INTUITIONISTIC CORRESPONDENCE THEORY

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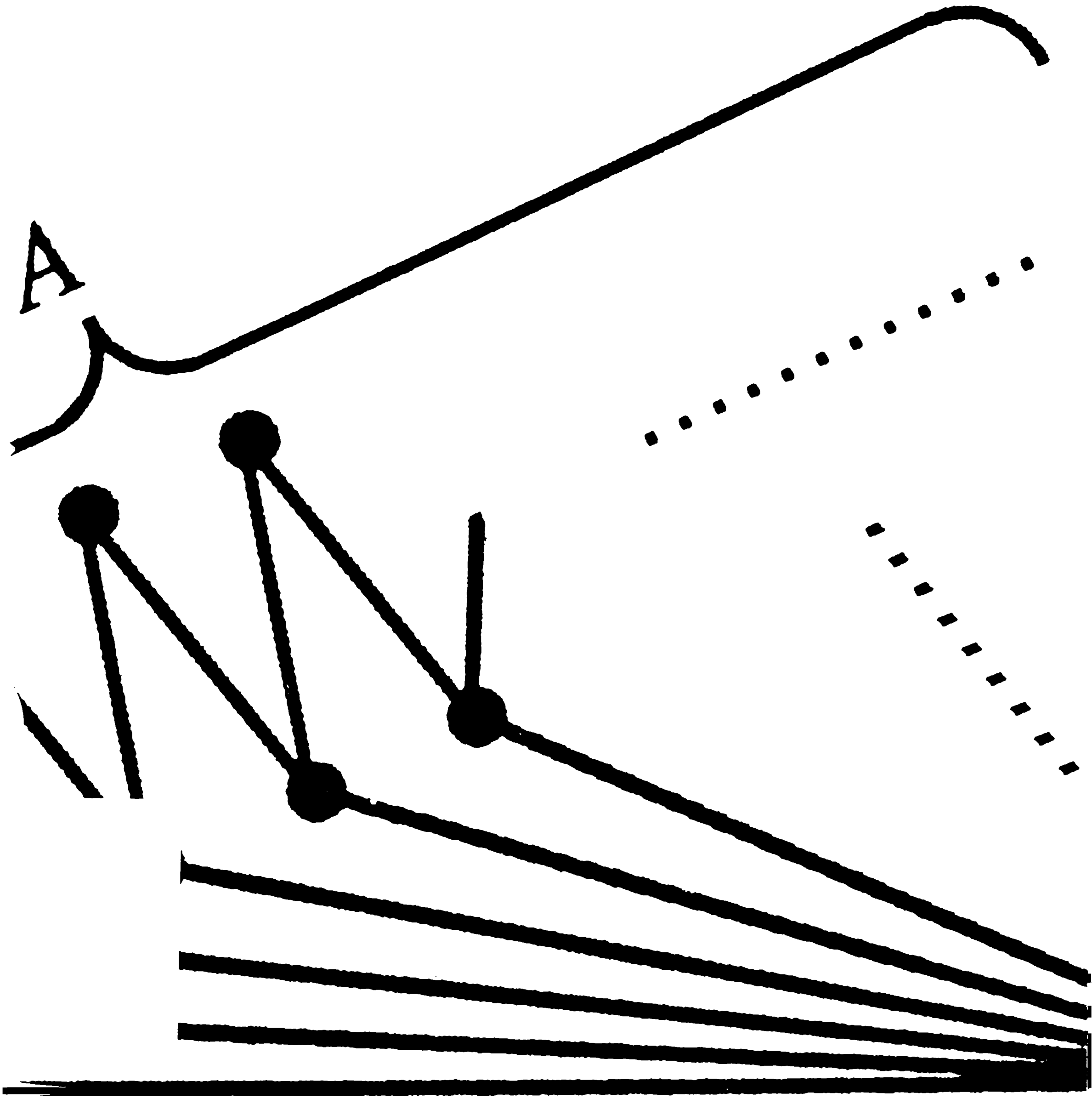

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CONTENTS

DANKWOORD

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I. Formulas of propositional logic as descriptions of frames

An introduction to the subject of this dissertation, with a synopsis of its contents, is contained in §1. Further preliminaries on Kripke semantics take up §2; §3 explains a notion of semantic tableaux that is central to part II. In §4 a number of fragments of intuitionistic logic are compared with regard to expressive power. §5 explores the relationship between intuitionistic logic and modal logic as means for describing frames.

§1. Introduction.

1.1 A short history of ideas.

Formal intuitionistic logic, as codified by Heyting [1930], is ultimately based on certain ideas of Brouwer about the way mathematics is created. In view of their later development, these may be sketched as follows.

Imagine somebody who enjoys mathematics (or a collective of mathematicians, if you consider that more realistic)[1]. As time flows, this person makes calculations, stipulations, and the various other things that mathematicians create — and of course, there may be periods during which he does nothing of mathematical interest. Now we assume, and this is what makes the temporal aspect important, that the mathematician has a certain freedom: the choice, at any time, to pursue one subject rather than another; and that his choices may affect the actual content of his findings. Thus given two mutually exclusive statements A and B, we need not know in advance whether the mathematician — if he is to make a pronouncement at all — will settle for A or for B. (On the other hand, we assume that our mathematician does not forget or blunder. So we do know, for example, that if A has been established, B will never be found true.)

We thus arrive at a picture of the combination of time and the mathematicians choices as a tree, or, more liberally, a partially ordered set. In the particular form of the 'theory of the creative subject', it was used by Brouwer to produce counterexamples to intuitionistically unacceptable statements (cf. Troelstra [1969], Dummett [1977]). It may also serve as a background to the intuitionistic explanation of the logical connectives in terms of proofs (the Brouwer-Heyting-Kreisel explanation)[2]. Philosophical niceties apart, this explanation runs as follows:

- A proof of 'A and B' ($A \wedge B$) is a pair of proofs, one for A and one for B.

-A proof of 'A or B' ($A \vee B$) is a construction which, depending on a parameter for which *some* value is sure to be found eventually, either gives a proof of A or a proof of B.

-A proof of 'if A, then B' ($A \rightarrow B$) is a construction which would turn any proof of A into a proof of B.[3]

There may be things that are known to be false. It suffices to postulate one statement, the falsum (symbol : $\perp$), which is always known to be false; that another statement A happens to be false may be expressed by $A \rightarrow \perp$.

Relational semantics turns the notion of a creative subject into precise mathematics, and the above explanation of the connectives into a definition of truth for intuitionistic logic. In the main, there are two approaches, associated with the names of Beth [1956] and Kripke [1965]. Of these, Beth's approach stays closest to the picture painted above. It is considered in an appendix; our chief concern will be Kripke's semantics. Further limiting the field, we shall deal only with propositional logic.

On both approaches, the possible stages in the development of an idealized mathematician are represented by a quasi-ordered set (more briefly : a **frame**) — i.e. a set A with a relation $\leq_A$ that satisfies the conditions

(i) $\forall\, a \in A \;.\; a \leq_A a$ (reflexivity)

(ii) $\forall a,b,c \in A :: a \leq_A b \;\&\; b \leq_A c \Rightarrow a \leq_A c$ (transitivity).

At the basis of Kripke's semantics lies an important simplification (Grzegorczyk [1964], Kripke [1965]), which is easy to explain in terms of the creative subject. The question is, what the elements of the quasi-ordered set (the **points** of the frame) stand for. One possible answer is that they stand for some combination of time and increasing knowledge; correspondingly, we envisage the mathematician as traveling along a path through the frame, entering new stages automatically as time goes on. This is the intuition behind Beth's semantics. In Kripke's semantics, on the other hand, the temporal component is weaker: again, the mathematician travels through the frame in the direction of the ordering, but now he may stay at any point arbitrarily long. As a consequence, the explanation of disjunction changes to: A or B is known at stage a if either A is known at a or B is known at a. For suppose the mathematician stays at a forever: that the parameter mentioned in the explanation of disjunction gets a value *eventually* then simply means that it already has one.

1.2 Kripke's Semantics.

Now let us fix a formalism, and have some precise definitions.

1.2.1 The language $\mathbb{I}$ of intuitionistic propositional logic has an infinite set $\mathbb{P}$ of proposition letters ($p,q,r,p_0,p_1,\ldots\ldots$ will be used to refer to them), binary connectives $\wedge$ (conjunction), $\vee$ (disjunction) and $\rightarrow$ (implication), and a nullary connective $\perp$ (falsity). Formulas are built from these in the usual way. The symbol $\mathbb{I}$ will also be used to denote the set of all $\mathbb{I}$-formulas. I shall employ $\varphi,\psi,\chi,\varphi_0,\varphi_1,\ldots\ldots$as variables over $\mathbb{I}$-formulas. Negation and truth are defined connectives: $\neg\varphi:=\varphi\rightarrow\perp$, $\top:=\neg\perp$.

If no confusion is likely to result, sub- and superscripts may be dropped without further warning. For instance, I write $a \leq b$ instead of $a \leq_A b$. As usual, $a \geq b$ is the same as $b \leq a$.

1.2.2 Definition: Let A be a frame. A subset $X \subseteq A$ is **upwards closed** if

$$\forall a \in A \ \forall x \in X \ (x \leq a \Rightarrow a \in X).$$

$\mathbb{U}(A)$ is the collection of all upwards closed subsets of A. If $a \in A$, I write $[a)_A$ for the set $\{a' \in A \mid a' \geq a\}$.

For example, $[a) \in \mathbb{U}(A)$. Observe that subsets $X \subseteq A$ are quasi-ordered by the restriction of $\leq$ to X, and thus may immediately be viewed as frames. If $x \in X \in \mathbb{U}(A)$, then clearly $[x)_X = [x)_A$.

1.2.3 Definition. Let A be a frame. A **valuation** on A is a function $V : \mathbb{P} \to \mathbb{U}(A)$.

For $p \in \mathbb{P}$, $V(p)$ is to be thought of as the set of all stages at which p is true. The requirement that $V(p)$ be upwards closed reflects the assumption above that the idealized mathematician never forgets.

1.2.4 Definition. A **model** is a pair $\mathfrak{A} := (A, V)$ of a frame A and a valuation V on A.

The valuation V is extended to a map of $\mathbb{I}$ into $\mathbb{U}(A)$ inductively, by

$$V(\perp) = \varnothing;$$
$$V(\varphi \wedge \psi) = V(\varphi) \cap V(\psi);$$
$$V(\varphi \vee \psi) = V(\varphi) \cup V(\psi);$$
$$V(\varphi \to \psi) = \{a \in A \mid \forall a' \geq a \ (a' \in V(\varphi) \Rightarrow a' \in V(\psi))\}.$$

The clauses of this definition are as we should have expected. In particular, given that we are not to talk about proofs, what comes closest to the existence of a proof at a of $\varphi \to \psi$ is the circumstance that as soon as φ becomes true, ψ does so too.

1.2.5 Definition. Let $\mathfrak{A} = (A, V)$ be a model, and $a \in A$. If $a \in V(\varphi)$, we say a **forces** φ (under valuation V on A). Notation:

$$(\mathfrak{A}, a) \Vdash \varphi.$$

If we have a particular model $\mathfrak{A}$ in mind, this will be shortened to $a \Vdash \varphi$.

To determine whether $a \Vdash \varphi$, we need only consider points $a' \geq a$. This fact may be put somewhat more generally.

1.2.6 Definition. (i) Let A be a frame. If $B \in \mathbb{U}(A)$, we call B, with the ordering inherited from A, a **generated subframe** of A. Notation: $B \subsetneq A$.
We say B is **generated by** $A_0 \subseteq A$ if B is the least generated subframe of A that contains A_0; and write $B = [A_0)$.
(ii) If $B \subsetneq A$ and $\mathfrak{A} = (A,V)$, $\mathfrak{B} = (B,V')$ are models such that for every $p \in \mathbb{P}$, $V'(p) = V(p) \cap B$, then $\mathfrak{B}$ is called a **generated submodel** of $\mathfrak{A}$. Notation: $\mathfrak{B} \subsetneq \mathfrak{A}$.

By induction over $\mathbb{I}$ one proves:

Lemma: If $(B,V') \subsetneq (A,V)$, then for all $\varphi \in \mathbb{I}$, $V'(p) = V(p) \cap B$.

1.3 Completeness.

Above, Kripke's semantics has been construed as a model for the intuitionistic conception of mathematics. Now the question arises to what extent this model is adequate, and an obvious test is: to see whether the formulas that are forced in every point of every Kripke model are the same as the theorems of the traditional formal systems of intuitionistic logic. To formulate the result, some new notation will be useful. If $\mathfrak{A} = (A,V)$ is a model, I shall write $\mathfrak{A} \Vdash \varphi$ (**φ is valid in $\mathfrak{A}$**) for: for all $a \in A$, $(\mathfrak{A},a) \Vdash \varphi$; and if $\Phi \subseteq \mathbb{I}$, $\mathfrak{A} \Vdash \Phi$ will mean that $\mathfrak{A} \Vdash \varphi$ for all $\varphi \in \Phi$. I shall assume some formal system of intuitionistic propositional logic (there are several variants; perhaps the easiest to use is the natural deduction system of Prawitz [1965]), and write $\Phi \vdash \psi$ for: there exists a deduction of ψ from assumptions in Φ.

Strong completeness theorem (Aczel [1968], Fitting [1969], Thomason [1968]): Let $\Phi \subseteq \mathbb{I}$ and $\psi \in \mathbb{I}$. Then $\Phi \vdash \psi$ iff for every model $\mathfrak{A}$, $\mathfrak{A} \Vdash \Phi$ implies $\mathfrak{A} \Vdash \psi$.

It is to be noted that the proof of this theorem is not intuitionistically acceptable. Accepting it, we decide to do classical mathematics. As it is, even our formulation of correspondence will be highly unintuitionistic[4]. It is an open question to what extent a truly intuitionistic correspondence theory is feasible. (Modulo a small modification of the forcing definition, intuitionistic completeness proofs exist: see Veldman [1976], de Swart [1976],[1977].)

1.4 Intermediate logics.

The proofs of completeness, in Kripke [1965] and in the stronger form that appears above, heralded a — classical — model theory for intuitionistic logic. A broad collection of results may be found in Gabbay's book [1981].

Of particular interest is a certain form of completeness theorem for intermediate logics - logics stronger than intuitionistic logic, but weaker than classical logic. For instance, the intermediate logic LC, obtained by adding the axiom $(p \to q) \vee (q \to p)$ (I shall call this formula LC as well) to a standard set of axioms and rules (including substitution) for intuitionistic propositional logic, is strongly complete for the class of all Kripke models on linearly ordered frames. Similarly, KC := $\neg p \vee \neg\neg p$ is strongly complete for the Kripke models whose frames are 'piecewise' (i.e. from each point onwards) directed (Smoryński [1973], see Gabbay [1981]). Through the interpretation of Kripke's semantics as a model of an intuitionistic philosophy of mathematics, such theorems connect intermediate axioms with possible conceptions of mathematics. Concretely, they allow us to study formal systems through the consideration of 'geometrical' properties of frames.

1.5 Validity.

Now we take a somewhat different view. We can interpret an axiom such as LC as expressing a property of frames. Formally: we can abstract from the parameters V and a in the definition of forcing (1.2.5), in three combinations. Suppression of a alone was explained above, in 1.3. Two abstractions remain.

Definition. Let A be a frame, $a \in A$, and φ an $\mathbb{I}$-formula.

(i) $(A,a) \Vdash \varphi$ (φ is **locally valid** in $a \in A$) iff for all valuations V on A, $(A,V,a) \Vdash \varphi$.

(ii) $A \Vdash \varphi$ (φ is **(globally) valid** on A) iff for all $a \in A$, $(A,a) \Vdash \varphi$.

If $(A,V,a) \nVdash \varphi$, I will sometimes say that V **refutes** φ in A and a; and if $A \nVdash \varphi$, that φ is **refutable** in A.

The local and global notions are correlated through lemma 1.2.6. Let a be a point in a frame A, and V any valuation on A. Define a valuation V_a on $[a)$ by

$$V_a(p) = V(p) \cap [a) \quad , \text{ for all } p \in \mathbb{P}.$$

Since $V_a(\varphi)$ is upwards closed, $([a),V_a) \Vdash \varphi$ iff $([a),V_a,a) \Vdash \varphi$; the latter statement is equivalent to $(A,V,a) \Vdash \varphi$ by lemma 1.2.6. Since every valuation on $[a)$ is of the form V_a for a valution V on A, we get immediately that $[a) \Vdash \varphi$ iff $(A,a) \Vdash \varphi$.

It can be shown (see 1.7 below) that for any frame A, $A \Vdash$LC iff for all $a \in A$, $[a)$ is linear (that is, A is **upwards linear**); and $A \Vdash$KC iff A is piecewise directed (see 2.6).

Thus, validity of these formulas corresponds to simple properties of frames. Such correspondences are the subject of correspondence theory. Broadly speaking, we interpret $\mathbb{I}$-formulas as statements about frames, and study the properties of frames that they may express. In particular, the language

$\mathbb{I}$ may be compared with other languages as a means for describing frames.

1.6 Correspondence with first order properties.

The properties mentioned above are simple in a particular way: they can be expressed in a first order language $\mathbb{L}_0$ with one relation symbol $\leq$. Frames are structures for $\mathbb{L}_0$, and we have

$$A \Vdash \mathrm{LC} \text{ iff } A \vDash \forall xyz\,(x\leq y \wedge x\leq z \rightarrow y\leq z \vee z\leq y);$$

$$A \Vdash \mathrm{KC} \text{ iff } A \vDash \forall xyz\,(x\leq y \wedge x\leq z \rightarrow \exists u(y\leq u \wedge z\leq u));$$

with $\vDash$ standing for the classical relation of satisfaction. Now, it is by no means obvious that *all* properties expressed by $\mathbb{I}$-formulas in this way are first order. Let us trace the definition of validity in terms of classical predicate logic.

First, then, the definition of forcing may be read as a translation of $\mathbb{I}$-formulas into formulas of an expansion $\mathbb{L}_1$ of $\mathbb{L}_0$: besides $\leq$, $\mathbb{L}_1$ has a unary predicate symbol for each element of $\mathbb{P}$ — we can use the same symbol in each case. Models (A,V) are structures for $\mathbb{L}_1$ with the predicate symbol p interpreted as $V(p)$. For each $\varphi \in \mathbb{I}$, a standard translation $\mathrm{St}(\varphi)$ may be defined as follows. ($\alpha[x:=y]$ will be the result of substituting the individual variable y for each occurence of the variable x in α).

Definition. Fix an individual variable x.

(i) $\mathrm{St}(p) = px$, for all $p \in \mathbb{P}$; $\mathrm{St}(\bot) = \bot$;

(ii) $\mathrm{St}(\psi\wedge\chi) = \mathrm{St}(\psi) \wedge \mathrm{St}(\chi)$;

(iii) $\mathrm{St}(\psi\wedge\chi) = \mathrm{St}(\psi) \vee \mathrm{St}(\chi)$;

(iv) $\mathrm{St}(\psi\rightarrow\chi) = \forall y(\, x\leq y \rightarrow (\, \mathrm{St}(\psi) \rightarrow \mathrm{St}(\chi))\, [x:=y])$, with y some individual variable distinct from x.

Clearly, for any model $\mathfrak{A} = (A,V)$, each $a \in A$ and each $\varphi \in \mathbb{I}$,

$$(\mathfrak{A},a) \Vdash \varphi \text{ iff } \mathfrak{A} \vDash \mathrm{St}(\varphi)[a].$$

Passing on to validity in frames, we need a second order language $\mathbb{L}_2$, obtained by allowing quantification over the unary predicate letters of $\mathbb{L}_1$. Then, if $p_1,\ldots,p_n$ are all the proposition letters in φ,

$$(A,a) \Vdash \varphi \text{ iff } A \vDash \forall p_1\ldots p_n\ \mathrm{St}(\varphi)[a],$$

with the understanding that the predicate variables of $\mathbb{L}_2$ range over the upwards closed sets. So at first sight, $\mathbb{I}$-formulas express second order properties of frames. To what extent, and under what conditions, $\mathbb{L}_2$-translations of $\mathbb{I}$-formulas reduce to $\mathbb{L}_0$-sentences, will be the main concern of part II of this dissertation.

1.7 Finding first order definitions.

How does one find an $\mathbb{L}_0$-equivalent of an $\mathbb{I}$-formula? The reader can easily convince himself by the above examples that standard translations are not very useful. Instead, we should try to isolate a pattern that must occur in every frame in which the given $\mathbb{I}$-formula is not valid. Let us take LC as an example.

Suppose, then, that $(A,V,a)\nVdash(p\to q)\vee(q\to p)$. By the definition of forcing — precisely, by the definition of V on disjunctions — we must have $a\nVdash p\to q$ and $a\nVdash q\to p$. Again by the definition, $a\nVdash p\to q$ reduces to the existence of a point $a'\geq a$ such that $a'\Vdash p$ and $a'\nVdash q$. For simplicity it would be nice if we could take $a'=a$; but, since $a'\Vdash q\to p$, we would fail to falsify LC. So a' and a are distinct. Let us write $x<y$ for ($x\leq y$ and not $y\leq x$): then similarly we must have $a''>a$ with $a''\Vdash q$ and $a''\nVdash p$. Actually, the important point is that a' and a'' be **incomparable**: not $a''\leq a'$ and not $a''\leq a'$ — that they are distinct from a then follows from $a\leq a',a''$. Now we may surmise that

$$(A,a)\nVdash\mathrm{LC}\ \text{ iff }\ A\vDash\exists yz(y\geq a\wedge z\geq a\wedge\neg y\leq z\wedge\neg z\leq y\ ).$$

We already have a proof for the direction from left to right, for the left hand side means that a valuation V on A exists such that $(A,V,a)\nVdash$LC. For the converse we must show that the right hand side implies the existence of such a V. Suppose b and c are incomparable successors of A. Then we can define V by: $V(p)=[b)$; $V(q)=[c)$; and the rest does not matter. Since not $c\leq b, b\nVdash q$, and similarly $c\nVdash p$; so by the heuristic reasoning above, $a\nVdash$LC.

Summing up: $A\nVdash\mathrm{LC}$ iff $\exists a\in A.\ (A,a)\nVdash\mathrm{LC}$

iff $\exists a\in A.\ A\vDash\exists yz\ (y\geq a\wedge z\geq a\wedge\neg y\leq z\wedge\neg z\leq y\ )$

iff $A\vDash\exists xyz\ (y\geq x\wedge z\geq x\wedge\neg y\leq z\wedge\neg z\leq y\ )$;

which may be rewritten as:

$$A\Vdash\mathrm{LC}\ \text{ iff }A\vDash\forall xyz\ (\ y\geq x\wedge z\geq x\to y\leq z\vee z\leq y).\ ^5$$

1.8 Semantic tableaux.

The search for 'refutation patterns' can be formalized by a method originally due to Beth, which was used by Kripke in [1965]. I shall briefly explain it in the form one finds in Fitting [1969]. The general situation in the search for a refutation pattern is that, in some point a, we deal with a finite set of formulas, some of which we want to come out true and others false. They can be marked accordingly: $T\varphi$ if φ is to come out true, $F\varphi$ if it should be false (the device of marking was introduced by Smullyan). Let us call a finite set of such **signed formulas** a **sequent**. There are obvious rules for expanding sequents.

if $T(\varphi\wedge\psi)$ occurs in a sequent, add $T\varphi$ and $T\psi$; similarly,

to $F(\varphi\wedge\psi)$, add either $F\varphi$ or $F\psi$;

to $T(\varphi\vee\psi)$, add $T\varphi$ or $T\psi$;

to $F(\varphi\vee\psi)$, add $F\varphi$ and $F\psi$;

to $T(\varphi\rightarrow\psi)$, add $F\varphi$ or $T\psi$.

These rules correspond to the definition of forcing. E.g. if $a \Vdash \varphi\rightarrow\psi$, then either $a \nVdash \varphi$ or $a \Vdash \psi$. In this particular case, the forcing definition says something about successors of a as well. To take that into account, it will suffice to carry the true formulas along when we create successors to a — something to be discussed presently.
Of course, we have no use for a sequent unless it is possible that in some model, some point a forces the formulas signed T, and does not force the formulas signed F (i.e. a **realizes** the sequent). A sequent is certainly not realizable if it contains $T\bot$, or, for some φ, both $T\varphi$ and $F\varphi$.
Now suppose a sequent Σ contains the signed formula $F(\varphi \rightarrow \psi)$. This situation requires, by the definition of forcing, a successor to the point realizing Σ, in which φ is true and ψ is false. It may be wise to try adding $T\varphi$ and $F\psi$ to Σ, and see of the result is realizable; but in general , one should start a new sequent Σ' consisting of the signed formulas in Σ that are marked T with $T\varphi$ and $F\psi$ added, and make a note to the effect that Σ' is to be associated with a successor of the point Σ is associated with – say, " Σ caused Σ' ".
The complex of sequents that results from applying these rules to a given initial sequent, with their causal relations, will be called a **semantic tableau**. Such a tableau is **closed** if some sequent in it contains $T\bot$ or, for some φ, both $T\varphi$ and $F\varphi$; **open** otherwise.
Semantic tableaux were devised as a method for deciding universal validity. This use derives from the fact that a sequent is realizable iff it can be developed to an open tableau. So in particular, φ is not universally valid iff $\{F\varphi\}$ can be so developed. (§3 contains a proof, of sorts; the matter is treated explicitly in Kripke [1965] and in Fitting's book.) But *we* can employ them to find refutation patterns, as the reader can illustrate by constructing an open tableau for $\{F(LC)\}$.

1.9 Limits of first order definability.

The question complementary to the one that was just considered at length is: how does one prove that a given $\mathbb{I}$-formula (or rather, the property of frames that it states) is *not* $\mathbb{L}_0$-definable? However, one might wonder if such $\mathbb{I}$-formulas exist; and here a few historical remarks are in order.

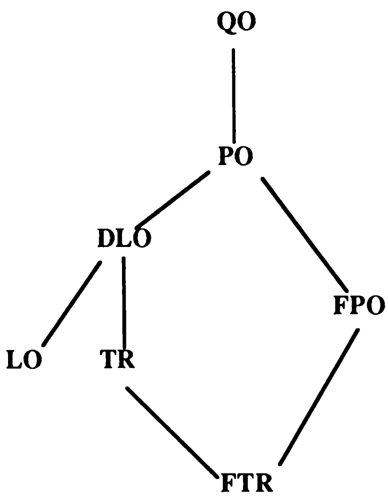
Just as Kripke semantics for intuitionistic logic derives from the relational semantics for modal logic (the development of which is sketched in Bull & Segerberg [1984]), so intuitionistic correspondence theory derives from modal correspondence theory. (The key reference for modal correspondence theory is van Benthem's chapter [1984] in the Handbook of Philosophical Logic.) In modal correspondence theory, a considerable divergence was found between modal definability and first order definability. The examples of this divergence typically exploited modal turns of speech that have no counterpart in intuitionistic logic. (By Gödel's translation [1932] – see §5 below – $\mathbb{I}$-formulas may be considered as a special kind of modal formulas.) This suggested the conjecture (van Benthem [1976a]) that all $\mathbb{I}$-formulas express first order properties of frames. Van Benthem also proposed semantic tableaux as the means to find these first order properties.

In the end, the conjecture was refuted, (see van Benthem [1984]). The proof used the Löwenheim-Skolem property, which had also been widely used in modal correspondence theory. Now, viewed from modal logic, intuitionistic logic represents not only a restriction on formulas, but also on frames (modal frames need not be quasi-orderings) and valuations (in modal semantics, $V(p)$ need not be upwards closed).

The question then arises what happens when the restrictions on frames are strengthened. Indeed, several more restricted classes have figured in the tradition of completeness theory. Some classes worth an abbreviative name are:

- **QO** : the class of all frames (quasi-orderings);
- **PO** : the class of all partial orderings, i.e. frames in which $a{=}b$ if both $a{\le}b$ and $b{\le}a$ (antisymmetry);
- **DLO**: the class of **downward linear orderings**, i.e. partial orderings in which points with a common successor are always comparable - $\exists x(a{\le}x \ \&\ b{\le}x) \Rightarrow a{\le}b \vee b{\le}a$;
- **TR**: the class of **trees**, by which we shall understand downwards linear orderings with a least element (the **root**) in which every interval $[a,b]$ $(=\{x \mid a{\le}x{\le}b\})$ is finite;
- **FPO**: the finite partial orderings;
- **FTR**: the finite trees;
- **LO**: the linear orderings.

(Note that the branches of our trees are all of type $\leq\omega$!)
Partially ordered by inclusion , these classes present the following picture:



Strong completeness holds from **TR** upwards, weak completeness (the statement of which differs from that of strong completeness by the requirement that Φ be finite) from **FTR.**
In Rodenburg [1982] it was shown that the formula SP_2:=

$$[\neg(p\wedge q)\vee\neg(p\wedge\neg q)\vee\neg(\neg p\wedge q) \rightarrow (p\wedge q)\vee(p\wedge\neg q)\vee(\neg p\wedge q)] \rightarrow$$

$$\neg(p\wedge q)\vee\neg(p\wedge\neg q)\vee\neg(\neg p\wedge q)$$

is not first order definable on **DLO**, and that every $\mathbb{I}$ -formula is first order definable on **FTR**. (See §§6,8 below.) The first result turned on the compactness property of first order logic, which, as it appeared, is much easier to handle than the Löwenheim-Skolem property. (Below we shall use the preservation of $\mathbb{L}_0$-formulas under ultraproducts).
In fact, these methods allow stronger conclusions: a formula such as SP_2 is not even definable by a *set* of first order sentences - in other words, it is not Δ-elementary[6]. There is, however, no point in mentioning this added strength in particular cases, since $\mathbb{I}$-formulas are either elementary or not even $\Sigma\Delta$-elementary, by an early result of van Benthem ([1976b], or see [1984]).
One naturally wonders whether overall first order definability holds for **TR** and **FPO.** (**LO** is simple: see 7.7.) These questions remained open for a while: clearly, sweeping methods such as compactness are of no avail here. Doets finally answered them, in the negative, by considering Ehrenfeucht games (see [B]). His results are stated in §§8 and 10.
The other kind of restriction, on formulas, is tightened in §7, where it is shown that semantic

tableaux always work as long as we avoid a certain sort of occurrences of disjunction, and in §11, on formulas in one proposition letter. Further syntactic observations may be found in §12. The expressiveness of 'fragments' of $\mathbb{I}$ is discussed on §4.

We are obviously very far removed from van Benthem's conjecture of 1976: first order definability in $\mathbb{I}$-formulas is a complex matter. Precisely how complex is difficult to say. It is shown in §7 that the tableau method, as developed in §3, does not work for all first order definable $\mathbb{I}$-formulas. I do not know an upper bound to the computational complexity of the set of all $\mathbb{I}$-formulas first order definable on **QO**. For the rest, there are just a few local answers: on **LO** and **FTR** the set of first order definable formulas is the entire set $\mathbb{I}$; on **TR** it is a decidable proper subset (§9).

1.10 $\mathbb{I}$-definability.

Part III is devoted to the question which first order properties of frames can be expressed by means of $\mathbb{I}$-formulas. As in part II, preservation properties – this time of $\mathbb{I}$-formulas, of course – function as a sieve. For example, $\forall xy(x \leq y \vee y \leq x)$ is not $\mathbb{I}$-definable since $\mathbb{I}$-formulas remain valid under the operation of taking disjoint unions (cf 2.4.3). Ideally, we should like to characterize the $\mathbb{I}$-definable classes of frames by closure under such operations, and then derive the typical forms of $\mathbb{L}_0$-formulas that are preserved under these operations. We shall find several obstacles in the way of this project, both in characterizing the $\mathbb{I}$-definable classes and in relating preservation properties to syntax.

As a warming-up, then, the problem which $\mathbb{L}_1$-sentences correspond with $\mathbb{I}$-formulas on models is dealt with in §13. §14 contains a short presentation of intuitionistic duality theory, and a characterization of the $\mathbb{I}$-definable classes of frames along the lines of the characterization of modally definable classes in Goldblatt & Thomason [1974]. The rest mirrors part II: both restrictions on frames and restrictions on $\mathbb{I}$-formulas are invoked to obtain elegant partial results.

1.11 Some other issues.

Comparison with first order logic is just one of several directions that an investigation of the expressiveness of $\mathbb{I}$-formulas may take. Two other directions are explored in part I: the expressiveness of *fragments* of $\mathbb{I}$ (§4); and a comparison with modal logic (§5).

Van Benthem [1984] names "three pillars of wisdom supporting the edifice of modal logic": completeness theory, correspondence theory and duality theory. Such pillars may also be thought to bear intuitionistic logic. The connections between the three, insofar as they are known, are similar. Duality has been mentioned above, and its relation to correspondence will appear in part III. The relation between completeness and correspondence is shadowy. Analogously to the modal case (van Benthem [1984]), it may be shown that the intuitionistic theory of an elementary class of frames is recursively axiomatizable. The related question: whether every first order definable axiom set is complete, is open. It is known that incomplete intermediate logics exist (Šehtman [1977]), but they are much harder to construct than their modal counterparts; in particular, van Benthem's example settling the above question for the modal case makes essential use of features that $\mathbb{I}$-formulas lack.

As was remarked before, we shall deal only with intuitionistic *propositional* logic. Some examples of correspondence in the realm of predicate logic may be found in van Benthem's Handbook article [1984].

[Footnotes to §1]

[1] Mutatis mutandis, a computing machine would serve as well.

[2] In itself, the BMK-explanation is independent of temporal considerations, as may be seen by its formalization as *realizability* (see Troelstra [1973]).

[3] The clauses for quantification are suppressed, since the body of this treatise deals only with propositional logic.

[4] See 1.7, and in particular note 5 below.

[5] From an intuitionistic standpoint, the relation between this equivalence and the method by which it was established is highly problematic. It would be preferable to state more precisely what our method gives us; it relates refutability of an $\mathbb{I}$-formula in A with the possibility of finding a certain pattern in A, so we should get a statement on the form

$$\varphi \text{ is refutable in } A \text{ iff } A \models \beta,$$

where β would begin with an existential quantifier.

[6] I shall call a class of structures elementary if it is definable by a single first order sentence; Δ-elementary if it is an intersection of elementary classes (i.e. definable by a set of first order sentences); and $\Sigma\Delta$-elementary if it is a union of Δ-elementary classes.

This terminology is in accordance with van Benthem [1986] (see ch. VIII). Chang & Keisler [1973], whose terminology for first order logic I will follow in almost all other respects, have 'basic elementary' for my 'elementary' and 'elementary' for my 'Δ-elementary'.

§2. Further examples and Kripke model theory.

This section begins with two lemmas elaborating minor points that were glossed over in the introduction. Next, the relation between the frame classes **QO** and **PO** is spelled out. The next subsection sums up the fundamental validity-preserving operations on frames. 2.5 - 2.10 contain a series of exemplary $\mathbb{I}$-formulas expressing simple properties of frames. We end with a theorem on a connection between embeddings and p-morphisms that will be applied in parts II and III.

2.1 In the example treated in §1, in defining a valuation to refute LC, only $V(p)$ and $V(q)$ were important; the value of V on other proposition letters did not matter. This fact may be stated generally, and proved by induction over $\mathbb{I}$-formulas:

Proposition: Let A be a frame, $\varphi \in \mathbb{I}$, and suppose V and V' are two valuations on A that agree on every proposition letter that occurs in φ. Then $V(\varphi) = V'(\varphi)$.

Hence if we only want to evaluate certain formulas in $p_0,........,p_n$, we need only specify $V(p_0)......V(p_n)$.

2.1 Substitution

If $\varphi,\varphi_1,....,\varphi_n \in \mathbb{I}$, $p_1,.....,p_n \in \mathbb{P}$, then $\varphi[p_1:=\varphi_1,.....,p_n:=\varphi_n]$ will denote the result of simultaneously substituting φ_1 for $p_1,.....,\varphi_n$ for p_n in φ.

Lemma: If $A \Vdash \varphi$, then $A \Vdash \varphi[p_1:=\varphi_1,.....,p_n:=\varphi_n]$.

Proof: Suppose $A \Vdash \varphi$, and let V be any valuation on A. Define a valuation V' on A by:

$$V'(p)=V(p) \text{ if } p \notin \{p_1,....,p_n\};$$

$$V'(p_i) = V(\varphi_i) \text{ for } 1 \leq i \leq n.$$

Clearly, for all $\psi \in \mathbb{I}$, $V'(\psi)=V(\psi[p_1:=\varphi_1,....,p_n:=\varphi_n])$. In particular, we may take $\psi=\varphi$, and note that , since $A \Vdash \varphi$, $V'(\varphi) = A$. □

2.3 Quasi-orderings and partial orderings.

A valuation on a frame A is a map into $\mathbb{U}(A)$. Consequently, if $a \le b$ and $a \Vdash \varphi$, $b \Vdash \varphi$ as well. If also $b \le a$, a and b force the same $\mathbb{I}$-formulas. Thus, as far as $\mathbb{I}$-formulas are concerned, a and b might as well be equal; and we are led to expect that the difference between **QO** and **PO** is inessential.

2.3.1 Definition. Let A be a frame, $a,b \in A$. If $a \le b$ and $b \le a$ we write $a \sim b$.

It is easily seen that $\sim$ is an equivalence relation. I shall denote the equivalence class of a point a by $\tilde{a}$.

2.3.2. Definition. Let A be a frame. The **contraction** C(A) of A is the quotient $A/\sim$, ordered by $\tilde{a} \le \tilde{b}$ iff $a \le_A b$.

The contraction is a partial ordering, and if $A \in$ **PO**, $A \cong \mathrm{C}(A)$. For $X \subseteq A$, define $\tilde{X} := \{\tilde{x} | x \in X\}$. In the description of the predicate languages $\mathbb{L}_0$, $\mathbb{L}_1$ and $\mathbb{L}_2$ in 1.6, equality was not mentioned. It is time now to get precise: these languages do *not* contain equality. The reason will be made clear in §6 (6.1). Remember that the set variables of $\mathbb{L}_2$ are supposed to run over the upwards closed sets. Thanks to these limitations, the following statement holds:

2.3.3 Theorem: Let A be a frame, $X_1, \ldots, X_m \in \mathbb{U}(A)$, and α an $\mathbb{L}_2$-formula. Then

$$A \models \alpha[X_1 \ldots X_m a_1 \ldots a_n] \quad \text{iff} \quad \mathrm{C}(A) \models \alpha[\tilde{X}_1 \ldots \tilde{X}_m \tilde{a}_1 \ldots \tilde{a}_n].$$

Proof: induction on α. In particular, if $\tilde{x} \in \tilde{X}$, then $y \in X$ for some $y \sim x$; then $y \le x$, and $x \in X$ since X is upwards closed. □

We saw in 1.6 that $\mathbb{I}$-formulas, as interpreted in frames, are a special sort of $\mathbb{L}_2$-formulas, so this theorem says in particular that in a frame and its contraction the same $\mathbb{I}$-formulas are valid.

2.4 Validity-preserving operations

Certain well-known constructions of new frames out of given ones have the property that all $\mathbb{I}$-formulas valid on the given frames are also valid on the new frame. One of these we met in 1.2.6.

2.4.1 Lemma: Let B be a generated subframe of the frame A, and $\varphi \in \mathbb{I}$. If $A \Vdash \varphi$, then also $B \Vdash \varphi$.

Proof: Any valuation V on B is also a valuation on A; and $(B,V) \subsetneq (A,V)$. Hence by lemma 1.2.6, $(B,V,b) \nVdash \varphi$ implies $(A,V,b) \nVdash \varphi$. It immediately follows that $B \nVdash \varphi$ implies $A \nVdash \varphi$. □

2.4.2 p-morphisms

p-morphisms may be described as monotonic funtions (homomorphisms) that are locally surjective. Precisely:

Definition. Let A and B be frames, and $f: A \to B$ a function.

(i) f is a **homomorphism** if it respects the ordering, i.e.
if $a \leq_A a'$, then $f(a) \leq_B f(a')$.

(ii) A homomorphism f is a p-morphism if it satisfies the p-morphism condition

$$\forall a \in A\ \forall b \in B\ \ (f(a) \leq b \ \Rightarrow\ \exists a' \geq a.\, f(a') = b)$$

Lemma: If $f: A \to B$ is a surjective p-morphism, then for any $\varphi \in \mathbb{I}$, $A \Vdash \varphi$ implies $B \Vdash \varphi$.

Proof: If $(B,V) \nVdash \varphi$, define for all $p \in \mathbb{P}$: $V'(p) = f^{-1}[V(p)]$. One shows by induction over $\mathbb{I}$-formulas ψ that for all $a \in A$, $(A,V',a) \Vdash \psi$ iff $(B,V,f(a)) \Vdash \psi$ – using the p-morphism condition in the case of implication. So in particular $(A,V') \nVdash \varphi$. □

If there exists a p-morphism of A onto B, we call B a **p-morphic image** of A. In any case:

Proposition: If $f: A \to B$ is a p-morphism, then $f[A] \subsetneq$ B.

Proof: $F[A] \in \mathbb{U}(B)$ by the p-morphism condition. □

2.4.3. Disjoint unions

In the disjoint union of sets that are quasi-ordered, the orderings may be carried along.

Definition: Let $(A_i \mid i \in I)$ be a family of frames. Let for all i, $A_i' = \{i\} \times A_i$, ordered by $(i,a) \leq_{A_i'} (i,a')$ iff $a \leq_{A_i} a'$. Then the **disjoint union** $\Sigma_{i \in I} A_i$ is the frame $\cup_{i \in I} A_i'$, ordered by the union of the orderings on the frames A_i'.

Lemma: If $\varphi \in \mathbb{I}$, and for all $i \in I$, $A_i \Vdash \varphi$, then $\Sigma_{i \in I} A_i \Vdash \varphi$.

Proof: A valuation refuting φ on $\Sigma_i A_i$ immediately reduces to a valuation refuting φ on some A_i .

□

Example: I mentioned in 1.10 that the $\mathbb{L}_0$-sentence $\forall xy\,(x \leq y \;\vee\; y \leq x)$ (=: α) is not $\mathbb{I}$-definable. This may be shown as follows: α holds in the trivial frame $\{0\}$ (with $0 \leq 0$, of course); but not in the disjoint union $\{0\}+\{0\}$, since the two copies of 0 are not comparable. By the lemma above, there cannot be an equivalent $\mathbb{I}$-formula.

2.4.4 Preservation

For a class **K** of frames and an $\mathbb{I}$-formula φ, we abbreviate $\forall A \in \mathbf{K}.\ A \Vdash \varphi$ to $\mathbf{K} \Vdash \varphi$ (similarly we get $\mathbf{K} \Vdash \Phi$ with $\Phi \subseteq \mathbb{I}$). Taking generated subframes, p-morphic images or disjoint unions may be considered as operations on classes of frames:

Definition. Let **K** be a class of frames. Then

$g\mathbf{K}$ is the class of generated subframes of elements of **K**;

$p\mathbf{K}$ is the class of p-morphic images of elements of **K**;

$d\mathbf{K}$ is the class of disjoint unions of families in **K**.

We can sum up the lemmas above as follows:

Proposition. Let **K** ba a class of frames, and $\varphi \in \mathbb{I}$. Then

$$\mathbf{K} \Vdash \varphi \quad \text{implies} \quad g\mathbf{K} \cup p\mathbf{K} \cup d\mathbf{K} \Vdash \varphi.$$

In words: $\mathbb{I}$-formulas are preserved under generated subframes, p-morphic images and disjoint unions.

2.5 Example. We call a frame A **atomic** if for all $a,a' \in A$, $a \le a'$ implies $a' \le a$. (The atoms are the equivalence classes $\tilde{a}$, that are unrelated in C(A) and cannot be split by $\mathbb{I}$-formulas.)
We identify models for classical propositional logic with valuations on the singleton frame $\{0\}$ — it may help to further identify $\{0\}$ with 1 (truth), and $\emptyset$ with 0 (falsity). We shall understand **tautology** in the sense of classical logic: φ is a tautology iff $\{0\} \Vdash \varphi$. We may write $V \models \varphi$ for $(\{0\},V) \Vdash \varphi$, and $\models \varphi$ for $\{0\} \Vdash \varphi$.

Claim: an $\mathbb{I}$-formula φ is a tautology iff φ is valid in all atomic frames.

Proof: the direction from right to left is obvious, since $\{0\}$ is atomic. Conversely, suppose $\models \varphi$. If A is atomic, then it can be written as a disjoint union of equivalence classes under $\sim$: $A \cong \Sigma(\tilde{a} | \tilde{a} \in C(A))$. For each $a \in A$, $C(\tilde{a}) \cong \{0\}$, so $C(\tilde{a}) \Vdash \varphi$. By theorem 2.3.3, $\tilde{a} \Vdash \varphi$, hence $\Sigma(\tilde{a} | \tilde{a} \in C(A)) \Vdash \varphi$ by lemma 2.4.3 , and $A \Vdash \varphi$ by the isomorphism.

2.6 Example. A frame A is **piecewise directed** if whenever $a \le_A b,c$, there exists $d \in A$ with $b,c \le d$. We claim (cf. 1.5) that $A \Vdash$ KC $(= \neg p \vee \neg\neg p)$ iff A is piecewise directed.

1° Suppose A is not piecewise directed: say $a \le b,c$ and there is no $d \in A$ with $b,c \le d$. Let $V(p)=[b)$. Then $b \nVdash \neg p$; and for all $d \ge c$, $d \ngeq b$, hence $d \nVdash p$, and $c \Vdash \neg p$, so $c \nVdash \neg\neg p$. We conclude that $a \nVdash$ KC.

2° Suppose $A \nVdash$ KC; say $(A,V,a) \nVdash$ KC. Then $b \Vdash p$ and $c \Vdash \neg p$ for some $b,c \ge a$. Points $d \ge b,c$ cannot exist, since they would force $\perp$. So A is not piecewise directed.

2.7 Definition. Let A be a frame.
(i) The quasi-ordering $\le_A$ determines a **strict** quasi-ordering $<_A$, defined by

$$a <_A b \quad \text{iff} \quad a \le b \;\&\; b \nleq a.$$

If $a < b$, b will be called a **strict** successor of a.
(ii) $X \subseteq A$ is a **chain** (in A) if $\forall x,y \in X$ ($x \le y$ or $y \le x$). X is a **strict** chain if $\forall x,y \in X$ ($x < y$ or $x = y$ or $y < x$).
(iii) The **height** of A is the least upper bound of the cardinalities of strict chains in A.

2.8 Example. A sequence of $\mathbb{I}$-formulas generalizing Peirce's Law can be defined as follows: let

$$P_0 = p_0,$$

$$P_{n+1} = [(p_{n+1} \to P_n) \to p_{n+1}] \to p_{n+1}.$$

(Peirce's Law is P_1.)

We claim that $A \Vdash P_n$ iff the height of A is at most n.

Proof, with induction over n: P_0 is never valid, as it should. Suppose the statement holds for n. If $A \nVdash P_{n+1}$, say $(A,V,a) \nVdash P_{n+1}$, then there must be $a_0 \geq a$ with

$$a_0 \Vdash (p_{n+1} \to P_n) \to p_{n+1}\ ,\ a_0 \nVdash p_{n+1}.$$

Then $a_0 \nVdash p_{n+1} \to P_n$, so we can find $a_1 \geq a_0$ with $a_1 \Vdash p_{n+1}$, $a_1 \nVdash P_n$. Since $V(p_{n+1})$ is upwards closed, $a_1 > a_0$. By induction hypothesis, $[a_1)$ has a chain $a_1 < ... < a_{n+1}$; with $a_0 < a_1$, A has height at least n+2.

Conversely, let $a_0 < a_1 < .... < a_{n+1}$ in A. Let $V(p_1),.....,V(p_{n+1}) \subseteq [a_1)$ be such that $([a_1, V) \nVdash P_n$; and $V(p_{n+1}) = [a_1)$. Then as above, $a_0 \nVdash P_{n+1}$.

2.9 Definition. Let A be a frame.

(i) We shall say a and b are **comparable** $(a,b \in A)$ if $a \leq b$ or $b \leq a$; **incomparable** otherwise.

(ii) An **antichain** in A is a set of mutually incomparable points.

(iii) The **width** of A is the least upper bound of the cardinalities of antichains in subframes $[a) \subseteq A$.

We write $a \leq X$, for a point a and a set X, as an abbreviation for $\forall x \in X.\ a \leq x$; and similarly $a < X$. The reason for the introduction of subframes $[a)$ in clause (iii) of the definition above is that antichains can only be relevant for the evaluation of $\mathbb{I}$-formulas if they have a predecessor.

We shall use the symbol $\bigwedge$ for iterated conjunction, with the convention that $\bigwedge\emptyset = \top$. Similarly, $\bigvee$ stands for iterated disjunction,and $\bigvee\emptyset = \bot$.

2.10 Example. Let W_n, for $n \in \mathbb{N}$, be the formula

$$\bigvee_{i \leq n} (\bigwedge(p_j \mid j \neq i \text{ and } j \leq n) \to p_i).$$

Note that LC = W_1.

Claim: $A \Vdash W_n$ iff the width of A is at most n. Indeed, if $(A,V) \nVdash W_n$, there must be $a \leq_A a_0$, ..., a_n with $a_i \Vdash \bigwedge_{j \neq i} p_j$ and $a_i \nVdash p_i$: then $\{a_0,.....,a_n\}$ must be an antichain of n+1 elements.

Conversely, if $[a)_A$ contains an antichain $\{a_0,\ldots\ldots,a_n\}$ of $n+1$ elements, define V on A by $V(p_i) = \cup_{j\neq i}[a_j)$; then $a_i \Vdash \wedge_{j\neq i} p_j$, $a_i \nVdash p_i$, and $a \nVdash W_n$.

2.11 p-retractions.

Within a restricted class of frames, the notions of 2.4 may become easier to handle. We conclude with two examples of this phenomenon (2.11.2, 2.11.6). They connect certain embeddings with surjective p-morphisms. We shall use them later on.

2.11.1 Definition. Let A,B be frames. Suppose f is a function from A to B, and $g:B\to A$ is a p-morphism such that $g\circ f = 1_A$ (the identity mapping of A).Then g will ba called a **p-retraction** of f. If $f\colon A\hookrightarrow B$ is the canonical embedding of a subframe of B, g may be called a p-retraction of B onto A, and A a **p-retract** of B.

2.11.2 Example. Suppose $a \in A \in$ **DLO**, and $a^* \geq a$ has no strict successors. Then $[a)$ is a p-retract of A: define $f\colon A \twoheadrightarrow [a)$ by $f(b) = b$ if $b \geq a$, $f(b) = a$ if $b \leq a$, and $f(b) = a^*$ if a and b are incomparable. The main reason why this works is that if b is not comparable with a, it is not comparable with any successor of a, by downward linearity.

2.11.3 Definition.

(i) Let A be a frame, $X \subseteq A$, and $a <_A X$. Then a **branches into** X if, whenever $x,x' \in X$ are incomparable, $a \leq b \leq x,x'$ implies $b < X$.

(ii) An (isomorphic) embedding $f\colon A \rightarrowtail B$ is **strong** if whenever $a \in A$ branches into some set $X \subseteq A$, $f(a)$ branches into $f[X]$.

The paths through a tree, as defined in §1, have type $\leq\omega$. Therefore:

2.11.4 Lemma Let $X \subseteq A \in$ **TR**; then X has a greatest lower bound in A.

Some lattice notation will be useful. Suppose a partial ordering A is given. If $X \subseteq A$ has a greatest lower bound, we denote it by $\wedge X$, and $\wedge\{a,b\} =: a\wedge b$; $\vee X$ is the least upper bound, if it exists, and $\vee\{a,b\} =: a\vee b$. (The symbols are the same as for conjunction and disjunction, but harmful confusions are not likely.)

A least element of the entire set A we call the **root** of A. A **cover** of a point $a \in X$ is a strict successor b such that $\forall x \in A\ (a < x \leq b \Rightarrow b \leq x)$.

2.11.5 Lemma. Let $A \in$ **TR** and $X \subseteq A$. If $\wedge X$ branches into X, then for each $x \in X$ there is a unique cover c_x of $\wedge X$ such that $\wedge X < c_x \leq x$. If $x,x' \in X$ are incomparable, then $c_x \neq c_{x'}$.

Proof: Since $\bigwedge X$ branches into X, $\bigwedge X < X$. As intervals $[\bigwedge X, x]$ are always finite, there exists for each $x \in X$ a cover c of $\bigwedge X$ such that $\bigwedge X < c \leq x$. This c is unique by downward linearity.
Now suppose $x,x' \in X$ are incomparable, and $c_x = c_{x'}$. Then since $\bigwedge X$ branches into X, $c_x \leq X$. But this implies $c_x \leq \bigwedge X$, contradicting that c_x covers $\bigwedge X$. □

For a point a in a frame A, let $\mathrm{Cov}_A(a)$ be the set of all covers of a in A.

2.11.6 Lemma. Suppose $A \in \mathbf{TR}$, $B \in \mathbf{DLO}$, and $f\colon A \rightarrowtail B$ is an embedding. Then f is strong iff

$$(*)\forall a \in A \ \ \forall c,c' \in \mathrm{Cov}_A(a) \ \ \forall b \in B \ \ (c \neq c' \ \&\ f(a) \leq b \leq f(c), f(c') \ \Rightarrow \ b \leq f[\mathrm{Cov}_A(a)]\,).$$

Proof: ($\Rightarrow$) Immediate by definition 2.11.4, since a branches into $\mathrm{Cov}_A(a)$.
($\Leftarrow$) Assume (*). Suppose $a \in A$ branches into X; let $x,x' \in X$, $f(x)$ and $f(x')$ incomparable, with $f(a) \leq b \leq f(x), f(x')$. Note that x and x' must be incomparable, since f is a homomorphism. We are to show that $b \leq f[X]$.
If $b \leq f(\bigwedge X)$, there is nothing to prove. By downward linearity, $b \nleq f[X]$ implies $f(\bigwedge X) < b$. So let us assume that $f(\bigwedge X) < b$.
Observe that $\bigwedge X \notin X$: since $a \leq \bigwedge X \leq x,x'$, we have $\bigwedge X < X$ by the definition of branching. In fact, $\bigwedge X$ branches into X. For if y,y' are incomparable elements of X, and $\bigwedge X \leq a' \leq y,y'$, then $a \leq a' \leq y,y'$ (as $a \leq \bigwedge X$), so $a' < X$.
Let c_x, $c_{x'}$ be covers of $\bigwedge X$ such that $\bigwedge X < c_x \leq x$ and $\bigwedge X < c_{x'} \leq x'$. By the above lemma, $c_x \neq c_{x'}$. Since f is an embedding, $f(c_x)$ and $f(c_{x'})$ are incomparable. Now $f(c_x), b \leq f(x)$; and $f(c_{x'}), b \leq f(x')$; hence, by downward linearity, b must be comparable with $f(c_x)$ and $f(c_{x'})$. Since $f(c_x) \nleq f(c_{x'})$ and $f(c_{x'}) \nleq f(c_x)$, the only arrangement possible is $b \leq f(c_x), f(c_{x'})$. By (*), $b \leq f[\mathrm{Cov}_A(\bigwedge X)]$. Since $\bigwedge X < X$, we conclude that $b \leq f[X]$. □

2.11.7 Lemma: Let $A \in \mathbf{TR}$ and $B \in \mathbf{DLO}$. Suppose $f\colon A \rightarrowtail B$ is a strong embedding, and $b \in B$. Then every cover of $\bigwedge f^{-1}[[b)]$ belongs to $f^{-1}[[b)]$.

Proof: Let C be the set of all covers of $\bigwedge f^{-1}[b)$ (we drop the outermost brackets); we must prove that $f[C] \geq b$. Observe that $\bigwedge f^{-1}[b)$ branches into C.
Since $f^{-1}[b)$ is upwards closed, $C \subseteq f^{-1}[b)$ is obvious if $\bigwedge f^{-1}[b) \in f^{-1}[b)$. So suppose $\bigwedge f^{-1}[b) \notin f^{-1}[b)$. Then there must be incomparable $a,a' \in f^{-1}[b)$, and distinct $c,c' \in C$ such that $\bigwedge f^{-1}[b) < c \leq a$ and $\bigwedge f^{-1}[b) < c' \leq a'$. We get

$$f(\bigwedge f^{-1}[b)) < b < f(c), f(c') :$$

for example, $b \leq f(a)$, so, since also $f(c) \leq f(a)$, $b \leq f(c)$ or $f(c) \leq b$ by downward linearity — the second of which would give $c \leq f^{-1}[b)$ since f is an embedding, and a contradiction. Now, since f

is strong, $f(\bigwedge f^{-1}[b))$ branches into $f[C]$; hence $b \leq f[C]$, as was to be shown. □

The following theorem generalizes one half of an unpublished theorem of de Jongh. The other half will appear as lemma 16.4. We write $(a]_A$ for $\{a' \in A \mid a' \leq_A a\}$.

2.11.8 Theorem. Let $A \in \mathbf{TR}$, with root a_0, and $B \in \mathbf{DLO}$; let $f\colon A \rightarrowtail B$ be a strong embedding. Suppose

(*a*) every $a \in A$ has a successor that is maximal in A;

(*b*) for every $b \in B, f^{-1}(b]$ is finite.

Then f has a p-retraction.

