TOPICS IN INTUITIONISM AND TOPOS THEORY

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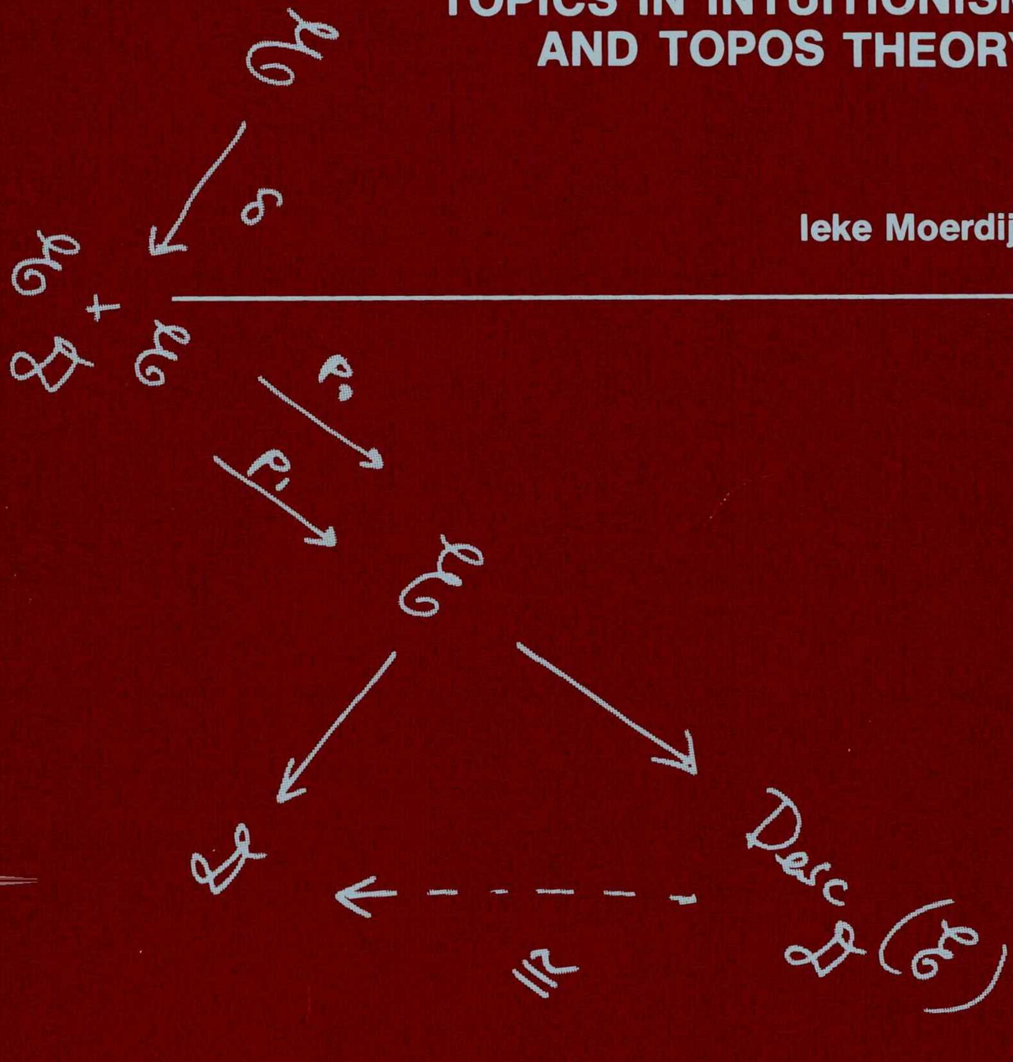

TOPICS IN INTUITIONISM AND TOPOS THEORY

ACADEMISCH PROEFSCHRIFT

ter verkrijging van de graad van
doctor in de Wiskunde en Natuurwetenschappen
aan de Universiteit van Amsterdam
op gezag van de Rector Magnificus
dr. D.W. Bresters,
hoogleraar in de faculteit der Wiskunde en Natuurwetenschappen,
in het openbaar te verdedigen in
de Aula der Universiteit
(tijdelijk in het Wiskundegebouw, Roetersstraat 15)
op woensdag 20 november 1985
te 15 uur precies

door

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geboren te Veenendaal

PROMOTOR: Prof.Dr A.S. Troelstra

ter herinnering aan mijn vader

PREFACE

This thesis consists of papers which have appeared or soon will appear in specialized mathematical journals, together with an introduction.
I wish to use this opportunity to express my indebtedness to all the people who have helped me understand something of intuitionism and topos theory.
I am especially indebted to my promotor A.S. Troelstra, for his constant encouragement, and for his expert advice on matters of intuitionistic analysis.
Obviously, I also owe a lot to my collaborators, G.F. van der Hoeven and G. Wraith; it has been a great pleasure to work with them.
Furthermore, at various stages I have benefited from conversations with D. van Dalen, J.M.E. Hyland, P.T. Johnstone (without his lucid writings I would never have learnt topos theory!), A. Joyal, G. Kreisel (much of the work with van der Hoeven is inspired by his pertinent questions), F.W. Lawvere, A.M. Pitts, and many others. I am grateful to all of them.

The research reported on here was carried out as part of ZWO-project 10-90-002, and I thank the ZWO-organization for their financial support.
Furthermore, I would like to thank Yvonne Voorn for much excellent typing, and "Espressobar HWL" for keeping me awake at crucial moments.

Amsterdam, September 9, 1985

CONTENTS

Introduction

It is by now a well-known and widely used fact that many constructions of a set-theoretical nature can be performed in the context of sheaves on a site (a site is a category equipped with a Grothendieck topology). More precisely, for many constructions of new sets from given ones there are corresponding constructions of new sheaves from given ones, having exactly the same defining universal properties. For example, for two sets X and Y the set Y^X of all functions from X to Y is completely determined by the property that for any third set Z, there is a bijection between functions $Z \to Y^X$ and functions $X \times Z \to Y$, denoted by

$$(1) \qquad \frac{Z \to Y^X}{X \times Z \to Y},$$

where $X \times Z$ is the Cartesian product (also definable by a universal property). For two *sheaves* X and Y, one can construct a sheaf Y^X, "the sheaf of sheaf-morphisms from X to Y", such that the correspondence (1) holds, when $X \times Z$ is now interpreted as the product of sheaves. This sheaf Y^X is completely determined by (1). As another example, one can construct *the sheaf* $P(X)$ *of subsheaves* of X from a given sheaf X, analogous to the construction of the *power set.*

In fact, all constructions which are intuitionistically meaningful as constructions of sets, are also meaningful as constructions of sheaves. In other words, each Grothendieck topos (or briefly topos, i.e. the category of sheaves on some site) can be regarded as a "set-theoretic universe", but a universe in which one has to work intuitionistically, i.e. in which one cannot freely use the principle of the excluded middle or the axiom of choice. There are several ways of making this more precise: one can list the categorical closure conditions corresponding to the basic operations on sets, such as the formation of the product $X \times Y$, the powerset $P(X)$, and the exponential Y^X, and define *elementary toposes* as the categories which satisfy these conditions, immediately observing that every Grothendieck topos is an elementary topos. From a more logical point of view, one would perhaps rather say that intuitionistic type theory (roughly, Heyting arithmetic with rules for function types and power types) is interpretable in any Grothendieck topos. (That is, one cannot only construct function types and power types in a Grothendieck topos as I indicated above, but the logical operations can also be interpreted. For example, if $R \subset X \times Y$ is a subsheaf - think of it as a two-place relation $R(x,y)$ - then there is a corresponding subsheaf $\forall_X R \subset Y$ having all the logical properties of the one-place predicate $\forall x \in X\, R(x,y)$ in intuitionistic type theory.) Or slightly differently, one can mimick the construction of the Van Neumann hierarchy $V = \bigcup_\alpha V_\alpha$ in any Grothendieck topos, and obtain a model of intuitionistic Zermelo Fraenkel set theory, IZF.

Classical logic or classical set theory is of course not excluded, but subsumed. In many important cases Grothendieck toposes provide set-theoretic universes where the principle of the excluded middle or the axiom of choice does hold. For example, this occurs in the case of the topos of sheaves on a Boolean algebra, or in the case of the topos of sets equipped with a continuous action of a topological group. In the first case, one will recover e.g. the Cohen-forcing models of set theory, and in the second case, e.g. the permutation models of set theory. In the non-classical case, one also recovers some traditional constructions from logic, such as Kripke models and Beth models, as special instances of Grothendieck toposes modelling intuitionistic set theory.

The relation between toposes and intuitionistic logic and set theory has a dual application. On the one hand, it has provided a very powerful tool for the study of problems in mathematical logic, mainly (but not only) those related to intuitionistic logic. On the other hand, however, the fact that any topos may be regarded as a universe of sets has turned out to be very useful in the study of general geometric problems of Grothendieck toposes themselves. Indeed, given a map of one Grothendieck topos $\mathcal{F}$ to another one $\mathcal{E}$, i.e. a geometric morphism $\mathcal{F} \xrightarrow{f} \mathcal{E}$, one may equivalently consider f as a topos constructed within the "universe" $\mathcal{E}$, i.e. properties of maps reduce to properties of Grothendieck toposes. Since $\mathcal{E}$ is an *intuitionistic* universe, one should really say that the study of maps of toposes reduces to the study of toposes themselves in an intuitionistic context. Or in a similar vein, maps of toposes $\mathcal{F} \to \mathcal{E}$ *over* a given base topos $\mathcal{S}$ correspond to maps of toposes *inside* $\mathcal{S}$, so that intuitionistically valid properties of maps of toposes generalize to the context of maps over a given base topos.

This last point of view is already useful in understanding constructions in classical topology. A map $Y \xrightarrow{f} X$ of topological spaces (let's assume them to be Hausdorff) corresponds to a geometric morphism $Sh(Y) \to Sh(X)$ between the toposes of sheaves, and hence to a topos in $Sh(X)$, or in fact a *space* in $Sh(X)$, denoted by Y_X. Y_X is not quite a topological space in $Sh(X)$, however, but a *space* in the generalized sense where one takes the lattice of *open subsets* as primitive, rather than the *points*. So one defines a *generalized space* or a *locale*, as a complete lattice A in which the distributive law

$$y \wedge \bigvee_i x_i = \bigvee_i y \wedge x_i$$

holds (in other words, A is a complete Heyting algebra), and a *map* $A \xrightarrow{f} A'$ of two such locales as a function $f^*: A' \to A$ which preserves finite meets and arbitrary sups. Every topological space X gives a locale $\mathcal{O}(X)$, and a continuous function $X \xrightarrow{f} Y$ gives a map $\mathcal{O}(X) \to \mathcal{O}(Y)$ of locales by $f^* = f^{-1}: \mathcal{O}(Y) \to \mathcal{O}(X)$. Conversely, every locale A gives rise to a topological space pt(A) of *points* of A (i.e. maps of locales $\{0,1\} \to A$). If $A \cong \mathcal{O}(\mathrm{pt}(A))$, A is said to have *enough points*.

Locales behave very much like topological spaces. In fact, in many respects locales behave much better than spaces, especially in a context without the axiom of choice, and even more so in an intuitionistic context.

Locales play a central role in topos theory, partly due to the fact that any topos $\mathcal{E}$ can be approximated from two sides by a locale: there exist locales A and B such that there are maps of toposes (i.e. geometric morphisms) f and g

$$\mathrm{Sh}(A) \xrightarrow{f} \mathcal{E} \xrightarrow{g} \mathrm{Sh}(B)$$

with very nice properties (technically: f can be chosen to be connected and locally connected, in fact even to have "acyclic fibres", and g can be chosen to be hyperconnected). The construction of A and B is completely constructive and explicit, so this generalizes to *maps* of toposes as just explained: given $\mathcal{F} \to \mathcal{E}$, there are locales A and B *in* $\mathcal{E}$ such that there are geometric morphisms f and g *over* $\mathcal{E}$

$$\begin{array}{ccccc} \mathrm{Sh}_{\mathcal{E}}(A) & \xrightarrow{f} & \mathcal{F} & \xrightarrow{g} & \mathrm{Sh}_{\mathcal{E}}(B) \\ & \searrow & \downarrow & \swarrow & \\ & & \mathcal{E} & & \end{array}$$

with the same properties as before. ($\mathrm{Sh}_{\mathcal{E}}(A)$, $\mathrm{Sh}_{\mathcal{E}}(B)$ are the toposes over $\mathcal{E}$ obtained by taking scheaves on A resp. B inside $\mathcal{E}$.)

I mentioned above what it means for a locale to have enough points. Since one often has to deal with locales in a given topos, as was just indicated, it is particularly important to note that for a specific locale the question of whether it has enough points or not may depend on the topos in which one works. One may formulate this slightly differently, by identifying locales with propositional (geometric) theories. A locale has enough points iff the corresponding theory has enough models, i.e. iff this theory is complete. The completeness proof of such a theory may depend on essentially non-constructive axioms, such as choice axioms. It is thus of interest to see exactly which axioms need to be added to intuitionistic logic in order to prove the completeness of a given propositional theory.

Let me give some examples: the completeness of the propositional theory corresponding to the Dedekind reals, i.e. the question whether the locale ("formal space") of Dedekind reals has enough points, is equivalent to the compactness of the unit interval, i.e. to the Heine-Borel theorem (HB). And the formal Cantor space (the Cantor space defined as a locale) has enough points iff the Fan Theorem (FT) holds, while formal Baire space has enough points iff Bar Induction (BI) holds. One can thus compare the strength of the completeness of the corresponding theories. In the paper "Heine-Borel does not imply the Fan Theorem "it will be shown, among other things, that the implications

$$(BI) \Rightarrow (FT) \Rightarrow (HB)$$

are the only ones that hold among these three principles (relative to intuitionistic set theory).

The other papers in this collection fall naturally into two parts, according to the dual interaction between logic and topos theory. While the first four articles are instances of applying topos theory to logic, the last four articles rather apply the idea that toposes can be regarded as universes of sets to the study of general topos theory.

The first group of articles is concerned with the construction of topos-theoretic models for theories of choice sequences. In intuitionistic mathematics, one can distinguish various ways of constructing sequences of natural numbers, by classifying the types of restrictions to be put on the choices of future values. For example, one can fix all values of the sequence at once, by requiring that the choices are made according to an algorithm for computing the next value from earlier ones; sequences of this type are "lawlike". At the other extreme, one finds the free choice sequences or lawless sequences; these are sequences constructed without any restrictions on the choice of values. There are many types of sequences in between these two, corresponding to intermediate restrictions such as the requirement that the sequence being constructed must lie in the image of a lawlike continuous function from N^N to itself, or lie in a so-called spread (spreads are certain closed subspaces of N^N).

These various types of construction processes are reflected in the logical properties of the corresponding sequences: for instance, lawlike sequences behave rather classically, but about a given lawless sequence we can never know more than some initial segment. This leads to the principle of *open data* for lawless sequences

$$A(\alpha) \rightarrow \exists n \forall \beta (\forall m \leqslant n\, \alpha m = \beta m \rightarrow A(\beta))$$

where α and β are lawless sequences (some restrictions on the predicate A are necessary, see the articles below and references given there).

All this may sound quite strange to a classically minded reader, but this makes it all the more interesting that these various types of choice sequences, with their different logical properties corresponding to the way they are constructed, occur in a natural way in certain Grothendieck toposes! This not only provides us with some mathematically natural models for theories of choice sequences, but also gives some insight in more syntactical and proof-theoretic methods used earlier in the metamathematical study of intuitionistic mathematics. For example, the elimination translations for the system $\underset{\sim}{LS}$ of lawless sequences and for the system $\underset{\sim}{CS}$ of Kreisel and Troelstra correspond literally to the *forcing relation* of certain topos models for these systems (see e.g. the first paper with G. van der Hoeven, below). And also at a more philosophical level, the informal ideas concerning construction processes for choice sequences now appear in a mathematically rigorous setting in the description of the *sites* which define the various models. This is particularly apparent in the article "Constructing choice sequences from lawless sequences of neighbourhood functions", reprinted below.

The idea of constructing a site as the category of "finite initial parts" of a construction process has many applications. An example which is particularly simple but extremely useful is that of adjoining an enumeration $\mathbb{N} \twoheadrightarrow X$ of a given (inhabited) set X to the "universe", say this universe is a topos $\mathcal{E}$ and X is an object of $\mathcal{E}$. One only needs to construct the locale in $\mathcal{E}$ presented by all possible finite initial segments of this enumeration. In this way one finds a "base-extension", i.e. a geometric morphism $\mathcal{F} \rightarrow \mathcal{E}$, such that in $\mathcal{F}$ the set X has become countable. The map $\mathcal{F} \rightarrow \mathcal{E}$ is an open surjection, so that all first-order properties of $\mathcal{E}$ are preserved and reflected along this base-extension (i.e. they are true in $\mathcal{E}$ iff they are true in $\mathcal{F}$). The same holds for many properties of locales in $\mathcal{E}$, i.e. many properties hold of a given locale A in $\mathcal{E}$ iff they hold for "the same locale" in $\mathcal{F}$ (technically, this is the locale in $\mathcal{F}$ corresponding to the pullback $\mathcal{F} \times_{\mathcal{E}} Sh_{\mathcal{E}}(A)$). Consequently, by adding suitable enumerations to the universe, one can for many purposes assume that the presentation of a given locale is countable. This makes locales behave not only very much like topological spaces, but in fact even like countably presented topological spaces! - within Hausdorff spaces, these are precisely the complete separable metric spaces.

This is one of the ideas that lie at the basis of the paper "Connected locally connected toposes are path-connected", written with G. Wraith, where we generalize the classical result that all connected locally connected complete separable metric spaces are path-connected to Grothendieck toposes. This result is improved upon in a subsequent note, entitled "Path-lifting for Grothendieck toposes". As is explained in the introduction of the paper with Wraith, these results seem to be an important first step if one wishes to develop the homotopy theory of Grothendieck toposes, in a way which generalizes and improves the homotopy theory of topological spaces.

The central role that locales play in topos theory has been emphasized earlier, but appears even more strikingly in the recent representation theorem of A. Joyal and M. Tierney. The assertion is that every Grothendieck topos is equivalent to the category of equivariant sheaves on a localic groupoid, i.e. a groupoid-object in the category of locales. This result is a rather immediate consequence of the so-called descent theorem for Grothendieck toposes. Joyal and Tierney prove this descent theorem by developing descent theory for "modules" over locales, similar to the classical descent theory for modules over commutative rings. The aim of the paper "An elementary proof of the descent theorem" is to give a direct proof of this result, again exploiting the fact that every Grothendieck topos can be regarded as a set-theoretic universe.

The last paper in this thesis, "Continuous fibrations and inverse limits of Grothendieck toposes", is concerned with the following problem: Suppose we have an inverse filtered system of toposes $(\mathcal{E}_i)_i$ and geometric morphisms $\mathcal{E}_i \xrightarrow{f_{ij}} \mathcal{E}_j$ $(i \geq j)$, with inverse limit $\mathcal{E}^\infty = \varprojlim \mathcal{E}_i$ and projections $\mathcal{E}^\infty \xrightarrow{p_i} \mathcal{E}_i$. Which properties of the "bonding mappings" f_{ij} are inherited by the projections p_i? It will be shown that some important properties are indeed inherited. (If all the f_{ij} are surjections (resp.

open surjections, connected locally connected, hyperconnected, or connected atomic) then so are all the p_i .)

The general methods that have been mentioned above again occur predominantly in this paper. First of all, the idea of regarding a topos $\mathcal{E}$ as a universe of sets, and a map $\mathcal{F} \to \mathcal{E}$ as a forcing extension of this universe, will lead to the construction of a very manageable presentation of the inverse limit $\mathcal{E}^{\infty} = \varprojlim \mathcal{E}_i$, which generalizes the construction of forcing extensions by iterated forcing with finite supports, familiar from logic. Second, the above mentioned fact that everything can be made countable by passing to a suitable base extension enables us to reduce the problems to the case of an inverse sequence $\ldots \to \mathcal{E}_2 \to \mathcal{E}_1 \to \mathcal{E}_0$ of toposes, rather than an arbitrary inverse system, a reduction which technically is of great value.

I will end the brief description of this collection of papers here. I hope that I have been able to give the non-specialist some idea of the unity and the central ideas underlying these papers, and I apologize to the specialist who will no doubt think that I have made matters only more obscure.

Annals of Pure and Applied Logic 27 (1984) 63–107
North-Holland

SHEAF MODELS FOR CHOICE SEQUENCES*

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Communicated by D. van Dalen
Received 29 April 1982; revised 1 December 1983

Contents

Introduction

Sheaf models and toposes have by now become an important means for studying intuitionistic systems. They provide a unifying generalization of earlier semantic notions, such as Kripke models, topological (Beth) models, and realizability interpretations. Moreover, higher order languages with arbitrary function- and power-types can be interpreted naturally in these models.

In this paper we investigate sheaf models for intuitionistic theories of choice sequences. We will be mainly concerned here with sheaf models for the theories LS and CS in the language of elementary analysis with variables for numbers and sequences. Both systems are theories for (parts of) intuitionistic Baire space. The part of CS not involving lawlike function variables coincides with the system FIM of [12], which was intended as a codification of intuitionistic mathematical practice.

* The results of this paper were first presented at the Peripatetic Seminar on Sheaves and Logic at the University of Sussex, November 1981.

The axioms of CS and LS are based on an analysis of how certain kinds of choice sequences are presented: thus, the conceptional viewpoint behind these systems is the 'analytic' one (as opposed to the 'holistic' viewpoint).

From the holistic viewpoint, the universe of choice sequences is grasped as a whole, and quantification over this domain is intuitively clear. From the analytic viewpoint, one sees choice sequences as individual objects, each given by a possibly non-predetermined construction process. Subdomains of choice sequences can be distinguished, according to the sort of information about a sequence that may become available at the various stages of its construction process. (For a discussion of holistic vs. analytic see [19].)

Extreme examples of subdomains of intuitionistic Baire space are the lawlike and the lawless sequences. Lawlike sequences are given by a set of rules which tell us how to construct a value for each given argument. These rules are the 'available data' on the sequence, they do not change during the construction process. The construction process of a lawless sequence, on the other hand, is comparable to the casting of an infinite-sided die, with the stipulation that an initial segment of the sequence may be deliberately fixed in advance. The available data on a lawless sequence consist at each stage of its construction of an initial segment of the sequence only.

LS is the formal theory of lawless sequences. The advantage of lawless sequences is that the relative simplicity of the available data makes it possible to justify rigorously (though informally) the validity of the traditional intuitionistic continuity axioms for this subdomain. The drawback of lawless sequences lies in the fact that the subdomain is not closed under any non-trivial continuous operation. LS is therefore not suited as a formal basis for intuitionistic analysis. The formal system CS is adequate for this purpose, it combines strong continuity axioms with closure under continuous operations. In general, on the analytic approach "one starts with (a conceptual analysis of) the idea of an individual choice sequence of a certain type (say τ) and attempts to derive from the way such a choice sequence is supposed to be *given* to us (i.e. from the type of data available at any given moment of its generation) the principles which should hold for the choice sequences of type τ" ([20, *p.* 5]).

The CS-axioms arise from the *presupposition* that there exists a notion of individual choice sequence for which the available data consist of lawlike continuous operations. The problem is to justify this presupposition, that is, to find a subdomain of intuitionistic Baire space for which the available data of its individual elements are the continuous operations (or any other subdomain of Baire space of which the CS-axioms can be seen to hold, cf. [10], [7]).

A common and important feature of LS and CS is, that their axioms give a full explanation of quantification over a subdomain of choice sequences in terms of quantification over lawlike objects. This is formally reflected in the elimination theorems for both systems.

Lawless sequences (of zero's and one's) first appear (as absolutely free sequences) in [13]. In [14] lawless sequences of natural numbers are treated, with a sketch of the elimination theorem. An extensive treatment of LS can be found in [18].

The elimination translation provides a model for LS: it is a syntactic interpretation of LS in its lawlike part, which is a subsystem of classical analysis. [17] gives an 'internal' model for LS: it is shown that there exists a universe of sequences $\mathcal{U}_\alpha$, constructed from a single lawless α, of which we can prove in LS that it is a model of LS. In [1] an LS-model is presented based on forcing techniques and Beth models. In the appendix to [1] it is shown that the 'internal model' construction of [17] is in fact equivalent to a Beth model construction.

CS was introduced and discussed extensively in [15]. A concise treatment can also be found in [18]. The elimination translation for CS (in [15]) gives a syntactic interpretation of this theory. [7] and [10] give models for relativized variants of CS. More specifically, universes $\mathcal{U}_\alpha$, constructed from a single lawless α, are presented for which one can prove in LS that they are models for variants of CS. Such projection models correspond to Beth models in the ordinary sense. The motivation behind the 'reductionist program' of constructing such internal models for complex notions of choice sequence inside LS is discussed in [10].

The emphasis in this paper lies with the system CS. In fact, our original aims were

(a) to see whether it was possible to obtain monoid models for the system CS (and possibly also LS),

(b) to deny or confirm the first impression that there might be a connection between monoid forcing and the elimination translation, and

(c) to try and simplify the construction of models for variants of CS as presented in [7].

We briefly outline the contents of the paper: in Section 1 we give the basic concepts relevant for the interpretation of intuitionistic theories in sheaves over a site $(\mathbf{M}, \mathcal{J})$, $\mathbf{M}$ a monoid, and $\mathcal{J}$ a Grothendieck topology on $\mathbf{M}$. In particular, we define 'Grothendieck topology $\mathcal{J}$ on a monoid $\mathbf{M}$', 'sheaf over $(\mathbf{M}, \mathcal{J})$', and we give the inductive clauses of Beth–Kripke–Joyal forcing over $(\mathbf{M}, \mathcal{J})$. The material in this section is standard, and proofs are not given in detail. Readers familiar with such interpretations can skip Section 1. It is intended for those less at home in toposes. We assume all readers to be familiar with interpretations in sheaves over complete Heyting algebras (or over topological spaces). Such models occur in Sections 4 and 5. A good introduction to such models is [4].

One of the main results of this paper is that sheaves over monoids give us a new and very simple model for the theory CS. This will be proved in Section 2, where we also show how to obtain similar models for variants of CS (Section 2.3).

There are essentially two ways to explain the naturalness and simplicity of these models. On the one hand it can be shown that forcing over the monoid of Section 2.2 coincides (at lower types) with the elimination translation of [15] (cf. 3.2), while the elimination translation is in fact the canonical interpretation prescribed by the axioms (cf. 3.1). On the other hand, the closure properties of the universe of choice sequences described by the CS-axioms (whatever that universe may be), can be captured in a geometric theory. The generic model in the classifying topos (in the sense of [16]) for this theory again coincides with the monoid model of Section 2.2. This correspondence will be worked out in [9]. (The relation between monoid models, elimination translations and classifying toposes described here for CS, also holds for the relativizations of CS discussed in 2.3.)

It should be remarked here that the techniques exploited in Section 2 can also be applied to theories which are analogous to CS or one of its relativized versions, but with Baire space replaced by the space of Dedekind reals. One then obtains models in which the Dedekind reals appear as the sheaf of continuous functions $\mathbb{R} \to \mathbb{R}$. As in Section 2.2, a model satisfying the axiom of real-analytic data may then be constructed; and as in Section 2.3, one can construct a model in which there is a dense subset D of $\mathbb{R}$ satisfying real open data

$$\forall d \in D(Ad \to \exists d_1, d_2\, (d_1 < d < d_2 \wedge \forall e \in D\, (d_1 < e < d_2 \to Ae))),$$

(this axiom has been considered in [8]).

We will not work out these models separately. One reason for this is that – as far as things go through – the proofs are completely analogous to those given in Section 2.2. Another, more fundamental, reason is that some of the results obtained in 2.2, like 'all functions from $\mathbb{R}$ to $\mathbb{R}$ are continuous' do *not* go through. One should not consider monoids of real functions, but sites of open subspaces of finite products $\mathbb{R}^n$, in order to be able to obtain a full parallel with Sections 2 and 3 of this paper.

Returning to the subject matter of Section 3, we stress that the elimination theorem (and hence also, monoid models) give us a *formal* interpretation of choice sequences ("quantification over choice sequences as a figure of speech" [18]). In this respect, the models presented in Section 2 are completely different from the ones in [7], which grew out of a conceptual analysis of a primitive notion of choice sequence. As explained in [19], from the conceptual point of view sheaf models for choice sequences over (subspaces of) Baire space are of a particular interest. Therefore we will prove in the fourth section that with each of the monoid models one may associate a spatial model which is first-order equivalent to it. For countable monoids, the space will be a subspace of Baire space.

In Section 5 models for lawless sequences are discussed. We will first explain that although approximations of LS can be modelled in sheaves over monoids, LS itself cannot. We will then give an LS-model in sheaves over a topological space instead. This model is inspired by the model discussed in the appendix of [1]. The proofs we give, however, are semantical (in contrast to Troelstra's original proofs), and our treatment works for a language with arbitrary function and powertypes (these are not contained in the original LS-language). We conclude this final section with a discussion of the elimination translation for LS, and its connection to projection models, and to our own model for LS.

How to read this paper. We repeat that readers who are familiar with forcing over sites can skip Section 1. As will be apparent from this introduction, we have made some efforts to explain the connections with the existing intuitionistic literature on choice sequences. This will be done in the expository Sections 3 and 5.3. Readers who are mainly interested in seeing classical models for intuitionistic theories of choice sequences are advised to read Sections 2, 4, 5.1 and 5.2 only.

1. Monoid models

In this section we present some basic definitions and facts of sheaf semantics, for the particular case of sheaves over a monoid. The material is standard, and proofs are omitted or only briefly outlined.

1.1. *Sheaves over monoids*

A *monoid* $\mathbf{M}$ is a category with just one object, or equivalently, a triple $\mathbf{M} = (M, \circ, 1)$, where M is a set with an associative binary operation $\circ$ which has a two-sided unit 1. If X is a set, an *action* of a monoid $\mathbf{M}$ on X is an operation

$$\upharpoonleft : M \times X \to X$$

such that for any $x \in X$ and $f, g \in M$,

(i) $x \upharpoonleft 1 = x$,
(ii) $(x \upharpoonleft f) \upharpoonleft g = x \upharpoonleft (f \circ g)$.

Such pairs $(X, \upharpoonleft)$ are called $\mathbf{M}$-*sets*; the element $x \upharpoonleft f$ of X is called the *restriction* of x to (or along) f. A *morphism* of $\mathbf{M}$-sets $(X, \upharpoonleft) \to (Y, \upharpoonleft)$ is a function $\alpha : X \to Y$ which preserves the action; i.e., $\alpha(x \upharpoonleft f) = \alpha(x) \upharpoonleft f$ for any $x \in X$, $f \in M$. A *sub*-$\mathbf{M}$-*set* of $(X, \upharpoonleft)$ is a subset $Y \subseteq X$ which is closed under $\upharpoonleft$; equivalently, a subset $Y \subseteq X$ with action $\upharpoonleft_Y$ such that the inclusion $Y \to X$ is a morphism of $\mathbf{M}$-sets.

We give some examples of $\mathbf{M}$-sets that will be used later. The set $\mathbb{N}$ of natural numbers can be made into an $\mathbf{M}$-set by giving it the trivial action: $n \upharpoonleft f = n$ for $n \in \mathbb{N}$, $f \in M$. All elements of $\mathbb{N}$ are 'constant' for this action. Another $\mathbf{M}$-set, which usually has hardly any constant elements, is the set of *sieves* (or *cribles*, or *right-ideals*) on $\mathbf{M}$: a *sieve* on $\mathbf{M}$ is a subset $S \subseteq M$ such that if $f \in S$ and $g \in M$ then also $f \circ g \in S$. The set of sieves is made into an $\mathbf{M}$-set by setting

$$S \upharpoonleft f = \{g \in M \mid f \circ g \in S\}.$$

Finally, note that M itself may be regarded as an $\mathbf{M}$-set, with action $f \upharpoonleft g = f \circ g$.

A (*Grothendieck*-) *topology* on $\mathbf{M}$ is a family $\mathcal{J}$ of sieves on $\mathbf{M}$ with the following properties:

(i) $M \in \mathcal{J}$,
(ii) if $S \in \mathcal{J}$ and $f \in M$, then $S \upharpoonleft f \in \mathcal{J}$,
(iii) if $R \subseteq M$, and there exists an $S \in J$ such that $\forall f \in S\ (R \upharpoonleft f \in \mathcal{J})$ then $R \in \mathcal{J}$.

The elements of $\mathscr{J}$ are called ($\mathscr{J}$-)*covers*, or ($\mathscr{J}$-)*covering sieves*. (When S is a subset of M (but not necessarily a sieve), we will often say that S is a cover while we actually mean that the sieve $\{s \circ f \mid s \in S,\ f \in M\}$ generated by S is a cover.) It can be shown from (ii) and (iii) that if S and S' are covering sieves, so is $S \cap S'$.

An **M**-set $(X, \upharpoonleft)$ is called ($\mathscr{J}$-)*separated* if for each $S \in \mathscr{J}$, $\forall f \in S\ (x \upharpoonleft f = y \upharpoonleft f)$ implies $x = y$, for all $x, y \in X$. We now define sheaves: a collection $(x_f \mid f \in S)$ of an **M**-set $(X, \upharpoonleft)$ indexed by a sieve $S \in \mathscr{J}$ is called *compatible* if for each $g \in M$, $x_f \upharpoonleft g = x_{f \circ g}$. Now an **M**-set $(X, \upharpoonleft)$ is a ($\mathscr{J}$-)*sheaf* if for each compatible collection $(x_f \mid f \in S)$ there exists a *unique* x (called the *join* of $(x_f \mid f \in S)$) with $x \upharpoonleft f = x_f$ for each $f \in S$. By the uniqueness of joins, sheaves are separated.

Conversely, with a separated **M**-set $(X, \upharpoonleft)$ we can associate a sheaf $L(X, \upharpoonleft)$ (the *sheafification* of $(X, \upharpoonleft)$ as follows: the elements of $L(X, \upharpoonleft)$ are equivalence-classes of compatible families $(x_f \mid f \in S)$ indexed by a cover S, where we identify two such families $(x_f \mid f \in S)$ and $(y_g \mid g \in T)$ if there exists a cover $R \subset S \cap T$ such that $x_f = y_f$ for each $f \in R$. The action of **M** on $L(X, \upharpoonleft)$ is defined by

$$(x_f \mid f \in S) \upharpoonleft g = (x_{g \circ h} \mid h \in S \upharpoonleft g).$$

$\upharpoonleft$ is well-defined on equivalence-classes, and $L(X, \upharpoonleft)$ is a sheaf. L is functorial, in the sense that a morphism $(X, \upharpoonleft) \xrightarrow{\alpha} (Y, \upharpoonleft)$ can be uniquely extended to a morphism $L\alpha : L(X, \upharpoonleft) \to L(Y, \upharpoonleft)$. (In fact, all this can be done also for **M**-sets which are not necessarily separated. For details, see [16].)

For a monoid **M** with a topology $\mathscr{J}$ on it, the collection of sheaves and morphisms between them form a category $\mathrm{Sh}(\mathbf{M}, \mathscr{J})$. This categroy is a topos, which means that it is possible to interpret higher-order intuitionistic logic in this category. Before we turn to this interpretation, let us indicate how to construct products, exponents, and powersets in $\mathrm{Sh}(\mathbf{M}, \mathscr{J})$.

The product of two **M**-sets $\mathbf{X} = (X, \upharpoonleft)$ and $\mathbf{Y} = (Y, \upharpoonleft)$ is simply the cartesian product $X \times Y$ with pointwise action, $(x, y) \upharpoonleft f = (x \upharpoonleft f,\ y \upharpoonleft f)$. It is easy to see that $\mathbf{X} \times \mathbf{Y}$ is a sheaf if $\mathbf{X}$ and $\mathbf{Y}$ are.

The exponent (function-space) $\mathbf{Y}^{\mathbf{X}}$ (or sometimes $(\mathbf{X} \to \mathbf{Y})$) is defined to be the set of morphisms

$$\alpha : \mathbf{M} \times \mathbf{X} \to \mathbf{Y}$$

(where **M** is regarded as an **M**-set), with action by $(\alpha \upharpoonleft f)(g, x) = \alpha(f \circ g, x)$.

This makes the *evaluation* $-(-) : \mathbf{Y}^{\mathbf{X}} \times \mathbf{X} \to \mathbf{Y}$, $\alpha(x) := \alpha(1, x)$ into a morphism of **M**-sets. One can check that $\mathbf{Y}^{\mathbf{X}}$ is a sheaf whenever $\mathbf{Y}$ is. There is a natural 1–1 correspondence between morphisms $\mathbf{Z} \to \mathbf{Y}^{\mathbf{X}}$ and morphisms $\mathbf{Z} \times \mathbf{X} \to \mathbf{Y}$ induced by the evaluation.

The **M**-set of *truthvalues* ('the *subobjectclassifier*') $\boldsymbol{\Omega}$ is the **M**-set of $\mathscr{J}$-closed sieves on $\mathbf{M}$: A sieve R on **M** is $\mathscr{J}$*-closed* if for any $f \in M$, $\exists S \in \mathscr{J}\ \forall s \in S\ (f \circ s \in R)$ implies $f \in R$. $\boldsymbol{\Omega}$ is a sub-**M**-set of the **M**-set of sieves on **M**, and $\boldsymbol{\Omega}$ is a sheaf. There is a natural 1–1 correspondence between morphisms $\mathbf{X} \xrightarrow{\alpha} \boldsymbol{\Omega}$ and subsheaves (sub-**M**-sets which are sheaves) $\mathbf{U} \subseteq \mathbf{X}$: given α, the corresponding $\mathbf{U}$ is defined by $x \in \mathbf{U} \leftrightarrow 1 \in \alpha(x)$. Conversely, given $\mathbf{U} \subseteq \mathbf{X}$, α is defined by $\alpha(x) = \{f \in M \mid x \upharpoonleft f \in U\}$. Powerobjects $\mathscr{P}(\mathbf{X})$ are now constructed as exponents $\boldsymbol{\Omega}^{\mathbf{X}}$.

1.2. *Forcing*

A language for higher-order logic consists of two parts, the set of *sorts* and the set of *constants*. The set of sorts can be built up inductively: the *basic sort* is the sort N of natural numbers; and if $s_1, \ldots, s_n$ and t are sorts, then so are $\mathcal{P}(s_1 \times \cdots \times s_n)$ (the sort of n-place relations taking arguments of sorts $s_1, \ldots, s_n$ respectively), and $t^{(s_1 \times \cdots \times s_n)}$ (the sort of n-place functions taking arguments of sorts $s_1, \ldots, s_n$ to a value of sort t). The other part is a set of constants $\{c_i \mid i \in I\}$, together with an assignment of a sort $\#(c)$ to each constant c. We also take the language to contain infinitely many variables of each sort.

A (*standard-*)*interpretation* $\mathcal{I}$ of such a language in a topos of sheaves on a monoid $\mathrm{Sh}(\mathbf{M}, \mathcal{J})$ assigns to each sort s a sheaf $\mathcal{I}(s)$, according to the following rules:

(i) $\mathcal{I}(N)$ is the sheafification of the constant $\mathbf{M}$-set $\mathbb{N}$ (we will usually write N for this sheaf).

(ii) $\mathcal{I}(\mathcal{P}(s_1 \times \cdots \times s_n)) = \mathcal{P}(\mathcal{I}(s_1) \times \cdots \times \mathcal{I}(s_n))$, and
$\mathcal{I}(t^{(s_1 \times \cdots \times s_n)}) = \mathcal{I}(t)^{\mathcal{I}(s_1) \times \cdots \times \mathcal{I}(s_n)}$.

Further, $\mathcal{I}$ assigns an element $\mathcal{I}(c)$ of $\mathcal{I}(\#c)$ to each constant c, which is a fixed point of the action on $\mathcal{I}(\#c)$ (this is the same as a morphism from the one-point $\mathbf{M}$-set $\mathbb{1}$ to $\mathcal{I}(\#c)$). By the correspondences given at the end of 1.1, one may also think of the interpretation as assigning a subsheaf of $\mathcal{I}(s_1) \times \cdots \times \mathcal{I}(s_n)$ to a constant of sort $\mathcal{P}(s_1 \times \cdots \times s_n)$, and a morphism $\mathcal{I}(s_1) \times \cdots \times \mathcal{I}(s_n) \to \mathcal{I}(t)$ to a constant of sort $t^{(s_1 \times \cdots \times s_n)}$. The empty product is $\mathbf{1}$, so the interpretation $\mathcal{I}(\mathcal{P}(\ ))$ is the $\mathbf{M}$-set of truthvalues $\boldsymbol{\Omega}$.

Terms of the language are built up as usual. Terms of sort $\mathcal{P}(\ )$ are called formulas. If $\tau(x_1, \ldots, x_n)$ is a term of sort t with free variables among x_i of sort s_i $(i = 1, \ldots, n)$, its interpretation (relative to $x_1, \ldots, x_n$) will be a morphism $\mathcal{I}(s_1) \times \cdots \times \mathcal{I}(s_n) \to \mathcal{I}(t)$, for which we write $[\![\tau]\!]_{x_1 \cdots x_n}$ (or, just $[\![\tau]\!]$). It is defined inductively. First consider terms built up from variables and non-logical constants: we let $[\![x_i]\!]_{x_1 \cdots x_n}$ be the projection $\mathcal{I}(s_1) \times \cdots \times \mathcal{I}(s_n) \to \mathcal{I}(s_i)$; and if $[\![\sigma]\!]$ and $[\![\tau_i]\!]$ have been defined for $i = 1, \ldots, n$, and σ and $\tau_1, \ldots, \tau_n$ are of the appropriate sorts, then we let $[\![\sigma(\tau_1, \ldots, \tau_n)]\!] = [\![\sigma]\!](\langle [\![\tau_1]\!], \ldots, [\![\tau_n]\!] \rangle)$. For formulas we also have the possibility of making new formulas by use of logical constants. If $A(x_1, \ldots, x_n)$ is a formula with x_i free, and $\mathcal{I}(\#x_i) = \mathbf{Y}_i$, $[\![A]\!]$ will be a morphism $\mathbf{Y}_1 \times \cdots \times \mathbf{Y}_n \to \boldsymbol{\Omega}$. Alternatively, $[\![A]\!]$ is interpreted as a subsheaf of $\mathbf{Y}_1 \times \cdots \times \mathbf{Y}_n$, and the corespondence is given by

$$\mathbf{y} = (y_1, \ldots, y_n) \in [\![A]\!] \quad \text{iff} \quad 1 \in [\![A]\!](y_1, \ldots, y_n).$$

We will write $\Vdash A(y_1, \ldots, y_n)$ for $1 \in [\![A]\!](y_1, \ldots, y_n)$. The definition of the interpretation can then be completed as follows: $\Vdash \sigma_1(y_1, \ldots, y_n) = \sigma_2(y_1, \ldots, y_n)$ iff $[\![\sigma_1]\!](y_1, \ldots, y_n) = [\![\sigma_2]\!](y_1, \ldots, y_n)$,

$$\Vdash R(\tau_1(y_1, \ldots, y_n), \ldots, \tau_k(y_1, \ldots, y_n)) \quad \text{iff} \quad (y_1, \ldots, y_n) \in [\![R(\tau_1, \ldots, \tau_k)]\!],$$

$\Vdash A \wedge B(\mathbf{y})$ iff $\Vdash A(\mathbf{y})$ and $\Vdash B(\mathbf{y})$,

$\Vdash A \vee B(\mathbf{y})$ iff there exists an $S \in \mathcal{J}$ such that for each $f \in S$ either $\Vdash A(\mathbf{y} \upharpoonright f)$ or $\Vdash B(\mathbf{y} \upharpoonright f)$,

$\Vdash \neg A(\mathbf{y})$	iff	for each $f \in M$, $\nVdash A(\mathbf{y} \restriction f)$,
$\Vdash A \to B(\mathbf{y})$	iff	for each $f \in M$ such that $\Vdash A(\mathbf{y} \restriction f)$, also $\Vdash B(\mathbf{y} \restriction f)$,
$\Vdash \forall x\, A(x)\,(\mathbf{y})$	iff	for each $a \in \mathcal{I}(\#x)$ and each $f \in M \Vdash A(a, \mathbf{y} \restriction f)$,
$\Vdash \exists x\, A(x)\,(\mathbf{y})$	iff	there exists an $S \in \mathcal{J}$ such that for each $f \in S$ we can find an $a_f \in \mathcal{I}(\#x)$ with $\Vdash A(a_f, \mathbf{y} \restriction f)$.

Finally, we list some properties of the interpretation; the easy proofs are left to the reader.

1.2.1. Lemma (i) $\Vdash A(y_1, \ldots, y_n)$ *implies* $\Vdash A(y_1 \restriction f, \ldots, y_n \restriction f)$, *for each* $f \in M$.
(ii) *If* $S \in \mathcal{J}$, *and for each* $f \in S$, $\Vdash A(y_1 \restriction f, \ldots, y_n \restriction f)$, *then also* $\Vdash A(y_1, \ldots, y_n)$
(iii) *For closed* A, *either* $\Vdash A$ *or* $\Vdash \neg A$.

$\Vdash A(y_1, \ldots, y_n)$ is defined as $(y_1, \ldots, y_n) \in [\![A]\!] \subseteq \mathbf{Y}_1 \times \cdots \times \mathbf{Y}_n$, so (i) says that $[\![A]\!]$ is a sub-**M**-set, (ii) says that it is in fact a subsheaf, while (iii) says that the one-point **M**-set **1** has only two subsheaves.

If **X** is a sheaf, a subset Y of X is said to *generate* **X** if every element of X is locally the restriction of an element of Y; that is, for each $x \in X$ we can find a cover $S \in \mathcal{J}$ such that

$$\forall f \in S\, \exists g \in M\, \exists y \in Y\, x \restriction f = y \restriction g.$$

Note that if the generating set Y is closed under restrictions, we may as a consequence of the preceding lemma restrict ourselves to Y when verifying whether a formula of the form $\forall x : XA$ or $\exists x : XA$ is forced. More precisely,

$\Vdash \forall x : XA(x)\,(\mathbf{p})$	iff	for all $y \in Y$ and all $f \in M$, $\Vdash A(x)\,(y, \mathbf{p} \restriction f)$,
$\Vdash \exists x : XA(x)\,(\mathbf{p})$	iff	there is a cover $S \in \mathcal{J}$ such that for each $f \in S$ we can find a $y_f \in Y$ with $\Vdash A(x)(y_f, \mathbf{p} \restriction f)$.

1.2.2. Lemma. *For any standard-interpretation,*

(i) $$\Vdash \forall x : s\, \exists !\, y : t\, A(x, y) \leftrightarrow \exists !\, f : t^s\, \forall x : s\, A(x, f(x)),$$
$$\Vdash \exists !\, y : \mathcal{P}(x)\, \forall x : s\, (A(x) \leftrightarrow y(x)).$$

(ii) *Adding constants* 0 *and* S *with their obvious interpretations, we obtain a model of higher-order Heyting's Arithmetic* (HAH) *with full induction:*

$$\Vdash \forall X : \mathcal{P}N\, (X(0) \wedge \forall n : N(X(n) \to X(Sn)) \to X = N).$$

2. Modelling CS and its relativizations

In this section we will describe monoid models for the system CS (Section 2.2), and for the relativizations of CS which are considered in [7] (Section 2.3). We shall reason classically about the models. Later on (in Section 3) we consider refinements using an intuitionistic metatheory. But first, we introduce some notation and state the CS-axioms.

2.1. *The theory* CS

CS was introduced and extensively discussed in [15]. The motivation behind its introduction was to give an adequate formal system for the foundation of intuitionistic analysis from the analytic viewpoint. The domain of choice sequences described by CS will be called B_C. We will write B_L for the domain of *lawlike sequences*. Before stating the axioms, we introduce some notation.

We use $k, n, m, \ldots$ as variables for natural numbers, $\varepsilon, \eta, \zeta, \ldots$ as variables for elements of B_C, $u, v, w, \ldots$ as variables for finite sequences of natural numbers, and $a, b, c, \ldots$ as variables for lawlike mappings from N to N, or from $N^{<N}$ to N. $x, y, z, \ldots$ are variables ranging over the whole of Baire space B, $\leq$ is used for the natural ordering between finite sequences, $*$ denotes concatenation. If $x \in B$ and $u \in N^{<N}$, then "$x \in u$" stands for "x has initial segment u", and we often write u for the basic open $\{x \mid x \in u\}$ of Baire space. If $x \in B$ and $n \in N$, then $\bar{x}(n)$ denotes the initial segment $\langle x(0), \ldots, x(n-1)\rangle$ of x of length n.

Besides B_C and B_L there is a third set playing an important role in the theory CS, namely the set K of lawlike inductive neighbourhood-functions (mappings from $N^{<N}$ to N). An element a of K has the following properties:

$$\forall x \in B\, \exists n \in N \quad a(\bar{x}(n)) > 0,$$

$$\forall u, v \in N^{<N}\, (u \leq v \,\&\, a(u) > 0 \rightarrow a(u) = a(v)).$$

Such a function a codes a continuous $\underset{\sim}{a} : N^N \rightarrow N^N$ by

$$\underset{\sim}{a}(x)(n) = m \quad \text{iff} \quad \exists k\, a(\langle n\rangle * \bar{x}(k)) = m + 1.$$

We put

$$\underset{\sim}{K} = \{\underset{\sim}{a} \mid a \in K\}.$$

Thus $\underset{\sim}{K} \subseteq \text{cts}(N^N, N^N)$, the set of continuous functions from Baire space to Baire space. In fact, classically, $\underset{\sim}{K}$ is the set of all continuous functions; intuitionistically, 'continuous' is in this context usually defined as 'being an element of $\underset{\sim}{K}$'.

The system CS consists of the following axioms and schemata.

1. (closure and pairing)

$$\forall f \in \underset{\sim}{K}\, \forall \varepsilon\, \exists \eta\, \eta = f(\varepsilon),$$

$$\forall \varepsilon, \eta\, \exists f, g \in \underset{\sim}{K}\, \exists \zeta\, (\varepsilon = f(\zeta) \wedge \eta = g(\zeta)).$$

2. (analytic data)

$$\forall \varepsilon\, (A(\varepsilon) \rightarrow \exists f \in \underset{\sim}{K}\, (\exists \eta\, (\varepsilon = f(\eta)) \wedge \forall \eta\, A(f(\eta)))).$$

3. (continuity for lawlike objects) For p ranging over N, B_L, or K:

$$\forall \varepsilon\, \exists p\, A(\varepsilon, p) \rightarrow \exists a \in K\, \forall u\, (au \neq 0 \rightarrow \exists p\, \forall \varepsilon \in u A(\varepsilon, p)).$$

4. ($\forall\varepsilon\,\exists\eta$ continuity)

$$\forall\varepsilon\,\exists\eta\, A(\varepsilon,\eta) \to \exists f \in \underline{K}\, \forall\varepsilon\, A(\varepsilon, f(\varepsilon)).$$

And finally a schema of lawlike countable choice

5. (AC–NF)

$$\forall n\, \exists a \in B_{\mathrm{L}}\, A(n, a) \to \exists \text{ lawlike } N \xrightarrow{F} B_{\mathrm{L}}\, \forall n\, A(n, Fn).$$

In the schemata, *there are no free variables except possibly lawlike ones.*

Observe that

(a) combination of CS3 and AC–NF yields a principle of continuous choice analogous to CS4;

(b) if $j: N\times N \to N$ is bijective, and h is the induced homeomorphism from $N^N \times N^N$ to N^N, $h(x, y)(n) = j(x(n), y(n))$, then $\pi_1 \circ h^{-1}$, $\pi_2 \circ h^{-1} \in \underline{K}$, and for all $f, g \in \underline{K}$, $h \circ \langle f, g\rangle \in \underline{K}$; hence $B_C \times B_C \cong B_C$ via h, by CS1.

2.2. *The model for* CS

Consider the monoid cts(B, B) of endomorphisms of Baire space, equipped with the open cover topology $\mathcal{J}$: for a sieve S we set $S \in \mathcal{J}$ iff there is an open cover $\{U_i : i \in I\}$ of B together with homeomorphisms $B \to U_i$ such that each of the composites $u_i : B \xrightarrow{\sim} U_i \hookrightarrow B$ is in S.

In connection with Section 3, we note the following. Let $\mathbb{K}$ be the set of external neighbourhoodfunctions, i.e. the set of functions $f: \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ which satisfy $\forall x \in B$ $\exists n \in \mathbb{N}$ $f(\bar{x}(n)) > 0$ and $\forall u, v\, (u \le v \wedge f(u) > 0 \to f(v) = f(u))$. Then each cover $S \in \mathcal{J}$ has a 'characteristic function' in $\mathbb{K}$, i.e. with each $S \in \mathcal{J}$ there is an $f_S \in \mathbb{K}$ such that for all $u \in \mathbb{N}^{<\mathbb{N}}$, there is a homeomorphism $B \xrightarrow{\sim} \{x \in B \mid x \in u\}$ in S whenever $f_S(u) \neq 0$. Conversely, with each $f \in \mathbb{K}$ we may associate a cover $S_f \in \mathcal{J}$, namely

$$S_f = \{g \mid \exists u\, (f(u) \neq 0 \wedge \mathrm{im}(g) \subseteq u)\}.$$

Our model will be the standard interpretation in Sh(cts(B, B), $\mathcal{J}$). We start by identifying the sheaf of natural numbers N and internal Baire space N^N in this model.

2.2.1. Lemma. (a) *N is isomorphic to the sheaf* cts$(B, \mathbb{N})$ *of continuous functions $B \to \mathbb{N}$, with the monoid action given by composition, $a \upharpoonright f = a \circ f$.*

(b) *N^N is isomorphic to* cts(B, B), *the monoid itself, with composition as the monoid action, $f \upharpoonright g = f \circ g$.*

Proof. (a) According to the definition of the standard interpretation given in Section 1.2, elements of N are equivalence classes of collections $(n_f \mid f \in S)$, $n_f \in \mathbb{N}$, $S \in \mathcal{J}$, which are compatible (i.e., $n_f = n_{f\circ g}$ for all $f \in S$ and $g \in$ cts(B, B)). If S is a cover, S contains continuous functions $u_i : B \xrightarrow{\sim} U_i \hookrightarrow B$ for some open cover $\{U_i\}$

of B, and with a compatible $(n_f \mid f \in S)$ we may associate a function $a : B \to \mathbb{N}$ by

$$a(x) = n_{u_i} \quad \text{iff} \quad x \in U_i.$$

Then a is well-defined (by compatibility), and continuous. Conversely, each continuous function $a : B \to \mathbb{N}$ determines an open cover $\{a^{-1}(n) \mid n \in \mathbb{N}\}$ of B, and hence a cover $S_a = \{f \mid \text{im}(f) \subseteq \text{some } a^{-1}(n)\} \in \mathscr{J}$, together with a compatible collection $(n_f \mid f \in S_a)$, where $n_f = m$ iff $\text{im}(f) \subseteq a^{-1}(m)$. These two constructions are each others inverses (up to equivalence) and they both preserve the monoid-action.

(b) The exponent N^N is the set of morphisms $\tau : \text{cts}(B, B) \times \text{cts}(B, \mathbb{N}) \to \text{cts}(B, \mathbb{N})$ with monoid-action given by $(\tau \upharpoonright f)(g, a) = \tau\,(f \circ g, a)$ (see 1.1).

With such a τ we associate the continuous function $f_\tau : B \to B$ defined by

$$f_\tau(x)(n) = \tau(1, \bar{n})(x),$$

where $\bar{n} : B \to \mathbb{N}$ is the constant function with value n. Conversely, with $f \in \text{cts}(B, B)$ we associate the morphism τ_f defined by

$$\tau_f(g, a)(x) = f(g(x))(a(x)).$$

As in part (a), these two constructions are inverse to each other, and they preserve the monoid-action. □

2.2.2. Remark. If $f \in N^N$, $a \in N$, then functional application in the model is given by $f(a) = \lambda x\,.\,f(x)(a(x))$. Thus $\Vdash f(a) = b$ iff for all $x \in B$, $f(x)\,(a(x)) = b(x)$.

$N^{<N}$, the sheaf of internal finite sequences of natural numbers, can be identified as $\text{cts}(B, \mathbb{N}^{<\mathbb{N}})$, in a way analogous to 2.2.1(a). If $f \in N^N$ and $a \in N$, then $\bar{f}(a)$, the initial segment of f with length a, is $\lambda x\,.\,\overline{f(x)}\,(a(x)) \in \text{cts}(B, \mathbb{N}^{<\mathbb{N}})$, and if $u \in N^{<N}$, then $\Vdash f \in u$ (i.e. f has initial segment u) iff for all x, $f(x) \in u(x)$. □

Next we turn to the interpretation of lawlike objects. Intuitively one may think of the application of the monoid-action to an element of a sheaf as a step in a construction process. For example, one may regard an element $f \in N^N$ as 'a choice sequence at some stage of its construction'. The information we have at that stage is, that the sequence lies in $\text{im}(f)$. After restricting f to g, we have the information that the sequence lies in $\text{im}(f \circ g)$.

Lawlike elements are elements whose construction is completed. Therefore we put

2.2.3. Definition. *Let* $\mathbf{X}$ *be any sheaf.* $\mathbf{X}_L$ *is the smallest subsheaf of* $\mathbf{X}$ *which contains the set* $\{x \in X \mid x \textit{ is invariant under the monoid-action}\}$, *i.e.* $x \in X_L$ *iff there is a cover* S *such that*

$$\forall f \in S\ \forall g \in \text{cts}(B, B)\ x \upharpoonright f = x \upharpoonright f \circ g.$$

We call the elements of X_L the *lawlike elements* of X.

Observe that $(N)_L = N$, $(N^{<N})_L = N^{<N}$; natural numbers and finite sequences of natural numbers are all lawlike. An element f of $N^N \cong \mathrm{cts}(B, B)$ is invariant under restrictions iff f is a constant function. Thus B_L is interpreted as the sheaf of locally constant functions from B to B.

If x is an element of a sheaf X, and $f: B \to B$ is constant, then $x \restriction f$ is lawlike. In other words, each element has a lawlike restriction, so

$$\Vdash \forall x \in X \quad \neg\neg(x \text{ is lawlike}).$$

An immediate consequence of this observation is the following *Specialization Property*:

(SP) $\Vdash \exists x \in X\, A(x) \to \exists x \in X_L\, A(x)$

for formulae A containing only lawlike parameters besides x (cf. 1.2.1). For $X = N^N$, this property was formulated in [15].

We now consider internal neighbourhoodfunctions. The exponent $N^{(N^{<N})}$ is the set of morphisms $f: \mathrm{cts}(B, B) \times \mathrm{cts}(B, \mathbb{N}^{<\mathbb{N}}) \to \mathrm{cts}(B, \mathbb{N})$ with restrictions defined as $(f \restriction g)(h, b) = f(g \circ h, b)$. We put

$$K_0 \text{ is the sheaf } \{f \in N^{(N^{<N})} \mid \Vdash \forall g \in N^N\, \exists a \in N\, f(\bar{g}(a)) > 0$$
$$\wedge \forall u, v \in N^{<N}\, (u \leqslant v \wedge f(u) > 0 \to f(v) = f(u))\},$$

and we interpret K as the sheaf $(K_0)_L$ of lawlike elements of K_0. Below we will show that the model satisfies Bar Induction, of which induction over K (and over K_0) is a well-known corollary.

Observe that an element of K which is invariant under restrictions is in fact a morphism $\mathrm{cts}(B, \mathbb{N}^{<\mathbb{N}}) \to \mathrm{cts}(B, \mathbb{N})$.

One easily proves the following. If $f \in \mathbb{K}$ (the external set of neighbourhood-functions) then $\tilde{f}: \mathrm{cts}(B, \mathbb{N}^{<\mathbb{N}}) \to \mathrm{cts}(B, \mathbb{N})$ defined by $\tilde{f}(b)(x) = f(b(x))$ is an element of K, and conversely, if $f \in K$ is invariant under restrictions, then $f = \tilde{g}$ for some $g \in \mathbb{K}$. Hence K is the sheaf of morphisms $\mathrm{cts}(B, B) \times \mathrm{cts}(B, \mathbb{N}^{<\mathbb{N}}) \to \mathrm{cts}(B, \mathbb{N})$ which are locally of the form $\tilde{f}$ for some $f \in \mathbb{K}$.

Let us look at internal functions on Baire space. The exponent $N^N \to N^N$ is the set of morphisms $F: \mathrm{cts}(B, B) \times \mathrm{cts}(B, B) \to \mathrm{cts}(B, B)$, with restrictions defined by $(F \restriction f)(g, h) = F(f \circ g, h)$. An $F \in N^N \to N^N$ preserves the monoid-action: $F(f \circ h, g \circ h) = F(f, g) \circ h$. Let $h: B \to B \times B$ be a homeomorphism, and write $\alpha = F(\pi_1 h, \pi_2 h)$. Then $F(f, g) = \alpha \circ h^{-1} \circ \langle f, g\rangle$ for any $f, g \in \mathrm{cts}(B, B)$, since $f = \pi_1 \circ h \circ h^{-1} \circ \langle f, g\rangle$ and $g = \pi_2 \circ h \circ h^{-1} \circ \langle f, g\rangle$. So F is completely determined by $F(\pi_1, h, \pi_2 h)$.

An $F \in N^N \to N^N$ which is invariant under restrictions is in fact a morphism from $\mathrm{cts}(B, B)$ to $\mathrm{cts}(B, B)$. Such an F is of the form $F(f) = \alpha \circ f$, where $\alpha = F(1)$. So lawlike elements of $N^N \to N^N$ are locally of the form $f \mapsto \alpha \circ f$ for some $\alpha \in \mathrm{cts}(B, B)$.

