COMPLETENESS AND DEFINABILITY

applications of the Ehrenfeucht game in second-order and intensional logic

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ACADEMISCH PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE WISKUNDE EN NATUURWETENSCHAPPEN AAN DE UNIVERSITEIT VAN AMSTERDAM OP GEZAG VAN DE RECTOR MAGNIFICUS, DR S.K. THODEN VAN VELZEN, HOOGLERAAR IN DE FACULTEIT DER GENEESKUNDE, IN HET OPENBAAR TE VERDEDIGEN IN DE AULA DER UNIVERSITEIT (OUDE LUTHERSE KERK, SINGEL 411, HOEK SPUI) OP

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HANS CORNELIS DOETS

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dedicated to Liesbeth

Contents

Preface

This dissertation contains results on classical first- and second-order logic (parts I and II) and their intensional colleagues: modal- tense- and intuitionistic (propositional) logic (part III).

One underlying theme is Ehrenfeucht's game and some of its variants.

Chapter 1 is an introduction to Ehrenfeucht game theory and its relation with (quantifierrank-) α-equivalence in (infinitary) logic. Section 1.0 intends to whet the appetite for the finite Ehrenfeucht game.

In chapter 2 the game is played on binary trees. A characterisation is obtained of all trees n-equivalent with the binary tree B_m all of whose branches have length m. In particular, it follows that B_m has *infinite* n-equivalents when $m \geqslant 2^n - 1$. This has been applied by Rodenburg [1986] to solve a problem in intuitionistic correspondence theory; the story is told in chapter 8.

Part II shows how to axiomatize certain monadic $\Pi^1{}_1$-theories, most of them dealing with well-foundedness. Chapter 3 is on linear orderings. One of the nicer results is in 3.3 where the effects of the Suslin property for the monadic $\Pi^1{}_1$-theory of $\mathbb{R}$ are isolated. Chapter 4 generalizes the method of 3 to the case of trees.

In part III, chapter 5 discusses Löwenheim-Skolem type problems in modal correspondence theory. It is shown that most examples of non-first-order definable modal formulas already cannot be first-order defined on *finite* frames. On the other hand, an example is given of a non-first-order definable formula which *is* first-order definable on all countable frames.

Chapter 6 modifies the Ehrenfeucht game for use in intensional logic; exact Kripke models are constructed universal with respect to finite partially ordered Kripke models.

Chapter 7 presents our version of $\mathbb{Z}$-completeness.

In chapter 9, games and the universal-exact Kripke model appropriate for one-variable intuitionistic formulas are applied to solve some problems in intuitionistic correspondence theory left open by Rodenburg [1982].

Appendix A constructs asymmetric linear orderings with lots of homogeneity-properties in each uncountable cardinal.

Appendix B reduces all of higher-order logic to monadic second-order logic - indicating the expressive possibilities of modal logic in the Kripke semantics.

To help the reader find his way, here is an indication what can be omitted without loss of understanding of the rest. In chapter 1, sections 4, 8 and 9 are not needed for the other chapters. Also, not much will be lost if, in the discussion of the α-game, the reader always assumes α to be *finite*. Section 2.4 can be read independently from the rest of chapter 2. Section 3.2 may be omitted. In part III, all chapters can be read individually (except for a couple of references where this is indicated.)

I am obliged to several people for different reasons; in particular I wish to thank here prof. Specker for a lecture featuring Ehrenfeucht games; Piet Rodenburg for the communication of his problems to which chapters 2/8/9 are devoted and the elimination of numerous mistakes in a previous version of this text; Anne Troelstra and Dick de Jongh for scientific support and the software used to produce this text on the Macintosh Plus.

However, above all, my gratitude concerns my thesis-advisor Johan van Benthem whose determination and persuasiveness eventually turned out to be irresistible.

Part I: DEFINABILITY

It were not best that we should all think alike; it is difference of opinion that makes horse-races. - *Pudd'nhead Wilson's Calendar*

1. Fraïssé-Ehrenfeucht theory for $L_{\infty\omega}$ and some of its fragments.

1.0 Introduction.

This chapter introduces five guises of α-equivalence between models, where α is an arbitrary ordinal.

For $\alpha=\omega$, this relation (called *elementary equivalence* and denoted by $\equiv$) is a basic one in model theory. For models of the same finite similarity-type, $A\equiv B$ just means that A and B have the same true (first-order) sentences. However, there are some uses for refinements, as is argued below.

α-Equivalence for *finite* α is explained game-theoretically as follows. Suppose A and B are models (of the same similarity type) and $n\in\mathbb{N}$. The *n-game on A and B, G(A, B, n),* has two players, I and II. They move alternately. I is allowed the first move; each player is allowed n moves. A **move** consists of an element in either A or B. However, if player I chooses an element in $\mathbf{A}$ (resp. $\mathbf{B}$) then player *II* has to counter in $\mathbf{B}$ (resp. $\mathbf{A}$). Therefore, a move of player *I* and the following counter-move of player *II* form an ordered pair in $A \times B$ (where A and B are the *universes* of $\mathbf{A}$ and $\mathbf{B}$ respectively).

When the game is over, the set of ordered pairs of moves is an at most n-element relation $h \subset A \times B$. *II* has **won** the play by definition if h is a *partial isomorphism* between $\mathbf{A}$ and $\mathbf{B}$, that is, if h is an injection on its domain which preserves the structure of the models. Of course, the larger n, the better *I*'s chances to defeat *II*.

Here is an example: $\mathbf{A}$ consists of the neighbour-relation R on the set of integers $\mathbb{Z}$ defined by nRm iff $|n-m|=1$; $\mathbf{B}$ is the quotient of this structure modulo (9) - this can be visualized as a circle with 9 points. Now, player *II* can win each game of length 3 between these models; however, *I* has a way to win the 4-game: his first three moves are chosen at equal distances in $\mathbf{B}$; his fourth is in $\mathbb{Z}$ not meeting any of *II*'s three moves there. Now whatever the answer of *II* in $B=\mathbb{Z}/(9)$, he cannot avoid one of the moves of *I*.

To explain the "meaning" of the game somewhat, let me remark that player *I*, in order to win, tries to point to *differences* between the models involved; while player *II* on the other hand tries to argue that the models are somehow *similar*.

Finally, $\mathbf{A}$ and $\mathbf{B}$ are called n-**equivalent** if *II* has a **winning strategy** for G($\mathbf{A}$, $\mathbf{B}$, n), that is, a method by which he can beat *I* no matter the choice of moves by *I*. So, in the example above, ($\mathbb{Z}$,R) and ($\mathbb{Z}$,R)/(9) are 3-equivalent but not 4-equivalent.

An easy case where *II has* such a winning strategy no matter the length of the game is when $\mathbf{A}$ and $\mathbf{B}$ are *isomorphic*: *II* then simply applies the isomorphism (or its inverse) to the moves of *I*.

n-Equivalence of A and B is denoted by $\mathbf{A} \equiv^n \mathbf{B}$.

The following well-known example will be used later on.

1.0.1 Proposition. *Finite linear orderings* $\mathbf{A}$ *and* $\mathbf{B}$ *are n-equivalent iff* $|A|=|B|$ *or* $|A|, |B| \geqslant 2^n-1$.

To prove 1.0.1, we employ a simple

1.0.2. Lemma. *Suppose that* $\mathbf{A}=(A, <)$ *and* $\mathbf{B}=(B, <)$ *are linear orderings. Then* $\mathbf{A} \equiv^{n+1} \mathbf{B}$ *iff ("back") for all* $b \in B$ *there exists* $a \in A$ *such that* $a\downarrow \equiv^n b\downarrow$ *and* $a\uparrow \equiv^n b\uparrow$ *and ("forth") for all* $a \in A$ *there exists* $b \in B$ *such that* $a\downarrow \equiv^n b\downarrow$ *and* $a\uparrow \equiv^n b\uparrow$.

(Here, $a\downarrow = \{a' \in A | a' < a\}$ and $a\uparrow = \{a' \in A | a < a'\}$ are identified with the ordered models on these sets.)

Notice that, for the lemma to make sense in general, *empty models* must be allowed explicitly.

Proof of 1.0.1. Induction on n. For $n=0$ there is nothing to prove. Assume the result for n and suppose that $|A|, |B| \geqslant 2^{n+1}-1$. Use 1.0.2. Suppose $a \in A$. Now, distinguish three cases. (i) $|a\downarrow| < 2^n-1$. Pick $b \in B$ such that $|b\downarrow| = |a\downarrow|$. Then $a\downarrow \equiv^n b\downarrow$. Furthermore, we must have $|a\uparrow|, |b\uparrow| \geqslant 2^n-1$; hence $a\uparrow \equiv^n b\uparrow$ by the induction hypothesis. (ii) $|a\uparrow| < 2^n-1$. This is similar to (i). (iii) $|a\downarrow|$, $|a\uparrow| \geqslant 2^n-1$. Pick $b \in B$ sucht that $|b\downarrow|, |b\uparrow| \geqslant 2^n-1$ (such a b necessarily exists) and apply the inductive hypothesis. The other parts of the result are left for the reader. ⊠

In the sequel, identify an ordinal with the well-ordered set of its predecessors. ω^* is the order-type of $(\omega,>)$ and $\zeta=\omega^*+\omega$ is the order type of the integers.

1.0.3 Example. (i) *If* $m \geqslant 2^n-1$, *then:* $m \equiv^n \omega+\omega^*$.
(ii) *For all* n: $\omega \equiv^n \omega+\zeta$.

Proof. Induction on n, using 1.0.2. For (i), use 1.0.1. (ii) follows from (i). ⊠

Below we shall encounter several situations in which either $\equiv$ is too fine or ordinary model theory is of no avail. In such cases, game theory provides a way out.

As to the first type, in part II we shall be confronted with models which have n-equivalents with a certain property for all n, while *elementary* equivalents with that property do not exist. As to the second, much of part III is concerned with model theory for *finite* models. In such a *non-elementary* domain, compactness and Löwenheim-Skolem theorems cannot be much of a help.

Chapter 1 is organized as follows. Section 1.1 fixes terminology. 1.2 defines α-equivalence for general α in the simplest of ways. (cf. Poizat's admirable [1985] logic-introduction for a treatment based on such an approach.) 1.3 is about games. For $\alpha>\omega$, this is due to Barwise [1973]. 1.4 is Karp's [1965]-characterisation generalizing Fraïssé's [1955] (one of the oldest references in the subject, save for Fraïssé's thesis). 1.5 gives the connection with (infinitary) logic and 1.6 finally defines Scott-sentences coding up the game-theoretic information on a sequence of elements. Remarkably, these were introduced at first for infinitary logic only, cf.

Chang [1968]. 1.7 specializes the discussion to (finite) first-order logic. In 1.8, the ordinal parameter α approaches absolute infinity. In game theory, this corresponds with not putting an upper bound on the length of the game, which seems to be the obvious way to "infinitize" games; however, 1.3 showed this to be a very crude step indeed. 1.9 finally collects some information on when $\equiv^\alpha$ equals $\equiv^\infty$.

1.1 Notation and terminology.

A **model** is a complex $\mathbf{A}=(A,\ldots)$ consisting of a set A (which, contrary to usual logical convention, often is allowed to be empty) together with any number of ("finitary") relations. Thus, functions (and, often, constants as well) are excluded from models.

A **language** (or **similarity-type**) is a set of relation-symbols, together with a specification of the number of arguments (the **arity**) for each symbol in the set.

If L is a language, an L-**model** is a model together with a surjection of L onto its set of relations such that arities are preserved. Thus, an L-model can be considered as a couple $\mathbf{A}=(A,{}^*)$ such that for each $R\in L$, if R is n-ary then $R^*\subset A^n$.

$h: A\to B$ is an **isomorphism** between the L-models $\mathbf{A}=(A,{}^*)$ and $\mathbf{B}=(B,{}^\circ)$ if it is bijective and preserves corresponding relations, i.e., if for each $R\in L$, if R is n-ary and $a_0,\ldots,a_{n-1}\in A$ then

$$R^*(a_0,\ldots,a_{n-1}) \text{ iff } R^\circ(ha_0,\ldots,ha_{n-1}).$$

If $\mathbf{A}=(A,{}^*)$ is an L-model and $B\subset A$ then $\mathbf{B}=\mathbf{A}|B$ is the L-model $(B,{}^\circ)$ where for each $R\in L$, if R is n-ary then $R^\circ=R^*|B=R^*\cap B^n$.

1.2 α-equivalence.

$h: A \to B$ is a **partial isomorphism** between the L-models $\mathcal{A}=(A,*)$ and $\mathcal{B}=(B,\circ)$ if Dom h is finite and h is an isomorphism between $\mathcal{A}|\text{Dom}\,h$ and $\mathcal{B}|\text{Ran}\,h$. Notice that each *finite* part of an isomorphism is a partial isomorphism but not every partial isomorphism is part of an isomorphism.

The next definition is basic for this chapter.

1.2.1 Definition. For L-models $\mathcal{A}$, $\mathcal{B}$ and ordinals α, $I_\alpha(\mathcal{A},\mathcal{B})$ is a set of partial isomorphisms between $\mathcal{A}$ and $\mathcal{B}$ defined as follows:

(i) $I_0(\mathcal{A},\mathcal{B})$ consists of all partial isomorphisms between $\mathcal{A}$ and $\mathcal{B}$;

(ii) $h \in I_{\alpha+1}(\mathcal{A},\mathcal{B})$ iff ("*back*") for all $b \in B$ there is an $a \in A$ such that $h \cup \{(a,b)\} \in I_\alpha(\mathcal{A},\mathcal{B})$ and ("*forth*") for all $a \in A$ there is a $b \in B$ such that $h \cup \{(a,b)\} \in I_\alpha(\mathcal{A},\mathcal{B})$;

(iii) for α a limit: $I_\alpha(\mathcal{A},\mathcal{B}) = \bigcap_{\xi<\alpha} I_\xi(\mathcal{A},\mathcal{B})$.

Suppose $\mathbf{a}=(a_0,\dots,a_{k-1}) \in A^k$ and $\mathbf{b}=(b_0,\dots,b_{k-1}) \in B^k$. $\mathbf{a}$ and $\mathbf{b}$ are called **α-equivalent** (notation: $\mathbf{a} \equiv^\alpha \mathbf{b}$ or, more explicitly, $(\mathcal{A},\mathbf{a}) \equiv^\alpha (\mathcal{B},\mathbf{b})$) iff the correspondence $(\mathbf{a},\mathbf{b}) = \{(a_i,b_i) \mid i<k\}$ is in $I_\alpha(\mathcal{A},\mathcal{B})$. ($\omega$-equivalence usually is called **elementary** equivalence.)