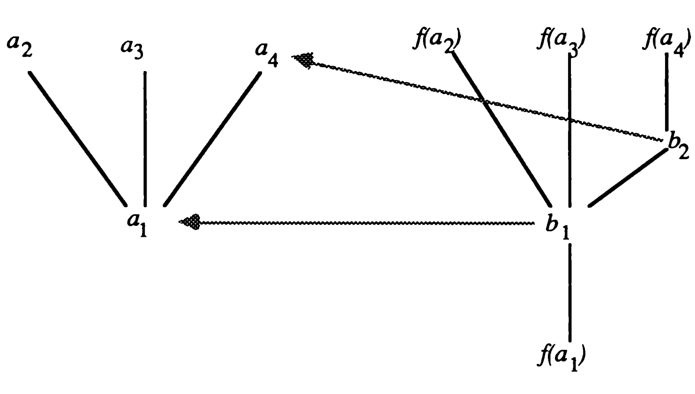
Proof: Suppose A, B and f are as stated, and satisfy conditions (*a*) and (*b*). We are to define a p-morphism $g\colon B \twoheadrightarrow A$ such that $g \circ f = 1_A$. Thus for some points $b \in B$, the value of g is fixed in advance: if $b = f(a)$, then $g(b) = a$. This can be generalized to some extent. Let

$$B_0 := \{b \in B \mid f^{-1}[b) \neq \emptyset\}.$$

Since g is to be a homomorphism, we must have $g(b) \leq f^{-1}[b)$; and it is not unreasonable to try

(i) if $b \in B_0$, then $g(b) = \bigwedge f^{-1}[b)$.

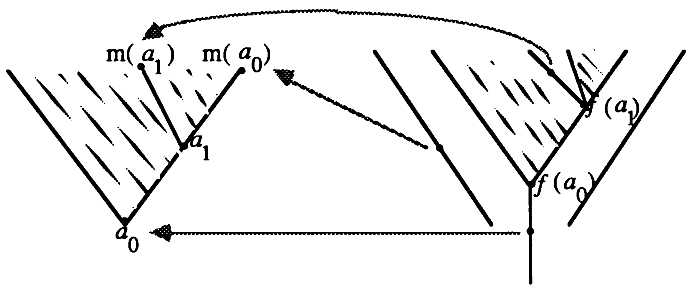
Because $f^{-1}[f(a)) = [a)$, this guarantees that $g \circ f = 1$: $g \circ f(a) = \bigwedge[a) = a$. Moreover, it is obvious that g is a homomorphism of B_0. Here is a diagram sketching some effects of clause (i); note that the triple branching (of a_1 into a_2, a_3 and a_4) must be preserved by f.



For the rest of B, we generalize the trick of the example above. This time a single maximal element of A might not suffice: it is possible that, though $f^{-1}[b) = \emptyset$, some $f(a)$ precedes b; and then we must be sure that $g(b) \geq a$. Fix for every $a \in A$ a maximal successor $\mathrm{m}(a) \in A$. Since B is downwards linear and f is an embedding, $f^{-1}(b]$ is linearly ordered; by condition (*b*) it is finite. Therefore it has a least upper bound $\bigvee f^{-1}(b]$ (we set $\bigvee\emptyset = a_0$). The proper generalization of the example is

(ii) if $b \in B - B_0$, then $g(b) = \mathrm{m}(\bigvee f^{-1}(b])$.

Illustration:



It remains to check that g is a p-morphism. There are two parts to this: (I) g is a homomorphism; (II) g satisfies the p-morphism condition.

I. Suppose $b_0 \leq_B b_1$. B_0 is downwards closed: $b \leq b' \in B_0$ implies $b \in B_0$. Hence for checking that g is a homomorphism there are three cases, one of which was dealt with above. The cases that remain are (i) $b_0 \in B_0$, $b_1 \notin B_0$; (ii) $b_0 \notin B_0$.

In case (i), $g(b_0) \in f^{-1}(b_1]$, so $g(b_0) \leq \bigvee f^{-1}(b_1] \leq \mathrm{m}(\bigvee f^{-1}(b_1]) = g(b_1)$.

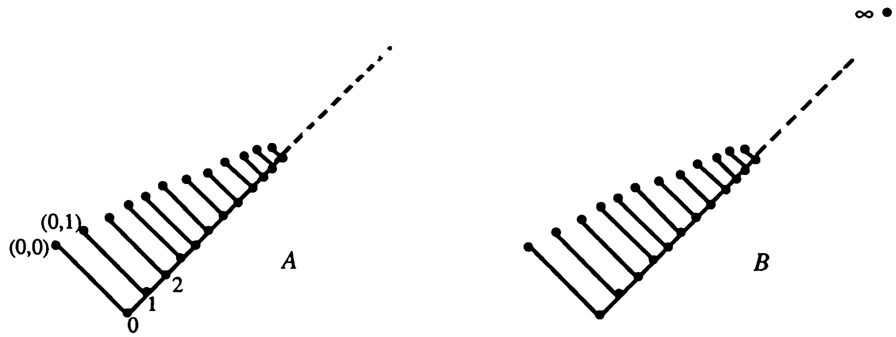
In case (ii), since $b_0 \leq b_1$, we have $f^{-1}(b_0] \subseteq f^{-1}(b_1]$. Suppose $f^{-1}(b_0] \neq f^{-1}(b_1]$; let $x \in f^{-1}(b_1] - f^{-1}(b_0]$. Then both $f(x) \leq b_1$ and $b_0 \leq b_1$, so by downward linearity $f(x) \leq b_0$ — contradicting $x \notin f^{-1}(b_0]$ — or $b_0 \leq f(x)$, contradicting $b_0 \notin B_0$. So $f^{-1}(b_0] = f^{-1}(b_1]$, whence $g(b_0) = g(b_1)$.

II. Suppose $g(b) \leq a$. We must find a successor b' of b (which may be b itself) with $g(b') = a$. If $b \notin B_0$, then $g(b)$ is maximal in A, so $g(b) = a$. If $b \in B_0$, then by 2.11.7, every cover, hence every strict successor, of $g(b)$ belongs to $f^{-1}[b)$. So if $g(b) \neq a$, still $f(a) \geq b$, and $g(f(a)) = a$. □

2.11.9(a) Example. The necessity of condition (b) is easily demonstrated. Let $A = \mathbb{N} \cup (\{0\} \times \mathbb{N})$, with $\leq_A$ extending the natural ordering on $\mathbb{N}$ by

$$n \leq_A (0,k) \text{ iff } n \leq_{\mathbb{N}} k;$$

Let ∞ be a new point; $B = A \cup \{\infty\}$, with the ordering of A extended by $\infty \geq \mathbb{N}$.



Then the canonical embedding $A \hookrightarrow B$ is strong, and condition (a) is satisfied; but it is easily checked that A is not a p-morphic image of B.

2.11.9(b) Remark. By II of the proof of the above theorem, $g(b) \leq_A a$ implies $g(b) = a$ or $b \leq_B f(a)$. Since $f(a) \geq b$ implies $a = gf(a) \geq g(b)$, the p-retraction g constructed in that proof has the property that

$$\{g(b)\} \cup f^{-1}[[b)_B] = [g(b))_A.$$

2.11.10 Corollary. If (i) $A \in$ **FTR** and $B \in$ **DLO**;

or (ii) $A,B \in$ **TR**, and every $a \in A$ has a successor that is maximal in A;

then every strong embedding of A into B has a p-retraction.

Proof. In either case it is immediate that condition (a) of the theorem is satisfied. Condition (b) holds in case (i) because A is finite, and in case (ii) because $(b]$ is finite. □

2.11.11 Definition. A tree A is binary if every point in A has at most 2 covers. More general, A is n-ary ($n \in \mathbb{N}$) if every point of A has at most n covers.

2.11.12 Corollary. If A and B satisfy either (i) or (ii) of corollary 2.11.10, and A is binary, then every embedding of A into B has a p-retraction.

Proof: Every embedding of a binary tree is strong. □

§3. Refutation patterns

The examples we have met so far were meant to suggest the following picture: there is a certain well-defined procedure, 'making semantic tableaux', that gives for each $\mathbb{I}$-formula φ a finite refutation pattern – or possibly a finite number of such patterns – with the property that φ is refutable in a frame A if and only if A exhibits, in some sense, one of these patterns. Several parts of this picture are still rather vague. This section will fill in the details: it contains a formal definition of semantic tableaux; and a precise description of the relation between tableaux and the frames in which the formula they treat is not valid, through the intermediary of multiple tableaux.

3.1 Signed formulas and sequents.

A signed formula is a pair (ξ,φ) with $\varphi \in \mathbb{I}$ and ξ one of the letters T,F. We always write $\mathrm{T}\varphi$, $\mathrm{F}\varphi$ instead of (T,φ), (F,φ). A finite set of signed formulas we call a **sequent**.
We use σ,τ as variables over signed formulas. Of a point a in a given model, we say a **realizes** $\mathrm{T}\varphi$ (notation $a \Vdash \mathrm{T}\varphi$) if $a \Vdash \varphi$; a realizes $\mathrm{F}\varphi$ ($a \Vdash \mathrm{F}\varphi$) if $a \nVdash \varphi$. A sequent Σ is realized in a ($a \Vdash \Sigma$) if for all $\sigma \in \Sigma$, $a \Vdash \sigma$.
If Σ is a sequent, then $\Sigma^{\mathrm{T}} = \{\mathrm{T}\varphi \mid \mathrm{T}\varphi \in \Sigma\}$ and $\Sigma_{\mathrm{T}} = \{\varphi \mid \mathrm{T}\varphi \in \Sigma\}$ ('T dropped'). Similarly we have Σ^{F} and Σ_{F}.

3.2 Definition. A sequent Σ is **full** if for all $\varphi,\psi \in \mathbb{I}$,
(i) $\mathrm{T}(\varphi\wedge\psi) \in \Sigma \Rightarrow \mathrm{T}\varphi, \mathrm{T}\psi \in \Sigma$; $\mathrm{F}(\varphi\wedge\psi) \in \Sigma \Rightarrow \mathrm{F}\varphi \in \Sigma$ or $\mathrm{F}\psi \in \Sigma$;
(ii) $\mathrm{T}(\varphi\vee\psi) \in \Sigma \Rightarrow \mathrm{T}\varphi \in \Sigma$ or $\mathrm{T}\psi \in \Sigma$; $\mathrm{F}(\varphi\vee\psi) \in \Sigma \Rightarrow \mathrm{F}\varphi, \mathrm{F}\psi \in \Sigma$;
(iii) $\mathrm{T}(\varphi\rightarrow\psi) \in \Sigma \Rightarrow \mathrm{F}\varphi \in \Sigma$ or $\mathrm{T}\psi \in \Sigma$; $\mathrm{F}(\varphi\rightarrow\psi) \in \Sigma \Rightarrow \mathrm{F}\psi \in \Sigma$.

3.3 Semantic tableaux.

Informally, a semantic tableau was defined as a set of sequents, together with information on the causal relations between them. Formally, we shall use a mapping S from the set X of the sequents that make up the tableau, to the power set $\mathbb{P}(X)$; $\mathrm{S}(x)$ may be read as 'the set of immediate successors of x', or 'the sequents caused by x'.

Definition. A **(semantic) tableau** is a pair $\mathfrak{X} = (X,\mathrm{S})$ of a finite set X of full sequents and a map

S: $X \to \mathbb{P}(X)$ such that for all $x \in X$

(i) if $y \in S(x)$, then $x^T \subseteq y^T$;

(ii) if $F(\varphi \to \psi) \in x$, then either $T\varphi \in x$ or $S(x)$ contains a sequent y such that either

$T\varphi, F\psi \in y$ or $F(\varphi \to \psi) \in y$ and $y^T \neq x^T$.

A tableau as defined here is the finished product of a tableau construction as described in 1.8. The elements of $S(x)$ represent the forcing behaviour of the nearest different successors of a point with behaviour x. Observe that since X is finite, and y^T in clause (ii) properly extends x^T, $F(\varphi \to \psi) \in x$ implies there are $x_0 = x, x_1, ..., x_n$ $(n \geq 0)$ with $x_{i+1} \in S(x_i)$ $(i<n)$ and $T\varphi, F\psi \in x_n$.
We call a tableau $\mathfrak{X} = (X,S)$ **open** if no $x \in X$ contains $T\bot$, or, for some $\varphi \in \mathbb{I}$, both $T\varphi$ and $F\varphi$; otherwise $\mathfrak{X}$ **closes**, or **is closed.** $\mathfrak{X}$ is **strict** if for all $x \in X$, $y \in S(x)$ implies $x^T \neq y^T$.

3.4 Examples. Tableau constructions serve to find refutation patterns for $\mathbb{I}$-formulas. Suppose one wants to refute φ: then it seems reasonable to consider only subformulas of φ, since these are the only formulas relevant to the evaluation of φ. Indeed, the sensible approach will be to start with a sequent $\{F\varphi\}$, and construct from it, step by step, a tableau that contains only what the rules (embodied in definitions 3.2 and 3.3) require.

(*a*) The search for an open tableau of F(KC) runs as follows. Start with F(KC) (= $F(\neg p \vee \neg\neg p)$), and expand this as far as possible using the rules of definition 3.2: add $F\neg p$, $F\neg\neg p$ and $F\bot$. Now definition 3.3 must be used, and there are several options; let us try them all. First attempt: simply add Tp. For $F\neg\neg p$, let us do the same: add $T\neg p$. Now we must add either Fp or $T\bot$, both of which make the tableau close. There were, however, other possibilities, which should be traced - but first let us write down what we have done in a concise way. We have expanded a sequent, adding signed formulas:

F(KC), $F\neg p$, $F\neg\neg p$, $F\bot$

and then we (repeatedly) chose one option out of several, and to help ourselves remember that there were other options, we mark the signed formulas involved:

......., Tp?, $T\neg p$?

Let us signify closure by underlining the last signed formula added:

......., $\underline{Fp}$?

Now we work back from right to left, taking the other alternative each time we meet a choice.

$\ldots\ldots, \mathrm{T}\neg p?, \underline{\mathrm{T}\bot}$

We conclude that the choice of $\mathrm{T}\neg p$ leads to closure. The alternative is to start a new sequent.

$$\begin{tikzcd}
\boxed{\mathrm{F(KC)}, \mathrm{F}\neg p, \mathrm{F}\neg\neg p, \mathrm{F}\bot, \mathrm{T}p?} \arrow[d] \\
\mathrm{T}p, \mathrm{T}\neg p, \mathrm{F}\bot
\end{tikzcd}$$

Now, this will lead to closure, as before. Given the choice of $\mathrm{T}p$, this development is unavoidable, so we should have

$$\begin{tikzcd}
& \boxed{\mathrm{F(KC)}, \mathrm{F}\neg p, \mathrm{F}\neg\neg p, \mathrm{F}\bot} \arrow[dl] \\
\boxed{\mathrm{T}p, \mathrm{F}\bot} &
\end{tikzcd}$$

Now we could again try $\mathrm{T}\neg p$ in the root sequent; but it would be handed on to the successor, and lead to closure. Likewise it does not make sense to try $\mathrm{T}\neg p$ in the successor sequent. So we end up with

$$\begin{tikzcd}
& \boxed{\mathrm{F(KC)}, \mathrm{F}\neg p, \mathrm{F}\neg\neg p, \mathrm{F}\bot} \arrow[dl] \arrow[dr] & \\
\boxed{\mathrm{T}p, \mathrm{F}\bot} & & \boxed{\mathrm{T}\neg p, \mathrm{F}\bot, \mathrm{F}p}
\end{tikzcd}$$

This is an open tableau, and with the observation that the successor sequents cannot have a common successor ($\mathrm{T}p$ and $\mathrm{T}\neg p$ give $\mathrm{T}\bot$), it gives off the refutation pattern of example 2.6.

We shall save a little on notation by leaving out signed formulas whose presence can be easily inferred, and that do not give rise to new steps in the construction. For example, with $\mathrm{F}\neg p$ and $\mathrm{F}\neg\neg p$ present, there was no need to repeat F(KC); and writing $\mathrm{F}\bot$ does not make sense ever.

(*b*) Recall that $\mathrm{P}_1 = ((p_1 \to p_0) \to p_1) \to p_1$, and $\mathrm{P}_2 = ((p_2 \to \mathrm{P}_1) \to p_2) \to p_2$. In constructing an economical open tableau for $\{\mathrm{FP}_2\}$, FP_2 reduces immediately to

$\mathrm{T}((p_2 \to \mathrm{P}_1) \to p_2), \mathrm{F}p_2$

(We could have started with a separate sequent $\{FP_2\}$; but then we would have continued with this one anyway.) Next, it is easy to choose between $F(p_2 \to P_1)$ and Tp_2: we get

$$(1)\quad T((p_2 \to P_1) \to p_2), Fp_2, F(p_2 \to P_1)$$

This must lead to a successor:

$$\begin{tikzcd}
(1) \arrow[d] \\
\boxed{Tp_2, FP_1}
\end{tikzcd}$$

FP_1 is attacked in the same way, resulting in

$$\begin{tikzcd}
(1) \arrow[d] \\
\boxed{Tp_2, T((p_1 \to p_0) \to p_1), Fp_1, F(p_1 \to p_0)} \arrow[d] \\
\boxed{Tp_1, Fp_0}
\end{tikzcd}$$

which is the pattern established by 2.8. (It is not necessary to repeat Tp_2 in the last sequent: it is inherited by a general rule.)

(c) The formula

$$((\neg\neg p \to p) \to p \vee \neg p) \to \neg p \vee \neg\neg p$$

is known as Scott's Axiom; we shall refer to it as SC. A tableau for $\{F(SC)\}$ may look as follows:

$$\begin{tikzcd}
& \boxed{T((\neg\neg p \to p) \to p \vee \neg p), F\neg p, F\neg\neg p, F(\neg\neg p \to p)} \arrow[dl] \arrow[dr] & \\
\boxed{T\neg p, Fp} & & \boxed{T\neg\neg p, Fp, F\neg p} \arrow[d] \\
& & \boxed{Tp}
\end{tikzcd}$$

If we had required, in (ii) of definition 3.3, that some sequent in $S(x)$ contain $\{T\varphi, F\psi\}$, the root of this tableau should have had three successors: $\{Tp\}$, besides what it already has; whereas two is clearly enough to represent a model in which SC is not valid.

(*d*) There need not be a single 'smallest' open tableau. A very simple example would be $KC \wedge P_2$, which leads to both the tableau in (*a*) and the one in (*b*).

3.5 Models from tableaux.

Some formal details apart, an open tableau is a model of the sort that the tableau construction was to help us find.

Definition. Let $\mathfrak{X} = (X,S)$ be an open tableau.
(i) $\leq_{\mathfrak{X}}$ is the reflexive and transitive closure of the relation $\{(x,y) \mid y \in S(x)\}$. We shall consider X as a frame, with quasi-ordering $\leq_{\mathfrak{X}}$.
(ii) The **model induced** by $\mathfrak{X}$ is the pair (X,V) with $V(p) = \{x \in X \mid Tp \in x\}$, for all $p \in \mathbb{P}$.

It is easy to see that the induced 'model' is indeed a model; in particular, each $V(p)$ is upwards closed by clause (i) of definition 3.3. If $\mathfrak{X}$ is strict, then $\leq_{\mathfrak{X}}$ is a partial ordering.

Proposition. Let $\mathfrak{X} = (X,S)$ be an open semantic tableau. Then in the model induced by $\mathfrak{X}$, every point realizes itself.

Proof. Induction on the complexity of σ will show that $\sigma \in x$ implies $x \Vdash \sigma$. Since $\mathfrak{X}$ is open, $T\bot \notin x$; $Fp \in x$ implies $Tp \notin x$, so $x \nVdash p$.
The induction steps are straightforward, except for implication. There $T(\varphi \to \psi) \in x$ implies, for all $y \geq x$, $T(\varphi \to \psi) \in y$; now $y \Vdash \varphi$ gives $F\varphi \notin y$ by induction hypothesis, hence $T\psi \in y$ by fullness, and $y \Vdash \psi$ by induction hypothesis; so $x \Vdash \varphi \to \psi$. On the other hand, $F(\varphi \to \psi) \in x$ gives $y \geq x$ with $T\varphi, F\psi \in y$, or $y > x$ with $F(\varphi \to \psi) \in y$, by 3.3(ii). Since X is finite, we must end up with $T\varphi, F\psi \in y \geq x$, so, by the induction hypothesis, $x \nVdash \varphi \to \psi$. □

Thus, our induced models are a pocket version of the standard Henkin models (cf. Aczel, Thomason [1968]). They are alike both in what they are made of (syntactic matter) and in their original purpose: to obtain counterexamples to formulas that are not deducible in some given calculus. The differences are in constructivity and size. Tableau constructions can be finished, whereas Henkin models are infinite; the ordering in a Henkin model would be defined globally, in a tableau successors are tailored to local needs. Finiteness and some freedom in the ordering will appear necessary for such purposes as establishing connections with first order logic.

In the meantime, it has not been stated how small the tableaux we deal with may be taken.

3.6.1 Definition. Let Σ be a sequent. The set of **signed subformulas** of Σ (notation: $Sf(\Sigma)$) is the smallest sequent $\Sigma' \supseteq \Sigma$ such that
(a) for $\xi \in \{T,F\}$, $(\xi, \varphi\wedge\psi) \in \Sigma'$ or $(\xi, \varphi\vee\psi) \in \Sigma'$ implies $(\xi,\varphi) \in \Sigma'$ and $(\xi,\psi) \in \Sigma'$;
(b) $T(\varphi \to \psi) \in \Sigma' \Rightarrow F\varphi, T\psi \in \Sigma'$; $F(\varphi \to \psi) \in \Sigma' \Rightarrow T\varphi, F\psi \in \Sigma'$.

3.6.2 Definition. If $\mathfrak{X} = (X,S)$ is a tableau, Σ is contained in some element of X, and $\cup X \subseteq Sf(\Sigma)$, we call $\mathfrak{X}$ a Σ-tableau. A **refutation** of φ is an open $\{F\varphi\}$-tableau.

3.7 It is easy to see that for any Σ, there are only finitely many Σ-tableaux. Many of these will contain sequents and connections that are not necessary. We shall define the minimal tableaux as the tableaux without frills.

3.7.1 Definition. Let $\mathfrak{X} = (X,S)$ and $\mathfrak{X}' = (X',S')$ be tableaux. $\mathfrak{X}$ is a **subtableau** of $\mathfrak{X}'$ if either $X \neq X'$ and there exists an injection $f: X \rightarrowtail X'$ such that for all $x \in X, x \subseteq f(x)$ and $\forall y \in S(x)$. $f(y) \geq_{\mathfrak{X}'} f(x)$; or $X = X'$ and $\forall x \in X$. $S(x) \subseteq S'(x)$.

A subtableau of a closed tableau may close no longer; but a subtableau of an open tableau is open, and this is what matters.

3.7.2 Definition. Let Σ be a sequent. A **minimal** Σ-tableau is a Σ-tableau no proper subtableau of which is a Σ-tableau. A **minimal refutation** of φ is a minimal open $\{F\varphi\}$-tableau.

A Σ-tableau is a finite constellation of finite sets, so it is clear that every Σ-tableau has minimal Σ-subtableaux. Minimality has a few simple consequences, illustrated in the examples above.

3.7.3 Proposition. If $\mathfrak{X} = (X,S)$ is a minimal Σ-tableau, then
(i) Exactly one element $x_0 \in X$ contains Σ; x_0 is the root of $(X, \leq_{\mathfrak{X}})$.
(ii) For every $x \in X - \{x_0\}$, there exist $y \in X$ and $F(\varphi \to \psi) \in y$ such that $x \in S(y)$, $F\psi \in x$, and $T\varphi \in x - y$.
(iii) $\mathfrak{X}$ is strict.

3.8 What is the connection between a frame in which φ is refutable and the – preferably minimal – refutations of φ? For the examples of §2, the following simple solution works:

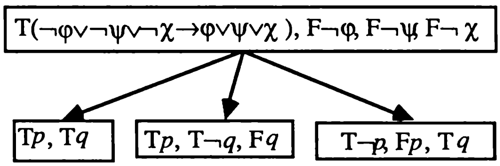
> $A \nVdash \varphi$ iff the induced frame of some minimal refutation of φ can be embedded into A - provided certain sequents are mapped to points $a_1, \ldots, a_n$ such that $[a_1) \cap \ldots \cap [a_n) = \emptyset$.

The proviso would be needed for $\{T\neg p, Fp\}$ and $\{Tp\}$ in 3.4(a). – This is a useful approach (cf. §7), but it does not work in general.

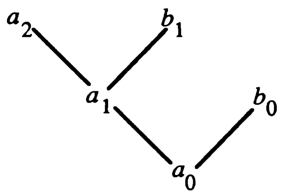
<u>Example.</u> Recall the formula SP_2 of 1.9: with φ for $p \wedge q$, $\psi := p \wedge \neg q$, and $\chi := \neg p \wedge q$, it reads

$$(\neg\varphi \vee \neg\psi \vee \neg\chi \to \varphi \vee \psi \vee \chi) \to \neg\varphi \vee \neg\psi \vee \neg\,\chi.$$

Its minimal refutation is, in a diagram omitting matters of course:

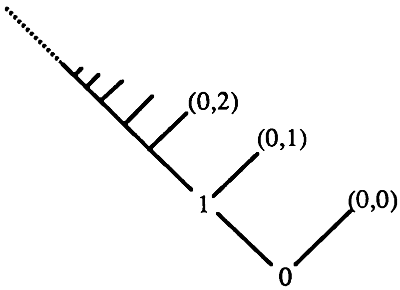


The induced frame can be embedded in a binary tree C of five nodes $a_0 < a_1, a_2, b_0, b_1$, with $a_1 < a_2, b_1$:



Take any valuation V on C, and suppose $(C,V) \nVdash SP_2$. Since φ, ψ and χ are mutually exclusive, a_0 must realize the bottom sequent of the tableau, with φ, ψ and χ each true in one top node. Say $a_2 \Vdash \varphi$ and $b_1 \Vdash \psi$. Then $a_1 \Vdash \neg\chi$; since $a_1 \Vdash \neg\varphi \vee \neg\psi \vee \neg\chi \to \varphi \vee \psi \vee \chi$, $a_1 \Vdash \varphi \vee \psi \vee \chi$. Whichever of φ, ψ and χ a_1 picks must hold in both a_2 and b_1: an impossibility.

It may be part of a refutation pattern that there are no points in certain positions. Here a_1 causes trouble, and we might think that such points do not occur in frames in which SP_2 is not valid. This would be a mistake, however. Consider the frame A of example 2.11.7 (on the next page):



Define: $V(p) = \{0\} \times \{3n, 3n{+}1 \mid n \in \mathbb{N}\}$; $V(q) = \{0\} \times \{3n, 3n{+}2 \mid n \in \mathbb{N}\}$. Then $(A,V) \nVdash SP_2$.

'Intermediate' points such as a_1 cannot simply be forbidden; they must be taken into consideration. This suggests that the general link between frames and tableaux is not to be thought of as embedding (tableaux into frames), but projection of frames onto tableaux. Indeed, the induced frame of the refutation of SP_2 is a p-morphic image of A. In general, however, p-morphism cannot be the right kind of projection: there are frames without finite p-morphic images (cf. Jankov [1968], in view of the duality explained in part III below).
We avoid this problem by generalizing the notion of tableau. The difficulty may be viewed as follows: we want to identify points that realize the same subset of some finite set of signed formulas, to guarantee a finite image. There may thus be several points realizing the same sequent x containing, say, some formula $F(p \rightarrow q \wedge r)$. In some points, $F(p \rightarrow q \wedge r)$ may be dealt with by a successor containing Fq; in others, by a successor containing Fr. In the image of the projection, this comes down to the existence of two kinds of successors for x. They should not be thrown into one successor set, for then the image would contain a pattern not to be found in the original.

3.9 Multitableaux.

3.9.1 Definition. A **multiple tableau** (short: **multitableau**) is a pair $(X,\mathfrak{S})$ of a finite set X of full sequents and a function $\mathfrak{S}: X \rightarrow \mathbb{P}(\mathbb{P}(X))-\{\emptyset\}$ such that
(i) if $y \in S \in \mathfrak{S}(x)$, then $x^T \subseteq y^T$;
(ii) if $F(\varphi \rightarrow \psi) \in x$, then either $T\varphi \in x$ or every $S \in \mathfrak{S}(x)$ contains a sequent y such that either $T\varphi, F\psi \in y$ or $F(\varphi \rightarrow \psi) \in y$ and $x^T \neq y^T$.

The old 'simple' tableaux will be regarded as multitableaux in which every collection $\mathfrak{S}(x)$ is a singleton: $\mathfrak{S}(x) = \{S(x)\}$.
Tableau terminology will be extended to multitableaux. Some extensions are entirely straightforward (open, Σ-multitableau, multirefutation). The canonical ordering $\leq_{\mathfrak{X}}$, for $\mathfrak{X} = (X,\mathfrak{S})$, is the reflexive and transitive closure of $\{(x,y) \mid y \in \cup\mathfrak{S}(x)\}$. $\mathfrak{X}$ is **strict** if $y \in S \in \mathfrak{S}(x)$

implies $x^{\mathrm{T}} \neq y^{\mathrm{T}}$. Minimality will be discussed indirectly in §7 (labeled frames).
A few more words on the relation between simple tableaux and multitableaux may be helpful.

3.9.2 <u>Definition.</u> By a **component** of a multitableau $\mathfrak{X} = (X, \mathfrak{S})$ I shall understand a tableau $\mathfrak{X}' = (X', S')$ such that $X' \subseteq X$, and
(i) the minimal elements of $(X', \leq_{\mathfrak{X}'})$ are minimal elements of $(X, \leq_{\mathfrak{X}})$;
(ii) for every $x \in X'$, $S'(x) \in \mathfrak{S}(x)$.

3.9.3 <u>Definition.</u> The **union** of a finite family $(\mathfrak{X}_i \mid i \in I)$ of multitableaux $(\mathfrak{X}_i = (X_i, \mathfrak{S}_i))$ is the multitableau $\mathfrak{X} = (X, \mathfrak{S})$ with $X = \cup_{i \in I} X_i$ and $\mathfrak{S}(x) = \cup(\mathfrak{S}_i(x) \mid x \in X_i)$.

One easily checks that $\mathfrak{X}$ indeed conforms to definition 3.9.1.
The relation between simple tableaux and multitableaux can now be stated as follows:

3.9.4 <u>Proposition.</u> Any multitableau is the union of its components.

3.9.5 <u>Example.</u> Multitableaux have practical use as a notation for alternative refutations of complex formulas. For instance, two open tableaux for

$$\{F([(p_3 \wedge (\neg p_4 \vee \neg\neg p_4 \to p_5) \wedge (P_2 \to p_6) \to p_5 \wedge p_6) \to p_3] \to p_3)\}$$

are represented in

$$\begin{tikzcd}
& \boxed{\begin{array}{r} T[(p_3 \wedge (\neg p_4 \vee \neg\neg p_4 \to p_5) \wedge (P_2 \to p_6) \to p_5 \wedge p_6) \to p_3], Fp_3, \\ F(p_3 \wedge (\neg p_4 \vee \neg\neg p_4 \to p_5) \wedge (P_2 \to p_6) \to p_5 \wedge p_6), Fp_5 \end{array}} \arrow[dl, "1"'] \arrow[dr, "2"] & \\
\boxed{\begin{array}{l} Tp_3, T(\neg p_4 \vee \neg\neg p_4 \to p_5), T(P_2 \to p_6), Fp_6 \\ Tp_5, FP_2, T((p_2 \to P_1) \to p_2), Fp_2, F(p_2 \to P_1) \end{array}} \arrow[d] & & \boxed{\begin{array}{l} Tp_3, T(\neg p_4 \vee \neg\neg p_4 \to p_5), T(P_2 \to p_6), \\ Fp_5, F\neg p_4, F\neg\neg p_4, Tp_6 \end{array}} \arrow[d] \\
\text{(continue as in 3.4(b))} & & \text{(continue as in 3.4(a))}
\end{tikzcd}$$

(The alternative successor sets - singletons - of the root sequent are signalled by the different labels of the arrows issuing there.)

3.10 Projections.

Definition. Let A be a frame; $\mathfrak{X} = (X,\mathfrak{S})$ a multitableau. A surjection $f: A \twoheadrightarrow X$ is a **projection** of A onto $\mathfrak{X}$ if

(i) f is a homomorphism, in the sense that $a \leq_A a'$ implies $f(a) \leq_{\mathfrak{X}} f(a')$;

(ii) for every $x \in X$, there exists for every $a \in f^{-1}\{x\}$ an $S \in \mathfrak{S}(x)$ such that $S \subseteq f[a)$.

Projections generalize p-morphisms:

Proposition. Let A be a frame; $\mathfrak{X} = (X,S)$ a simple tableau. Then $f: A \twoheadrightarrow \mathfrak{X}$ is a projection iff $f: A \twoheadrightarrow (X,\leq_{\mathfrak{X}})$ is a p-morphism.

Proof.

($\Rightarrow$) Let $f: A \twoheadrightarrow \mathfrak{X}$ be a projection; we must check the p-morphism condition.
Suppose $x \in \mathrm{S}(f(a))$. Since $a \in f^{-1}\{f(a)\}$, we have $\mathrm{S}(f(a)) \subseteq f[a)$ by the definition. That is to say: for some $a' \geq a, f(a') = x$. Now if $x \geq_{\mathfrak{X}} f(a)$, there must be a sequence $f(a) = x_0, x_1, ..., x_n = x$ ($n \geq 0$) with $x_{i+1} \in \mathrm{S}(x_i)$. Accordingly we find $a = a_0, ..., a_n$ with $f(a_i) = x_i$ (in particular $f(a_n) = x$) and $a_{i+1} \geq a_i$. By transitivity of $\leq$, $a_n \geq a$.
($\Leftarrow$) Let $f: A \twoheadrightarrow X$ be a p-morphism; again, there is only one condition to check. Suppose $x \in \mathrm{S}(f(a))$. Then $x \geq f(a)$, so by the p-morphism condition there exists $b \geq a$ with $f(b) = x$. It follows that $\mathrm{S}(f(a)) \subseteq f[a)$. □

3.11 Definition. Let $\mathfrak{A}$ be a model, and φ an $\mathbb{I}$-formula. Then $\Theta_\varphi^{\mathfrak{A}}$ is the function which assigns to each point a of $\mathfrak{A}$ the sequent $\Theta_\varphi^{\mathfrak{A}}(a) := \{\sigma \in \mathrm{Sf}\{\mathrm{F}\varphi\} \mid (\mathfrak{A},a) \Vdash \sigma\}$.

3.12 Theorem. Let A be a frame, and $\varphi \in \mathbb{I}$. Then $A \nVdash \varphi$ iff A can be projected onto a multirefutation of φ.

Proof. Suppose $(A,V) \nVdash \varphi$; let $\Theta = \Theta_\varphi^{(A,V)}$. It is easy to see that each $\Theta(a)$ is a full sequent. Let $X = \{\Theta(a) \mid a \in A\}$. Define $\mathfrak{S}$ by

$$\mathfrak{S}(x) = \{\Theta[a) \mid a \in \Theta^{-1}\{x\}\}.$$

Then (a) $(X,\mathfrak{S})$ (=: $\mathfrak{X}$) is an $\{\mathrm{F}\varphi\}$-multitableau, and (b) Θ projects A onto $\mathfrak{X}$.
Both these facts are direct consequences of the definitions:

(a) (i) Suppose $y \in S \in \mathfrak{S}(x)$; say $y = \Theta(b)$, $x = \Theta(a)$. We may assume by the definition of $\mathfrak{S}$ that $a \leq b$, whence $x^{\mathrm{T}} \subseteq y^{\mathrm{T}}$ is immediate.

(ii) If $\mathrm{F}(\psi\rightarrow\chi) \in \Theta(a)$, and $\mathrm{T}\psi \notin \Theta(a)$, then $a \nVdash \psi\rightarrow\chi$, $a \nVdash \psi$, and $b \Vdash \psi$, $b \nVdash \chi$ for some b

$> a$. Then by the definition of Θ, $T\psi$, $F\chi \in \Theta(b)$; and $\Theta(b) \in \Theta[a] \in \mathfrak{S}(\Theta(a))$.
(*b*) (i) Θ is a homomorphism since $a \le b$ implies $\Theta(b) \in \Theta[a] \in \mathfrak{S}(\Theta(a))$.
(ii) Trivially $\Theta[a] \subseteq \Theta[a]$.
For the converse, let $f: A \twoheadrightarrow \mathfrak{X}$ be a projection; define a valuation V on A by $V(p) = \{a \mid Tp \in f(a)\}$. We claim that for each $a \in A$, $a \Vdash f(a)$; hence $A \nVdash \varphi$.
That $a \Vdash f(a)$ is established by showing inductively that $\sigma \in f(a)$ implies $a \Vdash \sigma$. The case of implication is as follows:
If $T(\psi \to \chi) \in f(a)$, $a \le a' \Vdash \psi$, then, since f is a homomorphism, $f(a) \le_{\mathfrak{X}} f(a')$, so $T(\psi \to \chi) \in f(a')$. Then $F\psi \notin f(a')$ (otherwise $a' \nVdash \psi$ by induction hypothesis); so $T\chi \in f(a')$ by fullness. By induction hypothesis, $a' \Vdash \chi$. We conclude that $a \Vdash \psi \to \chi$.
If $F(\psi \to \chi) \in f(a)$, then either $T\psi \in f(a)$, or to every $S \in \mathfrak{S}(f(a))$ belongs some x containing $F(\psi \to \chi)$ with $x^T \ne f(a)^T$, or some x containing $T\psi$, $F\chi$. In the first case, $a \Vdash \psi$ and $a \nVdash \chi$ by induction hypothesis (using fullness), so $a \nVdash \psi \to \chi$. In the first subcase of the second case, take $S_0 \in \mathfrak{S}(f(a)$ such that $S_0 \subseteq f[a)$. We find $a' \ge a$ with $F(\psi \to \chi) \in f(a') \in S_0$, and $f(a')^T \ne f(a)^T$. Since X is finite, this can only be repeated finitely often; then we must have found $a'' \ge a$ with $T\psi$, $F\chi \in f(a'')$. Then as before, $a'' \nVdash \psi \to \chi$, and $a \nVdash \psi \to \chi$ since $V(\psi \to \chi)$ is upwards closed.

3.13 Remark. Since multiple tableaux are unions of simple components (3.9.4), we find, combining proposition 3.5 and theorem 3.12, that an $\mathbb{I}$-formula φ is valid in every frame iff every $\{F\varphi\}$-tableau closes. Because the set of all (minimal) $\{F\varphi\}$-tableaux can be effectively constructed, we can decide whether an $\mathbb{I}$-formula is universally valid. By 3.5 again, intuitionistic propositional logic has the finite model property: if φ is not universally valid, it is not valid in some finite model. Indeed, since (3.7) every refutation contains a minimal refutation, which is strict, and thus (3.5) gives rise to a partially ordered frame, we may state the finite model property in the form

$$\vdash \varphi \text{ iff } \mathbf{FPO} \Vdash \varphi.$$

(It is well known that **FPO** may even be replaced by **FTR**; see Smorynski [1973] or Gabbay [1981]. Actually, the proof of this fact will surface in §17.)

§4. Fragments

It is well known that in intuitionistic logic the connectives $\wedge$, $\vee$, $\rightarrow$ and $\perp$ are not interdefinable. For an $\mathbb{I}$-formula φ, let $\mathbf{Mod}(\varphi)$ be the class of all models in which φ is valid. Then non-interdefinability may be expressed in terms of the interpretation in Kripke models by statements such as : there exist $\mathbb{I}$-formulas φ such that for no $\mathbb{I}$-formula ψ not containing $\wedge$, $\mathbf{Mod}(\varphi)=\mathbf{Mod}(\psi)$.

Validity in *frames* does not have the same connection with intuitionistic logic. We shall investigate in this section to what extent connectives can be dropped without loss of expressive force with regard to frames. (A reduction of the sort we are seeking exists in modal logic: in van Benthem [1986] (Cor. 2.9) it is shown that for each modal formula ζ there exists a modal formula ζ^* with $\rightarrow$ and $\Diamond$ as its logical constants, such that for all modal frames $\mathfrak{A}=(A,R)$:$\forall a\in A$ ($\mathfrak{A}\models\zeta[a]$ iff $\mathfrak{A}\models\zeta^*[a]$).)

4.1 Definition. Let $c_1,\ldots,c_n$ be a sequence of connectives (not necessarily primitive); then $\mathbb{I}[c_1,\ldots,c_n]$ (the $\{c_1,\ldots,c_n\}$**-fragment**) is the set of all $\mathbb{I}$-formulas that can be built from $\mathbb{P}$ using only $c_1,\ldots,c_n$.

4.2 Definition. (i) Let φ be an $\mathbb{I}$-formula; then $\mathbf{Fr}(\varphi)$ is the class of all frames on which φ is valid.
(ii) $\mathbb{I}$-formulas φ and ψ are **equivalent** (notation: $\varphi\equiv\psi$) if $\mathbf{Fr}(\varphi)=\mathbf{Fr}(\psi)$.

These notions may be relativized to any class $\mathbf{K}$ of frames; $\mathbf{Fr}_{\mathbf{K}}(\varphi)$, then , is $\mathbf{Fr}(\varphi)\cap\mathbf{K}$, and $\varphi\equiv_{\mathbf{K}}\psi$ if $\mathbf{Fr}_{\mathbf{K}}(\varphi)=\mathbf{Fr}_{\mathbf{K}}(\psi)$.

As usual, we abbreviate $(\varphi\rightarrow\psi)\wedge(\psi\rightarrow\varphi)$ to $\varphi\leftrightarrow\psi$. We call φ and ψ **logically equivalent** if $\vdash\varphi\leftrightarrow\psi$ (equivalently, by completeness: $\varphi\leftrightarrow\psi$ is universally valid). Note that logical equivalence implies equivalence on frames.

We shall drop some parentheses in iterated implications, assuming association to the right: so $\varphi\rightarrow\psi\rightarrow\chi = \varphi\rightarrow(\psi\rightarrow\chi)$.

4.3 **The elimination of conjunction.**

4.3.1 Lemma. For each $\varphi\in\mathbb{I}$, there are $\varphi_1,\ldots,\varphi_n\in\mathbb{I}[\rightarrow,\vee,\perp]$ such that $\vdash\varphi\leftrightarrow\bigwedge_{1\leq i\leq n}\varphi_i$.

Proof: Move out conjunctions, using the logical equivalence of $\psi_1 \wedge \psi_2 \rightarrow \psi_3$ to $\psi_1 \rightarrow \psi_2 \rightarrow \psi_3$, $\psi_1 \rightarrow \psi_2 \wedge \psi_3$ to $(\psi_1 \rightarrow \psi_2) \wedge (\psi_1 \rightarrow \psi_3)$, $(\psi_1 \wedge \psi_2) \vee \psi_3$ to $(\psi_1 \vee \psi_3) \wedge (\psi_2 \vee \psi_3)$, and $\psi_1 \vee (\psi_2 \wedge \psi_3)$ to $(\psi_1 \vee \psi_2) \wedge (\psi_1 \vee \psi_3)$. □

4.3.2 Lemma. Let φ be an $\mathbb{I}$-formula. If the proposition letter p does not occur in φ, then $\varphi \equiv (\varphi \rightarrow p) \rightarrow p$.

Proof: Since $\vdash \varphi \rightarrow (\varphi \rightarrow p) \rightarrow p$, $A \Vdash \varphi$ implies $A \Vdash (\varphi \rightarrow p) \rightarrow p$. For the converse, substitute φ for p in $(\varphi \rightarrow p) \rightarrow p$. □

Now conjunction can be eliminated by a trick in which the adept will recognize the definition of conjunction in second order propositional logic (Prawitz [1965]).

4.3.3 Theorem. Every $\mathbb{I}$-formula is equivalent to a formula of the $\{\vee, \rightarrow, \bot\}$-fragment.

Proof: Suppose $\varphi \in \mathbb{I}$. By the first lemma, there are $\varphi_1, \ldots, \varphi_n \in \mathbb{I}[\vee, \rightarrow, \bot]$ such that $\vdash \varphi \leftrightarrow \bigwedge_{1 \leq i \leq n} \varphi_i$. Take a new proposition letter p. By the second lemma, $\bigwedge \varphi_i$ is equivalent to $(\bigwedge \varphi_i \rightarrow p) \rightarrow p$; the latter formula is logically equivalent to $(\varphi_1 \rightarrow \ldots \rightarrow \varphi_n \rightarrow p) \rightarrow p \in \mathbb{I}[\vee, \rightarrow, \bot]$. □

Note that the proof gives equivalents in $\mathbb{I}[\rightarrow, \bot]$ for formulas of the $\{\wedge, \rightarrow, \bot\}$-fragment. We go on to show that conjunction is the *only* connective that can be dispensed with.

4.4 Example. We saw in example 3.8 a frame C in which SP_2 is valid. In the subframe $C' = \{a_0, a_2, b_0, b_1\}$, SP_2 is *not* valid: C' is isomorphic to the induced frame of the minimal refutation of SP_2. It will be shown in §17 below that $\mathbb{I}[\wedge, \rightarrow, \bot]$-formulas are preserved in passing from C to C' (C' is a *directed subframe* of C, and $\mathbb{I}[\wedge, \rightarrow, \bot]$-formulas are *transparent*); hence SP_2 is not equivalent to an $\mathbb{I}[\wedge, \rightarrow, \bot]$-formula.

4.5 Lemma. Let A, A_0 be frames; $A = A_0 \cup \{a^*\}$, and $A_0 < a^*$. Suppose $\varphi \in \mathbb{I}[\wedge, \vee, \rightarrow]$, V_0 is a valuation on A_0 such that $(A_0, V_0) \nVdash \varphi$, and V is defined by $V(p) = V_0(p) \cup \{a^*\}$, for all $p \in \mathbb{P}$. Then $(A, V) \nVdash \varphi$.

Proof: By induction over ψ, $(A, V, a^*) \Vdash \psi$ for all $\psi \in \mathbb{I}[\wedge, \vee, \rightarrow]$. Using this and induction over $\psi \in \mathbb{I}[\wedge, \vee, \rightarrow]$, one proves that for all $a \in A_0$, $(A, V, a) \Vdash \psi$ iff $(A_0, V_0, a) \Vdash \psi$. □

The lemma implies that if some $\varphi \in \mathbb{I}[\wedge, \vee, \rightarrow]$ is not valid in a frame A, it remains not valid if we add to A a new top element a^*. Now KC is valid in frames with a top element, by 2.6. So if it were equivalent to a formula without $\bot$, it would be universally valid ; quod non. So KC $\not\equiv \varphi$ for all $\varphi \in \mathbb{I}[\wedge, \vee, \rightarrow]$.

4.6 Recall the notion of *height*, defined in 2.7.

Lemma. The height of the induced frame of a minimal refutation of an $\mathbb{I}[\wedge, \vee, \neg]$-formula is at most 2.

Proof: Let $\mathfrak{X}$ =(X,S) be a minimal refutation of $\varphi \in \mathbb{I}[\wedge, \vee, \neg]$. It suffices, by definition 3.5, to show that if $y \in S(x)$ for some $x \in X$, then $S(y) = \emptyset$.
Suppose there are x and z such that $y \in S(x)$ and $z \in S(y)$. Whenever x contains a signed formula of form $F(\psi \rightarrow \chi)$, we have $\chi = \bot$, since $x \subseteq Sf(F\varphi)$. By 3.7.3 (ii), $F\bot \in y$ and $F\bot \in z$. Moreover, $y^T \subseteq z^T$. Now let $X' = X - \{y\}$, and for $x \in X'$,

$$S'(x) = (S(x) \cup S(y)) - \{y\} \text{ if } y \in S(x),$$
$$= S(x) \text{ otherwise.}$$

Then (X', S') is still a refutation of φ, contradicting minimality. For by 3.7.3, $F\varphi \notin y$, so some $x \in X'$ contains $F\varphi$; and if $S'(x) \neq S(x)$, and $F\neg\psi \in x$, then $T\psi \in y$ implies that $T\psi$ belongs to all successors of y; while if $T\psi \notin y$, some $z \in S(y)$ contains either $\{T\psi, F\bot\}$ or $F\neg\psi$. □

If an $\mathbb{I}$-formula φ is not universally valid, it has a refutation (3.13), which may be taken minimal (3.7); φ is not valid in the induced frame of this refutation (3.5). So:

Corollary. If an $\mathbb{I}[\wedge, \vee, \neg]$-formula is valid in all frames of height at most 2, it is universally valid.

Now by 2.8, P_2 is valid in all frames of height at most 2, but not universally valid. So P_2 is not equivalent to an $\mathbb{I}[\wedge, \vee, \neg]$- formula.

4.7 Downwards linear orderings.

The argument of 4.5 fails if we consider only downwards linear frames. In fact, for equivalence on **DLO**, $\bot$ can be eliminated; as will appear presently.

4.7.1 Definition. Let $p \in \mathbb{P}$. For $\varphi \in \mathbb{I}$, we define $\varphi^p \in \mathbb{I}[\wedge, \vee, \neg]$ inductively:

(i) $q^p = q \vee p$, for all $q \in \mathbb{P}$; $\bot^p = p$.

(ii) $(\psi\wedge\chi)^p=\psi^p\wedge\chi^p$.
(iii) $(\psi\vee\chi)^p=\psi^p\vee\chi^p$.
(iv) $(\psi\rightarrow\chi)^p=\psi^p\rightarrow\chi^p$.

This definition is a version of the translation to minimal logic in Prawitz & Malmnäs [1968]. One easily proves by induction on the complexity of φ :

4.7.2 Lemma: Suppose $\varphi\in\mathbb{I}$. Then
(i) $\vdash p\rightarrow\varphi^p$;
(ii) if p does not occur in φ, then $\vdash\varphi^p[p:=\perp]\leftrightarrow\varphi$.

Now consider a minimal refutation $\mathfrak{X}=(X,S)$ of $\varphi^p\rightarrow p$. No $x\in X$ contains $\mathrm{T}p$: the one sequent (3.7.3) containing $\mathrm{F}(\varphi^p\rightarrow p)$ contains $\mathrm{F}p$, and again by proposition 3.7.3, any other $x\in X$ contains some $\mathrm{F}\psi^p\in\mathrm{Sf}(\mathrm{T}\varphi^p)$, hence (since $\vdash p\rightarrow\psi^p$, whence $\{\mathrm{T}p,\mathrm{F}\psi^p\}$ is not realizable) $\mathrm{T}p\notin x$. Therefore substituting $\perp$ for p in a minimal refutation of $\varphi^p\rightarrow p$ gives a (minimal, even) refutation of $\varphi^p[p:=\perp]\rightarrow\perp$. By (ii) of the lemma, if p does not occur in φ, this shows $\neg\varphi$ is not universally valid. Thus:

4.7.3 Lemma. Suppose φ is an $\mathbb{I}$-formula in which p does not occur. Then $\vdash\neg\varphi$ implies $\vdash\varphi^p\rightarrow p$.

4.7.4 Lemma (Glivenko): If $\neg\varphi$ is a tautology, then $\neg\varphi$ is universally valid.

Proof: Suppose $\models\neg\varphi$ and $A\in\mathbf{FPO}$; let V be a valuation on A. By example 2.5, $\neg\varphi$ is forced in every top element of A. Since $V(\varphi)$ is upwards closed, no element of A forces φ, hence $(A,V)\Vdash\neg\varphi$. So $\mathbf{FPO}\Vdash\neg\varphi$; by the finite model property (3.13), $\vdash\neg\varphi$. □

Corollary. Let Φ be a finite set of $\mathbb{I}$-formulas. If $\nvdash\neg\bigwedge\Phi$, then Φ is classically satisfiable.

Proof: If Φ is not classically satisfiable, we have $\models\neg\bigwedge\Phi$; then by Glivenko's theorem, $\vdash\neg\bigwedge\Phi$.□

If Φ is a set of $\mathbb{I}$-formulas, and $p\in\mathbb{P}$, $\Phi^p=\{\varphi^p|\varphi\in\Phi\}$. Observe that $(\bigwedge\Phi)^p=\bigwedge\Phi^p$.

4.7.5 Lemma. Suppose $\varphi\in\mathbb{I}$, and p does not occur in φ. If $A\in\mathbf{DLO}$ is rooted, and $(A,V)\nVdash\varphi^p$, then there is a valuation V^p on A such that $(A,V^p)\nVdash\varphi$.

Proof: let A,V,φ and p be as stated, and a_0 the root of A. Then $a_0\nVdash p$, since $\vdash p\rightarrow\varphi^p$ (by (i) of 4.7.2).
Let Φ be the set of all subformulas of φ. For every $a\in V(p)$, define

$$\Phi_a := \{\psi \in \Phi \mid \exists a' \leq a (a' \nVdash p \text{ and } a' \Vdash \psi^p)\}.$$

Since Φ is finite and $(a]-V(p)$ nonempty and linearly ordered, there exists $a' \leq a$ forcing $\Phi_a{}^p$. Hence $\vdash \neg \bigwedge \Phi_a$; for otherwise $\vdash \bigwedge \Phi_a{}^p \to p$ by lemma 4.7.3, implying $a' \Vdash p$.
By corollary 4.7.4, Φ_a is classically satisfiable. Fix for each Φ_a (*not* for each a!) a model $V_a \vDash \Phi_a$.
Now we define V^p on $\Phi \cap \mathbb{P}$:

$$V^p(q) = (V(q) - V(p)) \cup \{a \in V(p) \mid V_a \vDash q\}.$$

We shall abbreviate $(A, V^p, a) \Vdash \psi$ to $a \Vdash^p \psi$, and continue to use '$a \Vdash \psi$' for $(A, V, a) \Vdash \psi$.
Note that if $a \in V(p)$ and $\psi \in \Phi$, $a \Vdash^p \psi$ iff $V_a \vDash \psi$. We now establish the lemma by showing inductively that for all $\psi \in \Phi$, for all $a \in A - V(p)$, $a \Vdash^p \psi$ iff $a \Vdash \psi^p$: for, since $a_0 \nVdash p$ and $a_0 \nVdash \varphi^p$, we shall have $a_0 \nVdash^p \varphi$.
Most steps are simple; e.g. $a \nVdash \bot^p$ by $a \notin V(p)$, and for the same reason $a \Vdash q \vee p$ implies $a \Vdash^p q$. We check the case of implication. Let $\psi = \psi_1 \to \psi_2$.
Suppose $a \Vdash^p \psi$, $a \leq b \Vdash \psi_1{}^p$. If $b \Vdash p$, then, by $\vdash p \to \psi_2{}^p$, $b \Vdash \psi_2{}^p$. If $b \nVdash p$, by induction hypothesis $b \Vdash^p \psi_1$; so $b \Vdash^p \psi_2$, and by induction hypothesis $b \Vdash \psi_2{}^p$. So $a \Vdash \psi^p$.
For the converse, suppose $a \Vdash \psi^p$, $a \leq b \Vdash^p \psi_1$. If $b \nVdash p$, use the induction hypothesis. If $b \Vdash p$, then $\psi \in \Phi_b$ since $a \leq b$, $a \nVdash p$ and $a \Vdash \psi^p$. Since $b \Vdash^p \psi$, ψ_1, we have $V_b \vDash \psi$, ψ_1, so $V_b \vDash \psi_2$, and $b \Vdash^p \psi_2$.
Thus $a \Vdash^p \psi$. □

4.7.6 Theorem. For every $\varphi \in \mathbb{I}$, there exists $\psi \in \mathbb{I}[\vee, \to]$ such that $\varphi \equiv_{\mathbf{DLO}} \psi$.

Proof: Take $\chi \in \mathbb{I}[\vee, \to, \bot]$ equivalent to φ, by 4.3.3. Since for $A \in \mathbf{DLO}$, $A \nVdash \chi$ iff $[a)_A \nVdash \chi$ for some $a \in A$ iff , by the lemma, $[a)_A \nVdash \chi^p$ for some $p \in \mathbb{P}$ and $a \in A$ (p not occurring in χ), iff $A \nVdash \chi^p$, $\chi \equiv_{\mathbf{DLO}} \chi^p$; so take $\psi = \chi^p$. □

From 4.6 and 4.4, it is clear that further reductions are not possible on **DLO**.

4.8 Linear orderings

On **LO**, disjunction can be dropped by a 'logical' equivalence:

$$\mathbf{LO} \Vdash (\varphi \vee \psi) \leftrightarrow [(\varphi \to \psi) \to \psi] \wedge [(\psi \to \varphi) \to \varphi].$$

From a given $\mathbb{I}$-formula, one may successively eliminate $\bot$ (by 4.7), $\vee$ (by the above equivalence) and $\wedge$ (by 4.3). Consequently
Theorem: On **LO** , every $\mathbb{I}$-formula is equivalent to a formula of $\mathbb{I}[\to]$.

§5. Modal logic

As was noted in the introduction, modal logic allows a greater freedom of interpretation than intuitionistic logic. In general, a modal frame consists of a set A and a relation R on A — R need not be reflexive or transitive; and the sets in the range of a valuation need not be upwards closed in any sense. However, in comparing modal logic with intuitionistic logic as languages for talking about frames, restrictions on frames must be taken for granted; otherwise intuitionistic formulas could not be interpreted in the usual way. So in this section, as always, frames will be quasi-ordered sets. The restriction on valuations, on the other hand, can be limited to the interpretation of intuitionistic formulas. We shall say of an $\mathbb{I}$-formula that it is valid in a frame A if it is valid under all valuations that are appropriate for intuitionistic logic (just as we have done thus far), and of a modal formula if it holds everywhere in A under all valuations that are allowed for modal logic.