Each $\alpha \in \mathrm{cts}(B, B)$ has (externally) a neighbourhoodfunction $f_\alpha \in \mathbb{K}$, i.e. a function such that

$$\forall x\,(\alpha(x)(n)=m \;\Leftrightarrow\; \exists k\, f_\alpha(\langle n\rangle * \bar{x}(k))=m+1).$$

With $f_\alpha \in \mathbb{K}$ we may associate an internal neighbourhood function $\tilde{f}_\alpha \in K$ as above. One easily verifies that for $F_\alpha : g \mapsto \alpha \circ g$ in $(N^N \to N^N)_{\mathrm{L}}$,

$$\Vdash \text{“}\tilde{f}_\alpha \text{ is a neighbourhoodfunction for } F_\alpha\text{”}.$$

Hence for all $F \in (N^N \to N^N)_{\mathrm{L}}$,

$$\Vdash \text{“}F \text{ is continuous”}.$$

We will prove below that $\Vdash \forall F : N^N \to N^N$ (F is continuous).

$\underset{\sim}{K}$ is the sheaf of all lawlike mappings from N^N to N^N which have a neighbourhoodfunction in K. It will be clear from the foregoing that $\underset{\sim}{K} = (N^N \to N^N)_{\mathrm{L}}$.

The last step in the definition of the CS-model is the interpretation of the universe of choice sequences B_{C}. We interpret B_{C} as N^N, internal Baire space.

Under this interpretation, the axiom of closure (the first part of CS1) is obviously true. The verification of the axiom of pairing (the other half of CS1) is straightforward. We state this explicitly in the following lemma.

2.2.4. Lemma. *The standard interpretation in* Sh(cts$(B, B), \mathcal{J}$), *with the interpretation of* B_{L}, B_{C}, *and* K *as described above, gives a model of* CS1, *i.e.*

$$\Vdash \forall f \in \underset{\sim}{K}\; \forall \varepsilon \in B_C\; (f(\varepsilon) \in B_C),$$

$$\Vdash \forall \varepsilon, \eta \in B_C\; \exists f, g \in \underset{\sim}{K}\; \exists \zeta \in B_C\; (\varepsilon = f(\zeta) \wedge \eta = g(\zeta)). \quad \square$$

The following observation will help to simplify the proofs in the sequel.

2.2.5. Lemma. *Let* X *be a sheaf, and let* $A(p_1, \ldots, p_n, x)$ *be a formula (possibly containing parameters* $p_1, \ldots, p_n$*). Then if* $\Vdash \exists x \in X\, A(p_1, \ldots, p_n, x)$, *there exists a* $q \in X$ *such that* $\Vdash A(p_1, \ldots, p_n, q)$.

Proof. If $\Vdash \exists x \in X\, A(x)$, then there is an $f \in \mathbb{K}$ such that for all $u \in \mathbb{N}^{>\mathbb{N}}$,

$$f(u) \neq 0 \Rightarrow \exists x_u \in X \Vdash (A \upharpoonright \bar{u})(x_u),$$

where $\bar{u} : B \xrightarrow{\sim} \{y \mid y \in u\} \hookrightarrow B$, and $A \upharpoonright \bar{u}$ stands for the formula A with all parameters restricted to $\bar{u}$. Let $\{u_i\}_i$ be the set of minimal finite sequences such that $f(u_i) \neq 0$, and let S be the cover $\{g \in \mathrm{cts}(B, B) \mid \exists i\,(\mathrm{im}(g) \subseteq u_i)\}$. For each $g \in S$ there is a (unique) i and a (unique) $h \in \mathrm{cts}(B, B)$, such that $g = \bar{u}_i \circ h$. Let $x_g = x_{u_i} \upharpoonright h$. Then the collection $(x_g \mid g \in S)$ is compatible, so there is a unique $q \in X$ with $\forall g \in S : x_g = q \upharpoonright g$. For this q we have $\Vdash (A \upharpoonright g)(q \upharpoonright g)$ for each $g \in S$; hence also $\Vdash A(q)$ (cf. 1.2.1). $\square$

Each of the next three Theorems 2.2.6, 7, 9 consists of two parts, one stating the validity of a lawlike *schema* in the model, the other the validity of a related *axiom*. We will briefly consider the connection between these two parts below, cf. Remark 2.2.13. From now on, in this section "$\Vdash$" always refers to forcing in the interpretation in Sh(cts(B, B), $\mathcal{J}$) described above.

2.2.6. Theorem. (i) *The schema of lawlike countable choice* AC–N *holds* (*X any sort; besides x, A contains lawlike parameters only*):

$$\Vdash \forall n \in N\, \exists x \in X\, A(n, x) \rightarrow \exists F \in (X^N)_L\, \forall n\, A(n, Fn).$$

(ii) *The axiom of countable choice* AC–N* *holds* (*X any sort*):

$$\Vdash \forall P \in \mathcal{P}(N \times X)\, (\forall n\, \exists x\, P(n, x) \rightarrow \exists F \in X^N\, \forall n\, P(n, Fn)).$$

Proof. (i) Suppose $\Vdash \forall n \in N\; \exists x \in X\; A(n, x)$. By 2.2.5 and the specialization property, for each $n \in \mathbb{N}$ we can find an $x_n \in X_L$ with $\Vdash A(\underline{n}, x_n)$, and by SP we may even assume that x_n is invariant under restrictions. Let F: cts$(B, B) \times$ cts$(B, \mathbb{N}) \rightarrow X$ be the unique morphism determined by

$$F(1, \bar{n}) = x_n, \quad \text{for each } n \in \mathbb{N}.$$

Then F is lawlike, i.e. $\Vdash F \in (X^N)_L$, and $\Vdash \forall n\, A(n, Fn)$.

(ii) Choose P: cts$(B, B) \times$ cts$(B, \mathbb{N}) \times X \rightarrow \Omega$ with $\Vdash \forall n\, \exists x\, P(n, x)$. For each n we may find (by Lemma 2.2.5) an $x_n \in X$ such that $\Vdash P(\underline{n}, x_n)$, i.e. $P(1, \bar{n}, x_n) = T$. As in (i), let F: cts$(B, B) \times$ cts$(B, \mathbb{N}) \rightarrow X$ be the morphism determined by $F(1, \bar{n}) = x_n$. Then $\Vdash \forall n\, P(n, Fn)$, for if $n \in \mathbb{N}$ and $f \in$ cts(B, B), then

$$P(f, \bar{n}, (F \restriction f)(\bar{n})) = P(f, \bar{n}, x_n \restriction f) = P(1, \bar{n}, x_n) \restriction f = T \restriction f = T. \quad \square$$

2.2.7. Theorem. (i) *The schema of lawlike continuity for natural numbers* C–N *holds:* (*A has all non-lawlike parameters shown*)

$$\Vdash \forall \varepsilon \in N^N\, \exists n \in N\, A(\varepsilon, n) \rightarrow \exists F \in K\, \forall u \in N^{<N}$$
$$(Fu > 0 \rightarrow \exists n\, \forall \varepsilon \in u\, A(\varepsilon, n)).$$

(ii) *The axiom of continuity for natural numbers* C–N* *holds:*

$$\Vdash \forall P \in \mathcal{P}(N^N \times N)\, (\forall \varepsilon\, \exists n\, P(\varepsilon, n) \rightarrow \exists F \in K_0\, \forall u \in N^{<N}$$
$$(Fu > 0 \rightarrow \exists n\, \forall \varepsilon \in u\, P(\varepsilon, n))).$$

Proof. (i) Suppose $\Vdash \forall \varepsilon \in N^N\, \exists n \in N\; A(\varepsilon, n)$. Then in particular, choosing ε the identity mapping, we find a continuous $a: B \rightarrow \mathbb{N}$ such that $\Vdash A(1, a)$. Externally, a has a neighbourhoodfunction $g \in \mathbb{K}$ determined by

$$g(u) = m + 1 \quad \text{iff} \quad \forall x \in u\, a(x) = m.$$

Internalizing this neighbourhood function gives us the $F \in K$ with the required properties. More precisely, let

$$F: \text{cts}(B, \mathbb{N}^{<\mathbb{N}}) \rightarrow \text{cts}(B, \mathbb{N}), \quad F(\bar{u}) = g \circ \bar{u}.$$

Choose any $u \in \mathbb{N}^{<\mathbb{N}}$ such that $\Vdash F(\bar{u}) > 0$. Then a is constant on $\{x \in B \mid x \in u\}$, say with value n, and it follows easily from $\Vdash A(1, a)$ that $\Vdash \forall \varepsilon \in u\; A(1, \underline{n})$.

(ii) Choose $P:\text{cts}(B,B)\times\text{cts}(B,\mathbb{N})\times\text{cts}(B,B)\to\Omega$ such that $\Vdash\forall\varepsilon\,\exists n\,P(n,\varepsilon)$. Fix any homeomorphism $h:B\to B\times B$, and find a continuous $a:B\to\mathbb{N}$ such that $P(\pi_1 h, a, \pi_2 h)=T$. Let $\{u_i:B\to U_i\hookrightarrow B\}_i$ be a disjoint cover such that $a\circ u_i$ is constant, say with value n_i. We now have to find an $F:\text{cts}(B,B)\times\text{cts}(B,\mathbb{N}^{<\mathbb{N}})\to\text{cts}(B,\mathbb{N})$ such that

(1) $\Vdash F\in K_0$,

(2) $\Vdash\forall v\,(Fv\neq 0\to\exists n\,\forall\varepsilon\in v\,P(n,\varepsilon))$.

Define, for $f\in\text{cts}(B,B)$ and $v\in\mathbb{N}^{<\mathbb{N}}$,

$$F(f,\bar v)=\begin{cases}1, & \text{if for some } i,\ \overline{f(x)}(l\text{th}(v))\times v\subseteq h(U_i),\\ 0, & \text{otherwise.}\end{cases}$$

Note that $F(f,\bar v)$ is continuous, and that $F(fg,\bar v)=F(f,\bar v)\circ g$, so F determines a well-defined morphism $\text{cts}(B,B)\times\text{cts}(B,\mathbb{N}^{<\mathbb{N}})\to\text{cts}(B,\mathbb{N})$.

We show that now (1) and (2) hold:

For (1), the only thing that is perhaps not immediately clear is that $\Vdash\forall\varepsilon\,\exists u\,(\varepsilon\in u\wedge Fu\neq 0)$. To show this, choose f and g in $\text{cts}(B,B)$. Then $\forall x\in B\,\exists i\,\langle f(x),g(x)\rangle\in h(u_i)$, so

$$\forall x\,\exists i\,\exists u_x\ni f(x)\,\exists v_x\ni g(x)\;u_x\times v_x\subseteq h(U_i),$$

and we may assume $\text{lth}(u_x)=\text{lth}(v_x)$. Now choose for every x a neighbourhood w_x of x such that $\forall y\in w_x\,f(y)\in u_x$; then

$$\forall x\,\exists i\,\forall y\in w_x\,\overline{f(y)}(\text{lth}(v_x))\times v_x\subseteq h(U_i).$$

Thus, we have found a cover $\{w_j\}_j$ and finite sequences v_j such that for each x, $g(w_j(x))\in v_j$, and $\overline{f(w_j(x))}(\text{lth}(v_j))\times v_j\subseteq h(U_i)$, in other words

$$\Vdash\exists v\in\mathbb{N}^{<\mathbb{N}}\,(g\in v\wedge(F\restriction f)(v)\neq 0).$$

Hence (1) holds.

For (2), choose v and f such that $\Vdash F(f,\bar v)\neq 0$, i.e.

$$\forall x\,\exists i\,\overline{f(x)}(\text{lth}(v))\times v\subseteq h(U_i).$$

Now fix a cover $\{w_j:B\to W_j\hookrightarrow B\}_j$ such that for each j there is one particular U_i such that $\forall x\in W_j\,\overline{f(x)}(\text{lth}(v))\times V\subseteq h(U_i)$. It suffices to show that for each w_j,

$$\Vdash\exists n\,\forall\varepsilon\in\bar v\restriction w_j\,((P\restriction f)\restriction w_j)(n,\varepsilon).$$

To this end, let $n=n_i$, and choose g and k in $\text{cts}(B,B)$ such that $\Vdash k\in\bar v\restriction w_j\restriction g$, i.e. $\forall x\in B\;k(x)\in v$. Then $\forall x\in B\;\langle f\circ w_j\circ g(x),k(x)\rangle\in h(U_i)$, so we can find a continuous $\psi:B\to B$ such that $\langle fw_jg,k\rangle=H\circ u_i\circ\psi$.

But then

$$\begin{aligned}P(fw_jg,\bar n_i,k)&=P(\pi_1 hu_i\psi,au_i\psi,\pi_2 hu_i\psi)\\&=P(\pi_1 h,a,\pi_2 h)\restriction u_i\psi\\&=T\restriction u_i\psi=T,\end{aligned}$$

so $\Vdash(P\restriction fw_jg)\,(\bar n_i,k)$, which completes the proof of (2). □

Note that in the proofs of C–N and C–N* we did not use special properties of N, except that natural numbers are lawlike. Therefore,

2.2.8. Corollary. *The model satisfies the schema of lawlike continuity for lawlike objects, and the axiom of continuity for lawlike objects. In particular, the schema* CS3 *holds in the model.*

2.2.9. Theorem. (i) *The scheme of lawlike continuity for sequences* C–C *holds in the model (where* $A(\varepsilon, \eta)$ *is a formula with all non-lawlike parameters shown):*

$$\Vdash \forall \varepsilon \in N^N\, \exists \eta \in N^N\, A(\varepsilon, \eta) \rightarrow \exists F \in \underset{\sim}{K}\, \forall \varepsilon \in N^N\, A(\varepsilon, F\varepsilon).$$

(ii) *The axiom of continuity for sequences* C–C* *holds:*

$$\Vdash \forall P \in \mathcal{P}(N^N \times N^N)\, (\forall \varepsilon\, \exists \eta\, P(\varepsilon, \eta) \rightarrow \exists F : N^N \xrightarrow{\text{cts}} N^N\, \forall \varepsilon\, P(\varepsilon, F\varepsilon)).$$

Proof. (i) Suppose $\Vdash \forall \varepsilon\, \exists \eta\, A(\varepsilon, \eta)$. If we choose $\varepsilon = 1$, we find by Lemma 2.2.5 an $f \in \text{cts}(B, B)$ such that $\Vdash A(1, f)$. Hence also $\Vdash A(h, f \circ h)$ for all $h \in \text{cts}(B, B)$ (by Lemma 1.2.1, since all other parameters in A are lawlike). Thus letting F be the morphism "compose with f": $\text{cts}(B, B) \times \text{cts}(B, B) \rightarrow \text{cts}(B, B)$, $F(g, h) = f \circ h$, proves (i) (cf. the discussion of the internal set $\underset{\sim}{K}$ at the beginning of this subsection).

(ii) Choose a morphism $P : \text{cts}(B, B) \times \text{cts}(B, B) \times \text{cts}(B, B) \rightarrow \mathbf{\Omega}$; suppose that $\Vdash \forall \varepsilon\, \exists \eta\, P(\varepsilon, \eta)$. In particular, we find that $\Vdash \exists \eta\, (P \upharpoonright \pi_1 h)(\pi_2 h, \eta)$, for a homeomorphism $h : B \rightarrow B \times B$. By 2.2.5, there exists an $f \in \text{cts}(B, B)$ such that $\Vdash (P \upharpoonright \pi_1 h)(\pi_2 h, f)$, i.e. $P(\pi_1 h, \pi_2 h, f) \doteq T$. Define a morphism

$$F : \text{cts}(B, B) \times \text{cts}(B, B) \rightarrow \text{cts}(B, B)$$

by

$$F(g_1, g_2) = f \circ h^{-1} \circ \langle g_1, g_2 \rangle.$$

then for all $g_1, g_2 \in \text{cts}(B, B)$,

$$\begin{aligned} P(g_1, g_2, (F \upharpoonright g_1)(\dot{1}, g_2)) &= P(\pi_1 h h^{-1}\langle g_1, g_2\rangle, \pi_2 h h^{-1}\langle g_1, g_2\rangle, f h^{-1}\langle g_1, g_2\rangle) \\ &= (P(\pi_1 h, f, \pi_2 h) \upharpoonright h^{-1})\langle g_1, g_2\rangle = T, \end{aligned}$$

by choice of f. So

$$\Vdash \forall \varepsilon\, P(\varepsilon, F\varepsilon).$$

Finally, F is continuous, since all internal functions $N^N \rightarrow N^N$ are continuous, by Theorem 2.2.15 below. □

In the next theorem, we do not state the schema separately, since it follows from the axiom.

2.2.10. Theorem (Full Bar Induction BI*).

$$\Vdash \forall P \in \mathcal{P}(N^{<N})\,(\forall \varepsilon\, \exists u \in P\,(\varepsilon \in u) \wedge \forall u\,(\forall n\, u * \langle n\rangle \in P \leftrightarrow u \in P) \rightarrow \langle\ \rangle \in P).$$

In the proof of this theorem, we will externally use the principle of 'double Bar Induction', which says that if U is a subset of $\mathbb{N}^{<\mathbb{N}} \times \mathbb{N}^{<\mathbb{N}}$ barring each pair of sequences of natural numbers, and monotone and inductive in both arguments (separately), then $(\langle\ \rangle, \langle\ \rangle) \in \mathrm{U}$. This principle follows (constructively) from ordinary Bar Induction.

Proof of 2.2.10. Choose $P: \mathrm{cts}(B, B) \times \mathrm{cts}(B, \mathbb{N}^{<\mathbb{N}}) \rightarrow \boldsymbol{\Omega}$, such that

$$\Vdash \forall \varepsilon\, \exists u \in P\,(\varepsilon \in u) \wedge \forall u (\forall n\, u * \langle n\rangle \in P \leftrightarrow u \in P).$$

Now let

$$\hat{P} = \{(u, v) \in \mathbb{N}^{<\mathbb{N}} \times \mathbb{N}^{<\mathbb{N}} \mid P(\tilde{u}, \bar{v}) = T\}$$

(here T is the top-element of $\boldsymbol{\Omega}$, $\tilde{u}$ is the function $x \mapsto u * x \in \mathrm{cts}(B, B)$, and $\bar{v}$ is the constant function $B \rightarrow \mathbb{N}^{<\mathbb{N}}$ with value v).

Clearly, $\hat{P}$ is montone and inductive in each of its arguments. For each $x \in B$, let ε_x be the constant function $B \rightarrow B$ with value x. Then $\Vdash \exists n\, \bar{\varepsilon}_x(n) \in P$, hence for some continuous $a: B \rightarrow \mathbb{N}$, $\Vdash \bar{\varepsilon}_x(a) \in P$, i.e. $P(1, \lambda y \bar{x}(ay)) = T$. If y is any element of B, choose an initial segment u of y such that a is constant on $\{z \in B \mid z \in u\}$, say with value n. Then $P(\tilde{u}, \bar{x}(n)) = T$, i.e. $(u, \bar{x}(n)) \in \hat{P}$. This shows that $\hat{P}$ bars pairs of sequences, so by double Bar Induction, $(\langle\ \rangle, \langle\ \rangle) \in \hat{P}$, i.e. $\Vdash \langle\ \rangle \in P$. □

2.2.11. Theorem (i) (Analytic Data). *Let $A(\varepsilon)$ be a formula with all parameters lawlike, except for ε. Then*

$$\Vdash \forall \varepsilon \in N^N\,(A(\varepsilon) \rightarrow \exists F \in \underset{\sim}{K}\,(\exists \eta\, \varepsilon = F(\eta) \wedge \forall \zeta\, A(F(\zeta)))).$$

(ii) (Generalized Analytic Data). *Let $X_1, \ldots, X_n$ be arbitrary sorts, and let $A(x_1, \ldots, x_n)$ be a formula with all parameters lawlike, except for the variables x_i of sort X_i $(i = 1, \ldots, n)$. Then*

$$\Vdash \forall x_1 \cdots \forall x_n\,(A(x_1, \ldots, x_n) \rightarrow$$
$$\exists F \in ((X_1 \times \cdots \times X_n)^{N^N})_{\mathrm{L}}(\exists \eta\,(x_1, \ldots, x_n) = F\eta \wedge \forall \eta\, A(F(\eta)))).$$

Proof. Since $\underset{\sim}{K} = (N^N \rightarrow N^N)_{\mathrm{L}}$, (i) is a special case of (ii). To prove (ii), we may assume that $n = 1$, by taking the product $X = X_1 \times \cdots \times X_n$ of sheaves. So suppose $\Vdash A(x)$, with $x \in X$. Let $F: \mathrm{cts}(B, B) \times \mathrm{cts}(B, B) \rightarrow X$ be the morphism defined by $F(f, g) = x \upharpoonright g$. Then F is lawlike, $\Vdash F(1) = x$, and $\Vdash \forall \eta\, A(F\eta)$. □

Reviewing the properties of the model that have now been proved, we see that the CS-axioms are satisfied: CS1 was proved in 2.2.4, CS2 is 2.2.11(i), CS3 is 2.2.8, CS4 is 2.2.9(i), and AC–NF is a special case of 2.2.6(i). Thus,

2.2.12. Corollary. *The standard interpretation in* Sh(cts(B, B), $\mathcal{J}$) *(with* B_L, B_C *and* K *interpreted as described in the beginning of this section) gives a model of the theory* CS.

2.2.13. Remark. We promised above to say a word about the relation between the lawlike schemata and the axioms. We have given separate proofs of the starred axioms AC–N*, C–N*, C–C* and BI* here in order to make our discussion of the properties of the model self-contained. Readers familiar with [15] will be aware of the fact that variants of the schemata AC–N, C–N, C–C, and BI with an additional parameter of type N^N follow logically from the schemata without a parameter via analytic data. We shall indicate briefly how the starred axioms follow from the lawlike schemata via generalized analytic data. For example, consider the relation between C–N and C–N*: to prove C–N*

$$\Vdash \forall P \in \mathcal{P}(N^N \times N)\, (\forall \varepsilon\, \exists n\, P(\varepsilon, n)$$
$$\to \exists f \in K_0\, \forall u \in N^{<N}\, (fu \neq 0 \to \exists n\, \forall \varepsilon \in u\, P(\varepsilon, n))),$$

it suffices, by generalized analytic data, to show that

$$\Vdash \forall \text{ lawlike } F : N^N \to \mathcal{P}(N^N \times N)\, \forall \eta\, (\forall \varepsilon\, \exists n\, F(\eta)(\varepsilon, \nu)$$
$$\to \exists f \in K_0\, \forall u \in N^{<N}\, (fu \neq 0 \to \exists n\, \forall \varepsilon \in u\, F(\eta)(\varepsilon, n)));$$

in other words, it suffices to prove the schema

$$\Vdash \forall \varepsilon\, \exists n\, A(\eta, \varepsilon, n)$$
$$\to \exists f \in K_0\, \forall u \in N^{<N}\, (fu \neq 0 \to \exists n\, \forall \varepsilon \in u\, A(\eta, \varepsilon, n))$$

for formulas A with no other non-lawlike parameters than η and ε. In a similar way, AC–N* (C–C*, BI*) is reduced to AC–N (C–C, BI) with an additional parameter. The derivation of schemata with an additional parameter is treated in [15], Section 5.7. The proofs there use analytic data in the form

$$\Vdash \forall \eta\, (A\eta \to B\eta) \leftrightarrow \forall f \in \underline{K}\, (\forall \eta\, A(f(\eta)) \to \forall \eta\, B(f(\eta))). \quad \Box$$

A consequence of Theorem 2.2.7(ii) is that the model satisfies strong continuity principles for functions between metric spaces.

2.2.14. Theorem. *Let* $\langle X, \rho \rangle$ *be an internal metric space, which is separable, i.e.* $\Vdash \exists d \in X^N$ *(*$\{d_n \mid n \in \mathbb{N}\}$ *is dense in* X*). Then*

$$\Vdash \text{“all functions } N^N \to X \text{ are continuous”}.$$

Proof. Given a morphism $F \in X^{N^N}$, consider the predicates

$$P_k = \{(n, \varepsilon) \mid \rho(d_n, \mathrm{F}(\varepsilon)) < 2^{-k}\} \in \mathcal{P}(N \times N^N),$$

and apply C–N*. $\Box$

2.2.15. Theorem. $\Vdash$ "*if X is a complete separable metric space, and Y is a separable metric space, then all functions $X \to Y$ are continuous*".

Proof. In case X is Baire space, this is immediate from 2.2.14. The general case follows from the fact that AC–N* (logically) implies that every complete separable metric space X is a quotient of Baire space (i.e., $\Vdash$ "there exists a $g : N^N \to X$ such that for all $V \subset X$, V is open in X iff $g^{-1}(V)$ is open in N^N"), see e.g. [20]. □

Thus, for example, all functions from Baire space to itself, and all real functions are continuous in the model. It is perhaps illustrative to see what real numbers look like in the model: note that by countable choice (2.2.6) all reals are Cauchy. Since N appears as $\text{cts}(B, \mathbb{N})$, the sheaf Q of internal rationals is the sheaf of locally constant functions $B \to \mathbb{Q}$, for which we write $\text{loco}(B, \mathbb{Q})$. Sequences of rationals are morphisms $\alpha : \text{cts}(B, B) \times \text{cts}(B, \mathbb{N}) \to \text{loco}(B, \mathbb{Q})$, and we can show that such α are determined by their values $\alpha(1, \bar{n})$, so $Q^N \cong \text{loco}(B, \mathbb{Q})^{\mathbb{N}} \cong \text{loco}(B \times \mathbb{N}, \mathbb{Q})$. The sheaf of Cauchy sequences $C \subseteq Q^N$ is the subsheaf given by

$$\alpha \in C \quad \text{iff} \quad \Vdash \forall k\, \exists n\, \forall n' > n\ |\alpha n - \alpha n'| < 1/k,$$

while elements of C are identified according to

$$\Vdash \alpha \sim \beta \quad \text{iff} \quad \Vdash \forall k\, \exists n\, \forall n' > n\ |\alpha n' - \beta n'| < 1/k,$$

We write $\mathscr{R}$ for the sheaf of internal reals, which is the (internal) quotient $C/\sim$.

2.2.16. Proposition. *$\mathscr{R}$ is isomorphic to the sheaf of continuous real valued functions on Baire space.*

Proof. If $\alpha \in C$, then by AC–N, $\Vdash \exists f \in N^N\, \forall k\, \forall n' > fk\ |\alpha(fk) - \alpha n'| < 1/k$, hence (by 2.2.5) there exists a continuous $f : B \to \mathbb{N}^{\mathbb{N}}$ such that

$$\Vdash \forall k\, \forall n' > fk\ |\alpha(fk) - \alpha n'| < 1/k,$$

or equivalently,

$$(1) \qquad \forall x \in B\ \forall k \in \mathbb{N}\ \forall n' > f(x)(k)\ |\alpha(x, f(x)(k)) - \alpha(x, n')| < 1/k.$$

But (1) expresses that for each $x \in B$, $\{\alpha(x, n)\}_n$ is an (external) Cauchy-sequence, hence we have a function

$$F_\alpha : B \to \mathbb{R}, \quad x \mapsto \lim_n \alpha(x, n)$$

and it is straightforward to check that F_α is continuous, and that $\alpha \sim \beta$ implies that $F_\alpha = F_\beta$.

Conversely, for each continuous $g : B \to \mathbb{R}$ we can construct an internal Cauchy-sequence $\sigma_g \in \text{loco}(B \times \mathbb{N}, \mathbb{Q})$ such that

$$(2) \qquad \sigma_{F_\alpha} \sim \alpha, \quad \text{and} \quad F_{\sigma_g} = g,$$

as follows. Given g, fix for each n a cover $\mathcal{U}^n = \{U_k^n\}_k$ of B consisting of disjoint clopen subsets, with the property that

$$\forall x, g \in U_k^n \, |g(x) - g(y)| < 1/n, \quad \text{and} \quad \mathcal{U}^{n+1} \text{ refines } \mathcal{U}^n.$$

For each n and k we choose a rational $q(n, k)$ such that

$$\forall x \in U_k^n \, |g(x) - q(n, k)| < 2/n.$$

Now let

$$\sigma_g(x, n) = q(n, k(x)),$$

where $k(x)$ is the unique k with $x \in U_k^n$. Then $f: B \to \mathbb{N}^{\mathbb{N}}$ defined by $f(x)(n) = 4n$ is a modulus of convergence for σ_g, i.e.

$$\forall x \in B \, \forall k \, \forall n' > f(x)(k) \, |\sigma_g(x, f(x)(k)) - \sigma_g(x, n')| < 4/k,$$

so we have that $\Vdash$"σ_g is a Cauchy-sequence". Clearly, $\lim_{\to n} \sigma_g(x, n) = g(x)$, so the latter half of (2) holds. It is also straightforward to check that $\sigma_{F_\alpha} \sim \alpha$. To conclude the proof of the fact that σ and F are isomorphisms, it suffices to observe that they preserve the monoid-actions (the action on loco$(B \times \mathbb{N}, \mathbb{Q})$ is given by $\alpha \upharpoonright f = \alpha \circ (f \times 1)$), which is obvious. □

This concludes our discussion of the model. We will return to it from a different point of view in Section 3.

2.3. *Relativizations of* CS

In [7], relativizations of CS are studied, which are obtained by the following procedure: when $M \subseteq K$ is a monoid of neighbourhoodfunctions, (with a corresponding submonoid $\underline{M} = \{\underline{f} \mid f \in M\}$ of cts(B, B)), we can restrict the quantifiers over (lawlike) elements of $\underline{K}$ in the CS-axioms to $\underline{M}$. This leads to the following axioms:

CS(M) 1. a (closure of B_C) $\forall f \in \underline{M} \, \forall \varepsilon \, \exists \eta \, (f(\varepsilon) = \eta)$,
b (pairing) $\forall \varepsilon, \eta \, \exists f, g \in \underline{M} \, \exists \zeta \, (\varepsilon = f(\zeta) \wedge \eta = g(\zeta))$.
CS(M) 2. (analytic data) $\forall \varepsilon (A(\varepsilon) \to \exists f \in \underline{M} \, (\exists \eta \, \varepsilon = f(\eta) \wedge \forall \zeta \, A(f(\zeta))))$.
CS(M) 3. $=$ CS3
CS(M) 4. ($\forall \varepsilon \, \exists \eta$ continuity) $\forall \varepsilon \, \exists \eta \, A(\varepsilon, \eta) \to \forall \varepsilon \, \exists f \in \underline{M} \, A(\varepsilon, f(\varepsilon))$

and lawlike AC–NF as before.

For countable sets **M** these relativizations come up naturally if one tries to model CS in sheaves over Baire space (see Section 4 below).

CS(M)4 may seem rather unusual. Note first that it is non-trivial: elements of **M** are lawlike, and if η is a non-lawlike element of B_C, then there is no lawlike f such that $\forall \varepsilon \, f(\varepsilon) = \eta$. Secondly, an $f \in \underline{M}$ has a lawlike neighbourhoodfunction, so we can apply CS3 to $\forall \varepsilon \, \exists f \in \underline{M} \, A(\varepsilon, f(\varepsilon))$. This yields an open cover $\{u_i \mid i \in I\}$ of N^N by disjoint basic open sets, such that for all i there is an $f_i \in M$ satisfying

$\forall\varepsilon \in u_i\, A(\varepsilon, f_i(\varepsilon))$. Finally, through AC–NF we can piece the f_i's together and find an $f \in \underset{\sim}{K}$ such tat $\forall\varepsilon\, A(\varepsilon, f(\varepsilon))$. That is to say, CS($M$)4 + CS3 + AC–NF⊢CS4. A consequence of this is that CS(K) coincides with CS.

Note that the converse implication of CS(M)4 follows from CS(M)1a. If B_C is closed under application of elements of $\underset{\sim}{M}$ only and $\underset{\sim}{M}$ is a proper subset of $\underset{\sim}{K}$, then the converse of CS4 may fail.

In the sequel, j is some fixed bijection $N \times N \to N$ with inverses j_1 and $j_2 : N \to N$. This induces a homeomorphism $h : N^N \to N^N \times N^N$, $h(x)(m) = (j_1 x(m), j_2 x(m))$, with inverse h^{-1} such that $h^{-1}(x, y)(m) = j(x(m), y(m))$.

We call $\underset{\sim}{M}$ *pairing-closed* iff $\pi_1 \circ h$, $\pi_2 \circ h \in \underset{\sim}{M}$, and for all f and $g \in M$, $h^{-1} \circ \langle f, g\rangle \in \underset{\sim}{M}$. If $\underset{\sim}{M}$ is pairing-closed, then one can prove in CS(M) that $B_C \times B_C \cong B_C$ via h.

We shall briefly indicate how models for CS(M) can be obtained by the methods of Section 2.2.

Let $\mathbb{M}$ be a submonoid of cts(B, B), such that:

(1) For all finite sequences u, the function $\tilde{u} : x \mapsto u \mid x$ is in $\mathbb{M}$. ($u \mid x$ denotes the sequence obtained from x by replacing the initial segment $\bar{x}(\mathrm{lth}(u))$ of x by u.)

Let $\mathcal{J}$ be the collection of sieves $S \subseteq \mathbb{M}$ satisfying

(2) For all $x \in B$ there is a $\tilde{u} \in S$ such that $x \in \mathrm{im}(\tilde{u})$.

Then $\mathcal{J}$ is a Grothendieck topology on $\mathbb{M}$, and we interpret CS(M) in sheaves over ($\mathbb{M}$, $\mathcal{J}$).

Before we do so, however, a word on the condition (1) and the definition (2) seems in order. The open cover topology $\mathcal{J}$ on cts(B, B) is characterized by the fact that with each set $S \in \mathcal{J}$ there is a collection $\{U_i\}_i$ of opens of B which cover B, and such that for each i, S contains an embedding $B \xrightarrow{\sim} U_i \hookrightarrow B$. To preserve this characteristic property, we must restrict our attention to monoids which contain sufficiently many open embeddings. For reasons of simplicity we consider only monoids which satisfy (1), and we define the topology (2) accordingly.

In the model over ($\mathbb{M}$, $\mathcal{J}$), N, N^N, B_L, K and $\underset{\sim}{K}$ appear just as before (cf. Section 2.2, pp. 72–75). Let nbf($\mathbb{M}$) be the set of neighbourhoodfunctions for elements of $\mathbb{M}$. Then nbf($\mathbb{M}$) $\subseteq K$. M is interpreted as the set of locally constant maps $B \to$ nfb($\mathbb{M}$), and $\underset{\sim}{M}$ as the sheaf of morphisms cts(B, B) × cts(B, B) → cts(B, B) generated by the set of morphisms which are of the form $F(f, g) = h \circ g$, for some fixed $h \in \mathbb{M}$ (cf. the discussion of the interpretation of K and $\underset{\sim}{K}$ in Section 2.2). Note that $\underset{\sim}{M}$ has the same properties internally as $\mathbb{M}$ has externally; in particular, $\underset{\sim}{M}$ is closed under pairing iff $\mathbb{M}$ is, and $\underset{\sim}{M}$ contains j_1 and j_2 iff $\mathbb{M}$ does. Finally, B_C is interpreted as the smallest subsheaf of N^N (= cts(B, B)) which contains 1; in other words, B_C is interpreted as the set of functions in cts(B, B) which are locally in $\mathbb{M}$.

As in Section 2.2 we consider the specialization principle, continuity of lawlike functions $B_C \to N^N$, countable choice, $\forall\varepsilon\,\exists n$-continuity (C–N and C–N*), continuity of arbitrary functions $B_C \to N^N$, $\forall\varepsilon\,\exists\eta$-continuity (CS($M$)4) and bar induction (BI and BI*) in Sh($\mathbb{M}$, $\mathcal{J}$) under the interpretation described above.

If $\mathbb{M}$ contains all constant functions $B \to B$, i.e. if $B_L \subset B_C$, then the specialization property holds in Sh($\mathbb{M}$, $\mathcal{J}$) by the same argument as in 2.2.

All lawlike mappings $F: B_C \to N^N$ are locally of the form $F(f) = \alpha \circ f$, where $\alpha \in \text{cts}(B, B)$ is $F(1)$. So the elements of $(B_C \to N^N)_L$ are continuous and have a neighbourhoodfunction in K. Hence they can be extended naturally to continuous functions $N^N \to N^N$ in $\underline{K}$.

Sh($\mathbb{M}$, $\mathcal{J}$) is a model for the axiom of countable choice AC–N* (proof as in 2.2.6). If $\mathbb{M}$ contains all constant functions $B \to B$ then the schema AC–N holds in Sh($\mathbb{M}$, $\mathcal{J}$) (cf. 2.2.6). In any case, Sh($\mathbb{M}$, $\mathcal{J}$) is a model for the schema AC–N of lawlike countable choice to lawlike objects:

$$\Vdash \forall n\, \exists x \in X_L\, A(n, x) \to \exists \text{ lawlike } N \xrightarrow{F} X_L\, \forall n\, A(n, F(n)),$$

where X is an arbitrary sheaf.

The schema C–N is valid in Sh($\mathbb{M}$, $\mathcal{J}$) (cf. 2.2.7). If $\mathbb{M}$ is pairing-closed (i.e. if $B_C \times B_C \simeq B_C$) then the corresponding axiom C–N* holds as well, by the same argument as in 2.2.7.

If $\mathbb{M}$ is pairing-closed then *all* functions $F: B_C \to N^N$ are continuous. This follows immediately from C–N*.

In Sh($\mathbb{M}$, $\mathcal{J}$) the schema CS(M)4 holds:

$$\Vdash \forall \varepsilon\, \exists \eta\, A(\varepsilon, \eta) \to \forall \varepsilon\, \exists F \in \underline{M}\, A(\varepsilon, F(\varepsilon)).$$

The proof deviates slightly from the one for C–C in 2.2.9. Assume $\Vdash \forall \varepsilon\, \exists \eta\, A(\varepsilon, \eta)$, then in particular $\Vdash A(1, f)$ for some $f \in B_C$. This F is locally in $\mathbb{M}$, hence there is a cover $\{\bar{u}_i\}$ such that each $f \circ \bar{u}_i \in \mathbb{M}$, and of course $\Vdash A(\bar{u}_i, f \circ \bar{u}_i)$. Define $F_i \in \underline{M}$ by $F_i(g, h) = f \circ \bar{u}_i$, so $\Vdash A(\bar{u}_i, F_i(\bar{u}_i))$ for all i, and therefore $\Vdash \exists F \in \underline{M}\ A(1, F(1))$. Hence also $\Vdash \forall \varepsilon\, \exists F \in \underline{M}\ A(\varepsilon, F(\varepsilon))$.

If $\mathbb{M}$ is pairing-closed, then the axiom C-C* holds in the form

$$\Vdash \forall P \in \mathcal{P}(B_C \times B_C)\, (\forall \varepsilon\, \exists \eta\, P(\varepsilon, \eta) \to \exists F: B_C \to B_C\, \forall \varepsilon\, P(\varepsilon, F(\varepsilon))).$$

To see this, let $P \in \mathcal{P}(B_C \times B_C)$, i.e. $P: \mathbb{M} \times B_C \times B_C \to \boldsymbol{\Omega}$, and assume $\Vdash \forall \varepsilon\, \exists \eta\, P(\varepsilon, \eta)$. Then there is an $f \in B_C$ such that $P(\pi_1 \circ h, \pi_2 \circ h, f) = T$, where $h: B \xrightarrow{\simeq} B \times B$ is induced by j. Define the morphism $F: \mathbb{M} \times B_C \to B_C$ by $F(g_1, g_2) = f \circ h^{-1} \circ \langle g_1, g_2 \rangle$. One easily verifies that $\Vdash \forall \varepsilon\, P(\varepsilon, F(\varepsilon))$. Moreover, by the previous remark F is continuous.

Sh($\mathbb{M}$, $\mathcal{J}$) is a model for relativized analytic data and for generalized analytic data. Note that it suffices to prove CS(M)3 for the global elements of B_C, i.e. the elements of $\mathbb{M}$. Thus, assume that $\Vdash A(f)$ for some $f \in \mathbb{M}$, and define $F: \mathbb{M} \times B_C \to B_C$ by $F(g, h) = f \circ h$. Then trivially $\Vdash A(F(1))$, hence also $\Vdash \forall \varepsilon\, A(F(\varepsilon))$. Generalized analytic data is proved as in 2.2.11.

Finally, we consider BI and BI*. Sh($\mathbb{M}$, $\mathcal{J}$) is a model for the schema BI, independent of the properties of $\mathbb{M}$ (the proof is left to the reader). If $\mathbb{M}$ contains all constant functions $B \to B$, then BI* holds by the argument of 2.2.10. There is an alternative way of proving BI* however, which leads to the following result: if

$\mathbb{M}$ is pairing-closed, then BI* holds in Sh($\mathbb{M}$, $\mathcal{J}$). To see this, choose $P \in \mathcal{P}(N^{<N})$ such that

$$\Vdash \forall \varepsilon\, \exists u\, (\varepsilon \in u \wedge Pu) \wedge \forall u\, (\forall n\, P(u * \langle n \rangle) \leftrightarrow Pu).$$

Then in particular $\Vdash \exists u(\pi_1 \circ h \in u \wedge P(\pi_2 \circ h, u)$ (where $h : B \to B \times B$ is the homeomorphism induced by j), so we can find an $a \in \text{cts}(B, N^{<N})$ such that $\forall x \in B\ \pi_1 \circ h(x) \in a(x)$ and $P(\pi_2 \circ h, a) = T$. Let $\{\tilde{u}_i\}$ be a cover such that $a \circ \tilde{u}_i$ is constant for each i. For a finite sequence w, let us write w_1, w_2 for the finite sequences such that $h : w \xrightarrow{\sim} w_1 \times w_2$. We may without loss assume that $a \circ \tilde{u}_{i_1} = \bar{u}_{i_1}$, the constant function with value u_{i_1}. Obviously $\pi_2 \circ h \circ \tilde{u}_i = \tilde{u}_{i_2}$, so we have that $P(\tilde{u}_{i_2}, \bar{u}_{i_1}) = T$ for all i. Now consider the predicate $Qu = [P(\tilde{u}_2, \bar{u}_1) = T]$. Q now satisfies $\forall u\, (\forall n\, Q(u * \langle n \rangle) \leftrightarrow Qu)$ (note that $h : u * \langle n \rangle \xrightarrow{\sim} (u_1 * \langle j_1 n \rangle) \times (u_2 * \langle j_2 n \rangle))$. Hence we may apply BI to Q and find $P(\tilde{\langle\ \rangle}, \overline{\langle\ \rangle}) = T$, i.e. $\Vdash P(\langle\ \rangle)$.

As an immediate corollary to the observations just made, we obtain

2.3.1. Theorem. Sh($\mathbb{M}$, $\mathcal{J}$) *is a model for* CS(M). □

3. The connection with the elimination translation

We now want to investigate the connection between the interpretation of CS provided by the elimination translation of [15] and the monoid models. The interpretation of CS through the elimination translation is an interpretation in a constructive metatheory. Therefore we will first (Section 3.1) outline a constructive treatment of the monoid models presented earlier, before actually comparing the two interpretations (Section 3.2).

3.1. *Constructive metatheory*

We restrict ourselves to the interpretation of what we shall call the *minimal language*. This is a four-sorted language of predicate logic, with sorts N (natural numbers), B_L (lawlike sequences), K (lawlike inductive neighbourhoodfunctions) and B_C (choice sequences). It does not have a sort N^N. It is implicit in the rules of term-formation that both B_L and B_C are subsorts of N^N.

Note that there is a conceptual difference between the treatment of B_C and B_L as subsets of N^N and their treatment as separate sorts. Being of sort B_C or B_L is an intensional property of an object: it is given to us as an object of that sort. Being an element of the subset B_C or B_L is an extensional property of an object: from the way it is given to us we can prove that it satisfies the extensional $\in$-relation w.r.t. that subset.

The minimal language contains constants which make it possible to represent each primitive recursive $f : N^p \to N$ by a term $t[n_1, \ldots, n_p]$ in p numerical parameters. In particular there is a bijective $j : N^2 \to N$ in the language, with inverses $j_1, j_2 : N \to N$. Through j, j_1, j_2 elements of N can be viewed as codes for elements of N^p and $N^{<N}$. K is treated as a subsort of B_L, i.e. the domain $N^{<N}$ of the inductive neighbourhoodfunctions is coded in N.

We shall treat the minimal language rather loosely below. E.g. we use quantifiers $\forall f \in \underset{\sim}{K}$, $\exists f \in \underset{\sim}{K}$ and write equations $f(\varepsilon)(n) = m$ which are not in the minimal language. Note however that quantification over $\underset{\sim}{K}$ can be replaced by quantification over K and that atomic formulae $f(\varepsilon)(n) = m$ can be translated into their 'definition' $\exists k \ (a(\langle n \rangle * \bar{\varepsilon}(k)) = m + 1)$ where $a \in K$ is the neighbourhoodfunction of $f \in \underset{\sim}{K}$.

The treatment of CS in [15] is much more precise. The formal language used there to formulate the axioms in is an extension of the minimal language. The main difference is that it has constants $\text{app}_p : K \times B_C^p \to B_C$, where $\text{app}_p(a, \varepsilon_1, \ldots, \varepsilon_p)$ is written as $a \mid (\varepsilon_1, \ldots, \varepsilon_p)$. Among the CS-axioms in [15] is one specifying that $a \mid (\varepsilon_1, \ldots, \varepsilon_p) = f\nu_p(\varepsilon_1, \ldots, \varepsilon_p)$ where $f \in \underset{\sim}{K}$ has neighbourhoodfunction $a \in K$ and ν_p is a homeomorphism $(N^N)^p \to N^N$. Note that this axiom makes our CS1-axioms of closure and pairing redundant. In fact, closure and pairing are almost implicit in the presence of the constants app_p. The minimal language is entirely neutral in this respect. It can be used therefore to formulate all kinds of theories of choice sequences.

For our metatheory we use the theory IDB, or rather a definitional extension of this system. Strictly speaking, IDB is a two-sorted system, with variables $k, l, m, n, \ldots$ of sort $\mathbb{N}$, and variables $x, y, z, \ldots$ of sort B. The language has the same constants as the language of CS for the definition of primitive recursive functions from $\mathbb{N}^p$ to $\mathbb{N}$. In particular we have $j : \mathbb{N}^2 \to \mathbb{N}$ with inverses j_1 and $j_2 : \mathbb{N} \to \mathbb{N}$ in the language as above, so $\mathbb{N}^p$ and $\mathbb{N}^{<\mathbb{N}}$ can be treated codewise. We shall consider $\mathbb{N}^p$ and $\mathbb{N}^{<\mathbb{N}}$ as separate sorts here. Another constant of the language of IDB is the constant $\mathbb{K}$, for the set of neighbourhoodfunctions. Formally these are treated as maps from $\mathbb{N}$ to $\mathbb{N}$, but we refrain from this coding and continue to look upon external neighbourhoodfunctions as maps from $\mathbb{N}^{<\mathbb{N}}$ to $\mathbb{N}$. Working within IDB, continuous functions from B to B are the functions coded by elements of $\mathbb{K}$. We add a constant $\underset{\sim}{\mathbb{K}}$ to the language for these continuous functions. When working within IDB, we will often write $\text{cts}(B, B)$ for $\underset{\sim}{\mathbb{K}}$. ($\underset{\sim}{\mathbb{K}}$ is defined from $\mathbb{K}$ as $\underset{\sim}{K}$ is from K, see the beginning of 2.2.)

The axioms of IDB are the usual arithmetical axioms, the 'defining' axioms for its constants (in particular, the axiom of induction over $\mathbb{K}$), and the choice axiom AC–NF. Bar induction is *not* an axiom of IDB, nor does it have any of the typical intuitionistic continuity axioms for Baire space. Thus, IDB is just a subsystem of classical analysis.

We must adapt the interpretation of the language of CS in sheaves over $\underset{\sim}{\mathbb{K}} = \text{cts}(B, B)$ with the open cover topology to allow its treatment in IDB.

First we look at the definition of the open cover topology. As noted in Section 2, each open cover has a characteristic function in $\mathbb{K}$. In the constructive metatheory, we use this observation as the definition of open cover: a sieve

$S \subseteq \text{cts}(B, B)$ is a cover iff there is an $a \in \mathbb{K}$ such that for all $u \in \mathbb{N}^{<\mathbb{N}}$, $\lambda x \cdot u \mid x$ is in S whenever $a(u) \neq 0$. (Recall that $\lambda x \cdot u \mid x$ is the function "replace the initial segment of length $\text{lth}(u)$ by u". In [15] it is shown that this function has a neighbourhoodfunction in $\mathbb{K}$, i.e. $\lambda x \cdot u \mid x \in \text{cts}(B, B)$.)

The formal covers thus defined form a Grothendieck topology:

(i) $\lambda n \cdot m + 1 \in \mathbb{K}$, so $\text{cts}(B, B)$ is a cover.

(ii) If S is a cover with characteristic function $a \in \mathbb{K}$, and $f \in \text{cts}(B, B)$ has a neighbourhoodfunction $b \in \mathbb{K}$, then $S \upharpoonright f$ has characteristic function $a ; b \in \mathbb{K}$. (For $a ; b$ see [15]; $a ; b$ is defined in such a way that if $a ; b(v) \neq 0$, then there is a $u \in \mathbb{N}^{<\mathbb{N}}$ such that $a(u) \neq 0$ and $\text{im}(f \circ \lambda x \cdot v \mid x) \subseteq u$.)

(iii) If $R \subseteq \text{cts}(B, B)$ is a sieve, S a cover with characteristic function $a \in \mathbb{K}$, and $R \upharpoonright f$ is a cover with characteristic function $b_f \in \mathbb{K}$ for each $f \in S$, then R is a cover, with characteristic function a/b, where $b : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ is such that $\lambda v \cdot b(\langle u \rangle * v) = b_{\lambda x \cdot u \mid x}$. (For a/b see [15].)

Next we look at the sheaves that are needed to interpret the CS-language. As in Section 2, we interpret B_C as $\text{cts}(B, B)$. All other sorts and predicate constants are to be interpreted as sheaves of lawlike objects, i.e. objects which are locally invariant under restrictions. Such sheaves are completely determined by their global elements, the elements which are totally invariant under restrictions. Quantification over sheaves of lawlike objects reduces to quantification over the global elements of such sheaves, because

$$\Vdash \forall x \in X_L\, A(x) \quad \text{iff} \quad \forall x \in \bar{X}_L \Vdash A(x),$$

and

$$\Vdash \exists x \in X_L\, A(x) \quad \text{iff} \quad \exists S \in J\, \forall f \in S\, \exists x \in \bar{X}_L \Vdash (A \upharpoonright f)(x),$$

where $\bar{X}_L$ is the set of global elements of X_L. Consequently, we can interpret N, B_L and K by the external *sets* $\mathbb{N}$, B, and $\mathbb{K}$ (modulo coding of finite sequences), respectively.

Finally, we reformulate the forcing clauses. Prime formulas of CS are of the form $t = s$ (t and s numerical terms). Equations $t = s$ are basically of the form $\varepsilon n = m$ or of the form $an = m$, ε of sort B_C, a of sort B_L. (Constants are interpreted by 'themselves', more complex equations can be replaced by equivalent formulas in which only these simple equations occur.) B_C is interpreted as $\text{cts}(B, B)$, B_L as B, so we can put

$$\Vdash fn = m \quad \text{iff} \quad \forall x \in B\, f(x)(n) = m,$$

$$\Vdash xn = m \quad \text{iff} \quad xn = m.$$

We then proceed by induction:

$$\Vdash A \wedge B \quad \text{iff} \quad \Vdash A \text{ and } \Vdash B,$$

$$\Vdash A \vee B \quad \text{iff} \quad \exists a \in \mathbb{K}\, \forall u\, (au \neq 0 \Rightarrow (\Vdash A \upharpoonright (\lambda x \cdot u \mid x) \text{ or } \Vdash B \upharpoonright (\lambda x \cdot u \mid x)),$$

$$\Vdash A \to B \quad \text{iff} \quad \forall f \in \text{cts}(B, B)\, (\Vdash A \upharpoonright f \Rightarrow \Vdash B \upharpoonright f),$$

$\Vdash \forall p\, B(p)$ iff $\forall p \Vdash B(p)$, p of sort N, B_L or K,

$\Vdash \forall \varepsilon\, B(\varepsilon)$ iff $\forall f \in \mathrm{cts}(B, B) \Vdash B(f)$,

$\Vdash \exists p\, B(p)$ iff $\exists a \in \mathbb{K}\ \forall u\ (au \neq 0 \Rightarrow \exists p \Vdash B \upharpoonright (\lambda x \cdot u \mid x)(p))$,

$\Vdash \exists \varepsilon\, B(\varepsilon)$ iff $\exists f \in \mathrm{cts}(B, B) \Vdash B(f)$.

(In the last clause we have incorporated Lemma 2.2.5.)

The language restrictions make it rather tedious to verify that the proofs we gave for the validity of CS in sheaves over Cts(B, B) with the open cover topology in Section 2 can be given in IDB with respect to the adapted forcing definition above. It may be instructive to look at bar induction. Note first of all that the language does not permit the formulation of this principle as an axiom. Instead one can look at the schema with an additional parameter of sort B_C,

$$\forall \varepsilon\, \exists n\, A(\bar{\varepsilon}(n), \eta) \wedge \forall u (A(u, \eta) \leftrightarrow \forall n\, A(u * \langle n \rangle, \eta)) \rightarrow A(\langle\ \rangle, \eta).$$

To prove in IDB that this schema is forced one uses the same argument as in Section 2.3, except that external bar induction is replaced by induction over unsecured sequences (which is a corollary of induction over $\mathbb{K}$),

$$\forall a \in \mathbb{K}\ (\forall u\ (au \neq 0 \rightarrow B(u)) \wedge \forall u\ (B(u) \leftrightarrow \forall n\, B(u * \langle n \rangle)) \rightarrow B(\langle\ \rangle)).$$

Another problem here is that one has to show in IDB that the forcing interpretation is sound, in order to have a full constructive proof that forcing over cts(B, B) yields a CS model. Both the validity of the axioms and the soundness follow from the observations in the next subsection.

We close this subsection with the following remark. Let A be a lawlike sentence in the minimal language, i.e. all quantifiers in A are of sort N or sort B_L. Let A^* be the IDB-formula obtained by replacing quantifiers over K by quantifiers over $\mathbb{K}$. One easily verifies that

3.1.1. Lemma. IDB $\vdash A^*$ *iff* IDB $\vdash$ "$\Vdash$ A". □

In other words, the theory of the lawlike part of CS under the forcing interpretation is just IDB. Since in the definition of CS in [15] IDB *is* the lawlike part of CS, the treatment of forcing in an intuitionistic metatheory yields an interpretation which is in this respect more faithful than the classical treatment.

3.2. *Forcing and the elimination translation*

Convention. In this section we assume that all choice parameters in a formula are shown in notation.

In [15] a translation τ is defined which maps sentences of the language of CS to lawlike sentences. This translation is called the elimination translation. The elimination theorem shows that τ provides a sound interpretation of CS in IDB. We give a short account of this interpretation here.

The characteristic property of the CS-axioms is that they give an explanation of choice quantifiers in terms of quantifiers over lawlike objects. This characteristic property is exploited in the elimination translation.

Consider a formula $\exists\varepsilon\, A(\varepsilon)$. By the specialization property, it is equivalent to $\exists a \in B_L\, A(a)$. Thus existential quantification over B_C (in the absence of choice parameters) is explained as existential quantification over B_L. (In Section 2 we have shown that the specialization property is true under the forcing interpretation; in the form here, namely $\exists\varepsilon\, A(\varepsilon) \leftrightarrow \exists a \in B_L\, A(a)$, it follows logically from analytic data.)

Next we look at a formula $\forall\varepsilon\, \exists p\, A(\varepsilon, p)$ (p ranging over N, B_L, or K). By CS3 and CS1a it is equivalent to $\exists a \in K\, \forall u\, (au \neq 0 \rightarrow \exists p\, \forall\varepsilon\, A(u \mid \varepsilon, p))$, so universal quantification over B_C in the context of a lawlike existential quantifier is explained in terms of a lawlike quantifier over K and a universal choice quantifier over a formula of lower complexity. A similar observation holds for $\forall\varepsilon\, (A(\varepsilon) \vee B(\varepsilon))$.

By logic it follows that a universal choice quantifier in the context of a lawlike universal quantifier or a conjunction can be pushed inside, i.e.

$$\forall\varepsilon\, \forall p\, A(\varepsilon, p) \leftrightarrow \forall P\, \forall\varepsilon\, A(\varepsilon, p), \quad \text{and}$$
$$\forall\varepsilon\, (A(\varepsilon) \wedge B(\varepsilon)) \leftrightarrow (\forall\varepsilon\, A(\varepsilon) \wedge \forall\varepsilon\, B(\varepsilon));$$

so universal choice quantification in this context reduces to universal choice quantification over a formula of lower complexity.

Analytic data may equivalently be formulated as

$$\forall\varepsilon\, (A(\varepsilon) \rightarrow B(\varepsilon)) \leftrightarrow \forall f \in \underset{\sim}{K}\, (\forall\varepsilon\, A(f(\varepsilon)) \rightarrow \forall\varepsilon\, B(f(\varepsilon))),$$

so universal choice quantification in the context of an implication is explained in terms of lawlike universal quantification over $\underset{\sim}{K}$ and universal choice quantification over formulas of lower complexity.

By CS1b we have $\forall\varepsilon\, \forall\eta\, A(\varepsilon, \eta) \leftrightarrow \forall f, g \in \underset{\sim}{K}\, \forall\varepsilon\, A(f(\varepsilon), g(\varepsilon))$, i.e. a pair of universal choice quantifiers can be reduced to a single universal choice quantifier and a pair of lawlike quantifiers over $\underset{\sim}{K}$.

Consider a formula $\forall\varepsilon\, \exists\eta\, A(\varepsilon, \eta)$. By CS4 and CS1a this is equivalent to $\exists f \in \underset{\sim}{K}\, \forall\varepsilon\, A(\varepsilon, f(\varepsilon))$, i.e. universal choice quantification in the context of an existential choice quantifier is explained in terms of a lawlike quantifier over $\underset{\sim}{K}$ and a universal choice quantifier over a formula of lower complexity.

Finally a formula $\forall\varepsilon\, (f(\varepsilon)(n) = m)$, where $f \in \underset{\sim}{K}$, is easily seen to be equivalent to $\forall b \in B_L\, (f(b)(n) = m)$, so universal choice quantification over an atomic formula is explained as lawlike quantification.

One may summarize this by saying that the explanation of choice quantifiers consists of a procedure to push universal choice quantifiers over the other logical signs and to replace them eventually by universal lawlike quantifiers in front of equations $t = s$, and to replace existential choice quantifiers not in the scope of a universal one by existential lawlike ones straightaway.

This procedure *is* the elimination translation. τ namely is defined inductively as follows:

τ commutes with the connectives $\wedge, \vee, \rightarrow$, and the lawlike quantifiers $\forall p$ and $\exists p$,

$$\tau(\exists \varepsilon\, A(\varepsilon)) \equiv \exists a \in B_{\mathrm{L}}\, \tau(A(a)),$$

$$\tau(\forall \varepsilon\, (f(\varepsilon)(n) = m)) \equiv \forall a \in B_{\mathrm{L}}\, f(a)(n) = m,$$

$$\tau(\forall \varepsilon\, (A(\varepsilon) \wedge B(\varepsilon))) \equiv \tau(\forall \varepsilon\, A(\varepsilon)) \wedge \tau(\forall \varepsilon\, A(\varepsilon)),$$

$$\tau(\forall \varepsilon\, \forall p\, A(\varepsilon, p)) \equiv \forall p\, \tau(\forall \varepsilon\, A(\varepsilon, p)),$$

$$\tau(\forall \varepsilon\, (A(\varepsilon) \rightarrow B(\varepsilon))) \equiv \forall f \in \underline{K}\, (\tau(\forall \varepsilon\, A(f(\varepsilon))) \rightarrow \tau(\forall \varepsilon\, B(f(\varepsilon)))),$$

$$\tau(\forall \varepsilon\, (A(\varepsilon) \vee B(\varepsilon))) \equiv \exists a \in K\, \forall u\, (au \neq 0 \rightarrow \tau(\forall \varepsilon\, A(u \mid \varepsilon)) \vee \tau(\forall \varepsilon\, B(u \mid \varepsilon))),$$

$$\tau(\forall \varepsilon\, \exists p\, A(\varepsilon, p)) \equiv \exists a \in K\, \forall u\, (au \neq 0 \rightarrow \exists p\, \tau(\forall \varepsilon\, A(u \mid \varepsilon, p))),$$

$$\tau(\forall \varepsilon\, \forall \eta\, A(\varepsilon, \eta)) \equiv \forall f, g \in \underline{K}\, \tau(\forall \varepsilon\, A(f(\varepsilon), g(\varepsilon))),$$

$$\tau(\forall \varepsilon\, \exists \eta\, A(\varepsilon, \eta)) \equiv \exists f \in \underline{K}\, \tau(\forall \varepsilon\, A(\varepsilon, f(\varepsilon))).$$

(In [15], the clauses for $\vee$ and $\exists p$ contain an implicit application of AC–NF. Our presentation is slightly different from but equivalent to the one given in [15].)

The elimination theorem states that the interpretation of CS in IDB via τ is sound, i.e.

(a) $\mathrm{CS} \vdash A \Rightarrow \mathrm{IDB} \vdash \tau(A)$, *for all sentences A in the language of* CS,

and that is faithful, in the sense that

(b) $\mathrm{CS} \vdash A \leftrightarrow \tau(A)$, *for all sentences A in the language of* CS.

The obvious question to ask now is whether the forcing- and the elimination-interpretation are in any sense related to one another. The answer is given by the following theorem.

3.2.1. Theorem. *Let A be a sentence in the minimal language, and let $\tau(A)^*$ be obtained from the elimination translation $\tau(A)$ as indicated at the end of the preceding Section* 3.1. *Then $\tau(A)^*$ and $\Vdash A$ are provably equivalent in* IDB. *In fact one can show that $\tau(\forall \varepsilon\, A(f_1(\varepsilon), \ldots, f_n(\varepsilon)))^*$ is literally the same as $\Vdash A(f_1, \ldots, f_n)$.*

Proof. The second claim is proved by a straightforward induction on the logical complexity of $A(\varepsilon_1, \ldots, \varepsilon_n)$. From this, the equivalence of $\tau(A)^*$ and $\Vdash A$ for arbitrary sentences A follows easily, using the soundness of τ. □

This theorem shows that elimination and monoid forcing are essentially the same interpretation.

