial search afresh; this could not be iterated indefinitely, because d is well-founded.

Consider now the main inference rule of Σ_m, $\rho^{d,u*\langle m\rangle}$. By the subformula property of d we have to consider the following cases only.

Cases (i)-(iiia): contradiction to (3).

(i) $\rho^{d,u*\langle m\rangle} = [\bot]$; then $s^{d,u*\langle m,0\rangle} = \underline{a}\Rightarrow\bot$ and so $\underline{\text{Start}}(d,u*\langle m,0\rangle)$ contradicting (3).

(ii) $[\forall E]$;

$$(4)\qquad \frac{\underline{a} \Rightarrow E^k_j(\vec{s})}{[\forall E]\quad \underline{a} \Rightarrow \exists y E\langle \bar{m},y,i,n,\langle\vec{t}\rangle\rangle}\qquad \text{say.}$$

Recall that $E^k_j(\vec{s}) \equiv \forall x\exists yE\langle x,y,j,k,\langle\vec{s}\rangle\rangle$, and so necessarily $\langle i,n,\langle\vec{t}\rangle\rangle \equiv \langle j,k,\langle\vec{s}\rangle\rangle$ (syntactical identity). Therefore $s^{d,u*\langle m,0\rangle} \equiv s^{d,u}$ and so <u>Start</u>(d,u*⟨m,0⟩), contradicting (3) once again.

(iiia) $[\exists E^1]$; since d is normal, Σ_m must then have the following form (compare the first part of 4.5.4 below):

$$\frac{\begin{array}{l}\qquad\Delta\\ \qquad\underline{a}\Rightarrow E^k_j(\vec{s})\\ [\forall E]\ \ \underline{a}\Rightarrow\exists zCz\end{array}\qquad\left\{\begin{array}{c}\Gamma_p\\ \underline{a},C\bar{p}\Rightarrow\exists yE\langle\bar{m},y,i,n,\langle\vec{t}\rangle\rangle\end{array}\right\}_{p<\omega}}{\underline{a}\Rightarrow\exists yE\langle\bar{m},y,i,n,\langle\vec{t}\rangle\rangle}\tag{5}$$

First, if $\langle j,k,\langle\vec{s}\rangle\rangle \equiv \langle i,n,\langle\vec{t}\rangle\rangle$ then <u>Start</u>(d,u*⟨m,0,0⟩) as in (ii), contradicting (3).

<u>Cases (iiib),(iv): the search continues.</u>

(iiib) If, in (iiia), $\exists zCz$ is true, let $p := \mu z.Cz$, and consider – in place of Σ_m – its subderivation Γ_p (formally described by $d^{u*\langle m,p+1\rangle}$).

Before concluding the case $\rho^{d,u*\langle m\rangle} = [\exists E^1]$ let us turn first to case

(iv) If $\rho^{d,u*\langle m\rangle}$ is $[\exists E^*]$, then guided by lemma 4.1 we pick the first numeral $\bar{p}$ which does not occur in the sequents $s^{d,u*\langle m\rangle}$, $s^{d,u*\langle m,0\rangle}$, and we consider (as in case (iiib)) the subderivation $d^{u*\langle m,p+1\rangle}$.

<u>Cases (iiic),(v): happy ending.</u>

(iiic) If $\rho^{d,u*\langle m\rangle}$ is $[\exists E^1]$, and (iiia) and (iiib) do not apply, then in (5) $\langle j,k,\langle\vec{s}\rangle\rangle \neq \langle i,n,\langle\bar{t}\rangle\rangle$ and $\exists zCz$ is false; so we can extract from (5) the following derivation in A^∞ of (2):

$$\frac{\begin{array}{ll} & B[\langle i,n,\langle\vec{t}\rangle\rangle]\\ [\forall E] & E^k_j(\vec{s})\\ [\forall E] & \exists zCz\end{array}\qquad\left\{\begin{array}{ll} & C\bar{p}\\ [FE] & \bot\\ [\bot] & \exists yE\langle\bar{m},y,i,n,\langle\vec{t}\rangle\rangle\end{array}\right\}_{p<\omega}}{[\exists E^1]\qquad\exists yE\langle\bar{m},y,i,n,\langle\vec{t}\rangle\rangle}\tag{6}$$

(here we dropped the precedents of sequents).

(v) $[\exists I]$; by 2.1.2 $u*\langle m,0\rangle$ is then a top-node in d, and so $\rho^{d,u*\langle m,0\rangle}$ is either [T] or [TE]. In the first case $F^{d,u*\langle m,0\rangle}\in\underline{a}$; but all equations in $\underline{a}$ are true, so $F^{d,u*\langle m,0\rangle}$ is in any case a true equation.

These are all the cases in the absence of disjunction. Cases (i),(ii), (iiia) rule out possible failures of the construction; cases (iiib),(iv) allow the search to continue, while cases (iiic) and (v) yield the required derivation for (2).

Note that if E^* is true, then $\exists zCz$ in (6) is also true, and so case (iiic) is excluded. Our argument here must however be independent of E^* (cf. 3.4(4)-(6)), and so this case has to be considered throughout.

In order to clarify a bit the form of a search which proceeds through (iiib),(iv), let us consider for example the outcome of case (iv), and suppose that now case (ii) applies to Γ_p ($:\equiv$ the derivation formally described by $d^{u*\langle m,p+1\rangle}$). I.e., the following configuration occurs:

$$\begin{array}{llll}
 & & \qquad\quad \Gamma_p & \\
 & & \quad\ \underline{a},C\bar{p} \Rightarrow E^n_i(\vec{t}) & \Gamma_{p+1} \\
\underline{a} \Rightarrow \exists zCz & \cdots & [\forall E]\ \ \underline{a},C\bar{p} \Rightarrow \exists yE\langle\bar{m},y,i,n,\langle\vec{t}\rangle\rangle & \qquad \cdots \\ \hline
\text{the node } \boxed{u*\langle m\rangle}\!\longrightarrow & & [\exists E^*]\ \ \ \underline{a} \Rightarrow \exists yE\langle\bar{m},y,i,n,\langle\vec{t}\rangle\rangle &
\end{array}$$

Here (3) implies, as in (i)-(iii) ,

$$\neg\neg\underline{\mathrm{Pr}}_{L_1}A(\ulcorner\underline{a}\Rightarrow\exists zCz\urcorner) \qquad \text{and} \qquad \neg\neg\underline{\mathrm{Pr}}_{L_1}A(\ulcorner\underline{a},C\bar{p}\Rightarrow E^n_i(\vec{t})\urcorner)$$

which by 4.1(i) and the choice of $\bar{p}$ give

$$\neg\neg\underline{\mathrm{Pr}}_{L_1}A(\ulcorner\underline{a}\Rightarrow E^n_i(\vec{t})\urcorner)$$

contradicting $\underline{\mathrm{Crit}}_1(d,u)$. So we have adapted the argument of (ii) to the case that a search for a proof of (2) proceeds via case (iv). Other arguments are adapted in about the same way, and this allows the iteration of the search through (iiib),(iv) above.