5.1 The language $\mathbb{M}$ of propositional modal logic has the same proposition letters as $\mathbb{I}$, a binary connective $\rightarrow$ (implication), a unary operator $\Box$ (necessity), and a nullary connective $\perp$. Formulas are built as usual. We also denote by $\mathbb{M}$ the set of all $\mathbb{M}$-formulas; ζ, η, θ, ζ_0, ζ_1,... serve as variables over $\mathbb{M}$-formulas. Other connectives are defined from $\rightarrow$ and $\perp$ in the classical manner. The possibility operator $\Diamond$ is defined by $\Diamond\zeta{:=}\neg\Box\neg\zeta$.

5.2 <u>Definition</u>. If A is a frame, and $X\subseteq A$, we let $[X)_A= \cup_{x\in X}[x)_A$ (the **upward closure** of X in A).

For a simple example, we have $[\{x\})=[x)$.

5.3 Let A be a frame. A **modal valuation** (short: $\mathbb{M}$-valuation) on A is a mapping $V{:}\mathbb{P}\rightarrow\mathbb{P}(A)$. A **modal model** (short: $\mathbb{M}$-model) is a pair $\mathfrak{A}=(A,V)$ of a frame A and a modal valuation V on A. A valuation V is extended inductively to a map of $\mathbb{M}$ by

$$V(\perp)=\emptyset;$$
$$V(\zeta\rightarrow\eta)=(A-V(\zeta))\cup V(\eta);$$
$$V(\Box\zeta)=[V(\zeta)).$$

Note that the interpretation of $\rightarrow$ differs from the interpretation for $\mathbb{I}$. There is no $\mathbb{I}$-connective corresponding to this 'classical' implication.

5.4 Parallel to the forcing notation of 1.2.5, we may write $a \in V(\zeta)$ as $(\mathfrak{A},a) \vDash \zeta$; next, the parameters V and a may be abstracted from as in 1.5. Clearly, many notions defined for $\mathbb{I}$ can be carried over or extended to $\mathbb{M}$. Thus, for a modal formula ζ, $\mathbf{Fr}(\zeta)$ is the class of all frames in which ζ is valid.

5.5 As before, we have a notion of equivalence on frames: $\eta \equiv \zeta$ if $\mathbf{Fr}(\eta) = \mathbf{Fr}(\zeta)$. We want to discuss equivalence between $\mathbb{I}$-formulas and $\mathbb{M}$-formulas. $\varphi \equiv \zeta$ is defined as $\mathbf{Fr}(\varphi) = \mathbf{Fr}(\zeta)$. Note that on the modal side, more valuations are taken into account than on the intuitionistic side. Generally, in speaking of equivalence between formulas of different languages, we shall assume each formula is interpreted in the way proper to its language.

Now we shall consider how the expressive power of $\mathbb{M}$ with regard to models and frames compares with that of $\mathbb{I}$.

5.6 In the direction from $\mathbb{I}$ to $\mathbb{M}$, translations have been known since Gödel's paper [1932]. Define $\mathrm{M}: \mathbb{I} \to \mathbb{M}$ by

$$\mathrm{M}(p) = \Box p \text{ ; } \mathrm{M}(\bot) = \bot;$$
$$\mathrm{M}(\varphi \wedge \psi) = \mathrm{M}(\varphi) \wedge \mathrm{M}(\psi); \ \mathrm{M}(\varphi \vee \psi) = \mathrm{M}(\varphi) \vee \mathrm{M}(\psi);$$
$$\mathrm{M}(\varphi \to \psi) = \Box\,(\mathrm{M}(\varphi) \to \mathrm{M}(\psi)).$$

Then if $\varphi \in \mathbb{I}$ and (A,V) is a model, $a \Vdash \varphi$ iff $a \vDash \mathrm{M}(\varphi)$, for all $a \in A$. As to frames: since $\mathbb{I}$-valuations are special $\mathbb{M}$-valuations, it is immediate that $(A,a) \vDash \mathrm{M}(\varphi)$ implies $(A,a) \Vdash \varphi$. For the converse, suppose $(A,V,a) \nvDash \mathrm{M}(\varphi)$, where V is some modal valuation. Define V' by $V'(p) = V(\Box p)$ (for all $p \in \mathbb{P}$); then V' is an $\mathbb{I}$-valuation, and it is easily seen that $(A,V',a) \nVdash \varphi$.

Remark: Modal notation is an alternative to signed formulas: if $\mathfrak{A}$ is a model for $\mathbb{I}$, and a a point in $\mathfrak{A}$, then $(\mathfrak{A},a) \Vdash \mathrm{F}\varphi$ is equivalent to $(\mathfrak{A},a) \vDash \neg \mathrm{M}(\varphi)$.

5.7 In the converse direction, not every $\mathbb{M}$-formula has an equivalent in $\mathbb{I}$.

Example. $A \vDash \Box\Diamond p \to \Diamond\Box p$ iff every point in A has a maximal successor (van Benthem [1984], example 2.2.16; note that we assume frames are transitive). This implies that $\Box\Diamond p \to \Diamond\Box p$, though not universally valid, is valid in all finite frames; something no $\mathbb{I}$-formula can match, by the finite model property (3.13).

We shall not answer the general question which $\mathbb{M}$-formulas are equivalent, on frames, to $\mathbb{I}$-formulas. One should compare the characterization of $\mathbb{I}$-definable classes of frames (in part III) with that of $\mathbb{M}$-definable classes (Goldblatt and Thomason [1974]; cf. van Benthem [1986]). It may not be easy to derive an interesting necessary and sufficient criterion. A candidate is the *finite submodel property*:

$$\text{if } \mathfrak{A}\nvDash\zeta, \text{ then } \mathfrak{B}\nvDash\zeta \text{ for some finite } \mathfrak{B}\subseteq\mathfrak{A}.$$

If ζ is of form $M(\varphi)$, it has this property (Smoryński [1973]; cf. §7).
Instead, we shall give a characterization for equivalence on modal models, and derive from this a sufficient criterion for frames. The M-translation above produces formulas with two notable properties: the proposition letters are *boxed*, i.e. immediately preceded by a $\Box$ symbol; and so are the implications. In fact, we may consider every M-translation to be of form $\Box\zeta$; the equivalences $\Box\eta\wedge\Box\theta\leftrightarrow\Box(\eta\wedge\theta)$, $\Box\eta\vee\Box\theta\leftrightarrow\Box(\Box\eta\vee\Box\theta)$ and $\bot\leftrightarrow\Box\bot$ are easily seen to be universally valid (the latter two because our frames are reflexive and transitive).

Definition: The **degree** $d(\zeta)$ of an $\mathbb{M}$-formula ζ is defined inductively by

(i) $d(p) = 0$ for $p\in\mathbb{P}$; $d(\bot) = 0$;
(ii) $d(\eta\rightarrow\theta)= \max(d(\eta),d(\theta))$;
(iii) $d(\Box\eta) = d(\eta)+1$.

Lemma: Suppose $\eta\in\mathbb{M}$, and every occurrence of a proposition letter in $\Box\eta$ is boxed. Then there exists an $\mathbb{I}$-formula $I(\eta)$ such that $\mathbf{QO}\vDash\Box\eta\leftrightarrow M(I(\eta))$.

Proof: Induction on $d(\eta)$. If $d(\eta)=0$, then η is logically equivalent to $\bot$ or $\top$, or $\eta\in\mathbb{P}$, and we may take $I(\eta)=\eta$.
If $d(\eta)>0$, then η is a Boolean combination of formulas of form $\Box\theta$, and can be written (modulo logical equivalence) as

$$\bigwedge\nolimits_{1\leq i\leq m}(\bigvee(\Box\theta_{ij}|1\leq j\leq l_i)\vee\bigvee(\neg\Box\theta'_{ik}|1\leq k\leq l'_i))$$

Then

$$\vDash\Box\eta\leftrightarrow\bigwedge\nolimits_i\Box\,(\bigvee\nolimits_j\Box\theta_{ij}\vee\bigvee\nolimits_k\neg\Box\theta'_{ik}).$$

and it will suffice to consider the conjuncts of the right hand side. We have

$$\begin{aligned}\Box\,(\bigvee\nolimits_j\Box\theta_{ij}\vee\bigvee\nolimits_k\neg\Box\theta'_{ik}) &\leftrightarrow \Box(\bigwedge\nolimits_k\Box\theta'_{ik}\rightarrow\bigvee\nolimits_j\Box\theta_{ij})\\ &\leftrightarrow \Box\,(\bigwedge\nolimits_k M(I(\theta'_{ik}))\rightarrow\bigvee\nolimits_j M(I(\theta_{ij}))\end{aligned}$$

$$\leftrightarrow \quad \mathrm{M}(\bigwedge_k \mathrm{I}(\theta'_{ik}) \to \bigvee_j \mathrm{I}(\theta_{ij})),$$

using the induction hypothesis on θ'_{ik}, θ_{ij}. Let

$$\mathrm{I}(\eta) = \bigwedge_i(\bigwedge_k \mathrm{I}(\theta'_{ik}) \to \bigvee_j \mathrm{I}(\theta_{ij})). \qquad \square$$

Theorem: Let $\zeta \in \mathbb{M}$. There exists $\varphi \in \mathbb{I}$ such that $\mathbf{QO} \models \zeta \leftrightarrow \mathrm{M}(\varphi)$ iff
(*a*) $\mathbf{QO} \models \zeta \leftrightarrow \Box\zeta$; and
(*b*) if $p_1,...,p_n$ are all the proposition letters in ζ, then
$\mathbf{QO} \models \zeta \leftrightarrow \zeta[p_1 := \Box p_1, ..., p_n := \Box p_n]$.

Proof: ($\Leftarrow$) If an $\mathbb{M}$-formula ζ satisfies (*a*) and (*b*), we have $\mathbf{QO} \models \zeta \leftrightarrow \Box\zeta'$ for an $\mathbb{M}$-formula ζ' in which all occurrences of proposition letters are boxed (box the proposition letters by (*b*), and use that $\eta \leftrightarrow \theta$ implies $\Box\eta \leftrightarrow \Box\theta$). Now apply the lemma.
($\Rightarrow$) (*a*) and (*b*) hold for M-translations of $\mathbb{I}$-formulas: (*a*) by the remarks preceding the definition of degree, and (*b*) since $\Box p$ is equivalent to $\Box\Box p$ in models on quasi-orderings.

Corollary. If all occurrences of proposition letters in $\zeta \in \mathbb{M}$ are boxed, then there exists $\varphi \in \mathbb{I}$ such that $\zeta \equiv \varphi$.

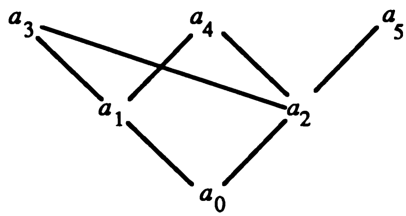
Proof: By the lemma, since $\zeta \equiv \Box\zeta$. $\square$

Note that the equivalence $\zeta \equiv \Box\zeta$ holds for *global* validity only, not for local validity. An $\mathbb{M}$-formula for which the local and global notions do not coincide, cannot have a local $\mathbb{I}$-equivalent (cf. 1.5). We end with an example of such a formula.

5.8 **Example**. Let φ, ψ and χ be as in 3.8; let ζ be

$$\mathrm{M}(\neg\varphi \vee \neg\psi \vee \neg\chi \to \varphi \vee \psi \vee \chi) \to \mathrm{M}(\neg\varphi \vee \neg\psi \vee \neg\chi).$$

Consider the frame $A := \{a_0, ..., a_5\}$ in wich a_0 is covered by a_1 and a_2, a_1 by a_3 and a_4, and a_2 by a_3, a_4 and a_5 (as in the diagram).



Then $(A,a_0)\models\zeta$; but $(A,a_2)\not\models\zeta$, as may be seen by taking $V(p)=\{a_3,a_4\}$, $V(q)=\{a_3,a_5\}$, (cf. 3.8). Since $\mathbb{I}$-formulas valid in a_0 should also be valid in a_2, ζ is not locally equivalent to any $\mathbb{I}$-formula.

II. First order definability

We say of an $\mathbb{I}$-formula φ that it is first order definable, or **elementary**, if there exists an $\mathbb{L}_0$-sentence α (for $\mathbb{L}_0$, see 1.6) such that

$$\text{for all frames } A, \quad A \Vdash \varphi \text{ iff } A \vDash \alpha.$$

In keeping with our use of $\equiv$ above, we shall abbreviate this to $\varphi \equiv \alpha$ (φ is **equivalent** to α). As before, these notions can be relativized to any subclass **K** of the class of all frames. In particular, φ is **elementary on K** if $\varphi \equiv_{\mathbf{K}} \alpha$ (that is: $\forall A \in \mathbf{K}$ ($A \Vdash \varphi$ iff $A \vDash \alpha$)) for some $\mathbb{L}_0$-sentence α. We denote by E(**K**) the class of those $\mathbb{I}$-formulas that are elementary on **K**.
In this second part we study the classes E(**K**). Sections 6 and 7 are mostly about E(**QO**). In §6, E(**QO**) is characterized by preservation properties (theorem 6.7.6); examples are presented of $\mathbb{I}$-formulas that are not elementary. Section 7 describes a method for finding first order equivalents that works for $\mathbb{I}$-formulas in which $\vee$ does not occur in certain positions.
If **K** and **K**' are classes of frames, and **K** is a subclass of **K**', then it is immediate from the definition of relativized equivalence that E(**K**') $\subseteq$ E(**K**). In particular, if E(**K**)=$\mathbb{I}$, then E(**K**') =$\mathbb{I}$ as well. So the property 'E(**K**)=$\mathbb{I}$' is inherited downwards in the hierarchy of classes of frames (see 1.9 for some sample classes). Section 8 explores conditions on frame classes that make E(**K**)=$\mathbb{I}$: first for **K**$\subseteq$**DLO**, establishing a procedure that produces Π^0_2-definitions; next, general restrictions on width and height are considered.
Section 9 investigates E(**TR**), and refines some of the results of §8. Section 10 gives examples of elements and nonelements of E(**FPO**).
Along the way, two sorts of observations are made on the complexity of E(**K**). Algorithmic complexity is calculated in one nontrivial case: for **K**=**TR**, E(**K**) is shown to be recursive in §9. As to quantifier complexity of first order definitions, the most sweeping result is at the end of §7, where it is shown that $\mathbb{I}$-formulas exist whose first order definitions essentially exceed Π^0_2 – and suggested that over all the complexity is unbounded.
The examples of nonelementary $\mathbb{I}$-formulas in §6 have two notable syntactic features. One is that they contain a certain sort of occurrences of $\vee$, which is shown to be necessary in §7. The other is that our example of a formula outside E(**DLO**) contains two distinct proposition letters. This is shown to be necessary in §11. The formulas in one proposition letter (the **monadic** formulas, as we shall call them) are a special case anyway: we possess an exhaustive description of the Lindenbaum algebra of monadic formulas (Rieger [1949]). This will enable us to give an exhaustive classification of the monadic formulas as to first order definability on **PO** and **FPO**.
Sections 8 through 11 make extensive use of theorems of Doets. I will state these without proof,

except when the proof is clearly within the scope of this dissertation. The interested reader is referred to Doets [A],[B].
We end with an overview, and answers to straightforward syntactic questions, in §12.

§6. Elementary $\mathbb{I}$-formulas

At first sight , the properties of frames defined by $\mathbb{I}$-formulas are second order (cf. 1.6). Often, however, these properties are not essentially second order. In this section we give first order definitions for a number of $\mathbb{I}$-formulas that were introduced earlier, together with some new examples. Next we prove a variant of a theorem of van Benthem (cf. his [1984], 2.2.10), characterizing the $\mathbb{I}$-formulas elementary on a given elementary class **K**. We use this in 6.8 to establish that Scott's axiom SC and the 2-stability principle SP_2 are not elementary.
First of all, however, we consider whether our first order language should contain equality. As it happens, the same consideration proves E(**QO**) = E(**PO**).

6.1 **Equality**

Let $\mathbb{L}_0[=]$ be the first order language obtained by expanding $\mathbb{L}_0$ with the equality symbol =. With ~ defined as in 2.3.1, the defining condition of partial orderings, as a class of frames, may be written

$$a{\sim}b \text{ iff } a{=}b.$$

For an $\mathbb{L}_0[=]$-formula α, let $\tilde{\alpha}$ be the $\mathbb{L}_0$-formula obtained from α by replacing every subformula of form $u{=}v$ by $u{\leq}v{\wedge}v{\leq}u$. Then the following clearly holds:

Lemma. If $A\in$ **PO**, and α is an $\mathbb{L}_0[=]$-sentence, then $A\models\alpha$ iff $A\models\tilde{\alpha}$.

In combination with theorem 2.3.3, this supports the view that equality has no part to play in intuitionistic correspondence theory. We say that a class **K** of frames is **closed under contraction** if $\forall A\in$ **K**. $C(A)\in$ **K**.

Theorem: Let **K** be a class of frames, closed under contraction; φ an $\mathbb{I}$-formula, and α an $\mathbb{L}_0[=]$-sentence. Then $\varphi\equiv_{\mathbf{K}}\alpha$ implies $\varphi\equiv_{\mathbf{K}}\tilde{\alpha}$.

Proof: Let **K**, φ and α be as stated, and $\varphi\equiv_{\mathbf{K}}\alpha$. Then for $A\in$ **K**,

$A\Vdash\varphi$ iff $C(A)\Vdash\varphi$, by theorem 2.3.3, and considering φ as an $\mathbb{L}_2$-formula;
iff $C(A)\models\alpha$, since $C(A)\in$ **K**;

iff $C(A)\models\tilde{\alpha}$, since $C(A)\in \mathbf{PO}$;
iff $A\models\tilde{\alpha}$, by theorem 2.3.3. □

Thus, modulo closure under contraction, $\mathbb{L}_0[=]$-definitions of $\mathbb{I}$-formulas can be replaced by $\mathbb{L}_0$-definitions.

6.2 Another corollary of theorem 2.3.3 is

Theorem. $E(\mathbf{PO}) = E(\mathbf{QO})$.

Proof: if $\varphi\in E(\mathbf{PO})$, then $\varphi\equiv_{\mathbf{PO}}\alpha$ for some $\alpha\in\mathbb{L}_0$. Then for $A\in\mathbf{QO}$, $A\Vdash\varphi$ iff $C(A)\Vdash\varphi$ (by 2.3.3) iff $C(A)\Vdash\alpha$ (since $C(A)\in\mathbf{PO}$) iff $A\Vdash\alpha$ (by 2.3.3). □

6.3 Examples: some first order definitions

(a) $p\vee\neg p \equiv\forall xy(x\leq y\rightarrow y\leq x)$. Suppose $A\not\models\forall xy(x\leq y\rightarrow y\leq x)$: say $a,b\in A, a<b$. Then with $V(p):=[b)$, $a\not\Vdash p$ since $b\not\leq a$, and $a\not\Vdash\neg p$ since $a\leq b\Vdash p$. Conversely, if $A\models\forall xy(x\leq y\rightarrow y\leq x)$, then A is atomic; and $A\Vdash p\vee\neg p$ by 2.5.

(b) $P_n\equiv\forall x_0\ldots x_n(\bigwedge_{i<n} x_i\leq x_{i+1}\rightarrow\bigvee_{i<n} x_{i+1}\leq x_i)$, by 2.8.

(c) $W_n\equiv\forall xx_0\ldots x_n(\bigwedge_{i\leq n}x\leq x_i\rightarrow\bigvee_{0\leq i\neq j\leq n}x_i\leq x_j)$, by 2.10.

6.4 Example: stability principles

Let $n\in\mathbb{Z}^+$; take the least number k such that $2^k\geq n+1$. Order $\{0,1\}^{\{0,\ldots,k-1\}}$ lexicographically as $f_0,\ldots,f_{2k-1}$, with 0 preceding 1. Let $(\neg)^0p_i=p_i$, $(\neg)^1p_i=\neg p_i$. Take for $j\leq n$, $\varphi_j:=\bigwedge_{i<k}(\neg)^{f_j(i)}p_i$. The n-stability principle SP_n is the formula

$$(\bigvee_{j\leq n}\neg\varphi_j\rightarrow\bigvee_{j\leq n}\varphi_j)\rightarrow\bigvee_{j\leq n}\neg\varphi_j.$$

We have met SP_2 on several occasions (1.9, 3.8). SP_1 is

$$(\neg p\vee\neg\neg p\rightarrow p\vee\neg p)\rightarrow\neg p\vee\neg\neg p,$$

which is logically equivalent to

$$(\neg\neg p \to p) \to p \vee \neg p.$$

$SP_1 \equiv KC$: for KC is logically equivalent to $SP_1[p := \neg p]$, hence $A \Vdash SP_1$ implies $A \Vdash KC$ by 2.2; and

$$\vdash \neg p \vee \neg\neg p \to (\neg\neg p \to p) \to p \vee \neg p.$$

Thus by 1.6 (cf. 2.6), SP_1 is first order definable. In fact, SP_1 is the only member of the sequence of stability principles that is first order definable. (See 6.8 below for a proof that SP_2 is not elementary; it adapts to $n>2$ in a straightforward manner.)
The meaning of SP_n is easily explained in terms of upwards closed sets: $A \nVdash SP_n$ iff for some $a \in A$, $[a)_A$ has a nonempty subset B such that $[a)-B$ can be partitioned into $n+1$ upwards closed blocks in such a way that every element of B has successors in each block. We shall see that this is a proper second order statement – even on **FPO** (§10).

6.5 Example: the Kreisel-Putnam axiom

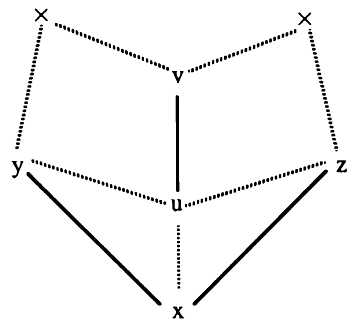
The Kreisel-Putnam axiom (short: KP) is the formula

$$(\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r).$$

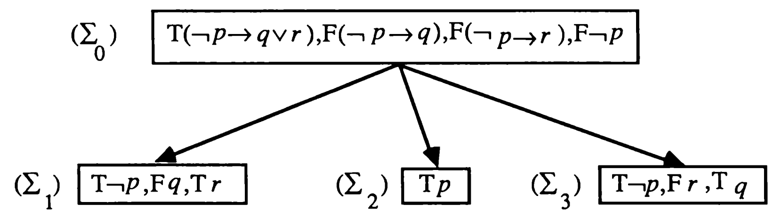
Van Benthem found an $\mathbb{L}_0$-equivalent to KP; it will look better if we use some abbreviations. We write $x \leq y,z$ instead of $x \leq y \wedge x \leq z$ (similarly, $x,y \leq z$, etc.); $\mathrm{Comp}(x,y)$ stands for $x \leq y \vee y \leq x$ (x and y are comparable); we use bounded quantification $\exists x \geq y$ (for $\exists x(x \geq y \wedge \ldots)$) and $\forall x \geq y$ (for $\forall x(x \geq y \to \ldots)$). Now take $\alpha :=$

$$\forall xyz(x \leq y,z \wedge \neg\mathrm{Comp}(y,z) \to \exists u \geq x[u \leq y,z \wedge \forall v \geq u \exists t \geq v(y \leq t \vee z \leq t)].$$

A diagram may help to clarify the meaning of $\neg\alpha$. Below, continuous lines correspond with existential quantification, broken lines with universal quantification combined with implication, and crosses represent types of points that are forbidden. ($\neg\alpha$ is written out in (*) below.)



KP has just one minimal refutation; it may be represented as follows (cf. 3.4, 3.7):



We claim that $\alpha\equiv$KP.

If $(A,V)\not\Vdash$KP, then there must be $x,y,z\in A$ with $x\le y,z$, and $x\Vdash\Sigma_0$, $y\Vdash\Sigma_1$, and $z\Vdash\Sigma_3$ (for Σ_i, see the diagram above). Then whenever $x\le u\le y,z$, $u\Vdash \mathrm{F}\neg p$, since $u\Vdash\neg p\to q\vee r$ and $u\not\Vdash q\vee r$. So all such u have a successor v with $v\Vdash\Sigma_2$. Since $\Sigma_1^{\mathrm{T}}\cup\Sigma_2^{\mathrm{T}}$ and $\Sigma_3^{\mathrm{T}}\cup\Sigma_2^{\mathrm{T}}$ are not realizable[1], neither v and y nor v and z have a successor in common. Thus

(*) $A\models\exists xyz(x\le y,z\wedge\neg\mathrm{Comp}(y,z)\wedge\forall u\ge x[u\le y,z\to\exists v\ge u\forall t\ge v(\neg y\le t\wedge\neg z\le t)])$,

that is, $A\models\neg\alpha$.

Conversely, if $A\models\neg\alpha$, pick suitable x,y,and z, and choose for every u such that $x\le u\le y,z$ a successor v_u such that $A\models\forall t\ge v_u(\neg y\le t\wedge\neg z\le t)$ (we apparently need the axiom of choice). Now define:

$$V(p)=\cup([v_u)|x\le u\le y,z)\ ;\ V(q)=A-(y]\ ;\ V(r)=A-(z].$$

It is straightforward to check that $(A,V,x)\not\Vdash$KP.

6.6 **Terminology**. We call a class **K** of structures for $\mathbb{L}_0[=]$ **elementary** if there exists a sentence α of $\mathbb{L}_0[=]$ such that any $\mathbb{L}_0[=]$-structure (A,R) belongs to **K** if and only if $(A,R)\models\alpha$. (Such **K** are called *basic* elementary in Chang& Keisler [1973].)
Obviously, the intersection of two elementary classes is again elementary. **QO** is elementary; hence, if $\mathbf{K}\subseteq\mathbf{QO}$ is such that a frame A belongs to **K** iff $A\models\alpha$, **K** is elementary. Therefore, $\varphi\in \mathrm{E}(\mathbf{QO})$ iff $\mathbf{Fr}(\varphi)$ is elementary.

6.7 A characterization of E(K) for elementary classes K

Let $(\mathfrak{A}_i|i\in I)$ be a family of structures for some first order language, and U an ultrafilter over I. We shall denote by f_U the equivalence class of f under the relation $\sim_U$ induced by U in $\prod_{i\in I}\mathfrak{A}_i$, and by $\prod_U\mathfrak{A}_i$ the reduced product of $(\mathfrak{A}_i)_{i\in I}$ modulo U(the ultraproduct over U). If $\mathfrak{A}_i$ is the same structure $\mathfrak{A}$ for all $i\in I$, $\prod_U\mathfrak{A}_i$ is called an ultrapower of $\mathfrak{A}$, and may be written $\prod_U\mathfrak{A}$.
We shall state a few well-known facts without proof; proofs are in Chang & Keisler. The important property of ultraproducts is expressed in

6.7.1 **Łoś's theorem**: For any formula α of the first order language appropriate to $(\mathfrak{A}_i|i\in I)$, and for any $f_U^{(1)},\ldots,f_U^{(n)}$ in the domain of $\prod_U\mathfrak{A}_i$,

$$\prod_U\mathfrak{A}_i\models\alpha[f_U^{(1)},\ldots,f_U^{(n)}] \text{ iff } \{i\in I|\mathfrak{A}_i\models\alpha[f^{(1)}(i)\ldots f^{(n)}(i)]\}\in U.$$

Since $I\in U$, first order sentences true in every $\mathfrak{A}_i$ are true in $\prod_U\mathfrak{A}_i$. Hence ultraproducts of frames are frames ($\forall x.x\leq x$ and $\forall xyz(x\leq y\wedge y\leq z\rightarrow x\leq z)$ are preserved); of models, models ($V(p)\in \mathbb{U}(A)$ is expressed by $\forall xy(px\wedge x\leq y\rightarrow py)$); and any structure $\mathfrak{A}$ is elementarily equivalent to its ultrapowers $\prod_U\mathfrak{A}$.

6.7.2 Lemma (Goldblatt). let $(A_i)_{i\in I}$ be a family of frames, U an ultrafilter over I. Then $\prod_U A_i$ is isomorphic to a generated subframe of the ultrapower $\prod_U\Sigma_{i\in I}A_i$.

Proof: Map f_U to $((i,f(i))_{i\in I})_U$. □

6.7.3 Corollary. Any class of frames closed under generated subframes, disjoint unions, isomorphic images and ultrapowers is closed under ultraproducts.

For $\Sigma^1{}_1$-formulas , one half of Łoś's theorem remains:

6.7.4 Lemma. Suppose α is a $\Sigma^1{}_1$-formula; $(\mathfrak{A}_i|i\in I)$ is a family of structures, and U an ultrafilter over I. Then

$$\{i\in I|\mathfrak{A}_i\models\alpha[f^{(1)}(i)\ldots f^{(n)}(i)]\}\in U \text{ implies } \Pi_U\mathfrak{A}_i\models\alpha[f_U^{(1)},\ldots,f_U^{(n)}]\,.$$

6.7.5 **Keisler's theorem**: Let **K** be a class of structures for some first order language. Then **K** is elementary iff both **K** and its complement are closed under ultraproducts and isomorphic images.

6.7.6 Theorem. Suppose **K** is an elementary class of frames, closed under disjoint unions and generated subframes; and $\varphi\in\mathbb{I}$. Then $\varphi\in E(\mathbf{K})$ iff $\mathbf{Fr}_{\mathbf{K}}(\varphi)$ is closed under elementary equivalence iff $\mathbf{Fr}_{\mathbf{K}}(\varphi)$ is closed under ultrapowers.

Proof: Let β be a sentence of $\mathbb{L}_0[=]$ such that for $\mathbb{L}_0$-structures A, $A\in\mathbf{K}$ iff $A\models\beta$. (So in particular, $\models\beta\rightarrow\forall x\, x\leq x \wedge\forall xyz(x\leq y\wedge y\leq z\rightarrow x\leq z)$.)
The implications from left to right are obvious (cf. 6.7.1). To close the circle, assume $\mathbf{Fr}_{\mathbf{K}}(\varphi)$ is closed under ultrapowers. Validity of φ is preserved under disjoint unions, generated subframes and isomorphic images (2.2.4); **K** is closed under these operations; hence so is $\mathbf{Fr}_{\mathbf{K}}(\varphi)$ .By 6.7.3, $\mathbf{Fr}_{\mathbf{K}}(\varphi)$ is closed under ultraproducts. Let $p_1,\ldots,p_n$ be all the proposition letters in φ. The complement of $\mathbf{Fr}_{\mathbf{K}}(\varphi)$ is defined by the $\Sigma^1{}_1$-sentence

$$\exists p_1\ldots p_n([\bigwedge_{1\leq i\leq n}\forall xy(x\leq y\wedge p_ix\rightarrow p_iy)\wedge\exists x\neg\mathrm{St}(\varphi)]\vee\neg\beta)$$

(cf. 1.6); it is closed under isomorphism and, by 6.7.4, under ultraproducts. So $\mathbf{Fr}_{\mathbf{K}}(\varphi)$ is elementary by Keisler's theorem, that is, $\varphi\in E(\mathbf{K})$.

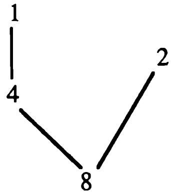
6.7.7 **Remark**. The fact that φ is not just an $\mathbb{M}$-formula, and that frames are quasi-orderings, is of no advantage in this proof. Thus, the theorem easily generalizes to modal logic; it then becomes a simple generalization of a theorem of van Benthem ([1984] 2.2.10).

6.8 Examples. We use theorem 6.7.6 to establish that some $\mathbb{I}$-formulas are not elementary.

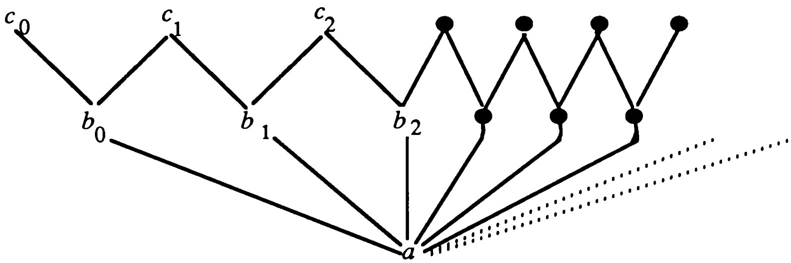
(*a*) **Scott's axiom**. This is the formula

$$\mathrm{SC} := [(\neg\neg p\rightarrow p)\rightarrow p\vee\neg p]\rightarrow\neg p\vee\neg\neg p$$

There is a diagram of its minimal refutation in 3.4 (*c*). Here we reproduce the induced frame:

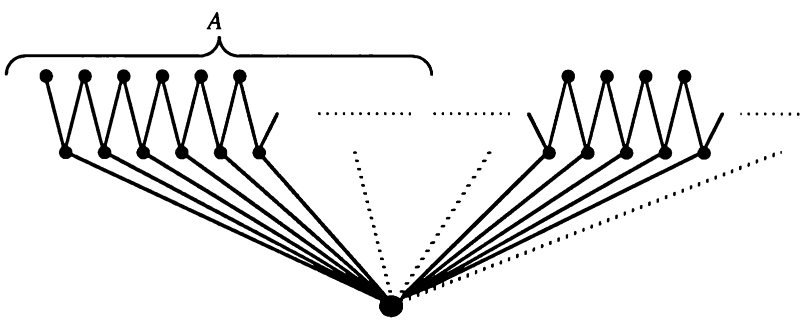


(The points are named in accordance with §11; we shall refer to this frame as $[8)_M$.)
Let $A:=\{a\}\cup\{b_n|n\in\mathbb{N}\}\cup\{c_n|n\in\mathbb{N}\}$, ordered thus : a is the root; b_n, $n\in\mathbb{N}$, are the covers of a; b_n is covered by c_n, c_{n+1}.



Claim: $A \Vdash$SC. Proof: let V be any valuation on A. If every c_n forces p, or every c_n forces $\neg p$, clearly $(A,V) \Vdash$SC. Otherwise, since in each endpoint either p or $\neg p$ is forced, there must be a pair c_n, c_{n+1} such that one forces p, the other $\neg p$. Then $b_n \Vdash \neg\neg p \to p$; hence neither b_n nor a forces $(\neg\neg p \to p) \to p \vee \neg p$. Since $b_k \Vdash$SC for all k, $(A,V) \Vdash$SC.

Now take a nonprincipal ultrafilter U over $\mathbb{N}$; consider the ultrapower $\Pi_U A$. By Los's theorem, $\Pi_U A$ is rooted, has height 3, every cover of the root has exactly two covers – in fact, we may picture $\Pi_U A$ as consisting of A with certain extra points, placed as the b_n's and c_n's in A, lying far to the right.



For example, since singletons do not belong to U, $(c_n|n\in\mathbb{N})_U$ is not identical with any $(c_k|n\in\mathbb{N})_U$ (for fixed k).

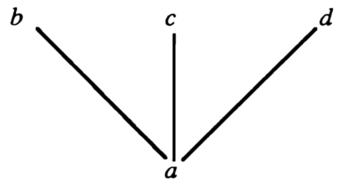
Claim: $\prod_U A \not\Vdash SC$. Proof: define f: $\prod_U A \to [8)_M$ by $f(c_n)=1$, $f(b_n)=4$, $f(a)=8$, and $f(x)=2$ for all points x outside (the copy of) A. f is easily seen to be a surjective p-morphism. Since $[8)_M \not\Vdash SC$, $\prod_U A \not\Vdash SC$ by lemma 2.4.2.

This shows that SC is not preserved under ultrapowers; so by theorem 6.7.6, SC$\notin$E(**PO**).

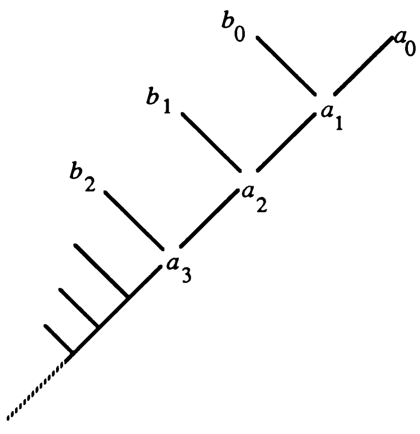
(*b*)**The 2-stability principle**. As before (3.8), we write

$$SP_2 = (\neg\varphi\vee\neg\psi\vee\neg\chi\to\varphi\vee\psi\vee\chi)\to\neg\varphi\vee\neg\psi\vee\neg\chi,$$

with $\varphi=p\wedge q$, $\psi=p\wedge\neg q$, $\chi=\neg p\wedge q$. We reproduce the induced frame B of its minimal refutation:



Let A be a downwards linear frame $\{a_n, b_n | n\in\mathbb{N}\}$ in which a_{n+1} is covered by a_n and b_n, and a_0 and all points b_n are endpoints.

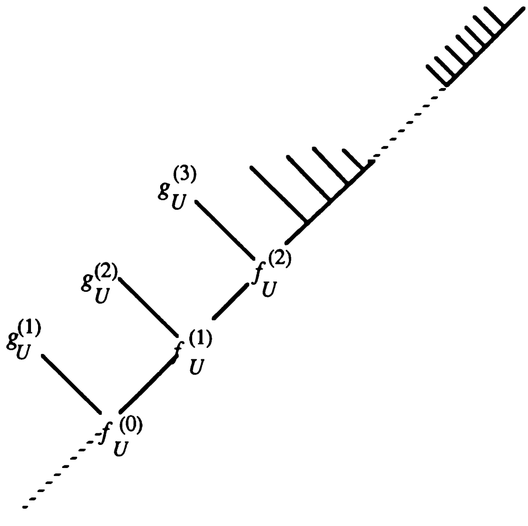


We claim that $A\Vdash SP_2$. Proof: take any valuation V on A. Since φ,ψ and χ are mutually exclusive, $b_n\Vdash\neg\varphi\vee\neg\psi\vee\neg\chi$ for every n, hence $b_n\Vdash SP_2$. Suppose $a_k\Vdash\neg\varphi\vee\neg\psi\vee\neg\chi\to\varphi\vee\psi\vee\chi$. Since a_1 has only two strict successors, $a_1\Vdash\neg\varphi\vee\neg\psi\vee\neg\chi$; so if $k\geq 1$, $a_1\Vdash\varphi\vee\psi\vee\chi$. Consequently, $a_2\Vdash\neg\varphi\vee\neg\psi\vee\neg\chi$. Again, if $k\geq 2$, this gives $a_2\Vdash\varphi\vee\psi\vee\chi$. Continuing in this way, we get $a_k\Vdash\neg\varphi\vee\neg\psi\vee\neg\chi$ .We conclude that $(A,V)\Vdash SP_2$.

Now take a nonprincipal ultrafilter U over $\mathbb{N}$, and consider $\prod_U A$. By Łoś's theorem, every point of $\prod_U A$ either is an endpoint, or has exactly two covers, one of which is an endpoint. Let $f^{(n)}$, $g^{(n)} \in A^I$ be defined as follows:

$$f^{(n)}(k)=a_{k \dot{-} n} \quad , \quad g^{(n)}(k)=b_{k \dot{-} n}$$

(Here $\dot{-}$ stands for cut-off subtraction: $k \dot{-} n$ equals 0 if $k<n$, $k-n$ otherwise.) So $f^{(n+1)}(k)$ and $g^{(n+1)}(k)$ cover $f^{(n)}(k)$ for all $k>n$.
Since cofinite sets belong to U, $f^{(n+1)}{}_U$ and $g^{(n+1)}{}_U$ cover $f^{(n)}{}_U$ by Łoś's theorem. Then $\prod_U A$ may be pictured as follows:



Claim: $\prod_U A \nVdash SP_2$. Proof: define h: $\prod_U A \twoheadrightarrow B$ by

$h(f_U)=a$ if $\exists n \in \mathbb{N}.\ f_U \leq f^{(n)}{}_U$;

$h(f_U)=b$ if $\exists n \in \mathbb{N}.\ f_U = g^{(3n)}{}_U$;

$h(f_U)=c$ if $\exists n \in \mathbb{N}.\ f_U = g^{(3n+1)}{}_U$;

$h(f_U)=d$ if $\exists n \in \mathbb{N}.\ f_U = g^{(3n+2)}{}_U$ or $\forall n \in \mathbb{N}.\ f^{(n)}{}_U \leq f_U$.

h is a p-morphism, so as in (a) it follows that $\prod_U A \nVdash SP_2$.

Since **DLO** is elementary, $SP_2 \notin$ E(**DLO**) by theorem 6.7.6.

6.9 **Remark.** We employed the ultrapower criterion because it gives rather perspicuous proofs . On inspection, however, one will see that, in either case, only a few first order properties and a countably repeating simple pattern are used. Both times we could have relied on the compactness theorem to produce a countable frame in which the $\mathbb{I}$-formula under consideration is lost. (Rodenburg [1982] in fact did this.)
From a proof that compactness always works, in proving $\mathbb{I}$-formulas nonelementary, and that countable frames suffice, one might hope to get estimates of the algorithmic complexity of E(**K**) for elementary **K**[2].

Footnotes:

[1] A sequent Σ is **realizable** if there exists a model $\mathfrak{A}$ and a point a of $\mathfrak{A}$ such that $(\mathfrak{A},a)\Vdash\Sigma$.

[2] Doets has found a modal formula that is elementary on the countable $\mathbb{M}$-frames, but not on all $\mathbb{M}$-frames. So if countable frames suffice for first order definability of $\mathbb{I}$-formulas, this would be an interesting difference between the modal case and the intuitionistic case.

§7. Labeled frames

According to 3.12, $A\nVdash\varphi$ iff A can be projected onto a multirefutation of φ; we have seen in the preceding section that the existence of such a projection need not correspond with a first order condition on frames. In this section we return to the idea of embedding semantic tableaux into frames, that was briefly considered in 3.8. We define labeled frames; and consider, essentially, embeddings of such frames. The existence of such an embedding is first order expressible; moreover, for a large class of $\mathbb{I}$-formulas, it guarantees refutability. The section ends with a discussion of the quantifier complexity of first order definitions. Among other things, it is shown that not all first order definitions can be found by means of labeled frames.

7.1 Partial projections

7.1.1 **Notation**. If f is a partial function from U to V, we write $f: U \dashrightarrow V$. The **domain** of f, abbreviated dom f, is the subset of U on which f is defined; the **range** of f, abbreviated ran$f$, is f[dom$f$]. If ran$f=V$, we write $f:U \dashrightarrow\!\!\!\!\twoheadrightarrow V$. If dom$f \subseteq$dom$f'$, and for all $u\in$ dom$f$, $f(u)=f'(u)$, we write $f\subseteq f'$.

7.1.2 Definition. Let A be a frame; $\mathfrak{X}=(X,\mathfrak{S})$ a multitableau. A partial surjection g:$A \dashrightarrow\!\!\!\!\twoheadrightarrow X$ is a **partial projection** of A onto $\mathfrak{X}$ if g is a projection of dom$g$, as a subframe of A, onto $\mathfrak{X}$.

We say g is a **Σ-projection** if g is a partial projection onto a Σ-multitableau. If g and g' are Σ-projections and $g\subseteq g'$, we shall say that g is a **Σ-subprojection** of g'.

The proof of the following lemma is basically Smoryński's proof for the finite submodel property.

7.1.3 Lemma. Let Σ be a sequent. Every Σ-projection has a finite Σ-subprojection.

Proof: Let g:$A \dashrightarrow\!\!\!\!\twoheadrightarrow \mathfrak{X}$ be a Σ-projection; $\mathfrak{X}=(X,\mathfrak{S})$. We define a sequence

$$B_0 \subseteq B_1 \subseteq .....$$

of finite subsets of dom$g$, as follows.
Pick $b_0 \in A$ such that $\Sigma\subseteq g(b_0)$; set $B_0=\{b_0\}$.
Suppose B_n has been defined. For each $b\in B_n-B_{n-1}$, take for every $x>_{\mathfrak{X}}g(b)$ that belongs to $g[b)$ an element $a_{x,b}$ in $[b)$ with $g(a_{x,b})=x$. Let

$$B_{n+1}=B_n\cup\{a_{x,b}\mid b\in B_n-B_{n-1},\, x>_{\mathfrak{X}}g(b) \text{ and } x\in g[b)\}.$$

Since chains in $(X,\leq_{\mathfrak{X}})$ are finite, there is a greatest N such that $B_N\neq B_{N-1}$. Set $g_0=g\lceil B_N$; i.e $\mathrm{dom}g_0=B_N$, and for all $b\in B_N$, $g_0(b)=g(b)$. Let $X'=\mathrm{ran}g_0$. For $x\in X'$, $S'\subseteq X'$, let

$$S'\in \mathfrak{S}'(x) \text{ iff } \exists b\in B_N : x=g(b) \text{ and } S'=g_0[b).$$

It is clear that g_0 is finite, and $g_0\subseteq g$. $\Sigma\subseteq g(b_0)\in X'$; so to prove that g_0 is a Σ-projection, it remains to show that (1) $\mathfrak{X}':=(X',\mathfrak{S}')$ is a multitableau, and (2) g_0 a partial projection.
(1) If $y\in\cup\mathfrak{S}'(x)$, then by the definition of $\mathfrak{S}'$, there are $a,b\in B_N$, with $a\leq b$ and $x=g(a)$, $y=g(b)$. Then $x\leq_{\mathfrak{X}}y$, hence $x^{\mathrm{T}}\subseteq y^{\mathrm{T}}$, and the first condition of definition 3.9.1 is satisfied. As to the second, suppose $\mathrm{F}(\varphi\rightarrow\psi)\in x\in X'$ and $\mathrm{T}\varphi\notin x$. Take any $S'\in\mathfrak{S}'(x)$; say $S'=g_0[b)$, with $x=g_0(b)$. By 3.10(ii), $\mathfrak{S}(x)$ contains some $S\subseteq g[b)$. By 3.9.1(ii), S has an element y such that $\mathrm{T}\varphi,\mathrm{F}\psi\in y$ or $\mathrm{F}(\varphi\rightarrow\psi)\in y$ and $y^{\mathrm{T}}\neq x^{\mathrm{T}}$, so $y>_{\mathfrak{X}}x$. Then $y=g_0(a_{y,b})\in S'$.
(2) g_0 trivially satisfies the conditions of definition 3.10, by the definition of $\mathfrak{S}'$ (cf. (*b*) in the proof of 3.12).

7.2 **Labeled frames.**

7.2.1 Definition. Let A be a frame, and $\mathfrak{X}$ an open Σ-multitableau. A Σ-projection $g{:}A\dashrightarrow\mathfrak{X}$ is a **Σ-labeled subframe** of A if whenever a set $B\subseteq\mathrm{dom}g$ has an upper bound in A, $\cup_{b\in B}g(b)^{\mathrm{T}}$ is realizable.

7.2.2 Definition. If g and h are Σ-labeled subframes of a given frame A, we write $g\leq_\Sigma h$ if

(i) $\mathrm{dom}g\subseteq\mathrm{dom}h$, and
(ii) $\forall a\in\mathrm{dom}g.\ g(a)\subseteq h(a)$.

Observe that $\leq_\Sigma$ is a quasi-ordering.
It is immediate that a Σ-subprojection of a Σ-labeled subframe is again a Σ-labeled subframe. Since a finite Σ-labeled subframe has only finitely many predecessors in $\leq_\Sigma$, lemma 7.1.3 implies that the class of Σ-labeled subframes has minimal elements. These have some convenient properties.

7.2.3 Lemma. Suppose $g{:}A\dashrightarrow\mathfrak{X}$ is a minimal Σ-labeled subframe of A. Then

(i) $\mathrm{dom}g$ is a rooted subframe of A; the root is the only point $a\in\mathrm{dom}g$ such that $\Sigma\subseteq g(a)$;
(ii) if $g(a_0)=g(a_1)$ and $(a_0]_{\mathrm{dom}g}=(a_1]_{\mathrm{dom}g}$, then $a_0=a_1$;

(iii) if $a_0 < a_1$ in dom$g$, then $g(a_0) \neq g(a_1)$.

Proof: If g does not satisfy (i)-(iii), we shall get a proper Σ-subprojection of g by discarding parts of dom$g$, making straightforward modifications in $\mathfrak{X}$ – as follows. If (i) is not satisfied, seek for $a \in$ dom$g$ with $\Sigma \subseteq g(a)$ and for all $a' >_{\mathrm{dom}g} a$, $\Sigma \not\subseteq g(a')$; restrict g to $[a)_{\mathrm{dom}g}$. If a_0, a_1 violate (ii) or (iii), drop a_0 from dom$g$. □

7.2.4 Definition. We call Σ-projections g and g' **equivalent** if there exists a frame-isomorphism f:dom$g \cong$domg' such that $\forall a \in$ dom$g$:$g'f(a) = g(a)$.

Given an open Σ-multitableau $\mathfrak{X}$, the lemma above restricts the construction of representatives of the equivalence classes of minimal Σ-labeled subframes onto $\mathfrak{X}$ in such a way that it is clear that there are only finitely many such equivalence classes. Let us assume unique representatives of these equivalence classes, and call them **Σ-labeled frames**. Lemma 7.2.3 also implies that Σ-labeled frames are finite. In sum, we have:

7.2.5 Proposition. Let Σ be a sequent. The Σ-labeled frames are finite, and finite in number.

A proper notion of subtableau for open multitableaux should imply that the minimal Σ-multitableaux are the ranges of Σ-labeled frames.

7.3 Transparency

By theorem 3.12, if an $\mathbb{I}$-formula φ is not valid in a frame A, there exists a projection g of A onto an open $\{F\varphi\}$-multitableau. By 7.1.3, g has a finite $F\varphi$-subprojection h (we drop the curly brackets). Since $h \subseteq g$, and g is a total projection, h is an $F\varphi$-labeled subframe of A. Now h has finitely many predecessors in $\leq_{F\varphi}$, so we conclude

Lemma. Let A be a frame, $\varphi \in \mathbb{I}$. If $A \not\Vdash \varphi$, then A has a minimal $F\varphi$-labeled subframe.

The converse is not generally true, as is shown by 3.8: the frame C in that example has an obvious $F(SP_2)$-labeled subframe, yet $C \Vdash SP_2$.

Definition. An $\mathbb{I}$-formula φ is **transparent** if φ is refutable in every frame that has an $F\varphi$-labeled subframe.

Suppose A is a frame. We denote by $\mathbb{L}_0[A]$ the expansion of $\mathbb{L}_0$ by distinct, unique individual constants for all elements of A. For elements of A and corresponding constants we shall use the same letter. A_A is the expansion of A, obtained by interpreting constants a of $\mathbb{L}_0[A]$ by the corresponding point $a \in A$. The **diagram** of A is the set

$$\{\alpha | \alpha \text{ is an atomic sentence of } \mathbb{L}_0[A] \text{ and } A_A \models \alpha\} \cup$$

$$\{\neg\beta | \beta \text{ is an atomic sentence of } \mathbb{L}_0[A] \text{ and } A_A \not\models \beta\}.$$

Theorem. Transparent formulas are first order definable.

Proof: Suppose φ is transparent. Let $g_1,...,g_n$ be the Fφ-labeled frames ($n \geq 0$). For $1 \leq i \leq n$, let δ_i be the conjunction of the diagram of domg_i. (By 7.2.5, the diagram of domg_i is finite.) Let $\varepsilon_i :=$

$$\delta_i \wedge \bigwedge (\neg \exists u \bigwedge_{b \in B} b \leq u \mid B \subseteq \mathrm{dom} g_i \text{ and } \cup_{b \in B} g(b)^{\mathrm{T}} \text{ is not realizable}).$$

Next, let ε'_i be the result of replacing the individual constants in ε_i by distinct new individual variables. If $v_1,...,v_m$ are all the free variables in $\bigvee_{1 \leq i \leq n} \varepsilon'_i$, let α be

$$\neg \exists v_1 ... v_m \bigvee_{1 \leq i \leq n} \varepsilon'_i.$$

We will show that $\varphi \equiv \alpha$.
If $A \not\models \alpha$, then for some i and $a_1,...,a_m \in A$,

$$(A, a_1,...,a_m) \models \varepsilon_i.$$

Let a_j ($1 \leq j \leq m$) be the interpretation of the constant b_j of $\mathbb{L}_0[\mathrm{dom}\, g_i]$. Define a partial function h of A by $h(a_j)=g(b_j)$. Then h is an Fφ-projection, because δ_i makes dom $h \cong$ dom g_i; and by the clauses $\neg\exists u \bigwedge b \leq u$ in ε_i, h is an Fφ-labeled subframe of A. Since φ is transparent, it follows that $A \not\Vdash \varphi$.
Conversely, if $A \not\Vdash \varphi$, then by the lemma, A has a minimal Fφ-labeled subframe h. Say h is equivalent to g_i by an isomorphism f: dom$h \cong$ domg_i. Let dom$h=\{a_1,...,a_m\}$; then $(A,a_1,...,a_m) \models \delta_i$, with a_j interpreting $f(a_j)$ ($1 \leq j \leq m$). The other conjuncts of ε_i hold in $(A,a_1,...,a_m)$ by definition 7.2.1. Thus $(A,a_1,...,a_m) \models \varepsilon_i$; hence $A \models \exists v_1 ... v_m \bigvee_{1 \leq i \leq n} \varepsilon_i$, that is, $A \models \neg\alpha$. □

Corollary. Transparent formulas are equivalent to Π^0_2-sentences.[1]

Proof: The sentence α in the above proof is easily seen to be logically equivalent to a Π^0_2-sentence. □

7.4 Deterministic formulas

Transparency is an abstract notion; we should like to know a property of $\mathbb{I}$-formulas that implies transparency, and that can be *seen* to hold. To find such a property, we consider the question how an $F\varphi$-labeled frame may help us to define a valuation that refutes φ.

An $F\varphi$-labeled subframe g constitutes a finite grid, the points of which are associated, in a regular fashion, with sequents in a multirefutation. If seems reasonable to expect that a valuation refuting φ is arrived at by associating sequents to the points outside the grid, and then extracting a valuation as in 3.12. Let us try to form such new sequents.

The sequent Σ of a point a outside dom$g$ should contain all signed formulas $F\psi$ that are associated to successors of a that belong to dom$g$ (in the resulting model, ψ is to be false in a), and all signed formulas $T\psi$ that belong to predecessors of a in dom$g$. Besides, Σ must be full. Most of the conditions for fullness are met automatically; e.g. if $F(\varphi_1 \vee \varphi_2)$ belongs to some successor, then $F\varphi_1$ and $F\varphi_2$ will turn up as well.

In fact, there is only one source of difficulty: the implications signed T. If $T(\psi \rightarrow \chi) \in \Sigma$, it may be that no predecessor of a has $T\chi$, and no successor $F\psi$. Apparently, we must add to Σ: either $F\psi$, or $T\chi$. But here a major snag appears. Maybe one predecessor of a has brought up $T(\psi \rightarrow \chi)$, and another $T\psi$. Then we must take $T\chi$, or Σ will certainly not be realizable. Now suppose $\chi = \chi_1 \vee \chi_2$: for fullness we must add $T\chi_1$ or $T\chi_2$ to Σ, and we cannot expect guidance from a's predecessors. Worse yet, a's successors in dom$g$ may make either addition impossible. They all carry $T\chi$; but some may opt for $T\chi_1$, and others for $T\chi_2$. In such a case, either choice in a will destroy the pattern.

The following definition singles out a class of $\mathbb{I}$-formulas that cannot lead to awkward choices.

Definition. (i) An implication $\psi \rightarrow \chi \in \mathbb{I}$ is **determinate** if $Sf(T\chi)$ contains no disjunctions signed T.
(ii) An $\mathbb{I}$-formula φ is **deterministic** if in any signed subformula $T(\psi \rightarrow \chi)$ of $F\varphi$, $\psi \rightarrow \chi$ is determinate.

Determinism is a syntactic property, that can be effectively checked for. Note by way of example that **KC** is deterministic; and that SC, SP_2 and KP are not.[2]

Lemma. Let A be a frame, and Σ a sequent. If g is a Σ-labeled subframe of A, then there exists a Σ-labeled subframe $h \supseteq g$ of A such that $\forall a \in A \exists a' \geq a. a' \in \mathrm{dom}h$.