As a corollary to the elimination theorem and Theorem 3.2.1 we now find that the monoid-forcing interpretation of CS (in the original CS-language) is classifying for CS, in the sense that

3.2.2. Corollary. IDB$\vdash$"$\Vdash A$" *iff* CS$\vdash A$. □

The monoid forcing interpretation is also classifying in the sense of [16]; this will be extensively discussed in [9].

We have thus shown that the elimination theorem is in fact a special case of the standard method of interpreting intuitionistic theories in sheaves over a category equipped with a Grothendieck topology. This result also shows that the elimination procedure is not just a syntactical trick.

(It should perhaps be remarked here that it is *not* claimed in [15] that the underlying idea is syntactical; the explanation of the elimination translation given above even suggests the contrary. The syntactic flavour of [15] rather seems inherent to the attention paid to the metatheory.)

A similar connection between monoid models and elimination translations can be formulated for relativizations of CS. We trust that, with the monoid models of Section 2.3 in mind, the interested reader can work out the details of an elimination translation "which expresses monoid forcing" for relativizations of CS.

4. Spatial models

We have now seen how CS and its relativizations can be interpreted in sheaves over (a submonoid of) cts(B, B) with the open cover topology. In the preceding section it has been shown that this interpretation corresponds to the elimination translation for CS, i.e. the interpretation is in a sense the one 'prescribed' by the axioms, and the monoid models are in a strong sense the classifying models for CS and its relativizations. But still, the monoid models do not help to solve the problem of finding an informally described class of construction processes (a subdomain of the universe of choice sequences) for which the validity of CS-axioms can be rigorously justified. As has already been said in the introduction, the monoid models are formally motivated, not conceptually.

It therefore remains of interest to find models for CS (or relativizations) which are spatial, and then preferably over spaces 'resembling' Baire space. The interest of such spaces lies in their relation to internal 'projection' models: a model over Baire space (treated in an intuitionistic metatheory) is equivalent to a projection model of the form $\mathcal{U}_\alpha = \{f(\alpha) \mid f \in S\}$, where S is a subset of cts(B, B) (cf. Section 5.3 below). Such a $\mathcal{U}_\alpha$ *is* a subdomain of intuitionistic Baire space, i.e. it is a 'conceptual model'. (For more discussion see [10] and especially [19].)

In fact, the Diaconescu cover [2] yields a general procedure for obtaining a cHa which is first-order equivalent to any given site (cf. [11]), but it seems to be difficult to describe the cHa's thus obtained in terms of familiar spaces. We will therefore not apply the Diaconescu cover here, but instead we give a more direct construction, which yields for each of the monoids $\mathbb{M}$ discussed in Section 2.3 a topological space $X_{\mathbb{M}}$ which is first-order equivalent to $\mathbb{M}$ with the open cover topology. For countable $\mathbb{M}$, $X_{\mathbb{M}}$ is homeomorphic to a subspace of Baire space. In general, $X_{\mathbb{M}}$ is a subspace of $\mathbb{M}^{\mathbb{N}}$ with the product topology, where $\mathbb{M}$ is regarded as a discrete space.

Let $\mathbb{M}$ be a submonoid of cts(B, B), of the form described in 2.3. If $F = (F_n)_n$ is a sequence of elements of $\mathbb{M}$, we define F_n^m by induction: $F_n^0 = 1$ (the identity-mapping) and $F_n^{m+1} = F_n^m \circ F_{n+m}$. (Thus, if $m > 0$, $F_n^m = F_n \circ \cdots \circ F_{n+m-1}$.) We will call a sequence *admissible* if for any composition F_n^m of m successive elements of F the first m numbers of the sequence $F_n^m(x)$, $x \in B$, do not depend on x; i.e. F is admissible iff for all m and n, $\lambda x\, F_n^{m+1}(x)(m) : B \to \mathbb{N}$ is constant.

For a sequence F, being admissible means that we can define points $\lim_n(F)$ of Baire space, for each $n \in \mathbb{N}$, by setting

$$\lim_n(F)(m) = F_n^{m+1}(x)(m), \quad \text{for some (all) } x \in B.$$

Let $X_{\mathbb{M}}$ be the space whose points are the admissible elements of $\mathbb{M}^{\mathbb{N}}$, with the product topology, regarding $\mathbb{M}$ as a discrete space; thus basic opens are the sets

$$\bar{F}n = \{G \mid G \text{ is admissible, and } F_i = G_i \text{ for } i = 0, \ldots, n-1\}.$$

Note that this topology makes the functions $\lim_n : X_{\mathbb{M}} \to B$ continuous.

The language that we will consider is the minimal language, with an addditional constant M (for a subset of K). Thus, we have a sort of natural numbers N, a sort of lawlike sequences B_L (a subsort of N^N), a sort of lawlike neighbourhoodfunctions K (a subsort of B_L), and a sort B_C of choice-sequences.

In sheaves over $X_{\mathbb{M}}$, Baire space N^N is interpreted as the sheaf of continuous B-valued functions. We will interpret B_C as the sheaf generated by (global) elements of the form

$$f \circ \lim_n : X_{\mathbb{M}} \to B,$$

where $n \in \mathbb{N}$, and f is (locally) an element of M (i.e. $\Vdash f \in B_C$ in the monoid model over $\mathbb{M}$, as in Section 2.3). The lawlike types are interpreted in sheaves over $X_{\mathbb{M}}$ as the sheaves of locally constant functions with the appropriate range.

We will show by formula-induction that forcing over the monoid $\mathbb{M}$ and forcing over the space $X_{\mathbb{M}}$ are equivalent (Theorem 4.3 below). But first we need a lemma to be able to compare covers in $\mathbb{M}$ and covers of $X_{\mathbb{M}}$.

4.1. Lemma. *Let $F \in X_{\mathbb{M}}$, $n \in \mathbb{N}$. Then for each $\alpha \in B$ there exists a sequence $G(\alpha)$ such that*

(i) *$G(\alpha)$ is admissible, and $G(\alpha) \in \bar{F}n$.*

(ii) *If $m > k \geq n$,* range$(G(\alpha)_k \circ \cdots \circ G(\alpha)_{m-1}) = V_{\bar{\alpha}(r)} := \{x \in B \mid \bar{\alpha}(r)$ is an initial segment of $x\}$, *for some r strictly increasing in m.*

(iii) *If $k \geq n$, then* $\lim_k G(\alpha) = \alpha$.

Proof. If $\alpha \in B$, then for each $k < n$ there exists a function g_k^α such that for each

m, the first $m+1$ values of $F_k \circ \cdots \circ F_{n-1}$ are constant on $V_{\bar{\alpha}(g_k^\alpha(m))}$, by continuity of $F_k \circ \cdots \circ F_{n-1}$ at α.

Let $g^\alpha(m) = \max_{k<n} g_k^\alpha(m)$. Then

(a) $$\forall k < n\, \forall x, x' \in V_{\bar{\alpha}(g^\alpha(n))}\, \forall i \leq m\, F_k \circ \cdots \circ F_{n-1}(x)(i) = F_k \circ \cdots \circ F_{n-1}(x')(\mathrm{i})$$

and without loss we may assume

(b) g^α is strictly increasing.

Now let

$$G(\alpha)_k(x) = F_k(x) \qquad \text{if } k < n,$$
$$G(\alpha)_k(x) = \bar{\alpha}(g^\alpha(k)) \mid x \quad \text{if } k \geq n.$$

(Recall that if $x \in B$, $U \in \mathbb{N}^{<\mathbb{N}}$, then $u \mid x$ denotes the sequence obtained from x by replacing the initial segment of x of length $\mathrm{lth}(u)$ by u.) Then (i)–(iii) hold: the only thing that is perhaps not immediately clear is that $G(\alpha)$ is admissible. Consider a composition $G(\alpha)_k \circ \cdots \circ G(\alpha)_{k+m}$ of $m+1$ successive elements of $G(\alpha)$: if $k+m<n$, then there is nothing to prove since F is admissible. If $k \geq n$, then $G(\alpha)_k \circ \cdots \circ G(\alpha)_{k+m}(x) = \bar{\alpha}(g^\alpha(k+m)) \mid x)$, and it is immediate from (b) that the first $m+1$ values of this output do not depend on x. And if $k < n \leq k+m$, then

$$G(\alpha)_k \circ \cdots \circ G(\alpha)_{k+m}(x) = F_k \circ \cdots \circ F_{n-1} \circ G(\alpha)_n \circ \cdots \circ G(\alpha)_{k+m}(x)$$
$$= F_k \circ \cdots \circ F_{n-1}(\bar{\alpha}(g^\alpha(k+m)) \mid x),$$

and by (a), the first $m+1$ values of this output do not depend on x. □

We now list some properties that we need in the inductive steps of Theorem 4.3 below:

4.2. Lemma. (a) *If* $U \subseteq \{\bar{G}m \mid G \in \bar{F}n, m \geq n\}$ *and* $\forall G \in \bar{F}n\, \exists m \geq n\, \bar{G}n \in U$, *then the set* $\{G_n^{m-n} \circ f \mid Fm \in U, f \in \mathbb{M}\}$ *is a cover of* $\mathbb{M}$.

(b) *If a sieve* S *covers in* $\mathbb{M}$, *then* S *bars each* $\bar{F}n$, *i.e.* $\forall G \in \bar{F}n\, \exists m \geq n\, G_n^{m-n} \in S$.

(c) *If* $g \in \mathbb{M}$, *then* $\{f \mid \exists m \geq n\, \exists G \in \bar{F}n\, G_n^{m-n} = g \circ f\}$ *is a cover of* $\mathbb{M}$.

Proof. (a) is immediate from Lemma 4.1; (b) follows trivially from the definition of admissibility; (c) is a combination of (a) and part (ii) of the definition of a Grothendieck topology. □

4.3. Theorem. *Let* $A(\varepsilon_1, \ldots, \varepsilon_p)$ *be a formula in the restricted language for* CS(M) *described above, (where* $\varepsilon_1, \ldots, \varepsilon_p$ *are the non-lawlike variables occurring in this formula,) and suppose* $m_1, \ldots, m_p \leq n$. *Then*

$$\bar{F}n \Vdash A(f_1 : m_1, \ldots, f_p : m_p) \quad \textit{iff} \quad \Vdash A(f_1 \circ F_{m_1}^{n-m_1}, \ldots, f_p \circ F_{m_p}^{n-m_p})$$

where we write $f_i : m_i$ *for* $f_i \circ \lim_{m_i}$. ($\Vdash$ *on the left is forcing in sheaves over the space* $X_{\mathbb{M}}$, $\Vdash$ *on the right is forcing over the monoid* $\mathbb{M}$ *with the open cover topology*).

Proof. By induction on A:

(1) $A(\varepsilon)$ is $\varepsilon m_1 = m_2$. Then if $k \leq n$,

$$\bar{F}n \Vdash \varepsilon m_1 = m_2[f:k] \quad \text{iff} \quad \forall G \in \bar{F}n\, f \circ \lim_k(G)(m_1) = m_2.$$

But $f \circ \lim_k = f \circ F_k^{n-k} \circ \lim_n$, so (using 4.1) this is equivalent to

$$\forall x \in B\, f \circ F_k^{n-k}(x)(m_1) = m_2, \quad \text{i.e.} \quad \Vdash \varepsilon m_1 = m_2[f \circ F_k^{n-k}].$$

(2) A is $B \wedge C$. This step is trivial.

(3) A is $B \vee C$. Then we have

$$\bar{F}n \Vdash B \vee C[f_i : m_i]$$
$$\text{iff} \quad \forall G \in \bar{F}n\, \exists m \geq n\, (\bar{G}m \Vdash B[f_i : m_i] \text{ or } \bar{G}m \Vdash C[f_i : m_i])$$
$$\text{iff} \quad \forall g \in \bar{F}n\, \exists m \geq n\, (\Vdash B[f_i \circ G_{m_i}^{m-m_i}] \text{ or } \Vdash C[f_i \circ G_{m_i}^{m-m_i}]).$$

But $G_{m_i}^{m-m_i} = F_{m_i}^{n-m_i} \circ G_n^{m-n}$, so by 4.2(a) and (b) this is equivalent to

$$\exists \text{ cover } S \text{ of } \mathbb{M}\, \forall f \in S \Vdash B[f_i \circ F_{m_i}^{n-m_i} \circ f] \text{ or } \Vdash C[f_i \circ F_{m_i}^{n-m_i} \circ f]$$
$$\text{iff} \quad \Vdash B \vee C[f_i \circ F_{m_i}^{n-m_i}].$$

(4) A is $B \to C$. In this case,

$$\bar{F}n \Vdash B \to C[f_i : m_i]$$
$$\text{iff} \quad \forall G \in \bar{F}n\, \forall m \geq n\, (\bar{G}m \Vdash B[f_i : m_i] \Rightarrow \bar{G}m \Vdash C[f_i : m_i])$$
$$\text{iff} \quad \forall G \in \bar{F}n\, \forall m \geq n\, (\Vdash B[f_i \circ F_{m_i}^{n-m_i} \circ G_n^{m-n}] \Rightarrow \Vdash C[f_i \circ F_{m_i}^{n-m} \circ G_n^{m-n}]$$

and by 4.2(c) this is equivalent to

$$\forall g \in \mathbb{M}\, (\Vdash B[f_i \circ F_{m_i}^{n-m_i} \circ g] \Rightarrow \Vdash C[f_i \circ F_{m_i}^{n-m_i} \circ g]),$$

i.e. $\Vdash B \to C[f_i : m_i]$.

(5), (6) The case of universal quantification over lawlike types is obvious. The case of existential lawlike quantifications is analogous to case (3) above.

(7) A is $\forall \eta\, B(\varepsilon_1, \ldots, \varepsilon_p, \eta)$. Now if $\bar{F}n \Vdash A[f_i : m_i]$, i.e.

$$\forall f \in B_C\, \forall m\, \bar{F}n \Vdash B[f_i : m_i, f : m],$$

then also

$$\forall f \in B_C\, \forall G \in \bar{F}n\, \forall m \geq n\, \bar{G}m \Vdash B[f_i : m_i, f : m]$$

(B_C is the sheaf of functions f with $\Vdash f \in B_C$ in the monoid-model); so by the induction hypothesis,

$$\forall f \in B_C\, \forall G \in \bar{F}n\, \forall m \geq n \Vdash B[f_i \circ F_{m_i}^{n-m_i} \circ G_n^{m-n}, f].$$

But then, if $f \in B_C$ and $g \in \mathbb{M}$ are arbitrary, we derive that for the cover S defined in 4.2(c),

$$\text{for each } f' \in S, \quad \Vdash B[f_i \circ F_{m_i}^{n-m_i} \circ g \circ f', f \circ f'],$$

hence also $\Vdash B[f_i \circ F_{m_i}^{n-m_i} \circ g, f]$. This shows that $\Vdash \forall \eta\, B(\varepsilon_i, \eta)\ [f_i \circ F_{m_i}^{n-m_i}]$. Conversely, if $\Vdash \forall \eta\, B(\varepsilon_i, \eta)\ [f_i \circ F_{m_i}^{n-m_i}]$, then if f and m are arbitrary, it follows immediately from the induction hypothesis that for each $G \in \bar{F}n$ and each $k \geq n$, $\bar{G}k \Vdash B\ [f_i : m_i, f : m]$. Hence also $\bar{F}n \Vdash B\ [f_i : m_i, f : m]$. Thus $\bar{F}n \Vdash \forall \eta\, B(\varepsilon_i, \eta)$ $[f_i \circ F_{m_i}^{n-m_i}]$.

(8) Finally, take A is $\exists \eta\, B(\varepsilon_1, \ldots, \varepsilon_p, \eta)$. Suppose that $\bar{F}n \Vdash \exists \eta\, B(\varepsilon_i, \eta)$ $[f_i : m_i]$, i.e.

$$\forall G \in \bar{F}n\ \exists m \geqslant n\ \exists k, f\ \bar{G}m \Vdash B\ [f_i : m_i, f : k].$$

We may assume $k \leqslant m$, so by induction hypothesis this is equivalent to

$$\forall G \in Fn\ \exists m \geqslant n\ \exists k, f \Vdash B\ [f_i \circ F_{m_i}^{n-m_i} \circ G_n^{m-n}, f \circ F_k^{n-k} \circ G_n^{m-n}].$$

Using Lemma 4.2(a), we then obtain $\Vdash \exists \eta\, B(\varepsilon_i, \eta)\ [f_i \circ F_{m_i}^{n-m_i}]$. Conversely, if $\Vdash \exists \eta\, B(\varepsilon_i, \eta)\ [f_i \circ F_{m_i}^{n-m_i}]$, Lemma 4.2(b) gives us for each $G \in \bar{F}n$ an $m \geqslant n$ and a function f_G such that $\Vdash B[f_i \circ F_{m_i}^{n-m_i} \circ G_n^{n-m}, f_G]$, or, using the induction hypothesis, $\bar{G}m \Vdash B[f_i : m_i, f_G : m]$. Thus $\bar{F}n \Vdash \exists \eta\, B(\varepsilon_i, \eta)\ [f_i : m_i]$. This completes the proof. □

5. Lawlessness

A universe of sequences which satisfies the CS-axioms has strong closure properties: it is closed under the application of all lawlike continuous operations. For sequences satisfying the CS(M)-axioms, these closure properties are somewhat weaker. On the far other end we find the universe of lawless sequences, which has no closure properties at all (application of a lawlike continuous operation other than the identity to a lawless sequence *never* yields a lawless sequence again!) An important axiom here is the axiom of *open data*, which roughly says that the extension of a property of lawless sequences is always an open subset of the space of lawless sequences (as a subspace of intuitionistic Baire space. For a precise formulation, see 5.2 below).

In this part of the paper, we first (Section 5.1) return to the models of Section 2.3, focusing attention on those which satisfy a version of open data. We also describe how to obtain models for the theory of lawless sequences LS by an internal model construction ('projection models', iterated forcing). The theory of lawless sequences is formulated here in a language without arbitrary function and power types (the minimal language), and the internal model-construction is essentially the construction of [17].

Unfortunately, the proof of the correctness of this construction in [17] involves a long formula induction, and is rather complex. Moreover, it is not easy to see whether this proof can be extended to a higher order language. Therefore we will in Section 5.2 present a sheaf model over (a space homeomorphic to) Baire space for the higher order theory of lawless sequences. The proofs given in 5.2 are purely semantical, and the model seems to be more perspicuous than the model for LS presented in [1].

Our construction of a model for the higher order theory of lawless sequences was actually inspired by Troelstra's appendix to [1], and it seems worth the effort of explaining this in more detail. This will be done in Section 5.3.

5.1. *Open data as analytic data*

As a first example of a CS(M)-model, consider the monoid $\mathbb{M}_h$ of local homeomorphisms of Baire space into itself. As has been said in 2.3, in sheaves over this monoid Baire space internally appears as cts(B, B), and the sheaf B_{M_h} of 'M_h-choice sequences' is interpreted as the subsheaf of cts(B, B) generated by 1, i.e. the sheafification of $\mathbb{M}_h$, which is just $\mathbb{M}_h$ in this case. In this particular model, analytic data 'is' open data (without choice parameters):

5.1.1. Proposition. *In the model just described,*

$$\Vdash \forall \varepsilon \, (A\varepsilon \to \exists u \, (u \ni \varepsilon \wedge \forall \eta \in u \, A\eta))$$

where A has all non-lawlike parameters shown.

Proof. We have to show that

$$\Vdash \forall \psi \in \underline{M}_h \, \forall \varepsilon \, (\exists \eta \, \varepsilon = \psi\eta \to \exists u \, (\varepsilon \in u \wedge \forall \xi \in u \, \exists \eta \, \xi = \psi\eta)).$$

It suffices to choose ψ with $\psi(f, g) = \hat{\psi} \circ g$ for some fixed local homeomorphism $\hat{\psi}$ (since such morphisms $\psi : M_h \times M_h \to M_h$ generate the sheaf $\underline{M}_h$ of internal lawlike operations $N^N \to N^N$, see 2.3). So suppose f and g are local homeomorphisms such that $\Vdash \exists \eta \, f = (\psi \restriction g)(\eta)$, i.e. (Lemma 2.2.3) there exists a local homeomorphism h such that $f = \hat{\psi} \circ h$. Find a cover $\{W_i : B \twoheadrightarrow W_i \subset B\}_i$ such that $f \restriction W_i$ is a homeomorphism, and let for each $x \in W_i$, $n^i_x \in \mathbb{N}$ be such that

$$u^i_x := \overline{f(x)}(n^i_x) \subseteq f(U_i),$$

and let v^i_x be an initial segment of x such that $f(v_x) \subseteq \overline{f(x)}(n^i_x)$. The $\{v^i_x\}_{i,x}$ form a cover, and $\Vdash f \restriction v^i_x \in u^i_x$ for each i and each $x \in W_i$. Also $\Vdash \forall \xi \in u^i_x \, \exists \eta \, \xi = \psi(\eta)$, for if k and l are local homeomorphisms such that $\Vdash k \in u^i_x \restriction l$, then $\forall y \in B \; k(y) \in u^i_x = \overline{f(x)}(n_i)$, so range($k$) $\subset U_i$, and therefore $k = \hat{\psi} h f^{-1} k$, i.e. $\Vdash \exists \eta \, k = \psi(\eta)$. $\square$

Sheaves over the monoid of local homeomorphisms were considered by Fourman in his talk at the Brouwer conference. He defined the subsheaf L of internal Baire space ($=$ cts(B, B)), his sheaf of 'lawless sequences', to be the sheaf of *local projections*. More precisely, let $j : B \times B \to B$ be a fixed homeomorphism, and define L to be the subsheaf of cts(B, B) generated by $j_1 = \pi_1 j$. Observe that this sheaf L becomes *definable* in the particular theory CS(M_h) under discussion, namely as $\{j_1\varepsilon \mid \varepsilon \in B_{M_h}\}$. Thus, this model may be regarded as a '*projection model*', projected from a CS(M)-model of the type described in 2.3. In this model, the sequences in L satisfy various conditions which are similar in character to the axioms of LS (as formulated in 5.2 below), but there are some striking differences.

For example, a crucial role is played by the notion of *independence*: two lawless sequences α and β are said to be independent iff $\langle\alpha, \beta\rangle := j \circ (\alpha, \beta) \in L$ (or in CS(M_h)-terms, $j_1\varepsilon$ and $j_1\eta$ are independent iff there exists a $\xi \in B_{M_h}$ such that $j_1\varepsilon = j_1 j_1 \xi$, $j_1\eta = j_1 j_2 \xi$). This notion of independence is necessary, for example, to formulate the multiple-parameter version of the open data axiom which is valid in this model. Thus, there is an essential difference between this axiom of open data, and the more traditional axiom, where instead of independence one has just inequality. It does not seem to be possible to modify this model so as to obtain a monoid model for 'ordinary' open data.

The problem is that sheaves over monoids have to satisfy some non-trivial closure-conditions (provided the sheaf and the monoid are non-trivial). For example in Fourman's model, the sheaf L is closed under projecting ($\Vdash \langle\alpha, \beta\rangle \in L \rightarrow \alpha \in L \wedge \beta \in L$). Such closure conditions are incompatible with the ordinary multiple-parameter version of open data. This strongly suggests that it is impossible to obtain monoid models for the theory LS.

Let us take a different approach for obtaining an LS-model, by starting with a monoid model for LS^1. (LS^1 is the theory with axioms (schemas) just like those of LS, but with the schemas LS3, LS4 restricted to formulas containing at most one parameter over choice sequences (lawless sequences), see [1], [18].) In fact, the sheaf L above is a domain satisfying the LS^1-axioms. A simpler LS^1-model can be obtained as follows: let $\mathbb{M}_0$ be the monoid of continuous functions of the form $\tilde{u}$, $\tilde{u}(x) = u \mid x$, for finite sequences u. The open cover topology is just the 'bar topology' ($\{\tilde{u}_i \mid i \in T\}$ covers iff $\{u_i \mid i \in I\}$ is a bar in $\mathbb{N}^{<\mathbb{N}}$). In Section 2.3 it was shown that sheaves over this monoid yield a model for this instance CS(M_0) of relativized CS, and it is clear that analytic data comes down to open data without (non-lawlike) parameters in this case, and the sheaf B_{M_0} of 'M_0-choice sequences' (the subsheaf of cts(B, B) generated by the identity) gives a model for the theory LS^1.

It is *not* an LS-model, of course, again because the monoid action on the sheaf gives us too many closure properties. For example, in sheaves over M_0, $\Vdash \exists\xi, \eta \in B_{M_0}$ $(\xi \neq \eta \wedge \exists n\, \forall k > n\, \bar{\xi}k = \bar{\eta}k)$ (take to different sequences u and v of equal length), which clearly contradicts (the two parameter case of) open data.

At this point, we may invoke a method of Troelstra's for constructing an LS-model from an LS^1-model: In [17], Troelstra shows that if L^1 is a subspace of Baire space satisfying the LS^1-axioms, then for each $\alpha \in L^1$,

$$\mathcal{U}_\alpha = \{u * \pi_u(\alpha) \mid u \in N^{<N}\}$$

can intuitionistically be shown to be a model of LS (here $\pi_u(\alpha)(n) = \alpha(u * \langle n\rangle)$).

Thus, *within* the monoid model Sh(M_0) under discussion, we have many LS-models, but they are not *definable* externally. An easy way out here is to construct *internally* the direct product $\mathcal{U} = \prod_{\alpha \in B_C} \mathcal{U}_\alpha$. Then Sh($M_0$) $\Vdash$ "$\mathcal{U} \Vdash$ LS" by Troelstra's result, and it is possible to reduce this two-step forcing to a single step. One then obtains a sheaf-model over a site $\mathbb{S}$ which is neither a monoid, nor a topological space. We will not describe the construction of $\mathbb{S}$ in detail: the reader who is familiar with models over sites will be able to work it out for himself.

It should be stressed that the proof of Troelstra's result uses induction on formulas, and holds only for the first-order language in which LS is usually formulated. We have not been able to find a direct proof of the validity of the open data axiom in sheaves over the site $\mathbb{S}$ *without* this restriction on the language.

5.2. *A sheaf model for* LS

We start by formulating the LS-axioms. They are formulated in a higher order language (with arbitrary function- and powersorts, as in Section 1), with in addition, sorts B_L for lawlike sequences, K for lawlike neighbourhoodfunctions, and L for lawless sequences; these are all subsorts of N^N. We use $\alpha, \beta, \gamma, \ldots$ as variables ranging over L. The axioms are

LS1 (decidable equality)

$$\forall \alpha, \beta \in L\, (\alpha = \beta \vee \alpha \neq \beta).$$

LS2 (density)

$$\forall u \in N^{<N}\, \exists \alpha\, \alpha \in u.$$

LS3 (higher order open data) For each $n \in \mathbb{N}$,

$$\forall \alpha_1 \cdots \forall \alpha_n\, (\neq(\alpha_1, \ldots, \alpha_n) \wedge A(\alpha_1, \ldots, \alpha_n) \rightarrow \exists u_1 \ni \alpha_1 \cdots \exists u_n \ni \alpha_n$$
$$\forall \beta_1 \in u_1 \cdots \forall \beta_n \in u_n\, (\neq(\beta_1, \ldots, \beta_n) \rightarrow A(\beta_1, \ldots, \beta_n)))$$

where $\neq(\alpha_1, \ldots, \alpha_n)$ abbreviates $\bigwedge_{1 \leq i < j \leq n} \alpha_i \neq \alpha_j$).

LS4 (higher order continuity) For each $n \in \mathbb{N}$,

$$\forall \alpha_1 \cdots \forall \alpha_n\, (\neq(\alpha_1, \ldots, \alpha_n) \rightarrow \exists a\, A(\alpha_1, \ldots, \alpha_n, a))$$
$$\rightarrow \exists e \in K_n\, \forall u_1 \cdots \forall u_n\, (e(u_1, \ldots, u_n) \neq 0$$
$$\rightarrow \exists a\, \forall \alpha_1 \in u_1 \cdots \forall \alpha_n \in u_n\, (\neq(\alpha_1, \ldots, \alpha_n) \rightarrow A(\alpha_1, \ldots, \alpha_n, a)))$$

(where K_n is the set of n-place lawlike neighbourhoodfunctions, defined in the obvious way).

In LS3 *and* LS4, *the formula* A *contains no other non-lawlike parameters than the ones shown.*

Our sheaf model will in fact be an interpretation in 'sheaves with a group action', as described in e.g. the appendix of [5]. Let $(v_n : n \in \mathbb{N})$ be an enumeration of $\mathbb{N}^{<\mathbb{N}}$ in which each sequence occurs infinitely many times. Let T be the space $\prod_{n \in \mathbb{N}} V_{v_n}$, equipped with the product topology. In this section, we will write V_u instead of just u for the basic open subset $\{x \mid x \in u\}$ of B, for u a finite sequence. If $u_1, \ldots, u_n$ are finite sequences, then we write $\langle V_{u_1}, \ldots, V_{u_n} \rangle$ for the basic open subset $\bigcap_{i=1}^{n} \pi_i^{-1}(V_{u_i})$ of T. T is obviously homeomorphic to B, but for present purposes T is notationally more convenient than B is.

We now define a group G of auto(homeo)morphisms of T as follows. Consider the following two types of automorphisms of T:

(1) For each $n, m \in \mathbb{N}$ and $u \in \mathbb{N}^{<\mathbb{N}}$ such that $v_n \leq u$ and $v_m \leq u$, the automorphism $h = h[n, m, u]$ of T which interchanges the nth and mth coordinate of a point $x \in T$, provided both coordinates begin with u; i.e.

$$h(x)_k = \begin{cases} x_m, & \text{if } k = n \text{ and } x_n \in u,\ x_m \in u, \\ x_n, & \text{if } k = m \text{ and } x_n \in u,\ x_m \in u, \\ x_k, & \text{otherwise.} \end{cases}$$

(2) For each $n \in \mathbb{N}$, $f, g : \mathbb{N} \to \mathbb{N}$ with

$$n < g0 < f0 < g1 < f1 < g2 < f2 < \cdots$$

such that

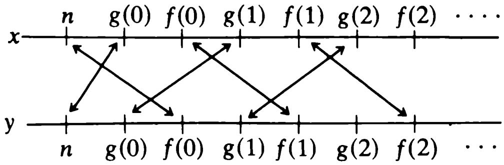
– $\{v_{g(m)}\}_m$ and $\{v_{f(m)}\}_m$ are constant sequences in $\mathbb{N}^{<\mathbb{N}}$

– $v_{g(0)}$ and $v_{f(0)}$ are incompatible extensions of v_n, the automorphism $h = h[n, f, g]$ of T defined as follows (cf. the picture below):

(a) If $x \in T$ is such that $x_n \in v_{f(0)}$ and $x_{g(0)} \in v_n$, then $h(x) = y$, where $y_n = x_{g(0)}$, $y_{f(0)} = x_n$, $y_{f(m+1)} = x_{f(m)}$ for each $m \in \mathbb{N}$, $y_{g(m)} = x_{g(m+1)}$ for each $m \in \mathbb{N}$, and $y_i = x_i$ for all $i \notin \{n\} \cup \{g(m) \mid m \in \mathbb{N}\} \cup \{f(m) \mid m \in \mathbb{N}\}$.

(b) If $y \in T$ is such that $y_n \in v_{g(0)}$ and $y_{f(0)} \in v_n$, then $h(y) = x$, where y and x are related as in (a).

(c) If $z \in T$ is a point to which neither (a) nor (b) applies, then $h(z) = z$.



G is the subgroup of the group of automorphisms of T generated by all homeomorphisms of the form (1) or (2).

Recall (cf. [5], appendix) that if G is a group of automorphisms of T, a 'sheaf with G-action' on T is a sheaf A on T with an action of G on the sections of A, written $a \mapsto a^g$, such that

$$a^1 = a, \qquad a^{g \circ h} = (a^g)^h, \qquad [\![a^g = b^g]\!] = g^{-1}([\![a = b]\!])$$

(and hence, $E(a^g) = g^{-1}(E(a))$, and $(a \restriction U)^g = a^g \restriction g^{-1}(U)$).

In the 'standard interpretation' in such sheaves with a group action, the sheaf of natural numbers N appears as the sheaf of continuous partial functions $U \to \mathbb{N}$, $U \in \mathcal{O}(T)$, with right composition

$$\left(U \xrightarrow{a} N\right) \mapsto \left(h^{-1}(U) \xrightarrow{a \circ h} N\right)$$

as action. Similarly, internal Baire space N^N appears as the sheaf of continuous partial functions $U \to B$ with right composition as action.

If A is a sheaf with G-action, a *global element* of A is a global section a of A which is invariant under the action of G ($a^g = a$ for $g \in G$). We define *the sheaf* A_L *of lawlike elements of* A to be the subsheaf of A generated by the global elements of A. (In fact, this is what we also did in Section 2.)

Our model will be the standard interpretation in sheaves over the space T with G-action, where the space T and the group G are as defined above. Further, we specify the interpretation of the additional constants: B_L and K are interpreted as the sheaf of locally constant partial functions $U \to B$ and $U \to \mathbb{K}$ respectively (where $\mathbb{K} \subset B$ is the set of external neighbourhoodfunctions), with right composition as action. The sheaf of lawless sequences L is the sheaf generated by the projections $\pi_n : T \to B$ ($n \in \mathbb{N}$), again with right composition as action. Note that each of the homeomorphisms in G locally either is the identity, or interchanges coordinates. Hence the sheaf of partial functions $U \to B$ ($U \in \mathcal{O}(T)$) which are locally some π_n is indeed closed under the action of G.

The rest of this section will consist of the proof of the following theorem.

5.2.1. Theorem. *The interpretation just described yields a model for the higher order theory* LS.

Verification of LS1 and LS2 is trivial. For the axioms of open data and continuity, however, we have to do some work. First note that if $A(\alpha_1, \ldots, \alpha_n, p_1, \ldots, p_k)$ is a formula with $\alpha_1, \ldots, \alpha_n$ as lawless parameters of sort L, and we interpret all other parameters $p_1, \ldots, p_k$ by global sections $\bar{p}_1, \ldots, \bar{p}_k$ of the appropriate sheaves, then

$$[\![A(\alpha_1, \ldots, \alpha_n, \bar{p}_1, \ldots, \bar{p}_k)]\!]$$

is a global section of the powersheaf $\mathcal{P}(L^n)$; that is a function $P : L^n \to \mathcal{O}(T)$ which is strict and extensional

$$(P(\langle \alpha_1, \ldots, \alpha_n \rangle) \subseteq E\langle \alpha_1, \ldots, \alpha_n \rangle,$$
$$P(\alpha_1 \upharpoonright U, \ldots, \alpha_n \upharpoonright U) = P(\alpha_1, \ldots, \alpha_n) \cap U,$$

and moreover preserves the action, i.e. $P(\alpha_1^g, \ldots, \alpha_n^g) = g^{-1}P(\alpha_1, \ldots, \alpha_n)$.

By the interpretation of lawlike elements described above, such functions generate the extensions of the formulas A occurring in the LS-axioms, and therefore we may restrict our attention to strict extensional functions P which preserve the action, as we do in the following two lemmas.

5.2.2. Lemma. *Let* $P : L^p \to \mathcal{O}(T)$ *be a global section of* $\mathcal{P}(L^p)$, *and let* x *and* y *be two points of* T *such that* $x_{n_i} = y_{n_i}$ *for each* $i = 1, \ldots, p$. *Then* $x \in P(\pi_{n_1}, \ldots, \pi_{n_p})$ *iff* $y \in P(\pi_{n_1}, \ldots, \pi_{n_p})$.

Proof. Suppose $x \in P(\pi_{n_1}, \ldots, \pi_{n_p})$, and choose sequences $u_1, \ldots, u_k$ such that

$x \in \langle V_{u_1}, \ldots, V_{u_k}\rangle \subseteq P(\pi_{n_1}, \ldots, \pi_{n_p})$. We may assume that each $u_i \geqslant v_i$, and that $k \geqslant n_p$. We now define an $h \in G$ and a point $z \in \langle V_{u_1}, \ldots, V_{u_k}\rangle$ such that $h(z) = y$ and $\pi_{n_i} \circ h = \pi_{n_i}$ for $i = 1, \ldots, p$. This suffices to prove the lemma since then

$$y = h(z) \in h\ (P(\pi_{n_1}, \ldots, \pi_{n_p})) = P(\pi_{n_1} \circ h^{-1}, \ldots, \pi_{n_p} \circ h^{-1}) = P(\pi_{n_1}, \ldots, \pi_{n_p}).$$

Let $\{a_1, \ldots, a_l\} = \{1, \ldots, k\} \backslash \{n_1, \ldots, n_p\}$. Choose for each $i \leqslant l$, two incompatible extension w_i and w_i' of u_{a_i}, and let $\{f^i(m)\}_m$ and $\{g^i(m)\}_m$ be sequences of natural numbers such that

$$v_{f^i(m)} = w_i \quad \text{and} \quad v_{g^i(m)} = w_i', \quad \text{for each } i \leqslant l,$$

and

$$k < g^i(0) < f^i(0) < g^i(1) < f^i(1) < \cdots, \quad \text{for each } i \leqslant l.$$

and such that the ranges of the g^i's, and those of the f^i's, are mutually disjoint. Now set

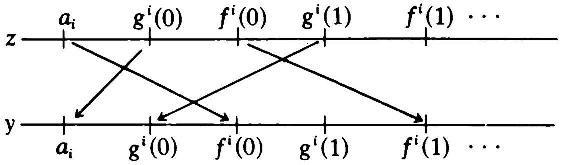
$$h = h[a_l, f^l, g^l] \circ \cdots \circ h[a_1, f^1, g^1]$$

and let z be the point defined by

$$z_{a_i} = y_{f^i(0)}, \quad z_{g^i(0)} = y_{a_i}, \quad z_{f^i(m)} = y_{f^i(m+1)} \quad \text{for each } m \in \mathbb{N},$$

$$z_{g^i(m+1)} = y_{g^i(m)} \quad \text{for each } m \in \mathbb{N},$$

$$z_n = y_n \quad \text{for all other } n.$$



Then $z \in \langle V_{u_1}, \ldots, V_{u_k}\rangle$, and $h(z) = y$. □

5.2.3. Lemma. *Let $P: L^P \to \mathcal{O}(T)$ be a global section of $\mathcal{P}(L^P)$, and let $U = \langle V_{u_1}, \ldots, V_{u_k}\rangle$ be a basic open subset of T with $U \subseteq P(\pi_{n_1}, \ldots, \pi_{n_p})$. If $f: \mathbb{N} \xrightarrow{1-1} \mathbb{N}$ is a function with $f(\{n_1, \ldots, n_p\}) \cap \{n_1, \ldots, n_p\} = \emptyset$, and $W = \langle V_{w_1}, \ldots, V_{w_l}\rangle$ is a basic open of T such that $w_{f(n_i)} \geqslant u_{n_i}$ $(i = 1, \ldots, p)$, then $W \subseteq P(\pi_{f(n_1)}, \ldots, \pi_{f(n_p)})$.*

Proof. Let U and f be as described in the lemma. By 5.2.2, we find

$$(1) \qquad \pi_{n_1}^{-1}(V_{u_{n_1}}) \cap \cdots \cap \pi_{n_p}^{-1}(V_{u_{n_p}}) \subseteq P(\pi_{n_1}, \ldots, \pi_{n_p}).$$

It suffices to show that

$$(2) \qquad \pi_{f(n_1)}^{-1}(V_{u_{n_1}}) \cap \cdots \cap \pi_{f(n_p)}^{-1}(V_{u_{n_p}}) \subseteq P(\pi_{f(n_1)}, \ldots, \pi_{f(n_p)}),$$

but by 5.2.2 again, (2) already follows from (3),

(3) $$\bigcap_{i=1}^{p} \pi_{f(n_i)}^{-1}(V_{u_{n_i}}) \cap \bigcap_{i=1}^{p} \pi_{n_i}^{-1}(V_{s_i}) \subseteq P(\pi_{f(n_1)}, \ldots, \pi_{f(n_p)}),$$

where s_i is the shortest sequence with

(4) $s_i \geqslant u_{n_i}$ and $s_i \geqslant v_{f(n_i)}$.

(We may of course assume that u_{n_i} and $v_{f(n_i)}$ are compatible, since otherwise (2) is trivially true.)

To prove (3), choose $y \in T$ such that

$$y_{f(n_i)} \in u_{n_i}, \quad y_{n_i} \in u_{n_i}, \quad \text{and} \quad y_{n_i} \in v_{f(n_i)} \qquad (i = 1, \ldots, p).$$

We will define a point $x \in \bigcap_{i=1}^{p} \pi_{n_i}^{-1}(V_{u_{n_i}})$, and an automorphism $h \in G$ such that $h(x) = y$, and $\pi_{f(n_i)} \circ h = \pi_{n_i}$ on a neighbourhood of x, $i = 1, \ldots, p$. This suffices to prove the lemma, since $x \in P(\pi_{n_1}, \ldots, \pi_{n_p})$ implies that also

$$x \in P(\pi_{f(n_1)} \circ h, \ldots, \pi_{f(n_p)} \circ h) = h^{-1}P(\pi_{f(n_1)}, \ldots, \pi_{f(n_p)}),$$

hence $y = h(x) \in P(\pi_{f(n_1)}, \ldots, \pi_{f(n_p)})$.

Let

$$h = h[n_p, f(n)_p, s_p] \circ \cdots \circ h[n_1, f(n_1), s_1].$$

$h \in G$, since $s_i \geqslant u_{n_i}$ (and without loss $u_{n_i} \geqslant v_{n_i}$), and $s_i \geqslant v_{f(n_i)}$. Let x be defined by

$$x_{n_i} = y_{f(n_i)}, \quad x_{f(n_i)} = y_{n_i}, \quad \text{for } i = 1, \ldots, p$$

$$x_m = y_m, \qquad \text{for other coordinates } m.$$

Then $x_{n_i} \in s_i$ and $x_{f(n_i)} \in s_i$, so $h[n_i, f(n_i), s_i]$ interchanges the n_ith and $f(n_i)$th coordinates on a neighbourhood of x. It is clear that $h(x) = y$, and that $x \in \bigcap_{i=1}^{n} \pi_{n_i}^{-1}(V_{u_{n_i}})$. □

5.2.4. Lemma. *As Lemma 5.2.3, but without the requirement that* $f(\{n_1, \ldots, n_p\}) \cap \{n_1, \ldots, n_p\} = \emptyset$.

Proof. This follows from Lemma 5.2.3 by factoring f as a composition of injections that do satisfy the hypothesis of 5.2.3. □

Proof of 5.2.1. It has already been observed that LS1 and LS2 are trivial, and the validity of open data (LS3) follows immediately from the preceding lemma. So we only have to check LS4. Now if $\forall \alpha_1 \cdots \forall \alpha_n\, \exists a\, A(\alpha_1, \ldots, \alpha_n, a)$ is a formula with all (lawlike) parameters interpreted by global elements, then $[\![\forall \alpha_1 \cdots \forall \alpha_p\, (\neq(\alpha_1, \ldots, \alpha_p) \to \exists a\, A(\alpha_1, \ldots, \alpha_p, a))]\!]$ is a global truth value, i.e. an open subset U of T such that $g^{-1}(U) = U$ for all automorphisms $g \in G$. But (using a composition of automorphisms of type (2)) it is easily seen that the only such U are $\emptyset$ and T.

We may thus assume that $[\![\forall \alpha_1 \cdots \forall \alpha_p\, (\neq(\alpha_1, \ldots, \alpha_p) \to \exists a\, A(\alpha_1, \ldots, \alpha_p, a))]\!]$

$= T$. In particular, if we let $n_1, \ldots, n_p$ be distinct natural numbers such that $v_{n_i} = \langle\ \rangle$ for $i = 1, \ldots, p$, we find that

$$T = [\![\exists a\, A(\pi_{n_1}, \ldots, \pi_{n_p}, a)]\!] = \bigcup_{a \in B_L} [\![A(\pi_{n_1}, \ldots, \pi_{n_p}, a)]\!].$$

Let e be a p-place (external) neighbourhoodfunction such that $e(w_1, \ldots, w_p) \neq 0$ implies that for some $a \in B_L$,

$$\langle V_{w_1}, \ldots, V_{w_p} \rangle \subseteq [\![A(\pi_{n_1}, \ldots, \pi_{n_p}, a)]\!].$$

Let $\bar{e}$ be the internationalization of e ($\bar{e}$ = "compose with e"). Then

$$[\![\forall w_1 \cdots \forall w_p\, (\bar{e}(w_1, \ldots, w_p) \neq 0 \rightarrow \exists a \in B_L\, \forall \beta_1 \in w_1 \cdots \forall \beta_p \in w_p \\ (\neq(\beta_1, \ldots, \beta_p) \rightarrow A(\beta_1, \ldots, \beta_p, a))]\!] = T. \quad (*)$$

To see this, choose $w_1, \ldots, w_p \in \mathbb{N}^{<\mathbb{N}}$ with $e(w_1, \ldots, w_p) \neq 0$. Choose $a \in B_L$ such that $\langle V_{w_1}, \ldots, V_{w_p} \rangle \subseteq [\![A(\pi_{n_1}, \ldots, \pi_{n_p}, a)]\!]$. Then by Lemma 5.3.3, it holds for any p-tuple of distinct natural numbers $m_1, \ldots, m_p$ that

$$\pi_{m_1}^{-1}(V_{w_1}) \cap \cdots \cap \pi_{m_p}^{-1}(V_{w_p}) = [\![\pi_{m_1} \in w_1 \wedge \cdots \wedge \pi_{m_p} \in w_p]\!] \\ \subseteq [\![A(\pi_{m_1}, \ldots, \pi_{m_p}, a)]\!]$$

Hence $(*)$ holds. □

5.3. *Projection models are Beth models*

In the foregoing we have used the word 'projection-model' to refer to universes of the form $\mathcal{U}_\alpha^M = \{f(\alpha) \mid f \in \underset{\sim}{M}\}$, where $\underset{\sim}{M}$ is a subset of $\underset{\sim}{K}$ and α is a lawless sequence or a sequence in a domain which satisfies the LS^1 axioms. In this section we give our own exposition of the fact that validity in such a projection model is equivalent to constructive validity in a topological model over (formal) Baire space (cf. [18], and the appendix to [1]). By doing so, we hope to clarify the remarks made in the introduction to Section 4, as well as to explain the relation between the model presented in Section 5.2 and the appendix to [1].

As in Section 3, we restrict ourselves to the four sorted minimal language. As formal language for the treatment of interpretations of this minimal language in projection models we take the same language, but with the sort B_C replaced by L (for lawless sequences). We use $\alpha, \beta, \gamma, \ldots$ as variables of sort L. Moreover, we add a constant $\underset{\sim}{K}$ for the sort of continuous functions $N^N \rightarrow N^N$ with neighbourhoodfunctions in K. As constructive metatheory for the treatment of Beth-models we use the system IDB (cf. Section 3).

Let $A(\varepsilon_1, \ldots, \varepsilon_n)$ be a formula in the minimal language, and let $\mathcal{U}_\alpha^M$ be a projection model. $\mathcal{U}_\alpha^M \Vdash A(f_1(\alpha), \ldots, f_n(\alpha))$ expresses that A holds if we interpret (a) the parameters $\varepsilon_1, \ldots, \varepsilon_n$ by $f_1(\alpha), \ldots, f_n(\alpha)$ respectively ($f_i \in \underset{\sim}{M}$); (b) the sort B_C by $\mathcal{U}_\alpha$, i.e. quantifiers over B_C are interpreted as quantifiers over $\mathcal{U}_\alpha$; and (c) the sorts B_L and N by themselves. (So the satisfaction sign $\models$ is treated in the traditional Tarskian sense here, be it within the theory LS^1, or within an LS^1-model). We will write $A^\alpha(f_1, \ldots, f_n)$ for the LS-formula in the single parameter α of sort L which denotes $\mathcal{U}_\alpha^M \models A(f_1(\alpha), \ldots, f_n(\alpha))$; thus $A^\alpha(f_1, \ldots, f_n)$ is obtained from $A(\varepsilon_1, \ldots, \varepsilon_n)$ by substituting $f_i(\alpha)$ for ε_i, $i = 1, \ldots, n$, replacing bound variables ε by suitably chosen $f(\alpha)$, and replacing the quantifiers $\forall\varepsilon$, $\exists\varepsilon$ by the corresponding $\forall f \in \underline{M}$, $\exists f \in \underline{M}$. We say that a sentence A holds in $\mathcal{U}_\alpha^M$ iff $LS^1 \vdash \forall\alpha\, A^\alpha$.

The LS^1-axioms provide a full explanation of universal lawless quantification over formulas $B(\alpha)$, in which α is the only choice parameter, and in which no quantifiers over L occur. This explanation proceeds along the same lines as the explanation of quantification over choice sequences in CS (cf. Section 3.2), but since we restrict ourselves to the explanation of universal quantifiers and avoid nested quantification, there is no need for the explanation of $\exists\alpha$, $\forall\alpha\,\exists\beta$, $\forall\alpha\,\forall\beta$. The main difference with the CS-explanation lies in the treatment of formulas of the form $\forall\alpha(A(\alpha) \to B(\alpha))$. By open data in a single parameter, $\forall\alpha\,(A(\alpha) \to B(\alpha))$ is equivalent to $\forall u(\forall\alpha \in u\; A(\alpha) \to \forall\alpha \in u\; B(\alpha))$, i.e., universal lawless quantification is explained in terms of universal quantification over finite sequences and universal lawless quantification over formulas of lower complexity.

The explanation leads to the following elimination translation for sentences $\forall\alpha \in u\, B(\alpha)$, $B(\alpha)$ not containing lawless quantifiers:

$$\begin{aligned}
\tau(\forall\alpha \in u\, f(\alpha)(n) = m) &\equiv \forall a \in u\, f(a)(n) = m,\\
\tau(\forall\alpha \in u\,(A(\alpha) \wedge B(\alpha))) &\equiv \tau(\forall\alpha \in u\, A(\alpha)) \wedge \tau(\forall\alpha \in u\, B(\alpha)),\\
\tau(\forall\alpha \in u\,(A(\alpha) \vee B(\alpha))) &\equiv \exists a \in K\, \forall v\,(av \neq 0\\
&\qquad \to (\tau(\forall\alpha \in u * v\, A(\alpha))\\
&\qquad \vee \tau(\forall\alpha \in u * v\, B(\alpha))),\\
\tau(\forall\alpha \in u\,(A(\alpha) \to B(\alpha))) &\equiv \forall v\,(\tau(\forall\alpha \in u * v\, A(\alpha))\\
&\qquad \to \tau(\forall\alpha \in u * v\, B(\alpha))),\\
\tau(\forall\alpha \in u\, \forall p\, A(\alpha, p)) &\equiv \forall p\, \tau(\forall\alpha \in u\, A(\alpha, p)),\ p \text{ of a lawlike sort},\\
\tau(\forall\alpha \in u\, \exists p\, A(\alpha, p)) &\equiv \exists a \in K\, \forall v\,(av \neq 0\\
&\qquad \to \exists p\, \tau(\forall\alpha \in u * v\, A(\alpha, p))).
\end{aligned}$$

The translation τ defined here is just a fragment of the full elimination translation for LS. τ has the following property (cf. [18]):

(1) *If $B(\alpha)$ is free of lawless quantifiers and lawless parameters other than α, then* $LS^1 \vdash (\forall\alpha\, B(\alpha) \leftrightarrow \tau(\forall\alpha\, B(\alpha)))$.

$\tau(\forall\alpha\, B(\alpha))$ is a formula of lawlike IDB (i.e. IDB with B_L for B, K for $\mathbb{K}$, $\underline{K}$ for cts(B, B), and the LS^1-axioms are conservative over IDB (in fact LS is conservative over IDB), so we also have

(2) *If $B(\alpha)$ is a formula as in* (1) *above, then*

$$LS^1 \vdash \forall\alpha\, B(\alpha) \quad \text{iff} \quad IDB \vdash \tau(\forall\alpha\, B(\alpha)).$$

Now let $A(\varepsilon_1, \ldots, \varepsilon_n)$ be a formula in the CS-language. Then $A^\alpha(f_1, \ldots, f_n)$ is a formula in a single lawless parameter, without lawless quantifiers. Hence we can apply the previous elimination theorem to $\forall\alpha\, A^\alpha(f_1, \ldots, f_n)$. Let us (suggestively) write $u \Vdash A(f_1, \ldots, f_n)$ for $\tau(\forall\alpha \in u\, A^\alpha(f_1, \ldots, f_n))$. The relation $u \Vdash A(f_1, \ldots, f_n)$ then satisfies the following equivalences (provable in IDB):

$$
\begin{aligned}
u \Vdash f(n) = m \quad &\text{iff} \quad \forall a \in B_L(a \in u \to f(a)(n) = m),\\
u \Vdash a(n) = m \quad &\text{iff} \quad an = m \ (a \text{ of sort } B_L),\\
u \Vdash A \wedge B \quad &\text{iff} \quad u \Vdash A \wedge u \Vdash B,\\
u \Vdash A \vee B \quad &\text{iff} \quad \exists a \in K\, \forall v\, (av \neq 0 \to (u * v \Vdash A \vee u * v \Vdash B)),\\
u \Vdash A \to B \quad &\text{iff} \quad \forall v\, (u * v \Vdash A \to u * v \Vdash B),\\
u \Vdash \forall p\, A(p) \quad &\text{iff} \quad \forall p\, u \Vdash A(p),\\
u \Vdash \exists_p A(p) \quad &\text{iff} \quad \exists a \in K\, \forall v\, (av \neq 0 \to \exists p\, u * v \Vdash A(p)),\\
u \Vdash \forall\varepsilon\, A(\varepsilon) \quad &\text{iff} \quad \forall f \in \underset{\sim}{M}\, u \Vdash A(f),\\
u \Vdash \exists\varepsilon\, A(\varepsilon) \quad &\text{iff} \quad \exists a \in K\, \forall v\, (av \neq 0 \to \exists f \in \underset{\sim}{M}\, u * v \Vdash A(f)).
\end{aligned}
$$

Inspection of these clauses shows that they are exactly the clauses defining 'formal' Beth-forcing for the minimal language, formulated in the language of lawlike IDB, where N, B_L and K are interpreted by themselves and B_C is interpreted as (the subsheaf of internal Baire space generated by) $\underset{\sim}{M}$. The word 'formal' in this context refers to the fact that the clauses for $\vee$ and $\exists$ are formulated in terms of existential quantification over K. The clauses are as for forcing in sheaves over Baire space, but we do not mention points. We just talk about finite sequences, and bars defined via K. In the absence of external bar induction this is a sensible adaption: instead of BI we can now use induction over unsecured sequences. The distinction between lawlike IDB and IDB itself is just a matter of notation. Hence the elimination theorem for LS^1 (properties (1) and (2) above) yields the following theorem.

5.3.1. Theorem. *Let $A(\varepsilon_1, \ldots, \varepsilon_n)$ be a formula in the CS-language. Then $\mathcal{U}_\alpha^M \models A(f_1(\alpha), \ldots, f_n(\alpha))$, i.e. $LS^1 \vdash \forall\alpha\, A^\alpha(f_1, \ldots, f_n)$, iff it is provable in IDB that $A(f_1, \ldots, f_n)$ holds in sheaves over formal Baire space, where N is interpreted as $\mathrm{cts}(B, \mathbb{N})$, B_L as the sheaf generated by the constant functions $B \to B$. K by the sheaf generated by the constant functions $B \to \mathbb{K}$, and B_C by the subsheaf of $\mathrm{cts}(B, B)$ generated by M.* □

A simple application of this result is the following. Let $\underset{\sim}{M}$ be the set {id}. Let $\forall\alpha\, A(\alpha)$ be an LS-sentence without other lawless quantifiers. Then $A(\alpha)$ and $A^\alpha(\mathrm{id})$ are equivalent in LS^1. So $LS^1 \vdash \forall\alpha\, A(\alpha)$ iff $A(\alpha)$ holds in sheaves over formal Baire space, where α is interpreted as the generic element id. In this sense lawless sequences are generic.

Another application is the one mentioned in the appendix to [1]: In [17] it is shown that for $\underset{\sim}{M} = \{f_n : \alpha \mapsto n * (\alpha)_n \mid n \in \mathbb{N}\}$, $\mathcal{U}_\alpha^M$ is an LS-model, provably in

LS^1. Hence the sheaf generated by $\underset{\sim}{M}$ is an LS-model over Baire space, provably in IDB. One easily verifies that there is a homeomorphism $h: B \to T'$, where T' is the product of all basic opens of Baire space (without repetitions), and that h can be chosen in such a way that $f_n \circ h^{-1} = \pi_n$. This is obviously the origin of the LS-model in 5.2 above.

Acknowledgements

Mike Fourman's talk at the Brouwer Symposium (June 1981) inspired our investigations into the subject. We would like to thank Robin Grayson for his helpful correspondence. Fourman, Grayson, and the authors independently and almost simultaneously (October 1981) observed that sheaves over the monoid of continuous functions from Baire space to itself with the open cover topology give a model for CS. Grayson also observed (independently from us) the connection with the elimination translation (cf. [3], [6]).

Additional notes (added September 1983)

(1) We would like to thank the referee, who spotted a large number of misprints in the original typescript.

(2) In Section 5.1 we discussed the model $\mathrm{Sh}(\mathbb{M}_h)$ for a theory of lawless sequences, which was the topic of M. Fourman's talk at the Brouwer conference. After this paper had been submitted, we received a copy of Fourman's article (Notions of choice sequence, in: A.S. Troelstra and D. van Dalen, eds., The L.E.J. Brouwer Centenary Symposium, North-Holland, Amsterdam, 1982), which is rather more general in content and, among other things, puts some of the results of this paper in a wider setting.

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On an independence result in the theory of lawless sequences

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Communicated by Prof. A.S. Troelstra at the meeting of October 25, 1982

The open data axiom LS3 for lawless sequences is actually an infinite list of axiom schemata: for each n we have

$$\mathrm{LS3}(n){:}\quad A(\alpha_1,\ldots,\alpha_n)\wedge\bigwedge_{i<j\le n}\alpha_i\ne\alpha_j\to\exists u_1\ni\alpha_1\ \ldots\ \exists u_n\ni\alpha_n$$
$$\forall\beta_1\in u_1\ \ldots\ \forall\beta_n\in u_n(\bigwedge_{i<j\le n}\beta_i\ne\beta_j\to A(\beta_1,\ldots,\beta_n));$$

here α_i, β_i range over lawless sequences, and the u_i range over finite sequences; '$\alpha\in u$' stands for 'α has initial segment u'. In [D] it was shown that LS3(1) does not imply LS3(2) by using Cohen generic sequences. In [DL], this method was used to show that LS3(2) does not imply LS3(3).

The aim of this note is to give simple proofs of these facts, by using the models described in [HM]. Our method also shows that LS3(3) does not imply LS3(4), but we have not been able to prove a similar independence result for larger n. For $n\ge 4$ a different approach seems necessary for showing LS3(n)$\not\to$LS3($n+1$).

We observe here that the models described below all satisfy the axioms LS1 (decidable equality) and LS2 (density), and that the models which show that LS3(n)$\not\to$LS3($n+1$), $n=1,2,3$, also yield the corresponding result LS4(n)$\not\to$LS4($n+1$) for the continuity axiom LS4. Thus we obtain

THEOREM. *For $n=1,2,3$, there exists a model satisfying* LS1, LS2, LS3(n), LS4(n), *but neither satisfying* LS3($n+1$), *nor* LS4($n+1$).

This note is far from selfcontained. All unexplained notation is as in [HM], and we assume familiarity with the LS-model described in section 5.2. of [HM]. As in [DL], it will be notationally more convenient to consider only $0-1$-sequences.

THE FIRST MODEL

Let M be the monoid of finite sequences of 0's and 1's, with the operation given by

$$u/v = v \text{ with the initial segment replaced by } u,$$

that is,

$$(u/v)n = \begin{cases} u(n), \text{ if } n < \text{lth}\,(u) \\ v(n), \text{ if lth}\,(u) \leq n < \text{lth}\,(v) \end{cases}$$

M may be regarded as a submonoid of the monoid $Cts(C,C)$ of continuous functions from Cantor Space to itself, by identifying u with the function $x \to u/x$ in $Cts(C,C)$. In sheaves over this monoid M equipped with the open cover topology, the internal exponent $2^{\mathbb{N}}$ appears as $Cts(C,C)$ (with restrictions given by right composition), and it was shown in [HM] (section 2.3) that if we interpret the domain of lawless sequences as the subsheaf of $Cts(C,C)$ generated by M, open data and continuity in a single parameter (LS3(1), LS4(1)) hold.

On the other hand, open data and continuity in two lawless parameters cannot hold, as follows easily from the observation that in this model

$$\Vdash \forall \alpha, \beta \exists n \forall m \geq n \alpha(m) = \beta(m).$$

To see this, choose two elements u and v of M, and let n be the maximum of lth (u) and lth (v). Then if $w \in M$ and $m \geq n$, we find for any $x \in C$

$$u/w/x(m) = v/w/x(m) = \begin{cases} w(m), \ m < \text{lth}\,(w) \\ x(m), \text{ otherwise.} \end{cases}$$

So $\Vdash u/w(m) = v/w(m)$. Thus $\Vdash \forall m \geq n\ u(m) = v(m)$. This proves

PROPOSITION 1. *There is a model for lawless sequences satisfying* (*decidable equality, density, and*) LS3(1), LS4(1), *but not* LS3(2), LS4(2).