By the well-foundedness of d the process must terminate, that is, one of cases (iiic),(v) ultimately appears, and we obtain a proof for (2), as desired.

4.2.4. *Disjunction reconsidered*

When disjunction does occur in the derivation d above, we must add to (i)-(v) above another case:

(vi) $\rho^{d,u*\langle m\rangle}$ is [vE]. We then consider simultanuously *both* minor premisses of $\rho^{d,u*\langle m\rangle}$, i.e., the nodes $u*\langle m,1\rangle$ and $u*\langle m,2\rangle$.

As in the last paragraph of 4.2.3, let us see what happens if case (ii) applies now to both $u*\langle m,1\rangle$ and $u*\langle m,2\rangle$. We have then the following configuration:

$$\begin{array}{lllll}
 & & \Gamma_1 & & \Gamma_2 \\
\Delta & & \underline{a},G_1 \Rightarrow E^n_i(\vec{t}) & & \underline{a},G_2 \Rightarrow E^n_i(\vec{t}) \\
\underline{a} \Rightarrow G_1 \vee G_2 & [\forall E] & \underline{a},G_1 \Rightarrow \exists yE\langle\ldots\rangle & [\forall E] & \underline{a},G_2 \Rightarrow \exists yE\langle\ldots\rangle \\
\hline
\boxed{u*\langle m\rangle}\longrightarrow & [\vee E] & \underline{a} \Rightarrow \exists yE\langle\bar{m},y,i,n,\langle\vec{t}\rangle\rangle & &
\end{array}$$

As in the last paragraph of 4.2.3,

$$\neg\neg\underline{Pr}_{L_1 A}(\ulcorner \underline{a}\Rightarrow G_1 \vee G_2\urcorner),\quad \neg\neg\underline{Pr}_{L_1 A}(\ulcorner \underline{a},G_1\Rightarrow E^n_i(\vec{t})\urcorner),\quad \neg\neg\underline{Pr}_{L_1 A}(\ulcorner \underline{a},G_2\Rightarrow E^n_i(\vec{t})\urcorner),$$

and so

$$\neg\neg\underline{Pr}_{L_1 A}(\ulcorner \underline{a}\Rightarrow E^n_i(\vec{t})\urcorner),\quad \text{contradicting}\quad \underline{Crit}_1(d,u).$$

This argument may be generalized to conclude that, at least for one successive choice of minors of $[\vee E]$ in the search described by (iiib),(iv), (vi) the construction leads to a node falling under one of the cases (iiic), (v), thus allowing a construction of a proof of A^∞ (incidently of A^∞_{rec}) for (2).

The assertion that this is the case is now seen quite easily to be formalizable as a Π^0_1 predicate (over d,u).

4.3. FORMALIZATION OF THE PREDICATE Crit

$\underline{Step}(d,w,p) \;:\equiv\; \bigvee_{i=1,2,3} \underline{Step}_i(d,w,p)$

where

$\underline{Step}_1(d,w,p) \;:\equiv\;$ "$\rho^{d,w} = [\exists E^1]$, and if $F^{d,w*\langle 0\rangle} \equiv: \exists zCz$ then $p = \mu z.Cz + 1$".

$\underline{Step}_2(d,w,p) \;:\equiv\;$ "$\rho^{d,w} = [\exists E^*]$, and if $F^{d,w*\langle 0\rangle} \equiv: \exists zCz$ then p is 1 + the value of the first numeral which does not occur in $s^{d,w}, s^{d,w*\langle 0\rangle}$".

$\underline{Step}_3(d,w,p) \;:\equiv\;$ "$\rho^{d,w} = [\vee E]$ and $1 \le p \le 2$".

These three predicates correspond to cases (iiib), (iv) and (vi) in 4.2.3/4, where the search described there proceeds to the p'th premise of the node w. It should be noted that $\underline{\mathrm{Step}}$ is a Δ^0_1 predicate. For example

$$\begin{aligned}\underline{\mathrm{Step}}_1(d,w,p) \ &\equiv\ \forall x,y\ [\ T(d,w,x)\ \&\ T(d,w{*}\langle 0\rangle,y)\ \longrightarrow\ A(x,y,p)\ ]\\ &\equiv\ \exists x,y\ [\ T(d,w,x)\ \&\ T(d,w{*}\langle 0\rangle,y)\ \&\ A(x,y,p)\ ]\end{aligned}$$

where

$$\begin{aligned}A(x,y,p)\ :\equiv\ (Ux)_0 = {}^{\ulcorner}\exists E^{1\urcorner}\ &\&\ \underline{\mathrm{Tr}_{QF}(\mathrm{inst}(\mathrm{antecedent}}((Uy)_1),p\dot{-}1))\\ &\&\ \forall q{<}p\ \neg\underline{\mathrm{Tr}}_{QF}(\underline{\mathrm{inst}}(\underline{\mathrm{antecedent}}((Uy)_1),q\dot{-}1)).\end{aligned}$$

$\underline{\mathrm{Tr}}_{QF}$ is a (Δ^0_1) truth predicate for equations, and $\underline{\mathrm{inst}}$ is a prim.rec. function which satisfies $\underline{\mathrm{inst}}({}^{\ulcorner}\exists xGx^{\urcorner},n) = {}^{\ulcorner}G\bar{n}^{\urcorner}$.

$$\underline{\mathrm{Selected}}(d,v)\ :\equiv\ \forall i{<}\underline{\mathrm{lth}}(v)\ \underline{\mathrm{Step}}(d,(v|i),(v)_i)$$

where

$$(v|i)\ :=\ \langle (v)_0,\ldots,(v)_{i\dot{-}1}\rangle \qquad (\text{for } i \le \underline{\mathrm{lth}}(v))$$

$$\underline{\mathrm{Final}}(d,v)\ :\equiv\ \bigvee_{i=1,2,3} \underline{\mathrm{Final}_i}(d,v)$$

where

$$\underline{\mathrm{Final}}_1(d,v)\ :\equiv\ \underline{\mathrm{Selected}}(d,v)\ \&\ \rho^{d,v} = [\bot] \text{ or } [\forall E]$$

$$\underline{\mathrm{Final}}_2(d,v)\ :\equiv\ \underline{\mathrm{Selected}}(d,v)\ \&\ \rho^{d,v} = [\exists E^1]$$

$$\underline{\mathrm{Final}}_3(d,v)\ :\equiv\ \underline{\mathrm{Selected}}(d,v)\ \&\ \rho^{d,v} = [\exists I].$$

These predicates correspond to the cases in 4.2.3 where the construction may stop, whether successfully or not.

$$\underline{\mathrm{Final}}^+(d,v,{}^{\ulcorner}A^{\urcorner})\ :\equiv\ \underline{\mathrm{Final}}_2^+(d,v,{}^{\ulcorner}A^{\urcorner})\ \vee\ \underline{\mathrm{Final}}_3(d,v)$$

where

$$\underline{\mathrm{Final}}_2^+(d,v,{}^{\ulcorner}A^{\urcorner})\ :\equiv\ \underline{\mathrm{Final}}_2(d,v)\ \&\ F^{d,v*\langle 0,0\rangle} \not\equiv A.$$

When for 4.2.3 $A \equiv E^n_i(\vec{t}) \equiv F^{d,u}$ then $\underline{\mathrm{Final}}^+(d,v,\ulcorner A\urcorner)$ expresses the conclusion of the construction by one of (iiic), (v), or possibly its continuation through (iiib). In any case, a "failure" through one of (i)-(iiia) is excluded. It is important to note that $\underline{\mathrm{Final}}$ and $\underline{\mathrm{Final}}^+$ are both Δ^0_1 predicates.