Proof: Let $\mathfrak{X} = (X, \mathfrak{S})$ be an open Σ-multitableau, and $g: A \dashrightarrow \mathfrak{X}$ a Σ-labeled subframe of A. Abbreviate $g[a] = g[a']$ to $a \approx a'$. The relation $\approx$ is an equivalence. We shall denote the equivalence class of a by $a^{\approx}$. Let

$$A_0 := \{a \in A \mid \forall a' \geq a.\ a' \approx a\}.$$

Since chains in X are finite, every element of A has successors in A_0.
By definition 7.2.1, $\cup(g(a')^{\mathrm{T}}|a'\in(a]\cap\mathrm{dom}g)$ is realizable, for every $a\in A$. Thus $\nvdash\neg\wedge\cup(g(a')_{\mathrm{T}}|a'\in(a]\cap\mathrm{dom}g)$; use corollary 4.7.4 to obtain for each $a^{\approx}$ a classical model $V_{a^{\approx}}$ of $\cup(g(a')_{\mathrm{T}}|a'\in(a]\cap\mathrm{dom}g)$. (It is important to do this by equivalence class, and not pointwise.) For $a\in A_0-\mathrm{dom}g$, define

$$h(a)=\{\mathrm{T}\psi\in\mathrm{Sf}(\Sigma)|V_{a^{\approx}}\vDash\psi\}\cup\{\mathrm{F}\chi\in\mathrm{Sf}(\Sigma)|V_{a^{\approx}}\vDash\neg\chi\};$$

then for $a\in\mathrm{dom}g$, $h(a)=g(a)$.
Each $h(a)$ is a full sequent. Define $\mathfrak{T}$: $\mathrm{ran}h\to\mathbb{PP}(\mathrm{ran}h)-\{\emptyset\}$ by

$$T\in\mathfrak{T}(x) \text{ iff } \exists a\in\mathrm{dom}h[h(a)=x \text{ and } T=h[a)].$$

Let $\mathfrak{P}=(\mathrm{ran}h,\mathfrak{T})$. Observe that if $a\in\mathrm{dom}h-\mathrm{dom}g$, then $\mathfrak{T}(h(a))=\{\{h(a)\}\}$.
Claim: $\mathfrak{P}$ is a Σ-multitableau. Proof:
(i) If $h(a)\in T\in\mathfrak{T}(x)$, then $\exists b\leq a{:}x=h(b)$. Then either $h(a)=x$; or $a\in\mathrm{dom}h-\mathrm{dom}g$ and $b\in\mathrm{dom}g$, and $h(a)^{\mathrm{T}}\supseteq h(b)^{\mathrm{T}}$ by definition of h and $V_{a^{\approx}}$; or $h(a)=g(a)$, $h(b)=g(b)$, $g(a)\geq_{\mathfrak{X}} g(b)$ by 3.10, and $g(a)^{\mathrm{T}}\supseteq g(b)^{\mathrm{T}}$ since $\mathfrak{X}$ is a multitableau.
(ii) If $\mathrm{F}(\psi\to\chi)\in x$, then either $x=h(a)$ for some $a\in\mathrm{dom}h-\mathrm{dom}g$, in which case $\mathrm{T}\psi\in x$ since $V_{a^{\approx}}$ is classical; or $x\in X$. In the latter case, if $\mathrm{T}\psi\notin x$, each $S\in\mathfrak{S}(x)$ contains some y with $\mathrm{T}\psi$, $\mathrm{F}\chi\in y$ or $\mathrm{F}(\psi\to\chi)\in y$. Now if $T\in\mathfrak{T}(x)$, $T=h[a)\supseteq g[a)$ for some $a\in g^{-1}\{x\}$. By definition 3.10, some $S\in\mathfrak{S}(x)$ is contained in $g[a)$; so $\exists y\in T($ $\mathrm{F}(\psi\to\chi)\in y$ or $\mathrm{T}\psi$, $\mathrm{F}\chi\in y)$.
Since $\mathrm{ran}h$ consists of sequents from $\mathfrak{X}$ and sequents determined by classical models, $\mathfrak{P}$ is open.
Trivially, h is a partial projection (cf. (b) in the proof of 3.12). To see that h is a Σ-labeled subframe of A, note that if $a\geq a_0,\dots,a_n\in\mathrm{dom}h$, and $a_1\notin\mathrm{dom}g$, then $\cup_{j\leq n}h(a_j)^{\mathrm{T}}=h(a_i)^{\mathrm{T}}$. □

<u>Theorem</u>. Deterministic formulas are transparent.

<u>Proof</u>: Suppose φ is deterministic, $\mathfrak{X}$ a multirefutation of φ, and $g{:}A\dashrightarrow\mathfrak{X}$ an $\mathrm{F}\varphi$-labeled subframe of A. By the preceding lemma, we may assume that $\forall a\in A\exists a'\geq a.\ a'\in\mathrm{dom}g$.
For each $a\in A$, let $\Sigma_0(a)$ be

$$\cup(g(a')^{\mathrm{T}}\mid a'\in(a]\cap\mathrm{dom}g)\cup\cup(g(a')^{\mathrm{F}}\mid a'\in[a)\cap\mathrm{dom}g).$$

Since g is a partial projection, $\Sigma_0(a)=g(a)$ for $a\in\mathrm{dom}g$. Let $\Sigma(a)$ be the closure of $\Sigma_0(a)$ under the rules

(1) If $\mathrm{T}(\psi\to\chi)\in\Sigma$ and $\mathrm{F}\psi\notin\Sigma$, add $\mathrm{T}\chi$ to Σ.

(2) If $T(\psi\wedge\chi)\in\Sigma$, add $T\psi$ and $T\chi$.

Since $g(a)$ always is a full sequent, $\Sigma(a)=g(a)$ for $a\in \mathrm{dom}g$.
As indicated above, $\Sigma_0(a)$ is full but for the decomposition of implications signed T, which is the subject of rule (1). Therefore rule (1) is used in the first step of the derivation of any element of $\Sigma(a)-\Sigma_0(a)$. Thus we can be sure that signed disjunctions $T(\psi\vee\chi)$ in $\Sigma(a)$ already belonged to $\Sigma_0(a)$: new formulas signed T come from the (T-signed) succedents of determinate implications. Since formulas signed F, and disjunctions signed T, are decomposed in $\Sigma_0(a)$, and the rest is covered by rules (1) and (2), each sequent $\Sigma(a)$ is full.
If $T\psi\in \Sigma_0(a)$, then $T\psi\in g(a')$ for some $a'\leq a$. It follows that $T\psi\in g(a'')$ for all $a''\geq a$ that belong to $\mathrm{dom}g$ – since $a''\geq a'$ and g is a partial projection. If $T\psi$ is added to $\Sigma(a)$ by rule (1) or (2) because some $T\psi'$ belonged to $\Sigma(a)$, and $T\psi'\in\cap(g(a'')|\ a''\geq a$ and $a''\in \mathrm{dom}g)$, then $T\psi\in\cap\ (g(a'')|a''\geq a$ and $a''\in\mathrm{dom}g)$ as well. Conclusion: $\Sigma\ (a)^{\mathrm{T}}\subseteq g(a'')^{\mathrm{T}}$ for all $a''\geq a$ that belong to $\mathrm{dom}g$.
Now since $[a)\cap\mathrm{dom}g\neq\emptyset$, and $\mathfrak{X}$ is open, it follows that $T\bot\notin\Sigma(a)$. Likewise, $T\psi\in\Sigma(a)$ implies $F\psi\notin\Sigma(a)$: for if $T\psi\in\Sigma(a)$, $T\psi\in g(a'')$ for all $a''\geq a$ in $\mathrm{dom}g$, implying that $F\psi\notin g(a'')$.
If $a\leq_A b$, then $\Sigma_0(a)^{\mathrm{T}}\supseteq\Sigma_0(b)^{\mathrm{T}}$ since $(a]\cap\mathrm{dom}g\subseteq(b]\cap\mathrm{dom}g$; and similarly $\Sigma_0(a)^{\mathrm{F}}\supseteq\Sigma_0(b)^{\mathrm{F}}$. Hence whenever rule (1) or (2) adds for a, it also adds for b. So $\Sigma(a)^{\mathrm{T}}\subseteq\Sigma(b)^{\mathrm{T}}$.
Now let $Y=\{\Sigma(a)|a\in A\}$; and for $y\in Y$ and $T\subseteq Y$,

$$T\in\mathfrak{T}(y)\ \text{ iff }\ \exists a\in\Sigma^{-1}\{y\}.\ T=\Sigma[a).$$

We want to show that $\mathfrak{P}:=(Y,\mathfrak{T})$ is a multitableau. For if it is , it is open, as shown above; $F\psi$ belongs to some sequent in Y; and trivially, Σ is a projection onto $\mathfrak{P}$, whence $A\not\Vdash\varphi$ by theorem 3.12.
Actually, only the second part of definition 3.9.1 remains to be checked. Suppose $F(\psi\to\chi)\in y$, and $T\psi\notin y$. Take any $T\in\mathfrak{T}(y)$; then $T=\Sigma[a)$ for some $a\in\Sigma^{-1}\{y\}$. Then some $y'\in g[a)$ contains $F(\psi\to\chi)$. Say $y'=g(a')$, with $a'\geq a$. Since g is a partial projection, there exists $a''\geq a'$ with $T\psi,F\chi\in g(a'')$ or $F(\psi\to\chi)\in g(a'')$ and $g(a'')^{\mathrm{T}}\neq g(a')^{\mathrm{T}}$. Then $g(a'')=\Sigma(a'')\in T$, and $g(a'')^{\mathrm{T}}\neq y^{\mathrm{T}}$ since $y^{\mathrm{T}}\subseteq g(a')^{\mathrm{T}}\subseteq g(a'')^{\mathrm{T}}$. □

With corollary 7.3 we get

<u>Corollary</u>. Deterministic formulas are equivalent to $\Pi^0{}_2$-sentences.

7.5 <u>Corollaries</u>: (i) $\mathbb{I}[\wedge,\to,\bot]\subseteq \mathrm{E}(\mathbf{PO})$
(ii) $\mathbb{I}[\wedge,\vee,\neg]\subseteq \mathrm{E}(\mathbf{PO})$

<u>Proof</u> of (ii): Implications in $\mathbb{I}[\wedge,\vee,\neg]$ have succedent $\bot$, hence are determinate. □

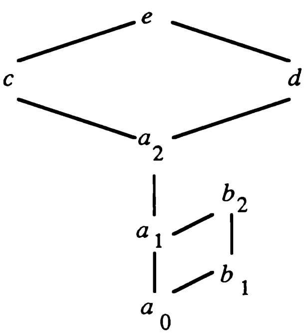
7.6 **Quantifier complexity**. First order definitions produced by the method of 7.3 are $\Pi^0{}_2$. One might wonder whether *all* elementary 𝕀-formulas are equivalent to $\Pi^0{}_2$-sentences. We will now show that KP is not; and construct a sequence of generalizations of KP whose first order definitions seem to require an ever increasing number of quantifier changes.

7.6.1 In 6.5 we gave a $\Pi^0{}_4$ equivalent for KP. There may be simpler first order equivalents; however, we will show that they cannot be as simple as $\Pi^0{}_2$, by exhibiting a chain

$$A_0 \subseteq A_1 \subseteq \ldots\ldots \subseteq A_n \subseteq \ldots\ldots \qquad (n \in \mathbb{N})$$

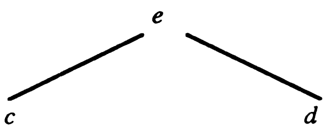
such that KP is valid in every A_n, but not in the union $A=\cup_{n\in\mathbb{N}} A_n$ (cf. Chang & Keisler, thm.3.2.3).

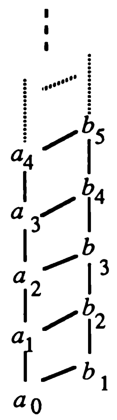
Let $A_n := \{a_i, b_j | 0 \leq i \leq n \text{ and } 1 \leq j \leq n\} \cup \{c,d,e\}$, partially ordered in such a way that e and b_n are endpoints, c and d are covered by e, a_n is covered by c and d, and for $i,j<n$, b_j is covered by b_{j+1} and a_i is covered by a_{i+1} and b_{i+1}. The diagram below shows A_2.



If $(A_n,V) \not\Vdash$ KP, some top node must force p, and another $\neg p$ (this immediately excludes $n=0$). Note that KP must hold in b_j, c, d, and e. Now, if $e \Vdash \neg p$, and $b_n \Vdash p$, we must have $a_n \Vdash \neg p$ and $a_n \Vdash \neg p \rightarrow q \vee r$; hence $a_n \Vdash q$ or $a_n \Vdash r$, and we would have $a_0 \Vdash$ KP. The other way round, with $e \Vdash p$ and $b_n \Vdash \neg p$, b_n is the only point where $\neg p$ is forced, and again $a_0 \Vdash$ KP. Since we obtained a contradiction either way, $A_n \Vdash$ KP.

Now consider A (see diagram on next page).





Define a valuation V on A by $V(p)=[b_1)$, $V(q)=[d)$ and $V(r)=[c)$. Then $(A,V)\nVdash\mathrm{KP}$.

7.6.2 Let p_n, q_n, r_n ($n\in\mathbb{N}$) be infinitely many distinct proposition letters. We define a sequence $(\mathrm{KP}_n)_{n\in\mathbb{N}}$ of $\mathbb{I}$-formulas as follows:

$$\mathrm{KP}_0:=\bot;$$
$$\mathrm{KP}_{n+1}:=[(p_n\to\mathrm{KP}_n)\to q_n\vee r_n]\to(\neg p_n\to q_n)\vee(\neg p_n\to r_n).$$

For a corresponding sequence of $\mathbb{L}_0$-formulas, let

$$\beta_0(x):=\top;$$
$$\beta_{n+1}(x):=\exists yz\geq x(\neg\mathrm{Comp}(y,z)\wedge\forall u\geq x(u\leq y,z\to\exists v\geq u(\beta_n(v)\wedge\neg\exists w\geq v(y\leq w\vee z\leq w))));$$

and $$\alpha_n:=\neg\exists x\beta_n(x).$$

We claim that $\mathrm{KP}_n\equiv\alpha_n$. For n=0 this is obvious; suppose it is true for n. Reasoning as in 6.5, we find that a multirefutation of KP_{n+1} must begin as indicated on the next page:

$$\begin{tikzcd}
 & \boxed{T((p_n \to KP_n) \to q_n \vee r_n), F(\neg p_n \to q_n), F(\neg p_n \to r_n)} \arrow[dl] \arrow[d] \arrow[dr] & \\
\boxed{T\neg p_n, Fq_n, Tr_n} & \boxed{Tp_n, FKP_n} & \boxed{T\neg p_n, Fr_n, Tq_n}
\end{tikzcd}$$

(note that $\vdash \neg p_n \to (p_n \to KP_n)$). Analogous to 6.5, and using that $(A,a) \nVdash KP_n$ iff $A \models \beta_n[a]$, we see that $KP_{n+1} \equiv \alpha_{n+1}$.
As to quantifier complexity, $\beta_0(x)$ is Δ^0_1, and β_{n+1} is Σ^0_{2n+2}; hence the α_{n+1} are Π^0_{2n+2}.

7.7 On restricted classes of frames, more formulas may become first order. In extreme cases, every $\mathbb{I}$-formula is first order; we already know about one of these, vz. **LO** (by 4.8 and 7.5). This phenomenon will be investigated further in the next section.

Footnotes

[1] §17 (17.8) contains some further considerations on the syntactic form of $\mathbb{L}_0$-equivalents of transparent formulas.

[2] Actually, deterministic formulas are equivalent to formulas in $\to,\bot$, as can be seen by a little second order propositional logic. As argued in §4, $\wedge$ can be eliminated. Next, $\vee$ to the left of $\to$ can be eliminated by the logical equivalence $\vdash(\varphi\vee\psi\to\chi)\leftrightarrow(\varphi\to\chi)\wedge(\psi\to\chi)$; the new $\wedge$ can be removed without introducing new disjunctions (4.3.3). So only disjunctions to the right of $\to$ (and disjunctions that are not subformulas of implications) can be problematic. Now in general, $\varphi\vee\psi$ is equivalent to the second order formula $\forall p((\varphi\to p)\to(\psi\to p)\to p)$ (p a new proposition letter); moreover, $(\varphi\to\forall p\psi)\leftrightarrow\forall p(\varphi\to\psi)$ is universally valid if p does not occur in φ. A formula is deterministic precisely when, after replacing the remaining disjunctions with quantifications over distinct new proposition letters, we can move all the quantifiers to the front, using the equivalence $(\varphi\to\forall p\psi)\leftrightarrow\forall p(\varphi\to\psi)$. Front universal quantifiers can be dropped: the definition of validity involves universal closure anyway.

§8. Classes of frames in which every $\mathbb{I}$-formula is first order definable

As was shown in 6.8(b), there exist $\mathbb{I}$-formulas that are not first order definable on **DLO**. We now consider a restriction additional to downward linearity, inverse wellfoundedness, under which every $\mathbb{I}$-formula becomes first order definable. We use a theorem of Doets [A] to indicate to what extent a restriction of this sort is necessary (some further details will appear in §9).
Finally, we consider restrictions on width and on height for the larger class of partial orderings.

8.1 Definition. We call a frame A **inversely well-founded** if every subset of A has maximal elements.

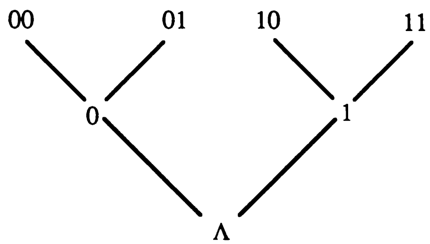
Clearly, A is inversely well-founded iff $(A,>)$ is well-founded in the ordinary sense. Another equivalent statement is: A does not contain an infinite ascending chain

$$a_0<a_1<\ldots<a_n<\ldots \qquad (n\in\mathbb{N}).$$

We shall denote the class of all inversely well-founded *downward linear orderings* by **IWD**.

8.2 Definition. Let A be a frame; $B\subseteq A$. We call B a **subtree** of A if B, with the ordering inherited from A, is a tree; a **strong** subtree if, moreover, the canonical embedding $B\hookrightarrow A$ is strong.

By way of example, consider the binary tree T of the sequences of zeros and ones of length at most 2 ordered by initial segments:



$T':=\{\Lambda,00,01,10\}$ is a subtree of T, but not a strong subtree; $T'\cup\{0\}$ is a strong subtree.

8.3 We introduce a partial ordering $\leq$ of sequents:

8.3.1 Definition. Let Σ and Θ be sequents. We write $\Sigma\leq\Theta$ if $\Sigma^T\subseteq\Theta^T$ and $\Theta^F\subseteq\Sigma^F$.

We write $<$ for the related strict ordering: $\Sigma<\Theta$ iff $\Sigma\leq\Theta$ and $\Sigma\neq\Theta$.

8.3.2 Definition. A multitableau $\mathfrak{X}=(X,\mathfrak{S})$ is **monotone** if for all $x,y\in X$, $y\in\cup\mathfrak{S}(x)$ implies $x<y$.

Monotony is an attribute comparable to strictness (3.3). Its use is, that if we climb through a monotone multitableau, passing from x to an element of $\cup\mathfrak{S}(x)$, we can take only a finite number of steps.

8.4 Σ-labeled trees

As in §7, we shall use partial projections to obtain first order equivalents of $\mathbb{I}$-formulas. The approach will be slightly different. In the definition of Σ-labeled subframe, we had to ensure that points with a common successor were assigned compatible sequents; for downwards linear orderings this difficulty does not exist. Since points with a common successor are linearly ordered, it is enough that each sequent is realizable. Instead we shall concentrate on difficulties caused by points *between* elements of the domain of a partial projection: points a such that there are a', $a''\in \mathrm{dom}g$ with $a'<a<a''$.

For a frame A, and $a\in A$, we denote by $\mathrm{Cov}_A(a)$ the set of all covers of a in A.

Definition. Let $A\in$ **DLO**, Σ a sequent, and $\mathfrak{X}=(X,\mathfrak{S})$ an open Σ-multitableau. A partial surjection $g:A\dashrightarrow\mathfrak{X}$ is a **Σ-labeled subtree** of A if

(i) $\mathrm{dom}g$ is a tree;

(ii) for every $a\in\mathrm{dom}g$, $g\restriction\mathrm{Cov}_{\mathrm{dom}g}(a)$ is a bijection onto some $S\in\mathfrak{S}(g(a))$.

We shall say g is **strong** if $\mathrm{dom}g$ is a strong subtree of A; **perfect** if g is strong, $\mathfrak{X}$ monotone, and the root of $\mathrm{dom}g$ is the only element of A such that $\Sigma\subseteq g(a)$.

By (ii) and 8.3 it is immediate that the domain of a perfect Σ-labeled subtree is finite – an upper bound for its size may be deduced from Σ.

As in 7.2, we assume unique representatives of the equivalence classes of perfect Σ-labeled subtrees; call them Σ-labeled trees. Given a suitable Σ-multitableau $\mathfrak{X}$, it is clear from the definition above how to construct Σ-labeled trees with range $\mathfrak{X}$. These trees are finite, and there are finitely many of them. Since there are finitely many Σ-multitableaux, we have

Lemma. For any sequent Σ, there are only finitely many Σ-labeled trees.

8.5 Lemma. Suppose $A\in\mathbf{IWD}$, and $\varphi\in\mathbb{I}$. If $A\nVdash\varphi$, then A has a perfect $\mathrm{F}\varphi$-labeled subtree.

Proof: Suppose $(A,V)\nVdash\varphi$. Clearly, a perfect labeled subtree of a generated subframe is a perfect labeled subtree; we may therefore assume that A has a root a_0. In addition, since $A\in\mathbf{IWD}$, we may assume that a_0 is the only point of (A,V) that does not force φ. For each $a\in A$, let

$$\Theta(a):=\Theta_\varphi^{(A,V)}(a)\ \ (=\{\sigma\in \mathrm{Sf}(\mathrm{F}\varphi)\mid a\Vdash\sigma\}).$$

We define a sequence

$$A_0\subseteq A_1\subseteq\dots\subseteq A_n\subseteq\dots \qquad\qquad (n\in\mathbb{N})$$

of finite subtrees of A, as follows. $A_0=\{a_0\}$. Suppose A_n has been defined. For each $a\in A_n-A_{n-1}$, take a set C_a of strict successors of a in A such that

(i) for every $b>a$, $\exists c\in C_a:\Theta(c)\leq\Theta(b)$;

(ii) if c,c' are distinct elements of C_a, then $\Theta(c)\nleq\Theta(c')$;

(iii) each $c\in C_a$ is maximal in $\{b\in A\mid \Theta(b)=\Theta(c)\}$.

Now let g be the restriction of Θ to $\cup_n A_n$, $X=\mathrm{ran}g$, and for $x\in X$,

$$S\in\mathfrak{S}(x)\ \text{ iff }\ \exists a\in g^{-1}\{x\}.\ S=g[C_a].$$

Put $\mathfrak{X}=(X,\mathfrak{S})$. We shall establish a sequence of claims, culminating in (6): g is a perfect $\mathrm{F}\varphi$-labeled subtree of A.

(1) If $a\in\mathrm{dom}g$, then a is maximal in $\{b\in A\mid\Theta(b)=\Theta(a)\}$.

a_0 is the only point that does not force φ, so $\{a_0\}=\{b\in A\mid \mathrm{F}\varphi\in\Theta(b)\}$. The other elements of dom$g$ are maximal by construction.

(2) If $y\in\cup\mathfrak{S}(x)$, then $y>x$.

If $y \in \cup\mathfrak{S}(x)$, then there are $a,c \in A$ such that $x=g(a)$, $y=g(c)$, and $c>a$. Then $\Theta(a) \leq \Theta(c)$ is immediate, and $\Theta(a) \neq \Theta(c)$ by (1).

(3) For each $a \in \mathrm{dom}g$, $C_a = \mathrm{Cov}_{\mathrm{dom}g}(a)$.

Suppose $c \in C_a$, and $a<b\leq c$, with $b \in \mathrm{dom}g$. Then $\Theta(a) \neq \Theta(b)$ by (1), and $\Theta(b) \leq \Theta(c)$. By (i) in the definition of C_a, there exists $c' \in C_a$, with $\Theta(c') \leq \Theta(b)$; by (ii) and transitivity of $\leq$, $c'=c$. Hence $\Theta(b)=\Theta(c)$, and $b=c$ by (1).

(4) $\mathfrak{X}$ is a multitableau.

By (2), $y \in \cup\mathfrak{S}(x)$ implies $x<y$, whence $x^{\mathrm{T}} \subseteq y^{\mathrm{T}}$ is immediate. If $\mathrm{F}(\psi \to \chi) \in x$, $\mathrm{T}\psi \notin x$, and $S \in \mathfrak{S}(x)$, then $S=g[C_a]$ for some $a \in g^{-1}\{x\}$. Then $a \nVdash \psi \to \chi$; so there exists $b \geq a$ with $b \Vdash \psi$ and $b \nVdash \chi$. Since $a \nVdash \psi$, $b>a$. So by (i), there is some $c \in C_a$ with $\Theta(c) \leq \Theta(b)$; $\Theta(a) \neq \Theta(c)$ by (1). Since $\mathrm{F}(\psi \to \chi) \in \Theta(b)$, $\mathrm{F}(\psi \to \chi) \in \Theta(c) = g(c) \in S$.

(5) $\mathfrak{X}$ is a monotone multirefutation of φ.

Since $a_0 \nVdash \varphi$, $\mathrm{F}\varphi \in g(a_0)$. By the nature of Θ, $\mathfrak{X}$ is open. Monotonicity is by (2).

(6) g is a perfect $\mathrm{F}\varphi$-labeled subtree of A.

By (3), (ii) and the definition of $\mathfrak{S}$, condition (ii) of definition 8.4 is satisfied. As for (i), it is immediate that dom$g$ is a rooted downwards linear ordering. Since $\mathfrak{X}$ is monotone, (3) and the definition of $\mathfrak{S}$ imply that chains in dom$g$ are finite; hence dom$g$ is a tree. Finally, to prove that g is strong, we check condition (*) of lemma 2.11.6 for the canonical embedding of dom$g$ into A. Suppose c, c' are distinct covers of a in dom$g$, and $a \leq b \leq c,c'$. Then $\Theta(b) \leq \Theta(c)$, $\Theta(c')$. By (ii), this implies that $\Theta(b) < \Theta(c)$, $\Theta(c')$. So by (i) and (ii), $a<b$ is impossible; hence $a=b$. Therefore $b \leq \mathrm{Cov}_{\mathrm{dom}g}(a)$; so dom$g$ is a strong subtree of A. □

8.6 Lemma. Suppose $A \in \mathbf{IWD}$ and $\varphi \in \mathbb{I}$. If A has a perfect $\mathrm{F}\varphi$-labeled subtree, then $A \nVdash \varphi$.

Proof: Suppose $g: A \twoheadrightarrow \mathfrak{X}$ is a perfect $\mathrm{F}\varphi$-labeled subtree of A. Since g is a projection of dom$g$ onto $\mathfrak{X}$, dom$g \nVdash \varphi$ by 3.12. Since the canonical embedding of dom$g$ into A is strong, and dom$g$ is finite, there exists by 2.11.10(i) a p-retraction $f: A \twoheadrightarrow \mathrm{dom}g$. Then $A \nVdash \varphi$ by lemma 2.4.2. □

8.7 Theorem. $E(\mathbf{IWD})=\mathbb{I}$.

Proof: Let $\varphi\in\mathbb{I}$ be given; we are to produce an $\mathbb{L}_0$-equivalent on **IWD**. Suppose $g_1,\ldots,g_n$ are all the $F\varphi$-labeled trees (by lemma 8.4, there are finitely many). Each g_i is finite, as observed in 8.4. Now let δ_i be the conjunction of the diagram of $\mathrm{dom}g_i$. Next, for each i ($1\leq i\leq n$) and $b\in\mathrm{dom}g_i$, let $\gamma_{i,b}$ be the formula

$$\bigwedge(v\leq b' \mid b'\in \mathrm{Cov}_{\mathrm{dom}g_i}(b)).$$

Finally, for $1\leq i\leq n$, let ε_i be

$$\delta_i\wedge\bigwedge(\forall v(b\leq v\leq b_1,b_2\rightarrow\gamma_{i,b}) \mid b\in\mathrm{dom}g_i \text{ and } b_1,b_2 \text{ are distinct elements of } \mathrm{Cov}_{\mathrm{dom}g_i}(b)).$$

Let ε_i' be the result of replacing the individual constants in ε_i by distinct new individual variables (so $\varepsilon_i'\in\mathbb{L}_0$). Suppose $v_1,\ldots,v_m$ are all the variables that occur free in $\bigvee_{1\leq i\leq n}\varepsilon_i'$; then let α be

$$\neg\exists v_1\ldots v_m\bigvee_{1\leq i\leq n}\varepsilon_i'.$$

This α is a first order equivalent of φ. Indeed, $A\not\models\alpha$ iff some ε_i ($1\leq i\leq n$) is true in an expansion of A, iff for some i, $\mathrm{dom}g_i$ is isomorphic to a subtree of A (by δ_i) which is strong by the rest of ε_i and lemma 2.11.6. Suppose A_0 is a strong subtree of A, and $f:A_0\rightarrow\mathrm{dom}g_i$ is an isomorphism. Then $g_i\circ f$ is a perfect $F\varphi$-labeled subtree of A, so $A\not\Vdash\varphi$ by lemma 8.6. Conversely, if $A\not\Vdash\varphi$, then A has a perfect $F\varphi$-labeled subtree f by lemma 8.5, which is equivalent to some g_i by assumption; so that in particular $\mathrm{dom}f\cong\mathrm{dom}g_i$. □

8.8 Remark: Observe that, in contrast to theorem 7.3, the first order equivalent is entirely determined by the domains of the $F\varphi$-labeled trees. In 7.3 we had to take the labeling into account.

8.9 Example. On **IWD**, SP_2 is equivalent to

$$\neg\exists xyzw(x<y,z,w\wedge\forall u[(x\leq u\leq y,z\rightarrow u\leq w)\wedge(x\leq u\leq y,w\rightarrow u\leq z)\wedge(x\leq u\leq z,w\rightarrow u\leq y)]).$$

The only relevant tree consists of a root with three covers (cf. 3.8).

8.10 The above first order definitions would still work if we allowed frames that can be obtained from inversely well-founded downwards linear orderings by replacing some points by infinite chains; with regard to a given formula φ, such chains behave essentially as finite chains with as many points as can be distinguished with subformulas of φ. Still some small extensions are possible (see §9); but soon, full first order definability is lost.

8.11 Definition: A binary tree A is **full** if every point in A has either two covers, or none.
The class of all full binary trees we denote by $\mathbf{TR}^{(2)}$.

We will show that $\mathrm{E}(\mathbf{TR}^{(2)})\neq\mathbb{I}$. As noted in the introduction to part II, an immediate consequence is that $\mathrm{E}(\mathbf{K})\neq\mathbb{I}$ for all frame classes $\mathbf{K}\supseteq \mathbf{TR}^{(2)}$.

8.12 Definition. The quantifier rank $\mathrm{rnk}(\alpha)$ of a first order formula α is defined inductively as follows:
1. $\mathrm{rnk}(\alpha)=0$ if α is atomic, $\alpha=\top$ or $\alpha=\bot$;
2. $\mathrm{rnk}(\neg\beta)=\mathrm{rnk}(\beta)$, $\mathrm{rnk}(\beta\wedge\gamma)=\mathrm{rnk}(\beta\vee\gamma)=\mathrm{rnk}(\beta\rightarrow\gamma)=\max(\mathrm{rnk}(\beta),\mathrm{rnk}(\gamma))$;
3. $\mathrm{rnk}\ (\exists x\beta)=\mathrm{rnk}(\forall x\beta)=\mathrm{rnk}(\beta)+1$.

Two structures $\mathfrak{A}$ and $\mathfrak{B}$ for the same first order language will be called **n-equivalent** (notation: $\mathfrak{A}\equiv^n\mathfrak{B}$) if they satisfy the same first order sentences of quantifier rank n.

8.13 Definition. Let A be a frame. A **path** through A is a maximal chain in A.

8.14 Definition. Let $n\geq 1$; then $P(n)$ is the conjunction of the following three conditions on full binary trees A:

P.1 Each point in A lies below a maximal point.

P.2(n) Each path through A has cardinality at least 2^n-2.

P.3(n) For all $a\in A$ and $m<2^n$, if some path through $[a)$ has cardinality m, then **every** path through $[a)$ has cardinality m.

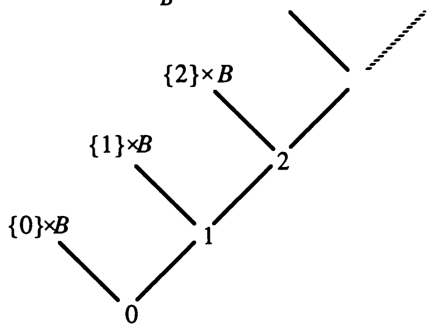
8.15 Proposition (Doets [A]). If A and B are full binary trees, and both satisfy $P(n)$, then $A\equiv^n B$.

8.16 Corollary. $SP_2 \notin E(\mathbf{TR}^{(2)})$.

Proof: Suppose $SP_2 \equiv_{\mathbf{TR}^{(2)}} \alpha$, and $\mathrm{rk}(\alpha)=n$. Let B be a full binary tree all of whose paths have cardinality 2^n-1. Since $B \Vdash SP_2$ by 8.9, and B satisfies $P(n)$, α (and hence SP_2) must hold in every full binary tree that satisfies $P(n)$. Now let $A=\mathbb{N}\cup(\mathbb{N}\times B)$, with $\leq_A$ extending the natural ordering of $\mathbb{N}$ as follows:

$n\leq(m,b)$ iff $n\leq m$;

$(n,b)\leq(m,b')$ iff $n=m$ and $b\leq_B b'$.



Then A is a full binary tree satisfying $P(n)$, hence $A\models\alpha$, and $A\Vdash SP_2$. But setting $V(p)=\cup(\{n\}\times B|n\not\equiv 2(\text{mod } 3))$ and $V(q)=\cup(\{n\}\times B|n\not\equiv 1(\text{mod } 3))$ (cf. 3.8) we find $(A,V)\nVdash SP_2$: a contradiction. □

8.17 **Width**

The width of a frame A was defined in 2.9 as the least upper bound of the cardinalities of antichains in rooted subframes of A. For $n\in\mathbb{N}$, we shall denote the class of all partial orderings of width at most n by $\mathbf{PO}_n$.

Hardly any width is needed to get nonelementary $\mathbb{I}$-formulas.

Example. The earlier example 6.8(b) can be adapted to show that $E(\mathbf{PO}_4)\neq\mathbb{I}$. Let $\psi_1:=p\wedge q$, $\psi_2:=p\wedge(q\to r)$, and $\psi_3:=(p\to r)\wedge q$; set

$$\varphi:=(\bigvee_{1\leq i\leq 3}(\psi_i\to r)\to \bigvee_{1\leq i\leq 3}\psi_i)\to \bigvee_{1\leq i\leq 3}(\psi_i\to r).$$

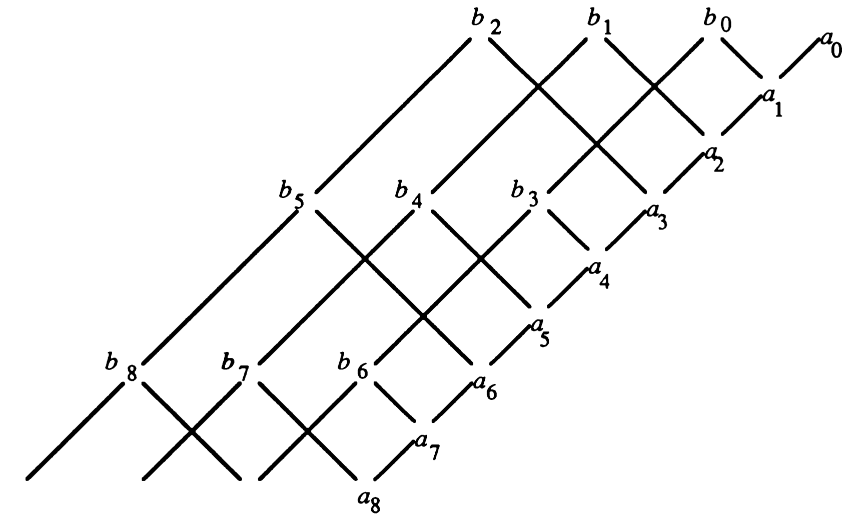
Note that $SP_2=\varphi[r:=\bot]$. In SP_2 we had $\vdash\psi\to\neg\chi$ and the like; here

(*) for $1 \leq i \neq j \leq 3$, $\vdash \psi_i \rightarrow \psi_j \rightarrow r$.

Let A be as in 6.8(b), except that the ordering is supplemented by

$b_m \leq b_n$ if $m \geq n$ and $m \equiv n$ (mod 3).

The top of the resulting frame is shown in the diagram below.



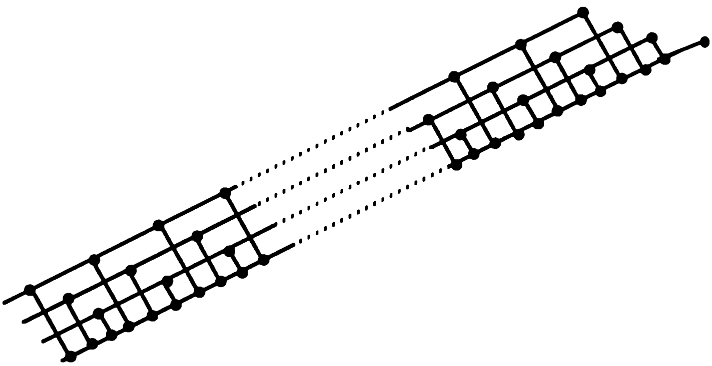
We want to show that $A \Vdash \varphi$. Take any valuation V on A.
Suppose i, j are distinct, $1 \leq i, j \leq 3$. Suppose some b_k realizes $\{T\psi_i, Fr\}$. By (*), $b_m \Vdash F\psi_j$ if $b_m \leq b_k$; and $b_m \Vdash r$ if $b_m \Vdash \psi_j$ and $b_m \geq b_k$. Therefore $b_m \Vdash \psi_j \rightarrow r$; $b_n \Vdash \bigvee_i(\psi_i \rightarrow r)$ for all $n \in \mathbb{N}$; and a fortiori $b_n \Vdash \varphi$.
Now suppose some a_k forces $\bigvee(\psi_i \rightarrow r) \rightarrow \bigvee\psi_i$. We want to prove that $a_k \Vdash \bigvee(\psi_i \rightarrow r)$; then we may conclude that $(A,V) \Vdash \varphi$. It will suffice to show that $a_m \Vdash \bigvee\psi_i$ for all $m \leq k$, since $\vdash \bigvee\psi_i \rightarrow \bigvee(\psi_i \rightarrow r)$ is an easy consequence of (*) (assume any ψ_l; then for $j \neq l, \psi_j \rightarrow r$ by (*), hence $\bigvee(\psi_i \rightarrow r)$). We have $a_0 \Vdash \bigvee(\psi_i \rightarrow r)$ because $\bigvee(\psi_i \rightarrow r)$ is a tautology and a_0 is an endpoint (cf. 2.5). Since $a_k \leq a_0$, $a_0 \Vdash \bigvee\psi_i$. Now suppose $m<k$, and $a_m \Vdash \bigvee\psi_i$. Since $b_m \Vdash \bigvee(\psi_i \rightarrow r)$, and $a_k \leq b_m$, $b_m \Vdash \bigvee\psi_i$ as well. By (*), there is some l such that both $a_m \Vdash \psi_l \rightarrow r$ and $b_m \Vdash \psi_l \rightarrow r$. Then $a_{m+1} \Vdash \psi_l \rightarrow r$; or $a \Vdash \psi_l$ and $a \nVdash r$ for some $a \geq a_{m+1}$, and then $a = a_{m+1}$ since a_m and b_m are the covers of a_{m+1}. In the first case, $a_{m+1} \Vdash \bigvee(\psi_i \rightarrow r)$, hence $a_{m+1} \Vdash \bigvee\psi_i$ since $a_k \leq a_{m+1}$; in the other case $a_{m+1} \Vdash \psi_l$, so $a_{m+1} \Vdash \bigvee\psi_i$ is immediate.

As before, take a nonprincipal ultrafilter U over $\mathbb{N}$, and consider $\prod_U A$. The ultrapower can be pictured clear enough with the help of Łoś's theorem. It has width 4, since it must satisfy the

$\mathbb{L}_0$-sentence

$$\forall x_1x_2x_3x_4 \bigvee(x_i\leq x_j|1\leq i,j\leq 4).$$

It ends in an isomorphic copy of A. For the rest, it consists of pieces which are like A, except that they have no endpoints. In a diagram (where the broken lines must suggest an infinity of points):



In particular, we can find c_n, d_n ($n\in\mathbb{N}$) in $\prod_U A$ such that each c_n is covered by d_n and c_{n+1}, and d_{n+3} covers d_n. Let $X=\bigcap_n[c_n)$.
Now we define a valuation V on $\prod_U A$ by

$$V(p)=[d_1)\cup[d_2)\cup X;\ V(q)=[d_0)\cup[d_1)\cup X;\ V(r)=X.$$

Then $d_{3n+i}\Vdash\{\mathrm{T}\psi_i,\mathrm{F}r\}$. Moreover, if $u\geq c_1$, then u is some d_{3n+i} forcing ψ_i, or $u\in X$ and $u\Vdash\psi_1$, or $u=c_n$ for some $n\in\mathbb{N}$, and $u\nVdash(\bigvee_{1\leq i\leq 3}(\psi_i\rightarrow r)$; hence $c_1\Vdash(\bigvee_{1\leq i\leq 3}(\psi_i\rightarrow r)\rightarrow\bigvee_{1\leq i\leq 3}\psi_i$. We conclude that $c_1\nVdash\varphi$, and therefore $\prod_U A\nVdash\varphi$. Hence φ is not elementary on $\mathbf{PO}_4$, by 6.7.6.

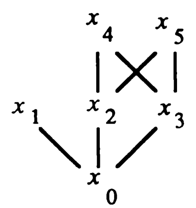
An open problem. Since $\mathbf{LO}\Vdash\varphi$ iff $\mathbf{PO}_1\Vdash\varphi$, $\mathrm{E}(\mathbf{PO}_1)=\mathbb{I}$ by 7.5 and the logical equivalence in 4.8. There is a gap between this result and the example above, which I have not been able to fill in.

8.18 **Height.** As example 6.8(a) shows, not every $\mathbb{I}$-formula is elementary on the class of all frames of height at most 3. If we reduce the maximal height by one, the problem of first order definability becomes trivial: take $\bot$, $\top$, or an $\mathbb{L}_0$-sentence stating that the number of strict successors of any point does not exceed a suitable finite bound.

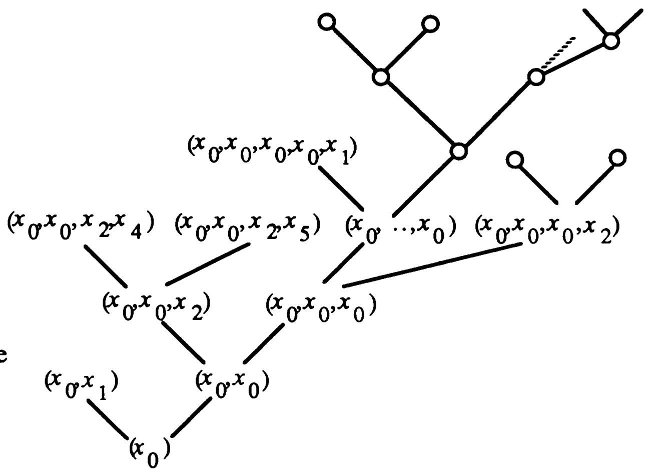
§9. Trees

We are going to show that E(**TR**) is decidable. The argument centers on binary trees (we establish that E($\mathbf{TR}^{(2)}$)= E(**TR**)), and the difference between validity on all binary trees and validity on finite binary trees.

9.1 Every $\mathbb{I}$-formula that is not universally valid, can be refuted on a binary tree; but not necessarily on a *finite* binary tree. The second of these well-known facts may be substantiated by SP_2: example 8.9 implies that SP_2 is universally valid on the finite binary trees.
As to the first, suppose $\nvdash\varphi$, and let (X,S) be a minimal refutation of φ, with root x_0. Then build a binary tree T of sequences of elements of X (ordered by initial segments) as follows: start with (x_0). If $(x_0,\ldots,x_n)$ has been put into T, and does not yet have strict successors, and $S(x_n)=\{x_{n+1},\ldots,x_k\}$, see if $k-n\leq 2$. If it is, add nodes $(x_0,\ldots,x_n,x_j)$ for $n<j\leq k$. If it is not, add $(x_0,\ldots,x_n,x_{n+1})$, $(x_0,\ldots,x_n,x_n)$, ..., $(x_0,\ldots,x_n,\ldots,x_n,x_k)$ (with x_n repeated $k-n$ times) and $(x_0,\ldots,x_n,\ldots,x_n)$ (with x_n repeated $k-n+1$ times). For example,



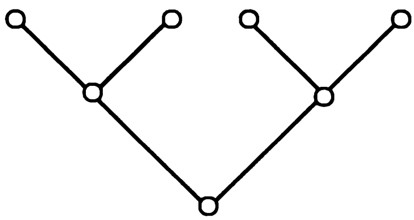
may become



Mapping each sequence to its last element gives a projection of T onto (X,S).
By proposition 3.7.3, (X,S) is strict; so x_{n+1} above contains *more* formulas signed T than x_n. Since X is finite, it follows that every point of T belongs to a finite path. In all, we have proven:

Lemma. If an $\mathbb{I}$-formula is refutable, then it is refutable in a binary tree in which every point has maximal successors.

9.2 Example. It is not so difficult to see that, for any tree C, SP_2 is refutable in C iff either some point of C has three covers (or more), or C contains a copy of the 'infinite comb' A of 2.11.9. The direction from right to left is standard: use 3.8, 2.11.10, and preservation under p-morphism (lemma 2.4.2). For the converse one uses a projection of C onto the tableau of 3.8 – if no $c\in C$ has three covers, there must be an infinite chain of points mapped to the root of the tableau, with side branches. Such a projection must exist, since the tableau of 3.8 is essentially the only multirefutation of SP_2. (Similarly, for $C\in\mathbf{DLO}$: $C\Vdash\!\!\!/\, SP_2$ iff some $c\in C$ branches into an antichain of three elements, or C contains an infinite comb.)
Now we have an easy intuitive reason why SP_2 is not elementary: in a first order language, we *can* say that some pattern recurs ad infinitum; but we have to give conditions under which it recurs, and these conditions must be finite in some way. One cannot decide whether a point belongs to an infinite comb by looking at patterns whose size remains below some fixed finite bound.
Some additional knowledge about a tree C may make all the difference. Suppose we know that C does not contain a copy of the tree



(cf. T in 8.2). Assume for simplicity that C is binary. Now if SP_2 is not valid in some point c_0 with two covers c_1 and c_2, we know which one of c_1, c_2 leads to an infinite comb: the one that has incomparable successors. And indeed, with $\mathrm{Comp}(u,v)=(u\le v\vee v\le u)$ as before, $C\Vdash\!\!\!/\, SP_2$ iff

$$C\models\exists x[\exists u,v\ge x.\neg\mathrm{Comp}(u,v)\wedge\forall y\ge x(\exists u,v\ge y.\neg\mathrm{Comp}(u,v)\rightarrow\exists z>y\exists u,v\ge z.\neg\mathrm{Comp}(u,v))].$$

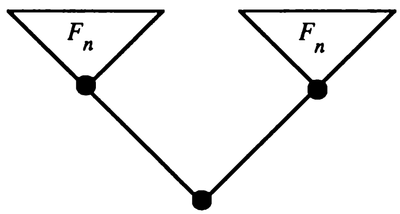
To prove that the second conjunct of the above $\mathbb{L}_0$-sentence is necessary, one uses that C does not contain a copy of T.

9.3 Definition. For each $n\in\mathbb{Z}^+$, we fix a full binary tree F_n in which every path has cardinality n.

The class of all trees into which F_n can *not* be embedded, we denote by $\mathbf{T}_n$.

Observe that each $\mathbf{T}_n$ is the intersection of **TR** with an elementary class. A defining axiom can be constructed from the diagram of F_n in a familiar way: let δ be the conjunction of the diagram of F_n; δ' the result of replacing the individual constants in δ by distinct variables $v_1,...,v_m$; then an $\mathbb{L}_0$-sentence defining $\mathbf{T}_n$ is $\neg\exists v_1...v_m\delta'$.

9.4 The approach of example 9.2 can be generalized. Let us assume that we have proved $\mathrm{E}(\mathbf{T}_n)=\mathbb{I}$. Let $\varphi\in\mathbb{I}$ be given; we are after a first order equivalent for φ on $\mathbf{T}_{n+1}$. Let $A\in\mathbf{T}_{n+1}$. If $[a)_A\in\mathbf{T}_n$, we have a first order sentence α such that $[a)\models\alpha$ iff $[a)\Vdash\varphi$, by hypothesis. Now consider the points a such that $[a)\notin\mathbf{T}_n$. Their disposition is constrained by



$= F_{n+1}$,

hence:

Lemma. If $A\in\mathbf{T}_{n+1}$, then $\{a\in A|[a)\notin\mathbf{T}_n\}$ is a chain.

Let us write F_A for $\{a\in A|[a)\notin\mathbf{T}_n\}$. F_A may be finite or infinite.
(A) If F_A is finite, then the difficult part of A is inversely wellfounded, and in view of §8, it seems we are all set if we can combine refutations on separate parts of A (cf. 9.6 below).
(B) If F_A if infinite, then it is a path in A. Again assuming that valuations can be properly combined, we should be able to give a first order definition along the lines of the example.

9.5 Definition. Let Σ be a sequent. We abbreviate $\bigwedge\Sigma_T\rightarrow\bigvee\Sigma_F$ to ψ_Σ.

The connection between Σ and ψ_Σ is: for any model (A,V), for any $a\in A$: $a\nVdash\psi_\Sigma$ iff $\exists a'\geq a.a'\Vdash\Sigma$. Hence $A\nVdash\psi_\Sigma$ iff Σ is realizable in A – that is, $(A,V,a)\Vdash\Sigma$ for some $a\in A$ and some valuation V on A.

9.6 We need a few lemmas to infer refutability from refutabilities on subframes. The first is an obvious fact about restriction to signed subformulas of a given sequent.

9.6.1 <u>Lemma</u>. Let $\mathfrak{X}=(X,\mathfrak{S})$ be a multitableau, and Σ a sequent. Define for $x\in X$:

$$x\lceil\Sigma=x\cap \mathrm{Sf}(\Sigma).$$

Now let $\mathfrak{X}\lceil\Sigma$ be $(X\lceil\Sigma, \mathfrak{S}\lceil\Sigma)$, with

$$X\lceil\Sigma=\{x\lceil\Sigma | x\in X\};$$

$$S\in(\mathfrak{S}\lceil\Sigma)(x\lceil\Sigma) \text{ iff } \exists y\in X\exists T\in\mathfrak{S}(y)[y\lceil\Sigma=x\lceil\Sigma \ \&\ S=\{z\lceil\Sigma | z\in T\}].$$

Then $\mathfrak{X}\lceil\Sigma$ is a multitableau; $\mathfrak{X}\lceil\Sigma$ is open if $\mathfrak{X}$ is open. If g is a projection onto $\mathfrak{X}$, then g', defined by $g'(a)=g(a)\lceil\Sigma$, is a projection onto $\mathfrak{X}\lceil\Sigma$.

9.6.2 <u>Lemma</u>. Suppose $A\in\mathbf{DLO}$; and Σ is a sequent. If $A\nVdash\psi_\Sigma$, then there exists a projection of A onto an open Σ-multitableau with root containing Σ.

<u>Proof</u>: If $A\nVdash\psi_\Sigma$, then by theorem 3.12 there exists a projection of A onto a multirefutation of ψ_Σ. By the preceding lemma, these can be turned into an open Σ-multitableau $\mathfrak{X}$ and a projection $g{:}A\twoheadrightarrow\mathfrak{X}$. Let $\mathfrak{X}=(X,\mathfrak{S})$; we regard X as a frame with ordering $\leq_{\mathfrak{X}}$.
Suppose $\Sigma\subseteq\Sigma_0\in X$. Fix a maximal successor Σ^* of Σ_0 in X. Let

$$A_0=\{a\in A|\exists a'\leq a.g(a')=\Sigma_0\}, \text{ and}$$
$$A_1=\{a\in A-A_0|\neg\exists a'\geq a.g(a')=\Sigma_0\}.$$

Obviously, A_0 is upwards closed; and A_1 is upwards closed because A is downwards linear. Define $\mathfrak{X}'=(X',\mathfrak{S}')$ by

$$X'=\{x\in X'|\Sigma_0\leq x\};$$

for $x\in X'$, $\mathfrak{S}'(x)=\{S\cap X'|S\in\mathfrak{S}(x)\}$.

Clearly, $\mathfrak{X}'$ is an open Σ-multitableau. Define h: $A\rightarrow X'$ by

$$h(a)=g(a) \text{ if } a\in A_0,$$
$$\quad =\Sigma^* \text{ if } a\in A_1,$$

$=\Sigma_0$ otherwise.

Then h is a projection of A onto $\mathfrak{X}'$. Since Σ_0 is the root of $\mathfrak{X}'$, we shall be done once we have proved this.

1° h is a homomorphism. Suppose $a\leq_A b$. Since A_0 is upwards closed, $a\in A_0$ implies $h(a)=g(a)\leq g(b)=h(b)$. Since A_1 is upwards closed, $a\in A_1$ implies $h(a)=h(b)$. Otherwise $h(a)$ is the root of X'.

2° h satisfies condition (ii) of definition 3.10 because g does. In particular, if $a\in A-(A_0\cup A_1)$, then $\exists a'\geq a.g(a)=\Sigma_0$; so if $S'\in\mathfrak{S}'(\Sigma_0)$, then $S'\subseteq g[a')\subseteq g[a)$. □

9.6.3 <u>Lemma</u>. Suppose $A\in\mathbf{DLO}$ and $\Sigma, \Sigma_1,\ldots, \Sigma_n$ are full realizable sequents such that $\Sigma^T\subseteq\Sigma_1{}^T,\ldots,\Sigma_n{}^T$, and whenever $F(\varphi\rightarrow\psi)\in\Sigma$ and $T\varphi\notin\Sigma$, there is some Σ_i $(1\leq i\leq n)$ such that $F(\varphi\rightarrow\psi)\in\Sigma_i$. Let $C\subseteq A$ be a chain, $B\subseteq A$ an antichain, such that $C\cap B\neq\emptyset$ and $A=C\cup\bigcup_{b\in B}[b)_A$. Suppose there exists a partition $\{B_1,\ldots,B_n\}$ of B such that

(i) $\forall i(1\leq i\leq n)\ \forall b\in B_i.[b)\not\Vdash\psi_{\Sigma_i}$;

(ii) $\forall c\in C\forall i(1\leq i\leq n)\ \exists b\in B_i.b\geq c$.

Then $A\not\Vdash\psi_\Sigma$.

<u>Proof</u>: By (i) and the preceding lemma, if $b\in B_i$ for some i $(1\leq i\leq n)$, then there exists an open Σ_i-multitableau $\mathfrak{X}_b=(X_b,\mathfrak{S}_b)$ with root $\Sigma_b\supseteq\Sigma_i$ and a projection $g_b:[b)\twoheadrightarrow\mathfrak{X}_b$. Define $\mathfrak{X}=(X,\mathfrak{S})$ by

$X=\{\Sigma\}\cup\bigcup_{b\in B}X_b$;

for $x\in X$, $\mathfrak{S}(x)=\bigcup(\mathfrak{S}_b(x)|b\in B\ \&\ x\in X_b)\cup\{\{\Sigma_b|b\in B\cap[c)\}|x=\Sigma\ \&\ c\in C\}$.

Then $\mathfrak{X}$ is an open tableau. First, $\Sigma^T\subseteq\Sigma_b{}^T$ for all $b\in B$, since for some i $(1\leq i\leq n)$, $\Sigma_i\subseteq\Sigma_b$. Second, if $F(\varphi\rightarrow\psi)\in\Sigma$ and $T\varphi\notin\Sigma$, then $F(\varphi\rightarrow\psi)$ belongs to some Σ_i, hence, by (ii), for any $c\in C$ there exist $b\geq c$ with $F(\varphi\rightarrow\psi)\in\Sigma_b$.

Define a mapping g of A onto X by

$g(a)=g_b(a)$ if $a\geq b\in B$;
$=\Sigma$ if $a\in C$.

Then g is a projection; in particular, $\{\Sigma_b|b\in B\cap[c)\}\subseteq g[c)$ for all $c\in C$.

But for some straightforward emendations (some signed subformulas of ψ_Σ may have to be added), $\mathfrak{X}$ is a multirefutation of ψ_Σ. So $A\not\Vdash\psi_\Sigma$ by 3.12. □

9.6.4 Lemma. Suppose $B\in\mathbf{TR}$, and $A\subseteq B$ is a strong subtree in which every point has maximal successors. Let $\{a_i|i\in I\}$ be a set of endpoints of A. Let $C:=A\cup\bigcup_{i\in I}[a_i)_B$. Then for all $\varphi\in\mathbb{I}$, $B\Vdash\varphi$ implies $C\Vdash\varphi$.

Proof: Let f be a p-retraction of B onto A such that

$$(1)\ \forall b\in B\colon \{f(b)\}\cup([b)_B\cap A)=[f(b))_A.$$

(Such f exist by 2.11.8; cf. 2.11.9(b), 2.11.10.) Suppose $C\nVdash\varphi$; let g be a projection onto a multirefutation $\mathfrak{X}=(X,\mathfrak{S})$ of φ, as in 3.12. Pick for each $i\in I$ a maximal sequent $x_i\geq_{\mathfrak{X}}g(a_i)$. Now define $h{:}B\to X$ by

$$\begin{aligned} h(b)&=g(b) \text{ if } \exists i\in I.b>a_i,\\ &=x_i \text{ if } b \text{ and } a_i \text{ are incomparable and } f(b)=a_i, \text{ for some } i\in I,\\ &=g(f(b)) \text{ otherwise.}\end{aligned}$$

We will show that h is a projection; then by 3.12, $B\nVdash\varphi$.

1° Suppose $b\leq_B b'$; we want to prove $h(b)\leq_{\mathfrak{X}}h(b')$. If $b>a_i$, then $b'>a_i$ as well, so $h(b)=g(b)\leq g(b')=h(b')$. If $f(b)=a_i$ and b is not comparable with a_i, then b' is incomparable with a_i by transitivity of $\leq$ and by downward linearity. Since $f(b')\geq f(b)$ and a_i is an endpoint, $h(b')=h(b)$. Otherwise, if $b'>a_i$, then $b\leq a_i$ by downward linearity, so $f(b)\leq f(a_i)=a_i$, whence $h(b)=g(f(b))\leq g(a_i)\leq g(b')=h(b')$. If $f(b')=a_i$, then $f(b)\leq a_i$; so if $h(b')=x_i$, $h(b)=g(f(b))\leq g(a_i)\leq x_i=h(b')$. Otherwise $h(b)\leq h(b')$ because both f and g are homomorphisms.