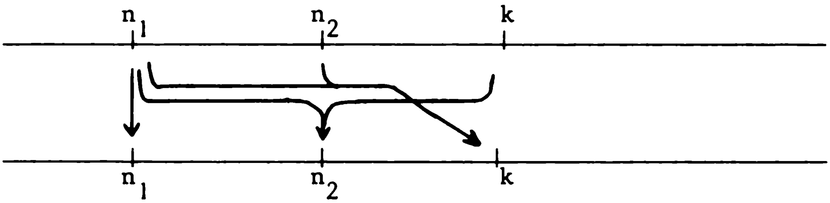
THE SECOND MODEL

We will now describe a model for LS3(2), LS4(2), in which LS3(3), LS4(3) do not hold. As in [HM] (section 5.2), we define a space T and a group G of automorphisms of T. Let $(v_n)_n$ be an enumeration of $2^{<\mathbb{N}}$ in which each sequence occurs infinitely many times. For $u \in 2^{<\mathbb{N}}$, let V_u denote the canonical basic open subset of C (Cantor space) determined by u. Now let $T = \Pi_n V_{v_n}$. (Observe that T is homeomorphic to C.)

We consider three types of homeomorphisms from T to itself. The first two types (1) and (2) are defined as in [HM], section 5.2 (but with Baire space replaced by Cantor space). In addition, we have a third type

(3) for each triple n_1, n_2, k of distinct natural numbers with $\upsilon_{n_1} + \upsilon_{n_2} = \upsilon_k$ (+ denotes pointwise addition modulo 2), a homeomorphism $h[n_1, n_2, k]$ defined by

$$h(x)_k = x_{n_1} + x_{n_2}$$
$$h(x)_{n_2} = x_k + x_{n_1}$$
$h(x)_m = x_m$ for all other m.



Thus, we have

$$\pi_k \circ h = \pi_{n_1} + \pi_{n_2}, \pi_{n_2} \circ h = \pi_{n_1} + \pi_k, \text{ and } \pi_m \circ h = \pi_m \text{ for } m \in \mathbb{N} \setminus \{k, n_2\}.$$

Now let G be the subgroup of the group of automorphisms of T generated by all homeomorphisms of types (1), (2), (3).

Our interpretation will be the standard interpretation in sheaves over T with a G-action (as in [HM], section 5.2.). The sort L of lawless $0-1$-sequences will be interpreted by the subsheaf (of internal Cantor space) generated by all functions $T \to C$ which are of the form $\pi_{n_1} + \ldots + \pi_{n_p}$, for a set of p distinct natural numbers $\{n_1, \ldots, n_p\}$.

Note that in this model the axioms of decidable equality and density for L are satisfied. We will now show that LS3(2) is also satisfied in this model. For this, we need two lemmas.

LEMMA 2. *If $\{n_1, \ldots, n_p\}$ and $\{m_1, \ldots, m_q\}$ are two distinct sets of natural numbers, listed without repetitions, then there are numbers i and j, $i \neq j$, and a homeomorphism $h \in G$ such that*

$$\pi_i \circ h = \pi_{n_1} + \ldots + \pi_{n_p}, \ \pi_j \circ h = \pi_{m_1} + \ldots + \pi_{m_q}.$$

PROOF. If $p = 1$, we can find a composition h of homeomorphisms of type (3) which leave π_{n_1} unchanged and add $\pi_{m_1}, \ldots, \pi_{m_q}$; i.e. for some $k \in \mathbb{N}$,

$$\pi_{n_1} \circ h = \pi_{n_1}, \ \pi_k \circ h = \pi_{m_1} + \ldots + \pi_{m_q}$$
(and $\pi_l \circ h = \pi_l$, all $l \in \mathbb{N} \setminus \{k, m_1, \ldots, m_q\}$).

If $p \neq 1$, first find an h which reduces $\pi_{m_1} + \ldots + \pi_{m_q}$ to a single projection, i.e.

$$\pi_k \circ h = \pi_{m_1} + \ldots + \pi_{m_q}.$$

Then apply the case $p = 1$ to the pair of (distinct!) sets $\{k\}$, $\{l_1, \ldots, l_r\}$, where $l_1, \ldots, l_r$ are such that $(\pi_{l_1} + \ldots + \pi_{l_r}) \circ h = \pi_{n_1} + \ldots + \pi_{n_p}$. □

LEMMA 3. *Let* $A(\alpha_1,\dots,\alpha_p)$ *be a formula with variables* $\alpha_1,\dots,\alpha_p$ *of sort* L, *and all other variables lawlike. Let* $U=\pi_0^{-1}(V_{u_0})\cap\dots\cap\pi_k^{-1}(V_{u_k})$ *be a basic open in* T *with* $U\subseteq[A(\pi_{n_1},\dots,\pi_{n_p})]$, *where* $n_1,\dots,n_p$ *are* p *mutually distinct natural numbers, and* $k\geq n_p$. *Then for each* p*-tuple mutually distinct numbers* $m_1,\dots,m_p$,

$$\pi_{m_1}^{-1}(V_{k_{n_1}})\cap\dots\cap\pi_{m_p}^{-1}(V_{k_{n_p}})\subseteq[A(\pi_{m_1},\dots,\pi_{m_p})].$$

PROOF. As lemma 5.2.4. in [HM]. □

COROLLARY 4. *In the model described above,* LS3(2) *holds, but* LS3(3) *does not hold.*

PROOF. It is clear that LS3(3) cannot hold, since we can find three lawless sequences α, β, γ in the model such that $\Vdash\alpha+\beta=\gamma$. LS3(2) does hold, i.e.

$$\Vdash\forall\alpha_1,\alpha_2(\alpha_1\neq\alpha_2\wedge A(\alpha_1,\alpha_2)\rightarrow\exists u_1\ni\alpha_1\exists u_2\ni\alpha_2$$
$$\forall\beta_1\in u_1\forall\beta_2\in u_2(\beta_1\neq\beta_2\rightarrow A(\beta_1,\beta_2))).$$

To see this, choose two distinct sections α_1, α_2 of the sheaf L. We may assume that α_1, α_2 are global sections of the form $\alpha_1=\pi_i$, $\alpha_2=\pi_j$, $i\neq j$, since such sections generate (by lemma 2). Now suppose $x\in[A(\pi_i,\pi_j)]$. By lemma 3 we can find finite sequences $u_1\ni x_i$, $u_2\ni x_j$ such that

(1) $$\pi_i^{-1}(V_{u_1})\cap\pi_j^{-1}(V_{u_2})\subseteq[A(\pi_i,\pi_j)],$$

i.e.

$$[\pi_i\in u_1\wedge\pi_j\in u_2\rightarrow A(\pi_i,\pi_j)]=T.$$

But then it holds that

$$[\forall\beta_1\in u_1\forall\beta_2\in u_2(\beta_1\neq\beta_2\rightarrow A(\beta_1,\beta_2))]=T.$$

For if $\{n_1,\dots,n_p\}$ and $\{m_1,\dots,m_q\}$ are distinct, we can find i',j' ($i'\neq j'$) and an $h\in G$ such that

(2) $$\pi_{i'}\circ h=\pi_{n_1}+\dots+\pi_{n_p},\ \pi_{j'}\circ h=\pi_{m_1}+\dots+\pi_{m_q}$$

(lemma 2), and by lemma 3, (1) implies that

(3) $$[\pi_{i'}\in u_1\wedge\pi_{j'}\in u_2\rightarrow A(\pi_{i'},\pi_{j'})]=T.$$

Hence also

$$[\pi_{n_1}+\dots+\pi_{n_p}\in u_1\wedge\pi_{m_1}+\dots+\pi_{m_q}\in u_2\rightarrow A(\pi_{n_1}+\dots$$
$$\dots+\pi_{n_p},\pi_{m_1}+\dots+\pi_{m_q})]$$
$$=[\pi_{i'}\circ h\in u_1(=u_1\circ h)\wedge\pi_{j'}\circ h\in u_2\rightarrow A(\pi_{i'}\circ h,\pi_{j'}\circ h)]$$
$$=h^{-1}[\pi_{i'}\in u_1\wedge\pi_{j'}\in u_2\rightarrow A(\pi_{i'},\pi_{j'})]=T.\quad\square$$

Summarizing, we have first shown that 'singleton projections' generate (lemma

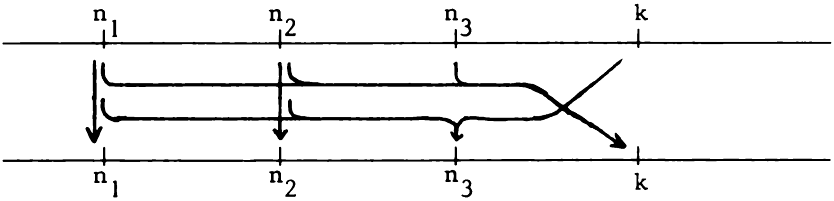
2). This enabled us to prove LS3(2) just as the full LS3 is proved in [HM] (lemma 3, corollary 4). In a similar way, we can show that LS4(2) holds in this model. LS4(3), however, cannot hold, since for three lawless sequences α_1, α_2, α_3, it cannot be decided on the basis of initial segments whether $\alpha_3 = \alpha_1 + \alpha_2$, or not. This shows that we have obtained the following

PROPOSITION 5. *There is a model for lawless sequences in which* LS3(2) *and* LS4(2) *hold, but* LS3(3) *and* LS4(3) *do not hold.*

THE THIRD MODEL

A slight modification of the model just described suffices to obtain a model for LS3(3), LS4(3) which is not a model for LS3(4), LS4(4). The space T remains the same, but the definition of the group G is different. Besides the homeomorphisms of types (1) and (2) from [HM], we now take as a third type all homeomorphisms of the form $h[n_1, n_2, n_3, k]$, where n_1, n_2, n_3, k are distinct natural numbers such that $\upsilon_k = \upsilon_{n_1} + \upsilon_{n_2} + \upsilon_{n_3}$. $h = h[n_1, n_2, n_3, k]$ is defined by

$$h(x)_k = x_{n_1} + x_{n_2} + x_{n_3}$$
$$h(x)_{n_3} = x_{n_1} + x_{n_2} + x_k$$
$h(x)_m = x_m$ for all other m.



The interpretation is again the standard interpretation in sheaves over T with G-action, but now L is the sheaf generated by the global elements $\pi_{n_1} + \ldots + \pi_{n_p}$, for $\{n_1, \ldots, n_p\}$, a set of p distinct natural numbers, *and p is odd*. Observe that this sheaf is closed under the action of G (i.e. right composition with elements of G preserves 'oddness').

In this model, LS3(4) does *not* hold, since we can find four distinct lawless sequences α, β, γ, δ such that $\Vdash \alpha + \beta + \gamma = \delta$. Similarly, LS4(4) does not hold, since we cannot continuously decide whether $\alpha + \beta + \gamma = \delta$ or not. LS3(3) and LS4(3), however, do hold. This is proved as for LS3(2) and LS4(2) in the second model described above, but one has to be slightly more careful now in showing that 'singleton-projections' generate for triples, i.e.

LEMMA 6. *Let $\{n_1, \ldots, n_p\}$, $\{m_1, \ldots, m_q\}$, $\{l_1, \ldots, l_r\}$ be distinct sets of natural numbers, listed without repetitions, and with p, q, r odd. Then there exists a homeomorphism $h \in G$ and (distinct) coordinates i, j, k such that*

$$\pi_i \circ h = \pi_{n_1} + \ldots + \pi_{n_p},\ \pi_j \circ h = \pi_{m_1} + \ldots + \pi_{m_q},\ \pi_k \circ h = \pi_{l_1} + \ldots + \pi_{l_r}.$$

SKETCH OF PROOF. As in lemma 2, the general case is easily reduced to the case $p=1$. Thus, we have three distinct sets

$$\{n\},\ \{m_1,\ldots,m_q\},\ \{l_1,\ldots,l_r\}.$$

And again, not bothering about the coordinates $l_1,\ldots,l_r$, but keeping n invariant, we may as well assume that $q=1$; i.e. we find a composition h of homeomorphisms of type (3) such that

$\pi_n \circ h = \pi_n$
$\pi_m \circ h = \pi_{m_1} + \ldots + \pi_{m_q}$ for some m,
while $\pi_{l_1} + \ldots + \pi_{l_r} = (\pi_{s_1} + \ldots + \pi_{s_{r'}}) \circ h$,
for some coordinates $s_1,\ldots,s_{r'}$, where r' is still odd.

Thus, we have three distinct sets of the form

$$\{n\},\ \{m\},\ \{s_1,\ldots,s_{r'}\}.$$

If $r'=1$, we are done. Otherwise, $r' \geq 3$, so the third set contains an element which is distinct both from m and from n. But in this case it is not difficult to see that we can find a composition h' of homeomorphisms of type (3) which reduces $\pi_{s_1} + \ldots + \pi_{s_{r'}}$ to a single projection, but leaves the coordinates n and m invariant. □

We have now obtained the following proposition:

PROPOSITION 7. *There is a model for lawless sequences in which* LS3(3) *and* LS4(3) *hold, but* LS3(4) *and* LS4(4) *do not hold.*

THE PROBLEM WITH MORE PARAMETERS

It is perhaps useful to indicate why this approach does not work in the case of more parameters. To obtain a model for LS3(4) $\not\rightarrow$ LS3(5) in a similar way, one is inclined to put sums of four lawless sequences in the sheaf L (to falsify LS3(5)), and to add homeomorphisms to G which reduce such sums to single projections. However, if one puts such sums in L, one has to do so homogeneously (in order to obtain open data in four parameters $\alpha_1,\ldots,\alpha_4$ for the formula $\exists\delta(\alpha_1+\alpha_2+\alpha_3+\alpha_4=\delta)$). But then one finds four-tuples of lawless sequences projected from the sets $\{n_1,n_2,n_3,n_4\}$, $\{n_3,n_4,n_5,n_6\}$, $\{n_5,n_6,n_7,n_8\}$, $\{n_1,n_2,n_7,n_8\}$, for example, i.e.

$$\exists\alpha_1\exists\alpha_2\exists\alpha_3\exists\alpha_4(\textstyle\bigwedge_{1\leq i<j\leq 4}\alpha_i \neq \alpha_j \wedge \alpha_1+\alpha_2+\alpha_3+\alpha_4=0)$$

will hold in the model. This clearly contradicts LS3(4).

The reason why the proof of lemma 6 above fails in this case lies in the fact that the analog of 'preservation of oddness' for the elements of G does not hold: if one adds sums of four tuples, one may find a sum $\pi_{n_1}+\pi_{n_2}+\pi_{n_3}+\pi_{n_4}$, where $\pi_{n_4}\circ h$ is the sum of, say, $\pi_{n_1}, \pi_{n_2}, \pi_{n_5}, \pi_{n_6}$, for some $h \in G$, which leaves the coordinates n_1, n_2, n_3 unchanged. Hence one has also added sums of three coordinates, since $(\pi_{n_1}+\pi_{n_2}+\pi_{n_3}+\pi_{n_4})\circ h = \pi_{n_1}+\pi_{n_2}+\pi_{n_3}+\pi_{n_1}+\pi_{n_2}+\pi_{n_5}+$

$+ \pi_{n_6} = \pi_{n_3} + \pi_{n_5} + \pi_{n_6}$. Therefore, in trying to prove the analog of lemma 6 one may, after having reduced three out of the four sets to singletons, end up with four sets which look like

$$\{n\},\ \{m\},\ \{k\},\ \{n,m,k\}.$$

In fact, such a situation *must* occur if one starts with the four sets $\{n_1, n_2, n_3, n_4\}$, $\{n_3, n_4, n_5, n_6\}$, $\{n_5, n_6, n_7, n_8\}$, $\{n_1, n_2, n_7, n_8\}$ considered above, since the relation $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$ must be preserved by the action of G on the elements of the sheaf L.

This suggests that for the case of more parameters, a totally different approach is needed.

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THE JOURNAL OF SYMBOLIC LOGIC
Volume 49, Number 3, Sept. 1984

ON CHOICE SEQUENCES DETERMINED BY SPREADS

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§1. Introduction. From the moment choice sequences appear in Brouwer's writings, they do so as elements of a spread. This led Kreisel to take the so-called axiom of spreaddata as the basic axiom in a formal theory of choice sequences (Kreisel [1965, pp. 133–136]). This axiom expresses the idea that to be given a choice sequence means to be given a spread to which the choice sequence belongs. Subsequently, however, it was discovered that there is a formal clash between this axiom and closure of the domain of choice sequences under arbitrary (lawlike) continuous operations (Troelstra [1968]). For this reason, the formal system CS was introduced (Kreisel and Troelstra [1970]), in which spreaddata is replaced by analytic data. In this system CS, the domain of choice sequences is closed under all continuous operations, and therefore it provides a workable basis for intuitionistic analysis. But the problem whether the axiom of spreaddata is compatible with closure of the domain of choice sequences under the continuous operations from a restricted class, which is still rich enough to validate the typical axioms of continuous choice, remained open. It is precisely this problem that we aim to discuss in this paper.

Recall that a *spread* is a (lawlike, inhabited) decidable subtree S of the tree $\mathbf{N}^{<\mathbf{N}}$ of all finite sequences, having all branches infinite:

(i) $$\forall u, v(u \in S \,\&\, v \leq u \rightarrow v \in S),$$

(ii) $$\forall u \exists n(u \in S \rightarrow u * \langle n \rangle \in S).$$

(Unless otherwise stated, all notational conventions are as in van der Hoeven and Moerdijk [1981], henceforth [HM]; so u, v range over finite sequences, n, m over natural numbers, α, β, ξ, η over elements of the domain of choice sequences, and $*$ is used for concatenation.) A spread S determines a subset of $\mathbf{N}^{\mathbf{N}}$, also called S, by

$$\alpha \in S \Leftrightarrow \forall n\, \bar{\alpha}(n) \in S.$$

Kreisel's axiom of *spreaddata* now reads

$$A(\alpha) \rightarrow \exists \text{ spread } S\, (\alpha \in S \,\&\, \forall \beta \in S\, A(\beta)),$$

where $A(\alpha)$ contains no free variables for choice sequences other than α; all other

Received February 21, 1983.

[1] Research supported by the Netherlands Organization for the Advancement of Pure Research (ZWO).

0022-4812/84/4903-0018/$01.90

parameters should be lawlike. Every spread contains sequences, i.e. we have the *density axiom*

$$\forall \text{ spread } S \, \exists \alpha \, \alpha \in S.$$

Other typical axioms are continuity principles for quantifier combinations of the form $\forall\alpha\exists n$ and $\forall\alpha\exists\beta$. (In the system CS the axiom of spreaddata is replaced by the axiom

$$A(\alpha) \to \exists \text{ lawlike continuous } f \, (\alpha \in \text{range}\,(f) \,\&\, \forall\beta \in \text{range}\,(f)\, A(\beta))$$

of *analytic data*. In other words, spreads are replaced by images of lawlike continuous functions. In CS, the density axiom is redundant, since the universe of choice sequences is closed under application of an arbitrary continuous operation.)

In this paper we will present a model for a theory of choice sequences containing the axiom of spreaddata. This model has all the desired properties: besides spreaddata and the density axiom, it satisfies $\forall\alpha\exists n$-continuity, $\forall\alpha\exists\beta$-continuity, bar induction, and the specialization property. Furthermore, the domain of choice sequences is closed under application of all lawlike continuous operations from a certain subclass $\mathbf{S} \subseteq \boldsymbol{K}$. Every spread is the image of a function in $\mathbf{S}$:

$$\forall \text{ spread } S \, \exists f \in \mathbf{S} \, \text{im}(f) = S.$$

The density axiom is an immediate consequence of this, and we also get relativized continuity principles for quantifier combinations of the form $\forall\alpha \in S\,\exists n$, $\forall\alpha \in S\,\exists\beta$, and relativized bar induction. Finally, an axiom of pairing holds in the model. The model will be similar to the models presented in [HM], and we will assume that the reader has some familiarity with the techniques used in §2 of that paper. As shown in [HM], the elimination translation for CS is a special case of such a model. The model we present here, however, does not lead to a similar elimination translation based on spreaddata rather than analytic data. This will be pointed out in a final section, where we will also briefly discuss the relation of this model to other models for spreaddata that have occurred in the literature.

Acknowledgements. We would like to thank G. Kreisel for drawing our attention to the problems discussed in this paper, and A. S. Troelstra, whose encyclopaedic knowledge of the subject was of great help to us.

§2. Description of the model. Our model will be similar to the sheafmodels for the systems CS(M) of [HM]. These systems contain an axiom of "relativized analytic data",

$$A(\alpha) \to \exists f \in M(\alpha \in \text{range}\,(f) \,\&\, \forall\beta \in \text{range}\,(f)\, A(\beta)),$$

where M is a fixed monoid of lawlike continuous operations. In these models, "lawlike" is interpreted as "external" or "constant" (that is, as lying in the image of the "constant sets functor" $\Delta\colon \text{Sets} \to \text{Sh}(\mathbf{C})$, $\mathbf{C}$ a site, Δ left adjoint to the global sections functor).

Assuming this interpretation of lawlike objects, a lawlike spread is just a spread given in Sets, and when working in a classical metatheory, in the models all lawlike spreads will automatically be decidable. In Sets, spreads correspond to closed nonempty subspaces of Bairespace: every spread $S \subseteq \mathbf{N}^{<\mathbf{N}}$ determines a closed subspace $\{x \in \mathbf{N}^{\mathbf{N}} \mid \forall n\, \bar{x}(n) \in S\}$ of Bairespace, and conversely, to each closed set $T \subseteq \mathbf{N}^{\mathbf{N}}$ we can assign a spread $\{\bar{x}(n) \mid n \in \mathbf{N}, x \in T\}$. These processes are inverse to each other.

We will begin the construction of our model by describing a class of mappings from Bairespace to itself which map spreads to spreads (i.e. are closed mappings) by *retracting* every spread onto its image:

DEFINITION 1. A closed continuous function $f: B \to B$ is called a *CHR-mapping* (closed-hereditary retraction mapping) if for any closed subset $F \subseteq B$, the restriction $f \restriction F: F \twoheadrightarrow f(F)$ has a continuous right inverse $i_F: f(F) \to F$; that is, $f \circ i_F = \mathrm{id}_{f(F)}$.

Note that if $f: B \to B$ is CHR, each inverse $i_F: f(F) \to F$ is also a closed mapping. For if G is a closed subset of $f(F)$ and $\{x_n\}_n$ is a sequence of points from G such that $\{i_F(x_n)\}_n$ converges to p, then $\{fi_F(x_n)\}_n = \{x_n\}_n$ converges to $f(p)$, so $f(p) \in G$, and $i_F f(p) = \lim_n i_F f i_F(x_n) = \lim_n i_F(x_n) = p$; hence also $p \in i_F(G)$.

Examples of CHR-mappings are closed homeomorphic embeddings and constant functions. In fact, the CHR-mappings form a monoid:

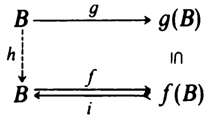
LEMMA 2. *The composition of two CHR-mappings is again a CHR-mapping.*

PROOF. If f and g are CHR and $F \subseteq B$ is a closed set, then we find right inverses $i_F: f(F) \to F$ and $j_{f(F)}: gf(F) \to f(F)$ for f and g respectively, so $i_F \circ j_{f(F)}$ is a right inverse for $g \circ f$. □

The key property of CHR-mappings is expressed by the following lemma.

LEMMA 3 (FACTORIZATION LEMMA). *Let f and $g: B \to B$ be CHR-mappings, and suppose $\mathrm{im}(g) \subseteq \mathrm{im}(f)$. Then there exists a CHR-mapping $h: B \to B$ such that $g = f \circ h$.*

PROOF. Let $i: f(B) \to B$ be a right inverse for $f: B \twoheadrightarrow f(B)$, and define h to be the function $i \circ g$.



Obviously $f \circ h = g$. To show that h is a CHR-mapping, choose a closed subset $H \subseteq B$. Since g is CHR, we can find a map $k: g(H) \to H$ such that the composite $g(H) \xrightarrow{k} H \xrightarrow{g \restriction H} g(H)$ is the identity map. Now let $j: h(H) = ig(H) \to H$ be the composite $k \circ f: ig(H) \to fig(H) = g(H) \xrightarrow{k} H$. Then $h \circ j = \mathrm{id}_{h(H)}$, for if $x \in h(H)$, then $x = i(y)$ for some $y \in g(H) \subseteq f(B)$, so $hj(x) = igkf(x) = igkfi(y) = igk(y) = i(y) = x$. Thus h is a CHR-mapping. □

In order to get a model which has the properties as described in the Introduction, we need a sufficient supply of CHR-mappings. For each spread S we will define a CHR-mapping $\tilde{S}$ which retracts B onto S.

The points of B carry a natural linear ordering given by $x < y$ iff $x(n) < y(n)$ for the smallest n at which x and y differ. If $x < y$ we will say that x is to the left of y.

Let S be a closed subspace of B. As noted earlier, S can also be regarded as a set of finite sequences $\{u \mid \exists x \in S\; x \in u\}$. We define the function $\tilde{S}$ as follows (for each $x \in B$ we define initial segments $\overline{\tilde{S}(x)}(n)$ of length n by induction). $\overline{\tilde{S}(x)}(0) = \langle\ \rangle$ of course, and

$$\overline{\tilde{S}(x)}(n+1) = \begin{cases} \bar{x}(n+1) & \text{if } \bar{x}(n+1) \in S, \\ \overline{\tilde{S}(x)}(n) * \langle m \rangle & \text{otherwise, where } m \text{ is the least number} \\ & \text{for which } \overline{\tilde{S}(x)}(n) * \langle m \rangle \in S. \end{cases}$$

Thus, when we think in terms of the tree $\mathbf{N}^{<\mathbf{N}}$, $\tilde{S}(x)$ is that path in S which is equal to x as long as this is possible, and then picks out the leftmost branch in S. (Later on, $\tilde{S}$ will also give an internal function from choice sequences to choice sequences, and in the model it will hold that $\alpha \in S \leftrightarrow \exists \beta\, \alpha = \tilde{S}(\beta)$.)

LEMMA 4. *For each closed $S \subseteq B$, $\tilde{S}$ is a uniformly continuous closed retraction of B onto S.*

PROOF. Uniform continuity of $\tilde{S}$ is clear, since we need only the initial segment $\bar{x}(n)$ of x to define $\overline{\tilde{S}(x)}(n)$. And if $x \in S$, $\tilde{S}(x) = x$, so $\tilde{S}$ retracts B onto S.

To see that $\tilde{S}$ is closed, suppose $F \subseteq B$ is closed, and $\{y_n\}_n$ is a sequence of points in F such that $\{\tilde{S}(y_n)\}_n$ converges to a point p. We need to show that $p \in \tilde{S}(F)$. Since $\tilde{S}(y_n) \to p$,

$$(*) \qquad \forall k \exists n_k \forall n \geq n_k\, \overline{\tilde{S}(y_n)}(k) = \bar{p}(k).$$

We now distinguish two cases:

1) If $\forall k \exists m_k \forall m \geq m_k\, \bar{y}_k(k) \in S$, then the sequence $\{y_n\}_n$ also converges to p. So p must lie in the closed set F, and $\tilde{S}(p) = p$.

2) Otherwise there exists a k_0 such that the set $M = \{m \mid \bar{y}_m(k_0) \notin S\}$ is infinite. Since S is a tree, also for each $k \geq k_0$ and $m \in M$, $y_m(k) \notin S$. By $(*)$ we find for this k_0 that $\forall n \geq n_{k_0}\, \overline{\tilde{S}(x_n)}(k_0) = \bar{p}(k_0)$. But then for $m \in M$, $m \geq n_{k_0}$, and $k \geq k_0$, $\overline{\tilde{S}(x_m)}(k)$ is the leftmost extension of $\bar{p}(k_0)$ in S, and hence no longer depends on m. Thus the sequence $\{\tilde{S}(y_n)\}_n$ contains a constant subsequence, necessarily having value p. Therefore also in this case, $p \in \tilde{S}(F)$. □

PROPOSITION 5. *For every closed $S \subseteq B$, the function $\tilde{S}$ is a CHR-mapping.*

PROOF. Let S be a closed subset of B. We want to define a right inverse $i_F : \tilde{S}(F) \to F$ for each restriction $\tilde{S} \restriction F : F \to \tilde{S}(F)$ of $\tilde{S}$ to a closed set F.

If $x \in S$, we call x a leftmost point in u if u is an initial segment of x and for each $n > \text{lth}(u)$, $x(n)$ is the smallest m such that $\bar{x}(n) * \langle m \rangle \in S$ (in other words, x is the leftmost branch in the tree $\{v \mid u \leq v \ \&\ v \in S\}$). To define i_F we consider three types of points in $\tilde{S}(F)$.

(1) If $x \in \tilde{S}(F)$ and for none of its initial segments u, x is a leftmost point in u, then $\tilde{S}^{-1}(x)$ consists of precisely one point, viz. x itself, so $x \in F$, and putting $i_F(x) = x$ is the only thing we can do.

(2) If $x \in \tilde{S}(F)$ and x is a leftmost point in one of its initial segments u, while x is not isolated in $\tilde{S}(F)$, we also put $i_F(x) = x$. Indeed, $x \in F$ in this case, since from the fact that x is not isolated in $\tilde{S}(F)$ we conclude that there is a sequence $\{y_n\}_n$ of points in F such that $\{\tilde{S}(y_n)\}_n$ converges to x, while for no n do we have $\tilde{S}(y_n) = x$. In particular, no subsequence of $\{\tilde{S}(y_n)\}_n$ is constant. Therefore it follows as in the proof of Lemma 4 that the points y_n also converge to x. Each y_n is in the closed set F, hence so is x.

(3) The remaining case: $x \in \tilde{S}(F)$, x is the leftmost point in one of its initial segments u, and x is isolated in $\tilde{S}(F)$. Then we let $i_F(x)$ be the leftmost point in $\tilde{S}^{-1}(x) \cap F$.

Clearly, $\tilde{S} \circ i_F$ is the identity on $\tilde{S}(F)$. We claim that i_F is continuous at each point $x \in \tilde{S}(F)$. To see this, choose a sequence $\{x_n\}_n$ in $\tilde{S}(F)$ converging to x. We have to show that $i_F(x_n) \to i_F(x)$ also. If x is a point of type (3) this is trivial. If x is a point of type (2), make any choice of points $y_n \in F$ with $\tilde{S}(y_n) = x_n$. Then again as in the proof of Lemma 4 it follows that the sequence $\{y_n\}$ also converges to x (provided we assume that for all n, $x_n \neq x$, which we can do without loss). In particular, $i_F(x_n) \to x = i_F(x)$. Finally, suppose x is a point of type (1), i.e. x is never leftmost in S. Without loss we may assume that the points x_n are all of the same type. If each x_n is of type (1) or each x_n is of type (2), we have $i_F(x_n) = x_n$ and $i_F(x) = x$, so trivially $i_F(x_n) \to i_F(x)$. So suppose all x_n are of type (3). For each n there exists a shortest sequence v_n such that x_n is the leftmost branch in S running through v_n. Choose any points y_n with $\tilde{S}(y_n) = x_n$. Then v_n must also be an initial segment of y_n. If $\text{lth}(v_n)$ converges to infinity, i.e. $\forall k\, \exists n_k\, \forall n \geq n_k\, \text{lth}(v_n) \geq k$, then clearly $y_n \to x$. In particular $i_F(x_n) \to x$. Otherwise there exists a subsequence of $\{x_n\}_n$ on which $\text{lth}(v_n)$ is constant, hence a subsequence of $\{x_n\}_n$ is constant. But all x_n were assumed to be of type (3) while x is of type (1), so this is impossible. □

Let us write S for the monoid of CHR-mappings. S can be equipped with a *Grothendieck topology*, as follows: A sieve of functions $\mathscr{W} \subseteq$ S is defined to be a *cover* if for some open cover $\{V_{u_i} \mid i \in I\}$ of Bairespace (i.e. $\forall x \in B\, \exists i \in I\, u_i$ is an initial segment of x), every function $\tilde{V}_{u_i}$ is a member of $\mathscr{W}$. To show that this indeed defines a Grothendieck topology, we need to verify that

(i) (transitivity) if $\mathscr{W}$ is a cover, and $\mathscr{R}$ is a sieve such that for each $f \in \mathscr{W}, f^*(\mathscr{R}) = \{g \mid f \circ g \in \mathscr{R}\}$ covers, then $\mathscr{R}$ also covers; and

(ii) (stability) if $\mathscr{W}$ is a cover and $f \in$ S then $f^*(\mathscr{W}) = \{g \mid f \circ g \in \mathscr{W}\}$ is a cover.

The proof of (i) uses the cancellation property of the mappings of the form $\tilde{S}$: if S and T are spreads and $S \subseteq T$, then $\tilde{S} \circ \tilde{T} = \tilde{T} \circ \tilde{S} = \tilde{S}$. If $\mathscr{W}$ is a cover, there is an open cover $\{V_{u_i} \mid i \in I\}$ of B with each $\tilde{V}_{u_i} \in \mathscr{W}$. By assumption, for each fixed i, $\tilde{V}^*_{u_i}(\mathscr{R})$ covers, so there is a cover $\{V_{v_j} \mid j \in J\}$ of Bairespace such that for each j, $\tilde{V}_{u_i} \circ \tilde{V}_{v_j} \in \mathscr{R}$. If w is an extension of some v_j which also extends u_i, then by cancellation $\tilde{V}_w = \tilde{V}_{u_i} \circ \tilde{V}_{v_j} \circ \tilde{V}_w \in \mathscr{R}$, so there exists an open cover $\{V_w\}_w$ of V_{u_i} with each corresponding $\tilde{V}_w \in \mathscr{R}$. This holds for each $i \in I$, so $\mathscr{R}$ is a covering sieve. Thus (i) holds.

To show (ii), pick $f \in$ S and suppose $\{V_{u_i} \mid i \in I\}$ covers B and each $\tilde{V}_{u_i} \in \mathscr{W}$. By continuity of f, there exists an open cover $\{V_{v_j} \mid j \in J\}$ of B such that each $f(V_{v_j}) \subseteq$ some V_{u_i}. From the factorization lemma it then follows that each $\tilde{V}_{v_i}$ is in $f^*(\mathscr{W})$.

This Grothendieck topology makes S into a site (also denoted by S), and we can interpret the higher order logic in Sh(S) as in [HM, §2]. Thus the natural numbers appear in the model as the sheaf $N = \text{Cts}(B, \mathbb{N})$, and internal Bairespace N^N is the sheaf $\text{Cts}(B, B)$. In both cases restrictions are given by composition, $x \upharpoonright f = x \circ f$. As in [HM], the *lawlike sequences* are interpreted as the subsheaf $B_L \subseteq \text{Cts}(B, B)$ of locally constant functions, while the *choice sequences* are interpreted by the subsheaf B_C of $\text{Cts}(B, B)$ generated by the identity. Thus our internal choice sequences are precisely the external functions from Bairespace to itself which are *locally* elements of S.

Any external continuous function $F: B \to B$ reappears internally as a continuous operation on Bairespace N^N, by $F(f) = F \circ f$. All internal lawlike continuous functions are (locally) of this form. In particular, if $\alpha: B \to B$ is an element of B_C, α induces a lawlike function $N^N \to N^N$ in the model, and it follows easily from the stability property (ii) above that this lawlike function restricts to a map of B_C into itself. Let us write $\hat{S}$ for the subsheaf of lawlike continuous operations on N^N generated by the elements of B_C in this way. That is, if F is an internal function $N^N \to N^N$ induced by an external continuous $F: B \to B$, then $\Vdash F \in \hat{S}$ iff $F \in B_C$, i.e. F is locally in S. Then in the model it holds that B_C is closed under the lawlike operations from $\hat{S}$,

$$\Vdash \forall\alpha\, \forall f \in \hat{S}\, \exists\beta\, f(\alpha) = \beta$$

(α and β range over B_C). Note that the functions in $\hat{S}$ do not map spreads to spreads, but they do so locally, i.e.

$$\Vdash \forall f \in S\, \forall \text{ spread } S\, \exists e \in K\, \forall u(e(u) \neq 0$$
$$\to \exists \text{ spread } S' \forall\alpha(\alpha \in S' \leftrightarrow \exists\beta \in u(\beta \in S \,\&\, \alpha = f(\beta)))).$$

Every function $F \in S$ appears in particular as an internal operation on N^N which is in $\hat{S}$, and we will also write S for the subsheaf of $\hat{S}$ generated by these internal mappings coming from an $F \in S$; so $\Vdash S \subseteq \hat{S}$.

Let us consider the relevant properties of this model. Many of the arguments that follow are analogous to the arguments in [HM, §2], and will only be indicated briefly.

First of all, as noted the universe of choice sequences is closed under operations from S, and this gives a *pairing axiom*

$$\Vdash \forall\alpha, \beta\exists\gamma\exists f, g \in S(\alpha = f(\gamma) \,\&\, \beta = g(\gamma)).$$

(Proof: if $\alpha, \beta \in B_C \subset N^N$, they are restrictions of the identity γ (locally), say $\alpha = \gamma \restriction f$, $\beta = \gamma \restriction g$, so $\Vdash \alpha = f(\gamma)$, $\beta = g(\gamma)$.)

Since S contains all constant functions, every $\alpha \in B_C$ has a restriction which is lawlike, so the model satisfies the *specialization property*,

$$\Vdash A(\alpha) \to \exists \text{ lawlike } a\ A(a),$$

where all parameters other than α in $A(\alpha)$ are lawlike.

The *density axiom* $\forall$ spread $S\, \exists\alpha\, \alpha \in S$ holds in the model: again by using constant functions, or alternatively, by observing that $\Vdash \hat{S} \in S$. We will come back to this below, and formulate an axiom of *strong density*.

If $\alpha \in B_C$, α is externally given as (locally) an element $f \in S$, and every $f(\beta)$ is a restriction of α. Hence if $\Vdash A(\alpha)$ and all other parameters in A are lawlike, it follows that $\Vdash \forall\beta\, A(f(\beta))$. As in [HM] this yields *analytic data relativized* to S,

$$\Vdash A(\alpha) \to \exists f \in S(\forall\beta\, A(f(\beta)) \,\&\, \alpha \in \mathrm{im}(f)).$$

From this, we immediately obtain *spreaddata* by an application of the factorization lemma: if $\alpha \in B_C$, then, on a suitable cover, $\mathrm{im}(\alpha) = S$ is a closed subset of B, and every other $\beta \in B_C$ with $\mathrm{im}(\beta) \subseteq S$ can be written as $\alpha \circ \gamma$ for some γ, i.e. as a restriction of α. Thus

(*spreaddata*) $\Vdash A(\alpha) \to \exists$ spread $S(\alpha \in S \,\&\, \forall \beta \in S\, A(\beta))$.

Continuity principles follow as in [HM] by considering the generic element $\tilde{B} = \mathrm{id} \in B_C$. For example, for $\forall\alpha\exists\beta$-continuity suppose $\Vdash \forall\alpha\exists\beta\, A(\alpha, \beta)$ (all non-lawlike parameters in $A(\alpha, \beta)$ shown). Then in particular there is a $\beta \in B_C$ such that $\Vdash A(\mathrm{id}, \beta)$. β acts internally by composition as a lawlike operation F on B_C which is in $\hat{\mathrm{S}}$, and we obtain

$$\Vdash \forall\alpha\exists\,\beta\, A(\alpha, \beta) \to \exists F \in \hat{\mathrm{S}}\, \forall\alpha\, A(\alpha, F(\alpha)).$$

A similar argument gives $\forall\alpha\exists n$-continuity.

Since B_C contains all constant functions, Bar Induction holds in the form BI* (see [HM, §2]).

Note that in the axiom of spreaddata as just formulated, we cannot economize on spreads, i.e. there is no proper subclass P of the class of all spreads such that spreaddata holds with "$\exists$ spread S" replaced by "$\exists$ spread $S \in P$". This follows by taking $A(\alpha)$ to be $\alpha \in S$, and choosing $\alpha = \tilde{S}$.

This is how it should be, since given any spread S, there is no a priori reason why S cannot occur as "complete information at a certain stage", i.e. why we cannot construct a sequence α such that at a certain stage of its construction the only information we have about α is that $\alpha \in S$. One way of formalizing this as an axiom is to say that for any spread S there is a step in a construction process consisting of a single application of a lawlike continuous operation f (under which the universe of choice sequences is closed), such that after applying this step to the universal sequence α about which we have not yet gained any knowledge, we know that $\alpha \in S$ and nothing more. We call this axiom the axiom of *strong density*, since it is a strengthening of the ordinary density axiom ($\forall S \exists\alpha\, \alpha \in S$).

STRONG DENSITY AXIOM. $\forall$ spread $S\, \exists f \in \mathrm{S}\, \forall\alpha(\alpha \in S \leftrightarrow \exists\beta\, \alpha = f(\beta))$.

Observe that the strong density axiom is satisfied in our model, since

$$\Vdash \forall \text{ spread } S\, \mathrm{im}(\tilde{S}) = S.$$

Since the mappings $\tilde{S}$ are retractions, i.e. $\Vdash \tilde{S} \circ \tilde{S} = \tilde{S}$, we obtain relativized forms of continuity,

(relativized $\forall\alpha\exists\beta$-continuity) $\Vdash \forall\alpha \in S\, \exists\beta\, A(\alpha, \beta)$

$$\to \exists \text{ lawlike continuous } F: S \to B_C\, \forall\alpha \in S\, A(\alpha, F\alpha),$$

(relativized $\forall\alpha\exists n$-continuity) $\Vdash \forall\alpha \in S\, \exists n\, A(\alpha, n)$

$$\to \exists \text{ lawlike continuous } F: S \to N\, \forall\alpha \in S\, A(\alpha, F\alpha).$$

(By definition, a lawlike continuous operation $F: S \to B_C$ comes from a neighbourhood function $N^{<N} \xrightarrow{F} N^{<N}$ such that for all n, $\{u \mid \mathrm{length}\, F(u) \geq n\}$ is a(n inductive) bar for S. Similarly for functions $S \to N$. The relativized versions follow easily from the global ones together with the fact that

$$\Vdash \forall \text{ spread } S\, \exists f \in \mathrm{S}(\mathrm{im}(f) = S \,\&\, f \circ f = f).)$$

We also conclude that a relativized form of Bar Induction holds in the model: for any spread S,

(BI_S) $\Vdash \forall P \subseteq N^{<N}(P$ is a monotone inductive bar for $S \to \langle\,\rangle \in P)$.

PROOF. This follows from the global version BI* and strong density. Suppose P is monotone ($u \geq v \in P \to u \in P$), inductive ($\forall n(u * \langle n\rangle \in S \to u * \langle n\rangle \in P) \to u \in P$) and bars S ($\forall \alpha \in S\, \exists n\, \bar{\alpha}(n) \in P$). Let $f \in \mathrm{S}$ be such that $\mathrm{im}(f) = S$, let $P' = \{v \mid \forall \alpha \in v\, f(\alpha)(\mathrm{1th}(v)) \in P\}$, and apply BI* to P' to conclude that $\langle\,\rangle \in P$.

For the record, let us sum up the properties of the model. (The axiom of pairing as formulated below can actually be strengthened by replacing $\hat{\mathrm{S}}$ by S; countable choice is proved just as in [HM].)

THEOREM 6. *The interpretation in* Sh(S) *described above yields a model in which there is a monoid* $\hat{\mathrm{S}}$ *of internal lawlike continuous functions, satisfying the following axioms*:

countable choice:	$\forall n \exists m\, A(n,m) \to \exists f \in N^N \forall n\, A(n, fn)$;
pairing and closure:	$\forall \alpha, \beta \exists \gamma\, \exists f, g \in \hat{\mathrm{S}}\, \alpha = f(\gamma)\, \&\, \beta = g(\gamma)$,
	$\forall \alpha\, \forall f \in \hat{\mathrm{S}}\, \exists \beta\, f(\alpha) = \beta$;
specialization:	$\exists \alpha\, A(\alpha) \to \exists a\, A(a)$;
spreaddata:	$A(\alpha) \to \exists$ spread $S(\alpha \in S\, \&\, \forall \beta \in S\, A(\beta))$;
strong density:	$\forall$ spread $S\, \exists f \in \hat{\mathrm{S}}\, \mathrm{im}(f) = S$;
Bar Induction:	$\forall$ spread $S\, \forall P \subseteq N^{<N}$ (P is a monotone inductive bar for $S \to \langle\,\rangle \in P$);
$\forall\alpha\exists n$-*continuity*:	$\forall \alpha \in S\, \exists n\, A(a, n) \to \exists$ lawlike continuous $F: S \to N\, \forall \alpha \in S\, A(\alpha, F\alpha)$;
$\forall\alpha\exists\beta$-*continuity*:	$\forall \alpha \in S\, \exists \beta\, A(\alpha, \beta) \to \exists F \in \hat{\mathrm{S}}\, \forall \alpha \in S\, A(\alpha, F\alpha)$.

(Except for countable choice, all nonlawlike variables are shown in notation.) □

§3. Concluding remarks. One of the first models for spreaddata seems to be the projection model in van Dalen and Troelstra [1970] (see also Troelstra [1970]). Essentially the same model can be obtained as an analog of the LS-model presented in §5.2 of [HM], which one obtains by replacing Bairespace by the space of decreasing sequences of spreads $(S_n)_n$ such that $\bigcap_n S_n$ consists of a single point, with the product topology (finite initial segments topology). Our LS-model from [HM] is essentially equivalent to the LS-model presented in Fourman [1982, §2.2]. If in Fourman's model one replaces basic opens $U_1 \times \cdots \times U_n \subset B^n$ of finite products of Bairespace by finite products of spreads $S_1 \times \cdots \times S_n \subset B^n$, one obtains an analog of Fourman's LS-model in which the axiom of spreaddata holds (this seems to be the model indicated in Fourman [1982, §2.4]).

The main differences between these models and the model presented in this paper are caused by the fact that in the former models, the universe of choice sequences is not closed under application of nontrivial lawlike operations. Consequently, $\forall\alpha\exists\beta$-continuity does not hold in the form described in Kreisel [1965]. (In the spreaddata-analog of our LS-model from [HM], $\forall\alpha\exists\beta$-continuity holds in the form

$$\forall \alpha \exists \beta\, A(\alpha, \beta) \to \exists f \in K\, \forall u (f(u) \neq 0 \to (\forall \alpha \in u\, A(\alpha, \alpha) \vee \exists \beta \forall \alpha \in u\, A(\alpha, \beta)))\text{.)}$$

On the other hand, these other models can be constructed within a constructive metatheory (IDB), and hence are equivalent to elimination translations into IDB, whereas our present model cannot: the statement that for all spreads S, the mapping $\tilde{S}$ is a closed hereditary retraction contradicts Church's thesis. It remains an open question whether a model for spreaddata with the properties as described in Theorem 6 above can be constructed within a constructive metatheory.

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CONSTRUCTING CHOICE SEQUENCES FROM LAWLESS SEQUENCES OF NEIGHBOURHOOD FUNCTIONS

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1. INTRODUCTION

The aim of this paper is to illustrate how various notions of choice sequence can be derived from, or reduced to, the notion of a lawless sequence. More accurately, we will construct a sequence of models, starting with a model for lawless sequences of neighbourhood functions, and arriving by subsequent modifications at a model for the theory CS of Kreisel & Troelstra(1970).

Such a process of gradually transforming a model for the theory of lawless sequences into a model for the theory of CS provides an answer to the question posed in Kreisel(1968), p.243, "How fundamental are lawless sequences", in the sense that it shows that many concepts of choice sequence can be derived from a notion of lawlessness.

The first model to be discussed in section 4 will be a model for lawless sequences of neighbourhood functions, which is completely analogous to a model for the theory LS of lawless sequences of natural numbers (for LS, see Kreisel(1968), Troelstra(1977)). This model for LS will be presented in section 3, after a short introduction to forcing over sites given in section 2.

With a lawless sequence of neighbourhood functions ξ one can associate a "potential" sequence of natural numbers α: given an initial segment $(f_o,\ldots,f_n)$ of ξ, the information we have about α is that it lies in the image of $f_o \circ \ldots \circ f_n$ (where the f_i's are regarded as lawlike continuous operations $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, so composing them makes sense).

A first modification of the site serves to eliminate an intensional aspect of the information we have about such a potential sequence α: two initial segments $(f_o,f_1,\ldots,f_n)$ and $(f_o \circ f_1, f_2,\ldots,f_n)$ represent the same information about α, and should therefore be identified, given that α is the sequence we are interested in, rather than the lawless sequence of neighbourhood functions that α is constructed from.

A next modification turns these potential sequences of natural numbers into actual ones, simply by refining the Grothendieck topology of the underlying site. We will see that the universe of choice sequences obtained at this stage is of little interest.

This situation changes radically if we modify the site once more, this time in order to obtain closure properties of the universe of choice sequences, and, in a next step, eliminate the intensional aspects introduced with these closure properties. We then have a model in which the universe of choice sequences satisfies the CS-axioms of *analytic data* and $\forall\alpha\exists n$ - *continuity*, i.e.

$$\forall\alpha(A(\alpha) \to \exists F(\alpha \in im(F) \wedge \forall\beta \in im(F)A(\beta)))$$

$$\forall\alpha\exists nA(\alpha,n) \to \exists F\forall\alpha A(\alpha,F\alpha),$$

where F ranges over lawlike continuous operations $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, and $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ respectively.

Moreover, this model has a natural notion of independence, which is decidable (i.e. $\alpha \# \beta \vee \neg\alpha \# \beta$ is valid, where we write $\alpha \# \beta$ for " α is independent from β "). Using this notion, we can formulate several variants of $\forall\alpha\exists\beta$ - continuity which are valid in this model, such as

$$\forall\alpha\exists\beta(\neg\alpha\#\beta \wedge A(\alpha,\beta)) \to \exists F\forall\alpha A(\alpha,F\alpha)$$

$$\forall\alpha\exists\beta(\alpha\#\beta \wedge A(\alpha,\beta)) \to \exists e \in K\forall u(e(u) \neq 0 \to \exists\beta\forall\alpha \in u(\alpha\#\beta \to A(\alpha,\beta))).$$

A multiple parameter version of analytic data also holds:

$$\forall\alpha_1,\ldots\alpha_n(\#(\alpha_1,\ldots,\alpha_n) \wedge A(\alpha_1,\ldots,\alpha_n) \to \exists F_1,\ldots,F_n(\bigwedge_{i\leq n} \alpha_i \in im(F_i) \wedge$$
$$\wedge \forall\beta_1,\ldots,\beta_n(\#(\beta_1,\ldots,\beta_n) \to A(F\beta_1,\ldots,F\beta_n))),$$

where $\#(\alpha_1,\ldots,\alpha_n)$ abbreviates $\bigwedge\{\alpha_i\#\alpha_j \mid 1\leq i<j\leq n\}$.

However, the usual version of $\forall\alpha\exists\beta$ - continuity (which is an axiom of CS),

$$\forall\alpha\exists\beta A(\alpha,\beta) \rightarrow \exists F\forall\alpha A(\alpha,F\alpha),$$

does not hold.

Finally, we modify the site by introducing the possible creation of certain dependencies between sequences. This is done in two steps. After the first step, we obtain validity of the usual CS-version of $\forall\alpha\exists\beta$ - continuity. So the only thing that is missing for a CS-model is the axiom of *pairing*,

$$\forall\alpha,\beta\exists\gamma\exists F,G(\alpha = F\gamma \wedge \beta = G\gamma).$$

A second step will accomplish the validity of this axiom, and we have arrived at a model for CS.

The constructions of these models and the proofs of their properties can be performed in an intuitionistic system like IDB (see Kreisel & Troelstra(1970)). This means that the theories of choice sequences that we provide models for are all consistent with Church's thesis ("all *lawlike* sequences are recursive") and lawlike countable choice.

As we said above, this sequence of models illustrates how various concepts of choice sequence can be reduced to the concepts of lawlessness. There is an interesting parallel here between the material of this paper and the program of "imitating" notions of choice sequence by means of "projections of lawless sequences" (cf. van Dalen & Troelstra(1970), van der Hoeven & Troelstra(1980), van der Hoeven (1982)), which has a similar purpose of reducing arbitrary choice sequences to lawless ones.

For example, in van der Hoeven(1982) a restricted version of CS is modelled by sequences constructed from a lawless sequence of neigbourhood functions and two lawless sequences of natural numbers, of which the latter two serve to make potential sequences into actual ones and to create dependencies between members of the universe of choice sequences.

There are some important differences between these two approaches, however, the main one being that here we obtain new notions of choice sequence by modifying the underlying site, that is, by modifying the notion of truth, whereas on the projections approach new universes of choice sequences are constructed by applying more complex continuous operations to lawless sequences.

Our present approach is technically simpler, because the changes in the forcing definition really make the intensional differences between sequences invisible. Using projections, the forcing definition remains the same, but long formula inductions are needed to show that for formulas in the language of analysis the property of being forced is independent of intensional differences in the parameters.

On the other hand, choice sequences projected from lawless sequences give a clearer picture of a construction process. In the sites we discuss here there are obvious representatives of steps in such a process ("going back along the arrows"), but the process as a whole is not explicitly presented.

Summarizing the results of this paper, then, we find models which have properties similar to the models for the CS-like systems constructed by projections. In particular, our last model but one, in which all of CS except pairing holds, is closely related to the models of van der Hoeven & Troelstra(1980). Technically, however, the projections approach is much more involved than the present one. The full generality of the models we obtain here has (so far) not been achieved along the projections approach: the projection models are all models of restricted variants of the theories we model here. Moreover, we obtain some new models for - so it seems to us - interesting systems with a primitive relation of independence.

In our paper van der Hoeven & Moerdijk(to appear) we constructed two models for the system CS by using forcing over sites, as we do here. Especially the first model (section 2.2 of that paper) is in some sense much simpler than the present one, but its construction is not motivated by a "reduction to lawless sequences" and, contrary to the present approach, we do not meet interesting (sub-)systems on the way of the construction of that model. The second CS-model in that paper (section 4) bears a relation to lawless sequences, but since it is constructed from the first one simply by considering what would be needed to prove it first order equivalent, this relation is less natural as a reduction. (See the remarks in Troelstra(1983), pp. 245-6.)

Thus, the constructions of these three CS-models are motivated rather differently, and the relation between these models needs closer investigation. This is a problem, however, that we do not touch upon in this paper.

2. FORCING OVER SITES

To make this paper accessible to readers who are less familiar with forcing over sites, we will review some of the basic notions of this theory, otherwise known as sheaf semantics or Beth-Kripke-Joyal semantics.

Let $\mathbb{C}$ be a category. If C is an object of $\mathbb{C}$, a *sieve* on C is a collection of morphisms S with codomain C which is closed under right composition, i.e. if $D \xrightarrow{f} C \in S$ and $E \xrightarrow{g} D$ is any morphism of $\mathbb{C}$ then $f \circ g \in S$.

A *Grothendieck topology* on $\mathbb{C}$ is a function which associates to every object C of $\mathbb{C}$ a family $J(C)$ of sieves on C, called *covering sieves*, such that

(i) (trivial cover) For each C, the maximal sieve
$\{f \mid \mathrm{codomain}(f) = C\} \in J(C)$.

(ii) (stability) If $S \in J(C)$ and $D \xrightarrow{f} C$ is a morphism of $\mathbb{C}$ then
$f^*(S) = \{E \xrightarrow{g} D \mid f \circ g \in S\} \in J(D)$.

(iii) (transitivity) If $R \in J(C)$ and S is a sieve on C such that for each $D \xrightarrow{f} C \in R$, $f^*(S) \in J(D)$, then $S \in J(C)$.

A *site* is a category equipped with a Grothendieck topology. A site is called *consistent* if $\phi \notin J(C)$ for some $C \in \mathbb{C}$, i.e. at least one object is not covered by the empty family.

(i), (ii), (iii) are closure conditions, so the intersection of a family of Grothendieck topologies is again a Grothendieck topology. Consequently, if for *some* objects $C \in \mathbb{C}$ we specify a couple of families (not necessarily sieves) $\{C_i \xrightarrow{f_i} C\}_i$ with codomain C ("basic covering families"), then there exists a smallest Grotendieck topology J with the property that for each of these selected objects C, and for each sieve S on C, $S \in J(C)$ whenever S contains one of these basic covering families. This smallest topology J is called the topology *generated* by the basic covering families.

In general, it is rather hard to keep track of what a collection of basic covers generates, in particular, it is hard to see whether the generated Grothendieck topology is consistent. For this reason, it is more convenient to work with basic covers of the following form: for *each* object C we specify a collection $K(C)$ of families $\{C_i \xrightarrow{f_i} C\}_i$ such that

(i') (trivial cover) The one-element family $\{C \xrightarrow{id} C\} \in K(C)$.

(ii') (stability) If $C_i \xrightarrow{f_i} C\}_i \in K(C)$ and $D \xrightarrow{g} C$ is a morphism of $\mathbb{C}$, then there is a family $\{D_j \xrightarrow{h_j} D\}_j \in K(D)$ such that for each j there is an i and a morphism k with $f_i \circ k = g \circ h_j$.

(iii') (transitivity) If $\{C_i \xrightarrow{f_i} C\}_i \in K(C)$ and for each i we have a family $\{C_{ij} \xrightarrow{g_{ij}} C_i\}_j \in K(C_i)$, then $\{C_{ij} \xrightarrow{f_i \circ g_{ij}} C\}_{i,j} \in K(C)$.

If we have a family of *basic covers* $K(C)$ for each $C \in \mathbb{C}$ satisfying (i')-(iii'), then the Grothendieck topology J generated by K is defined by

$$R \in J(C) \leftrightarrow \exists S \in K(C) S \subseteq R.$$

In particular, J is consistent iff $\phi \notin K(C)$ for some object C (we say that K is consistent).

In section 4, we will define (models over) sites by some basic covers which in general do not satisfy (i')-(iii'). So the way to show that our models are consistent is to find a bigger collection K of basic covers which does satisfy (i')-(iii'), and is consistent. This is rather straightforward in all cases, and will in general not be shown in detail.

A *domain* X on a site $(\mathbb{C},J)$ is a functor $\mathbb{C}^{op} \to$ Sets, i.e. a collection of sets $\{X(C) \mid C \in \mathbb{C}\}$ together with *restriction maps*

$$X(D) \to X(C), \qquad x \longmapsto x|f,$$

for every morphism $C \xrightarrow{f} D$ of $\mathbb{C}$, such that $(x|f)\,|\,g = x\,|\,(f \circ g)$, and $x\,|\,id = x$. The elements of X(C) are to be thought of as *partially constructed* members of the domain X, C is the "stage" of construction, and by the restriction along $D \xrightarrow{f} C$ we gain more information about such a partially constructed member of X, i.e. we perform a construction step.

A *lawlike domain* (more precisely, a domain of lawlike objects) is a domain which consists of *complete* objects: there is nothing to be constructed. So X is lawlike if X(C) = a fixed set X , and all restriction maps are identities. Thus, for each "external" set X there is a corresponding lawlike domain, also denoted by X, with X(C) = X. The main examples that occur in section 4 are the lawlike domain of natural numbers ($\mathbb{N}(C) = \mathbb{N}$ for all C), and the domain of lawlike neighbourhood functions K (K(C) = K, the set of inductively defined neighbourhood functions).

Given a collection of domains on $(\mathbb{C},J)$ we define forcing for a many sorted language L. Each sort of L is identified with a certain domain. And each constant c of L, of sort X say, is identified with a family of elements $c(D) \in X(D)$, D an object of $\mathbb{C}$, coherent in the sense that $c(D) \mid f = c(E)$ for any morphism $E \xrightarrow{f} D$.

Moreover, we assume to be given an interpretation of each relation symbol R (taking n arguments of sorts $X_1, \ldots, X_n$ say). The interpretation of R is an assignment of a subset $R(C) \subseteq X_1(C) \times \ldots \times X_n(C)$ to each object C, such that for $D \xrightarrow{f} C$,

$$(x_1,\ldots,x_n) \in R(C) \rightarrow (x_1|f,\ldots,x_n|f) \in R(D).$$

The *forcing relation*

$$C \Vdash \varphi(x_1,\ldots,x_n),$$

where φ has free variables among $v_1, \ldots v_n$, v_i of sort X_i, and $x_i \in X_i(C)$, is now defined by induction. For atomic formulas we have

$$C \Vdash x = y \leftrightarrow \text{there is an } S \in J(C) \text{ such that } x|f = y|f \text{ for all } f \in S$$

$$C \Vdash R(x_1,\ldots,x_n) \leftrightarrow \text{there is an } S \in J(C) \text{ such that } (x_1|f,\ldots,x_n|f) \in R(D) \text{ for all } D \xrightarrow{f} C \in S.$$

Furthermore,

$$C \Vdash \perp \leftrightarrow \phi \in J(C)$$

$$C \Vdash \varphi \wedge \psi(x_1,\ldots,x_n) \leftrightarrow C \Vdash \varphi(x_1,\ldots,x_n) \text{ and } C \Vdash \psi(x_1,\ldots,x_n)$$

$$C \Vdash \varphi\wedge\psi(x_1,\ldots,x_n) \Longleftrightarrow C \Vdash \varphi(x_1,\ldots,x_n) \text{ and } C \Vdash \psi(x_1,\ldots,x_n)$$

$$C \Vdash \varphi\vee\psi(x_1,\ldots,x_n) \Longleftrightarrow \{D \xrightarrow{f} C \mid D \Vdash \varphi(x_1|f,\ldots,x_n|f) \text{ or } D \Vdash \psi(x_1|f,\ldots,x_n|f)\} \in J(C)$$

$$C \Vdash \varphi\to\psi(x_1,\ldots,x_n) \Longleftrightarrow \text{for all morphisms } D \xrightarrow{f} C, \text{ if } D \Vdash \varphi(x_1|f,\ldots,x_n|f) \text{ then } D \Vdash \psi(x_1|f,\ldots,x_n|f).$$

and for variables v of sort Y we have

$$C \Vdash \exists v\varphi(v,x_1,\ldots,x_n) \Longleftrightarrow \{D \xrightarrow{f} C \mid \exists y\in Y(D)\ D \Vdash \varphi(y,x_1|f,\ldots,x_n|f)\} \in J(D)$$

$$C \Vdash \forall v\varphi(v,x_1,\ldots,x_n) \Longleftrightarrow \text{for all } D \xrightarrow{f} C \text{ and all } y \in Y(D),\ D \Vdash \varphi(y,x_1|f,\ldots,x_n|f).$$

By induction, one can show that the forcing relation has the important properties of being *monotone* and *local*:

(monotone) If $D \xrightarrow{f} C$ and $C \Vdash \varphi(x_1,\ldots,x_n)$ then $D \Vdash \varphi(x_1|f,\ldots,x_n|f)$.