The main purpose of this chapter is to present four or five characterisations of α-equivalence, most of them connecting it with $L_{\infty\omega}$ and one in terms of the Ehrenfeucht game. Before we introduce the last one however, we give a simple lemma.

1.2.2 Lemma. *If* $\alpha<\beta$ *then* $I_\beta(\mathcal{A},\mathcal{B}) \subset I_\alpha(\mathcal{A},\mathcal{B})$.

Proof. Induction on β. For limit β, this is obvious, so assume $\beta=\gamma+1$. Let

$h \in I_\beta(A,B)$; we prove $h \in I_\alpha(A,B)$ using induction on α. For $\alpha=0$ and limit α this is clear. Thus, assume $\alpha=\delta+1$ and $h \in I_\delta(A,B)$. To see that $h \in I_{\delta+1}(A,B)$ we check the forth-condition only: let $a \in A$ be arbitrary. Since $h \in I_{\gamma+1}(A,B)$, there exists $b \in B$ such that $h \cup \{(a, b)\} \in I_\gamma(A,B)$. Now $\alpha=\delta+1 \leqslant \gamma$, hence, by induction-hypothesis on γ, $h \cup \{(a,b)\} \in I_\delta(A,B)$. ⊠

1.3 Ordinal-bounded Ehrenfeucht games.

We now turn to the first characterisation of α-equivalence. This is the imaginative reformulation in terms of games, due to Ehrenfeucht, which, in practical applications, seems to work best.

Suppose that A and B are models for the same language, that $h \in I_0(A,B)$ and that α is some ordinal. $G(A, B, h, \alpha)$ is the following game for two players *I* and *II*: *I* and *II* make moves alternately as follows. *I* begins. His first move consists of *three* things: (i) an ordinal $\alpha_0 < \alpha$; (ii) one of the models A, B; (iii) one element of the model chosen under (ii). *II* now is allowed, as a counter-move, to choose one element from the model not chosen by *I*. Next, it is *I*'s turn again. His second move again consists of three things: (i) an ordinal $\alpha_1 < \alpha_0$; (ii) one of the models, and (iii) an element of the model chosen. Again, *II* picks an element from the other model. This goes on with the proviso that the sequence of ordinals picked by *I* must be strictly descending. Of course, after a *finite* number of moves and counter-moves in this fashion, player *I* must pick the ordinal 0 eventually. Player *II* is allowed one last counter-move, and this is where the play of the game ends. Suppose the moves of *I* have been as follows: $(\alpha_0, C_0, c_0), \ldots, (\alpha_{k-1}, C_{k-1}, c_{k-1})$ (where C_i is either A or B and $c_i \in A$ if $C_i = A$,

$c_i \in B$ if $C_i = B$) and *II* has countered with the sequence $d_0, \ldots, d_{k-1}$ (thus, $d_i \in A$ if $C_i = B$ and $d_i \in B$ if $C_i = A$). Let $(a_i, b_i) = (c_i, d_i)$ in case $c_i \in A$ and $d_i \in B$; $(a_i, b_i) = (d_i, c_i)$ otherwise $(i < k)$. We shall say that player *II* **has won** the play of the game when $h \cup \{(a_i, b_i) \mid i < k\} \in \mathrm{I}_0(A, B)$. Otherwise, player *I* **has won**.

This description of the game is inadequate when one of the models is empty, or if $\alpha = 0$. Therefore we agree that *II* wins the play when $\alpha = 0$ or if both models are empty; and *I* wins if *II* has no answer since the model he has to choose from is empty.

Again, the "meaning of the game" is explained somewhat by noting that it is the "intention" of *I* to show that A and B (more accurately, $(A, a)_{a \in \mathrm{Dom}(h)}$ and $(B, ha)_{a \in \mathrm{Dom}(h)}$) are *different* in some sense, while it is the "intention" of *II* to argue that they are not.

Clearly, the larger α, the easier the task of *I*, since the play can be somehow longer. Though each play necessarily must be finite, *I* can make it last as long as he wants without telling *II* - at least some of the time. Only if α is *finite* it is clear beforehand that the play will end after α moves for both players at worst. If $\alpha = \omega$, the *first* move of *I* betrays his intentions as to the length of the play. But if, say, $\alpha = \omega + 3$, *I* has three moves to go for free before he must inform *II* how many more he'll use, etc.

Example: 1.0.3(ii) shows that $\omega \equiv^{\omega} \omega + \zeta$. However, $\omega \not\equiv^{\omega+1} \omega + \zeta$. To see this, let *I* start with the move $(\omega, \omega + \zeta, 0_\zeta)$ (where by 0_ζ I mean the 0 of the ζ-part of $\omega + \zeta$). Now if *II* answers with $n \in \omega$ then *I* chooses $(n+1, \omega + \zeta, -1_\zeta)$, starting a decreasing sequence of length $n+1$ in $\omega + \zeta$.

For finite α, the game was invented by Ehrenfeucht [1961] as a reformulation of the Fraïssé characterisation (cf. below); for general α, cf. Barwise [1973].

1.3.1 Example. An ordinal is identified with the well-ordered set of its predecessors. If $\beta<\alpha$ then player *I* has a *method* to win each play of G(α, β, $\emptyset$, α). *Proof*: the first move of *I* is (β, α, β). If *II* answers with $\gamma\in\beta$, *I* copies this in his second move as (γ, α, γ). As $\gamma<\beta$, *II* will answer with some $\delta<\gamma$ in β which is copied by *I* in his third move, and so on. Eventually, *II* will pick 0 (his sequence of ordinals must be strictly descending if he is not to lose: remember *II* has to take care that the alternate moves build a partial isomorphism). This is copied by *I* – and *II* now is without answer. ⊠

A *method* for one of the players to win each play of the game is called a **winning strategy** for that player. More precisely, a **winning strategy for *II*** is a function σ defined on sequences of odd length such that each play $(a_0, b_0,\dots,a_{k-1}, b_{k-1})$ of the game (where the successive moves of *I* are $a_0,\dots,a_{k-1}$; those of *II* being $b_0,\dots,b_{k-1}$) with the property that for all $i<k$: $b_i = \sigma(a_0, b_0,\dots,a_i)$, is a win for *II*. **Winning strategies for *I*** are defined similarly.

1.3.2 Theorem. *In each game* G(*A*, *B*, *h*, α), *one of the players has a winning strategy.*

Proof. Immediate from the *Gale-Stewart theorem*, since all plays have finite length. However, here is the argument:

First, we define a **position** in G(*A*, *B*, *h*, α) to be a finite set $g\subset A\times B$ together with some $\beta<\alpha$. By the phrase "*I* and *II* have reached the position (g, β) after k moves each in their play" we shall understand that (i) g is the $\leqslant k$-element set of pairs which consist of an element played by

I and the counter-move of *II* corresponding to this element (in the right order) and that (ii) β is the *last* ordinal played by *I*.

A position (g, β) now is defined to be a **win** for *I* obviously in case he has a winning strategy for $G(A, B, h \cup g, \beta)$.

Now, we can prove 1.3.2 using these notions as follows. Suppose that *I* has *no* winning strategy for $G(A, B, h, \alpha)$. This clearly means that the *initial position* ($\emptyset$, α) is no win for *I*. Let *II* play according to the following method: if possible he picks his moves in such a way that after each of his moves a position (g, β) remains which is no win for *I*. By induction, we verify that *II* always has a counter move which just performs this: first, since ($\emptyset$, α) is no win for *I*, whatever the first move of *I*, *II* has an answer leaving no win for *I*. Similarly, if in the play a position (g, β) is reached which is no win for *I* then, whatever *I* does next, *II* can counter leaving no win for *I*. Eventually (and this is where the *finiteness* of the game comes into play), this sequence of moves comes to an end with a position (f, 0). Being no win for *I*, we must have $f \cup h \in I_0(A,B)$. Hence, the method described is winning for *II*. ⊠

Now comes the first characterisation of α-equivalence.

1.3.3 Theorem. *For* $h \in I_0(A,B)$ *and* α *an ordinal the following are equivalent:*

(i) $h \in I_\alpha(A,B)$;

(ii) II has a winning strategy for $G(A, B, h, \alpha)$.

Proof.

(i) ⇒ (ii). Assume $h \in I_\alpha(A,B)$. The strategy of *II* consists in *II* trying to

pick his moves in such a fashion that in each position (g, β) of the play he has $h \cup g \in I_\beta(A,B)$. This is satisfied at the starting-position $(\emptyset, \alpha)$ by assumption. If the play ends with $(f, 0)$ and $h \cup f \in I_0(A,B)$, then *II* has won by definition. Lastly, *II* can keep this requirement during the play: Suppose a position (g, β) has been reached with $h \cup g \in I_\beta(A,B)$ and *I* chooses, say, (γ, A, a) as his next move, where $\gamma < \beta$. Then $\gamma + 1 \leqslant \beta$, hence $h \cup g \in I_{\gamma+1}(A,B)$ by 1.2.2. Therefore, $b \in B$ exists such that $h \cup g \cup \{(a,b)\} \in I_\gamma(A,B)$ - and this will be the answer of *II*, of course.
(ii) $\Rightarrow$ (i). Let σ be a winning strategy for *II* in $G(A, B, h, \alpha)$; we show $h \in I_\alpha(A,B)$ by induction on α:
If $\alpha = 0$ there is nothing to prove since $h \in I_0(A,B)$ by hypothesis. If α is a limit, σ wins every play in $G(A, B, h, \beta)$ with $\beta < \alpha$ for *II*, therefore $h \in \bigcap_{\beta \in \alpha} I_\beta(A,B) = I_\alpha(A,B)$. Finally, assume $\alpha = \beta + 1$. We check the forth-condition: let $a \in A$ be arbitrary. Consider (β, A, a) as a first move of *I*. Suppose σ provides the answer $b \in B$ for *II*. Clearly, (a modification of) σ is a winning strategy for *II* in $G(A, B, h \cup \{(a, b)\}, \beta)$. By induction hypothesis then, $h \cup \{(a, b)\} \in I_\beta(A,B)$. ⊠

1.3.4 Example. In 3.4.1 below, we re-prove Ehrenfeucht's result that ω^ω is ω-equivalent with the ordered structure Ω of *all* ordinals. However, ω^ω is *not* $(\omega+1)$-equivalent with this structure, which can be seen as follows.

According to Rosenstein [1982] corollary 6.19 p.106, if $\alpha < \omega^{n+1}$ then $\alpha \not\equiv^{2n+2} \omega^{n+1}$.(*) Now, let *I* play $(\omega, \Omega, \omega^\omega)$ as a first move. Suppose *II* counters with $\alpha < \omega^{n+1}$. Then, by (*), $(2n+2, \Omega, \omega^{n+1})$ is a winning move for *I*.

1.3.5 Example. Karp [1965] has shown (cf. Rosenstein [1982] thm.14.29 p.357) that if $\omega^\delta = \delta$ then $\delta \equiv^\delta \delta + \delta \cdot \tau$ for all (not necessarily well-ordered) order types τ.

Hence $\equiv^\alpha$ will never coincide with isomorphism on the ordinals as there always are $\delta \geqslant \alpha$ (of the *same* power as α) such that $\omega^\delta = \delta$.

1.4 Fraïssé-Karp sequences.

A **Karp sequence** for A, B, h, α is a sequence $\langle I_\xi \mid \xi \leqslant \alpha \rangle$ of length $\alpha + 1$ such that the following hold:

1. (i) $I_\xi \subset I_0(A,B)$ for all $\xi \leqslant \alpha$;
   (ii) $\xi < \delta \leqslant \alpha \Rightarrow I_\delta \subset I_\xi$;
   (iii) $h \in I_\alpha$;
2. for all $\xi < \alpha$ and $g \in I_{\xi+1}$:
   ("*back*") for all $b \in B$ there exists $a \in A$ such that $g \cup \{(a,b)\} \in I_\xi$;
   ("*forth*") for all $a \in A$ there exists $b \in B$ such that $g \cup \{(a,b)\} \in I_\xi$.

Karp sequences for *finite* α are due to Fraïssé [1955] which is one of the oldest references in the subject. For general α, they are due to Karp [1965]. They provide a second characterisation:

1.4.1 Theorem. *$h \in I_\alpha(A,B)$ iff there is a Karp sequence for* A, B, h, α.

Proof. If $h \in I_\alpha(A,B)$ then obviously $\langle I_\xi(A,B) \mid \xi \leqslant \alpha \rangle$ is the required Karp sequence, by 1.2.2. For the converse, assume $\langle I_\xi \mid \xi \leqslant \alpha \rangle$ is a sequence as required. Now, copy the proof of 1.3.3.(i) $\Rightarrow$ (ii), replacing $I_\xi(A,B)$ by I_ξ $(\xi \leqslant \alpha)$ (notice that 1.2.2 has been *built into* the definition of Karp sequence). This shows *II* to have a winning strategy in

$G(A, B, h, \alpha)$, which suffices by the other half of 1.3.3. ⊠

1.5 Logic.

Now, it is time for logic.

Let L be any language. Fix a countably infinite set of variables. *Atomic formulas* of L are all identities $x_0 = x_1$ and expressions $R(x_0, \ldots, x_{k-1})$ where $R \in L$ is k-ary and $x_0, \ldots, x_{k-1}$ are variables. The set $L_{\infty\omega}$ of (infinitary) formulas of L is the least one such that

1. all atomic formulas are in $L_{\infty\omega}$;
2. if $\varphi \in L_{\infty\omega}$ then $\neg\varphi \in L_{\infty\omega}$;
3. if $\Phi \subset L_{\infty\omega}$ is any set then $\bigwedge\Phi$ and $\bigvee\Phi$ are in $L_{\infty\omega}$;
4. if $\varphi \in L_{\infty\omega}$ and x is a variable then $\forall x\varphi$ and $\exists x\varphi$ are in $L_{\infty\omega}$.

I assume known the basic notions and semantics concerning these objects.