Let us use the binary encodement of finite sets of numbers. The predicates $n \in x$, $x = \emptyset$ etc. are then just prim.rec. arithmetical expressions.

$$\underline{\mathrm{Bar}}(d,x) \quad :\equiv \quad x \neq \emptyset \quad \&$$

$$\forall w \in x \ \{ \ \underline{\mathrm{Final}}(d,w) \quad \& \quad \forall u,y<x \ [\ \rho^{d,u} = [vE]$$

$$\& \quad w = u*\langle{}^1_2\rangle*y \ \longrightarrow \ \exists w' \in x \exists z<x \ w' = u*\langle{}^2_1\rangle*z \ ] \ \}.$$

I.e., a "bar" for d is a finite non-empty set of "final" nodes, which intersects both minor subderivations of each instance of vE if it intersects one of them.

$$\underline{\mathrm{Crit}}_2(d,u) \quad :\equiv \quad \forall m,x \ [\ \underline{\mathrm{Bar}}(d^{u*\langle m\rangle},x) \ \longrightarrow \ \exists w \in x \ \underline{\mathrm{Final}}^+(d^{u*\langle m\rangle},w,\ulcorner F^{d,u}\urcorner) \ ]$$

$$\underline{\mathrm{Crit}}(d,u) \quad :\equiv \quad \underline{\mathrm{Crit}}_1(d,u) \quad \& \quad \underline{\mathrm{Crit}}_2(d,u).$$

Note that $\underline{\mathrm{Crit}}$ is intuitionistically equivalent to a Π^0_1 predicate.

$$\underline{\mathrm{Final}}^{++}(d,v,\ulcorner A\urcorner) \quad :\equiv \quad \underline{\mathrm{Final}}_2^{++}(d,v,\ulcorner A\urcorner) \ \vee \ \underline{\mathrm{Final}}_3^{++}(d,v)$$

where

$$\underline{\mathrm{Final}}_2^{++}(d,v,\ulcorner A\urcorner) \quad :\equiv \quad \underline{\mathrm{Final}}_2^{+}(d,v,\ulcorner A\urcorner) \quad \& \quad \neg\underline{\mathrm{Tr}}_{\Sigma^0_1}(\ulcorner F^{d,v*\langle 0\rangle}\urcorner)$$

$$\underline{\mathrm{Final}}_3^{++}(d,v) \quad :\equiv \quad \underline{\mathrm{Final}}_3(d,v) \quad \& \quad \underline{\mathrm{Tr}}_{QF}(\ulcorner F^{d,v*\langle 0\rangle}\urcorner).$$

$\underline{\mathrm{Final}}^{++}$ corresponds to a real termination of the search described in 4.2.3. In contrast to $\underline{\mathrm{Final}}^{+}$ however, $\underline{\mathrm{Final}}^{++}$ is a Π^0_1 predicate, and not a Δ^0_1 one.

4.4 - 4.6. PROOF OF 3.4(1): THE EXISTENCE OF A CRITICAL NODE
(first part of the proof theoretic reductions)

4.4. LEMMA.

$$\vdash_{V_0+\mathrm{BI}} [\ E^* \ \&\ \underline{\text{E-Der}}(d)\ \&\ \underline{\text{Start}}(d,u)\ ] \ \rightarrow\ \neg\neg\exists w \succ u\ \underline{\text{Crit}}_1(d,w).$$

PROOF. Denote the formula to be proven by $R(u)$. First, we prove below by BI using the well-foundedness of the proof-tree d the (open) formula

$$S(u) :\equiv\ [\ E^* \ \&\ \underline{\text{E-Der}}(d)\ \&\ \underline{\text{Start}}(d,u)\ \&\ \neg R_4(d,u)\ ] \ \rightarrow$$
$$\neg\neg\exists w \succ u\ \underline{\text{Start}}(d,w).$$

Assuming $\forall u S(u)$, $R(u)$ follows easily by a second use of BI, where $S(u)$ is to be used for the induction step.

Towards proving $S(u)$ by BI assume the premiss of $S(u)$, assume $\forall n S(u*\langle n\rangle)$, and consider cases for $\rho^{d,u}$ which by the normality of d can only be one of the following:

(i) $\rho^{d,u}$ is [T]. This contradicts $R_1(d,u)$. $\rho^{d,u}$ is also not [TE] by $R_3(d,u)$.

(ii) $\rho^{d,u}$ is [FE]. As in 2.1.2 the normality of d implies then that $\rho^{d,u*\langle 0\rangle}$ is [T], and so $F^{d,u} \in \underline{a}^{d,u}$, contradicting $R_2(d,u)$.

(iii) $\rho^{d,u}$ is a propositional rule, [$\exists$I] or [$\forall$E]. If $\neg\neg\underline{\text{Pr}}_{L_1A}(\ulcorner s^{d,u*\langle n\rangle}\urcorner)$ for all $n<3$, then of course $\neg\neg\underline{\text{Pr}}_{L_1A}\ulcorner s^{d,u}\urcorner$, since all the rules considered in this case are rules of L_1. This contradicts $R_1(d,u)$. So $\neg\neg\exists n<3\ \neg\underline{\text{Pr}}_{L_1A}(\ulcorner s^{d,u*\langle n\rangle}\urcorner)$. For the cases considered the subformula property of d implies trivially $R_j(d,u) \rightarrow R_j(d,u*\langle n\rangle)$ for $j = 2,3$, and so we conclude that $\neg\neg\exists n<3\ \underline{\text{Start}}(d,u*\langle n\rangle)$.

