STUDIES ON NATURAL LOGIC AND CATEGORIAL GRAMMAR

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To
Julienne
Benjamín
Vica
and
Mirtala

And now, kind friends, what I have wrote,
I hope you will pass o'er,
And not criticize as some have done
Hitherto herebefore.

Julia Moore: *The Sweet Singer of Michigan*

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INTRODUCTION

1. GENERAL CLAIM. This dissertation combines a systematical and a historical search for a Natural Logic: a proof system tailored directly for natural language. The goal of the systematical search is the construction of a Natural Logic, while the goal of the historical search is to discover principles useful for this logic. This work aims to show that there isn't an opposition between the semantical and the inferential approach to natural language: the proof system begins where semantics leaves off. Philosophers should handle with caution one-liners like *the central issue of linguistic theory is the construction of a theory of truth for natural language* (cf. Montague, 1970a); or the *central issue of linguistic theory is the construction of a proof system for natural language* (cf. Lakoff, 1972). We claim that the combination of semantics and proof theory is necessary for a proper understanding of natural language inference.

2. HISTORY AND NATURAL LOGIC. Traditional logic is generally considered a Natural Logic because its syntactical forms are in close accord with natural language sentences. Given that there are inferences which fall beyond the scope of syllogistic we pose the question: how did Traditional Logic cope with those inferences?

In our historical search we shall focus on the work of the classical authors Ockham, Leibniz, De Morgan and Peirce. These authors, we shall conclude, used monotonicity to explain non-syllogistic inference. The study of Peirce's logic will reveal further principles useful in the construction of a Natural Logic: the principle of conservativity and the principle of anaphoric identification.[1]

From Ockham to Leibniz, and from Leibniz to De Morgan, we see a decreasing understanding of syntactical issues: the historical use of monotonicity displays a successive decline in the conception of the appropriate vehicles of inference (*'logical forms'*).[2] One can say that Leibniz and De Morgan had a poor theory of logical form, while Ockham had an unpractical one. We shall also see that a syntactical criterion for monotonicity was missing in the proposals of De Morgan and Leibniz. This absence caused their attempts to trespass the syllogistic borders to fail. Ockham did have a criterion: the Medieval *suppositio* theory enabled him to define precisely the contexts in which monotonicity rules can be used. But the proverbial complexity of the *suppositio* theory renders Ockham's strategy unpractical.

The understanding of syntactical issues improved at the threshold of modern logic in the work of Peirce: he gave a manageable syntactic characterization of monotonicity. Peirce's version of monotonicity, however, refers only to formal languages. In this dissertation we shall show that it is possible to extend Peirce's analysis of monotonicity to natural language.

To resume, the historical search will reveal the following principles of Natural Logic :

- monotonicity
- conservativity
- anaphoric identification

And two related desiderata:

- an adequate theory of '*logical form*'
- a criterion for monotonicity

3. MODERN PROPOSALS FOR NATURAL LOGIC

3.1. LOGIC AND GRAMMATICAL FORM. Suppes (1979) presents a theory of inference for a fragment of English. In Suppes' proof system the vehicles of inference are the analysis trees generated by a context-free grammar. Broadly speaking, this system uses monotonicity as inference rule, supplemented by reductio ad absurdum. Suppes lists all the schemata in which monotone substitution is allowed. We select here two examples:

$$\frac{\text{Some N'' + TV + All + N'} \quad \text{All N' VP}}{\text{Some N'' + TV + All + N'}}$$

$$\frac{\text{All N'' +Aux + Neg+ TV + ALL+ N'} \quad \text{All N' are N}}{\text{All N'' +Aux + Neg+ TV + ALL+ N}}$$

These schemata generates the following inferences:

1.

$$\frac{\text{Some boys love all vegetables} \quad \text{All cabbages are vegetables}}{\text{Some boys love all cabbages}}$$

2.

$$\frac{\text{All boys do not love all cabbages} \quad \text{All cabbages are vegetables}}{\text{All boys do not love all vegetables}}$$

Suppes' system is on all accounts a proof system tailored directly for natural language: in it grammatical form and logical form coincide. Moreover, the system includes 'a good deal more than the classical syllogism.' But the form of this pioneer system is not very insightful. For instance, Suppes introduces about 75 rules of inference, which could be reduced to a common pattern:

- expressions occurring in upward monotone positions may be replaced by terms with a larger denotation

• expressions occurring in downward monotone positions may be replaced by expressions with a smaller denotation.

Obviously, the system would improve if supplemented with a theory of monotonicity marking (cf. Chapter V).

3.2. LOGIC AND MONOTONICITY MARKING. In some American universities research is going on trying to rehabilitate pre-Fregean logic: the Sommers' project of Natural Logic. Sommers claims that

> traditional formal logic is especially suited to the task of making perspicuous the logical form of sentences in natural languages that are actually used in deductive reasoning.

Sommers' system consists of a monotonicity calculus, algebraic rules and rules for the manipulation of quantifiers. The monotonicity calculus resembles a mechanism of monotonicity marking: logical constants are represented by constructions displaying their monotone properties. But there is a main difference between our conception of Natural Logic and Sommers' approach: Sommers' system is not tailored directly for natural language. The vehicles of inference of Sommers' natural logic are expressions of a formal language scarcely resembling natural language. For instance, the representation of *Some sailor is giving every child a toy* is

$$+S + G^3 -C + T$$

From our point of view, interesting as Sommers logic is, it does not qualify as Natural Logic.

4. OUR SYSTEMATICAL PROPOSAL. The main goal of this dissertation is the construction of a Natural Logic.[3] Our Natural Logic is based on Categorial Grammar and on a mechanism allowing the systematic computation of monotonicity. To achieve this we define a variant of Categorial Grammar, the *Lambek Grammar*, henceforth LG.

LG is an implicational logic formulated as a natural deduction calculus (cf. Prawits 1965). The derivations of LG are disambiguated objects which may be seen as :

- syntactical analyses
- recipes for the computation of lambda terms
- vehicles of inference

Our system, like Suppes, is tailored directly for natural language. But via the lambda terms we are able to extend Peirce's analysis of monotonicity to natural language: we have a theory of monotonicity marking as well.

In principle there is no difference between our approach to natural language inference and the standard one: the vehicles of inference of logical systems are expressions of an disambiguated language. The main difference lies in the relation between the formal objects and the natural language objects. In the standard case the logical forms need not to reflect the grammatical form. In our case the vehicles of inference are the grammatical forms themselves.

At this part our twofold goal is reached: after extracting useful principles from the history of logic, we employ them in our construction of a Natural Logic based on categorial grammar. However, a new question arises: is our Natural Logic founded on a sound linguistic basis?

5. THE LINGUISTIC STATUS OF THE NON-DIRECTED LAMBEK CALCULUS

5.1. AMBIGUATING RELATION. To make our proof system a more realistic one we shall take a closer look at the status of LG itself. LG is a variant of the *Lambek Calculus*, henceforth LP, discussed in Van Benthem (1986). LP is generally seen as a crude instrument for linguistic purposes. We analyse this assessment of LP using the structure of proofs as a point of reference in the linguistic discussion. The conclusion of our investigation will be that the rejection of LP is based on the identification of the system with a particular ambiguating relation: the ordering of the lexical items in the analysis trees determines the corresponding string.[4] In fact, the arguments intended to show the inadequacy of LP, only show that *this* ambiguating relation is inadequate.

5.2. AMBIGUATING RELATION FOR LG. We reject the idea that an LG proof cannot be a parse because the lexical items are not presented in the correct sequence. Instead of the ordering of the lexical material, we take the function-argument structure as guide. Since not all natural language expressions combine directly (cf. Geach 1972), we make use of 'shadow' assumptions: assumptions which are proxies of lexical items. As long as the shadow assumption is 'active' the corresponding lexical assumption is stored away. The withdrawal of a shadow assumption is followed directly by the use of the lexical item it was a proxy for.[5] The shadow assumptions allow us to define a more realistic ambiguating relation for LG.

6. BEYOND NON-DIRECTED LAMBEK CALCULI. The perspective adopted in this dissertation leads to new logical and linguistical questions. By relaxing LP as follows:

an assumption is used at least once in a derivation

we obtain the categorial system called LPC. In this dissertation we shall argue that strengthening LP does not necessarily imply choosing for LPC. One can choose from several possibilities:

- •LP+ unrestricted right to identify occurrences of assumptions (LPC)
- •LP + unrestricted right to identify some kind assumptions (cf. Prijatelj 1989)
- •LP + controlled identification of assumptions (Chapter VII)

In our view, identification of assumptions is triggered by special lexical items -hence the name '*controlled identification*'. By using proof-structure for linguistical description, we can show that LP + controlled identification of assumptions can deal with the ubiquitous nature of the Boolean particles 'and' and 'or'. We interpret the Booleans as such triggers, and by doing so we are able to capture their polymorphic nature. The same strategy allows us to incorporate a modest mechanism of anaphorical binding in LG.

7. OVERVIEW. Chapter I contains a discussion of the notion of Natural Logic. There we comment on several objections which have been raised against natural language as vehicle of inference and as illustration of our approach we define a syllogistic Natural Logic. Chapter II is an historical investigation into extensions of syllogistic. Chapter III is a study on Peirce's treatment of first-order inference from a specific point of view: what can Natural Logic learn from Peirce's work. Chapter IV is about the vehicles of inference on which Natural Logic works. Here LP and LG are defined. Chapter V completes the work preparatory to our system of Natural Logic. In this chapter we introduce the mechanism of monotonicity marking. Chapter VI contains the system of Natural Logic. Chapter VII discusses the adequacy of LG as a linguistic theory.

NOTES TO THE INTRODUCTION

[1]These principles have a Medieval history as well (cf. Prior 1962, Geach 1962), but for lack of space we shall not elaborate on it.

[2]Surprisingly, Aristotle's focussing on to the so-called '*categorical sentences*' is to be seen as a wise decision, given the lack of an adequate general syntactical theory of natural language.

[3]Our Natural Logic differs from Suppes' system because we incorporate in the grammar a mechanism which allows us to mark monotone sensitive positions. And it differs from Sommers' because our vehicles of inference are the grammatical forms themselves.

[4]In Montague's Universal Grammar a language is defined as a pair $\langle L, R \rangle$ where L is a disambiguated language and R is the '*ambiguating relation*'. R relates the expressions of the disambiguated language with strings of basic expressions of L (Montague 1970b).

[5]This mechanism resembles closely Cooper's storage mechanism but originates in a natural way by reflecting on the structure of proofs and their 'derivational' history (cf. Cooper 1983). Incidentally, in the interpretation of expressions in whose construction shadow assumptions occur, there is no semantically significant role for the shadows. They become bound variables. But even more important, the LG derivations in which shadow assumptions are marked, are structurally similar to the logical forms of transformational grammar. Our shadow hypotheses correspond to the traces left behind by the moved elements. Our mechanism of referring to the withdrawn assumption resembles the indexing mechanism that keeps track of the traces and the elements they are traces of. These similarities are at least suggestive.

CHAPTER I

THE IDEAL OF A NATURAL LOGIC

DESCRIPTION OF THE CONTENTS OF THE CHAPTER. The first section introduces the theme of this chapter. The second section consists of a discussion of the notion of Natural Logic. The third section comments on several objections which have been raised against natural language as vehicle of inference. The fourth section defines a fragment of English for which we construct a simple syllogistic Natural Logic.

INTRODUCTION

1.1. THE IDEAL OF NATURAL LOGIC. Recent developments in formal semantics have led to a revival of the ideal of a proof system in close contact with natural language: the so-called '*Natural Logic*'. Our view on Natural Logic is unfolded in this chapter. We propose a working definition of Natural Logic as a proof system based on grammatical form. The emphasis on grammatical form should not be seen as opposed to a semantic approach to natural language. On the contrary. Several results from formal semantics must be used in the construction of any realistic proof system for natural language.

In the second section, we consider and, subsequently, reject the idea that either Traditional logic or first-order Logic or Montague Grammar is an adequate basis for a theory of natural language inference.

In the third section we meet several objections to natural language as vehicle of inference. Our conclusion is that Natural Logic is feasible. We argue that the problem of ambiguity can be avoided. However, we point out that there are inferences which operate without eliminating scope ambiguity beforehand.

In the fourth section, we construct a syllogistic Natural Logic as an example of the current enterprise. The vehicles of inference of this system are grammatical forms based on the analysis: Noun Phrase + Verb Phrase.

We illustrate the logical strength of the system by proving that it generates all the traditional syllogisms. In fact, we prove that traditional syllogistic depends only on *monotonicity* and *conversion*. For a comparison of this result with previous semantical analyses of syllogistic inference, we refer the reader to Van Eijck (1985b).

2. NATURAL LOGIC

2.1. SYLLOGISTIC AS A BASIS FOR A NATURAL LOGIC. Any theory which gives a *systematic* account of inferences in natural language can be called *Natural Logic*. According to this view, the classical logical systems could be Natural Logics. Consider, for example, Aristotle's logic of categorical sentences, i.e. the logic of the expressions: *Every S is a P* , *Some S is a P* , *No S is a P* and *Not every S is P* . A Natural Logic based on Aristotle's theory may take the following form. We have a logical theory of the categorical sentences. Therefore, we can give an account of inferences by relating the sentences involved with the categorical forms.

2.2. SYLLOGISTIC AND GRAMMATICAL FORM. At first sight, Traditional Logic is a serious candidate for the role of Natural Logic since its grammatical forms are in close accord with natural language -it is often said that syllogistic respects the subject-predicate form. This suggests that syllogistic constructions have the form: Noun Phrase + Verb Phrase. But this is a mistake. The categorical parsing and the grammatical parsing of the sentence *Every logician is a philosopher* are not the same. The Aristotelian will distinguish a quantifier (*every*), a subject (*logician*), a copula (*is*) and a predicate (*philosopher*). In other words, the Subject of the traditional logician is not the Noun Phrase of the linguist: Traditional Logic is a logic of Common Nouns.[1]

There is an accord between syllogistic and natural language forms, but this accord is not an identity of syntactical forms. Sometimes sentences have to be brought into categorical form. Thus, for syllogistic purposes the sentence *Every philosopher wanders* will be expanded in the unlovely paraphrase *Every philosopher is a thing that wanders* . Of course, this strategy is quite legitimate. The paraphrases are only a small nuisance and they can be avoided by correlating sentences directly with syllogistic forms - as Aristotle himself did. Instead of saying that (a) is valid because (b) is valid, one can say that (a) is valid because (c) is valid:

(a)

Every logician is a philosopher
Every philosopher wanders
———
Every logician wanders

(b)

Every logician is a philosopher
Every philosopher is a thing that wanders
———
Every logician is a thing that wanders

(c)

All L are P
All P are W
———
All L are W

2.3. LIMITATIONS OF SYLLOGISTIC. Syllogistic gives a systematic account of inferences involving sentences similar to the categorical ones. But the paucity of the Aristotelian sentence forms limits the scope of logic. This is the main reason why it cannot be a *realistic* Natural Logic. Several classical authors consciously tried to expand the scope of Aristotle's system. But, as we shall see in the next Chapter, their theory of grammatical form was not sound. In the end, their disregard for the structure of the vehicles of inference is to be held responsible for the failure of their attempts.

2.4. PREDICATE LOGIC AS A BASIS FOR NATURAL LOGIC. In an analogous way, first-order logic can be incorporated into a theory of Natural Logic. However, in this case, the problem signaled in the last paragraph becomes more acute. To put it mildly, the syntactic structure of natural language sentences is not captured in any *systematic* way by their first-order counterparts. The correlation between natural language sentences and first-order formulas is not a *simple* correlation between syntactical objects. But, of course, no translation -not even a reliable one- ever is.

The problem is not so much that predicate logic has variables at places where natural language has none. This is a small problem, comparable to the fact that some languages lack articles while others have them. The real problem is that we do not have a systematic translation from natural language into predicate logic. Correlations which work well in one context, fail to do so in another. For instance the sentences *All men are perceptive* and *Every man is perceptive* correspond to the same first-order formula. One may expect that the same should hold for the sentences *All men are not perceptive* and *Every man is not perceptive* . But this is not the case. Some native speakers of English would interpret *All men are not perceptive* as $\neg\forall x(Mx \rightarrow Px)$, while they would interpret *Every man is not perceptive* as $\forall x(Mx \rightarrow \neg Px)$.

2.5. GRAMMATICAL FORM AND NATURAL LOGIC. Both for syllogistic and for predicate logic the following holds. In the evaluation of natural language inferences, grammatical form is not the most important factor. Suppose one wants to evaluate the claim that the sentence ϕ is implied by the set of sentences Γ. One grasps the truth conditions of the sentences involved and one translates these sentences into the appropriate formulas of the logical system, i.e. formulas with the same truth-conditions. After that, one focuses on the translations. If the translations of Γ imply the translation of ϕ then one concludes that Γ itself implies ϕ . One can argue that the grammatical form does indeed play an important role in the understanding of the truth conditions, but not in the translation itself. This is the reason why the classical logical systems may fail to give a systematic account of inferences in natural language: the relation between natural language and logical language is ad hoc.

We can liberalize our conception of Natural Logic by demanding that in the evaluation of arguments, the grammatical form should merely play a relevant role. This means that a grammatical theory of natural language will be one of the components of Natural Logic, because such a theory provides expressions with grammatical form. The objects which *represent* grammatical forms, however, may be expressions of a logical language. In this case, the task of constructing a Natural Logic consists of choosing an adequate logical language and of finding a systematic correlation between grammatical forms and expressions of this language.

2.6. BEYOND FIRST-ORDER LOGIC. This view on Natural Logic does not eliminate first-order logic as the logical engine of the theory. To see this point we only need to think of the correlations between grammatical deep structure and first-order formulas (cf. Harman, 1972). But due to the limitations of expressive power of first-order logic, this move may be too restricted. This was realized by Reichenbach who employed higher-order logic in his analysis of natural language (Reichenbach, 1947). But Reichenbach did not give an analysis of the grammatical form of natural language sentences, nor did he give an analysis of the relationship of grammatical form and higher-order expressions. This has finally come into being in Montague (1973), which has been described as a synthesis of categorial grammar and higher-order logic.

2.7. MONTAGUE GRAMMAR AND NATURAL LOGIC. In Montague (1973) we find a procedure correlating natural language with a formal language for which we already have a logic. We evaluate inferences in natural language by translating them into higher-order proofs. The translation is no longer ad hoc and the target language has a great expressive power. At first sight, Montague's grammar could be a Natural Logic. However, as Thomason (1974) and Van Benthem (1981) pointed out, Montague's uniform strategy complicates the evaluation of inferences. For instance, this strategy compels us to deem (a) invalid because it has the same *form* as (b):

(a)	(b)
Every rodent is an animal	Every price is a number
some rodents hibernate	some prices are changing
some animals hibernate	some numbers are changing

In the same way, we are obliged to reject (c) because it has the same form as (d):

(c)	(d)
John runs	The temperature rises
John is the mayor of New York	The temperature is ninety
The mayor of New York runs	Ninety rises

With Montague Grammar we have a successful theory providing a method for computing the denotation of natural language expressions, but a poor theory of natural language deduction.[2]

2.8. DIRECT INTERPRETATION AND NATURAL LOGIC. For Montague's immediate goal, the construction of a rigourous semantics for natural language, the use of higher-order logic is superfluous. The higher-order language is only the passage to the mathematical objects which constitute the meaning of natural language expressions. We have the following picture:

natural language ⇒ translation into higher-order logic ⇒ interpretation into mathematical objects

In principle, as Montague (1970a) elaborates, the relation between natural language and the universe of denotations, may take the simpler form:[3]

natural language ⟹ interpretation into mathematical objects

But from this perspective, the notion of Natural Logic becomes problematic. Higher-order logic, being semantically superfluous, is the essential component of *Montague Grammar as Natural Logic*: it is in this medium that inference takes place. Semantically superfluous though it may be, the logical language is the inferential mechanism; without it there seems to be no Natural Logic.

2.9. A PROOF SYSTEM FOR NATURAL LANGUAGE. There is an alternative, however. Church (1951) notices that, in principle, there is no difference between formal languages and natural languages. This has been taken to mean that it is possible to construct a coherent semantics for natural language. The syntax yields the well-formed expressions, the semantics provides these expressions with a denotation. But one can also pursue the comparison in another familiar direction. With a formal language and its semantics one usually has a proof system. The similarity between formal and natural languages may be taken to mean that a coherent proof system for natural language is possible. Montague's pioneering work showed that a rigourous local semantics for natural language is feasible. The contention of this dissertation is that an adequate proof system for natural language is feasible as well.

3. THE CASE AGAINST NATURAL LOGIC

3.1. TRADITIONAL OBJECTIONS AGAINST NATURAL LOGIC. Any plea for Natural Logic defies the ideological basis of modern logic. In the Fregean tradition natural languages are considered very poor vehicles of inference indeed. Expressions of this view are the following:

- Frege himself advocates the elimination of natural language in the formulation of mathematical proofs.
- Tarski is often seen as *implicitly* rejecting the very possibility of a rigourous definition of entailment for natural language.
- The *misleading form* thesis for natural language denies the possibility of devising a logic for direct use, at least for linguistic objects other than expressions of logical languages.

In the following sections we shall briefly consider these views, as well as a more modern rejection of an inferential approach to natural language.

3.2. FREGE. In the beginning, Frege tried to use natural language in the reduction of mathe matical concepts to logical ones, but he eventually abandoned this approach. As far as precision is concerned, Frege found natural language to be inadequate. In the first place, vernacular languages are not suitable for expressing mathematical statements in a conspicuous way. In the second place, inferences in natural language do not always live up to elementary exigencies of rigour. The point is that reasoning in ordinary language admits transitions licensed by tacit (semantical or syntactical) properties of the expressions involved. In the end, Frege constructed a formal language in which the inferential steps can be rigourously checked. As he puts it, natural language relates to his formalized language as the eye to the microscope. Natural language is a versatile instrument but, as soon as rigour demands *great sharpness of resolution*, it proves to be inadequate.

Frege's rejection of natural language in Frege (1879), seems to be a practical one. The daily practice of mathematicians proves that this rejection is not compulsory. There are also practical disadvantages in the *total* elimination of natural language as vehicle of inference. Ironically, formalized proofs are difficult to read; the obligation of writing down any single step in the proof makes the whole hardly perspicuous.

3.3. TARSKI. Of a more theoretical nature is Tarski's *implicit* rejection of natural language as a vehicle of inference. Tarski very often stresses the difficulty in constructing a coherent semantics for natural language. By putting together Tarski's assertion that *truth* is not definable for natural languages and the semantic definition of consequence, we could infer that it is impossible to define the notion of consequence for natural languages. [4]

This is a negative but definite result about natural language. However, its definiteness contains more than Tarski himself would accept. To see this point we only need to remember that the expression 'formalized language' refers to languages having a clear syntactical basis. Whereas for most formalized languages the notion of well-formedness is well-defined (and for almost all these languages this notion is effective as well), this is not the case for natural languages.[5] Since natural language is not a formalized language, and the precise results obtained in Tarski (1936) refer only to such languages, it follows that nothing definite can be said about everyday languages in this framework.

This view on the syntax of natural language seems to rule out a Natural Logic based on grammatical form. If the notion of well-formedness is not clear then all the notions based on it will be obscure. In particular, inference rules which make reference only to the syntactic structure will not be of great use. However, it is important to remember that the first systems of mathematical logic lacked a clear syntactic basis as well; this did not block the construction of powerful calculi of inference.

3.4. MISLEADING FORM THESIS. The ideal of Natural Logic is opposite to the so-called *Misleading Form Thesis for Natural Language*. This is the thesis that the *grammatical form* of sentences is at variance with their *logical form*. If we presuppose an intuitive notion of sentence, truth and entailment, then logical form may be introduced in the following way:[6]

> 'Fix a logical system L. Correlate with each sentence natural language sentence ψ a well-formed expression $\tau(\psi)$ in L. Then $\tau(\phi)$ is the logical form of ϕ iff $\tau(\Gamma) \models_L \tau(\phi)$ whenever the set of sentences Γ intuitively entails ϕ'.

We interpret the defenders of the misleading form thesis as saying that the grammatical form of ψ is of no use when we try to give a precise formulation of $\tau(\psi)$, because:

> 'the fact that two expressions belong to the same grammatical category does not entitle us to believe that their logical representation will be the same'.

3.5. LOGICAL BEHAVIOUR AND GRAMMATICAL STRUCTURE. This mismatch between logical behaviour and grammatical structure has led to the rejection of a *form* of Natural Logic. Quine resumes the situation by saying that

> 'If we were to devise a logic of ordinary language for direct use on sentences as they come, we would have to complicate our rules of inference in sundry unilluminating ways'. Quine (1960 : 158)

Even those who are unsympathetic to the misleading form thesis will agree with Quine. It seems impossible to devise a Natural Logic working on sentences as they come. This point can be elaborated a little more. The difficulties facing a naive conception of Natural Logic can be classified under the following headings:

- structural ambiguity
- poor correspondence of grammatical form with validity.

We shall consider these themes in the next two sub-sections

3.5.1. STRUCTURAL AMBIGUITY. It is well-known that some sentences may be parsed in several ways. This fact is supposed to be problematic for a Natural Logic: one parse may sanction some inferences which are not acceptable under another grammatical analysis. For example, the sentence *old men and women walk* entails *old women walk* if we associate it with the bracketing *old (men and woman) walk*. But it fails to do so if we associate it with the bracketing *((old men) and women) walk*. This example shows that natural language sentences seen as linear strings of words, are not reliable as instruments of inference.

For the Natural Logic project we are engaged in, structural ambiguity will be a fact of life and not a problem we have to address. We have chosen our Natural Logic as vehicle of inference entities which lack the feature of structural ambiguity: the proper objects of the calculus are *natural language sentences cum grammatical analysis*, things similar to the bracketed entities appearing above.

3.5.1.1. REMARK. Inference is not an all-or-none matter. Given sentences ϕ, ψ we may say that

' ϕ strongly entails ψ iff ϕ entails ψ for each bracketing of ϕ '.

Thus *old men and women walk* would strongly imply *old men walk* . This observation can be generalized. The sentence *Every man loves a bright girl* is considered semantically ambiguous. There is an interpretation in which *a bright girl* occurs in the scope of *Every man* , and there is another interpretation in which the relative scope of the Noun Phrases is reversed. But in both readings *Every man loves a bright girl* entails *Every man loves a girl* and *Every dull man loves a bright girl* . Thus, there is a kind of inference which yields conclusions without having to resolve the scope ambiguities. These observations suggest a question to be addressed: can we describe a Natural Logic which can also yield conclusions in ambiguous cases? This question will be addressed in Chapter VI.

3.5.2. VALIDITY AND GRAMMATICAL FORM. The choice of syntactic analyses as vehicles of inference, however, is not the end of the matter. The thesis that Natural Logic is concerned with grammatical forms, could be taken as implying that *an argument is valid in virtue of the grammatical form of the sentences involved.*

We could then imagine that the task of a Natural Logic is similar to the task of Traditional Logic. Both of them should try to classify patterns of valid argument schemas. But the former should classify patterns which can be described in linguistic terms. This is, nevertheless, a step we cannot take, in virtue of the poor match of grammatical form with validity. For instance, the following arguments will probably be considered valid:

$$\frac{\text{Abelard barely cried}}{\text{Abelard cried}} \qquad \frac{\text{Abelard wearily cried}}{\text{Abelard cried}} \qquad \frac{\text{Abelard certainly cried}}{\text{Abelard cried}}$$

However, we can not express this intuition by saying that *any* sentence analyzed syntactically as **NP + Adverb + Verb** entails another sentence analyzed syntactically as **NP + Verb**. A counter-example to this putative inference rule is:

$$\frac{\text{Abelard allegedly cried}}{\text{Abelard cried.}}$$

This observation against grammatical form as carrier of validity is not conclusive. An analysis a little more fine-grained would yield a sub-categorization of adverbs. We could say that *allegedly* is an intensional adverb, while *wearily* , *barely* , *certainly* are extensional adverbs.

And we could say that only extensional adverbs support the above kind of inference. However, the situation is a little more complicated. Certain Noun Phrases block the generalization. For instance, we cannot say that *No logician wearily cried* entails *No logician cried*.

The proper response to this last problem requires further subtleties which will be introduced in due course. For the moment, it is enough to know that our notion of Natural Logic is not committed to the view that arguments in natural language are valid in virtue of grammatical form. Roughly speaking we can say the following. Natural Logic is based on the grammatical form of sentences, in the sense that the forms associated with sentences are the vehicles of inference. But they are not all there is to the notion of Natural Logic. Natural Logic will take the grammatical analyses of *Abelard wearily cried* and *No logician wearily cried* as a starting point. The same engine used to produce these analyses, will be employed to generate, in an algorithmic manner, markings of inference sensitive positions. We shall then be able to read off from the syntactic analyses that in *No logician wearily cried*, *wearily cried* cannot be replaced by *cried*, while such a replacement will be allowed in *Abelard wearily cried*.

3.6. A MODERN OBJECTION AGAINST NATURAL LOGIC. There is still another objection against the notion of Natural Logic -an objection of a mathematical nature.[7] There are appealing arguments in favour of higher-order logic as the appropriate logic for natural language. For instance, at first sight, the logical representation of the sentences *Abelard has all the properties of a logician*, *Abelard cries easily*; *Abelard is a minor logician*, asks for a higher-order language. This has far-reaching consequences, since higher-order logic lacks some of the metamathematical properties of ordinary logic. The text-book Dowty et al. (1981) says that the semantic study of natural language renders superfluous the construction of a proof system for natural language. This is based on the opinion that a reason

> 'for preferring the semantic method to the deductive is that certain logics cannot be given axiomatic definitions of validity and entailment, though model-theoretic definitions of these notions are perfectly feasible for them.' Dowty et al. (1981 : 52)

Dowty et al. suggest that, from a linguistic point of view, the semantic approach to inference is more adequate. The incompleteness of higher-order logic forces one to prefer a semantic approach to natural language inference to a proof-theoretical one. But we disagree with the way in which these authors interpret the mathematical data and with their implicit assessment of contemporary logical culture.

First, as Kreisel (1952 : 120) remarks, we should 'not speak of the completeness of a formal system, but of the completeness of a certain interpretation of the formal system'. Whether a higher-order logic is complete or not, depends on the interpretation one chooses.[8] If one works with Henkin's general models, then completeness is achievable (cf. Henkin 1950). This means that to support a preference for the semantic approach over the proof-theoretic approach, one needs to establish first that the generalised models are inappropriate for natural language.[9]

Second, logical practice shows that incompleteness does not mean superfluity. Even accepting that proof theory is narrower than semantics, one might wonder if Kreisel's question has some relevance for natural language:

> 'what more do we know if we have proven a theorem by restricted means than if we merely know that it is true?'

In this context, Dummett's approach to meaning is highly relevant. Dummett (1978) emphasises at several places the idea that the theory of meaning should be determined by proof theory: the meaning of a sentence is determined by its proof conditions.

Incidentally, in Chapter IV and V we shall show how the *proof* that a natural language expression belongs to a linguistic category can be used to provide that expression with a denotation. Thus, through the proof we know that the expression belongs to a particular category and we know its denotation.

Thirdly, behind the opinion of Dowty et al., there is a dramatic change in the conception of logic. In formal semantics, a logic is usually identified with a language and its interpretation (cf. Chang and Keisler, 1973). In this conception there is no place for the inference rules as essential part of logic. This view reflects the abstract characterization of logic due to Lindström (1969). But formal semanticists should be aware that the highlight of syntax lies in the immediate future:

> 'The tradition called 'syntactic' - for want of a nobler title- never reached the level of its rival. In recent years, during which the algebraic tradition has flourished, the syntactic tradition was not of note and would without doubt have disappeared in one or two more decades, for want of any issue of methodology. The disaster was averted because of computer science -that great manipulator of syntax- which posed it some very important problems.
> Some of these problems (such as questions of algorithmic complexity) seem to require more the letter than the spirit of logic. On the other hand all the problems concerning correctness and modularity of programs appeal in a deep way to the syntactic tradition of *proof theory*.' (Girard et al. 1989 : 4)

These observations suggest that the view expressed by Dowty et al. is not a good enough reason for the rejection of a proof-theoretic approach to natural language inferences. In the next section we give a small example of the way in which semantics can be used to construct a proof system for natural language. The scope of this system covers the ground traditionally assigned to syllogistic system. The main difference between our proof system and syllogistic is first, that we base our inference rules on semantic properties of denotations, and second that we stay closer to the grammatical form than syllogistic does.

4. A SYLLOGISTIC NATURAL LOGIC

4.1. SYLLOGISTIC AND GENERALIZED QUANTIFIERS. In this section we give a uniform account of simple inferences involving the classical determiners. In particular, we shall consider sentences of the form NP VP, where NP is any expression consisting of one of the classical determiners followed by a suitable English expression. The setting of our considerations will be the generalized quantifier perspective of natural language quantification.
The Aristotelian system is a theory for such inferences. Instead of simply taking over the Aristotelian system we want to show that syllogistic inference can be related in a uniform way to the basic semantic notion of *monotonicity*. We shall give an explanation of syllogistic inference by treating it as consisting of monotone replacements. Earlier explanations treat syllogistic inference as consisting of replacements of common nouns or of replacements of verb phrases. To these replacements we now add the replacement of full noun phrases.

The view that syllogistic inference is related to monotonicity has been advanced by several writers active within the generalized quantifier framework; cf. Van Eijck (1985b), Van Benthem (1986 : Chapter 7), Zwarts (1986 : Chapter 4) and Westerståhl (1990).

4.1.1. THE RULE OF QUALITY. Positive *universal* categorical sentences play a crucial role in syllogistic inferences, for they give a *cue* for the existence of an inclusion relation. In our treatment of syllogistic inference we generalize this observation. Both *universal* and *particular* sentences are crucial in those inferences, for they both give a cue for the existence of an inclusion relation. This clearly distinguishes the role played by negative and positive sentences in syllogistic inferences. The interesting paper Van Eijck (1985b) devoted to the syllogistic, concludes with an open question:

> 'Can our generalized quantifier perspective be used to provide an illuminating motivation for the success of the combined Distribution/Quality test? Classical arguments in this area consist in mere combinatorial checking of all possible cases: one would like to replace that by a more semantic analysis.' Van Eijck (1985b : 18)

Our treatment of syllogistic offers a semantic explanation of the role of the Rule of Quality.

4.2. AN ARISTOTELIAN FRAGMENT OF ENGLISH. To show that a Natural Logic can account for syllogistic inferences, we define a small fragment of English.

4.2.1. THE LEXICON. The basic expressions of the language fall into the following categories:
(1) The set of common nouns, CN = { LOGICIAN, MAN, WOMAN, PHILOSOPHER }.
(2) The set of intransitive verbs, IV = { WANDERS, RUNS, THING}
(3) The copula IS.
(4) The operator THAT.
(5) The set of classical determiners, DET = { EVERY, SOME, A, NO, NOT EVERY }.

4.2.2. THE SYNTACTIC RULES.

(4) If A is a common noun , then QA is a noun phrase, where Q ranges over EVERY, SOME, A, NO and NOT EVERY.

(Noun phrases will be designed by **NP**).

(5) If B is a noun phrase then IS A B is a verb phrase.

(6) If A is an intransitive verb, then A is a verb phrase.

(Verb Phrases will be designed by **VP**).

(7) If A is verb phrase, then THING THAT A is a common noun.

(8) If A is a noun phrase and B is a Verb Phrase, then A B is a sentence.

(Sentences will be designed by the schema **NP VP**).

4.3. NOUN PHRASES AS GENERALIZED QUANTIFIERS. Fix a set of individuals D_e and a classical set of truth-values D_t. Common Nouns will take their denotation in the set D_{et} of functions from D_e into D_t. If A is a Common Noun, $[\![A]\!]$ will designate its denotation. Noun Phrases take their denotation one step further in the hierarchy, namely in the set $D_{(e,t),t}$ of functions from CN denotations into truth-values. In this section we identify D_{et} with the set of sets of D_e, and $D_{(e,t),t}$ with the set of families of sets of D_{et}. Denotations of NP's are the Generalized Quantifiers first studied by Mostowski and explicitly introduced in the study of natural language semantics by Barwise & Cooper (1981).

4.3.1. DENOTATIONS. The denotation of B, $[\![B]\!]$, is defined as follows:

(1) If B is a member of CN or IV then $[\![B]\!] \subseteq D_{e,t}$

(2) If B is a member of CN, then $[\![\text{IS A B}]\!] = [\![\text{B}]\!]$

(3) If B is a member of VP, then $[\![\text{THING THAT B}]\!] = [\![\text{B}]\!]$

(4) If B is a member of CN , then

(4.1) $[\![\text{EVERY B}]\!] = \{ X_{e,t} \mid [\![\text{B}]\!] \subseteq X\}$

(4.2) $[\![\text{NO B}]\!] = \{ X_{e,t} \mid [\![\text{B}]\!] \cap X = \varnothing \}$

(4.3) $[\![\text{NOT EVERY B}]\!] = \{ X_{e,t} \mid [\![\text{B}]\!] \subsetneq X\}$

(4.4) $[\![\text{SOME B}]\!] = [\![\text{A B}]\!] = \{ X_{e,t} \mid [\![\text{B}]\!] \cap X \neq \varnothing \}$

(5) $[\![\text{NP VP}]\!] = 1 \iff [\![\text{VP}]\!] \in [\![\text{NP}]\!]$.

These definitions allow us to express in a uniform manner the conditions under which classical sentences are true. For example, EVERY WOMAN IS A LOGICIAN, SOME WOMAN IS A LOGICIAN, NO WOMAN IS A LOGICIAN, NOT EVERY WOMAN IS A LOGICIAN will be true if $[\![\text{LOGICIAN}]\!] \in [\![\text{EVERY WOMAN}]\!]$; $[\![\text{LOGICIAN}]\!] \in [\![\text{SOME WOMAN}]\!]$; $[\![\text{LOGICIAN}]\!] \in [\![\text{NO WOMAN}]\!]$ and $[\![\text{LOGICIAN}]\!] \in [\![\text{NOT EVERY WOMAN}]\!]$.

Notice that if a sentence like EVERY WOMAN WANDERS is true, then $[\![\text{WOMAN}]\!] \subseteq [\![\text{WANDERS}]\!]$ must be the case. But then also $[\![\text{IS A WOMAN}]\!] \subseteq [\![\text{WANDERS}]\!]$ is the case, since the denotation of $[\![\text{IS A WOMAN}]\!]$ and the detonation of $[\![\text{WOMAN}]\!]$ are the same.

4.3.2. COMPLEMENT NP'S.

(6) $[\![\text{NO B}]\!] = \{ X_{e,t} \mid X \notin [\![\text{SOME B}]\!] \} = D_{(e,t),t} - [\![\text{SOME B}]\!]$.

(7) $[\![\text{NOT EVERY B}]\!] = \{ X_{e,t} \mid X \notin [\![\text{EVERY B}]\!] \} = D_{(e,t),t} - [\![\text{EVERY B}]\!]$.

4.4. MONOTONICITY PROPERTIES. In this section we introduce the notions of structural and lexical monotonicity. We shall show that these notions are very useful for our Natural Logic project.

4.4.1. STRUCTURAL MONOTONICITY. The set-theoretical relation $x \in y$ is **upward monotone** in y, i.e. for all z with $y \subseteq z$, $x \in y$ entails $x \in z$. We will say that any expression of the form NP VP is monotone in NP, for if $[\![\text{NP VP}]\!]$ denotes the truth, then $[\![\text{NP VP}]\!]$ corresponds to an expression of the form $x \in y$.

4.4.2. LEXICAL MONOTONICITY. The classical NP's have some closure properties which can be described in terms of monotonicity.

(a) $[\![\text{EVERY B}]\!]$ and $[\![\text{SOME B}]\!]$ are closed under supersets, i.e. if $x \in [\![\text{EVERY B}]\!]$ and $x \subseteq y$, then $y \in [\![\text{EVERY B}]\!]$; if $x \in [\![\text{SOME B}]\!]$ and $x \subseteq y$, then $y \in [\![\text{SOME B}]\!]$. Any set closed under supersets is called **upward monotone.**

(b) $[\![\text{NOT EVERY B}]\!]$ and $[\![\text{NO B}]\!]$ are closed under subsets, i.e. if $x \in [\![\text{NOT EVERY B}]\!]$ and $y \subseteq x$, then $y \in [\![\text{NOT EVERY B}]\!]$; if $x \in [\![\text{SOME B}]\!]$ and $y \subseteq x$, then $y \in [\![\text{SOME B}]\!]$. Any set closed under subsets is called **downward monotone.**

4.4.3. MONOTONICITY AND TRADITIONAL LOGIC. As van Benthem (1986) points out, (lexical) monotonicity lies at the the heart of Traditional Logic. On the one hand, upward monotonicity reflects the classical Dictum De Omni:

'whatever is true of every X is true of what is X'.

On the other hand, downward monotonicity reflects what traditional logic called *distributed* occurrence of terms:

'a term is distributed in a sentence if the sentence is true about *all of the predicate*.'

These analogies, first noticed in Van Eijck (1985b), suggest that a monotone explanation of syllogistic inference is possible.

4.5. PROOF SYSTEM. In this section we introduce a sizeable number of monotonicity rules which are adequate for the treatment of syllogistic inferences.

- STRUCTURAL MONOTONICITY RULE **M**

$$\frac{NP_1\ VP \quad [\![NP_1]\!] \subseteq [\![NP_2]\!]}{NP_2\ VP}\ M$$

- LEXICAL MONOTONICITY RULES

$$\frac{\text{EVERY A } VP_1 \quad [\![VP_1]\!] \subseteq [\![VP_2]\!]}{\text{EVERY A } VP_2}\ M1$$

$$\frac{\text{SOME A } VP_1 \quad [\![VP_1]\!] \subseteq [\![VP_2]\!]}{\text{SOME A } VP_2}\ M2$$

$$\frac{\text{NO A } VP_1 \quad [\![VP_2]\!] \subseteq [\![VP_1]\!]}{\text{NO A } VP_2}\ M3$$

$$\frac{\text{NOT EVERY A IS C} \quad [\![B]\!] \subseteq [\![C]\!]}{\text{NOT EVERY A IS B}}\ M4$$

The soundness of these lexical rules is a direct consequence of the definitions in 4.4 and the set-theoretical properties of the operations and relations involved.

4.5.1. THE ROLE OF POSITIVE SENTENCES. The Rule of Quality says that a valid syllogism must have at least one positive premiss. Now we show that any positive categorical sentence is a cue for the existence of an inclusion-relation between NP's. This is not exactly an answer to van Eijck's question, since this fact cannot be interpreted as a criterion of validity. We only offer a motivation for the Quality Rule. According to our analysis, the role of a positive sentence is to fuel the monotonicity engine. We can see, for instance, that if EVERY A IS A B is true, then a property of EVERY B is also a property of EVERY A. This is represented by $[\![\text{EVERY B}]\!] \subseteq [\![\text{EVERY A}]\!]$. By applying M to $[\![\text{EVERY B}]\!] \subseteq [\![\text{EVERY A}]\!]$ and EVERY B VP, we obtain EVERY A VP. The following proposition captures the relation between positive sentences and inclusion between NP's.

PROPOSITION 1. ANY POSITIVE SENTENCE IMPLIES AN INCLUSION RELATION.
Proof
We give here two examples.
(1) EVERY A IS A B entails $[\![\text{EVERY A}]\!] \supseteq [\![\text{EVERY B}]\!]$.

By definition, $[\![\text{EVERY A IS A B}]\!] = 1 \Leftrightarrow [\![B]\!] \in [\![A]\!] \Leftrightarrow [\![B]\!] \supseteq [\![A]\!]$. Moreover, $x \in [\![\text{EVERY B}]\!] \Leftrightarrow x \supseteq [\![B]\!]$. Hence, $x \supseteq [\![A]\!]$ i.e. $x \in [\![\text{EVERY A}]\!]$. Therefore

$[\![\text{EVERY A}]\!] \supseteq [\![\text{EVERY B}]\!]$.

(2) SOME A IS A B entails $[\![\text{EVERY A}]\!] \supseteq [\![\text{SOME B}]\!]$.

By definition, $[\![\text{SOME A IS A B}]\!] = 1 \Leftrightarrow [\![\text{B}]\!] \in [\![\text{A}]\!] \Leftrightarrow [\![\text{B}]\!] \cap [\![\text{A}]\!] \neq \varnothing$. Moreover, $x \in [\![\text{EVERY B}]\!] \Leftrightarrow x \supseteq [\![\text{B}]\!]$. Hence, $x \cap [\![\text{A}]\!] \neq \varnothing$ i.e. $x \in [\![\text{SOME A}]\!]$. Therefore $[\![\text{SOME A}]\!] \supseteq [\![\text{EVERY B}]\!]$.

□

As a corollary to Proposition 1, we obtain the following inference rules:

$$\frac{\text{EVERY A IS B}}{[\![\text{EVERY B}]\!] \subseteq [\![\text{EVERY A}]\!]}\text{P1} \qquad \frac{\text{EVERY A IS B}}{[\![\text{SOME A}]\!] \subseteq [\![\text{SOME B}]\!]}\text{P2}$$

$$\frac{\text{SOME A IS B}}{[\![\text{EVERY A}]\!] \subseteq [\![\text{SOME B}]\!]}\text{P3} \qquad \frac{\text{SOME A IS B}}{[\![\text{EVERY B}]\!] \subseteq [\![\text{SOME A}]\!]}\text{P4}$$

$$\frac{\text{EVERY A IS B}}{[\![\text{NOT EVERY A}]\!] \subseteq [\![\text{NOT EVERY B}]\!]}\text{P5} \qquad \frac{\text{EVERY A IS B}}{[\![\text{NO B}]\!] \subseteq [\![\text{NO A}]\!]}\text{P6}$$

$$\frac{\text{SOME A IS B}}{[\![\text{NO B}]\!] \subseteq [\![\text{NOT EVERY A}]\!]}\text{P7} \qquad \frac{\text{SOME A IS B}}{[\![\text{NO A}]\!] \subseteq [\![\text{NOT EVERY B}]\!]}\text{P8}$$

$$\frac{\text{EVERY A VP}}{[\![\text{IS A(N) A}]\!] \subseteq [\![\text{VP}]\!]}\text{P9}$$

A proof for these rules can be found in appendix 1.

4.5.2. REMARK. Notice that P1-8 can be formulated in a more general way. For instance, P3 can take the form

$$\frac{\text{SOME A VP}}{[\![\text{EVERY A}]\!] \subseteq [\![\text{SOME THING THAT VP}]\!]}\text{P'3}$$

In later sections we shall take this general formulation for granted.

4.6. EXAMPLES OF DERIVATIONS. Many syllogistic inferences can now be explicated in an easy way. For convenience, we introduce the abbreviation:

$I(X,Y) =: [\![\text{X}]\!] \subseteq [\![\text{Y}]\!]$.

(1) NO LOGICIAN WANDERS, SOME WOMAN IS A LOGICIAN ⟹ NOT EVERY WOMAN WANDERS.

$$\dfrac{\text{NO LOGICIAN WANDERS} \qquad \dfrac{\text{SOME WOMAN IS A LOGICIAN}}{\text{I(NO LOGICIAN, NOT EVERY WOMAN)}}\,P7}{\text{NOT EVERY WOMAN WANDERS}}\,M$$

(2) EVERY LOGICIAN IS A PHILOSOPHER, EVERY PHILOSOPHER WANDERS ⟹ EVERY LOGICIAN WANDERS.

$$\dfrac{\text{EVERY PHILOSOPHER WANDERS} \qquad \dfrac{\text{EVERY LOGICIAN IS A PHILOSOPHER}}{\text{I(EVERY PHILOSOPHER, EVERY LOGICIAN)}}\,P1}{\text{EVERY LOGICIAN WANDERS}}\,M$$

(3) EVERY LOGICIAN IS A POET, NOT EVERY LOGICIAN RUNS ⟹ NOT EVERY POET RUNS.

$$\dfrac{\text{NOT EVERY LOGICIAN RUNS} \qquad \dfrac{\text{EVERY LOGICIAN IS A POET}}{\text{I(NOT EVERY LOGICIAN, NOT EVERY POET)}}\,P5}{\text{NOT EVERY POET RUNS}}\,M$$

4.7. ACCOUNT OF TRADITIONAL SYLLOGISMS: THE TWO FIRST FIGURES.

The examples considered in the previous sections are instances of classical syllogisms. The format of our monotonicity explanations can be used to explain several syllogistic inference figures.

PROPOSITION 2. THE SYLLOGISMS OF THE FIRST AND SECOND FIGURE ARE DERIVABLE BY MONOTONICITY ONLY.