The *quantifier rank* $\mathrm{qr}(\varphi)$ of $\varphi \in L_{\infty\omega}$ is an ordinal recursively defined as follows:

1. $\mathrm{qr}(\varphi) = 0$ if φ is atomic;
2. $\mathrm{qr}(\neg\varphi) = \mathrm{qr}(\varphi)$;
3. $\mathrm{qr}(\bigwedge\Phi) = \mathrm{qr}(\bigvee\Phi) = \sup\{\mathrm{qr}(\varphi) \mid \varphi \in \Phi\}$;
4. $\mathrm{qr}(\forall x\varphi) = \mathrm{qr}(\exists x\varphi) = \mathrm{qr}(\varphi) + 1$.

Here is the logical characterisation of α-equivalence:

1.5.1 Theorem. *$h \in I_\alpha(A,B)$ iff for every $\varphi \in L_{\infty\omega}$ with $qr(\varphi) \leqslant \alpha$ and every valuation f of the free variables of φ into* Dom *h: $A \models \varphi[f]$ iff $B \models \varphi[h \circ f]$ (i.e., h* preserves satisfaction *of quantifier-rank $\leqslant \alpha$ - formulas).*

Proof. I only give one half here - the other has to wait for the next (and last) characterisations.

Suppose φ has $qr(\varphi) \leqslant \alpha$ and φ, h do *not* satisfy the equivalence. Say, $A \models \varphi[f]$, but $B \models \neg\varphi[h \circ f]$. If φ is atomic, then $h \notin I_0(A,B)$. If $\varphi = \bigwedge\Psi$, there is a $\psi \in \Psi$ such that $A \models \psi[f]$ and $B \models \neg\psi[h \circ f]$. The same goes when $\varphi = \bigvee\Psi$. If $\varphi = \neg\psi$, then $A \models \neg\psi[f]$ and $B \models \psi[h \circ f]$. Repeating this, we find a quantified subformula, $\forall x\psi$ say, of φ, satisfied by f resp. $h \circ f$ in *exactly one* of the models, A say. Hence, for some $b \in B$ $B \models \neg\psi[(h \circ f) \cup \{(x,b)\}]$; on the other hand, for all $a \in A$ $A \models \psi[f \cup \{(x,a)\}]$. Now we can give the first move in a winning strategy for *I* in $G(A, B, h, \alpha)$: *I* plays the triple $(qr(\psi), B, b)$. Whatever *II*'s counter-move $a \in A$, *I* proceeds to find a quantified subformula of ψ satisfied by f and a (resp. $h \circ f$ and b) in exactly one of the models and repeats the procedure.

Ultimately, an *atomic* formula and a position $(0, g)$ are reached such that, necessarily, $h \cup g \notin I_0(A,B)$.

Summarizing: *I* can use φ as a guide to find a winning sequence of moves.

This gives one half of the theorem by 1.3.3. ⊠

1.6 Scott-sentences.

For the last couple of characterisations of α-equivalence a definition is needed which is due to Scott (cf. Chang [1968]).

1.6.1 Definition. Fix an enumeration $v_0, v_1, v_2, \ldots$ of all variables.

For $\mathbf{A} = (A, \ldots)$, $\mathbf{a} = (a_0, \ldots, a_{k-1}) \in A^k$ and α an ordinal, define the formula $[\![\mathbf{a}]\!]^\alpha = [\![(\mathbf{A}, \mathbf{a})]\!]^\alpha$ (the **α-characteristic** of $\mathbf{a}$ in $\mathbf{A}$) as follows:

1. $[\![\mathbf{a}]\!]^0$ is the conjunction of all atomic or negated atomic formulas in $v_0, \ldots, v_{k-1}$ satisfied by $\mathbf{a}$ in $\mathbf{A}$;
2. $[\![\mathbf{a}]\!]^{\alpha+1} = \bigwedge_{a \in A} \exists v_k [\![\mathbf{a}\, a]\!]^\alpha \wedge \forall v_k \bigvee_{a \in A} [\![\mathbf{a}\, a]\!]^\alpha$;
3. $[\![\mathbf{a}]\!]^\alpha = \bigwedge_{\xi < \alpha} [\![\mathbf{a}]\!]^\xi$ when α is a limit.

(Here, *juxtaposition* denotes *prolongation*: $\mathbf{a}\, a = (a_0, \ldots, a_{k-1}, a)$.)

To appreciate this definition, notice that it is (in a sense to be explained) a formalisation of the definition of α-equivalence, 1.2.1.

1.6.2 Lemma. *$[\![\mathbf{a}]\!]^\alpha$ has quantifier rank α, free variables $v_0, \ldots, v_{k-1}$ and it is satisfied by $\mathbf{a}$ in $\mathbf{A}$.*

The following completes the sequence of theorems 1.3.3, 1.4.1, 1.5.1:

1.6.3 Theorem. *For $\mathbf{a} \in A^k$ and $\mathbf{b} \in B^k$ the following are equivalent:*

1. $\mathbf{a} \equiv^\alpha \mathbf{b}$;
2. $\mathbf{B} \models [\![\mathbf{a}]\!]^\alpha [\mathbf{b}]$;
3. $[\![\mathbf{b}]\!]^\alpha$ ($= [\![(\mathbf{B}, \mathbf{b})]\!]^\alpha$) $= [\![\mathbf{a}]\!]^\alpha$.

Proof, and proof second half of 1.5.1.

The condition from 1.5.1 in terms of sequences is:

0. if $qr(\varphi) \leqslant \alpha$ *and* φ *has free variables* $v_0, \ldots, v_{k-1}$ *at most then:* $A \models \varphi[a] \iff B \models \varphi[b]$.

We are going to show $1 \Rightarrow 0 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.

Using the Ehrenfeucht game-characterisation, $1 \Rightarrow 0$ has been proved above.

$0 \Rightarrow 2$: this is an immediate consequence of 1.6.2.

$2 \Rightarrow 3$: induction on α. For $\alpha = 0$ and limit α, this is clear. Assume the implication for α and all $\boldsymbol{a}$ and $\boldsymbol{b}$. Now, let $B \models [\![\boldsymbol{a}]\!]^{\alpha+1}[\boldsymbol{b}]$. (*) In order that $[\![\boldsymbol{a}]\!]^{\alpha+1} = [\![\boldsymbol{b}]\!]^{\alpha+1}$ it suffices that $\{[\![\boldsymbol{a}\,a]\!]^{\alpha} \mid a \in A\} = \{[\![\boldsymbol{b}\,b]\!]^{\alpha} \mid b \in B\}$.

$\subset$: let $a \in A$. By (*), $b \in B$ exists such that $B \models [\![\boldsymbol{a}\,a]\!]^{\alpha}[\boldsymbol{b}\,b]$. By induction hypothesis, $[\![\boldsymbol{a}\,a]\!]^{\alpha} = [\![\boldsymbol{b}\,b]\!]^{\alpha}$. Hence, $[\![\boldsymbol{a}\,a]\!]^{\alpha} \in \{[\![\boldsymbol{b}\,b]\!]^{\alpha} \mid b \in B\}$. The $\supset$-part is similar.

$3 \Rightarrow 1$: induction on α. For $\alpha = 0$ this is clear. For α limit, notice that if $[\![\boldsymbol{a}]\!]^{\alpha} = [\![\boldsymbol{b}]\!]^{\alpha}$ then $[\![\boldsymbol{a}]\!]^{\xi} = [\![\boldsymbol{b}]\!]^{\xi}$ for all $\xi < \alpha$ by 1.6.2; now, apply the induction-hypothesis.

Lastly, assume $3 \Rightarrow 1$ for α and all $\boldsymbol{a}$, $\boldsymbol{b}$ and let $[\![\boldsymbol{a}]\!]^{\alpha+1} = [\![\boldsymbol{b}]\!]^{\alpha+1}$. To see that $\boldsymbol{a} \equiv^{\alpha+1} \boldsymbol{b}$, we check the forth-condition:

Pick $a \in A$. Since $[\![\boldsymbol{a}\,a]\!]^{\alpha}$ is a quantifier rank-α subformula of $[\![\boldsymbol{a}]\!]^{\alpha+1}$, it must occur in $[\![\boldsymbol{b}]\!]^{\alpha+1}$ as well; hence $[\![\boldsymbol{a}\,a]\!]^{\alpha} = [\![\boldsymbol{b}\,b]\!]^{\alpha}$ for some $b \in B$. By induction hypothesis then, $\boldsymbol{a}\,a \equiv^{\alpha} \boldsymbol{b}\,b$. ⊠

Remark. The equivalent 1.6.3.3 shows that the α-characteristics are *canonical objects* associated with the α-equivalence classes. If one does not care about them being $L_{\infty\omega}$*-formulas* at the same time, their definition can be simplified, leaving out logic almost completely, by

1.6.1.2'. $[\![\boldsymbol{a}]\!]^{n+1} = \{[\![\boldsymbol{a}\,a]\!]^{n} \mid a \in A\}$.

This remark is due to Shelah [1975]. Of course, this can be extended into the transfinite by putting, for limits γ,

3' $[\![\boldsymbol{a}]\!]^{\gamma} = \bigcup_{\xi<\gamma} [\![\boldsymbol{a}]\!]^{\xi}$.

1.7 The finite case

Specializing to the *finite fragment* $L_{\omega\omega}$ of $L_{\infty\omega}$, first-order logic, there is a *slight* problem. Notice that, though each finite formula has finite quantifier rank, not every finite quantifier rank $L_{\infty\omega}$-formula is finite. However, we have the (in the sequel, often used)

1.7.1 Lemma. *If the language of **A** is finite then, for all $k,n \in \mathbb{N}$, there are only finitely many n-characteristics belonging to sequences of length k.*

Proof. Induction on n. When we have a finite language, there are only finitely many atomic formulas in $v_0, \dots, v_{k-1}$ for each k. If their number is s_k, it is clear that there are at most 2^{s_k} possible formulas $[\![a]\!]^0$ for a of length k. Hence, there are $\leqslant$ exp exp s_{k+1} formulas $[\![a]\!]^1$, and so on. ⊠

Therefore, we can specialize 1.5.1 to

1.7.2 Theorem. *For models **A, B** of the same* finite *language, when $a \in A^k$, $b \in B^k$, $n \in \omega$, the following are equivalent:*

1. $\boldsymbol{a} \equiv^n \boldsymbol{b}$
2. *for all finite formulas φ of quantifier-rank $\leqslant n$ in the appropriate number of free variables:* $\boldsymbol{A} \models \varphi[a] \Leftrightarrow \boldsymbol{B} \models \varphi[b]$.

Proof. $1 \Rightarrow 2$: by 1.5.1. $2 \Rightarrow 1$: if 2. holds then, by 1.6.3 and 1.7.1, $\boldsymbol{B} \models [\![a]\!]^n [b]$. The result follows by 1.6.3. ⊠

1.8 The unbounded case.

Suppose that $\boldsymbol{A}$ and $\boldsymbol{B}$ are models. Let us construct a Karp sequence for $\boldsymbol{A}$, $\boldsymbol{B}$ which is as long as possible: $I_0 \supset I_1 \supset I_2 \supset \ldots \supset I_\alpha \supset \ldots$; for instance, we might have $I_\alpha = I_\alpha(\boldsymbol{A},\boldsymbol{B})$. By a cardinality-argument (cf. 1.9.1), the sequence must become stationary at some place α, i.e., $I_\alpha = \bigcap_\xi I_\xi$. Now, I_α either is empty, or it isn't. Either way, it has the following **back-and-forth property**: if $h \in I_\alpha$ then (*back*) for all $b \in B$ there is $a \in A$ such that $h \cup \{(a,b)\} \in I_\alpha$, and (*forth*) for all $a \in A$ there is $b \in B$ such that $h \cup \{(a,b)\} \in I_\alpha$. This simply is due to the fact that $I_\alpha = I_{\alpha+1}$.

I_α can be looked at as the *greatest fixed point* of the *monotone operator* defined on subsets of $\mathrm{I}_0(\boldsymbol{A},\boldsymbol{B})$ which associates with each $X \subset \mathrm{I}_0(\boldsymbol{A},\boldsymbol{B})$ the set $\{g \mid \forall a \in A \exists b \in B (g \cup \{(a,b)\} \in X) \wedge \forall b \in B \exists a \in A (g \cup \{(a,b)\} \in X)\}$.

Now, $\boldsymbol{A}$ and $\boldsymbol{B}$ are called **partially isomorphic** if a *non-empty* set I of partial isomorphisms exists with this back-and-forth property.

In the games $\mathrm{G}(\boldsymbol{A}, \boldsymbol{B}, h, \alpha)$, the ordinal α is a means to *preserve finiteness* of all plays, *without* giving a fixed upper bound on their lengths (at least when $\alpha \geqslant \omega$). Somehow, the larger α, the longer player $\boldsymbol{I}$ can keep the play going (and the better his chances to defeat $\boldsymbol{II}$). The natural limit is the game $\mathrm{G}(\boldsymbol{A}, \boldsymbol{B}, h) = \mathrm{G}(\boldsymbol{A}, \boldsymbol{B}, h, \Omega)$ where $\boldsymbol{I}$ can go on forever, and plays have length ω. Of course, if $\boldsymbol{I}$ is to win, this becomes apparent after a finite number of moves already. Hence, the *Gale-Stewart determinacy theorem* still applies. We now have the following "limiting case" of our characterisations, which is presented without parameters for simplicity.

1.8.1 Theorem. *The following are equivalent:*

1. $\bigcap_\alpha I_\alpha(A,B) \neq \emptyset$;
2. *II has a winning strategy for* $G(A, B, \emptyset)$;
3. *A and B are partially isomorphic ;*
4. $A \equiv_{\infty\omega} B$ *(i.e., they have the same* $L_{\infty\omega}$*-theory).*

Proof.

1⇒2. Suppose $I_{\alpha+1}(A,B) = I_\alpha(A,B) \neq \emptyset$. *II* plays in such a way that only positions h are reached for which some $g \in I_\alpha(A,B)$ exists with $h \subset g$.

2⇒3. The required set is the one of all positions h reached during every possible play in which *II* uses his winning strategy.

3⇒4. If I is a set witnessing 3., the *constant* sequence $I, I, I, \ldots$ of length α is a Karp sequence. Apply 1.4.1/1.5.1.