(local) If $S \in J(C)$ and $D \Vdash \varphi(x_1|f,\ldots,x_n|f)$ for every $D \xrightarrow{f} C$ in S, then $C \Vdash \varphi(x_1,\ldots,x_n)$.

A formula $\varphi(v_1,\ldots,v_n)$ is called *valid* (notation: $\models \varphi(v_1,\ldots,v_n)$) if for each object C and each n-tuple $(x_1,\ldots,x_n) \in X_1(C) \times \ldots \times X_n(C)$,

$$C \Vdash \varphi(x_1,\ldots,x_n).$$

Function symbols F of L, taking n arguments of sorts $X_1, \ldots, X_n$ to a value of sort Y, are treated as $n+1$-place relation symbols such that $\forall x_1 \ldots x_n \exists! y\ F(x_1,\ldots,x_n) = y$ is valid.

This interpretation makes all of intuitionistic predicate calculus valid, and when higher order sorts (exponentials and powersets) are properly defined it provides a model for intuitionistic type theory with full comprehension. When the sort $\mathbb{N}$ of natural numbers is interpreted by the corresponding lawlike domain, we obtain a model for (higher order) intuitionistic arithmetic (These are well-known facts, but they are not needed for the understanding of the rest of this paper.) The first order part of arithmetic is classical if we work in a classical metatheory, since (as is easily shown by induction) we have

$$C \Vdash \varphi(x_1,\ldots,x_n) \leftrightarrow \varphi(x_1,\ldots,x_n) \quad \text{is true}$$

if the sorts X_i are lawlike, and all quantifiers in φ range over lawlike sorts.

3. A MODEL FOR LS

As a preparation to the next section, which is the core of this paper, we will now describe a model for the theory LS of lawless sequences of natural numbers. This model is not new, and was first described in Fourman(1982).

The underlying site of the model has as *objects* finite products of basic open subspaces of Baire space. We write such objects as

$$V_{u_1} \times \ldots \times V_{u_n},$$

where $u_i \in \mathbb{N}^{<\mathbb{N}}$ and $V_{u_i} = \{x \in \mathbb{N}^{\mathbb{N}} \mid x \text{ has initial segment } u_i\}$. The empty product, which is the one point space, is denoted by *1*. *Morphisms* from one such object to another

$$\Phi\colon V_{u_1} \times \ldots \times V_{u_n} \to V_{v_1} \times \ldots \times V_{v_m},$$

are continuous maps induced by injections $\varphi\colon \{1,\ldots,m\} \rightarrowtail \{1,\ldots,n\}$ such that $u_{\varphi(i)}$ extends v_i, via $\Phi(x_1,\ldots,x_n) = (x_{\varphi(1)},\ldots,x_{\varphi(m)})$. The *Grothendieck topology* is generated by basic covers of two sort ($*$ denotes concatenation):

(i) (open covers) $\{V_{u*n} \hookrightarrow V_u\}_{n\in\mathbb{N}}$ is a cover.

(ii) (projections) the singleton $\{V_u \times V_v \to V_u\}$ is a cover.

Classically, the generated Grothendieck topology can be described as: a family $\{\Phi_i\colon U_i \to U\}$ covers iff the images $\Phi_i(U_i) \subseteq U$ form an (open) cover of U. In an intuitionistic metatheory like IDB, we do not get all open covers, but only the inductively defined ones (cf. the remarks at the end of this section).

The relevant domains over this site are the following: we have the lawlike domains $\mathbb{N}$ of natural numbers and K of neighbourhood functions, the lawlike domain of continuous operations $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ corresponding to neighbourhood functions, and the lawlike domain $\mathbb{N}^{\mathbb{N}}$ of lawlike sequences (so all the "external" sequences appear in the model as lawlike sequences). The domain L of lawless sequences is the domain of projections,

$$L(V_{u_1}\times\ldots\times V_{u_n}) = \{\pi_i : V_{u_i}\to \mathbb{N}^{\mathbb{N}} \mid i=1,\ldots n\},$$

with restrictions defined by composition: If $\Phi: V_{u_1}\times\ldots\times V_{v_n} \to V_{v_1}\times\ldots\times V_{v_m}$ is a morphism induced by φ as above, then $\pi_i|\Phi = \pi_i\circ\Phi = \pi_{\varphi(i)}$ $(i=1,\ldots,m)$. If U is an object of the site and $\alpha\in L(U)$, then α is interpreted as a sequence of natural numbers by

$$(1)\quad U \Vdash \alpha(n) = m \Longleftrightarrow \forall x\in U\; \alpha(x)(n) = m,$$

in other words, if $U = V_{u_1}\times\ldots\times V_{u_n}$ and $\alpha = \pi_i$ then $U \Vdash \alpha\in v$ iff u_i extends v, for any finite sequence v (as usual, $\alpha\in v$ stands for $\forall i < \ell th(v)$ $\alpha(i) = v(i)$). Note that definition (1) is monotone and local, i.e. if $\Phi: W\to U$ and $U \Vdash \alpha(n) = m$ then $W \Vdash (\alpha|\Phi)(n) = m$, and if $\{\Phi_i : W_i\to U\}_i$ covers and each $W_i \Vdash (\alpha|\Phi_i)(n) = m$ then $U \Vdash \alpha(n) = m$.

This completes the description of the model.

The validity of the two simpler LS-axioms, *density*: $\forall v\exists\alpha(\alpha\in v)$ and *decidable equality*: $\forall\alpha,\beta(\alpha=\beta \vee \neg\alpha=\beta)$ is easily verified. For density, take a $v\in\mathbb{N}^{<\mathbb{N}}$ and an object $V_{u_1}\times\ldots\times V_{u_n}$. Then the projection $V_{u_1}\times\ldots\times V_{u_n}\times V_v \to V_{u_1}\times\ldots\times V_{u_n}$ covers, and $V_{u_1}\times\ldots\times V_{u_n}\times V_v \Vdash \pi_{n+1}\in v$. So $V_{u_1}\times\ldots\times V_{u_n} \Vdash \exists\alpha(\alpha\in v)$. Furthermore, it is easily seen that

$$V_{u_1}\times\ldots\times V_{u_n} \Vdash \pi_i\neq\pi_j \quad \text{iff} \quad i\neq j,$$

from which decidable equality follows immediately.

Before we prove the validity of open data and continuity in the model we state three simple observations about the forcing relation.

<u>Observation 1</u>. *If* $A(\alpha_1,\ldots,\alpha_n)$ *is a formula which has all its non-lawlike parameters among* $\alpha_1,\ldots,\alpha_n$, *and* $i_1,\ldots,i_n$ *are distinct numbers in* $\{1,\ldots,k\}$, *then*

$$V_{u_1} \times \ldots \times V_{u_k} \Vdash A(\pi_{i_n}, \ldots, \pi_{i_n}) \;\; iff \;\; V_{u_{i_1}} \times \ldots \times V_{u_{i_n}} \Vdash A(\pi_{i_1}, \ldots, \pi_{i_n})$$

(proof: $\Vdash$ is local and monotone (section 2), and the projection

$\Phi: V_{u_1} \times \ldots \times V_{u_k} \to V_{u_{i_1}} \times \ldots \times V_{u_{i_n}}$, $\Phi(x_1, \ldots, x_k) = (x_{i_1}, \ldots, x_{i_k})$, is a cover.)

Observation 2. *If* A *has only lawlike parameters, then* $\models A$ *iff for some object* U, $U \Vdash A$.

(proof: if $U \Vdash A$ then by observation 1, $1 \Vdash A$, so by monotonicity, $V \Vdash A$ for any object V since 1 is terminal, i.e. there is a unique morphism $V \to 1$ in the site.)

Observation 3. *Let* U *be any object in the site. Then*

$U \Vdash \underline{\forall}\alpha_1 \ldots \underline{\forall}\alpha_n(A(\alpha_1, \ldots, \alpha_n) \to B(\alpha_1, \ldots, \alpha_n))$ *iff for all* $u_1, \ldots, u_n \in \mathbb{N}^{<\mathbb{N}}$,

$V_{u_1} \times \ldots \times V_{u_n} \Vdash A(\pi_1, \ldots, \pi_n)$ *implies* $V_{u_1} \times \ldots \times V_{u_n} \Vdash B(\pi_1, \ldots, \pi_n)$.

Here $\underline{\forall}\alpha_1 \ldots \underline{\forall}\alpha_n$ (..) abbreviates $\forall\alpha_1 \ldots \alpha_n(\bigwedge_{i<j} \alpha_i \neq \alpha_j \to (..))$ and $A(\alpha_1, \ldots, \alpha_n)$, $B(\alpha_1, \ldots, \alpha_n)$ have all their non-lawlike parameters among $\alpha_1, \ldots, \alpha_n$.

(proof: by observation 2, we may assume $U = 1$. But if $W = V_{w_1} \times \ldots \times V_{w_k}$ is any object, and $i_1, \ldots, i_n$ are indices such that $W \Vdash \bigwedge_{\ell<\ell'} \pi_{i_\ell} \neq \pi_{i_{\ell'}} \wedge A(\alpha_{i_1}, \ldots, \alpha_{i_n})$, then $i_1, \ldots i_n$ are all distinct, and by observation 1, $V_{v_1} \times \ldots \times V_{v_n} \Vdash A(\pi_1, \ldots, \pi_n)$ where $v_j = w_{i_j}$. So we may restrict ourselves to the case $W = V_{v_1} \times \ldots \times V_{v_n}$, as was to be shown.)

Note that as a consequence of observation 3 we have:

Genericity lemma. $\models \underline{\forall}\alpha_1 \ldots \underline{\forall}\alpha_n A(\alpha_1, \ldots, \alpha_n)$ *iff* $V_{()} \times \ldots \times V_{()} \Vdash A(\pi_1, \ldots, \pi_n)$ (n-fold product).

Using these observations, validity of the open data axiom and the axiom of continuity is easily established.

Open data reads

$$\underline{\forall}\alpha_1 \ldots \underline{\forall}\alpha_n(A(\alpha_1, \ldots, \alpha_n) \to \exists u_1, \ldots, u_n(\alpha_1 \epsilon u_1 \wedge \ldots \wedge \alpha_n \epsilon u_n \wedge$$
$$\wedge \underline{\forall}\beta_1 \epsilon u_1 \ldots \underline{\forall}\beta_n \epsilon u_n A(\beta_1, \ldots, \beta_n))).$$

By observation 3, it suffices to show that for any n-tuple $u_1, \ldots, u_n$, if

(1) $V_{u_1} \times \ldots \times V_{u_n} \Vdash A(\pi_1, \ldots, \pi_n)$,

then also

(2) $V_{u_1} \times \ldots \times V_{u_n} \Vdash \underline{\forall}\beta_1 \in u_1 \ldots \underline{\forall}\beta_n \in u_n\, A(\beta_1, \ldots, \beta_n)$.

So assume (1). To prove (2), it suffices by observation 3 again to show that if

(3) $V_{w_1} \times \ldots \times V_{w_n} \Vdash \pi_1 \in u_1 \wedge \ldots \wedge \pi_n \in u_n$

then also

(4) $V_{w_1} \times \ldots \times V_{w_n} \Vdash A(\pi_1, \ldots, \pi_n)$.

But if (3) holds, then w_i extends u_i so we have an inclusion morphism $V_{w_1} \times \ldots \times V_{w_n} \hookrightarrow V_{u_1} \times \ldots \times V_{u_n}$, restriction along which shows that (4) now follows from (1).

The axiom of continuity is

$$\underline{\forall}\alpha_1 \ldots \underline{\forall}\alpha_n \exists m\, A(\alpha_1, \ldots, \alpha_n, m) \rightarrow \exists F \underline{\forall}\alpha_1 \ldots \underline{\forall}\alpha_n\, A(\alpha_1, \ldots, \alpha_n, F(\alpha_1, \ldots, \alpha_n)),$$

where F ranges over lawlike continuous operations $\mathbb{N}^{\mathbb{N}} \times \ldots \times \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}$ (induced by neighbourhood functions $\mathbb{N}^{<\mathbb{N}} \times \ldots \times \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}$), and $A(\alpha_1, \ldots, \alpha_n, m)$ has no non-lawlike parameters other than $\alpha_1, \ldots, \alpha_n$. By observation 2, proving continuity is equivalent to showing that if

(1) $\models \underline{\forall}\alpha_1 \ldots \underline{\forall}\alpha_n \exists m\, A(\alpha_1, \ldots, \alpha_n, m)$

then also

(2) $\models \exists F \underline{\forall}\alpha_1 \ldots \underline{\forall}\alpha_n\, A(\alpha_1, \ldots, \alpha_n, F(\alpha_1, \ldots, \alpha_n))$.

So assume (1). Then in particular for the n-fold product of $V_{(\,)}$,

$$V_{(\,)} \times \ldots \times V_{(\,)} \Vdash \exists m\, A(\pi_1, \ldots, \pi_n, m),$$

so we find a cover $\{\phi_i : W_i \rightarrow V_{(\,)} \times \ldots \times V_{(\,)}\}$ and natural numbers m_i such that for each i,

$$W_i \Vdash A(\pi_1 \mid \phi_i, \ldots, \pi_n \mid \phi_i, m_i).$$

By observation 1, we may assume Φ_i to be a canonical inclusion $W_i = V_{w_{1,i}} \times \ldots \times V_{w_{n,i}} \hookrightarrow V_{()} \times \ldots \times V_{()}$, and by passing to a disjoint refinement of the inductive open cover $\{W_i\}_i$ of $V_{()} \times \ldots \times V_{()}$, we can define F by

$$F(x_1,\ldots,x_n) = m_i \quad \text{iff} \quad (x_1,\ldots,x_n) \in W_i.$$

Then $W_i \Vdash F(\pi_1,\ldots,\pi_n) = m_i$, so $W_i \Vdash A(\pi_1,\ldots,\pi_n, F(\pi_1,\ldots,\pi_n))$. Since the W_i cover, it follows that $V_{()} \times \ldots \times V_{()} \Vdash A(\pi_1,\ldots,\pi_n, F(\pi_1,\ldots,\pi_n))$. So by the genericity lemma, $\models \underline{\forall}\alpha_1 \ldots \underline{\forall}\alpha_n A(\alpha_1,\ldots,\alpha_n, F(\alpha_1,\ldots,\alpha_n))$, hence (2) holds.

The treatment of the model above is completely constructive, i.e. can be performed in an intuitionistic metatheory like IDB. (By definition of the Grothendieck topology, every cover has a corresponding characteristic neighbourhood function in K, so the map F defined in the proof of the continuity axiom can indeed be defined in IDB.) It is a corollary of the elimination theorem (Troelstra(1977)) that any interpretation of LS in IDB is equivalent to the elimination translation. In particular, for first order sentences A in the language of LS,

$$\text{IDB} \vdash (\Vdash A) \leftrightarrow \tau(A),$$

where $\Vdash$ is forcing over the model described above (formalized in IDB) and τ is the elimination translation. In fact, a simple formula induction shows that $V_{u_1} \times \ldots \times V_{u_n} \Vdash A(\pi_1,\ldots,\pi_n)$ and $\tau(\underline{\forall}\alpha_1 \in u_1 \ldots \underline{\forall}\alpha_n \in u_n A(\alpha_1,\ldots \alpha_n))$ are literally the same (A not containing lawless parameters other than $\alpha_1, \ldots, \alpha_n$), provided one includes observation 2 in the forcing definition to get rid of vacuous quantifiers.

4. CHOICE SEQUENCES CONSTRUCTED FROM LAWLESS SEQUENCES OF NEIGHBOURHOOD FUNCTIONS

4.1. Lawless sequences of neighbourhood functions.

As usual, K denotes the (inductively defined) class of neighbourhood functions $\mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$. Neighbourhood functions induce lawlike continuous operations $F: \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ and $F: \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ (cf. Troelstra(1977)), and we will often identify F and f, writing things like $F \in K$, or $f \circ g$ for the composition of the corresponding continuous operations $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, etc.

Completely analogous to the model for lawless sequences described in section 3, we may construct a model for lawless sequences of neighbourhood functions from K (or lawless sequences of lawlike continuous operations $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$). *Objects* of the site are now finite products of basic open subsets of $K^{\mathbb{N}}$,

$$V_{\xi_1} \times \ldots \times V_{\xi_n} \quad (n \geq 0),$$

where $\xi_i \in K^{<\mathbb{N}}$, $\xi_i = (f_{i1}, \ldots f_{ik_i})$ (and each f_{ij} is identified with the corresponding continuous operation $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$). *Morphisms* of the site are functions

$$\Phi\colon V_{\xi_1} \times \ldots \times V_{\xi_n} \to V_{\zeta_1} \times \ldots \times V_{\zeta_m}$$

which are induced by injections $\varphi\colon \{1,\ldots,m\} \rightarrowtail \{1,\ldots,n\}$ such that $\xi_{\varphi(i)}$ extends ζ_i $(i=1,\ldots,m)$, and $\Phi(x_1,\ldots,x_n) = (x_{\varphi(1)},\ldots,x_{\varphi(m)})$. The *Grothendieck topology* is generated by open covers and projections, just as in section 3. And as in this preceding section, lawless sequences are interpreted as projections. The logical properties of the model are of course exactly the same as the properties of the model of section 3. For the record:

Theorem. *The site described above gives a model for the theory of lawless sequences of neighbourhood functions, i.e. it satisfies the axioms of density, decidable equality, open data, and continuity* (the continuity axiom has to be rephrased using inductively defined neighbourhood functions on the tree $K^{<\mathbb{N}}$).

4.2. Potential choice sequences of natural numbers.

With each of the lawless sequences π_i $(i=1,\ldots,n)$ at an object $V_{\xi_1} \times \ldots \times V_{\xi_n}$ of the site of 4.1 we can associate a potential sequence α_i of natural numbers, by setting

$$V_{\xi_1} \times \ldots \times V_{\xi_n} \Vdash \alpha_i(n) = m \quad \text{iff} \quad \forall x \in \mathbb{N}^{\mathbb{N}} \;\; f_{i1} \circ \ldots \circ f_{ik_i}(x)(n) = m.$$

(Here $\xi_i = (f_{i1},\ldots,f_{ik_i})$, and the f_{ij} are regarded as operations $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, so composing them makes sense.) α_i is not an actual sequence, i.e.

$$V_{\xi_1} \times \ldots \times V_{\xi_n} \not\Vdash \forall n \exists m \, \alpha_i(n) = m,$$

since the extensions $(f_{i1},\ldots,f_{ik_i}, g_1,\ldots,g_\ell)$ of ξ_i such that $f_{i1}\circ\ldots\circ g_\ell(x)(n)$ is constant in x do not form a cover of V_{ξ_i}. However, we can always extend a sequence in $K^{<\mathbb{N}}$ by a constant function (constant when regarded as an operation $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$), and therefore

$$V_{\xi_1}\times\ldots\times V_{\xi_n} \Vdash \neg\neg\forall n\exists m\, \alpha_i(n) = m .$$

The information we have about such a potential sequence α at an object $V_{(f_1,\ldots f_k)}$ is that (in case α ever turns out to be a real sequence) $\alpha \in \operatorname{im}(f_1\circ\ldots\circ f_k)$. Thus, since α is the only non-lawlike element at $V_{(f_1,\ldots,f_n)}$ that we are interested in here, the two objects $V_{(f_1,f_2,f_3,\ldots,f_n)}$ and $V_{(f_1\circ f_2,f_3,\ldots,f_n)}$ represent equivalent information. Therefore we will identify two such objects by passing to a quotient space of $K^{\mathbb{N}}$:

<u>Definition</u>. *Let* $\sim$ *be the equivalence relation on* $K^{\mathbb{N}}$ *generated by*

$$(f_1,f_2,f_3,\ldots) \sim (f_1\circ f_2,f_3,\ldots).$$

The space X *is the quotient space* $K^{\mathbb{N}}/\sim$, *with the quotient topology.*

<u>Lemma</u>. *The canonical projection* $p\colon K^{\mathbb{N}} \to X$ *is open.*

<u>Proof</u>. The equivalence relation $\sim$ can also be described by

$$(f_n)_n \sim (g_n)_n \quad \text{iff} \quad \exists k,\ell(f_1\circ\ldots\circ f_k = g_1\circ\ldots\circ g_\ell \;\&\; \forall n\geq 1\; f_{k+n} = g_{\ell+n}).$$

So

$$p^{-1}p(V_{(f_1,\ldots,f_n)}) = \{(g_n)_n \mid \exists\ell\exists h\in K\; g_1\circ\ldots\circ g_\ell = f_1\circ\ldots\circ f_n\circ h\},$$

which is open in $K^{\mathbb{N}}$. □

We will now modify the definition of our site, by replacing each basic open $V_{(f_1,\ldots,f_k)}$ of $K^{\mathbb{N}}$ by the corresponding open $p(V_{(f_1,\ldots,f_k)})$ of X. Note that $p(V_{(f_1,\ldots,f_k)}) = p(V_{(f_1\circ\ldots\circ f_k)})$, so we only need to work with sequences of length 1. We will write

$$V_f = p(V_{(f)}),$$

and sometimes by abuse of notation for sequences of length other than 1, $V_{(f_1,\ldots,f_k)} = p(V_{(f_1,\ldots,f_k)})$, so $V_{()}$ denotes $V_{id} = X$.

As our site we now take the site obtained by applying p to everything in the site of 4.1. So *objects* are now finite products

$$V_{f_1} \times \ldots \times V_{f_n},$$

and *morphisms* $\Phi: V_{f_1}\times\ldots\times V_{f_n} \to V_{g_1}\times\ldots\times V_{g_m}$ come from injections $\varphi: \{1,\ldots,m\} \to \{1,\ldots,n\}$ such that for each $i = 1,\ldots,m$ there exists an $h_i \in K$ with $g_i \circ h_i = f_{\varphi(i)}$, and $\Phi(x_1,\ldots,x_n) = (x_{\varphi(1)},\ldots,x_{\varphi(n)})$ as before, except that if is now a function on equivalence classes (note that this is well defined). The *Grothendieck topology* is generated by basic covers of two kinds:

(i) (open inclusions) $\{V_{f\circ g} \hookrightarrow V_f\}_{g\in K}$ is a cover

(ii) (projections) $V_f \times V_g \to V_f$ is a cover.

4.3. Actual choice sequences of natural numbers.

Our next modification will be to force the potential sequences α of 4.2 to become real sequences by allowing to pass to a bar in $\mathbb{N}^{<\mathbb{N}}$ to find the n^{th} value of α. Each finite sequence u induces a lawlike continuous operation $\bar{u}: \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ defined by

$$\begin{aligned}\bar{u}(x) &= u \mid x \quad (\text{"overwrite } u\text{"}) \\ &= (u(0),\ldots,u(\ell th(u)-1),\ x(\ell th(u)),\ x(\ell th(u)+1),\ldots),\end{aligned}$$

and we now add to the site of 4.2 as new basic covers the families of inclusions

$$\{V_{f\circ\bar{u}_i} \hookrightarrow V_f\}_i$$

for each inductive bar $\{u_i\}_i$ for $\mathbb{N}^{\mathbb{N}}$. (Note that this makes the covers of type (i) in 4.2 redundant.) By stability and transitivity of the induced Grothendieck topology, this means that for each inductive bar $\{u_i^1 \times \ldots \times u_i^k\}_i$ for $\mathbb{N}^{\mathbb{N}} \times \ldots \times \mathbb{N}^{\mathbb{N}}$, we have a corresponding family

$$\{V_{f_1\circ\bar{u}_i^1}\times\dots\times V_{f_k\circ\bar{u}_i^k} \hookrightarrow V_{f_1}\times\dots\times V_{f_k}\}_i$$

in the site.

Observe that covers of this form are stable. For example, if $\{V_{f\circ\bar{u}_i} \to V_f\}_i$ is a cover and $V_g \to V_f$ is an inclusion in the site (so $g = f\circ h$ for some h) then by continuity of h there is an inductive cover $\{v_j\}_j$ of $\mathbb{N}^{\mathbb{N}}$ such that h maps each v_j into some u_i (i.e. if $x \in \mathbb{N}^{\mathbb{N}}$ extends v_j then $h(x)$ extends u_i, we write this as $h(v_j) \subseteq u_i$), so $\bar{u}_i\circ h\circ\bar{v}_j = h\circ\bar{v}_j$, $g\circ\bar{v}_j = f\circ h\circ\bar{v}_j = f\circ\bar{u}_i\circ h\circ\bar{v}_j$, and hence there is a commutative diagram

$$\begin{array}{ccc} V_{f\circ\bar{u}_i} & \longrightarrow & V_f \\ \uparrow & & \uparrow \\ V_{g\circ\bar{v}_j} & \longrightarrow & V_g \end{array}$$

We now have obtained actual sequences:

Proposition. *In this model* $\models \forall\alpha\forall n\exists m\,\alpha(n) = m$, *i.e. at each object* $V_{f_1}\times\dots\times V_{f_k}$, *for each choice sequence* $\alpha_i (i=1,\dots,k)$, $V_{f_1}\times\dots\times V_{f_k} \Vdash \forall n\exists m\,\alpha_i(n) = m$

proof. It suffices to take $k = 1$ (cf. observation 1, section 3), i.e. to show that for all n, $V_f \Vdash \exists m\,\alpha(n) = m$. But f is continuous (as a map $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$) so there is an inductive bar $\{v_j\}_j$ for $\mathbb{N}^{\mathbb{N}}$ such that for all j there exists an m_j with $f(x)(n) = m_j$ whenever $x \in v_j$. But then $V_{f\circ\bar{v}_j} \Vdash \alpha(n) = m_j$ for each j, so $V_f \Vdash \exists m\,\alpha(n) = m$. □

The universe of choice sequences of natural numbers we have now obtained models a rather poor theory. Most importantly (since we are on our way to a model for CS) the universe is not closed under application of lawlike continuous operations. We do not have analytic data, or $\forall\alpha\exists\beta$ - continuity. Also, of the LS-like properties not much is left. For example, the model does not satisfy decidable equality for choice sequences.

On the positive side, we have

Proposition. *In the model under consideration, the following are valid:*

(i) (*density*) $\forall\alpha\neg\neg\exists a\alpha = a$

(α ranges over choice sequences, a over lawlike sequences)

(ii) ($\forall\alpha\exists n$ - *continuity*) $\forall\alpha\exists nA(\alpha,n) \to \exists F\forall\alpha A(\alpha,F\alpha)$

(here, as usual, A does not contain non-lawlike parameters other than α; and F ranges over lawlike continuous operations $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$).

In the proof we use

Lemma. (*genericity of the choice sequence* α *at* $V_{()}$) *Let* $A(\alpha)$ *be a formula with* α *as its only non-lawlike parameter. If* $V_{()} \Vdash A(\alpha)$ *then* $\models \forall\alpha A(\alpha)$.

proof. Take a choice sequence α_i at $V_{f_1} \times \ldots \times V_{f_n}$ ($i \le n$). Since all other parameters in $A(\alpha)$ are (interpreted by) constant (elements), it suffices to show that α_i is a restriction of the sequence α at $V_{()}$ (by monotonicity of $\Vdash$). But restricting α along

$$V_{f_1}\times\ldots\times V_{f_n} \xrightarrow{\pi_i} V_{f_i} \hookrightarrow V_{()}$$

yields α_i. □

proof of proposition. (i) We show $V_{()} \Vdash \neg\neg\exists a\alpha = a$. By observation 1 of section 2 it suffices to show that for each f, $V_f \not\Vdash \neg\exists a\alpha = a$. But this is indeed the case, since if $g: \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is constant with value b, then $V_{f\circ g} \Vdash \alpha_i = a$ when we let $a = f(b)$.

(ii) By observation 2 of section 2, we have to show that if $\models \forall\alpha\exists nA(\alpha,n)$ then $\models \exists F\forall\alpha A(\alpha,F\alpha)$. But if $\models \forall\alpha\exists nA(\alpha,n)$ then $V_{()} \models \exists nA(\alpha,n)$, so (cf. observation 1) there are an inductive cover $\{u_i\}_i$ and numbers n_i such that $V_{\bar{u}_i} \Vdash A(\alpha,n_i)$. We may assume the u_i to be disjoint (incompatible), so we can define a lawlike continuous operation $F: \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ with value n_i on u_i, i.e. $F(x) = n_i$ if x extends u_i. Then $V_{\bar{u}_i} \Vdash A(\alpha,F\alpha)$, hence since $\{V_{\bar{u}_i} \to V_{()}\}_i$ is a cover, $V_{()} \Vdash A(\alpha,F\alpha)$. By the genericity lemma, $\models \forall\alpha A(\alpha,F\alpha)$. □

4.4. Closure under lawlike continuous operations.

We will now enlarge our universe of choice sequences, by "projecting" from the lawless sequence $(f_1,f_2,\ldots)$ of neighbourhood functions not only the single sequence α defined by $\alpha(n) = m$ iff $\exists k \forall x \in \mathbb{N}^{\mathbb{N}}\ f_1 \circ \ldots \circ f_k(x)(n) = m$ as in 4.2, but using one lawless sequence to generate an indefinite number of sequences of the form $b(\alpha)$, where b is a lawlike continuous operation $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$.

At the level of the site this means that instead of having finite products of spaces V_f as objects, we now take finite products of objects of the form

$$V_f^{a_1,\ldots,a_k},$$

where $a_1,\ldots,a_k \in K$. If $\{b_1,\ldots,b_m\} \subset \{a_1,\ldots,a_k\}$ we add a morphism

$$V_f^{a_1,\ldots,a_k} \to V_f^{b_1,\ldots,b_m}$$

to our site. On the underlying spaces, this is just the identity map. Going back along this morphism corresponds to the "step in the construction" by which we decide to consider some more choice sequences projected from the single lawless sequence about which we know that it starts with f.

Since we should be able to consider an arbitrary (finite) number of such choice sequences without narrowing our information about the sequences we already had, we should declare this morphism a *cover* in the site. (Stability of this new type of basic cover is trivial, since the underlying function of topological spaces is the identity.)

The model we obtain in this way indeed satisfies closure, that is

$$\forall \alpha \forall F \exists \beta\ F(\alpha) = \beta$$

is valid, where α,β range over the new domain of choice sequences, and F over lawlike continuous operations, but the extra logical properties that we obtain are rather uninteresting. The reason is that the information we have about the single choice sequence $\beta = b(\alpha)$ at the object $V^b_{f\circ g}$ say, is too intensional. This information expresses that $\beta \in \mathrm{im}(b \circ f \circ g)$, and that f and g are the first two elements in our lawless sequence of neighbourhood functions, while b is the lawlike operation that we apply to extract the sequence $\beta = b(\alpha)$. We want to abstract from the different rôles played by b and f in the construction of β, i.e. to pass to a stage of information where f is regarded as an operation used for closing off. This means adding a morphism

$$V_g^{b\circ f} \to V_{f\circ g}^{b}$$

to the site, which on the level of underlying spaces is defined by concatenation, $x \longmapsto f{*}x$ (this is obviously well-defined on equivalence classes $x \in X = K^{\mathbb{N}}/\sim$, and it does not depend on the choice of f, i.e. if f' would be another function such that $b\circ f = b\circ f'$ and $f\circ g = f'\circ g$, the same morphism is defined). Since we wish to ignore the "intensional difference" between the information at $V_{f\circ g}^{b}$ and the information at $V_g^{b\circ f}$ completely, we should moreover declare this morphism $V_g^{b\circ f} \to V_{f\circ g}^{b}$ to cover.

(Digression: A similar abstraction is made in the theory of lawless sequences. A lawless sequence α is usually conceived of as constructed by fixing a finite initial segment u, and then starting to make free choices (throwing a die). At each stage of the construction, the information we have about α is an initial segment $u{*}v$, but we abstract from the extra "intensional" information that u is the initial "deliberate" placings of the die, whereas v comes from making free choices. See Troelstra(1977).)

When compared to the earlier models of this section 4, the properties that the universe of choice sequences has in this model are much richer and much more interesting.

Before we investigate some of these properties, however, we give an explicit description of the site that we have obtained at this stage. For easy reference, we call this site $\mathbb{K}$. *Objects* of the site $\mathbb{K}$ are finite products

$$V_{f_1}^{\bar{a}_1} \times \ldots \times V_{f_n}^{\bar{a}_n}$$

where $f_i \in K$ and $\bar{a}_i \in K^{<\mathbb{N}}$. *Morphisms* of $\mathbb{K}$ are best described by: all compositions of morphisms of the various types mentioned above. A more explicit but rather tiresome description is as follows: a morphism

$$\Phi\colon V_{f_1}^{\bar{a}_1}\times\ldots\times V_{f_n}^{\bar{a}_n} \to V_{g_1}^{\bar{b}_1}\times\ldots\times V_{g_m}^{\bar{b}_m}$$

with $\bar{a}_i = (a_{i1},\ldots,a_{ik_i})$ and $\bar{b}_j = (b_{j1},\ldots,b_{j\ell_j})$ is induced by an injection $\varphi\colon \{1,\ldots,m\} \rightarrowtail \{1,\ldots,n\}$ together with injections $\rho_j\colon \{1,\ldots,\ell_j\} \rightarrowtail \{1,\ldots,k_{\varphi(j)}\}$ for each $j = 1,\ldots,m$, such that there are maps h_j and k_j $(j=1,\ldots,m)$ with

$$h_j \circ f_{\varphi(j)} = g_j \circ k_j$$

and for each $p = 1,\ldots,\ell_j$,

$$b_{jp} \circ h_j = a_{\varphi(j)\rho_j(p)}.$$

On the underlying spaces we have

$$\Phi(x_1,\ldots,x_n) = (h_1 * x_{\varphi(1)},\ldots,h_m * x_{\varphi(m)})$$

($*$ for concatenation; this is well-defined on equivalence classes and does not depend on the choice of h_j, k_j). The *Grothendieck topology* of $\mathbb{K}$ is generated by basic covers of four kinds:

(i) ("open covers") $\{V^{\bar{a}}_{f\circ\bar{u}_i} \to V^{\bar{a}}_f\}_i$ is a cover, for each inductive bar $\{u_i\}_i$ for $\mathbb{N}^{\mathbb{N}}$.

(ii) (projections) $V^{\bar{a}}_f \times V^{\bar{b}}_g \to V^{\bar{a}}_f$ is a cover.

(iii) (adding choice sequences) $V^{\bar{a}}_f \to V^{\bar{b}}_f$ is a cover for the map which is the identity on the level of spaces, and is induced by an inclusion of the sequence $\bar{b}$ as a subsequence of $\bar{a}$.

(iv) (abstraction) $V^{\bar{a}\circ f}_g \to V^{\bar{a}}_{f\circ g}$ is a cover, where $\bar{a}\circ f = (a_1\circ f,\ldots,a_k\circ f)$ if $\bar{a} = (a_1,\ldots,a_k)$.

The *universe of choice sequences* at an object

$$V_{g_1}^{\bar{b}_1} \times \ldots \times V_{g_m}^{\bar{b}_m}$$

consists of all sequences of the form $\beta = b_{jp} \circ \alpha_j$ $(p=1,\ldots,\ell)$, where $\bar{b}_j = (b_{j1},\ldots,b_{jp})$, and β is a sequence by

$$V_{g_1}^{\bar{b}_1}\times\ldots\times V_{g_m}^{\bar{b}_m} \Vdash \beta(n) = m \text{ iff } \forall x \in \mathbb{N}^{\mathbb{N}}\ b_{jp}\circ g_j(x)(n) = m.$$

Restriction of choice sequences along morphisms is defined in the obvious way. If Φ is a morphism as just described and $\beta = b_{jp} \circ \alpha_j$ as above, then the restriction of β along Φ is the sequence

$$\beta \mid \Phi = b_{\varphi(j)\rho_j(p)} \circ \alpha_{\varphi(j)}$$

at the object $V_{f_1}^{\bar{a}_1} \times \ldots \times V_{f_n}^{\bar{a}_n}$. This definition of restrictions is compatible with the definition of $\Vdash \beta(n) = m$ in the sense that for a morphism Φ with codomain W and domain U, $W \Vdash \beta(n) = m$ implies $U \Vdash (\beta|\Phi)(n) = m$, and if $\{\Phi_i : U_i \to W\}_i$ is a cover in $\mathbb{K}$ such that $U_i \Vdash (\beta|\Phi)(n) = m$ for each i, then also $W \Vdash \beta(n) = m$.

In this model, all one parameter axioms of the theory CS are valid:

Theorem 1. *In the model over* $\mathbb{K}$ *just described, the following are valid:*

(i) (*closure*) $\forall\alpha\forall F\exists\beta\ \beta = F(\alpha)$

(ii) (*analytic data*) $\forall\alpha(A(\alpha) \to \exists F(\alpha \in im(F) \wedge \forall\beta \in im(F)A(\alpha)))$

(iii) ($\forall\alpha\exists n$ - *continuity*) $\forall\alpha\exists nA(\alpha,n) \to \exists F\forall\alpha A(\alpha,F\alpha)$.

Here, as usual, F ranges over lawlike continuous operations (into $\mathbb{N}$ or $\mathbb{N}^{\mathbb{N}}$), α,β over choice sequences, and all non-lawlike parameters in A are shown.

Before we prove the theorem, let us reformulate the genericity lemma of 4.3:

Genericity lemma. *Let* $A(\alpha)$ *be a formula with* α *as its only non-lawlike parameter. Then in the model over* $\mathbb{K}$, $V_{()}^{id} \Vdash A(\alpha)$ *implies* $\models \forall\alpha A(\alpha)$.

proof. As before, we have to show that every choice sequence at an object $V_{f_1}^{\bar{a}_1} \times \ldots \times V_{f_n}^{\bar{a}_n}$ is the restriction of the (single) choice sequence α at $V_{()}^{id}$. But if $\beta = a_{ij} \circ \alpha_i$ is a choice sequence at $V_{f_1}^{\bar{a}_1} \times \ldots \times V_{f_n}^{\bar{a}_n}$, then $\beta = \alpha \mid \Phi$ where Φ is the composite

$$V^{\bar{a}_1}_{f_1}\times\ldots\times V^{\bar{a}_n}_{f_n} \xrightarrow{\pi_i} V^{\bar{a}_i}_{f_i} \longrightarrow V^{a_{ij}}_{f_i} \hookrightarrow V^{a_{ij}}_{(\,)} \longrightarrow V^{id}_{a_{ij}} \hookrightarrow V^{id}_{(\,)}. \qquad \Box$$

<u>proof of theorem 1.</u>

(i) By the lemma, it suffices to show $V^{id}_{(\,)} \Vdash \forall F \exists \beta\ \alpha = F(\beta)$. But if f is a law-like continuous operation, then $V^{id,f}_{(\,)} \to V^{id}_{(\,)}$ covers, and at $V^{id,f}_{(\,)}$ we have two sequences α and β with $V^{id,f}_{(\,)} \Vdash \alpha = f(\beta)$.

(ii) Suppose $V^{\bar{a}_1}_{f_1}\times\ldots\times V^{\bar{a}_n}_{f_n} \Vdash A(\beta)$ for some sequence $\beta = a_{ij} \circ \alpha_i$. Since $V^{\bar{a}_1}_{f_1}\times\ldots\times V^{\bar{a}_n}_{f_n} \to V^{a_{ij}1}_{f_i}$ covers, we find $V^{a_{ij}}_{f_i} \Vdash A(\beta)$. In other words, we may assume that $n = 1$ and $\bar{a}_1$ has length 1, and we can write $V^a_f \Vdash A(\beta)$. Then also $V^{a\circ f}_{(\,)} \Vdash A(\beta)$ by restricting along the morphism $V^{a\circ f}_{(\,)} \to V^a_f$, so if we add a choice sequence α corresponding to id and put $F = a\circ f$ we find $V^{af,id}_{(\,)} \Vdash A(F\alpha)$. Hence since $V^{af,id}_{(\,)} \to V^{id}_{(\,)}$ covers, $V^{id}_{(\,)} \Vdash A(F\alpha)$. By the genericity lemma, $\models \forall\alpha A(F\alpha)$, so a fortiori $V^a_f \Vdash \forall\alpha A(F\alpha)$, while $V^{af,id}_{(\,)} \Vdash \beta = F\alpha$, so $V^a_f \Vdash \beta \in \mathrm{im}(F)$ since $V^{af,id}_{(\,)} \to V^a_f$ is a cover.

(iii) The proof of $\forall\alpha\exists n$ - continuity is analogous to the one we gave in 4.3. $\Box$

<u>Corollary</u>. *Let* A *be a sentence of the language of* CS *containing only one choice variable. Then* A *is valid in this model over* $\mathbb{K}$.

<u>proof</u>. From the proof of the elimination theorem for CS (Kreisel & Troelstra(1970)) we conclude that if $CS \vdash A$ then $CS^1 \vdash A$, where CS^1 is the theory axiomatized by the axioms of CS which contain only one choice variable. But the theorem above states that these axioms are valid over $\mathbb{K}$. $\Box$

The model does not satisfy the axioms for CS in more choice variables, notably the *pairing axiom* $\forall\alpha,\beta\exists F,G\exists\gamma(\alpha=F\gamma \wedge \beta=G\gamma)$, and $\forall\alpha\exists\beta$ - continuity.

The properties of the universe of choice sequence in this model can be more closely analysed if we introduce a primitive predicate of *independence*

$\alpha \# \beta$ "α and β are independent choice sequences"

into the language. The interpretation of $\#$ in the model is given by

$$V_{f_1}^{\bar{a}_1}\times\ldots\times V_{f_n}^{\bar{a}_n} \Vdash a_{ij}\circ\alpha_i \# a_{k\ell}\circ\alpha_\ell \quad \text{iff} \quad i \neq \ell,$$

i.e. two choice sequences are independent if they come from different factors of the product $V_{f_1}^{\bar{a}_1} \times \ldots \times V_{f_n}^{\bar{a}_n}$, meaning that they are extracted from distinct processes for choosing a lawless sequence of neighbourhood functions. Obviously, $\#$ is *decidable*, i.e.

$$\alpha \# \beta \vee \neg \alpha \# \beta$$

is valid in the model (just as decidable equality in section 3).

With this primitive $\#$ added to the language, the model can be shown to satisfy a set of axioms that allows elimination of choice sequences. For example, we have multiple parameter versions of analytic data and $\forall\alpha\exists n$ - continuity:

<u>Theorem 2</u>. *The model over* $\mathbb{K}$ *satisfies the following multiple parameter versions of analytic data and continuity, where* $\#(\alpha_1,\ldots,\alpha_n)$ *abbreviates* $\bigwedge_{i<j} \alpha_i \# \alpha_j$.

$$\forall\alpha_1\ldots\alpha_n(\#(\alpha_1,\ldots,\alpha_n) \wedge A(\alpha_1,\ldots,\alpha_n) \rightarrow \exists F_1\ldots F_n$$
$$(\bigwedge_{i=1}^{n} \alpha_i \in \mathrm{im}(F_i) \wedge \forall\beta_1\ldots\beta_n(\#(\beta_1,\ldots,\beta_n) \rightarrow A(F\beta_1,\ldots F\beta_n))))$$
$$\forall\alpha_1\ldots\alpha_n(\#(\alpha_1,\ldots,\alpha_n) \rightarrow \exists m\, A(\alpha_1,\ldots,\alpha_n, m)) \rightarrow$$
$$\rightarrow \exists F\forall\alpha_1\ldots\alpha_n(\#(\alpha_1,\ldots,\alpha_n) \rightarrow A(\alpha_1,\ldots,\alpha_n, F(\alpha_1,\ldots,\alpha_n))).$$

the proofs are easy modifications of the proofs given for the one-parameter case, using a "genericity lemma" for independent n-tuples, saying that the independent n-tuple $(\alpha_1,\ldots,\alpha_n)$ at $V_{()}^{id} \times \ldots \times V_{()}^{id}$ (n-fold product) is the generic such.

We do have *dependent* versions of pairing and $\forall\alpha\exists\beta$ - continuity:

<u>Theorem 3</u>. *The model over* $\mathbb{K}$ *satisfies*

(i) $\forall\alpha,\beta(\neg\alpha\#\beta \rightarrow \exists F,G\exists\gamma(\alpha=F\gamma \wedge \beta=G\gamma))$

(ii) $\forall\alpha\exists\beta(\neg\alpha\#\beta \wedge A(\alpha,\beta)) \rightarrow \exists F\forall\alpha A(\alpha,F\alpha)$.

<u>proof</u>. (i) If α,β are given at $V_{f_1}^{\bar{a}_1} \times \ldots \times V_{f_n}^{\bar{a}_n}$ and are not dependent, then $\alpha = a_{ij}\circ\alpha_i$, $\beta = a_{ik}\circ\alpha_i$ for some i and some neighbourhood functions a_{ij}, a_{ik} occurring in $\bar{a}_i$. Take $F = a_{ij}$, $G = a_{ik}$, $\gamma = id\circ\alpha_i$, which exists at the object

$V_{f_1}^{\bar{a}_1} \times \ldots \times V_{f_i}^{\bar{a}_i, id} \times \ldots \times V_{f_n}^{\bar{a}_n}$ covering $V_{f_1}^{\bar{a}_1} \times \ldots \times V_{f_n}^{\bar{a}_n}$.

(ii) Suppose $\forall\alpha\exists\beta(\neg\alpha\#\beta \wedge A(\alpha,\beta))$ is forced somewhere, or equivalently (since there are no non-lawlike parameters) everywhere. Then $V_{(\,)}^{id} \Vdash \exists\beta(\neg\alpha\#\beta \wedge A(\alpha,\beta))$, and from this it follows that there is a disjoint inductive bar $\{u_i\}_i$ and elements $f_i \in K$ such that $V_{\bar{u}_i}^{id,f_i} \Vdash A(\alpha,\beta)$ (where α still corresponds to id, and β to f_i). Let $F\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be the lawlike continuous operation with $F|u_i = f_i$, i.e. $F(x) = f_i(x)$ if x extends u_i. Then $V_{\bar{u}_i}^{id,f_i} \Vdash \beta = F(\alpha)$, so $V_{\bar{u}_i}^{id,f_i} \Vdash A(\alpha,F\alpha)$, and hence since the objects $V_{\bar{u}_i}^{id,f_i}$ cover $V_{(\,)}^{id}$, also $V_{(\,)}^{id} \Vdash A(\alpha,F\alpha)$. By genericity, $\Vdash \forall\alpha A(\alpha,F\alpha)$. □

The continuity axiom for the quantifier combination $\forall\alpha_1 \ldots \forall\alpha_n\exists\beta$ in fact splits into several variants. Without proof we state some for $n = 1$.

Theorem 4. *The following versions of* $\forall\alpha\exists\beta$ *-continuity are valid in the model over* $\mathbb{K}$.

(i) (*uniformity*) $\forall\alpha\exists\beta(\alpha\#\beta \wedge A(\alpha,\beta)) \to \exists e \in K\forall u(e(u) \neq 0 \to \exists\beta\forall\alpha \in u(\alpha\#\beta \to A(\alpha,\beta)))$

(ii) $\forall\alpha\exists\beta A(\alpha,\beta) \to \exists e \in K\forall u(e(u) \neq 0 \to (\exists F\forall\alpha \in uA(\alpha,F\alpha) \vee \exists\beta\forall\alpha \in u(\alpha\#\beta \to A(\alpha,\beta))))$.

(By analytic data and continuity for the quantifier combination $\forall\alpha\exists f \in K$, (i) may equivalently be formulated as

(i') $\forall\alpha\exists\beta(\alpha\#\beta \wedge A(\alpha,\beta)) \to \forall\alpha\exists F\forall\beta(\alpha\#\beta \to A(\alpha,F\beta))$.)

4.5. Identifying independent processes.

The obstruction to having the usual form of $\forall\alpha\exists\beta$ - continuity at the end of 4.4 lies in the fact that there are independent "parallel" processes for constructing lawless sequences of neighbourhood functions. As a further abstraction, we will now allow identification of independent processes which "until now" have yielded the same result. This abstraction is formalized by adding more morphisms to the site $\mathbb{K}$ of 4.4: we add morphisms

$$V_f^{\bar{a},\bar{b}} \to V_f^{\bar{a}} \times V_f^{\bar{b}}$$

which, at the level of the underlying spaces, are just the diagonal maps. We thus obtain a new site which has as morphisms all compositions of morphisms of the type described in 4.4 and these new diagonal maps, and with the same basic covers (i)-(iv) above generating the Grothendieck topology. We call this new site $\mathbb{L}$. The model over $\mathbb{L}$, with the universe of choice sequences defined as for the model over $\mathbb{K}$, now comes very close to validating the CS-axioms:

<u>Theorem 1</u>. *In the model over the site* $\mathbb{L}$ *just described, the following are valid:*

(i) (*closure*) $\forall\alpha\forall F\exists\beta(\beta=F\alpha)$

(ii) (*analytic data*) $\forall\alpha(A\alpha\to\exists F(\alpha\in \mathrm{im}(F)\wedge\forall\beta\in \mathrm{im}(F)A\beta))$

(iii) ($\forall\alpha\exists n$ - *continuity*) $\forall\alpha\exists nA(\alpha,n)\to\exists F\forall\alpha A(\alpha,F\alpha)$

(iv) ($\forall\alpha\exists\beta$ - *continuity*) $\forall\alpha\exists\beta A(\alpha,\beta)\to\exists F\forall\alpha A(\alpha,F\alpha)$

<u>proof</u>. (i)-(iii) are proved just as in 4.4 (cf. theorem 1 of 4.4), since the genericity lemma remains valid over $\mathbb{L}$. For $\forall\alpha\exists\beta$ - continuity, suppose $\Vdash\forall\alpha\exists\beta A(\alpha,\beta)$ (at every object, if at any). Then in particular $V^{id}_{(\,)}\Vdash\exists\beta A(\alpha,\beta)$. From this it follows that there is a cover $\{\phi_i\colon U_i\to V^{id}_{(\,)}\}_i$ where either $U_i=V^{id}_{\bar{u}_i}\times V^{f_i}_{g_i}$ and $U_i\Vdash A(\alpha,\beta_i)$ (with α coming from $V^{id}_{\bar{u}_i}$ and β_i from $V^{f_i}_{g_i}$), or $U_i=V^{id,h_i}_{\bar{u}_i}$ and $U_i\Vdash A(\alpha,\beta_i)$ (with α corresponding to id, β_i to h_i), all this for some inductive bar $\{u_i\}_i$. But if U_i is of the first type, we can restrict along

$$V^{id,f_i\circ g_i}_{\bar{u}_i}\longrightarrow V^{id}_{\bar{u}_i}\times V^{f_i\circ g_i}_{\bar{u}_i}\longrightarrow V^{id}_{\bar{u}_i}\times V^{f_i\circ g_i}_{(\,)}\longrightarrow V^{id}_{\bar{u}_i}\times V^{f_i}_{g_i},$$

and we conclude that $V^{id,f_i\circ g_i}_{\bar{u}_i}\Vdash A(\alpha,\beta_i)$. In other words, without loss all U_i are of the second type. Now let $F\colon\mathbb{N}^{\mathbb{N}}\to\mathbb{N}^{\mathbb{N}}$ be the lawlike continuous operation such that $F\,|\,u_i=h_i$. Then $V^{id,h_i}_{\bar{u}_i}\Vdash F(\alpha)=\beta_i$, so $V^{id,h_i}_{\bar{u}_i}\Vdash A(\alpha,F\alpha)$. Since $\{V^{id,h_i}_{\bar{u}_i}\to V^{id}_{(\,)}\}_i$ is a cover, it follows that $V^{id}_{(\,)}\Vdash A(\alpha,F\alpha)$, so by the genericity lemma, $\models\forall\alpha A(\alpha,F\alpha)$. □

The model over $\mathbb{L}$ is not a model for CS, since the pairing axiom is not satisfied. With one small modification, however, we obtain pairing. Let $\mathbb{M}$ be the site with the same objects and morphisms as $\mathbb{L}$, and with the Grothendieck topology generated by the basic covers (i)-(iv) of 4.4, and in addition for each object $V_{f_1}^{\bar{a}_1} \times \ldots \times V_{f_n}^{\bar{a}_n}$ of the site,

(v) (diagonals) the collection of morphisms

$$\{V_{(\ )}^{\bar{a}_1 f_1 h_1} \times \ldots \times V_{(\ )}^{\bar{a}_n f_n h_n} \longrightarrow V_{f_1}^{\bar{a}_1} \times \ldots \times V_{f_n}^{\bar{a}_n} \mid \text{all} \quad (h_1,\ldots,h_n) \in K^n\}$$

is a cover, where on the underlying spaces the morphism for an n-tuple $h_1, \ldots, h_n$ is given by

$$(x_1,\ldots,x_n) \longmapsto (f_1 \circ h_1 * x_1,\ldots,f_n \circ h_n * x_n).$$

Over the site $\mathbb{M}$, we define a universe of choice sequences exactly as in 4.4. We then obtain

Theorem 2. *The model over* $\mathbb{M}$ *is a model for* CS.

proof. All the axioms are verified exactly as for theorem 1 above, except for pairing. But pairing is trivially forced to hold by the new covers of type (v). □

Footnotes.

1. The second author acknowledges financial support by the Netherlands Organization for the Advancement of Pure Research (ZWO).

2. The contents of this paper differs from the lecture given by the second author at the Logic Colloquium in Aachen, where some results from van der Hoeven & Moerdijk (to appear), (to appear 2) were presented.

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THE JOURNAL OF SYMBOLIC LOGIC
Volume 49, Number 2, June 1984

HEINE-BOREL DOES NOT IMPLY THE FAN THEOREM

IEKE MOERDIJK

Introduction. This paper deals with locales and their spaces of points in intuitionistic analysis or, if you like, in (Grothendieck) toposes. One of the important aspects of the problem whether a certain locale has enough points is that it is directly related to the (constructive) completeness of a geometric theory. A useful exposition of this relationship may be found in [1], and we will assume that the reader is familiar with the general framework described in that paper.

We will consider four formal spaces, or locales, namely formal Cantor space C, formal Baire space B, the formal real line R, and the formal function space R^R being the exponential in the category of locales (cf. [3]). The corresponding spaces of points will be denoted by pt(C), pt(B), pt(R) and pt(R^R). Classically, these locales all have enough points, of course, but constructively or in sheaves this may fail in each case. Let us recall some facts from [1]: the assertion that C has enough points is equivalent to the compactness of the space of points pt(C), and is traditionally known in intuitionistic analysis as the *Fan Theorem* (FT). Similarly, the assertion that B has enough points is equivalent to the principle of (monotone) *Bar Induction* (BI). The locale R has enough points iff its space of points pt(R) is locally compact, i.e. the unit interval $\mathrm{pt}[0,1] \subset \mathrm{pt}(R)$ is compact, which is of course known as the *Heine-Borel Theorem* (HB). The statement that R^R has enough points, i.e. that there are "enough" continuous functions from R to itself, does not have a well-established name. We will refer to it (not very imaginatively, I admit) as the principle (EF) of Enough Functions.

As is well known, (BI) $\Rightarrow$ (FT) $\Rightarrow$ (HB). A possible way of explaining that (BI) implies (FT) is by observing that B is homeomorphic to the exponential C^C, as has recently been pointed out by Hyland [3]. In the present context, therefore, the exponential R^R is a natural object of study. Note that (BI) $\Rightarrow$ (EF) since R^R is countably presented, and hence a continuous image of B. The principle (FT) holds in every spatial topos, but (BI) does not, so the implication (BI) $\Rightarrow$ (FT) is not reversible (cf. [2]).

In [1, §4.11], it was asked whether (HB) implies (FT). We will show that this is not the case by proving that R has enough points in sheaves over the locale $K(\mathbf{R}^2)$ of coperfect open subsets of $\mathbf{R}^2$. Hyland has asked whether (BI) or (FT) follows from the assertion that R^R has enough points. We will show that in the same topos of

Received October 5, 1982.

0022-4812/84/4902-0015/$01.60

sheaves over $K(\mathbf{R}^2)$, (EF) holds, thus answering Hyland's question negatively (§2 below). The converse implication (FT) $\Rightarrow$ (EF) will also be seen to be false, but (EF) $\Rightarrow$ (HB) is true (§1).

Thus, our results complete the picture of valid implications in intuitionistic analysis, or in toposes, between the four statements (FT), (BI), (HB), and (EF): in the diagram below, the implications indicated are the only ones that hold.

$$\begin{array}{ccc} (\mathrm{BI}) & \Rightarrow & (\mathrm{FT}) \\ \Downarrow & & \Downarrow \\ (\mathrm{EF}) & \Rightarrow & (\mathrm{HB}) \end{array}$$

§1. Internal locales. Let us begin with some conventions. There has been some confusion concerning the use of the terms locale, frame, space, etc. In this paper, the words *locale* and *frame* are used as in [1], and the elements of the frame corresponding to a locale A will be called the *opens* of A. A *space* is a locale with enough points, or equivalently, a sober topological space. The product sign $\times$ will always denote the product in the category of locales. If X and Y are spaces their product in the category of topological spaces is denoted by $X \times_s Y$; so $X \times_s Y \cong \mathrm{pt}(X \times Y)$. As pointed out above, C, B and R all have enough points classically, and we will use $\mathbf{C}$, $\mathbf{B}$, and $\mathbf{R}$ to refer to the corresponding *external* spaces. $\mathbf{R}^{\mathbf{R}}$ refers to the external exponential, i.e. the space of continuous functions $\mathbf{R} \to \mathbf{R}$ with the compact-open topology.

If A is an external locale, (internal) locales in $\mathrm{Sh}(A)$ are locales over A, i.e. continuous maps $X \to A$ of locales in the real world of sets, and internal continuous maps from X to Y correspond to external continuous maps over A. As is well known, formal Cantor space, formal Baire space, and the formal real line interpreted in $\mathrm{Sh}(A)$ are represented by the projections $\mathbf{C} \times A \to A$, $\mathbf{B} \times A \to A$, and $\mathbf{R} \times A \to A$. Since R^R is defined as the exponential in the category of locales, it follows that its interpretation in $\mathrm{Sh}(A)$ is presented by $\mathbf{R}^{\mathbf{R}} \times A \to A$.

The following very useful observation is due to Hyland, and is also mentioned in [1].

1.1. LEMMA. *Let X and Y be sober spaces, i.e. locales with enough points. Then the internal locale Y_X represented by the projection $Y \times X \to X$ has enough points in* $\mathrm{Sh}(X)$ *iff the locale product $Y \times X$ has enough points externally, i.e. coincides with $Y \times_s X$. (This happens for example if either X or Y is locally compact.)*

Recall that a sublocale A_j of a locale A is called *closed* if j is of the form $(-) \vee a$ for some open a of A. It is easily seen that classically, a closed sublocale of a space is again a space. (This need not be true intuitionistically! Cf. 2.6 below.) We now immediately derive

1.2. THEOREM. (a) *In spatial toposes,* (EF) *holds if and only if* (BI) *does.*

(b) *There exists a spatial topos in which* (EF) *fails, hence in particular* (FT) *does not imply* (EF).

PROOF. (a) Note that in sets, $\mathbf{B} = \mathbf{N}^{\mathbf{N}}$ is homeomorphic to a closed subspace of $\mathbf{R}^{\mathbf{R}}$. Now suppose X is any space such that $\mathrm{Sh}(X)$ satisfies (EF). Then by Lemma 1.1, $\mathbf{R}^{\mathbf{R}} \times X$ is spatial. But since the pullback of a closed sublocale is again closed, it follows from the remark above that $\mathbf{N}^{\mathbf{N}} \times X$ must be spatial, so by applying 1.1 again we find that $\mathrm{Sh}(X)$ satisfies (BI). Conversely, the implication (BI) $\Rightarrow$ (EF) holds in any topos, as pointed out in the Introduction.

(b) (BI) fails in sheaves over the space $\mathbf{Q}$ of rationals, but (FT) holds in any spatial topos (cf. [2]). □

Let us observe that R is a continuous image of R^R: evaluation at a point of R gives a surjective map $R^R \to R$ of locales. Hence (EF) implies (HB). Thus the only questions still open are whether (EF) implies (FT) and whether (HB) implies (FT). The first implication, and hence also the second, will be seen to be false in sheaves over $K(\mathbf{R}^2)$, to which we now turn.

§2. The model over $K(\mathbf{R}^2)$. Let j be the smallest nucleus (J-operator, closure-operator) on $\mathcal{O}(\mathbf{R}^2)$ which identifies U and $U\setminus\{t\}$, for each open U and each point t. The resulting sublocale of $\mathbf{R}^2$ is denoted by $K(\mathbf{R}^2)$; its lattice of opens is precisely the lattice of complements of perfect closed subsets of $\mathbf{R}^2$. Obviously, $K(\mathbf{R}^2)$ cannot have any points. In [2], it was shown that (FT) fails in sheaves over $K(\mathbf{R}^2)$. We will show that (EF), and hence (HB), do hold over $K(\mathbf{R}^2)$, thus answering the remaining open questions mentioned above. To illustrate how small $K(\mathbf{R}^2)$ really is, we begin with the following observation.