Proof

BARBARA: EVERY M VP, EVERY S IS A M ⟹ EVERY S VP

$$\dfrac{\text{EVERY M VP} \qquad \dfrac{\text{EVERY S IS A M}}{\text{I(EVERY M, EVERY S)}}\,P1}{\text{EVERY S VP}}\,M$$

DARII: EVERY M IS P, SOME S IS A M ⟹ SOME S VP

$$\dfrac{\text{EVERY M VP} \qquad \dfrac{\text{SOME S IS A M}}{\text{I(EVERY M, SOME S)}}\,P4}{\text{SOME S VP}}\,M$$

CELARENT: NO M VP, EVERY S IS A M ⟹ NO S VP

$$\dfrac{\text{NO M VP} \qquad \dfrac{\text{EVERY S IS M}}{\text{I(NO M, NO S)}}\,P6}{\text{NO S VP}}\,M$$

FERIO: NO M VP, SOME S IS A M ⟹ NO S VP

$$\frac{\text{NO M VP} \qquad \dfrac{\text{SOME S IS M}}{\text{I(NO M, NOT EVERY S)}}\,P7}{\text{NOT EVERY S VP}}\,M$$

DATISI: EVERY M VP, SOME M IS S ⟹ SOME S VP

$$\frac{\text{EVERY M VP} \qquad \dfrac{\text{SOME M IS S}}{\text{I(EVERY M, SOME S)}}\,P3}{\text{SOME S VP}}\,M$$

FERISON NO M VP, SOME M IS S ⟹ NOT EVERY S VP

$$\frac{\text{NO M VP} \qquad \dfrac{\text{SOME M IS S}}{\text{I(NO M, NOT EVERY S)}}\,P8}{\text{NOT EVERY S VP}}\,M$$

DISAMIS: SOME M VP, SOME M IS S ⟹ SOME S VP.

$$\frac{\text{SOME M VP} \qquad \dfrac{\text{SOME M IS S}}{\text{I(SOME M, SOME S)}}\,P2}{\text{SOME S VP}}\,M$$

BOCARDO: SOME M VP, SOME M IS S ⟹ SOME S VP

$$\frac{\text{NOT EVERY M VP} \qquad \dfrac{\text{SOME M IS S}}{\text{I(NOT EVERY M, NOT EVERY S)}}\,P2}{\text{SOME S VP}}\,M$$

□

Observe that in these schemas the structure of the NP and the VP is not relevant. The monotonicity rules we have used operate at a very broad level without demanding a finer syntactic analysis. But other schemas demand a less global approach. The following section takes care of those cases.

4.8. ACCOUNT OF TRADITIONAL SYLLOGISMS: THE THIRD FIGURE.

To explain the syllogisms of the third figure, one must take into account the particular logic of SOME and NO. We do so by introducing new rules which rest on the particular logical properties of these items. Thus we add the conversion rules to our system .

- CONVERSION RULES

$$\frac{\text{SOME A IS A B}}{\text{SOME B IS A(N) A}}\,\text{C1} \qquad \frac{\text{NO A IS A B}}{\text{NO B IS A(N) A}}\,\text{C2}$$

More in general,

$$\frac{\text{SOME A VP}}{\text{SOME THING THAT VP IS A(N) A}} \qquad \frac{\text{NO A VP}}{\text{NO THING THAT VP IS A(N) A}}$$

Our definition of derivation must be extended to cover the case in which C1-2 is one of the rules employed in the derivation.

PROPOSITION 2. THE SYLLOGISMS OF THE THIRD FIGURE ARE DERIVABLE BY MONOTONICITY AND CONVERSION.

Proof

CAMESTRES: EVERY P VP, NO S VP ⇒ NO S IS A P

$$\frac{\text{NO S VP} \qquad \dfrac{\text{EVERY P VP}}{\text{I(IS A P, VP)}}}{\text{NO S IS A P}}\,\text{M}$$

BAROCO: EVERY P VP, NOT EVERY S VP ⇒ NOT EVERY S IS A P

$$\frac{\text{NOT EVERY S VP} \qquad \dfrac{\text{EVERY P VP}}{\text{I(IS A P, VP)}}}{\text{NOT EVERY S IS A P}}\,\text{M}$$

CESARE: NO P VP, EVERY S VP ⇒ NO S IS A P

$$\dfrac{\dfrac{\text{NO P VP} \qquad \dfrac{\text{EVERY S VP}}{\text{I(IS A S, VP)}}}{\text{NO P IS A S}}\,\text{M}}{\text{NO S IS A P}}\,\text{C2}$$

FESTINO : NO P VP, SOME S VP ⇒ NOT EVERY S IS A P

$$\frac{\dfrac{\text{NO P VP}}{\text{NO THING THAT VP IS A P}}\,\text{C2} \qquad \dfrac{\text{SOME S VP}}{\text{I(NO THING THAT VP, NOT EVERY S)}}\,\text{P7}}{\text{NOT EVERY S IS A P}}\,\text{M}$$

□

This completes the derivation of the Aristotelian modi valid in modern settings. The two central features in these set-theoretical proofs are the use of positive sentences to obtain information about inclusion relations, and the application of monotonicity substitution rules at the broadest level possible. According to our analysis of the syllogistic forms, these inference patters rests on two principles: monotonicity and conversion. The latter can be seen as a more *algebraic* property. We shall see in Chapter III that Peirce's theory of logical inference also makes use of monotonicity and of certain algebraic principles.

4.9. PROPER NAMES IN SYLLOGISTIC INFERENCES. Is there any intrinsic interest in this logic of Noun Phrases? At the very least, we can give a semantic explanation of inferences which are treated in traditional logic in an awkward way. More importantly, we can do this while respecting the grammatical form of the sentences involved. Hence, our explanation warrants soundness while avoiding the mismatch between grammatical form and logical representations. We shall elaborate this point by working out concrete examples.

Consider the following arguments:

EVERY PHILOSOPHER IS A LOGICIAN	ABELARD IS A LOGICIAN
ABELARD IS A PHILOSOPHER	ABELARD IS A PHILOSOPHER
ABELARD IS A LOGICIAN	SOME PHILOSOPHER IS A LOGICIAN

The first-order explanation of their validity is given through proofs involving sentences like $\forall x(P(x) \rightarrow L(x))$ and $\exists x(P(x) \wedge L(x))$. For traditional logic, the situation is a little more complicated. The original Aristotelian system lacks a *natural* form for *Abelard is a logician*. Some traditional logicians solved this problem by correlating this sentence to *Every Abelard is a logician*. We obtain then the syllogisms

EVERY EUROPEAN IS A LOGICIAN	EVERY ABELARD IS A LOGICIAN
EVERY ABELARD IS A PHILOSOPHER	EVERY ABELARD IS A PHILOSOPHER
EVERY ABELARD IS A LOGICIAN	SOME LOGICIAN IS A PHILOSOPHER

These syllogisms are proven to be valid through the Aristotelian valid syllogisms Barbara en Darapti. The validity of Darapti rests on the presupposition of existential import. Sommers, a traditional logician who rejects this presupposition, considers sentences like *Abelard is a philosopher* as systematically ambiguous (cf. Sommers, 1982). It may be correlated to *Every Abelard is a philosopher* and to *Some Abelard is a philosopher*. His explanation of the second inference then takes the form

EVERY ABELARD IS A LOGICIAN
SOME ABELARD IS A PHILOSOPHER
___
SOME LOGICIAN IS A PHILOSOPHER

and this is a valid inference resting on Datisi.[10]

The traditional paraphrases and the first-order translation show a major departure from the syntactic structure of the given sentences. Presently we shall give another explanation of the validity of those inferences, without violating this structure.

4.10. A GENERALIZED QUANTIFIER EXPLANATION. The common interpretation of a sentence like ABELARD IS A PHILOSOPHER is the following. To the lexical object ABELARD we assign an object, s, from D_e; and to the expression IS A PHILOSOPHER we assign a member, $[\![\text{PHILOSOPHER}]\!]$, of the set D_{et}. Then we say

$$[\![\text{ABELARD IS A PHILOSOPHER}]\!] = 1 \iff s \in [\![\text{PHILOSOPHER}]\!]$$

In the Montague tradition, however, we collect all those members of D_{et} which map s into 1, forming in this way the set $I_s = \{P \in D_{et} | s \in P\}$. I_s is a member of $D_{(et)t}$ and we design this NP denotation by $[\![\text{ABELARD}]\!]$. We now say $[\![\text{ABELARD IS A B}]\!] = 1 \Leftrightarrow [\![\text{B}]\!] \in [\![\text{ABELARD}]\!]$.

The motivation behind Montague's analysis of proper names is that it will treat proper names and quantified expressions as members of the same syntactical category. This idea can be exploited in the context of Natural Logic. If proper names and quantified expressions belong to the same category, then they are mutually replaceable. The substitution rule we want to use is the structural monotonicity rule M. Thus we need to show that a positive sentence of the form PROPER NAME IS A B, is a clue for the existence of an inclusion relation between NP's. The two one-premiss rules which we obtain will be called UI and EI.

$$\frac{\text{ABELARD IS A B}}{[\![\text{EVERY B}]\!] \subseteq [\![\text{ABELARD}]\!]}\text{UI} \qquad \frac{\text{ABELARD IS A B}}{[\![\text{ABELARD}]\!] \subseteq [\![\text{SOME B}]\!]}\text{EI}$$

These two rules allow us to give a simple explanation of the inferences of section 2.5:

1.

$$\frac{\text{EVERY LOGICIAN IS A PHILOSOPHER} \qquad \dfrac{\text{ABELARD IS A LOGICIAN}}{\text{I(EVERY LOGICIAN, ABELARD)}}\text{U1}}{\text{ABELARD IS A PHILOSOPHER}}\text{M}$$

2.

$$\frac{\text{ABELARD IS A PHILOSOPHER} \qquad \dfrac{\text{ABELARD IS A LOGICIAN}}{\text{I(ABELARD, SOME LOGICIAN)}}\text{U2}}{\text{SOME LOGICIAN IS A PHILOSOPHER}}\text{M}$$

5. CONCLUDING REMARKS. In this chapter we have discused the notion of a logic based on grammatical form. We met several objections to a proof system for natural language, and have also given an example of the form a Natural Logic may take.

The notion of monotonicity allowed us to give a *systematic* account of syllogistic -the central theory of Traditional Logic. The question may now arise whether Natural Logic has no more strength than the old syllogistic. In the next chapter we shall be concerned with the historical question whether Traditional Logic ever used monotonicity, under the guise of Dictum de Omni and Distribution, to trespass the borders of syllogistic inference. The answer to the historical question can be interpreted as a provisional assessment of the strength of Natural Logic.

NOTES TO CHAPTER I

[1] In section 4 we shall show that it is possible to define syllogistic as a logic of Noun Phrases -thus bringing rumour in harmony with reality.

[2] To save the theory from contradicting our intuitions too overtly, Montague introduced '*meaning postulates*'. In the end, suitable meaning postulates will warrant that (a) and (c) hold good. In the construction of our own Natural Logic we shall make use of meaning postulates as well.

[3] Of course, one still needs a language for speaking about the mathematical objects. For instance, one can say that the interpretation of EVERY LOGICIAN is $\{ X \mid \text{'Logician'} \subseteq X \}$. In this case we interpret the NP directly; the object into which that expression is interpreted, however, is not present to our senses. The linguistic objects are correlated with *linguistic representations* of the mathematical objects. The direct interpretation should not be confounded with the project described in Swift (1726, Part III): 'An expedient was therefore offered, that since words are only names for things, it would more convenient for all men to carry about them such things as were necessary to express the particular business they are to discourse on."

[4] It is worth noticing that the view usually attributed to Tarski is not his last word on the subject of natural language and semantics. In Tarski (1969) he says:

'I should like to emphasize that, when using the term 'formalized languages", I do not refer exclusively to linguistic systems that are formulated entirely in symbols, and I do not have in mind anything essentially opposed to natural languages. On the contrary, the only formalized languages that seem to be of real interest are fragments of natural languages (fragments provided with complete vocabularies and precise syntactical rules) or those which can at least be adequately translated into natural language'.

[5] A typical example of a language in which the notion of well-formedness is not decidable, is Hilbert and Bernays expansion of the usual logical language with the definite description symbol, τ. In their system an expression of the form $(\tau x)A$ is well-formed if $\exists x A x$ and $\forall xy(Ax \wedge Ay \rightarrow x = y)$ are provable. Since provability is not decidable, neither is well-formedness in Hilbert and Bernays 's system. Of course, when this was written, Church had not yet published his note on the *Entscheidungsproblem*.

[6] Of course, this explanation has an air of precision which is not supported by the facts. It yields at most a criterion of adequacy. We need to know more about the correlation τ of sentences and expressions in L.

[7] This objection is not formulated as a rejection of natural logic, but as a rejection in general of a proof system for natural language.

[8] This holds even for ordinary predicate logic. If one chooses only finite models, then this logic is incomplete, (Tratenbrot, 1963).

[9] Incidentally, it is worth noticing that the text-book, Gamut (1991, Section 6.5) suggests that lack of completeness is not a serious drawback for the construction of higher-order proof systems useful in the treatment of natural language inferences.

[10] For a thorough analysis of the history and the logic of singular sentences within traditional logic, see the rich Barth (1974).

CHAPTER II

TRADITIONAL LOGIC AND NON-SYLLOGISTIC INFERENCES

DESCRIPTION OF THE CONTENTS OF THE CHAPTER. In the first section we describe the abstract form of the traditional monotonicity rules. In the second section we outline the context in which De Morgan introduced non-syllogistic arguments. In the third section we consider Leibniz' use of monotonicity in order to justify non-syllogistic inferences. In the fourth section we consider the oldest versions of monotone rules which we have as yet been able to find, namely Ockham's formulation in terms of the medieval supposition theory.

1. INTRODUCTION

1.1. TRADITIONAL MONOTONICITY RULES. The first chapter ended with the question whether the notions of *Distribution* and the *Dictum de Omni* -the guise in which monotonicity penetrates the syllogistic- were employed in accounts of non-syllogistic inference. The natural place to start our research is the work of those traditional logicians who tried to explain non-syllogistic inferences. Our working hypothesis is that *particular forms* of the lexical monotonicity rules (see I.4.4.3.) can be found in the works of the classical authors De Morgan, Leibniz and Ockham.

As we pointed out, traditional logic is a logic of Nouns. Hence, we would expect the monotonicity rules to take the following form:

UPWARD MONOTONICITY (M↑)	DOWNWARD MONOTONICITY (M↓)
Every A is B F(A)	Every A is B F(B)
F(B)	F(A)

where A, B are Nouns and F(A), F(B) are sentences containing these Nouns.

We shall show that in their account of non-syllogistic inference, De Morgan, Leibniz and Ockham employed versions of M↑ and M↓. However, we shall also point out that De Morgan's and Leibniz' description of the contexts F(A) and F(B), was not adequate. Ockham's version of the monotone rules is formulated in terms of the medieval *suppositio* theory. In fact, Ockham uses this theory in order to determine monotone sensitive occurrences of terms. But he does not go far enough: parts of complex terms are not marked as inferentially sensitive.

We shall start our brief historical journey with a general analysis of De Morgan's use of monotonicity in his explanation of non-syllogistic inference.

2. DE MORGAN

2.1. THE GENERAL SETTING. In De Morgan (1847), De Morgan introduced a symbolic language for the representation of inference forms. The categorical sentences are symbolized as follows:

Every S is P ⟹ S)P
No S is P ⟹ S.P
Some S is P ⟹ SP
Some S is not P ⟹ S:P

De Morgan also used lower case letters for negative terms. The conjunctive term X and Y is rendered as *XY* or *X-Y* whereas the disjunctive *X or Y* is rendered as *X,Y*.

De Morgan extended the range of application of logic by increasing its expressive power. On the systematic side, the negative terms suggested the notion of a *universe of discourse* :

> 'In logic, it is desirable to consider names of inclusion with the corresponding names of exclusion; and this I intend to do to a much greater extent than is usual; inventing names of exclusion by the prefix not, as in tree and not-tree, man and not-man. Let these be called *contrary* or *contradictory* names. Let us take a pair of contrary names, as *man* and *no-man.* It is plain that between them they represent everything imaginable or real in the universe. But the contraries of common language usually embrace, not the whole universe, but some one general idea. Thus, of men, Briton and alien are contraries; every man must be one of the two, no man can be both. *Not-Briton* and *alien* are identical names, and so are not-alien and Briton. . . In order to express this, let us say that the whole idea under consideration is the *universe* (meaning merely the whole of which we are considering parts) and let names which have nothing in common, but which between them contain the whole idea under consideration, be called contraries *in, or with respect to that universe* '. De Morgan (1847 : 37-8)

And the interplay of the new terms suggested the so-called *laws of De Morgan*:

> 'The complexity consists in the terms being conjunctively or disjunctively formed from other terms, as in PQ, that to which both the names P and Q belong conjunctively; and as in P,Q that to which one (or both) of the names P and Q belong disjunctively. The contrary of PQ is p,q; that of P,Q is pq. *Not both* is either not one or not the other, or not either. *Not either P nor Q* is logically 'not P and not Q' or pq : and this is then the contrary of P,Q '. De Morgan (1847 : 118)

On the practical side, with this notation we are able to build up a large reservoir of valid non-syllogistic schemata. Examples of valid schemata expressible in De Morgan's notation are:

$$\frac{\text{Every A is B} \qquad \text{Every A is C}}{\text{Every A is B and C}} \qquad \frac{\text{Every A is B and C}}{\text{Every A is B and every A is C}}$$

2.1.1. REMARK. (A) As a matter of fact, De Morgan himself did not accept that the introduction of complex terms yields new inference schemes. He writes:

> 'Accordingly X)P + X)Q = X) P-Q is not a syllogism, nor even an inference, but only the assertion of our right to use at our pleasure either one of two ways of saying the same thing '. De Morgan (1847 : 117).

(B) Observe that De Morgan formulated the laws which bear his name in the context of a logic of terms, and not in the context of propositional logic. The original *propositional* formulation of those laws, is to be found in the medieval studies on *consequentiae* (Kneale and Kneale, 1962: 294).

2.1.2. RELATIVE TERMS. De Morgan (1847) discussed the so-called '*relative terms*' such as *tail of a horse* . De Morgan himself is supposed to be the first logician who was conscious of the validity of inferences involving relatives. Well-known is the so-called '*De Morgan's example*':

$$\frac{\text{Every horse is an animal}}{\text{Every tail of a horse is a tail of an animal}}$$

commonly seen as an inference which exposes the weakness of traditional logic.

However, the relatives are not represented in the symbolism of De Morgan (1847). In fact it took him some years to devise a symbolism in which relatives could be expressed, see De Morgan (1966). In the meantime, his treatment of relative inferences had to differ from the schematic approach, since he had no schemata having the required form at his disposal.

2.1.3. DIRECT APPROACH TO NATURAL LANGUAGE INFERENCE. To explain relative inferences De Morgan chose a direct approach. He formulated versions of $M\uparrow$ and $M\downarrow$ which can be applied directly to natural language sentences:

> 'For every term used universally less may be substituted, and for every term used particularly, more. The species may take the place of the genus, when all the genus is spoken of; the genus may take the place of the species when some of the species is mentioned or the genus, used particularly, may take the place of the species used universally. De Morgan (1847 : 115).

2.2. SYLLOGISTIC FULLNESS THESIS. De Morgan rejects the syllogistic fullness thesis, i.e. the thesis that any valid inference is either an immediate inference or a classical syllogism, or is reducible to one of these. He felt that several interesting inferences were unaccounted for in the more traditional logic:

> 'Observing that every inference was frequently declared to be reducible to syllogism, with no exception unless in the case of mere transformations, as in the deduction of *No X is Y* from *No Y is X*, I gave a challenge in my work on formal logic, to deduce syllogistically from *Every man is an animal, Every head of a man is the head of an animal*'. De Morgan (1966 : 29).

2.2.1. REMARK. The argument that De Morgan actually used may seem questionable, because of his use of the expression *the* in the conclusion:

$$\frac{\text{Every man is an animal}}{\text{Every head of a man is the head of an animal}}$$

However his intentions are clear. In 2.1.2. we introduced De Morgan's example. The reader may notice that the generosity of history has replace *the* by *a* , thus correcting De Morgan's mistake.

2.2.2. THE ORIGINAL EXAMPLES. In De Morgan (1847 : 114) the inferences were:

(a)

$$\frac{\text{Man is animal}}{\text{The head of a man is the head of an animal}}$$

(b)

$$\frac{\text{Every man is an animal}}{\text{He who kills a man kills an animal}}$$

It is supposed to be obvious that neither argument is an instance of any classical scheme. De Morgan also argued that (b) is not reducible to syllogisms either. He considered this argument to be equivalent to:

(c)

$$\frac{\text{Every man is an animal} \quad \text{Some one kills a man}}{\text{Some one kills an animal}}$$

But this move fails to yield a syllogism, for *some one kills a man* and *some one kills an animal* are not syllogistic sentences. One can apply to (c) the paraphrase strategy, thus obtaining:

(d)

$$\frac{\text{Every man is an animal}\quad\text{Some one is a killer of a man}}{\text{Some one is a killer of an animal}}$$

But (d) is not a syllogism since it lacks a middle term. It is true that *man* occurs in both premisses, but in the second one it occurs embedded in the complex term *killer of man* .

2.2.3. REMARKS. The usual interpretation is that De Morgan wanted to show that traditional logic cannot handle relative arguments. It is of some importance to make two qualifications:
(A) De Morgan himself spoke of compound expressions in general as a problem for the syllogistic fullness thesis. And even though he actually formulated his two arguments with relational expressions, the point he wanted to make can equally well be made without them. The following one does it :

$$\frac{\text{Every horse is an animal}\quad\text{Some brown horse runs}}{\text{Some brown animal runs}}$$

(B) The expanded inference (c) does not count as a syllogism by De Morgan standards, but this is due to his restricted notion of a syllogisms. Aristotle's own conception of the syllogism is broader than De Morgan's, as the following passage shows:

> 'That the first term belongs to the middle, and the middle to the extreme, must not be understood in the sense that they can always be predicated of one another or that the first term will be predicated of the middle in the same way as the middle is predicated of the last term. It happens sometimes that the first term is stated of the middle, but the middle is not stated of the first term, e.g. if wisdom is knowledge, and wisdom is of the good, then conclusion is that there is knowledge of the good '. Aristotle Book I, 36.

In fact, (d) is an instance of the so-called syllogisms *ex obliquis* studied in medieval logic:

> 'Deinde. . . supponamus quod aliquando in syllogizando ex obliquis non oportet quod extremitas syllogistica vel medium syllogisticum sit extremitas alicuis praemissae . . . . Verbi gratia, bonus est syllogismus *Homo omnem equum est videns; Brunellus est equus; ergo Homo Brunellum est videns*. In hoc autem syllogismo iste terminus *equus* est medium, qui nec est subjectum nec praedicatum in maiore propositione '. Buridan (1976 : 100)

2.3. DE MORGAN'S INFERENCE RULES. Having thus disposed of the *fullness thesis*, De Morgan introduced the rules which would take care of relative inference (cf. 2.1.3.). The rules he gave are neither unproblematic in their applications nor felicitously worded:

D↑ : The genus may take the place of the species *when some of the species is mentioned.*
D↓ : The species may take the place of the genus *when all the genus is spoken of.*
However, at first sight these rules seem to be up to the work required. For instance, with the aid of D↓ we can explain the acceptability of this inference:

$$\frac{\text{A man sees every animal}}{\text{A man sees every horse}}$$

To achieve that goal we use semantical facts. We know that ⟦animal⟧ , the denotation of *animal*, is the genus of ⟦ horse⟧ , the denotation of *horse* ; and by the same token that ⟦horse⟧ is a species of ⟦animal⟧ . We therefore use D↓ in order to substitute *horse* for *animal* in the given premiss; in doing this we reach the desired sentence as a conclusion.

2.3.1. A SCHEMATIC FORMULATION OF DE MORGAN'S RULES. In this section we give an abstract formulation of D↑ and D↓. De Morgan's wording of the rules seems to imply that we need *truths*. If so, his logic flies into the face of a logical principle; for it would rule out valid arguments with false premisses. For instance, consider the following argument:

$$\frac{\text{Every animal is a horse} \quad \text{A man sees every horse}}{\text{A man sees every animal}}$$

This argument is valid, but the actual genus - species relationship which exists between the denotations of *horse* and *animal* , precludes the rules from acknowledging this.
We know, however, that De Morgan accepted the principle that a valid argument could have false premisses, see De Morgan (1847 : 1). Hence we have to bring our interpretation in line with this principle. We shall not require the relevant denotations to behave as genus and species. We shall just assume that they do. This can be done because universal sentences like *Every A is B* are employed to assert that A is species of B, and that B is genus of A . This is why universal sentences are an essential element in the arguments generated by De Morgan's rules (cf. Chapter I. 4.1.1.) This interpretation of universal positive sentences is De Morgan's own interpretation:

> 'when X)Y, the relation of X to Y is well understood as that of species to genus'.
> De Morgan (1847 : 75)

The preceding discussion will be resumed in the following formulation of De Morgan's rules:

D↑

$$\frac{\text{Every X is Y} \quad \text{F(X)}}{\text{F(Y)}}$$

provided some of the denotation of X is spoken of in F(X)

D↓

$$\frac{\text{Every X is Y} \quad \text{F(Y)}}{\text{F(X)}}$$

provided all of the denotation of Y is spoken of in F(Y)

2.3.2. DE MORGAN'S RULES AND DICTUM DE OMNI. De Morgan called D↑ above, a version of the *Dictum de Omni* (cf. I.4.4.3.). As we pointed out, this classical dictum has often been thought of as the central syllogistic principle. Central in the use of the Dictum is the role assigned to universal sentences. Given the sentence *Every A is B* and the further information that some or every C is A , the Dictum entitles us to infer that some or every C is A. And that is exactly what D↑ is supposed to do. For instance, assumes that in *Every S is M* and *Some S is M* , 'some of the denotation of S is spoken of'. Then D↑ yields:

$$\frac{\text{EVERY M IS P} \quad \text{EVERY S IS M}}{\text{EVERY S IS P}} \qquad \frac{\text{EVERY M IS P} \quad \text{SOME S IS M}}{\text{SOME S IS P}}$$

The rule D↓ above is the mirror image of the dictum de Omni. Assume that in *No A is B* and in *Not every A is B* , 'all of the denotation of B is spoken of'. Then D↓ yields these syllogisms:

$$\frac{\text{EVERY S IS M} \quad \text{NO M IS P}}{\text{NO S IS P}} \qquad \frac{\text{EVERY P IS M NOT} \quad \text{EVERY S IS M}}{\text{NOT EVERY S IS P}}$$

2.3.3. DICTUM DE OMNI AND NON-SYLLOGISTIC INFERENCES. De Morgan also applied D↑ and D↓ outside the categorical fragment. This is why we are interested in those rules. For instance, the application of D↑ to (c) is quite direct. The universal sentence *Every man is an animal* establishes a genus-species relationship. Assume that in *Some one kills a man* some of the denotation of *man* is spoken of. Then, in accordance with D↑, the conclusion follows.

But it is rather disappointing that we ignore how De Morgan coped with the inferences (a) and (b). We have isolated the contribution of the universal sentence given as a premiss in (a) and (b).[1] We know for sure that substitutions have to occur. But what we do not know is in which sentences the substitutions are to be carried out. De Morgan does not indicate this explicitly.

2.3.4. A RECONSTRUCTION. Choose as a premiss *He who kills a man kills a man* . Assume that in the second occurrence of *man* some of the denotation of *man* is spoken of. Then the following inference is generated by using D↑ with regard to that specified occurrence of *man* :

$$\frac{\text{Every man is an animal} \quad \text{He who kills a man kills a man}}{\text{He who kills a man kills an animal}}$$

We can also see why De Morgan could have considered (a) a valid inference. Assume that in *The head of a man* some of the denotation of *man* is spoken of, then one application of D↑ to *The head of a man is the head of a man* would yields *The head of a man is the head of an animal*.

2.4. SHORTCOMINGS IN DE MORGAN'S APPROACH. As illustrated above, De Morgan's logic seems stronger than syllogistic logic, since D↑ and D↓ can generate inferences involving relatives. There are, however, a few problems. It is not sufficient that the denotation of the relevant expressions be given as genus and species. It is just as important that the expressions themselves be used in a particular way. Before substituting one expression for another, we have to be certain that the 'genus [is] being spoken universally of' in one case and that 'some of the species [is] being mentioned' in the other.

2.4.1. CONDITIONS ON DE MORGAN'S RULES. Up till now, we have assumed that the expressions of our examples fulfil those restrictions. This is a simplification, since we have not yet given any criterion to determine whether this is the case. Once more, we are in the dark about De Morgan's real choice. Our hypothesis is that he took expressions of generality as a guide-line. Speaking about categorical sentences, he said that the words of the sentences indicate whether the subject is 'spoken of universally' or not:

> 'In such propositions as *Every X is Y, Some Xs are Y* &c., X is called the subject and Y the predicate. It is obvious that the words of the proposition point out whether the subject is spoken of universally or partially, but not so of the predicate." De Morgan (1847 : 6)

The generalization of this remark results in the following criteria

C_1: In the context F(an A) some of A is mentioned.

C_2 : In the context F(every A) all of A is spoken of.

However, a little reflection shows these criteria to be inadequate. It is true that (b) can be generated by using C_1 and D↑. But the same holds for the following invalid inference:

$$\frac{\text{Every man is an animal} \quad \text{He who kills a man kills a man}}{\text{He who kills an animal kills a man}}$$

This inference shows conclusively that the combination of C_1 and $D\uparrow$ is unsound: the premisses are true and the conclusion is false. C_1 does not permit differentiation between the two occurrences of *man* in the tautological premiss; therefore *animal* is in both cases substitutable for *man* , and the first part of $D\uparrow$ does the rest.

2.4.2. THE CONDITIONS IN TERMS OF DISTRIBUTION. De Morgan's treatment of his non-monadic arguments thus fails, but it is worth emphasizing that this is not due to the abstract format of the rules. It is rather the restrictions C_1 and C_2 which have proved wanting: the conditions *when all the genus is spoken of, when some of the species is mentioned* are not effective. We cannot tell whether a given expression obeys them or not. The use of C_1 and C_2, which seems implicit in De Morgan's strategy, makes the restrictive conditions applicable. But these criteria are clearly not adequate. At this point we may consider abandoning the literal reading of De Morgan's rules, and instead, try to interpret them in terms of the traditional doctrine of distribution. This doctrine can be seen as providing the means needed for the description of the contexts in which substitution is allowed.

In fact, the description Prior gave of distribution suggests a connection between the traditional doctrine of distribution and De Morgan's original rules:

> 'It is often said . . .that a distributed refers to all and an undistributed term to only a part, of its extension . . . What the traditional writers were trying to express seems to be something of the following sort: a term I is distributed in a proposition f(I) if and only if it is replaceable in f(I), without loss of truth, by any term 'falling under it' in the way that a species falls under a genus '. Prior (1967: 39).

This is not all too farfetched, since De Morgan himself identifies the expressions *universally spoken* of and *distributed*:

> 'It is usual in modern works to say that a term which is universally spoken of is distributed. . . The manner in which the subject is spoken of is expressed; as to the predicate, it is universal in negatives but particular in affirmatives '. De Morgan (1966 : 6)

Let us give a syntactic characterization of distribution:

- The terms A, C, D are distributed in
  Every A is B, No C is D, Not Every B is A
- The terms B, C, D are un-distributed in
  Every A is B, Some C is D, Not Every B is A

With the backing of his own identification, we can re-word De Morgan's rules in the following form:

D↑	D↓
Every A is B F(A)	Every A is B F(B)
F(B)	F(B)
provided that A occurs non-distributively in F(A).	provided that B occurs distributively in F(A).

But this does not work. The distribution doctrine only says that *kills a man* has two different values within the tautological premiss; it says nothing at all about the distribution value of *man* therein. If we want to complement De Morgan's rules with the distribution doctrine, then this doctrine will have to be extended itself so as to include the elements of compound expressions. What we need is a procedure for computing distribution values, starting from basic expressions and using distribution values induced by the expressions of generality. De Morgan himself does not, however, appear to have recognized the need for such a systematic procedure.

As we pointed out, in his treatment of non-syllogistic inference Leibniz, like De Morgan, made use of what we have called monotonicty rules. In the next section we shall consider how monotonicity was used by Leibniz.

3. LEIBNIZ

3.1. JUNGIUS' NON-SYLLOGISTIC INFERENCES. Leibniz' interest in non-syllogistic inferences was aroused by Jungius (1957). Jungius' book is seen as one of the few 17th century logical texts which deserve any attention, mainly because he recognized certain non-syllogistic patterns of inference. These patterns are the following (see Jungius 1957: 89, 122, 151-154):

- The inferences *a compositis ad divisa*:

Plato est philosophus eloquents	Plato est philosophus eloquents
Plato est philosophus	Plato est eloquens

- The inferences *a divisis ad composita*

Omnis planeta per zodiacum movetur Omnis planeta est stella
___
Omnis planeta est stella quae per zodiacum movetur

- The inferences *per inversionem relationis*

Salomon est filius Davidis	David est pater Salomonis
David est pater Salomonis	Salomon est filius Davidis

- The syllogisms *ex obliquis*
- The inferences a rectis ad obliqua

We already encountered the first two patterns in section 2.2.2. We comment on the syllogisms *ex obliqui* in section 2.2.1. We shall come back to them in our consideration of Ockham's monotonicity rules.

3.1.1. THE INFERENCES A RECTIS AD OBLIQUA. The inferences *a rectis ad obliqua*, on the other hand, are now of greater importance. These are immediate inferences whose premisses contain expressions in the nominative case (for instance *man* and *animal* in *Every man is an animal*) , whereas in the conclusion these expressions are transferred into one of the non-nominative cases by some *notio respectiva* (for instance, the notion of killing transferred the expressions *man* and *animal* into the accusative case in *He who kills a man kills an animal*). The inference which Jungius used to illustrate this pattern is a variant of De Morgan's second inference:

Omnis circulus est figura
____
Quicumque circulum describit figuram describit

Jungius never tried to explain, however, why those patterns yield valid inferences; he simply took it for granted that they do.
Leibniz, on the other hand, considered this lack of justification a gap which had to be filled. Well-known is his treatment of the inferences *per inversionem relationis*; see Kneale & Kneale (1962 : 324-25). We shall see that Leibniz failed to tackle the problem of multiple generality in all its comprehensiveness, but that he did strive to give non-trivial demonstrations of some arguments commonly considered to involve multiple generality and calling for procedures from first-order predicate logic; cf. Dummett (1973 : 8).

3.2. LEIBNIZ' MONOTONE INFERENCE RULE. In Leibniz (1768, VI : 38-9), Leibniz describes how to cope with inferences *a rectis ad obliqua*. He introduced some rules of inference and showed that according to these rules, the conclusion follows from the premisses. This strategy depends on the principle of substitutivity of equivalents, and also on a syntactical generalization of the Dictum de Omni. To get some insight in Leibniz' method we shall apply it to this Latin version of De Morgan's example:

$$\frac{\text{Omnis equus est animal}}{\text{Omnis cauda equi est cauda animalis}}$$

Leibniz' rules are the following:

L↑: Esse prædicatum in propositione universali affirmativa, idem est, ac salva veritate loco subjecti substitui posse in omnia alia propositione affirmativa, ubi subjectum illud prædicati vice fungitur. Exempli causa: quia graphice est ars, si habemus *rem quæ est graphice*, substituere poterimus *rem quæ est ars*.

PR: Obliquo speciali aequipollet obliquos generalis cum speciali recto, ideo sibi mutuo substitui possunt. Verbi gratia, pro termino qui discit graphicem substitui potest, qui discit rem quae est graphicem. Et contra, pro termino qui discit rem quae est graphicem substitui potest qui discit graphicem.

L↑ says that given the sentence *Omnis S est P*, *P* is substitutable for *S* in any affirmative sentence ϕ in which *S* occurs as predicate; clearly Leibniz allowed ϕ to be a relative sentence. Therefore it should be evident that Leibniz' generalization of the Dictum de Omni does not affect the kind of expression which may be involved in the substitutions; it allows instead for a new context of substitutions: relative sentences.

It is thus not possible to apply L↑ to expressions in one of the oblique cases directly, since it only concerns expressions in the nominative case. We have to bring oblique expressions into the range of the dictum. The paraphrase strategy takes care of this situations. PR says that if *A* is an expression with *B* as one of its non-nominative cases, then *B* is equivalent to the complex expression *R quae est A* , where *R* is the word *res* in the same case as *B* . For instance, *equi* and *animalis* are equivalent to *rei quae est equus* and *rei quae est animal* . In virtue of this equivalence, oblique expressions can be brought into the scope of L↑. To do this, Leibniz must appeal to the principle of substitutivity of equivalents; this is what he does in saying that *B* and the complex expression *R quae est A* are substitutable for each other.

3.2.1. LEIBNIZ' RULES AND DE MORGAN'S EXAMPLE. Now we proceed to work out the derivation of *Omnis cauda equi est cauda animalis* from *Omnis equus est animal* , making use of a tautological premiss:

$$\cfrac{\cfrac{\cfrac{\text{Omnis cauda equi est cauda equi}}{\text{Omnis cauda equi est cauda rei quae est equus}}\,\text{PR} \quad \text{Omnis equus est animal}}{\text{Omnis cauda equi est cauda rei quae est animal}}\,\text{L}\uparrow}{\text{Omnis cauda animalis est cauda equi}}\,\text{PR}$$

3.3. THE SHORTCOMINGS IN LEIBNIZ' APPROACH. Leibniz' strategy is to some extent more elegant than De Morgan's solution; moreover it is systematically applicable and this is more than what we can say of De Morgan's proposal. However, Leibniz' treatment of the inferences *a rectis ad obliqua* unfortunately overlooks a few things:

(A) The only constraint Leibniz imposed upon $L\uparrow$ is that S must occur as a predicate in the context of substitution. If he had limited himself to the standard categorical sentences, then he could, by implication, have derived a constraint. For in this case, S has to appear non-distributively: the predicates of affirmative categorical sentences occur, per definition, in this way. But Leibniz went beyond the categorical fragment by permitting certain relative sentences to be equivalent to oblique expressions, and thereby making these substitutable for each other. Without undergoing a generalization, the traditional doctrine of distribution, however, does not predict which distribution values *equus* has in *Omnis cauda equi est cauda rei quae est equus* , since *equus* is neither a predicate here nor a subject of any categorical sentence.

(B) The lack of constraints on $L\uparrow$ becomes a problem when we look at PR. Here, there is no mention of contexts where the non-nominative expression may occur, and no mention of contexts in which the substitution of the complex expression for the oblique expression should not be carried out.

As a result of Leibniz' overlooking those points the rules are unsound. The following sequence constructed in accordance with PR and $L\uparrow$ proves it:

Omnis cauda equi est cauda equi
———————————————— PR
Omnis cauda rei quae est equus est cauda equi   Omnis equus est animal
———————————————————————————————— $L\uparrow$
Omnis cauda rei quae est animal est cauda equi
———————————————————— PR
Omnis cauda animalis est cauda equi

3.4. REMARKS. The previous example illustrates that Leibniz' strategy is unsatisfactory. In the sentence *Omnis cauda rei quae est equus est cauda equi* , the expression *equus* is the predicate of the relative sentence *rei quae est equus* . Hence the paraphrase strategy has brought it within the range of $L\uparrow$. This is, of course, a mistake. A more attentive theory of grammatical form could have avoided this mistake. Occurrences of the predicates of relative sentences are not automatically replaceable by terms with a larger denotation. But occurrences of those predicates in the *Verbal Phrases* of affirmative sentences are indeed replaceable.

We started out by promising to study the role of monotonicity in the work of De Morgan, Leibniz and Ockham. We have so far discussed De Morgan and Leibniz. In the next section we close our historical journey: we shall survey Ockham's use of the *suppositio* theory as the structural guide for his use of monotonicity.

4. OCKHAM

4.1. SUPPOSITIO AND DISTRIBUTION. The shortcomings of Leibniz' strategy and De Morgan's proposal show that their generalizations of the Dictum de Omni have to be supplemented. In the first case with effective constraints restricting the context of substitution, and in the second case with effective definitions of this context. The distribution doctrine stops short of yielding those features needed because it is restricted to categorical sentences; furthermore, it sees all categorical expressions as logically simple even when they are syntactically complex.

The doctrine of distribution, however, is considered to be a simplification, or even worse, to be an impoverished version of medieval *suppositio* theory. With the help of this theory, medieval logicians were able to handle arguments which lie beyond the scope of first-order predicate monadic logic. We have already stated that indeed Willian of Ockham formulated inference rules which may be seen as precursors to the rules offered by De Morgan, i.e. which may be seen as instantiations of M↑ and M↓. We shall see presently that he formulated his rules in terms of supposition theory. Because of this, we give a short description of some aspects of this theory. Of course, we do not pursue the supposition theory in all its complexity (and its richness), taking instead the supposition assignments as primitive. For convenience, we also consider only two syncategorematical expressions, namely *omnis* and *non* ; and we shall call all transitive verbs *copula.*

4.2. BRIEF SKETCH OF THE SUPPOSITION THEORY. Roughly speaking, *suppositio* is a property which *occurrences* of terms may have. A fair description of this marking of occurrences is given by Kneale and Kneale:

> 'Every distributive sign (i.e. a sign with the sense of *all* or *none*) gives *suppositio confusa et distributiva* to the term to which is directly adjoined, and a negative sign does the same also for the remote term, but an affirmative sign gives *suppositio confusa tantum* to the remote term'.
>
> 'A term which occurred with existential quantification and not preceded by any term with universal quantification was said to have *suppositio determinata* . . .'.
> Kneale and Kneale (1962 : 258, 260)

Observe that the *suppositio* of a term depends on its relative position in a sentence with regard to expressions which trigger *suppositio* on their environment.

4.2.1. EXAMPLES.

(1) Consider the following sentences:

(i) Omnis homo est animal.

(ii) Non homo est animal.

(iii) Homo est animal.

(iv) Homo non est animal.

From the quotations it follows that *suppositio confusa et distributiva* belongs to the occurrences of *homo* in (i), (ii) and likewise to *animal* in (ii), (iii). *Suppositio determinata*, on the other hand, is possessed by *homo* in (iii), (iv). But *animal* has *suppositio confusa tantum* in (i).

(2) Consider the following sentences:

(v) Asinum omnis homo videt.

(vi) Omnis homo videt asinum

According to the definitions *asinum* has *suppositio determinata* in (v) and *suppositio confusa tantum* in (vi).

4.2.2. DISTRIBUTION AND *SUPPOSITIO* THEORY. It will be clear that within the categorical fragment *suppositio confusa et distributiva* belongs to expressions which according to the distribution doctrine occur distributively and conversely. It is also the case that any expression having supposition determinata occurs non-distributively. But the inverse does not follow: *animal* appears non-distributively in (i) according to the distribution doctrine, and has suppositio confusa tantum according to the medieval theory.
This distinction between *suppositio confusa tantum* and *suppositio determinata* which distribution theory obliterates, gains importance when we look at sentences and arguments outside the categorical framework. For instance, the distribution doctrine is unable to distinguish, in pure distribution terms, between the two occurrences of *asinum* in the apparently equivalent sentences (v) and (vi) of example (B).

As we saw in example (B) the *suppositio* theory distinguishes between these two occurrences. Moreover, medieval logicians formulated an inference rule based on this distinction: from a sentence having an expression A with suppositio determinata, say (v), and *asinum* in it we can confidently move to another one, differing from the first in that A occurs now with supposition confusa tantum, thus (vi). The converse inference, however, was explicitly rejected; situations were described showing that moving the other way around could be moving from the true into the false; cf. Kneale & Kneale (1962 : 259).

4.3. OCKHAM'S MONOTONE RULES. Ockham's version of M↑ and M↓ is the following:

O↑ : Ab inferiori ad superius sine distributione, sive illud superius supponat confusa tantum sive determinata, est consequentia bona. Ockham (1951 : 274)

O↓ : A superiori distributo ad inferius distributum est bona consequentia. Bird (1961 : 69)

The examples which illustrate the working of these rules make clear that they rest on universal sentences: the subject of such a sentence is called the *inferior* and the predicate the *superior* expression. This observation permits us to represent the above rules as follows:

O↑

$$\frac{\text{Omnis S is P} \quad \text{F(S)}}{\text{F(P)}}$$

Provided that S occurs in F(S) with *suppositio* non-distributiva

O↓

$$\frac{\text{Omnis S is P} \quad \text{F(P)}}{\text{F(S)}}$$

Provided that P occurs in F(P) with *suppositio confusa et distributiva*

The rules O↑ and O↓ operate on the surface form of sentences. In applying these rules one uses the *suppositio* theory to determine whether an expression occurs in an inferential position or not. This proof system is then a Natural Logic based on monotonicity and on a theory of monotonicity marking. Due to this marking theory, Ockham's natural logic is able to explain the inferences with which De Morgan and Leibniz unsuccessfully wrestled. They had monotone rules, but lacked a theory of monotonicity marking. This historical fact makes clear that monotone rules without a theory of marking, are not an improvement vis-a-vis the old syllogistic.

4.3.1. EXAMPLES.

(1) The following sequence is sanctioned by O↓

$$\frac{\text{Omnis equus est animal} \quad \text{Omne animal videt hominem}}{\text{Omnis equus videt hominem}}$$

In *Omne animal videt hominem* the expression *hominem* is the remote term with regard to the affirmative sign *omnis* . Hence it has *suppositio confusa tantum*.. Therefore O↓ can be applied.

(2) The following sequence is sanctioned by O↑

$$\frac{\text{Omnis equus est animal} \quad \text{Equus videt hominem}}{\text{Animal videt hominem}}$$

In *Equus videt hominem* the expression *equus* is not preceded by any universal sign. Hence it has *suppositio determinata*. Therefore O↑ can be applied.

4.4. OCKHAM ON SYLLOGISMS EX OBLIQUIS. We have emphasized in section 2.2.1 that not all early logicians would reject the validity of the arguments (c) or (d). Some logicians could see this arguments as valid syllogisms *ex obliquis*. These are two-premiss arguments in which transitive verbs may play the role of the copula and in which oblique expressions are allowed to appear as (part of) syllogistic terms.

Ockham himself listed a great number of oblique syllogisms, claiming that the Dictum de Omni is the logical principle governing their validity.[2] We must remark that Ockham did not resort to O↑ or O↓ in justifying oblique syllogisms: the version of the dictum he employed in this connection contains no mention of supposition.

But the fact that Ockham did not use his rules for the generation of oblique syllogisms, need not stop us from doing so. The reason for this is that we are not primarily interested in his treatment of this kind of syllogisms, but in stressing the relative superiority of O↑ and O↓ to M↑ and M↓. Consider this Latin version of (c):

(d)

$$\frac{\text{Omnis homo es animal} \quad \text{Hominem aliquis necat}}{\text{Animal aliquis necat}}$$

We have a universal sentence giving the superior-inferior characterization of *homo* and *animal* . Furthermore, in the other premiss *hominem* occurs with *suppositio* non-distributiva. Thus, in accordance with O↑ the substitution of *animal* for *hominem* yields the given conclusion.

One might object to this demonstration, because the expression *homo* and not *hominem* is initially given in the superior-inferior relationship. But Ockham himself allows for the possibility of having the inferior expression in one of the oblique cases in the context of substitution and, as a result of that, also the superior after the substitution has taken place:

> 'Tamen aliquando consequentia valet, quia aliquando non possunt tales partes ordinari secundum superius et inferius nisi etiam tota extrema sic ordinentur vel possunt sic ordinari; sicut patet hic *homo albus-animal album*; *zidens hominem-zidens animal* '.. Ockham (1951 : 188-9).

4.5. OCKHAM'S RULES AND DE MORGAN'S EXAMPLE. De Morgan introduced (c) as the syllogistic expansion of his second inference. We have shown that Ockham knew of inference rules which may be used to justify that inference. Furthermore, we have pointed out that Ockham would recognize the validity of (c), although his justification might be different from ours.

However, we have not yet touched upon the question whether O↑ could be used equally well in reference to the original inference (b). Let the premisses below be given:

(1) Omnis equus est animal.

(2) Omnis cauda equi est cauda equi.

We have already seen what role universal sentences like (1) play. But in order to apply O↑ or O↓ to (2) we need to know which supposition belongs to the occurrences of *equi* therein. We certainly know that the complex expression *cauda equi* has *suppositio confusa et distributiva* in its first and *suppositio confusa tantum* in its second occurrence. But the question that interests us is the supposition of the expressions which make up the complex one. Ockham's answer is conclusive: neither *equi* or *cauda* has supposition in (2). *Suppositio*, he says, adheres to the extremes of a sentence and not to the expressions making up subjects or predicates:

> 'Solum categorema quod est extremum propositionis . . . supponit personaliter'. Ockham (1951 : 188).
>
> 'Per illam particulam *extremum propositionis* excluditur pars extremi, quantumcumque sit nomen et categorema. Sicut hic *homo albus est animal* nec *homo*, nec *albus* supponit set totum extremum supponit'. Ockham loc. cit.