4⇒1. Let α be such that $I_\alpha(A,B) = \bigcap_\xi I_\xi(A,B)$. By 1.5.1, $\emptyset \in I_\alpha(A,B)$. ⊠

1.8.2 Corollary. (Barwise [1973]) *Countable partially isomorphic models are isomorphic.*

Proof. In $G(A, B, \emptyset)$, let *I* enumerate all elements of $A \cup B$. If *II* uses a winning strategy, the result of the play is an isomorphism as required. ⊠

1.8.3 Example. *On the ordinals, partial isomorphism coincides with isomorphism.*

1.9 Basis results

As part of the above, we saw that, for all A and B, there is an α such that $A \equiv_{\infty\omega} B$ iff $A \equiv^{\alpha} B$. As to how large such an α must be

1.9.1 Theorem.

1. *If* $|A| = n < \omega$ *then* $A \equiv_{\infty\omega} B$ *iff* $A \cong B$ *iff* $A \equiv^{n+1} B$;
2. *If* A *and* B *are infinite then* $A \equiv_{\infty\omega} B$ *iff* $A \equiv^{\alpha} B$, *where* α *is the* least *ordinal of power* $> |A|, |B|$.

Proof. 1. In $G(A, B, n+1)$, let I first enumerate A. If II uses a winning strategy, the position reached must be surjective (otherwise, let I play a new element of B). 2. Notice that $|I_0(A,B)| = \max(|A|, |B|)$. Furthermore, the first α for which $I_\alpha(A,B) = I_{\alpha+1}(A,B)$ has $|\alpha| \leqslant |I_0(A,B)|$, hence is less than the first one of power $> |A|, |B|$. ⊠

Of course, 1.9.1.2 is quite weak. A much better result is *Nadel's basis theorem* from Nadel [1974]. Here, an **admissible set** is a transitive model of the theory KP, which is a weak version of ZF at least sufficient to carry out the proof of 1.9.2; cf. Barwise [1975]. To see that the next theorem strengthens 1.9.1, note that, by the *Löwenheim-Skolem theorem*, each transitive infinite set is a member of lots of admissibles of the same power.

1.9.2 Theorem. *Let* $\mathbb{A}$ *be an admissible set such that* $L, A, B \in \mathbb{A}$ *and let* $\alpha = \mathbb{A} \cap \mathrm{OR}$ *be the set of ordinals in* $\mathbb{A}$. *If* $A \equiv^{\alpha} B$ *then* $A \equiv_{\infty\omega} B$.

Proof. Since $\equiv_{\infty\omega}$ can be defined inductively, this is immediate from

Gandy's theorem (see for instance Barwise [1975] chapter VI). However, here is a direct proof:

Suppose that φ_0 is any $L_{\infty\omega}$-sentence such that, say, $A \models \varphi_0$ but $B \models \neg\varphi_0$. Let $\beta = qr(\varphi_0)$. We know that I has a winning strategy σ for $G(A, B, \emptyset, \beta)$. Let T be the set of all positions $((a, b), \xi)$ $(\xi \leqslant \beta)$ which can be reached in some play where I uses his strategy σ.

Claim: *for every $((a, b), \xi) \in T$ there exists $\varphi \in \mathbb{A} \cap L_{\infty\omega}$ in the appropriate number of free variables such that $A \models \varphi[a]$ and $B \models \neg\varphi[b]$.*

As $((\emptyset, \emptyset), \beta) \in T$ to begin with, this claim establishes the result. It is proved by induction on ξ as follows:

If $((a, b), 0) \in T$, we cannot have $a \equiv^0 b$, since I was using a *winning* strategy. Therefore there is a formula φ as required which is either atomic or negated atomic. Now, let $((a, b), \xi) \in T$ and $\xi > 0$. ($a \in A^k$) Suppose σ prescribes the move (δ, A, a) for I in this position. Then $\delta < \xi$, and, by induction hypothesis,

$$\forall b \in B\, \exists \psi \in \mathbb{A}\, [A \models \psi[a, a] \;\wedge\; B \models \neg\psi[b, b]\,].$$

By Σ-*collection* (cf. Barwise [1975] thm.4.4 p.17) on $\mathbb{A}$, obtain $\Phi \in \mathbb{A}$ such that

1. $\forall b \in B\, \exists \psi \in \Phi\, [A \models \psi[a, a] \;\wedge\; B \models \neg\psi[b, b]\,]$, and
2. $\forall \psi \in \Phi\, \exists b \in B\, [A \models \psi[a, a] \;\wedge\; B \models \neg\psi[b, b]\,]$.

Now, $\varphi = \exists x_k \bigwedge\Phi$ is the formula required. (When σ prescribes a move in B, a similar argument finds φ of the form $\forall x_k \bigvee \Phi$.) ⊠

The **Scott-rank** $sr(A)$ of A is the least ordinal α such that for all k and $a, b \in A^k$: if $a \equiv^{\alpha} b$ then $a \equiv^{\alpha+1} b$. (Thus, A is *ω-homogeneous* iff $sr(A) \leqslant \omega$.)

1.9.3 Theorem. *If $\mathbb{A}$ is admissible and $A \in \mathbb{A}$ then* $\mathrm{sr}(A) \leqslant \mathbb{A} \cap \mathrm{OR}$.

(In particular, $\mathrm{sr}(A)$ *exists.)*

Proof. Apply 1.9.2 to $(A, a), (A, b) \in \mathbb{A}$. ⊠

For $\alpha = \mathrm{sr}(A)$, the **Scott-sentence** of A is

$$\sigma_A = [\![\emptyset]\!]^\alpha \wedge \bigwedge_{a} \forall x\, ([\![a]\!]^\alpha \rightarrow [\![a]\!]^{\alpha+1}).$$

A one-sided version of 1.9.1 is part 3 of the theorem below:

1.9.4 Theorem. *1.* $\mathrm{qr}(\sigma_A) = \mathrm{sr}(A) + \omega$ *;*

2. $A \models \sigma_A$ *;*

3. $B \models \sigma_A$ *iff* $B \equiv_{\infty\omega} A$.

Proof of 3.: $\Leftarrow$ by 1, 2. $\Rightarrow$: If $B \models \sigma_A$ then in particular $B \models [\![\emptyset]\!]^\alpha$, hence $B \equiv^\alpha A$. (*) But also, $B \models \bigwedge_{a} \forall x\, ([\![a]\!]^\alpha \rightarrow [\![a]\!]^{\alpha+1})$; therefore *II* can win $\mathrm{G}(A, B, \emptyset, \gamma)$ for each γ as follows: he simply moves in such a way that only positions $((a, b), \xi)$ are reached with $a \equiv^\alpha b$. By (*), this is right at the initial position. If $((a, b), \xi)$ is reached and $a \equiv^\alpha b$, then $B \models [\![a]\!]^\alpha [b]$ and hence $B \models [\![a]\!]^{\alpha+1} [b]$ by assumption. Now if *I* plays, say, $a \in A$, there is $b \in B$ with $B \models [\![a\, a]\!]^\alpha [b\, b]$, etc. ⊠

2. On n-equivalence of binary trees.[1]

2.1 Introduction

For $m>0$, B_m is the binary tree all of whose branches have length m.

This chapter characterizes the binary trees n-equivalent with B_m (for each pair n, m). It grew out of corollary 2.2.4 below which says that *finiteness* of binary trees is not first-order expressible. This fact is used in Rodenburg [1986] to solve a problem in correspondence theory for intuitionistic propositional calculus, cf. chapter 8 below. Section 2.4 contains the much stronger result that *finiteness* of binary trees is not even Σ^1_1 - be it with a less informative proof. For an application of the Ehrenfeucht game played on *structures of* trees, cf. Tiuryn [1984].

Playing on trees is appreciably less easy than it was on finite linear orderings, due to the fact that a handy decomposition lemma like 1.0.2 is not available: decomposing a tree yields much more than just two

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[1] 2.1-3 *are reproduced from Doets [198?] (with slight changes)*.

trees.

Let me define the notions involved.

A **tree** is a non-empty partially ordered set $T=(T, <)$ where, for each $a \in T$, the set $a\downarrow = \{x \in T \mid x < a\}$ is linearly ordered by $<$; in this chapter, moreover, it is required always that the sets $a\downarrow$ are *finite*.

A *least* element of a tree is called its **root**. The tree $(T, <)$ is **binary** if it has a root and each non-empty set $a\uparrow = \{x \in T \mid a < x\}$ $(a \in T)$ has *exactly two* minimal elements, in other words, each non-maximal $a \in T$ has exactly two immediate successors. A **branch** through T is a maximal linearly ordered subset. A **branch above** $a \in T$ is a branch in the subtree $a\uparrow$ of T. The **length** of a branch is its order type; the order-type of $a\downarrow$, denoted by $h(a)$, is called the **height** of a in T.

2.2 Playing in trees.

For each $n \geqslant 1$, define the class Q(n) of binary trees as follows.

2.2.1 Definition. Let $n \geqslant 1$. The binary tree T **satisfies Q(n)** iff the following conditions are met:

Q.1(n) if $n=2$ then T has a maximal element; if $n \geqslant 3$, there is a maximal element above any given one.

Q.2(n) every branch through T has length $\geqslant 2^n - 2$.

Q.3(n) some branch through T has length $\geqslant 2^n - 1$.

Q.4(n) for all $x \in T$ and $m < 2^{n-1} - 1$: if *some* branch above x has length m then *every* branch above x has length m.

Notice that every binary tree satisfies Q(1): Q.1(n)-Q.3(n) only demand something if $n \geqslant 2$; and Q.4(n) is non-trivial for $n \geqslant 3$ only.

The next theorem shows that Q(n) is included in an n-equivalence class. In section 2.3 it is shown that Q(n) actually can be expressed by a quantifier rank-n sentence; whence it can be used to characterize n-equivalence with B_m for any m.

2.2.2 Theorem. *If the binary trees T^1 and T^2 satisfy* Q(n), *then* $T^1 \equiv^n T^2$.

Proof. Induction on n.
The case $n=1$ is clear. Assuming 2.2.2 to hold for n we check it for $n+1$ using Ehrenfeucht-games. Thus, suppose $t \in T^1$ is the first move of player ***I*** in the $(n+1)$-game between T^1 and T^2. There are *three* cases to consider.

Case 1. $h(t) < 2^n-1$.
Decompose T^1 in:
(i) two top-subtrees t^1 and t^2, final sections of T^1, the roots of which are the two minimal elements of $t\!\uparrow$;
(ii) the linear ordering $t\!\downarrow$ of type $h(t)$;
(iii) the trees t_a (where $a < t$), final sections of T^1; the root of t_a being the immediate successor of $a \in t\!\downarrow$ which is not $\leqslant t$.
(The items under (ii) and (iii) are absent in case t is the root of T^1.)
Since T^1 satisfies Q($n+1$), it is clear that all trees in this decomposition satisfy Q(n). For instance, Q.1(n) is inherited from Q.1($n+1$) by final sections (this is true even if $n=1$ or $n=2$). Q.4($n+1$) implies Q.4(n); and if T^1 satisfies Q.4($n+1$) then so do his final sections. By Q.2($n+1$), each branch in, say, t^1 has length $\geqslant 2^{n+1}-2-h(t)-1 > 2^{n+1}-2-(2^n-1)-1 = 2^n-2$, i.e., has length $\geqslant 2^n-1$. Thus, t^1 has Q.2(n) and Q.3(n). The same goes

for the other subtrees.

Now player *II* answers t with some $s \in T^2$ for which $h(s) = h(t)$.

Let j be the isomorphism between $t\downarrow$ and $s\downarrow$. s induces a decomposition of T^2 similar to the one described for t in which all trees satisfy $Q(n)$. By induction hypothesis, corresponding trees in the decompositions are n-equivalent. Therefore, *II* can win the remaining n-game using the following strategy: *above* t or s he uses winning strategies between t^i and s^i ($i=1, 2$). *Below* t or s he answers using the isomorphism j. Finally, a move in some t_a ($a<t$) by *I* is answered using a winning strategy between t_a and $s_{j(a)}$ and vice versa.

This strategy is clearly winning for *II* since the union of partial isomorphisms between corresponding substructures in the decompositions is a partial isomorphism between T^1 an T^2.

Case 2. There is *no* branch of length $\geqslant 2^n-1$ above t.

By Q.4(n+1) there exists $u \leqslant t$ such that *all* branches above u have length 2^n-2. Hence, u is the root of a final section B_u of T^1 in which all branches have length 2^n-1.

Since T^2 satisfies Q.1(n+1) (for, $n+1 \geqslant 2$) and Q.4(n+1), there exists $v \in T^2$ which is the root of a final section B_v of T^2 isomorphic with B_u. (1)

By Q.2(n+1), $u\downarrow$ and $v\downarrow$ have order types $\geqslant 2^{n+1}-2-(2^n-1) = 2^n-1$; hence $u\downarrow \equiv^n v\downarrow$ by proposition 1.0.1. (2)

If $a<u$, branches above a containing u have length $\geqslant 2^n-1$; by Q.4(n+1) therefore, *all* branches above a have length $\geqslant 2^n-1$; in particular, all branches through u_a have length $\geqslant 2^n-1$. Thus, u_a (defined as under case 1.(iii)) satisfies $Q(n)$. The same goes for the v_b ($b<v$).

By induction hypothesis, $u_a \equiv^n v_b$ whenever $a<u$ and $b<v$. (3)

Now *II* uses the following strategy. First, he answers t using the isomorphism (1). The remaining n-game is dealt with as follows. Between B_u and B_v, *II* goes on using the isomorphism (1). Below u or v he uses the winning strategy (2). If *I* makes a move x in some u_a ($a<u$) for the first time while a has not been played yet, *I* is granted the *extra move* a as well. Then *II* answers a by some $b<v$ using (2) and next answers x by some $y \in v_b$ using (3).

Of course, if a has been played before, b has been fixed already and no extra move is granted (this occurs in particular when x is not the first move in u_a by either player).

Case 3. $h(t) \geqslant 2^n-1$ and some branch above t has length $\geqslant 2^n-1$.

By Q.4(n+1) then, *all* branches above t have length $\geqslant 2^n-1$. Hence, in the decomposition described under 1. above, t^1 and t^2 satisfy Q(n). If $a<t$, branches above a containing t and, hence, *all* branches above a, have length $\geqslant 2^n-1$; thus t_a satisfies Q(n).

Since T^2 satisfies Q.3(n+1) and $2^{n+1}-1 = 2(2^n-1)+1$, *II* can find $s \in T^2$ such that $h(s) = 2^n-1$ while some branch above s has length $\geqslant 2^n-1$. It follows that s^1, s^2 and all s_b ($b<s$) satisfy Q(n).