(iv) $\rho^{d,u}$ is [$\exists E^*$]. Let $\bar{p}$ be the first numeral which does not occur in $s^{d,u}$, $s^{d,u*\langle 0\rangle}$, and prove

$$(*)\quad \neg\neg[\ \underline{\text{Start}}(d,u*\langle 0\rangle)\ \ v\ \ \underline{\text{Start}}(d,u*\langle p+1\rangle)\ ]$$

like in (iii), using 4.1(i). That is, for the u considered

$$\begin{aligned} \neg\underline{\mathrm{Start}}(d,u*\langle j\rangle) &\rightarrow \neg R_1(d,u*\langle j\rangle) \\ &\rightarrow \neg\neg\underline{\mathrm{Pr}}_{L_1A}\ulcorner s^{d,u*\langle j\rangle}\urcorner \end{aligned}$$

while by the choice of p and 4.1(i)

$$\underline{\mathrm{Pr}}_{L_1A}\ulcorner s^{d,u*\langle 0\rangle}\urcorner \ \& \ \underline{\mathrm{Pr}}_{L_1A}\ulcorner s^{d,u*\langle p+1\rangle}\urcorner \rightarrow \underline{\mathrm{Pr}}_{L_1A}\ulcorner s^{d,u}\urcorner \rightarrow \neg\underline{\mathrm{Start}}(d,u).$$

Since this contradicts the assumed premiss of S(u), one gets (*) by intuitionistic prop. logic (cf. KLEENE [52], p.119,*60i,g).

(v) $\rho^{d,u}$ is [∀I]. Let $\bar{p}$ be the first numeral which does not occur in $s^{d,u}$, and proceed to prove $\neg\neg\underline{\mathrm{Start}}(d,u*\langle p+1\rangle)$ like in (iii), using 4.1(ii).

(vi) $\rho^{d,u}$ is $[\exists E^1]$, $F^{d,u*\langle 0\rangle} \equiv: \exists zCz$, where Cz is q.f.. Since $R_1(d,u)$, i.e., $\neg\underline{\mathrm{Pr}}_{L_1A}\ulcorner s^{d,u}\urcorner$, we get from 3.2.1 $\forall mR_1(d,u*\langle m+1\rangle)$. $R_3(d,u)$ implies $\forall mR_3(d,u*\langle m+1\rangle)$ trivially. Finally, for each m $R_2(d,u)$ and $C\bar{m}$ imply outright $R_2(d,u*\langle m+1\rangle)$. Summing up we hence get

$$(*) \quad \underline{\mathrm{Start}}(d,u) \ \& \ \exists zCz \rightarrow \exists z\, \underline{\mathrm{Start}}(d,u*\langle z\rangle).$$

But by the subformula property of d $\exists zCz$ is a subformula of the Π^0_2 sentence E^*, and so $E^* \rightarrow \exists zCz$, while by the assumed $\forall nS(u*\langle n\rangle)$,

$$\underline{\mathrm{Start}}(d,u*\langle z\rangle) \rightarrow \neg\neg\exists w \succ u*\langle z\rangle\ \underline{\mathrm{Start}}(d,w).$$

So we get from (*)

$$\underline{\mathrm{Start}}(d,u) \ \& \ E^* \rightarrow \neg\neg\exists w \succ u\ \underline{\mathrm{Start}}(d,w)$$

as wished. □

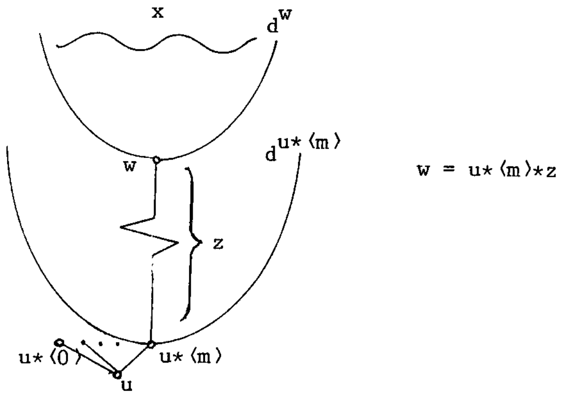
4.5.1. LEMMA.

$$\vdash_{\mathcal{V}_0+BI} \underline{\text{E-Der}}(d) \ \& \ \underline{\mathrm{Crit}}_1(d,u) \ \& \ \neg\exists v \succ u\ \underline{\mathrm{Start}}(d,v) \rightarrow \underline{\mathrm{Crit}}_2(d,u).$$

We prove this lemma as a corollary of

<u>4.5.2. LEMMA</u>. *Let* A *be an* E-*sentence. Then*

$$
\begin{aligned}
\vdash_{\mathcal{Y}_0+BI}\ & \underline{\mathrm{E\text{-}Der}}(d) \ \& \ \underline{\mathrm{Crit}}_1(d,u) \ \& \ w = u*\langle m\rangle*z \ \& \ \underline{\mathrm{Selected}}(d^{u*\langle m\rangle},z) \\
& \& \ \forall v \succ u \ \neg\underline{\mathrm{Start}}(d,v) \ \& \ \underline{\mathrm{Bar}}(d^{w},x) \\
& \& \ \forall y\in x \ \neg\underline{\mathrm{Final}}^{+}(d^{w},y,\ulcorner F^{d,u}\urcorner) \\
& \rightarrow \ \neg\neg\underline{\mathrm{Pr}}_{L_1A}(\ulcorner \underline{a}^{d,w}\Rightarrow F^{d,u}\urcorner).
\end{aligned}
$$



4.5.3. *Proof that 4.5.2 implies 4.5.1*

Assume the premiss of 4.5.1. For each $m<\omega$ this implies the first five conjuncts of 4.5.2 for $w = u*\langle m\rangle$, $z = \langle\rangle$, and also

$$\neg\underline{\mathrm{Pr}}_{L_1A}(\ulcorner \underline{a}^{d,u*\langle m\rangle}\Rightarrow F^{d,u}\urcorner)$$

since $\underline{a}^{d,u*\langle m\rangle} = \underline{a}^{d,u}$ here. So, by the contrapositive form of 4.5.2, and quantifying over m,

$$\forall m,x\ [\ \underline{\mathrm{Bar}}(d^{u*\langle m\rangle},x) \ \rightarrow \ \exists y\in x\ \underline{\mathrm{Final}}^{+}(d^{u*\langle m\rangle},y,\ulcorner F^{d,u}\urcorner)\ ]$$

(note that $\underline{\mathrm{Final}}^{+}$ is decidable); i.e., $\underline{\mathrm{Crit}}_2(d,u)$ as required. □

4.5.4. *Proof of 4.5.2*

Write S(w) for the closure over x,z of the formula to be proved. By BI the problem reduces to showing

$$\forall n\, S(w * \langle n \rangle) \rightarrow S(w).$$

So assume

(1) $\forall n\, S(w * \langle n \rangle)$ and

(2) the premiss $S^{-}(w)$ of $S(w)$.

Note first that the definition of Selected implies, by a trivial induction on lth(w)

(3) $F^{d,w} \equiv F^{d,u*\langle m \rangle} \equiv \exists y\, E\langle \bar{m},y,i,n,\langle \vec{t} \rangle\rangle$

(4) $R_2(d,w) :\equiv$ "all equations in $\underline{a}^{d,w}$ are true".

Consider now cases for $\rho^{d,w}$.

(i) $[\top]$. Then $F^{d,w} \in \underline{a}^{d,w}$. But by the subformula property of d no Σ^0_1 sentence may be discharged in d, because an E-sentence has no subformula of the form $G \vee H$, $G \rightarrow H$ or $\exists z G$ where G is Σ^0_1. So this case is ruled out. A similar argument excludes the cases [&E] and [$\rightarrow$E].