Ockham did not make clear why we should deny any supposition to the elements of *cauda equi*. But we can try to understand his motivation, to some extent following Buridan's remarks on this question.[3] We take first the soundness of O↑ and O↓ as given. Therefore, if one of those rules assisted by certain assumptions yields an invalid inference, then we conclude that at least one of the assumptions has to be rejected. Consider now:

(3) Omnis asinus logici currit.

Suppose that both *asinus* and *logici* have supposition in (3). Then we have three possibilities: they have either *suppositio determinata* or *suppositio confusa tantum* or *suppositio confusa et distributiva*.

However, neither of them could have *suppositio determinata*, since both *asinus* and *logici* occur after *omnis* . But *logici* might have *supposition confusa tantum*. Thus, according to O↑ the following sequence would be valid:

$$\frac{\text{Omnis logicus est homo} \quad \text{Omnis asinus logici currit}}{\text{Omnis asinus hominis currit}}$$

But it is not. Let every donkey owned by a logician be running; let every logician be a man and let some donkey owned by a non-logician be resting. In such a situation both premisses are true and the conclusion is false. Therefore, *logici* cannot have *suppositio confusa tantum* in (10).

So, if *logici* has any supposition in (3), it must be *suppositio confusa et distributiva*. This latter supposition has to be attributed to *asinus* in that sentence because *omnis* is adjoined to it directly. Hence, the next inference would be valid according to O↓:

(e)

$$\frac{\text{Omnis asinus albus est asinus} \quad \text{Omnis asinus logici currit}}{\text{Omnis asinus albus logici currit}}$$

We accept this inference as valid. Ockham and Buridan, however, did not. Ockham introduced the convention that an affirmative sentence with an empty subject is false.[4]
From this point of view, we might describe a situation refuting the claim that (e) is valid. Let, again, every logician's donkey be running; let also white donkeys exist, all of which are wild. Then there is no white donkey belonging to a logician. Thus *asinus albus logici* is an empty expression and this makes the conclusion false, according to Ockham's convention. So (e) turns out to be invalid.

At this point, Ockham and Buridan might reject the supposition claim made on behalf of *asinus* or the convention on affirmative sentences. Clearly, both of them abandoned the supposition claim. So, *asinus* has no supposition in (3). The claim that *logici* has *suppositio confusa et distributiva* in (3) can be discredited in the same way. The conclusion is that neither *logici* nor *asinus* has supposition in *Omnis asinus logici currit*.

Strictly speaking, the preceding inference only shows that *logici* and *asinus* have no supposition when they make up the subject of a universal affirmative sentence. Consequently, in the view of Ockham, *cauda* and *equi* lack any supposition in their first occurrence in (3). But we do not know which kind of considerations brought Ockham to deny them any supposition in their second occurrence as well.

Assume that one attributes *suppositio confusa tantum* to the second occurrence of *equi* in (3). Applying O$\uparrow$ to (1) and (2) he would get as a conclusion *Omnis cauda equi est cauda animalis*. But the two-premisses inference:

$$\frac{\text{Omnis equus est animal} \quad \text{Omnis cauda equi est cauda equi}}{\text{Omnis cauda equi est cauda animalis}}$$

would be as far as Ockham would get. It was not open to him to treat (2) as a ladder that could be kicked away after having reached the conclusion, thus obtaining:

$$\frac{\text{Omnis equus est animal}}{\text{Omnis cauda equi est cauda animalis}}$$

And this certainly is not due to the lack of metalogical backing. There is not much anachronism in using a rule of Buridan which, to the modern reader, allows the elimination of tautological premisses:

> 'Ad quamcumque propositionem cum aliqua necessaria sibi apposita ... sequitur aliqua conclusio ad eadem propositionem solam sequitur eadem conclusio, sine appositione illius necessaria. . .'.
> Buridan (1976 : 36)

The real trouble is that (2), for Ockham nor for Buridan, can count as a necessary proposition: Ockham's convention allows it to be false.

4.6. REMARKS ON DE MORGAN AND OCKHAM. Ockham's convention throws new light on the background of the inference:

Every horse is an animal
_____
Every tail of a horse is a tail of an animal

the validity of which we take for granted, as De Morgan himself would have done. According to that convention, De Morgan's example has to be considered formally invalid, in the usual sense that there is an interpretation under which the premisses are true and the conclusion false. Let every horse be an animal; let no horse have a tail, then *tail of a horse* would be empty and, hereby, *every tail of a horse is a tail of a horse* would be false.

We see thus in which way De Morgan's rejection of the syllogistic fullness thesis could fail to impress a holder of it. If De Morgan's example is not formally valid, then the fact that it is not reducible to a syllogism cannot be regarded as proving the inadequacy of the thesis. This point was brought home to De Morgan by Mansel; this logician tried to prove that De Morgan's relative arguments are not formally valid. To do this he devised an inference, essentially similar to

Every guinea pig is an animal
_____
Every tail of a guinea pig is a tail of an animal

which had to play the counter-example role: given that guinea pigs do not have tails it follows, on Ockham's convention, that the premiss is true and the conclusion false.

In his reply to this criticism, De Morgan abandoned Ockham's convention. However, he did not adopt the modern interpretation of universal categorical sentences, making the conclusion of the last inference trivially true. He abandoned instead the principle of bivalence in reference to sentences containing empty expressions. On the schema *The tail of a S is P* he said

> A guinea pig, for instance, puts this proposition out of the pale of assertion, and equally out of that of denial; the tail of non tailed animal is beyond us. De Morgan (1966 : 252-3).

To sustain the validity of his non-monadic arguments, he seems to resort to a new view on formal validity:

> Again, let X be an existing animal, it follows that the tail of X is the tail of an animal. Is this consequence formal or material? Formal, because this is true whatever a tail may be, so long as there is a tail; and it cannot be refused assertion except when X has no tail." De Morgan loc. cit.

Mansel's criticism and De Morgan's answer show that relative expressions like *tail of a horse* , *killer of a man* , because of their possible emptiness, force a dilemma upon the syllogistic. In order to accept or deny the validity of arguments involving this kind of expression, some traditional principle will have to be abandoned. The way-out for Mansel and De Morgan was to deny the general validity of *Every A is A* . In contrast to this, the modern interpretation seems a more superficial revision of the traditional framework, for even if *A* is empty, the law of thought *Every A is A* is trivially true.

5. CONCLUDING REMARKS. In this chapter we have pursued the history of the lexical monotone rules. We have shown that several attempts at explaining the validity of non-syllogistic natural language inferences made use of substitution rules which revolve around monotonicity.

In a certain sense, the historical use of monotonicity can be seen as an extrapolation of the Dictum de Omni from the formal system of syllogistic schemas into natural language. Furthermore, we have pointed out that some formulations of those rules are not adequate. The reasons for this inadequacy have been localized in the defective characterization of the contexts in which the substitution must take place.

The reading of De Morgan and Leibniz, makes the prospect of a Natural Logic very gloomy indeed. The situation changes when one takes a look at the Medieval logicians. The best formulation of the rules has been found in the work of Ockham. We have shown that the soundness and fruitfulness of his approach is due to the underlaying *suppositio* theory. In terms of this theory monotone inferential occurrences are characterized. We notice also that Ockham restricts the scope of his theory by denying supposition to the parts of complex terms. In this sense, Ockham's marking mechanism does not go far enough. Nevertheless, our historical excursion shows that a theory of monotonicity marking is indispensable for Natural Logic. The complexity of the medieval *suppositio* theory, however, may suggest that pursuing such a marking theory is not a realistic goal. The exclusive concentration of the *suppositio* theory on the surface forms of sentences, makes a highly complicated enterprise of the monotonicity marking.

In the next chapter we shall see how this situation improved on the threshold of modern logic, especially in the work of C.S. Peirce. There we shall see that for formal languages there is an effective characterization of the substitution contexts on which the monotonicity rule can be applied. Later on, we shall also show that this characterization can be generalized to natural language contexts.

NOTES TO CHAPTER II

[1]We identify, as De Morgan did, ***man is animal*** with ***Every man is an animal***.

[2]Ockham (1954 : 351-2, 360, 365-7).

[3]Buridan (1976 : 98-9).

[4]'As far as presently known the first logician to consider the question of existential import or to propose a tenable theory of it was William of Ockham, who holds that the affirmative categorical propositions are false and the negative true when the subject term is empty'. Church (1965 : 420).

CHAPTER III

NATURAL LOGIC AND C.S.PEIRCE

DESCRIPTION OF THE CONTENTS OF THE CHAPTER. The first section introduces the theme of the chapter. The second section is concerned with Peirce's treatment of propositional logic. It introduces the propositional part of the *System of Existential Graphs* (SEG) -the so-called *Alpha Graphs*. Furthermore, it argues that some principles which Peirce employs here could be used in Natural Logic. The fourth section consists of a presentation of the predicate logic part of SEG -the so-called *Beta Graphs*. Again, the usefulness of Peirce's notions for Natural Logic is argued.

1. INTRODUCTION

1.1. PRELIMINARY REMARKS. In the previous chapters we pointed out that the notion of monotonicity is important for Natural Logic. We also showed that this notion was central to attempts of trespassing the syllogistic bounds. Moreover, we saw that a criterion for monotonicity was missing in De Morgan's and Leibniz' proposals. Ockham did have a criterion. However, we pointed out that the proverbial complexity of the *suppositio* theory renders Ockham's strategy infelicitous.

The difficulties facing the *suppositio* theory have been attributed to its exclusive concentration on the surface forms of sentences, rather than on their construction (cf. Geach 1962; Dummett 1973). As is well-known, modern logic pays close attention to the way in which formulas are constructed, and this is the way to recognize monotone occurrences.

In this chapter we introduce Peirce's treatment of formal inference in which monotonicity plays a central role: the *system of existential graphs*. In this proof system we discern a generalizable syntactic characterization of monotone occurrences. Although Peirce's use of monotonicity is confined to formal languages, we think that the *study* of Peirce's logic, is relevant for Natural Logic. It will enable us to identify some principles which, like monotonicity, are useful for Natural Logic.

2. THE SYSTEM OF EXISTENTIAL GRAPHS

2.1. GENERAL CHARACTERIZATION OF SEG. The SEG is set up with the intention of giving 'a satisfactory logical analysis of the reasoning in mathematics'. A part of this project consists of the construction of a language in which proofs and inference principles can be represented. The following passage by Peirce reminds us of Frege's project:

> What is requisite is to take really typical mathematical demonstrations, and state each of them in full, with perfect accuracy, so as not to skip any step, and then to state the principle of each step so as perfectly to define it, yet making this principle as general as possible... If we attempt to make the statement in ordinary language, success is practically impossible. . . At all times, the burden of language is felt severely, and leaves the mind with no energy for its main work. It is necessary to devise a system of expression for the purpose which shall be competent to express any proposition whatever without being embarrassed by its complexity, which shall be absolutely free from ambiguity, perfectly regular in its syntax, free from all disturbing suggestions, and come as nearer to a clear skeleton diagram of that element of the fact which is pertinent to the reasoning as possible.
> Peirce (1976. III. 406)[1]

2.2. SPECIAL PROPERTIES OF SEG. SEG is a proof system with special properties. In the first place, this system is based on a **non-linear** propositional language. In the second place, in the construction of the predicate logic part of the Existential Graphs, Peirce abandoned the quantification symbols. In this system we find what has been called a system of **implicit quantification**. This system represents the most radical implementation of Quine's analysis of quantification theory: it is the role of the variables and not the symbolizing of quantifiers which distinguishes predicate logic from earlier systems of logic. In the third place, the system embodies a few global inference rules which allow us to draw consequences from given premisses without having to resolve them into smaller parts. Roughly speaking, the effects of the rules are the following:

- within any context we may introduce c.q. eliminate double negations;
- within specified syntactic positions we are allowed to insert c.q. to delete (occurrences of) formulas;
- within certain syntactic configurations we are allowed to copy c.q. eliminate (occurrences of copied) formulas; and finally
- within certain syntactic configurations we are allowed to identify c.q. diversify argument places.

2.3. THE STUDY OF SEG. In the present chapter we hope to show the importance of the study of SEG. Peirce himself described SEG as his *chef d'oeuvre*, thus estimating SEG as being even more important than his earlier contributions to logic. This fact makes SEG already worth studying from a historical point of view. However, SEG itself constitutes the result of a historical development and in the next section we intend to point out some aspects of this process. Firstly, we shall survey the form which monotonicity takes in the so-called *Qualitative*

Logic. It will then become apparent that Peirce sought to reduce inference to simple substitution procedures - a goal reminding Jevon's program of mechanical inference. [2] Next we shall argue that the operations of deletion and insertion introduced in SEG rest on monotonicity.

2.4. MONOTONICITY RULES. In the *Qualitative Logic* Peirce describes a formal language and a proof system for propositional logic:

> 'So far, we have a language but still no algebra. For an algebra is a language with a code of formal rules for the transformation of expressions, by which we are enabled to draw conclusions without the trouble of attending to the meaning of the language we use.' Peirce (1976, IV, p. 107)

With regard to this system, Peirce says that inference must be seen as elimination and introduction of formulas:

> 'We require that the rules should enable us to dispense with all reasoning in our proofs except the mere substitution of particular expressions in general formulae.' Peirce (1976, IV, p. 108)

To achieve this ideal, Peirce takes Modus Ponens as a primitive inference rule and proceeds to generalize this rule as a general substitution rule:

> 'The general rule of substitution is that if $\neg a \vee b$, then b may be substituted for a under an even number of negations, while under an odd number a may be substituted for b.' Peirce (1976, IV, p. 108)

2.4.1. A FORMAL FORMULATION OF PEIRCE'S RULES. Peirce's general rule of substitution can be brought into formats similar to the substitution schemes introduced in Chapter II. 1.1.

UPWARD MONOTONICITY (M↑)

$$\frac{\neg\phi \vee \psi \quad \Gamma(\phi)}{\Gamma(\psi)}$$

provided that ϕ occurs in $\Gamma(\phi)$ within the scope of an even number of negations.

DOWNWARD MONOTONICITY (M↓)

$$\frac{\neg\phi \vee \psi \quad \Gamma(\psi)}{\Gamma(\phi)}$$

provided that ψ occurs in $\Gamma(\phi)$ within the scope of an odd number of negations.

2.4.1.1. EXAMPLES. Several well-known inferences rest on the monotonicity rules.[3]

(1) Modus Ponens is an instance of M↑:

$$\frac{\neg\phi \vee \psi \quad \phi}{\psi}$$

(2) Modus Tollens is an instance of M↓:

$$\frac{\neg\phi \vee \psi \quad \neg\psi}{\neg\phi}$$

(3) Transitivity is an instance of M↓ :

$$\frac{\neg\phi \vee \psi \quad \neg\psi \vee \chi}{\neg\phi \vee \chi}$$

2.4.2. REMARKS. It can be demonstrated that monotonicity alone does not account for all inferences in propositional reasoning.

(A) There are several inferences in which monotonicity is involved, but which require additional principles:
Monotonicity yields the following derivation:

$$\frac{\neg\psi \vee \chi \qquad \dfrac{\neg\phi \vee \chi \quad \psi \vee \phi}{\psi \vee \chi}}{\chi \vee \chi}$$

To derive χ from one $\chi \vee \chi$ one needs the additional (Boolean) identity $\chi \vee \chi = \chi$.

(B) There also are several elementary inferences in which monotonicity is not involved. The first inference which one may think of in this connection is the introduction of the conjunction:

$$\frac{\phi \quad \psi}{\phi \wedge \psi}$$

The above examples suggest that Peirce's substitution rules are to be augmented with various other types of principles to obtain a proof system as strong as ordinary propositional logic. Peirce himself was aware of the limitations of a purely monotone proof system -in the Existential Graphs extra principles are added to monotonicity. The system of inference rules thus obtained can be used as the basis for the axiomatization of propositional reasoning. (The first published proof of this assertion is to be found in Roberts, 1973).

This observation is relevant for our Natural Logic. It follows from our examples, that purely monotone Natural Logic will not systematize *natural* inference principles. But Peirce's work suggests to us ways of improving the strength of that logic - ways that are worth considering. In the next section we turn to the identification of Peirce's additional principles.

2.5. THE LANGUAGE OF THE ALPHA GRAPHS. In this section we introduce the alpha graphs. A more precise treatment of Peirce's system is given in Sánchez Valencia (1989). In this dissertation we shall concentrate on the most salient properties of the system.

The language of these graphs consist of:

(a) the space we use while writing sentences. Every part of this space is called the '*blank*'. The blank is interpreted as a symbol denoting the Truth.
The interpretation of all the parts of this space as a symbol for a special sentence allows Peirce to define an unrestricted deletion rule: if ϕ is a sentence then the result of deleting ϕ will still be a sentence, namely Truth.

(b) a symbol for negation ; negation is represented by us by means of a box surrounding the sentence to be negated. Since the blank is a sentence, we can negate it. The result is □. The meaning of this symbols is Falsity since it is the negation of Truth.

(c) Sentences which are written in the same space are considered as one sentence, namely their conjunction.

CONVENTIONS. The following conventions are introduced to enable the reader to understand Peirce's formulation of the inference rules:

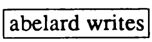
(d) The box will be called **sep** or **enclosure**.

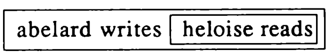
(e) The result of enclosing a sep is called a **double enclosure**.

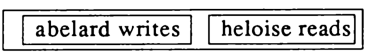
(f) sentences are called **graphs**.

Essentially, the language of the alpha graphs can be described as a propositional language augmented with a propositional constant, and based on the set of connectives { $\neg$, $\wedge$ }. Notice, for instance, that one sep is negation and a double enclosure is double negation.

Examples
(1) abelard writes heloise reads.

(2) 

(3) 

(4) 

The *'meaning'* of the above graphs is, respectively:

(1) Abelard writes $\wedge$ Heloise reads;

(2)$\neg$ (Abelard writes);

(3) $\neg$(Abelard writes $\wedge$ $\neg$(Heloise reads)), i.e. Abelard writes $\rightarrow$ Heloise reads;

(4) $\neg$($\neg$(Abelard writes) $\wedge$ $\neg$(Heloise reads)), i.e. Abelard writes $\vee$ Heloise reads;

We see from these examples that the alpha language is translatable into propositional logic. The translation goes also the other way around. But for our present purposes this is not quite relevant. See, however, appendix 2.

2.6. As we pointed out, Peirce formulates a few global inference rules which, supplemented with the blank (or T as symbol for verum) as an axiom, yield the full propositional logic. The way in which Peirce introduces the rules is the following:

2.6.1 DELETION AND INSERTION

> 'Within an even finite number (including none) of seps, any graph may be erased; within an odd number any graph may be inserted.'Peirce (4 : 492).

2.6.2. COPYING RULE

> 'Any graph may be iterated within the same or additional seps, or if iterated, a replica may be erased, if the erasure leaves another outside the same or additional seps.' Peirce (4 : 492).

2.6.3. DOUBLE NEGATION RULE

> 'Anything can have double enclosures added or taken away, provided there be nothing within one enclosure but outside the other.'Peirce (4 : 379).

Thereafter, we shall refer to 2.6.1/2.6.3 as 'the alpha rules'

2.6.4. EXAMPLES. We shall work out some concrete examples. For convenience, we assume that the relation of deducibility ($\Rightarrow$) has been defined.

(1) abelard writes heloise reads $\Rightarrow$ abelard writes
Proof
Apply the deletion rule to the graph *heloise reads*.

(2) abelard writes ⇒ [[abelard writes] [heloise reads]]

Proof

Apply the double negation rule to the graph *abelard writes*.This gives [[abelard writes]].
Use the insertion rule to introduce [heloise reads].

The result is [[abelard writes] [heloise reads]].

(3) [[abelard writes] [abelard writes]] ⇒ abelard writes

Proof

Apply the copy rule to the two occurrences of the graph [abelard writes] in the graph [[abelard writes] [abelard writes]]. The result is [[abelard writes]]. Finally, one application of the double negation rule yields *abelard writes*.

(4) [[abelard writes] [abelard writes]] ⇔ abelard writes

Proof

This follows form (2) and (3).

2.6.5. REMARK. The previous examples show that the alpha rules allow us to generate the one-premiss propositional inferences mentioned in 2.5. We have here a hint of the strength of the system. There is, however, a small subtlety. The easy monotone inferences become relatively complex ones. It requires several steps to prove:

abelard writes [abelard writes [heloise reads]] ⇒heloise reads

Proof

1. abelard writes [abelard writes [heloise reads]] ⇒ (copy rule)

2. abelard writes [[heloise reads]] ⇒ (deletion rule)

3. [[heloise reads]] ⇒ (double negation rule)

4. heloise reads

One should not think that the complexity of modus ponens indicates that the system is not based on monotonicity. In the next section we shall be concerned with the logical analysis of the global logical principles embodied in SEG. We claim that deletion and insertion can be seen as restricted cases of the monotonicity rules schematized in 2.5. We shall also argue that the real new principles added to the monotonicity rules are algebraic principles and the copying rule -a special case of the principle of conservativity.

3.1. THE ALPHA RULES AND MONOTONICITY. We are going to show that the deletion and insertion rules are restricted forms of monotonicity rules. We assume in our discussion that logical principles are invariant under changes of notation. In the discussion we use a propositional language based on the constants $\{ T, \neg, \wedge \}$. We assume also that the other constants have been defined as usual. Moreover, if ϕ occurs in $\Gamma(\phi)$ in the scope of an even (odd) number of negations, we will say that ϕ occurs positively (negatively) in $\Gamma(\phi)$.

(A) *Deletion*. Notice that the deletion of the sentence ϕ leaves the blank. But the blank is interpreted as T(ruth). If we abstain from the special notation of the alpha graphs, then the proper way of looking at deletion is the following:

'any formula that occurs positively may be replaced by T'.

We can schematize this rule in the following way

$$\frac{\Gamma(\phi)}{\Gamma(T)}$$

provided that ϕ occurs positively in $\Gamma(\phi)$.

Let us see what we need to add to this rule to generate $\phi \wedge \psi \Rightarrow \phi$. Application of deletion to ψ gives $\phi \wedge T$. By using the Boolean identity $\phi \wedge T = \phi$, one obtains ϕ.

(B) *Insertion*. Notice that the insertion of the sentence ϕ, replaces the blank. Thus, the proper way of looking at insertion is this:

'any negative occurrence of T may be replaced by an arbitrary formula ϕ'.

We can schematize this rule in the following way

$$\frac{\Gamma(T)}{\Gamma(\phi)}$$

provided that T occurs negatively in $\Gamma(T)$

Let us see what becomes of the inference $\psi \Rightarrow \phi \vee \psi$. Application of double negation to ψ yields $\neg\neg\, \psi$. By using the Boolean identity $T \neg\, \psi \Leftrightarrow \neg\psi$ one obtains $\neg(T \neg\, \psi)$. By inserting $\neg\, \phi$, we have $\neg(\neg\, \phi \neg\, \psi)$. But then, by definition, we have derived $\phi \vee \psi$.

(C) *Double negation*. We have seen that deletion and insertion should be considered as restricted monotonicity rules, in the sense that the implication premiss of the rules is restricted to formulas of the form $\phi \rightarrow T$. To do now what we were able to do with the original rules, we add to monotonicity the Boolean identities:

$$\text{(a) } \phi \wedge T \Leftrightarrow \phi; \quad \text{(b) } \phi \Leftrightarrow \neg\neg \phi$$

Double negation was already part of the Alpha graphs. The need for (a) is not felt in the alpha graphs. This is due to the special features of the language in which the system is formulated. Observe, also, that we assume the principle of unrestricted substitution of equivalents.

(D) *Copy.* Now we turn on to the copy rule. We schematize this rule as follows

$$\phi \wedge \Gamma(\psi) \Leftrightarrow \phi \wedge \Gamma(\phi \wedge \psi)$$

This rule is not completely independent from monotonicity. There is some redundancy in the alpha rules (Peirce himself noticed this redundancy):

(1) Suppose one has a formula of the form $\phi \wedge \Psi(\psi)$. By Peirce's copy rule one obtains at once $\phi \wedge \Psi(\phi \wedge \psi)$. But suppose that ψ occurs negatively in $\Psi(\psi)$. We first replace ψ by $T \wedge \psi$, and then insert ϕ . This is permissible since the occurrence of T herein satisfies the condition for the insertion rule. Thus, in this case the copy rule is dispensable.

(2) Suppose one has a formula of the form $\phi \wedge \Psi(\phi \wedge \psi)$. By Peirce's copy rule one obtains at once $\phi \wedge \Psi(\psi)$. But suppose that $\phi \wedge \psi$ occurs positively in $\Psi(\psi)$. Then this occurrence of ϕ must be positive. We first replace ϕ by T . This yields $T \wedge \psi$. Hence, we may introduce now the equivalent formula ψ .

Suppose that ψ occurs positively in $\Psi(\psi)$. As above, we can derive $\phi \wedge \Psi(T \wedge \psi)$. But the occurrence of T therein does not satisfy the condition on the insertion rule. Hence, monotonicity is not applicable. This is a situation in which the copy rule does something monotonicity alone cannot do. For instance, monotonicity cannot prove $\phi \Rightarrow \phi \wedge \phi$. But the copy rule can. Similarly, we can argue that if $\phi \wedge \psi$ is negative in $\phi \wedge \Psi(\phi \wedge \psi)$, monotonicity is helpless, while the copy rule yields at once $\phi \wedge \Psi(\psi)$. For instance, monotonicity alone cannot prove $\phi \wedge \neg(\phi \wedge \psi) \Rightarrow \phi \wedge \neg\psi$, but the copy rule can.

Of course, the copy rule is not indispensable in an absolute sense. One only needs to think of the several axiomatizations of propositional logic (or Boolean algebra) in which the copy rule is not used. The above observations are only meant to indicate the relative indispensability of the copy rule *vis-a-vis* monotonicity.

We have been discussing the copy rule, because we want to assess the importance of adding principles analogous to the copy rule to a monotone logic. This is important for Natural Logic because it relies mainly on monotonicity. To illustrate the extra inferential strength of monotonicity plus the copy rule, we want to point out that the combined work of these rules allows us to prove:

(c) $\phi \wedge \neg \phi \Leftrightarrow \neg T$

(d) $\phi \wedge \phi \Leftrightarrow \phi$

(e) $\phi \wedge \psi \Leftrightarrow \psi \wedge \phi$

(f) $\phi \wedge (\psi \vee \chi) \Leftrightarrow (\phi \wedge \psi) \vee (\phi \wedge \chi)$

3.2. DIGRESSION. The proof of (f) rests on (e), monotonicity and the copy rule. The proof of (e) rests also on the copy rule, monotonicity and a notational trick. If we define the copy rule along the line: $\phi \wedge \Psi(\psi) / \phi \wedge \Psi(\psi \wedge \phi)$, the proof of (e) is:

$$\phi \wedge \psi \Rightarrow \text{[copy rule]}\ \phi \wedge (\psi \wedge \phi) \Rightarrow \text{[deletion]}\ \psi \wedge \phi .$$

If we choose to define the rule in such a way that the relative ordering of the formulas remains the same: $\phi \wedge \Psi(\psi) / \phi \wedge \Psi(\phi \wedge \psi)$, then there is no way of deriving commutativity from deletion and the copy rule.

Thus, one can define the copy rules in such a way that some *algebraic* principles become easily derivable. For practical purposes this is a legitimate strategy. However, one cannot say that the principles thus obtained have been analyzed. On the contrary. The independent character of the principles thus obtained is obscured.

3.3. THE COPY RULE AND CONSERVATIVITY. In section 3.1 we have seen that there are uses of the copy rule which are independent of monotonicity. In this section we want to show that the copy rule corresponds to a property of the implication known as conservativity. By showing this correspondence, we want to make plausible that the copy rule - in some form - can be incorporated into natural logic, because conservativity is a property shared by one of the most important logical items of the natural language: the determiners. In this section we investigate the relationship between conservativity and the copy rule.

The conservativity of the implication is captured in the following schema:

$$\text{(a) } \phi \rightarrow \psi \Leftrightarrow \phi \rightarrow (\phi \wedge \psi)$$

In standard logic one derives from the above schema the following schema for the conjunction:

$$\text{(b) } \phi \wedge \neg(\phi \wedge \psi) \Leftrightarrow \phi \wedge \neg\psi$$

The conservative schema (b) shall be called the one step copy rule.

ASSERTION 1

The one-step copy rule, the insertion rule and double negation imply the copy rule.

Proof

We shall prove one special case which contains all the necessary information.

First case: $\phi \wedge \psi \Leftrightarrow \phi \wedge (\phi \wedge \psi)$

$\phi \wedge \psi \Leftrightarrow \phi \wedge \neg\neg \psi$	Double negation
$\Leftrightarrow \phi \wedge \neg (\phi \wedge \neg \psi)$	Insertion
$\Leftrightarrow \phi \wedge \neg (\phi \wedge \neg (\phi \wedge \psi))$	One step copy applied on $\phi \wedge \neg \psi$
$\Leftrightarrow \phi \wedge \neg \neg (\phi \wedge \psi)$	One step copy applied on $\phi \wedge \neg (\phi \wedge \neg (\phi \wedge \psi))$
$\Leftrightarrow \phi \wedge (\phi \wedge \psi)$	Double negation

3.4. CONCLUDING REMARKS ON THE ALPHA GRAPHS. The previous sections demonstrate the importance of Peirce's work for Natural Logic. In the first place we have seen that it is possible to give a syntactic characterization of monotone sensitive positions. In the second place, we have seen that the alpha graphs are stronger than a pure monotone calculus by the presence of the copy rules. We have also established a connection between conservativity and the copy rules. As we remarked above, this connection is encouraging for Natural Logic. There are lexical items which are conservative. For instance the semantical behaviour of *all* determiners is such that $[\![\text{Det X}]\!]$ Y is the same object as $[\![\text{Det X}]\!]$ $Y \wedge X$. This semantical information is available. In the previous section we have seen that in the propositional case, the addition of conservativity to monotonicity yields a stronger proof system. We are entitled to hope that the addition of conservativity to Natural Logic will result in a more realistic proof system.

Actually, Peirce's contribution to the theory of quantification is as important as his contribution to propositional reasoning. One natural question would be: can Natural Logic learn something from Peirce's treatment of predicate logic? We turn to this question in the next section.

4. THE BETA GRAPHS.

4.1. PEIRCE AND THE THEORY OF QUANTIFICATION. What is called *theory of quantification* is the result of a development involving Peirce's logical work, in particular Peirce (1885). In one unpublished addendum to Peirce (1885), we found new principles governing the behaviour of quantifiers in deductions (Peirce, 3 : 403 E). These principles, which regulate identifications and diversifications of variables, are the following:

- Identification of variables.
  1. $\forall x \forall y\ \phi\ (x,y) \rightarrow \forall x \phi\ (x,x)$;
  2. $\exists x \forall y\ \phi\ (x,y) \rightarrow \exists x \phi\ (x,x)$.

- Diversification of variables.
  3. $\forall x \phi\ (x,x) \rightarrow \forall x \exists y \phi\ (x,y)$;
  4. $\exists x \phi\ (x,x) \rightarrow \exists y \exists x \phi\ (x,y)$.

In the next section we shall see that these principles were incorporated into the predicate logic part of the SEG -the *Beta Graphs*. We claim that the treatment of diversification and identification of variables to be found in the Beta Graphs, rests on monotonicity. Moreover, we think that the way in which this is done is relevant for Natural Logic, because it suggests analogous operations on the natural language counterparts of variables: pronouns.

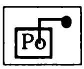
In the next section we give a general description of the Beta Graphs. Next we turn to a discussion of Peirce's principles of quantification. For convenience, we formulate the rules in a standard language. Finally, we present a tentative assessment of the relevance of the Beta principles for Natural Logic.

4.2. THE BETA GRAPHS. The vocabulary of the beta graphs contains the parentheses, n-ary predicate letters and a quantifier symbol: ____ , the quantifier line. This line is a protean element in the system. In the first place it is an atomic graph interpreted as saying that some object exists. In the second place it is also used to show that argument places are no longer empty. In this section we will try to keep these three functions of the line as separate as possible. For the line as a sentence we will use a simple horizontal line; for the line entering one argument place we will use a vertical line; to represent the quantifier line in its identifying role we will use a kind of vine diagram, writing the line as the branch from which joined graphs hang down.

The quantifier line standing alone means that some indeterminate object exists. The line keeps to some extent this existential import when attached to argument places:

> 'Thus the interpretation of (___is beautiful) to mean "Something is beautiful" is decidedly the more appropriate.' Peirce (4 : 440)

The combination of the above reading of the quantifier line with the interpretation of the alpha graphs makes universal sentences expressible. For instance, the following expression means that P holds of all objects:



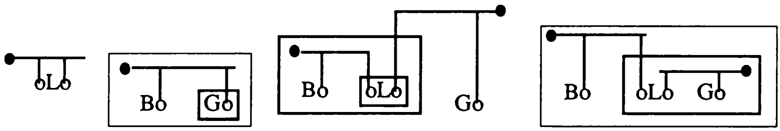
Peirce expresses these facts by saying that a line occurring positively has existential and a line occurring negatively has universal import:

> ' . . . and any line of identity whose outermost part is evenly enclosed refers to something, and any whose outermost part is oddly enclosed refers to anything there may be.' Peirce (4 : 457)

To read more complex graphs we introduce the convention that the least negated expression has to be read first. The general idea is that in the interpretation of a given line, we have to interpret all the graphs we find in our way to the end of the line- giving priority to horizontal lines above other graphs when those occur in the scope of the same negations:

> 'And the interpretation must be given outside of all seps and proceed inward.'
> Peirce (4 : 457)

4.3. EXAMPLES. With L for *loves* , B for *boy* and G for *girl* , the reading of the following graphs is, respectively, *Somebody loves himself* , *Every boy is a girl* , *Some girl is loved by every boy* , and *Every boy loves some girl* :



4.4.CONVENTION. The following convention is introduced to enable the reader to understand Peirce's formulation of the quantification rules:

- The quantifier line is called the *line of identity* or *ligature*

- The space we use when writing sentences of this language is called *the sheet of assertion.*

4.5.THE QUANTIFICATION RULES. the quantification rules are introduced in the following manner.

4.5.1. SCOPE

> 'Permission No 3. is to be understood as permitting the extension of a line of identity on the sheet of assertion to any onoccupied part of the sheet of assertion.'
> Peirce (4 : 417)

4.5.2. DIVERSIFICATION OF ARGUMENT PLACES

> 'This rule permits any ligature, where evenly enclosed, to be severed'
> Peirce (4 : 505)

4.5.3. IDENTIFICATION OF ARGUMENT PLACES

> 'This rule permits . . . any two ligatures, oddly enclosed in the same seps, to be joined.'Peirce (4 : 505)

4.5.4. COPY RULE

> 'The rule of iteration must now be amended as follows:
> Rule 4 (amended). Anything can be iterated under the same enclosures or under additional ones, its identical connections remaining identical.' Peirce (14 : 386)

From now on, we shall refer to those rules as *the Beta Rules*. In appendix 2 we collect some examples of derivations in the Beta Graphs.

4.6. In this section we shall concentrate on the logical points of Peirce's monotone treatment of predicate logic inference. In appendix 2 we give a fair description of the Beta Graphs. In the main text we shall not use the notation of the beta graphs but resort instead to the standard notation of predicate logic.

4.6.1. THE LANGUAGE. To obtain a system similar to the Beta Graphs, one takes a formulation of predicate logic based on $\{\exists, \wedge, \neg\}$ and state:

(i) If ϕ is a sentence, then $\exists x\phi$ is a sentence defined as $\exists x(x=x) \wedge \phi$.

(ii) If ϕ is a sentence, then ϕ is called a **beta graph**.

4.6.2. PEIRCE'S SYSTEM OF RULES. We formulate now the inference rules introduced by Peirce. The rules do not apply to all the formulas of the language, but only the to beta graphs. These rules can be divided into two parts: propositional and quantificational rules. For convenience we repeat:

THE PROPOSITIONAL PRINCIPLES

(i) $\neg\neg\phi \Leftrightarrow \phi$

(ii) $\phi \wedge T \Leftrightarrow \phi$

(iii) $\phi \wedge \Psi(\phi \wedge \psi) \Leftrightarrow \phi \wedge \Psi(\psi)$, provided that no free variable of ϕ is bound in $\Psi(\phi \wedge \psi)$.

(iv) $$\frac{\Gamma(\phi)}{\Gamma(T)}$$

provided that ϕ occurs positively in $\Gamma(\phi)$

(v) $$\frac{\Gamma(T)}{\Gamma(\phi)}$$

provided that T occurs negatively in $\Gamma(T)$

To these principles we add :

THE PRINCIPLES OF QUANTIFICATION

(vii) ELIMINATION OF VACUOUS QUANTIFICATION.

$$\exists x(x=x) \wedge \phi \Leftrightarrow \phi$$

(viii) ALPHABETIC VARIANTS.

$$\exists x\phi \Leftrightarrow \exists y[y/x]\ \phi$$

In the following rules we define ϕ' by: $\phi' \equiv [y/x]\ \phi$

(ix) PASSAGE RULE.

$$\exists x\phi \wedge \exists y\psi \Leftrightarrow \exists x(\phi \wedge \exists y\psi).$$

(x) DIVERSIFICATION OF VARIABLES.

$$\frac{\Gamma(\phi)}{\Gamma(\exists x\phi')}$$

provided that ϕ occurs positively in $\Gamma(\phi)$.

(xi) IDENTIFICATION OF VARIABLES.

$$\frac{\Gamma(\exists x\phi)}{\Gamma(\phi')}$$

provided that $\exists x\phi$ occurs negatively in $\Gamma(\phi)$, and $\exists x\phi$ occurs within the scope of an occurrence of $\exists y$.

We can also derive diversification and identification rules in which the universal quantifier is introduced or eliminated:

(xii) DIVERSIFICATION OF VARIABLES.

$$\frac{\Gamma(\phi)}{\Gamma(\forall x\phi')}$$

provided that ϕ occurs negatively in $\Gamma(\phi)$.

(xiii) IDENTIFICATION OF VARIABLES.

$$\frac{\Gamma(\forall x\phi)}{\Gamma(\phi')}$$

provided that $\forall x\phi$ occurs positively in $\Gamma(\phi)$,
and $\forall x\phi$ occurs within the scope of an occurrence of $\exists y$ or $\forall y$.

4.6.3. EXAMPLES. Before commenting on the rules, let us give some examples of the way they work. To save space, we assume the passage rule for the universal quantifier.

(1) $\forall x(\phi \rightarrow \psi) \Rightarrow \forall x\phi \rightarrow \forall x\psi$

Proof (observe that ϕ is negative in $\forall x(\phi \rightarrow \psi)$: bring the formula into primitive notation)

$\forall x(\phi \rightarrow \psi) \Rightarrow$ (diversification)

$\forall x(\forall y[y/x]\phi \rightarrow \psi) \Rightarrow$ (passage)

$\forall y[y/x]\phi \rightarrow \forall x\psi \Rightarrow$ (alphabetic variant)

$\forall x\phi \rightarrow \forall x\psi$

(2) $\forall y \forall x\ \phi \Rightarrow \forall x[x/y]\phi$

$\forall y \forall x\ \phi \Rightarrow$ (identification)

$\forall y[x/y]\ \phi$

(3) $\exists x\ (\phi \wedge \psi) \Rightarrow \exists x\phi \wedge \exists x\psi$

$\exists x\ (\phi \wedge \psi) \Rightarrow$ (diversification)

$\exists x(\phi \wedge \exists y[y/x]\psi) \Rightarrow$ (passage + alphabetic variant)

$\exists x\phi \wedge \exists x\psi$

(4) $\exists y(Girl(y) \wedge \forall x\ (Boy(x) \rightarrow Loves(x,y)) \Rightarrow \forall x(Boy(x) \rightarrow \exists y\ (Girl(y) \wedge Loves(x,y)))$

Proof

$\exists y(Girl(y) \wedge \forall x\ (Boy(x) \rightarrow Loves(x,y)) \Rightarrow$ (copy rule)

$\exists y(Girl(y) \wedge \forall x\ (Boy(x) \rightarrow (Girl(y) \wedge Loves(x,y))) \Rightarrow$ (deletion)

$\exists y(\forall x(Boy(x) \rightarrow (Girl(y) \wedge Loves(x,y))) \Rightarrow$ (diversification)

$\exists y(\forall x\ (Boy(x) \rightarrow \exists z(Girl(z) \wedge Loves(x,z))) \Rightarrow$ (vacuous quantification)

$\forall x\ (Boy(x) \rightarrow \exists z(Girl(z) \wedge Loves(x,z))$

4.7. IDENTIFICATION, DIVERSIFICATION AND MONOTONICITY. Reflecting on the validity of the identification and diversification rules, one sees that they are monotone uses of existential generalization: $\phi \rightarrow \exists x[x/y]\phi$. Peirce's rules are instances of the monotonicity rules -instances in which a tautologous premiss does not show up. The rules (x) and (xii) are sound, because (a) and (b) below are:

(a)

$$\frac{\phi \rightarrow \exists x\phi' \quad \Gamma(\phi)}{\Gamma(\exists x\phi')}$$

provided that ϕ occurs positively in $\Gamma(\phi)$, $\phi' = [x/y]\ \phi$.

(b)

$$\frac{\phi' \rightarrow \exists x\phi \quad \Gamma(\exists x\phi)}{\Gamma(\phi')}$$

provided that $\exists x\phi$ occurs negatively in $\Gamma(\phi)$,
$\phi' = [y/x]\,\phi$ and $\exists x\phi$ occurs within the scope of an occurrence of $\exists y$.
This observation is meant to sustain our claim that Peirce's treatment of quantification in the Beta Graphs rests on monotonicity. Monotonicity is a prominent part of Natural Logic, thus we are entitled to believe that Peirce's operations could be made available for a Natural Logic.

4.8. IDENTIFICATION AND DIVERSIFICATION IN NATURAL LANGUAGE. The quantification rules (vii) -(ix) are combinatorial devices without evident relevance for Natural Logic. But the operations of diversification are another matter. We comment here on some examples that suggest that Peirce's operations are relevant for the study of natural language inference. We shall assume that Peirce's operations are defined for a system of restricted quantification. Restricted quantification is more like natural language quantification: full NP's and not determiners are involved in the latter. Thus, from this point of view, example (2) takes the form:

$$\forall_{x \in Z} \forall_{y \in Z} \phi \rightarrow \forall_{x \in Z} [x/y]\phi$$

In the examples we assume that the bold expressions are bound together. Consider first the following sentences:
(1) **every man** knows that mary loves **him**
(2) **some man** knows that mary loves **him**
(3) **some man** loves **himself**
(4) **every man** loves **himself**

There is a theory of monotonicity marking according to which the occurrences of the pronouns in those sentences are positive (see Chapter V). By using Peirce's diversification operation, we should have as a result:
(1) every man knows that mary loves some one.
(2) some man knows that mary loves some one.
(3) some man loves some one.
(4) every man loves some one.
And this seems to be correct.
Consider the following sentences:
(9) **no man** knows that mary loves **him**
(10) **some man** doesn't know that mary loves **him**
(11) **some man** doesn't love **himself**
(12) **no man** loves **himself**

According to the theory of monotonicity marking we mentioned, the pronouns occur negatively in those sentences. Hence, Peirce's rule does not allow us to infer:

(13) no man knows that mary loves some one
(14) some man doesn't know that mary loves some one
(15) some man doesn't love some one
(16) no man loves some one
And again this seems to be correct.

On the other hand, if *some one* occurs negatively in (13)-(14), then they imply (9)-(12). This again seems to be correct.

These examples suggest that incorporating Peirce's diversification and identification principles to Natural Logic can only improve it. Whether this is possible or not, remains to be seen. The above examples indicate that monotonicity marking augmented with diversification and identification could be combined in a profitable way.

Let us consider some more examples. *Some man loves some man* does not imply *Some man loves himself* . But *Every man loves every man* implies *Every man loves himself* . Notice that Peirce's rules combined with monotonicity marking tells us why this should be the case: the second occurrence of *Every man* is positive in *Every man loves every man* . Furthermore, it occurs in the scope of other occurrence of *Every man* . This situation is analogous to the structural conditions mentioned in (xiii). Thus we identify the argument places to which these two occurrences of *Every man* are connected. On the other hand, the second occurrence of *Some man* in *Some man loves some man* does not satisfy the structural conditions mentioned in (x): it does not occur in a negative position.

Incidentally, observe that one can not derive *Himself loves every man* although the eliminated occurrence of *every man* is positive -this sentence is not even well-formed. But, again, this occurrence does not satisfy the structural condition mentioned in (xiii): it does not occur in the scope of other occurrence of *every man* .

5. CONCLUDING REMARKS ON THE BETA GRAPHS. In this part of the chapter we have studied Peirce's treatment of logic in his system of Beta Graphs. We have pointed out that his operations of diversification and identification of variables rest on an appropriate characterization of the syntactic positions in which this can be done. In standard terms these operations revolve around appropriate monotone substitutions. We have also shown that we are entitled to believe that Peirce's operations can be used in the treatment of natural language anaphora in inferences. Essential to these operations is the information whether the relevant items occur positively or negatively. Thus, one needs a theory of grammatical form in which this information can be encoded.

6. THE ARCHITECTURE OF PEIRCE'S SYSTEM. Our analysis of the syllogistic used the semantic principle of monotonicity and the algebraic property of conversion. The study of Peirce's

System of Existential Graphs reveals a finer composition of inference principles. The SEG appears to be a system based on the following principles:

- Monotonicity -restricted to the case in which True is introduced into negative contexts and eliminated from positive ones
- The copy rule -proven by us to be a combination of monotonicity and conservativity
- Algebraic principles - like double negation and commutativity
- The identification and diversification of bound variables.

The architecture of this system is interesting for Natural Logic because it coincides with current views on the semantics of natural language. The principles of Conservativity and Monotonicity are central to the study of natural language quantification. In Chapter VI we shall show that the addition of Conservativity to Natural Logic has an effect similar to the one in Peirce's system: the inferential power increases. Furthermore, the variables in Peirce's system demand an antecedent quantifier. In this sense they resemble the kind of pronouns which demand an antecedent Noun Phrase. Moreover, Peirce's operations on variables suggest a Natural Logic mechanism for the handling of typical first order inferences.

NOTES TO CHAPTER III

[1] References to Peirce (1931-38) will follow the convention of referring first to the volume, and then to the paragraph. Similarly, references to Peirce (1976) will follow the convention of referring first to the series, then to the volume, and finally to the page number.

[2]There remains an aspect of SEG that we will not consider in the present chapter, although this aspect renders SEG an interesting system for modern readers. Sowa (1984), a book devoted to cognitive science and artificial intelligence, claims that SEG is more adequate than standard systems for the representation of knowledge. In particular, Sowa takes SEG as the logical base for the construction of a theory of *conceptual graphs* , from the perspective of artificial intelligence research. Readers interested in this aspect of SEG are referred to Sowa's own work.

[3]For a more general discussion of the formal properties of these monotonicity rules, we refer the reader to Chapter V.

CHAPTER IV

LAMBEK CALCULUS

DESCRIPTION OF THE CONTENTS OF THE CHAPTER. In section 1 we describe the theme of this chapter. In section 2 we describe the language of the so-called *'non-directional Lambek Calculus'* (LP). In section 3 we describe an implicational system of Natural Deduction (ND). In section 4 we motivate the restrictions that imposed on ND yield LP as one of its sub-systems. In section 5 we introduce the typed lambda calculus. In section 6 we introduce the so-called *Lambek terms* - to be connected with derivations in LP. In section 7 we give an explicit proof of the correspondence between derivations in LP and Lambek terms. In section 8 we show that *normalisation* is provable for LP. Finally, in section 9 we introduce a variant to LP, the *Lambek Grammar*.

1. INTRODUCTION

1.1. NATURAL LOGIC AND CATEGORIAL GRAMMAR. A salient feature of modern logic is the precise definition of vehicles of inference: modern logic gives rise to a rigourous theory of logical form. Natural Logic should learn from this -we have already seen that Natural Logic without a coherent syntactical theory fails.

In principle, several linguistic theories can function as a syntactical basis for Natural Logic. But we are convinced that the categorical approach is most adequate because it is a theory in which syntax and semantics are integrated. We believe that a grammar based on the calculus of *category combination* described in Lambek (1958) qualifies as linguistic basis for Natural Logic.

1.2. CLASSICAL CATEGORIAL GRAMMAR. The Lambek systems of category combination are extensions of classical CG. Classical CG is a language recognition device first described in Ajdukiewicz (1935). In its original form, the device works as follows. We have at our disposal primitive categories and a category functor, '/', the so-called right looking functor. Complex categories are built from two basic categories e (proper names) and t (sentences) and a recursive procedure:

if β and α are categories, then so is (β/α) .

The recognition work starts with assigning each word to a category. To determine the category of a complex expression we first write the categories of its elements. After this, we read the string of categories from left to right. When we find the first sub-string of the form $(\beta/\alpha)\alpha$ we replace this sub-string by β . This yields a new string of categories. We then rescan the new string, applying the reduction rule wherever possible. The complex expression belongs to the category γ only if successive applications of the procedure finally lead to the string γ .