For the remaining n-game, *II* uses a strategy similar to the one used under 2. above; except that above s or t he uses that $s^i \equiv^n t^i$ ($i=1, 2$). ⊠

2.2.3 Examples. The following trees satisfy Q(n).

1. The binary tree B_m all of whose branches have length $m \geqslant 2^n-1$.
2. *Infinite* binary trees provided that, along every infinite branch, (i) there occur infinitely many side-trees, and (ii) all finite side-trees occurring are of type 1 (i.e., are of the form B_m for some $m \geqslant 2^n-1$).

(Of course, there are more complicated infinite binary trees satisfying

Q(*n*) as well.)

These examples make manifest the following

2.2.4 Corollary. Finiteness *of trees is not a first-order property on the class of all binary trees.*

Noteworthy also is

2.2.5 Corollary. *"Every branch has length* $\geqslant 2^n-1$*" and its negation some branch has length* $\leqslant 2^n-2$*" ($n>1$) cannot be expressed by first-order sentences of quantifier-rank n on the class of (finite) binary trees.*

2.3 Characterizing *n*-equivalence with B_m.

By 2.2.3, B_m satisfies Q(*n*) whenever $m \geqslant 2^n-1$; hence 2.2.2 gives one half of the following

2.3.1 Theorem. *Let* $m \geqslant 2^n-1$. *A binary tree T satisfies* Q(*n*) *iff* $T \equiv^n B_m$.

The other half is established by propositions 2.3.2-4 below.

These results show that Q(*n*) can be expressed by a first-order sentence of quantifier rank *n*. And this can be used to construct a simple quantifier rank-*n* logical equivalent of $[\![B_m]\!]^n$ for $m \geqslant 2^n-1$: the theory of binary trees has a straightforward quantifier rank-4 axiomatisation (instead of *finiteness* of the $x\downarrow$, we only require discreteness with first and last element of these orderings – to see this works, use chapter 4).

Notice that Q.1(n) can be expressed by a first-order sentence of quantifier rank $\leqslant n$. Q.2(n)-Q.4(n) are dealt with by 2.3.4, 2.3.2 and 2.3.3, respectively.

In the sequel, $\varphi^{<x}$ and $\varphi^{>x}$ denote the formulas obtained from φ by *restricting quantifiers* to the sets $\{y \mid y<x\}$ resp. $\{y \mid x<y\}$ (and changing bound veriables when necessary).

2.3.2 Proposition. *Define the sentences* φ_n *by:*

$$\varphi_1 \quad \text{is } \exists x\,(x=x)$$

$$\varphi_{n+1} \quad \text{is } \exists x\,(\varphi_n^{<x} \wedge \varphi_n^{>x}).$$

Then φ_n *has quantifier rank n and it holds in a tree iff there is a branch of length* $\geqslant 2^n-1$.

Proof. Obvious. ⊠

In view of 2.2.5, the next result is not entirely trivial.

2.3.3 Proposition. *Let k be any integer* $\geqslant 1$ *and T a binary tree such that* $T \equiv^{n+1} B_k$. *Then T satisfies* Q.4(n+1).

Proof. We may assume that $n>1$ since otherwise Q.4(n+1) is trivially satisfied.

Suppose that T fails to satisfy Q.4(n+1). Let $m<2^n-1$ be minimal such that for some $t \in T$ and branches α and β above t we have $|\alpha|=m$ and $|\beta|>m$.

By minimality of m, $\alpha \cap \beta = \emptyset$.

By the same token, if u is the immediate successor of t which is the least element of α, then *all* branches through the top-tree t^1 of which u is the root have length m.

Let t^2 be the other top-tree of T through which β is a branch. Again by minimality of m, all branches through t^2 have length $\geqslant m$. Finally, we may as well assume β to have *finite* length since $n+1 \geqslant 3$ and Q.1(3) is a quantifier rank-3-sentence true in B_k and hence valid in T.

Now, choose $x \in B_k$ with $(T, u) \equiv^n (B_k, x)$ (1).

By (a variation on) proposition 1.0.1 it follows that all branches through $B_x = \{x' \in B_k \mid x \leqslant x'\}$ have length m. Let $y \in B_k$ be the element $\neq x$ with $y\downarrow = x\downarrow$. Then all branches through $B_y = \{y' \in B_k \mid y \leqslant y'\}$ have length m too.

Notice that if $s \in \beta$ and $(T, u, s) \equiv^{n-1} (B_k, x, z)$ then $z \geqslant y$, since $n-1 \geqslant 1$ and and $\neg s \leqslant u \wedge \forall w < u (w < s)$ holds and has quantifier rank 1.

The proof is finished by indicating how I can defeat II in the n-game between (T, u) and (B_k, x), contradicting (1).

If $m < 2^{n-1}$ then I, by picking the largest element s of β, wins the n-game: II has to answer with a maximal element $z \geqslant y$, whence there remain $m-1 < 2^{n-1}-1$ elements in $\{w < z \mid y \leqslant w\} = \{w < z \mid \neg w \leqslant x\}$ and I can defeat II in $n-1$ more moves by playing on β below s (use 1.0.1).

If $2^{n-1} \leqslant m$ then $2^{n-1} < |\beta|$ and I picks $s \in \beta$ such that $\{v \in \beta \mid s < v\}$ has $2^{n-1}-1$ elements. On penalty of losing (cf. 2.3.2) II must answer with a $z \geqslant y$ above which there are branches of length $\geqslant 2^{n-1}-1$. But then $\{w < z \mid y \leqslant w\}$ has $\leqslant m - 2^{n-1} < 2^n - 1 - 2^{n-1} = 2^{n-1}-1$ elements left and I needs only $n-1$ more moves on β below s to defeat II. ⊠

2.3.4 Proposition. *Suppose that* $k \geqslant 2^{n+1}-2$. *Let* T *be a binary tree such that* $T \equiv^{n+1} B_k$. *Then* T *satisfies* Q.2$(n+1)$.

Proof. Suppose that some branch α through T has length $m < 2^{n+1}-2$.

Since the quantifier rank-(n+1)-sentence

$$\forall x\,(\,\neg\varphi_n^{<x} \;\rightarrow\; \exists y\,(x<y)\,)$$

(φ_n defined in 2.3.2) holds in B_k (for, $2^n \leqslant 2^{n+1}-2$), α has an element t of height 2^n-2.

Now $\{s \in \alpha \mid t < s\}$ is a branch above t of length $m-(2^n-1) < 2^{n+1}-2-(2^n-1) = 2^n-1$, hence, by 2.3.3, *every* branch above t has length $m-(2^n-1)$.

But, the quantifier rank-(n+1)-sentence $\forall x\,(\,\varphi_n^{<x} \vee \varphi_n^{>x}\,)$ is satisfied in B_k; on the other hand, $x=t$ is a counter-example in T. ⊠

2.3.5 Proposition. *For each $m < 2^n-1$ there is a sentence of quantifier rank $\leqslant n$ which is satisfied in a tree iff all of its branches have length m.*

Proof. Left to the reader. ⊠

2.3.6 Corollary. *If $m<2^n-1$ and T is a binary tree such that $T \equiv^n B_m$ then $T \cong B_m$.*

For the next corollary, compare 1.0.1.

2.3.7 Corollary. $B_m \equiv^n B_k$ *iff $m=k$ or $m, k \geqslant 2^n-1$.*

Hence, summarizing,

2.3.8 Theorem. *The binary tree $(T, <)$ is n-equivalent with B_m iff either $m<2^n-1$ and $T \cong B_m$ or $m \geqslant 2^{n-1}$ and T satisfies $Q(n)$.*

2.4. Finiteness of binary trees is not $\Sigma^1{}_1$.

Corollary 2.2.4 says that *finiteness* of binary trees is not a first-order property. Obviously, it is (monadically) $\Pi^1{}_1$: by König's lemma, a binary tree is infinite iff it has a branch without endpoint. This section is devoted to a proof of the

2.4.1 Theorem. Finiteness *of binary trees is not monadically $\Sigma^1{}_1$-definable.*

To prove this, you need a simple

2.4.2 Lemma. *For all $k,n \in \mathbb{N}$ there are $p,q \in \mathbb{N}$ such that (i) $p<q$ and (ii) for all $X_1, \dots, X_k \subset B_p$ there are $Y_1, \dots, Y_k \subset B_q$ with $(B_p, X_1, \dots, X_k) \equiv^n (B_q, Y_1, \dots, Y_k)$.*

Proof. Let Σ be the (by 1.7.1, finite) set of n-characteristics of models $(T, <, X_1, \dots, X_k)$ where $(T, <)$ is any finite binary tree and define $h: \mathbb{N} \to \mathbf{P}(\Sigma)$ by $h(p) = \{ \tau \in \Sigma \mid B_p \models \exists X_1 \dots \exists X_k \tau \}$.

By the pigeon-hole principle, there are $\Delta \subset \Sigma$ and an infinite $A \subset \mathbb{N}$ such that each $p \in A$ has $hp = \Delta$. Now, choose $p<q$ in A. ⊠

Proof of 2.4.1. Suppose that the $\Sigma^1{}_1$-sentence $\exists X_1 \dots \exists X_k \tau$ defines *finiteness* of binary trees. Let n be the quantifier rank of τ. Let $p<q$ be given by 2.4.2. Choose $X_1, \dots, X_k \subset B_q$ such that $(B_q, X_1, \dots, X_k) \models \tau$.

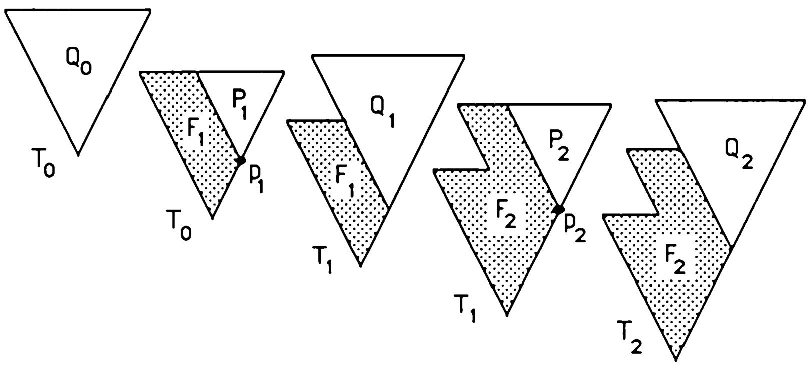
I shall construct an infinite n-equivalent of $(B_q, X_1, \dots, X_k)$, contradicting the assumption.

The construction involves a series of pairs $(T_0, Q_0), (T_1, Q_1), (T_2, Q_2), \dots$

where T_i is a finite binary tree (with k unary relations) and $Q_i \subset T_i$ is an upward closed subset of T_i, the ordering on which is an isomorph of B_q.

This is done as follows: first, $T_0 = (B_q, X_1, \ldots, X_k)$; $Q_0 = T_0$. If (T_i, Q_i) has been constructed, choose $p_{i+1} \in Q_i$ such that the tree on $P_{i+1} = \{ x \in T_i \mid p_{i+1} \leqslant x \}$ is an isomorph of B_p. By 2.4.2, choose an n-equivalent Q_{i+1} of P_{i+1}, the tree on which is an isomorph of B_q. T_{i+1} is obtained from T_i by replacing P_{i+1} by Q_{i+1}.

Put $F_{i+1} = T_i \backslash P_{i+1} = T_{i+1} \backslash Q_{i+1}$. Look at the picture to see how you get as far as T_2:



I shall need the following three simple facts about this construction.

Claim 1. If all elements of the sequence $\mathbf{a}$ belong to F_{i+1} then $(T_i, \mathbf{a}) \equiv^n (T_{i+1}, \mathbf{a})$.

Proof. T_{i+1} is obtained from T_i by exchanging P_{i+1} for Q_{i+1}. Now, $Q_{i+1} \equiv^n P_{i+1}$; let σ be a winning strategy for *II* in the n-game between these models. Consider the n-game between $(T_i, \mathbf{a})$ and $(T_{i+1}, \mathbf{a})$. *II* can win this by using σ when *I* plays in P_{i+1} or Q_{i+1} and copying *I* on F_{i+1}. ⊠

Claim 2. If $j < l$ and $\mathbf{a}$ belongs to F_j then $(T_j, \mathbf{a}) \equiv^n (T_l, \mathbf{a})$.

Proof. Induction on l using the argument of the proof of claim 1. ⊠

Claim 3. If $j \leqslant i$, $\mathbf{a}$ belongs to F_j and $(Q_j, \mathbf{a}) \equiv^{n-1} (T_i \setminus F_j, \mathbf{b})$ then

$$(T_j, \mathbf{a}, a) \equiv^{n-1} (T_i, \mathbf{a}, b).$$

Proof. T_i can be considered as obtained from T_j by exchanging Q_j for $T_i \setminus F_j$. Argue again as in the proof of claim 1. ⊠

To make the construction work, the selection of p_{i+1} in Q_i requires some care. To explain how this is done, let Σ be the (finite) set of $(n-1)$-characteristics of models $(A, <, Y_1, \ldots, Y_k, a)$ where $(A, <)$ is a finite tree. Choose an enumeration $f: \mathbb{N} \to \Sigma \times \mathbb{N}$ such that for all $(\sigma, j) \in \Sigma \times \mathbb{N}$ there is an $i \geqslant j$ with $f(i) = (\sigma, j)$. (This happens for instance if f assumes each value (σ, j) an infinite number of times.) We now require that

$[\neq]_i$ if $f(i) = (\sigma, j)$, $j \leqslant i$ and for some $a^1 \in T_i \setminus F_j$: $\sigma = [\![(T_i \setminus F_j, a^1)]\!]^{n-1}$, then p_{i+1} has been picked such that for some $a^2 \in T_i \setminus F_j$: $\sigma = [\![(T_i \setminus F_j, a^2)]\!]^{n-1}$ and $\neg a^2 \geqslant p_{i+1}$ – i.e., $a^2 \in F_{i+1}$.

(To see that this can be effected, act as follows. If $a^1 \in F_i$ or if $a^1 \in Q_i$ and $\{y \in Q_i \mid a^1 \leqslant y\}$ has height $> p$, then any choice of p_{i+1} will do. If $a^1 \in Q_i$ and $\{y \in Q_i \mid a^1 \leqslant y\}$ has height $\leqslant p$, pick $x < a^1$ in Q_i such that $\{y \mid x \leqslant y\}$ has height $p+1$. Let p_{i+1} be the immediate successor of x which is not $\leqslant a^1$.)

Now, let $T = \bigcup_i F_i$. Then T is an infinite binary tree. I claim that $T \equiv^n T_0$.