2.1. PROPOSITION. *Let A be any subset of $\mathbf{R}^2$ which does not contain a perfect closed set. Then the inclusion $K(\mathbf{R}^2) \hookrightarrow \mathbf{R}^2$ of locales factors through the inclusion $\mathbf{R}^2\setminus A \hookrightarrow \mathbf{R}^2$ of spaces, i.e. $K(\mathbf{R}^2)$ is a sublocale of $\mathbf{R}^2\setminus A$.*

PROOF. If U is open in the subspace $\mathbf{R}^2\setminus A$, then $U = \mathrm{Int}(U \cup A)\setminus A$ (where the interior is taken in $\mathbf{R}^2$). For such a $U \in \mathcal{O}(\mathbf{R}^2\setminus A)$, we denote $\mathrm{Int}(U \cup A)$ by $\hat{U}$.

$$\begin{array}{ccc} \mathcal{O}(\mathbf{R}^2\setminus A) & \xleftarrow{(-)\cap(\mathbf{R}^2\setminus A)} & \mathcal{O}(\mathbf{R}^2) \\ & \searrow^{\varphi^*} & \downarrow j \\ & & K(\mathbf{R}^2) \end{array}$$

Now in the above diagram of frames, define φ^* by $\varphi^*(U) = j(\hat{U})$. We claim that

(a) *the diagram commutes, i.e. $j(V) = j(\mathrm{Int}(V \cup A))$ for each open $V \subseteq \mathbf{R}^2$, and*

(b) *φ^* is a frame map, i.e. it preserves finite meets and arbitrary joins.*

PROOF OF (a). Obviously, $j(V) \subseteq j(\mathrm{Int}(V \cup A))$. $\mathbf{R}^2\setminus j(V)$ is the largest perfect closed subset of $\mathbf{R}^2\setminus V$, and similarly for $\mathbf{R}^2\setminus j(\mathrm{Int}(V \cup A))$. So to prove that $j(\mathrm{Int}(V \cup A)) \subseteq j(V)$, it suffices to show that if F is a perfect closed subset of $\mathbf{R}^2\setminus V$, we also have $F \subseteq \mathbf{R}^2\setminus\mathrm{Int}(V \cup A) = \overline{\mathbf{R}^2\setminus(V \cup A)}$. Now if x is a point of such an F, let W_x be an open neighbourhood of x, and let C be a copy of the Cantor set, $C \subseteq W_x \cap F$. Then $C \cap V = \varnothing$, so $W_x \subseteq V \cup A$ would imply $C \subseteq A$, contradicting the assumption on A. Hence $W_x \nsubseteq V \cup A$, i.e. $x \in \overline{\mathbf{R}^2\setminus(V \cup A)}$.

PROOF OF (b). It is clear that φ^* preserves the top and bottom elements, as well as binary intersections. Now let $\{U_i \mid i \in I\}$ be a collection of open subsets of $\mathbf{R}^2\setminus A$. We need to show that

$$j\left(\bigcup_{i\in I}\mathrm{Int}(A \cup U_i)\right) = j\left(\mathrm{Int}\left(A \cup \bigcup_{i\in I} U_i\right)\right).$$

But by (a), it suffices to show that $A \cup \bigcup_{i\in I}\mathrm{Int}(A \cup U_i) = A \cup \mathrm{Int}(A \cup \bigcup_{i\in I} U_i)$, which is obviously true.

Thus, we have a map of locales $K(\mathbf{R}^2) \xrightarrow{\varphi} \mathbf{R}^2\setminus A$, giving the required factorization. □

We now turn to $\mathrm{Sh}(K(\mathbf{R}^2))$. If X is a space (externally), we obtain an internal locale represented by the projection $X \times K(\mathbf{R}^2) \to K(\mathbf{R}^2)$. $X \times K(\mathbf{R}^2)$ is a sublocale of $X \times \mathbf{R}^2$ (note that $X \times \mathbf{R}^2 = X \times_s \mathbf{R}^2$, since $\mathbf{R}^2$ is locally compact), and we have a pullback of locales

$$\begin{array}{ccc} X \times K(\mathbf{R}^2) & \xrightarrow{\pi_2} & K(\mathbf{R}^2) \\ \big\downarrow & & \big\downarrow \\ X \times \mathbf{R}^2 & \xrightarrow{\pi_2} & \mathbf{R}^2 \end{array}$$

We write $1 \otimes j : \mathcal{O}(X \times \mathbf{R}^2) \to \mathcal{O}(X \times K(\mathbf{R}^2))$ for the frame-map corresponding to the inclusion $X \times K(\mathbf{R}^2) \hookrightarrow X \times \mathbf{R}^2$. Thus each open of $X \times K(\mathbf{R}^2)$ is the form $1 \otimes j(W)$ for some subset W of $X \times \mathbf{R}^2$, and we use W as a *name* for this open of $X \times K(\mathbf{R}^2)$. Recall that the (global) opens of the internal locale $X \times K(\mathbf{R}^2) \to K(\mathbf{R}^2)$ are just the opens of $X \times K(\mathbf{R}^2)$. The crucial part of the proof is the following lemma.

2.2. LEMMA. *Let (X,d) be a complete metric space, and W an open subset of $X \times \mathbf{R}^2$ such that $\mathrm{Sh}(K(\mathbf{R}^2)) \not\models W = \top$ when W is regarded as an open of the internal locale $X \times K(\mathbf{R}^2) \to K(\mathbf{R}^2)$. Then there exist a copy $D \subseteq \mathbf{R}^2$ of the Cantor set and a continuous function $f: D \to X$ such that the graph of f is disjoint from W.* ($\top$ *denotes the top element of any locale.*)

PROOF. It is notationally convenient to assume that complete metrics have been fixed in X, $\mathbf{R}^2$, and $X \times \mathbf{R}^2$, which are bounded by 1, and such that $A \subseteq X$ and $B \subseteq \mathbf{R}^2$, $\mathrm{diam}(A \times B) \geq \max(\mathrm{diam}(A), \mathrm{diam}(B))$. By induction we will first define for each finite sequence $u \in 2^{<\mathbf{N}}$ a nonempty basic open subset $O_u = U_u \times V_u$ of $X \times \mathbf{R}^2$ such that for all sequences u and v:

1) $\bar{O}_u \subseteq O_v$ if u extends v, $\bar{O}_u \cap \bar{O}_v = \varnothing$ if u and v are incompatible, and $\mathrm{diam}(O_u) \leq 2^{-\mathrm{lth}(u)}$; and hence also

2) $\bar{V}_u \subseteq V_v$ if u extends v, $\bar{V}_u \cap \bar{V}_v = \varnothing$ if u and v are incompatible, and $\mathrm{diam}(V_u) \leq 2^{-\mathrm{lth}(u)}$; furthermore

3) $j(V_u) \not\models O_u \leq W$ (where j is the nucleus of 2.1).

For the case $u = \langle\ \rangle$, the empty sequence, we just take $U_u = X$, $V_u = \mathbf{R}^2$. For the induction step, the following observation is needed:

(*) *For any sentence φ, and any nonempty open $V \subseteq \mathbf{R}^2$, if $j(V) \not\models \varphi$ then there are nonempty opens $V_0, V_1 \subseteq V$ with $\bar{V}_0 \cap \bar{V}_1 = \varnothing$, such that $j(V_0) \not\models \varphi, j(V_1) \not\models \varphi$.*

PROOF OF (*). If $j(V) \not\models \varphi$, then $V - [\![\varphi]\!]$ must contain a perfect closed set, and hence a copy of the Cantor set K. Write K as the disjoint sum of two copies K_0 and K_1 of the Cantor set, and let V_0 and V_1 be open neighbourhoods of K_0 and K_1 such that $\bar{V}_0 \cap \bar{V}_1 = \varnothing$. Then $K_0 \subseteq V_0 - [\![\varphi]\!]$, so $j(V_0) \not\leq [\![\varphi]\!]$, i.e. $j(V_0) \not\models \varphi$. Similarly $j(V_1) \not\models \varphi$. (The referee pointed out that this argument is somewhat simpler than my original proof of (*).)

(*) being established, let us suppose that the O_u have been defined for all sequences of length n, and pick one such sequence u. We will now define O_{u*0} and O_{u*1}. Using (*), we may choose two nonempty open sets $V_0, V_1 \subseteq V_u$ such that $\bar{V}_0 \cap \bar{V}_1 = \varnothing$, and $j(V_0) \not\models O_u \leq W, j(V_1) \not\models O_u \leq W$. That is, in $X \times K(\mathbf{R}^2)$, we have for each $i = 0, 1$,

$$O_u \wedge \pi_2^{-1}(V_i) \not\leq W \wedge O_u \wedge \pi_2^{-1}(V_i),$$

or equivalently $U_u \times V_i \nleq W$. Now keep i fixed, and cover $U_u \times V_i$ in $X \times \mathbf{R}^2$ by open cubes $B_i^\zeta = U_i^\zeta \times V_i^\zeta$ of diameter $< 2^{-(n+1)}$ (ζ ranging over some indexing set), such that $\overline{B_i^\zeta} \subseteq U_u \times V_i$. Then the B_i^ζ also form a cover of $U_u \times V_i$ in $X \times K(\mathbf{R}^2)$, so for some ζ, say ζ_i, we must have that in $X \times K(\mathbf{R}^2)$, $B_i^\zeta \nleq W$. Now let $O_{u*i} = B_i^{\zeta_i} = U_i^{\zeta_i} \times V_i^{\zeta_i}$, so $V_{u*i} = V_i^{\zeta_i} \subseteq V_i$. Then from the fact that $B_i^{\zeta_i} \nleq W$ in $X \times K(\mathbf{R}^2)$ it follows that $O_{u*i} \nleq W \cap \pi_2^{-1}(V_{u*i})$ in $X \times K(\mathbf{R}^2)$, which just means that $j(V_{u*i}) \nvDash O_{u*i} \leq W$, so condition 3) is satisfied. Conditions 1) and 2) are obvious.

We will now build our Cantor set D. Let us write $F = (X \times \mathbf{R}^2)\backslash W$, and $F_u = O_u \backslash W = (U_u \times V_u)\backslash W$. By condition 3), each F_u is nonempty (in fact, from Proposition 2.1 we may even derive that $\pi_2(F_u)$ must contain a copy of the Cantor set). Hence we can choose a point $y_u \in F_u$ for each $u \in 2^{<\mathbf{N}}$. Write $x_u = \pi_2(y_u) \in V_u$. By condition 2), we find that for each $\alpha \in 2^{\mathbf{N}}$, $\{x_u\}_{u \leq \alpha}$ is a Cauchy sequence, so it converges to a point $x_\alpha \in \mathbf{R}^2$. Note that also by 2), all the x_α must be different, since $x_\alpha \in \bar{V}_u$ if $u \leq \alpha$. Let $D = \{x_\alpha \mid \alpha \in 2^{\mathbf{N}}\}$. Then as a subspace of $\mathbf{R}^2$, D is homeomorphic to the Cantor space. (The canonical map $\alpha \mapsto x_\alpha$ is a continuous bijection, hence a homeomorphism, from the Cantor space to D.)

By condition 1), each sequence $\{y_u\}_{u \leq \alpha}$ also is a Cauchy sequence. Hence it converges to a point y_α, which must necessarily lie in the closed set F. Now let $g: D \to F \subseteq X \times \mathbf{R}^2$ be the function defined by $g(x_\alpha) = y_\alpha$. Then g is continuous, and since $\pi_2(y_\alpha) = \lim_{u \leq \alpha} \pi_2(y_u) = \lim_{u \leq \alpha} x_u = x_\alpha$, g must be of the form (f, id), thus giving us the required continuous function $f: D \to X$. □

Recall that a (Hausdorff) space X is called an *absolute neighbourhood retract* (ANR) for a space Y if for any closed subset $G \subseteq Y$, every continuous function $G \to X$ has a continuous extension over some open subset of Y containing G.

2.3. THEOREM. *If X is a complete metrizable space and an* ANR *for $\mathbf{R}^2$, then the internal locale represented by $X \times K(\mathbf{R}^2) \to K(\mathbf{R}^2)$ has enough points in sheaves over $K(\mathbf{R}^2)$.*

PROOF. Let W and W' be two open subsets of $X \times \mathbf{R}^2$ with $W \subseteq W'$, such that if we regard them as names for internal opens of the locale $X \times K(\mathbf{R}^2) \to K(\mathbf{R}^2)$, then $\mathrm{Sh}(K(\mathbf{R}^2)) \vDash \mathrm{pt}(W') \subseteq \mathrm{pt}(W)$. We want to show that $\mathrm{Sh}(K(\mathbf{R}^2)) \vDash W' \subseteq W$, and for this it suffices to show that for every cube $U \times V$ which is contained in W', $\mathrm{Sh}(K(\mathbf{R}^2)) \vDash U \times V \subseteq W$. If not, then we can apply Lemma 2.2 with X replaced by U and $\mathbf{R}^2$ replaced by V to obtain a continuous function $g: D \to U$, where D is perfect closed, such that for each $p \in D$, $(g(p), p) \notin W$. We can then extend g to a continuous function $N \xrightarrow{f} U$ with open domain $N \subseteq V$, and the function $(f, \mathrm{id}): N \to X \times \mathbf{R}^2$ then restricts to an internal point q of $X \times K(\mathbf{R}^2) \to K(\mathbf{R}^2)$ defined over $j(N)$. Since $(f, \mathrm{id})^{-1}(W) \cap D = \varnothing$, we cannot have $j(N) \vDash q \in W$, although we do have $j(N) \vDash q \in W'$. This contradicts the fact that $\mathrm{Sh}(K(\mathbf{R}^2)) \vDash \mathrm{pt}(W') \subseteq \mathrm{pt}(W)$. □

2.4. COROLLARY. (a) *In* $\mathrm{Sh}(K(\mathbf{R}^2))$, *$R^R$ and R have enough points, i.e.* $\mathrm{Sh}(K(\mathbf{R}^2))$ *satisfies* (EF) *and* (*hence*) (HB).

(b) (EF) *does not imply* (FT) (*and hence neither does* (HB)).

PROOF. As remarked at the beginning of this section, (b) follows from (a). (a) is immediate from the preceding theorem, since $\mathbf{R}^{\mathbf{R}}$ and $\mathbf{R}$ satisfy the hypothesis of 2.3 by Tietze's extension theorem for normal spaces. □

The proof above does not use the full strength of Tietze's theorem. It is obvious that the same argument gives a more general version of 2.3 and 2.4: for example, we could replace $\mathbf{R}^2$ by any complete metric dense-in-itself space Y and prove the analogous results for sheaves over $K(Y)$. I do not know any application of this more general fact.

2.5. REMARK. It follows from Proposition 2.1 that if W is an open subset of $X \times \mathbf{R}^2$ such that $\mathrm{Sh}(K(\mathbf{R}^2) \not\models W = T$, $\pi_2(X \times \mathbf{R}^2 \backslash W)$ must contain a copy of the Cantor set. The converse is false, however. For example, let $C \subseteq \mathbf{R}^2$ be a fixed copy of the Cantor set, and let X be the set C with the discrete topology. Then if we let $W = X \times \mathbf{R}^2 \backslash \{(x, x) \mid x \in C\}$, W is open, but the image of W under the quotient map of frames $1 \otimes j : \mathcal{O}(X \times \mathbf{R}^2) \to \mathcal{O}(X \times K(\mathbf{R}^2))$ is the top-element $\top$; in other words, $\mathrm{Sh}(K(\mathbf{R}^2)) \models W = \top$. To see this, let $B_x = \{x\} \times \mathbf{R}^2 \in O(X \times \mathbf{R}^2)$. Then for each $x \in C$, $\mathrm{Sh}(K(\mathbf{R}^2)) \models B_x \leq W$ (just omit the point x from R^2); hence since the sets B_x form an open cover of $X \times \mathbf{R}^2$, $\mathrm{Sh}(K(\mathbf{R}^2)) \models \top = \bigvee_x B_x \leq W$.

This explains why the Cantor set D had to be constructed quite carefully in the proof of 2.2, and could not be obtained by just applying 2.1. However, if X is compact, the converse does hold, and a considerably easier proof can be given of Lemma 2.2 for this case. Consequently, Theorem 2.3 can be proved much more easily for locally compact X. This covers the special case $\mathrm{Sh}(K(\mathbf{R}^2)) \models$ (HB) of Corollary 2.4, but does not apply to the case of R^R, of course. □

Let me end this paper by drawing attention to another curious phenomenon in intuitionistic analysis illustrated by the model over $K(\mathbf{R}^2)$.

2.6. COROLLARY. *(In localic toposes) A compact closed sublocale of a space need not be a space.*

PROOF. Look at R and C in $\mathrm{Sh}(K(\mathbf{R}^2))$. □

Acknowledgements. This paper was written in June 1982, when I was visiting the University of Cambridge. I would like to thank Martin Hyland, and the logic section of the Philosophy Department of the University of Amsterdam for making this visit possible. I am also grateful to Martin Hyland for drawing my attention to the questions that this paper answers.

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Connected locally connected toposes are path-connected

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Introduction

The proposition stated by the title was conjectured by A. Joyal in 1983 during a seminar at Columbia University. Every topologist knows that a connected locally connected topological space is not necessarily path-connected. The natural numbers with the cofinite topology is an example, and so is "the long segment". However, it is true that all connected locally connected complete metric spaces are path-connected (Menger (1929), Moore (1932)).

Toposes are generalizations of (sober) topological spaces, if we identify a topological space X with the topos of sheaves on X. The notions of connectedness and local connectedness were defined in SGA 4 (Grothendieck and Verdier (1972)) for toposes in a way that extends the usual versions of these concepts for topological spaces. How then can Joyal's conjecture be true ? The explanation lies in the correct interpretation of what path-connectedness means for a topos $\mathcal{E}$. It does not mean that "for every pair of points x_0, x_1 of $\mathcal{E}$ there is a path $I \xrightarrow{f} \mathcal{E}$ with $f(0) = x_0$, $f(1) = x_1$". This is an inappropriate definition in as much as toposes do not necessarily have points. Instead, one has to construct the "space of paths in $\mathcal{E}$" as being again a topos. More precisely, a topos $\mathcal{F}$ is exponentiable if the 2-functor $\mathcal{F} \times (-)$ has a right 2-adjoint $(-)^{\mathcal{F}}$, and $\mathcal{E}^{\mathcal{F}}$ is interpretable as the topos of maps from $\mathcal{F}$ to $\mathcal{E}$. Points of $\mathcal{E}^{\mathcal{F}}$ correspond to maps from $\mathcal{F}$ to $\mathcal{E}$. The (topos of sheaves on the) unit interval I is an exponentiable topos, so for any topos $\mathcal{E}$ we may form the topos $\mathcal{E}^I$ of paths in $\mathcal{E}$. The inclusion of the end-points $\{0,1\} \to I$ induces a map of toposes

$$\mathcal{E}^I \to \mathcal{E} \times \mathcal{E},$$

* supported by the Netherlands Organisation for the Advancement of Pure Research (ZWO).

and it is natural to say that $\mathcal{E}$ is path-connected if this map is a surjection. We will prove the following slightly stronger result (over an arbitrary base topos).

Theorem. For any connected locally connected topos $\mathcal{E}$, the canonical map $\mathcal{E}^I \to \mathcal{E} \times \mathcal{E}$ is an open surjection; so in particular, $\mathcal{E}$ is path-connected.

The explanation of what "goes wrong" for spaces like the long segment L is straightforward. The topos L^I has no points corresponding to non-trivial paths reaching the end-point (see e.g. Steen and Seebach (1978), Engelking (1977)), but $L^I \to L \times L$ is nevertheless a surjective map of toposes.

In attempting to prove a result of this kind, two approaches are available. One is to manipulate directly with a site for $\mathcal{E}^I$ (or a suitable site "covering" this topos). In this rather algebraic approach, one generally "stays at one place" (one base topos). The other approach is more geometrical: the strategy is to use adequate extensions of the base topos available from general topos theory, which enable one to follow classical arguments about points of separable metric spaces rather closely. Although both approaches are equivalent, we will follow the second one in this paper, because it shows more clearly the interplay between general topos theory and arguments (somewhat similar to those) from topology. (But we will also give a brief description of the maps of sites involved in the "algebraic approach", see 2.6 below.)

Apart from the element of surprise, and as an illustration of the slogan that generalized spaces are better behaved than topological spaces, what can this result be used for ? One answer is: homotopy theory for toposes. Homotopy groups of topological spaces are really topological groups (which usually, but not always, turn out to be discrete), so it is hardly revolutionary to insist that homotopy groups (or groupoids, or other gadgets) of toposes are themselves toposes. The point is made in SGA 4 that the right notion of quotient by an equivalence relation for toposes is to take the topos of descent data. If $X.$ denotes a simplicial topos, $\Pi_0(X.)$ will denote the topos of descent data; that is, its objects are pairs (A, σ) where A is an object of X_0 and σ is an isomorphism $d_0^*A \to d_1^*A$ in X_1, satisfying the usual coherence conditions. We have a surjective map of toposes $X_0 \to \Pi_0(X_0)$. Let $\Delta.$ denote the cosimplicial topos given by the standard simplices. For any topos E, we have the simplicial topos $E^{\Delta.}$ and we define $\pi_0(E)$ to be the topos $\Pi_0(E^{\Delta.})$. This is the topos of connected components of E.

Of course, Δ_1 is just the unit interval I. Let us denote by $P(E)$ the $E \times E$ - topos $E^I \to E \times E$. We denote by $\Gamma(E)$ the $E \times E$ - topos $\Gamma_1(E) \to E \times E$ given by $\pi_0(P(E))$, got by applying π_0 in the context of $E \times E$ - toposes. We assert that $\Gamma_1(E) \to E \times E$ is a groupoid topos, and is the fundamental groupoid topos of E. Pulling back along the diagonal $E \to E \times E$ gives the fundamental group $\pi_1(E)$ as an E - topos (this takes care of the base point). We hope to say more about this in a later paper.

P. Johnstone has pointed out to us an example of a topological space having trivial fundamental group as a topological space, but a non-trivial fundamental group as a generalized space, a topos. The example is a "long loop" (obtained from the long segment by identifying the two end-points), which admits no non-trivial maps from the circle, but - being connected and locally connected - has a non-trivial, but pointless, generalized "space" of loops. Since the homotopy relation for it is given by an open equivalence relation, its fundamental group as a generalized space will be discrete and isomorphic to $\mathbb{Z}$.

Both authors would like to thank the organizers of the category theory conference at Löwenberg Manor, where the actual writing of this article got started, and P. Johnstone, A. Joyal and A. Kock for valuable conversations.

1. Preliminaries

In this paper, all toposes are Grothendieck toposes over a fixed, but arbitrary, base topos S (thought of as "the" category of sets).

1.1 Spaces and locales. Our terminology concerning locales, spaces, etc. will be as in Joyal and Tierney (1982). So a locale is a complete Heyting algebra, and a map of locales is a function which preserves finite meets and arbitrary sups; spaces are the duals of locales. For a space X, $\mathcal{O}(X)$ denotes the corresponding locale, the elements of which are called the opens of X. A (sober) topological space is a space with enough points.

A presentation of a space X is a poset $\mathbb{P}$ equipped with a stable system of covering families, such that $\mathcal{O}(X)$ is isomorphic to the set of downwards closed subsets of $\mathbb{P}$ which are closed for the system of covers, i.e. $\mathcal{O}(X) \cong \{S \subseteq \mathbb{P} \mid (p < q \in S \Rightarrow p \in S)$, and $(T$ covers p, $T \subseteq S \Rightarrow p \in S)\}$. (Equivalently, $\mathbb{P}$ is a site for the topos of sheaves on X.)

For general information about spaces and locales, see Isbell (1972), Johnstone (1982), Joyal & Tierney (1982), and Hyland (1981).

1.2 Open maps. A geometric morphism $F \xrightarrow{\varphi} E$ is open if φ^* preserves first-order logic. φ is open iff its localic part (its spatial reflection) is, iff the unique $\wedge\bigvee$ - map $\Omega_E \to \varphi_*(\Omega_F)$ in E has an internal left-adjoint. (A topos F is called open if the canonical map $F \to S$ is an open geometric morphism.) An important characterization states that $F \overset{\varphi}{\to} E$ is open iff there is a site $\mathbb{C}$ for F in E such that (in E it holds that) all covers in $\mathbb{C}$ are inhabited. We can take $\mathbb{C}$ to have a terminal object iff $F \to E$ is also a surjection. In particular, a space X is open (and surjective) iff it has a presentation $\mathbb{P}$ (with a top-element 1) the covers of which are all inhabited.

For some proofs and more information, see Johnstone (1980), Joyal and Tierney (1982).

1.3 Connected locally connected maps. A geometric morphism $F \xrightarrow{\varphi} E$ is connected if φ^* is full and faithful. φ is called locally connected (or molecular) if φ^* commutes with Π-functors. $F \overset{\varphi}{\to} E$ is locally connected iff there is a site $\mathbb{C}$ for F in E all whose covers are inhabited and connected, and we may take $\mathbb{C}$ to have a terminal iff φ is also connected. In particular, a space X is connected and locally connected iff it has a presentation $\mathbb{P}$ with a top-element 1, whose covers are inhabited and connected. For $\mathbb{P}$ we can take the connected open subspaces of X, so we may without loss of generality assume that $\mathbb{P}$ is closed under sups of chains. (A chain in $\mathbb{P}$ is a sequence $(V_1,\ldots,V_k)$ of elements of $\mathbb{P}$ such that for each $i = 1,\ldots,k-1$, there is a $W_i \in \mathbb{P}$ with $W_i < V_i$, $W_i < V_{i+1}$; equivalently, since all covers in $\mathbb{P}$ are inhabited, $V_i \wedge V_{i+1}$ is a surjective (open) space.) We call such a presentation $\mathbb{P}$ of X, with $1 \in \mathbb{P}$, $\mathbb{P}$ closed under sups of chains, and all covers inhabited and connected, a molecular presentation of X.

For proofs and further information, see Barr and Paré (1980), and the appendix of Moerdijk (1984).

The properties of being an open (surjective) map, and of being a (connected) locally connected map are closed under composition. Moreover, as is clear from the characterizations in terms of sites, these properties are preserved by pulling back along an arbitrary geometric morphism.

1.4 Exponentiability. A topos F is exponentiable if the functor $F \times (-)$ of Grothendieck toposes over S has a right-adjoint $(-)^F$ (in the appropriate

2-categorical sense). Any compact regular space X is exponentiable as a topos, and if Y is any space, Y^X is the topos of sheaves on the space Y^X, i.e. the exponential in the category of spaces (so there is no harm in not distinguishing the two notationally). The construction of the exponential space Y^X in S is stable; that is, if $S' \overset{\varphi}{\to} S$ is a geometric morphism, then $\varphi^{\#}(Y^X) \cong \varphi^{\#}(Y)^{\varphi^{\#}(X)}$, as spaces in S'.

For exponentials of toposes see Johnstone and Joyal (1982); the case of spaces is dealt with in Hyland (1981).

1.5 The unit interval. By the unit interval I we will always mean the unit interval as defined as a locale, as a "formal space" (see e.g. Fourman and Grayson (1982)). Thus, in any topos S, I is a compact regular space, and hence exponentiable as a space and as a topos. Moreover, the construction of I as a formal space is stable; i.e. for a geometric morphism $S' \overset{\varphi}{\to} S$, $\varphi^{\#}(I_S) = I_{S'}$ (where the subscript denotes where I is constructed; by stability, this subscript can be suppressed).
(I need not coincide with the corresponding topological space of Dedekind cuts; in fact it does iff this topological space is compact. Since we work over an arbitrary base topos, we have to deal with the formal space, rather than the topological space.)

1.6 Some base extensions. We will use the following three types of base extension.

Lemma A. (Joyal) Let E be a given topos over S. Then there exists a space X in S and a geometric morphism $X \to E$ which is connected and locally connected. Thus, if E is itself connected locally connected, so is X.

proof. See Johnstone (1984).

Lemma B. Let S be an object of $\mathcal{S}$. Then there exists an open surjection $\mathcal{E} \overset{\gamma}{\to} \mathcal{S}$ such that S is countable in $\mathcal{E}$, i.e. there is an epimorphism $\mathbb{N} \to \gamma^*(S)$ in $\mathcal{E}$. ($\mathcal{E}$ may be taken to be a space.)

proof. See Joyal and Tierney (1982), § V.3.

Axioms of choice are generally not available in a topos. However, the following lemma says that we can apply an axiom of dependent choices in the context of toposes, provided we allow for a change of base.

Lemma C. Let S be an object of $\mathcal{S}$, and let T be an inhabited tree of finite sequences from S "all whose branches are infinite":

(i) $< \ > \in T$,

(ii) $u \prec v$ and $u \in T \Rightarrow v \in T$ ($u \prec v$ means that u extends v),

(iii) $u \in T \Rightarrow \exists s \in S \ u * s \in T$ ($*$ for concatenation).

Then there exists an open surjection $\mathcal{E} \overset{\gamma}{\to} \mathcal{S}$ such that T has a branch in $\mathcal{E}$, i.e. there is a function $\mathbb{N} \overset{\alpha}{\to} \gamma^*(S)$ in $\mathcal{E}$ such that (in $\mathcal{E}$ it holds that) $\forall n \in \mathbb{N} < \alpha(0),\ldots,\alpha(n) > \in \gamma^*(T)$.

proof. We introduce a generic branch in the standard way: consider T as a poset, and make it into a presentation of a space X by equipping it with the covering system generated by

$\{u * s \mid s \in S\}$ covers u, for each $u \in T$.

T has a top-element (i), and all covers are inhabited (iii), so if we take $\mathcal{E}$ to be the topos of sheaves on X, $\mathcal{E} \to \mathcal{S}$ is an open surjection.

2. Proof of the theorem.

2.1 Reduction to the case of spaces. As a first remark, let us point out that it suffices to prove the theorem stated in the introduction for the special case that $\mathcal{E}$ is the topos of sheaves on a space. Indeed, if $\mathcal{E}$ is a connected locally connected topos, there exists a connected locally connected map $X \to \mathcal{E}$, where X is a connected locally connected space (1.6, lemma A). If the theorem is true for spaces, then $X^I \to X \times X$ is an open surjection, and hence $\mathcal{E}^I \to \mathcal{E} \times \mathcal{E}$ must be one, by the following commutative diagram (cf. Joyal and Tierney (1982), prop. VII.1.2).

$$\begin{array}{ccc} X^I & \longrightarrow & X \times X \\ \downarrow & & \downarrow \\ \mathcal{E}^I & \longrightarrow & \mathcal{E} \times \mathcal{E} \end{array}$$

We will first prove a slightly weaker version of the theorem, namely

2.2 Proposition. Let X be a connected locally connected space. Then $X^I \to X \times X$ is a stable surjection.

As said in the introduction, our strategy will be to extend the base topos $\mathcal{S}$ sufficiently so as to be able to perform a classical argument (in 2.5 below) somewhat similar to Menger (1929), Moore (1932) (see also Engelking (1977), exercise 6.3.11). To this end, we first introduce a generic pair of points (in 2.3), and then we force some countability conditions (in 2.4).

2.3 The generic pair of points. Let X be a given connected locally connected space in $\mathcal{S}$. Let $\mathcal{F} = Sh(X \times X) \xrightarrow{p} \mathcal{S}$, and write $Y = p^*(X)$. Y is a connected locally connected space in $\mathcal{F}$, and p is an open surjection (in fact p is connected locally connected). In $\mathcal{F}$, there is a generic pair of points $(y_0, y_1) : 1 \to Y \times Y = p^{\#}(X \times X)$, corresponding to the projections. A simple diagram-argument shows that to prove that $X^I \to X \times X$ is a stable surjection in S, it now suffices to find an open surjection $G \overset{q}{\to} F$ such that in G there is a map of spaces $I \overset{f}{\to} q^{\#}(Y)$ with $f(0) = x_0$, $f(1) = x_1$ (I is the formal unit interval in G).

<u>2.4 Countability conditions</u>. Let $\mathbb{P}$ be the presentation by connected opens of Y in F, so $\mathbb{P}$ is a molecular presentation as in 1.3. For each $W \in \mathbb{P}$, let

$$< \mathcal{U}_i(W) : i \in I_W >$$

be the family of covers of W in $\mathbb{P}$. Adjoining surjective functions $\mathbb{N} \to I_W$ to E (for each $W \in \mathbb{P}$) as in 1.6, lemma B, we find an open surjection $F' \overset{r}{\to} F$ such that in F', $Z = r^{\#}(Y)$ has a molecular presentation in which for each element the family of "basic" covers of this element is countable.

Similarly, we can adjoin surjections $\mathbb{N} \to \{U \in \mathbb{P} \mid y_0 \in U\}$ and $\mathbb{N} \to \{U \in \mathbb{P} \mid y_1 \in U\}$. So in F', the points y_0, y_1 of $r^{\#}(Y)$ each have a countable neighbourhood base consisting of elements of the molecular presentation.

So in F', we now have the following data: a connected locally connected space Z and two points z_0, z_1 of Z, with a molecular presentation $\mathbb{P}$ of Z such that

(i) for all $W \in \mathbb{P}$, $< \mathcal{U}_n(W) : n \in \mathbb{N} >$ enumerates the covers of W in $\mathbb{P}$,

(ii) $< N_n(Z_0) : n \in \mathbb{N} >$ enumerates the elements of $\mathbb{P}$ which contain z_0,

(iii) $< N_n(z_1) : n \in \mathbb{N} >$ enumerates the elements of $\mathbb{P}$ which contain z_1.

2.5 <u>proof of proposition 2.2.</u> After these preparations, we can now construct an extension $G \to F'$ such that in G there actually is a path $I \to s^{\#}(Z)$ from z_0 to z_1 (s will be an open surjection). We work in F' with the data as in 2.4.

A <u>chain from</u> z_0 <u>to</u> z_1 is a chain $(V_1,\dots,V_k)$ of elements of $\mathbb{P}$ (see 1.3) such that $z_0 \in V_1$ and $z_1 \in V_k$. Consider the tree T of pairs of finite sequences

$$< (V_1^m, \dots, V_{k(m)}^m)_{m<n}, (\rho^m)_{m<n} > ,$$

where the $(V_1^m, \dots, V_{k(m)}^m)$ are chains from z_0 to z_1, and the $\rho^m : \{1,\dots,k(m+1)\} \to \{1,\dots,k(m)\}$ are functions such that

(a) $j < j' \Rightarrow \rho^m(j) < \rho^m(j')$, and $V_j^{m+1} < V_{\rho^m(j)}^m$;

(b) for each $m' < m$ and each $j < k(m)$, V_j^m is contained in an element of $\mathcal{U}_n(V^{m'}_{\rho^{m'} \circ \dots \circ \rho^{m-1}(j)})$, for each $n < m$;

(c) V_1^m is contained in $N_n(z_0)$ for each $n < m$, and $V_{k(m)}^m$ is contained in $N_n(z_1)$ for each $n < m$;

(d) Given $i < k(m)$, suppose $\rho^m(j) = i$ for $j = j_0, j_0+1, \dots, j_0+k$.
Then $V_{j_0}^{m+1} < V_{i-1}^m$ (unless $j_0 = 0$, i.e. $i = 0$), and $V_{j_0+k}^{m+1} < V_{i+1}^m$ (unless $j_0 + k = k(m+1)$, i.e. $i = k(m)$).

It follows from the molecularity of $\mathbb{P}$ that any such pair of finite sequences satisfying (a)-(d) can be extended to a longer one. Explicitly: suppose we are given $< (V_1^m, \dots, V_{k(m)}^m)_{m<n}, (\rho^m)_{m<n} >$

as above. Cover each V_j^n $(j = 1,\dots,k(n))$ by a common refinement $\mathcal{W}_j$ of the covers $\mathcal{U}_{n+1}(V^{m'}_{\rho^{m'}\circ\dots\circ\rho^{n-1}(j)})$, $m' < n$. Choose a $\tilde{W}_0 \ni z_0$ in $\mathbb{P}$ such that $\tilde{W}_0 <$ some element of $\mathcal{W}_1$, and $\tilde{W}_0 < N_{n+1}(z_0)$. Similarly choose a $W_{k(n+1)} \ni z_1$ such that $W_{k(n+1)} <$ some element of $\mathcal{W}_{k(n)}$ and $W_{k(n+1)} < N_{n+1}(z_1)$.

Now for each $j < k(n)$, some $W_j \in \mathcal{W}_j$ must have positive intersection with some $\tilde{W}_{j+1} \in \mathcal{W}_{j+1}$ (i.e. $\exists U \in \mathbb{P}$ $U < W_j$ and $U < \tilde{W}_{j+1}$). Now let $V_1^{n+1} = \tilde{W}_0$, $V_2^{n+1},\dots, V_{k-1}^{n+1} = W_1$ be a chain in $\mathcal{W}_1$, and let V_k^{n+1} be an element of $\mathbb{P}$ with $V_k^{n+1} < W_1$ and $V_k^{n+1} < \tilde{W}_2$ (so in particular $V_k^{n+1} < V_1^n$ and $V_k^{n+1} < V_2^n$, for condition (d)). Let $\rho^n(i) = 1$ for $1 < i < k$. Now define the next bit $V_{k+1}^{n+1},\dots, V_{k+\ell}^{n+1}$ in a similar way: let $V_{k+1}^{n+1} = V_k^{n+1}$, let $V_{k+2}^{n+1} = \tilde{W}_2$, $V_{k+3}^{n+1},\dots, V_{k+\ell-1}^{n+1} = W_2$ be a chain from $\tilde{W}_2$ to W_2 in $\mathcal{W}_2$, and let $V_{k+\ell}^{n+1}$ be an element of $\mathbb{P}$ with $V_{k+\ell}^{n+1} < W_2$ and $V_{k+\ell}^{n+1} < \tilde{W}_3$.

Let $\rho^n(i) = 2$ for $k < i < k + \ell$; etc., etc., until $V_{k(n+1)}^{n+1} := W_{k(n+1)} \ni z_1$.

By **lemma** C of 1.6, there is an open surjection $G \to F'$ such that the tree T has an infinite branch in G. Replacing F' by G, we work within G with this <u>fixed</u> branch which we will denote by

$$< (V_1^m,\dots, V_{k(m)}^m)_{m \in \mathbb{N}} \;,\; (\rho^m)_{m \in \mathbb{N}} >$$

We now mimic this branch of chains of V_j^m's by consecutive rational intervals in $I = [0,1]$. (Notational convention: the open interval (p,q) stands for $[p,q)$ if $p = 0$, and for $(p,q]$ if $q = 1$.) Let $p_0^{-1} = 0$, $q_1^{-1} = 1$. Suppose we have defined (p_i^m , q_i^m) for $1 < i < k(m)$,

$$0 = p_1^m < q_1^m = p_2^m < q_2^m = \dots < q_{k(m)}^m = 1.$$

Define (p_j^{m+1}, q_j^{m+1}) for $1 < j < k(m+1)$ as follows. If $\rho^m(j) = i$ for $j = j_0, j_0+1,\dots, j_0+k$, choose rationals $r_1,\dots, r_k$ with $p_i^m < r_1 < \dots < r_k < q_i^m$, and let

$$p_{j_0}^{m+1} = p_i^m \text{ , } q_{j_0}^{m+1} = p_{j_0+1}^{m+1} = r_1, \ldots,$$
$$q_{j_0+k-1}^{m+1} = p_{j_0+k}^{m+1} = r_k \text{ , } q_{j_0+k}^{m+1} = q_i^m .$$

So for each m, $\{[p_j^m , q_j^m] : j = 1,\ldots,k(m)\}$ is a "cover" of $[0,1]$ by consecutive closed intervals having one point in common, and the cover for $m+1$ refines the one for m according to the function ρ^m.

Define a function $f^* : \mathbb{P} \to \mathcal{O}(I)$ by

$$f^*(U) = \bigvee\{(p_j^m , q_{j'}^m) \mid j < j' \text{ and } V_j^m \vee\ldots\vee V_{j'}^m < U\}.$$

We claim that f^* defines a continuous map of spaces $I \xrightarrow{f} Z$, and that $f(0) = z_0$, $f(1) = z_1$. Proof of this claim:

(i) $f^*(1) = (p_0^0 , q_{k(0)}^0) = [0,1]$ (by the notational convention).

(ii) f^* preserves binary meets; more precisely, since $\mathbb{P}$ does not have meets, if $W < f^*(U)$ and $\tilde{W} < f^*(\tilde{U})$ then $W \wedge \tilde{W} < f^*(V)$ for some $V \in \mathbb{P}$ with $V < U$, $V < \tilde{U}$. Indeed, suppose $W = (p_j^m , q_{j'}^m) < f^*(U)$ because $V_j^m \vee\ldots\vee V_{j'}^m < U$, and $\tilde{W} = (p_{\tilde{j}}^{\tilde{m}} , q_{\tilde{j}'}^{\tilde{m}}) < f^*(\tilde{U})$ because $V_{\tilde{j}}^{\tilde{m}} \vee\ldots\vee V_{\tilde{j}'}^{\tilde{m}} < \tilde{U}$. Let us say $\tilde{m} > m$, and $p_j^m < p_{\tilde{j}}^{\tilde{m}} < q_{j'}^m < p_{\tilde{j}'}^{\tilde{m}}$ (other cases are symmetric or trivial). By construction, $q_{j'}^m = q_i^{\tilde{m}}$ for some $i > \tilde{j}$, and

$$V_{\tilde{j}}^{\tilde{m}} \vee\ldots\vee V_i^{\tilde{m}} < V_j^m \vee\ldots\vee V_{j'}^m < U.$$

So $W \wedge W' = (p_{\tilde{j}}^{\tilde{m}} , q_i^{\tilde{m}}) < f^*(V_j^{\tilde{m}} \vee\ldots\vee V_i^{\tilde{m}})$, and $V_j^{\tilde{m}} \vee\ldots\vee V_i^{\tilde{m}} \in \mathbb{P}$ since $\mathbb{P}$ is closed under sups of chains, and moreover $V_j^{\tilde{m}} \vee\ldots\vee V_i^{\tilde{m}} <$ both U and $\tilde{U}$.

(iii) f^* maps basic covers in $\mathbb{P}$ to sups in $\mathcal{O}(I)$: Let $\{U_\alpha\}_\alpha$ be a cover of $U \in \mathbb{P}$, and suppose $(p_j^m , q_{j'}^m) < f^*(U)$ because $V_j^m \vee\ldots\vee V_{j'}^m < U$. For each k, $j < k < j'$, we have by stability a cover $\{W_\beta^k\}$ of V_k^m in $\mathbb{P}$ such that $W_\beta^k < U_\alpha$. Say $\{W_\beta^k\} = \mathcal{U}_{n_k}(V_k^m)$.
Let $\tilde{m} = \max(n_j,\ldots,n_{j'})$. Now consider the chain $(V_1^{\tilde{m}},\ldots, V_{k(\tilde{m})}^{\tilde{m}})$ from z_0 to z_1. By definition, there are $1 < \ell_k < \ell_k' < k(\tilde{m})$ such that

$$\rho^m \circ \ldots \circ p^{\tilde{m}-1}(i) = k \Leftrightarrow \ell_k < i < \ell'_k$$

(for $k = 1,\ldots,k(m)$, but only $k = j,\ldots,j'$ are relevant). So for $\ell_k < i < \ell'_k$,

$$V_i^{\tilde{m}} < \text{some } W_\beta^k < \text{some } U_\alpha .$$

Hence $(p_i^{\tilde{m}}, q_i^{\tilde{m}}) < f^*(U_\alpha)$, and therefore

$$(p^{\tilde{m}}_{\ell_j}, q^{\tilde{m}}_{\ell_j}) \vee \ldots \vee (p^{\tilde{m}}_{\ell_{j'}}, q^{\tilde{m}}_{\ell_{j'}}) < \bigvee_\alpha f^*(U_\alpha).$$

Since $p_j^n = p^{\tilde{m}}_{\ell_j}$ and $q^m_{j'} = q^{\tilde{m}}_{\ell_{j'}}$, this <u>almost</u> means that $(p_j^m, q_{j'}^m) < \bigvee_\alpha f^*(U_\alpha)$, but we miss the boundary points! To make up for those, however, it suffices to note the following consequence of condition (d): Given any V_i^n, there is an $n' > n$ such that $V^n_{j'}, \ldots, V^{n'}_{i'} < V_i^m$ and $\rho(j) < i < \rho(j')$, where $\rho = \rho^n \circ \ldots \circ \rho^{n'-1}$ (i.e. for chains which are sufficiently fine, we get over the boundary).

This completes the proof that f^* defines a map $f : I \to Z$ of spaces.

(iv) Finally, $f(0) = z_0$, $f(1) = z_1$: Clearly, if $f(0) \in U$ then $V_0^m < U$ for some m, so $z_0 \in U$. Conversely, if $z_0 \in U$ then $U = N_n(z_0)$ for some n, so $V_0^m < U$ for $m > n$; hence $f(0) \in U$. Thus $f(0) = z_0$ as points of Z. Similarly $f(1) = z_1$.

This completes the proof of proposition 2.2. In 2.7 we will show that $X^I \to X \times X$ is in fact an <u>open</u> surjection.

<u>2.6. Remark.</u> As said in the introduction, one can also give a more "algebraic" proof, by working directly with sites (presentations). We briefly describe the sites involved. Let X be a connected locally connected space, with a molecular presentation $\mathbb{P}$. Hyland (1981) gives a presentation for the space X^I. It is not hard to see that in the present case, it suffices to consider elements in the presentation of the form

$$\bigwedge_{i=1}^{n} [(p_i , p_{i+1}), U_i] ,$$

where $0 = p_1 < .. < p_n = 1$ are rationals, and $(U_1,...,U_n)$ is a chain in $\mathbb{P}$ (Hyland would write $[(p_i , p_{i+1}) << f^*(U_i)]$ for our $[(p_i , p_{i+1}), U_i]$). Let $\mathbb{Q}$ be a presentation of X^I with underlying poset consisting of opens of X^I of this form. Let $\mathbb{P}\otimes\mathbb{P}$ denote the presentation of $X \times X$ obtained in the obvious way from the presentation $\mathbb{P}$ of X. The inverse image $\mathcal{O}(X)\otimes\mathcal{O}(X)\xrightarrow{F}\mathcal{O}(X^I)$ of the map $X^I \to X \times X$ of proposition 2.2 is induced by the functor

$$\mathbb{P}\otimes\mathbb{P}\xrightarrow{F}\mathcal{O}(X^I),$$
$$F(V\otimes W) = \bigvee\{ \bigwedge_{i=1}^{n} [(p_i, p_{i+1}), U_i] \in \mathbb{Q} | U_1 < V , U_n < W\} .$$

To show that F induces an open surjection, one checks that $\mathcal{O}(X \times X)\xrightarrow{F}\mathcal{O}(X^I)$ has a left-inverse, left-adjoint $G : \mathcal{O}(X^I)\to\mathcal{O}(X \times X)$ described in terms of presentations by

$$\mathbb{Q}\xrightarrow{G}\mathbb{P}\otimes\mathbb{P}$$
$$G(\bigwedge_{i=1}^{n} [(p_i , p_{i+1}), U_i] = U_1\otimes U_n ,$$

and that the Frobenius-law $G(U \wedge F(V)) = G(U) \wedge V$ holds.

<u>2.7. Openness of the map $X^I\to X \times X$.</u> Given the fact that X^I has a presentation $\mathbb{Q}$ as in 2.6, our proof of 2.2 actually shows that $X^I\to X \times X$ is an open surjection. We argue again in the geometric style, using base extensions to enable ourselves to reason about points.

In general, a map $B\xrightarrow{f}A$ of spaces in $\mathcal{S}$ (or in any topos) is open iff the image $f(V)$ is an open subspace of A, for all V in some basis (some presentation) of B (see Joyal and Tierney (1982)). If we allow for change of base, images can be described in terms of points, just as in topology: if $V \in \mathcal{O}(B)$ and $U \in \mathcal{O}(A)$, then $f(V) = U$ iff for any geometric morphism

$G \longrightarrow S$ and any point $p \in \varphi^{\#}(A)$, we have (writing V for $\varphi^{\#}(V)$, U for $\varphi^{\#}(U)$)

$$(*) \qquad p \in U \Leftrightarrow \text{ there is a surjection } H \xrightarrow{\psi} G \text{ and a point } q \in V \subset \psi^{*}\varphi^{*}(B) \text{ such that in } H,\ p = f(q).$$

Let us consider the special case where $B \xrightarrow{f} A$ is the map $X^{I} \longrightarrow X \times X$ of proposition 2.2. Take a basic open $U = \bigwedge_{i=1}^{r} [(p_i , p_{i+1}), U_i]$ of X^I as in 2.6. We claim that the image of U is the open subspace $U_1 \times U_n$ of $X \times X$. To show the equivalence $(*)$, choose $G \xrightarrow{\varphi} S$ and a point $p = (x_0 , x_1) \in \varphi^{\#}(U_1) \times \varphi^{\#}(U_n)$ in G. Since $(\varphi^{\#}(U_1),\ldots,\varphi^{\#}(U_n))$ is a chain in G, there is an open surjection $G_0 \longrightarrow G$ such that, writing φ_0 for the composite $G_0 \longrightarrow G \xrightarrow{\varphi} S$, there are points $y_i \in \varphi_0^{\#}(U_i \wedge U_{i+1}) = \varphi_0^{\#}(U_i) \wedge \varphi_0^{\#}(U_{i+1})$ $(i = 1,\ldots,n-1)$. Let $y_0 = x_0$, $y_n = x_1$ in G_0. Since each $\varphi_0^{\#}(U_i)$ is a connected locally connected space in G_0, our proof of 2.2 shows that there exists an open surjection $H \xrightarrow{\psi} G_0$ such that in H there are paths $f_i : I \longrightarrow \psi^{\#}\varphi_0^{\#}(U_i)$ with $f_i(0) = y_i$, $f_i(1) = y_{i+1}$ $(i = 0,\ldots,n-1)$. Putting these paths together, we obtain a map $I \xrightarrow{f} \psi^{\#}\varphi_0^{\#}(X)$ with $f(\frac{i}{n}) = y_i$ $(i = 0,\ldots,n)$. This shows $\Rightarrow$ of $(*)$ for this particular case.

The other implication $\Leftarrow$ is obvious.

This completes the proof of the theorem as stated in the introduction.

<u>2.8. Remark</u>. Finally, we point out that openness of the map $E^{I} \longrightarrow E \times E$ can be of interest, even if this map is not surjective. In fact, this generalizes the notion of semi-local path-connectedness for topological spaces: one easily shows that for a topological space X, the map $X^{I} \longrightarrow X \times X$ (of topological spaces, not of toposes) is open iff X is semi-locally path-connected. (We are indebted to P.T. Johnstone for this observation.)

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PATH-LIFTING FOR GROTHENDIECK TOPOSES

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In [MW] we proved that every connected locally connected topos E is path-connected, in the sense that the canonical map

$$(1) \qquad E^I \to E \times E$$

given by the inclusion of the endpoints $\{0,1\} \subset I$ is a surjection (actually, we showed it to be an open surjection). Here I, the unit interval, is identified with the topos of sheaves on I, and E^I is the "path-space" of E, i.e. there is an equivalence

$$\mathrm{Hom}_S(F,E^I) \simeq \mathrm{Hom}_S(F \times_S I, E)$$

of categories of geometric morphisms over the base topos S, which is natural in F.

The aim of this note is to point out that a much stronger result can in fact be proved. Let $F \xrightarrow{f} E$ be a connected locally connected map of toposes, i.e. F is connected and locally connected as an E-topos (intuitively, the fibers of f are connected and locally connected). We will prove that for every path α in E^I and points y_0, y_1 of F with $f(y_0) = \alpha(0)$, $f(y_1) = \alpha(1)$, there exists a lifting $\beta \in F^I$ of α with $\beta(i) = y_i$. More precisely, if we form the topos

$$E^I \times_{(E \times E)} (F \times F)$$

of such triples (α, y_0, y_1) by pulling back $F \times F \xrightarrow{f \times f} E \times E$ along the map in (1), then the result can be stated as follows:

<u>Theorem 1.</u> Let $F \xrightarrow{f} E$ be a connected locally connected map of toposes. Then the canonical map

$$F^I \longrightarrow E^I \times_{(E \times E)} (F \times F)$$

given by the map $F^I \to E^I$ induced by f, and $F^I \to F \times F$ as in (1), is a stable surjection.

For the special case of spatial toposes, or equivalently, for the case of spaces (in the generalized sense of e.g. [JT], otherwise known as locales!) we can do slightly better.

Theorem 2. Let $Y \xrightarrow{f} X$ be a connected locally connected map of (generalized) spaces. Then the canonical map

$$Y^I \to X^I \times_{(X \times X)} (Y \times Y)$$

is an open surjection.

I conjecture that theorem 1 can also be strengthened by replacing "stable surjection" with "open surjection", but I don't quite see how to prove this right now.

Finally, it may be worthwhile to state the more down-to-earth case of metric spaces explicitly. Recall that an open map $Y \xrightarrow{f} X$ of topological spaces is o-acyclic if Y has a basis of open sets which intersect all fibers of f in a connected (if non-empty) set (see [MV]). We will see that the following corollary is just a special case of theorem 2, with the base topos S taken as the category of classical sets.

Corollary. Let $Y \xrightarrow{f} X$ be a 0-acyclic map of complete separable metric spaces, and assume f has connected fibers. Let $Y^I \longrightarrow X^I$ be the induced map of function spaces (with the compact-open topology), and let

$$S = \{(\alpha, y_0, y_1) \mid \alpha : I \to X, \ \alpha(0) = f(y_0), \ \alpha(1) = f(y_1)\}$$

topologized as a subspace of $X^I \times Y \times Y$. Then the map

$$Y^I \to S, \quad \beta \mapsto (f \circ \beta, \beta(0), \beta(1))$$

is an open surjection.

Preliminaries, notational conventions. This note is written as a sequel to [MW], and we assume that the reader is familiar with that paper. All the basic results and the notation that we use here can be found in section 1 of [MW].

§1. Reduction to the generic case.

The only technically difficult thing to be proved is the following lemma.

1.1 Lemma. Let $Y \xrightarrow{f} I$ be a connected locally connected map of a space Y into the unit interval I, in "the" base topos S, and let $y_0 \in f^{-1}(0)$, $y_1 \in f^{-1}(1)$ be two points of Y. Then there exists an open surjection $G \longrightarrow S$ such that in the topos G, f has a section $I \xrightarrow{s} Y$ with $s(0) = y_0$, $s(1) = y_1$. In other words, there is a commutative diagram of toposes and geometric morphisms

$$\begin{array}{ccc} G\times_S\{0,1\} & \xrightarrow{(y_0,y_1)} & Y \\ \downarrow & \overset{s}{\nearrow} & \downarrow f \\ G\times_S I & \xrightarrow{\pi_2} & I \end{array} .$$

This lemma will be proved in section 2.

1.2 Proof of Theorem 1. Let $F \xrightarrow{f} E$ be a connected, locally connected map of toposes. It suffices to show that for any map $G \longrightarrow E^I\times_{(E\times E)}(F\times F)$ there exists an open surjection $H \to G$ and a map $H \to F^I$ such that

(1)
$$\begin{array}{ccc} H & \longrightarrow & F^I \\ \downarrow & & \downarrow \\ G & \longrightarrow & E^I\times_{(E\times E)}(F\times F) \end{array}$$

commutes. By working in G, i.e. by replacing S by G, we may assume $G = S$ (the map $F \longrightarrow E$ remains connected locally connected after this change of base). So suppose we are given a map $I \xrightarrow{\alpha} E$ and two points x_0, x_1 of F, with $f(x_i) = \alpha(i)$.

Let $Y \xrightarrow{g} F$ be a connected locally connected map, where Y is a space (identified with the corresponding topos $Sh(Y)$), as in [MW], §1.6. By extending S, i.e. by replacing S with S' where $S' \longrightarrow S$ is an open surjection, we may assume that in S there are points $y_0, y_1 \in Y$ with $g(y_i) = x_i$. Consider the pullback of toposes over S

$$\begin{array}{ccc} Z & \xrightarrow{\beta} & Y \\ & & \downarrow g \\ \downarrow h & & F \\ & & \downarrow f \\ I & \xrightarrow{\alpha} & E \end{array}$$

(Z is spatial over S, since Y is spatial over S, so a fortiori over E.) Let z_0, z_1 be the points of Z with $\beta(z_i) = y_i$, $h(z_i) = i$. By lemma 1.1 there exists an open surjection $H \rightarrow S$ such that in H there is a map $I \xrightarrow{s} Z$ with $h \circ s = id$ and $s(i) = z_i$. Compsosing with $g \circ \beta$ and transposing, this gives the map $H \rightarrow F^I$ required in (1).

□

<u>1.3. Proof of Theorem 2</u>. Let $Y \xrightarrow{f} X$ be a connected locally connected map of spaces. It follows from theorem 1 that

$$Y^I \longrightarrow X^I \times_{(X \times X)} (Y \times Y) = S \quad .$$

is a stable surjection. To prove that $Y^I \longrightarrow S$ is in fact open, one shows that the image of a basic open of the form $\bigwedge_{i=0}^{n-1} [(p_i, p_{i+1}), (U_i]$, where $p_0 = 0 < p_1 < \ldots < p_n = 1$ are rationals and $(U_0, \ldots, U_{n-1})$ is a chain of opens in Y, is the open $\left(\bigwedge_{i=0}^{n-1} [(p_i, p_{i+1}), f(U_i)]\right) \times_{(X \times X)} (U_0 \times U_{n-1})$ of S. This is completely similar to [MW] §2.7.

□

1.4. Proof of the Corollary. It is observed in [J] that an open map $X \to Y$ of topological spaces is 0-acyclic iff it is locally connected as a map of (generalized) spaces, or equivalently, as a map of toposes. If X and Y are complete metric spaces in Sets, then the corresponding locales $\mathcal{O}(X)$, $\mathcal{O}(Y)$ are countably presented, and hence so are the locales corresponding to the exponentials X^I, Y^I, and the pullback $X^I \times_{(X\times X)} (Y\times Y)$ as generalized spaces. Consequently, these generalized spaces have enough points ([MR]), i.e. they coincide with the corresponding topological spaces. By these general facts, the corollary follows immediately from theorem 2.

□

1.5. Remark on path-connectedness. Note that by taking $X = 1$ in theorem 2, it follows that if Y is a connected locally connected space, the map $Y^I \longrightarrow Y\times Y$ is an open surjection. This is the special case proved in [MW]. As explained there, the corresponding fact for toposes, stated in the first lines of this paper, follows easily (cf. [MW], §2.1).

Similarly, by taking $X = 1$ in the corollary, we obtain the old result of K. Menger and R.L. Moore, saying that all connected and locally connected complete separable metric spaces are path-connected (for references, see [MW]).

□

§2. The generic case.

We will now prove lemma 1.1. A crucial preliminary result is the following:

2.1 Proposition. Let $Y \xrightarrow{f} X$ be a locally connected map of spaces, and let $\mathbb{P}$ be a presentation (a poset with a stable covering system) for X. Then there exists a presentation $\mathbb{Q}$ for Y such that taking direct images induces a function

$$\mathbb{Q} \xrightarrow{f(-)} \mathbb{P} \qquad U \longmapsto f(U) = \exists_f(U)$$

with the following property: Given a cover $\{U_\alpha\}_\alpha$ of $U \in \mathbb{Q}$, and $V \in \mathbf{P}$ with $V \leq f(U_\alpha) \cap f(U_\beta)$ for two given indices α and β, there exists a cover $\mathcal{W}$ of V in $\mathbf{P}$ such that for each $W \in \mathcal{W}$ there is a chain $U_\alpha = U_{\alpha_1}, \ldots, U_{\alpha_n} = U_\beta$ from U_α to U_β in $\mathbb{Q}$ with for $i = 1,\ldots,n-1$ $W \leq f(U_{\alpha_i} \wedge U_{\alpha_{i+1}})$ (or more precisely, since $\mathbf{P}$ may not have meets, $W \leq f(\tilde{U}_i)$ for some $\tilde{U}_i \in \mathbf{P}$, $\tilde{U}_i \leq U_{\alpha_i} \wedge U_{\alpha_{i+1}}$). Moreover, if $Y \xrightarrow{f} X$ is also a connected map of spaces, we may take $f^{1}(V) \in \mathbb{Q}$ for each $V \in \mathbf{P}$, so $Y \xrightarrow{f} X$ is determined by the adjoint pair $\mathbb{Q} \underset{f^{-1}}{\overset{f(-)}{\rightleftarrows}} \mathbf{P}$.

<u>proof</u>. This is really a special case of lemma 2.5 from [M]. But it can also be proved directly, by starting with a molecular presentation of Y, where Y is considered as an internal space in $Sh(\mathbf{P})$.

□

<u>2.2 Remark</u>. Intuitively, the elements of $\mathbb{Q}$ are the opens of Y all whose non-empty fibers are connected. Clearly, we cannot assume that $\mathbb{Q}$ is closed under unions of chains, i.e. we need not have $(U,U' \in \mathbb{Q})$ and $Pos(U \wedge U') \Rightarrow U \vee U' \in \mathbb{Q})$. It is easy to see, however, that when $\mathbb{Q}$ comes from a molecular presentation of Y in $Sh(\mathbf{P})$, it has the following property:

(1) If $U,V \in \mathbb{Q}$ with $Pos(U \wedge V)$ and $f(U \wedge V) = f(U) \wedge f(V)$, then $U \vee V \in \mathbb{Q}$.

Moreover, it will hold that

(2) If $U \in \mathbb{Q}$, $V \in \mathbf{P}$ and $V \leq f(U)$, then $U \wedge f^{-1}(V) \in \mathbb{Q}$.

□

<u>**2.3 Proof of lemma 1.1**</u>. Let $Y \xrightarrow{f} I$ be a connected, locally connected map of spaces in $\mathcal{S}$, and let $\mathbf{P}$ be the presentation of I by rational intervals. (As in [MW], we take (p,q) to stand for $[p,q)$ if $p = 0$, and for $(p,q]$ if $q = 1$.) Let $\mathbb{Q}$ be a presentation for Y as in 2.1, coming from an internal molecular presentation of Y in $Sh(\mathbf{P})$; so (1) and (2) of 2.2 will hold. Thus $f: Y \rightarrow I$

is induced by the pair

$$\mathbb{Q} \underset{f^{-1}}{\overset{f}{\rightleftarrows}} \mathbb{P}$$

and $\mathbb{Q}$ is molecular since $\mathbb{P}$ is (cf. also [M], lemma in section 4).