(ii) $[\bot]$. Then $s^{d,w*\langle 0 \rangle} = \underline{a}^{d,w} \Rightarrow \bot$, while $\neg$Start$(d,w*\langle 0 \rangle)$ implies (by (4))

$$\neg\neg\underline{\mathrm{Pr}}_{L_1 A}(\ulcorner \underline{a}^{d,w} \Rightarrow \bot \urcorner),$$

so

$$\neg\neg\underline{\mathrm{Pr}}_{L_1 A}(\ulcorner \underline{a}^{d,w} \Rightarrow F^{d,u} \urcorner).$$

(iii) $[\forall E]$. Then (3) implies

(5) $F^{d,w*\langle 0 \rangle} \equiv F^{d,u}$.

On the other hand $\neg$Start$(d,w*\langle 0 \rangle)$ implies

(6) $\neg\neg\underline{\mathrm{Pr}}_{L_1 A}(\ulcorner s^{d,w*\langle 0 \rangle} \urcorner)$.

Here $\underline{a}^{d,w*\langle 0 \rangle} = \underline{a}^{d,w}$ so (5) and (6) yield $\neg\neg\underline{\mathrm{Pr}}_{L_{\underline{1}} A}(\ulcorner \underline{a}^{d,w} \Rightarrow F^{d,u} \urcorner)$.

(iv) $[\exists E^1]$, $F^{d,w*\langle 0 \rangle} \equiv: \exists z C z$. Let Bar$(d^w,x)$.

Subcase [a]. $\langle\rangle \in x$. Then $\neg$Final$^{+}(d^w,\langle\rangle,\ulcorner F^{d,u} \urcorner)$ by $S^{-}(w)$, and so by the definition of Final^{+} for this case $F^{d,w*\langle 0,0 \rangle} \equiv F^{d,u}$, and we get as in (iii)

$$\neg\neg\underline{\mathrm{Pr}}_{L_1 A}(\ulcorner \underline{a}^{d,w} \Rightarrow F^{d,u} \urcorner).$$

Subcase [b]. $\langle\rangle \notin x$. Then, since $x \neq \emptyset$ by the definition of *Bar*, we must have for some p $\underline{\mathrm{Step}}(d^{w},\langle\rangle,p)$. This means that the premiss of $S(w{*}\langle p\rangle)$ is satisfied, and hence by the BI hyp. (1) applied to $w{*}\langle p\rangle$

$$\neg\neg\underline{\mathrm{Pr}}_{L_1 A}(\ulcorner \underline{a}^{d,w}, C(\bar{p}) \Rightarrow F^{d,u} \urcorner).$$

But $C(\bar{p})$ is here a true equation, so by 3.2.1

$$\neg\neg\underline{\mathrm{Pr}}_{L_1 A}(\ulcorner \underline{a}^{d,w} \Rightarrow F^{d,u} \urcorner).$$

(v) $[\exists E^{*}]$, $F^{d,w*\langle 0\rangle} \equiv: \exists z Cz$. Let $\bar{p}$ be the first numeral which does not occur in $s^{d,w}, s^{d,w*\langle 0\rangle}$. We have then as in (iv)[b]

(7) $$\neg\neg\underline{\mathrm{Pr}}_{L_1 A}(\ulcorner s^{d,w*\langle p+1\rangle} \urcorner) \equiv \neg\neg\underline{\mathrm{Pr}}_{L_1 A}(\ulcorner \underline{a}^{d,w}, C(\bar{p}) \Rightarrow F^{d,u} \urcorner),$$

and as in (iii) we get

$$\neg\neg\underline{\mathrm{Pr}}_{L_1 A}(\ulcorner s^{d,w*\langle 0\rangle} \urcorner) \equiv \neg\neg\underline{\mathrm{Pr}}_{L_1 A}(\ulcorner \underline{a}^{d,w} \Rightarrow \exists z Cz \urcorner)$$

which together with (7) and lemma 4.1 yields

$$\neg\neg\underline{\mathrm{Pr}}_{L_1 A}(\ulcorner \underline{a}^{d,w} \Rightarrow F^{d,u} \urcorner).$$

(vi) $[\vee E]$, $F^{d,w*\langle 0\rangle} \equiv: G_1 \vee G_2$. Let

$$x^{(j)} :\equiv \{\, y \mid \langle j\rangle * y \in x \,\} \qquad (j=1,2).$$

Then, by the definition of *Bar*, $S^{-}(w)$ implies

$$\underline{\mathrm{Bar}}(d^{w*\langle j\rangle}, x^{(j)}) \quad \& \quad \forall y \in x^{(j)}\ \neg\underline{\mathrm{Final}}^{+}(d^{w*\langle j\rangle}, y, \ulcorner F^{d,u} \urcorner)$$

while trivially

$$\underline{\mathrm{Selected}}(d^{u*\langle w\rangle}, z*\langle j\rangle) \qquad (j=1,2).$$

Apply now, as in (iv) and (v), the BI hyp. (1) to $w*\langle j\rangle$ $(j=1,2)$, to yield

(8) $$\neg\neg\underline{\mathrm{Pr}}_{L_1 A}(\ulcorner \underline{a}^{d,w}, G_j \Rightarrow F^{d,u} \urcorner) \qquad (j=1,2).$$

On the other hand we get as in (iii)

$$\neg\neg\underline{\mathrm{Pr}}_{L_1 A}(\ulcorner s^{d,w*\langle 0\rangle} \urcorner) \equiv \neg\neg\underline{\mathrm{Pr}}_{L_1 A}(\ulcorner \underline{a}^{d,w} \Rightarrow G_1 \vee G_2 \urcorner)$$

which together with (8) yields

$$\neg\neg\underline{\mathrm{Pr}}_{L_1 A}(\ulcorner \underline{a}^{d,w} \Rightarrow F^{d,u}\urcorner).$$

(vii) [∃I]. Then the definition of <u>Bar</u> implies

$$\text{(9)} \qquad \underline{\mathrm{Bar}}(d^w, x) \;\longrightarrow\; x = \{\langle\rangle\}.$$

For this case, $\underline{\mathrm{Final}}^+(d^w, \langle\rangle, \ulcorner F^{d,u}\urcorner)$ automatically, while by $S^-(w)$ (9) implies $\neg\underline{\mathrm{Final}}^+(d^w, \langle\rangle, \ulcorner F^{d,u}\urcorner)$, so this case is ruled out. □

<u>4.6. PROPOSITION.</u>

$$\vdash_{\mathcal{V}_0+\mathrm{BI}} E^* \;\;\&\;\; \underline{\text{E-Der}}(d) \;\;\&\;\; \underline{\mathrm{Start}}(d,u) \;\longrightarrow\; \neg\neg\exists v \succ u \;\underline{\mathrm{Crit}}(d,v).$$

<u>PROOF</u>. Straightforward from 4.4 and 4.5.1 using BI and the well-foundedness of the proof-tree d. □

Applying proposition 4.6 to u = ⟨⟩ we get assertion 3.4.(1).