To see this, consider the Ehrenfeucht n-game on these models. For ***II*** to win, it suffices that he chooses his counter moves in such a way that after his m-th move a position $(\mathbf{a}, \mathbf{b})$ is reached such that, if the elements of $\mathbf{a}$ all belong to F_j, then

$$[*]_j \qquad (T_j, \mathbf{a}) \equiv^{n-m} (T_0, \mathbf{b}).$$

This obviously is correct for the initial position (0,0) of the game, since there, $j=0$.

Now, suppose that after m moves for each player a position $(\mathbf{a}, \mathbf{b})$ has been reached such that $[*]_j$ holds. Since the game is played between T and T_0, player I can choose his next move in either one of these models.

(i). Suppose that I plays a next move $a \in T$. Since the F_j are cumulative, we may assume, without restricting the generality of the argument, that $a \in F_l$ for some $l \geqslant j$. Now, by claim 2, $[*]_l$ holds as well. Therefore, II can find $b \in T_0$ such that $(T_l, \mathbf{a}, a) \equiv^{n-m-1} (T_0, \mathbf{b}, b)$ and $[*]_l$ has been secured.

(ii). Now, suppose I plays $b \in T_0$. By $[*]_j$, $a \in T_j$ exists with $(T_j, \mathbf{a}, a) \equiv^{n-m-1} (T_0, \mathbf{b}, b)$. If $a \in F_j$, then II can use a as his answer, obtaining $[*]_j$ for the new position. So, suppose that $a \in T_j \backslash F_j = Q_j$. Let $\sigma = [\![(Q_j, a)]\!]^{n-1}$. Pick $i \geqslant j$ with $f(i) = (\sigma, j)$.

By construction, $Q_j \equiv^n T_i \backslash F_j$; hence, $a^1 \in T_i \backslash F_j$ exists with $\sigma = [\![(T_i \backslash F_j, a^1)]\!]^{n-1}$. By $[\neq]_i$, there is an $a^2 \in F_{i+1}$ such that $\sigma = [\![(T_i \backslash F_j, a^2)]\!]^{n-1}$.

Now, notice that, by claim 1, $(T_{i+1}, \mathbf{a}, a^2) \equiv^n (T_i, \mathbf{a}, a^2)$ and, by claim 3, $(T_i, \mathbf{a}, a^2) \equiv^{n-1} (T_j, \mathbf{a}, a)$. Therefore, combining, $(T_{i+1}, \mathbf{a}, a^2) \equiv^{n-1} (T_j, \mathbf{a}, a)$ $\equiv^{n-m-1} (T_0, \mathbf{b}, b)$ (by choice of a above). Thus II, by playing a^2, establishes $[*]_{i+1}$ for the new position. ⊠

PART II: COMPLETENESS

We know all about the habits of the ant, we know all about the habits of the bee, but we know nothing at all about the habits of the oyster. It seems almost certain that we have been choosing the wrong time for studying the oyster. - *Pudd'nhead Wilson's Calendar*

3. Monadic $\Pi^1{}_1$-theories of $\Pi^1{}_1$-properties: linear orderings.

3.1 Introduction; ω and finite orderings.

Natural axioms of a number of theories are of the second-order ($\Pi^1{}_1$-) form $\forall R\,\varphi(R)$, where φ is first-order and R is a second-order variable. For instance, the *induction principle* of arithmetic, *completeness* of the reals, Zermelo's *Aussonderungsaxiom* and the Fraenkel-Skolem *replacement-axiom* in set theory are of this type.

As to *first-order versions* of these principles, the natural option is to require $\varphi(R)$ not for *all* R but for *parametrically first-order definable* R only, thus replacing the second-order axiom by its corresponding first-order *schema*.

Obviously, the new theory will have models not allowed by the old one (by the *Löwenheim-Skolem-Tarski theorem* for instance) and hence it may turn out to be strictly weaker than its second-order companion. For instance, second-order arithmetic is categorical, hence it implies first-order sentences beyond the scope of the first-order induction schema.

On the other hand, if the language is restricted sufficiently, conservation may occur. This chapter contains a number of examples. They all concern theories of linear orderings (but see chapter 4 below where one of our examples is generalized to trees); conservation is proved with respect to monadic $\Pi^1{}_1$-sentences. The method of proof consists in showing how to transfer counter-examples to a $\Pi^1{}_1$-sentence on a "non-standard" model to a standard model. To illustrate this method, I present the simplest case, the ordering of the natural numbers ω, in what follows.

It is clear what it means for an ordered set to *satisfy complete induction* when there is a least element and every element has an immediate successor. **Definable induction** requires that every *definable* set containing the least element and closed under immediate successors contains every element. *Complete induction* is the usual $\Pi^1{}_1$-instrument transforming a suitable set of first-order quantifier rank $\leqslant 3$ axioms into a categorical description of the order type ω. *Definable induction* doesn't come close to this (cf. example 1.0.3(ii)), but it suffices for the monadic $\Pi^1{}_1$-theory:

3.1.1 Theorem. *If 1.* $(M,<) \equiv^3 (\omega, <)$ *and*

2. $\mathbf{M} = (M, <, X_1, \dots, X_k)$ *satisfies definable induction*

then $\mathbf{M}$ *has n-equivalents of order type* ω *for every n.*

Proof. By the Löwenheim-Skolem theorem, we may assume M to be countable. Define $X = \{a \in M \mid \forall b < a\,([b, a)$ has a *finite* n-equivalent$)\}$. (Notice that a sub*set* is identified here with the sub*model* with that universe.) Now, X is a *definable* set: $[b, a)$ has a finite n-equivalent iff it satisfies an n-characteristic belonging to a finite model; and of these, there are only finitely many (see 1.7.1). Hence, X is defined by the formula $\forall y < x \bigvee \{\tau^{[y,x)} \mid \tau \in \Sigma\}$, where Σ is the set of such characteristics and $\tau^{[y,x)}$ denotes relativisation of quantifiers in τ to the interval $[y, x)$. Trivially, X contains the least element of M. Also, X is closed under immediate successors: if S is a finite n-equivalent of $[b, a)$ and c is the immediate successor of a then it is clear that the ordered sum $S+\{a\}$ is the required finite n-equivalent of $[b, c)$. By definable induction then, $X = M$. Let a_0 be the least element of M and choose $a_0 < a_1 < a_2 < \ldots$ cofinal in M (which we have assumed to be countable!). Choose a finite n-equivalent S_i of $[a_i, a_{i+1})$ for each i. Then $S = \Sigma_i S_i$ is the required n-equivalent of order type ω. (For the handling of ordered sums, cf. below, in particular 3.1.7.) ⊠

Virtually the same proof works for the class of *finite* ordered models.

Notice that a linear ordering $(M, <)$ is finite if it contains a least and a greatest element, every non-maximal element has an immediate successor and **restricted induction** is satisfied, which says that every set containing the least element and closed under immediate successors (insofar as they exist) contains the greatest element as well. (Of course, other characterisations work as well.) Restricted induction brings along its first-order companion: **definable restricted induction**.

3.1.2 Theorem. *If the linearly ordered model* $\boldsymbol{M}$:

1. has least and greatest element and every non-maximal element has an immediate successor;

2. satisfies definable restricted induction

then $\boldsymbol{M}$ *has finite n-equivalents for all n.*

Proof. Begin as in the proof of 3.1.1. Definable restricted induction now shows X to contain the greatest element b of $\boldsymbol{M}$. Thus, $[a, b)$ has a finite n-equivalent and so does $[a, b] = \boldsymbol{M}$, as required. ⊠

3.1.3 Examples. The following models show that we cannot strengthen the conclusions of 3.1.1-2 to: $\boldsymbol{M}$ has an *elementary* equivalent (i.e., a model n-equivalent with $\boldsymbol{M}$ for *all n simultaneously*) which has order-type ω (resp. which is finite). For the second, let $\boldsymbol{M}$ have order type $\omega+\omega^*$ and let the X_i be empty (use 1.0.3(i) to see that this satisfies definable restricted induction). For the first, consider $\boldsymbol{M}+\boldsymbol{N}$ where $\boldsymbol{M}$ is the previous model and $\boldsymbol{N}$ has order type ω - but $X_0=N$ this time (again, use 1.0.3(i)). The bigger n, the longer an n-equivalent of $\boldsymbol{M}$ has to be (namely, at least 2^n-1 by 1.0.1.) and hence the larger the first element of X_0 in an n-equivalent of $\boldsymbol{M}+\boldsymbol{N}$.

3.1.4 Remark. The direct method of proof of 3.1.1-2 works for the class of well-ordered models too; instead, I shall derive that result from the corresponding one for the order-complete models in section 3.3.

In the following basic theorem, Σ is a set of first-order sentences in a language L and $\forall R\varphi(R)$ is a $\Pi^1{}_1$-sentence over L.

Let $X_1,\ldots,X_k$ be new unary relation-symbols and $L_k = L\cup\{X_1,\ldots,X_k\}$.

(L_k-) **definably-**φ is, by definition, the set of universal closures of L_k-formulas obtained from φ by replacing each occurrence $R(t_1, \dots, t_m)$ by some fixed L_k-formula $\eta(t_1, \dots, t_m)$ (taking measures against clash of variables). Thus, L_k-definably-φ intuitively requires $\varphi(R)$ only when R is (parametrically) first-order definable in the language L_k.

The union over all k of these schemata is the **first-order schema corresponding to** $\forall R\, \varphi(R)$.

3.1.5 Theorem. *The following two conditions are equivalent:*

(i) *for each first-order formula* $\psi = \psi(X_1, \dots, X_k)$ *in the language* L_k*: if* $\Sigma + \forall R\, \varphi(R) \models \forall X_1 \dots \forall X_k\, \psi$ *, then* $\Sigma + L_k$*-definably-*$\varphi \models \psi(X_1, \dots, X_k)$ *;*

(ii) *each model* $(M, U_1, \dots, U_k)$ *of* $\Sigma + L_k$*-definably-*φ *has an* n*-equivalent satisfying* $\Sigma + \forall R\, \varphi(R)$ *for each* n*.*

Proof. (i) $\Rightarrow$ (ii): let $(M, U_1, \dots, U_k) \models \Sigma + L_k$-definably-$\varphi$ have the n-characteristic $[\![\emptyset]\!]^n = \tau(X_1, \dots, X_k)$. We want a model of $\Sigma + \forall R\, \varphi(R) + \exists X_1 \dots \exists X_k \tau$. If such a model does not exist, then $\Sigma + \forall R \varphi(R) \models \forall X_1 \dots \forall X_k \neg\tau$; hence by (i) $\Sigma + L_k$-definably-$\varphi \models \neg\tau(X_1, \dots, X_k)$, contradicting the assumptions on $(M, U_1, \dots, U_k)$.

(ii) $\Rightarrow$ (i): assume $\Sigma + \forall R\, \varphi(R) \models \forall X_1 \dots \forall X_k\, \psi$ and let $(M, U_1, \dots, U_k)$ be a model of $\Sigma + L_k$-definably-φ. By (ii), there is an n-equivalent satisfying $\forall R \varphi(R) + \Sigma$ where we take n to be the quantifier rank of ψ. By assumption, this n-equivalent satisfies $\forall X_1 \dots \forall X_k\, \psi$, hence $\psi(X_1, \dots, X_k)$ is also satisfied, so $(M, U_1, \dots, U_k)$ must satisfy this formula as well. ⊠

Below, the Σ of the theorem will always be finite. Therefore we may require that the n-equivalent of (ii) satisfies $\forall R \varphi(R)$ only, without invalidating the truth of (ii) $\Rightarrow$ (i): simply let n be at least the maximum of quantifier ranks of formulas in Σ.

In what follows, results of type (ii) are proved. According to the theorem, this shows that, in the context of Σ, *the first-order schema corresponding to* $\forall R\varphi(R)$ *suffices to prove all monadic* $\Pi^1{}_1$*-consequences of this second-order statement*. (Actually, theorem 3.3.9 below does a little better.)

All models encountered here will have the form $\mathbf{M} = (M, <, U_1, \ldots, U_k)$, where $<$ linearly orders M and $U_1, \ldots, U_k \subset M$. If $X \subset M$, then $\mathbf{X}$ (and sometimes X as well) denotes the submodel of $\mathbf{M}$ with universe X, i.e., $\mathbf{X} = \mathbf{M}|X$. $I \subset M$ is an **interval** if $x<y<z$ and $x, z \in I$ imply $y \in I$; notations like (x, z) and $[x, z)$ denote specific intervals as usual.

If $\mathbf{I}$ is an ordered set and $\mathbf{m}$ a function on I associating a model $\mathbf{m}(i)$ with every $i \in I$, we may form the **ordered sum** $\Sigma_{i\in I}\, \mathbf{m}(i)$, being the model obtained from the $\mathbf{m}(i)$ by glueing (disjoint copies of) them one after the other according to the ordering of I. Formally, $\Sigma_{i\in I}\, \mathbf{m}(i)$ can be defined as the model with universe $\bigcup_{i\in I}(m(i) \times \{i\})$ with the ordering defined by: $(a, i) < (b, j)$ iff $i <_I j$, or: $i = j$ and $a <_i b$ (here, $<_i$ and $<_I$ denote the orderings of $\mathbf{m}(i)$ resp. $\mathbf{I}$); and if U_n^i is the n-th unary relation of $\mathbf{m}(i)$ $(1 \leqslant n \leqslant k)$ then $U_n = \bigcup_{i\in I} (U_n^i \times \{i\})$ is the corresponding one of the ordered sum.

A **condensation** of an ordered model $\mathbf{M}$ is a partition of M into intervals. Any condensation P of $\mathbf{M}$ inherits an ordering from $\mathbf{M}$ by putting, for $p, q \in P$: $p<q$ iff for some $a \in p$ and $b \in q$ (equivalently, iff for *all* $a \in p$ and $b \in q$): $a<b$. Hence, a condensation P of $\mathbf{M}$ is nothing but a way to write $\mathbf{M}$ as an ordered sum $\mathbf{M} = \Sigma_{p\in P}\, \mathbf{p}$.

If the condensation P is induced by the equivalence $\sim$, (such equivalences are called *congruences* by some) we write $P = M/\sim$.

3.1.6 Lemma. *Let R be any transitive binary relation on the ordered model M. Define $\sim\ =\ \sim_R$ by: $a \sim b$ iff one of the following holds:*

(i) $a = b$

(ii) $a < b$ *and for all c, d such that* $a \leqslant c < d \leqslant b$: cRd

(iii) $b < a$ *and for all c, d such that* $b \leqslant c < d \leqslant a$: cRd.