Categorial Grammar was further developed in Bar-Hillel (1953). In this paper a new functor is introduced - the so-called *left looking* functor. The associated formation rule says that

if β and α are categories, then so is $(\beta\backslash\alpha)$.

The cancellation procedure is also enriched with a rule which allows the substitution of β for any string of the form $\alpha(\alpha\backslash\beta)$.

1.2.1. EXAMPLE OF CATEGORY ASSIGNMENTS.

Parts of sentences	category
Proper Names	e
Intransitive Verbs	e\t
Transitive Verbs	(e\t)/e
Noun Phrases	t/(e\t)
Adverbs	(e\t)\(e\t)

1.2.2. EXAMPLES OF RECOGNITIONS.

1. *Abelard cries* belongs to t :

$$\begin{array}{cc} \text{Abelard} & \text{cries} \\ \text{e} & \text{e\\t} \\ \hline \multicolumn{2}{c}{\text{t}} \end{array} \quad \text{result first scanning}$$

2. *Abelard cries bitterly* belongs to t :

$$\begin{array}{lll} \text{Abelard} & \text{cries} & \text{bitterly} \\ \text{e} & \underline{\text{e\\t} \quad \text{(e\\t)\\(e\\t)}} & \\ \text{e} & \text{e\\t} & \text{result first scanning} \\ \hline \text{t} & & \text{result second scanning} \end{array}$$

3. *Every man likes Abelard* belongs to the category t :

$$\begin{array}{lll} \text{Every man} & \text{likes} & \text{Abelard} \\ \text{t/(e\\t)} & \underline{\text{(e\\t)/e} \qquad \text{e}} & \\ \text{t/(e\\t)} & \text{(e\\t)} & \text{result first scanning} \\ \hline \text{t} & & \text{result second scanning} \end{array}$$

The reader can find more details about CG in Buszkowski et al. (1988), Oerhle et al. (1988).

1.2.3. LAMBEK CATEGORIAL GRAMMAR. The recognition pattern of the previous examples resembles Natural Deduction trees: the substitution of adjacent categories can be seen as the result of application of Modus Ponens. The analogy with the logical calculus was completed in Lambek (1958). Lambek constructed a calculus of *sequents* of the form

$$\alpha_1, \ldots, \alpha_n \Rightarrow \beta$$

meaning that $\alpha_1, \ldots, \alpha_n$ reduces to β.

One formulation of the rules and axioms for the Lambek Calculus (L) is the following

(1) $\alpha \Rightarrow \alpha$

(2) $(\alpha\beta)\gamma \Rightarrow \alpha(\beta\gamma)$ $\qquad \alpha(\beta\gamma) \Rightarrow (\alpha\beta)\gamma$

(3) $\dfrac{\gamma\alpha \Rightarrow \beta}{\gamma \Rightarrow \beta/\alpha}$ $\qquad \dfrac{\alpha\gamma \Rightarrow \beta}{\gamma \Rightarrow \alpha\backslash\beta}$

(4) $\dfrac{\gamma \Rightarrow \beta/\alpha}{\gamma\alpha \Rightarrow \beta}$ $\qquad \dfrac{\gamma \Rightarrow \alpha\backslash\beta}{\alpha\gamma \Rightarrow \beta}$

(4) $\dfrac{\alpha \Rightarrow \beta \quad \beta \Rightarrow \psi}{\alpha \Rightarrow \psi}$

The recognition mechanism is described as follows:

(1) If A belongs to the category α, and B belongs to the category β/α, then BA belongs to the category β.

(2) If A belongs to the category α, and B belongs to the category $\alpha\backslash\beta$, then AB belongs to the category β.

(3) If A belongs to the category α, $\alpha \Rightarrow \beta$ is derivable in L, then A belongs to the category β as well.

1.3. NON-DIRECTIONAL LAMBEK CALCULUS. Van Benthem (1986) introduced a Lambek Calculus in which there is only one category functor: $\rightarrow$. The main difference between this functor and the old one is that $\rightarrow$ lacks directionality. This is why this grammar is known as '*non-directional Lambek Calculus*'. One formulation of the rules and axioms for this calculus is the following (Notation: capital letters stand for sequences of categories):

(1) $\alpha \Rightarrow \alpha$

(2) $\alpha \rightarrow \beta\, \alpha \Rightarrow \beta$ $\qquad \alpha\;\; \alpha \rightarrow \beta \Rightarrow \beta$ $\qquad$ (elimination rules)

(3) $$\frac{\Gamma\ \alpha \Rightarrow b}{\Gamma \Rightarrow \alpha \rightarrow \beta} \qquad \frac{\alpha\ \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow \beta}$$ (introduction rules)

(4) $$\frac{\Gamma \Rightarrow \alpha}{\Delta_1\Gamma\Delta_2 \Rightarrow \Delta_1\alpha\Delta_2}$$ (replacement)

(4) $$\frac{\alpha \Rightarrow \beta \quad \beta \Rightarrow \psi}{\alpha \Rightarrow \psi}$$ (transitivity rule)

As a matter of fact, this system becomes equivalent to L if we add to the latter the permutation rule

$$\frac{\Delta_1\gamma\alpha\Delta_2 \Rightarrow \beta}{\Delta_1\alpha\gamma\Delta_2 \Rightarrow \beta}$$ (permutation rule)

This explains why Van Benthem's CG is also known as LP.
In this dissertation we shall be concerned with LP - albeit formulated in the format of Natural Deduction. The formulation of LP in a Natural Deduction style is not obligatory, but for Natural Logic the natural deduction format is the more convenient one.

1.4. NATURAL LOGIC AND CATEGORIAL GRAMMAR. Lambek's original system is more syntactic than semantic. LP is more semantic than syntactic, it is constructed with a direct view towards semantical interpretation.

For a Natural Logic based on Categorial Grammar, it becomes necessary to find the right mixture of syntax and semantics. Natural Logic should navigate between the Scylla of the more syntactic approach and the Charybdis of the more semantic approach.

Since Natural Logic is based on grammatical form and its semantical interpretation, it will be necessary to direct our attention to LP first. Later we shall define a variant to LP, called Lambek Grammar (LG), which shall provide us with the vehicles of inference for Natural Logic. The empirical adequacy of this variant will be considered in a later chapter.

2. THE LANGUAGE OF LP

2.1. THE CATEGORIES IN LP. We define the language of LP -essentially an implicational language with a finite number of primitives - as follows:

2.2. DEFINITION. The set of categories is the smallest set C such that
Basis. $\{e, t, p\}$ is contained in C.
Inductive step. if α , β are members of C, so is $(\alpha \rightarrow \beta)$.

2.2.1. CONVENTION.
$\alpha_1 \to \alpha_2 \to \ldots \to \alpha_{n-1} \to \alpha_n$ abbreviates $\alpha_1 \to (\alpha_2 \to \ldots \to (\alpha_{n-1} \to \alpha_n \ldots)$.

2.2.2. REMARK ON SYNTACTIC INTERPRETATION. (A) The category to which Proper Names are assigned is e ; the category of Sentences is t and p is the category of Common Nouns; $\alpha \to \beta$ is the category of expressions which form a complex expression of category β when combined with expressions of category α .

(B) The choice of three basic categories instead of the usual two, has a practical motivation: some principles of Natural Logic are more easily formulated if we distinguish between the category of Common Nouns and the category of Verb Phrases.[1]

2.3. RULES OF TYPE CHANGE. Roughly speaking, LP can be described as an implicational calculus that supports linguistically sensible rules of type change. An example of a rule of category change is the so-called *Montague Rule*:

> Expressions assigned to category α may also be assigned to category $(\alpha \to \beta) \to \beta$, for arbitrary category β .

This rule allows one to give a homogeneous treatment to coordination. In usual practice Proper Names are assigned to the category e, noun phrases are assigned to the category $(e \to t) \to t$. These assignments make it difficult to explain that *Abelard and every man* forms a syntactic unity, since *and* coordinates only expressions of the same category. However, by applying the Montague Rule to e one deduces that Proper Names belong also to the category of Noun Phrases.
Another example of category change principles is the so-called *Geach rule*:

> Expressions assigned to category $\alpha \to \beta$ may also be assigned to category $(\gamma \to \alpha) \to (\gamma \to \beta)$, for an arbitrary category γ .

This rule allows one to give a categorial explanation of the construction of Verb Phrases. Usually, Noun Phrases are assigned to the category $(e \to t) \to t$, Transitive Verbs are assigned to the category $e \to e \to t$. These assignments make it difficult to explain that *loves every man* forms a syntactic unity. But by applying Geach to $(e \to t) \to t$, one deduces that *every man* belongs also to the category $(e \to e \to t) \to e \to t$.

In the next section we develop in detail a system of Natural Deduction as a necessary step towards the definition of LP.

3. A CONSTRUCTIVE IMPLICATIONAL SYSTEM OF NATURAL DEDUCTION

3.1. NOTATION. Derivations in the Natural Deduction Calculus (ND) are labelled trees. The top nodes of derivations are formulas indexed by numeral. We write the top nodes as α^n, but they are in fact ordered pairs of the form $\langle\alpha, n\rangle$. We use D, D_i, D', D'_i for arbitrary derivations. We write

$$\begin{array}{c} D \\ \beta \end{array}$$

to indicate that D is a derivation with β as its conclusion. We use $[\alpha^n]$ for the (possibly empty) set of occurrences of α^n in a derivation; thus, the following tree is a deduction D, with conclusion β, containing the set of occurrences of α^n among the '*open*' ('*undischarged*', '*alive*') assumptions:

$$\begin{array}{c} [\alpha^n] \\ D \\ \beta \end{array}$$

3.2. DEFINITION. The set of derivations is the smallest set T such that
Basis. The one-node derivation α^n of α from the open assumption α^n belongs to T.
Inductive step. (i) Assume D_1 and D_2 are derivations and that B, A are their respective sets of open assumptions. Then

$$\begin{array}{cc} D_1 & D_2 \\ \alpha \to \beta & \alpha \end{array} \in T \Rightarrow \frac{\begin{array}{cc} D_1 & D_2 \\ \alpha \to \beta & \alpha \end{array}}{\beta}\to E \in T$$

The open assumptions set of the new derivation is $A \cup B$.

(ii) Assume D_1 is a derivation with A as its set of open assumptions. Then

$$\begin{array}{c} [\alpha^n] \\ D_1 \\ \beta \end{array} \in T \Rightarrow (n)\,\frac{\begin{array}{c} [\alpha^n] \\ D_1 \\ \beta \end{array}}{\alpha \to \beta}\to I \quad \in T$$

The set of open assumptions of the new derivation is $A - \{\alpha^n\}$. All the members of $[\alpha^n]$ are called *discharged.*

3.3. DEFINITION. We say that $\Delta \vdash \beta$ iff there is a derivation with conclusion β and open assumptions B, where $\{\alpha \mid \langle\alpha, n\rangle \in B\}$, for a certain n, is a subset of the *set* of formulas Δ.

3.3.1. EXAMPLE. The rules of category-change mentioned in 2.3 can be expressed by $e \vdash (e \to t) \to t$ and $(e \to t) \to t \vdash (e \to e \to t) \to (e \to t)$ respectively. The derivations justifying these type changes are (a) and (b) below:

$$\dfrac{\dfrac{e \to t^3 \quad e^2}{t}\,(3)}{(e \to t) \to t} \qquad\qquad \dfrac{\dfrac{\dfrac{(e \to t) \to t^3 \qquad \dfrac{e \to e \to t^1 \quad e^2}{e \to t}}{t}\,(2)}{e \to t}\,(1)}{(e \to e \to t) \to e \to t}$$

(a) (b)

3.3.2. REMARK ON THE INDICES. The derivations generated by this system are over-annotated. In an use of the →I-rule (*Introduction rule*, *Conditionalization*) we refer back to a set of occurrences; indices are then used to mark the node(s) we refer to. But adorning the nodes is necessary only when the use of the rule is threatened by ambiguity. Derivations which are unequivocal are seldom annotated. This is the policy followed in Troelstra & Van Dalen (1988): annotated derivations are introduced only when it makes a difference. We argue in the next section that for the definition of LP it is indispensable to work with fully adorned trees, because we want to restrict the use of the →E-rule (*Elimination rule*, *Modus Ponens*). When the form of the formulas makes the restriction redundant, we shall avoid unnecessary indices or references to the rules employed.

We have given a standard presentation of Natural Deduction. However, some variation is possible according to our choices in the following matters:

(a) Should we keep track either of formulas or occurrences of formulas in derivations?

(b) In the definition of the relation of entailment $\Delta \vdash \phi$ should we say that Δ is a set, a multiset or a list?

(c) In the definition of the relation of entailment $\Delta \vdash \phi$, should we say that all the members of Δ must be used in the relevant derivation, or should we say that the used premisses must be contained in Δ?

As it happens, LP is a variant of ND in which we keep track of occurrences, and in which the relation of entailment solely holds between the list of open assumptions and the conclusion of a derivation.

4. TOWARDS A DEFINITION OF LP

4.1. THE NEED FOR RESTRICTING THE INFERENCE RULES. Given the linguistic motivation of LP, this calculus can not coincide with ND. ND contains, for example, the derivations

(a)

$$(2)\ \frac{\beta^{1}}{\alpha \to \beta}\to I$$

(b)

$$\cfrac{\cfrac{\alpha \to (\alpha \to \beta)^{1} \quad \alpha^{2}}{\alpha \to \beta \qquad \alpha^{2}}\to E}{(2)\ \cfrac{\beta}{\alpha \to \beta}\to I}\to E$$

(c)

$$(1)\ \frac{\alpha^{1}}{\alpha \to \alpha}\to I$$

These derivations would justify the rules

- Expressions assigned to category β, may also be assigned to category $\alpha \to \beta$, for arbitrary category α.
- Expressions assigned to category $\alpha \to (\alpha \to \beta)$, may also be assigned to category $\alpha \to \beta$.
- Expressions assigned to the empty category, may also be assigned to category β, for arbitrary provable β.

Unrestricted use of these rules does not make linguistic sense. According to the first one, sentences would be intransitive verbs; according to the second one, conjunctions would also be sentence modifiers. Therefore, to be linguistically relevant LP must fail to generate (a) and (b). The third derivation is odd in this context. There is no empty category. It is true that in linguistics one works with empty elements, but they do not belong to the empty category. They are combinatorial devices assigned to specific linguistic domains. The observation that these rules do not make sense in general does not imply, of course, that they not make sense in particular cases.

4.2. RESTRICTING THE RULES. It is obvious that we need to restrict the strength of ND to obtain a linguistically relevant calculus. This is done in the current section. We shall first discuss the strategy we shall follow. Next, we summarize the discussion in a more formal definition of LP.

4.2.1. TO AVOID (a). ND generates (a) because the elimination rule does not need to refer successfully to the expression indexed by the numeral. The initial characterisation of $[\alpha^n]$ as a possibly empty set makes this vacuous reference possible. If $[\alpha^n]$ is empty, then the use of *n* in the Conditionalization, fails to refer. To avoid derivations like (a) it suffices to ask that $[\alpha^n]$ should be non-empty in any application of this rule.

4.2.2. TO AVOID (b). In ND each time one uses a formula as an assumption, one must write it. The uses of a formula are then represented by distinct objects: two formula occurrences are two different objects - even if they are occurrences of the same formula. But we have the unlimited right to identify these occurrences. The use of this right, combined with conditionalization generates (b). To avoid (b) one can restrict Conditionalization: any introduction should eliminate at most one assumption. Alternatively, one can say that an assumption should be used only once in →E applications. We choose for the second alternative. The reason is the following. In ND a formula occurrence is used only once in an application of modus ponens. Occurrences may become identified and then eliminated. But identification must have taken place before conditionalization; they are independent operations. Consider the ambiguous derivation D:

$$\cfrac{\cfrac{\cfrac{\alpha \to (\alpha \to \beta) \quad \alpha}{\alpha \to \beta \qquad \alpha}{\scriptstyle \to E}}{\beta}{\scriptstyle \to E}}{\alpha \to \beta}{\scriptstyle \to I}$$

We could annotate this tree in several ways. The restriction we shall impose has as a result that one of these ways is not longer available: we want to exclude the situation in which different occurrences of a formula have the same name. To avoid inferences like (b) we formulate →E so that two derivations are combined into one derivation, only when their sets of open assumptions are disjoint. This means that the relevant part of D can not be annotated as (d), but only, for instance, as (e):

$$\cfrac{\cfrac{\alpha \to (\alpha \to \beta) \quad \alpha^2}{\alpha \to \beta \qquad \alpha^2}{\scriptstyle \to E}}{\beta}{\scriptstyle \to E} \qquad\qquad \cfrac{\cfrac{\alpha \to (\alpha \to \beta) \quad \alpha^3}{\alpha \to \beta \qquad \alpha^2}{\scriptstyle \to E}}{\beta}{\scriptstyle \to E}$$

(d) (e)

Henceforth, the elimination rule can only be applied to (e), since (d) will not be available. In this case it eliminates at most one occurrence of α^n : there is at most one in the tree.

Note that a restriction on conditionalization allows (d) and (e). But (d) is not linguistically sensible: one proper name and a transitive verb do not, in general, combine into a sentence.

Somewhere, a filter must be imposed on the derivations to get rid of (d). We prefer to impose the filter at the modus ponens stage, barring (d) and (b) simultaneously.

4.2.3. AVOIDING (c). To avoid this kind of derivation we simply restrict the use of elimination: no set of open assumptions may be empty after an application of conditionalization.

We now define LP as a proper subset of ND derivations.

4.3. DEFINITION OF THE NON-DIRECTED LAMBEK CALCULUS. LP is the set of all the ND deductions D which satisfy

(i) D is a one node derivation.
(ii) In each use in D of the elimination rule the open sets of assumptions A, B are disjoint.
(iii) In each use in D of the introduction rule neither $A - [\alpha^n]$ nor $[\alpha^n]$ are empty.

4.4. DEFINITION. We say that $\Delta \vdash \beta$ iff there is a derivation with conclusion β and open assumptions B, where Δ is the multiset $\{\alpha \mid \langle\alpha,n\rangle \in B\}$, for certain n .

4.4.1. REMARKS. 4.4 differs from Definition 3.3 in two ways. (A) According to 3.3, derivation (e) proves $\{ e, e, e \to e \to t \} \vdash e \to t$. Thus, (e) would also prove $\{e, e \to e \to t \} \vdash e \to t$. But this is one of the results we have tried to avoid. To keep intact the fruits of the new inference rules, we have to covert Δ into a multiset.

(B) Δ coincides now with B. According to Definition 3, if one has $\Delta \vdash \beta$, one will also have $\Delta \cup \Delta' \vdash \beta$. This will not be the case in the new setting. Each formula listed in Δ (the premisses), has exactly one occurrence in the derivation (as an open assumption).

However, the two definitions share an important property:

(C) An ordering of the premisses does not impose a particular ordering on the assumptions.

4.5. We collect some simple observations in the following:

Lemma 1
1) Elimination of non existent assumptions is not available in LP.
2) Assumptions of modus ponens premisses are disjoint.
3) Every conclusion depends upon at least one assumption.
4) LP is closed under sub-derivations.
5) Every LP derivation has at least one open assumption.
6) Repeated use of an open assumption is not allowed.
7) Withdrawal of assumptions is limited to one at the time.

8) The identity of the indices is immaterial.

Proof

Directly from the definition.

□

4.5.1. REMARK. Van Benthem defines LP as a calculus of occurrences of premisses satisfying the constraint:

> 'each application of the introduction rule eliminates exactly one formula occurrence'.

The above lemma shows that our definition is equivalent to Van Benthem's. In 4.2.2 we have argued in favour of our definition; but see 6.2.1.

4.6. SIMILARITY OF DERIVATIONS. Apart from syntactic identity and isomorphy, there is another structural relation between derivation trees. Two derivations are called '*similar*' if:
they are the same tree but have a different indexing of assumptions
Consequently, (d) and (e) are isomorphic but not similar: (d) has one open assumption, (e) has two. Consider (d). Choose an arbitrary numeral and substitute it for *2*. The result is a similar derivation which is not in LP. Consider (e). Choose any pair of different numerals. Substitute them for *2* and *3*. The result is a similar derivation which is in LP. More in general, one can prove by induction that if D and D' are similar, then $D \in LP$ iff $D' \in LP$.

As we pointed out in 1.1 the LP derivations are systematically linked to expressions of the typed lambda calculus. In the next section we consider the language of this calculus. We shall identify terms which correspond one-to-one with the LP derivations. These terms will be called '*Lambek Terms*'.

5. THE LANGUAGE OF THE TYPED LAMBDA CALCULUS

5.1. DERIVATIONS AND TERMS. There is a systematic connection between derivations in intuitionistic implication logic and typed lambda terms, first noticed by Curry, and further elaborated by Howard. Van Benthem (1986) shows a similar connection between derivations in LP and a class of terms -here suggestively called *Lambek terms*. Van Benthem gives a procedure which applied to a derivation in LP yields a Lambek term. Conversely, this procedure applied to a Lambek term yields a derivation in LP. In this section we first define the language of the typed lambda calculus. In the next one we shall define the class of Lambek terms.

5.2. THE LANGUAGE OF THE TYPED LAMBDA CALCULUS. The following definitions determine the language of the typed lambda Calculus, henceforth '*lambda calculus*'.

5.2.1. DEFINITION. The set of types is the smallest set A given by

(i) e , p and t are in T;

(ii) $(\alpha \to \beta)$ is in T if α and β are.

5.2.2. DEFINITION. We assume that we have at our disposal an infinite supply of variables written X_{α}^{n}, where n is a natural number and α a type. The set of terms is the smallest set A satisfying:

(i) Any variable of type α is in A ;

(ii) **Application**: If N_{α} and $M_{\alpha \to \beta}$ are in A, then also $(M_{\alpha \to \beta}N_{\alpha})$;

(iii) **Abstraction**: If M_{β} is in A and X_{α} is a variable, then also $[\lambda X_{\alpha}.M_{\beta}]$.

Note that this definition is similar to Definition 3.2.

5.2.3. CONVENTION

(i) X_{α}, X, Y_{α}, Y, . . . denote arbitrary variables of type α .

(ii) M_{α}, M, N_{α}, N, . . . denote arbitrary terms of type α .

(iii) Outermost parentheses are not written.

(iv) $M_1, \ldots .M_n$ is short for $(\ldots((M_1M_2)M_3)\ldots M_n)$.

(v) '≡' denotes syntactic identity.

(iv) In the same context (comment, definition, proof) M_{α}, M denote the same term: M_{α} introduces the term and M is used for the purpose of cross-reference.

5.2.4. DEFINITION. The set of free variables in N_{α} , FV(N) , is given by

(i) $FV(X) = \{X\}$;

(ii) $FV(MN) = FV(M) \cup FV(N)$;

(iii) $FV(\lambda X.M) = FV(M)-\{X\}$.

5.2.5. CONVENTION.

(i) If $M_1, \ldots, M_n$ is a term used in a proof or a definition, then we assume that all free variables are different from the bound variables.

(ii) A term without free variables shall be called a '*closed*' term; a term which is not closed shall be called '*open*.

(iii) $M_{\alpha}(X_{\alpha_1}, \ldots, X_{\alpha_n})$ denotes a term in which X_{α_i} occurs free for $1 \leq i \leq n$.

5.2.6. DEFINITION. If N_{α} is a term and X_{β} is a variable of the same type as M_{γ}, then the result of substituting M for the free occurrences of X in N (Notation: N[X:= M]) is given by:

(i) $X[X:= M] \equiv N$;

(ii) $Y[X:= M] \equiv Y$, if $Y \not\equiv X$;

(iii) $(NP)[X:= M] \equiv N[X:= M]\, P[X:= M]$;

(iv) $(\lambda Y.\ N)[X:= M] \equiv \lambda Y.\ N[X:= M]$.

6. The Lambek terms

In this section we introduce the typed terms which shall be connected with derivations in LP.

6.1. DEFINITION. The set of Lambek Terms (LT) is the set of all the terms P which satisfy:

(i) P is a variable.

(ii) $P \equiv M_{\alpha \to \beta} N_{\alpha}$ is in LT iff $FV(M) \cap FV(N) = \emptyset$.

(iii) $P \equiv \lambda X_{\alpha}.N$ is in LT iff $X_{\alpha} \in FV(N)$, $FV(N)-\{ X \} \neq \emptyset$.

Notice that this definition is similar to 4.3.

6.2. OBSERVATIONS. In the following lemma we show some consequences of 6.1. They are intended to facilitate the comparison with the LP derivations.

Lemma 2

1) If $\lambda X_{\beta}.N_{\gamma}$ is in LT, then so is N . Furthermore X is a free variable of N .

2) If $N_{\beta \to \alpha} P_{\beta}$ is in LT, then N and P are in LT. Furthermore, $FV(N) \cap FV(P) = \emptyset$.

3) If $\lambda X_{\beta}.N_{\gamma}$ is in LT, then $\lambda X_{\beta}.N_{\gamma}$ is open.

4) LT is closed under sub-terms.

5) Every member of LT is open.

6) A variable occurs at most once as free variable in a Lambek Term.

7) λ binds exactly one variable occurrence in a Lambek Term.

8) The identity of the free variables is immaterial.

Proof

Directly from the definition.

□

These observations stress the structural analogy between Lambek derivations and LT. In the next section we will establish the specific relation obtaining between these derivations and terms.

6.2.1. REMARK. Van Benthem defines the Lambek terms as terms satisfying the constraint:

'each occurrence of a λ binds exactly one free variable occurrence'.

The previous lemma shows that our definition is equivalent to Van Benthem's. We think that Van Benthem's definition of the Lambek terms is more revealing than ours. But our definition corresponds better to our definition of LP.

7. Derivations in the Lambek Calculus and Lambek Terms

7.1. THE BASIS FOR THE CORRESPONDENCE. In this section we explicitly establish the correspondence between LP and LT. To read off lambda-terms from the derivations in LP, we use a correspondence between the Clauses of 3.2 and the Clauses of 5.2.2:

- The one-node derivations correspond to Clause (i) of Definition 5.2.2;
- $E_{\rightarrow}$ corresponds to Application;
- $I_{\rightarrow}$ corresponds to Abstraction.

7.2. FROM LAMBEK DERIVATIONS TO LAMBEK TERMS. It is known, that to each derivation in ND there is a corresponding a typed term. A fortiori, to each derivation in LP there is a corresponding typed term. However, this term need not be a Lambek term. An independent proof is needed to be sure that the term belongs to LT.

The following picture represents the essential features of a transformation of a proof of the Montague Rule into a Lambek Term:

$$(2)\dfrac{\dfrac{e \rightarrow t^2 \quad e^1}{t}}{(e \rightarrow t) \rightarrow t} \Rightarrow (2)\dfrac{\dfrac{X_{e \rightarrow t} \quad Y_e}{t}}{(e \rightarrow t) \rightarrow t} \Rightarrow (2)\dfrac{\dfrac{X_{e \rightarrow t} \quad Y_e}{XY}}{(e \rightarrow t) \rightarrow t} \Rightarrow \dfrac{\dfrac{X_{e \rightarrow t} \quad Y_e}{XY}}{\lambda X.\ XY}$$

Assumptions match atoms, elimination reflects application, and introduction reflects abstraction.

7.2.1. PROPOSITION 1. There is an effective procedure for obtaining from an LP derivation D with conclusion β and open assumptions $B = \{(\alpha^i)_1, \ldots, (\alpha^i)_n\}$ a Lambek term M_β, such that

1. $FV(M) = \{X_{\alpha_1}, \ldots, X_{\alpha_n}\}$;
2. α_j is the category which appears in $(\alpha^i)_j$;
3. $X_{\alpha_j} \not\equiv X_{\alpha_k}$, for $1 \leq j,k \leq n$.

Proof

Let D be a derivation of β from $\{(\alpha^i)_1, \ldots, (\alpha^i)_n\}$.

Basis. If D is a one node derivation, then take any variable of type β as M.

Assume that the assertion holds for Lambek derivations of less complexity than D.

Inductive step. (i) Suppose that the last rule employed in D is $E_{\rightarrow}$. Then D has the form

$$\dfrac{\begin{matrix} D_1 \\ \gamma \rightarrow \beta \end{matrix} \qquad \begin{matrix} D_2 \\ \gamma \end{matrix}}{\beta} E_{\rightarrow}$$

Without loss of generality, assume that $\{(\alpha^i)_1, \ldots, (\alpha^i)_m\}$ and $\{(\alpha^i)_{m+1}, \ldots, (\alpha^i)_n\}$ are the assumptions of D_1 and D_2 respectively. According to 4.5.2, both D_1 and D_2 are Lambek derivations. Hence, by the inductive hypothesis, there is a Lambek term $N_{\gamma\rightarrow\beta}$ with $FV(N) = \{X_{\alpha_1}, \ldots, X_{\alpha_m}\}$ corresponding to D_1, and a Lambek term P_γ with $FV(P) = \{X_{\alpha_{m+1}}, \ldots, X_{\alpha_n}\}$ corresponding to D_2. These sets of variables need not be disjoint. But according to 6.2.8 we shall always be able to find Lambek terms $N'_{\gamma\rightarrow\beta}$, P'_γ such that $FV(N') \cap FV(P') = \emptyset$. Therefore a Lambek term corresponding to D can be found by putting $M_\beta \equiv N'P'$.

(ii) Suppose that the last rule employed is D $I_\rightarrow$. Then $\beta \equiv \gamma \rightarrow \delta$ and D has the form

$$\cfrac{\begin{array}{c}[\gamma^k]\\ D_1\\ \delta\end{array}}{\gamma \rightarrow \delta}\,(k)\ I\rightarrow$$

According to observations 4.5.1 and 4.5.3, D_1 is a Lambek derivation, with open assumptions $\{(\alpha^i)_1, \ldots, (\alpha^i)_n, \gamma^k\}$; furthermore, $\{(\alpha^i)_1, \ldots, (\alpha^i)_n\}$ is not empty.

By the induction hypothesis there is a Lambek term N_δ with $FV(N) = \{X_{\alpha_1}, \ldots, X_{\alpha_n}, Y_\gamma\}$. By assumption, Y_γ is different from all the X_{α_i}. Hence, since $\{(\alpha^i)_1, \ldots, (\alpha^i)_n\}$ is not empty, neither is $FV(N)- \{Y_\gamma\}$. A Lambek term corresponding to D can now be found by putting $M_\beta \equiv \lambda Y.N$.

□

7.2.2. CONVENTION. The term M found in this way will be called the '*meaning*' of D.

7.3. FROM LAMBEK TERMS TO LAMBEK DERIVATIONS. The proof of the next assertion results in a procedure leading from (the construction tree of) a Lambek Term to a derivation in LP. A pictorial representation of this procedure in action is the following:

$$\cfrac{\cfrac{X_{e\rightarrow t}\quad Y_e}{XY}}{\lambda X.\, XY} \Rightarrow \cfrac{\cfrac{e\rightarrow t^2\quad t^1}{XY}}{\lambda X.XY} \Rightarrow \cfrac{\cfrac{e\rightarrow t^2\quad t^1}{t}}{\lambda X.XY} \Rightarrow \cfrac{\cfrac{e\rightarrow t^2\quad t^1}{t}}{(e\rightarrow t)\rightarrow t}\,(2)$$

Assumptions match atoms, elimination reflects application, and introduction reflects abstraction.

7.3.1. PROPOSITION 2. There is an effective procedure to obtain from a Lambek term M_β with $FV(M) = \{X_{\alpha_1}, \ldots, X_{\alpha_n}\}$ an LP derivation D with conclusion β such that

1. $\{(\alpha^i)_1, \ldots (\alpha^i)_n\}$ is its set of open assumptions;
2. $\alpha^i{}_j$ is the type of X_{α_j} and

3. $(\alpha^i)_j \not\equiv (\alpha^i)_k$, for $1 \leq j,k \leq n$.

Proof

Assume M_β is in LT.

Basis. If $M \equiv X_\beta$, then the one-node derivation β^i is the desired derivation.

Inductive step. Suppose that the assertion holds for Lambek terms of less complexity than M.

(i) Let $M \equiv P_{\gamma \to \beta} N_\gamma$. Then, again without loss of generality, we can assume

(a) $FV(P) = \{X_{\alpha_1}, \ldots, X_{\alpha_m}\}$ and

(b) $FV(N) = \{X_{\alpha_{m+1}}, \ldots, X_{\alpha_n}\}$

By induction, we have derivations D_1 and D_2

$$\begin{array}{cc} D1 & D2 \\ \gamma \to \beta & \gamma \end{array}$$

where the assumptions sets of D_1 and D_2 are, respectively, $\{(\alpha^i)_1, \ldots, (\alpha^i)_m\}$ and $\{(\alpha^i)_1, \ldots, (\alpha^i)_n\}$. If these assumptions sets are not disjoint, we can always find similar D'_1 and D'_2 derivations in which this is the case (this is the point made in 4.5.8). The desired derivation can be found by way of applying the elimination rule.

(ii) Let $M \equiv \lambda Y_\gamma . N_\delta$. Since N_δ is a Lambek Term it follows that $FV(N_\delta)-\{Y_\gamma\} \neq \emptyset$ and $FV(N) = \{X_{\alpha_1}, \ldots, X_{\alpha_n}, Y_\gamma\}$. By induction, we have a Lambek derivation D_1 of δ with assumptions $\{(\alpha^i)_1, \ldots, (\alpha^i)_m, \gamma^k\}$:

$$\begin{array}{c} [\gamma^k] \\ D1 \\ \\ \delta \end{array}$$

By assumption γ^k is distinct from all the other assumptions. Hence, since $FV(N_\delta)-\{Y_\gamma\}$ is not empty, neither is $\{(\alpha^i)_1, \ldots, (\alpha^i)_m\}$. The derivation D we were looking for can now be obtained by using $I_\to$ and withdrawing γ^k.

□

7.3.2. CONVENTION. The derivation D obtained in this way will be called the '*tree*' of M .

7.4. REMARK. (A) Notice that we can not say that the procedures from 7.2 and 7.3 are inverse operations. The following situation is not excluded:

$$\dfrac{\dfrac{e \to t^2 \quad t^1}{t}}{(e \to t) \to t}(2) \Rightarrow \lambda X_{e \to t}.XZ \Rightarrow \dfrac{\dfrac{e \to t^2 \quad t^3}{t}}{(e \to t) \to t}(2)$$

The derivations are not the same because of the choice of indices -they are only similar. However, for some uses of the correspondence established, similarity is sufficient. (see 4.6)

(B) The following situation isn't excluded either:

$$\lambda X_{e \to t}.XY \Rightarrow \dfrac{\dfrac{e \to t^2 \quad t^1}{t}\,(2)}{(e \to t) \to t} \Rightarrow \lambda X_{e \to t}.XZ$$

The terms are not the same, they are only congruent. However, for some uses of the correspondence established, congruency is sufficient.

8. Normalisation for LP derivations and for Lambek terms

8.1. NORMALISATION IN LP. In this section we comment on the fact that some derivations which are different as syntactical objects, can be shown to be only different wrappings of a common structure: their normal form. This can be proven by making use of the notion of normalisation. We shall prove directly that LP derivations have a normal form; (cf 8.3).

8.2. DETOURS AND REDUCTS. This section introduces some notions used in the proof that every derivation in LP has a normal form.

8.2.1. DEFINITION.

(i) If β and $\beta \to \gamma$ are the premisses of an application of $E_{\to}$ then β is called the '*minor*' and $\beta \to \gamma$ the '*major*' premiss of the application.

(ii) A *detour* in a derivation consists of an application of $I_{\to}$ followed directly by an application of $E_{\to}$ where an occurrence of the formula eliminated by $I_{\to}$ is used as the minor premiss in this application of $E_{\to}$. Thus, a detour will have the form:

$$\dfrac{\dfrac{\begin{matrix}[\delta^n]\\ D_1\\ \gamma\end{matrix}}{\delta \to \gamma}\,(n)\to I \qquad \begin{matrix}D_2\\ \delta\end{matrix}}{\gamma}\to E$$

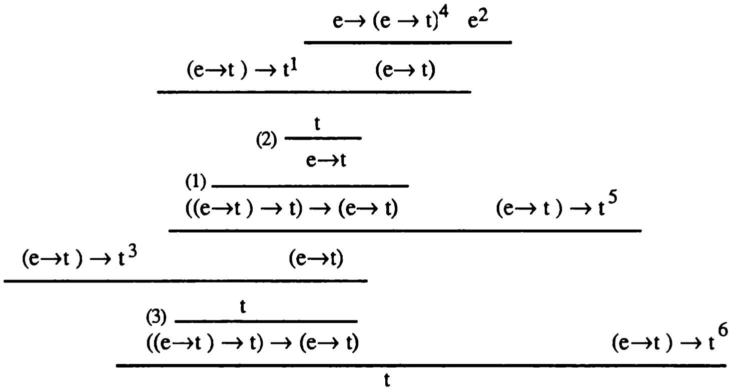
(iii) Let D be a derivation with a detour like the previous one. A derivation D' in which the following derivation replaces the detour is called a *reduct* of D:

$$\begin{array}{c} D_2 \\ \delta \\ D_3 \\ \gamma \end{array}$$

where D_3 is a derivation like D_1 except that the top node δ^n has been changed into the conclusion of D_2. (We assume that the derivations are Lambek derivations, so that there is exactly one occurrence of δ^n to be taken care of). In fact, one should prove that the result of a detour elimination is an LP derivation, but this can be done (see 8.5).

(iv) A derivation without any detour is said to be in **normal form.**

8.2.2. EXAMPLE. Consider derivation (a):



In this derivation we first withdraw assumption 1 and then apply the result directly to the assumption 5. Afterwards we withdraw 3 and then apply the result to the assumption 6. Elimination of the lowest detour yields the reduct (b):

$$\cfrac{(e\to t)\to t^{6} \qquad \cfrac{\cfrac{\cfrac{\cfrac{(e\to t)\to t^{1} \qquad \cfrac{e\to (e\to t)^{4} \quad e^{2}}{(e\to t)}}{t}}{e\to t}\,(2)}{((e\to t)\to t)\to (e\to t)}\,(1) \qquad (e\to t)\to t^{5}}{(e\to t)}}{t}$$

Elimination of the remaining detour now yields derivation (c):

$$\cfrac{(e\to t)\to t^{6} \qquad \cfrac{\cfrac{(e\to t)\to t^{5} \qquad \cfrac{e\to (e\to t)^{4} \quad e^{2}}{e\to t}}{t}}{e\to t}\,(2)}{t}$$

This last derivation is said to be in normal form. In fact, with the previous example we have proven:

PROPOSITION 3

Every LP derivation has a Normal Form.

8.3. NORMALISATION FOR THE LAMBEK TERMS. In this section we comment on the fact that some terms which are different as syntactical objects, can be shown to be only different wrappings of a common structure: their normal form. This can be proven by making use of the notion of normalisation -a conservative operation on terms. We shall prove directly that Lambek terms derivations have a normal form. (cf. 8.1)

8.4. REDEXES, REDUCTION AND CONTRACTION. We introduce here notions concerning terms which corresponding to the notions from 8.2.

8.4.1. DEFINITION.

(i) A term of the form $[\lambda X_\alpha.N]M_\alpha$ is called a *redex* and the term $N[X:=M]$ is called its *contractum*.

(ii) If a term M contains a redex and we replace this redex by its contractum and the result is N, then we say that M contracts to N.

(iii) M reduces to N if there is a sequence of terms $N_1, \ldots, N_n$ such that $M \equiv N_1, N_n \equiv N$ and N_i contracts to N_{i+1}.

8.4.2. EXAMPLE. Consider the meaning of the above derivation (a) :

(a') $[\lambda U_{(e \to t) \to t}.U([\lambda V_{(e \to t) \to t}.(\lambda Z_e.V(Y_{e \to e \to t}Z)]X_{(e \to t) \to t})]W_{(e \to t) \to t}$
replacing the outermost redex yields:

(b') $W_{(e \to t) \to t}([\lambda V_{(e \to t) \to t}.(\lambda Z_e.V\,(Y_{e \to e \to t}Z)]X_{(e \to t) \to t})$;
the subsequent elimination of the remaining redex yields:

(c') $W_{(e \to t) \to t}(\lambda Z_e.X_{(e \to t) \to t}(Y_{e \to e \to t}Z)$.

Observe that to each detour in the derivations there is a corresponding redex. On the other hand, to each redex in the terms there is a corresponding a detour. Thus, we have

8.5. To each detour in a derivation there is a corresponding redex in its meaning.
8.6. To each redex in a term there is a corresponding a detour in its tree.

We want to show that the Lambek terms are closed under β-conversion. Similar things should be checked for the LP derivations but we can do this by looking at the corresponding meanings.

8.7. THE SET OF LAMBEK TERMS IS CLOSED UNDER SUBSTITUTION OF DISJOINT TERMS.
Proof
Suppose N_β, M_α are disjoint Lambek Terms. We want to show that $N[X_\alpha := M]$ is also a Lambek Term. Remember that by the observations from 6.2, $\lambda X.N$ and M are in LT. Furthermore, these terms are open and they do not share any free variable.

Basis step. If N_β is a variable then the result follows since in this case $N[X_\alpha := M]$ is M or N .
Inductive step. Assume the result holds for $Q_{\gamma \to \beta}$, P_γ.

(i) $N \equiv QP$. Then $N[X := M] \equiv Q[X := M]P[X := M]$. X must occur in Q or in P but not in both. Suppose X occurs in Q . By induction hypothesis $Q[X := M]$ is in LT. Notice that neither P and Q nor M and P have free variables in common. Hence $Q[X := M]$ and P have no variables in common and therefore $Q[X := M]P \equiv N[X := M]$ is in LT. If X occurs in P we argue in the same way.

(ii) $N \equiv \lambda Y.Q$. Then $N[X := M] \equiv \lambda Y.Q[X := M]$. By induction hypothesis $Q[X := M]$ is in LT. If X occurs free in Q, then by convention $Y \not\equiv X$. If X does not occur (free) in Q , $Q[X := M] \equiv Q$. In both cases we have that Y is a free variable in $Q[X := M]$. Furthermore, M is open, since it is a Lambek Term. But by assumption Y is not a free variable of M . Thus, M must have free variables distinct from Y . Hence $\lambda Y.Q[X := M]$ is a Lambek Term.
□

Corollary

8.8. THE CONTRACTUM OF A LAMBEK TERM IS A LAMBEK TERM.

Proof

Direct from 8.7.

□

8.9. THE EXPANSION OF A LAMBEK TERM IS A LAMBEK TERM. One can also prove that if M_α is a *proper* sub-term of N_β and X_α does not occur in N_β, then $[\lambda X_\alpha.N'_\beta]M_\alpha$ is a Lambek term, where N'_β is like N_β except that it contains X_α instead of M_α. We shall sketch an argument proving this assertion. M_α is in LT: LT is closed under sub-terms, and N_β is in LT. If $\lambda X_\alpha.N'_\beta$ is not in LT, then it is closed, since X_α must occur in it. But if it is closed then $X_\alpha \equiv N'_\beta$ and so $N_\beta \equiv M_\alpha$. However this is not possible since M_α must be a proper sub-term of N_β. Obviously $[\lambda X_\alpha.N'_\beta]$ and M_α have no variables is common. So $[\lambda X_\alpha.N'_\beta]M_\alpha$ is a Lambek Term.

8.10. NORMAL FORM. An important property of a typed term is that it *has* a normal form. More accurately, if M_α is a term, then there is a redex-free term N of type α such that $M_\alpha = N_\alpha$ is provable in the lambda calculus. Typed terms have the additional property that the order in which redexes are eliminated is immaterial: all alternative reductions lead to the same term.

8.10.1. DEFINITION. A term which contains no redexes is called a **normal form term.** If M contracts to N and N is a normal form term, then N is called a normal form of M.

8.10.2. DEFINITION. A term M is said to be **strongly normalisable** iff there is no infinite sequence

$$M = M_1 \Rightarrow M_2 \Rightarrow \dots \Rightarrow M_n \Rightarrow \dots$$

where M_i contracts to M_{i+1}.

8.10.3. IF M IS A LAMBEK TERM AND N IS A REDUCT OF M, THEN N IS SHORTER THAN M.

Proof

Assume than M contains a sub-term of the form $[\lambda X_\alpha.P_\beta]Q_\alpha$. By the construction of the Lambek Terms, it follows that X occurs exactly once in P and that Q occurs exactly once in M. This means that in P[X:= Q] , Q occurs exactly once. Thus P[X:= Q] will be less complex than $[\lambda X_\alpha.P_\beta]Q_\alpha$: it has one 'λ' less. Consequently, a reduct of M will be less complex than M itself.

COROLLARY

8.10.4. LAMBEK TERMS ARE STRONGLY NORMALISABLE.

Having thus developed LP and compared its derivations with typed terms, we shall define the variant of LP that will generate the vehicles of inference for Natural Logic. This is the theme of the next section.

9. LAMBEK GRAMMAR

9.1. A VARIANT TO LP. In this section we define a variant of LP which we call *Lambek Grammar* (LG). LG generates derivations whose indices may be expressions of a natural language. Natural language expressions will be identified with the derivations which LG provides, i.e. with their *syntactical analysis*. In a later chapter we shall say that a string of natural language expressions implies another one, if the analysis of the former implies the analysis of the latter.

9.2. MOTIVATING THE LAMBEK GRAMMAR. Let us sketch the reasons for introducing LG. In the construction of our proof system, the *denotation* of the vehicles of inference plays an important role. Since we identify expressions with their analyses, this means that the denotation of the expressions plays an important role. Remember that to each derivation D of LP, there is a corresponding Lambek Term T. It is only natural to take the interpretation of the term T as the denotation of the derivation D to which the term corresponds. For instance, we could say that the meaning of the derivation (a) below is the interpretation of the term (b):

$$\frac{p \to ((e \to t) \to t) \qquad p}{(e \to t) \to t} \quad \Rightarrow \quad X_{p \to ((e \to t) \to t)} Y_p$$

(a) (b)

But if we see (a) as the analysis of *every logician* then we cannot say that the interpretation of the term (b) should be taken as the denotation of (a). The interpretation of (b) is restricted only by one condition: $[\![XY]\!]$ should be a member of a specific set of functions. The assignments which take care of the free variables are not constrained in the particular choice of the denotation of expressions. Therefore it is possible that a particular choice turns out to be unfortunate -in the sense that the object chosen may show a behaviour we do not want to associate with a decent denotation of the analysis of *every logician*. For instance, it is possible to correlate (b) with a function EVERY GIRL, such that the inference (c) below becomes valid while (d) becomes invalid:

$$\frac{\text{EVERY LOGICIAN IS A WOMAN} \qquad \text{EVERY GIRL IS A WOMAN}}{\text{EVERY LOGICIAN IS A GIRL}}$$

(c)

$$\frac{\text{EVERY LOGICIAN IS A GIRL} \quad \text{EVERY GIRL IS A WOMAN}}{\text{EVERY LOGICIAN IS A WOMAN}}$$

(d)

The point of this observation is a familiar one, namely that natural language is an interpreted language. In this sense, the semantics of natural language is severely constrained by an obvious adequacy condition: formal interpretations are constrained by pre-theoretical interpretations. Thus, by itself the denotation of the term XY may be an arbitrary function of the adequate type. But the denotation of this term - taken as fixing the denotation of the derivation of *every logician* - must be a quite specific function. To provide the analyses with the right denotations we introduce LG. LG will provide *every logician* with the analysis

$$\frac{\overset{\text{every}}{p \to ((e \to t) \to t)} \quad \overset{\text{logician}}{p}}{(e \to t) \to t}$$

(e)

To fix the meaning of (e), we consider a typed language the vocabulary of which contains the constants $\text{EVERY}_{p \to ((e \to t) \to t)}$ and LOGICIAN_p. The general idea is that the meaning of a derivation in LG should be found by mapping lexical indices into constants and numerical indices into variables. In this view the interpretation of the terms corresponding to basic lexical items is no longer dependent on the assignment functions, but on the interpretation functions. From the possible interpretation functions we select the function which codifies our semantical knowledge. Having done so, we can finally say that the denotation of (e) is the interpretation of EVERY(LOGICIAN).[2]

9.3. CONVENTIONS. In LG a basic assignment statement is considered to be an indexed category, i.e. a category with a word as index. Thus, instead of writing $s \in \alpha$, we shall simply write α^s, or $\overset{s}{\alpha}$. We shall adapt LP to the presence of **lexical indices**. Derivations in LG are labelled trees. The top nodes of derivations are formulas with a numeral or an English expression as index. We write those top nodes as α^n, α^s or $\overset{s}{\alpha_{\supseteq}}$. An (open) assumption with a numeral as index will be called a **numerical assumption**. An (open) assumption with an English expression as index will be called a **lexical assumption**. We assume that the set of numerical indices and the set of lexical indices are disjoint. For example, if n is a numeral, then $[\alpha^n]$ can not refer to a set of occurrences of a lexical expression. We reserve n for the function of variable for numerals and s for the function of variable for lexical items. The notation $(\alpha_1{}^s, \alpha_2{}^s, \ldots, \alpha_n{}^s)$ will be used to list the lexical assumptions of the category α with lexical index s. Thus

$$\begin{array}{c}(\alpha_1{}^s, \alpha_2{}^s, \ldots, \alpha_n{}^s) \\ D_1 \\ \beta\end{array}$$

denotes a derivation with conclusion β from the assumptions $\alpha_1{}^s, \alpha_2{}^s, \ldots, \alpha_n{}^s$. When we use this notation we assume that D_1 has no numerical hypotheses.