4.7 - 4.11. PROOF OF 3.4(4). (Second part of the proof-theoretic reduction)

<u>4.7. LEMMA</u>.

$$\vdash_{\mathcal{V}_0+\mathrm{BI}} \underline{\text{E-Der}}(d) \;\;\&\;\; R_5(d,w) \;\longrightarrow\; \exists x\; \underline{\mathrm{Bar}}(d^w, x).$$

<u>PROOF</u>. Straightforward by BI and the well-foundedness of d. □

4.8.1. Let us use the following ad hoc terminology.

"x is a bar" $:\equiv\; \forall u,v \in x\;\; u \not\prec v$

"x is in d" $:\equiv\; \forall u \in x\;\; \{d\}u \not\simeq 0$

$x \models y \;:\equiv\;$ "x and y are bars" &

$\qquad\qquad \forall u \in y\; \exists v \in x\; (v \preccurlyeq u) \;\;\&\;\; \exists u \in y\; \exists v \in x\; (v \prec u)$

$x \models^d y \;:\equiv\;$ "x and y are in d" & $x \models y$.

LEMMA (in $\mathcal{V}_0$+BI). *If* d *is well-founded, i.e.,*

$$\forall\chi\exists n\ \{d\}(\bar{\chi}(n)) \simeq 0,$$

then $|\!\subset^{d}$ *is well-founded:*

$$\forall\psi\exists n\ \neg\ [\ \psi(n)\ |\!\subset^{d}\ \psi(n+1)\ ].$$

PROOF. Apply BI to

$$S(u)\ :\equiv\ \forall\psi\ \{\{\langle u\rangle\}\ |\!\subset^{d}\ \psi(0)\ \rightarrow\ \exists n\ \neg[\ \psi(n)\ |\!\subset^{d}\ \psi(n+1)\ ]\ \}.\qquad \square$$

4.8.2. PROPOSITION.

$$\vdash_{\mathcal{V}_0+\mathrm{BI}}\ \underline{\mathrm{E\text{-}Der}}(d)\ \ \&\ \ \underline{\mathrm{Crit}}(d,u)\ \rightarrow$$

$$\forall m\neg\neg\exists w\ \underline{\mathrm{Final}}^{++}(d^{u*\langle m\rangle}, w, \ulcorner F^{d,u}\urcorner).$$

PROOF. Assume E-Der(d) and Crit(d,u), and fix m. We shall apply BI to the well-ordering $|\!\subset^{d}$ of 4.8.1. We prove below that

(1) $\qquad \forall y\ {}^{d}\!\supset\!|\, x\ S(y)\ \rightarrow\ S(x)$

for

$$S(x)\ :\equiv\ \underline{\mathrm{Bar}}(d^{u*\langle m\rangle}, x)\ \rightarrow\ \neg\neg\exists w\ \underline{\mathrm{Final}}^{++}(d^{u*\langle m\rangle}, w, \ulcorner F^{d,u}\urcorner).$$

Then, by BI on ${}^{d}\!\supset\!|$ applied to S we get

(2) $\qquad \forall x\ S(x).$

But by 4.7 we have

(3) $\qquad \underline{\mathrm{Bar}}(d^{u*\langle m\rangle}, x)\qquad$ for some x

and so by (2)

$$\neg\neg\exists w\ \underline{\mathrm{Final}}^{++}(d^{u*\langle m\rangle}, w, \ulcorner F^{d,u}\urcorner)$$

as required.

Towards proving (1) assume $\forall y \overset{d}{\dashv} x\ S(y)$ and the premiss of $S(x)$, $\underline{\text{Bar}}(d^{u*\langle m\rangle},x)$. By $\underline{\text{Crit}}_2(d,u)$ then

$$(4) \qquad \exists w\in x\ \underline{\text{Final}}^{+}(d^{u*\langle m\rangle},w,\ulcorner F^{d,u}\urcorner).$$

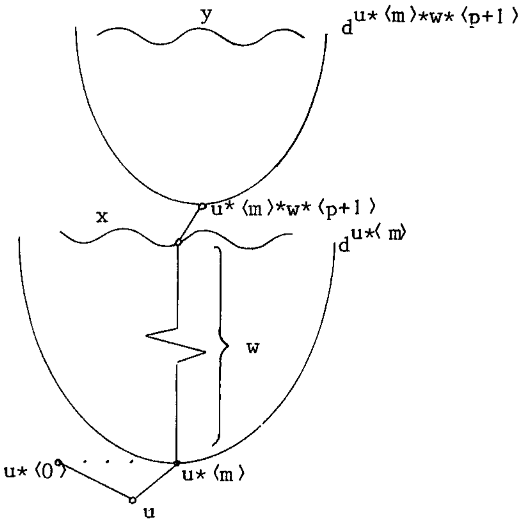
Observe the two possible cases for $\rho^{d,u*\langle m\rangle*w} =: \rho$.

(i) If ρ is $[\exists E^1]$, $F^{d,u*\langle m\rangle*w*\langle 0\rangle} \equiv: \exists zCz$, assume

$$(5) \qquad \exists zCz \vee \neg\exists zCz$$

If $\neg\exists zCz$ then $\underline{\text{Final}}_2^{++}(d^{u*\langle m\rangle},w,\ F^{d,u}\ )$ by definition. If $\exists zCz$ let $p := \mu z.Cz$. By 4.7., for some y

$$\underline{\text{Bar}}(d^{u*\langle m\rangle*w*\langle p+1\rangle},y).$$



Let

$$y' \text{ encode the set } \{\, w*\langle p+1\rangle*v \mid v\in y \,\}\ ;$$

then we have quite trivially

$$\underline{\mathrm{Bar}}(d^{u*\langle m\rangle},x')$$

for $x' := (x\backslash\{w\}) \cup y'$ (set theoretic operations), and also $x' \overset{d}{\dashv} x$. So by BI hyp.

$$\neg\neg\exists w\ \underline{\mathrm{Final}}^{++}(d^{u*\langle m\rangle},w,\ulcorner F^{d,u}\urcorner). \tag{6}$$

Here (6) depends on (5), but since (6) is negated (5) is eliminable (cf. KLEENE [52], p.119 *58b-c,*51a).