2° We must show that h satisfies condition (ii) of definition 3.10; i.e.,

$$\forall b\in B\exists S\in\mathfrak{S}(h(b)).S\subseteq h[[b)_B].$$

The only nontrivial case is the one in which $h(b)$ is defined by the last clause of the definition of h. So suppose $h(b)=g(f(b))$, and

$$(2)\ \text{if } f(b)=a_i, \text{ then } b\leq a_i.$$

Since g is a projection, there exists $S\in\mathfrak{S}(h(b))$ such that $S\subseteq g[[f(b))_C]$. It will suffice to show that $g[[f(b))_C]\subseteq h[[b)_B]$.

Suppose $c\geq_C f(b)$. If $c\in A$, then by (1) either $c=f(b)$ or $c\geq b$. If $c\notin A$, then $c>a_i$ for some i with $a_i\geq f(b)$. By (1), $a_i=f(b)$ or $a_i\geq b$; by (2), $a_i\geq b$. So if $c>a_i\geq f(b)$, then $c>b$. We conclude that

$$[f(b))_C\subseteq\{f(b)\}\cup[b)_B.$$

Since f is the identity on A, $h(c)=g(c)$ for all $c\in C$. Therefore

$$g[[f(b))_C]=h[[f(b))_C]\subseteq h[\{f(b)\}\cup[b)_B]=h\{f(b)\}\cup h[[b)_B]=h[[b)_B],$$

since $hf(b)=gf(b)=h(b)$. □

9.7 To combine first order definitions, we must relativize $\mathbb{L}_0$-sentences.

Definition. Let $\alpha\in\mathbb{L}_0$, and let u be an individual variable that does not occur bound in α. The relativization α^u of α is defined inductively as follows:

(i) $\alpha^u=\alpha$ for atomic α and $\alpha=\bot,\top$;
(ii) $(\beta\to\gamma)^u=\beta^u\to\gamma^u$, and similarly for $\wedge$, $\vee$, $\neg$;
(iii) $(\forall v\beta)^u=\forall v(v\geq u\to\beta^u)$ and $(\exists v\beta)^u=\exists v(v\geq u\wedge\beta^u)$.

Observe that $A\vDash\alpha^u[a]$ iff $[a)_A\vDash\alpha$.

9.8 Lemma. Let $\mathbf{K}$ be a class of frames, and $\vdash\varphi\leftrightarrow\psi\wedge\chi$. Then if $\psi,\chi\in \mathrm{E}(\mathbf{K})$, $\varphi\in \mathrm{E}(\mathbf{K})$ as well.

Proof: If $\psi\equiv_{\mathbf{K}}\alpha$ and $\chi\equiv_{\mathbf{K}}\beta$, then $\varphi\equiv_{\mathbf{K}}\alpha\wedge\beta$. □

9.9 Theorem. For all $n\in\mathbb{Z}^+$, $\mathrm{E}(\mathbf{T}_n)=\mathbb{I}$.

Proof: Induction over n. Since $\mathbf{T}_1=\emptyset$, the basis is trivial.
Suppose that $\mathrm{E}(\mathbf{T}_n)=\mathbb{I}$, and $\varphi\in\mathbb{I}$. We want to prove that $\varphi\in \mathrm{E}(\mathbf{T}_{n+1})$.
Let $\mathbb{S}$ be the collection of all full realizable subsequents $\Sigma\subseteq\mathrm{Sf}(\varphi)$. $\mathbb{S}$ is partially ordered by the relation $\preceq$ defined in 8.3.1. We shall prove

$$\forall\Sigma\in\mathbb{S}.\psi_\Sigma\in \mathrm{E}(\mathbf{T}_n)$$

by induction on the number of strict successors of Σ in $(\mathbb{S},\preceq)$. This will suffice by 9.8 and

(1) Let $\mathbb{S}'=\{\Sigma\in\mathbb{S}|\mathrm{F}\varphi\in\Sigma\}$. Then $\vdash\varphi\leftrightarrow\bigwedge_{\Sigma\in\mathbb{S}'}\psi_\Sigma$.

This statement holds because for any model $\mathfrak{A}$ and any point a in $\mathfrak{A}$, $a\not\Vdash\varphi$ iff $\exists\Sigma\in\mathbb{S}'.a\Vdash\Sigma$. (Take $\Sigma=\Theta_\varphi^{\mathfrak{A}}(a)$.) Since if $a\Vdash\varphi$ no $a'\geq a$ realizes any $\Sigma\in\mathbb{S}'$, $\vdash\varphi\to\bigwedge_{\Sigma\in\mathbb{S}'}\psi_\Sigma$. Conversely, $a\Vdash\bigwedge_{\Sigma\in\mathbb{S}'}\psi_\Sigma$

implies that $a \not\Vdash \Sigma$ for all $\Sigma \in \mathbb{S}'$.
Now let $\Sigma \in \mathbb{S}$. Let $\mathbb{S}_1,\ldots,\mathbb{S}_k$ be all the subsets $\mathbb{S}''$ of $\mathbb{S}$ such that

(i) $\Sigma \prec \mathbb{S}''$;

(ii) if $F(\psi \to \chi) \in \Sigma$ and $T\psi \notin \Sigma$, then some $\Sigma' \in \mathbb{S}''$ contains $F(\psi \to \chi)$.

By our major induction hypothesis – $E(\mathbf{T}_n) = \mathbb{I}$ – we have an $\mathbb{L}_0$-sentence β such that

$$(2)\ \neg\beta \equiv_{\mathbf{T}_n} \psi_\Sigma.$$

The second induction hypothesis is

(3) for all $\Sigma' \in \cup_{1\le i\le k}\mathbb{S}_i$, we have an $\mathbb{L}_0$-sentence $\beta_{\Sigma'}$ such that $\neg\beta_{\Sigma'} \equiv_{\mathbf{T}_{n+1}} \psi_{\Sigma'}$.

Let β_n be an $\mathbb{L}_0$-sentence such that

(4) $\forall A \in \mathbf{TR} : A \vDash \beta_n$ iff $A \notin \mathbf{T}_n$.

Suppose $\mathbb{S}_i = \{\Sigma_{i,1},\ldots,\Sigma_{i,m_i}\}$, $1 \le i \le k$. Using (3) we can construct $\mathbb{L}_0$-sentences $\delta_i = \delta_i(u, v_1,\ldots,v_{m_i})$ such that

(5) $\forall A \in \mathbf{TR} : A \vDash \delta_i[a, a_1,\ldots,a_{m_i}]$ iff $a_1,\ldots,a_{m_i}$ are distinct covers of a,
and for $1 \le j \le m_i$, $[a_j)_A \vDash \beta_{\Sigma_{i,j}}$

Take δ_i=

$$\bigwedge_{1\le j\le m_i} [u<v_j \wedge \forall v(u \le v < v_j \to v \le u) \wedge$$

$$\bigwedge(\neg v_j \le v_j \mid 1 \le j_1 \ne j_2 \le m_i) \wedge \bigwedge_{1\le j\le m_i}(\beta_{\Sigma_{i,j}})^{v_j}].$$

The formulas δ_i correspond with case (A) of 9.4.
Using (3) and (4) we can construct $\mathbb{L}_0$-sentences $\varepsilon_i = \varepsilon_i(u, v,\ldots,v_{m_i})$, for $1 \le i \le k$, such that

(6) $\forall A \in \mathbf{TR} : A \vDash \varepsilon_i[a, a_1,\ldots,a_{m_i}]$ iff $\{a, a_1,\ldots,a_{m_i}\}$ is an antichain,
$[a_j) \vDash \beta_{\Sigma_{i,j}}$ for $1 \le j \le m_i$, $[a)_A \vDash \beta_n$,
and every a_j ($1 \le j \le m_i$) covers a predecessor of a.

Take $\varepsilon_i =$

$$\bigwedge(\neg\mathrm{Comp}(u,v_j)|\ 1\leq j\leq m_i) \wedge \bigwedge(\neg\mathrm{Comp}\ (v_j,v_l)|\ 1\leq j<l\leq m_i) \wedge$$

$$\bigwedge(\ (\beta_{\Sigma_{i,j}})^{v_j}\ |\ 1\leq j\leq m_i\ ) \wedge (\beta_n)^u \wedge$$

$$\bigwedge(\forall v(v<v_j\rightarrow v<u)\ |\ 1\leq j\leq m_i\ ).$$

The ε_i are meant for case (B) of 9.4.
Now let γ be

$$\exists w[\neg\beta_n\wedge\beta]^w \vee$$
$$[\beta_n\wedge\bigvee_{1\leq i\leq k}(\ \exists uv_1,...,v_{m_i}\delta_i \vee\forall w(\ \beta_n\rightarrow\exists uv_1,...,v_{m_i}\varepsilon_i\ )^w)]$$

(The first disjunct of γ deals with the possibility that φ is refutable in a subframe $[a)_A\in \mathbf{T}_n$.) Our aim is to prove that

$$\psi_\Sigma\equiv_{\mathbf{T}_{n+1}}\neg\gamma.$$

Let $A\in \mathbf{T}_{n+1}$; and $F_A=\{a\in A\mid [a)\notin \mathbf{T}_n\}$, as in 9.4.

(I) Suppose $(A,V)\nVdash\psi_\Sigma$, and $A\vDash\neg\exists w[\neg\ \beta_n\wedge\beta]^w$. Then $F_A\supseteq\{a\in A\ |a\ \nVdash\psi_\Sigma\}$, by (2). Let $F^*:=\{a\in A|a\Vdash\Sigma\}$. By 9.5, F^* is a subset of F_A. For $a\in A$, let

$$\Theta(a)=\Theta_\varphi{}^{(A,V)}(a)=\ \{\sigma\in \mathrm{Sf}(\mathrm{F}\varphi)|a\Vdash\sigma\};$$

so $\Theta(a)\in \mathbb{S}$ for all $a\in A$. We distinguish two cases:

1. F^* has a maximal element a^*. Collect covers $a_1,...,a_m$ of a^* such that

(a) if $\mathrm{F}(\psi\rightarrow\chi)\in\Sigma$ and $\mathrm{T}\psi\notin\Sigma$, then for some a_j $(1\leq j\leq m)$, $\mathrm{F}(\psi\rightarrow\chi)\in\ \Theta(a_j)$;

(b) if $1\leq l\neq j\leq m$, then $\Theta(a_l)\neq\Theta(a_j)$.

Then for some i between 1 and k, $\{\Theta(a_1),...,\Theta(a_m)\}=\mathbb{S}_i$, and since by (3) $[a_j)\vDash\beta_{\Theta(a_j)}$ for $1\leq j\leq m$, $A\vDash\delta_i[a^*,a_1,...,a_m]$ by (b) and (5).

2. There is no maximal element in F^*; then F^* is an infinite upwards closed chain in A, by 9.4 and since $F^*\subseteq F_A$. Suppose $b_0\Vdash\Sigma$. Then $a_1,...,a_m\notin F^*$ may be found satisfying (a) and (b) of case 1,

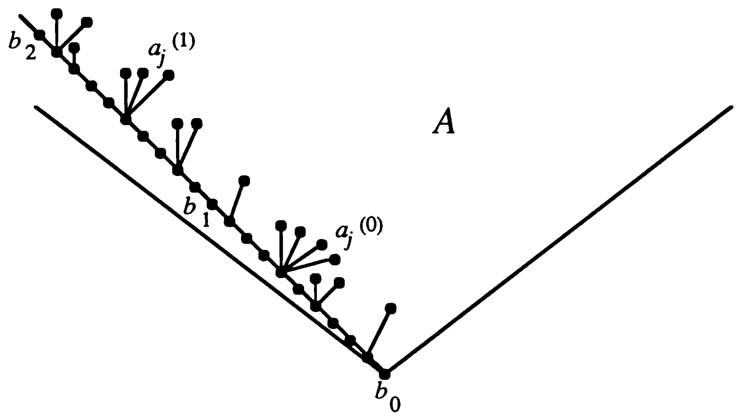
and covering elements of $F^*\cap[b_0)$. Then $a_1,...,a_m$ are mutually incomparable; by (3), $[a_j)\models\beta_{\Theta(a_j)}$ $(1\leq j\leq m)$. Since F^* is infinite and A is a tree, there exists $b\in F^*$ incomparable with $a_1,...,a_m$. Again, for some i $(1\leq i\leq k)$, $\{\Theta(a_1),...,\Theta(a_m)\}=\mathbb{S}_i$. By (6), we have

$$A\models\varepsilon_i\,[b_1,a_1,...,a_m].$$

This process may be repeated with b_1 instead of b_0, and so on ad infinitum. At least one $\mathbb{S}_i$ must recur infinitely often; then

$$A\models\forall w(\beta_n\rightarrow\exists uv_1...v_{m_i}\varepsilon_i)^w.$$

(II) Suppose $A\models\gamma$; we must show that $A\nVdash\psi_\Sigma$.
If $A\models\exists w\,[\neg\beta_n\wedge\beta]^w$, then $A\nVdash\psi_\Sigma$ by (2) and 2.4.1.
If $A\models\delta_i\,[a,a_1,....,a_{m_i}]$, then we can apply lemma 9.6.3 with $C=\{a\}$ and $B=\{a_1,...,a_{m_i}\}$, proving $\{a\}\cup\bigcup([a_j)|1\leq j\leq m_i)\nVdash\psi_\Sigma$. By lemma 2.11.6, $\{a,a_1,....,a_{m_i}\}$ is a strong subtree of A; hence by lemma 9.6.4, $A\nVdash\psi_\Sigma$.
If $A\models\beta_n\wedge\forall w(\beta_n\rightarrow\exists uv_1,...,v_{m_i}\varepsilon_i)^w$, we can construct a strong subtree B of A as follows. Let b_0 be the root of A; then $b_0\in F_A$, and there are $a_1^{(0)},...,a_{m_i}^{(0)}$, $b_1\geq b_0$ such that $A\models\varepsilon_i[b_1,a_1^{(0)},...,a_{m_i}^{(0)}]$. Then $B_0:=(b_1]\cup\{a_1^{(0)},...,a_{m_i}^{(0)}\}$ is a strong subtree of A, with endpoints $a_1^{(0)},...,a_{m_i}^{(0)}$ and b_1, and $b_1\in F_A$. (The canonical embedding $B_0\hookrightarrow A$ is strong by lemma 2.11.6.) Continuing with b_1, and so on, we find $(B_n|n\in\mathbb{N})$ such that $B:=\bigcup_nB_n$ is a strong subtree of A, as shown in the picture, with endpoints $a_1^{(n)},...\,a_{m_i}^{(n)}$ ($n\in\mathbb{N}$).



Since $[a_j^{(n)})\models\beta_{\Sigma_{i,j}}$, $[a_j^{(n)})\nVdash\psi_{\Sigma_{i,j}}$ by (3). Since $\mathbb{S}_i$ satisfies (i) and (ii) above, we get

$$B \cup \bigcup ([a_j^{(n)})_A \mid n \in \mathbb{N},\ 1 \le j \le m_i) \not\Vdash \psi_\Sigma$$

by lemma 9.6.3. By lemma 9.6.4 we conclude that $A \not\Vdash \psi_\Sigma$. □

9.10 Definition. The class of all finite binary trees will be denoted by $\mathbf{FTR}_2$.

9.11 Theorem. If $\mathbf{TR}^{(2)} \subseteq \mathbf{K} \subseteq \mathbf{TR}$, then $E(\mathbf{K}) = \{\varphi \in \mathbb{I} \mid \vdash \varphi$ or $\mathbf{FTR}_2 \not\Vdash \varphi\}$.

Proof: Let $\Phi := \{\varphi \in \mathbb{I} \mid \vdash \varphi$ or $\mathbf{FTR}_2 \not\Vdash \varphi\}$. Suppose $\mathbf{TR}^{(2)} \subseteq \mathbf{K} \subseteq \mathbf{TR}$. Then obviously $E(\mathbf{TR}) \subseteq E(\mathbf{K}) \subseteq E(\mathbf{TR}^{(2)})$. We claim that (1) $E(\mathbf{TR}^{(2)}) \subseteq \Phi$, and (2) $\Phi \subseteq E(\mathbf{TR})$.
As to (1), let $\psi \notin \Phi$, and suppose $\psi \equiv_{\mathbf{TR}^{(2)}} \alpha$, with $\mathrm{rnk}(\alpha) = n$; set $m = 2^n - 1$. Since $\not\vdash \psi$, by lemma 9.1 there is a binary tree A in which ψ is not valid (since $\psi \notin \Phi$, A must be infinite), and in which every point has maximal successors. We may extend A to a full binary tree satisfying $P(n)$ (cf 8.14); then still $B \not\Vdash \psi$, since A is a p-retract of B(2.11.12). Hence $B \not\models \alpha$. Since F_m satisfies $P(n)$, $F_m \not\models \alpha$ by 8.15. Consequently, $F_m \not\Vdash \psi$, contradicting $\psi \notin \Phi$.
To settle (2), suppose $\varphi \in \Phi$. If $\vdash \varphi$, then $\varphi \equiv \top$. If $\mathbf{FTR}_2 \not\Vdash \varphi$, then for some m, $F_m \not\Vdash \varphi$ (as before, extending a given binary tree and using 2.11.12). By 9.9, there is some $\alpha \in \mathbb{L}_0$ such that $\varphi \equiv_{\mathbf{T}_m} \alpha$. Take $\beta_m \in \mathbb{L}_0$ such that $A \models \beta_m$ iff $A \notin \mathbf{T}_m$: then $\varphi \equiv_{\mathbf{TR}} \alpha \wedge \neg \beta_m$. For suppose $A \models \beta_m$; then F_m is a p-morphic image of A by 2.11.12, hence $A \not\Vdash \varphi$. □

9.12 Corollary. If $\mathbf{TR}^{(2)} \subseteq \mathbf{K} \subseteq \mathbf{TR}$, then $E(\mathbf{K})$ is decidable.

Proof: We have seen in §3 how to decide whether $\varphi \in \mathbb{I}$ is universally valid. A related procedure decides whether $\mathbf{FTR}_2 \Vdash \varphi$: try to construct a monotonic refutation of φ in which $|S(x)| \le 2$ for each sequent x. If there is such a refutation, then the induced model can be unfolded to a finite binary tree in the standard way. Conversely, if $A \in \mathbf{FTR}_2$ and $(A,V) \not\Vdash \varphi$, then a refutation as described can be obtained from (A,V) by the method of §8. Begin with a maximal node $a_0 \not\Vdash \varphi$, and let $\Theta(a_0) = \Theta_\varphi^{(A,V)}(a_0)$ be the root of the tableau. If a_1, a_2 are the covers of a_0, $\Theta(a_0) \prec \Theta(a_1), \Theta(a_2)$ is guaranteed by the maximality of a_0. Set $S(\Theta(a_0)) = \{\Theta(a_1), \Theta(a_2)\}$. Repeat this for maximal a_i' with $\Theta(a_i') = \Theta(a_i)$, for $i=1,2$ – for $i=1$ only if $\Theta(a_1) = \Theta(a_2)$. Continue in this way with new sequents $\Theta_\varphi^{(A,V)}(a)$, up to the endpoints of A. Since in a simple tableau each sequent has just one successor set, you need not look at successors of a if $\Theta(a)$ has been processed earlier in the construction.

□

9.13 Examples

(a) Recall the stability principles (6.4). Example 8.9 implies that $\mathbf{FTR}_2 \Vdash \mathrm{SP}_2$. By very similar reasoning, $\mathbf{FTR}_2 \Vdash \mathrm{SP}_n$ for all $n>1$. Hence by 9.11 and 6.4, $\mathrm{SP}_n \in \mathrm{E}(\mathbf{TR})$ iff $n=1$.

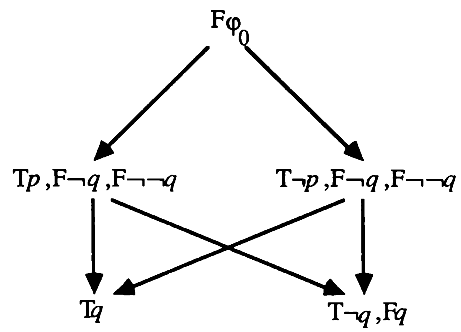
(b)Let $p_0,\dots,p_n$ $(n\geq 1)$ be distinct proposition letters. The n-ary branching restriction BR_n is the formula

$$\bigwedge_{i\leq n}((p_i \to \bigvee_{j\neq i} p_j) \to \bigvee_{j\neq i} p_j) \to \bigvee_{i\leq n} p_i.$$

(De Jongh & Gabbay [1974]). It is known (and may be checked in a straightforward manner) that BR_n is refutable in a finite tree A iff A is not n-ary. Consequently, $\mathrm{BR}_n \in \mathrm{E}(\mathbf{TR})$ iff $n=1$.

9.14 One might have hoped that the above procedure would also work for **DLO**. It does not. In particular, $\mathrm{E}(\mathbf{DLO}) \neq \mathrm{E}(\mathbf{TR})$, and if we let $\mathbf{D}_n$ be the class of those downwards linear orderings in which F_n cannot be embedded, $\mathrm{E}(\mathbf{D}_3) \neq \mathbb{I}$. (N.B.: $\mathrm{E}(\mathbf{D}_2)=\mathrm{E}(\mathbf{LO})=\mathbb{I}$, since frames in $\mathbf{D}_2$ are disjoint unions of linear orderings.)

Example. Let $\varphi_0 := (p \to \neg q \vee \neg\neg q) \vee (\neg p \to \neg q \vee \neg\neg q)$. We sketch a refutation of φ_0:



φ_0 is deterministic. Projection of F_3 onto the tableau above is the only $\mathrm{F}\varphi_0$-labeled frame with downwards linear domain. By lemma 7.3 and theorem 7.4, for $A \in \mathbf{DLO}$ we have

$A \not\Vdash \varphi_0$ iff A has a minimal $\mathrm{F}\varphi_0$-labeled subframe iff F_3 can be embedded in A

iff $A \notin \mathbf{D}_3$.

Hence $\mathbf{FTR}_2 \nVdash \varphi_0 \wedge SP_2$. By 9.11, $\varphi_0 \wedge SP_2 \in E(\mathbf{TR})$. But $\varphi_0 \wedge SP_2 \notin E(\mathbf{D}_3)$: consider example 6.8(*b*). In the frame A in that example, $\varphi_0 \wedge SP_2$ is valid; $\mathbf{D}_3$ is elementary, and closed under disjoint unions and generated subframes; so, since $\varphi_0 \wedge SP_2$ is not valid the ultrapower $\prod_U A$, $\varphi_0 \wedge SP_2 \notin E(\mathbf{D}_3)$ by 6.7.6.

§10. Finite frames

This section just contains a few examples. We show that the formulas BR_n, of which only the first belongs to E(**DLO**), are elementary on **FPO**. A theorem of Doets [B] (stated in 10.3) provides a method to prove nonelementarity. We shall find that not only E(**FPO**)$\not\subseteq$E(**DLO**), but E(**DLO**)$\not\subseteq$ E(**FPO**) as well.

10.1 Branching restrictions

Let $\beta_n(u,v_0,...,v_n)$ $(n \geq 1)$ be the $\mathbb{L}_0$-formula

$$\bigwedge_{i\leq n} u<v_i \wedge \bigwedge (\neg v_i \leq v_j \mid i,j\leq n \text{ and } i\neq j) \wedge \\ \bigwedge_{i<j\leq n} \forall w\,(u\leq w\leq v_i,v_j \to \bigwedge_{k\leq n} w\leq v_k).$$

Then $A \models \beta_n[a,a_0,...,a_n]$ iff $\{a_0,...,a_n\}$ is an antichain and a branches into $\{a_0,...,a_n\}$. Set

$$\alpha_n := \forall u\ v_0 \ldots v_n\ \neg\beta_n.$$

Recall the formulas BR_n $(n\geq 1)$ of 9.13(b). Quite analogously to that example, it may be shown that BR_n is refutable in a finite partial ordering A iff $A \models \exists u\ v_0 \ldots v_n\ \beta_n$; thus $\mathrm{BR}_n \equiv_{\mathbf{FPO}} \alpha_n$. (Cf. the remark below.)

<u>Remark.</u> On the class of all frames, the formulas BR_n are essentially second order. Yet in a way they are not very complex. We can state a necessary and sufficient condition for BR_n to be refutable, in terms of only points and the ordering:

(*) $\quad A \not\Vdash \mathrm{BR}_n$ iff $A \Vdash \exists u\ v_0 \ldots v_n\ \beta_n$ or A contains an infinite comb.

(*) can be formulated in the infinitary language $L\omega_1\omega_1$. We will show how to construct a valuation refuting BR_n if $A \Vdash \exists u\ v_0 \ldots v_n\ \beta_n$; the other case is similar (cf. the treatment of SP_2 in 3.8). Suppose $A \models \beta_n[a,a_0,...,a_n]$. We may assume that a is the root of A (2.4.1). We are to define a valuation V on A such that

$$(A,V,a) \not\Vdash \bigwedge_{i\leq n} ((p_i \to \bigvee_{j\neq i} p_j) \to \bigvee_{j\neq i} p_j) \to \bigvee_{i\leq n} p_i.$$

For $a' \in A$, let

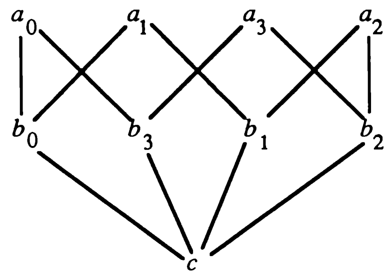
$$a' \notin V(p_i) \text{ iff } \exists j{\neq}i\text{: } a' \leq a_j.$$

Then $V(p_i)$ is upwards closed, for if $a' \notin V(p_i)$, then a' precedes some a_j with $j{\neq}i$; then so does every predecessor of a'.

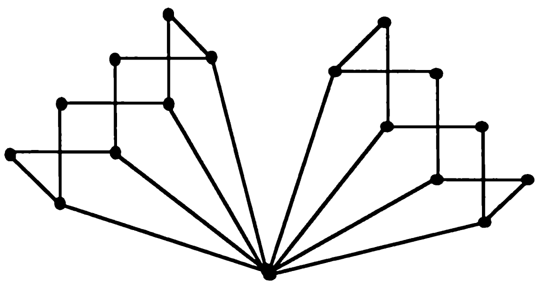
It is easy to see that $a \nVdash p_i$, for all $i \leq n$; and that $a_i \Vdash p_i$, while $a_i \nVdash p_j$ for all $j{\neq}i$.

By β_n, any $a' \in A$ either precedes all the a_i ($i{\leq}n$), or precedes at most one a_i. If $a' \leq a_i$, then $a' \nVdash p_i \rightarrow \bigvee_{j\neq i} p_j$. Hence if $a' \leq a_0,\ldots,a_n$, $a' \nVdash p_i \rightarrow \bigvee_{j\neq i} p_j$ for all $i{\leq}n$. If a' precedes only a_i, then $a' \Vdash p_i$, so $a' \Vdash \bigvee_{j\neq i'} p_j$ for every $i'{\neq}i$. Finally, if a' does not precede any of $a_0,\ldots,a_n$, then $a' \Vdash p_i$ for all $i{\leq}n$. Therefore $a \Vdash \bigwedge_{i\leq n} ((p_i \rightarrow \bigvee_{j\neq i} p_j) \rightarrow \bigvee_{j\neq i} p_j)$. Since $a' \nVdash \bigvee_{i\leq n} p_i$, $a' \nVdash \mathrm{BR}_n$.

10.2 For each $n \in \mathbb{Z}^+$ we define a sequence of frames A_n, A_n', A_n'', ... as follows. A_n consists of nodes a_i ($i{<}n$), b_i ($i{<}n$) and c; c is the root, Cov(c) (the set of covers of c) is $\{b_0,\ldots,b_{n-1}\}$, Cov(b_i) $= \{a_i,a_j\}$, where $j{\equiv}i{+}1$ (mod n), and the a_i are endpoints. Below is a diagram of A_4.



Let us write s$^{(m)}$ for s with m primes. The frame $A_n^{(m)}$ consists of $m{+}1$ copies of A_n, with the roots identified. Formally, we put $A_n^{(m)} = \{c\} \cup \{a_i^{(k)},b_i^{(k)} \mid k{\leq}m, i{<}n\}$. The order extends that on A_n by Cov(c) $= \{b_i^{(k)} \mid k{\leq}m, i{<}n\}$, Cov($b_i^{(k)}$) $= \{a_i^{(k)}, a_j^{(k)}\}$ with $j{\equiv}i{+}1$ (mod n); the $a_i^{(k)}$ are endpoints. Below is a diagram of A_4':



10.3 Proposition (Doets). If $l \geq 2^{n+2}-3$, then $A_l \equiv^n A_n'$.

10.4 Example (Doets). Recall from 6.8(a) that

$$\mathrm{SC}:= [(\neg\neg p \to p) \to p \vee \neg p] \to \neg p \vee \neg\neg p.$$

The frame A in 6.8(a) is one half of an infinite version of the frames A_l defined above. In particular, $A_l \Vdash$ SC for the same reason as $A \Vdash$ SC: if $(A_l,V) \nVdash$ SC, then $c \Vdash (\neg\neg p \to p) \to p \vee \neg p$, and there must be a_i, a_j with $j \equiv i+1$ (mod l), one of which forces p, while the other forces $\neg p$. Then $b_i \Vdash \neg\neg p \to p$, hence $b_i \Vdash p \vee \neg p$, so $b_i \Vdash p$ or $b_i \Vdash \neg p$. But whatever b_i forces, a_i and a_j must *both* force: which they fail to do.
On the other hand, it is easy to see that $A_l' \nVdash$ SC: let $V(p) = \{a_i \mid i<l\}$; then $b_i \nVdash \neg\neg p \to p$, $b_i' \Vdash \neg p$, and $c \Vdash (\neg\neg p \to p) \to p \vee \neg p$, $c \nVdash \neg p \vee \neg\neg p$.
Now suppose that SC is elementary on **FPO**; say SC $\equiv_{\mathbf{FPO}}$ α. Let rnk(α) = n and $m \geq 2^{n+2}-3$. Then $A_m \models \alpha$, so $A_m' \models \alpha$ by 10.3; hence $A_m' \Vdash$ SC by the assumed equivalence, contrary to what has just been shown. We conclude that SC $\notin$ E(**FPO**).

Remark: We will prove in the next section that SC $\in$ E(**DLO**) (11.4). Thus, the example shows that E(**DLO**) $\not\subseteq$ E(**FPO**).

10.5 Further examples

The proof of proposition 10.3 consists of a consideration of Ehrenfeucht games. Roughly, as l increases it takes longer ($\approx$ takes more quantifiers) to tell that an a_i' is further removed from an a_j than any a_k. This suggests the following, trivial, generalization:

Corollary. If $l \geq 2^{n+2}-3$, then $A_l \equiv^n A_l^{(m)}$.

(a) Recall that SP_2 is

$$(\neg\varphi\vee\neg\psi\vee\neg\chi \to \varphi\vee\psi\vee\chi) \to \neg\varphi\vee\neg\psi\vee\neg\chi$$

with $\varphi = p\wedge q$, $\psi = p\wedge\neg q$ and $\chi = \neg p\wedge q$. Similarly to the example above, one can show that $A_l \Vdash \mathrm{SP}_2$: if $c \Vdash \neg\varphi\vee\neg\psi\vee\neg\chi \to \varphi\vee\psi\vee\chi$, then since each b_i has only two strict successors, $b_i \Vdash \varphi\vee\psi\vee\chi$. If $b_i \Vdash \varphi$, and a_i, a_j are the covers of b_i, then a_i and a_j both force φ; then b_j must force φ as well. Proceeding in this way, we find that *all* b_i force φ (or all force ψ, or all force χ). So $A_l \Vdash \mathrm{SP}_2$, and since $A_l^{(2)} \nVdash \mathrm{SP}_2$ (cf. 8.9), it follows that $\mathrm{SP}_2 \notin$ E(**FPO**).

(*b*) SP_2 was generalized to a sequence $(SP_n)_{n\in\mathbb{Z}^+}$ in 6.4. It can be shown, by an argument resembling that of (*a*), that $A_l \Vdash SP_n$ for $n>2$; it is straightforward that $A_l^{(n)} \nVdash SP_n$. Thus $SP_n \notin E(\mathbf{FPO})$.

Recall that SP_n was defined as

$$(\bigvee_{j\leq n} \neg\varphi_j \rightarrow \bigvee_{j\leq n} \varphi_j) \rightarrow \bigvee_{j\leq n} \neg\varphi_j,$$

with $\varphi_j = \bigwedge_{i<k} (\neg)^{f_j(i)} p_i$, where k is the least number such that $2^k \geq n+1$, $f_0, \ldots, f_{2^k-1}$ are the functions from $\{0,\ldots,k-1\}$ into $\{0,1\}$ in lexicographical order (0 preceding 1), and $(\neg)^0 p_i = p_i$, $(\neg)^1 p_i = \neg p_i$. A necessary and sufficient condition for SP_n to be refutable in an arbitrary frame A seems essentially harder to formulate than for BR_n. The following would do:

> for some $a \in A$, there is a partition $\{A_0,\ldots,A_n,B\}$ of $[a)_A$ in which $A_0,\ldots,A_n$ are upwards closed, $a \in B$, and every element of B has successors in each A_j ($j\leq n$).

The idea is to make $A_j = V(\varphi_j)$. This condition is more complex than (*) in 10.1 in that it involves quantification over sets.

10.6 On **DLO** every $\mathbb{I}$-formula has an equivalent in $\mathbb{I}[\vee,\rightarrow]$ (theorem 4.7.6); on **FPO** this is not true (cf. 4.5). Now the contrast between the formulas BR_n (first order definable by 10.1) and the formulas SP_n raises a question, which does not seem easy to answer:

> are all $\mathbb{I}[\vee,\wedge,\rightarrow]$-formulas in E(**FPO**)?

The answer might help to clarify the role of negation.

10.7 Δ-definability

Every subclass of **FPO** that is closed under isomorphism is Δ-elementary. For suppose $\mathbf{K} \subseteq \mathbf{FPO}$ is closed under isomorphism. For every $A \in \mathbf{FPO} - \mathbf{K}$, we can take the conjunction δ_A of the diagram of A, and turn this into an $\mathbb{L}_0$-sentence δ_A' by exchanging constants for variables and quantifying existentially. For every $n \in \mathbb{N}$, there is an $\mathbb{L}_0$-sentence κ_n such that, for $A \in \mathbf{FPO}$,

$$A \models \kappa_n \quad \text{iff} \quad |A| = n.$$

Then **K** is defined by $\{\neg(\delta_A' \wedge \kappa_{|A|}) \mid A \in \mathbf{FPO} - \mathbf{K}\}$.

It follows that on **FPO** every **I**-formula is Δ-elementary. So in **FPO** Δ-elementary $\mathbb{I}$-formulas need not be elementary; in contrast to the elementary frame classes (cf. 1.9).

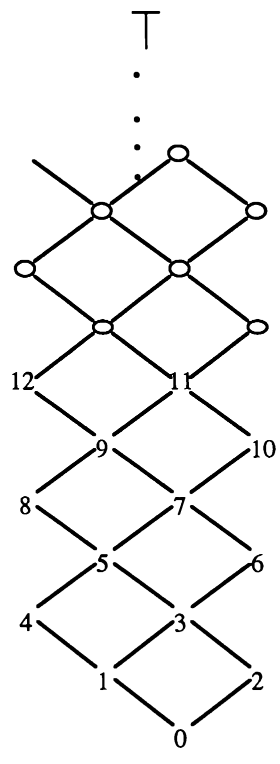
§11. Monadic formulas

For a proposition letter p, we let $\mathbb{I}(p)$ be the set of all $\mathbb{I}$-formulas that contain no proposition letters other than p. The logical structure of these 'monadic' formulas is completely known. Using this structure, we prove that all monadic formulas are elementary on **DLO** (11.4), and establish for the classes **PO** and **FPO** which monadic formulas are elementary on them. The results are tabulated in 11.9.

11.1 The Rieger-Nishimura Lattice

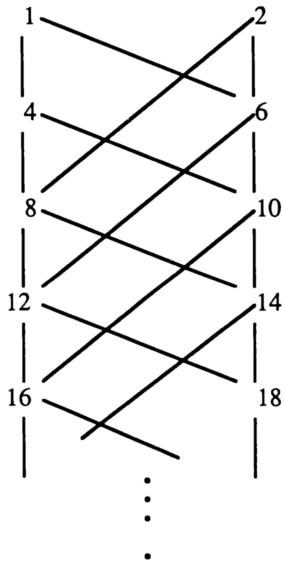
Let $(M_n \mid n \in \mathbb{N})$ be the following sequence of $\mathbb{I}(p)$-formulas:

$$M_0 = \bot,\ M_1 = p,\ M_2 = \neg p;$$
$$\text{for odd } n>2,\ M_n = M_{n-2} \vee M_{n-1};$$
$$\text{for even } n>2,\ M_n = M_{n-2} \rightarrow M_{n-3}.$$



It may be shown, by patient induction, that every $\mathbb{I}(p)$-formula is logically equivalent to some M_n, or universally valid (logically equivalent to $\top$). (This was first done by Rieger [1949], and independently by Nishimura [1960]. An idea of the proof can also be gained from Gabbay [1981].) Similarly it may be established that $\vdash M_n \to M_k$ precisely when this is (inductively) obvious from the definition. We get the following neat picture of the Lindenbaum algebra of intuitionistic propositional logic restricted to $\mathbb{I}(p)$ (the free Heyting algebra on one generator[1], sometimes referred to as the Rieger-Nishimura lattice). (On the previous page.)

There is a general method for turning complete Heyting algebras (= pseudo-Boolean algebras) such as this into equivalent[2] frames (Raney [1952]). It consists in selecting those elements of the Heyting algebra that are not the join of all strictly lower elements, and inverting the ordering. The above then becomes (observe that $0 = \bigvee\emptyset$) the following frame M:



11.2 Examples. Several $\mathbb{I}(p)$-formulas have appeared in earlier sections. The principle of excluded middle, $p\vee\neg p$ (6.3(a)) is M_3. By the rules for constructing the sequence $(M_n)_n$, M_4 would be $\neg p\to p$. Here, however, a simplification is in order: $\vdash(\neg p \to p)\leftrightarrow\neg\neg p$, hence we may take $M_4 = \neg\neg p$. A more sweeping simplification is as follows: for odd $n\geq 5$, $M_n=M_{n-1}\vee M_{n-2}= (M_{n-3}\to M_{n-4})\vee M_{n-3}\vee M_{n-4}$; since $\vdash M_{n-4} \to (M_{n-3} \to M_{n-4})$, we may reduce M_n to $M_{n-1}\vee M_{n-3}$. Then $M_5=\neg p\vee\neg\neg p$ =KC (1.4, 2.6). Next, M_6 would be $\neg\neg p\to p \vee\neg p$; simplifying once more, we let $M_6=\neg\neg p\to p$. In fact, for *all* even $n\geq 6$ we may take $M_n=M_{n-2}\to M_{n-5}$: for $M_n=M_{n-2}\to M_{n-4}\vee M_{n-5}$, and $M_{n-2}=M_{n-4}\to M_{n-5}$. (To algebraists all these reductions may have been clear from the diagram.) Then finally, SC $= [(\neg\neg p\to p)\to p \vee\neg p]\to\neg p \vee\neg\neg p =$

$[M_6 \to M_3] \to M_5 = M_8 \to M_5 = M_{10}$.

11.3 Minimal refutations.

There is an intimate connection between the frame M and the minimal refutations of the monadic formulas.

Define a valuation V on M by $V(p) = \{1\}$. Then $k \Vdash M_n$ iff $\vdash M_k \to M_n$, as can be shown by induction over the sequence $(M_n)_n$. Indeed, the cases n=0,1,2 are easily calculated. If $n>2$ and n is odd, $k \Vdash M_n$ iff $k \Vdash M_{n-1}$ or $k \Vdash M_{n-2}$. So by induction hypothesis, $k \Vdash M_n$ implies $\vdash M_k \to M_{n-1}$ or $\vdash M_k \to M_{n-2}$, hence $\vdash M_k \to M_n$. Conversely, if $\vdash M_k \to M_n$, then by the Rieger-Nishimura latice, $k=1$ and $n \geq 3$, whence $k \Vdash M_n$ by $k \Vdash M_k$; or k is even, and $n=k+1$ or $n>k+2$. If $n=k+1$, $M_n = M_k \vee M_{k-1}$, and $k \Vdash M_k$ by induction hypothesis. If $n>k+2$, then $\vdash M_k \to M_{n-1}$, and $k \Vdash M_k$ by induction hypothesis.

Let $\mathfrak{M} := (M,V)$. Define a refutation $\mathfrak{X}_n = (X_n,S_n)$ of M_n as follows. Let k be the highest element of M that does not force M_n in $\mathfrak{M}$. Then let X_n be the image under $\Theta^{\mathfrak{M}}{}_{FM_n}$ of $[k)_M$; and $x \in S_n(\Theta^{\mathfrak{M}}{}_{FM_n}(l))$ iff x is the image of an immediate successor of l in $[k)_M$.

In fact, $\mathfrak{X}_n$ is the *unique* minimal refutation of M_n. Note that k in the definition of $\mathfrak{X}_n$ is 1 if $n \in \{0,2\}$, $n+1$ if n is odd, and $n-2$ if $n \geq 4$ and n is even. The minimal refutation of M_3 is presented by

$$\begin{tikzcd} \boxed{F(p \vee \neg p), Fp, F\neg p} \arrow[d] \\ \boxed{Tp} \end{tikzcd}$$

If $n>4$ is odd, then FM_n decomposes to FM_{n-1}, FM_{n-3}; which gives rise to distinct strict successors $\{TM_{n-3}, FM_{n-4}\}$ and (if $n \geq 7$) $\{TM_{n-5}, FM_{n-6}\}$ ($\{Fp\}$ if $n=5$). If $n>4$ is even, FM_n reduces to $\{TM_{n-2}, FM_{n-3}\}$, with successors as for FM_{n-3}. Clearly $(X_n, \leq_{\mathfrak{X}_n}) \cong [k)_M$.

Thus each M_n has a single minimal refutation $\mathfrak{X}_n = (X_n, S_n)$. Moreover, if $x,y \in X_n$, and $x^T \subseteq y^T$, then $x \leq_{\mathfrak{X}_n} y$; therefore the induced ordering in *any* tableau (X_n, S) is $\leq_{\mathfrak{X}_n}$. Now suppose $f\colon A \twoheadrightarrow \mathfrak{X}$ is a minimal FM_n-labeled subframe of a frame A. Then every component of $\mathfrak{X}$ has $\mathfrak{X}_n$ as a subtableau. By minimality, each component has the same sequents as $\mathfrak{X}_n$ (cf. definition 7.2.2). Since $\leq_{\mathfrak{X}_n}$ is maximal, the induced frames of $\mathfrak{X}$ and $\mathfrak{X}_n$ must be the same.

Since every point of M has at most two covers, the result of unfolding a frame $[k)_M$ to a tree is binary. Let us denote this tree by T_k. Now suppose $A \in$ **DLO**, $\varphi \in \mathbb{I}(p)$, and $A \nVdash \varphi$. For some $n \in \mathbb{N}$, $\vdash \varphi \leftrightarrow M_n$, so $A \nVdash M_n$. By lemma 7.3, A has a minimal FM_n-labeled subframe $f\colon A \twoheadrightarrow \mathfrak{X}$. We have seen that we may assume $\mathfrak{X} = \mathfrak{X}_n$. Now if $[k)_n \cong X_n$, dom$f \cong T_k$, since $A \in$ **DLO**. This proves:

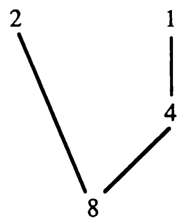
Lemma: For each $\varphi \in \mathbb{I}(p)$, there is some $k \in \mathbb{N}$ such that for all $A \in \mathbf{DLO}$, $A \not\Vdash \varphi$ only if T_k can be embedded in A.

11.4 As we have just seen, each formula M_n is refutable in a finite binary tree; so by theorem 9.11, M_n is elementary on the class of trees. The above lemma allows something stronger, however.

Theorem: $\mathbb{I}(p) \subseteq \mathrm{E}(\mathbf{DLO})$.

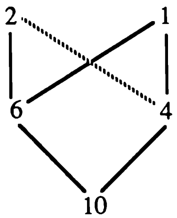
Proof: Suppose $\varphi \in \mathbb{I}(p)$; let k be as in the lemma. There is an $\mathbb{L}_0$-sentence δ_k such that for $A \in \mathbf{DLO}$, $A \vDash \delta_k$ iff T_k can be embedded in A. We show that $\varphi \equiv_{\mathbf{DLO}} \neg\delta_k$.
($\Rightarrow$) Suppose $A \in \mathbf{DLO}$, $A \vDash \delta_k$. Then by 2.11.12 there exists a p-morphism $g: A \twoheadrightarrow T_k$. Since $T_k \not\Vdash \varphi$, $A \not\Vdash \varphi$ by 2.4.2.
($\Leftarrow$) If $A \in \mathbf{DLO}$ and $A \not\Vdash \varphi$, then by the lemma T_k can be embedded in A; hence $A \vDash \delta_k$. □

11.5 We know from 6.8 that M_{10} is not elementary on $\mathbf{PO}$. Let us compare the frame A of 6.8(a) with $[8)_M$ (in which, by 11.3, M_{10} is not valid). We may picture $[8)_M$ thus:



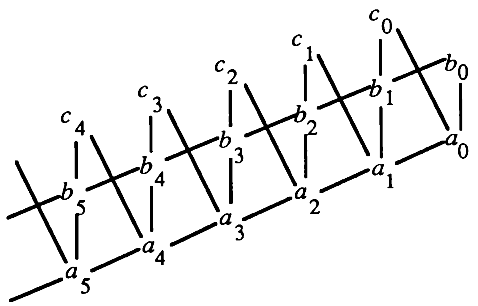
$[8)_M$

Now A may be regarded as an infinite series of copies of $[8)_M$ glued together, in which every time between 8 and 2 a predecessor of 1 has been interpolated. As shown in 6.8(a), these interpolations make it impossible to refute M_{10} in A. Now $[10)_M$, like $[8)_M$, is not wholly trivial; we show that it can be treated similarly. The germ of the construction is pictured below. (On the next page.)



As the reader can check, extending the ordering of $[10)_M$ by $4 \le 2$ (broken line) produces a frame in which M_{12} is valid.

Let $B = \{a_n, b_n, c_n \mid n \in \mathbb{N}\}$, ordered as follows: the covers of a_0 are b_0 and c_0, and for $n>0$, $\mathrm{Cov}(a_n) = \{a_{n-1}, b_n, c_n\}$; b_0 is a maximal element, and for $n>0$, $\mathrm{Cov}(b_n) = \{b_{n-1}, c_{n-1}\}$; every c_n is maximal. (See the diagram below.)



B

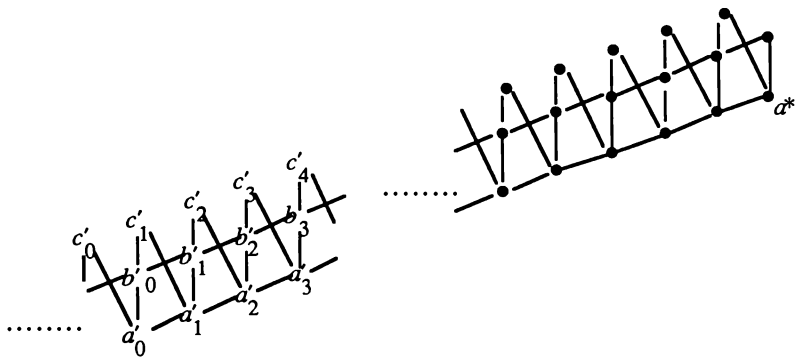
We show that $B \Vdash M_{12}$. Recall that M_{12} is

$$([(\neg\neg p \to p) \to p \vee \neg p] \to \neg p \vee \neg\neg p) \to \neg\neg p \vee (\neg\neg p \to p).$$

Suppose that $(B,V) \nVdash M_{12}$. Then there must be a partial projection f of B onto the minimal refutation of M_{12} – the induced frame of which is $[10)_M$ – with each $b \in \mathrm{dom}g$ realizing $f(b)$. In particular, there must be some $a \in B$ forcing M_{10} ($= [(\neg\neg p \to p) \to p \vee \neg p] \to \neg p \vee \neg\neg p$), and some $b>a$ forcing $\neg\neg p$ and not forcing p. a must be some a_n, since in $[b_n)$, $b_n{<}x{<}y \,\&\, b_n{<}x'{<}y'$ implies that x and x' are comparable – which excludes $\mathrm{dom}f \subseteq [b_n)$. a has successors forcing $\neg p$, so b may be assumed minimal: if $b'<b$, then $b' \nVdash \neg\neg p$. There exists $a_i < b$ with $a_{i-1} \nless b$; $a \le a_i$. Suppose $a_i \le x \Vdash \neg\neg p \to p$. Then $x>a_i$, because of b; hence any maximal successor y of x succeeds b, so $y \Vdash p$. It follows that $x \Vdash \neg\neg p$, so $x \Vdash p$. Therefore $a_i \Vdash (\neg\neg p \to p) \to p \vee \neg p$. Because $a \le a_i$, $a_i \Vdash M_{10}$; so $a_i \Vdash \neg p \vee \neg\neg p$. Since $a_i<b$, $a_i \nVdash \neg p$. So $a_i \Vdash \neg\neg p$, contradicting the minimality of b. We conclude that $B \Vdash M_{12}$.

As in 6.8, take a nonprincipal ultrafilter U over $\mathbb{N}$, and consider $B' = \prod_U B$. Reasoning parallel to

6.8(b), we see that B' ends in an isomorphic copy of B, below which a set $\{a'_n, b'_n, c'_n \mid n \in \mathbb{N}\}$ may be found such that $\mathrm{Cov}_{B'}(a'_n) = \{c'_n, b'_n, a'_{n+1}\}$, $\mathrm{Cov}_{B'}(b'_n) = \{c'_{n+1}, b'_{n+1}\}$, and each c'_n is maximal (as in the diagram below).



B'

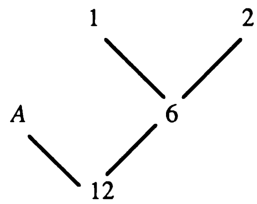
Let $a^* \in B'$ be the element of the isomorphic copy of B that corresponds with $a_0 \in B$. Define a valuation V' on B' by

$$\text{for all } b' \in B',\ b' \in V'(p) \text{ iff } b' \nleq a^* \text{ and for all } n \in \mathbb{N}, b' \nleq c'_n.$$

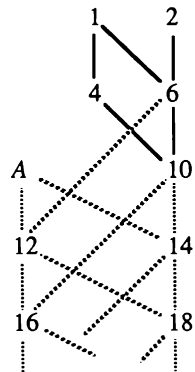
We will show that $(B',V',a'_0) \nVdash \mathrm{M}_{12}$. We consider $[a'_0)$ only. Let $W = \{a'_n, b'_n, c'_n\}_{n\in\mathbb{N}}$. Then $c'_n \Vdash \neg p$, for all $n \in \mathbb{N}$, and elements of $[a'_0)-W$ either precede a^*, or force p. Since $a^* \Vdash \{\mathrm{T}\neg\neg p, \mathrm{F}p\}$, $a'_0 \nVdash \neg\neg p \to p$; by $c'_n \Vdash \neg p$, $a'_0 \nVdash \neg\neg p$. Similarly, $b'_n \nVdash \neg\neg p$. It follows that $b'_n \nVdash \neg\neg p \to p$: if $b' \geq b'_n$ and $b' \Vdash \neg\neg p$, then $b' \nleq a^*$ and $b' \notin W$, so $b' \Vdash p$. Now suppose $b'' \geq a'_0$ and $b'' \Vdash (\neg\neg p \to p) \to p \vee \neg p$. Then $b'' \nleq b'_n$, for all $n \in \mathbb{N}$, since $b'_n \Vdash \neg\neg p \to p$ and $b'_n \nVdash p \vee \neg p$. So b'' is a c'_n, and forces $\neg p$; or $b'' \notin W$ and $b'' \Vdash \neg\neg p$. Therefore $a'_0 \Vdash [(\neg\neg p \to p) \to p \vee \neg p] \to \neg p \vee \neg\neg p$. We have shown that $a'_0 \nVdash \mathrm{M}_{12}$.
We conclude by 6.7.6 that $\mathrm{M}_{12} \notin \mathrm{E}(\mathbf{PO})$.

11.6 Theorem: For all $n \geq 10$, $\mathrm{M}_n \notin \mathrm{E}(\mathbf{PO})$.

Proof: The cases n=10, n=12 have been dealt with, in 6.8(a) and 11.5 respectively. We will assume that the frames A defined in 6.8(a) and M defined in 11.1 are disjoint.
Suppose n=11. Let $C_{12} = \{1,2,6,12\} \cup A$, with the ordering inherited from M and A extended by $12 \leq A$ (see diagram on next page).



Obviously, $[b] \Vdash \mathrm{M}_{10}$; we know from 6.8($a$) that $A \Vdash \mathrm{M}_{10}$. Thus, by a consideration in 11.3, $C_{12} \Vdash \mathrm{M}_{11}$: for $(C_{12}, V, 12) \nVdash \mathrm{M}_{11}$ requires that M_{10} be not forced in a strict successor of 12. Taking an ultrapower $\prod_U C_{12}$ over a nonprincipal ultrafilter U over $\mathbb{N}$ is tantamount to replacing A in C_{12} by $\prod_U A$, by Łoś's theorem (6.7.1). Now, there exists a p-morphism $f\colon \prod_U A \twoheadrightarrow [8)_M$, as shown in 6.8($a$). Extending f with the identity on $C_{12}-A$, we get a p-morphism $\prod_U C_{12} \twoheadrightarrow [12)_M$. Since $[12)_M \nVdash \mathrm{M}_{11}$ (as noted in 11.3), $\prod_U C_{12} \nVdash \mathrm{M}_{11}$ as well (by 2.4.2).
For n=14, the same frames and p-morphism work: since $\mathrm{M}_{14}=\mathrm{M}_{12}\to\mathrm{M}_{11}$, and $C_{12}\Vdash\mathrm{M}_{11}$, $C_{12}\Vdash\mathrm{M}_{14}$; and $[12)_M \nVdash \mathrm{M}_{14}$ by 11.3.
For even k>12, let $C_k := ([k)_M - \{8\}) \cup A$, with the ordering inherited from M and A extended by: $m \leq_{C_k} A$ iff $m \leq_M 8$. The diagram below may help to visualize this frame:



Let $C_8 := A$, $C_{10} := [10)_M$. In C_{10}, M_{12} can be refuted, but M_{10} is valid. As to the latter claim: observe that C_{10} is isomorphic to $[8)_M$ with the interpolation mentioned in 11.5. If $(C_{10}, V)\nVdash\mathrm{M}_{10}$, then $6 \nVdash \neg p \vee \neg\neg p$ and $6 \Vdash (\neg\neg p \to p) \to p \vee \neg p$. It is easily seen that 6 must force $\neg\neg p \to p$, from which a contradiction follows.
We may continue inductively. Let n=k–1 (so n is odd); suppose M_{k-4} is valid in C_{k-4} and C_{k-6}. As noted in 11.3, $k \nVdash \mathrm{M}_n$ implies that M_{n-3} is not forced in some strict successor of k. Since $n-3=k-4$, $C_k \Vdash \mathrm{M}_n$. Taking an ultrapower $C_k' = \prod_U C_k$ over a nonprincipal ultrafilter U over $\mathbb{N}$ (that is, replacing A by $\prod_U A$), and extending the p-morphism $f\colon \prod_U A \twoheadrightarrow [8)_M$ by the identity on C_k-A, we get a p-morphism $C_k' \twoheadrightarrow [k)_M$, so $C_k' \nVdash \mathrm{M}_n$, by 2.4.2 and a remark in 11.3. Similarly, $C_k' \nVdash \mathrm{M}_{k+2}$; while $C_k \Vdash \mathrm{M}_{k+2}$ is immediate by $C_k \Vdash \mathrm{M}_n$ (n=k–1; $\mathrm{M}_{k+2}=\mathrm{M}_k\to\mathrm{M}_{k-1}$).
Thus no M_n with $n\geq 10$ is preserved under ultrapowers. By 6.7.6, $\mathrm{M}_n \notin \mathrm{E}(\mathbf{PO})$ for $n\geq 10$. □

11.7 Theorem. For all $n<10$, $\mathrm{M}_n \in \mathrm{E}(\mathbf{PO})$.

Proof: This follows immediately from corollary 7.4 since, by 11.2, for $n<10$, M_n may be written as a deterministic formula.

11.8 First order definability of $\mathbb{I}(p)$-formulas on **FPO** was investigated by Doets [B]. There appears a difference of one formula. To begin with, we know from 10.4 that $\mathrm{M}_{10} \notin \mathrm{E}(\mathbf{FPO})$. Then the argument of 11.6 may be adapted to **FPO**, using frames A_l and A_l' from §10 instead of A and $\prod_U A$, and considerations of quantifier rank instead of the overall elementary equivalence of frames to their ultrapowers. We state without further proof:

11.8.1 Theorem (Doets): If $n \geq 10$ and $n \neq 12$, $\mathrm{M}_n \notin \mathrm{E}(\mathbf{FPO})$.

11.8.2 Unlike A of 6.8(a), the frame B of 11.5 has infinite height. This difference proves essential.

Theorem (Doets): If $n<10$ or $n=12$, $\mathrm{M}_n \in \mathrm{E}(\mathbf{FPO})$.