We define Lambek Grammar as a set of derivations in which numerical and lexical indices are employed:

9.4. DEFINITION. The set of LG derivations is the smallest set T such that

Basis. The one-node derivations α^n of α from the open assumption α^n belong to T; the one-node derivations α^s of α from the open assumption α^s belong to T.

Inductive step. (i) Assume that D_1 and D_2 are derivations and that B, A are their respective sets of open assumptions. Then

$$\begin{array}{cc} D_1 & D_2 \\ \alpha \rightarrow \beta & \alpha \end{array} \in T \Rightarrow \dfrac{\begin{array}{cc} D_1 & D_2 \\ \alpha \rightarrow \beta & \alpha \end{array}}{\beta} \rightarrow E \in T$$

The set of open assumptions of the new derivation is $A \cup B$.

Restriction: the set of *numerical* assumptions in A and the set of *numerical* assumptions in B must be disjoint.

(ii) Assume D_1 is a derivation with A as its set of open assumptions. Then

$$\begin{array}{c} [\alpha^n] \\ D_1 \\ \beta \end{array} \in T \Rightarrow (n) \dfrac{\begin{array}{c} [\alpha^n] \\ D_1 \\ \beta \end{array}}{\alpha \rightarrow \beta} \rightarrow I \in T$$

The open assumptions set of the new derivation is $A - \{\alpha^n\}$. Here, $[\alpha^n]$ is called *discharged.*

Restriction: The sets $A - \{\alpha^n\}$, $[\alpha^n]$ are not empty.

9.4.1. REMARK. According to the definition of LG lexical assumptions are never eliminated. On the other hand, numerical assumptions are not present in the *analyses*. Numerical assumptions may be seen as empty elements, not realized phonetically. Later we shall see that the combination of numerical and lexical assumptions can be profitably used to solve some adequacy problems in linguistic description.

9.5. EXAMPLES. The derivation (a) below does not belong to LG while (b) does:

$$(\text{Abelard})\,\dfrac{\dfrac{e \to t^1 \qquad \overset{\text{Abelard}}{e}}{t}}{e \to t}$$

(a)

$$\dfrac{\dfrac{\overset{\text{and}}{t \to t \to t} \qquad \begin{array}{c}\text{loves} \quad \text{Abelard} \quad \text{Heloise}\\ e \to e \to t \quad e \quad e\\ D\\ t\end{array}}{t \to t} \qquad \begin{array}{c}\text{admires} \quad \text{Abelard} \quad \text{Heloise}\\ e \to e \to t \quad e \quad e\\ D\\ t\end{array}}{t}$$

(b)

(a) is not an LG derivation since an assumption that is lexically indexed has been withdrawn. By obliterating the distinction between the two kinds of indices, we could say that (a) may be generated by the original LP. On the other hand, (b) satisfies the conditions imposed on derivations of LG. Note that this time the derivation could not have been generated by LP, since the premisses of a Modus Ponens application share some indices.

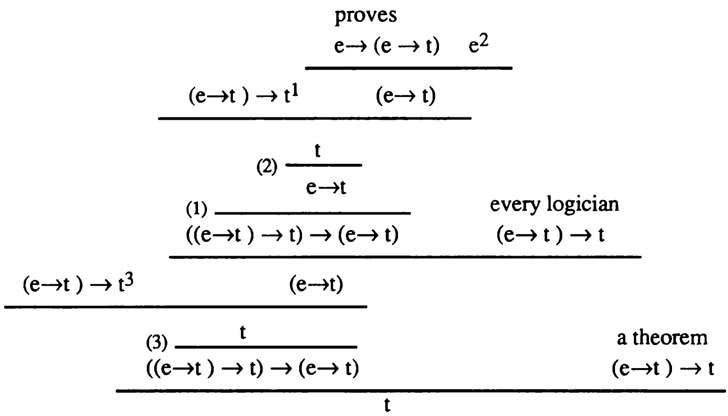
9.6. DEFINITION. Let $s_1\, s_2 \ldots s_n$ be a string of expressions over Σ, and $B = \{s_1 \in \alpha_1, s_2 \in \alpha_2, \ldots, s_n \in \alpha_n\}$ a set of lexical indices. If there is a derivation D with B as its set of open assumptions, then we say that D is an analysis of $s_1\, s_2 \ldots s_n$.

The following trees are analyses of the string *every logician proves a theorem.*

(a)

$$\dfrac{\overset{\text{a theorem}}{(e \to t) \to t} \qquad (2)\,\dfrac{\dfrac{\overset{\text{every logician}}{(e \to t) \to t} \qquad \dfrac{\overset{\text{proves}}{e \to (e \to t)} \quad e^2}{e \to t}}{t}}{e \to t}}{t}$$

(b)



9.7. DIGRESSION. Our notation for the assignments is similar to the type assignment formulas used in the type assignment to (ordinary) lambda terms (cf. Hindley & Seldin, 1986). In this approach one has three assignment rules: the primitive assignment and the recursive steps:

$$\frac{y \in \alpha \to \beta \quad x \in \alpha}{yx \in \beta} \qquad \frac{\begin{matrix}[x \in \alpha] \\ D \\ M \in \beta\end{matrix}}{\lambda x\, M \in \alpha \to \beta}\ \text{withdrawal of } x \in \alpha$$

We can pursue the analogy a little more by using two examples:

1.

$$\frac{\dfrac{\text{every} \in e \to t \to ((e \to t) \to t) \quad \text{logician} \in e \to t}{\text{every logician} \in (e \to t) \to t} \qquad \text{wanders} \in e \to t}{\text{every logician wanders} \in t}$$

2.

$$\frac{\text{a theorem} \in (e \to t) \to t \qquad \dfrac{\dfrac{\text{every logician} \in (e \to t) \to t \qquad \dfrac{x \in e \quad \text{proves} \in e \to e \to t}{\text{provesx} \in e \to t}}{\text{every logician provesx} \in t}}{\lambda x.\ \text{every logician provesx} \in e \to t}}{\text{a theorem } (\lambda x.\ \text{every logician provesx}) \in t}$$

9.8. FROM LG DERIVATIONS INTO TERMS. The correspondence between LG and the lambda calculus can be established as in 7.2. But now we are able to pay more attention to natural language details. Suppose we are focussing on a fragment of a natural language. Assume further that the initial assignment has been established. Then by each initial assignment $s \in \alpha$ we extend the vocabulary of the lambda calculus by adding the constant S of type α. The idea is now that every derivation with lexical assumptions as the only open assumptions, should be made to correspond to a closed term M. Thus the meaning of derivation (a) from 9.6. will be the term

A THEOREM (λX. EVERY LOGICIAN PROVEX).

Please, note the similarity between this term and the conclusion of derivation (b) from the previous section.

10. CONCLUDING REMARKS. In this chapter we have defined a Lambek system, LG, which will make Natural Logic possible. As a necessary preparation, we introduced the reader to LP. We have also introduced the Lambek Terms and provided an explicit proof of the correspondence between derivations in LP and LT. We have shown that strong normalisation is easily provable for the Lambek Terms, and that normalisation holds for LP.

The Lambek Grammar is the system which provides the vehicles of inference for Natural Logic. We have said that Natural Logic needs the combined development of syntax and semantics. We have also claimed that this variant of LP allows us to integrate syntax and semantics. Since we want to use LG derivations as vehicles of inference, we shall have to be precise about the semantics of the non-directed systems. This will be a central issue in the next chapter.

NOTES TO CHAPTER IV

[1] Of course, this syntactical difference does not affect the semantical interpretation. We shall let Common Nouns and Verb Phrases take their denotation in the same domain.

[2] The interpretation functions will express our '*meaning postulates*'.

CHAPTER V

MONOTONICITY AND MONOTONICITY MARKING

DESCRIPTION OF THE CONTENTS OF THE CHAPTER. In section 1 we introduce the themes to be considered. In section 2 we comment on the proper generalization of Peirce's syntactical criterion for semantic monotonicity. In section 3 we introduce the semantics of the typed lambda calculus. We prove that positive (negative) occurrence implies upward (downward) monotonicity in this calculus. We also comment on the natural inversion of this result: do positive (negative) positions exhaust all monotone sensitive positions? In Sections 4 and 5 we describe the tools that make LG a suitable basis for Natural Logic. In section 4 we describe a lambda language with monotone constants. This language will provide LG derivations with a *meaning* that mirrors their *denotation*. In Section 5 we describe a mechanism for the transformation of ordinary LG derivations into vehicles of inference: an algorithm for the marking of monotone sensitive positions.

1. INTRODUCTION

1.1. The previous Chapter was an introduction to the linguistic theory of Natural Logic. In the present Chapter we shall describe the additional tool which allows us to combine LG with inference: monotonicity marking. We have seen that Peirce uses a syntactical criterion for monotonicity: expressions which occur in syntactical *positive* position may be replaced by (semantically) stronger expressions; expressions which occur in syntactical *negative* positions may be replaced by (semantically) weaker expressions.
In this chapter we show that in the lambda calculus, too, monotonicity is tied up with properly defined positive and negative positions. Hence, it is possible to define a syntactic condition on terms, warranting the soundness of monotone substitutions.

We will also address a technically inverse question: does monotonicity imply positive (negative) occurrence?

As a preparation for the next chapter, we consider a lambda calculus with special constants: they are meant to be counterparts of natural language expressions. We then use constrained interpretation functions (in other words, '*meaning postulates*') to assure that the denotation of these constants reflects our semantical knowledge.

The correlation between derivations and terms allows us to transfer Peirce's criterion to the derivations themselves. We develop a mechanism which makes monotonicity visible at the level of grammatical forms. We show that a monotone position in a derivation corresponds with a monotone position in the associated term. This warrants the soundness of monotone substitution at the level of grammatical form, thus resolving the problem which vexed De Morgan and Leibniz. Moreover, the mechanism is a simple tool built into the construction of the expressions, thus avoiding the limitations of *suppositio* theory.

To provide the proper background to the theme of this chapter, we shall first consider monotonicity in the context of first-order logic -as a better-known pilot system.

2. GENERALIZING PEIRCE'S CRITERION FOR MONOTONICITY

2.1. MONOTONICITY IN FIRST-ORDER LANGUAGES. A survey of the literature shows that there are several alternative ways of defining *monotonicity* of a formula ϕ with some parameter R. One can define it with respect to

- either all models of ϕ or only one model;
- either all occurrences of R or only one of its occurrences;
- either arbitrary sub-formulas of ϕ or only predicates.

For first-order logic, Lyndon elucidated the precise relation between monotonicity and positive (negative) occurrences. In Lyndon (1959)

> monotonicity is a property which formulas may have with respect to *all* models and all the occurrences of a *predicate*in them.

2.1.1. DEFINITION OF GLOBAL MONOTONICITY. In Lyndon (1959) the notion of *upward monotone* (which he calls '*increasing*') first-order formulas is introduced:

> $\Gamma(R)$ is upward monotone in R iff $\forall x(R(x_1 \ldots x_n) \rightarrow S(x_1 \ldots x_n))$, $\Gamma(R) \models \Gamma(S)$, where $\Gamma(S)$ is the result of replacing the predicate R in $\Gamma(R)$ by S.
> (Notation : '$\forall x\, \phi$' denotes an arbitrary universal closure of ϕ).

REMARKS. This kind of monotonicity is called '*global*' because it may be read as saying that

> for *all models* $\mathfrak{A} = \langle A, I \rangle$ and interpretations I' if $\mathfrak{A} \models \Gamma(R)$ and $I(R) \subseteq I'(R)$, then $\langle A, I' \rangle \models \Gamma(R)$.

After having introduced this definition, Lyndon proves that there is a syntactic condition on $\Gamma(R)$ and R that implies upward monotonicity of $\Gamma(R)$ with respect to R. The syntactic condition is Peirce's definition of '*polarity*'. An occurrence of R in $\Gamma(R)$ is positive iff R occurs under an even number of negation symbols. A sentence $\Gamma(R)$ is positive in R if every occurrence of R in ϕ is positive. Then we have:

Proposition 1

If $\Gamma(R)$ is positive (negative) in R, then $\Gamma(R)$ is upward (downward) monotone in R.

Proof.

Induction on the complexity of $\Gamma(R)$.

□

Hence, positiveness implies upward monotonicity in first-order logic.
Similarly, negativeness implies downward monotonicity. However, the *direct* converse does not hold: for instance, $p \wedge \neg p$ is upward monotone in p but ϕ is not positive in p. Up to logical equivalence, however, Lyndon proves a converse of the above proposition:

Proposition 2 (Lyndon's Theorem)
If $\Gamma(R)$ is upward monotone in R, then $\Gamma(R)$ is equivalent to a sentence $\Gamma'(R)$ positive in R.
Proof
See Lyndon (1959).
□

In the remaining discussion we shall concentrate on upward monotonicity.

2.1.2. DEFINITION OF LOCAL MONOTONICITY. One may define monotonicity in the following way. Let $\mathfrak{A} = \langle A, I \rangle$, then

> $\Gamma(R)$ upward monotone w.r.t. R in $\mathfrak{A}$ if $(\mathfrak{A}, R) \models \Gamma$, $I(R) \subseteq I(R')$ implies $(\mathfrak{A}, R') \models \Gamma$.

this might hold for some special $\mathfrak{A}$, even if it does not hold uniformly.

EXAMPLE: $\exists x\, Qx \rightarrow \forall x \neg Rx$, with $\mathfrak{A} \not\models \exists x\, Qx$.

QUESTION (LOCAL LYNDON THEOREM): will there be a formula positive in R which is at least equivalent to $\Gamma(R)$ on all models $(\mathfrak{A}, R')$ for varying R' (but this fixed $\mathfrak{A}$)?

Van Benthem conjectures that although the answer may be negative in general, such a result holds on *finite* models.

2.1.3. MONOTONICITY AND OCCURRENCES. Lyndon's definition of monotonicity refers to all occurrences of a *predicate* in a formula. Analogous to his definition, one can define

> $\Gamma(R)$ as upward monotone in a *specified occurrence* of R iff
> $\forall x\, (R(x_1 \ldots x_n) \rightarrow S(x_1 \ldots x_n)), \Gamma(R) \models \Gamma(S)$, where $\Gamma(S)$ is the result of replacing the specified occurrence of the predicate R in $\Gamma(R)$ by S.

But this new notion can be reduced to the old one. We sketch a proof of this reduction here. Extend the language with the new predicate R' defined by $\forall x\, (R(x_1 \ldots x_n) \leftrightarrow R'(x_1 \ldots x_n))$. Now we have (cf. Kleene 1967 : 122):

> $\models \Delta(R) \leftrightarrow \Delta(R')$, where $\Delta(S)$ is the result of replacing a specified occurrence of the predicate R in $\Delta(R)$ by R'.

One can see that if $\Gamma(R')$ is upward monotone in R', then $\Gamma(R)$ is upward monotone in a specified occurrence of R.

2.1.4. REMARK. In the previous paragraph we used the notion of *'specified occurrence'* in a formula. A *specified occurrence* of a predicate R in $\Gamma(R)$ can be defined as the predicate R together with a code of its position in the construction tree of $\Gamma(R)$. An elegant and precise formulation can be found in Troelstra & Van Dalen (1988).

2.1.5. GLOBAL MONOTONICITY AND ARBITRARY FORMULAS. Until now we have restricted our attention to monotonicity in predicate letters. But we can say that for all formulas ψ

> a formula $\Gamma(\phi)$ is upward monotone in ϕ if $\forall \mathbf{x}(\phi \rightarrow \psi), \Gamma(\phi) \models \Gamma(\psi)$, where $\Gamma(\psi)$ is the result of replacing ϕ by ψ in $\Gamma(\phi)$, and ϕ is free for ψ in $\Gamma(\phi)$.

Like in the previous case we can reduce this definition to Lyndon's original formulation. We introduce new predicate letters F, Y with as many argument places as the formulas have free variables. We define F, Y by $\forall \mathbf{x}(\phi \leftrightarrow F(\mathbf{x}))$ and $\forall \mathbf{x}(\psi \leftrightarrow Y(\mathbf{x}))$ respectively. We have then $\models \forall \mathbf{x}(\phi \rightarrow \psi) \leftrightarrow \forall \mathbf{x}(F(\mathbf{x}) \rightarrow Y(\mathbf{x}))$, $\models \Gamma(\phi) \leftrightarrow \Gamma(F)$ and $\models \Gamma(\psi) \leftrightarrow \Gamma(Y)$. It follows that if $\Gamma(F)$ is (global)upward monotone in F, then $\Gamma(\phi)$ is upward monotone in ϕ.

2.1.6. GLOBAL MONOTONICITY AND SPECIFIED OCCURRENCE. Finally, one can also extrapolate to arbitrary formulas:

> $\Gamma(\phi)$ is upward monotone in a specified occurrence of ϕ if $\forall \mathbf{x}(\phi \rightarrow \psi), \Gamma(\phi) \models \Gamma(\psi)$, where $\Gamma(\psi)$ is the result of replacing the specified occurrence of ϕ by ψ in $\Gamma(\phi)$, and ϕ is free for ψ in $\Gamma(\phi)$.

But this case can be reduced to the previous one in the manner explained in 2.1.3. So, one can derive as a corollary to Proposition 1:

Proposition 3a

If some specified occurrence of ϕ is positive in $\Gamma(\phi)$, then $\Gamma(\phi)$ is upward monotone in that specified occurrence of ϕ.

One establishes in a similar manner:

Proposition 3b

If some occurrence of ϕ is negative in $\Gamma(\phi)$, then $\Gamma(\phi)$ is downward monotone in that specified occurrence of ϕ.

In Section 3 we shall define semantical monotonicity for the typed language in the format of Lyndon's definition. In practical cases, however, one employs monotonicity with respect to specified occurrences of arbitrary expressions. The discussion above shows that this case can be reduced to the most general one.

2.1.7. DIGRESSION. Proposition 3 can be used to prove the soundness of the following monotonicity rules:

M↑

$$\frac{\forall x(\phi \rightarrow \psi) \quad \Gamma(\phi)}{\Gamma(\psi)}$$

with ϕ positive in $\Gamma(\phi)$

M↓

$$\frac{\forall x(\phi \rightarrow \psi) \quad \Gamma(\psi)}{\Gamma(\phi)}$$

with ψ negative in $\Gamma(\psi)$

These rules belong to the folklore of modern logic, and they can be seen as generalizations of Modus Ponens and Modus Tollens. Peirce's monotonicity rules described in Chapter 3 are in fact special cases of M↑ and M↓. Modern writers have also occupied themselves with these rules. They are for example the *Dictum de Omni* in Sommers (1982 : 184), the *Semisubstitutivity of Conditional Rules* in Zeman (1967 : 484), and Theorem *24* in Kleene (1967 : 124). Even some years earlier, the rules were mentioned in Kleene (1952 : 154). In this last book we are referred to Curry (1939 : 290-91) for another version of the rules. Curry, in turn, refers us to Herbrand (1930) and Maclane (1934). Finally, the first post-Fregean reference to the rules which we have found is Behmann (1922 : 172-174).

In general, these versions of the monotonicity rules show two main divergencies. First, there is a weak version in which the premisses of the rules are provable formulas (Behmann; Curry; Herbrand; Kleene, 1967; Zeman). There is also a strong version in which the premisses are assumptions (Kleene, 1952; Maclane; Sommers).

Secondly, there is a version in which the substitution affects all the occurrences of ϕ in $\Gamma(\phi)$ (Herbrand, Maclane) and another in which the substitution affects only one specified occurrence of ϕ (Behmann; Curry; Kleene; 1952, 1967; Sommers; Zeman).

2.2. GENERALIZING PEIRCE'S CRITERION. Counting negations is a coarse criterion for monotonicity - even in the context of first-order logic. For instance in $p \rightarrow q$, the formula p is positive in the sense that it occurs under no negations - but it is obviously not upward here. As a matter of fact, the above definition of positiveness is confined to languages with $\neg$, $\wedge$ and $\vee$ as their only constants. If we wish to incorporate $\rightarrow$ as a constant, a definition of '*polarity*' has better take the following form:

DEFINITION OF POLARITY

(i) R occurs positively in $R(t_1, \ldots, t_n)$.

(ii) If R occurs positively (negatively) in ϕ, then R occurs positively (negatively) in $\phi \wedge \psi$, $\phi \vee \psi$, $\psi \rightarrow \phi$, $\forall x\phi$, $\exists x\phi$.

(iii) If R occurs positively (negatively) in ϕ, then R occurs negatively (positevely) in $\neg\phi$, $\phi \rightarrow \psi$.

Given this definition, the polarity names 'positive' or 'negative' lose the direct connotation of occurrence under negations. The picture that emerges is that of expressions occurring under the scope of expressions denoting monotone functions. This idea allows one to generalize polarity to other languages. Consider a language having the symbols A, B and a as its vocabulary. The formation rules are:

(i) A is a formula.
(ii) If ϕ is a formula, then $A(\phi)$, $B(\phi)$ are formulas.

Let A denote an upward monotone function and B denote a downward monotone function. We define polarity of a formula occurrence as follows:

(i) ϕ is positive in ϕ.
(ii) If ϕ is positive in $F(\phi)$, then it is positive in $A(F(\phi))$ and negative in $B(F(\phi))$.
(iii) If ϕ is negative in $F(\phi)$, then it is negative in $A(F(\phi))$ and positive in $B(F(\phi))$.

The discussion on monotonicity in the typed lambda calculus, will use the format (i)-(iiii).

3. MONOTONICITY IN THE TYPED LAMBDA CALCULUS

3.1. SEMANTICS FOR THE LAMBDA CALCULUS. Up till now, we have been preoccupied with monotonicity in first-order logic. To say something about monotonicity and the typed lambda calculus, we need to introduce the semantics of the lambda calculus. Before doing so, let us give a broad characterization of the intended semantics.

Expressions belonging to the basic types take their denotation in fixed domains; expressions belonging to complex types are considered as functions. Well-formed expressions consisting of the concatenation of two expressions, are interpreted as the result of applying the denotation of the left expression to the denotation of the right one. More precisely:

3.2. DEFINITION OF THE FREGEAN UNIVERSE. Let D be a non-empty set. Then D_α is given for all types α by the following recursion:
(i) $D_e = D$
(ii) $D_t = \{0,1\}$
(iii) $D_p = D_{e \to t}$
(iv) $D_{\alpha \to \beta} = {D_\beta}^{D_\alpha}$, the collection of set theoretic functions from D_α to D_β.

3.3. DEFINITION OF AN ORDERING OF THE FREGEAN UNIVERSE. To compare the denotations of expressions, we partially order the sets D_α by a relation $\leq_\alpha$ as follows:
(i) If c, $d \in D_e$ then $c \leq_e d$ iff $c = d$.
(ii) If c, $d \in D_t$ then $c \leq_t d$ iff $c = 0$ or $d = 1$.
(iii) If c, $d \in D_{\alpha \to \beta}$ then $c \leq_{\alpha \to \beta} d$ iff for each $a \in D_\alpha$, $c(a) \leq_\beta d(a)$.

3.4. DEFINITION OF MODELS. A **typed model** is a pair $\langle \{D_\alpha\}, I \rangle$, where
(i) $\{D_\alpha\}$ is a Fregean universe.
(iii) I is a function on the set of all constants such that $I(C_\alpha) \in D_\alpha$.

3.5. DEFINITION OF ASSIGNMENTS.
(i) An assignment is a function f on the set of all variables, such that $f(X_\alpha) \in D_\alpha$.
(ii) If Y is any variable, f an assignment, then [a/Y]f is the assignment given by

$$[a/Y]f = \left| \begin{array}{l} [a/Y]f(X) = f(X) \text{ if } X \neq Y \\ [a/Y]f(X) = a \text{ if } X = Y \end{array} \right.$$

(iii) The set of all assignments will be called Ass.

3.6. DEFINITION OF THE DENOTATION FUNCTION. We define the notion of the denotation of an expression M of type α with regard to a model $\mathfrak{A}$ and an assignment f, (Notation : $[\![M]\!]_f$). This denotation will always belong to D_α.
$[\![M]\!]_f$ is given by the following recursion:
(i) $[\![M]\!]_f = f(M)$ when M is a variable.
(ii) $[\![M]\!]_f = I(M)$ when M is a constant.
(iii) $[\![MN]\!]_f = [\![M]\!]_f([\![N]\!]_f)$, when M has type $\alpha \to \beta$ and N type α.
(iv) When M is of type β, and X is type α then $[\![\lambda X.M]\!]_f$ is that function in $D_{\alpha \to \beta}$ such that for all $a \in D_\alpha$: $[\![\lambda X.M]\!]_f(a) = [\![M]\!]_{[a/X]f}$.

3.7. DEFINITION OF STRUCTURES. A **structure** S is a pair $\langle A, [\![\ ]\!] \rangle$ of a model and a denotation function into it.

We now turn to some notions of monotonicity in this setting.

3.8. UPWARD MONOTONE FUNCTIONS. A function $z \in D_{\alpha \to \beta}$ is upward monotone iff for every $x, y \in D_\alpha$, $x \leq_\alpha y$ entails $z(x) \leq_\beta z(y)$.

3.9. DOWNWARD MONOTONE FUNCTIONS. A function $z \in D_{\alpha \to \beta}$ is downward monotone iff for every $x, y \in D_\alpha$, $x \leq_\alpha y$ entails $z(y) \leq_\beta z(x)$.

The definition of monotone functions is language independent. In a natural way, expressions associated with these functions can be called *monotone*. We are going to give a definition of monotonicity similar to the definition introduced by Lyndon for predicate logic.
Assume that N'_α is like N_α except for containing an occurrence of M'_β *wherever* N_α contains M_β:

3.10.1. UPWARD MONOTONE TERMS. N_α is upward monotone in M_β iff for all models and assignments, $[\![M]\!]_f \leq_\beta [\![M']\!]_f$ entails $[\![N]\!]_f \leq_\alpha [\![N']\!]_f$.

3.10.2. DOWNWARD MONOTONE TERMS. N_α is downward monotone in M_β iff for all models and assignments, $[\![M']\!]_f \leq_\beta [\![M]\!]_f$ entails $[\![N]\!]_f \leq_\alpha [\![N']\!]_f$.

3.10.3. REMARK. The convention on variables warrants that M is free for M' in N. Notice that this means that N' may differ from N not only with respect to M, but also with respect to the bound variables. Thus, strictly spoken, we should say that N' is like N, modulo α-conversion, except for containing an occurrence of M'_β wherever N_α contains M_β.

The definition of monotone terms corresponds to the notion introduced in 2.1.1. It is also possible to define monotonicity with regard to specified occurrences. Assume that N'_α is like N_α except for containing an occurrence of M'_β where N_α contains a specified occurrence of M_β. We shall refer to this specified occurrence of M in N by **M**. We have the following definitions:

3.11. UPWARD MONOTONE OCURRENCE. N_α is *upward monotone* in $\mathbf{M}_\beta$ iff for all models and assignments, $[\![M]\!]_f \leq_\beta [\![M']\!]_f$ entails $[\![N]\!]_f \leq_\alpha [\![N']\!]_f$.

3.12. DOWNWARD MONOTONE OCCURRENCE. N_α is *downward monotone* in M_β iff for all models and assignments, $[\![M']\!]_f \leq_\beta [\![M]\!]_f$ entails $[\![N]\!]_f \leq_\alpha [\![N']\!]_f$.

The new definitions boil down to the previous ones: one adds to the system the equation $M = M^*$, for a new M^*. This means that one has the right to replace *any* occurrence of M in a term by M^* and vice-versa. Let N^* be the the term that results from replacing M^* for an specified occurrence of M in N. It follows that if N^* is monotone in M^* in the old sense, then N is monotone in **M** in the new sense.

In the following definition we provide the basis for the definition of monotone syntactic positions.

3.13. DEFINITION OF ACTIVE OCCURRENCE. An occurrence of M_β, $\mathbf{M}_\beta$, is called *active* according to the following clauses:

(i) **M** is active in **M**.

(ii) **M** is active in PQ iff **M** is active in P.

(iii) **M** is active in $\lambda X.P$ iff **M** is active in P and $X \notin FV(M)$.

3.13.1. REMARK. Van Benthem (1986) defines the positive occurrences of a free variable X_α in a term Q as follows:

(i) X_α occurs positively in X_α.

(ii) If X_α occurs positively in Q then also in QP .

(iii) If X_α occurs positively in Q and $X \neq Y$, then also in $\lambda Y.\, Q$.

In fact, our definition of active occurrence is a generalization of Van Benthem's definition to occurrences of terms. If X_α occurs positively in Q , then X_α is active in this term. The converse does not hold because we allow complex terms to be active.

3.14. ACTIVE OCCURRENCE IMPLIES UPWARD MONOTONICITY. Van Benthem (1986) shows that if X_α occurs positively in N , then N is upward monotone in this variable. Van Benthem's proof can be adapted to cases in which an arbitrary term is active.

Proposition 4.
If $\mathbf{M}_\beta$ is active in N_α, then N_α is upward monotone in $\mathbf{M}_\beta$.

Proof

(i) *Basis step.* If $M_\beta \equiv N_\alpha$ then the assertion holds evidently.

(ii) *Inductive step.* Assume $[\![M]\!]_f \leq_\beta [\![M']\!]_f$. There are two cases to consider.

(A) Let $N_\alpha \equiv P_{\gamma \to \alpha} Q_\gamma$. Since **M** is active in N , by definition, **M** will be active in P . By the inductive hypothesis, P is upward monotone in **M** . Therefore, by definition, $[\![P]\!]_f \leq_{\gamma \to \alpha} [\![P']\!]_f$. Hence by definition of $\leq_{\gamma \to \alpha}$, and $[\![\ ]\!]$, $[\![PQ]\!]_f \leq_\alpha [\![P'Q]\!]_f$.

(B) Let $N_\alpha \equiv \lambda X_\gamma.\, P_\delta$. Since **M** is active in N , **M** is active in P and $X_\gamma \notin VV(M)$.
By inductive hypothesis P is upward monotone in **M** . It is to be shown that $[\![\lambda X_\gamma.\, P]\!]_f \leq_{\gamma \to \delta} [\![\lambda X_\gamma.\, P']\!]_f$. By definition of $\leq_{\gamma \to \alpha}$, and truth, it suffices to show that $[\![P]\!]_{[a/X]f} \leq_\delta [\![P']\!]_{[a/X]f}$ for all $a \in D\gamma$. But the latter follows by the inductive hypothesis, applied to P , P' and $[a/X]f$.

□

3.15. COROLLARY 1. If X_γ is active in N_δ then the denotation of $\lambda X.N$ is an upward monotone function.

In a sense, Proposition 4 generalizes the *structural monotonicity* introduced in Chapter I. 4.4.1. We said there that the expressions $x \in y$ are upward monotone in y -independently of the particular monotone properties of y itself. We have obtained the result that MN is upward monotone in M -independently of the monotone properties of the object that one assigns to M . Notice that Proposition 4 is not quite analogous to Propositions 1 or 3. An analogous proposition for the typed language, must allow that terms occurring in (embedded) argument positions could have monotone properties.

We shall give some definitions that enable us to prove in our system a result analogous to Proposition 3.

3.16.1. POLARITY OF OCCURRENCES. Assume that the language contains constants denoting monotone functions. A specified occurrence $\mathbf{M}_\beta$ of M_β is called *positive (negative)* according to the following clausules:

(i) **M** is positive in **M** .

(ii) **M** is positive (negative) in PQ if **M** is positive (negative) in P .

(iii) **M** is positive (negative) in PQ if **M** is positive (negative) in Q , and P denotes an upward monotone function.

(iv) **M** is negative (positive) in PQ if **M** is positive (negative) in Q , and P denotes a downward monotone function

(v) **M** is positive (negative) in $\lambda X.P$ if **M** is positive (negative) in P and $X \notin FV(\mathbf{M})$.

3.16.2. POLARITY OF OCCURRENCES. A term N is positive (negative) in M iff all the occurrences of M in N are positive (negative).

3.17. POLARITY IMPLIES MONOTONICITY.

Proposition 5

If N_β is positive (negative) in M_α, then N_β is upward (downward) monotone in M_α.

Proof

The only new cases as compared to Proposition 4 are argument positions:

Assume the proposition for terms of smaller complexity than N .

(A) $N_\beta \equiv P_{\gamma\to\beta}Q_\gamma$. There are two cases to consider.

(a) N is positive in M . We want to prove that N is upward monotone in M . We have two sub-cases to consider.

(a.1.) M is a sub-term of P but not of Q . Since N is positive in M , it follows that P is positive in M . By inductive hypothesis we have that P is upward monotone in M . The rest is as in Proposition 4.

(a.2.) M is a sub-term of Q but not of P . There are again two sub-cases.

(a.2.1.) Q is positive in M . Since N is positive in M , it follows that P must denote an upward monotone function. By the inductive hypothesis, $[\![M]\!]_f \leq [\![M']\!]_f$ entails $[\![Q]\!]_f \leq [\![Q']\!]_f$. Therefore $[\![PQ]\!]_f \leq [\![PQ']\!]_f$, since $[\![P]\!]_f$ is upward monotone.

(a.2.2.) Q is negative in M . Since N is positive in M , it follows that P must denote a downward monotone function. By the inductive hypothesis $[\![M]\!]_f \leq [\![M']\!]_f$ entails $[\![Q']\!]_f \leq [\![Q]\!]_f$. But $[\![P]\!]_f$ is downward monotone. Therefore $[\![PQ]\!]_f \leq [\![PQ']\!]_f$.

(a.3.) M is a sub-term of P and Q . Hence, as above, P is upward monotone in M . Thus $[\![P]\!]_f \leq [\![P']\!]_f$. By definition $[\![PQ']\!]_f \leq [\![P'Q']\!]_f$. As in (a.2.2.) we have two possibilities:

(a.3.1.) $[\![Q]\!]_f \leq [\![Q']\!]_f$ and P denote an upward function. Hence $[\![PQ]\!]_f \leq [\![PQ']\!]_f$. Therefore $[\![PQ]\!]_f \leq [\![P'Q']\!]_f$.

(a.3.2.) $[\![Q']\!]_f \leq [\![Q]\!]_f$ and P denote a downward function. Hence $[\![PQ]\!]_f \leq [\![PQ']\!]_f$. Therefore $[\![PQ]\!]_f \leq [\![P'Q']\!]_f$.

(b) N is negative in M . Similar.

□

3.18. COROLLARY.

Proposition 5 implies Proposition 4.

□

3.19. COROLLARY.

If X is positive (negative) in N , then $\lambda X.N$ is upward (downward) monotone.

Proof

Similar to the proof of 2.15., but one uses Proposition 5 rather than 4.

□

3.20. PRESERVATION. Having answered the question whether positive (negative) occurrence implies upward (downward) monotonicity, we can ask conversely whether the syntactic notion of polarity exhaustively describes all semantically monotone positions. For instance, does the following analogy of Lyndon's theorem hold in the lambda calculus:

> M_α is upward monotone in X_β iff there is a term M'_α equivalent to M_α such that X_β is active in M'_α.

Van Benthem (1991) shows that this assertion fails for the full lambda calculus: the term $X_{t \to t}(X_{t \to t} Y_t)$ is upward monotone in $X_{t \to t}$ without being definable by a term in which $X_{t \to t}$ occurs only actively. What can be shown however, is a similar Lyndon Theorem for Lambek terms M_α with regard to variables X_β of a '*Boolean type*':

$$\beta \equiv \gamma_1 \to \gamma_2 \to \ldots \to t .$$

Another related question is whether the following proposition holds for the full lambda calculus with Boolean parameters:

> M_α is upward monotone in X_β iff there is a term M'_α equivalent to M_α such that M'_α is *positive* in X_β.

Answering this question may be extremely tricky. Van Benthem (personal communication) devised the following example of a term $X_{t\rightarrow t}(X_{t\rightarrow t}Y_t)$ upward monotone in $X_{t\rightarrow t}$ but definable by a term C in which $X_{t\rightarrow t}$ occurs only positively:

1. $X_{t\rightarrow t}(X_{t\rightarrow t}(0)) = X_{t\rightarrow t}(0) \wedge X_{t\rightarrow t}(1)$ [A]

2. $X_{t\rightarrow t}(X_{t\rightarrow t}(1)) = X_{t\rightarrow t}(1) \vee X_{t\rightarrow t}(0)$ [B]

3. $X_{t\rightarrow t}(X_{t\rightarrow t}Y_t) = (Y_t \wedge B) \vee (\neg Y_t \wedge A)$ [C]

Nevertheless, Van Benthem conjectures that the preservation theorem will fail for the full lambda calculus with Boolean constants, but that it will go through for Lambek terms with Boolean constants.

4. AN APPLIED TYPED LANGUAGE

4.1. MONOTONICITY IN FORMAL SEMANTICS. We have by now established a connection between polarity and monotonicity. If we want to use this connection, we need to know which terms denote monotone functions and which natural language expressions take such functions as denotation. In the field of formal semantics the notion of monotonicity has been used primarily to characterize the behaviour of determiners and noun phrases. Monotonicity is interesting for Natural Logic, however, because its role is not confined to one category: almost all linguistic categories contain monotone items. In this section we collect monotonicity information about selected lexical items. We shall do this by sketching a typed language which contains constants with monotone properties. In the next chapter we shall see that this extended type language is a necessary tool for the construction of Natural Logic.

4.2. THE TYPED LANGUAGE. Consider a typed language containing the following constants:

constants	type
(1) ABELARD, HELOISE	e
(2) LOGICIAN, THEOREM, THING, MAN, MEN, HEAD	p
(3) WANDER, WALK, RUN	$e \rightarrow t$
(4) NOT	$t \rightarrow t$
(5) FEMALE, MALE, TALL, SMALL	$p \rightarrow p$

(6) PROVE, LOVE, IS $e \to e \to t$

(7) EVERY, A, NO, MOST, FEW, THE $p \to (e \to t) \to t$

(8) THAT $p \to (e \to t) \to p$

(9) OF $(e \to t) \to t) \to p \to p$

(10) IN, AT, ON, WITH, WITHOUT $((e \to t) \to t) \to (e \to t) \to (e \to t)$

Expressions which can be constructed with this language do resemble English expressions. An expression of this language is λx_e.A LOGICIAN(IS X) . This expression may be seen as the meaning of a derivation of *is a logician*. We can also exploit the fact that the typed language has a semantics to provide the derivations with a denotation:

4.2.1. DEFINITION. If D is a derivation from LG and N is the term that corresponds to D, then $[\![N]\!]$ will be called the denotation of D.

EXAMPLE. We say that the meaning of a derivation of *is a logician* is λx_e.A LOGICIAN(IS X) and that $[\![\lambda x_e$.A LOGICIAN (IS X)$]\!]$ is its denotation.

4.3. INTERPRETATION OF THE NEW CONSTANTS. In this section we use the interpretation function I to codify our semantical knowledge.

(1)
I(EVERY) , I(A) , I(NO) , I(MOST) , I(FEW) , I(THE) are those functions in $D_{p \to (e \to t) \to t}$ such that for any $x \in D_p$:
(a)I(EVERY)(x)(y) = 1 iff $x \subseteq y$.

(b) I(A)(x) (y) = 1 iff $x \cap y \neq \emptyset$.

(c) I(NO)(x)(y) = 1 iff $x \cap y = \emptyset$.

(d)I(MOST)(x)(y) =1 iff $\mathrm{card}| x \cap y | > \mathrm{card}| x - y |$.

(e) I(FEW)(x)(y) =1 iff $\mathrm{card}| x \cap y | < \mathrm{card}| x - y |$.

(f) I(THE)(x)(y) =1 iff $\mathrm{card}| x | = 1$ and $x \subseteq y$.

(2)
(a) I(THING) = D_p .

(b) I(IS) is that function on $D_{e \to (e \to t)}$ such that for $x, y \in D_e$, I(IS)(x)(y) = 1 iff $x = y$.

(c) I(THAT) is that function in $D_{(e \rightarrow t) \rightarrow (p \rightarrow p)}$ such that for any $x \in D_{e \rightarrow t}$, $y \in D_p$.

$I(THAT)(x)(y) = x \cap y$.

(d) I(NOT) is that function in $D_{t \rightarrow t}$ such that for any $x \in D_t$, $I(NOT)(x) = 1 - x$.

(3)
(a) I(FEMALE), I(MALE) are introspective upward functions on $D_{(e \rightarrow t) \rightarrow (e \rightarrow t)}$.

(b) I(SMALL), I(TALL) are introspective functions on $D_{(e \rightarrow t) \rightarrow (e \rightarrow t)}$.

(4)
(a) I(IN), I(AT), I(WITH), I (OF) are upward functions on
$D_{((e \rightarrow t) \rightarrow t) \rightarrow ((e \rightarrow t) \rightarrow (e \rightarrow t))}$ such that
I(IN)(x), I(AT)(x), I(WITH)(x) are upward introspective functions and I(OF)(x) is an upward function.

(b) I(WITHOUT) is a downward function in $D_{((e \rightarrow t) \rightarrow t) \rightarrow ((e \rightarrow t) \rightarrow (e \rightarrow t))}$ such that I(WITHOUT)(x) is an upward introspective function.

(5)
If A_α is a constant not mentioned in (1)-(4), then I(A) is simply fixed as a member of D_α.

4.3.1. REMARKS ON THE INTERPRETATION. (A) The interpretation of the *determiners* is the standard interpretation in the generalized quantifier approach to natural language quantification. It follows that the denotation of the *determiners* have specific monotone properties. The interpretations of NO and EVERY are downward monotone functions, while the denotation of A is an upward monotone function. On the other hand, the interpretations of the *noun phrases* EVERY X, A X, MOST X, THE X are upward monotone functions, while the denotations of NO X, FEW X are downward monotone functions.

Incidentally, notice that if I(MOST)XY denotes the truth, then X can not be the empty set. If it were then we would have the absurdity that $0 > 0$. This makes plain that this definition of MOST is not compatible with the intuition that MOSTXX always denotes the truth, (cf. Barwise and Cooper 1973). We stick to the above definition, since we consider a sentence like *most unicorns are unicorns* to be false.

(B) The interpretation of THING is the set D_p, which we have defined as being the same as the set $D_{e \rightarrow t}$. The interpretation of IS is the diagonal on D_e. The interpretation of THAT allows us to give a denotation to combinations of *nouns* and *intransitive verbs*. A more insightful representation can be given by assuming: $I(THAT)ZY = [\![\lambda X_e\, Z_{e \rightarrow t}(X) \wedge V_p(X)]\!]$. For combinatorial reasons we need a denotation for THING. We know, for instance, that I(A)XY is the same as I(A)YX. But whereas *a woman wanders* is English, *a wanders woman* is not. The best we can obtain *is a thing that wanders is a woman*. The interpretation of THING warrants that the denotations of (AWANDERS)WOMAN will be the denotation of a derivation of *a thing that wanders is a woman*. The denotation of NOT is a downward monotone function.

(C) The denotations of the *adjectives* S MALL , TALL , FEMALE and MALE are introspective functions.[1] This definition embodies the familiar insight that, for instance, ELEPHANT is entailed by SMALL ELEPHANT as well as by MALE ELEPHANT. The difference between the *absolute adjectives* MALE an d SMALL lies in their monotonicity. All absolute adjectives denote upward functions. This definition embodies the familiar insight that whereas MALE ELEPHANT entails MALE ANIMAL , SMALL ELEPHANT does not entail SMALL ANIMA L.

(D) The denotation of the prepositions reflect the intuition that (most) *adverbial phrases* are introspective. One agrees that the sentence *Abelard works in the convent with Heloise* entails *Abelard works in the convent* as well *Abelard works with Heloise* and *Abelard works*. We have also taken the denotations of WITH , OF and IN t

o be upward monotone. We want to explain the relation between *works with Heloise* and *works with a woman*: the first entails the second. On the other hand, we want to explain that *without a knife* entails *without a sharp knife*. We accomplish this by assuring that WITHOUT denotes a downward function.

We now have at our disposal a set of terms that we know the monotone properties of. This allows us to predict, for instance, that NOT(EVERY MAN)LOVES is upward monotone in MAN and downward monotone in LOVES. The next section makes this information available at the level of LG derivations.

5. MONOTONICITY MARKING

5.1. MOTIVATION BEHIND THE RULES. We have established that in the lambda calculus positive occurrences are monotone sensitive positions. In this section we describe a mechanism which transfers that information to LG objects. We identified some constants of the typed language as having monotone properties. The corresponding derivation will display this information. Let us give the motivation behind the marking mechanism to be described presently. Through the correspondence with the Lambek terms (see Chapter IV), we know that the major of a Modus Ponens application, corresponds to the head of an application. As we already know, this head is upward monotone. This explains the convention introduced at 5.3. (i). The argument of an upward monotone function occurs in an upward monotone sensitive position, and the argument of a downward monotone function occurs in an downward monotone position. This explains the conventions introduced in 5.3. (ii). The introduction of an abstraction does not alter the monotonicity of the terms occurring in the *body* of the abstraction.

5.2. INTERNAL MONOTONICITY MARKING. We introduce a notation which allows us to use the assignments to produce the intended interpretations of natural language expressions:

(i) if α, β are categories, then $\alpha^{+} \to \beta$, $\alpha^{-} \to \beta$, and $\alpha \to \beta$ are categories.

(ii) If A is an expression assigned to the category $\alpha^{+} \to \beta$, then the meaning of the derivation of A, is an upward monotone function in $D_{\alpha \to \beta}$.

(iii) If A is an expression assigned to the category $\alpha^{-} \to \beta$, then the meaning of the derivation of A, is a downward monotone function in $D_{\alpha \to \beta}$.

(iv) If A is an expression assigned to the category $\alpha \to \beta$, then the meaning of the derivation of A, is an arbitrary function in $D_{\alpha \to \beta}$.

5.3. EXTERNAL MONOTONICITY MARKING. We will introduce a procedure which allows us to transform ordinary Lambek derivations into derivations which display the polarity of the assumptions used in the derivation.

(i) The major premiss in a Modus Ponens application is positive in the relevant derivation:

$$\frac{\alpha \to \beta \quad \alpha}{\alpha} \quad \Rightarrow \quad \frac{\underset{+}{\alpha \to \beta} \quad \alpha}{\alpha}$$

(ii) The minor premise in an application is positive if the major is the category $\alpha^{+} \to \beta$:

$$\frac{\alpha^{+} \to \beta \quad \alpha}{\beta} \quad \Rightarrow \quad \frac{\underset{+}{\alpha^{+} \to \beta} \quad \underset{+}{\alpha}}{\beta}$$

(iii) The minor premise in an application is negative if the major is the category $\alpha^{-} \to \beta$:

$$\frac{\alpha^{-} \to \beta \quad \alpha}{\beta} \quad \Rightarrow \quad \frac{\underset{+}{\alpha^{-} \to \beta} \quad \underset{-}{\alpha}}{\beta}$$

(iv) The withdrawal of a numerical index leaves the previous marking unchanged. This will be indicated by putting a + symbol below the last but one node:

$$\begin{array}{c} [\alpha^{i}] \\ D_1 \\ \text{(i)}\dfrac{\beta}{\alpha \to \beta} \end{array} \quad \Rightarrow \quad \begin{array}{c} [\alpha^{i}] \\ D_1 \\ \beta \\ \text{(i)}\dfrac{+}{\alpha \to \beta} \end{array}$$

(v) Assume that each node from α^i to β is either positive or negative. If the number of positive nodes is even, then

$$\begin{array}{c}[\alpha^i]\\ D_1\\ (i)\dfrac{\beta}{\alpha \to \beta}\end{array} \quad \Rightarrow \quad \begin{array}{c}[\alpha^i]\\ D_1\\ \beta\\ (i)\dfrac{+}{\alpha^{+} \to \beta}\end{array}$$

(vi) Assume that each node from α^i to β is either positive or negative. If the number of positive nodes is odd, then

$$\begin{array}{c}[\alpha^i]\\ D_1\\ (i)\dfrac{\beta}{\alpha \to \beta}\end{array} \quad \Rightarrow \quad \begin{array}{c}[\alpha^i]\\ D_1\\ \beta\\ (i)\dfrac{+}{\alpha^{-} \to \beta}\end{array}$$

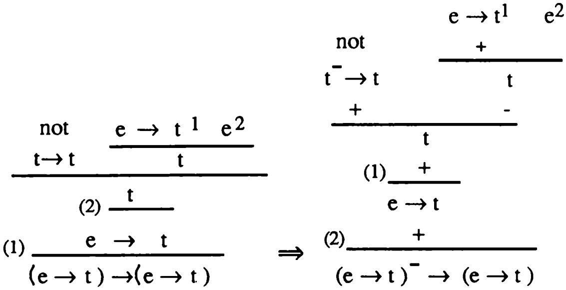
5.4. EXAMPLES. The following examples show the result of marking LG derivations. If *a* is a lexical item, then in the meaning of the derivation *A* stands for the Lambek term corresponding to *a*.