(ii) If ρ is [$\exists$I] by our definition of normality (cf. A.1.1) $\rho^{d,u*\langle m\rangle*w*\langle 0\rangle}$ cannot be other than [T]. But we have $R_2(d,u*\langle m\rangle*w*\langle 0\rangle)$ because of $R_2(d,u*\langle m\rangle)$ and $\underline{\mathrm{Selected}}(d^{u*\langle m\rangle},w)$.
Hence $\underline{\mathrm{Tr}}_{QF}(\ F^{d,u*\langle m\rangle*w*\langle 0\rangle}\ )$ and so $\underline{\mathrm{Final}}_3^{++}(d^{u*\langle m\rangle},w,\ F^{d,u}\ )$. □

4.9. LEMMA. *There are prim.rec. functions* f_j (j=2,3) s.t.

$$\vdash_{\mathcal{V}_0} \underline{\text{E-Der}}(d)\ \ \&\ \ \underline{\mathrm{Final}}_j^{++}(d,v,\ulcorner E_i^n(\vec t)\urcorner)\ \ \&\ \ \text{"}F^{d,v} \equiv E_i^n(\vec t)\text{"}$$

$$\rightarrow\ \underline{\mathrm{Pr}}_{rec}^{\infty}(f_j(d,v,\langle i,n,\langle\vec t\rangle\rangle),\ulcorner B[\langle i,n,\langle\vec t\rangle\rangle]\Rightarrow F^{d,v}\urcorner).$$

PROOF.

(i) Let $\{f_2(d,v,\langle i,n,\langle\vec t\rangle\rangle)\}$ describe the tree

$$[T]\ B[\,] \Rightarrow B[\langle i,n,\langle\vec t\rangle\rangle]$$

$$\dfrac{\begin{array}{c}\dfrac{[\forall E]\ B[\,]\Rightarrow \underline{\mathrm{Ineq}}(\langle j,k,\langle\vec s\rangle\rangle,\langle i,n,\langle\vec t\rangle\rangle)\to F^{d,v*\langle 0,0\rangle} \qquad [TE]\ \ B[\,]\Rightarrow \underline{\mathrm{Ineq}}(\ )}{[\to E]\ \ B[\,]\Rightarrow F^{d,v*\langle 0,0\rangle}} \\ [\forall E]\ \ B[\,]\Rightarrow F^{d,v*\langle 0\rangle}\end{array} \qquad \{\Gamma_m\}_{m<\omega}}{[\exists E^1]\ \ B[\langle i,n,\langle\vec t\rangle\rangle]\Rightarrow F^{d,v}}$$

where $F^{d,v*\langle 0,0\rangle} \equiv E^k_j(\vec{s})$ and $F^{d,v*\langle 0\rangle} \equiv \exists z Cz$, and where

$$\Gamma_m \;:\equiv\; \begin{array}{ll} [T] & B[\langle i,n,\langle\vec{t}\rangle\rangle],C\bar{m} \Rightarrow C\bar{m} \\ [FE] & B[\langle i,n,\langle\vec{t}\rangle\rangle],C\bar{m} \Rightarrow \bot \\ [\bot] & B[\langle i,n,\langle\vec{t}\rangle\rangle],C\bar{m} \Rightarrow F^{d,v} \end{array}$$

(ii) Let $\{f_3(d,v,\langle i,n,\langle\vec{t}\rangle\rangle)\}$ describe the tree

$$\begin{array}{ll} [TE] & B[\langle i,n,\langle\vec{t}\rangle\rangle] \Rightarrow F^{d,v*\langle 0\rangle} \\ [\exists I] & B[\langle i,n,\langle\vec{t}\rangle\rangle] \Rightarrow F^{d,v} \end{array}$$

$f_j(\ldots)$ are indices of functions recursive in $\{d\}$, and by the s.m.n.-theorem f_j are indeed prim.rec. functions. The proof of the lemma for these functions is now straightforward. The only less trivial detail is the correctness of the [TE] inferences in the definition of f_2. From $\underline{\text{Final}}^{++}_2(d,v,\ulcorner E^n_i(\vec{t})\urcorner)$ we only know that $E^k_j(\vec{s})$ and $E^n_i(\vec{t})$ are not syntactically identical, but this does not exclude, prima facie, that $\vec{s}$ and $\vec{t}$ are numerically equal. Recall, however, that by our definition of E-Der in 3.1 $\vec{t}$ and $\vec{s}$ are tuples of numerals, and therefore their numerical equality would imply their syntactical identity. □

4.10. COROLLARY.

$$\vdash_{\mathcal{V}_0+BI} \underline{\text{E-Der}}(d) \;\&\; \underline{\text{Crit}}(d,u) \;\&\; \text{"}F^{d,u} \equiv E^n_i(\vec{t})\text{"}$$
$$\rightarrow \forall m \neg\neg\exists x\; \underline{\text{NPrf}}^{\infty}_{\text{rec}}(x,\ulcorner B[\langle i,n,\langle\vec{t}\rangle\rangle] \Rightarrow F^{d,u*\langle m\rangle}\urcorner).$$

PROOF. Immediate from 4.8 and 4.9. □

4.11. PROPOSITION (= 3.4.(4)).

$$\vdash_{\mathcal{V}_0+BI+AC^-_{00}} \underline{\text{E-Der}}(d) \;\&\; \underline{\text{Crit}}(d,u) \;\&\; \text{"}F^{d,u} \equiv E^n_i(\vec{t})\text{"}$$
$$\rightarrow \neg\neg\exists\phi\; \underline{\text{NPrf}}^{\infty}(\phi,\ulcorner s[\langle i,n,\langle\vec{t}\rangle\rangle]\urcorner).$$

PROOF. Assume the premiss; then by 4.10

$$\forall m \neg\neg\exists x \ \underline{\mathrm{NPrf}}^{\infty}_{\mathrm{rec}}(x, \ulcorner B[\langle i,n,\langle\vec{t}\rangle\rangle] \Rightarrow F^{d,u*\langle m\rangle}\urcorner)$$

and so by AC^{-}_{00}

$$\text{(1)} \qquad \neg\neg\exists\psi \ \underline{\mathrm{NPrf}}^{\infty}_{\mathrm{rec}}(\psi m, \ulcorner B[\langle i,n,\langle\vec{t}\rangle\rangle] \Rightarrow F^{d,u*\langle m\rangle}\urcorner).$$

Define now ϕ recursively in ψ:

$$\phi\langle\rangle := \langle \ulcorner \forall I \urcorner, \ulcorner s[\langle i,n,\langle\vec{t}\rangle\rangle]\urcorner \rangle$$

$$\phi(\langle m\rangle * u) := \{\psi m\}(u).$$

The matrix of the positive form of (1) for ψ obviously implies

$$\underline{\mathrm{NPrf}}^{\infty}(\phi, \ulcorner s[\langle i,n,\langle\vec{t}\rangle\rangle]\urcorner),$$

and so (1) implies the succedent of the proposition. □

B.5. SOLUTION OF THE REDUCED PROBLEM FOR L_1 (proof of 3.5(10))

5.1. PROPOSITION (= 3.5(10)). *Let S be a* Σ_2^0 *enumerated theory (with provability-predicate* $\exists x \forall y\ \underline{\mathrm{Prf}}_S(x,y,\ulcorner F \urcorner)$ *say) which is* Σ_2^0 *complete. Then there is a q.f. formula* E(x) *s.t., in the notation of* 3.5,

$$\vdash_{A+\underline{\mathrm{Con}}(S)+\underline{\mathrm{Comp}}_{\Sigma_2^0}(S)} \forall x \neg \underline{\mathrm{Pr}}_S \ulcorner s^E(x) \urcorner \ \ \& \ \ \neg\neg E^*.$$

The proof given below is based on KRIPKE [63].