Proof: For $n<10$, the statement follows from 11.7. So only $n=12$ remains to be considered. As before, $u \leq v \vee v \leq u$ will be abbreviated to Comp(u,v); we also introduce CS(u,v) for $\exists w\ (u \leq w \wedge v \leq w)$ (u and v have a common successor). Let $\beta_1, \beta_2, \beta_3, \beta_4$ be the following formulas:

$$\beta_1 := x_{10} < x_6, x_4 \wedge x_4 < x_1 \wedge x_6 < x_2, x_1'$$
$$\beta_2 := \forall y\ (x_{10} \leq y \leq x_6, x_4 \ \rightarrow\ y \leq x_{10})$$
$$\beta_3 := \forall y\ (x_{10} < y \wedge \mathrm{Comp}(y,x_4) \ \rightarrow\ \neg\mathrm{CS}(y,x_2))$$
$$\beta_4 := \forall y\ (x_6 < y \wedge \mathrm{Comp}(y,x_1') \ \rightarrow\ \neg\mathrm{CS}(y,x_2)).$$

Let $\beta := \beta_1 \wedge \beta_2 \wedge \beta_3 \wedge \beta_4$; and $\overline{\beta}$ be the existential closure of β. We want to show that $\mathrm{M}_{12} \equiv_{\mathbf{FPO}} \neg\overline{\beta}$.

($\Rightarrow$) Suppose $A \in$ **FPO** and $A \vDash \beta[a_{10}, a_6, a_4, a_2, a_1, a_1']$ (a_{10} corresponding with x_{10}, etc.). Define f: $[a_{10})_A \rightarrow [10)_M$ by

$$\begin{aligned} f(a) &= 10 \ \text{ if } a = a_{10}; \\ &= 4 \ \text{ if } a_{10} < a \leq a_4; \\ &= 6 \ \text{ if } a_{10} < a \leq a_6; \\ &= 2 \ \text{ if } a \text{ and } a_2 \text{ have a common successor;} \\ &= 1 \ \text{ otherwise.} \end{aligned}$$

Then f is a (surjective) p-morphism. First, f is a homomorphism by β_1 and since, by β_3, $a_1, a_4 \not\leq a_2$, and, by β_4, $a_1' \not\leq a_2$. Second, the p-morphism condition is satisfied: $a_1 \geq a_4$, and $f(a_1) = 1$; $a_1' \geq a_6$, and $f(a_1') = 1$; $a_2 \geq a_6$, and $f(a_2) = 2$. Since $[10)_M \not\Vdash \mathrm{M}_{12}$ by 11.3, $A \not\Vdash \mathrm{M}_{12}$ by 2.4.2.

($\Leftarrow$) Suppose $(A,V) \not\Vdash \mathrm{M}_{12}$. Choose $a_{10} \Vdash \{\mathrm{TM}_{10}, \mathrm{FM}_7\}$ such that no $a > a_{10}$ realizes $\{\mathrm{TM}_{10}, \mathrm{FM}_7\}$. Since by 11.3 there must be a partial projection g of $[a_{10})_A$ onto the minimal refutation $\mathfrak{X}_{12}=(X_{12},$

S_{12}) of M_{12}, β_1 is satisfied by a_{10} and certain $a_6, a_4, a_2, a_1, a_1' \geq a_{10}$. Let us identify $(X_{12}, \leq_{\mathfrak{X}_{12}})$ with $[10)_M$ (cf. 11.3), and assume $g(a_i^{(')}) = i$. To check β_2, note that $a_{10} \leq a \leq a_6, a_4$ implies $g(a) \leq_{\mathfrak{X}_{12}} g(a_6), g(a_4)$, hence $g(a) = g(a_{10})$, and by maximality of a_{10}, $a = a_{10}$. For β_3, note that $a_2 \Vdash \neg p$, $a_4 \Vdash \{T\neg\neg p, Fp\}$, and $a_{10} < a$ implies $a \Vdash \neg\neg p \vee (\neg\neg p \rightarrow p)$ by maximality of a_{10}. Finally, β_4 must hold since we may assume that a_6 is maximal among the points $a \geq a_{10}$ that realize $\{T(\neg\neg p \rightarrow p), F(p \vee \neg p)\}$; then $\mathrm{Comp}(a, a_1')$ implies $a \Vdash p$. $\square$

11.9 The table below sums up the results of this section. Plus means elementary, minus nonelementary.

Classes \ M_n	$n<10$	$n=12$	$n=10,11$ or $n>12$
PO	+	–	–
FPO	+	+	–
DLO	+	+	+

Footnotes:

[1] There will be more on Heyting algebras in §14 below.

[2] In a sense to be explained in §14.

§12. Syntactic closure properties and proper inclusions of classes E(K)

In some respects, the results of the preceding sections are rather scattered and incomplete. We now tie up some loose ends from two points of view: syntax of $\mathbb{I}$-formulas (12.1-4) and proper inclusions of E(**K**) for different classes **K** of frames (12.5).

12.1 Theorem: Let **K** be a class of frames. Then

(a) if $\varphi, \psi \in E(\mathbf{K})$, $\varphi\wedge\psi \in E(\mathbf{K})$;

(b) if φ and ψ have no proposition letters in common and $\psi \in E(\mathbf{K})$, then $\varphi\to\psi \in E(\mathbf{K})$;

(c) if φ and ψ belong to E(**K**) and have no proposition letters in common, then $\varphi\vee\psi \in E(\mathbf{K})$.

Proof:

(a) follows from 9.8.

(b) If $\vdash \neg\varphi$, then $\vdash \varphi\to\psi$, so $\varphi\to\psi \equiv \top$. If $\nvdash \neg\varphi$, then by 4.7.4, φ is classically satisfiable; say $V_0 \models \varphi$. It will suffice to show that $\psi \equiv \varphi\to\psi$.

($\Rightarrow$) If $A \Vdash \psi$, then obviously $A \Vdash \varphi\to\psi$.

($\Leftarrow$) Suppose $A \nVdash \psi$; let V be a valuation such that $(A,V) \nVdash \psi$. Since φ and ψ do not have any proposition letters in common, we may assume, using 2.1, that for each proposition letter p occurring in φ, $V(p) = A$ or $V(p) = \emptyset$, according as $V_0 \models p$ or not. Then $(A,V) \Vdash \varphi$, hence $(A,V) \nVdash \varphi\to\psi$.

(c) If on $A \in \mathbf{K}$ there exist valuations V_1, V_2 such that, for some $a \in A$, $(A,V_1,a) \nVdash \varphi$ and $(A,V_2,a) \nVdash \psi$, we can combine V_1 and V_2, by 2.1, to a valuation V such that $(A,V,a) \nVdash \varphi$ and $(A,V,a) \nVdash \psi$. Suppose that $\varphi \equiv \alpha$ and $\psi \equiv \beta$. Relativize α and β to α^u and β^u (as in 9.7). Then it is easy to see that $\varphi\vee\psi \equiv \forall u\,(\alpha^u \vee \beta^u)$. □

12.2 Corollary: For any class **K** of frames and all $\varphi \in \mathbb{I}$, $\neg\varphi \in E(\mathbf{K})$.

Proof: Recall that $\neg\varphi = \varphi\to\bot$. Since $\bot$ contains no proposition letters, and may be considered to belong to $\mathbb{L}_0$, we can apply (b) of the above theorem. □

12.3 Theorem: The following implications do *not* generally hold for $\varphi,\psi \in \mathbb{I}$:

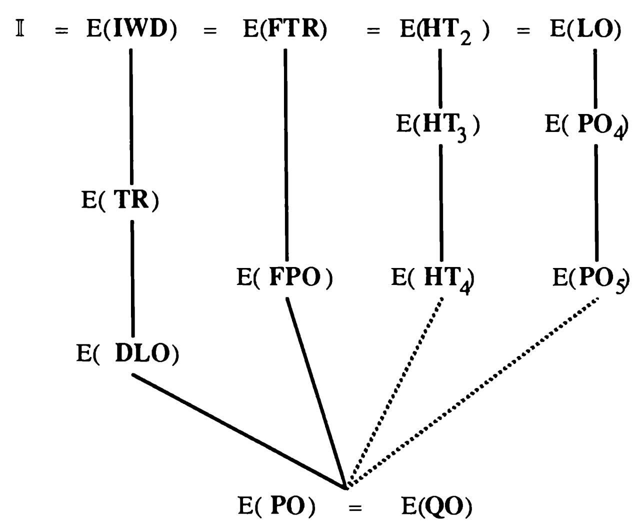
(i) $\varphi,\psi \in E(\mathbf{PO}) \;\Rightarrow\; \varphi\to\psi \in E(\mathbf{TR})$;

(ii) $\varphi \in$ E(**PO**) $\Rightarrow$ $\varphi[r{:=}\neg r] \in$ E(**TR**);
(iii) $\varphi\wedge\psi \in$ E(**PO**) $\Rightarrow$ $\varphi \in$ E(**PO**) or $\psi \in$ E(**PO**).

Proof:
(i) Consider SP_2 (see 1.9). It is not elementary on **TR**, by 8.16 and $\mathbf{TR}^{(2)} \subseteq \mathbf{TR}$; by corollary 7.4, both its antecedent and its succedent are elementary on **PO**.
(ii) Let $\varphi_0 = (\neg\neg r \rightarrow r) \wedge SP_2$, with r a proposition letter not occurring in SP_2. If $A \Vdash \neg\neg r \rightarrow r$, then $A \Vdash SP_2$; hence (cf. 6.3(a)) $\varphi_0 \equiv \forall xy\, (x{\le}y \rightarrow y{\le}x)$. But $\varphi_0[r{:=}\neg r]$ is logically equivalent to SP_2, which does not belong to E(**TR**).
(iii) By 11.1, $\vdash M_7 \rightarrow M_{10}\wedge M_{12}$. By 11.6, M_{10} and M_{12} do not belong to E(**PO**); by 11.7, M_7 does. □

12.4 Remark: The implication (ii) above may be weakened to: if $\varphi \in$ E(**PO**), and φ' results from φ by replacing every proposition letter p in φ by its negation $\neg p$, then $\varphi' \in$ E(**PO**). Even this weaker version is not valid; as is shown by the same example φ_0. Let SP_2' be $SP_2[p{:=}\neg p, q{:=}\neg q]$. Since $\mathbf{FTR}_2 \Vdash SP_2$, $\mathbf{FTR}_2 \Vdash SP_2'$ by substitution (2.2). Now the proof of 8.16 may be seen to go through, if we change $V(p)$ to $\cup(\{n\}{\times}B \mid n{\equiv}2 \pmod 3)$ and $V(q)$ to $\cup(\{n\}{\times}B \mid n{\equiv}1 \pmod 3)$.

12.5 If **K**, **K'** are classes of frames, and $\mathbf{K} \subseteq \mathbf{K'}$, then E(**K'**) $\subseteq$ E(**K**). In the preceding sections, we have proved several noninclusions. Below we give a schema, in which a connecting line represents a proper inclusion of the lower set in the higher. We denote the class of all frames of height at most n by $\mathbf{HT}_n$.



That E(**IWD**) = $\mathbb{I}$ is shown in §8 (theorem 8.7); for $\mathbf{HT}_2$, see 8.18. **FTR** is contained in **IWD**. That E(**LO**) = $\mathbb{I}$ is noted in 7.7. E(**PO**) = E(**QO**) by theorem 6.2. The other inclusions shown by lines or broken lines in the diagram are obvious.

E(**FPO**) $\not\subseteq$ E(**TR**) by examples 9.13(b) and 10.1 ($BR_2 \in$ E(**FPO**) – E(**TR**)). By 11.4 and 10.4, SC (= M_{10}) $\in$ E(**DLO**) – E(**FPO**).

It was noted in 8.18 that E($\mathbf{HT}_3$) $\neq \mathbb{I}$. To see that E($\mathbf{HT}_{n+1}$) $\neq$ E($\mathbf{HT}_n$) ($n\geq 3$), suppose the proposition letter p does not occur in P_n (2.8), and consider $\varphi_n := P_n \vee SC$. By 2.8, $\mathbf{HT}_n \Vdash \varphi_n$; $\varphi_n \notin$ E($\mathbf{HT}_{n+1}$) by a simple adaptation of the argument in 6.8(a) (add a chain of n–2 points under the roots of the frames involved).

That E($\mathbf{PO}_4$) $\neq \mathbb{I}$ is shown in 8.17. A proof that E($\mathbf{PO}_n$) $\neq$ E($\mathbf{PO}_{n+1}$) ($n\geq 4$) runs parallel to the above proof of E($\mathbf{HT}_n$) $\neq$ E($\mathbf{HT}_{n+1}$), using φ of 8.17 and the formulas W_n of 2.10 with wider versions of the frame in 8.17 (where still every point has at most two covers).

This shows that the inclusions in the vertical paths of the diagram are proper. We also noted some relations between these paths. Here are some more: SC $\in$ E(**DLO**) – E($\mathbf{HT}_3$); E($\mathbf{HT}_n$) $\not\subseteq$ E(**TR**) and E($\mathbf{HT}_n$) $\not\subseteq$ E(**FPO**) for all n (use $P_n \vee SP_2$); E($\mathbf{PO}_n$) $\subseteq$ E(**FPO**) for all n (use $W_n \vee SP_2$). It may be that E(**FPO**) $\subseteq$ E($\mathbf{HT}_n$) and E(**FPO**) $\subseteq$ E($\mathbf{PO}_n$) for all n, but this would require further analysis. We have not shown that E(**DLO**) $\not\subseteq$ E($\mathbf{PO}_4$), though it seems rather likely.

III. $\mathbb{I}$-definability

This part contains results relating to the question complementary to that of part II: which first order sentences correspond with $\mathbb{I}$-formulas? Mostly, however, we will be concerned with a problem that is, in a sense, preliminary, viz. to characterize $\mathbb{I}$-definable classes of models and frames in terms of closure under certain operations.

§13. Models

Let Φ be a set of $\mathbb{I}$-formulas. We shall denote the class of all models $\mathfrak{A}$ in which every formula in Φ is valid (notation: $\mathfrak{A} \Vdash \Phi$) by **Mod**(Φ). We are going to characterize the classes **Mod**(Φ) in terms of closure under certain operations known from modal correspondence theory.

13.1 Definition. Let $\mathfrak{A} = (A,V)$ and $\mathfrak{A}' = (A',V')$ be models. A **p-relation** between $\mathfrak{A}$ and $\mathfrak{A}'$ is a relation $R \subseteq A \times A'$ such that dom$R$ (= $\{a \in A \mid \exists a' \in A'. Raa'\}$) = A and ran$R$ (= $\{a' \in A \mid \exists a' \in A. Raa'\}$) = A', and moreover

(i) for all proposition letters p, $\forall a \in A, a' \in A'$: $Raa' \Rightarrow (a \in V(p) \Leftrightarrow a' \in V(p))$;

(ii) if $a_1 \leq_A a_2$ and $Ra_1a'_1$, then $\exists a'_2 \geq_{A'} a'_1$: $Ra_2a'_2$;

(iii) if $a'_1 \leq_{A'} a'_2$ and $Ra_1a'_1$, then $\exists a_2 \geq_A a_1$: $Ra_2a'_2$.

It is clear from the definition that p-relations are symmetric: a p-relation between $\mathfrak{A}$ and $\mathfrak{A}'$ is also a p-relation between $\mathfrak{A}'$ and $\mathfrak{A}$.

13.2 Invariance. We say that a property P is **invariant** for an operation O if P can neither be gained nor lost by applying O (equivalently, if P is preserved under both O and its inverse); we say of a formula that it is invariant for O if the property of validating it (or forcing or satisfying it in a fixed element, as the case requires) is thus invariant.

13.3 p-relations are cognate to p-morphisms. Indeed, we began the proof of lemma 2.4.2 (preservation of validity under p-morphic images) by defining, given a surjective p-morphism f: $A \twoheadrightarrow B$ and a valuation V on B, a valuation V' on A such that (A,V') and (B,V) are p-related by the graph of f. The rest of that proof can be generalized to show that $\mathbb{I}$-formulas are invariant under p-relations:

Lemma. If there exists a p-relation between $\mathfrak{A}$ (= (A,V)) and $\mathfrak{A}'$ (= (A',V')), then for any $\varphi \in \mathbb{I}$, $\mathfrak{A} \vDash \varphi \Leftrightarrow \mathfrak{A}' \vDash \varphi$.

Proof: A straightforward induction over $\mathbb{I}$-formulas φ shows that for all $a \in A$ and $a' \in A'$, Raa' implies that $(\mathfrak{A},a) \Vdash \varphi$ if and only if $(\mathfrak{A}',a') \Vdash \varphi$. Conditions (ii) and (iii) are used in the implication step. Since $\mathrm{ran}R = A'$, $\mathfrak{A} \models \varphi$ then implies $\mathfrak{A}' \models \varphi$; and conversely, $\mathfrak{A}' \models \varphi \Rightarrow \mathfrak{A} \models \varphi$ because $\mathrm{dom}R = A$. □

13.4 Definition. Let $(\mathfrak{A}_i \mid i \in I)$ be a family of models $\mathfrak{A}_i = (A_i,V_i)$. The **disjoint union** $\Sigma_{i\in I}\mathfrak{A}_i$ is the model $\mathfrak{A} = (A,V)$ in which $A = \Sigma_{i\in I} A_i$ (2.4.3) and V is defined by $V(p) = \{(i,a) \in A \mid a\in V_i(p)\}$.

Preservation of $\mathbb{I}$-formulas under disjoint unions is straightforward:

Lemma. If $(\mathfrak{A}_i \mid i \in I)$ is a family of models, $\varphi \in \mathbb{I}$, and $\forall i \in I.\ \mathfrak{A}_i \Vdash \varphi$, then $\Sigma_{i\in I}\mathfrak{A}_i \Vdash \varphi$.

13.5 Lemma. Let $\mathfrak{A} = (A,V)$ be a model. For each $a \in A$, define a valuation V_a on $[a)_A$ by

$$\forall p \in \mathbb{P}\colon V_a(p) = V(p) \cap [a)_A\,,$$

and let $\mathfrak{A}_a = ([a)_A, V_a)$. Then there is a p-relation between $\mathfrak{A}$ and $\Sigma_{a\in A}\mathfrak{A}_a$.

Proof: Take $R := \{(a', (a,a')) \mid a \leq_A a'\}$. □

13.6 Ultraproducts. In 1.6, models are taken as structures for $\mathbb{L}_1$. This identification fixes the notion of ultraproducts for models: if $\mathfrak{A}_i = (A_i,V_i)$ for $i \in I$, and U is an ultrafilter over I, then the ultraproduct $\Pi_U\mathfrak{A}_i$ is the model $(\Pi_U A_i, V)$ with $f_U \in V(p)$ iff $\{i \in I \mid f(i) \in V_i(p)\} \in U$ (cf. 6.7). We give translations of two lemmas of 6.7, in which $(\mathfrak{A}_i \mid i \in I)$ and U are as above. Define $\mathrm{St}(\mathrm{T}\varphi) := \mathrm{St}(\varphi)$, $\mathrm{St}(\mathrm{F}\varphi) := \neg\mathrm{St}(\varphi)$ (cf. 1.6).

13.6.1 Lemma. For all signed formulas σ, $(\Pi_U\mathfrak{A}_i,f_U) \Vdash \sigma$ iff $\{i \in I \mid (\mathfrak{A}_i,f(i)) \Vdash \sigma\} \in U$.

Proof: By 1.6 and 6.7.1,

$$\begin{aligned}(\Pi_U\mathfrak{A}_i,f_U) \Vdash \sigma \quad &\text{iff}\quad \Pi_U\mathfrak{A}_i \models \mathrm{St}(\sigma)[f_U]\\ &\text{iff}\quad \{i \in I \mid \mathfrak{A}_i \models \mathrm{St}(\sigma)[f(i)]\} \in U\\ &\text{iff}\quad \{i \in I \mid (\mathfrak{A}_i, f(i)) \Vdash \sigma\} \in U.\end{aligned}$$

□

As before, we call $\prod_U \mathfrak{A}_i$ an **ultrapower** if $\mathfrak{A}_i$ is the same model $\mathfrak{A}$ for all i, and use the notation $\prod_U \mathfrak{A}$. We call $\mathfrak{A}$ an **ultraroot** of $\prod_U \mathfrak{A}$. An immediate consequence of the last lemma is

13.6.2 Lemma. $\mathbb{I}$-formulas are invariant for ultrapowers of models.

13.6.3 Lemma. $\prod_U \mathfrak{A}_i$ is isomorphic to a generated submodel of the ultrapower $\prod_U \Sigma_{i\in I} \mathfrak{A}_i$.

Proof: Define F: $\prod_U \mathfrak{A}_i \to \prod_U \Sigma_{i\in I} \mathfrak{A}_i$ by $F(f_U) = ((i,f(i))_{i\in I})_U$. Let V' be the valuation on the ultrapower; after 6.7.2, it remains only to show that $f_U \in V(p)$ iff $F(f_U) \in V'(p)$. Both sides easily reduce to $\{i \in I \mid f(i) \in V_i(p)\} \in U$. □

13.7 Saturation. Let Γ be a set of first order formulas in which a single individual variable v may occur free. A structure $\mathfrak{A}$ **realizes** Γ if there is an element a of the domain of $\mathfrak{A}$ such that $\mathfrak{A}\models\gamma[a]$ for all $\gamma \in \Gamma$ (short: $\mathfrak{A} \models \Gamma[a]$).
Suppose $\mathfrak{A}$ is a structure for a given first order language $\mathbb{L}$, with domain A. For a subset $X \subseteq A$, $\mathbb{L}[X]$ is the language obtained by extending $\mathbb{L}$ with distinct constants $\underline{x}$ for all $x \in X$, and $\mathfrak{A}_X$ the expansion of $\mathfrak{A}$ to a structure for $\mathbb{L}[X]$ in which each $\underline{x}$ is interpreted as x. $\mathfrak{A}$ is **countably saturated** if for every finite $A_0 \subseteq A$, the expansion $\mathfrak{A}_{A_0}$ realizes every set $\Gamma(v)$ of $\mathbb{L}[A_0]$-formulas (with only v occurring free) that is consistent with the first order theory of $\mathfrak{A}_{A_0}$. ($\Gamma(v)$ is consistent with a theory T if T has a model that realizes $\Gamma(v)$.)

13.7.1 An ultrafilter is said to be **countably incomplete** if it is not closed under countable intersections. In particular, a free ultrafilter over $\mathbb{N}$ is countably incomplete: it does not contain any singleton $\{n\}$, so it contains all their complements; but it does not contain $\emptyset = \bigcap_{n\in\mathbb{N}} (\mathbb{N} - \{n\})$.
We shall use the following fact (see Chang & Keisler [1973], Ch. 6):

Lemma. Let $\mathbb{L}$ be a countable first order language, U a countably incomplete ultrafilter over a set I, and $(\mathfrak{A}_i \mid i \in I)$ a family of structures for $\mathbb{L}$. Then the ultraproduct $\prod_U \mathfrak{A}_i$ is countably saturated.

13.7.2 Let $\mathfrak{A} = (A,V)$ be a model, $a \in A$, and $\varphi \in \mathbb{I}$. Then obviously $(\mathfrak{A},a) \Vdash \mathrm{F}\varphi$ iff $\mathfrak{A} \models \neg\mathrm{St}(\varphi)[a]$. Thus, every signed formula may be written as an $\mathbb{L}_1$-formula; and if Σ^* is a (possibly infinite) set of signed formulas, $(\mathfrak{A},a) \Vdash \Sigma^*$ (i.e. $(\mathfrak{A},a) \Vdash \sigma$ for all $\sigma \in \Sigma^*$, or a **realizes** Σ^*) iff a realizes $\mathrm{St}[\Sigma^*] := \{\mathrm{St}(\sigma) \mid \sigma \in \Sigma^*\}$. Accordingly, the above lemma may be specialized as follows:

Lemma. Let Σ^* be a set of signed formulas, $(\mathfrak{A}_n \mid n \in \mathbb{N})$ a family of models, and U a free ultrafilter over $\mathbb{N}$. Let a be any point in $\prod_U \mathfrak{A}_n$. If every sequent $\Sigma \subseteq \Sigma^*$ is realized in a successor of a, then some successor of a realizes Σ^*.
Proof: By lemma 13.7.1. That every finite $\Sigma \subseteq \Sigma^*$ is realized in a successor of a means that $\{a \leq x\}$

$\cup$ St[Σ*] is finitely satisfiable in $(\prod_U \mathfrak{A}_n, a)$. □

13.8 Recall the notion of generated submodel (1.2.6). By the **root** of a model (A,V), we shall mean the root of the underlying frame A.

Theorem. Let $\mathbf{M}$ be a class of models. There exists a set $\Phi \subseteq \mathbb{I}$ such that $\mathbf{M} = \mathbf{Mod}(\Phi)$ iff $\mathbf{M}$ is closed under p-relations, generated submodels, disjoint unions, ultrapowers and ultraroots.

Proof: One direction is by lemmas 3, 1.2.6, 4 and 13.6.2. We concentrate on the other: suppose $\mathbf{M}$ satisfies the stated closure conditions. Observe that by 13.6.3, $\mathbf{M}$ is also closed under ultraproducts. Let

$$\Phi := \{\varphi \in \mathbb{I} \mid \forall \mathfrak{D} \in \mathbf{M}.\ \mathfrak{D} \Vdash \varphi\}.$$

Suppose $\mathfrak{A} \Vdash \Phi$, $\mathfrak{A} = (A,V)$. We want to show that $\mathfrak{A} \in \mathbf{M}$. By 1.2.6, given the closure conditions on $\mathbf{M}$, we may suppose that A has a root a_0.
Let

$$\Sigma^* := \{\sigma \mid \sigma \text{ is a signed formula and } a_0 \Vdash \sigma\}.$$

Every sequent $\Sigma \subseteq \Sigma^*$ is realized somewhere in some model in $\mathbf{M}$, for if not, we would have $\mathbf{M} \Vdash \psi_\Sigma$ $(= \bigwedge \Sigma_T \to \bigvee \Sigma_F$, cf. 9.5), hence $\psi_\Sigma \in \Phi$; since $a_0 \nVdash \psi_\Sigma$, this would contradict $\mathfrak{A} \Vdash \Phi$. Using closure under generated submodels, we may take for every sequent $\Sigma \subseteq \Sigma^*$ a rooted $\mathfrak{M}_\Sigma \in \mathbf{M}$ the root of which realizes Σ. We suppose M_Σ is the frame of $\mathfrak{M}_\Sigma$.
Let $\mathbb{S}$ be the collection of all sequents contained in Σ^*. Let X_Σ, for $\Sigma \in \mathbb{S}$, be the set $\{\Sigma' \in \mathbb{S} \mid \Sigma' \supseteq \Sigma\}$. Since $X_{\Sigma'} \cap X_{\Sigma''} = X_{\Sigma' \cup \Sigma''} \neq \emptyset$, the collection $U_0 := \{X_\Sigma \mid \Sigma \in \mathbb{S}\}$ has the finite intersection property. Consequently, there exists an ultrafilter U extending U_0. Let $f \in \prod_{\Sigma \in \mathbb{S}} M_\Sigma$ be such that for each Σ, $f(\Sigma)$ is the root of M_Σ. Then by Łoś's theorem (6.7.1), f_U is the root of $\mathfrak{M} := \prod_U \mathfrak{M}_\Sigma$; and by 13.6.1, $(\mathfrak{M}, f_U) \Vdash \sigma$ for every $\sigma \in \Sigma^*$, since, as $\{\Sigma \mid \mathfrak{M}_\Sigma \Vdash \sigma\} \supseteq X_{\{\sigma\}} \in U_0 \subseteq U$, $\{\Sigma \mid \mathfrak{M}_\Sigma \Vdash \sigma\} \in U$. Moreover, $\mathfrak{M} \in \mathbf{M}$ by closure under ultraproducts.
Take a free ultrafilter U' over $\mathbb{N}$, and consider the ultrapowers $\prod_{U'} \mathfrak{A} =: (B_1, V_1)$ and $\prod_{U'} \mathfrak{M} =: (B_2, V_2)$. By Łoś's theorem, B_1 and B_2 are rooted, and both roots realize Σ^*. Define a relation $R \subseteq B_1 \times B_2$ by

$$Rb_1b_2 \quad \text{iff} \quad \forall \varphi \in \mathbb{I}\colon\ (B_1,V_1,b_1) \Vdash \varphi \text{ iff } (B_2,V_2,b_2) \Vdash \varphi.$$

We show that R is a p-relation. First, 13.1(i) holds by definition. Second, for 13.1(ii), suppose $b_1 \leq_{B_1} b'_1$ and Rb_1b_2. Let Θ^* be the set of all signed formulas that are realized in b'_1. Then every

sequent $\Theta \subseteq \Theta^*$ is realized in a successor of b_2, for otherwise we have $b_2 \Vdash \psi_\Theta$, from which $b_1 \Vdash \psi_\Theta$ would follow – contradicting $b_1 \leq b'_1 \Vdash \Theta$. By lemma 13.7.2, there exists $b'_2 \geq_{B_2} b_2$ realizing the entire set Θ^*; then $Rb'_1b'_2$. Condition 13.1(iii) is symmetric. Since the roots of B_1 and B_2 are related, (ii) and (iii) imply that dom$R = B_1$ and ran$R = B_2$.
There remains only a walk around the diagram:

$$\begin{tikzcd}
\mathfrak{A} \arrow[d, dash] & \mathfrak{M} \arrow[d, dash] \\
\prod_{U'} \mathfrak{A} \arrow[r, dash, "R"] & \prod_{U'} \mathfrak{M}
\end{tikzcd}$$

By closure under ultrapowers, $\prod_{U'} \mathfrak{M} \in$ **M**. By closure under p-relations, $\prod_{U'} \mathfrak{A} \in$ **M**. By closure under ultraroots, $\mathfrak{A} \in$ **M**. □

13.9 An $\mathbb{L}_1$-structure is a model iff it satisfies the following axioms:

$$\forall x.\ x \leq x$$

$$\forall xyz\ (x \leq y \wedge y \leq z \rightarrow x \leq z)$$

$$\forall xy\ (px \wedge x \leq y \rightarrow py), \text{ for all } p \in \mathbb{P}.$$

Let us denote the first order 'theory of models' determined by these axioms by *Mod*. By the completeness theorem for first order logic, an $\mathbb{L}_1$-sentence α is true in all models iff $Mod \vdash \alpha$. In particular, two $\mathbb{L}_1$-sentences α and β are equivalent on models iff $Mod \vdash \alpha \leftrightarrow \beta$.
The theory *Mod* is not finitely axiomatizable. So unlike the class of all frames, the class of all models is not elementary. As a consequence of this, we must distinguish between elementary classes of models (classes **M** such that for some $\mathbb{L}_1$-sentence α, for all models $\mathfrak{A}$: $\mathfrak{A} \in$ **M** iff $\mathfrak{A} \models \alpha$) and elementary classes of $\mathbb{L}_1$-structures.
By the complement of a class **M** of models we shall mean the class of all *models* that do not belong to **M**, not the complement in the universe of all $\mathbb{L}_1$-structures.

<u>Lemma</u> (Separation Theorem). If $\mathbf{M}_1$ and $\mathbf{M}_2$ are disjoint classes of structures for the same first order language, both closed under ultraproducts and isomorphism, then there exists an elementary class $\mathbf{K} \supseteq \mathbf{M}_1$ that is disjoint with $\mathbf{M}_2$.

Proof: see Chang & Keisler [1973], corollary 6.1.17. □

Theorem. Let $\mathbf{M}$ be a class of models. There exists an $\mathbb{I}$-formula φ such that $\mathbf{M} = \mathbf{Mod}(\varphi)$ iff $\mathbf{M}$ is closed under p-relations, generated submodels, disjoint unions and ultrapowers, and the complement of $\mathbf{M}$ is closed under ultraproducts.

Proof: If $\mathbf{M} = \mathbf{Mod}(\varphi)$, $\mathbf{M}$ is closed under the operations listed in theorem 8. The complement of $\mathbf{M}$ is defined by $Mod \cup \{\neg\forall x\, \mathrm{St}(\varphi)\}$, hence closed under ultraproducts by Łoś's theorem.
For the converse, suppose $\mathbf{M}$ and its complement satisfy the stated closure conditions. Since $\mathbf{M}$ is closed under p-relations, it is closed under isomorphism (hence: so is its complement). By 13.6.3, $\mathbf{M}$ is closed under ultraproducts. $\mathbf{M}$ and its complement are disjoint classes of $\mathbb{L}_1$-structures, so by the separation theorem, there is an $\mathbb{L}_1$-sentence α such that for all models $\mathfrak{A}$, $\mathfrak{A} \in \mathbf{M}$ iff $A \models \alpha$. Since the complement of $\mathbf{M}$ is closed under ultraproducts, $\mathbf{M}$ is closed under ultraroots; so by theorem 8, $\mathbf{M} = \mathbf{Mod}(\Phi)$ for some $\Phi \subseteq \mathbb{I}$. So

$$Mod \cup \{\forall x\, \mathrm{St}(\varphi) \mid \varphi \in \Phi\} \models \alpha;$$

by compactness, there exists a finite set $\Phi_0 \subseteq \Phi$ such that

$$Mod \cup \{\forall x\, \mathrm{St}(\varphi) \mid \varphi \in \Phi_0\} \models \alpha.$$

Then $Mod \models \alpha \leftrightarrow \bigwedge_{\varphi \in \Phi_0} \forall x\, \mathrm{St}(\varphi)$, and $\mathbf{M} = \mathbf{Mod}(\bigwedge \Phi_0)$. □

Corollary. Let α be an $\mathbb{L}_1$-sentence. There exists an $\mathbb{I}$-formula φ such that $Mod \vdash \alpha \leftrightarrow \forall x\, \mathrm{St}(\varphi)$ iff α is preserved under p-relations, generated submodels and disjoint unions.

Proof: Let $\mathbf{M}$ be the class of all models $\mathfrak{A}$ such that $\mathfrak{A} \models \alpha$. If $Mod \vdash \alpha \leftrightarrow \forall x\, \mathrm{St}(\varphi)$, for some $\varphi \in \mathbb{I}$, then $\mathbf{M} = \mathbf{Mod}(\varphi)$, hence $\mathbf{M}$ is closed under p-relations, generated submodels and disjoint unions by the theorem above – so α is preserved under these operations. If, conversely, α is preserved under p-relations, generated submodels and disjoint unions, then $\mathbf{M}$ is closed under these operations. Since $\mathbf{M}$ and its complement are Δ-elementary classes of $\mathbb{L}_1$-structures, they are closed under ultraproducts and isomorphism. By the theorem, there exists $\varphi \in \mathbb{I}$ such that $\mathbf{M} = \mathbf{Mod}(\varphi)$; then $Mod \models \alpha \leftrightarrow \forall x\, \mathrm{St}(\varphi)$ by the completeness theorem. □

Remark. The same argument was used for the modal case in an unpublished note by R. Woodrow. It applies globally as well as locally; except that for local definability, invariance must be used rather than preservation.

§14. $\mathbb{I}$-definable classes of frames

For sets Φ of $\mathbb{I}$-formulas, let us denote the class of all frames A in which every formula in Φ is valid (in symbols: $A \Vdash \Phi$) by $\mathbf{Fr}(\Phi)$. A class $\mathbf{K}$ of frames is **$\mathbb{I}$-definable** if $\mathbf{K} = \mathbf{Fr}(\Phi)$ for some $\Phi \subseteq \mathbb{I}$. We shall characterize the $\mathbb{I}$-definable classes of frames in terms of closure under certain operations on frames. We arrive at this characterization by way of Birkhoff's theorem from universal algebra, using a generalized version of frames as intermediary between frames and algebras.

Universal algebraic notions that I leave unexplained may be looked up in Grätzer's book [1968] or in Balbes & Dwinger [1974]. Specific references for Heyting algebras are Rasiowa & Sikorski [1963] and Balbes & Dwinger [1974].

14.1 Definition: A **Heyting algebra** is an algebra $\mathfrak{U} = (U, \wedge, \vee, \perp, \rightarrow)$ of type (2,2,0,2) (i.e. $\wedge$,$\vee$ and $\rightarrow$ are binary operations, and $\perp$ is a nullary operation, on the set U), in which $(U, \wedge, \vee, \perp)$ is a distributive lattice with least element $\perp$ (we write $\leq_{\mathfrak{U}}$, or $\leq$, for the lattice ordering on U; $\perp \leq u$ for all $u \in U$), and $\rightarrow$ is a relative pseudo-complement for $\leq$; i.e. for any $u,v \in U$, $u \rightarrow v$ satisfies

$$(*) \qquad \forall x \in U\colon\ x \wedge u \leq v \ \text{ iff } \ x \leq u \rightarrow v.$$

The class of all Heyting algebras we shall denote by **Ha**.

For any elements u,v of a Heyting algebra $\mathfrak{U}$, $u \wedge v \leq v$, hence by (*): $u \leq v \rightarrow v$. So $v \rightarrow v$ is the greatest element of $\mathfrak{U}$. We shall denote it by $\top$ (usually, Heyting algebras are introduced with $\top$ as one more nullary operation). For a variable-free definition, take $\top := \perp \rightarrow \perp$ as in 1.2.1.

14.2 Examples. One very particular Heyting algebra has been diagrammed in 11.1. Henceforward, I shall use $\mathfrak{R}$ to denote it.

In general, for any frame A, $\mathbb{U}(A)$, the collection of upwards closed subsets of A, may be viewed as a Heyting algebra. The operations of $\mathbb{U}(A)$ are $\cap$ (intersection), $\cup$ (union), $\emptyset$ (the empty set), and an operation $\Rightarrow$ defined by

$$(**) \quad X \Rightarrow Y = \{a \in A \mid [a) \cap X \subseteq Y\}$$

(cf. the evaluation of $\varphi \rightarrow \psi$ in 1.2.4). To see that $\Rightarrow$ is indeed a relative pseudocomplement for $(\mathbb{U}(A), \cap, \cup)$, observe that (**) is equivalent to

$$X \Rightarrow Y = \cup\{Z \in \mathbb{U}(A) \mid Z \cap X \subseteq Y\},$$

which is a reformulation of (*) in 14.1.

Actually, our first example was a special case of this construction: $\mathfrak{K} \cong \mathbb{U}(M)$, for the frame M of 11.1.

14.3 Definition: The notions of subalgebra, homomorphism and product of Heyting algebras are straightforward. When $\mathfrak{U}, \mathfrak{V} \in$ **Ha**, and $\mathfrak{U}$ is a subalgebra of $\mathfrak{V}$, we write $\mathfrak{U} \subseteq \mathfrak{V}$. The product of $(\mathfrak{U}_i \mid i \in I)$ is written as $\prod_{i \in I} \mathfrak{U}_i$.

If $\mathbf{L} \subseteq$ **Ha** is a class of Heyting algebras, we shall write $\boldsymbol{S}(\mathbf{L})$ for the class of subalgebras of elements of $\mathbf{L}$, $\boldsymbol{H}(\mathbf{L})$ for the class of homomorphic images of such elements, and $\boldsymbol{P}(\mathbf{L})$ for the class of products of subfamilies of $\mathbf{L}$.

14.4 Equations. Heyting algebras may be considered as structures for a first order language $\mathbb{L}_\mathrm{H}$ with equality, and function symbols interpreted as $\wedge, \vee, \bot$ and $\rightarrow$. Since we are only interested in $\mathbb{L}_\mathrm{H}$ for its atomic formulas, there will be no confusion if we just take $\wedge, \vee, \bot$ and $\rightarrow$ for function symbols of $\mathbb{L}_\mathrm{H}$. Another useful convention is that we shall let the proposition letters of $\mathbb{I}$ be the individual variables of $\mathbb{L}_\mathrm{H}$. This way, an atomic formula of $\mathbb{L}_\mathrm{H}$ is an equation $\varphi=\psi$, with $\varphi, \psi \in \mathbb{I}$. Terms φ, ψ may be evaluated as usual (as before, we use notation from Chang & Keisler [1973]). If the list $p_1, \ldots, p_n$ contains all the proposition letters in φ and ψ, and $\mathfrak{U} = (U, \wedge, \vee, \bot, \rightarrow)$ is a Heyting algebra, the equation $\varphi=\psi$ is **valid** in $\mathfrak{U}$ (notation: $\mathfrak{U} \models \varphi=\psi$) if for all $u_1, \ldots, u_n \in U$, $\mathfrak{U} \models \varphi=\psi\ [u_1 \ldots u_n]$ (iff $\varphi[u_1 \ldots u_n] = \psi[u_1 \ldots u_n]$).
If Γ is a set of equations, we write $\mathfrak{U} \models \Gamma$ for $\forall \gamma \in \Gamma.\ \mathfrak{U} \models \gamma$.

14.5 Definition: Let $\mathbf{L}$ be a class of algebras of the same type; that is, in model theoretic terms, a class of structures for the same language $\mathbb{L}$ without relation symbols. Then $\mathbf{L}$ is called a **variety** (or equational class) if there exists a set Γ of equations of $\mathbb{L}$ such that for any algebra $\mathfrak{A}$ of the appropriate type, $\mathfrak{A} \models \Gamma$ iff $\mathfrak{A} \in \mathbf{L}$.

14.6 Proposition: **Ha** is a variety.

Proof: A set of equations defining **Ha** is obtained by adding to a set of equations defining the distributive lattices:

$$p \wedge \bot = \bot \quad (\bot \text{ is the least element})$$

$$p \to p = \top \quad (:= \bot \to \bot)$$
$$p \wedge (p \to q) = p \wedge q$$
$$p \wedge q \to r = p \to (q \to r)$$
$$(p \to q) \wedge q = q$$ □

14.7 Proposition (Birkhoff's Theorem): Let **L** be a class of algebras of the same type. Then ***HSP*(L)** is a variety, and the smallest variety containing **L**.

A proof may be found in the standard texts. Observe that in particular, for a variety **L**, ***H*(L)** $\subseteq$ **L**, ***S*(L)** $\subseteq$ **L** and ***P*(L)** $\subseteq$ **L**. Thus, the proposition implies that validity of an equation is preserved under ***H***, ***S*** and ***P***.

14.8 Varieties of Heyting algebras and $\mathbb{I}$-definable classes of frames. There is a simple connection between validity of $\mathbb{I}$-formulas on a frame A and validity of equations in the Heyting algebra $\mathbb{U}(A)$.

(a) Let $p_1,\dots,p_n$ contain all the proposition letters in φ. Let V be a valuation on a frame A. Observe that definition 1.2.4 closely parallels the evaluation of terms in the Heyting algebra $\mathbb{U}(A)$; indeed, it is immediate that $V(\varphi) = \varphi[V(p_1)\dots.V(p_n)]$. Thus

$$\begin{aligned} A \Vdash \varphi \quad &\text{iff} \quad \text{for every valuation } V \text{ on } A\text{: } (A,V) \Vdash \varphi \\ &\text{iff} \quad \text{for every valuation } V \text{ on } A\text{: } V(\varphi) = A \\ &\text{iff} \quad \text{for all } X_1,\dots,X_n \in \mathbb{U}(A)\text{: } \varphi[X_1\dots X_n] = A \\ &\text{iff} \quad \mathbb{U}(A) \models \varphi = \top. \end{aligned}$$

(b) An equation need not have the form $\varphi=\top$. However, the relative pseudocomplement presents a way to get around this difficulty. For elements u,v of any Heyting algebra, we have $u \le v$ iff $u \to v = \top$ (use (*) of 14.1: $u = \top \wedge u$, since $\top$ is the greatest element). Hence $u=v$ iff $(u \to v) \wedge (v \to u) = \top \wedge \top = \top$. As a consequence, we may assume equations to be of the form $\varphi = \top$. Moreover, it is clear by (a) that

$$\mathbb{U}(A) \vDash \varphi = \psi \text{ iff } \mathbb{U}(A) \vDash \varphi \leftrightarrow \psi = \top \quad \text{(abbreviating } (\varphi\to\psi)\wedge(\psi\to\varphi) \text{ as before)}$$
$$\text{iff } A \Vdash \varphi \leftrightarrow \psi.$$

We conclude that a class **K** of frames is $\mathbb{I}$-definable iff there exists a subvariety $\mathbf{L} \subseteq \mathbf{Ha}$ such that any frame A belongs to **K** iff $\mathbb{U}(A) \in \mathbf{L}$.

14.9 To exploit Birkhoff's Theorem, we must find operators on classes of frames matching the operators ***H***, ***S*** and ***P***. I shall give an example of the kind of parallel that we are looking for.

14.9.1 Recall the definition of the disjoint union of a family of frames (2.4.3).

Proposition. Let $(A_i \mid i \in I)$ be a family of frames. Then $\mathbb{U}(\Sigma_{i\in I} A_i) \cong \prod_{i\in I} \mathbb{U}(A_i)$.

Proof: An upwards closed subset X of $\Sigma_{i\in I} A_i$ is uniquely determined as a union of sets $\{i\} \times X_i$, with $X_i \in \mathbb{U}(A_i)$. The mapping $X \mapsto (X_i)_{i\in I}$ is the desired isomorphism. To see that $\to$ is preserved, note that the ordering of the disjoint union is such that $(i,a) \in X \Rightarrow Y$ (where $Y = \bigcup_{i\in I}(\{i\} \times Y_i)$) iff $a \in X_i \Rightarrow Y_i$. □

14.9.2 It is far less simple to find analogues to ***H*** and ***S***. In fact, at least for ***S***, it is impossible. We want a construction which, given a frame A and a subalgebra $\mathbb{V} \subseteq \mathbb{U}(A)$, produces a frame B such that $\mathbb{U}(B) \cong \mathbb{V}$. Let A be an infinite set, ordered by

$$a \le a' \text{ iff } a = a'.$$

Then $\mathbb{U}(A) = \mathbb{P}(A)$, and for $X,Y \subseteq A$, $X \Rightarrow Y = (A - X) \cup Y$. (This is an instance of a general fact: any Boolean algebra $(U,\wedge,\vee,\bot,\neg,\top)$ gives rise to a Heyting algebra $(U,\wedge,\vee,\bot,\to)$ with $u\to v = \neg u \vee v$.)
Let $\mathbb{V}$ consist of all the finite and cofinite subsets of A, i.e. for $X \subseteq A$,

$$X \in \mathrm{V} \text{ iff either } X \text{ is finite or } A - X \text{ is finite.}$$

Then $\mathbb{V}$ is a subalgebra of $\mathbb{U}(A)$, and $|\mathbb{V}| = |A|$. If $\mathbb{U}(B) \cong \mathbb{V}$, then every singleton $\{a\} \in \mathbb{V}$ must correspond with a distinct singleton $\{b_a\} \in \mathbb{U}(B)$, since a set corresponding to $\{a\}$ cannot have proper nonempty subsets. As these b_a must be maximal in B, we obtain, for distinct $X,X' \subseteq A$, distinct sets $\{b_a \mid a \in X\}$ and $\{b_{a'} \mid a' \in X'\}$ in $\mathbb{U}(B)$. Then $|\mathbb{U}(B)| \ge 2^{|A|}$, contradicting the assumption that $\mathbb{U}(B) \cong \mathbb{V}$.

To accommodate this problem, we loosen the notion of frame.

14.10 Definition: A **generalized frame** is a pair $\mathbb{A} = (A, \mathbb{V})$ consisting of a frame A and a subset $\mathbb{V}$ of $\mathbb{U}(A)$ which contains $\emptyset$ and is closed under $\cap, \cup$ and the relative pseudocomplement $\Rightarrow$ of $\mathbb{U}(A)$.

In other words, we require that $\mathbb{V}$ is a subalgebra of $\mathbb{U}(A)$. This clearly is a sufficient adaptation to the difficulty of 14.9.2.

Frames may be considered as generalized frames by identifying A with $(A, \mathbb{U}(A))$. Accordingly, we generalize our use of the symbol $\mathbb{U}$: for any generalized frame $\mathbb{A} = (A, \mathbb{V})$, $\mathbb{U}(\mathbb{A})$ is the collection $\mathbb{V}$, considered as a Heyting algebra.

Having extended the notion of frame, we next adapt some concepts introduced in 1.2, 1.5 and 2.4.

14.11 Thus far, generalized frames have been motivated by mathematical expediency. A more philosophical motivation might run as follows. In 1.2, every upwards closed set counted as a proposition – in the down-to-earth sense that it could be selected as interpretation of a proposition letter. One might not want to be so liberal. In that case, propositions would still be upwards closed sets (as no information is ever lost); and the universe of propositions would have to be a subalgebra of the collection of all upwards closed sets, to make sure that every formula could be interpreted. In short, we shall have generalized frames $\mathbb{A} = (A, \mathbb{V})$, and valuations must give values in the propositional domain $\mathbb{V}$. [1]

Definition. A **valuation** on a generalized frame $\mathbb{A} = (A, \mathbb{V})$ is a function $V: \mathbb{P} \to \mathbb{V}$. (As before, $\mathbb{P}$ is the set of proposition letters.)

By the closure properties of $\mathbb{V}$, a valuation V may be extended to a map of $\mathbb{I}$ into $\mathbb{V}$ as in 1.2.4. Validity is defined as in 1.5: with $\mathbb{A} = (A, \mathbb{V})$,

$$\mathbb{A} \Vdash \varphi \quad \text{iff} \quad (A, V) \Vdash \varphi \text{ for every valuation } V \text{ on } \mathbb{A}.$$

Obviously, if $\mathbb{V} = \mathbb{U}(A)$, we have $\mathbb{A} \Vdash \varphi$ iff $A \Vdash \varphi$ in the sense of 1.5.

In line with the notation introduced for frames at the beginning of this section, we shall write $\mathbb{A} \Vdash \Phi$, for $\Phi \subseteq \mathbb{I}$, if $\forall \varphi \in \Phi.\ \mathbb{A} \Vdash \varphi$, and denote the class of all generalized frames in which every formula belonging to Φ is valid by $\mathbf{Gfr}(\Phi)$. A class $\mathbf{K}$ of generalized frames is $\mathbb{I}$-definable if $\mathbf{K} = \mathbf{Gfr}(\Phi)$ for some $\Phi \subseteq \mathbb{I}$. We may write $\mathbf{K} \Vdash \varphi$ if $\forall \mathbb{A} \in \mathbf{K}: \mathbb{A} \Vdash \varphi$.

The argument of 14.8 works just as well for generalized frames. We sum up the conclusions:

Proposition. (i) Let $\mathbb{A}$ be a generalized frame, and $\varphi,\psi \in \mathbb{I}$. Then

(a) $\mathbb{A} \Vdash \varphi$ iff $\mathbb{U}(\mathbb{A}) \models \varphi = \top$;

(b) $\mathbb{U}(\mathbb{A}) \models \varphi = \psi$ iff $\mathbb{A} \Vdash \varphi \leftrightarrow \psi$.

(ii) A class $\mathbf{K}$ of generalized frames is $\mathbb{I}$-definable iff there exists a subvariety $\mathbf{L} \subseteq \mathbf{Ha}$ such that any generalized frame $\mathbb{A}$ belongs to $\mathbf{K}$ iff $\mathbb{U}(\mathbb{A})$ belongs to $\mathbf{L}$.

14.12 As shown in 14.9.1, disjoint unions of frames correspond with products of Heyting algebras. Subsequently it turned out that no construction of frames can correspond to taking subalgebras. In the opposite direction, however, a parallel can be found: there is an operation on frames A which produces subalgebras of $\mathbb{U}(A)$ (modulo isomorphism), and one that produces homomorphic images.

14.12.1 Proposition. Suppose $f: A \twoheadrightarrow B$ is a surjective p-morphism. Then $\mathbb{U}(B)$ is isomorphic to a subalgebra of $\mathbb{U}(A)$.

Proof: Consider the inverse function f^{-1} on $\mathbb{U}(B)$. Because f is a homomorphism, f^{-1} maps $\mathbb{U}(B)$ into $\mathbb{U}(A)$: if $X \in \mathbb{U}(B)$, and $a' \geq a \in f^{-1}[X]$, then $f(a') \geq f(a) \in ff^{-1}[X] = X$; since X is upwards closed, we get $f(a') \in X$ and $a' \in f^{-1}[X]$.
As an inverse function, f^{-1} preserves $\cap, \cup$ and $\emptyset$; f^{-1} is injective since f is surjective.
To show that f^{-1} preserves $\Rightarrow$, we need the p-morphism condition. The crucial point is

(*) for $a \in A$, and $X,Y \in \mathbb{U}(B)$, $[f(a)) \cap X \subseteq Y$ iff $[a) \cap f^{-1}[X] \subseteq f^{-1}[Y]$.

Since f is a homomorphism, $f[[a)] \subseteq [f(a))$; hence $[a) \subseteq f^{-1}f[[a)] \subseteq f^{-1}[[f(a))]$. So if $[f(a)) \cap X \subseteq Y$, $[a) \cap f^{-1}[X] \subseteq f^{-1}[[f(a))] \cap f^{-1}[X] \subseteq f^{-1}[Y]$. For the converse, suppose $[a) \cap f^{-1}[X] \subseteq f^{-1}[Y]$. If $b \geq f(a)$, then by the p-morphism condition there exists $a' \geq a$ such that $f(a') = b$. So if $f(a) \leq b \in X$, we have $a' \in [a) \cap f^{-1}[X]$ with $f(a') = b$. Then $a' \in f^{-1}[Y]$; so $b = f(a') \in ff^{-1}[Y] = Y$.
Now,

$$\begin{aligned} a \in f^{-1}[X \Rightarrow Y] \ &\text{iff}\ f(a) \in X \Rightarrow Y \\ &\text{iff}\ [f(a)) \cap X \subseteq Y \end{aligned}$$

$$\begin{aligned}&\text{iff } [a) \cap f^{-1}[X] \subseteq f^{-1}[Y], \text{ by } (*),\\ &\text{iff } a \in f^{-1}[X] \Rightarrow f^{-1}[Y].\end{aligned}$$ □

14.12.2 Proposition. Suppose $A \subsetneq B$. Then $\mathbb{U}(A)$ is a homomorphic image of $\mathbb{U}(B)$.

Proof: Define f: $\mathbb{U}(B) \to \mathbb{U}(A)$ by $f(X) = X \cap A$. Since actually $\mathbb{U}(A)$ is a subset of $\mathbb{U}(B)$, f is surjective. Preservation of $\cap$,$\cup$ and $\emptyset$ is immediate.
Let $\Rightarrow_A$ denote the relative pseudocomplement in $\mathbb{U}(A)$, $\Rightarrow_B$ in $\mathbb{U}(B)$. Then for all $X,Y \in \mathbb{U}(B)$

$$\begin{aligned}(X \Rightarrow_B Y) \cap A &= A \cap \{b \in B \mid [b) \cap X \subseteq Y\}\\ &= \{a \in A \mid [a) \cap X \subseteq Y\}\\ &= \{a \in A \mid [a) \cap X \cap A \subseteq Y \cap A\}\\ &= X \cap A \Rightarrow_A Y \cap A.\end{aligned}$$

So f preserves $\Rightarrow$ as well. □

Thus we may hope that suitable generalizations of p-morphisms and generated subframes will do the job.

14.13 Definition. (i) Let $\mathbb{A} = (A, \mathbb{V})$ and $\mathbb{B} = (B, \mathbb{W})$ be generalized frames. A p-morphism f: $A \to B$ is a **p-morphism** from $\mathbb{A}$ to $\mathbb{B}$ if $\forall X \in \mathbb{W}. f^{-1}[X] \in \mathbb{V}$.
(ii) Let $\mathbb{A} = (A, \mathbb{V})$ and $\mathbb{B} = (B, \mathbb{W})$ be generalized frames. $\mathbb{A}$ is a **generated subframe** of $\mathbb{B}$ (notation: $\mathbb{A} \subsetneq \mathbb{B}$) if $A \subsetneq B$ and $\mathbb{V} = \{X \cap A \mid X \in \mathbb{W}\}$.
(iii) Let $\mathbb{A}_i = (A_i, \mathbb{V}_i)$ be generalized frames, for all $i \in I$. The **disjoint union** $\Sigma_{i\in I} \mathbb{A}_i$ is the generalized frame $(\Sigma_{i\in I} A_i, \mathbb{V})$ in which $X \subseteq \Sigma A_i$ belongs to $\mathbb{V}$ iff $\forall i \in I. X \cap (\{i\} \times A_i) \in \mathbb{V}_i$.

$\mathbb{B}$ will be called a **p-morphic image** of $\mathbb{A}$ if there exists a p-morphism from $\mathbb{A}$ to $\mathbb{B}$ that is a surjection for the underlying frames.
Note that disjoint unions of frames are a special case of disjoint unions of generalized frames. Also, if $A \subsetneq B$, then $\mathbb{U}(A) = \{X \cap A \mid X \in \mathbb{U}(B)\}$, so $(A, \mathbb{U}(A)) \subsetneq (B, \mathbb{U}(B))$. As to p-morphisms: if f: $A \to B$ is a homomorphism, $f^{-1}[X]$ is upwards closed for every $X \in \mathbb{U}(B)$. So f: $A \to B$ is a p-morphism iff f is a p-morphism from $(A, \mathbb{U}(A))$ to $(B, \mathbb{U}(B))$.