(1)

$$(1)\dfrac{\dfrac{e \to t^1 \quad \overset{\text{abelard}}{e}}{t}}{(e \to t) \to t} \quad \Rightarrow \quad (1)\dfrac{\dfrac{\overset{+}{e \to t^1} \quad \overset{\text{abelard}}{e}}{t}\ +}{(e \to t)^{+} \to t}$$

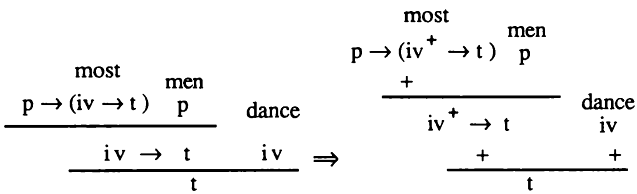
Meaning: $\lambda Y_{e \to t}.Y(\text{ABELARD})$

(2)



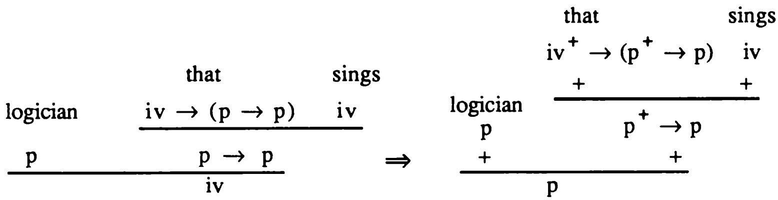
Meaning: $\lambda Y_{e \rightarrow t}.\lambda Z_e.\text{NOT}(YZ)$

(3)



Meaning: (MOST MEN) DANCE

(4)



Meaning: (THAT SINGS) LOGICIAN

5.5. POLARITY OF SUBDERIVATIONS. Let D be a derivation with conclusion α.

(i) A **node** γ has polarity iff all the nodes in the path from γ to α are marked and

(ii) A **node** γ is positive iff (a) γ has polarity, and (b) the number of nodes marked by '-' is even.

(iii) A **node** γ is negative iff (a) γ has polarity, and (b) the number of nodes marked by '-' is odd.

(iv) A **proper sub-derivation** D_1 with conclusion β is positive(negative) if β is positive (negative).

5.6. EXAMPLES. We consider the examples from 5.4. For perspicuity's sake, we shall refer to the sub-derivations by using the *lexical assumption.* We shall also compare the results of definition 3.16.

In (1) *Abelard* is negative nor positive. In the meaning of (1) ABELARD is neither positive nor negative.

In (2) *not* is positive. In the meaning of (2) NOT is positive. Notice that in λZ_e.NOT(YZ), Y is negative. Hence $\lambda Y_{e \rightarrow t}$.NOT(YZ) denotes a downward function.

In (3) *most, most men* and *dance* are positive; while *men* is positive nor negative. In the meaning of (3) MEN is positive nor negative. On the other hand, DANCE, MOST and MOST MEN are positive.

The definition of polarity of subderivations is intended to warrant the following propositions.

Let D be a derivation with conclusion α and let N_α be its meaning. Let D_1 be a sub-derivation of D with conclusion β, and M_β its meaning.

Proposition 8

(i) If D_1 is positive in D, then N_α is upward monotone in M_β.
(ii) If D_1 is negative in D, then N_α is downward monotone in M_β.
We will give a sketch of a proof of (i):
If D_1 is a positive sub-derivation of D, then the path from β to α is marked with an even number of minus symbols. This means that an even number of categories $\delta^- \rightarrow \gamma$ has been applied to get from β into α. The terms corresponding to these categories denote downward functions. Hence M is positive in N. By using Proposition 2 we deduce that N is upward in M.

□

Let us conclude our preliminary work on Natural Logic by showing that withdrawal of positive(negative) assumptions corresponds to the construction of an upward(downward) monotone function.

Assume D be a derivation with conclusion $\alpha \rightarrow \beta$. Assume that the last step in D is the withdrawal of α. Then D contains a subderivation D_1 with conclusion β, and the open assumption α. Let M_β be the meaning of D_1. One can prove

Proposition 9

(i) If α is positive in D_1, then $\lambda X_\alpha.M_\beta$ is an upward function.

(ii) If α is negative in D_1, then N_α is a downward function M_β .

proof

This follows from Proposition 8, and 3.19.

□

6. CONCLUDING REMARKS. In this chapter we discussed the notion of monotonicity as it is in first order logic. We also discussed a generalization of Peirce's criterion for monotonicity to the lambda calculus. We proved that in the lambda calculus positive (negative) occurrence implies upward (downward) monotonicity. It is important to notice that Proposition 5 is the cornerstone of a monotone Natural Logic. We must emphasise that this proposition has been proven for the full lambda system. Therefore, the monotone mechanism will still be available for *extensions* of the Lambek mechanism -a theme in Chapter VII. Besides, we have discussed the possibility of proving the converse of the above result for the Lambek terms. We noticed that Van Benthem (1991) resolved this question for the full lambda system and for the Lambek fragment.

We were able to extend Peirce's criterion to the objects of the Lambek systems as well - which is due to the correspondence between terms and derivations. Finally, we have described a mechanism for the transformation of ordinary LG derivations into vehicles of inference. The basis for Natural Logic has been laid down. In the next Chapter we develop our proof system for natural language.

Notes to Chapter V

[1]To say that f is an introspective function symply means that f(X) ≤ X. This property is called '*restrictive*' in Keenan & Faltz (1985).

CHAPTER VI

A SYSTEM OF NATURAL LOGIC

DESCRIPTION OF THE CONTENTS OF THE CHAPTER. In the first section we describe the purpose of this chapter: the construction of a Natural Logic. In the second section we define a fragment of English for which we construct a Natural logic. In the third section we define Natural Logic itself. In the fourth section we list some examples of inferences produced by the proof system. In the fifth section we incorporate conservativity in Natural Logic. In the sixth section we consider inferences in which scope becomes a relevant factor. In the seventh section we show the combined effect of monotonicity and conservativity.

1. Introduction.

1.1. GENERAL DESCRIPTION. The purpose of this chapter is the construction of a Natural Logic. Our Natural Logic uses the inferential information carried by the marked LG derivations. In a marked LG derivation we see immediately which expressions are replaceable. An upward monotone (+) expression will be replaceable by an expression with a larger denotation. Similarly, a downward monotone (−) expression will be replaceable by an expression with a smaller denotation. This will be the prime mechanism of our Natural Logic: every substitution uses the monotonicity information encoded in the syntactical construction, and the ≤-information provided by the underlaying semantics. Hence, the inference rules need both syntactic information about monotonicity and semantic information about the ≤-relation. The way in which we obtain this information has been described in the previous chapter.

In the second section we define a disambiguated language called Formal English (FE) whose expressions are LG derivations. The semantics of FE is obtained by mapping the lexical items of FE into the constants of the typed language introduced in Chapter V. In fact, the association of LG derivations with expressions of that language takes care of the semantics. In the third section we discuss the appropriate formulation of the monotonicity rules and we introduce a couple of abbreviations which allows for a simple formulation. In the fourth section we illustrate the strength of the proof system. In the fifth section we add to Natural Logic the principle of conservativity -one of Peirce's principles of inference. In the sixth section we discuss the problem that scope poses to Natural Logic. Finally, in the last section we show the effect of adding conservativity to the logic.[1]

2. A FRAGMENT OF FORMAL ENGLISH.

2.1. A FRAGMENT OF FORMAL ENGLISH. In this section we introduce a fragment of English (FE) for which we shall define a Natural Logic.The expressions of FE are the derivations which LG generates by using the initial statements as assumptions. Expressions of FE are, for instance, the derivations (a) and (b) below:

$$\text{Abelard} \quad e \qquad\qquad \cfrac{\cfrac{e \to t^{2} \qquad \overset{\text{Abelard}}{e}}{t}\;(2)}{(e \to t) \to t}$$

(a) (b)

2.1.1. REMARK. Treating assignment statements or LG derivations as expressions of a natural language will appear to be unnatural. For instance, 'Abelard ∈ e' should be seen as an expression about English and not as an expression of English. This is the reason behind our pedantic characterization of FE as formal English.

2.2. VOCABULARY OF FE. The vocabulary of FE consists of proper names, determiners, common nouns, verbs, absolute and non-absolute adjectives, adverbs, prepositions.

(1) Proper names: {abelard, heloise}.

(2) Determiners: {every, all, a, some, no, most, few, the}.

(3) Common nouns: {logician(s), person(s), theorem(s), thing(s), man, men, head}.

(4) Intransitive verbs:{wander(s), walk(s), run(s)}.

(5) Transitive verbs:{prove(s), love(s), see(s), is, are}.

(6) Auxiliary verb: {do(es) }.

(6) Adjectives: {female, male, tall, small }.

(7) Adverbs: {clumsily, passionately, quickly, sharply}.

(8) Prepositions: {in, at, on, with, without , of}.

(9) relative pronoun; {that, who, which}.

2.3. INITIAL ASSIGNMENTS. In the characterization of the initial assignments we shall use $\{s_1, \ldots, s_n\} \subseteq C_\alpha$ as abbreviation for the conjunction of the initial assignments $s_1 \in \alpha, \ldots, s_n \in \alpha$.

(1) {abelard, heloise} $\subseteq C_e$.

(2) {every, all, a, some, no, most, few,the} $\subseteq C_{p \to ((e \to t) \to t)}$ ·

(3){logician(s), person(s), theorem(s), thing(s), man, men, head(s)} $\subseteq C_p$.

(4){prove(s), love(s), see(s), is, are} $\subseteq C_{e \to e \to t}$ ·

(5) {wander(s), walk(s), run(s)} $\subseteq C_{e \to t}$ ·

(6) {female, male, small, tall} $\subseteq C_{p \to p}$ ·

(7) {clumsily, passionately, do(es) } $\subseteq C_{(e \to t) \to (e \to t)}$ ·

(8) {in, at, with, without } $\subseteq C_{((e \to t) \to t) \to (e \to t) \to (e \to t)}$ ·

(9) {of} $\subseteq C_{((e \to t) \to t) \to (p \to p)}$ ·

(10) {that} $\subseteq C_{(e \to t) \to (p \to p)}$ ·

2.4. EXPRESSIONS OF FE. In the following definition we assume the notions of *derivation* and *open assumptions introduced* in the description of LG.

(1) The initial assignments are expressions of FE.

(2) If each of $s_1 \in \alpha, \ldots, s_n \in \alpha$ is in FE, and there is a derivation D in LG containing these assignments among its open assumptions, then D is a member of FE.

2.4.1. ABBREVIATIONS. Writing the expressions of FE takes up a large amount of space. It is true, as Frege remarked, that the comfort of the typesetter is not the summum bonum. Except, of course, when we are the typesetters. For convenience, we introduce some abbreviations:

category	abbreviation
$e \to t$	iv
$e \to (e \to t)$	$e \to$ iv
$(e \to t) \to t$	iv $\to$ t , np
$p \to (e \to t) \to t$	det, p $\to$ np
$((e \to t) \to t) \to (p \to p)$	np $\to$ (p $\to$ p)
$(e \to t) \to (e \to t)$	iv $\to$ iv
$((e \to t) \to t) \to ((e \to t) \to (e \to t))$	np $\to$ (iv $\to$ iv)

2.4.2. EXAMPLE. The derivation (a) below is a normal analysis of *a head of a horse* , while (b) is an abbreviated version:

$$\dfrac{\overset{a}{p \rightarrow ((e \rightarrow t) \rightarrow t)} \qquad \dfrac{\dfrac{\overset{of}{((e \rightarrow t) \rightarrow t) \rightarrow (p \rightarrow p)} \qquad \dfrac{\overset{a}{p \rightarrow ((e \rightarrow t) \rightarrow t)} \quad \overset{horse}{p}}{(e \rightarrow t) \rightarrow t}}{p \rightarrow p} \qquad \overset{head}{p}}{p}}{(e \rightarrow t) \rightarrow t}$$

(a)

$$\dfrac{\overset{a}{p \rightarrow np} \qquad \dfrac{\dfrac{\overset{of}{np \rightarrow (p \rightarrow p)} \qquad \dfrac{\overset{a}{p \rightarrow np} \quad \overset{horse}{p}}{np}}{p \rightarrow p} \qquad \overset{head}{p}}{p}}{np}$$

(b)

2.5. MONOTONICITY MARKING IN FE. In Chapter V.4 we have established that the denotations of certain constants are monotone functions. Thus, we know that the denotation of FEMALE PROFESSOR is a monotone function. Furthermore, the denotation of the determiners may also have monotone properties. We will make these information available at the level of the assignment statements. For instance EVERY is a downward monotone function while the denotation of EVERY N is an upward monotone function.

To make this information available at the level of the primitive assignments, we shall no longer simply say that the category of *every* is $p \rightarrow (e \rightarrow t) \rightarrow t$. Instead, we shall use a category with internal monotonicity marking: $p \rightarrow ((e \rightarrow t)^{+} \rightarrow t)$.

This example will be extended to cover the relevant expressions:

(1) every, all $\in p^{-} \rightarrow ((e \rightarrow t)^{+} \rightarrow t)$.

(2) some, a $\in p^{+} \rightarrow ((e \rightarrow t)^{+} \rightarrow t)$.

(3) no $\in p^{-} \rightarrow ((e \rightarrow t)^{-} \rightarrow t)$.

(4) most $\in p \rightarrow ((e \rightarrow t)^{+} \rightarrow t)$.

(5) few $\in$ $p \rightarrow ((e \rightarrow t)^{-} \rightarrow t)$.

(6) the $\in$ $p \rightarrow ((e \rightarrow t)^{+} \rightarrow t)$.

(7) female, male $\in$ $p^{+} \rightarrow p$.

(8) clumsily, passionately, do(es) $\in$ $(e \rightarrow t)^{+} \rightarrow (e \rightarrow t)$.

(9) in, at, with $\in$ $((e \rightarrow t) \rightarrow t)^{+} \rightarrow ((e \rightarrow t)^{+} \rightarrow (e \rightarrow t))$.

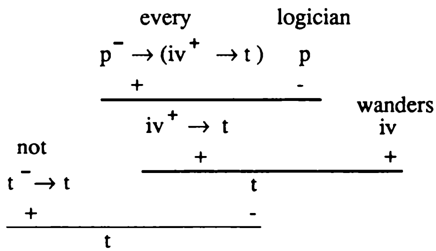
(10) without $\in$ $((e \rightarrow t) \rightarrow t)^{-} \rightarrow ((e \rightarrow t)^{+} \rightarrow (e \rightarrow t))$.

(11) of $\in$ $((e \rightarrow t) \rightarrow t)^{+} \rightarrow (p^{+} \rightarrow p)$.

(12) that $\in$ $(e \rightarrow t)^{+} \rightarrow (p^{+} \rightarrow p)$.

2.5.1. FL EXPRESSIONS WITH MONOTONICITY MARKING.

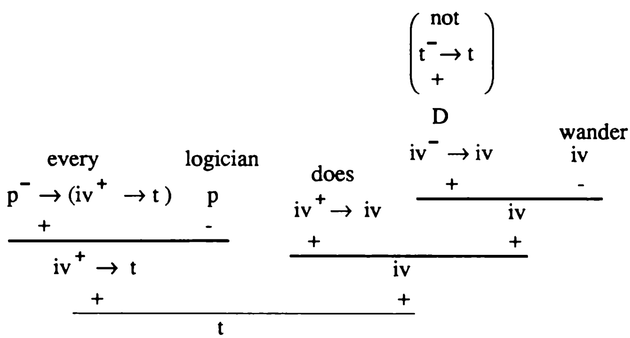
(1)



The monotonicity information can be summed up as follows:

$\text{not}^{+}((\text{every}^{-}\ \text{logician}^{+})^{-}\text{wanders}^{-})^{-}$.

(2) In the following example we use a self-explanatory extension of the notation explained in Chapter IV. 3.11.



Monotonicity marking: $(\text{every}^{+}\ \text{logician}^{-})^{+}\ (\text{does}^{+}\ (\text{not}^{+}\ \text{wander}^{-})^{+}\ )^{+}$.

(3)

$$\dfrac{\underset{+}{\overset{\text{head}}{p}} \qquad \underset{+}{\dfrac{\underset{+}{\overset{\text{of}}{np^{+}\to(p^{+}\to p)}} \qquad \underset{+}{\dfrac{\underset{+}{\overset{\text{a}}{p^{+}\to np}} \quad \underset{+}{\overset{\text{horse}}{p}}}{np}}}{p^{+}\to p}}}{p}$$

Abbreviation of the marking:

$\text{head}^{+}\ (\text{of}^{+}\ \text{a}^{+}\ \text{horse}^{+})^{+}$.

(4)

$$\dfrac{\underset{+}{\overset{\text{lives}}{iv}} \qquad \underset{+}{\dfrac{\underset{+}{\overset{\text{without}}{np^{-}\to(iv^{+}\to iv)}} \qquad \underset{-}{\dfrac{\underset{+}{\overset{\text{a}}{p^{+}\to np}} \quad \underset{+}{\overset{\text{child}}{p}}}{np}}}{iv^{+}\to iv}}}{iv}$$

Abbreviation: $\text{lives}^{+}\ (\text{without}^{+}(\text{a}^{-}\ \text{child}^{-})^{-})$.

2.6. SEMANTICS OF THE ENGLISH FRAGMENT. We can define a semantics for FE by making use of the typed language described in Chapter V. 4. We assume that if c is a member of the vocabulary, then the denotation of $c\in\alpha$ is the denotation of C_{α}. The denotation of a derivation D will be the denotation of the corresponding term, i.e. if D is an LG derivation, and M is its corresponding term, then $[\![M]\!]$ is the denotation of D.

3. THE SYSTEM OF NATURAL LOGIC

3.1. MONOTONICITY IN FE. In this section we bring together the syntactical and the semantical approach to FE. Let us start utilizing the fact that if an occurrence, $\mathbf{M}_\alpha$, of M_α is positive in Q_β then Q_β is upward monotone in $\mathbf{M}_\alpha$. This fact can be used to pass from the term N of type t containing **M** into the term N' in which **M** has been replaced by a term with a larger denotation. In a self-explanatory notation we can codify this monotonicity rule as follows:

$$\frac{\llbracket M \rrbracket \leq \llbracket M' \rrbracket \quad \llbracket N(M) \rrbracket}{\llbracket N(M') \rrbracket}$$

Remember that for each of the Lambek terms there is a corresponding derivation. Hence, we can interpret this rule with regard to the derivations themselves: if a derivation D_1 occurs positively in D , and the denotation of D_2 is larger than the denotation of D_1 , then the denotation of D is smaller than the denotation of D' , if D' is obtained from D by substituting D_1 for D_2.

ILLUSTRATION. Consider (1) and (2) below.

(1) $\llbracket \lambda X_{e \to t}.\ X\ \text{HELOISE}_e \rrbracket \leq_{(e \to t) \to t} \llbracket \text{SOME}\ (\text{NUN}_p) \rrbracket$

(2)

$$\dfrac{\dfrac{\dfrac{\underset{+}{e \to t} \qquad \overset{\text{Heloise}}{e}}{\underset{+}{t}}}{\underset{+}{(e \to t)^{+} \to t}} \qquad \underset{+}{\overset{\text{wanders}}{e \to t}}}{t}$$

The derivation (a) below occurs positively in (2). From (1) we know that denotation of (a) is smaller than the denotation of the (b) below:

$$\dfrac{\dfrac{\underset{+}{e \to t} \qquad \overset{\text{Heloise}}{e}}{\underset{+}{t}}}{(e \to t)^{+} \to t} \Rightarrow \lambda X_{e \to t}.\ X\ \text{HELOISE}_c \qquad\qquad \dfrac{\underset{+}{\overset{\text{Some}}{p^{+} \to ((e \to t)^{+} \to t)}} \qquad \underset{+}{\overset{\text{nun}}{p}}}{(e \to t)^{+} \to t} \Rightarrow \text{SOME(NUN)}$$

(a) (b)

The result of replacing (a) by (b) is (c) below:

$$\dfrac{\dfrac{\overset{\text{Some}}{\underset{+}{p^{+} \to ((e \to t)^{+} \to t)}} \qquad \overset{\text{nun}}{\underset{+}{p}}}{\underset{+}{(e \to t)^{+} \to t}} \qquad \overset{\text{wanders}}{\underset{+}{e \to t}}}{t}$$

(c)

3.2. CONVENTIONS. The important thing about the derivations as vehicles of inference is the monotonicity information. Hence, strings carrying the relevant information may be used instead of the full derivations.

3.2.1. ILLUSTRATION. The above derivation may now take the following form:

$$\dfrac{[\![\lambda X_{e \to t}.\, X\,(\mathrm{HELOISE}_e)]\!] \leq_{np} [\![(\mathrm{SOME\ NUN})_{np}]\!] \qquad \text{heloise}^{+}\ \text{wanders}}{\text{some nun wanders}}$$

We shall abbreviate the terms themselves as well as the inclusion relation. The result will be more perspicuous.

3.2.3. ABBREVIATIONS OF TYPED TERMS. We list a small number of schema's for abbreviations of typed meanings of derivations. These abbreviations are intended for the ease of reading and writing

Term	Abbreviation
$\lambda X_{e \to t} X\,(\mathrm{HELOISE}_e)$	H_{np}
$\lambda X_e .\ \mathrm{A}\ N_p\ (\mathrm{IS}\ X)$	IS A N_p
THAT $N_{e \to t}$ M_p	M_p THAT $N_{e \to t}$
$N_{(e \to t) \to (e \to t)}\ M_{e \to t}$	$M_{e \to t}\ N_{(e \to t) \to (e \to t)}$
SOME M_p (λX_e.A N_p (IS X))	SOME M_p IS A N_p
NO M_p (λX_e.A N_p (IS X))	NO M_p IS A N_p
EVERY M_p (λX_e.A N_p (IS X))	EVERY M_p IS A N_p
SOME(THAT $N_{e \to t}$ M_p)	SOME M_p THAT $N_{e \to t}$ IS A M_p
NO(THAT $N_{e \to t}$ M_p)	NO M_p THAT $N_{e \to t}$ IS A M_p
EVERY(THAT $N_{e \to t}$ M_p)	EVERY M_p THAT $N_{e \to t}$
EVERY M_p (λX_e .A (THAT $N_{e \to t}$ M_p) (IS X))	EVERY M_p IS A M_p THAT $N_{e \to t}$

3.2.4. ABBREVIATION OF THE INCLUSION RELATION.
$I_\alpha(M, N) =: [\![M_\alpha]\!] \leq_\alpha [\![N_\alpha]\!]$.

3.2.4. ILLUSTRATION. Using the above conventions, our example takes the form:

$$\frac{I_{np}(\text{HELOISE, SOME NUN}) \quad \text{heloise}^{+} \text{ wanders}}{\text{some nun wanders}}$$

3.3. THE INFERENCE RULES. According to the abbreviations introduced so far, the monotonicity rules are:

M↑

$$\frac{I_\alpha(M, M') \quad n(m^{+})}{n(m')}$$

M↓

$$\frac{I_\alpha(M, M') \quad n(m'^{-})}{n(m)}$$

3.3.1. REMARKS ON THE USE OF THE RULES. (A) The monotonicity rules can be used to show that several inferences which are valid from a pre-theoretical point of view, are also valid from the point of view of Natural Logic. We can, for instance, establish that *some black horses run* follows from *most black horses run* . The following inference is an instance of M↑:

$$\frac{I_{det}(\text{MOST, SOME}) \text{ most}^{+} \text{ black horses run}}{\text{some black horses run}}$$

(a)

(B) Not all inferences are direct ones. We can show that *Some horses run* follows from *Most black horses run* by extending derivation (a) to the following derivation:

$$\frac{I_{P}(\text{BLACK HORSES, HORSES}) \quad \dfrac{I_{det}(\text{MOST, SOME}) \text{ most}^{+} \text{ black horses run}}{\text{some (black horses)}^{+} \text{ run}}}{\text{some horses run}}$$

(b)

The semantical premisses employed in (a) and (b), namely $[\![$ MOST $]\!] \leq [\![$ SOME$]\!]$, and $[\![$ BLACK HORSES $]\!] \leq [\![$ HORSES $]\!]$ are not triggered by visible syntactical information. These premisses are derived from the properties of the denotations of the expressions involved. The denotation of MOST is an existential determiner, while the denotation of BLACK is an introspective function.

(C) One can say that *Abelard loves Heloise* entails *Abelard loves a woman* . In our framework we capture this inference as follows:

$$\frac{I_{np}(\text{HELOISE, A WOMAN}) \quad \text{abelard loves heloise}^{+}}{\text{abelard loves a woman}}$$

(c)

But the semantic premiss ⟦ HELOISE ⟧ ≤ ⟦ A WOMAN⟧ is not derived from the properties of the denotations of HELOISE and of A WOMAN. However, this premiss can be derived from the unstated assumption: *Heloise is a woman*. The full explanation is the following:

$$\frac{\dfrac{\text{heloise is a woman}}{I_{np}(\text{HELOISE, A WOMAN})} \quad \text{abelard loves heloise}^{+}}{\text{abelard loves a woman}}$$

From the previous remarks it follows that our proof system must

• compile a number of interesting semantical inclusions which can be used as semantical input for the rules;

• compile a number of interesting sentences which trigger semantical inclusions.

3.4. SOME IDENTITIES AND INCLUSIONS. In this section we list several inclusions and identities which will be widely used in our natural logic. Their validity can be easily checked.
(A) The following inclusions will be called *analytic* sentences, since their truth depends on the denotation of the expressions involved.
(o) Every M_p is a M_p.

(i)

(1) I_{det}(MOST, SOME).

(2) I_{det}(FEW, SOME).

(3) I_{det}(THE, A).

(ii)

(1) I_p(IS A N, N).

(2) I_{iv}(THING THAT N, $N_{e \to t}$).

(3) I_p (M THAT N, M).

(4) I_p (M, THING).

(5) I_{iv} (IS A M THAT N, N) .

(iii)

(1) I_p ($N_{p \rightarrow p}$ M , M) .

(2) I_{iv}($N_{iv \rightarrow iv}$ M, M) .

(3) I_{iv}(DO M, M) = I_{iv}(M, DOM) .

(iv)

(1) I_{α}(NOT $M_{\alpha \rightarrow t}$) = D_{α} - ⟦ $M_{\alpha \rightarrow t}$ ⟧ .

(2) I_t(NO M N, NOT (SOME MN)) .

(B) The following entailments will be called *analytic* entailments, since their validity depends on the denotations of the premisses involved.

(**1**) EVERY M_p IS A N_p entails

(a) I_{np}(EVERY N, EVERY M) ,

(b) I_{np}(SOME M, SOME N) , and

(c) I_p(M, N) .

(**2**) EVERY M_p $N_{e \rightarrow t}$ entails

(a) I_{np}(EVERY THING THAT N, EVERY M) ,

(b) I_{np}(SOME M_p , SOME THING THAT N),

(c) I_{iv}(IS A THING THAT M, N) , and

(d) I_p(M, THING THAT N) .

(**3**) SOME M_p IS A N_p entails

(a) I_{np}(EVERY M, SOME N) ,

(b) I_{np}(EVERY N, SOME M), and

(c) I_p(M, N) .

(**4**) SOME M_p $N_{e \rightarrow t}$ entails

(a) I_{np}(EVERY M, SOME THING THAT N) ,

(b) I_{np}(EVERY THING THAT N, SOME M)

(**5**) H_e IS A N_p entails

(a) I_{np}(EVERY N, H) ,

(b) I_{np}(H, SOME N)

(6) MOST M_p $N_{e \to t}$ entails

(a) I_{np}(MOST M_p , SOME THING THAT N)

(b) I_{np}(EVERY THING THAT N, MOST M)

(7) FEW M_p $N_{e \to t}$ entails

(a) I_{np}(EVERY THING THAT N, FEW M)

(8) THE M_p $N_{e \to t}$ entails

(a) I_{np}(EVERY THING THAT N, EVERY M),

(b) I_{np}(SOME M, SOME THING THAT $N_{e \to t}$]],

(c) I_{iv}(IS A THING THAT M, N)

(d)I_{iv}(M, THING THAT N)

(9) THE M_p IS A N_p entails

(a) I_{np}(EVERY N, EVERY M)

(b) I_{np}(SOME M, SOME N)

(c) I_p(M, N)

3.5.1. REMARK. It is possible to give a precise definition of the relation of deducibility ($\Longrightarrow$). In this dissertation we shall rely on an intuitive use of this relation.

4. EXAMPLES OF NATURAL LOGIC INFERENCE

We have already seen that

(1) Heloise is a woman, Abelard loves Heloise $\Rightarrow$ Abelard loves a woman.

(2) Most black horses run $\Rightarrow$ Some black horses run.

(3) Most black horses run $\Rightarrow$ Some horses run.

To these examples we now add the following.

(4) Abelard works with Heloise, Heloise is a woman $\Rightarrow$ Abelard works with a woman.

Proof

$$\dfrac{\dfrac{\text{heloise is a woman}}{I_{np}(\text{HELOISE, A WOMAN})} \quad \text{abelard works with heloise}^{+}}{\text{abelard works with a woman.}}$$

(5) John works with heloise $\Rightarrow$ John works.

$$\frac{I_{iv}(\text{WORKS WITH HELOISE, WORKS})\ \text{john (works with heloise)}^{+}}{\text{john works}}$$

(6) Heloise is a woman $\Rightarrow$ A woman is a woman

$$\frac{\dfrac{\text{heloise is a woman}}{I_{np}(\text{HELOISE, A WOMAN})}\quad \text{heloise}^{+}\ \text{is a woman}}{\text{a woman is a woman.}}$$

(7) Some nun is a woman $\Rightarrow$ Some woman is a nun

$$\frac{\dfrac{\text{some nun is a woman}}{I_{np}(\text{EVERY NUN, SOME WOMAN})}\quad (\text{every nun})^{+}\ \text{is a nun}}{\text{some woman is a nun}}$$

(8) Most men dance skilfully $\Rightarrow$ Most men dance

$$\frac{I_{np}(\text{DANCE SKILFULLY, DANCE})\ \ \text{most men (dance skilfully)}^{+}}{\text{most men dance}}$$

(9) Most men dance, Most men sing $\Rightarrow$ Some things that sing dance.

$$\frac{\dfrac{\text{most men dance}}{I_{np}(\text{MOST MEN, SOME THINGS THAT DANCE})}\ (\text{most men})^{+}\ \text{sing}}{\text{some things that dance sing}}$$

(10) Every horse is an animal $\Rightarrow$ Every head of a horse is a head of an animal

$$\frac{\dfrac{\text{Every horse is an animal}}{I_{np}(\text{HORSE, ANIMAL})}\quad \text{every head of a horse is a head of an horse}^{+}}{\text{every head of a horse is a head of an animal}}$$

(11) $\Rightarrow$ Every man that sings sings.

$$\frac{I_{iv}(\text{IS A MAN THAT SINGS, SINGS})\quad \text{every man that sings (is a man that sings)}^{+}}{\text{every man that sings sings}}$$

(12) $\Rightarrow$ Every man that sings is a man.

$$\frac{I_{iv}(\text{MAN THAT SINGS, MAN}) \quad \text{every man that sings is a (man that sings)}^{+}}{\text{every man that sings is a man}}$$

(13) The man is a logician, The logician is a musician $\Rightarrow$ The man is a musician.

$$\frac{\dfrac{\text{the logician is a musician}}{I_{np}(\text{A LOGICIAN, A MUSICIAN})} \quad \text{the man is (a logician)}^{+}}{\text{the man is a musician}}$$

4.2. A DERIVED RULE. Given a precise definition of the entailment relation, one can establish that for sentences ϕ and ψ, if $\phi \Rightarrow \psi$, then $[\![\phi]\!] \leq [\![\psi]\!]$. We prefer to add our proof-system this entailment as a primitive inference rule.

4.3. EXAMPLES

(14) Some nun is a woman $\Rightarrow$ Some woman is a nun, entails
$[\![\text{SOME NUN IS A WOMAN}]\!] \leq [\![\text{SOME WOMAN IS A NUN}]\!]$.

(15) Most men dance $\Rightarrow$ Some men dance,entails
$[\![\text{MOST MEN DANCE}]\!] \leq [\![\text{SOME MEN DANCE}]\!]$.

(16) No woman is a nun $\Rightarrow$ Not (some nun is a woman)

$$\frac{I_t(\text{NO WOMAN IS A NUN, NOT (SOME WOMAN IS A NUN)}) \qquad \text{(no woman is a nun)}^{+}}{\text{not (some woman is a nun)}}$$

(17) No nun is a woman $\Rightarrow$ No woman is a nun

$$\frac{\dfrac{I_t(\text{SOME NUN IS A WOMAN, SOME WOMAN IS A NUN}) \quad \dfrac{\text{no woman is a nun}}{\text{not (some woman is a nun)}^{-}}}{\text{not (some nun is a woman)}}}{\text{no nun is a woman}}$$

(18) No men dance $\Rightarrow$ Not most men dance

$$\frac{I_t(\text{MOST MEN DANCE, SOME MEN DANCE}) \qquad \dfrac{\text{no men dance}}{\text{not (some men dance)}^{-}}}{\text{not most men dance}}$$

The above inferences show the way in which a monotone Natural Logic works. Next we want to discuss the effects of adding to the system a new principle.

5. CONSERVATIVITY

5.1. CONSERVATIVITY AND DISTRIBUTION. At several places we have stressed the importance of the notion of conservativity. One should not think, however, that monotonicity exhausts the logical properties of expressions. For instance, the usual interpretation of *every man* is upward monotone. But it has additional properties which are independent of its monotonicity. Two of these properties are:

- CONSERVATIVITY. EVERY MAN X entails EVERY MAN (MAN ∩ X).
- DISTRIBUTION. EVERY MAN X AND EVERY MAN Y entail EVERY MAN (Y ∩ X).

Natural logic may significantly gain in strength if it is extended to cover such additional principles. But it is not easy to accomplish this. Trying to capture conservativity puts our semantical imagination to the test. We must be able to see that ⟦ MAN ⟧ ∩ ⟦ X ⟧ can be the denotation of the Verb Phrases *is a man that x* , *is a man and x* or of *is a x man* . And that, consequently, the following inferences obey the same principle:

$$\frac{\text{every man wanders}}{\text{every man is a man that wanders}} \qquad \frac{\text{every man is mortal}}{\text{every man is a mortal man}}$$

$$\frac{\text{every man wanders}}{\text{every man is a man and wanders}}$$

Trying to capture distribution is testing our syntactical imagination. We must be able to see that *and* in *every man wanders and every man sings* must be a sentence operator. However, in *every man wanders and sings* , *and* must be a Verb Phrase operator. Thus, in the case of distribution even the category of one of the items must be changed in the course of the derivation. Distribution can be seen as the combination of the deletion of one item and the raising of the category of *and* . This means that distribution will complicate the grammar underlying our natural logic. As we have pointed out in the previous chapter, the transition from $t \rightarrow (t \rightarrow t)$ into $(e \rightarrow t) \rightarrow (e \rightarrow t) \rightarrow (e \rightarrow t)$ is not a Lambek transition. But at the end of chapter 7 we shall show a way in which Boolean items can be incorporated into a Lambek Calculus.

5.2. CONSERVATIVITY. Van Benthem (1991) shows how to compute the effects of conservativity in lambda terms. A matter of further research is to try to discover an adequate algorithm for the marking of conservativity sensitive positions: a full use of Conservativity needs a '*theory of goverment*' determining which predicates restrict which argument places. As Van

Benthem points out, in transitive sentences like *Every boy loves a girl* there is no Conservativity in the sense of getting an equivalence with *Every boy loves a girl who is boy* . What one does have is a restriction of argument roles (cf. Van Benthem 1991).

In this dissertation, we shall incorporate the principle of conservativity as a lexical principle. We add to our list of identities the following one:

$$[\![Q_{det} M_p \; N_{vp}]\!] = [\![Q_{det} \; M_p \text{ IS A } M_p \text{ THAT } N_{vp}]\!].$$

This identity generates almost directly the following inferences:

(19) No nun wanders ⟹ No nun is a nun that wanders

(20) Some nun wanders ⟹ Some nun is a nun that wanders

(21) Some nun wanders ⟹ Some nun is a nun that wanders

(22) Every nun wanders ⟹ Every nun is a nun that wanders

(23) The nun wanders ⟹ The nun is a nun that wanders

The principle of conservativity can now be used to incorporate some other inferences.

(24) Every nun that wanders thinks, Every nun wanders ⟹ Every nun thinks

$$\dfrac{\dfrac{\dfrac{\text{every nun wanders}}{\text{every nun is a nun that wanders}}}{I_p(\text{NUN, NUN THAT WANDERS})} \quad \text{every (nun that wanders)}^{-} \text{ thinks}}{\text{Every nun thinks}}$$

(24) Some nun wanders ⟹ Some nun is a nun

$$\dfrac{\dfrac{\text{some nun wanders}}{\text{some nun is a (nun that wanders)}^{+}} \quad I_p(\text{NUN THAT WANDERS, NUN})}{\text{some nun is a nun}}$$

(25) Most nuns wander ⟹ Most nuns are nuns

$$\dfrac{\dfrac{\text{Most nuns wander}}{\text{Most nuns are (nuns that wander)}^{+}} \quad I_p(\text{NUN THAT WANDERS, NUN})}{\text{Most nuns are nuns}}$$

6. SCOPE OF PROPER NAMES IN NATURAL LOGIC

In this section we take a closer look at derivations in which scope plays a major role. We shall show that although proper names are generally considered to be scopeless semantically, they are not so inferentially. Consider the following inferences:

(a) Abelard loves no woman, Abelard is a man $\Longrightarrow$ Some man loves no woman

$$\dfrac{\dfrac{\text{abelard is a man}}{I_{np}(\text{ABELARD, SOME MAN})} \quad \text{abelard}^{+}\ \text{loves no woman}}{\text{some man loves no woman}}$$

This inference is possible in virtue of the derivation:

$$
\dfrac{
\begin{array}{c}\text{abelard}\\ (e\to t)^{+}\to t\\ +\end{array}
\qquad
\dfrac{\dfrac{
\begin{array}{c}\text{no woman}\\ (e\to t)^{-}\to t \quad e^{1}\\ +\end{array}
\quad
\begin{array}{c}\dfrac{\begin{array}{c}\text{loves}\\ e\to e\to t \quad e^{1}\\ +\end{array}}{e\to t}\\ -\end{array}
}{t}\;(1)\;+}{\begin{array}{c}e\to t\\ +\end{array}}
}{t}
$$

The meaning of this derivation is: ABELARD (λX.NO WOMAN(LOVES(X))). Therefore, the meaning of the conclusion is: SOME MAN (λX. NO WOMAN(LOVES(X))).

(b) Abelard loves no woman, Abelard is a man $\Longrightarrow$ Every man loves no woman.

$$\dfrac{\dfrac{\text{abelard is a man}}{I_{np}(\text{EVERY MAN,ABELARD})} \quad \text{abelard}^{-}\ \text{loves no woman}}{\text{every man loves no woman}}$$

This inference is possible in virtue of the derivation:

loves
$e \rightarrow e \rightarrow t \quad e^1$
abelard $\quad$ +
$(e \rightarrow t)^{+} \rightarrow t \qquad e \rightarrow t$
+ $\qquad$ +
t
no woman $\qquad$ (1) +
$(e \rightarrow t)^{-} \rightarrow t \quad e^1 \qquad e \rightarrow t$
+ $\qquad$ -
t

The meaning of this derivation is the term:

NO WOMAN(λX.ABELARD(LOVES(X)).

The meaning of the conclusion is the term:

NO WOMAN(λX. EVERY MAN(LOVES(X)).

The conclusions of these inferences are not equivalent. Hence, sometimes it does matter in which position a proper name is interpreted.[2] This point can be elaborated with a couple of new inferences:

(c) Heloise doesn't love Abelard, Abelard is a man $\Rightarrow$ Heloise doesn't love a man.

abelard is a man
I_{np}(ABELARD, A MAN) $\quad$ heloise doesn't love abelard^{+}
heloise doesn't love a man

(d) Heloise doesn't love Abelard, Abelard is a man $\Rightarrow$ Heloise doesn't love every man.

abelard is a man
I_{np}(EVERY MAN, ABELARD) $\quad$ heloise doesn't love abelard^{-}
heloise doesn't love every man

As in the previous examples, the conclusions of these inferences are not equivalent.

6.2. REMARK. Even with the semantically scopeless proper names we seem to have to disambiguate natural language sentences before inference takes place. But as far as monotonicity goes, this is not quite true. In Chapter I we said that there are inferences which operate without eliminating scope ambiguity beforehand. In our Natural Logic these inferences can be identified in a easy way.

The following inferences are indifferent to the way in which the premiss is parsed. The changing of the scope of the proper name does not yield non equivalent conclusions:

Abelard loves every woman		Some man loves every woman
Abelard loves the woman		Some man loves the woman
Abelard loves a woman	Abelard is a man $\Rightarrow$	Some man loves a woman
Abelard loves two women		Some man loves two women
Abelard loves most women		Some man loves most women

The point is that if in $F(m^+)$ only upward monotone expressions occur, then for any meaningful '*permutation*' F^*, of $F(m)$, m will be positive in F^*. This is the reason why for a monotonicity inference one does not need to disambiguate a string like *Some man loves every woman* . In the two standard readings of this sentence *man* and *woman* will be positive.
The presence of downward monotone expressions like *no* and *not* makes ambiguity a pressing matter for a monotone logic. For instance, inferences (b) and (c) above are made plausible only by giving the relevant LG derivations. They are acceptable only as inferences between the disambiguated LG objects. But they do not qualify as plausible inferences operating on the surface forms of the sentences. We shall discuss this question once again in the next chapter

7. SCOPE AND CONSERVATIVITY

Sometimes it is important to indicate relative scope. Hence, with regard to multiple quantification in sentences it is advisable to resort to another abbreviation.
Consider derivations corresponding to A THEOREM(λX.EVERY LOGICIAN(PROVES(X))) and EVERY LOGICIAN (λX. A THEOREM (PROVES(X))) .
We shall abbreviate the derivations corresponding to these terms by

a theorem [every logician proves]
every logician[proves a theorem] .

We are now in the position to illustrate the combined work of monotonicity and conservativity. We show that from
(1) every logician [proves (every theorem that every logician proves)$^+$], and
(2) a theorem[every logician proves],
follows:
(3) every logician [proves a theorem].
Proof
Assume (1) and (2). Conservativity and (2) yield
(4) a theorem[is a theorem that every logician proves]
By using (4) we establish the inclusion
(5) $[\![\text{EVERY THEOREM THAT EVERY LOGICIAN PROVES}]\!] \leq [\![\text{A THEOREM}]\!]$.

By applying monotonicity to (5) and (1) we obtain (3).

In fact we can argue to have brought about that (3) follows from (2).
(1) corresponds to the term:

EVERY LOGICIAN(λZ.EVERY(THAT(EVERY LOGICIAN (PROVES X))THEOREM) (PROVES Z Y)).

In predicate logic notation this term can be written as:

$$\forall x(Lx \to \forall y((Ty \wedge \forall z(Lz \to zPy)) \to xPy)) .$$

One can check the validity of this sentence and so the term below can be added harmlessly to the stock of analytic sentences:

EVERY N_p [M_{tv} (EVERY P_p THAT EVERY N_p M_{tv})] .

That this sentence is rather productive can be concluded form the fact that the following permutation is derivable:

every theorem [every logician proves] $\Rightarrow$ every logician [proves every theorem]

Proof
From the assumption we have
(1) Every theorem is a theorem that every logician proves.
This sentence entails the inclusion
(2) ⟦every theorem that every logician proves⟧ ≤ ⟦every theorem ⟧.
An instance of the new analytic sentence is
(3) Every logician [proves (every theorem that every logician proves)$^{+}$]
From (2) and (3) by monotonicity we obtain:
(4) Every logician [proves every theorem]

From the last two examples we see that adding conservativity to the proof system makes it possible to generate interesting inferences -inferences usually considered beyond the reach of natural logics.

8. CONCLUDING REMARKS. In this chapter we have described our Natural Logic. The range of the inferences covers some of the ground occupied by predicate logic. We have shown that a '*grammatical*' approach to inference can reach beyond syllogistic: inferences often considered manageable only with the apparatus of predicate logic are manageable within our proof system (see the example in the last section). However, as some of the examples show, the inferential power of our natural logic is incomparable with that of predicate logic. We are able to generate inferences involving non classical quantifiers like MOST and FEW. So, maybe it is the wrong idea to seek natural language systems which are comparable with predicate logic.

There are, however, inferences which can be easily dealt with in predicate logic while they escape our Natural Logic, namely inferences involving anaphoric connections and inferences involving non-sentential use of Boolean particles. This kind of inference can not be considered within our system without changing the logical part of Lambek Grammar in some way. This theme will be considered in the next chapter.

We have also shown that the use of derivations as vehicles of inference, instead of the plain strings, is a necessary component of the logic. However as our treatment of scope suggest, the wording of the inferences in terms of strings may sometimes sound counter-intuitive. For example, one doesn't think that *Every man loves no woman* follows from *Abelard loves no woman* . Our Natural Logic claims this because it gets the input

Abelard$^-$ loves no woman'.

The inferential mechanism work is all right. But is the underlying grammar a realistic one? Have we founded Natural Logic on shaky ground? A discussion of these questions will be found in the next chapter. We shall show that LG needs not to produce the above marking of *Abelard loves no woman* .

NOTES TO CHAPTER VI

[1] Intuitively, natural language expressions will be identified with LG derivations i.e. with their *syntactical analysis*. We shall say that a string of expressions implies another one if the analysis of the former implies the analysis of the latter. As we pointed out in the introduction: in our proof system grammatical form and logical form coincide.

[2] Notice the curious fact that the inferences (a) and (b) seen as inferences at the level of strings, do not have the same plausibility. Derivation (a) is acceptable as string derivation. But (b) is not. Apparently, the preferred reading of the sentence *Abelard loves no woman* takes *Abelard* as positive -which is possible only if we raise the proper name to the category of noun phrases. This means that the Montague rule has interesting logical effects, and not only combinatorial ones. In fact, we think that the situation is more general. There is a strong tendency in natural language in favour of interpreting positively the initial part of a string. We shall consider this point in Chapter vii.

CHAPTER VII

EMPIRICAL PHENOMENA

DESCRIPTION OF THE CONTENTS OF THE CHAPTER. The first section contains a survey of the rest of the chapter. In the second section we shall discuss and reject the usual association of strings with Lambek derivations. In the third section we describe a new strategy for linking strings with derivations. In the fourth section we show that sentences of the form NP TV NP have at most two readings in LG. In the fifth section we introduce the generalized Boolean expressions *and* and *or* to LG. Elaborating further on the strategy of the fourth section, we consider anaphorical phenomena in the last section.

1. INTRODUCTION

1.1. NATURAL LOGIC AND THE ADEQUACY OF ITS GRAMMAR. Natural Logic cannot be disconnected from the linguistic discussion on the empirical adequacy of its underlying grammar. This question is especially pressing for us since undirected Lambek systems are considered crude devices for linguistic description. A typical remark is the following:

> 'LP is inadequate as linguistic theory because whenever a sequent $T \Rightarrow x$ is derivable, then for all permutations $\pi(T)$ of T, $\pi(T) \Rightarrow x$ is derivable'.[1]

Moreover, LP is considered inadequate as *semantical* tool because it assigns several interpretations to unambiguous sentences. It could also be seen as an inadequate basis for Natural Logic because it can not cope with the Boolean particles *and* and *or* , nor with anaphorical phenomena.

We are convinced that the inadequacy of non-directed Lambek systems has been proclaimed too soon. In this chapter we shall discuss the adequacy of LG and LP, focussing on the following themes: the association of Lambek derivations with strings; and the strengthening of the logical machinery of LG. If we may seem to adhere much importance to the linking between strings and derivations, this can be explained by our desire to make LG, and thereby Natural Logic, more realistic. To be more precise the point we want to make is the following. Suppose one associates the string *no student attended any inspiring lecture* with the term

$$\text{NO STUDENT } (\lambda X_e \,.\text{ANY INSPIRING LECTURE}(\lambda Y_e.\text{ATTENDED}(Y, X)))\ ,$$

and suppose that one adds to LG the basic assignment:

$$\text{any} \in (e \to t)^- \to (e \to t)^+ \to t\,.$$

By constructing the corresponding tree we obtain the following (abbreviated) monotonicity marking: *No student attended any inspiring lecture*$^{+}$. Then we shall have the following derivation

$$\frac{\text{no student attended any inspiring lecture}^{+}}{\text{no student attended any lecture}}$$

Given the analysis associated with the string, the derivation is correct. But against the background of our intuitions something is wrong. The prominent reading of this sentence blocks the inference.[2] This situation is similar to the situation discussed in Chapter VI. 6: Natural Logic yields the result that *Abelard loves no woman* entails *Every man loves no woman* . The prominent reading of the second sentence, however, blocks this inference.