5.2. LEMMA. *For S as above, there exists a* Σ_2^0 *predicate* J(x) *s.t.*

(i) $\vdash_A \forall x,y\ [\ J(x)\ \ \&\ \ J(y)\ \ \rightarrow\ \ x=y\ ]$

(ii) $\nvdash_S \neg J(\bar{m})$ *for every numeral* $\bar{m}$.

PROOF. Let $\underline{\mathrm{neg}}$ and $\underline{\mathrm{sub}}_2$ be prim.rec. functions s.t. for every formula F

$$\underline{\mathrm{neg}}(\ulcorner F \urcorner) = \ulcorner \neg F \urcorner$$

$$\underline{\mathrm{sub}}_2(\ulcorner F \urcorner, x, y) = \ulcorner F[\bar{x}/a][\bar{y}/b] \urcorner$$

where $\bar{x}$ is the numeral with value x, and where F[t/a] is the formula which comes from F by replacing every occurrence of the parameter a by (the closed term) t. Define

$$K(x,n,m) \ :\equiv \ \forall y\ \underline{\mathrm{Prf}}_S(x,y,\underline{\mathrm{neg}}(\underline{\mathrm{sub}}_2(n,n,m)))$$

$$L \ \equiv \ L(a,b) \ :\equiv \ \exists x\ [\ K(x,a,b)\ \ \&\ \ \forall z{<}x \forall w{<}z\ \neg K(z,a,w)\ ]$$

$J(m) \;:\equiv\; L(\ulcorner L \urcorner, m)$ (here the g.n. $\ulcorner L \urcorner$ is the code of the fixed formula $L(a,b)$, while the defining symbol L is understood as a <u>predicate</u>)

We may assume w.l.g. that the g.n. of a proof is larger than the g.n. of the formula it derives, because $\underline{Prf}_S$ can be replaced by $\underline{Prf}'_S(x,y,z) :\equiv$ $:\equiv \exists x' < x\ \underline{Prf}_S(x',y,z) \ \&\ x = 2^{x'} \cdot 3^z$. This change is harmless in all other respects. Hence

$$(1) \qquad L(m,n) \leftrightarrow L^*(m,n)$$

is provable in A, where L^* is defined like L except that the bounded quantifier $\forall w < z$ is replaced by an unbounded $\forall w$; and so

$$(2) \qquad \vdash_A \forall x,y\ [\ J(x)\ \&\ J(y) \ \rightarrow\ x=y\ ].$$

Next suppose

$$(3) \qquad \vdash_S \neg J(\bar{m}) \qquad\qquad \text{for some } \bar{m},$$

i.e.,

$$\exists m \exists x \forall y\ \underline{Prf}_S(x,y,\ulcorner \neg L(\ulcorner L \urcorner, \bar{m}) \urcorner).$$

Then

$$(4) \quad \neg\neg\exists m \exists x\ [\ \forall y\ \underline{Prf}_S(x,y,\ulcorner \neg L(\ulcorner L \urcorner,\bar{m})\urcorner)\ \ \&\ \ \forall z < x \forall w \exists y\ \neg\underline{Prf}_S(z,y,\ulcorner \neg L(\ulcorner L \urcorner,\bar{w})\urcorner)\ ]$$

which is just $\neg\neg\exists m\ L(\ulcorner L \urcorner, m)$ by (1) and the definition of L.
But by $\underline{Comp}_{\Sigma^0_2}(S)$

$$(5) \qquad \forall m\ [\ L(\ulcorner L \urcorner, m) \ \rightarrow\ \exists x \forall y\ \underline{Prf}_S(x,y,\ulcorner L(\ulcorner L \urcorner, \bar{m})\urcorner)\ ],$$

while the definition of L implies

$$(6) \qquad \forall m\ [\ L(\ulcorner L \urcorner, m) \ \rightarrow\ \exists x \forall y\ \underline{Prf}_S(x,y,\neg\ulcorner L(\ulcorner L \urcorner, \bar{m})\urcorner)\ ],$$

so (4), (5), (6) together imply $\neg\neg\exists x \forall y\ \underline{Prf}_S(x,y,\ulcorner \bot \urcorner)$, contradicting $\underline{Con}(S)$. □

5.3.1. LEMMA. *For* S *as above there is a* Σ^0_2 *predicate* $M(x)$, *s.t. for every q.f. predicate* $P(x)$

$$\nvdash_S \neg\forall x\ [\ M(x) \leftrightarrow P(x)\ ].$$

PROOF. Let $U(n,x)$ be a binary q.f. predicate which enumerates all unary q.f. predicates (by Kleene's enumeration theorem, cf. e.g. KLEENE [52], §58), and let J be as in 5.2. Define

$$M(x) \ :\equiv\ \exists y\ [\ J(y)\ \&\ U(y,x)\ ].$$

By 5.2(i) then

$$J(\bar{m}) \vdash_A \forall x\ [\ M(x) \leftrightarrow U(\bar{m},x)\ ] \qquad \text{for every numeral } \bar{m}.$$

But by 5.2(ii)

$$\nvdash_S \neg J(\bar{m}),$$

so

$$\nvdash_S \neg\forall x\ [\ M(x) \leftrightarrow U(\bar{m},x)\ ] \qquad \text{for every } m, \text{ as desired.} \ \square$$

5.3.2. LEMMA. *Lemma 5.3.1 holds also when* M *is required to be* Π^0_2.

PROOF. Replace the $M \equiv \exists y \forall z\ M_0(x,y,z)$ defined above by $\forall y \exists z\ \neg M_0(x,y,z)$. $\square$

5.4. PROOF OF 5.1 (concluded). Let $M(z)$ be given by 5.3.2, and write $M(z)$ as $\forall x \exists y\ E\langle x,y,z\rangle$.

(i) Assume now $\mathrm{Pr}_S\ulcorner s^E(\bar{n})\urcorner$ for some n (i.e., $\exists x \forall y\ \mathrm{Prf}_S(x,y,\ulcorner s^E(\bar{n})\urcorner)$). By the form of the sequent $s^E(\bar{n})$ we have then

$$\vdash_S \forall z{\neq}\bar{n} M(z) \rightarrow M(\bar{n})$$

and therefore

$$\vdash_S \neg\forall z\ [\ z{\neq}\bar{n} \leftrightarrow M(z)\ ]$$

contradicting 5.3.2.

(ii) Assume $\neg E^*$, i.e., $\neg\forall z M(z)$. Then, by $\mathrm{Comp}_{\Sigma^0_2}(S)$, $\neg\neg\mathrm{Pr}_S(\ulcorner\neg\forall z M(z)\urcorner)$. But taking $P(z) :\equiv z{=}z$ in 5.3.2 we get $\nvdash_S \neg\forall z M(z)$, a contradiction. So $\neg\neg E^*$. $\square$

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INDICES

References are to sections. When a number of references is given for the same item the most relevent one is occasionally underlined.

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