14.14 The proofs of propositions 14.9.1 and 14.12 are easily generalized. We get

Proposition. (i) If $(\mathbb{A}_i \mid i \in I)$ is a family of generalized frames, then $\mathbb{U}(\Sigma_{i\in I} \mathbb{A}_i) \cong \Pi_{i\in I} \mathbb{U}(\mathbb{A}_i)$.
(ii) If $\mathbb{A} \subsetneq \mathbb{B}$, then $\mathbb{U}(\mathbb{A})$ is a homomorphic image of $\mathbb{U}(\mathbb{B})$.

(iii) If $\mathbb{B}$ is a p-morphic image of $\mathbb{A}$, then $\mathbb{U}(\mathbb{B})$ is isomorphic to a subalgebra of $\mathbb{U}(\mathbb{A})$.

14.15 We have associated with every generalized frame $\mathbb{A}$ a Heyting algebra $\mathbb{U}(\mathbb{A})$. We also want a construction in the opposite direction.

14.15.1 Definition. Let $\mathfrak{U} = (U,\wedge,\vee,\perp,\rightarrow)$ be a Heyting algebra. A **filter** in $\mathfrak{U}$ is a set $\nabla \subseteq U$ such that

(i) $v \geq u \in \nabla$ implies $v \in \nabla$ (in other words, ∇ is upwards closed);

(ii) if $u,v \in \nabla$, then $u \wedge v \in \nabla$.

A filter ∇ is **prime** if it is proper (i.e. equals neither U nor $\emptyset$) and satisfies

(iii) if $u \vee v \in \nabla$, then $u \in \nabla$ or $v \in \nabla$.

The following will be useful:

14.15.2 Lemma: Let $\mathfrak{U} = (U,\wedge,\vee,\perp,\rightarrow)$ and $\mathfrak{V} = (V,\wedge,\vee,\perp,\rightarrow)$ be Heyting algebras; $f\colon \mathfrak{U} \rightarrow \mathfrak{V}$ a homomorphism. Then, if ∇ is a prime filter in $\mathfrak{V}$, $f^{-1}[\nabla]$ is a prime filter in $\mathfrak{U}$.

Proof: Suppose ∇ is a prime filter in $\mathfrak{V}$; we check (i) - (iii) above for $f^{-1}[\nabla]$.
(i) Homomorphisms preserve $\leq$. So if $u' \geq u \in f^{-1}[\nabla]$, we get $f(u') \geq f(u) \in \nabla$, hence ($\nabla$ being upwards closed) $f(u') \in \nabla$.
(ii) If $u,u' \in f^{-1}[\nabla]$, $f(u \wedge u') = f(u) \wedge f(u') \in \nabla$; so $u \wedge u' \in f^{-1}[\nabla]$.
(iii) If $u \vee u' \in f^{-1}[\nabla]$, then $f(u) \vee f(u') = f(u \vee u') \in \nabla$. Hence $f(u) \in \nabla$ or $f(u') \in \nabla$; accordingly, $u \in f^{-1}[\nabla]$ or $u' \in f^{-1}[\nabla]$. □

Stone [1937] used prime filters to obtain topological representations of distributive lattices, and a fortiori of Heyting algebras. There is a natural ordering on the resulting spaces which makes them into generalized frames, with certain open sets as propositions. Very roughly, the idea may be formulated as follows. The points in a generalized frame $(A,\mathbb{V})$ determine prime filters $\nabla_a := \{X \in \mathbb{V} \mid a \in X\}$ in $\mathbb{V}$. Then, if there are no points, we can at least take the prime filters and pretend they are of form ∇_a.

14.15.3 Definition: Let $\mathfrak{U} = (U,\wedge,\vee,\perp,\rightarrow)$ be a Heyting algebra. Let A be the set of all prime filters of $\mathfrak{U}$, ordered by inclusion; and for each $u \in U$, define $X_u \subseteq A$ by $X_u := \{a \in A \mid u \in a\}$. Then $\mathbb{F}(\mathfrak{U})$, the **prime filter representation** of $\mathfrak{U}$, is $(A, \{X_u \mid u \in U\})$.

It is straightforward to check that $\mathbb{F}(\mathfrak{U})$ is a generalized frame. An example may be of help.

14.16 Example: Let $\mathfrak{R}$ be the Heyting algebra pictured in 11.1. It is clear from the diagram that every filter in $\mathfrak{R}$ has a least element; thus, every filter can be written as $[r)$, with r in the domain R of $\mathfrak{R}$. A filter $[r)$ is prime precisely if r cannot be decomposed as $r = r_1 \vee r_2$ with $r_1, r_2 < r$; and $[r) \subseteq [r')$ iff $r' \leq r$. Since $[\bot) = R$, the frame of $\mathbb{F}(\mathfrak{R})$ is isomorphic to the frame M of 11.1, extended with a least element $\top$. (Because $\top = \bigvee(R - \{\top\})$, it did not belong to M; but $\top$ cannot be represented as a *finite* supremum of strictly less elements.)
Modulo an isomorphism, we may write $\mathbb{F}(\mathfrak{R}) = (M', \mathbb{V})$, with $M' = M \cup \{\top\}$ as above. Then $\mathbb{V}$ *almost* equals $\mathbb{U}(M')$: only M is missing. On the other hand, $\mathbb{U}(M) \cong \mathfrak{R}$.

The following theorem is proved in the literature on Heyting algebras.

14.17 Representation Theorem (Stone): Let $\mathfrak{U}$ be a Heyting algebra; define $F_{\mathfrak{U}}: \mathfrak{U} \to \mathbb{U} \circ \mathbb{F}(\mathfrak{U})$ by $F_{\mathfrak{U}}(u) = X_u$ (in the notation of 14.15.3). Then $F_{\mathfrak{U}}$ is an isomorphism.

We state the crucial lemma, for later reference. To do so, we need the duals of filters:

Definition: An **ideal** in a Heyting algebra $\mathfrak{U} = (U, \wedge, \vee, \bot, \to)$ is a downwards closed set closed under finite joins, i.e. $\Delta \subseteq U$ is an ideal iff

(i) $u' \leq u \in \Delta$ implies $u' \in \Delta$;

(ii) $u_0, u_1 \in \Delta$ implies $u_0 \vee u_1 \in \Delta$.

Lemma (prime filter theorem for Heyting algebras): Let $\mathfrak{U}$ be a Heyting algebra, ∇ a filter in $\mathfrak{U}$, and Δ an ideal such that $\nabla \cap \Delta = \varnothing$. Then there exists a prime filter ∇^* of $\mathfrak{U}$ such that $\nabla^* \supseteq \nabla$ and $\nabla^* \cap \Delta = \varnothing$.

(The proof uses Zorn's lemma; the prime filter theorem is equivalent to the prime ideal theorem for Boolean algebras.)

14.18 Generalized frames are not so well-behaved: it is not generally true that $\mathbb{F} \circ \mathbb{U}(\mathbb{A}) \cong \mathbb{A}$. For one, there may be too few sets in $\mathbb{U}(\mathbb{A})$ to localize every point of $\mathbb{A}$: take for instance $\mathbb{A} = (\{0,1\}, \{\{0,1\}, \varnothing\})$. Here, $\mathbb{U}(\mathbb{A})$ is the two-element Boolean algebra (which we shall denote by **2**), and **2** has only one prime filter, viz. $\{\top\}$. A different example is in 14.11: $M \not\cong M'$. We thus have a construction that may lead to genuinely new frames.
There is a nice special case. Since $\mathbb{U} \circ \mathbb{F}(\mathfrak{U}) \cong \mathfrak{U}$ by the representation theorem, we have

14.18.1 Proposition: Let $\mathfrak{U}$ be a Heyting algebra. Then $\mathbb{F} \circ \mathbb{U}(\mathbb{F}(\mathfrak{U})) \cong \mathbb{F}(\mathfrak{U})$.

The isomorphism takes each prime filter ∇ in $\mathbb{UF}(\mathfrak{U})$ to the prime filter $\{u \mid X_u \in \nabla\}$. It is a rather special feature of generalized frames of the form $\mathbb{F}(\mathfrak{U})$: ordinarily, homomorphisms from $\mathbb{FU}(\mathbb{A})$ to $\mathbb{A}$ involve an essential loss of information. E.g., homomorphisms from M' to M have finite range.

A generalized frame $\mathbb{A}$ will be called a **descriptive frame** if $\mathbb{A} \cong \mathbb{FU}(\mathbb{A})$. If $\mathbb{FU}(\mathbb{A}) = (B, \mathbb{V})$, I shall call B the **prime filter extension** of $\mathbb{A}$, in symbols: $B = \mathrm{pe}(\mathbb{A})$. Note that frames are embedded in their prime filter extensions by the map $a \mapsto \nabla_a$ (with $\nabla_a = \{X \in \mathbb{U}(\mathbb{A}) \mid a \in X\}$).

14.18.2 Proposition. For all $\varphi \in \mathbb{I}$, $\mathbb{A} \Vdash \varphi$ iff $\mathbb{FU}(\mathbb{A}) \Vdash \varphi$.

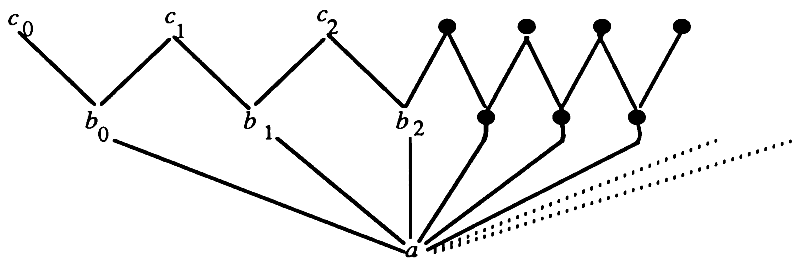
Proof: By proposition 14.11, since $\mathbb{U}(\mathbb{A}) \cong \mathbb{U}(\mathbb{F}\circ\mathbb{U}(\mathbb{A}))$. □

In particular, valid formulas are preserved under $\mathbb{F}\circ\mathbb{U}$. This contrasts with taking prime filter extensions of frames: we will show in the next example that valid formulas may be lost that way. Here we note that anti-preservation remains:

14.18.3 Proposition: Let $\mathbb{A}$ be a generalized frame, and $\varphi \in \mathbb{I}$. Then $\mathrm{pe}(\mathbb{A}) \Vdash \varphi$ implies $\mathbb{A} \Vdash \varphi$.

Proof: Suppose $\mathrm{pe}(\mathbb{A}) \Vdash \varphi$. Then since $\mathbb{U}(\mathbb{F}\circ\mathbb{U}(\mathbb{A})) \subseteq \mathbb{U}(\mathrm{pe}(\mathbb{A}))$, $\mathbb{F}\circ\mathbb{U}(\mathbb{A}) \Vdash \varphi$. By the corollary above, $\mathbb{A} \Vdash \varphi$. □

14.19 Example. Let A be the frame defined in 6.8(a). We reproduce the relevant diagram:



Let $C := \{c_n \mid n \in \mathbb{N}\}$. We consider the prime filters of $\mathbb{U}(A)$.

First, then, there are the filters $\nabla_d = \{X \in \mathbb{U}(A) \mid d \in X\}$, for all $d \in A$. Since every X in $\mathbb{U}(A)$ not of form $[d)_A$ can be decomposed as $X = Y \cup Z$ with $Y, Z \in \mathbb{U}(A)$ proper subsets of X, the filters ∇_d are the only principal prime filters. Since $\nabla_d \subseteq \nabla_{d'}$ iff $d \le d'$, A lies embedded in $\mathrm{pe}(A)$. Now for the nonprincipal prime filters. Observe

(1) If a prime filter ∇ contains a subset of C, ∇ is maximal in $\mathrm{pe}(A)$.

The reason is, that if a prime filter ∇ contains $X \subseteq C$ (hence contains C, by 14.15.1(i)), it must contain either Y or $C-Y$ for every $Y \subseteq C$, by (iii) of the definition. Suppose $C \in \nabla$ and ∇' is a filter properly extending ∇. Say $Z \in \nabla' - \nabla$. Then $Z \cap C \in \nabla'$; since $Z \supseteq Z \cap C$, $Z \cap C \notin \nabla$. So $C-Z \in \nabla$, and $\emptyset = (C-Z) \cap Z \in \nabla'$; that is, ∇' is not a proper filter.

(2) If ∇ is a prime filter, $\nabla \neq \{A\}$, and ∇' is a prime filter properly extending ∇, then ∇' is maximal in pe(A).

Similarly to the case of (1), there will be an ultrafilter U over $\mathbb{N}$ such that

$$\forall X \subseteq \mathbb{N}: \ \cup_{n \in X} [b_n)_A \in \nabla \ \text{ iff } \ X \in U.$$

So a proper filter extending ∇ must contain a subset of C.
Thus, pe(A) is not as forbidding as it may have seemed: at least it is not higher than A. Now we come to a crucial point:

(3) Let ∇_d, ∇ be maximal in pe(A); suppose ∇ is not principal. Then if $\nabla' \subseteq \nabla_d \cap \nabla$ is a prime filter, $\nabla' = \{A\}$.

Suppose $\nabla' \subseteq \nabla_d \cap \nabla$ is a filter, and $A - \{a\} \in \nabla'$. Say that $d = c_k$. Let $X = [b_{k-1}) \cup [b_k)$ (or just $[b_k)$ if k=0), and $Y = \cup\{[b_n) \mid n{\neq}k-1,k\}$. Then $A - \{a\} = X \cup Y$; so if ∇' is prime, $X{\in}\nabla'$ or $Y{\in}\nabla'$. Now $X \notin \nabla$: for then ∇ would also contain either $[b_{k-1})$ or $[b_k)$, and be principal. On the other hand, $d \notin Y$, so $Y \notin \nabla_d$. It follows that ∇' is not a prime filter.

We have seen in 6.8(a) that $A \Vdash$ SC (= $[(\neg\neg p \to p) \to p \vee \neg p] \to \neg p \vee \neg\neg p$). For pe($A$) we can define a projection onto the frame $[8)_M$ (cf. 11.1, 6.8(a)) by putting:

$$f(\{A\}) = 8, f(\nabla_{b_n}) = 4 \text{ for all } n \in \mathbb{N}, f(\nabla_c) = 1 \text{ for all } c \in C,$$
$$\text{and } f(\nabla) = 2 \text{ for nonprincipal } \nabla.$$

Therefore pe(A) $\not\Vdash$ SC.

14.20 The statement symmetric to 14.14(i) is: "if $(\mathfrak{A}_i \mid i \in I)$ is a family of Heyting algebras, then $\mathbb{F}(\prod_{i\in I} \mathfrak{A}_i) \cong \Sigma_{i\in I} \mathbb{F}(\mathfrak{A}_i)$". Unfortunately, this is false.

<u>Example:</u> Let $\mathfrak{A}_n = \mathbf{2}$ (= the Boolean algebra $\{\bot,\top\}$) for all $n \in \mathbb{N}$. Since $|\mathbb{F}(\mathbf{2})| = 1$, $|\Sigma_{n\in\mathbb{N}}\mathbb{F}(\mathfrak{A}_n)| = \aleph_0$. But $\prod_n \mathfrak{A}_n$ is isomorphic to the Boolean algebra $\mathbb{P}(\mathbb{N})$; the prime filters in $\mathbb{P}(\mathbb{N})$ are the

ultrafilters over $\mathbb{N}$, of which there are $2^{2^{\aleph_0}}$ (Tarski; see e.g. Bell & Slomson [1969] Ch. 6, thm. 1.5); so $|\mathbb{F}(\prod_n \mathfrak{A}_n)| = 2^{2^{\aleph_0}}$. Then obviously $\Sigma_n \mathbb{F}(\mathfrak{A}_n) \ncong \mathbb{F}(\prod_n \mathfrak{A}_n)$.

This also shows that a disjoint union of descriptive frames need not be a descriptive frame: since

$$\mathbb{FU}(\Sigma_n \mathbb{F}(\mathfrak{A}_n)) \cong \mathbb{F}(\prod_n \mathbb{UF}(\mathfrak{A}_n)) \cong \mathbb{F}(\prod_n \mathfrak{A}_n) \ncong \Sigma_n \mathbb{F}(\mathfrak{A}_n),$$

$\Sigma_n \mathbb{F}(\mathfrak{A}_n)$ is not descriptive. We do have, as a consequence of 14.14(i) and the representation theorem,

<u>Proposition.</u> If $(\mathfrak{A}_i \mid i \in I)$ is a family of Heyting algebras, then $\mathbb{F}(\prod_{i\in I} \mathfrak{A}_i) \cong \mathbb{FU}(\Sigma_{i\in I} \mathbb{F}(\mathfrak{A}_i))$.

14.21 <u>Lemma.</u> Let $\mathfrak{A}$ and $\mathfrak{B}$ be Heyting algebras, and f: $\mathfrak{A} \to \mathfrak{B}$ a homomorphism. Define g on the set of prime filters of $\mathfrak{B}$ by

$$g(\nabla) = f^{-1}[\nabla].$$

Then g is a p-morphism from $\mathbb{F}(\mathfrak{A})$ to $\mathbb{F}(\mathfrak{B})$.

<u>Proof:</u> Let $\mathfrak{A} = (U,\wedge,\vee,\bot,\rightarrow)$, $\mathfrak{B} = (V,\wedge,\vee,\bot,\rightarrow)$, A the frame of prime filters of $\mathfrak{A}$, and B the same for $\mathfrak{B}$. For every $b \in B$, $g(b) \in A$ by lemma 14.15.2. g is a homomorphism by an elementary property of inverse functions. In notation from 14.17, $b \in g^{-1}[F_{\mathfrak{A}}(u)]$ iff $g(b) \in F_{\mathfrak{A}}(u)$ iff $u \in g(b)$ iff $f(u) \in b$ iff $b \in F_{\mathfrak{B}}(f(u))$; so $g^{-1}[F_{\mathfrak{A}}(u)] \in F_{\mathfrak{B}}[V]$. By definition 14.13, it will now suffice to show that g:$B \to A$ satisfies the p-morphism condition .
So, suppose $a \geq_A g(b_0)$ $(=f^{-1}[b_0])$. We must find $b \geq_B b_0$ such that $g(b) = a$. Let ∇ be the filter generated by $f[a] \cup b_0$, and Δ the ideal generated by $f[U - a]$:

$$\nabla = \{v \in V \mid \exists u \in a\, \exists v_1 \in b_0.\ v \geq f(u) \wedge v_1\},$$

$$\Delta = \{w \in V \mid \exists u_1 \dots u_m \in U - a.\ w \leq \textstyle\bigvee_{1\leq j\leq m} f(u_j)\}.$$

By the prime filter theorem, it suffices to show that $\nabla \cap \Delta = \emptyset$: for then there is a prime filter $b \supseteq \nabla$ with $b \cap \Delta = \emptyset$, whence $f^{-1}[b] = a$, and $b \supseteq b_0$ (i.e. $b \geq_B b_0$) by the definition of ∇.
Suppose $v_0 \in \nabla \cap \Delta$. Then there exist $u \in a$, $v_1 \in b_0$, $u_1,\dots,u_m \in U - a$, with

$$f(u) \wedge v_1 \leq v_0 \leq \textstyle\bigvee_{1\leq j\leq m} f(u_j),$$

so $f(u) \wedge v_1 \leq \bigvee_{1\leq j\leq m} f(u_j)$. Consequently,

$$v_1 \leq f(u) \to \bigvee f(u_j) = f(u \to \bigvee u_j).$$

Since $v_1 \in b_0$, $f(u \to \bigvee u_j) \in b_0$, and

$$u \to \bigvee u_j \in f^{-1}[b_0] \subseteq a.$$

Since $u \in a$, we get $\bigvee u_j \geq u \wedge (u \to \bigvee u_j) \in a$, so $\bigvee u_j \in a$. Since a is prime, some u_j must belong to a: a contradiction. □

14.22 Proposition. If $\mathfrak{U}$ and $\mathfrak{V}$ are Heyting algebras and $\mathfrak{U} \subseteq \mathfrak{V}$, then $\mathbb{F}(\mathfrak{U})$ is a p-morphic image of $\mathbb{F}(\mathfrak{V})$.

Proof: Let U be the domain of $\mathfrak{U}$, V that of $\mathfrak{V}$; let A be the frame of prime filters of $\mathfrak{U}$, B that of $\mathfrak{V}$. Define g on B by

$$g(b) = b \cap U.$$

Then g is the inverse of the canonical embedding i: $\mathfrak{U} \hookrightarrow \mathfrak{V}$. Since i is a homomorphism, g is a p-morphism from $\mathbb{F}(\mathfrak{V})$ to $\mathbb{F}(\mathfrak{U})$ by the lemma above. So we need only prove that g is surjective. Let $a \in A$; we must find $b \in B$ such that $a = g(b)$ $(= b \cap U)$. Let ∇ be the filter in $\mathfrak{V}$ generated by a, and Δ the ideal in $\mathfrak{V}$ generated by $U - a$:

$$\nabla = \{v \in V \mid \exists u \in a.\ v \geq u\}$$

$$\Delta = \{w \in V \mid \exists u_1 \dots u_m \in U - a.\ w \leq \bigvee_{1 \leq j \leq m} u_j\}.$$

As in the proof of the lemma, it will suffice to show that $\nabla \cap \Delta = \emptyset$. Say that $v_0 \in \nabla \cap \Delta$: then we have $u \in a$, $u_1, \dots, u_m \in U - a$, with

$$u \leq v_0 \leq \bigvee_{1 \leq j \leq m} u_j.$$

Since $u \in a$, this implies $\bigvee_{1 \leq j \leq m} u_j \in a$; which is impossible because a is prime. □

14.23 Proposition. If $\mathfrak{U}$ and $\mathfrak{V}$ are Heyting algebras and $\mathfrak{V}$ is a homomorphic image of $\mathfrak{U}$, then $\mathbb{F}(\mathfrak{V})$ is isomorphic to a generated subframe of $\mathbb{F}(\mathfrak{U})$.

Proof: Let f: $\mathfrak{U} \twoheadrightarrow \mathfrak{V}$ be a surjective homomorphism of Heyting algebras. Define g as in lemma 21.

Define A, B, U, V as before. By lemma 21 and proposition 2.4.2, $g[B] \subsetneq A$. g is one-to-one: for if $b, b' \in B$ and $b \leq b'$, i.e. $b \subseteq b'$, then $f^{-1}[b] \subseteq f^{-1}[b']$ since f is surjective. So $B \cong g[B]$. It remains to show that

$$\{g[\mathrm{F}_{\mathfrak{V}}(v)] \mid v \in V\} = \{g[B] \cap \mathrm{F}_{\mathfrak{U}}(u) \mid u \in U\}$$

(notation from 14.17).

Observe that, by surjectivity, every $v \in V$ is $f(u)$ for some $u \in U$. Thus it would suffice if $\forall u \in U$. $\mathrm{F}_{\mathfrak{U}}(u) \cap g[B] = g[\mathrm{F}_{\mathfrak{V}}(f(u))]$. This equality is seen to hold by the following calculation:

$$b \in \mathrm{F}_{\mathfrak{V}}(f(u)) \text{ iff } f(u) \in b \text{ iff } u \in f^{-1}[b] = g(b) \text{ iff } g(b) \in \mathrm{F}_{\mathfrak{U}}(u).$$

□

14.24 Theorem. A class $\mathbf{K}$ of generalized frames is $\mathbb{I}$-definable iff $\mathbf{K}$ and its complement are closed under $\mathbb{F} \circ \mathbb{U}$, and $\mathbf{K}$ is closed under disjoint unions, generated subframes and p-morphic images.

Proof:

($\Rightarrow$) Suppose $\mathbf{K}$ is $\mathbb{I}$-definable. By proposition 14.18.2, $\mathbb{I}$-formulas are invariant under $\mathbb{FU}$. By proposition 14.11(ii), there exists a variety $\mathbf{L}$ of Heyting algebras such that any generalized frame $\mathbb{A}$ belongs to $\mathbf{K}$ iff $\mathbb{U}(\mathbb{A})$ belongs to $\mathbf{L}$. Suppose $\mathbb{A}_i \in \mathbf{K}$ for all $i \in I$. Then $\mathbb{U}(\mathbb{A}_i) \in \mathbf{L}$, for all $i \in I$; since $\mathbf{L}$ is a variety, this implies $\prod_{i \in I} \mathbb{U}(\mathbb{A}_i) \in \mathbf{L}$. By 14.14(i), $\prod \mathbb{U}(\mathbb{A}_i) \cong \mathbb{U}(\Sigma \mathbb{A}_i)$. Since $\mathbf{L}$, as a variety, is closed under isomorphism, $\mathbb{U}(\Sigma \mathbb{A}_i) \in \mathbf{L}$; hence $\Sigma \mathbb{A}_i \in \mathbf{K}$. We conclude that $\mathbf{K}$ is closed under disjoint unions. Closure under generated subframes and p-morphic images can be proved similarly, with applications of the rest of 14.14 .[2]

($\Leftarrow$) Suppose $\mathbf{K}$ and its complement are closed under $\mathbb{FU}$, and $\mathbf{K}$ is closed under disjoint unions, generated subframes and p-morphic images. Let $\mathbf{L}$ be the closure of $\mathbb{U}[\mathbf{K}]$ under isomorphism. Suppose $\mathbb{U}(\mathbb{A}) \in \mathbf{L}$: then $\mathbb{U}(\mathbb{A}) \cong \mathbb{U}(\mathbb{B})$ for some $\mathbb{B} \in \mathbf{K}$, hence $\mathbb{FU}(\mathbb{A}) \cong \mathbb{FU}(\mathbb{B})$; and $\mathbb{FU}(\mathbb{B}) \in \mathbf{K}$ by closure under $\mathbb{FU}$. Since $\mathbf{K}$ is closed under isomorphism, $\mathbb{FU}(\mathbb{A}) \in \mathbf{K}$; and because the complement of $\mathbf{K}$ is closed under $\mathbb{FU}$, we must have $\mathbb{A} \in \mathbf{K}$. We conclude that $\forall \mathbb{A}$ $(\mathbb{A} \in \mathbf{K} \Leftrightarrow \mathbb{U}(\mathbb{A}) \in \mathbf{L})$.

Now it will suffice, by proposition 14.11(ii), to show that $\mathbf{L}$ is a variety. Suppose $\mathfrak{U}_i \in \mathbf{L}$ for all $i \in I$. Say $\mathfrak{U}_i \cong \mathbb{U}(\mathbb{A}_i)$, with $\mathbb{A}_i \in \mathbf{K}$. Then

$$\prod_{i \in I} \mathfrak{U}_i \cong \prod_{i \in I} \mathbb{U}(\mathbb{A}_i) \cong \mathbb{U}(\Sigma_{i \in I} \mathbb{A}_i)$$

by 14.14(i), and $\Sigma \mathbb{A}_i \in \mathbf{K}$ by closure under disjoint unions. So $\mathbf{L}$ is closed under products. If $\mathfrak{U} \subseteq \mathfrak{V} \cong \mathbb{U}(\mathbb{A})$, with $\mathbb{A} \in \mathbf{K}$, then $\mathbb{F}(\mathfrak{U})$ is a p-morphic image of $\mathbb{F}(\mathfrak{V})$ (proposition 22), and, since $\mathbb{F}(\mathfrak{V}) \cong \mathbb{FU}(\mathbb{A})$, of $\mathbb{FU}(\mathbb{A})$. By closure of $\mathbf{K}$ under $\mathbb{FU}$ and p-morphic images, $\mathbb{F}(\mathfrak{U}) \in \mathbf{K}$. Since

$\mathfrak{A} \cong \mathbb{FU}(\mathfrak{A})$ by the representation theorem, $\mathfrak{A} \in \mathbf{L}$. So $\mathbf{L}$ is closed under subalgebras. Finally, suppose $\mathfrak{B}$ is a homomorphic image of $\mathfrak{A} \in \mathbf{L}$; say $\mathfrak{A} \cong \mathbb{U}(\mathbb{A})$, $\mathbb{A} \in \mathbf{K}$. Then $\mathbb{F}(\mathfrak{B})$ is isomorphic to a generated subframe of $\mathbb{F}(\mathfrak{A})$ (proposition 23), and a fortiori of $\mathbb{FU}(\mathbb{A})$. By the closure properties of $\mathbf{K}$, $\mathbb{F}(\mathfrak{B}) \in \mathbf{K}$, and as before, $\mathfrak{B} \in \mathbf{L}$. So $\mathbf{L} = \boldsymbol{HSP}(\mathbf{L})$; therefore $\mathbf{L}$ is a variety, by Birkhoff's theorem. □

14.25 Now we return to $\mathbb{I}$-definable classes of frames. For a class $\mathbf{K}$ of frames, let $\mathrm{Th}_{\mathbb{I}}(\mathbf{K}) := \{\varphi \in \mathbb{I} \mid \mathbf{K} \Vdash \varphi\}$. We shall denote the least $\mathbb{I}$-definable class of frames containing $\mathbf{K}$ by $\mathbf{Fr}_{\mathbb{I}}(\mathbf{K})$ – clearly, then, $\mathbf{Fr}_{\mathbb{I}}(\mathbf{K}) = \mathbf{Fr}(\mathrm{Th}_{\mathbb{I}}(\mathbf{K}))$.
Let A be a frame, and $\mathbf{K}$ a class of frames. Then

$$A \in \mathbf{Fr}_{\mathbb{I}}(\mathbf{K}) \text{ iff } A \Vdash \mathrm{Th}_{\mathbb{I}}(\mathbf{K})$$

$$\text{iff } \mathbb{U}(A) \models \{\varphi = \top \mid \mathbf{K} \Vdash \varphi\} \quad \text{(by proposition 14.11(i))}$$

$$\text{iff } \mathbb{U}(A) \models \{\varphi = \psi \mid \mathbb{U}[\mathbf{K}] \models \varphi = \psi\} \quad (\text{by 14.8(b)}, \ \mathbb{U}[\mathbf{K}] \models \varphi = \psi \text{ iff } \mathbb{U}[\mathbf{K}] \models \varphi \leftrightarrow \psi = \top)$$

$$\text{iff } \mathbb{U}(A) \in \boldsymbol{HSP}(\mathbb{U}[\mathbf{K}]), \text{ by Birkhoff's theorem.}$$

So if $A \in \mathbf{Fr}_{\mathbb{I}}(\mathbf{K})$, there will be a family $(B_i \mid i \in I)$ of frames in $\mathbf{K}$, and a Heyting algebra $\mathfrak{B}$, such that $\mathbb{U}(A)$ is a homomorphic image of $\mathfrak{B}$, and $\mathfrak{B}$ a subalgebra of $\prod_{i \in I} \mathbb{U}(B_i)$. By proposition 14.14(i), we may suppose that $\mathfrak{B}$ is a subalgebra of $\mathbb{U}(\Sigma_{i \in I} B_i)$.

14.26 The next step is to investigate the relation between A and $\mathbb{F}(\mathfrak{B})$, given that $\mathbb{U}(A)$ is a homomorphic image of $\mathfrak{B}$.

<u>Lemma.</u> Suppose $\mathbb{U}(A)$ is a homomorphic image of $\mathfrak{B}$; let C be the frame of prime filters in $\mathfrak{B}$, and V the domain of $\mathfrak{B}$. Then A is isomorphic to a subframe B of C such that

$$\mathbb{U}(B) = \{B \cap \mathrm{F}_{\mathfrak{B}}(v) \mid v \in V\}.$$

<u>Proof:</u> Let $f\colon \mathfrak{B} \twoheadrightarrow \mathbb{U}(A)$ be a surjective homomorphism. Define $g\colon A \to C$ by $g(a) = f^{-1}[\nabla_a]$. That indeed $g(a) \in C$ follows from lemma 14.15.2. Because $a \leq_A a'$ implies $\nabla_a \subseteq \nabla_{a'}$, g is a homomorphism. If $a \nleq_A a'$, then $a' \notin [a)$, so $[a) \in \nabla_a - \nabla_{a'}$; suppose $[a) = f(v)$, then $v \in g(a) - g(a')$, so $g(a) \nleq_C g(a')$. Take $B = g[A]$.
Now if $U \in \mathbb{U}(B)$, there is some $u \in V$ such that $U = g[f(u)] = \{f^{-1}[\nabla_a] \mid a \in f(u)\}$. For any $b \in B$,

$$b \in U \text{ iff } \exists a \in f(u).\, b = g(a) = f^{-1}[\nabla_a]$$

$$\text{iff } \exists a \in f(u).\, \forall v \in V\,(v \in b \Leftrightarrow f(v) \in \nabla_a)$$

$$\text{iff } \exists a \in f(u).\, \forall v \in V\,(v \in b \Leftrightarrow a \in f(v))$$

iff $u \in b$ (down: take $v{=}u$; up: take $a{=}g^{-1}(b)$, then $u \in b \Rightarrow$
$u \in g(a) \Rightarrow u \in f^{-1}[\nabla_a] \Rightarrow f(u) \in \nabla_a \Rightarrow a \in f(u)$; and $b{=}g(a){=}f^{-1}[\nabla_a]$)

iff $b \in \mathrm{F}_{\mathfrak{V}}(u)$.

This shows that $U = B \cap \mathrm{F}_{\mathfrak{V}}(u)$; we conclude that indeed $\mathbb{U}(B) = \{B \cap \mathrm{F}_{\mathfrak{V}}(v) \mid v \in V\}$. □

14.27 The characterization of $\mathbb{I}$-definable classes of frames uses generalized frames, as was to be expected after the example in 14.9.2. The concept of generalized frame is hidden in the notion of **subalgebra-based** (cf. Goldblatt & Thomason [1974]).

Definition. A frame A is **subalgebra-based** on a frame B if there exists a descriptive frame $\mathbb{F}(\mathfrak{V}) = (C, \mathbb{V})$ such that

(i) $\mathfrak{V}$ is a subalgebra of $\mathbb{U}(B)$;
(ii) A is a subframe of C;
(iii) $\mathbb{U}(A) = \{A \cap X \mid X \in \mathbb{V}\}$.

Theorem. Let **K** be a class of frames, and A a frame. Then $A \in \mathbf{Fr}_{\mathbb{I}}(\mathbf{K})$ iff A is isomorphic to a frame that is subalgebra-based on the disjoint union of a subfamily of **K**.

Proof: As noted in 14.25, $A \in \mathbf{Fr}_{\mathbb{I}}(\mathbf{K})$ iff there is a subfamily $(B_i \mid i \in I)$ of **K** and a subalgebra $\mathfrak{V}$ of $\mathbb{U}(\Sigma_{i\in I} B_i)$ such that $\mathbb{U}(A)$ is a homomorphic image of $\mathfrak{V}$. Suppose $\mathbb{F}(\mathfrak{V}) = (C, \mathbb{V})$. By lemma 26, A is isomorphic to a subframe B of C such that $\mathbb{U}(B) = \{X \cap B \mid X \in \mathbb{V}\}$. By the definition above, B is subalgebra-based on ΣB_i. □

Corollary. A class **K** of frames is $\mathbb{I}$-definable iff it is closed under isomorphism and disjoint unions, and contains every frame subalgebra-based on some element of **K**.

14.28 We end with another consequence of 14.25, to be used in the next section. If $\mathfrak{V}$ is a subalgebra of $\mathbb{U}(\Sigma B_i)$, then $\mathbb{F}(\mathfrak{V})$ is isomorphic to a p-morphic image of $\mathbb{F}\mathbb{U}(\Sigma B_i)$, by proposition

22. If $\mathbb{U}(A)$ is a homomorphic image of $\mathfrak{V}$, then $\mathbb{FU}(A)$ is isomorphic to a generated subframe of $\mathbb{F}(\mathfrak{V})$ by proposition 23. Taking an isomorphic copy $\mathfrak{V}'$ of $\mathfrak{V}$ such that $\mathbb{FU}(A) \subsetneq \mathbb{F}(\mathfrak{V}')$, we get

Proposition. Let A be a frame, and $\mathbf{K}$ a class of frames. Then if $A \in \mathbf{Fr}_{\mathbb{I}}(\mathbf{K})$, there is a subfamily $(B_i \mid i \in I)$ of $\mathbf{K}$ such that $\mathbb{FU}(A)$ is a generated subframe of a p-morphic image of $\mathbb{FU}(\sum_{i \in I} B_i)$.

Footnotes:

[1] This move is comparable to the nonstandard interpretation of higher order logic (e.g. Henkin [1950]). The parallel extends to completeness theory: intuitionistic propositional logic is complete for generalized frames, in the sense that an $\mathbb{I}$-formula φ can be deduced from the set of all substitution instances of a set $\Psi \subseteq \mathbb{I}$ iff φ is valid in every generalized frame in which Ψ is valid.

[2] One can also prove preservation theorems directly, on the pattern of 2.4.

§15. $\mathbb{I}$-definable elementary classes

In this section we consider the question which $\mathbb{L}_0$-sentences are $\mathbb{I}$-definable – precisely, for which $\mathbb{L}_0$-sentences α there is a set Φ of $\mathbb{I}$-formulas such that for any frame A, $A \Vdash \Phi$ iff $A \vDash \alpha$. As in §14, the answers are adaptations of results of modal correspondence theory: theorem 3 derives from Goldblatt & Thomason [1974]; lemma 2 and the discussion in 4 are inspired by, respectively copied from, van Benthem [1986].

15.1 Lemma 13.7.1 can be generalized to languages of arbitrary cardinality. The generalization requires that the index set be sufficiently large and the ultrafilters of a special kind (see Chang & Keisler, §6.1); but these conditions do not figure in the corollary that we shall use:

Lemma. Let $\mathfrak{A}$ be a structure for some first order language. Then $\mathfrak{A}$ has a countably saturated ultrapower.

If $\mathfrak{A}$ is a structure for the first order language $\mathbb{L}$, we denote by Th($\mathfrak{A}$) the first order theory of $\mathfrak{A}$, i.e. the set of all $\mathbb{L}$-sentences true in $\mathfrak{A}$. For $\mathbb{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$, $\mathfrak{A} \equiv \mathfrak{B}$ will mean that $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent (that is, Th($\mathfrak{A}$)=Th($\mathfrak{B}$)).

15.2 Lemma. For any frame A, the prime filter extension pe(A) is a p-morphic image of an ultrapower of A.

Proof. Let A be a frame. Add to $\mathbb{L}_0$ distinct unary predicate letters P_X for all $X \in \mathbb{U}(A)$; expand A to $\mathfrak{A}=(A,X)_{X \in \mathbb{U}(A)}$, with X as the denotation of P_X. Take a countably saturated ultrapower $\mathfrak{B}=(B,X')_{X \in \mathbb{U}(A)}$ of $\mathfrak{A}$ (with X' as interpretation of P_X), by the lemma above. Observe that $\mathfrak{B} \equiv \mathfrak{A}$ by Łoś's Theorem. Define a function f on B by

$$f(b)=\{X \in \mathbb{U}(A) \mid b \in X'\}.$$

(Note that $b \in X'$ iff $\mathfrak{B} \vDash P_X b$.) We shall prove that f is a surjective p-morphism from B to pe(A).
(1) $f(b)$ is a prime filter in $\mathbb{U}(A)$, since for all X and $Y \in \mathbb{U}(A)$,

$$X \subseteq Y \text{ implies } \mathfrak{A} \vDash \forall x(P_X x \to P_Y x), \text{ hence } \mathfrak{B} \vDash \forall x(P_X x \to P_Y x),$$

so if $X \in f(b)$, equivalently $\mathfrak{B} \models P_X b$, we get $\mathfrak{B} \models P_Y b$, whence $Y \in f(b)$;

$$\mathfrak{A} \models \forall x(P_X x \wedge P_Y x \rightarrow P_{X \cap Y} x),$$

and $X,Y \in f(b)$ implies $\mathfrak{B} \models P_X b \wedge P_Y b$, hence, since $\mathfrak{B} \equiv \mathfrak{A}$, $\mathfrak{B} \models P_{X \cap Y} b$ and $X \cap Y \in f(b)$; and similarly

$$\mathfrak{A} \models \forall x(P_{X \cup Y} x \rightarrow P_X x \vee P_Y x)$$

makes $X \cup Y \in f(b)$ imply $X \in f(b)$ or $Y \in f(b)$.
(2) f is a homomorphism. Suppose $b_1 \leq_B b_2$. We have for all $X \in \mathbb{U}(A)$,

$$\mathfrak{A} \models \forall uv(P_X v \wedge v \leq u \rightarrow P_X u);$$

consequently $\mathfrak{B} \models P_X b_1 \rightarrow P_X b_2$, i.e. $b_1 \in X'$ implies $b_2 \in X'$, and $f(b_1) \subseteq f(b_2)$.
(3) f is onto, by the saturation of $\mathfrak{B}$. For, let ∇ be any prime filter of $\mathbb{U}(A)$. Then

$$\Gamma_\nabla := \{P_X v \mid X \in \nabla\} \cup \{\neg P_Y v \mid Y \notin \nabla\}$$

is consistent with $\mathrm{Th}(\mathfrak{B})$ – for suppose it is not, then since $\mathfrak{B} \equiv \mathfrak{A}$ we get $X \in \nabla$ and $Y_1,\ldots,Y_m \notin \nabla$ such that

$$\mathfrak{A} \models \forall v(P_X v \rightarrow \bigvee_{1 \leq j \leq m} P_{Y_j} v);$$

i.e. $X \subseteq \cup_j Y_j$, so $\cup_j Y_j \in \nabla$, and since ∇ is prime, some Y_j must belong to ∇: a contradiction. Since $\mathfrak{B}$ is countably saturated, there exists b in B realizing Γ_∇; and $\mathfrak{B} \models \Gamma_\nabla[b]$ implies $f(b) = \nabla$.
(4) f satisfies the p-morphism condition. Suppose $\nabla \geq_{\mathrm{pe}(A)} f(b_1)$. Then

$$\Gamma := \{P_X v \mid X \in \nabla\} \cup \{\neg P_Y v \mid Y \notin \nabla\} \cup \{b_1 \leq v\}$$

is consistent with $\mathrm{Th}((\mathfrak{B},b_1))$. For, suppose not: then there are $X \in \nabla$ and $Y_1,\ldots,Y_m \notin \nabla$ such that

$$\mathfrak{B} \models \forall v \geq b_1\, (P_X v \rightarrow \bigvee_{1 \leq j \leq m} P_{Y_j} v),$$

or, with $Z := X \Rightarrow \cup_j Y_j$, and since $b_1 \geq b_1$,

$$\mathfrak{B} \models P_Z b_1.$$

Thus by the definition of f, $Z \in f(b_1)$. But since $f(b_1) \subseteq \nabla$, and $X \in \nabla$, this implies $\cup_j Y_j \in \nabla$, and a

contradiction as in (3).
So Γ is realized in $\mathfrak{B}$; and if $\mathfrak{B}\models\Gamma[b_2]$, we have $b_1 \leq b_2$ and $f(b_2)=\nabla$. □

15.3 Theorem. Let **K** be a Δ-elementary class of frames. Then **K** is $\mathbb{I}$-definable iff **K** is closed under disjoint unions, p-morphic images and generated subframes, and the complement of **K** is closed under prime filter extensions.

Proof: ($\Rightarrow$) Suppose $\Phi\subseteq\mathbb{I}$, and $\mathbf{K}=\mathbf{Fr}(\Phi)$ is Δ-elementary. Since $A\not\Vdash\Phi$ implies $\mathrm{pe}(A)\not\Vdash\Phi$, by proposition 14.18.3, the complement of **K** is closed under prime filter extensions. $\mathbb{I}$-formulas are preserved under disjoint unions, p-morphic images and generated subframes by proposition 2.2.4; since **K** is $\mathbb{I}$-definable, it is closed under these operations.
($\Leftarrow$) Suppose **K** is Δ-elementary, and **K** and its complement satisfy the closure conditions as stated. Let $A\in\mathbf{Fr}_{\mathbb{I}}(\mathbf{K})$. By proposition 14.28, $\mathbb{FU}(A)$ is a generated subframe of a p-morphic image of $\mathbb{FU}(\Sigma_{i\in I}B_i)$ for some family $(B_i|i\in I)$ of frames in **K**. Say

$$\mathbb{FU}(A)=(A',\mathbb{V})\subsetneq(B,\mathbb{W})\xleftarrow{\text{p-m}}(C,\mathbb{X})=\mathbb{FU}(\Sigma B_i).$$

Since **K** is Δ-elementary, it is closed under ultrapowers; since it is also closed under p-morphic images, **K** is closed under prime filter extensions, by lemma 2. Now, by closure under disjoint unions, $\Sigma B_i\in\mathbf{K}$; by closure under prime filter extensions, $C=\mathrm{pe}(\Sigma B_i)\in\mathbf{K}$. By closure under p-morphic images, $B\in\mathbf{K}$. By closure under generated subframes, $A'\in\mathbf{K}$ (cf. definition 14.13). Since $A'=\mathrm{pe}(A)$, and the complement of **K** is closed under prime filter extensions, $A\in\mathbf{K}$. So $\mathbf{Fr}_{\mathbb{I}}(\mathbf{K})\subseteq\mathbf{K}$, and this implies that **K** is $\mathbb{I}$-definable. □

15.4 The above theorem characterizes the $\mathbb{I}$-definable $\mathbb{L}_0$-sentences by their preservation properties: an $\mathbb{L}_0$-sentence is $\mathbb{I}$-definable iff it is preserved under p-morphic images, generated subframes and disjoint unions, and its negation is preserved under prime filter extensions. For some of these preservation properties, syntactic criteria are known.

Definition. The set of **restricted positive** $\mathbb{L}_0$-formulas is the least set Γ of $\mathbb{L}_0$-formulas such that
(i) atomic formulas belong to Γ;
(ii) $\perp\in\Gamma$; and if $\alpha,\beta\in\Gamma$, then $\alpha\wedge\beta$ and $\alpha\vee\beta$ belong to Γ;
(iii) if $\alpha\in\Gamma$, and u and v are distinct individual variables, then $\exists v(u\leq v\wedge\alpha)$ and $\forall v(u\leq v\rightarrow\alpha)$ belong to Γ.

The following theorem is proved for modal frames in chapter 15 of van Benthem [1986]:

Theorem (van Benthem). An $\mathbb{L}_0$-sentence α is preserved under generated subframes, p-morphic images and disjoint unions iff α is logically equivalent to an $\mathbb{L}_0$-sentence $\forall u\beta$ with β a restricted positive $\mathbb{L}_0$-formula.

Our a priori restriction to quasi-orders corresponds with replacing the second occurrence of α by $\alpha\wedge\forall v.v\leq v\wedge\forall v\forall w(v\leq w\rightarrow\forall u(w\leq u\rightarrow v\leq u))$.
So if we had syntactic criteria for anti-preservation of $\mathbb{L}_0$-formulas under prime filter extensions – that is, necessary and sufficient criteria for $\mathrm{pe}(A)\models\alpha$ to imply $A\models\alpha$ – we would have syntactic criteria for $\mathbb{I}$-definability. As things are, we have neither.

15.5 As with $\mathbb{L}_0$-definability for $\mathbb{I}$-formulas, there are two sorts of limitation that can make the $\mathbb{I}$-definability problem easier: we can specialize to particular kinds of frames, and to particular kinds of formulas. The concluding sections present an example of either sort.

§16. $\mathbb{I}$-definable classes of finite frames

Let **K** be a class of frames – e.g., **K**=**FPO** or **K**=**FTR**. Suppose **K**'$\subseteq$**K**. We say **K**' is $\mathbb{I}$-definable in **K** if there is a set Φ of $\mathbb{I}$-formulas such that **K**'=**Fr**(Φ)$\cap$**K**, and **strongly** $\mathbb{I}$-definable if Φ may be taken to consist of a single formula. We characterize in this section the classes $\mathbb{I}$-definable in, respectively, **FPO** and **FTR**, by means of operations on frames.

16.1 The salient fact about finite frames is Jankov's theorem [1968]. For the reader's convenience, we present the proof that Gabbay gives in Chapter 4 §3 of his book.

Lemma (Jankov's theorem): If $A\in$ **FPO** is rooted, there exists an $\mathbb{I}$-formula ψ_A such that for any frame B, $B\nVdash\psi_A$ iff for some $b\in B$, A is a p-morphic image of $[b)_B$.

Proof: Let $A\in$ **FPO** be given, with root a_0. Take distinct proposition letters p_a for all $a>_A a_0$. Define a valuation V on A by $V(p_a)=[a)$. Let $P:=\{p_a|a>a_0\}$, and $P_a:=\{p\in P|a\Vdash p\}$, for all $a\in A$.
For each $Q\subseteq P$, let $\psi_Q=\bigwedge Q\rightarrow\bigvee(P-Q)$. When $a<_A a'$, let $\chi_{aa'}$ be

$$(\chi_{aa'})\qquad \bigwedge P_a\rightarrow\psi_{P_{a'}}\rightarrow\bigvee(P-P_a).$$

Now let $\psi_A:=\varphi_0\rightarrow\varphi_1$, where

$$\varphi_0:=\bigwedge(\psi_Q|\neg\exists a\in A.Q=P_a)\wedge\bigwedge(\chi_{aa'}|a,a'\in A \text{ and } a<a');$$
$$\varphi_1:=\bigvee_{a\in A}\psi_{P_a}.$$

It is easy to see that $a_0\Vdash\varphi_0$ and $a_0\nVdash\varphi_1$. Hence if A is a p-morphic image of $[b)_B$, $B\nVdash\psi_A$ by 2.4.4. For the converse, suppose $(B,V',b_0)\Vdash\varphi_0$ and $(B,V',b_0)\nVdash\varphi_1$. Define $f:[b_0)_B\rightarrow A$ by

$$f(b)=a \text{ iff } \forall p\in P.b\Vdash p\Leftrightarrow a\Vdash p.$$

This is a good definition: if $f(b)$ is defined, it is unique, for the points of A are uniquely determined by the $p\in P$ that they force. And f is defined everywhere on $[b_0)_B$ since, if $Q\subseteq P$ is not P_a for any $a\in A$, $b_0\Vdash\psi_Q$, whence for $b\geq b_0$ the set $\{p\in P|b\Vdash p\}$ cannot equal Q. Also, $a\leq a'$ iff $P_a\subseteq P_{a'}$, which makes f a homomorphism.
Since $b_0\nVdash\varphi_1$, f is surjective: for each ψ_{P_a}, there must be some $b\geq b_0$ with $b\Vdash\bigwedge P_a$ and $b\nVdash\bigvee(P-P_a)$, and no b can take care of more than one P_a.

It remains to prove the p-morphism condition. Suppose $a>f(b)$. Since $b_0 \Vdash \chi_{f(b)a}$, and $b \Vdash \bigwedge P_{f(b)}$, $b \Vdash \psi_{P_a} \to \bigvee(P-P_{f(b)})$. Now $b \nVdash \bigvee(P-P_{f(b)})$, so $b \nVdash \psi_{P_a}$ – hence there must be some $b' > b$ with $b' \Vdash \bigwedge P_a$ and $b' \nVdash \bigvee(P-P_a)$, i.e. $f(b')=a$. □

16.2 Lemma: Let A be any frame, and $(A_i|i\in \mathbb{I})$ a family of generated subframes of A such that $\cup_{i\in I}A_i=A$. Then A is a p-morphic image of $\Sigma_{i\in I}A_i$.

Proof: Map (i,a) to a (similar to 13.5). □

16.3 Theorem. A class **K** of finite partially ordered sets is $\mathbb{I}$-definable in **FPO** iff **K** is closed under p-morphic images, generated subframes and disjoint unions of finite families.

Proof: ($\Rightarrow$) Immediate from 2.4.4.
($\Leftarrow$) Let **K** satisfy the above closure conditions. Set $\Phi:=\mathrm{Th}_{\mathbb{I}}(\mathbf{K})$. Suppose $A\in\mathbf{FPO}$, and $A\Vdash\Phi$. We shall prove that $A\in\mathbf{K}$. Since $A=\cup_{a\in A}[a)_A$, A is a p-morphic image of $\Sigma_{a\in A}[a)$; so it suffices to show that $[a)\in\mathbf{K}$ for each $a\in A$.
Suppose $[a)\notin\mathbf{K}$. Then the Jankov-formula $\psi_{[a)}$ belongs to Φ: for otherwise $\psi_{[a)}$ is not valid in some $B\in\mathbf{K}$, and $[a)$ is a p-morphic image of a subframe $[b)_B\subsetneq B$, hence $[a)\in\mathbf{K}$. But since $A\Vdash\Phi$, this would mean $A\Vdash\psi_{[a)}$, which is impossible by 2.4.4 and Jankov's theorem. □

16.4 We can improve on the above result by further restricting the class of frames under consideration.

Let A and B be finite trees. We shall write $A\prec B$ for: A is a p-morphic image of B. The relation $\prec$ is a quasi-ordering on the class **FTR**. If $|A|=|B|$ and $f{:}B\twoheadrightarrow A$ is a surjective p-morphism, then f is in fact an isomorphism. Hence the equivalence relation "$A\prec B$ and $B\prec A$" is isomorphism.

Lemma (De Jongh). Let $A, B\in\mathbf{FTR}$. Every surjective p-morphism $f{:}A\twoheadrightarrow B$ is a p-retraction for some strong embedding $g{:}B\rightarrowtail A$.

Proof: Suppose $f{:}A\twoheadrightarrow B$ is a surjective p-morphism. We define $g{:}B\to A$ in such a way that for any $b\in B$, $g(b)$ is a maximal element in $f^{-1}[b]$, by induction up B. For the root b_0 of B, $g(b_0)$ is some maximal element of $f^{-1}[b_0]$. Now suppose b covers b', and $g(b')$ has been defined. Then since $fg(b')=b'$, and f is a p-morphism, some $a>g(b')$ must belong to $f^{-1}[b]$; take as $g(b)$ a maximal such a.
It is clear that g is an injective homomorphism. Suppose b branches into $X\subseteq B$. We must show that

$g(b)$ branches into $g[X]$: so suppose $g(b) \leq a \leq g(x), g(x')$, with x and x' incomparable elements of X. Then $b \leq f(a) \leq x, x'$, since f is a homomorphism and $f \circ g = 1_B$. So $f(a) < X$ by definition 2.11.3, and a$< g[X]$. □

Corollary: $A \prec B$ iff there exists a strong embedding of A into B.

16.5 The following lemma is usually formulated in terms of strong embeddings. An elegant proof may be found in Nash-Williams [1963].

Lemma (Kruskal's theorem): Any subclass of **FTR** has only finitely many $\prec$-minimal elements, modulo isomorphism. (In other words, **FTR** is a well-quasi-ordering.)

Suppose **K** is a class of finite trees, downwards closed in the sense that $A \prec B \in \mathbf{K}$ implies $A \in \mathbf{K}$. Then its complement **FTR**–**K** is upwards closed. Let $\{A_1, \ldots, A_n\}$ be a maximal set of mutually nonisomorphic minimal elements of **FTR**–**K**. Take their Jankov formulas ψ_{A_i}, and consider $\psi = \bigwedge_{1 \leq i \leq n} \psi_{A_i}$.

If $A \in \mathbf{FTR}$, and $A \Vdash \psi$, then $A_i \prec A$ is impossible, for $1 \leq i \leq n$, by lemma 2.4.2 and Jankov's theorem. So $A \in \mathbf{K}$. On the other hand, if $A \in \mathbf{K}$, then $A \Vdash \psi_i$ for all i $(1 \leq i \leq n)$, as $A_i \not\prec A$. So $\mathbf{K} = \{A \in \mathbf{FTR} | A \Vdash \psi\}$. In view of lemma 2.4.2, we have proved

Theorem: Let **K** be a class of finite trees. Then the following statements are equivalent:

(i) **K** is $\mathbb{I}$-definable;
(ii) **K** is downwards closed in $\prec$;
(iii) **K** is strongly $\mathbb{I}$-definable.

16.6 Limiting the class of frames as we have done has its price: characterizations such as we invoked in 15.4 are lost. (This is not to say that they cannot be regained, but it would require a different proof.) We know that the $\mathbb{I}$-definable $\mathbb{L}_0$-formulas are equivalent, on **FTR**, to $\mathbb{L}_0$-formulas of a particular form: that of the $\mathbb{L}_0$-translations given in §8 (theorem 7). We should like to prove that every $\mathbb{L}_0$-formula that is $\mathbb{I}$-definable in **FTR** is reducible to this form. The reduction must be constructive in some sense; typically, one would expect it to be a proof of equivalence in some first order theory. It is not self-evident that such reductions are possible.
The main difficulty here is that **FTR** is not Δ-elementary. A study of Δ-elementary classes, such as **DLO**, might yield interesting results. I have not pursued this.

§17. Classes of frames definable with transparent formulas

Let $\mathbb{T}\subseteq\mathbb{I}$ be the class of transparent formulas, as defined in 7.3. We have shown in §7 that every transparent formula is equivalent to an $\mathbb{L}_0$-sentence; and that many $\mathbb{I}$-formulas are transparent (in particular the deterministic formulas of 7.4). In this section we derive closure conditions on classes $\mathbf{Fr}(\Phi)$ for sets $\Phi\subseteq\mathbb{T}$ – in other words, for $\mathbb{T}$-definable classes of frames. We end with a syntactic characterization of the $\mathbb{L}_0$-formulas that are equivalent to a transparent formula.

17.1 Definition: Let A be a frame. A subframe B of A is a **directed subframe** of A if for finite $B_0\subseteq B$, if B_0 has an upper bound in A, B_0 also has an upper bound in B. Notation: $B\subseteq_d A$.

For example, generated subframes are directed subframes.

Lemma: Let A, B, C be frames.
(i) $A\subseteq_d B\subseteq_d C$ implies $A\subseteq_d C$ (i.e., $\subseteq_d$ is transitive).
(ii)If $A\subseteq B\subseteq C$ and $A\subseteq_d C$, then $A\subseteq_d B$.