We now proceed in the style of [MW], §2.4 and 2.5. First of all, by replacing S by an (open surjective) extension $S' \to S$ of S, we may assume that we have the following enumerations:

(i) for each $U \in \mathbb{Q}$, $(\mathcal{U}_n(U) \mid n \in N)$ enumerates the covers of U in $\mathbb{Q}$;

(ii) $(N_n(y_0) \mid n \in N)$ and $(N_n(y_1) \mid n \in N)$ enumerate the elements of $\mathbb{Q}$ which contain y_0 and y_1, respectively,

just as in [MW], §2.4.

We now build a tree T of pairs of finite sequences

$$((V_1^m, \ldots, V_{k(m)}^m)_{m \leq n}, (p_0^m, \ldots, p_{k(m)}^m)_{m \leq n}),$$

where each $(V_1^m, \ldots, V_{k(m)}^m)$ is a chain in $\mathbb{Q}$ from y_0 to y_1, and $0 = p_0^m < p_1^m < \ldots < p_{k(m)}^m = 1$ are rationals, such that

(a) for each $m < n$, $\{p_0^m, \ldots, p_{k(m)}^m\} \subset \{p_0^{m+1}, \ldots, p_{k(m+1)}^{m+1}\}$, and the chain $(V_1^{m+1}, \ldots, V_{k(m+1)}^{m+1})$ refines $(V_1^m, \ldots, V_{k(m)}^m)$ accordingly, i.e. for $1 \leq i \leq k(m)$, $1 < j < k(m+1)$, $(p_{j-1}^{m+1}, p_j^{m+1}) \subset (p_{i-1}^m, p_i^m) \Rightarrow V_j^{m+1} \leq V_i^m$;

(b) for each $m' < m \leq n$ and each $j \leq k(m)$, V_j^m is contained in an element of $\mathcal{U}_\ell(V_i^{m'})$ for each $\ell \leq n$, where $i \leq k(m')$ is the index with $(p_{j-1}^m, p_j^m) \subset (p_{i-1}^{m'}, p_i^{m'})$;

(c) V_1^m is contained in $N_\ell(z_0)$ for each $\ell \leq m$, and $V_{k(m)}^m$ is contained in $N_\ell(z_1)$ for each $\ell \leq m$;

(d) given $1<i \leq k(m)$, suppose $p^{m+1}_{j-1} = p^{m}_{i-1}$, $p^{m+1}_{j+k} = p^{m}_{i}$ (so $(p^{m+1}_{j-1},p^{m+1}_{j}),\ldots,(p^{m+1}_{j+k-1},p^{m+1}_{j+k})$ are all contained in (p^{m}_{i-1},p^{m}_{i}), and accordingly $V^{m+1}_{j}\vee\ldots\vee V^{m+1}_{j+k} \leq V^{m}_{i}$, see (a) above). Then $V^{m+1}_{j} \leq V^{m}_{i-1}$ (unless $j+k = k(m+1)$, i.e. $i = k(m)$);

(e) for $1 \leq i \leq j \leq k \leq k(m)$,

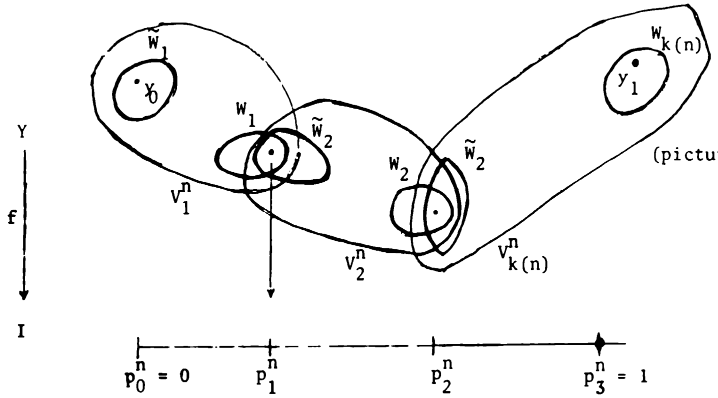
$$f(V^{m}_{i}\vee..\vee V^{m}_{j}) \wedge f(V^{m}_{j+1}\vee..\vee V^{m}_{k}) = f((V^{m}_{i}\vee..\vee V^{m}_{j}) \wedge (V^{m}_{j+1}\vee..\vee V^{m}_{k}))$$

(f) for $i = 1,..,k(m)$, $[p^{m}_{i-1},p^{m}_{i}] \leq f(V^{m}_{i})$, i.e., there are rationals r,r', $r<p^{m}_{i-1}<p^{m}_{i}<r'$, with $(r,r') \leq f(V^{m}_{i})$.

We claim that by molecularity of $\mathbb{P}$ and $\mathbb{Q}$, and by 2.2, 2.3 above, any pair of sequences satisfying (a) - (f) can be extended to a longer one. To see this, suppose we are given $((V^{m}_{1},..,V^{m}_{k(m)})_{m\leq n}, (p^{m}_{0},..,p^{m}_{k(m)})_{m\leq n})$ as above. Cover each V^{n}_{j} $(j=1,..,k(n))$ by a common refinement $\mathcal{W}_{j}$ of the covers $\mathcal{U}_{n+1}(V^{m}_{i_m})$ ($m\leq n$, $i_m \leq k(m)$ the index such that $(p^{n}_{j-1},p^{n}_{j}) \subset (p^{m}_{i_m-1}, p^{m}_{i_m})$). Choose $\tilde{W} \ni y_0$ in $\mathbb{Q}$ such that $\tilde{W}_1 \leq$ some element of $\mathcal{W}_1$ and $\tilde{W}_1 \leq N_{n+1}(y_0)$, and similarly choose $W_{k(n)} \ni y_1$ in $\mathbb{Q}$ such that $W_{k(n)} \leq$ some element of $\mathcal{W}_{k(n)}$ and $W_{k(n)} \leq N_{n+1}(y_0)$. $\tilde{W}_1$ and $W_{k(n)}$ having been chosen, also fix for each of $j<k(n)$ a $W_j \in \mathcal{W}_j$ and a $\tilde{W}_{j+1} \in \mathcal{W}_{j+1}$ which have positive intersection. This can be done by molecularity of $\mathbb{Q}$. Moreover, we choose W_j and $\tilde{W}_{j+1}$ in such a way that there is an $U_j \in \mathbb{Q}$ with

(1) $U_j \leq W_j$ and $U_j \leq \tilde{W}_{j+1}$ and $p_j^n \in f(U_j)$,

since by (e) and (f), $p_j^n \in f(V_j^n \wedge V_{j+1}^n)$.



(picture for $k(n) = 3$)

So $0 = p_0^n \in f(\tilde{W}_1)$, $p_j^n \in f(W_j \wedge \tilde{W}_{j+1})$ $(0<j<k(n))$, and $1 = p_{k(n)}^n \in f(W_{k(n)})$. Since f is an open surjection, there are for each j, $1 \leq j \leq k(n)$, opens

$$\tilde{W}_j = O_1^j, \ldots, O_{u_j}^j = W_j$$

in $\mathcal{W}_j$ such that $(f(O_1^j), \ldots, f(O_{u_j}^j))$ is a chain in $\mathbf{P}$ from p_{j-1}^n to p_j^n.

By proposition 2.1 we can find rational intervals $(q_1^j, r_1^j), \ldots, (q_{u_j-1}^j, r_{u_j-1}^j)$ with

$$p_{j-1}^n < q_1^j < r_1^j < \ldots < q_{u_j-1}^j < r_{u_j-1}^j < p_j^n \quad (j=1,\ldots,k(n)),$$

and for each $s < u_j$ a chain

(2) $O_s^j = O_{s,1}^j, \ldots, O_{s,t_{s,j}}^j = O_{s+1}^j$

in $\mathcal{W}_j$ from O_s^j to O_{s+1}^j $(s=1,\ldots,u_j-1)$ with $(q_s^j, r_s^j) \leq f(O_{s,t}^j \wedge O_{s,t+1}^j)$ for each $s < u_j$ and $t < t_{s,j}$.

By 2.2(2), we can refine this chain (2) by letting

$$A^j_{s,t} = O^j_{s,t} \wedge f^{-1}(q^j_{s,t}, r^j_{s,t}) \qquad 1 < t < t_{s,j}, \quad 1 \le s < u_j$$

$$A^j_{1,1} = O^j_1 \wedge f^{-1}(p^n_{j-1}, r^j_1) = \tilde{W}_j \wedge f^{-1}(p^n_{j-1}, r^j_1)$$

$$A^j_{s,t_s} = O^j_s \wedge f^{-1}(q^j_s, r^j_{s+1}) = A^j_{s+1,1} \qquad 1 < s < u_j - 1$$

$$A^j_{u_j-1,t_{u_j-1}} = O^j_{u_j} \wedge f^{-1}(q^j_{u_j-1}, p^n_j) = W_j \wedge f^{-1}(q^j_{u_j-1}, p^n_j) \ .$$

So now we have for each $j = 1,..,k(n)$ a chain

$$(3)^j \qquad A^j_{1,1}, A^j_{1,2}, \ldots, A^j_{1,t_1} = A^j_{2,1}, A^j_{2,2}, \ldots, A^j_{u_j-1,t_{u_j-1}}$$

with $A^j_{1,1} \le \tilde{W}_j, A^j_{u_j-1,t_{u_j-1}} \le W_j$.

To define $(V^{n+1}_1,..,V^{n+1}_{k(n+1)})$ we take the chains $(3)^j$, one after the other, but with an open inserted "glue" the $(3)^j$-chain to the $(3)^{j+1}$ one, as follows. Fix j, $1 \le j < k(n)$, and choose nationals a_j, b_j with

$$(4) \qquad r^j_{u_j-1} < a_j < p^n_j < b_j < q^{j+1}_1, \text{ and } (a_j,b_j) \le f(U_j),$$

$$B_j = U_j \wedge f^{-1}(a_j,b_j), \qquad 1 \le j < k(n),$$

and define $(V^{n+1}_1,\ldots,V^{n+1}_{k(n+1)})$ to be the chain

$$(5) \qquad (3)^1, B_1, B_1, (3)^2, B_2, B_2, \ldots, (3)^{k(n)-1}, B_{k(n)-1}, B_{k(n)-1}, (3)^{k(n)},$$

where $(3)^j$ abbreviates the chain in formula $(3)^j$ above.

Finally, we define the rationals $(p^{n+1}_0,\ldots,p^{n+1}_{k(n+1)})$, refining the sequence $(p^n_0,..,p^n_{k(n)})$. For each $j = 1,..,k(n)$ and $s = 1,..,u_j-1$, we have a chain $(A^j_{s,t} | 1<t<t_{s,j})$ over the interval (q^j_s, r^j_s), and we subdivide this interval into

$t_{s,j}-2$ pieces accordingly: take rationals $c^j_{s,t}$: $1 < t < t_{s,j}-1$ with

$(6)^j_s \quad q^j_s < c^j_{s,2} <..< c^j_{s,t_{s,j}-2} < r^j_s$.

To refine $(p^n_0,..,p^n_{k(n)})$, we replace (p^n_{j-1},p^n_j) for $1 < j < k(n)$ by

$(7)_j \quad (p^n_{j-1},b_{j-1},(6)^j_1,..,(6)^j_{u_j-1},a_j,p^n_j)$, where $(6)^j_s$ abbreviates the $(t_{s,j}-1)$-tuple in formula $(6)^j_s$ above; for $j = 1$, $j = k(n)$, we replace (p^n_0,p^n_1) and $(p^n_{k(n)-1},p^n_{k(n)})$ respectively by a sequence as in $(7)_j$, but without the a_j and the b_{j-1}.

This, finally, defines $((V^{n+1}_1,..,V^{n+1}_{k(n+1)}), (p^{n+1}_0,..,p^{n+1}_{k(n+1)}))$. We leave it to the reader to check that (a) -(f) are indeed satisfied.

After this rather tedious part of the proof, we can finish quite straightforwardly, just as in [MW]. By lemma C of §1.6 of that paper, there is an open surjection $G \longrightarrow S$ in which the tree T has an infinite branch. We replace S by G, and work in G with this fixed branch

$$((V^m_1,...,V^m_{k(m)})_{m\in N}, (p^m_0,..,p^m_{k(m)})_{m\in N}) .$$

Using this branch, we can define a map $s: I \longrightarrow Y$ of generalized spaces by the function

$$s^*: \mathbf{Q} \longrightarrow \mathcal{O}(I)$$

$$s^*(U) = \bigvee\{(p^m_j,p^m_{j'}) \mid j < j' \text{ and } V^m_{j+1}\vee..\vee V^m_{j'} \leq U\} .$$

Then just as in [MW] it follows easily from (a) - (e) that s^* preserves $\wedge$ and $\bigvee$, and that $s(0) = y_0$, $s(1) = y_1$. (By condition (e), sups of the form $V^m_{j+1}\vee..\vee V^m_j$ are in $\mathbf{Q}$ (cf.2.2) - this is used to prove that s^* preserves $\wedge$. In [MR], sups of chains were automatically elements of the presentation.)

Finally, we have to show that s is a section of f, i.e., $f \circ s$ is the identity on I. But by condition (f), we have $\forall U \in \mathbb{Q}\ s^{-1}(U) = s^*(U) \leq f(U)$, so in particular

(8) $\forall V \in \mathbf{P}\ (f \circ s)^{-1}(V) = s^{-1}f^{-1}(V) \leq ff^{-1}(V) = V.$

But in any topos, the (formal) unit interval I is a T_1-space, in the appropriate sense of generalized spaces (see e.g. [F]), i.e.

(9) for any pair $X \underset{\psi}{\overset{\varphi}{\rightrightarrows}} I$ of maps of generalized spaces,

$$\forall V \in \mathcal{O}(I)\ \varphi^{-1}(V) \leq \psi^{-1}(V) \text{ implies } \varphi = \psi.$$

So for our particular case we conclude that $f \circ s = id$.

This completes the proof of lemma 1.1.

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Journal of Pure and Applied Algebra 37 (1985) 185-191
North-Holland

AN ELEMENTARY PROOF OF THE DESCENT THEOREM FOR GROTHENDIECK TOPOSES

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Communicated by F.W. Lawvere
Received August 1984

The key theorem of Joyal & Tierney [1] is the descent theorem for geometric morphisms of Grothendieck toposes (over a fixed base topos $\mathscr{S}$). This theorem says that open surjections are effective descent morphisms – a fact which has remarkable consequences (see *loc. cit.*). Joyal and Tierney prove the descent theorem by first developing descent theory for 'modules' (suplattices) over locales, parallel to descent theory for commutative rings. In this way they provide an algebraic explanation for the theorem. The purpose of this note is to give a direct proof of the descent theorem.

1. Formulation of the descent theorem (see Joyal & Tierney [1])

Let $\mathscr{E} \xrightarrow{f} \mathscr{F}$ be a geometric morphism of Grothendieck toposes over $\mathscr{S}$, and consider the diagram

$$\begin{array}{ccccccc} \mathscr{E} \times_{\mathscr{F}} \mathscr{E} \times_{\mathscr{F}} \mathscr{E} & \overset{p_{12}}{\underset{p_{13}}{\overset{\xrightarrow{p_{23}}}{\rightrightarrows}}} & \mathscr{E} \times_{\mathscr{F}} \mathscr{E} & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & \mathscr{E} & \xrightarrow{f} & \mathscr{F}. \\ & & \uparrow{\scriptstyle \delta} & & & & \\ & & \mathscr{E} & & & & \end{array}$$

Descent-data on an object $X \in \mathscr{E}$ consists of a morphism $\theta : p_1^*(X) \to p_2^*(X)$ such that $\delta^*(\theta) = \mathrm{id}$ and $p_{13}^*(\theta) = p_{23}^*(\theta) \circ p_{12}^*(\theta)$ (the cocycle condition). Des(f) denotes the category of pairs (X, θ), θ descent-data on $X \in \mathscr{E}$, where morphisms $(X, \theta) \to (X', \theta')$ are morphisms $X \xrightarrow{f} X'$ in $\mathscr{E}$ which commute with descent-data in the obvious way. Any object $f^*(D)$, $D \in \mathscr{F}$, can be equipped with descent-data in a canonical way, and this gives a commutative diagram

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & \mathrm{Des}(f) \\ & \underset{f^*}{\searrow} & \downarrow U \\ & & \mathcal{E} \end{array}$$

where U is the forgetful functor. f is called an *effective descent morphism* if $\mathcal{D} \to \mathrm{Des}(f)$ is an equivalence of categories. The descent theorem states that every open surjection is an effective descent morphism.

Note that by working inside $\mathcal{D}$, it suffices to prove this theorem for the special case that $\mathcal{E} \xrightarrow{f} \mathcal{D}$ is the canonical geometric morphism $\mathcal{E} \xrightarrow{\gamma} \mathcal{S}$; accordingly, we will only consider this case.

2. Some preliminary remarks

Let $\mathcal{E} = \mathrm{Sh}(\mathbb{C})$, $\mathbb{C}$ a site in $\mathcal{S}$. Then a site for $\mathcal{E} \times \mathcal{E} = \mathcal{E} \times_{\mathcal{S}} \mathcal{E}$ is given by the product-category $\mathbb{C} \times \mathbb{C}$ with the coarsest topology making the projections

$$\mathbb{C} \times \mathbb{C} \underset{P_2}{\overset{P_1}{\rightrightarrows}} \mathbb{C}$$

continuous, i.e. the topology is generated by covers of the form

$$\{(C_i, D) \xrightarrow{(f_i, \mathrm{id})} (C, D)\}_i \quad \text{and} \quad \{(C, D_j) \xrightarrow{(\mathrm{id}, g_j)} (C, D)\}_j,$$

where $\{C_i \xrightarrow{f_i} C\}_i$ and $\{D_j \xrightarrow{g_j} D\}_j$ are covers in $\mathbb{C}$. The inverse image p_1^* of the geometric morphism $\mathcal{E} \times \mathcal{E} \xrightarrow{p_1} \mathcal{E}$ comes from composing with P_1, followed by sheafification. Similarly for p_2^*. The inverse image δ^* of the diagonal $\mathcal{E} \xrightarrow{\delta} \mathcal{E} \times \mathcal{E}$ comes from composing with $\Delta : \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ followed by sheafification: given $Y \in \mathrm{Sh}(\mathbb{C} \times \mathbb{C}) = \mathcal{E} \times \mathcal{E}$, $\delta^*(Y)$ is the sheaf associated to the presheaf $C \mapsto Y(C, C)$. So for $Y = p_1^*(X)$, $\delta^* p_1^*(X) \cong X$, and we have a canonical natural transformation η, $\eta_C : p_1^*(X)(C, C) \to X(C)$, which is the unit of the associated sheaf adjunction. Similarly for p_2^*.

3. The case of connected locally connected geometric morphisms

As a warming up exercise, let us point out that the descent theorem is trivial when $\mathcal{E} \longrightarrow \mathcal{S}$ is connected, locally connected (this is not needed for the proof of the general case). Indeed, let $\mathbb{C}$ be a molecular site for $\mathcal{E}$ (with a terminal, since γ is connected). Constant presheaves on $\mathbb{C}$ are sheaves, and p_1^*, p_2^* are just given by composition with P_1 and P_2 respectively (no sheafification needed). Now suppose X is a sheaf on $\mathbb{C}$, with descent-data $X \circ P_1 \xrightarrow{\theta} X \circ P_2$. This means that we are given functions $\theta_{CD}: X(C) \to X(D)$ for every pair of objects C and D of $\mathbb{C}$. Naturality of θ means that for any $C' \xrightarrow{f} C$, $D' \xrightarrow{g} D$, $X(g) \circ \theta_{CD} = \theta_{C'D'} \circ X(f)$. $\delta^*(\theta) = \mathrm{id}$ means that for any C, $\theta_{CC}: X(C) \to X(C)$ is the identity. And the cocycle condition means that for any triple C, D, E of objects of $\mathbb{C}$, $\theta_{DE} \circ \theta_{CD} = \theta_{CE}$. So in particular, taking $C = E$, θ_{CD} is inverse to θ_{DC}, i.e. θ is an isomorphism. From this it easily follows that X is isomorphic to the constant sheaf $\gamma^*(X(1))$: define

$$X \underset{\varphi}{\overset{\psi}{\rightleftarrows}} \gamma^*(X(1))$$

by the components $\varphi_C = \theta_{1C}$; $\psi_C = \theta_{C1}$. φ and ψ are inverse to each other, and are natural in C by naturality of θ. It remains to show that any morphism $\gamma^*(T) \xrightarrow{\tau} \gamma^*(T')$ which is compatible with the canonical descent-data comes from a map $T \to T'$. But this is clear from the fact that γ^* is full and faithful.

4. A proof of the descent theorem

This is essentially the same as 3, but we have to keep track of sheafification all the time. Let $\mathscr{E} \xrightarrow{\gamma} \mathscr{S}$ be an open surjection, and let $\mathbb{C}$ be an open site for $\mathscr{E}$; i.e. $\mathbb{C}$ has a terminal object 1, and every cover in $\mathbb{C}$ is inhabited. We have to show that

(a) every object $X \in \mathscr{E}$ equipped with descent-data is isomorphic to a constant sheaf;

(b) every morphism $\gamma^*(T) \longrightarrow \gamma^*(T')$ which commutes with the canonical descent-data is of the form $\tau = \gamma^*(f)$.

To prove (a), choose $X \in \mathscr{E}$ with descent-data θ. Write $\mathscr{E} \times \mathscr{E} \xrightarrow{p_i} \mathscr{E}$ and $\mathscr{E} \times \mathscr{E} \times \mathscr{E} \xrightarrow{\bar{p}_i} \mathscr{E}$ for the projections. Identifying $p_2^*(X)(C, D)$ with $p_1^*(X)(D, C)$ in the canonical way, we may regard θ as a system of functions (in $\mathscr{S}$)

$$\theta_{CD}: p_1^*(X)(C, D) \to p_1^*(X)(D, C)$$

which are natural in C, D: for $C' \to C$ and $D' \to D$,

$$\begin{array}{ccc} p_1^*(X)(C, D) & \xrightarrow{\theta_{CD}} & p_1^*(X)(D, C) \\ \downarrow & & \downarrow \\ p_1(X)(C', D') & \xrightarrow{\theta_{C'D'}} & p_1^*(X)(D', C') \end{array}$$

commutes. This implies that θ_{CD} is determined by its restriction $\theta_{CD} \circ i_1$,

$$X(C) \xhookrightarrow{i_1} p_1^*(X)(C, D) \xrightarrow{\theta_{CD}} p_1^*(X)(D, C)$$

for which we also write θ_{CD}. The condition $\delta^*(\theta) = \mathrm{id}$ means that

$$\begin{array}{ccc} p_1^*(X)(C,C) & \xrightarrow{\theta_{CC}} & p_1^*(X)(C,C) \\ & {\scriptstyle\eta_C}\searrow \quad \swarrow{\scriptstyle\eta_C} & \\ & X(C) & \end{array}$$

commutes for every C, while the cocycle condition means that

$$\begin{array}{ccc} \bar{p}_1^*(X)(C,D,E) & \xrightarrow{\theta_{CD(E)}} & \bar{p}_1^*(X)(D,C,E) \\ \Big\downarrow{\scriptstyle\theta_{CE(D)}} & & \Big\downarrow{\scriptstyle\theta_{DE(C)}} \\ \bar{p}_1^*(X)(E,C,D) & \xrightarrow{\sim} & \bar{p}_1^*(X)(E,D,C) \end{array}$$

where $\theta_{CD(E)}$ is the obvious map induced by θ_{CD}, etc.

We will use the following lemma, to be proved below.

Lemma. *For $X\in\mathscr{E}=\mathrm{Sh}(\mathbb{C})$, and objects C,D,E of $\mathbb{C}$, the canonical square*

$$\begin{array}{ccc} p_1^*(X)(C,D) & \hookrightarrow & \bar{p}_1^*(X)(C,D,E) \\ \Big\uparrow & & \Big\uparrow \\ X(C) & \hookrightarrow & p_1^*(X)(C,E) \end{array}$$

is a pullback in $\mathcal{S}$.

Let $S=\{x\in X(1)\mid \theta_{11}(i_1(x))=i_1(x)\}$, where $i_1:X(1)\hookrightarrow p_1^*(X)(1,1)$ as above. We claim that $X\cong\gamma^*(S)$ via

$$X \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \gamma^*(S),$$

where φ is the transpose of $S\to X(1)$, and ψ is the map defined by the components

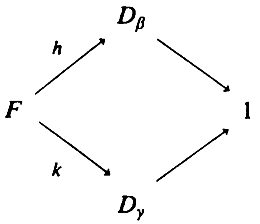
$$\begin{array}{ccc} X(C) & \overset{\psi_C}{\dashrightarrow} & \gamma^*(S)(C) \\ \Big\downarrow{\scriptstyle i_1} & & \Big\downarrow{\scriptstyle j_C} \\ p_1^*(X)(X,1) & \xrightarrow{\theta_{C1}} & p_1^*(X)(1,C) \end{array}$$

where j_C is the obvious embedding, natural in C. The nontrivial thing is to show that ψ_C is well-defined, i.e. that $\theta_{C1}\circ i_1$ factors through j_C. (Naturality of ψ_C is then obvious.) So take $x\in X(C)$, and write $y=\theta_{C1}(i_1(x))\in p_1^*(X)(1,C)$. We have to show that y "locally does not depend on the C-coordinate". y is given as a compatible family $\{y_\alpha\}_\alpha$, $y_\alpha\in X(D_\alpha)$, for a cover $\{(D_\alpha,C_\alpha)\xrightarrow{(D_\alpha,f_\alpha)}(1,C)\}_{\alpha\in I}$ in $\mathbb{C}\times\mathbb{C}$.

Fix α, and let $x_\alpha = x \upharpoonright f_\alpha \in X(C_\alpha)$. Then $\theta_{C_\alpha D_\alpha}(x_\alpha) = y_\alpha$, and by the cocycle condition, we have for any object E of $\mathbb{C}$ that $\theta_{C_\alpha E}(x_\alpha) = \theta_{D_\alpha E}(y_\alpha)$ in $\bar{p}_1^*(X)(E, D_\alpha, C_\alpha)$. So by the lemma,

$$\theta_{C_\alpha E}(x_\alpha) = \theta_{D_\alpha E}(y_\alpha) \in X(E).$$

Choosing $E = C_\alpha$, we find that $\theta_{C_\alpha C_\alpha}(x_\alpha) \in X(C_\alpha)$, and hence since η_{C_α} is the identity on $X(C_\alpha) \xrightarrow{i_1} p_1^*(X)(C_\alpha, C_\alpha)$, that $\theta_{C_\alpha C_\alpha}(x_\alpha) = x_\alpha$. Now let E run over all the objects D_β, $\beta \in \mathcal{A}$. Clearly by naturality of θ, if



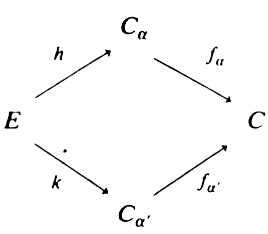
then $\theta_{C_\alpha D_\beta}(x_\alpha) \upharpoonright h = \theta_{C_\alpha E}(x_\alpha) = \theta_{C_\alpha D_\gamma}(x_\alpha) \upharpoonright k$, so since $\{D_\beta \to 1\}_{\beta \in \mathcal{A}}$ is a cover in $\mathbb{C}$ (by openness), there is a unique $z_\alpha \in X(1)$ with $z_\alpha \upharpoonright D_\beta = \theta_{C_\alpha D_\beta}(x_\alpha)$. So by naturality of θ again,

$$z_\alpha = \theta_{C_\alpha 1}(x_\alpha) \in X(1),$$

while moreover since $\theta_{C_\alpha C_\alpha}(x_\alpha) = x_\alpha$,

$$z_\alpha \upharpoonright C_\alpha = x_\alpha \in X(C_\alpha).$$

We claim that $\{z_\alpha\}_\alpha$ determines an element $z \in \gamma^*(S)(C)$. (Note that clearly if this is so, $j_C(z) = \theta_{C1}(x)$.) Indeed, the z_α are compatible in the sense that if



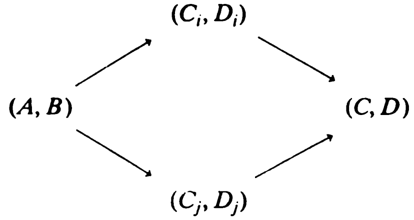
commutes, then $z_\alpha = z_{\alpha'} \in X(1)$ – this is obvious from naturality of θ. Moreover, each $z_\alpha \in S$. For if E is any object of $\mathbb{C}$, we have $\theta_{1E}(z_\alpha) = \theta_{C_\alpha E}(x_\alpha)$ in $\bar{p}_1^*(X)(1, C_\alpha, E)$ by the cocycle condition, so by the lemma, $\theta_{1E}(z_\alpha) \in X(E)$. Since $\eta_1 | X(1)$ is the identity, we find for $E = 1$ that $\theta_{11}(z_\alpha) = z_\alpha$. This proves that ψ_C is well-defined.

It is now clear that φ and ψ are inverse to each other: One way round, it suffices to show that $\psi_1 \varphi_1(s) = s$ for $s \in S$. But $\varphi_1(s) = s \in X(1)$, and $\theta_{11}(s) = i_1(s)$ by definition of S, so this is clear. The other way round, take $x \in X(C)$. Then

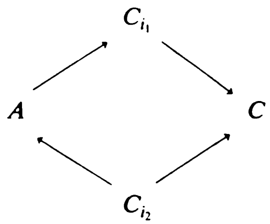
$\psi_C(x) \in \gamma^*(S)(C)$ is the element z as above with $z \upharpoonright f_\alpha = z_\alpha \in S \subset X(1)$. So by definition, $\varphi_C(z) \in X(C)$ is given by $\varphi_C(z) \upharpoonright f_\alpha = z_\alpha \upharpoonright C_\alpha$. But $z_\alpha \upharpoonright C_\alpha = x_\alpha$ as we have seen. So $\varphi_C(z) = x$, i.e. $\varphi_C \psi_C = \mathrm{id}$. This proves (a).

To prove (b), suppose $\gamma^*(T) \xrightarrow{\tau} \gamma^*(T')$ is compatible with the canonical descent-data θ and θ' on $\gamma^*(T)$, $\gamma^*(T')$. It is trivial to check that $T = \{t \in \gamma^*(T)(1) \mid \theta_{11}(t) = t\}$, and similarly for T'. So if $t \in T \subset \gamma^*(T)(1)$, then $\theta'_{11}\tau_1(t) = \tau_1(\theta_{11}(t)) = \tau_1(t)$, so $\tau_1(t) \in T'$. Therefore τ comes from a map $T \to T'$, proving (b).

It remains to prove the lemma. To this end, suppose $x \in p_1^*(X)(C, D)$ and $y \in p_1^*(X)(C, E)$ are equal in $\bar{p}_1^*(X)(C, D, E)$. Write $x = \{x_\alpha\}_\alpha$, $x_\alpha \in X(C_\alpha)$ a compatible family for a cover $\mathscr{U} = \{(C_\alpha, D_\alpha) \to (C, D)\}_{\alpha \in \mathscr{A}}$ in $\mathbb{C} \times \mathbb{C}$, and $y = \{y_\beta\}_\beta$, $y_\beta \in X(C_\beta)$, a compatible family for a cover $\mathscr{V} = \{(C_\beta, E_\beta) \to (C, E)\}_{\beta \in \mathscr{B}}$ in $\mathbb{C} \times \mathbb{C}$. Equality of x and y in $\bar{p}_1^*(X)(C, D, E)$ means that there is a common refinement $\mathscr{W} = \{(C_i, D_i, E_i) \to (C, D, E)\}_{i \in I}$ of $\{(C_\alpha, D_\alpha, E) \to (C, D, E)\}_\alpha$ and $\{(C_\beta, D, E_\beta) \to (C, D, E)\}_\beta$ in $\mathbb{C} \times \mathbb{C} \times \mathbb{C}$ on which x and y agree. Replacing $\mathscr{U}$ by $\{(C_i, D_i) \to (C, D)\}$ and $\mathscr{V}$ by $\{(C_i, E_i) \to (C, E)\}_i$ we get the following notationally more manageable situation: we are given $x_i \in X(C_i)$, $y_i \in X(C_i)$, such that whenever we have a commutative diagram



then $x_i \upharpoonright A = x_j \upharpoonright A$, and a similar condition for compatibility of $\{y_i\}$ with D replaced by E. Moreover, since x and y agree on the cover $\mathscr{W}$, $x_i = y_i$ for every i. We now have to show that $x = \{x_i\}$ comes from an element of $X(C)$, i.e. that $\{x_i\}$ is compatible for the cover $\{C_i \to C\}$ in $\mathbb{C}$. So suppose



commutes. Take a cover $\{(P_\alpha, Q_\alpha, R_\alpha) \to (A, D_{i_1}, E_{i_2})\}_\alpha$ refining $\mathscr{W}$; i.e. for each α there is a $j_\alpha \in I$ such that

$$\begin{array}{ccc}
(P_\alpha, Q_\alpha, R_\alpha) & \longrightarrow & (A, D_{i_1}, E_{i_2}) \\
\downarrow & & \downarrow \\
(C_{j_\alpha}, D_{j_\alpha}, E_{j_\alpha}) & \longrightarrow & (C, D, E)
\end{array}$$

commutes. By openness, $\{(P_\alpha, Q_\alpha) \to (A, D_{i_1})\}_\alpha$ is a cover in $\mathbb{C} \times \mathbb{C}$, while moreover,

$$\begin{aligned}
x_{i_1} 1 P_\alpha = x_{j_\alpha} 1 P_\alpha & \quad \text{(by compatibility of } \{x_i\} \text{ over } (C, D)) \\
= y_{j_\alpha} 1 P_\alpha & \quad \text{(by } x = y \text{ over } (C, D, E)) \\
= y_{i_2} 1 P_\alpha & \quad \text{(by compatibility of } \{y_i\} \text{ over } (C, E)) \\
= x_{i_2} 1 P_\alpha & \quad \text{(by } x = y \text{ over } (C, D, E)).
\end{aligned}$$

The family $\{P_\alpha \to A\}_\alpha$ covers A, so $x_{i_1} 1 A = x_{i_2} 1 A$. This completes the proof of the lemma.

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CONTINUOUS FIBRATIONS AND INVERSE LIMITS OF TOPOSES

by

Ieke Moerdijk

Introduction.

We will discuss some preservation properties of limits of filtered inverse systems of (Grothendieck) toposes. If $(E_i)_i$ is such a system, with geometric morphisms $f_{ij} : E_i \to E_j$ $(i \geq j)$, and E^∞ is the inverse limit with projection morphisms $\pi_i : E^\infty \to E_i$, we say that a property of geometric morphisms is preserved if whenever each f_{ij} has the property then so does each π_i. It will be proved that some of the important properties of geometric morphisms are preserved by filtered inverse limits, notably surjections, open surjections, hyperconnected geometric morphisms, connected locally connected geometric morphisms, and connected atomic morphisms (definitions and references will be given below).

Filtered inverse limits of toposes have been considered by Grothendieck and Verdier (SGA 4(2), exposé IV, §8). Here we will take a slightly different - more "logical" - approach, by exploiting more explicitly the possibility of regarding a topos as a set theoretic universe (for a constructive set theory, without excluded middle and without choice), and a geometric morphism $F \to E$ of toposes as a topos F constructed in this universe E. This will also bring some parallels with iterated forcing (with finite supports) in set theory to the surface.

Already in the first section this parallel becomes apparent when we show that any geometric morphism of toposes can be represented by a morphism of the underlying sites which possesses some special properties that will be very useful for studying inverse limits. Such a morphism will be called a continuous fibration. In the second section, we will characterize some properties of geometric morphisms in terms of continuous fibrations. This enables us to prove the preservation properties for inverse sequences of toposes (sections 3 and 4). In the final section we will show how the results can be generalized to arbitrary (small) filtered systems.

Throughout this paper, S denotes a fixed base topos, all toposes are assumed to be Grothendieck toposes (over S), and all geometric morphisms are taken to be bounded (over S). Moreover, 2-categorical details will be suppressed for the sake of clarity of exposition. Such details are not really relevant for the preservation properties under consideration, but the meticulous reader should add a prefix "pseudo-" or a suffix "up to canonical isomorphism" at the obvious places.

Acknowledgements. The results of sections 1 and 2 were obtained during my stay in Cambridge, England (Spring 1982), where I profited from stimulating discussions with M. Hyland. My original motivation for applying this to inverse limits came from a rather different direction: a question of G. Kreisel led me to consider inverse sequences of models for theories of choice sequences of the type considered in van der Hoeven and Moerdijk(1984).

I would also like to thank A.Pitts for some useful comments on a preliminary version of this paper.

The author is financially supported by the Netherlands Organization for the Advancement of Pure Research (ZWO).

1. Iteration and continuous fibrations.

If $\mathcal{E}$ is a topos and $\mathbb{C}$ is a site in $\mathcal{E}$, we write $\mathcal{E}[\mathbb{C}]$ for the category $Sh_{\mathcal{E}}(\mathbb{C})$ of sheaves on $\mathbb{C}$, made in $\mathcal{E}$. So $\mathcal{E}[\mathbb{C}]$ is a (Grothendieck) topos over $\mathcal{E}$, and every topos over $\mathcal{E}$ is of this form. Now suppose we have geometric morphisms

$$\mathcal{F} \to \mathcal{E} \to \mathcal{S},$$

where $\mathcal{E} = \mathcal{S}[\mathbb{C}]$ and $\mathcal{F} = \mathcal{E}[\mathbb{D}]$, for sites $\mathbb{C}$ in $\mathcal{S}$ and $\mathbb{D}$ in $\mathcal{E}$. We will construct a site $\mathbb{C} \ltimes \mathbb{D}$ in $\mathcal{S}$ such that $\mathcal{F} \to \mathcal{S}$ is equivalent to $\mathcal{S}[\mathbb{C} \ltimes \mathbb{D}] \to \mathcal{S}$, and $\mathcal{F} \to \mathcal{E}$ corresponds to a flat continuous functor $T: \mathbb{C} \to \mathbb{C} \ltimes \mathbb{D}$. The <u>objects</u> of $\mathbb{C} \ltimes \mathbb{D}$ are pairs (C,D), with C an object of $\mathbb{C}$ and D an object of $\mathbb{D}$ over C ($D \in \mathbb{D}_0(C)$, where $\mathbb{D}_0 : \mathbb{C}^{op} \to \mathcal{S}$ is the sheaf on $\mathbb{C}$ of objects of $\mathbb{D}$). <u>Morphisms</u> $(C,D) \to (C',D')$ of $\mathbb{C} \ltimes \mathbb{D}$ are pairs (f,g), $f: C \to C'$ in $\mathbb{C}$, and $g: D \to D' \mid f$ over C (i.e. $g \in \mathbb{D}_1(C)$, where $\mathbb{D}_1$ is the sheaf on $\mathbb{C}$ of morphisms of $\mathbb{D}$, and $D' \mid f = \mathbb{D}_0(f)(D')$, the restriction of D' along f). Composition is defined in the obvious way: if $(f,g) : (C,D) \to (C',D')$ and $(f',g') : (C',D') \to (C'',D'')$, then $(f',g') \circ (f,g)$ is the pair $(f' \circ f,\ (g' \mid f) \circ g)$. The <u>Grothendieck topology</u> on $\mathbb{C} \times \mathbb{D}$ is defined by: $\{(C_i,D_i) \xrightarrow{(f_i,g_i)} (C,D)\}_i$ covers iff the subsheaf S of $\mathbb{D}_1$ at C generated by the conditions $C_i \Vdash g_i \in S$ satisfies $C \Vdash$ "S covers D". (It looks as if we only use the topology of $\mathbb{D}$, but the topology of $\mathbb{C}$ comes in with the definition of S as the subsheaf $S \in P(\mathbb{D}_1)(C)$ "generated by" these conditions.) One easily checks that this indeed defines a Grothendieck topology on $\mathbb{C} \ltimes \mathbb{D}$.

If X is an object of $\mathcal{S}[\mathbb{C}]$, with a sheaf structure $(E, |)$ for $\mathbb{D}$ ($E: X \to \mathbb{D}_o$ and $|: X \times_{\mathbb{D}_o} \mathbb{D}_1 \to X$ the maps of extent and restriction), we can construct a sheaf $\tilde{X}$ on $\mathbb{C} \ltimes \mathbb{D}$ as follows: $X(C,D) = \{x \in X(C) \mid E_C(x) = D\}$, and for $(f,g) : (C',D') \to (C,D)$ and $x \in X(C,D)$, $\tilde{X}(f,g)(x) = X(f)(x) \mid g$.

Conversely, if Y is a sheaf on $\mathbb{C} \ltimes \mathbb{D}$, we first construct a sheaf $\bar{Y}$ on $\mathbb{C}$, by

$$\bar{Y}(C) = \coprod\{Y(C,D) \mid D \in \mathbb{D}_o(C)\}$$

and restrictions along $C' \xrightarrow{f} C$ given by

$$\bar{Y}(f)(y) = Y(f,1)(y),$$

where $y \in Y(C,D)$, 1 the identity on $D \mid f$ (over C'). The sheaf $\bar{Y}$ carries a canonical sheaf structure for $\mathbb{D}$ in $\mathcal{S}[\mathbb{C}]$: extent is given by the components

$$E_C : \bar{Y}(C) \to \mathbb{D}_o(C), \quad y \longmapsto D \text{ if } y \in Y(C,D),$$

while restrictions are given by the components

$$|_C : \bar{Y}(C) \times_{\mathbb{D}_o(C)} \mathbb{D}_1(C) \to \bar{Y}(C):$$

if $y \in \bar{Y}(C)$ with $E_C(y) = D$, and $C \Vdash f: D' \to D$, then $y \,|_C\, f = Y((C,D') \xrightarrow{(1,f)} (C,D)(y) \in Y(C,D')$.

It is clear that $X \longmapsto \tilde{X}$ and $Y \longmapsto \bar{Y}$ define functors which are inverse to each other (up to natural isomorphism). Moreover, we have a canonical projection functor $P: \mathbb{C} \ltimes \mathbb{D} \to \mathbb{C}$, and this yields

<u>1.1. Proposition</u>. F is equivalent to $\mathcal{S}[\mathbb{C} \ltimes \mathbb{D}]$, in other words

$$S[\mathbb{C}][\mathbb{D}] \simeq S[\mathbb{C} \ltimes \mathbb{D}]$$

and the geometric morphism $F \to E$ is induced by the functor $P: \mathbb{C} \ltimes \mathbb{D} \to \mathbb{C}$. ∎

If we assume that the category (underlying) $\mathbb{D} \in E = S[\mathbb{C}]$ has a terminal object $1 \in \mathbb{D}_o$, with components $1_C \in \mathbb{D}_o(C)$, then $P: \mathbb{C} \ltimes \mathbb{D} \to \mathbb{C}$ has a right adjoint

$$T: \mathbb{C} \to \mathbb{C} \ltimes \mathbb{D}, \quad T(C) = (C, 1_C).$$

T is a flat and continuous functor of sites, so it induces a geometric morphism $F = E[\mathbb{D}] \to E$ by Diaconescu's theorem, and this is again the given geometric morphism that we started with.

Even without assuming that $\mathbb{D}$ has a terminal object (in E), P locally has a right adjoint. For each object $(C,D) \in \mathbb{C} \ltimes \mathbb{D}$, the functor

$$P/(C,D) : \mathbb{C} \ltimes \mathbb{D} \,/\, (C,D) \to \mathbb{C}/C$$

has a right adjoint

$$T_{(C,D)} : \mathbb{C}/C \to \mathbb{C} \ltimes \mathbb{D}/(C,D), \quad T_{(C,D)}(C' \xrightarrow{f} C) = (C', D \mid f).$$

So $P/(C,D) \circ T_{(C,D)}$ is the identity on $\mathbb{C}/C$. If $\mathbb{C}$ and $\mathbb{D}$ have terminal objects 1, then $T = T_{(1,1)}$.

Thus we have obtained a fibration $\mathbb{C} \ltimes \mathbb{D} \to \mathbb{C}$. In general, if we are given a geometric morphism $F \to E$ and a site $\mathbb{C}$ in S for E, we can express F as sheaves on a site (with a terminal object) in E, and then "take this site out" to obtain a site in the base topos S, as described in proposition 1.1. In this way, we get the following theorem (by taking for $\mathbb{D}$ in 1.2 the site $\mathbb{C} \ltimes \mathbb{D}$ of 1.1).

1.2. Theorem. Let $F \to E$ be a geometric morphism over S, and let $\mathbb{C}$ be a site for E. Then there exists a site $\mathbb{D}$ in S for F, $F = S[\mathbb{D}]$, such that the geometric morphism $F \to E$ is induced by a pair of functors

$$\mathbb{D} \underset{T}{\overset{P}{\rightleftarrows}} \mathbb{C}$$

with the following properties: T is right adjoint right inverse to P, T is flat and continuous, and for each $D \in \mathbb{D}$, $P/D : \mathbb{D}/D \to \mathbb{C}/PD$ has a flat and continuous right adjoint right inverse T_D. ∎

We will refer to a pair $\mathbb{D} \rightleftarrows \mathbb{C}$ with these properties as a continuous fibration. The geometric morphism

$$p: S[\mathbb{D}] \to S[\mathbb{C}]$$

that such a pair induces is decribed in terms of presheaves by

p_* = compose with T

p^* = compose with P, then sheafify.

If $\mathbb{C}$ and $\mathbb{D}$ have finite inverse limits, then so does $\mathbb{C} \ltimes \mathbb{D}$, and P and T both preserve them. P cannot be assumed to preserve covers, however, since in that case it would give rise to a pair of geometric morphisms $F \rightleftarrows E$; see Moerdijk & Reyes(1984), theorem 2.2.

1.3. Examples. (a) (Iterated forcing in set theory). If $\mathbb{P}$ is a poset in S (with $p < p'$ iff $p \to p'$, iff "p extends p' "), and $\mathbb{Q}$ is a poset in $S^{\mathbb{P}^{op}}$, then $\mathbb{P} \ltimes \mathbb{Q}$ is the poset in S of pairs (p,q) with $p \in \mathbb{P}$, $p \Vdash q \in \mathbb{Q}$, and $(p,q) < (p',q')$ iff $p < p'$ and $p \Vdash q < q'$. If $E = Sh_S(\mathbb{P}, \neg\neg)$, and $\mathbb{Q}$ is a poset in E, $F = Sh_E(\mathbb{Q}, \neg\neg)$, then $F \simeq Sh_S(\mathbb{P} \ltimes \mathbb{Q}, \neg\neg)$. In other words, $(\mathbb{P}, \neg\neg) \ltimes (\mathbb{Q}, \neg\neg) \cong (\mathbb{P} \ltimes \mathbb{Q}, \neg\neg)$.

(b) Let G be a group in S (thought of as a category with one object), and $E = S^G$ be the topos of left G-sets. If H is a group in E, H can be identified with a group in S on which G acts on the left (preserving unit and multiplication of H). According to proposition 1.1 above, $(S^G)^H \simeq S^{G \times H}$, where $G \times H$ is the product of G and H with group action given by $(g_1,h_1)(g_2,h_2) = (g_1 g_2, h_1 g_1 h_2)$ (since G acts on the left, $h_1 g_1 h_2$ can only be read as $h_1(g_1 h_2)$). This is just the semidirect product of G and H, i.e.

$$(S^G)^H \simeq S^{G \rtimes H} .$$

(c) Let $\mathbb{C}$ be a site in S, and A a locale in $S[\mathbb{C}]$. A gives rise to the following data (see e.g. Joyal & Tierney(1982)): for each $C \in \mathbb{C}$ a frame $A(C)$ in S, and for each morphism $f: C \to D$ of $\mathbb{C}$ a frame map ($\wedge\bigvee$-map) $A(D) \xrightarrow{A(f)} A(C)$ with a left adjoint Σ_f. The formula for internal sups of A (or rather, in the frame of opens of A) is as follows: If $S \in \Omega^A(C)$ is a subsheaf of A over C, then $\bigvee_A(S) \in A(C)$ is the element $\bigvee_{A(C)} \bigcup_{f:D\to C} \{\Sigma_f(x) \mid x \in S(D \xrightarrow{f} C) \subseteq A(D)\}$. So a site $\mathbb{C} \ltimes A$ for $S[\mathbb{C}][A]$ has as objects the pairs (a,C), $a \in A(C)$; as morphisms $(a,C) \xrightarrow{f} (b,D)$, where $C \xrightarrow{f} D$ in $\mathbb{C}$ and $a < A(f)(b)$ in $A(C)$; and $(a_i,C_i) \xrightarrow{f_i} (b_i,D)$ covers iff $\bigvee_{A(D)}\{\Sigma_{f_i}(a_i) \mid i\} = b$.

If G is a topological group and A is a locale in the topos $C(G)$ of continuous G-sets, one may apply this procedure to a site for $C(G)$, so as to obtain an explicit site for $C(G)[A]$ as used in Freyd(1979). (A site for $C(G)$ is the atomic site with the following underlying category: objects are quotients G/U, U an open subgroup of G, and morphisms $\varphi: G/U \to G/V$ are maps of left G-sets, or equivalently, cosets gV with $U \subset gVg^{-1}$.)

2. Some types of geometric morphisms.

We will express some properties of geometric morphisms in terms of their corresponding continuous fibrations of sites in S.

We start with the case of surjections. Recall that a geometric morphism $F \xrightarrow{\varphi} E$ is a surjection iff φ^* is faithful, iff the localic part of φ, $Sh_E(\varphi_*(\Omega_F)) \to E$ is a surjection, iff the unique $\wedge\bigvee$-map $\Omega_E \to \varphi_*(\Omega_F)$ in E is monic.

2.1. Lemma. Let $F \xrightarrow{\varphi} E$ be a geometric morphism, and $\mathbb{C}$ a site for E, i.e. $E = S[\mathbb{C}]$. Then φ is a surjection iff φ is induced by a continuous fibration $\mathbb{D} \underset{T}{\overset{P}{\rightleftarrows}} \mathbb{C}$ with the property that P preserves covers of objects in the image of T (i.e., if $\{D_i \xrightarrow{f_i} TC\}_i$ is a cover in $\mathbb{D}$, then $\{PD_i \xrightarrow{\hat{f}_i} C\}_i$ covers in $\mathbb{C}$, where $Pf_i = \hat{f}_i$ = the transposed of f_i.)

Proof. (⇒) Let $\mathbb{A}$ be a site for F in E, with a terminal object $1_{\mathbb{A}}$. As is well-known, $F = E[\mathbb{A}] \to E$ is a surjection iff $E \models$ "every cover of $1_{\mathbb{A}}$ is inhabited". Under the construction $\mathbb{D} = \mathbb{C} \ltimes \mathbb{A}$ of section 1, this translates precisely into the righthand side of the equivalence stated in the lemma.

(⇐) There are several ways of proving this direction. We choose to compute the canonical map of frames $\Omega_E \xrightarrow{\lambda} \varphi_*(\Omega_F)$ in E (since we will need to consider this map anyway), and show that it is monic.

Since φ_* comes from composing with T,

$$\varphi_*(\Omega_F)(C) = \text{closed cribles on } TC \text{ in } \mathbb{D}.$$

The components $\lambda_C : \Omega_E(C) \to \varphi_*(\Omega_F)(C)$ of the map λ are given by

$$\lambda_C(K) = \text{the closed crible generated by } TK, \text{ or equivalently,}$$
$$\text{by } \{D \xrightarrow{f} TC \mid Pf: PD \to C \in K\}.$$

We will denote the closure of a crible for a given Grothendieck topology by brackets [.], so $\lambda_C(K) = [TK] = [\{f \mid Pf \in K\}]$. The (internal) right adjoint ρ of λ has components

$$\rho_C : \varphi_*(\Omega_F)(C) \to \Omega_E(C)$$
$$\rho_C(S) = \{C' \xrightarrow{f} C \mid Tf \in S\}$$

(this is a closed crible already, since T preserves covers).

So if $C' \xrightarrow{f} C \in \rho_C\lambda_C(K)$, that is $TC' \xrightarrow{Tf} TC \in \lambda_C(K)$, then there is a cover $D_i \xrightarrow{g_i} TC'$ in $\mathbb{D}$ such that $P(Tf \circ g_i) \in K$ for each i. But $P(Tf \circ g_i) = f \circ Pg_i$, and $\{Pg_i : D_i \to C'\}$ covers by assumption, so $f \in K$. ∎

Next, let us consider open surjections. $F \xrightarrow{\varphi} E$ is <u>open</u> iff φ^* preserves first order logic. As with surjections, φ is open iff its localic part is, iff the unique $\wedge\bigvee$-map $\Omega_E \to \varphi_*(\Omega_F)$ in E has an internal left adjoint. See Johnstone(1980), Joyal & Tierney (1982) for details, and various other equivalents.

<u>2.2. Lemma</u>. Let $F \xrightarrow{\varphi} E$ and $E = S[\mathbb{C}]$ as in 2.1. Then φ is an open surjection iff φ is induced by a continuous fibration $\mathbb{D} \rightleftarrows \mathbb{C}$ such that P preserves covers.

<u>Proof</u>. ($\Rightarrow$) If φ is open, there is a site $\mathbb{A}$ for F in E such that $E \models$ "$\forall A \in \mathbb{A}$: all covers of A are inhabited" (see Joyal & Tierney(1982)). Moreover, if φ is a surjective, we may assume that $\mathbb{A}$ has a terminal. So by constructing $\mathbb{D} = \mathbb{C} \ltimes \mathbb{A}$, the implication from left to right is clear.

($\Leftarrow$) We will show that the unique internal frame map in E, $\lambda: \Omega_E \to \varphi_*(\Omega_F)$, has an internal left adjoint μ. First note that since P preserves covers, the components of λ (cf. the proof of 2.1) can now be described by

$$\lambda_C(K) = \{D \xrightarrow{f} TC \mid Pf: PD \to C \in K\}$$

for $C \in \mathbb{C}$, K a closed crible on C in $\mathbb{C}$. Now define $\mu: \varphi_*(\Omega_F) \to \Omega_E$ by setting for $C \in \mathbb{C}$ and S a closed crible on TC,

$$\mu_C(S) = [PS], \text{ the closed crible generated by } \{Pf: PD \to C \mid f: D \to TC \in S\},$$

μ is indeed a natural transformation, for suppose we are given $\alpha: C' \to C$ in $\mathbb{C}$, $S \in \Omega_F(TC)$. We have to check that

$$\begin{array}{ccc} \Omega_F(TC) & \xrightarrow{\mu_C} & \Omega_E(C) \\ \downarrow{\scriptstyle (T\alpha)^{-1}} & & \downarrow{\scriptstyle \alpha^{-1}} \\ \Omega_F(TC') & \xrightarrow{\mu_{C'}} & \Omega_E(C') \end{array}$$

$\alpha^{-1}(\mu_C(S)) = \mu_{C'}((T\alpha)^{-1}(S))$, i.e.

$$\{C_o \xrightarrow{f} C \mid \alpha f \in [PS]\} = [\{Pg\colon PD \to C' \mid g\colon D \to TC',\ T\alpha \circ g \in S\}].$$

$\supseteq$ is clear, since $P(T\alpha \circ g) = \alpha \circ Pg : PD \to C$. Conversely, suppose $C_o \xrightarrow{f} C' \xrightarrow{\alpha} C \in [PS]$. Then there is a cover $C_i \xrightarrow{h_i} C_o$ such that $\alpha f h_i = P(k_i)$ for some $k_i : D_i \to TC \in S$. But k_i can be factored as

$$\begin{array}{ccc} D_i & \xrightarrow{k_i} & TC \\ & {\scriptstyle u_i} \searrow \quad \nearrow {\scriptstyle T\alpha} & \\ & TC' & \end{array}$$

such that $Pu_i = f \circ h_i$, by adjointness $P \dashv T$, from which the inclusion $\subseteq$ follows immediately.

Finally, μ and λ are indeed adjoint functors, since as one easily checks,

$$\mu_C \lambda_C(K) \subset K, \qquad \lambda_C \mu_C(S) \supset S,$$

for each $C \in \mathbf{C}$ and closed cribles S on TC, K on C. ∎

Recall that $F \xrightarrow{\varphi} E$ is <u>hyperconnected</u> iff its localic part is trivial, i.e. $\varphi_*(\Omega_F) \cong \Omega_E$ in E (see Johnstone(1981), Joyal & Tierney(1982)).

2.3. Lemma. Let $F \xrightarrow{\varphi} E$, $E = S[\mathbb{C}]$ be as before. Then φ is hyperconnected iff φ is induced by a continuous fibration $\mathbb{D} \underset{T}{\overset{P}{\rightleftarrows}} \mathbb{C}$ such that P preserves covers, and moreover every unit morphism $D \xrightarrow{\eta_D} TPD$ is a singleton-cover in $\mathbb{D}$.

Proof. ($\Rightarrow$) Write φ as $E[\mathbb{A}] \to E$, $\mathbb{A}$ a site for F in E. φ is an open surjecion, so we may assume that it holds in E that if $\{A_i \to A\}_{i \in I}$ is a cover then I must be inhabited, and moreover that $\mathbb{A}$ contains a terminal object. φ is hyperconnected iff the canonical framemap $P(1) \to \varphi_*(\Omega_F)$ (= the poset of closed cribles on $\mathbb{A}$) is an isomorphism in E, iff its left adjoint

$$\mu(K) = [\![K \text{ is inhabited}]\!]$$

is an isomorphism. So K is inhabited iff $1 \in K$, and hence every map $A \to 1$ is a cover in $\mathbb{A}$. $\Rightarrow$ now follows by taking $\mathbb{D} = \mathbb{C} \ltimes \mathbb{A}$, as before.

($\Leftarrow$) Recall $\lambda: \Omega_E \to \varphi_*(\Omega_F)$ and its right adjoint ρ from the proofs of 2.1 and 2.2. By 2.1, we have $\rho\lambda = \text{id}$. Conversely, if $C \in \mathbb{C}$ and $S \in \Omega_F(TC)$ is a closed crible on TC in $\mathbb{D}$, then

$$\begin{aligned} \lambda_C \rho_C(S) &= \{D \xrightarrow{f} TC \mid Pf : PD \to C \in \rho_C(S)\} \\ &= \{D \xrightarrow{f} TC \mid TPD \xrightarrow{TPf} TC \in S\}. \end{aligned}$$

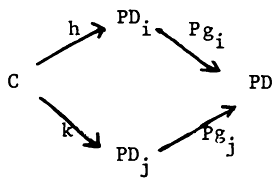
But if $D \xrightarrow{\eta} TPD$ covers, then $TPf : TPD \to TC \in S$ iff $TPf \circ \eta = D \xrightarrow{f} TC \in S$, i.e. $\lambda\rho = \text{id}$. So $\varphi_*(\Omega_F) \simeq \Omega_E$. ∎

A geometric morphism $F \xrightarrow{\varphi} E$ is connected if φ^* is full and faithful. φ is called atomic if φ^* is logical, or equivalently, if $F = E[\mathbb{D}]$ for some atomic site $\mathbb{D}$ in E (see Barr & Diaconescu (1980), Joyal & Tierney(1982)). Such an atomic morphism φ is connected iff it is of the form $E[\mathbb{D}]$ for some atomic site $\mathbb{D}$ having a terminal object (1 is an atom in F).

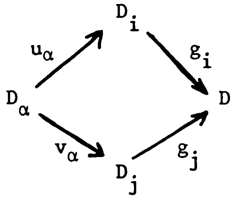
<u>2.4. Lemma</u>. Again, let $F \xrightarrow{\varphi} E$ be a geometric morphism, $\mathbb{C}$ a site for E in S. Then φ is atomic connected iff φ is induced by a continuous fibration $\mathbb{D} \underset{T}{\overset{P}{\rightleftarrows}} \mathbb{C}$ where P preserves and reflects covers.

<u>Proof</u>. (⇒) Let $\mathbb{A}$ be an atomic site for F in E, with a terminal object. It suffices to take $\mathbb{D} = \mathbb{C} \ltimes \mathbb{A}$.

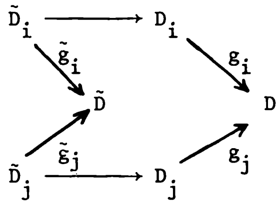
(⇐) Let us first note that if X is a sheaf on $\mathbb{C}$, $C \circ P$ is a sheaf on $\mathbb{D}$. For in that case φ^* = compose with P, and $\varphi_*\varphi^* \simeq$ id, so φ is certainly a connected surjection. To see this, suppose $\{D_i \xrightarrow{g_i} D\}_i$ is a cover in $\mathbb{D}$ and $x_i \in X(PD_i)$ is a compatible family of elements - compatible over $\mathbb{D}$, that is. $\{PD_i \xrightarrow{Pg_i} PD\}_i$ is a cover in $\mathbb{C}$, so it suffices to show that the family $\{x_i\}$ is also compatible over $\mathbb{C}$. That is, for each i,j and each commutative square



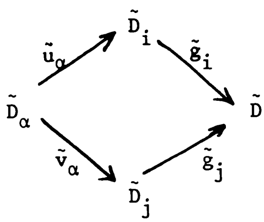
in $\mathbb{C}$, $x_i \mid h = x_j \mid k$. Since $\{x_i\}$ is $\mathbb{D}$-compatible and X is a sheaf on $\mathbb{C}$, it is sufficient to find a cover $PD_\alpha \xrightarrow{f_\alpha} C$ and for each α a commutative square



with $P(u_\alpha) = hf_\alpha$, $P(v_\alpha) = kg$. Write $e: C \to PD$ for $Pg_i \circ h = Pg_j \circ k$, and let $\tilde{D} \xrightarrow{\tilde{e}} D = T_D(e)$. Applying T_{PD_i} and T_{PD_j} we find a diagram



with $P\tilde{g}_i = P\tilde{g}_j = id_C$. P reflects covers, so $\tilde{D}_i \xrightarrow{\tilde{g}_i} D$ is a cover in $\mathbb{D}$ (consisting of one element). Pulling back this cover along $\tilde{g}_j$ yields a family of commutative squares



such that the $\tilde{v}_\alpha$ cover $\tilde{D}_j$. We need only set $f_\alpha = P(\tilde{g}_i \circ \tilde{u}_\alpha) = P(\tilde{g}_j \circ \tilde{v}_\alpha)$ to obtain the desired cover of C.