Then $\sim$ induces a condensation.

Proof. Straightforward. ⊠

All condensations used in the sequel are defined in this fashion.

3.1.7 Lemma. *If for all $i \in I$ $\mathbf{m}(i) \equiv^n \mathbf{m}'(i)$, then $\Sigma_{i \in I}\mathbf{m}(i) \equiv^n \Sigma_{i \in I}\mathbf{m}'(i)$.*

Proof. It is straightforward to describe a winning strategy for the second player in the Ehrenfeucht n-game between these sums under the condition given. ⊠

The following generalisation of 3.1.7 is needed in section 3.3.

3.1.8 Lemma. *Suppose that $\mathbf{I}$ and $\mathbf{J}$ are ordered sets and that $\mathbf{m}$ and $\mathbf{m}'$ associate ordered models $\mathbf{m}(i)$ resp. $\mathbf{m}'(j)$ to each $i \in I$ resp. $j \in J$ such that:*

(*) $(I, \{\, i \mid \mathbf{m}(i) \models \sigma \,\})_{\sigma \in \Sigma} \equiv^n (J, \{\, j \mid \mathbf{m}'(j) \models \sigma \,\})_{\sigma \in \Sigma}$

where Σ is the set of n-characteristics.

Then $\Sigma_{i \in I}\, \mathbf{m}(i) \equiv^n \Sigma_{j \in J}\, \mathbf{m}'(j)$.

Proof. Use the Ehrenfeucht game-technique. If the first player chooses, say, $a \in \Sigma_i m(i)$, the second player locates the $i \in I$ for which $a \in m(i)$, then uses (*) to find a j corresponding to i; in particular, $\mathbf{m}'(j) \equiv^n \mathbf{m}(i)$, and a counter move is readily found; etc. ⊠

3.1.9 Examples where 3.1.5(ii) fails.

1. (van Benthem.) Consider the $\Pi^1{}_1$-statement $\forall X\varphi(X)$ in the language of $<$ where $\varphi(X)$ says: X and its complement cannot both be cofinal. Obviously, every ordered model of $\forall X\varphi(X)$ has a greatest element. On the other hand, the first-order schema corresponding to φ does not imply this. A counter-model is $(\omega, <)$: notice that each *definable* set here is either *finite* or *cofinite*. (*Proof:* Use 1.0.3(ii). Let $\psi(x)$ be any formula in the free variable x. If *no* $a \in \zeta$ satisfies ψ in $\omega+\zeta$ then $\exists y \forall x(y < x \rightarrow \neg\psi)$ holds in $\omega+\zeta$; therefore it holds in ω by 1.0.3(ii) and the set defined by ψ in ω must be finite. On the other hand, If *some* $a \in \zeta$ satisfies ψ in $\omega+\zeta$ then *every* $a \in \zeta$ satisfies ψ in $\omega+\zeta$ - this is because for each pair $a, b \in \zeta$ there is an automorphism h of $\omega+\zeta$ such that $ha=b$. Hence, $\exists y \forall x(y < x \rightarrow \psi)$ holds in $\omega+\zeta$; therefore, it holds in ω and the set defined by ψ in ω must be cofinite. ⊠)

2. In theories defining a pairing, the restriction to *monadic* languages is only apparent and results like ours can fail badly. We mentioned the case of arithmetic; also, each model of set theory certainly is *definably well-founded*, nevertheless such models need not have a well-founded n-equivalent for n large enough: well-founded models have standard integers, therefore they are arithmetically correct; but, Gödel-sentences *are* arithmetical.

3.1.10 Question. (van Benthem) Suppose that $M \equiv (V_\omega, \in)$ and $(M, U_1, \ldots, U_k)$ satisfies *definable well-foundedness* (cf. chapter 4). Must it have a well-founded n-equivalent for each n? (The method of chapter 4 does not suffice; the objection of 3.1.9.2 does not apply.)

3.2 Monadic Π^1_1-theory of scattered orderings.

A linear ordering $M=(M, <)$ is called **scattered** if it does not embed the ordering $(\mathbb{Q},<)$ of the rationals.

$\mathbb{Q}$ embeds *every* countable ordering; in particular, it embeds ω^*. It follows that every well-ordering is scattered. I shall need the

3.2.1 Lemma. *A scattered ordered sum of scattered orderings is scattered.*

Proof. Suppose $\mathbb{Q} \subset \Sigma_{i \in I} m(i)$. If some $\mathbb{Q} \cap m(i)$ contains at least *two* rationals, it contains the interval between them and so $m(i)$ cannot be scattered. Hence, sending $p \in \mathbb{Q}$ to the $i \in I$ for which $p \in m(i)$ embeds $\mathbb{Q}$ in I, a contradiction. ⊠

There is more than one way to formalize scatteredness into a Π^1_1-statement and not every formalisation is a good one from our point of view.

3.2.2 Example. Let $\eth$ express that $<$ is a dense ordering containing at least two elements. $\varphi(X)$ is the formula obtained from $\neg\eth$ by relativizing quantifiers to membership in (the set) X. Clearly, $\mathcal{M}$ is scattered iff it satisfies $\forall X\varphi(X)$. Here is an example of a model $\mathcal{M}=(M, <, X, Y, Z)$ which is *definably*-φ but has *no* scattered 3-equivalent. Partition $\mathbb{Q}$ into dense subsets R, S, T and put $\mathcal{M}=\Sigma_{q\in\mathbb{Q}} M_q$, where $M_q=(\mathbb{Z}, <, X^q, Y^q, Z^q)$ and $X^q=\mathbb{Z}$ if $q\in R$; X^q is empty otherwise; similarly, $Y^q=\mathbb{Z}$ or $=\emptyset$ depending on whether $q\in S$ or not; and $Z^q=\mathbb{Z}$ or $=\emptyset$ depending on whether $q\in T$. Notice that each interval M_q of $\mathcal{M}$ is a set of indiscernibles of $\mathcal{M}$ (use automorphisms of M_q) hence, if A is a *definable* set of $\mathcal{M}$, either $A\cap M_q=\emptyset$ or $M_q\subset A$. Therefore, no non-empty definable set of $\mathcal{M}$ is densely ordered and it follows that $\mathcal{M}$ is *definably*-φ. On the other hand, the fact that $\mathcal{M}$ satisfies sentences such as $\forall x\in X\forall y\in Y(x<y \rightarrow \exists z\in Z(x<z<y))$ shows that no 3-equivalent of $\mathcal{M}$ can be scattered. ⊠

A "good" formalisation of scatteredness should avoid this counter-example.

3.2.3 Lemma. *An ordering is scattered iff it has no densely ordered condensation.*

Proof. Only if: use the axiom of choice. *If*: suppose that $\mathbb{Q}\subset M$. Define $\sim$ by way of 3.1.6 where aRb iff $a<b$ and $(a, b)\cap\mathbb{Q}$ is finite. It is easy to see that $\sim$ induces a dense condensation. ⊠

The (dyadic!) Π^1_1-characterisation of scatteredness contained in the lemma is a "good" one according to the following theorem, where we call a model **definably scattered** if no *definable* equivalence partitions $\mathcal{M}$ into a dense ordering of intervals. Notice that the model of 3.2.2 is *not* definably scattered in this sense.

3.2.4 Theorem. *If M is definably scattered, then it has scattered n-equivalents for each n.*

Proof. I use what Rosenstein [1982] calls a **condensation-argument**, originating with Hausdorff. Define $\sim$ in the fashion of 3.1.6 with aRb meaning that (a, b) has a scattered n-equivalent (if $a<b$). By 3.2.1, R is transitive. Hence, $\sim$ induces a condensation by 3.1.6.

Also, $\sim$ is *definable* (compare the proof of 3.1.1): there are only *finitely many* n-characteristics; let Γ be the (finite) set of n-characteristics belonging to scattered models. Then (c, d) has a scattered n-equivalent iff $M \models \bigvee_{\tau\in\Gamma} \tau^{(c,d)}$ - where $\tau^{(c,d)}$ is obtained from τ by relativizing quantifiers to membership in (c, d). It is now clear that $\sim$ can be defined as well.

Claim 1: *each equivalence class has a scattered n-equivalent.*
Proof: let I be an equivalence class and $a\in I$.

(i) I has a greatest element b. Then $a\sim b$ and $I^{\geqslant a} = \{x\in I \mid a\leqslant x\} = [a, b]$ has a scattered n-equivalent by definition.

(ii) If not, choose a sequence $a_0 = a < a_1 < \ldots < a_\xi < \ldots (\xi<\alpha)$ cofinal in I. Each $(a_\xi, a_{\xi+1})$ and, hence, each $[a_\xi, a_{\xi+1})$ has a scattered n-equivalent A_ξ. Hence $I^{\geqslant a} = \Sigma_{\xi<\alpha} [a_\xi, a_{\xi+1})$ has the n-equivalent $\Sigma_{\xi<\alpha} A_\xi$ by 3.1.7 which, by 3.2.1, is scattered.

Argue similarly for $I^{<a} = \{x\in I \mid x<a\}$; so, $I = I^{<a} + I^{\geqslant a}$ has a scattered n-equivalent. ⊠

Claim 2: *the induced ordering of the equivalence classes is dense.*
Proof: suppose $I<J$ are equivalence classes and no equivalence class is between I and J. Let $a\in I$ and $b\in J$; suppose that $a\leqslant c<d\leqslant b$. Then (c, d) has

a scattered n-equivalent: if $c,d \in I$ or $c,d \in J$ this is clear; and if $c \in I$ and $d \in J$ we know from the argument above that $I^{>c}$ and $J^{<d}$ have scattered n-equivalents; but, $(c, d) = I^{>c} + J^{<d}$. Therefore, $a \sim b$, a contradiction. ⊠

Since M is definably scattered, $\sim$ cannot have more than *one* equivalence class: M itself. Consequently, M must have a scattered n-equivalent by the first claim. ⊠

3.2.5 Remark. By 3.2.2 and 3.2.3, we have two Π^1_1-formalisations of scatteredness; however, the first-order schema corresponding to the second one (definable scatteredness) is strictly stronger than the first-order schema belonging to the first.

3.3. Monadic Π^1_1-theory of complete orderings, of well-orderings and of the reals.

The ordering $(M, <)$ is **complete** if each non-empty set with an upper bound has a *least* upper bound (a *sup*). Hence, M is called **definably complete** if this holds for *definable* sets.

3.3.1 Theorem. *If M is definably complete, it has complete n-equivalents for each n.*

Before proving 3.3.1, here is an example and a corollary.

3.3.2 Example. The following model shows it is impossible to strengthen the conclusion of 3.3.1 to requiring an elementary equivalent of M.

Choose rationals $q_0 < q_1 < q_2 < \ldots$ and $r_0 > r_1 > r_2 > \ldots$ such that $\lim q_i = \lim r_i$ is *irrational*; take $A = \{q_i \mid i \in \mathbb{N}\} \cup \{r_i \mid i \in \mathbb{N}\}$ and consider $M = (\mathbb{Q}, <, A)$. For each n, the models $(\mathbb{R}, <, \{1, \ldots, m\})$ for $m \geqslant 2^n - 1$ are n-equivalents of M (use 1.0.1). On the other hand, suppose that $(N, <, B)$ is a complete elementary equivalent of M. It follows that B has order-type $\omega + \alpha$ for some α. N must contain a sup of the first ω elements of B. However, M lacks an element which is a limit of A's - a contradiction. ⊠

M is **(definably) well-ordered** if each non-empty subset of M (which is parametrically first-order definable on M) has a least element.

The following trivial lemma may look surprising, as *completeness* usually is considered only in the context of *dense* orderings.

3.3.3 Lemma. *M is (definably) well-ordered iff it is (definably) complete, has a least element, and every non-maximal element has an immediate successor.*

Proof. Suppose $\emptyset \neq X \subset M$ and X has no minimum. Put $Y = \{y \in M \mid \forall x \in X (y < x)\}$. Y is definable if X is definable. Since the least element of M must be in Y, Y is non-empty; moreover, every $x \in X$ is an upper bound of Y. Thus, Y has a *sup* y. If $y \in Y$, the immediate successor of y is minimal in X. Hence, $y \notin Y$. But then, y must be minimal in X, a contradiction. ⊠

3.3.4 Corollary. *If M is definably well-ordered, it has well-ordered n-equivalents for each n.*

Proof. Notice that 3.3.3 defines *well-order* as *completeness* plus a quantifier rank 3-statement. By 3.3.3, M is definably complete. Thus, let $m=\max(n, 3)$ and take N to be a complete m-equivalent of M by 3.3.1. By 3.3.3 again, N is the model required. ⊠

We say that the sum $\Sigma_{i\in I}\, m(i)$ is **completely ordered** if the ordering of I is complete.

3.3.5 Lemma. *(i) Completely ordered sums of complete orderings with endpoints are complete.*

(ii) Well-ordered sums of complete orderings with least elements are complete.

Proof. (i): Let $X\subset \Sigma_{i\in I} m(i)$ have an upper bound in $m(i_0)$. Then $J=\{\, j \mid X\cap m(j)\neq\emptyset\,\}$ has the upper bound i_0. Let $j=\sup J$. (a) $j\in J$. Then $\max m(j)$ is an upper bound for $X\cap m(j)$ and $\sup X=\sup(X\cap m(j))$. (b) $j\notin J$. Then $\sup X=\min m(j)$. (ii): similar. ⊠

Proof of 3.3.1. Define $\sim$ in the fashion of 3.1.6 with aRb meaning: $a<b$ and (a, b) has a complete n-equivalent.

Notice that R is transitive. Hence, $\sim$ induces a condensation by 3.1.6. (N.B. this would not have been so obvious in case we would have defined $x\sim y$ to mean that (x, y) had a complete n-equivalent only.)

Furthermore, $\sim$ is definable: compare the proof of 3.2.4. Hence, the equivalence classes are definable as well.

Claim 1: *each equivalence class with an upper (lower) bound has a greatest (resp. least) element and each equivalence class has a* *complete n-equivalent.*

Proof: let I be an equivalence class and $a \in I$.