17.2 Recall the definitions of partial projection and Σ-labeled subframe (7.1, 7.2). Observe that if $\mathfrak{X}$ is an open multitableau, $B\subseteq_d A$, and $g{:}B\dashrightarrow\mathfrak{X}$ is a Σ-projection, then g is a Σ-labeled subframe of A iff g is a Σ-labeled subframe of B.

Lemma: Suppose $\varphi\in\mathbb{T}$, and A is a frame. Then $A\nVdash\varphi$ iff φ is refutable in a finite directed subframe of A.

Proof: By definition 7.3,

$$A\nVdash\varphi \text{ iff } A \text{ has an } F\varphi\text{-labeled subframe.}$$

Suppose $A\nVdash\varphi$; $\mathfrak{X}=(X,\mathfrak{S})$ is a multirefutation of φ, and $g{:}A\twoheadrightarrow\mathfrak{X}$ a minimal Fφ-labeled subframe of A. As remarked in 7.2, dom$g$ is finite. Now consider the subsets $U\subseteq$dom$g$ with the following property:

U has an upper bound in A, and

if $U' \subseteq \mathrm{dom}g$ properly extends U, then U' does *not* have an upper bound in A.

For each such U, pick an upper bound $a_U \geq U$, and let

$$A_0 := \mathrm{dom}g \cup \{a_U\}_U.$$

Then A_0 is a finite directed subframe of A. Since g is also an Fφ-labeled subframe of A_0, $A_0 \not\Vdash \varphi$. Conversely, suppose $A_0 \subseteq_d A$ and $A_0 \not\Vdash \varphi$. Then A_0 has an Fφ-labeled subframe g; this g is also an Fφ-labeled subframe of A. □

The last paragraph of the above proof also establishes:

Corollary. If $B \subseteq_d A$, then for each $\varphi \in \mathbb{T}$: $A \Vdash \varphi$ implies $B \Vdash \varphi$.

17.3 Definition: A set $\{A_i \mid i \in I\}$ of frames is **directed** if

$$\forall i,j \in I\ \exists k \in I: A_i \subseteq A_k \text{ and } A_j \subseteq A_k\,.$$

So if $\{A_i \mid i \in I\}$ is a directed set of frames, elements of $\cup_{i \in I} A_i$ are ordered in the same way in every A_i in which they occur together. Moreover, for every pair of elements there is some A_i in which they occur together. Thus we may safely consider $\cup_{i \in I} A_i$ as a frame, the **union** of $\{A_i | i \in I\}$, ordered by

$$a \leq b \text{ iff } \exists\, i \in I.\ a \leq_{A_i} b.$$

If a class **K** of frames is closed under the operation of taking unions of directed subsets of **K**, we shall say **K** is **closed under directed unions**.

17.4 Lemma. Let $\{A_i \mid i \in I\}$ be a directed set of frames, and $\varphi \in \mathbb{T}$. Then

$$\forall i \in I.\ A_i \Vdash \varphi \text{ implies } \cup_{i \in I} A_i \Vdash \varphi.$$

Proof. Suppose $\cup_{i \in I} A_i \not\Vdash \varphi$. By lemma 2, φ is refutable in some finite $B \subseteq_d \cup_i A_i$. By directedness, there must be some $i \in I$ such that $B \subseteq A_i$. By lemma 1, $B \subseteq_d A_i$, so $A_i \not\Vdash \varphi$ by lemma 2. □

17.5 <u>Lemma</u> For every rooted $B \in$ **FPO** there exists a formula $\varphi_B \in \mathbb{I}[\wedge,\to,\bot]$ such that for every frame A:

$$A \Vdash\!\!\!/\, \varphi_B \text{ iff } B \text{ is a p-morphic image of a directed subframe of } A.$$

<u>Proof</u>: Let a finite frame B be given, with root w. Take distinct proposition letters p_b, q_b, for every $b \in B$.

We define for each $b \in B$ a formula $\varphi_b \in \mathbb{I}[\wedge,\to,\bot]$ with $>$-recursion. Assume $\varphi_{b'}$ has been defined for every $b' \in \mathrm{Cov}(b)$. Then $\varphi_{b'} :=$

$$q_b \wedge \bigwedge\{\neg q_{b'} \mid \neg\exists b''.\, b'' \geq b', b\} \wedge \bigwedge_{b' \geq b} p_{b'} \wedge \bigwedge_{b' \in \mathrm{Cov}(b)}(\varphi_{b'} \to p_b) \to p_b$$

(recall that $\bigwedge\emptyset = \top$).

We take $\varphi_B := \varphi_w$. Note that φ_B is transparent, by theorem 7.4.

Define a valuation V on B by

$$V(p_b) = \{b' \mid b' \not\leq b\} \;\; ; V(q_b) = [b).$$

We shall prove that for all b and b', $(B,V,b') \Vdash \varphi_b$ iff $b' \not\leq b$, with $>$-induction on b. In particular, it will follow that $B \Vdash\!\!\!/\, \varphi_B$; by lemma 2.4.2 and corollary 2, then, $A \Vdash\!\!\!/\, \varphi_B$ if B is a p-morphic image of a directed subframe of A.

If $b' \not\leq b$, then $b' \Vdash p_b$, hence $b' \Vdash \varphi_b$. For the converse it will suffice to prove $b \Vdash\!\!\!/\, \varphi_b$. Observe that $b \Vdash q_b$; $b'' \Vdash\!\!\!/\, q_{b'}$ if $b' \not\leq b''$, so $b \Vdash \neg q_{b'}$ if b and b' have no successors in common. If $b' \not\geq b$, then $b \Vdash p_{b'}$. Thus if $b' \in \mathrm{Cov}(b)$, $b' \Vdash p_b$; moreover $b \Vdash\!\!\!/\, \varphi_{b'}$ by induction hypothesis; so

$$b \Vdash \bigwedge_{b' \in \mathrm{Cov}(b)}(\varphi_{b'} \to p_b).$$

So b forces the antecedent of φ_b. Since $b \Vdash\!\!\!/\, p_b$, $b \Vdash\!\!\!/\, \varphi_b$.

It now remains to prove that $A \Vdash\!\!\!/\, \varphi_B$ implies that B is a p-morphic image of a directed subframe of A. In the sequel we shall say that a point x, under a given valuation, **refutes** an implication $\psi \to \chi$ if $x \Vdash \psi$ and $x \Vdash\!\!\!/\, \chi$.

Let B^* be the tree of all sequences $(v_0,\ldots,v_k)$ of elements of B ($k \geq 0$), with $v_0 = w$ and $v_{i+1} \in \mathrm{Cov}(v_i)$ ($i<k$), ordered by initial segments. $\mathrm{e}: B^* \to B$ is the projection to the last element.

Suppose $(A,V) \Vdash\!\!\!/\, \varphi_B$. We define a function $f: B^* \to A$, with induction over B^*, in such a way that $f(\ldots,b)$ refutes φ_b, for all $b \in B$.

- $f(w)$ is an arbitrary element of A that refutes φ_B (under V).
- Suppose $f(v_0,\ldots,v_k)$ has been defined, and $\mathrm{Cov}_B(v_k) = \{b_1,\ldots,b_n\}$. Suppose, moreover, that $f(v_0,\ldots,v_k)$ refutes φ_{v_k}. Then $f(v_0,\ldots,v_k) \Vdash\!\!\!/\, p_{v_k}$, so $f(v_0,\ldots,v_k) \Vdash\!\!\!/\, \varphi_{b_i}$, $1 \leq i \leq n$. Take

$a_1,\dots,a_n \geq f(v_0,\dots,v_k)$ that refute $\varphi_{b_1},\dots,\varphi_{b_n}$ respectively, and define: $f(v_0,\dots,v_k,b_i)=a_i$.
Now $e\circ f^{-1}$ is a partial function from A onto B. For suppose $a=f(\dots,b)$ and $a'=f(\dots,b')$ with $b\neq b'$. Then a refutes φ_b, and a' refutes $\varphi_{b'}$. Since $B\in$ **FPO**, $b\nleq b'$ or $b'\nleq b$. In the first case $a\Vdash p_{b'}$, $a'\nVdash p_{b'}$; the other case is symmetric. Hence $a\neq a'$.
We can extend ran$f$ to a directed subframe of A, as we did with dom$g$ in the proof of lemma 2: for each $U\subseteq$ran$f$ such that U has an upper bound in A, but not in ran$f$, and no $U'\subseteq$ran$f$ that properly extends U has an upper bound in A, we pick an upper bound $a_U\in A$. Let A' be the result of adding these points a_U to ran$f$. Note that $U\neq U'$ implies $a_U\neq a_{U'}$, and that every a_U is maximal in A'.
Choose for every a_U a maximal upper bound b_U of $e\circ f^{-1}[U]$. Such upper bounds exist: suppose $b=e\circ f^{-1}(a)$, $b'=e\circ f^{-1}(a')$. If $\{b,b'\}$ does not have an upper bound, then, since a refutes φ_b and a' refutes $\varphi_{b'}$, $a\Vdash q_b$ and $a'\Vdash\neg q_b$; this is impossible if a and a' have a common successor.
Clearly $A'\subseteq_d A$. Define $g:A'\to B$ by

$$g(a)=e\circ f^{-1}(a) \text{ if } a\in \text{ran}f\,;\; g(a_U)=b_U.$$

By construction, g is surjective. We shall prove that g is a p-morphism. Recall that if $a\in$ran$f$, a refutes $\varphi_{g(a)}$ (under V).
(i) g is a homomorphism. If $a<_{A'}a_U$, then $a\in U$, hence

$$g(a) = e\circ f^{-1}(a) \leq b_U = g(a_U).$$

If $a<_{A'}a'\in$ ran$f$, and $g(a)\nleq g(a')$, then (since a refutes $\varphi_{g(a)}$) $a\Vdash p_{g(a')}$; which would make it impossible for a' to refute $\varphi_{g(a')}$.
(ii) g satisfies the p-morphism condition. Suppose $g(a)\leq b$. If a is one of the additional upper bounds a_U, then $g(a)$ is maximal in B, and there is nothing to prove. Otherwise $a\in$ran$f$. Suppose $g(a)\neq b$. If $b\in \text{Cov}_B(g(a))$, $g(a)=e(v_0,\dots,v_k)$ for some $(v_0,\dots,v_k)\in B^*$, we may take $a'=f(v_0,\dots,v_k,b)$, and find $b=g(a')$, $a'>a$. In general, we shall find $a'>a$ with $g(a)=b$ in finitely many moves of this kind. □

17.6 Corollary. For every rooted $B\in$ **FPO** there exists a formula $\psi_B\in\mathbb{I}[\to,\bot]$ such that for every frame A:

$$A\nVdash\psi_B \text{ iff } B \text{ is a p-morphic image of a directed subframe of } A.$$

Proof: The conjunctions in ψ_B may be eliminated by repeated applications of the logical equivalence

$$\vdash(\varphi\wedge\psi\to\chi)\leftrightarrow(\varphi\to\psi\to\chi). \qquad \square$$

17.7 The use of the above is analogous to that of Jankov's theorem in §16.

Theorem. Let **K** be a class of frames. The following statements are equivalent:

(i) **K** is $\mathbb{I}[\rightarrow,\perp]$-definable;

(ii) **K** is $\mathbb{T}$-definable ;

(iii) **K** is closed under p-morphic images, directed unions, directed subframes and disjoint unions.

Proof:

(i) $\Rightarrow$(ii) since $\mathbb{I}[\rightarrow,\perp]$-formulas are transparent, by theorem 7.4.

(ii) $\Rightarrow$(iii) by 2.4.2, lemma 4, corollary 2 and 2.4.3.

(iii) $\Rightarrow$(i) : let **K** be closed under p-morphic images, directed unions, directed subframes and disjoint unions. Set

$$\Phi:=\{\varphi\in\mathbb{I}[\rightarrow,\perp] \mid \mathbf{K}\Vdash\varphi\}.$$

We will show that $\mathbf{K}=\mathbf{Fr}(\Phi)$. Suppose $A\Vdash\Phi$: we are to prove that $A\in\mathbf{K}$. Let **A** be the set of all finite directed subframes of A. Let $B\in\mathbf{A}$, and suppose $b_1,...,b_n$ are the minimal elements of B. Then $[b_i)_B$ is a directed subframe of A, $1\leq i\leq n$. Take formulas $\psi_{[b_i)}\in\mathbb{I}[\rightarrow,\perp]$, by the corollary above, such that for any frame C, $C\nVdash\psi_{[b)}$ iff $[b_i)_B$ is a p-morphic image of a directed subframe of C. Then $A\nVdash\psi_{[b_i)}$; since $A\Vdash\Phi$, $\psi_{[b_i)}\notin\Phi$. So there are $K_1,...,K_n\in\mathbf{K}$ with $K_i\nVdash\psi_{[b_i)}$, $1\leq i\leq n$; by the corollary , then, each $[b_i)_B$ is a p-morphic image of a directed subframe of a frame in **K**, hence $[b_i)_B\in\mathbf{K}$. Since B is a p-morphic image of $\sum_{1\leq i\leq n}[b_i)$, by lemma 16.2, $B\in\mathbf{K}$. Thus $\mathbf{A}\subseteq\mathbf{K}$. Finally, **A** is a directed set of frames, and $A=\cup\mathbf{A}$. Since **K** is closed under directed unions, we have $A\in\mathbf{K}$. □

Remark. This theorem implies that to every set of $\mathbb{T}$-formulas there exists a set of $\mathbb{I}[\rightarrow,\perp]$-formulas which is valid in the same frames.[1] I do not know whether something similar holds for intermediate logics; to be precise, whether an intermediate logic axiomatized by $\mathbb{T}$-formulas (on top of a formal system for intuitionistic propositional logic, with substitution as a provability rule) can always be axiomatized with $\mathbb{I}[\rightarrow,\perp]$-formulas , or even $\mathbb{I}[\wedge,\rightarrow,\perp]$-formulas .

17.8 By theorem 7.3, every $\mathbb{T}$-formula is equivalent to an $\mathbb{L}_0$-formula. Indeed, 7.3 effectively constructs a unique $\mathbb{L}_0$-transtation $\mathrm{Tr}(\varphi)$ for each $\mathbb{T}$-formula φ. Let **cs-formulas** be formulas

$$CS(y_1,...,y_n) : \exists x\,(y_1 \leq x \wedge ... \wedge y_n \leq x) \qquad (n \geq 2)$$

with x a variable distinct from $y_1,...,y_n$ (cf. 11.8.2). The translations defined in 7.3 are, modulo logical equivalence, conjunctions of sentences $\neg\exists y_1...y_k\,\beta_j$, in which each β_j is a conjunction of atomic formulas, negations of atomic formulas, and negations of cs-formulas; each β_j describes an $F\varphi$-labeled frame.

We can stylize Tr(φ), turning the descriptions β_j into descriptions of trees. Let $g{:}A \twoheadrightarrow \mathfrak{X}$ be the labeled frame that β_j describes. Let A^* be the tree of finite sequences $(a_0,...,a_n)$ of elements of A ($n \geq 0$) in which a_0 is the root of A and $\forall i<n.\ a_{i+1} \in \mathrm{Cov}_A(a_i)$, ordered by initial segments; and $\mathrm{e}{:}A^* \twoheadrightarrow A$ the projection to the last element. Let α_j be the $\mathbb{L}_0[A^*]$-sentence

$$\bigwedge(a \leq a' \mid a,a' \in A^* \ \&\ a' \in \mathrm{Cov}_{A^*}(a)) \ \wedge\ \bigwedge(\neg a \leq a' \mid \mathrm{e}(a) \nleq_A \mathrm{e}(a'))$$

$$\wedge\ \bigwedge(\neg CS(a_1,...,a_{k'}) \mid \cup_{1 \leq i \leq k'} g \circ \mathrm{e}(a_i)^{\mathrm{T}} \text{ is not realizable}).$$

Let $\beta_j{}^*$ be the result of substituting distinct new variables for the constants in α_j. Then it is easy to see that $\beta_j{}^*$ is satisfiable in any frame in which β_j is satisfiable. So if $\mathrm{Tr}(\varphi) = \bigwedge_{1 \leq j \leq m} \neg\exists y_1...y_k \beta_j$, then

$$\mathbf{QO} \vDash \bigvee_{1 \leq j \leq m} \exists y_1...y_k\,\beta_j \rightarrow \bigvee_{1 \leq j \leq m} \exists y_1...y_l\,\beta_j{}^*.$$

Conversely, if $\beta_j{}^*$ is satisfiable in some frame B, say by points $b_1,...,b_l$ corresponding with elements $a_1,...,a_l$ of A^* in order, then we define an $F\varphi$-labeled subframe $h{:}B \dashrightarrow \mathfrak{X}$ by $h(b_i) = g \circ \mathrm{e}(a_i)$ ($1 \leq i \leq l$). Since $\varphi \in \mathbb{T}$, $B \nVdash \varphi$. So we may take Tr(φ) to be $\bigwedge_{1 \leq j \leq m} \neg\exists y_1...y_l\,\beta_j{}^*$.

$\beta_j{}^*$ begins with a sequence of atomic conjuncts. We may suppose that these are ordered in such a way that there is only one variable whose first occurrence is at the left hand side of a $\leq$. (Make the ordering from left to right agree with the ordering of the corresponding points in the tree.) Now we can move some existential quantifiers to the right, rewriting $\beta_j{}^*$ in the form

existential quantifier – sequence of bounded existential quantifiers $(\exists v \geq u)$ –
conjunction of negations of atomic formulas and negations of cs-formulas.

Negating this, and rewriting, produces an $\mathbb{L}_0$-sentence $\forall x\,\gamma_j$ in which γ_j consists of a sequence of bounded universal quantifiers $(\forall v \geq u)$ followed by a disjunction of atomic formulas and cs-formulas. We shall call a formula of this form, containing at most one variable x free, a **t-formula** .

We have proven the following refinement of theorem 7.3:

Theorem. For every transparent $\mathbb{I}$-formula φ, an $\mathbb{L}_0$-equivalent $\mathrm{Tr}(\varphi)$ can be effectively constructed. $\mathrm{Tr}(\varphi)$ is a conjunction $\bigwedge_{1\le j\le m}\forall x\,\gamma_j$, in which each γ_j is a t-formula.

17.9 Call an $\mathbb{L}_0$-sentence α $\mathbb{T}$-**definable** if there exists a set $\Phi\subseteq\mathbb{T}$ such that for every frame A,

$$A\models\alpha \text{ iff } A\Vdash\Phi,$$

that is, if the class of quasi-ordered models of α is $\mathbb{T}$-definable in the earlier sense.

Proposition. Let α be an $\mathbb{L}_0$-sentence. Then the following are equivalent:

(i) for some $\varphi\in\mathbb{I}[\to,\bot]$, $\alpha\equiv\varphi$;

(ii) α is $\mathbb{T}$-definable;

(iii) α is preserved under p-morphic images, directed unions, directed subframes and disjoint unions.

Proof: (i)$\Rightarrow$(ii) since $\mathbb{I}[\to,\bot]\subseteq\mathbb{T}$ by theorem 7.4.
(ii)$\Rightarrow$(iii) since $\mathbb{T}$-definable classes are closed under the operations of (iii) by theorem 7.
(iii)$\Rightarrow$(i): Let $\mathbf{K}$ be the class of quasi-ordered models of α. By theorem 7, $\mathbf{K}$ is $\mathbb{I}[\to,\bot]$-definable; suppose $\mathbf{K}=\mathbf{Fr}(\Phi)$, with $\Phi\subseteq\mathbb{I}[\to,\bot]$. Let QO be the first order theory of quasi-order. We have $QO\cup\mathrm{Tr}[\Phi]\models\alpha$; hence by compactness, $QO\models\bigwedge\mathrm{Tr}[\Phi_0]\to\alpha$ for some finite $\Phi_0\subseteq\Phi$. Then $\alpha\equiv\bigwedge\Phi_0$. Since $\bigwedge\Phi_0$ is an $\mathbb{I}[\wedge,\to,\bot]$-formula, eliminating negation produces an equivalent $\mathbb{I}[\to,\bot]$-formula – as noted under 4.3.3. □

17.10 Lemma. If $\mathbb{L}_0$-sentences α and β are $\mathbb{T}$-definable, then so is $\alpha\wedge\beta$.

Proof: Suppose $\alpha\equiv\varphi$ and $\beta\equiv\psi$, with $\varphi,\psi\in\mathbb{T}$. Then obviously $\alpha\wedge\beta\equiv\varphi\wedge\psi$. We may assume that φ and ψ belong to $\mathbb{I}[\to,\bot]$, by the proposition above: then $\varphi\wedge\psi\in\mathbb{I}[\wedge,\to,\bot]\subseteq\mathbb{T}$ by theorem 7.4. □

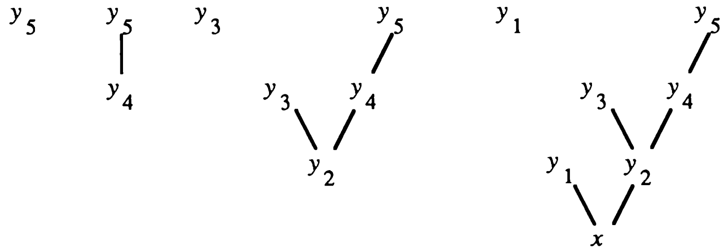
17.11 Theorem. Suppose $\alpha=\forall x\bigwedge_{1\le i\le m}\beta_i$, with each β_i a t-formula with free variable x. Then α is $\mathbb{T}$-definable.

Proof: By the lemma, it suffices to show that $\forall x\beta_i$ is $\mathbb{T}$-definable, $1\le i\le m$. We shall construct $\mathbb{T}$-formulas by a method resembling that of van Benthem [1986] lemma 14.5.
Let β_i be a t-formula, consisting of a sequence of bounded universal quantifiers followed by a disjunction γ. In 17.8 above, the bounded quantifiers were derived from a finite tree; we can recover the form of this tree from the bounded quantifiers. Indeed, we shall construct trees $T(y)$ for all variables y occurring in β_i, by induction from right to left in the quantifier prefix.

If for some variable y occurring in β_i, there is no bounded quantifier of the form $\forall v{\geq}y$ preceding γ, we let the tree $T(y)$ consist of a single point y. Otherwise, consider all the quantifiers $\forall v_1{\geq}y,\ldots,\forall v_n{\geq}y$ in the prefix: $T(y)$ is the union of the trees $T(v_1),\ldots,T(v_n)$ (they happen to be disjoint) with y added as a root.
For example, if β_i is

$$\forall y_1{\geq}x\forall y_2{\geq}x\forall y_3{\geq}y_2\forall y_4{\geq}y_2\forall y_5{\geq}y_4 \quad (y_5{\leq}y_1 \vee y_3{\leq}y_4 \vee y_4{\leq}y_2 \vee\ y_2{\leq}x \ \vee\ \mathrm{CS}(y_1,y_3,y_5))$$

we get trees



For each node y in the tree $T(x)$, take a distinct proposition letter p_y. We define $\mathbb{I}$-formulas ψ_y by induction down $T(x)$, as follows. Let ψ be the disjunction of clauses

ψ_z, for every cover z of y in $T(x)$;

p_u, for every clause $u{\leq}y$ in γ.

(As always, the empty disjunction is $\perp$.) We take $\psi_y{=}p_y{\rightarrow}\psi$.
So for our example we get, with $p_x{=}p$ and $p_{y_i}{=}q_i$:

y_5: $q_5{\rightarrow}\perp$,
y_4: $q_4{\rightarrow}\neg q_5{\vee}q_3$,
y_3: $q_3{\rightarrow}\perp$,
y_2: $q_2{\rightarrow}\neg q_3{\vee}(q_4{\rightarrow}\neg q_5{\vee}q_3){\vee}q_4$,
y_1: $q_1{\rightarrow}q_5$,
x: $p{\rightarrow}(q_1{\rightarrow}q_5){\vee}(q_2{\rightarrow}\neg q_3{\vee}(q_4{\rightarrow}\neg q_5{\vee}q_3){\vee}q_4){\vee}q_2$.

Finally, let χ be the conjunction of the formulas $\neg(p_{u_1}{\wedge}\ldots{\wedge}p_{u_k})$ with $\mathrm{CS}(u_1,\ldots,u_k)$ a clause of γ; and take $\varphi_i{=}\chi{\rightarrow}\psi_x$. It is straightforward to check that φ_i is deterministic – hence, by theorem 7.4, $\varphi_i{\in}\mathbb{T}$. The $\mathbb{T}$-equivalent of $\forall x\beta_i$ is φ_i: since $A{\models}\neg\forall x\beta_i$ iff A has an Fφ_i-labeled subframe (as,

hopefully, the example will help the reader to see), $\forall x\beta_i \equiv \varphi_i$ by transparency of φ_i.
With 17.8 above, this theorem implies that an $\mathbb{L}_0$-sentence is $\mathbb{T}$-definable iff it is equivalent, on **QO**, to the universal closure of a conjunction of t-formulas.

Remark. Since the essentials of a rooted finite partial order can be put in a t-formula, one would expect a greater similarity between the formulas φ_B of theorem 5 and the formulas φ_i constructed here. There are two reasons for the difference. One is that φ_i contains $\vee$: eliminating it (cf. §7 footnote 2) might introduce a number of new proposition letters. The other is that φ_B is defined locally, unlike φ_i. In the latter case, all the cs-constituents were dealt with at once.

Footnote

[1] Equivalence is proved directly in §7, footnote 2.

Appendix

Beth semantics

There is another well-known semantics for intuitionistic logic, devised by E.W. Beth. As far as propositional logic is concerned, the difference with Kripke semantics lies mainly in the treatment of disjunction.

A1 Let frames and valuations be as usual: so in particular, if V is a valuation on a frame A, and $p \in \mathbb{P}$, $V(p)$ is upwards closed in A. Recall that a path through A is a maximal chain $C \subseteq A$. A path through a, for a point $a \in A$, will be a path through A that contains a.

Definition: Let A be a frame, $a \in A$ and $X \subseteq A$. . Then X **bars** a if every path through a intersects X.

A2 At first sight, Beth forcing will seem to model another notion of constructivity than Kripke forcing. The definition below appears to distinguish between having calculated a proposition – that is, being at a point within $V(p)$ – and knowing that p is true, in the sense that, whichever way we continue, p will come out true.

Definition: Let A be a frame, V a valuation on A, and $a \in A$. Then

(i) $a \Vdash p$ iff $V(p)$ bars a;

(ii) $a \Vdash \varphi \wedge \psi$ iff $a \Vdash \varphi$ and $a \Vdash \psi$;

(iii) $a \Vdash \varphi \rightarrow \psi$ iff $\forall a' \geq a$: if $a' \Vdash \varphi$, then $a' \Vdash \psi$;

(iv) $a \Vdash \varphi \vee \psi$ iff a is barred by a set X such that $\forall x \in X$: $x \Vdash \varphi$ or $x \Vdash \psi$;

(v) $a \nVdash \bot$.

All the same, ordinary intuitionistic logic is sound and complete for the Beth semantics. An accessible proof, via Kripke semantics, is in Kripke [1965]. It turns finite Kripke models into Beth models on finitely branching trees. A slight refinement of the transformation (almost as described in 9.1) will give binary trees, and more:

Proposition: Let $\varphi \in \mathbb{I}$. Then $\vdash \varphi$ iff φ is valid in the Beth semantics on all binary trees in which every point is succeeded by endpoints.

(Validity in a frame is defined formally as in 1.4, with Beth's forcing instead of Kripke's.)

A3 The correspondence theory is very different from that for Kripke semantics. In order to show how, we begin by deriving some consequences of the truth definition.

Lemma: Let A be a frame, $a \in A$, and suppose $X \subseteq A$ bars a.

(i) Let every $x \in X$ be barred by a set Y_x. Then $\cup_{x \in X} Y_x$ bars a.

(ii) If X is upwards closed and $a \leq b$, then X bars b.

Proof: (i) Let C be a path through a. C intersects X; thus for some $x \in X$, C is a path through x, and must intersect Y_x.
(ii) Let C be a path through b. There must be a path C' through a and b that coincides with C from b upwards. Since X bars a, $C' \cap X \neq \emptyset$. Suppose $c \in C' \cap X$. Then if $c < b$, $b \in X$ since X is upwards closed; thus in any case, X bars b. □

A4 In the following lemma, and everywhere below, the forcing sign "$\Vdash$" will stand for Beth forcing, as defined in A2, and validity based on that definition.

Lemma: Let (A,V) be a model; $a,b \in A$.
(i) If $a \leq b$, and $a \Vdash \varphi$, then $b \Vdash \varphi$.
(ii) $a \Vdash \varphi$ iff a is barred by a set of points forcing φ.

Proof: By induction on the complexity of φ, with lemma 3. (ii) from left to right is trivial, since $\{a\}$ bars a. The other direction uses (i). □

A5 By a tautology we understand, as before, a propositional formula valid in classical logic. Beth semantics emphatically lacks the finite model property:

Theorem: In finite frames every tautology is valid.

Proof: For maximal points a, definition 2 is just the truth definition of classical logic. So in endpoints every tautology is valid. But in a finite frame every point is barred by the endpoints: hence the tautologies hold everywhere. □

The proof allows a stronger statement:

Corollary: In frames in which the set of endpoints bars every point, every tautology is valid.

Anyway, this shows that there is no interesting correspondence theory for finite frames – $\top$ and $\bot$ are the only relevant first order definitions.

A6 The same holds for linear frames. They are equivalent to one-point frames: if $V(p)\neq\emptyset$, then it bars every node, hence p is forced everywhere.

A7 The next class in complexity is $\mathbf{TR}^{(2)}$, the class of full binary trees. (By A6, infinite branches without side-branches are equivalent to endpoints.) As to correspondence with first order logic, it is the last class as well:

Theorem: For $\mathbb{I}$-formulas φ, either

(i) φ is not a tautology, and $\varphi\equiv_{\mathbf{TR}^{(2)}}\bot$; or

(ii) φ is a tautology that is not valid intuitionistically, and φ has no first order equivalent on $\mathbf{TR}^{(2)}$; or

(iii) $\vdash\varphi$, and $\varphi\equiv_{\mathbf{TR}^{(2)}}\top$.

Proof: The one point that is not trivial is that refutable tautologies have no first order equivalents on $\mathbf{TR}^{(2)}$, i.e. that the first part of (ii) implies the second.

Suppose φ is a tautology, and $\nvdash\varphi$. By A5, φ is valid on all *finite* trees. If φ has a first order equivalent α of quantifier rank $\mathrm{rnk}(\alpha)=n$, then by theorem 8.15, α holds in all full binary trees satisfying $P(n)$. So φ is valid in all full binary trees satisfying $P(n)$.

By proposition 2, φ is Beth-refutable in some binary tree A in which every point is succeeded by endpoints. Then A can be extended to a full binary tree A' in which $P(n)$ holds and φ is still refutable. (If a has just one cover a', replace $[a')$ by two copies $\{0\}\times[a')$ and $\{1\}\times[a')$, putting $(i,a'')\in V(p)$ iff $a''\in V(p)$; repeat this process level by level, starting at the root of A. Replace some endpoints by finite full binary trees, to satisfy $P.2(n)$ and $P.3(n)$.) This contradicts the conclusion of the previous paragraph: so φ cannot be equivalent to a first order formula on $\mathbf{TR}^{(2)}$. □

As with the Kripke semantics, if $\varphi\equiv_{\mathbf{K}}\alpha$, under the Beth semantics, and $\mathbf{K}'\subseteq\mathbf{K}$, then $\varphi\equiv_{\mathbf{K}'}\alpha$ as well. So we have

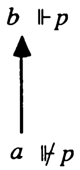
<u>Corollary</u>: Let $\varphi \in \mathbb{I}$, and suppose **K** is a class of frames containing all full binary trees. Then if $\models\varphi$ and $\not\models\varphi$, φ has no first order equivalent on **K**.

This means that the conclusion of A5 holds generally: for Beth semantics, correspondence with $\mathbb{L}_0$-formulas is a non-subject.

A8 We can try to get around this difficulty by modifying Beth semantics or extending $\mathbb{L}_0$. We shall discuss both possibilities, and a combination of the two.

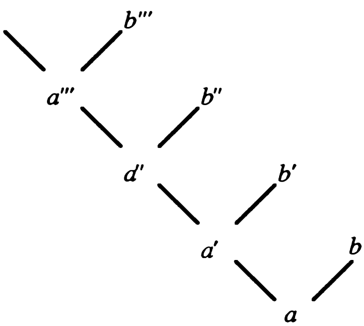
A9 Though refuting any at all interesting formula in the Beth semantics requires an infinite model, such a model will exhibit a pattern that can be finitely described. As an example, take $\neg\neg p \to p$. To refute it, we need a point a forcing $\neg\neg p$ and not forcing p. Then $a \not\Vdash \neg p$, so there must be $b > a$ forcing p. We have:

(1)

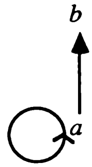


This cannot be all, for a would be barred by $V(p)$, and consequently force p. So a must have another successor a', forcing $\neg\neg p$ and not forcing p, and not preceding b. Next, a' will give rise to new successors b' and a'', like b and a respectively, and so on ad infinitum. This process may be taken to generate an infinite comb:

(2)



Nonetheless, almost all points in (2) are the same. In fact, we can represent the a's and b's and their arrangement by a finite graph:

(3) 

The paths through (3) correspond with the paths through (2): either one takes a finite number of a's and finishes with a b, or one loops through a forever – corresponding with the spine of (2).
Such finite graphs resemble the finite models that Kripke used to construct Beth models from. They are not exactly the same, though: Kripke would have had a loop at b as well. It might be attempted to do Beth semantics with frames in which the ordering need not be reflexive – we shall not pursue this now.

A10 The truth definition in A2 contains second order clauses: (i) and (iv) quantify over certain subsets of the frame. As a consequence, we would be hard put even to define first order equivalents of $\mathbb{I}$-formulas on Beth *models*, parallel to the standard translations of §1.[1] When we abstract from valuations, we add another layer of second order quantifiers, this time over upwards closed sets (cf. 1.6). It would be something already if we knew to what extent the second layer can be eliminated.
We propose to take a look at the correspondence on Beth frames between $\mathbb{I}$-formulas and formulas of $\mathbb{L}_2$, with the set variables ranging over paths. We shall use π,ρ as informal variables over paths. In the formal language $\mathbb{L}_2$, we write X, Y, Z etc. for sets, instead of proposition letters as in 1.6.

A11 In fact, we have no real business with the second order theory of paths. We can easily generalize Beth frames to two-sorted structures, consisting of an ordinary frame A and an explicit domain of paths in A. From now on

A Beth frame is a pair $\mathbb{A}=(A,\Pi)$ of a frame A and a collection Π of paths through A satisfying the following existence axioms:

(i) $\exists\pi.\alpha\in\pi$ (there are paths through every point)
(ii) $b\geq a\in\pi \Rightarrow \exists\rho(b\in\rho \;\&\; \forall c\in\pi(c\leq a \Rightarrow c\in\rho))$
(iii) $b\leq a\in\pi \Rightarrow \exists\rho(b\in\rho \;\&\; \forall c\in\pi(c\geq a \Rightarrow c\in\rho))$

$\mathbb{L}_2$ will be regarded as a two-sorted first order language; structures for $\mathbb{L}_2$ have two sorts, one of which (the first, say) is a frame. Of course, it cannot be guaranteed that the second sort actually *is* a collection of paths through the first; but we can make sure that it is isomorphic to a set of paths by a few additional axioms:

(iv) $\forall a(a\in\pi \Leftrightarrow a\in\rho) \Rightarrow \pi=\rho$	(extensionality: paths are subsets of the frame)
(v) $a,b\in\pi \Rightarrow a\leq b \vee b\leq a$	(paths are linearly ordered)
(vi) $b\notin\pi \Rightarrow \exists a\in\pi\ (b\ngeq a \ \&\ b\nleq a)$	(paths are maximal)

A12 With valuations as before, we get the following truth definition:

<u>Definition</u>: Let $\mathbb{A}=(A,\Pi)$ be a Beth frame, V a valuation on A, and $a\in A$. Then

(i) $a\Vdash p$ iff $\forall\pi\in\Pi(a\in\pi \Rightarrow \exists b\in\pi.\ b\in V(p))$;

(ii) $a\Vdash\varphi\wedge\psi$ iff $a\Vdash\varphi$ and $a\Vdash\psi$;

(iii) $a\Vdash\varphi\rightarrow\psi$ iff $\forall a'\geq a$: if $a'\Vdash\varphi$, then $a'\Vdash\psi$;

(iv) $a\Vdash\varphi\vee\psi$ iff $\forall\pi\in\Pi(a\in\pi \Rightarrow \exists b\in\pi\ (b\Vdash\varphi$ or $b\Vdash\psi))$;

(v) $a\nVdash\bot$.

Note that (ii), (iii) and (v) are the same as in A2.

A13 The results of A3-5 hold for generalized Beth frames. In particular, the path construction for 3(ii) can be carried through by axiom (iv). By a straightforward verification, intuitionistic propositional logic is sound for the generalized Beth semantics. Since ordinary Beth models qualify as generalized Beth models, completeness is a trivial consequence of completeness for ordinary Beth semantics.

We conclude with two examples of correspondence on generalized Beth frames. Since the standard frames are a subclass of the generalized frames, the correspondences hold for the standard frames as well.

A14 <u>Example</u>: Let $\mathbb{A}=(A,\Pi)$ be a Beth frame. Then

$$\mathbb{A}\Vdash\neg\neg p\rightarrow p \quad \text{iff} \quad \mathbb{A}\vDash\forall X\exists x(\ Xx\wedge\forall y\geq x\exists z\geq y Xz\ )$$

The proof is by contraposition, in both directions.

($\Rightarrow$) Suppose $\mathbb{A}\nvDash\forall X\exists x(Xx\wedge\forall y\geq x\exists z\geq y.Xz)$, i.e. for some $\pi\in\Pi$,

$$(*)\ \forall a\in\pi\exists b\geq a.[b)\cap\pi=\emptyset.$$

Let $V(p)=\{b\in A|[b)\cap\pi=\emptyset\}$. Take $a\in\pi$. Then $a\Vdash\neg\neg p$: for if $a'\geq a$, then either $[a')\cap\pi=\emptyset$, and $a'\Vdash p$; or $\exists a''\geq a'.a''\in\pi$, and then by (*) for some $b\geq a$, $b\Vdash p$; so in either case $a'\nVdash\neg p$. On the other hand, $a\nVdash p$; for $a\in\pi$, and π does not intersect $V(p)$.
($\Leftarrow$) Suppose $(\mathbb{A},V,a)\Vdash\neg\neg p$ and $(\mathbb{A},V,a)\nVdash p$. By the truth definition, there must be some path π through a that does not intersect $V(p)$, but every point of which is followed by elements of $V(p)$. Elements of $V(p)$, of course, cannot have successors on π. Winding up:

$$\mathbb{A}\vDash\exists X\forall x(Xx\rightarrow\exists y\geq x\forall z\geq y.\neg Xz).$$

$\square$

A15 The above example may give the impression that (generalized) Beth correspondence is more complicated than Kripke correspondence. This is not a correct impression, as will appear from the next, and last, example.
The essence of deterministic formulas comes out in the proof of theorem 7.4 at the point where it is argued that, if $T(\psi\vee\chi)\in\Theta(a)$, $T(\psi\vee\chi)$ cannot have been added by the closure conditions. The difficulty in the construction of that proof, avoided by determinism, is this: in some point a, we may be forced to make some formula $\varphi_1\rightarrow\varphi_2\vee\varphi_3$ true. Furthermore, it may be that in some successors of a, φ_2 is true and φ_3 false; while in other successors φ_3 is true and φ_2 false; and no provision has been made, above a, for making φ_1 false. Then φ_1 must be true in a, and the Kripke semantics would have us choose in a whether φ_2 is true or φ_3. We obviously cannot choose either, and that is why the proof does not work for all formulas. In Beth semantics, however, there is no problem: in the situation just sketched, a may still be barred by points forcing either φ_2 or φ_3, and thus force $\varphi_2\vee\varphi_3$. We shall illustrate this with SP_2 (recall: SP_2 is

$$(\neg\varphi\vee\neg\psi\vee\neg\chi\rightarrow\varphi\vee\psi\vee\chi)\rightarrow\neg\varphi\vee\neg\psi\vee\neg\chi,$$

with $\varphi=p\wedge q$, $\psi=p\wedge\neg q$ and $\chi=\neg p\wedge q$) on **DLO**.

<u>Example</u>: Let $\mathbb{A}=(A,\Pi)$ be a Beth frame, with $A\in$ **DLO**. Then

$$\mathbb{A}\Vdash SP_2 \quad \text{iff} \quad \mathbb{A}\vDash\forall X\exists x[\,Xx\wedge\forall y_1y_2y_3\geq x(\textstyle\bigvee_{i\neq j}y_i\leq y_j\vee\bigvee_{1\leq i\leq 3}Xy_i)]$$

The proof is, as always, by contraposition.

($\Leftarrow$) Suppose $(\mathbb{A},V)\not\Vdash SP_2$. Then somewhere in A, $\neg\varphi\vee\neg\psi\vee\neg\chi$ is not forced. So there is a path $\pi\in\Pi$ on which $\neg\varphi$, $\neg\psi$ and $\neg\chi$ are never forced. Thus if $a\in\pi$, then $a\not\Vdash\neg\varphi$, $a\not\Vdash\neg\psi$, and $a\not\Vdash\neg\chi$. So every $a\in\pi$ has successors $b\Vdash\varphi$, $c\Vdash\psi$ and $d\Vdash\chi$, and since φ, ψ and χ are mutually incompatible (that is, $\varphi\to\neg\psi\wedge\neg\chi$, etc.), b, c and d must be mutually incomparable and off π.
($\Rightarrow$) Suppose $\pi\in\Pi$ is such that each $a\in\pi$ has successors b, c and d that are pairwise incomparable and off π. Then there are pairwise disjoint upwards closed sets U, V and W not intersecting π, such that every point of π is succeeded by elements of all three. Their construction is straightforward, but it does require some bookkeeping. Note that π has no greatest element: if $a\in\pi$, then there exists $b\geq a$ with $b\notin\pi$, so π must have elements incomparable with b (axiom (vi) in A11); these must be higher than a.

(a) Take $a_0\in\pi$. Choose b_0, c_0, $d_0\in[a_0)-\pi$ pairwise incomparable.

(b) Suppose a_ξ, b_ξ, c_ξ and d_ξ have been chosen, for some ordinal ξ, with $a_\xi\in\pi$ and b_ξ, c_ξ and $d_\xi\notin\pi$. With axiom (vi) of A11, we can find $a_{\xi+1}\in\pi$ such that $a_{\xi+1}\nleq b_\xi$, $a_{\xi+1}\nleq c_\xi$ and $a_{\xi+1}\nleq d_\xi$. Choose $b_{\xi+1}$, $c_{\xi+1}$, $d_{\xi+1}\in[a_{\xi+1})-\pi$ pairwise incomparable.

(c) Suppose a_ξ, b_ξ, c_ξ and d_ξ have been chosen for all ξ less than some limit ordinal λ, with $a_\xi\in\pi$, b_ξ, c_ξ, $d_\xi\in[a_\xi)-\pi$, and all b_ξ, c_ξ and d_ξ pairwise incomparable. Then if $\{a_\xi|\xi<\lambda\}$ is cofinal in π, the construction is finished. Otherwise we continue with $a_\lambda>\{a_\xi|\xi<\lambda\}$ on π, and pairwise incomparable b_λ, c_λ and d_λ in $[a_\lambda)-\pi$.

Since A is a set, the construction finishes at some ordinal λ; and then we may take $U=\cup_{\xi<\lambda}[b_\xi)$, $V=\cup_{\xi<\lambda}[c_\xi)$ and $W=\cup_{\xi<\lambda}[d_\xi)$. By downward linearity, U, V and W are disjoint.
Now for $a\in A$, let a belong to $V(p)$ *unless* a has successors in W, and to $V(q)$ unless a has successors in V. Then for all $v\in V$, $v\Vdash p\wedge\neg q$ $(=\psi)$; for all $w\in W$, $w\Vdash\neg p\wedge q$ $(=\chi)$; and for all u without successors in V or W, $u\Vdash p\wedge q$ $(=\varphi)$. Moreover, no point of π forces $\neg\varphi$ (because of U), $\neg\psi$ (because of V), or $\neg\chi$ (because of W); so if $a\in\pi$, then $a\not\Vdash\neg\varphi\vee\neg\psi\vee\neg\chi$.
We shall be done once we have shown that some $a\in\pi$ forces $\neg\varphi\vee\neg\psi\vee\neg\chi\to\varphi\vee\psi\vee\chi$; since $\forall a\in\pi$: $a\not\Vdash\neg\varphi\vee\neg\psi\vee\neg\chi$, it is sufficient to show that every $a'\notin\pi$ forces $\varphi\vee\psi\vee\chi$. So take $a'\notin\pi$, and consider any path ρ through a'. We have seen above that the c_ξ force $p\wedge\neg q$, and $d_\xi\Vdash\neg p\wedge q$. We shall prove that if ρ does not contain any c_ξ or d_ξ, there is an a'' on ρ without successors in V or W. Then $a''\Vdash p\wedge q$; and it follows that a' is barred by points forcing φ, ψ or χ, so that $a'\Vdash\varphi\vee\psi\vee\chi$.
Suppose that ρ does not pass through points c_ξ or d_ξ. Suppose that $a'<c_\xi$. Then there must be some $x\in\rho$ that is not comparable with c_ξ. Then $x>a'$. Now suppose that $x<c_{\xi'}$. Then since $x\notin\pi$, $x>a_{\xi'}$ by downward linearity. Likewise, since $a'\notin\pi$, we know that $a'>a_\xi$. Since $x>a'$, $a_\xi<c_{\xi'}$. So $\xi<\xi'$ by construction. But then $a_{\xi'}<a'<c_\xi$ since a' and $a_{\xi'}$ both precede x and $a'\notin\pi$; whereas $a_{\xi'}\nless c_\xi$ by construction. Thus, x has no successors $c_{\xi'}$, and by downward linearity, no successors in V. If

necessary, we can repeat this argument with d_ξ's, and x instead of a'. We end up with a'' as desired. □

The reader will have noticed that not all of SP_2 was employed in the first part of the above proof. Let us abbreviate $\forall X\exists x(Xx\wedge\forall y_1y_2y_3\geq x(\bigvee_{i\neq j}y_i\leq y_j\vee\bigvee_{1\leq i\leq 3}Xy_i))$ to α. We have in fact shown:

if $\mathbb{A}\nvDash\alpha$, then $\mathbb{A}\nVdash SP_2$, and

if $\mathbb{A}\nVdash\neg\varphi\vee\neg\psi\vee\neg\chi$, then $\mathbb{A}\nvDash\alpha$;

and observed in passing that $\mathbb{A}\nVdash SP_2$ implies $\mathbb{A}\nVdash\neg\varphi\vee\neg\psi\vee\neg\chi$. Thus we have established an equivalence between $\mathbb{I}$-formulas:

Corollary: With φ, ψ and χ as above, SP_2 is equivalent with $\neg\varphi\vee\neg\psi\vee\neg\chi$ on downwards linear Beth frames.

This contrasts sharply with Kripke validity (4.6): the expressive power of disjunction has become quite different. With Kripke validity, disjunction without implication was fairly trivial, whereas with implication it soon became unmanageable; under Beth's definition, it would seem that disjunction is already complex with $\neg$ and $\wedge$, but does not react so violently to implication.

A16 So there may be an interesting correspondence theory for Beth semantics after all. Inspection of the above example gives rise to the following conjectures: with set variables interpreted as paths, either in a predetermined domain of paths, or ranging over all paths in the frame,

I on **DLO**, every $\mathbb{I}$-formula is equivalent to an $\mathbb{L}_2$-formula;

II on **PO**, SP_2 is not equivalent to an $\mathbb{L}_2$-formula.

Footnote

[1] In one rather natural class of Beth models, the quantification over bars (or paths) can be replaced by quantification over numbers. Consider the models on finitely branching trees (i.e. in which Cov(a) is finite for all a): a point a in such a tree is barred by an upwards closed set X iff for some $n\in\mathbb{N}$, all successors of a that can be reached from a in n steps, from a point to one of its covers, belong to X. The nontrivial direction in this equivalence is proved by an application of König's Lemma.

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INDEX OF SYMBOLS

INDEX OF DEFINITIONS

Samenvatting

De Kripke-semantiek voor de intuïtionistische logika induceert een verband tussen intermediaire axioma's en quasi-ordeningen (frames): men kan een intermediair axioma opvatten als een bewering over een frame, die geldt voor een gegeven frame juist als het axioma daarin geldig is. Daarmee wordt de taal van de propositielogika een medium voor de beschrijving van frames. De vraag rijst nu wat voor eigenschappen van frames op deze manier uitdrukbaar zijn. In het bijzonder kan men onderzoeken of formules van de propositielogika "corresponderen" met formules van een andere logische taal (met bijbehorende semantiek), in de zin dat ze dezelfde eigenschap van frames uitdrukken.

Dit proefschrift handelt voornamelijk over correspondenties tussen formules van de taal $\mathbb{I}$ van de intuïtionistische propositielogika en klassiek geïnterpreteerde formules van een predikaatlogische taal $\mathbb{L}_0$ met één binaire relatie. In deel II wordt bewezen dat er $\mathbb{I}$-formules bestaan die niet eerste orde definieerbaar zijn (i.e. niet corresponderen met $\mathbb{L}_0$formules). Uit resultaten van Doets volgt dat sommige $\mathbb{I}$-formules zelfs niet eerste orde definieerbaar zijn op relatief overzichtelijke klassen van frames, zoals bomen (met alle paden van type $\leq\omega$), of eindige partiële ordeningen. De grenzen van de eerste orde definieerbaarheid worden in twee opzichten onderzocht: zekere beperkingen op de vorm van $\mathbb{I}$-formules garanderen dat men een corresponderende $\mathbb{L}_0$-formule kan vinden; aan de andere kant worden langs verschillende wegen frameklassen afgebakend waarop elke $\mathbb{I}$-formule eerste orde definieerbaar is. De $\mathbb{I}$-formules in één propositieletter worden geclassificeerd naar eerste orde definieerbaarheid (§11). Er wordt aangegeven hoe men kan beslissen of een $\mathbb{I}$-formule eerste orde definieerbaar is op de klasse der bomen.

Deel III onderzoekt de afsluitingseigenschappen van $\mathbb{I}$-definieerbare klassen van frames; en welke $\mathbb{L}_0$-formules corresponderen met $\mathbb{I}$-formules. Bekende resultaten van de modale correspondentietheorie worden overgezet naar het intuïtionistische geval.

In twee uitweidingen in deel I worden fragmenten van de taal $\mathbb{I}$ bestudeerd, en enige opmerkingen gemaakt over correspondentie tussen $\mathbb{I}$-formules en formules van de modale propositielogika.

Een alternatieve interpretatie van $\mathbb{I}$-formules in frames gaat terug op Beth. Voor de Beth-semantiek kan men dezelfde soort vragen stellen als hier is aangeduid met betrekking tot de semantiek van Kripke. De appendix bespreekt de vraag naar eerste orde definieerbaarheid. $\mathbb{L}_0$-definieerbare eigenschappen geven geen inzicht in de 'klassieke' Beth-interpretatie. Er bestaat echter een redelijke, meer handelbare variant van de Beth-semantiek.

STELLINGEN
bij het proefschrift
Intuitionistic Correspondence Theory
van P.H. Rodenburg.

I. Laat voor algebra's $\mathfrak{A}=(A;F)$, $\mathbb{S}(\mathfrak{A})$ de collectie zijn van alle deelverzamelingen van A die gesloten zijn onder de operaties in F; noem twee algebra's $(A;F)$ en $(A;F')$ *equivalent* als ze dezelfde polynomen in >0 variabelen hebben; en definieer $\mathbb{S}^+(\mathfrak{A})$ als

$$\bigcap(\mathbb{S}(\mathfrak{A}') \mid \mathfrak{A}' \text{ is equivalent met } \mathfrak{A}\ ).$$

Dan bestaat er, voor een niet-lege verzameling A en een algebraïsch afsluitingssysteem $\mathbb{S}$ over A, een algebra $\mathfrak{A}$ met drager A waarvoor $\mathbb{S}=\mathbb{S}^+(\mathfrak{A})$ desda A oneindig is, of $\emptyset\notin\mathbb{S}$, of $|\bigcap(\mathbb{S}-\{\emptyset\}|\neq 1$.
(Zie P. Rodenburg, Characterization of the algebraic closure systems that can be represented by $\mathbb{S}^+$, Algebra Universalis 14 (1982) pp.263-4.)

II. Zij $\mathbb{L}$ de taal van de infinitaire modale propositielogika, waarin conjuncties zijn toegestaan van willekeurig grote verzamelingen formules. De collectie van equivalentieklassen modulo S4.3 van formules van $\mathbb{L}$ in één propositionele variabele p is geen verzameling.
(Cf. D.H.J. de Jongh, A class of intuitionistic connectives, in: J. Barwise, H.J. Keisler and K. Kunen, eds., The Kleene Symposium, Amsterdam 1980.)

III. In een klasse van eindige frames waarvan de breedte een vaste eindige bovengrens heeft, is elke $\mathbb{I}$-formule elementair.
(Zulks in contrast met §10 van dit proefschrift.)

IV. De intermediaire logika geaxiomatiseerd door SP_n is beslisbaar, voor elke $n\in\mathbb{Z}^+$.
(Zie voorbeeld 6.4 in dit proefschrift voor de definitie van SP_n.)

V. Zij $\mathbb{Q}\otimes\mathbb{Q}$ de structuur der paren van rationale getallen, strict geordend door

$$(q,r) < (q',r') \text{ desda } q<q' \text{ en } r<r'.$$

Definieer

$$x_1x_2x_3 \ := \ \forall u\,(\,u<x_1 \wedge u<x_3 \rightarrow u<x_2\,)$$

en voor elke $n>3$,

$$x_1x_2\ldots x_{n-1}x_n \ := \ \bigwedge\nolimits_{1\leq i\leq n} x_1\ldots x_{i-1}x_{i+1}\ldots x_n\,.$$

Zij T de universele theorie van $\mathbb{Q}\otimes\mathbb{Q}$. Dan wordt T geaxiomatiseerd door een stel axioma's voor de theorie der stricte halfordeningen, met toegevoegd de axioma's

$$\forall x_1 \ldots x_n \bigvee (x_{\rho(1)} \ldots x_{\rho(n)} \mid \rho \text{ is een permutatie van } \{1, \ldots, n\})$$

voor alle $n \geq 3$.
(Dit beantwoordt een vraag in: J.F.A.K. van Benthem, The logic of time, Dordrecht 1983 — z. l.c. p.28.)

VI. Een halfordening Q is *splitsend* als er voor elke $a \in Q$ elementen a_u en a^v zijn zo dat

$$Q = (a] \cup [a_u) = [a) \cup (a^v] \text{ , en}$$
$$(a] \cap [a_u) = [a) \cap (a^v] = \emptyset \, .$$

Noem een splitsende halfordening Q *van type (iii)* als Q niet bestaat uit twee onderling ongeordende elementen (type (i)) en niet isomorf is met $\mathbb{Z}$ (type (ii)), en Q geen deelordening heeft die een lineaire som is van twee of meer splitsende verzamelingen. Definieer voor functies $\phi: Q \to Q$: $\phi^0(x)=x$; $\phi^{n+1}(x)=\phi(\phi^n(x))$.
Een splitsende halfordening Q is van type (iii) desda er een familie $(Q_i | i \in \mathbb{Z})$ bestaat van paarsgewijs disjuncte deelordeningen van Q, met isomorfismen $\phi_i: Q_i \cong Q_{i+1}$, zo dat $Q = \bigcup_{i \in \mathbb{Z}} Q_i$ en $\forall i \in \mathbb{Z}$ $\forall x,y \in Q_i$:

(i) $x \leq_Q \phi_i(y) \Leftrightarrow y \nleq_{Q_i} x$;

(ii) $\forall j \geq 2$: $x \leq \phi_i^j(y)$.

(Dit beantwoordt een vraag in: Ph. Dwinger, Unary operations on completely distributive complete lattices, z. Stephen D. Comer ed., Universal Algebra and Lattice Theory, Charleston 1984 (Berlijn 1985), p.73.)

Errata in Intuitionistic Correspondence Theory

p. 5, regel 8 van boven (kort: +8): voor *p*, lees φ (bis)

p. 7, regel 15 van beneden (kort: –15): occurrence

p. 7, regel 10 van beneden (kort: –10): St($\psi\vee\chi$)

p. 8, +14: not $a''\leq a'$ and not $a'\leq a''$

p. 9, –12: see if the result

p. 10, +16: refute (see

p. 11, –4: ~~clearly,~~

p. 13, +13: BHK

p. 19, –6: disjunction, and

p. 37, –1: φ_i

p. 39, –2: $\mathbb{I}[\blacktriangleleft, \blacktriangleright, \blacksquare$

p. 41, +3: $\not\vdash\neg\blacktriangleleft\Phi_a$

p. 50, –6: $\blacktriangleleft_{i<k}(\neg)^{f_j(i)}p_i$

p. 55, –6: Łos's theorem

p. 98, +9: lattice

p. 126, –8: generated by $f[a]$ [since b_0 $\underline{\mathbf{U}}$ $f[a]$]

p. 136, –2: for each ψ_{P_a},

p. 145, –13: eliminating conjunction produces