Having the information that $\varphi^*(X) = X \circ P$ for every $X \in S[\mathbb{C}]$, we easily prove that φ^* is logical.

First, φ^* preserves the subobject classifier: we always have a canonical map $\Omega_F \xrightarrow{\sigma} \varphi^*(\Omega_E)$ in F with an adjoint τ. In this case the components are described by

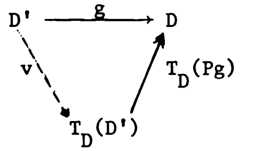
$$\sigma_D : \Omega_F(D) \longrightarrow \Omega_E(PD)$$

$$\sigma_D(K) = \{C \xrightarrow{f} PD \mid T_D(f) \in K\} = PK$$

$$\tau_D : \Omega_E(PD) \longrightarrow \Omega_F(D)$$

$$\tau_D(S) = \{D' \xrightarrow{g} D \mid Pg \in S\}.$$

Note that $\sigma_D(K)$ as described is indeed a closed crible, and that σ and τ are natural. We claim that σ and τ are inverse to each other. If S is a closed crible on PD, then $\sigma_D\tau_D(S) = P(\{D' \xrightarrow{g} D \mid Pg \in S\})$, so clearly $\sigma_D\tau_D(S) \subset S$. Conversely if $C \xrightarrow{f} PD \in S$, then $T_D(f) \in \tau_D(S)$ and $PT_D(f) = f$, so $S \subset \sigma_D\tau_D(S)$. And if K is a closed crible on D, we have $\tau_D\sigma_D(K) = \{D' \xrightarrow{g} D \mid T_D(Pg) \in K\}$, so $K \subset \tau_D\sigma_D(K)$ since there is a factorization



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But P maps this morphism $v: D' \to T_D(D') :=$ domain $T_D(Pg)$ to the identity on PD', so v covers since P reflects covers. Thus $K \supset \tau_D\sigma_D(K)$.

Finally, we show that φ^* preserves exponentials, i.e. for $X,Y \in E$, $\varphi^*(Y^X) \cong \varphi^*(Y)^{\varphi^*(X)}$. Let us describe the isomorphism $\alpha : \varphi^*(Y^X) \to \varphi^*(Y)^{\varphi^*(X)}$ with inverse β explicitly: $Y^X(C) =$ the set of natural transformations $X \to Y$ over C, so $\varphi^*(Y^X)(D) = Y^X(PD)$, while $\varphi^*(Y)^{\varphi^*(X)}(D)$ = the set of natural transformations $X \circ P \to Y \circ P$ over D. The canonical map $\alpha : \varphi^*(Y^X) \to \varphi^*(Y)^{\varphi^*(X)}$ has components α_D described as follows. Given $\tau : X \to Y$ over PD, $\alpha_D(\tau) : X \circ P \to Y \circ P$ over D is defined by

$$\alpha_D(\tau)_f(x) = \tau_{Pf}(x)$$

where $D' \xrightarrow{f} D$ and $x \in X(PD')$. The components β_D of $\beta : \varphi^*(Y)^{\varphi^*(X)} \to \varphi^*(Y^X)$ are described as follows. Given $\sigma : X \circ P \to Y \circ P$ over D, $\beta_D(\sigma) : X \to Y$ over PD is defined by

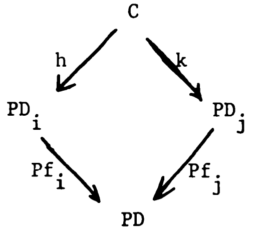
$$\beta_D(\sigma)_g(x) = \sigma_{T_D(g)}(x),$$

for $C \xrightarrow{g} PD$, (so $T_D(g) : D' \to D$ with $PD' = C$), and $x \in X(C)$. It is clear that $\beta \circ \alpha$ is the identity. The other way round, $\alpha \circ \beta = id$ follows from the fact that if $D' \xrightarrow{f} D$, then the unit $u : D' \to$ domain $T_D(Pf)$, with $T_D(Pf) \circ u = f$ is mapped by P to the identity on PD'. For if $x \in X(PD')$ then $\alpha_D \circ \beta_D(\sigma)_f(x) = \beta_D(\sigma)_{Pf}(x)\ \sigma_{T_D(Pf)}(x) = \sigma_f(x)$, the last equality by naturality of σ, since $Pu = id$. ∎

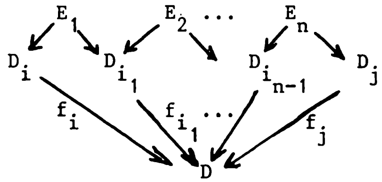
Finally, we consider the case of connected locally connected geometric morphisms. This class has been studied in Barr & Paré(1980). For some equivalent descriptions see the Appendix (This Appendix has been added since we will need the main characterization of locally connected morphisms of Barr & Paré in a slightly different form later on, and moreover for the bounded case there is a rather short and self-contained proof of this characterization, which we give in this Appendix.)

Let $\mathbb{D} \underset{T}{\overset{P}{\rightleftarrows}} \mathbb{C}$ be a continuous fibration, and let $\{D_i \xrightarrow{f_i} D\}_{i \in I}$ be a cover in $\mathbb{D}$. Given a commutative square

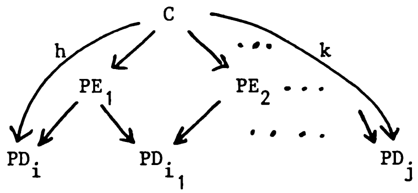
(*)



we call D_i and D_j C - connected (for the cover $\{D_i \xrightarrow{f_i} D\}$) if there are $i = i_o, i_1, \ldots, i_n = j$ in I such that there is a commuting zig-zag



the P - image of which commutes under C:



2.5. Lemma. Let $F \xrightarrow{\varphi} E$, $E = S[\mathbb{C}]$ be as before. Then φ is connected locally connected iff it is induced by a continuous fibration $\mathbb{D} \underset{T}{\overset{P}{\rightleftarrows}} \mathbb{C}$ such that P preserves covers, and for every cover $\{D_i \xrightarrow{f_i} D\}$ in $\mathbb{D}$ and every commutative square of the form (*), the family of maps $C' \to C$ such that D_i, D_j are C' - connected is a cover of C.

Proof. (⇒) Let $\mathbb{A}$ be a molecular site (see Appendix) in $\mathcal{E}$, with terminal object $1_{\mathbb{A}}$. A straightforward argument by forcing over $\mathbb{C}$ shows that $\mathbb{D} = \mathbb{C} \ltimes \mathbb{A}$ satisfies the required conditions.

(⇐) This is really completely similar to the proof of ⇐ of lemma 2.4. The condition on $\mathbb{D} \underset{T}{\overset{P}{\rightleftarrows}} \mathbb{C}$ is exactly what we need to show that if X is a sheaf on $\mathbb{C}$, $X \circ P$ is a sheaf on $\mathbb{D}$. So again φ^* = compose with P and $\varphi_*\varphi^* \cong \mathrm{id}$. So φ is connected. Moreover, it is now straightforward to check that φ^* commutes with Π-functors (or, since the conditions on $\mathbb{D} \rightleftarrows \mathbb{C}$ are stable under localization, just check that φ^* preserves exponentials), as in the proof of 2.4. ∎

3. A description of inverse limits.

In the special case of inverse sequences of geometric morphisms we can easily express the inverse limit explicitly as sheaves over a site using continuous fibrations. Suppose we are given a sequence

$$\cdots\ \mathcal{E}_{n+1} \xrightarrow{f_n} \mathcal{E}_n \longrightarrow \cdots \longrightarrow \mathcal{E}_1 \xrightarrow{f_o} \mathcal{E}_o$$

of toposes and geometric morphisms over $\mathcal{S}$. By the results of sections 1, we can find sites $\mathbb{C}_n$ for $\mathcal{E}_n$ in $\mathcal{S}$, that is $\mathcal{E}_n = \mathcal{S}[\mathbb{C}_n]$, such that f_n is represented by a continuous fibration

$$\mathbb{C}_{n+1} \underset{T_n}{\overset{P_n}{\rightleftarrows}} \mathbb{C}_n .$$

We will write $P_{nm} : \mathbb{C}_n \to \mathbb{C}_m$ for $P_{m-1} \circ \cdots \circ P_n$, $T_{nm} : \mathbb{C}_m \to \mathbb{C}_n$ for $T_{m-1} \circ \cdots \circ T_n$. (So P_{nn}, T_{nn} are identity functors.)

We construct a site $\mathbb{C}^{\infty} = \varprojlim_n \mathbb{C}_n$ as follows. Objects of $\mathbb{C}^{\infty}$ are sequences $(C_n)_n$, C_n an object of $\mathbb{C}_n$, such that $P_n(C_{n+1}) = C_n$ and $\exists m \forall n \geq m$ $C_{n+1} = T_n(C_n)$. Morphisms $(C_n)_n \to (C'_n)_n$ of $\mathbb{C}^{\infty}$ are just sequences $f = (f_n)_n$, $f_n : C_n \to C'_n$ a morphism of $\mathbb{C}_n$, such that $P_n(f_{n+1}) = f_n$.

There is a canonical fibration

$$\mathbb{C}^{\infty} \underset{T_m^{\infty}}{\overset{P_m^{\infty}}{\rightleftarrows}} \mathbb{C}_m$$

for each m: $P_m^{\infty}((C_n)_n) = C_m$, and for $C \in \mathbb{C}_m$,

$$T_m^{\infty}(C)_n = \begin{cases} P_{mn}(C) & \text{if } m \geq n \\ T_{nm}(C) & \text{if } m \leq n. \end{cases}$$

$\mathbb{C}^{\infty}$ is made into a site by equipping it with the coarsest Grothendieck topology making all the functors T_m^{∞} continuous. That is, the topology on $\mathbb{C}^{\infty}$ is generated by covers of the form

$$\{T_m^{\infty}(C_i \xrightarrow{f_i} C)\}_i, \quad \text{for} \quad \{C_i \xrightarrow{f_i} C\}_i \text{ a cover in } \mathbb{C}_m.$$

Note that this is a stable generating system for the topology. This topology makes each pair $P_m^{\infty}, T_m^{\infty} : \mathbb{C}^{\infty} \rightleftarrows \mathbb{C}_m$ into a continuous fibration. Moreover, the $P_m^{\infty}, T_m^{\infty}$ are coherent in the sense that $T_{m+1}^{\infty} \circ T_m = T_m^{\infty}$, $P_m \circ P_{m+1}^{\infty} = P_m^{\infty}$.

Let $E^{\infty} = S[\mathbb{C}^{\infty}]$. The functors T_m^{∞} are flat and continuous, so they induce geometric morphisms

$$\pi_m : E^{\infty} \to E_m$$

over S. We claim that $E^\infty = \varprojlim E_m$. Indeed, suppose we are given geometric morphisms $g_n : F \to E_n$ with $f_n \circ g_{n+1} = g_n$, each n (or really only up to canonical isomorphism, since E^∞ is a pseudo-limit; but cf. the introduction). g_n comes from a flat and continuous functor $\mathbb{C}_n \xrightarrow{G_n} F$, and $G_{n+1} \circ T_n = G_n$. Define $G^\infty : \mathbb{C}^\infty \to F$ by $G^\infty((C_n)_n) = G_m(C_m)$, for m so large that $C_{n+1} = T_n(C_n)$, $\forall n > m$. (This does not depend on the choice of m, since $G_{n+1} \circ T_n = G_n$. Moreover, the canonical maps $G_{m+1}(C_{m+1}) \to G_m(C_m)$ become isomorphisms eventually, so we could equivalently have set $G^\infty((C_n)_n) = \varprojlim G_m(C_m)$. This also looks more coherently functorial.) Observe that G^∞ is a flat and continuous functor $\mathbb{C}^\infty \to F$, and that $G^\infty \circ T_m^\infty = G_m$. So we obtain a geometric morphism

$$g^\infty : F \to E^\infty, \text{ with } \pi_m \circ g^\infty = g_m.$$

g^∞ is obviously unique up to natural isomorphism, since $(C_n)_n = \varprojlim_m T_m^\infty(C_m)$ in $\mathbb{C}^\infty$, and this inverse limit is eventually constant, so really a finite inverse limit. Therefore, any flat and continuous functor $H: \mathbb{C}^\infty \to F$ with $H \circ T_m^\infty = G_m$ for all m must satisfy $H((C_n)_n) = \varprojlim (G_m(C_m))$. For the record,

<u>3.1. Theorem</u>. If $\ldots E_{n+1} \xrightarrow{f_n} E_n \longrightarrow \ldots \longrightarrow E_1 \xrightarrow{f_o} E_o$ is a sequence of geometric morphisms, with f_n induced by a continuous fibration $\mathbb{C}_{n+1} \underset{T_n}{\overset{P_n}{\rightleftarrows}} \mathbb{C}_n$, then $\mathbb{C}^\infty$ as constructed above is a site for $E^\infty = \varprojlim E_n$, and the canonical projections $E^\infty \xrightarrow{\pi_m} E_m$ correspond to the continuous fibrations $\mathbb{C}^\infty \underset{T_m^\infty}{\overset{P_m^\infty}{\rightleftarrows}} \mathbb{C}_m$. ∎

Suppose we are given an inverse sequence of continuous fibrations

$$\cdots\ \mathbb{C}_{n+1} \underset{T_n}{\overset{P_n}{\rightleftarrows}} \mathbb{C}_n \ \cdots \rightleftarrows \mathbb{C}_2 \underset{T_1}{\overset{P_1}{\rightleftarrows}} \mathbb{C}_1 \underset{T_o}{\overset{P_o}{\rightleftarrows}} \mathbb{C}_o$$

as above. Let A_n be the preordered reflection of $\mathbb{C}_n$. That is, the objects of A_n are the same as those of $\mathbb{C}_n$, and we put $C < C'$ in A_n iff there is a morphism $C \to C'$ in $\mathbb{C}_n$. A family $\{C_i < C\}_i$ covers in A iff there is a covering family $\{C_i \to C\}_i$ in $\mathbb{C}_n$. So $S[\mathbb{C}_n] \to S[A_n] \to S$ is the hyperconnected - localic factorisation of $S[\mathbb{C}_n] = E_n \to S$. There are canonical projection functors $\mathbb{C}_n \xrightarrow{\tau_n} A_n$ and the continuous fibration $\mathbb{C}_{n+1} \underset{T_n}{\overset{P_n}{\rightleftarrows}} \mathbb{C}_n$ is mapped down by τ_n to a continuous fibration $A_{n+1} \rightleftarrows A_n$. So we obtain a diagram

$$\begin{array}{ccccccc}
\mathbb{C}^\infty & \cdots \rightleftarrows & \mathbb{C}_2 & \rightleftarrows & \mathbb{C}_1 & \rightleftarrows & \mathbb{C}_o \\
\downarrow{\scriptstyle \tau^\infty} & & \downarrow{\scriptstyle \tau_2} & & \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_o} \\
A^\infty & \cdots \rightleftarrows & A_2 & \rightleftarrows & A_1 & \rightleftarrows & A_o
\end{array}$$

and as is immediate from the construction, A^∞, the limit of the lower sequence as described before theorem 3.1, is the preordered reflection of the site $\mathbb{C}^\infty$. So $S[\mathbb{C}^\infty] \to S[A^\infty] \to S$ is the hyperconnected - localic factorisation of $E^\infty \to S$.

Thus we have the following corollary.

<u>3.2. Corollary</u>. The localic reflection preserves limits of inverse sequences. ∎

This will be generalized in section 5.

4. Preservation properties of inverse sequences.

We are now ready to prove preservation under limits of inverse sequences of toposes for the classes of geometric morphisms that were analysed in section 2. In the next section, this will be generalized to arbitrary filtered systems.

4.1. Theorem. Let $\ldots \mathcal{E}_{n+1} \xrightarrow{f_n} \mathcal{E}_n \longrightarrow \ldots \longrightarrow \mathcal{E}_1 \xrightarrow{f_o} \mathcal{E}_o$ be a sequence of Grothendieck toposes and geometric morphisms over $\mathcal{S}$, and let $\mathcal{E}^\infty = \varprojlim \mathcal{E}_n$ be the inverse limit of this sequence, with canonical projections $\mathcal{E}^\infty \xrightarrow{\pi_n} \mathcal{E}_n$. Then

(i) If each f_n is a surjection, then so is each π_n;

(ii) If each f_n is an open surjection, then so is each π_n;

(iii) If each f_n is hyperconnected, then so is each π_n;

(iv) If each f_n is connected atomic, then so is each π_n;

(v) If each f_n is connected locally connected, then so is each π_n.

Proof. I postpone the proof of (i). For (ii), choose by repeated application of lemma 2.2 a sequence of sites $\mathbb{C}_n$ for $\mathcal{E}_n$, i.e. $\mathcal{E}_n = \mathcal{S}[\mathbb{C}_n]$, such that f_n is induced by a continuous fibration

$$\mathbb{C}_{n+1} \underset{T_n}{\overset{P_n}{\rightleftarrows}} \mathbb{C}_n$$

where P_n preserves covers. By lemma 2.2 again, we need to show that each projection $P_n^\infty : \mathbb{C}^\infty \to \mathbb{C}_n$ as described in section 3 preserves covers. Since the family of covers generating the topology of $\mathbb{C}^\infty$ is stable, it is sufficient to show that P_n^∞ preserves these generating covers. But if $\{T_m^\infty(C_i) \xrightarrow{T_m^\infty(g_i)} T_m^\infty(C)\}_i$ is a basic cover of $\mathbb{C}^\infty$ coming from a cover $\{C_i \xrightarrow{g_i} C\}_i$ in $\mathbb{C}_m$, then

$P_n^\infty(T_m^\infty(g_i)) = P_{mn}(g_i)$ if $n < m$, and $P_n^\infty(T_m^\infty(g_i)) = T_{nm}(g_i)$ if $n > m$, so P_n^∞ preserves this cover since both T_{nm} and P_{nm} preserve covers. This proves (ii).

The proofs of (iv) and (v) are entirely similar. For example to show for (iv) that P_m^∞ reflects covers if all the P_n do, one reason as follows: suppose we have a family of morphisms $g^i = (g_n^i)_n$ in $\mathfrak{C}^\infty$ with common codomain,

$$(C_n^i)_n \xrightarrow{g_i} (C_n)_n,$$

and suppose $\{P_m^\infty(g^i)\}_i$, that is $\{C_m^i \xrightarrow{g_m^i} C_m\}_i$, is a cover in $\mathfrak{C}_m$. Choose an $n_o > m$ such that $C_{n+1} = T_n(C_n)$ for $n > n_o$. $P_{n_o m}$ reflects covers since all the P_m do, so $\{C_{n_o}^i \xrightarrow{g_{n_o}^i} C_{n_o}\}_i$ is a cover in $\mathfrak{C}_{n_o}$. Thus $\{T_{n_o}^\infty(C_{n_o}^i) \xrightarrow{h^i} T_{n_o}^\infty(C_{n_o})\}_i$ is a cover in $\mathfrak{C}^\infty$, where $h^i = T_{n_o}^\infty(g_{n_o}^i)$. But $T_{n_o}^\infty(C_{n_o}) = (C_n)_n$ since $m > n_o$, so we need only show that for a fixed i, the canonical map

$$k^i : (C_n^i)_n \to T_{n_o}^\infty(C_{n_o}^i), \qquad (k_n^i = g_n^i \quad \text{for} \quad n < n_o)$$

is a cover. Choose $n_i > n_o$ so large that $(C_n^i)_n = T_{n_i}(C_{n_i}^i)$. Then k^i is the map

$$k^i = T_{n_i}^\infty(k_{n_i}^i) : T_{n_i}^\infty(C_{n_i}^i) \to T_{n_i}^\infty(T_{n_i n_o}(C_{n_o}^i)),$$

which covers because the map $C_{n_i}^i \to T_{n_i n_o}(C_{n_o}^i)$ covers in $\mathfrak{C}_{n_i}$, as follows from the fact that $P_{n_i n_o}$ reflects covers.

For (v), suppose each continuous fibration $\mathfrak{C}_{n+1} \underset{T_n}{\overset{P_n}{\rightleftarrows}} \mathfrak{C}_n$ is of the form as described in lemma 2.5. We wish to show for each m that the continuous fibration $\mathfrak{C} \underset{T_m^\infty}{\overset{P_m^\infty}{\rightleftarrows}} \mathfrak{C}_m$ is again of this form.

It suffices to consider the basic covers in $\mathbb{C}^\infty$ which generate the topology when verifying the conditions of lemma 2.5. So let $\{T_n^\infty(C_i \to C)\}_i$ be such a basic cover, coming from a cover $\{C_i \to C\}_i$ in $\mathbb{C}_n$. Since $T_n^\infty = T_{n'}^\infty \circ T_{n'n}$ (any $n' > n$) and $T_{n'n}$ preserves covers, we may without loss assume that $n > m$. It is now clear that we only need to show that the composite continuous fibration $\mathbb{C}_n \underset{T_{nm}}{\overset{P_{nm}}{\rightleftarrows}} \mathbb{C}_m$ again satisfies the conditions of lemma 2.5.
Thus, arguing by induction, the following lemma completes the proof of case (v) of the theorem.

<u>Lemma</u>. Let $\mathbb{D} \underset{T}{\overset{P}{\rightleftarrows}} \mathbb{C} \underset{S}{\overset{Q}{\rightleftarrows}} \mathbb{B}$ be two continuous fibrations, each as described in lemma 2.5. Then the composite continuous fibration $\mathbb{D} \underset{TS}{\overset{QP}{\rightleftarrows}} \mathbb{B}$ also satisfies the conditions of lemma 2.5.

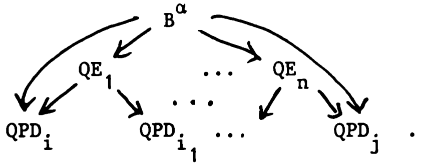
<u>Proof of lemma</u>. Clearly if P and Q preserve covers then so does QP. Suppose $\{D_i \to D\}$ is a cover in $\mathbb{D}$, and for fixed indices i,j we are given $B \in \mathbb{B}$ and a commutative square

$$\begin{array}{ccccc} & & B & & \\ & \swarrow & & \searrow & \\ QPD_i & & & & QPD_j \\ & \searrow & & \swarrow & \\ & & QPD & & \end{array}$$

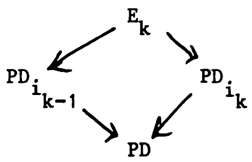
Applying the conditions of lemma 2.5 to $\mathbb{D} \to \mathbb{C}$, we find a cover $\{B^\alpha \to B\}_\alpha$ of B, and for each α connecting zig-zags ($n = n_\alpha$ depending on α)

$$\begin{array}{ccccccccc} & & E_1 & & \cdots & & E_n & & \\ & \swarrow & & \searrow & & \swarrow & & \searrow & \\ PD_i & & & & PD_{i_1} \;\cdots\; PD_{i_{n-1}} & & & & PD_j \\ & & \searrow & & \searrow \;\; \swarrow & & \swarrow & & \\ & & & & PD & & & & \end{array}$$

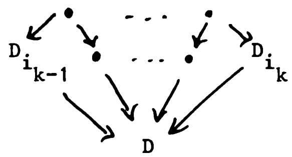
such that there Q-images commute under B^{α}:



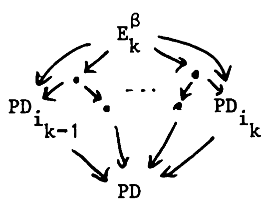
Now fix α, and apply the conditions of 2.5 to the continuous fibration $\mathbb{D} \rightleftarrows \mathbb{C}$ for each square $(0<k<n_{\alpha})$



separately. This gives for each $k<n_{\alpha}$ a cover $\{E_k^{\beta} \to E_k\}$ and connecting chains



the P-images of which commute under E_k^{β}:



For each $k < n_\alpha$, $\{QE_k^\beta \to QE_k\}$ covers in $\mathbb{B}$, and pulling back this cover along $B^\alpha \to QE_k$ gives a cover of B^α. It now suffices to take a common refinement of these n_α - many covers of B^α, say $\{B^{\alpha\gamma} \to B^\alpha\}_\gamma$; then the family of composites $\{B^{\alpha\gamma} \to C\}_{\alpha,\gamma}$ is a cover of C such that D_i and D_j are $B^{\alpha\gamma}$-connected (each α,γ), thus proving the lemma.

We now complete the proof of the theorem. (iii) can be proved exactly as (ii),(iv) and (v), but for (i) I do not see an argument of this type. But in any case, (i) and (iii) follow immediately from Corollary 3.2: Let A_n be the localic reflection of $\mathcal{E}^n \to \mathcal{S}$, so $A^\infty = \varprojlim A_n$ is the localic reflection of $\mathcal{E}^\infty$.

$$\begin{array}{ccccccc} \mathcal{E}^\infty & \cdots & \xrightarrow{f_1} & \mathcal{E}_1 & \xrightarrow{f_o} & \mathcal{E}_o \\ \downarrow & \cdots & & \downarrow & & \downarrow \\ Sh(A^\infty) & \cdots & \xrightarrow{g_1} & Sh(A_1) & \xrightarrow{g_o} & Sh(A_o) \end{array} .$$

For (iii), suppose each f_n is hyperconnected, and choose m. We want to show that π_m is hyperconnected. By working in $\mathcal{E}_m$, we may assume that $m = 0$ and that $\mathcal{E}_o = \mathcal{S}$. Since each f_n is hyperconnected, each A_n is the one-point locale 1 (i.e. $\mathcal{O}(A_n) = \mathcal{P}(1)$), hence so is A^∞, hence $\mathcal{E}^\infty \to \mathcal{S}$ is hyperconnected.

For (i), suppose each f_n is surjective. To prove that each π_m is surjective, it is again sufficient to assume $m = 0$ and $\mathcal{S} = \mathcal{E}_o$. But then by proposition IV.4.2 of Joyal & Tierney(1982), $A^\infty \to A_o = 1$ is surjective, hence so is $\mathcal{E}^\infty \to \mathcal{S}$

This completes the proof of the theorem. ∎

5. Preservation properties of filtered inverse systems.

In this section we will generalize theorem 4.1 to inverse limits of arbitrary (small) filtered systems of toposes, and prove the main theorem. (Recall that a poset I is filtered (or directed) if for all $i,j \in I$ there is a $k \in I$ with $k \geq i$, $k \geq j$.)

5.1. Theorem. Let $(\mathcal{E}_i)_{i \in I}$ be an inverse system of toposes over $\mathcal{S}$, indexed by a filtered poset I in $\mathcal{S}$, with transition mappings $f_{ij} : \mathcal{E}_i \to \mathcal{E}_j$ for $i \geq j$. Let $\mathcal{E}^\infty = \varprojlim \mathcal{E}_i$ be the inverse limit of this system, with projection mappings $\mathcal{E}^\infty \xrightarrow{\pi_i} \mathcal{E}_i$. Then

(i) if each f_{ij} is a surjection, then so is each π_i;

(ii) if each f_{ij} is an open surjection, then so is each π_i;

(iii) if each f_{ij} is hyperconnected, then so is each π_i;

(iv) if each f_{ij} is connected atomic, then so is each π_i;

(v) if each f_{ij} is connected locally connected, then so is each π_i.

Corollary 3.2 will also be generalized to arbitrary filtered systems. A main ingredient involved in the proofs is the following theorem, which is of independent interest.

5.2. Theorem. All the types of geometric morphisms involved in theorem 5.1 are reflected down open surjections. More explicitly, if

$$(*) \qquad \begin{array}{ccc} \mathcal{E}' & \xrightarrow{q} & \mathcal{E} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathcal{S}' & \xrightarrow{p} & \mathcal{S} \end{array}$$

is a pullback of toposes, where p is an open surjection, then

(i) if f' is a surjection, then so is f;

(ii) if f' is an open surjection, then so is f;

(iii) if f' is hyperconnected, then so is f;

(iv) if f' is (connected) atomic, then so is f;

(v) if f' is (connected) locally connected, then so is f.

Proof of 5.2. (i) is trivial. Note that the types of geometric morphisms in (ii)-(v) are stable under pullback, so by using a localic cover $\mathcal{S}[A] \to \mathcal{S}'$ (A a locale in $\mathcal{S}$, $\mathcal{S}[A] \to \mathcal{S}'$ an open surjection; see Diaconescu(1976), Johnstone(1980), Joyal & Tierney(1982)), we may assume that $\mathcal{S}' \to \mathcal{S}$ is localic whenever this is convenient.

Case (ii) follows from proposition VII.1.2 of Joyal & Tierney (1982).

For case (iii), let $\mathcal{S}' = \mathcal{S}[A]$, A an open surjective locale in $\mathcal{S}$, and write $B = f_*(\Omega_{\mathcal{E}})$, $B' = f'_*(\Omega_{\mathcal{E}'})$ for the localic reflections of $\mathcal{E}$ in $\mathcal{S}$ and of $\mathcal{E}'$ in $\mathcal{S}'$. Then we have a pullback of open surjective locales

$$\begin{array}{ccc} B' & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & 1 \end{array}$$

in $\mathcal{S}$, such that the map $B' \to A$ is an isomorphism. So $\mathcal{O}(B') = \mathcal{O}(A) \otimes \mathcal{O}(B)$, and in terms of frames we get

$$\begin{array}{ccc} \mathcal{O}(A) \otimes \mathcal{O}(B) & \underset{\exists_{\pi_1}}{\overset{\pi_2^*}{\rightleftarrows}} & \mathcal{O}(B) \\ {\scriptstyle \pi_1^*}\uparrow\downarrow{\scriptstyle \exists_\pi} & & {\scriptstyle \exists_g}\downarrow\uparrow{\scriptstyle B^*} \\ \mathcal{O}(A) & \underset{A^*}{\overset{\exists_A}{\rightleftarrows}} & \mathcal{P}(1) \end{array}$$

where the maps $\exists_{(.)}$ are left adjoint to the frame maps $(.)^*$, and a Beck condition holds (Joyal & Tierney(1982), §V.4):

$$A^*\exists_B = \exists_{\pi_1}\pi_2^*, \qquad B^*\exists_A = \exists_{\pi_2}\pi_1^* .$$

By assumption π_1^* is an isomorphism, so $\exists_{\pi_1}$ is its inverse. We claim that B^* is an isomorphism, with inverse $\exists_B$. $\exists_B \circ B^* = \mathrm{id}$ since $B \to 1$ is an open surjection. The other way round, we have $\pi_2^* B^* \exists_B = \pi_1^* A^* \exists_B = \pi_1^* \exists_{\pi_1} \pi_2^* = \pi_2^*$, and π_2 is an open surjection, so π_2^* is $1-1$, so $B^*\exists_B = \mathrm{id}$. This proves case (iii).

For case (iv), let us first prove that atomicity by itself is reflected. This actually follows trivially from the characterization of atomic maps by Joyal & Tierney(1982), Ch.VII: a map $E \xrightarrow{f} S$ is atomic iff both f and the diagonal $E \xrightarrow{\Delta} E \times_S E$ are open. For consider the following diagram:

$$\begin{array}{ccccc} E' & \xrightarrow{\Delta'} & E' \times_{S'} E' & \longrightarrow & S' \\ \downarrow & & \downarrow & & \downarrow \\ E & \xrightarrow{\Delta} & E \times_S E & \longrightarrow & S \end{array}$$

Since (*) in the statement of the theorem is a pullback and the right-hand square above is a pullback, so is the left-hand sqaure. Therefore by case (ii) of the theorem, if Δ' is open so is Δ. Hence if $E' \xrightarrow{f'} S'$ is atomic, so is $E \xrightarrow{f} S$.

Next, we show that if f' in the diagram (*) is <u>stably</u> connected (i.e. the pullback of f' along a geometric morphism is still connected, as is the case for connectedness in combination with local connectedness, and hence in combination with atomicity, as in (iv) and (v) of the theorem), then f must be connected. To do this, we use descent theory (Ch.VIII of Joyal & Tierney(1982): As f' is stably connected, so is the map $\mathcal{E}' \times_{\mathcal{E}} \mathcal{E}' \xrightarrow{f' \times f'} \mathcal{S}' \times_{\mathcal{S}} \mathcal{S}'$, since we have a pullback

$$\begin{array}{ccc} \mathcal{E}' \times_{\mathcal{E}} \mathcal{E}' & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow f \\ \mathcal{S}' \times_{\mathcal{S}} \mathcal{S}' & \longrightarrow & \mathcal{S} \end{array},$$

and similarly $\mathcal{E}' \times_{\mathcal{E}} \mathcal{E}' \times_{\mathcal{E}} \mathcal{E}' \longrightarrow \mathcal{S}' \times_{\mathcal{S}} \mathcal{S}' \times_{\mathcal{S}} \mathcal{S}'$ is connected. Now clearly, if f'^* is faithful so is f^* (this is case (i) above). To show that f^* is full, let $\beta: f^*(X) \to f^*(Y)$ be a map in $\mathcal{E}$. $\mathcal{E}' \xrightarrow{q} \mathcal{E}$ is an open surjection, hence an effective descent morphism, so β is equivalent to a map $\gamma = q^*(\beta) : q^*f^*(X) \to q^*f^*(Y)$ compatible with descent data in $\mathcal{E}'$, i.e. $\gamma: f'^*p^*(X) \to f'^*p^*(Y)$. Descent data lives in $\mathcal{E}'$, $\mathcal{E}' \times_{\mathcal{E}} \mathcal{E}'$, and $\mathcal{E}' \times_{\mathcal{E}} \mathcal{E}' \times_{\mathcal{E}} \mathcal{E}'$, so since f', $f' \times f'$, $f' \times f' \times f'$ are all connected, as just pointed out, γ must come from a map $\delta: p^*(X) \to p^*(Y)$ compatible with descent data in $\mathcal{S}'$, $\gamma = f'^*(\delta)$. $\mathcal{S}' \xrightarrow{p} \mathcal{S}$ is an effective descent morphism, so $\gamma = p^*(\alpha)$ for some $\alpha: X \to Y$ in $\mathcal{S}$. Then also $\beta = f^*(\alpha)$. This proves that f^* is full.

It remains to show that if f' is locally connected, so is f. Since f and f' are open surjections, it suffices to prove (cf. the Appendix) that for a generating collection of objects $X \in \mathcal{E}$, the locale $B = f_*(X_{dis})$ (corresponding to the frame $f_*(P(X))$ of subobjects of X) is locally connected. Thus, assuming $\mathcal{S}' = \mathcal{S}[A] \xrightarrow{p} \mathcal{S}$ is localic over $\mathcal{S}$, it suffices to show for open surjective locales A and B in $\mathcal{S}$ that if $A \times B \xrightarrow{\pi_1} A$ is locally connected, i.e. $Sh(A) \models$ "$p^{\#}(B)$ is a locally connected locale", then B is locally connected in S. We use proposition 3 of the Appendix (or rather, the first lines of its proof). Let $S, T \in S$, and let $\alpha: f^*(S) \to f^*(T)$ be a map in $Sh(B)$. Consider $\beta = \pi_2^*(\alpha) : \pi_2^* f^*(S) \to \pi_2^* f^*(T)$, i.e. $\beta : \pi_1^* p^*(S) \to \pi_1^* p^*(T)$ in $Sh(A \times B) = Sh_{Sh(A)}(p^{\#}(B))$. Since π_1 is locally connected, it holds in $Sh(A)$ that β locally (in the sense of $p^{\#}(B)$) comes from a map $\gamma : p^*(S) \to p^*(T)$. As seen from S, this means that there is a cover $\{U_i \times V_i\}_i$ of $A \times B$, $U_i \in \mathcal{O}(A)$ and $V_i \in \mathcal{O}(B)$, and maps $\gamma_i : p_i^*(S) \to p_i^*(T)$ over U_i such that $\pi_1^*(\gamma_i) = \beta = \pi_2^*(\alpha)$ in $Sh(U_i \times V_i)$:

$$\begin{array}{ccc} Sh(U_i \times V_i) & \xrightarrow{\pi_2} & Sh(V_i) \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle f_i} \\ Sh(U_i) & \xrightarrow{p_i} & S \end{array} \quad .$$

Translating this back into S, we have continuous maps (of locales) $\alpha: V_i \to S^T$ and $\gamma_i : U_i \to S^T$ (where S^T denotes the product of T copies of the discrete locale S, cf. the Appendix), such that $U_i \times V_i \xrightarrow{\pi_2} V_i \xrightarrow{\alpha} S^T = U_i \times V_i \xrightarrow{\pi_1} U_i \xrightarrow{\gamma_i} S^T$. The following lemma then shows that α must locally (namely, for the cover $\{V_i\}$ of B) come from a map S^T in S.

<u>Lemma</u>. Let

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{\pi_1} & X \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{q} & Z \end{array}$$

be a pullback of locales in S, with p and q open surjections. Then this square is also a pushout of locales.

Proof of lemma. This follows once more from the Beck condition mentioned above: we have a pushout of frame maps $(.)^*$

$$\begin{array}{ccc} \mathcal{O}(X) \otimes_{\mathcal{O}(Z)} \mathcal{O}(Y) & \underset{\exists_{\pi_1}}{\overset{\pi_1^*}{\rightleftarrows}} & \mathcal{O}(X) \\ \pi_2^* \Big\uparrow \Big\downarrow \exists_{\pi_2} & & \exists_p \Big\downarrow \Big\uparrow p^* \\ \mathcal{O}(Y) & \underset{q^*}{\overset{\exists_q}{\rightleftarrows}} & \mathcal{O}(Z) \end{array}$$

with left adjoints $\exists_{(-)}$, and $q^* \exists_p = \exists_{\pi_2} \pi_1^*$, $p^* \exists_q = \exists_{\pi_1} \pi_2^*$. Now suppose we have continuous maps $h: X \to P$ and $k: Y \to P$ into some locale P such that $h\pi_1 = k\pi_2$. It suffices to show that $h^* = p^* \exists_p h^*$ and $k^* = q^* \exists_p h^*$. For the first equality, we have $\pi_1^* p^* \exists_p h^* = \pi_2^* q^* \exists_p h^* = \pi_2^* \exists_{\pi_2} \pi_1^* h^* = \pi_2^* \exists_{\pi_2} \pi_2^* k^* = \pi_2^* k^* = \pi_1^* h^*$. π_1^* is $1-1$, so $h^* = p^* \exists_p h^*$. The other equation is verified similarly.

This completes the proof of the lemma.

Applying this argument to each open of B, we find that B is a locally connected locale. Thus we have proved theorem 5.2. ∎

5.3. Lemma. Let I be a directed poset in S. Then there exists a localic open surjection $S[A] \xrightarrow{p} S$ and a $\varphi: \mathbb{N} \to p^*(I)$ in $S[A]$ such that φ is cofinal in I, i.e.

$S[A] \models \forall i \in p^*(I) \ \exists n \in \mathbb{N} \ \ \varphi(n) > i$.

Proof. We adjoin a generic cofinal sequence to the universe in the standard way: Let $\mathbb{P}$ be the poset of finite sequences $s = (i_1,\ldots,i_n)$ from I with $i_1 < \ldots < i_n$, partially ordered by $s < t$ iff s extends t, and let the locale A be defined by the following covering system on $\mathbb{P}$: for each $(i_1,\ldots,i_n) \in \mathbb{P}$ and each $j \in I$, the family

$$\{(i_1,\ldots,i_n,\ldots,i_m) \mid j < i_m\}$$

of extensions of $(i_1,\ldots,i_n)$ covers $(i_1,\ldots,i_n)$. Since I is directed, this is a stable family of covers (i.e. if S covers s and $t < s$ then $\{r \in S \mid r < t\}$ covers t), and moreover each cover is inhabited. Thus if A is the locale defined by this poset with covering system on $\mathbb{P}$, $\mathcal{S}[A] \to \mathcal{S}$ is an open surjection. ∎

<u>Proof of theorem 5.1</u>. Let $(\mathcal{E}_i)_i$ be the given system in $\mathcal{S}$, and let $(\mathcal{F}_i)_i$ denote the corresponding system obtained by change of base along the map $\mathcal{S}[A] \to \mathcal{S}$ of 5.3. Let $\mathcal{F}^\infty$ and $\mathcal{E}^\infty$ denote the inverse limits. So we have a pullback

$$\begin{array}{ccc} \mathcal{F}^\infty & \longrightarrow & \mathcal{E}^\infty \\ \downarrow & & \downarrow \\ \mathcal{S}[A] & \longrightarrow & \mathcal{S} \end{array}$$

$\mathcal{F}^\infty = \varprojlim_i \mathcal{F}_i = \varprojlim \mathcal{F}_{\varphi(n)}$, where φ is the cofinal sequence of lemma 5.3. Since the geometric morphisms of types (ii)-(v) are all preserved by pullback, the result now follows immediately from theorems 5.2 and 4.1 in these cases. The case (i) of surjections follows from corollary 5.4 below and proposition IV.4.2 of Joyal & Tierney(1982), just as for inverse sequences (see section 4). ∎

5.4. Corollary. The localic reflection preserves limits of filtered inverse systems of toposes.

Proof. The hyperconnected - localic factorisation is preserved by pullback, so this follows from corollary 3.2, using a change of base by lemma 5.3 as in the proof of 5.1. ∎

5.5. Remark. In Joyal & Tierney(1982) it was noted that the localic reflection preserves binary products. By corollary 5.4 this can be extended to arbitrary (small) products. It is not true, however, that the localic reflection preserves all inverse limits. Here is a simple example of a case where it does not preserve pullbacks: Let $\mathbb{C}$ be the category with two objects C,D and only two non-identity arrows f and g: C → D

$$C \cdot \underset{g}{\overset{f}{\rightrightarrows}} \cdot D$$

Let $\mathbb{D}_1$ be the category (poset) in $S^{\mathbb{C}^{op}}$ with $\mathbb{D}_1(D) = \{0<1\}$ $\mathbb{D}_1(C) = \{0<1\}$, $\mathbb{D}_1(f)(1) = \mathbb{D}_1(f)(0) = 1$, $\mathbb{D}_1(g) = id$. Let $\mathbb{D}_2$ be exactly the same, but with f and g interchanged. Construct the product-poset $\mathbb{D}_1 \times \mathbb{D}_2$ in $S^{\mathbb{C}^{op}}$. So we have a pullback

$$\begin{array}{ccc} S^{(\mathbb{C} \ltimes (\mathbb{D}_1 \times \mathbb{D}_2))^{op}} & \longrightarrow & S^{(\mathbb{C} \ltimes \mathbb{D}_1)^{op}} \\ \downarrow & & \downarrow \\ S^{(\mathbb{C} \ltimes \mathbb{D}_2)^{op}} & \longrightarrow & S^{\mathbb{C}^{op}} \end{array}$$

The poset-reflection of the diagram

$$\begin{array}{ccc} \mathbb{C} \ltimes (\mathbb{D}_1 \times \mathbb{D}_2) & \longrightarrow & \mathbb{C} \ltimes \mathbb{D}_1 \\ \downarrow & & \downarrow \\ \mathbb{C} \ltimes \mathbb{D}_1 & \longrightarrow & \mathbb{C} \end{array}$$

however, do not give a pullback of posets, as is easily verified, and therefore it does not give a pullback of the corresponding locales of downwards closed sets. ∎

Appendix: Locally connected geometric morphisms.

Let S be an arbitrary topos, which we fix as our base topos from now on. In M. Barr & R. Paré(1980) several characterizations are given of toposes which are locally connected (they say: molecular) over S. The purpose of this Appendix is to give an alternative proof of these characterizations for the special case of Grothendieck toposes over S. So in the sequel, geometric morphism means bounded geometric morphism. Our proof uses locale theory, and familiarity with the paper of Barr and Paré is not presupposed. Also, this Appendix can be read independently from the rest of this paper.

For the case of Grothendieck toposes over S, the result of Barr and Paré can be stated as follows.

Theorem. The following conditions on a geometric morphism $\gamma: E \to S$ are equivalent:

(1) There exists a locally connected site $\mathbb{C} \in S$ such that $E = S[\mathbb{C}]$, the topos of sheaves on $\mathbb{C}$ ("locally connected site" is defined below).

(2) There exists a site $\mathbb{C} \in S$ such that $E = S[\mathbb{C}]$, with the property that all constant presheaves on $\mathbb{C}$ are sheaves.

(3) The functor $\gamma^*: S \to E$ (left adjoint to the global sections functor $\gamma_*: E \to S$) has an S-indexed left adjoint.

(4) γ^* commutes with Π-functors, i.e. for each $S \xrightarrow{\alpha} T$ in S we have a commutative diagram

$$\begin{array}{ccc} S/S & \xrightarrow{\gamma^*/S} & E/\gamma^*(S) \\ \downarrow \Pi_\alpha & & \downarrow \Pi_{\gamma^*(\alpha)} \\ S/T & \xrightarrow{\gamma^*/T} & E/\gamma^*(T) \end{array}$$

A geometric morphism satisfying these equivalent conditions is called locally connected. Recall that a site $\mathbb{C}$ is called locally connected (or molecular) if every covering sieve of an object $C \in \mathbb{C}$ is connected (as a full subcategory of $\mathbb{C}/S$) and inhabited.

The implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are easy. The difficult part is (4) ⇒ (1), and it is here that our approach differs from the one taken by Barr and Paré. To prove (4) ⇒ (1), choose first a site $\mathbb{C}$ for E in S, closed under finite limits and subobjects, say. It suffices to show that for each $C \in \mathbb{C}$, the locale (in S) $Sub_E(C)$ of subobjects of C is locally connected (see definition 1 below). If $S \in S$, a section of $\gamma^*(S)$ over C is nothing but a continuous map from the locale $Sub_E(C)$ into the discrete locale S, i.e. a global section of the constant object S in the localic topos $Sh(Sub_S(C))$, so it follows that for each $C \in \mathbb{C}$, the inverse image of the canonical geometric morphism $Sh(Sub_S(c) \to S$ preserves Π-functors. So we only need to show (4) ⇒ (1) of the theorem for locales, i.e. to show proposition 3 below. But let us recall some definitions first.

Definition 1. Let A be a locale in S.

(i) an element $a \in A$ is called positive (written pos(a)) if every cover of a is inhabited. A is positive if $pos(1_A)$.

(ii) $a \in A$ is called connected if a is positive, and every cover of a by positive elements is positively connected. (A cover $U = \{b_i\}_i$ of a is positively connected if for every $b_i, b_j \in U$ there is a chain $b_i = b_{i_0}, \ldots b_{i_n} = b_j$ in U such that $\forall k < n \ pos(b_{i_k} \wedge b_{i_{k+1}})$.) A is connected if 1_A is connected.

(iii) A is <u>open</u> if for every $a \in A$, $a = \bigvee\{b < a \mid pos(b)\}$.

(iv) A is <u>locally connected</u> if for every $a \in A$, $a = \bigvee\{b < a \mid b$ is connected$\}$

Lemma 2. Let A be a positive open locale in S. The following conditions are equivalent:

(1) A is connected.

(2) For every $S \in S$, every continuous map $A \xrightarrow{f} S$ into the discrete locale S is constant, i.e. $\exists s \in S\ f^{-1}(s) = 1_A$.

(3) Every continuous map $A \to \Omega$ into the discrete locale $\Omega = P(1)$ is constant.

Proof. (1)⇒(2): Let $A \xrightarrow{f} S$, and put $a_s = f^{-1}(\{s\})$. Then $\bigvee_{s \in S} a_s = 1$, so also $\bigvee\{a_s \mid pos(a_s)\} = 1$ since A is open. This latter cover must be positively connected. But if $pos(a_s \wedge a_{s'})$, then since $a_s \wedge a_{s'}$ is covered by $f^{-1}(s) \wedge f^{-1}(s') = f^{-1}(\{s\} \cap \{s'\}) = \{f^{-1}(s) \mid s = s'\}$, it follows that $s = s'$. So $\exists s \in S\quad a_s = 1$.

(2)⇒(3) is clear.

(3)⇒(1): Suppose $U = \{b_i\}_{i \in I}$ is a cover of 1_A, with $pos(b_i)$ for all i. Fix $i_o \in I$ and define $p_j \in \Omega$ for each $j \in I$ to be the value of the sentence

$$\exists \text{ chain } b_{i_o}, b_{i_1}, \ldots, b_{i_n} = b_j \text{ in } U \text{ such that } \forall k < n \quad pos(b_{i_k} \wedge b_{i_{k+1}}).$$

Now let $f^{-1} : P(\Omega) \to O(A)$ be defined by

$$f^{-1}(V) = \bigvee\{b_j \in U \mid p_j \in V\}.$$

We claim that f^{-1} is an $\wedge\bigvee$-map, i.e. defines a continuous map

$A \to \Omega$. Indeed, f^{-1} preserves $\bigvee$ by definition, and $f^{-1}(\Omega) = \bigvee \mathcal{U} = 1_A$. To show that f^{-1} preserves $\wedge$, it suffices by openness of A that $\mathrm{pos}(b_j \wedge b_{j'})$ implies $p_j = p_{j'}$, which is obvious from the definition of p_j. By (3), f is constant. But $b_{i_0} < f^{-1}(\{\top\})$, so $1_A < f^{-1}(\{\top\})$, i.e. $\mathcal{U}$ connected.

Note that from this proof we can extract that for a positive open locale A, a continuous function $A \xrightarrow{f} S$ (S discrete) corresponds to giving a sequence $\{a_s \mid s \in S\}$ of elements of A such that $\bigvee_{s \in S} a_s = 1$ and $\mathrm{pos}(a_s \wedge a_{s'}) \Rightarrow s = s'$. We will call such a cover $\{a_s \mid s \in S\}$ of A discrete.

Proposition 3. Let A be a locale in $\mathcal{S}$, and suppose that the inverse image functor γ^* of the geometric morphism $\mathrm{Sh}(A) \to \mathcal{S}$ preserves Π-functors (hence exponentials). Then A is locally connected.

Proof. As is well-known and easy to prove, the assumption implies that A is open (e.g. γ^* preserves universal quantification, and use Joyal & Tierney(1982), §VII.1.2). We will show that if A is open and γ^* preserves exponentials, then A is locally connected.

Preservation of exponentials means that for any $S, T \in \mathcal{S}$ and any $a \in A$ (viewed as a sublocale of A), an $f: a \to S^T$ is continuous for the product locale S^T iff it is continuous for the discrete locale S^T.

Let for $a \in A$, F_a be the set of continuous maps $a \to \Omega$. Then we have a canonical map

$$\varphi_a : a \to \Omega^{F_a}$$

which by assumption is continuous as a map into the discrete locale Ω^{F_a}. Thus, writing $a_\alpha = \varphi_a^{-1}(\alpha)$, $C(a) = \{a_\alpha \mid \alpha \in \Omega^{F_a}\}$ is a discrete cover of a, and since A is open, so is $C^+(a) = \{c \in C(a) \mid \mathrm{pos}(c)\}$. We claim that the elements of $C^+(a)$ are the "connected components" of a.

First, suppose we are given a continuous $a \xrightarrow{g} S$ into a discrete locale S. Then for each $a_\alpha \in C^+(a)$ we have

$$\text{(i)} \quad \exists! s \in S \quad a_\alpha < g^{-1}(s).$$

Indeed, the set $U = \{s \in S \mid \mathrm{pos}(a_\alpha \wedge g^{-1}(s))\}$ is inhabited since a_α is positive and covered by $\{g^{-1}(s) \mid s \in U\}$. Suppose $s \in U$, and let $a \xrightarrow{f} \Omega$ be the composite

$$a \xrightarrow{g} S \xrightarrow{[\![\cdot = s]\!]} \Omega.$$

Then $g^{-1}(s) = f^{-1}(\top) = \bigvee\{a_\beta \in C^+(a) \mid \beta_f = \top\}$. Since $\mathrm{pos}(a_\alpha \cap g^{-1}(s))$, there is a $\beta \in \Omega^{F_a}$ with $\beta_f = \top$ such that $\mathrm{pos}(a_\alpha \wedge a_\beta)$, so $\alpha = \beta$, so $\beta_f = \top$, i.e. $a_\alpha < g^{-1}(s)$. Since $s \in U$ was arbitrary, this shows that the inhabited set U can have <u>at most</u> one element, proving (i).

In other words, there is a natural $1-1$ correspondence

$$\text{(ii)} \quad \frac{a \longrightarrow S \quad \text{(of locales in } \mathcal{S}\text{)}}{C^+(a) \to S \quad \text{(of "sets" in } \mathcal{S}\text{)}}$$

between continuous maps into discrete locales S and maps $C^+(a) \to S$ in $\mathcal{S}$: given $a \xrightarrow{g} S$, define $\varphi\colon C^+(a) \to S$ by $\varphi(c) =$ the unique s with $c < g^{-1}(s)$ (see (i)); and given $\varphi\colon C^+(a) \to S$ define f by $f^{-1}(s) = \bigvee \varphi^{-1}(s)$.

Now we show that each $b \in C^+(a)$ is connected. By lemma 2, it suffices to show that each continuous map $b \to \Omega$ is constant. This would follow from (i) if we show that a continuous $b \to \Omega$ can always be extended to a continuous $a \to \Omega$. So take $b_o \in C^+(a)$ and $b_o \xrightarrow{g} \Omega$, corresponding to $\varphi: C^+(b_o) \to \Omega$ by (ii). Write $a = \bigvee C^+(a) = \bigvee\{c \mid \exists b \in C^+(a) : c \in C^+(b)\}$. The set $\mathcal{U} = \{c \mid \exists b \in C^+(a) c \in C^+(b)\}$ is a discrete cover (i.e. $\forall c,c' \in \mathcal{U}$ $\mathrm{pos}(c \wedge c') \to c = c'$) of a by positive elements, and $C^+(b_o) \subset \mathcal{U}$. Ω is injective, so φ can be extended to a function $\psi: \mathcal{U} \to \Omega$, which by discreteness of $\mathcal{U}$ gives a continuous $f: a \to \Omega$, defined by $f^{-1}(p) = \bigvee \psi^{-1}(p)$.
f is the required extension of g.

This completes the proof of proposition 3, and hence of the implication $(4) \Rightarrow (1)$ of the theorem.

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SAMENVATTING

Dit proefschrift bevat een aantal artikelen waarin het verband tussen intuitionistische verzamelingenleer en topos theorie onderzocht wordt. Kortweg komt dit verband erop neer dat men een gegeven Grothendieck topos hetzij als meetkundig objekt (een gegeneraliseerd schema of topologische ruimte, zoals gebruikelijk in de Grothendieck-school), hetzij als een universum, oftewel een model, voor intuitionistische verzamelingenleer op kan vatten. De toepassingen hiervan werken natuurlijk in twee richtingen: enerzijds kan men de meetkundige konstrukties van Grothendieck topossen gebruiken om modellen te konstrueren voor in de mathematische logika onderzochte intuitionistische theorieën, maar anderzijds kan men de intuitionistische verzamelingenleer gebruiken bij meetkundige konstrukties in de theorie van Grothendieck topossen.

De eerste vijf artikelen in dit proefschrift zijn toepassingen van het eerste type. In de artikelen met G.F. van der Hoeven worden modellen gekonstrueerd voor de theorie van keuzerijen, en wordt het verband aangegeven met de klassieke literatuur over dit onderwerp waarin géén gebruik wordt gemaakt van topos theorie, maar van meer traditionele methoden uit de mathematische logika. In het vijfde artikel wordt onder meer een topos gekonstrueerd waarin het eenheidsinterval kompakt is, maar de Cantor ruimte niet!

De overige vier artikelen zijn veeleer toepassingen van het tweede type: hier wordt voortdurend gebruik gemaakt van het feit dat een topos opgevat kan worden als een intuitionistisch universum. In het eerste artikel worden enkele eigenschappen van gefilterde inverse limieten van Grothendieck topossen onderzocht. Het belangrijkste resultaat is dat als $(\mathcal{E}_i)_{i\in I}$ een gefilterd invers systeem is, met geometrische morfismen $f_{ij}: \mathcal{E}_i \to \mathcal{E}_j$ $(i \leqslant j)$, en alle f_{ij} zijn surjekties (resp. open surjekties, samenhangende lokaal samenhangende morfismen, hyper-samenhangende morfismen, samenhangende atomaire morfismen) dan zijn alle projektie-morfismen $\varprojlim \mathcal{E}_i \xrightarrow{p_j} \mathcal{E}_j$ dat ook. In het tweede artikel wordt een zeer kort nieuw bewijs gegeven voor de zogenaamde Descent stelling voor Grothendieck topossen. De laatste twee artikelen vormen een aanzet tot een homotopie theorie van topossen in "topologische stijl". In het artikel met Wraith wordt bewezen dat als $\mathcal{E}$ een samenhangende lokaal samenhangende topos is, het kanonieke evaluatiemorfisme $\mathcal{E}^I \to \mathcal{E} \times \mathcal{E}$ een open surjektie is. Hierbij is $\mathcal{E}^I$ de "weg-topos" van $\mathcal{E}$. Met andere woorden, elke samenhangende-lokaal samenhangende topos is ook weg-samenhangend. Dit resultaat laat teven zien dat zelfs gewone topologische ruimten eigenlijk betere eigenschappen hebben wanneer men ze als topos beschouwt. In het laatste artikel wordt de studie van de weg-topos voortgezet, en onder meer een verscherping van het zo juist genoemde resultaat met Wraith bewezen.

S T E L L I N G E N

behorende bij het proefschrift

Topics in Intuitionism and Topos Theory

van I. Moerdijk

1. Er zijn goede redenen om <u>behalve</u> niet - zoals in de standaard literatuur beweerd wordt - als voegwoord te klassificeren, maar als voorzetsel. (Zie F. Landman & I. Moerdijk: <u>Behalve</u> als voorzetsel, Spectator <u>9</u> (4), 1980.)

2. De op zichzelf voor de hand liggende opmerking dat de kracht van het kompositionaliteits-principe afhangt van andere restrikties op de vorm van de regels van de grammatika, heeft zeer belangrijke gevolgen voor de inrichting van grammatika's van een Montague-achtig type. (Zie F. Landman & I. Moerdijk: Compositionality and the analysis of anaphora, Linguistics & Philosophy <u>6</u>, 1983; en ook: Compositional semantics and morphological features, Theoretical Linguistics <u>10</u>, 1983.)

3. Let $\mathbb{S} \subset \mathbb{C}$ be a sieve in a category $\mathbb{C}$, and let $\mathbb{C} \xrightarrow{F} \mathbb{D}$ be a functor. Then the canonical morphism

$$N(\mathbb{S}) \coprod_{N(\mathbb{C})} N(\mathbb{D}) \longrightarrow N(\mathbb{S} \coprod_{\mathbb{C}} \mathbb{D})$$

is an anodyne extension (N denotes the nerve functor). From this fact it follows easily that N induces an equivalence between the homotopy category of categories and the usual homotopy category of simplicial sets by inverting the weak equivalences. (This latter consequence is originally due to Quillen; see L. Illusie, Complex cotangent et déformation II, Springer LNM 283, 1972.)

4. A topological space X is called <u>universal</u> for intuitionistic predicate logic (IPC) if whenever a formula ϕ is not derivable in IPC from a set of formulas Γ, there exists a sheaf model over X in which all formulas from Γ hold, but ϕ does not. It can be shown that every metrizable space without isolated points is universal for IPC. (See I. Moerdijk: Some topological spaces which are universal for intuitionistic predicate logic, Indag.Math. <u>44</u>, 1982.)

5. Every C^∞-ring which is a domain is a local ring; and every local C^∞-ring is Henselian and has a real closed residue field. (See I. Moerdijk & G. Reyes: Rings of smooth functions and their localizations I, to appear in J. of Algebra.)

6. Let $C^\infty(M)$ be the ring of smooth functions on a given manifold M, and let P be a prime ideal in $C^\infty(M)$. Then the localization $C^\infty(M)_P$ in the category of C^∞-rings, i.e. the ring $\varinjlim_{f \notin P} C^\infty(M-Z(f))$, is a local ring iff P is a z-ideal. (See I. Moerdijk, Ngo van Quê, G. Reyes: Rings of smooth functions and their localizations II, to appear.)

7. Although the square root of a smooth function need not be smooth, the function $\mathbb{R} \xrightarrow{t^2} \mathbb{R}_{\geq 0}$ is nonetheless a stable effective epimorphism in the category of C^∞-schemes (or duals of finitely generated C^∞-rings), as follows from the following result: let M be a smooth manifold, and let $f,g \in C^\infty(M)$. Then $f \in (g(x) - t^2) \subset C^\infty(M\times\mathbb{R})$ iff there is a $p \in C^\infty(\mathbb{R})$ which vanishes on $\mathbb{R}_{\geq 0}$, such that $f(x) \in (p(g(x))) \subset C^\infty(M)$. (See I. Moerdijk, Ngo Van Quê, G. Reyes: Forcing smooth square roots and integration, to appear.)

8. There is a smooth analogon of the usual Zariski topos, which is a very adequate model for infinitesimal analysis as used by Lie, E. Cartan, Darboux, etc. as well as for a constructive version of non-standard analysis. (See I. Moerdijk & G. Reyes: A smooth version of the Zariski topos, to appear in Advances in Mathematics.)

9. There is an algebraic proof of the theorem of Ambrose, Palais, and Singer on the 1-1 correspondence between affine connections and sprays on a smooth manifold. (See W. Ambrose, R.S. Palais, I.M. Singer: Sprays, Anais da Acad.Bras. de Ciências 32, 1960.)
This new proof actually gives a much more general result: the 1-1 correspondence not only holds for (finite-dimensional) smooth manifolds, but also for manifolds with singularities, for spaces of smooth functions, and for algebraic schemes. (See I. Moerdijk & G. Reyes: On the relation between connections and sprays, to appear.)

10. De laatste stelling bij het proefschrift van A.J.M. van Engelen is onjuist.