Whatever is wrong here, it is not Natural Logic itself. If we think that a criterion of adequacy for Natural Logic requires that its inferences coincide with our intuitions about scope, then it becomes important that the vehicles of inference reflect those intuitions. By guiding the process of linking between derivations and strings we are able to keep the grammar from running against the wall of our intuitions.

Before entering into details we describe our global attitude to the inadequacy of non-directed Lambek systems, and to possible ways of strengthening them.

1.2. STRENGTHENING THE NON-DIRECTED LAMBEK SYSTEMS. There are several reasons why one would like to strengthen LP. One of them is the following. Within LP we can capture the *polymorphic* nature of negation. Starting from the basic assignment

$$\text{not} \in t \rightarrow t$$

we can establish that *not* lives on all categories

$$(\alpha \rightarrow t) \rightarrow (\alpha \rightarrow t) .$$

We would like to capture in the same way the polymorphic nature of other Boolean expressions. But, for example, starting from the basic assignment

$$\text{or} \in t \rightarrow t \rightarrow t$$

we cannot prove in Lambek systems that *or* lives on all categories

$$(\alpha \rightarrow t) \rightarrow (\alpha \rightarrow t) \rightarrow (\alpha \rightarrow t) .$$

As a matter of fact, the transition from the basic category of *or* into the higher-order ones can be brought about in the relevance fragment of implicational logic: if suffices to allow in LP for the identification of assumptions. For historical reasons, LP + identification of assumptions is called LPC. We shall show that strengthening LP to explain the polymorphic nature of *and* and *or* does not necessarily imply choosing for LPC. In Section 5 we shall show that by using controlled identification of assumptions, LG can capture the polymorphic nature of the Boolean expressions without collapsing into LPC. Controlled identification shall also be used in our treatment of anaphorical phenomena in LG. But first we shall prepare our proposal on the relation between strings and derivations with a note on:

1.3. THE ASSOCIATION OF LAMBEK DERIVATIONS WITH STRINGS. In the literature on LP the relation between derivations and strings of expressions has been neglected and incorrectly so. So far nobody defined correspondence between strings of English expressions and LP derivations, unless one takes the relative position of the assumptions in a derivation as the criterion of correspondence. In our presentation of LG we said that a derivation is an analysis of a string if the members of the string appear as the only open assumptions of the derivation. These characterizations are not sophisticated enough. We shall make some new proposals for the association of LG derivations with strings. But independent of our proposal, we intend to impose adequacy conditions on any possible association. We think that any association should have as a consequence:

(a) If a derivation D corresponds to a string S, then the normal form of D also corresponds to S.

(b) To different strings correspond different derivations.

Let us describe our motivation for these conditions. We have taken the LG derivations as vehicles for our Natural Logic partially because they are unambiguous objects. But if a derivation is associated with two strings, then the unambiguous character of the derivations is lost -at least at intuitive level. For instance, a derivation D associated both with *Every man does not love a woman* and with *Not every man loves a woman* will hardly be called unambiguous. Demanding that the association of derivations with strings should determine a (partial) function from LG derivations into strings of English expressions, would be a natural solution. But then, as a consequence we should have that two derivations that are the same would have to correspond to the same string. We shall also say that

'two derivations are the same if they have the same normal form'.

This criterion of identity for derivations is taken from proof theory.

Those conditions (a) and (b) will feature in our discussion on the inadequacy of LP ascribed to the unwanted number of readings it assigns to strings. We turn to this discussion in the next section.

2. DERIVATIONS, STRINGS AND READINGS

2.1. A REJECTION OF LAMBEK GRAMMAR. As we said before, one complaint directed against LP concerns the number of non-equivalent derivations (and thus meanings) which the system attaches to sentences. For instance, the simple sentence *Heloise loves Abelard* is associated with two different derivations. This would be a problem, because the sentence is not ambiguous. One then concludes that the system produces one derivation to much and its inadequacy seems to have been established conclusively.

In this section we analyse the way in which this multiplicity of readings comes about. The point we are going to make is that from the multi-set {abelard, heloise, loves} two non- equivalent derivations can be constructed -which does not imply, however, that they should be associated with the same string.

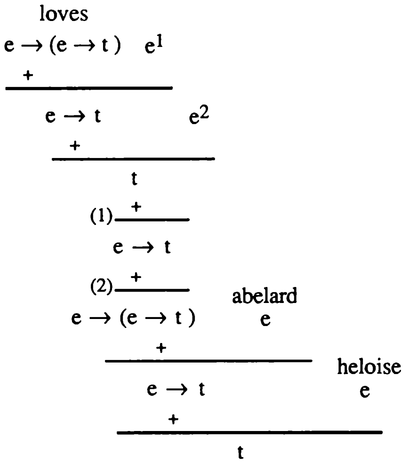
Consider the derivation:

$$\dfrac{\dfrac{\overset{\text{loves}}{e \to (e \to t)} \qquad \overset{\text{abelard}}{e}}{e \to t}\;+ \qquad \overset{\text{heloise}}{e}}{t}\;+$$

(a)

The following question may arise: which string of English words do we want to associate with (a). A standard answer has (a) corresponding to *Heloise loves Abelard* . The reasons for this choice are evident. In simple English sentences of the form NP_1 (TV $NP_2)_{VP}$, the verb combines with the object first . Subsequently, the verb-phrase combines with the subject. This choice is uncontroversial.

Consider the derivation:



(b)

In the proofs of the inadequacy of the non-directed Lambek systems, (b) is associated with the string *Heloise loves Abelard* because it has the same ordering of assumptions as (a). But now we get ourselves into a predicament. The meaning of (a) is

LOVE(HELOISE, ABELARD),

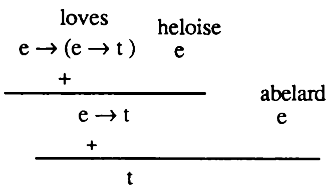
while the (simplified) meaning of (b) is

LOVE(ABELARD, HELOISE).

In general, these meanings will be associated with different denotations. Hence we have constructed the string *Heloise loves Abelard* as an ambiguous expression.

One way out of this predicament consists in changing the logical structure of LP (cf. Hendriks, 1987). We want to argue that the bad result noted above is not due to the logical properties of the Lambek systems. Instead, we think that it is the way in which derivations are associated with strings which has to be revised.

Consider (b) again. By bringing this derivation in normal form we obtain:

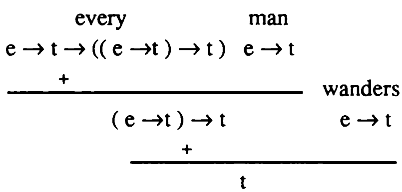


(c)

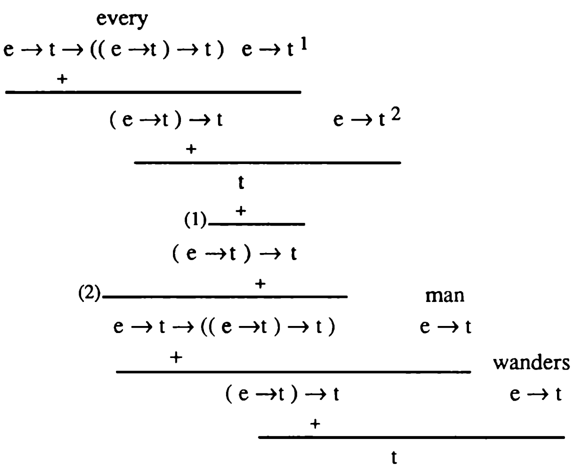
The reasons that led us to associate (a) with *Heloise loves Abelard* should lead us now to associate (c) with *Abelard loves Heloise* . But (c) is the *normal form* of (b): according to our adequacy conditions they should be associated with the same string. They are not, however. The strings *Heloise loves Abelard* and *Abelard loves Heloise* are different. Evidently we were mistaking when we said that (b) could be associated with *Heloise loves Abelard* because of the ordering of the assumptions.

The previous discussion makes clear that one should not claim that the Lambek Grammar gives two readings to the string *Heloise loves Abelard* , without having shown that the derivations which yield the readings are indeed derivations of *that string*. Evidently, the relative position of the lexical assumptions is not a safe guide. This point will be elaborated next with the help of two further examples.

(A) Assume that the ordering of the assumptions determines the association of derivations with strings. Consider the derivations (d) and (e) below with meanings EVERY MAN WANDERS and EVERY WANDERS MAN , respectively:



(d)



(e)

Since the ordering of the assumptions in (d) and (e) is the same, one associates them with the same string, namely *Every man wanders* . The evidence for the semantical inadequacy of LP and LG is strengthened in this way. The argument is not conclusive, though.
Consider (e) once again. By eliminating the detours in (e), we obtain (f) below:

$$\dfrac{\dfrac{\overset{\text{every}}{e \to t \to ((e \to t) \to t)} \quad \overset{\text{wanders}}{e \to t}}{(e \to t) \to t}{+} \qquad \overset{\text{man}}{e \to t}}{t}{+}$$

(f)

But (f) would correspond to the string *Every wanders man* and not to *Every man wanders* . Hence, the same would hold for (e): they are the same derivation.

(B) Finally we consider one example in which normal forms do not seem to play a role, while geometrical illusion does (in fact these derivations are not official derivations, but a similar effect can be achieved by introducing detours).
Consider (g):

$$\dfrac{\overset{\text{every man}}{(e \to t) \to t} \qquad \dfrac{\dfrac{\dfrac{\overset{\text{loves}}{e \to (e \to t)} \quad e^1}{e \to t}{+} \qquad \overset{\text{a woman}}{(e \to t) \to t}}{t}{+}}{e \to t}{(1)+}}{t}{+}$$

(g)

One assumes that (g) is to be associated with *Every man loves a woman* , because the ordering of the assumptions is the ordering of the string. But (g) is the *same* derivation as (h):

$$\dfrac{\dfrac{\dfrac{\overset{\text{a woman}}{(e \to t) \to t} \qquad \dfrac{\overset{\text{loves}}{e \to (e \to t)} \quad e^1}{e \to t}\,+}{t}\,+}{e \to t}\,(1)+ \qquad \overset{\text{every man}}{(e \to t) \to t}}{t}\,+$$

(h)

However, according to the ordering criterion, (h) should be associated with the string *A woman loves every man* . Thus, once again, the same derivation is made to correspond with different strings.

Resuming, we have argued that non-directed Lambek systems are not lost for linguistics. They do not necessarily produce too many readings for strings. We introduced the adequacy conditions regulating the connexion between strings and derivations, thus enabling ourselves to reject the ordering of assumptions as a criterion of association. The rejection of LP is based on the identification of the systems with a particular *ambiguating relation*: the ordering of the assumptions determines the corresponding string.[3] The arguments intended to show that undirected Lambek systems are inadequate only show that this ambiguating relation is inadequate for LP.

3. ASSOCIATION OF DERIVATIONS AND STRINGS

3.1. ASSOCIATION MECHANISMS. In the previous section we have severed the ties between strings and LG derivations. Now we are going to connect them again. We are convinced that it is not wise to look for a general association mechanism linking derivations and *arbitrary* strings. The next task is to lay down an ambiguating procedure for linking *particular* strings with LG derivations.

Our working hypothesis is that TV's wear on their faces: 'I combine with my object ', while VP's wear on their faces: 'I combine with my subject'. This uncontroversial reading allows us to formulate a tentative:

ASSOCIATION MECHANISM (AM)

> 'A LG derivation D corresponds to the string NP (TV NP')$_{vp}$ if in D (the category depending on) TV is combined with (the category depending on) NP', and (the category depending on) VP is combined with NP.'

AM warrants that the derivations below, correspond to the strings *Heloise loves Abelard* and *Abelard loves Heloise* , respectively:

$$\dfrac{\dfrac{\overset{\text{loves}}{e \to (e \to t)} \quad \overset{\text{abelard}}{e}}{e \to t}\;+ \qquad \overset{\text{heloise}}{e}}{t}\;+ \qquad\qquad \dfrac{\dfrac{\overset{\text{loves}}{e \to (e \to t)} \quad \overset{\text{heloise}}{e}}{e \to t}\;+ \qquad \overset{\text{abelard}}{e}}{t}\;+$$

(1) (2)

Although AM works well for (1) and (2) it still needs revision as we shall show presently.

3.2. SHADOW ASSUMPTIONS. At first sight it seems difficult to apply AM in the cases in which the NP is a complex noun phrase -the categories of TV's and NP's do not combine directly. Consider derivation (3):

$$\dfrac{\dfrac{\dfrac{\overset{\text{a woman}}{(e \to t) \to t} \qquad \dfrac{\overset{\text{loves}}{e \to (e \to t)} \quad e^1}{e \to t}\;+}{t}\;+}{e \to t}\;(1)+ \qquad \overset{\text{every man}}{(e \to t) \to t}}{t}\;+$$

(3)

In (3) *loves* is not combined with a lexical hypothesis, but with the numerical assumption e^1. The withdrawal of this assumption is followed immediately by the application of the lexical hypothesis *every man* . We will define the numerical assumptions related with lexical assumptions in the described way as:

SHADOW ASSUMPTIONS

> The numerical assumption e^i is a shadow assumption of the lexical assumption z iff the withdrawal of e^i is followed by the use of z as the major of a Modus Ponens application.

Now we can reword AM:

ASSOCIATION MECHANISM (AM)

> 'A Lambek derivation D corresponds to the string X (TV Y)$_{vp}$ if TV is combined with Y or its shadow, and VP is combined with X or its shadow.'

The AM mechanism allows us to associate derivation (3) with the string *A woman loves every man* . This derivation corresponds to the term:

EVERY MAN (λX_e A WOMAN (LOVES X))

in which EVERY MAN has wide scope. In the next section we show that there is another derivation corresponding to the same string in which A WOMAN has wide scope. But first we give two examples of our ambiguating relation.

3.2.1. EXAMPLES (In the following examples we use a self-explanatory notation for the shadow hypotheses)

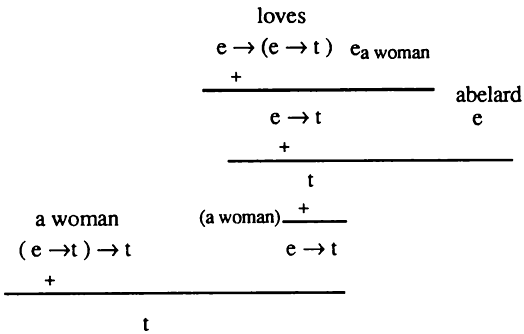
1.

$$\frac{\begin{matrix}\text{a woman}\\ (e \to t) \to t\end{matrix} \qquad \dfrac{\begin{matrix}\text{loves} & \text{abelard}\\ e \to e \to t & e\end{matrix}}{e \to t}\,+}{t}\,+$$

This derivation corresponds to the string *A woman loves Abelard* with associated meaning:

A WOMAN (LOVES(ABELARD)) .

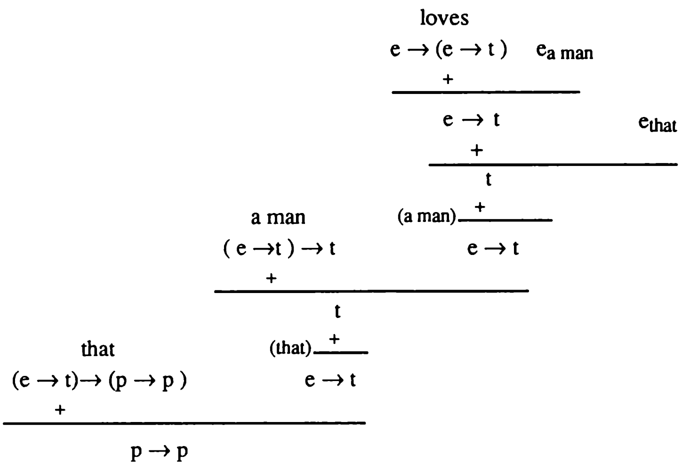
2.



This derivation corresponds to the string *Abelard loves a woman* , with associated meaning:

A WOMAN (λX_e. LOVES(X,ABELARD)).

3.



According to AM this derivation corresponds to the string *that loves a man* , and has as meaning the term:

THAT (λY_e.A MAN (λX_e. LOVES(X,Y))) .

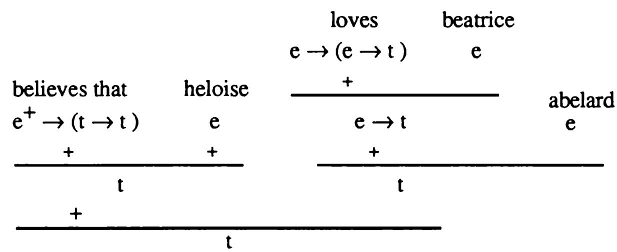
3.3. POLARITY AND THE ASSOCIATION MECHANISM. In sentences of the form *Heloise believes-that Abelard loves a woman* , *A woman believes-that Abelard loves Heloise* , the NP's in subject position are not considered to occur in the scope of the 'opaque' expression '*believes-that*'. More in general, in Dowty (1979) it is claimed that the subject always has wider scope than the auxiliaries. We can capture this demand by using the mechanism of monotonicity marking. We shall not require the subject always to be active, since we want to attach two readings to sentences like *Every man loves a woman* . However, a weaker condition will work:

POLARITY CONVENTION

In the derivation corresponding to the string x Y, x must be positive.

3.3.1. EXAMPLES

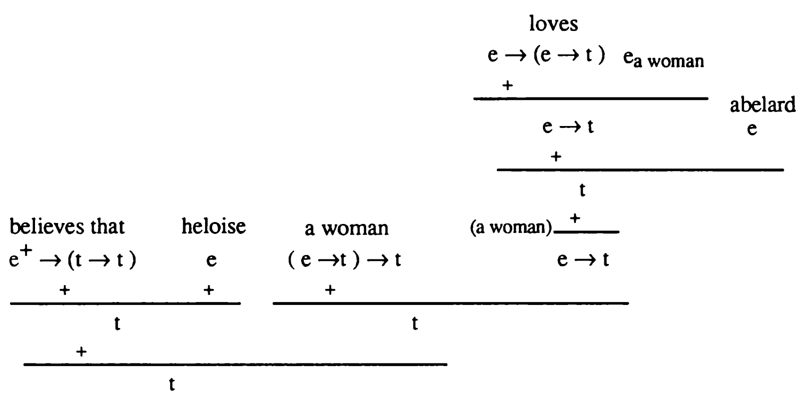
1.



This derivation corresponds to the string *Heloise believes that Abelard loves Beatrice* , with associated meaning

BELIEVES-THAT (HELOISE, LOVES(BEATRICE, ABELARD)).

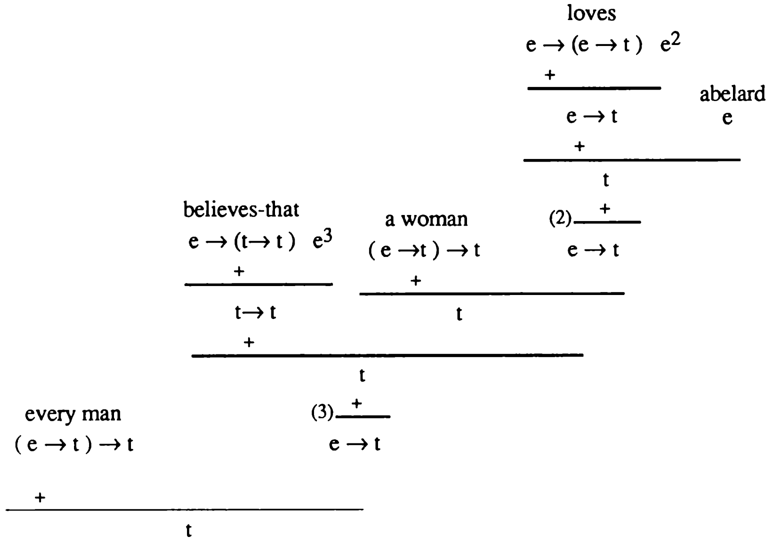
2.



This derivation corresponds to the string *Heloise believes that Abelard loves a woman* , with associated meaning:

BELIEVES-THAT (HELOISE, (A WOMAN (λY_e.LOVES(Y, ABELARD))).

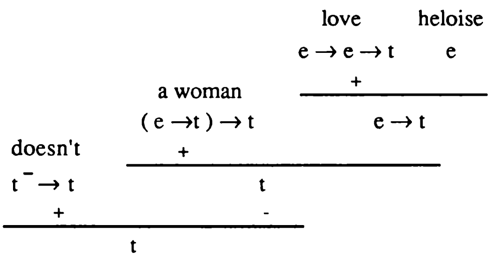
3.



This derivation corresponds to the string *Every man believes that Abelard loves a woman* , with associated meaning

EVERY MAN (λX_e .BELIEVES(X, (A WOMAN (λY_e.LOVES(ABELARD, Y))))).

The polarity convention does also cover cases other than the one above. For instance, the following derivation cannot be associated with *A woman doesn't love Heloise* , because *A woman* is not positive in the derivation:



3.3.2. REMARK. To our knowledge there are English counter-examples to the polarity convention. To begin with, the Shakespearian sentence: *All that glitters is not gold* . In this sentence the negation may be interpreted as the expression with the widest scope. However, we think that this case should be attributed to special properties of the English *all* . Properties which are not shared by other English determiners.

Another counter-example is formed by '*numerical noun phrases*', witness *Two men won't be sufficient to carry the piano* (Ladusaw 1980). But the impact of this case is less clear than the previous one. Here, the predicate *sufficient* seems to trigger the wide scope reading of the negation. In the sentence *Two men won't be at the party* the wide scope reading of *wont* is not present.

Be it as it may be, the polarity convention will fail under restricted circumstances. But we think that they are a seizable number of marked cases which can be explicitly ruled out.

3.4. ACTIVITY AND THE ASSOCIATION MECHANISM. In the philosophical literature on natural language quantification one often reads that certain NP's always have wide scope. Examples of such NP's are *a certain woman* and *any woman* . Their existence invites an extension of our mechanism:

ACTIVITY CONVENTION

> 'In the derivation corresponding to the strings X any z Y; X a certain z Y the determiner 'any' and 'a certain' must be active'.

Our new convention establishes that a derivation associated with the term

$$\text{DOESN'T (A CERTAIN WOMAN } (\lambda Y_e.\text{LOVES}(Y, \text{ABELARD}))$$

can not be associated with the string *Abelard doesn't love a certain woman* . The term A CERTAIN is not active -and consequently the same will hold for *a certain* in the corresponding derivation.

Similarly, the string *No student attended any lecture* , cannot correspond to a derivation associated with the term

$$\text{NO STUDENT } (\lambda X_e\ .\text{ANY LECTURE}(\lambda Y_e.\text{ATTENDED}(Y, X))).$$

The activity convention predicts that only the derivation corresponding to the following term will qualify[4]:

$$\text{ANY LECTURE } (\lambda Y_E\ .\text{NO STUDENT}(\lambda X_E.\text{ATTENDED}(Y, X))).$$

3.5. SURVEY OF THE MECHANISM. We have suggested an ambiguating relation between LG derivations and strings, based on the following general principles:

POLARITY CONVENTION

> In the derivation corresponding to the string x Y , x must be positive.

SHADOW ASSUMPTIONS

The numerical assumption e^i is a shadow assumption of the lexical assumption z iff the withdrawal of e^i is followed by the use of z as the major of a Modus Ponens application.

For particular linguistic constructions these general principles need additional constraints such as:

ASSOCIATION MECHANISM

A Lambek derivation D corresponds to the string X $(TV\ Y)_{vp}$ if TV is combined with Y or its shadow, and VP is combined with X or its shadow.

ACTIVITY CONVENTION

In the derivation corresponding to the strings X any z Y; X a certain z Y the determiner 'any' and 'a certain' must be active.

In the next section we use our ambiguating mechanism to discuss Van Benthem's proof that LP assigns four readings to the string *Every man loves a woman* .

4. The readings of NP TV NP

4.1. THE FOUR READINGS OF NP TV NP. Van Benthem (1991) has shown that in LP there are exactly four non-equivalent derivations with conclusion t , depending on the multiset

$$\{(e \to t) \to t\ ,\ e \to (e \to t),\ (e \to t) \to t\ \}.$$

After linking these derivations with the string *Every man loves a woman* , Van Benthem concludes that this string has four readings. The point we are going to make is that there are indeed four derivations indexed by the members of the set

{every man, loves, a woman}.

However, these four derivations are not derivations of the same string, they are rather pairwise similar derivations of two different strings.

First consider the following derivation (1):

```
                                          loves
                                     e → (e → t)   e¹
                                      +
                                     ______________
                                        e → t          e²
                                            +
                                     __________________
                                            t
                                        (?)_+__
                   ( e →t ) → t ³            e → t
                          +
                   _________________________
                              t
                          (??)_+__
( e →t ) → t ⁴                 e → t
          +
______________________________
              t
```

This derivation is ambiguous. The elimination of the ambiguity results in two different derivations, (2) and (3) :

```
                                          loves
                                     e → (e → t)   e¹
                                      +
                                     ______________
                                        e → t          e²
                                            +
                                     __________________
                                            t
                                        (2)_+__
                   ( e →t ) → t ³            e → t
                          +
                   _________________________
                              t
                          (1)_+__
( e →t ) → t ⁴                 e → t
          +
______________________________
              t
             (2)
```

$$\dfrac{(e \to t) \to t^4 \quad + \quad \dfrac{\dfrac{\dfrac{(e \to t) \to t^3 \quad + \quad \dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{loves}}{e \to (e \to t)} \quad + \quad e^1}{e \to t} \quad + \quad e^2}{t}}{e \to t}\,(1)}{t}}{e \to t}\,(2)}{t}}$$

(3)

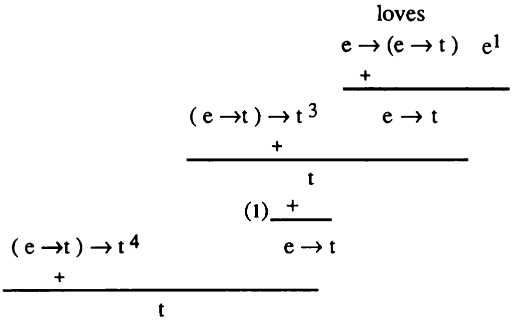
4.2. DERIVATION 2. Let us concentrate on derivation (2) first. The derivations (4) and (5) below have the same extensional meaning:

$$\dfrac{\dfrac{\dfrac{\overset{\text{loves}}{e \to (e \to t)} \quad + \quad e^1}{e \to t} \quad + \quad e^2}{t}}{e \to t}\,(2) \qquad\qquad \dfrac{\overset{\text{loves}}{e \to (e \to t)} \quad + \quad e^1}{e \to t}$$

(4) (5)

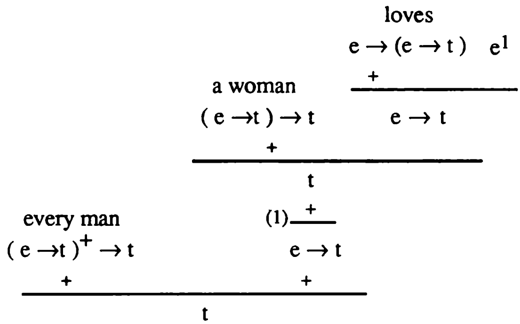
The meaning of (4) is $\lambda Y_e.\text{LOVES } X_eY$, while the meaning of (5) is $\text{LOVES } X_e$. But these terms are extensionally equal.The point is that $\lambda N.MN$ is extensionally equal to M , if N does not belong to the set of free variables of M . Since $\text{LOVES } X_e$ is a Lambek Term, the variable Y_e , which represents e^2 can not occur in $\text{LOVES } X_e$. Therefore $\lambda Y_e.\text{LOVES } X_eY_e = \text{LOVES } X_e$. And we are allowed to replace $\text{LOVES } X_e$ by $\lambda Y_e.\text{LOVES } X_eY_e$ and vice versa. Given the variable convention, this replacement can take place in arbitrary contexts. Focussing on the derivations themselves, this substitution potential implies that derivations like (4) and (5) are always inter-changeable.[5]

Consequently we can simplify (2), obtaining:

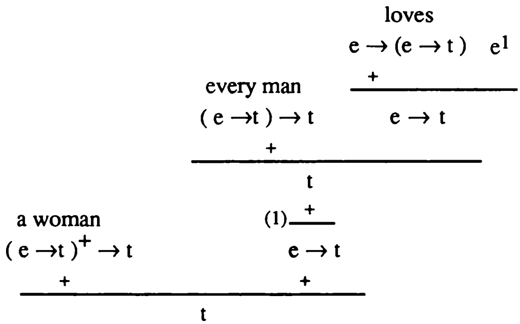


(6)

Given {every man, a woman}, there are two ways in which we can annotate (6). We obtain the similar derivations (7) and (8) below:



(7)

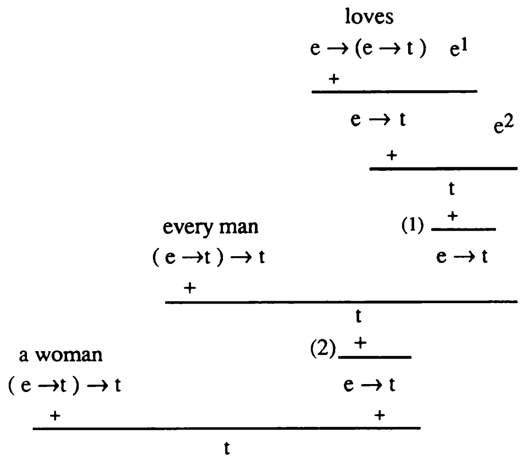


(8)

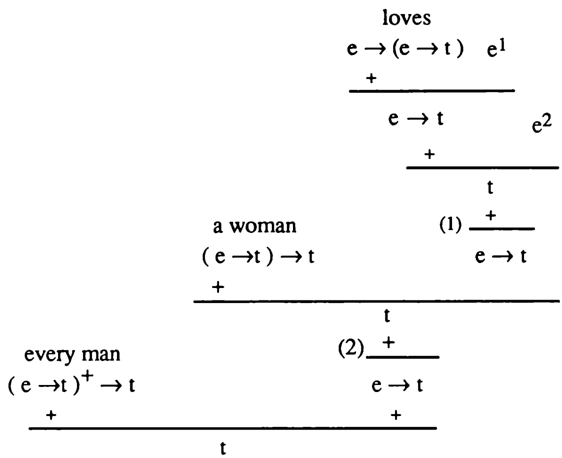
According to our ambiguating mechanism (7) corresponds to the string *A woman loves every man* , while (8) corresponds to the string *Every man loves a woman* .

So far we have two similar derivations corresponding to two different strings. Let us see whether the same holds for derivation (3).

4.3. DERIVATION 3. By using the set of indices {every man, a woman} we obtain again two different derivations:



(9)



(10)

Once again, according to our association mechanism, these are similar derivations to be associated with two different strings. Derivation (9) corresponds to the string *A woman loves every man* , while (1) corresponds to *Every man loves a woman* .

The previous discussion shows that the four derivations that can be obtained with the indices {every man, loves, a woman}, do not correspond to the same string. An explosion of readings does not necessarily arise at the level of the strings NP TV NP. There is no such thing in LP: derivations in themselves have nothing to do with strings until one defines the ambiguating relation.

The concept of shadow assumptions was crucial in our argument against the explosion of readings in LP and LG. They are not solely devised for this purpose, however. The next section will demonstrate -by given other applications- that the shadow assumptions are not an ad-hoc device.

5. GENERALIZED BOOLEAN OPERATIONS.

5.1 BOOLEAN EXPRESSIONS. The particles 'and' and 'or' are central in natural language. However, our Natural Logic did not say anything about them. In this section we want to make up for that omission. As we pointed out in the introduction the Boolean particles are particularly problematic for LP. The derivation of higher-order coordination from sentence coordination requires the use of mechanisms excluded from those systems. For instance, in order to derive the category of predicate disjunction

$$(e \rightarrow t) \rightarrow (e \rightarrow t) \rightarrow (e \rightarrow t)$$

from the category of sentential disjunction

$$t \rightarrow t \rightarrow t$$

one needs to use an assumption twice. At first sight it would seem as if one has to allow unrestricted identification of assumptions to explain higher order coordination. This would be ill-conceived because unrestricted identification is not possible in natural language. The standard solution consists in adding to the system the so-called *'generalized Boolean operations'* (c.f. Keenan & Faltz, 1985; Partee & Rooth, 1983; Hendriks, 1987).

In this section we shall show that it is possible to extend LG to cases in which the Boolean particles are involved -thus making them available for Natural Logic. We introduce 'controlled' identification of assumptions triggered by Boolean particles and pronouns. Our identifications are controlled because they are not available for all the items of the vocabulary.

In the literature on natural deduction there is an interesting proposal treating the identification of assumptions in systems in which an assumption may be used only once (cf. Curry 1958). The main idea can be illustrated by the following picture:

$$\dfrac{\dfrac{\dfrac{e \to e \to t \quad e^1}{e \to t} \quad e^2}{t}}{t}\,I(1,2)$$

Given a derivation in which the same formula has been used more than once, we are allowed to identify the indices by which we distinguish the occurrences of that formula. Before the identification, the conclusion t in (a) depends on e^1 and e^2. After the identification the conclusion depends on only one of the indices, say e^1. The other assumption is eliminated from the set of open assumptions. In the corresponding term we can describe this situation as follows: identification allows us to pass from the term

$$X_{e \to e \to t}\, Y_e\, Z_e$$

to

$$X_{e \to e \to t} Y_e\, Y_e.$$

It will be clear that unrestricted use of identification yields the system LC.

5.2. CONTROLLED IDENTIFICATION. Here we shall describe a way in which identification of assumptions can be incorporated into LG without collapsing into LC. To this end we introduce the following lexicalized identification rules:

5.2.1. IDENTIFICATION CONTROLLED BY CONJUNCTION

$$\dfrac{\dfrac{\dfrac{\overset{\text{and}}{t \to t \to t} \quad \overset{\overset{\alpha^m}{D_1}}{t}}{t \to t} \quad \overset{\overset{\alpha^n}{D_2}}{t}}{t}}{t}\,I(m, n)$$

5.2.2. IDENTIFICATION CONTROLLED BY DISJUNCTION

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{or}}{t \to t \to t} \qquad \dfrac{\alpha^m \;\; D_1}{t}}{t \to t} \qquad \dfrac{\alpha^n \;\; D_2}{t}}{t}}{t}\, I(m, n)$$

Immediately after 'm' is identified with 'n', α^n is withdrawn from the set of open assumptions. We also assume that if α^m is the shadow of x, then after identification α^n becomes a new shadow of x.

Derivations made with the help of these lexicalized identification rules can be associated with strings by the following coordination principles

5.3. COORDINATION PRINCIPLES

A derivation D without numerical indices corresponds to the string 'X and Y' if 'and' is combined with $(XZ)_t$ and $(\text{and } XZ)_{t \to t}$ is combined with $(YV)_t$. [Similar convention for 'or'].

A derivation D without numerical indices corresponds to the string (X TV and Y TV')Z if TV and TV' are combined with Z or its shadow. [Similar convention for 'or'].

5.3.1. EXAMPLES (we use here a self-explanatory notation for the identification rule)

1.

$$\dfrac{\dfrac{\text{every player}}{(e \to t)^+ \to t^{+}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\text{and}}{t^+ \to t^+ \to t^{+}} \qquad \dfrac{\dfrac{\text{failed}}{e \to t^{+}} \;\; e^1}{t^{+}} \qquad \dfrac{\dfrac{\text{cried}}{e \to t^{+}} \;\; e^2}{t^{+}}}{t^{+}}\, I(1,2)}{t^{+}}\,(1)}{e \to t^{+}}}{t}$$

The 'first part' of the derivation ending, with the withdrawal of e^1, has as meaning:

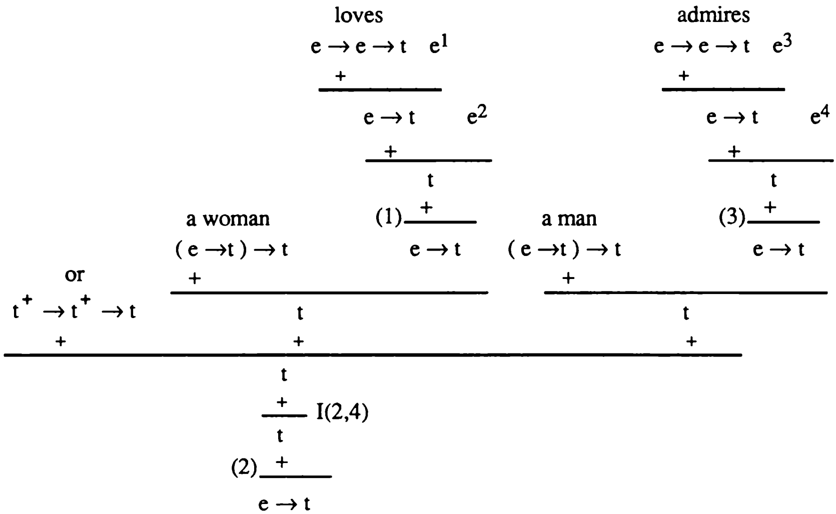
$$\lambda x.\ \text{AND}\ (\text{FAILED}\ x,\ \text{CRIED}\ x)\ .$$

This derivation corresponds to the string 'failed and cried'. The meaning of the whole derivation is:

EVERY PLAYER (λX. AND (FAILED X, CRIED X)) .

By AM the derivation corresponds to the string *Every player failed and cried* .

2.



Before the identification, the meaning of this derivation is

OR (A WOMAN λX. LOVES(X, Y), A MAN λV. ADMIRES(V, Z)).

After the identification the meaning becomes:

OR (A WOMAN λX. LOVES(X, Y), A MAN λV. ADMIRES(V, Y)).

After the withdrawal of the identified assumption the meaning is:

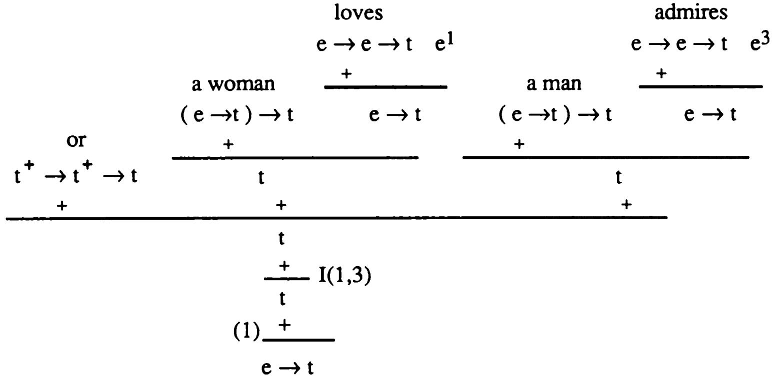
λY.OR (A WOMAN λX. LOVES(X, Y), A MAN λV. ADMIRES(V, Y))

The corresponding derivation is associated with the string *loves a woman or admires a man* . By applying EVERY BOY to this term we obtain the term:

EVERY BOY (λY.OR (A WOMAN λX. LOVES(X, Y), A MAN λV. ADMIRES(V, Y))).

The corresponding derivation is associated with the string *Every boy loves a woman or admires a man* .

(3)



The meaning of this derivation is

λY.OR (A WOMAN λX. LOVES(Y, X), A MAN λV. ADMIRES(Y, V)).

A new term is obtained when we apply EVERY BOY . The question is now: to which string does this term correspond? Notice that in the derivation itself e^1 and e^3 become shadows of *Every boy* . Hence by the coordination principles the derivation corresponds to the string *A woman loves or a man admires every boy* .

5.3.2. COORDINATION AND NATURAL LOGIC. The monotonicity marking of the the above examples is abbreviated by

every player (failed and cried)$^+$,
every boy (loves a man or admires a woman)$^+$
(a woman loves or a man admires)$^+$ every boy

Hence, we can now generate the inferences

1.

$$\frac{I_{vp}(\text{FAILED AND CRIED, FAILED})\ \ \text{every player (failed and cried)}^+}{\text{every player failed}}$$

2.

$$\frac{I_{vp}(\text{LOVES A MAN, LOVES A MAN OR ADMIRES A WOMAN})\ \ \text{every boy (loves a man)}^+}{\text{every boy loves a man or admires a woman}}$$

3.

$$\frac{I_{vp}(\text{A WOMAN LOVES, A WOMAN LOVES OR A MAN ADMIRES })\ \ \text{(a woman loves)}^+\text{every boy}}{\text{a woman loves or a man admires every boy}}$$

These inferences show that the controlled unification strengthen Natural Logic. But there is a problem posed by Proper Names. We turn to this question in the next section.

5.4. COORDINATION AND PROPER NAMES. Notice that a proper name has no shadow : Proper Names aren't major premisses of Modus Ponens applications. This limitation doesn't seem to be a problem since one can always apply the Montague rule and get things straight. However, there are extra problems. Consider once again the term

$$\lambda Y_e.\text{OR (A WOMAN } \lambda Z_e.\ \text{LOVES}(Y, Z), \text{ A MAN } \lambda V_e.\ \text{ADMIRES}(Y, V)) .$$

By applying

$$\lambda X_{e \to t}.X(\text{ABELARD})$$

we obtain

$$\lambda X_{e \to t}.X(\text{ABELARD})[\lambda Y_e.\text{OR (A WOMAN } \lambda Z_e.\ \text{LOVES}(Y, Z), \text{ A MAN } \lambda V_e.\text{ADMIRES}(Y, V))] .$$

According to our convention the term corresponds to the string *A woman loves or a man admires Abelard*. But by eliminating the redexes of this term we obtain

$$\text{OR (A WOMAN } \lambda Z_e.\text{LOVES(ABELARD}, Z), \text{ A MAN } \lambda_e.\text{ADMIRES(ABELARD}, V)) .$$

But this term should correspond to the sentence *A woman loves Abelard or a man admires Abelard* . Hence, our derivation corresponds to different strings, thereby violating our adequacy condition.

A solution to this problem would consist in assigning Proper Names *directly* to the category of NP's without having to raise them locally. By applying ABELARD we then obtain the term already in normal form:

$$\text{ABELARD}[\lambda Y_e.\text{OR (A WOMAN } \lambda Z_e.\ \text{LOVES}(Y, Z), \text{ A MAN } \lambda_e.\text{ADMIRES}(Y, V))] .$$

The specific properties of ABELARD would be dealt with at the same level in which we deal with the special properties of complex Noun Phrases: the semantics of the typed language. We think that this solution is not ad hoc, because the raising of Proper Names is especially relevant in the context of coordination (cf. Moortgat 1988).

5.4. A MISSING READING. In Montague Grammar the sentence *John caught and ate a fish* is given two interpretations. In one of them John caught and ate the same fish; in the other interpretation John caught a fish and ate another one. It is not clear whether this sentence indeed has the second reading. But there are pragmatic arguments in favour to make the second reading a sensible one (cf. Hendriks 1987).

Be it as it may, our treatment of Boolean coordination is not strong enough to produce a derivation associated with *John caught and ate a fish* having the meaning:

$$\text{JOHN}[\lambda Z.(\text{AND (A FISH } [\lambda V.\ \text{CAUGHT}(V, Z)], \text{ A FISH } [\lambda W.\text{ATE}(W, Z)]))] .$$

But a simple change in our characterization of shadow assumptions makes this reading available:

SHADOW* ASSUMPTIONS

> The numerical assumption e^i is a shadow* assumption of the lexical assumption z iff the withdrawal of e^i is followed by the use of z as the major or the **minor** of a Modus Ponens application.

Let us explain how we get the missing reading. We shall use terms instead of trees because they encode all relevant information.
At first we obtain the term:

$$\text{AND}\ (X_{(e \to t) \to t}[\lambda V.\ \text{CAUGHT}(V,Z)],\ Y_{(e \to t) \to t}\ [\lambda W.\text{ATE}(W,\ K)]).$$

By using the identification rule, we have:

$$\text{AND}\ (X_{(e \to t) \to t}[\lambda V.\ \text{CAUGHT}(V,Z)],\ X_{(e \to t) \to t}\ [\lambda W.\text{ATE}(W,\ K)]).$$

Withdrawal of the identified assumption, and subsequent application to *a fish* yield:

$$\lambda X_{(e \to t) \to t}.\text{AND}\ (X_{(e \to t) \to t}[\lambda V.\ \text{CAUGHT}(V,Z)],\ X_{(e \to t) \to t}\ [\lambda W.\text{ATE}(W,\ K)])\ (\text{A FISH}).$$

This term corresponds to the string *caught and ate a fish* . The elimination of the redex gives the term

$$\text{AND}\ (\text{A FISH}\ [\lambda V.\ \text{CAUGHT}(V,Z)],\ \text{A FISH}\ [\lambda W.\text{ATE}(W,\ K)]).$$

Identification of the remaining free variables, abstraction and use of JOHN, result in the desired term:

$$\text{JOHN}[\lambda Z.(\text{AND}\ (\text{A FISH}\ [\lambda V.\ \text{CAUGHT}(V,\ Z)],\ \text{A FISH}\ [\lambda W.\text{ATE}(W,\ Z)]))].$$

We have reasons for avoiding the shadow* hypotheses, however. The term:

$$\text{AND}\ (\text{A FISH}\ [\lambda V.\ \text{CAUGHT}(V,Z)],\ \text{A FISH}\ [\lambda W.\text{ATE}(W,\ K)])$$

corresponds to the string *caught a fish and ate a fish* . If this term also corresponds to *caught and ate a fish* , we have a defective ambiguating mechanism: one and the same derivation would correspond to different strings.

In the light of the previous discussion it could appear that our adequacy conditions are too strong after all. They are incompatible with the shadow* convention needed for the generation of the above reading. But we think that this is a marginal case and we are not eager to incorporate it in our system.

We shall not pursue this matter in this dissertation. Instead we show the flexibility of controlled identification. Until now, we have only admitted the identification of numerical assumptions. However, in natural language some occurrences of pronouns are prone to identification. In the next section we make a modest proposal for the treatment of anaphoric phenomena in LG.

6. ANAPHORIC BINDING

6.1. REFLEXIVES . Van Benthem (1991) has shown that it is possible to incorporate reflexive pronominal binding into non-directed Lambek systems. For instance *herself* is considered a 'relation reducer' belonging to the category $(e \rightarrow e \rightarrow t) \rightarrow (e \rightarrow t)$. The meaning of reflexives is given by the term $\lambda X_{e \rightarrow e \rightarrow t}. \lambda Y. X(Y,Y)$. The sentences *Heloise sees herself* and *Every woman sees herself* are associated with the terms SEES(HELOISE, HELOISE) , EVERY WOMAN λX_e. SEES (X,X), respectively.

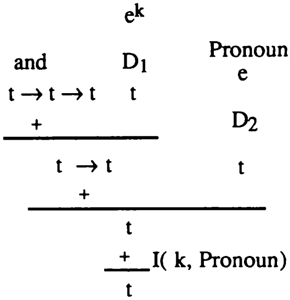
6.2. IDENTIFICATION OF ASSUMPTIONS WITH PRONOUNS. Consider the sentence *Abelard loves Heloise and admires her* . There is a reading of this sentence in which Heloise is the person whom Abelard loves and admires. But in this case the lexical solution is not available for this reading is optional. Therefore we cannot treat non-reflexive pronouns in the same way as reflexives; another approach is called for.

In Montague grammar one obtains the anaphoric reading of our sentence by applying to HELOISE the term

$$\lambda X.\ \text{LOVES}(X, \text{ABELARD}) \wedge \text{ADMIRES}(X, \text{ABELARD})\ .$$

Notice that this term corresponds also to an LG derivation of *Abelard loves and admires* -a derivation guided by the identification mechanism. It is suggestive to think that the same mechanism can be used for the construction of a derivation of *Abelard loves Heloise and admires her* . We shall sketch a procedure for the construction of LG derivations corresponding to such expressions. To this end we extend the identification mechanism allowing pronouns to be identified with numerical assumptions:

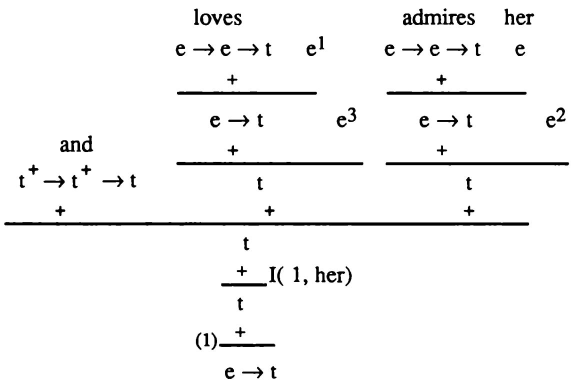
6.2.1. IDENTIFICATION OF PRONOUNS I



After the identification the assumption indexed by a pronoun is eliminated.

6.3. EXAMPLES

(1)



Before the identification, the meaning of the derivation is:

AND(LOVES(X, Y), ADMIRES(HER,Z)).

After the identification the meaning of the derivation becomes:

AND(LOVES(X,Y), ADMIRES(X,Z)) .

The withdrawal yields:

λX.AND(LOVES(X,Y), ADMIRES(X,Z)) .

To the above derivation we can apply any NP (of the appropriate gender). Among others, we obtain derivations corresponding to the following strings (the bold expressions are anaphorically linked):

(1) loves **a certain woman** and admires **her**
(3) loves **a woman** and admires **her**
(5) loves **heloise** and admires **her**
(7) loves **the girl** and admires **her**
(9) loves **one girl** and admires **her**

Evidently, controlled identification of assumptions engenders derivations we would not have otherwise. And as a consequence we can obtain the multiple readings which we intuitively attach to the sentences listed above.

6.4. REMARKS. (A) Since the application of the NP in the derivation is not constrained, we shall also get derivations for the string *loves **no woman** and admires **her***. This is an undesirable result which is also obtained in Montague Grammar. Van Eijck (1985 : 192-3) passingly suggests the idea of restricting Montagues's quantifying-in at VP level to upward monotone NP's.[6] In our framework this proposal would take the form:

> after withdrawal of an identified variable, only NP's of category $(e \to t)^{+} \to t$ should be applied.

This excludes at once a derivation for the string *loves **no woman** and admires **her*** on the basis of our identification rule. Further research is needed in order to determine whether this restriction is practical or not. But notice that Van Eijck's suggestion belongs in a grammar in which monotonicity information can be encoded in the categorization of lexical items, i.e. LG.

(B) In the earlier linguistic treatment of anaphorical binding, the sentences of the list below, were considered to be systematically linked to the previous list:

(2) loves **a certain woman** and admires **a certain woman**
(4) loves **a woman** and admires **a woman**
(6) loves **heloise** and admires **heloise**
(8) loves **the girl** and admires **the girl**
(10) loves **one girl** and admires **one girl**

The members of the previous list (see 6.3) were considered to be transformations of the members of the present one. Pronouns were seen as useful devices for avoiding repetition of Noun

Phrases -hence the name 'pronouns of laziness'.[7] Assumed was that the sentences (n) and (n +1) have the same meaning.

Geach (1964) conclusively argues that reflexives are not pronouns of laziness: 'Only Satan pities Satan' is not equivalent with 'Only Satan pities himself'.

An important question, from the point of view of Natural logic, is whether we have at least entailment -from the first list to the second or the other way around. For instance, it seems that the entailment relation holds between the members of both lists: (n) entails (n +1). This is not limited to the so-called existential NP's because *loves* ***any woman*** *and admires* ***her*** also entails *loves any woman and admires any woman* .

The reader may have noticed that the entailments mentioned can be predicted from Peirce's analysis: the bound pronouns occur positively, hence diversification may then take place. According to Peirce, one should only have, for instance, that *loves* ***any woman*** *and admires* ***her*** entails *loves any woman and admires some one* . It remains to be seen whether it is some kind of conservativity which generates the inference:

$$\frac{\text{loves } \textbf{any woman} \text{ and admires } \textbf{her}^{+}}{\text{loves any woman and admires any woman}}$$

instead of the more general

$$\frac{\text{loves } \textbf{any woman} \text{ and admires } \textbf{her}^{+}}{\text{loves any woman and admires some one}}$$

Controlled identification is, of course, a modest mechanism *vis-a-vis* the complexity of anaphorical phenomena. And we do not entertain the hope that anaphorical binding will be reduced to controlled identification. But we also think that this mechanism yields something more than an ad hoc restatement of the facts. To illustrate this point we conclude this chapter by discussing two further syntactical constructions in which our context -sensitive mechanism may be employed. The counter-examples are left to the reader.

6.5. FURTHER CONTEXT OF IDENTIFICATIONS. (A) A new type of syntactical construction can be described in which identification of pronouns and numerical assumptions is allowed. Intuitively, the bold expressions in the following sentences may be anaphorically bound:

(1) **Every man** believes that **he** dances;
(2) **No man** believes that **he** dances;
(3) **Every man** believes that **he** loves Heloise;
(4) **Every man** believes that Heloise loves **him**.

The identification pattern one can infer from these sentences is the following:

6.5.1. IDENTIFICATION OF PRONOUNS II

$$\dfrac{\dfrac{\dfrac{\overset{\text{X-that}}{e \to t \to t} \quad e^{k}}{t \to t}\,{\scriptstyle +} \qquad \dfrac{\overset{\text{Pronoun}}{e}}{t}\,D}{t}\,{\scriptstyle +}}{t}\,{\scriptstyle +}\ I(k, \text{Pronoun})$$

EXAMPLE. In a familiar way, we can construct a derivation with the meaning:

BELIEVES-THAT(X, DANCES (HE)).

By applying the identification rule we obtain:

BELIEVES-THAT(X, DANCES (X)).

Withdrawal and application of EVERY MAN yields the term:

EVERY MAN(λX_e. BELIEVES-THAT(X, DANCES (X)).

This term corresponds to a derivation associated with ***Every man believes that he dances*** . Similarly, we obtain derivations corresponding to the other strings. Notice that in this case *Every man believes that he dances* does not entail *Every man believes that every man dances* . This fact is taken into account in our system because we shall have

believes-that $\in e^{+} \to t \to t$.

Hence, after the application of $t \to t$ we stop the flow of monotonicity: the pronoun occurring within its scope lacks monotonicity marking at the end of the derivation.

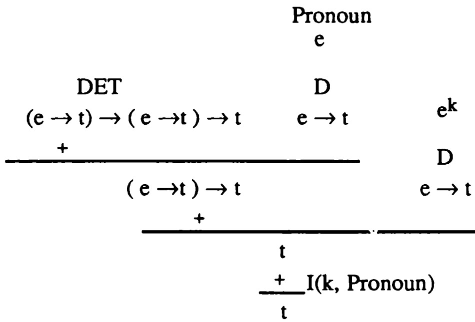
(B) Another familiar pattern of anaphorical binding is instantiated in the following sentences:

(1) A man loves a woman that he admires.

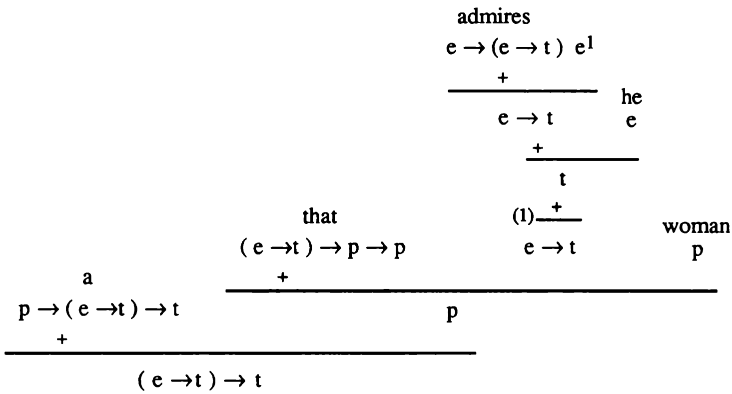
(2) A man loves a woman that admires him.

The identification pattern suggested is:

6.5.1. IDENTIFICATION OF PRONOUNS III.



EXAMPLE. The following derivation is associated with the string *a woman that he admires* :



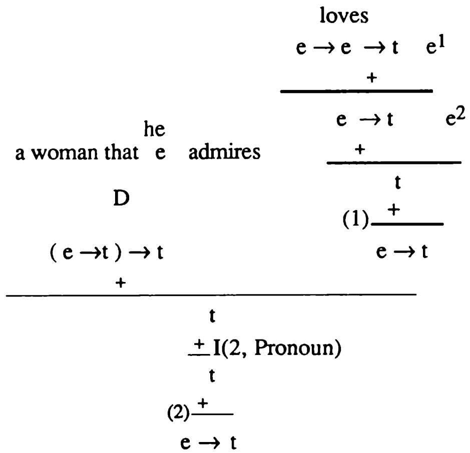
We shall abbreviate this derivation by:

he
a woman that e admires

D

(e →t) → t

Now, we can continue in this way :



Let us turn to the terms. Before the identification takes place, the meaning of the derivation is:

A (WOMAN(X) $\wedge$ ADMIRES (X, HE)) [λZ.LOVES (Z,V)].

After the identification and the withdrawal we have a derivation with the meaning:

λV_E.A (WOMAN(X) $\wedge$ ADMIRES (X, V)) [λZ.LOVES (Z,V)].

Application of *every man* will give a derivation associated with the string: ***Every man loves*** *a woman that* ***he*** *admires* . In a similar manner, we can construct a derivation associated with the string ***Every man loves a woman that loves him*** .

With this example we conclude our tentative approach to anaphorical binding in LG. We think that the previous discussion shows that the controlled identification of assumptions may turn out to be a fruitful addition to the mechanism of non-directed Lambek calculus.

7. CONCLUDING REMARKS. In this chapter we have considered the question of the empirical adequacy of the Lambek Grammar. We have shown that it is a misconception that this system would be inadequate on account of the many readings it produces. This criticism is not valid as long as no one has explicitly defined the ambiguating relation for the non-directed systems. We have introduced adequacy conditions for any definition of the ambiguating relation. We have then argued that reading the corresponding string from the ordering of the assumptions, is an unsatisfactory definition of the ambiguating relation. We have also made a new proposal for the linking of LG derivations with strings. Whether the particular solution we advocate is tenable or

not, is independent of the fact that rejection of non-directed Lambek systems assumes an ambiguating relation that has not been explicitly defined yet.

We have pointed out that there is no need to weaken the underlying logic in order to explain scope phenomena. This can be done by introducing the notion of *shadow hypotheses*. We have also considered the possibility of extending the Lambek grammar by adding the connectives *and* and *or*. With this purpose in mind we have introduced mechanisms which allow us to identify assumptions under certain conditions. Finally we have considered the possibility of generating sentences with the right anaphorical bindings. We have introduced simple schemas for the identification of pronouns with lexical hypotheses. In this way, we were able to generate sentences with a reading that the original system did not provide. Whether this approach is felicitous, is a matter of further research. We have merely shown that the structures used as vehicles of inference by Natural Logic *may* also be useful for explaining several other phenomena - anaphoric binding being one of them.

NOTES TO CHAPTER VII

1 One wonders how the following off-hand rejection of transformational grammar would be received:

'Transformational Grammar is inadequate as linguistic theory because it has the rule 'move α', for all categories α'.

2 See, however, note 4.

3 See the introduction for an explanation of the expression 'ambiguating relation'.

4 In fact, with regard to 'any' the situation is not completely clear. The study of the so-called 'polarity items' suggests that the first term is the correct one and that the second one is wrong: the negative polarity item 'any' must occur in negative syntactic position (cf. Ladusaw 1980). However, the monotonicity marking of the wide scope 'any' and the polarity 'any' will coincide. The polar item is interpreted as an existential noun phrase. And we must mark this 'any' as follows:

$$\text{any} \in (e \rightarrow t)^{+} \rightarrow (e \rightarrow t)^{+} \rightarrow t\,.$$

5 Of course, these considerations do not hold in the full lambda system: if LOVES x_e were not a Lambek term, we would lack the guarantee that Y_e does not occur in LOVES x_e.

6 Van Eijck himself rejects his own proposal on grounds which are not relevant for our present purposes.

7 'In the earliest transformational grammars, it was suggested that a pronominalization transformation optionally replaces a repeated noun phrase by a personal pronoun.

On such a view, all anaphoric pronouns were regarded as what Geach (1964) calls 'pronouns of laziness'; reference was considered irrelevant for the syntax.' Partee (1972 : 421)

EPILOGUE

The preceding chapters contain our view on Natural Logic and define such a logic for natural language. The historical search we were engaged in revealed the principle of monotonicity as a fundamental principle for syllogistic and non-syllogistic reasoning. Accordingly, we have based our Natural Logic on monotonicity, using a theory of monotonicity marking which makes monotone sensitive positions syntactically recognizable.

The architecture of the Natural Logic discussed in this dissertation displays the principles of

- monotonicity
- conservativity
- anaphoric identification

To incorporate such principles in Natural Logic one must have a

- theory of monotonicity marking
- theory of government
- theory of anaphoric binding

This dissertation provides a theory of monotonicity marking, as well as some proposals for the treatment of anaphorical binding. However, the use of conservativity presupposes a theory of government which is as yet non-existent. Further research on this topic should (and can) result in such a theory. This addition would make Natural Logic more fitted for its task, namely to systematize natural language inference.

Appendix 1

Proof of P1

$$\dfrac{\dfrac{\dfrac{\text{EVERY A IS A B}}{[\![B]\!] \in [\![\text{EVERY A}]\!]}}{[\![A]\!] \subseteq [\![B]\!]}{\scriptstyle (4.1)} \qquad \dfrac{X \in [\![\text{EVERY B}]\!]}{[\![B]\!] \subseteq X}{\scriptstyle (4.1)}}{\dfrac{[\![A]\!] \subseteq X}{X \in [\![\text{EVERY A}]\!]}{\scriptstyle (4.1)}}$$

Proof of P2

$$\dfrac{\dfrac{\dfrac{\text{EVERY A IS A B}}{[\![B]\!] \in [\![\text{EVERY A}]\!]}}{[\![A]\!] \subseteq [\![B]\!]}{\scriptstyle (4.1)} \qquad \dfrac{X \in [\![\text{SOME A}]\!]}{[\![A]\!] \cap X \neq \emptyset}{\scriptstyle (4.1)}}{\dfrac{[\![B]\!] \cap X \neq \emptyset}{X \in [\![\text{SOME B}]\!]}{\scriptstyle \text{def}}}$$

Proof of P3

$$\dfrac{\dfrac{\dfrac{\text{SOME A IS A B}}{[\![B]\!] \in [\![\text{SOME A}]\!]}}{[\![A]\!] \cap [\![B]\!] \neq \emptyset}{\scriptstyle (4.3)} \qquad \dfrac{X \in [\![\text{EVERY A}]\!]}{[\![A]\!] \subseteq X}{\scriptstyle (4.3)}}{\dfrac{X \cap [\![B]\!] \neq \emptyset}{X \in [\![\text{SOME B}]\!]}{\scriptstyle (4.3)}}$$

Proof of P4

$$\dfrac{\dfrac{\dfrac{\text{SOME A IS A B}}{[\![B]\!] \in [\![\text{SOME A}]\!]}}{[\![A]\!] \cap [\![B]\!] \neq \emptyset}{\scriptstyle (4.3)} \qquad \dfrac{X \in [\![\text{EVERY B}]\!]}{[\![B]\!] \subseteq X}{\scriptstyle (4.3)}}{\dfrac{[\![A]\!] \cap X \neq \emptyset}{X \in [\![\text{SOME A}]\!]}{\scriptstyle (4.3)}}$$

Proof of P5

$$\dfrac{\dfrac{\dfrac{\text{EVERY A IS A B}}{[\![\text{EVERY B}]\!] \subseteq [\![\text{EVERY A}]\!]}{\scriptstyle \text{P1}}}{D_{(et)t} - [\![\text{EVERY A}]\!] \subseteq D_{(et)t} - [\![\text{EVERY B}]\!]}}{[\![\text{NOT EVERY A}]\!] \subseteq [\![\text{NOT EVERY B}]\!]}{\scriptstyle (7)}$$

Proof of P6

$$\dfrac{\dfrac{\dfrac{\text{EVERY A IS A B}}{[\![\text{SOME A}]\!] \subseteq [\![\text{SOME B}]\!]}\,\text{P2}}{D_{(et)t} - [\![\text{SOME B}]\!] \subseteq D_{(et)t} - [\![\text{SOME A}]\!]}}{[\![\text{NO B}]\!] \subseteq [\![\text{NO A}]\!]}\,(6)$$

Proof of P7

$$\dfrac{\dfrac{\dfrac{\text{SOME A IS A B}}{[\![\text{EVERY A}]\!] \subseteq [\![\text{SOME B}]\!]}\,\text{P3}}{D_{(et)t} - [\![\text{SOME B}]\!] \subseteq D_{(et)t} - [\![\text{EVERY A}]\!]}}{[\![\text{NO B}]\!] \subseteq [\![\text{NOT EVERY A}]\!]}\,(6),\ (7)$$

Proof of P8

$$\dfrac{\dfrac{\dfrac{\text{SOME A IS A B}}{[\![\text{EVERY B}]\!] \subseteq [\![\text{SOME A}]\!]}\,\text{P4}}{D_{(et)t} - [\![\text{SOME A}]\!] \subseteq D_{(et)t} - [\![\text{EVERY B}]\!]}}{[\![\text{NO A}]\!] \subseteq [\![\text{NOT EVERY B}]\!]}\,(6),\ (7)$$

Proof of P9

$$\dfrac{\dfrac{\text{EVERY A VP}}{[\![\text{A}]\!] \subseteq [\![\text{VP}]\!]} \quad [\![\text{A}]\!] = [\![\text{IS A(N) A}]\!]}{[\![\text{IS A(N) A}]\!] \subseteq [\![\text{VP}]\!]}$$

Proof of P'3

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{SOME A VP}}{[\![\text{VP}]\!] \in [\![\text{SOME A}]\!]}}{[\![\text{A}]\!] \cap [\![\text{VP}]\!] \neq \emptyset}\,(4.3) \quad [\![\text{THING THAT VP}]\!] = [\![\text{VP}]\!]}{[\![\text{A}]\!] \cap [\![\text{THING THAT VP}]\!] \neq \emptyset} \qquad \dfrac{X \in [\![\text{EVERY A}]\!]}{[\![\text{A}]\!] \subseteq X}\,(4.1)}{X \in [\![\text{SOME THING THAT VP}]\!]}$$

Proof of UI

$$\dfrac{\dfrac{\dfrac{\text{ABELARD IS A B}}{s \in [\![\text{B}]\!]} \qquad \dfrac{X \in [\![\text{EVERY B}]\!]}{[\![\text{B}]\!] \subseteq X}}{s \in X}}{X \in [\![\text{ABELARD}]\!]}$$

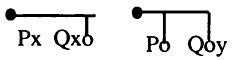
Proof of EG

$$\dfrac{\dfrac{\dfrac{\text{ABELARD IS A B}}{s \in [\![\text{B}]\!]} \qquad \dfrac{X \in [\![\text{ABELARD}]\!]}{s \in X}}{[\![\text{B}]\!] \cap X \neq \emptyset}}{X \in [\![\text{SOME B}]\!]}$$

Appendix 2
THE BETA GRAPHS

1. QUANTIFIER LINE AND SCOPE. To prove some properties of the beta graphs we need an inductive definition of the language.The definition we give here is *partly* based on the selectives employed by Peirce himself.

As a preliminary step we define the '*quantifier line*' as a simple or broken horizontal line with a marked free extreme adhering to one of its horizontal parts. By the '*application*' of a quantifier line L to a selective β we understand the result of the following simultaneous operations: (1) L is written above β in such a way that all its horizontal parts occur outside negations in β; (2) each occurrence of the variable x_i in β is substituted by a vertical line joined to a horizontal part of L. The result of this operation will be designated by $/\sigma/(L_i/x_i)$ and the expression occurring under L_i or joined to it, is called its '*scope*'. For instance, if Px xQy is a selective, the following expressions are results of the defined operation:



The afore mentioned line will inherit the index from the variable it replaces and (sometimes) we will write that index on its marked extreme. If σ does not contain any occurrence of the variable x_i then $/\sigma/(L_i/x_i)$ will not be defined.

2. THE SELECTIVE LANGUAGE.

A more formal characterization of this language is the following.

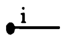
(i) The blank is a selective.

(ii) If P is an n-ary predicate letter, then $Px_1 \ldots x_n$ is an atomic selective.

(iii) If σ, σ' are a selectives, so are $\boxed{\sigma}$, $\sigma\sigma'$.

(iv) If σ is a selective and the variable x_i occurs in σ, then also $/\sigma/(L_i/x_i)$ is a selective.

3. THE BETA GRAPHS. We define a graph as a special selective: if σ is a selective lacking individual variables, then it will be called a *beta graph.* We use the constant T to define the quantifier line as a sentence:

 := Tx(L/x)

4. POLARITY. As usual we define polarity by counting negations:

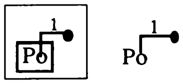
(i) Any selective σ occurs in σ in the scope of 0 negations.

(ii) If σ occurs in ψ in the scope of n negations, then it occurs in $\boxed{\psi}$ in the scope of n +1 negations.

(iii) If σ occurs in $\psi\phi$ in the scope of n negations, then also in $\psi\phi$, $/\psi/(L_i/x_i)$.

Now we say that a line L occurs positively (negatively) in ψ, if its marked extreme occurs in ψ in the scope of an even(odd) number of negations.

Observe that in (a) below L_1 occurs negatively its marked extreme, occurs within 1 negation, whereas in (b) L_1 occurs positively since its marked extreme occurs within 0 negations.



(a) (b)

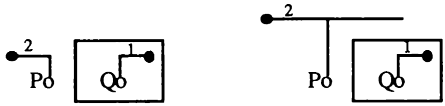
5. REMARKS.(A)This selective language differs in an important respect from the original graphs. We construe quantifier-free expressions by using the selectives, whereas by Peirce there are not such formulas. Any expression of its language is taken as quantified according to the polarity involved:

> A selective at its first occurrence shall be asserted in the mode proper to the compartment in which it occurs. If it be on that occurrence evenly enclosed, it is only affirmed to exists under the same conditions under which any graph in the same closs is asserted. . . If, however, at its first occurrence, it be oddly enclosed, then, in the disjunctive mode of interpretation, it will be denied, subject to the conditions proper to the close in which it occurs, so that its existence being disjunctively denied, a non-existence will be affirmed and as subject, it will be universal. Peirce (4 : 461)

(B) Notice that a standard language can easily be translated into the selective language. Given the expressive completeness of the set $\{\neg, \wedge, \exists\}$ we can focus the translation on formulas constructed with these symbols. If we have a translation $\pi(\phi)$ of ϕ into the selective language, and the variable x occurs free in this formula, then the translation of $\exists x\phi$ is nothing else but the application of a quantifier line to $\pi(\phi)$.

6. A SYSTEM OF RULES . The rules for the graphs fall out in three parts: propositional rules, quantificational rules and mixed rules. The propositional rules consist of the alpha rules considered earlier on. The quantificational rules regulate certain operations which may be carried out on the quantifier line. These operations consist in the lengthening or shortening, cutting or joining of the line. The mixed rules allow for the introduction of double negation on the quantifier line and the copy of expressions or elimination of copied expressions bound to the same quantifier lines in the same way. We comment on the quantificational and mixed rules, and point out their standard predicate logic counter-parts.

6.1. LENGTHENING AND SHORTENING OF QUANTIFIER LINES. Any line (vertical or horizontal) may be extended whenever no negation is crossed in this operation. This operation will be called a *legal extension of lines*. Thus if we have the graph (a) to start with, then we can extend the vertical part of P's line and afterwards the horizontal part of it, obtaining in this way the graph (b):



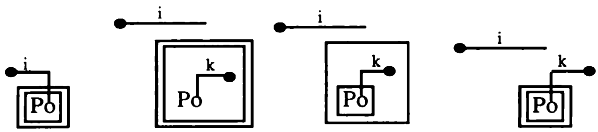
(a) (b)

In the graph (b) the negated expression belongs to the scope of P's line. But this corresponds to the construction in which P's line is applied to the selective consisting of Px and the negated expression. Similarly, we can start from the graph (b) and shorten P's lines to obtain (a). These two manipulations of the quantifier line correspond, according to our translation, to the process of widening and shortening the scope of the existential quantifier, i.e. to one of the scope principles listed in the section about the 1885 paper. The above transformations correspond to the substitution of the equivalent expressions $\exists x Px \wedge \neg\exists y Qy$, $\exists x(Px \wedge \neg\exists y Qy)$.

6.2. THE CUTTING OF A LINE. We also have the operation of cutting lines on *certain* parts. The idea is that any vertical part occurring in the scope of an even number of negations may be cut into two parts. The result of this cutting must be a graph, hence we must ensure that in this operation no vertical line appears which is not joined to a quantifier line. To this end we will use the notion of *cut*. Let l be a vertical line attached to a horizontal line L of σ. A cut of l consists in the result of the following simultaneous operation: first of cutting l into two parts leaving an un-marked extreme pertaining to the original horizontal line, secondly of adding a new horizontal line with a heavy point to the other extreme and finally of eliminating the empty vertical extreme.This new line will have an index distinct from the index of the original line and its horizontal part will not cross any negation . For instance, a cut of the quantification line i in (a), will be the pair of lines i and k in (b):

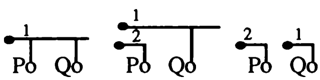
(a) (b)

A cut of a quantification line does not need to be carried out in a positive context. The negations enclosing Po in the the graph (a) divide the vertical line into three segments. The first segment occurs in the scope of 2 negations; the second segment occurs in the scope of 1 negation; the third occurs in the scope of 0 negations. The results of cutting are the graphs (b), (c) and (d):



(a) (b) (c) (d)

A **legal cut** of a quantification line is a cut carried out on a vertical positive part of a quantification line. Hence, among the above graphs, only (b) and (d) qualify as results of legal cuts. Here is a more substantive example of a legal cut. If we have the graph (e), then we can legally cut one of the vertical lines obtaining (f) as a result. In predicate logic terms, this means that we can pass from $\exists x(Px \wedge Qx)$ into $\exists x(\exists yPy \wedge Qx)$. But lines may be shortened. Hence, at the end, we may derive (g), i.e. $\exists yPy \wedge \exists xPx$:

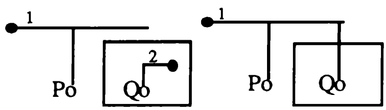


(e) (f) (g)

Notice that this operation corresponds to one of the diversification principles quoted in section 4.1.

6.3. JOINING LINES. We have also the converse operation of joining two lines together. The idea in this case is that any quantifier line L_j occurring negatively may be joined to any other line L_i under which (scope) it occurs. To implement this idea we need the notion of *legal joint*. Let L_i, L_j be two horizontal lines such that L_j occurs in the scope of L_j. **The legal joint of Lj to Li** is the result of the operation of attaching the straightened marked extreme of L_j to L_i, eliminating both L_j's marked point and its index directly.

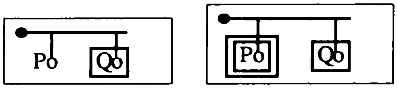
Thus this rule allows us to transform the graph (a) into the graph (b). According to our translation we have the correct inference of $\exists x(Px \wedge \neg Qx)$ from $\exists x(Px \wedge \neg \exists yQx)$.



(a) (b)

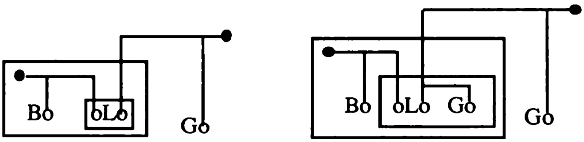
Notice that this operation corresponds to one of the identification principles quoted earlier on.

6.4. THE MIXED RULES. (A) In the first place we have a rule which allows for the introduction and elimination of double negation on any part of the scope of a quantifier line. For instance, we can pass from (a) into (b) and conversely. This amounts to the inference of ∀x(¬¬Px → Qx) from ∀x(Px → Qx):



(a) (b)

(B) We also have a generalization of the copying rule for sub-formulas bound to a given quantifier line. The general idea is that we are allowed to copy c.q. to delete sub-formulas which are identical except by their position inside a given expression. Thus we can pass from the graph (a) below into (b) and vice versa. In predicate logic terms, this means that we can pass from ∃x(ϕ(x) ∧ Γ(ψ)) into ∃x(ϕx ∧ Γ(ϕ(x) ∧ψ)) and from ∃x(ϕx ∧ Γ(ϕ(x) ∧ψ)) into ∃x(ϕ(x) ∧ Γ(ψ)).



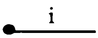
(a) (b)

In the context of standard symbolisms, we must assume that in Γ(ψ), ψ does not occur in the scope of a quantifier ∃x or ∀x. This can be achieved by introducing a variable convention. In a formula Γ each quantifier binds a distinct variable.

7. THE BETA SYSTEM. The rules discussed in the previous section can be summarized as follows:

7.1. AXIOMS

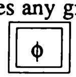
7.1.1 The blank

7.1.2. 

7.2.Rules

7.2.1. DELETION RULE (DR). Within an even number of boxes any graph may be deleted.

7.2.2. INSERTION RULE (IR). Within an odd number of boxes any graph may be inserted.

7.2.3. DOUBLE NEGATION RULE (DNR). The graphs ϕ and 

are mutually interchangeable.

7.2.4. COPYING RULE (CR). The graphs ϕΓ(ψ) ; ϕΓ(ϕψ) are mutually interchangeable.

7.2.5. SCOPE RULE (SR). Any horizontal quantifier line L may be legally lengthened or shortened.

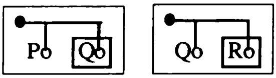
7.2.6. CUT RULE (CUTR). On any positive part of a vertical line L a cut may be carried out.

7.2.7. JOIN RULE (JR). Any line L_j occurring negatively may be legally joined to any other horizontal line L_i if L_j occurs in the scope of L_i.

7.1. EXAMPLES.

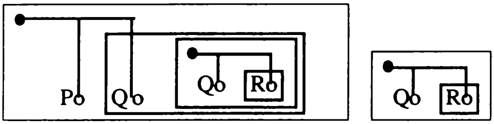
(a) We give here Peirce's proof of Barbara:

1.



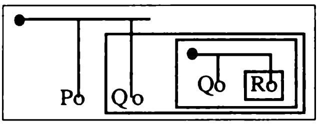
Assumptions

2.



CR

3.



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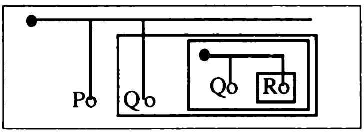
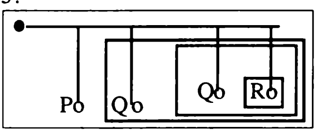
4.



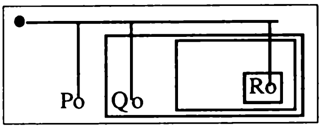
SR

5.



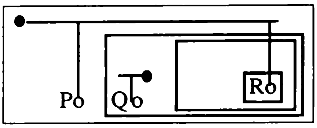
JR

6.



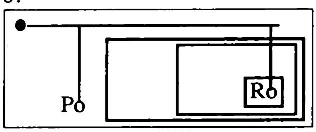
CR

7.



CutR

8.



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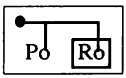
9.



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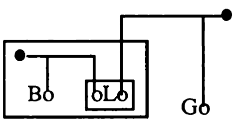
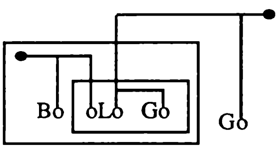
(b) A classical example is the inference of *Every boy loves some girl* from *There is a girl loved by every boy*. In this system the inference is proven as follows:

1.



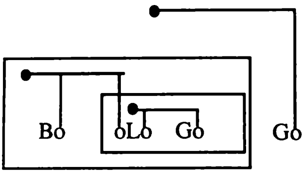
Assumption

2.



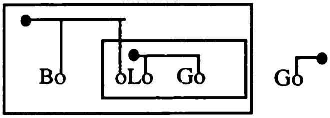
CR

3.



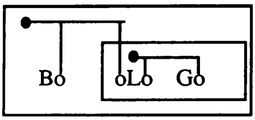
CutR

4.



SR

5.



ER

7.2. THE SOUNDNESS OF PEIRCE'S RULES. In this section we consider the question as to whether these rules are sound or not. Our strategy is as follows: given the translation π of the graphs into standard logic, we point out that the rule in question can be seen as derived inference rules within standard systems.

1. The soundness of the alpha rules has been proved in Sánchez Valencia (1989).

2. The effect of SR consists in giving wider(narrower) scope to a quantifier. We have here in fact a rule based on the so-called passage rule: $\exists x\phi(x) \wedge \psi \leftrightarrow \exists x(\phi(x) \wedge \psi)$. This equivalence holds whenever ψ does not contain the variable x free. In the selective language such a condition is automatic fulfilled.

3. Consider now CutR. Remember that a part of a vertical line is positive if the legal cut on that part leaves a positive line. But positive lines are translated into existential quantifiers. Hence the effect of this rule is the introduction an existential quantifier at certain designed positions within a formula. Thus in standard terms this rule has the reading:

$$\frac{\Gamma(\phi)}{\Gamma(\exists x\phi')}$$

provided that ϕ occurs positively in $\Gamma(\phi)$, where $\phi' = [x/y]\ \phi$.

Notice that the soundness of this rule can be explained in terms of monotonicity. One can show that in standard systems the following rule is a derived inference rule:

$$\frac{\phi \rightarrow \exists x\phi' \quad \Gamma(\phi)}{\Gamma(\exists x\phi')}$$

provided that ϕ occurs positively in $\Gamma(\phi)$, where $\phi' = [x/y]\ \phi$.

4.Consider now JR. Notice that a line which occurs negatively is translated into an existential quantifier occurring negatively. Hence the effect of this rule consists of the elimination of an existential quantifier at certain designed positions within a formula.
Thus in standard terms the rule has this reading:

$$\frac{\Gamma(\exists x\phi)}{\Gamma(\phi')}$$

provided that $\exists x\phi$ occurs negatively in $\Gamma(\phi)$.

Notice that $\exists x\phi(x)$ can occur in the scope of a quantifier $\exists y$. If this is the case, the the introduction of $\phi(y)$ has the effect of binding the variable to the quantifier $\exists y$. To assure that this is always the case, we can add to the rule the condition: $\exists x\phi$ must occur within the scope of an occurrence of $\exists y$.
This rule can also be seen as a special case of the earlier considered monotonicity rule. Thus we can show that the following rule is a derived rule in standard systems:

$$\frac{\phi' \rightarrow \exists x\phi \qquad \Gamma(\exists x\phi)}{\Gamma(\phi')}$$

provided that $\exists x\phi$ occurs negatively in $\Gamma(\phi)$,
$\exists x\phi$ occurs within the scope of an occurrence of $\exists y$ and $\phi' = [y/x]\phi$.

7.4. THE COMPLETENESS OF PEIRCE'S RULES. Peirce's system of quantification can be compared with the axiomatization of first-order inference given in Quine (1947). In this discussion we use the notation for the Beta Graphs introduced in the main text of this chapter. We show that Quine's axioms are derivable.

(1) $\forall x(\phi \rightarrow \psi) \rightarrow \forall x\phi \rightarrow \forall x\phi\psi$

proof

1. $\forall x(\phi \rightarrow \psi) \rightarrow \forall x(\phi \rightarrow \psi)$	Alpha theorem
2. $\forall x(\phi \rightarrow \psi) \rightarrow \forall x(\forall y\phi' \rightarrow \psi)$	Diversification
3. $\forall x(\phi \rightarrow \psi) \rightarrow (\forall y\phi' \rightarrow \forall x\psi)$	Passage
3. $\forall x(\phi \rightarrow \psi) \rightarrow (\forall x\phi \rightarrow \forall x\psi)$	Alphabetic variant

(2) $\forall x \forall y\ \phi \rightarrow \forall y \forall x \phi$

proof

1. $\forall y \forall x\ \phi \rightarrow \forall y \forall x \phi$	Alpha theorem
2. $\forall y \forall x\ \forall z \phi' \rightarrow \forall y \forall x \phi$	Diversification
3. $\forall x\ \forall z \phi' \rightarrow \forall y \forall x \phi$	Vacuous quantification
4. $\forall x\ \forall y \phi \rightarrow \forall y \forall x \phi$	Alphabetic variant

(3) $\forall x\ \phi \rightarrow \phi'$, where ϕ' is like ϕ except for containing free occurrences of y wherever ϕ contains free occurrences of x.

proof

1. $\forall x\ \phi \rightarrow \forall x \phi$	Alpha theorem
2. $\forall x\ \phi \rightarrow \phi'$	Identification

(3) $\phi \rightarrow \forall x \phi$, where x is not free in ϕ.

proof

1. $\phi \rightarrow \phi$	Alpha theorem
2. $\neg(\phi\ \wedge \neg\ \phi)$	Definition
3. $\neg(\phi\ \wedge \neg\neg\neg\ \phi)$	Double negation
4. $\neg(\phi\ \wedge \neg\neg(\exists x(x{=}x)\ \ \wedge \neg\ \phi))$	Insertion
5. $\neg(\phi\ \wedge \neg\neg(\exists x(x{=}x)\ \ \wedge \neg\ \phi))$	Definition
6. $\neg(\phi\ \wedge \neg\neg \exists x \neg\ \phi)$	Definition
7. $\neg(\phi\ \wedge \neg \forall x\ \phi)$	Definition
8. $\phi \rightarrow \forall x \phi$	Definition

Of course, the system is closed under Modus Ponens. But Quine's axioms are universal closures of formulas. So one needs to demand that the alpha theorems which start the proof are universal closures. This isn't a trivial condition, since it takes the role of the rule:

'If ϕ is a theorem, then so is $\forall x\ \phi$'

There is, however, no evidence that Peirce was aware of this rule. Alternatively, one can say that Peirce's system lacks a *real* rule of introduction for the universal quantifier and a *real* elimination rule for the existential quantifier.

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Index of Names

SAMENVATTING IN HET NEDERLANDS

Het doel van dit proefschrift is te laten zien dat een natuurlijke logica, d.w.z. een rechtstreeks op natuurlijke taal gebaseerd inferentiesystem mogelijk is. We onderzoeken de geschiedenis van de logica op inferentiebeginselen die toepasbaar zijn op de natuurlijke taal. Het bestuderen van Ockham, Leibniz, De Morgan en Peirce levert diverse nuttige beginselen op. Een van deze beginselen is het zogenaamde principe van monotonie. Tegelijkertijd ontdekken we dat, op Peirce na, het werk van deze auteurs een (naar moderne maatstaven) gebrekkig begrip vertoont van de notie 'logische vorm'.

Om te voorkomen dat onze natuurlijke logica aan hetzelfde euvel zou leiden als haar illustere voorgangers, definiëren we natuurlijke logica als een inferentiesysteem gebaseerd op grammaticale vorm. Op deze manier bereiken we een nauwe aansluiting tussen logische en grammaticale vorm: in ons systeem vallen deze twee noties samen.

Onze definitie van natuurlijke logica veronderstelt de aanwezigheid van een theorie van grammaticale vormen. We menen deze gevonden te hebben in de categoriale grammatica -een linguistisch beschrijvingsmodel waarvan de ontstaansgeschiedenis tot Frege terug te voeren is. De specifieke categoriale grammatica die we gebruiken is een variant op de zogenaamde Lambek Calculus. De combinatie van deze grammatica met de historische inferentiebeginselen resulteert in onze natuurlijke logica.

De geografie van deze dissertatie is de volgende:
Hoofdstuk I bevat een discussie van de notie van natuurlijke logica.We bespreken hier verschillende bezwaren tegen natuurlijke taal als inferentiemedium. Hoofdstuk II is een historisch onderzoek naar uitbreidingen van de syllogistiek. Hoofdstuk III is is een studie van Peirce's werk vanuit een speciefieke standpunt: wat kan natuurlijke logica van Peirce leren? Hoofdstuk IV is gewijd aan de logische vormen waar natuurlijke logica mee werkt. Het is in dit hoofstuk waarin we onze eigen categoriale grammatica definieren. Hoofdstuk V beeindigt het voorbereidend werk voor de constructie van onze natuurlijke logica. In dit hoofstuk introduceren we een mechanisme voor het markeren van plaatsen waarin monotone substituties plaats kunnen hebben. Hoofdstuk VI is gewijd aan de constructie van de natuurlijke logica. zelf. Hoofdstuk VII bevat een discussie over de linguistische adequaatheid van onze categoriale grammatica..

RESUMEN EN CASTELLANO

Esta disertación persigue un ideal ha menudo mencionado en la literatura: la construcción de un sistema de inferencia applicable directamente a los lenguages vernaculares, esto es, la construcción de una *lógica natural.* Asumiendo que los sistemas lógicos pre-fregeanos son esencialmente lógicas naturales, hemos planteado una pregunta histórica con motivos puramente canibalescos: ¿ Qué se puede aprender de las lógicas naturales del pasado? Al final de cuentas el estudio de Ochkam, Leibniz, De Morgan y Peirce nos ha permitido aislar tres principios de inferencia natural: monotonía, conservatividad e identificación anafórica.

Pero al mismo tiempo hemos podido descubrir que la ausencia de una sólida base sintáctica marró los intentos de Leibniz, De Morgan y Ockham. Esta limitación es ausente en la obra de Peirce, aunque sus observaciones son, a primera vista, relevantes solamente para lenguages artificiales.

Para evitar que nuestra lógica sufriese de la misma debilidad que sus predecesoras, hemos optado por definir la lógica natural como un sistema en el cual los vehículos de inferencia son las formas gramaticales mismas. De esta manera logramos una estrecha correspondencia entre forma lógica y forma gramatical: en nuestro sistema estas formas coinciden. Al mismo tiempo, hemos sido capaces de aplicar las observaciones de Peirce al idioma natural.

Nuestra concepción de la lógica natural presupone la existencia de una teoría de formas gramaticales. No es ninguna sorpresa que hayamos escogido la gramática categorial -esencialmende inventada por Frege- como la base lingüística de nuestra lógica. La gramática categorial que hemos escogido es aquella engendrada por Lambek y resucitada por Van Benthem. La combinación de esta gramática con los principios de inferencia extraídos del pasado forman la arquitectura de nuestra lógica natural.

La geografía de esta disertación es la siguiente:

El Capítulo I discute la idea de una lógica natural. Aquí nosotros rechazamos diversas objeciones contra una lógica natural. El Capítulo II es una investigación histórica acerca de extensiones de la silogística. El Capítulo III es un estudio de la lógica de Peirce desde un punto de vista especial : ¿ Qué puede aprender de Peirce la lógica natural ? El Capítulo IV está dedicado a la definición de los vehículos de inferencia de nuestra lógica. En este capítulo introducimos la gramática categorial debida a Lambek y Van Benthem. El Capítulo V finaliza nuestro trabajo preparatorio. Aquí describimos una manera de marcar lugares en los cuales substituciones monotónicas son permisibles. El Capítulo VI contiene nuestro sistema lógico propiamente dicho. El Capítulo VII, por otra parte, está dedicado a una discusión de la propiedad lingüística de la gramática de Lambek y Van Benthem.

Finalmente: esta disertación es en Inglés pero no es acerca del Inglés. Justamente por haber empleado la gramática (universalisable) de Lambek y Van Benthem nuestras observaciones pueden ser trasplantadas facilmente a otros idiomas, por ejemplo al Kakchikel o al Castellano.

Stellingen behorende bij het proefschrift
'Studies on Natural Logic and Categorial Grammar', Víctor Sánchez, 30 januari 1991.

1. Beschouw de logische taal L(Q, Q') verkregen door de toevoeging van de monotone kwantoren Q en Q' aan een standaard predicaatlogische taal L. Een verzameling axioma's voor L(Q, Q') is de volgende:
(0) De theorema's van L.
(1) $\forall x(\phi \rightarrow \psi) \rightarrow (Qx\phi \rightarrow Qx\psi)$.
(2) $Qx\phi \leftrightarrow \neg Q'x \neg \phi$
(3) $Qx\phi \leftrightarrow Qy\psi$, waar ϕ een formule is waarin y niet vrij voorkomt en ψ uit ϕ wordt verkregen door elke vrij voorkomen van x te vervangen door y.
Henkin's bewijs van de Lyndon stelling laat zich generaliseren tot een bewijs van deze stelling voor L(Q, Q'). Bovendien volgt hieruit dat een zin uit L(Q, Q') monotoon stijgend is in een predicaatletter R desda hij equivalent is met een zin waarin R uitsluitend positief voorkomt. Dit is een eerste stap op weg naar de algemene preservatiestelling besproken in Hoofdstuk V van dit proefschrift.

Referenties:
'An Extension of the Craig-Lyndon interpolation theorem', L. Henkin, *Journal of Symbolic Logic* 28, 201-216.
'A Proof of Lyndon Interpolation Theorem in a Logic with Monotone Quantifiers', V. Sánchez, manuscript.

2. Genten's originele sequenten calculus voor de implicatie bevat de volgende regels:
Axioma's : $A \vdash A$

Logische Regels:

$$\text{R1}\ \frac{A, \Gamma \vdash B}{\Gamma \vdash A \rightarrow B} \qquad \text{L1}\ \frac{\Delta \vdash A \quad B, \Gamma \vdash C}{A \rightarrow B, \Delta, \Gamma \vdash C}$$

$$\text{CUT}\ \frac{\Delta \vdash C \quad C, \Gamma \vdash A}{\Delta, \Gamma \vdash A}$$

Structurele Regels:

$$\frac{\Delta, A, B, \Gamma \vdash C}{\Delta, B, A, \Gamma \vdash C}\ \text{(permutatie)}$$

$$\frac{\Delta, A, A, \Gamma \vdash C}{\Delta, A, \Gamma \vdash C}\ \text{(contractie)}$$

$$\frac{\Gamma \vdash C}{\Gamma, A \vdash C}\ \text{(verzwakking)}$$

De permutatieregel wordt bij deze opzet een afgeleide regel.
Dit is Gentzen waarschijnlijk bekend geweest, getuige zijn opmerking:
"The schemata are not all mutually independent, i.e., certain schemata could be eliminated with the help of the remaining ones. Yet if they were left out, the 'Hauptsatz' could not longer be valid." Szabo (1969, p. 85).
(Zie Hoofdstuk IV. 1. 3. over de permutatieregel en de Lambek Calculus)

Referentie:
Szabo (1969) : *Collected papers of Gerhard Gentzen*, M. Szabo (ed), North Holland, Amsterdam.

3. Quine's variabelenvrije predicaten logica zoals geaxiomatiseed in Bacon (1985) is logisch equivalent met Tarski's subtitutievrije predicaten logica zoals geaxiomatiseerd in Monk (1965).

Referenties:
Bacon (1985): 'The Completeness of a Predicate-functor Logic', *Journal of Symbolic Logic*, 50, 903-926
Monk (1965): 'Subtitutionless predicate logic with identity', D, Monk *Archiv für mathematische Logik und Grundlagenforschung* 7, 102-121.
'A New Completeness Proof of Bacon's Predicate-functor Logic', V. Sánchez, manuscript.

4. Anders dan Hawkins (en Peirce zelf) hebben gesuggereerd, is Dedekinds definitie van oneindigheid onafhankelijk van Peirces behandeling van dit begrip.

Referenties:
The Collected Papers of Charles Sanders Peirce. C. S. Peirce, Cambridge,Mass.: Harvard University Press.
'A Compendium of C.S. Peirce's 1866-1885 work ', B.S. Hawkins, *Notre Dame Journal of Formal Logic*, Volume XVI, Number 1.
'Peirce e Dedekind: La Definizione di Insieme Finito', F.Gana, *Historia Mathematica* 12 (1985) 203-218.
'Peirce and arguments valid only in finite domains', V. Sánchez, manuscript.

5. F. Gana schrijft dat "Non risulta che Peirce abbia mai usato ne conosciuto l'assioma di scelta'". Maar de volgende passage laat zien dat Peirce een impliciet gebruik heeft gemaakt van het keuze axioma:
"We have two collections, The M's and the N's. . . To begin with, there are vast multitudes of relations such that taking any one of them, r, every M is r to an N and every N is r'd by an M. Each one of the r relations could also be modified as to reduce to what we can call an x-to-one relation, by running through the M's and cutting the connection of each M with every N but one. Call such a resulting relation t. Then every M would be t to a single N."

Referenties:
The Collected Papers of Charles Sanders Peirce. C. S. Peirce, Cambridge,Mass.: Harvard University Press.
'Peirce e Dedekind: La Definizione di Insieme Finito', F.Gana, *Historia Mathematica* 12 (1985) 203-218.
'Peirce and arguments valid only in finite domains', V. Sánchez, manuscript.

6. L. E. J. Brouwer is de meest bekende Nederlandse wiskundige uit de twintigste eeuw. Vandaar dat het nieuwe wiskunde gebouw van de Universiteit van Amsterdam "Euclides" heet.

7. De ontboezemingen van gewone mensen in praatprogramma's laten zien dat de uitdrukking "doe gewoon dan doe je gek genoeg" in Nederland letterlijk juist is.