If I has no upper bound, choose $a_0 = a < a_1 < a_2 < \ldots < a_\xi < \ldots (\xi < \alpha)$ cofinal in I. Choose a complete n-equivalent N_ξ of $[a_\xi, a_{\xi+1})$ for each $\xi < \alpha$. Then $\Sigma_{\xi<\alpha} N_\xi$ is a complete n-equivalent of $\Sigma_\xi [a_\xi, a_{\xi+1}) = \{x \in I \mid a \leqslant x\} = I^{\geqslant a}$. If I has an upper bound, it must have a *sup* s by definable completeness. I claim that $s \in I$ (and so, s is the maximum of I, $I^{\geqslant a} = [a, s]$ and hence (a, s) and, therefore, $I^{\geqslant a}$ as well, have a complete n-equivalent by definition). For if not, choose $a_0 = a < a_1 < a_2 < \ldots < a_\xi < \ldots$ cofinal in I again to show that (a, s) has a complete n-equivalent as before.

Much the same goes for the other half $I^{<a} = \{x \in I \mid x < a\}$ of I, and so the claim has been proved. ⊠

Claim 2: *the induced ordering on the class $M/\sim$ of equivalence classes is dense.*

Proof: suppose $I < J$ are neighbours in $M/\sim$.
Then $a = \sup I$ and $b = \inf J$ are neighbours in M; moreover, $a \in I$ and $b \in J$. Hence, (a, b) is empty; therefore, $a \sim b$ - a contradiction. ⊠

If there is but *one* equivalence class, we are done. So, assume not. The rest of the proof works towards a contradiction.

The following argument is taken from Rosenstein [1982] (thm.7.17, p.117). Choose a complete n-equivalent $\tau(I)$ for each I with $I \in M/\sim$ in such a way that $T = \{\tau(I) \mid I \in M/\sim\}$ is finite (this is possible by 1.7.1). Now, if for *some* $\sigma \in T$, $\{I \in M/\sim \mid \tau(I) = \sigma\}$ is *not* dense in the ordering of $M/\sim$, there must be a proper interval $C_0 \subset M/\sim$ such that no $I \in C_0$ has $\tau(I) = \sigma$. Repeating this argument (first with C_0 and $T \setminus \{\sigma\}$ etc.) using induction on the finite cardinal $|T|$, one ultimately arrives at the following

Claim 3: *there is a proper (open) interval D of $M/\sim$ and a set $\Sigma \subset T$ such that (i) every $I \in D$ has $\tau(I) \in \Sigma$, and (ii) if $\sigma \in \Sigma$ then $\{ I \in D \mid \tau(I) = \sigma \}$ is dense in D.*

The contradiction aimed for is contained in the next

Claim 4: *D has but one element.*

Proof: suppose that $a, b \in \bigcup D$ and $a < b$. We need to show that (a, b) has a complete n-equivalent. Suppose that $a \in I$, $b \in J$. If $I = J$, there is nothing to prove. Let E be the interval (I, J) in D. Now, $(a, b) = I^{>a} + \bigcup E + J^{<b}$; therefore it suffices to show that these components have complete n-equivalents. For $I^{>a}$ and $J^{<b}$, this is known already (cf. the proof of claim 1). Therefore, it remains to show that $\bigcup E$ has such an n-equivalent as well.

First, notice that claim 3 remains valid if we replace D by E.

Now, construct a complete n-equivalent $\boldsymbol{N}$ of the submodel $\bigcup E = \Sigma_{I \in E} I$ of $\boldsymbol{M}$ as follows: let $h: \mathbb{R} \to \Sigma$ be any partition of $\mathbb{R}$ into $|\Sigma|$ classes $\{ x \in \mathbb{R} \mid h(x) = \sigma \}$ $(\sigma \in \Sigma)$ each of which is dense in $\mathbb{R}$ and put $\boldsymbol{N} = \Sigma_{x \in \mathbb{R}} h(x)$. By 3.3.5 and claim 1, $\boldsymbol{N}$ is completely ordered. It remains to show that $\boldsymbol{N}$ is n-equivalent to $\bigcup E$.

First, notice that the models $(E, <, \{ I \in E \mid \tau(I) = \sigma \})_{\sigma \in \Sigma}$ and $(\mathbb{R}, <, \{ x \in \mathbb{R} \mid h(x) = \sigma \})_{\sigma \in \Sigma}$ (with $|\Sigma|$ unary relations each) are partially isomorphic and a fortiori n-equivalent. (The argument for dense orderings is well-known; the extra structure involved here - partitions into $|\Sigma|$-many dense sets - doesn't complicate it terribly much.) The result now follows from 3.1.8. ⊠

This finishes the proof of 3.3.1. ⊠

The most prominent type of (dense) complete ordering is λ, the order type of the set of reals. The following example shows that we cannot strengthen the conclusion of 3.3.1 by requiring the n-equivalent to be of type λ under the assumption that the ordering of M is dense.

3.3.6 Example. For $x \in \mathbb{R}$, let $m(x) = ([0,1], <, \emptyset)$ if x is rational, and $m(x) = ([0,1], <, [0,1])$ otherwise. Consider $M = \Sigma_{x \in \mathbb{R}} m(x)$.

M has the complete order type $(1+\lambda+1)\cdot\lambda$ (cf. 3.3.5) - so it certainly is definably complete. On the other hand, the proof of lemma 3.3.8 below shows that it lacks a 5-equivalent of order-type λ: each complete 5-equivalent of M has a definable equivalence splitting the model in an uncountable number of proper intervals - contradicting the *Suslin property* of $\mathbb{R}$. ⊠

Hence, the Suslin property of $\mathbb{R}$ *contributes* to its monadic $\Pi^1{}_1$-theory. The following definition, suggested by 3.3.6, isolates this contribution:

3.3.7 Definition. M has **property I** if each densely ordered condensation of M has a dense set of singletons.

3.3.8 Lemma. *Models of order type* λ *and, more generally, all complete orderings with the Suslin property* have *property* I.

Proof. Suppose that P is a densely ordered condensation of a Suslin ordering. Suppose that $p<q$ in P but (p,q) does not contain a singleton. By Suslinity, (p,q) must be countable; hence, it has the order type of the rationals. Therefore, (p,q) has as many bounded sets without *sup* as there are irrationals. Let K be such a set. Then $\bigcup K$ is a bounded set in the

original ordering without *sup*. ⊠

3.3.9 Theorem. *If M is definably-I, definably complete and densely ordered without endpoints, then it has n-equivalents of order type λ for each n.*

Proof. First, we follow the proof of 3.3.1 with some slight modifications. To begin with, we may assume by the Löwenheim-Skolem theorem that M is only countable.
Now, define $\sim$ by the scheme of 3.1.6 with aRb meaning: $a<b$ and (a,b) has an n-equivalent of order type λ. Again, R is transitive, so $\sim$ induces a condensation by 3.1.6.

Claim 1: *each equivalence class has an n-equivalent of one of the following types:* 1, $\lambda+1$ *(if it begins M),* $1+\lambda$ *(if it ends M),* λ *(if it does both) or* $1+\lambda+1$.
Proof: much as before. Notice that we need to form *countable* sums only since M is countable, thus preserving separability of the models involved - thereby guaranteeing one of the order types required. ⊠

Claim 2: *$M/\sim$ is densely ordered.*
Proof: use that the ordering of M is dense! ⊠

Again, it suffices to show that M is the only class in $M/\sim$. Suppose it is not.

Claim 3: *there is a proper (open) interval D of $M/\sim$ and a finite set Σ of models of order type either* 1 *or* $1+\lambda+1$ *such that*

(i) every $I \in D$ has an n-equivalent in Σ; and

(ii) if $\sigma \in \Sigma$ then $\{ I \in D \mid I \equiv^n \sigma \}$ is dense in D.

Proof: as before. ⊠

In order to reach the desired contradiction, and stepping over some obvious details (cf. the proof of 3.3.1, claim 4), construct an n-equivalent $\boldsymbol{N}$ of $\bigcup D$ of order type λ as follows:

Since M is definably-I, $M/\sim$ has a dense set of singletons; hence Σ contains a singleton model τ_0. Take $h: \mathbb{R} \to \Sigma$ partitioning $\mathbb{R}$ into $|\Sigma|$ classes $\{ x \in \mathbb{R} \mid h(x) = \sigma \}$ $(\sigma \in \Sigma)$ each of which is dense and such that $\{ x \in \mathbb{R} \mid h(x) = \tau_0 \}$ happens to be the set of *irrationals*.

Put $\boldsymbol{N} = \Sigma_{x \in \mathbb{R}} h(x)$.

By 3.3.5, $\boldsymbol{N}$ is complete as before and it is easy to see that $\boldsymbol{N}$ has a countable dense set this time - whence $\boldsymbol{N}$ has the order type λ.

That $\boldsymbol{N} \equiv^n \bigcup D$ follows as before, using 3.1.8. ⊠

3.3.10 Corollary. *Every ordering which has* I, *is complete and is densely ordered without endpoints satisfies the monadic* $\Pi^1{}_1$*-theory of* $\mathbb{R}$.

3.3.11 Example. For each ordinal α, $\lambda+(1+\lambda)\cdot\alpha$ has the required properties (and differs from λ for $\alpha \geqslant \omega_1$, since ω_1 is not $\leqslant \lambda$).

3.3.12 Remark. Kunen [1968] §17 contains a $\Pi^1{}_1$-characterization of $(\mathbb{R}, <)$.

3.4. Appendix: strengthening 3.2.4 and 3.3.4.

Let M_0 be the smallest class of order types such that

(1) $1 \in M_0$

(2) $\alpha, \beta \in M_0 \Rightarrow \alpha+\beta \in M_0$

(3) $\alpha \in M_0 \Rightarrow \alpha\cdot\omega, \alpha\cdot\omega^* \in M_0$.

By 3.2.1, all types in M_0 are scattered.

By a theorem of Laüchli and Leonard (cf. Rosenstein [1982] thm.7.9 p.115) M_0 contains *n*-equivalents for each scattered ordering and for all *n*. (Shelah [1975] notes that almost the same proof shows this to be the case relative to the monadic second-order language as well.)

Their method of proof shows that the extra unary relations of our models do not spoil this situation:

3.2.4' Theorem. *If M is definably scattered, it has (scattered) n-equivalents with order type in* M_0 *for each n.*

Proof. On M, define $\sim$ by way of 3.1.6 with aRb meaning: $a<b$ and (a,b) has an *n*-equivalent with order type in M_0. By (1), (2), R is transitive, so $\sim$ induces a condensation.

Claim 1: *each equivalence class has an n-equivalent with order type in* M_0.

Proof: if the class I is unbounded, choose $a<a_0<a_1<a_2<\ldots$ cofinal in I (by Löwenheim-Skolem, assume M is countable). For $i<j$, let $h(i,j)$ be the *n*-characteristic of $[a_i, a_j)$. By *Ramsey's theorem,* there is an infinite set $A\subset\mathbb{N}$ and a σ such that if $i<j$ and $i, j\in A$ then $h(i,j)=\sigma$. Let $i=\min A$ and

choose $N_0 \equiv^n [a, a_i)$ and $N \models \sigma$ with order types in M_0. Then $I^{\geqslant a} \equiv^n N_0 + N \cdot \omega$ and this model has order type in M_0 by (2), (3).
The same goes for $I^{<a}$, etc. ⊠

As before, $M/\sim$ must either be dense or consist of one class only; since the first alternative cannot obtain, the proof is finished. ⊠

Next, let K be the smallest class of order types such that

(1) $1 \in \mathsf{K}$

(2) $\alpha, \beta \in \mathsf{K} \Rightarrow \alpha+\beta \in \mathsf{K}$

(3) $\alpha \in \mathsf{K} \Rightarrow \alpha \cdot \omega \in \mathsf{K}$.

Clearly, $\mathsf{K} \subset \mathsf{M}_0$. All types in K are well-ordered and it is easy to see (using Cantor normal forms) that $\alpha \in \mathsf{K}$ iff $0 < \alpha < \omega^\omega$.

K contains n-equivalents for each well-ordering and for all n (it is easy to see that no smaller class has this property).

Again, extra unary relations do not change this state of affairs:

3.3.4' Theorem. *If M is definably well-ordered, it has (well-ordered) n-equivalents with order-type in* K *for each n.*

Proof. Define $X = \{a \mid \forall b < a [b, a)$ has an n-equivalent with order type in $\mathsf{K}\}$. X is definable, hence if $X \neq M$ then $M \setminus X$ has a least element a. Pick $b < a$ such that $[b, a)$ has no n-equivalent with type in K. By (1), (2), a cannot be a successor. Now, choose $b < b_0 < b_1 < \ldots$ cofinal in $[b, a)$ and argue as in the previous proof. ⊠

3.4.1 Corollary. (Ehrenfeucht - cf. Rosenstein [1982] thm.6.22 p.108)

$\omega^\omega \equiv (\mathrm{OR}, <)$ *(where* OR *is the class of all ordinals).*

Proof. If ***I*** starts an $(n+1)$-game with a move $\alpha \in \mathrm{OR}$, ***II*** answers with an n-equivalent β in ω^ω. (Notice that $\alpha\uparrow \cong \mathrm{OR}$ and $\beta\uparrow \cong \omega^\omega$.) ⊠

The classes $\mathbf{M}_0$ and **K** were inductively defined by closure properties obtained by looking at what it takes to prove 3.2.4 and 3.3.4. In the same way, one may find, e.g., a class **C** of order types such that each completely and densely ordered model without endpoints has n-equivalents with types in **C**; closure properties needed here are

(1) $\lambda \in \mathbf{C}$

(2) $\alpha, \beta \in \mathbf{C} \Rightarrow \alpha + 1 + \beta \in \mathbf{C}$

(3) $\alpha \in \mathbf{C} \Rightarrow (\alpha+1)\cdot\omega, (1+\alpha)\cdot\omega^* \in \mathbf{C}$

(4) if $h\colon \mathbb{R} \to \mathbf{C}$ has $h[\mathbb{R}]$ finite and all $\{x \in \mathbb{R} \mid h(x) = \sigma\}$ $(\sigma \in h[\mathbb{R}])$ dense in $\mathbb{R}$ then $\Sigma_{x\in\mathbb{R}} (1 + h(x) + 1) \in \mathbf{C}$.

4. Monadic Π^1_1-theory of well-founded trees.

The previous chapter dealt with linearly ordered models only; the scope is widened here somewhat to the notion of a *tree*.

A partially ordered set $M=(M, <)$ is called a **tree** iff, for each $m \in M$, the set $m{\downarrow} = \{ m' \in M \mid m' < m \}$ is linearly ordered.

The Π^1_1-property considered here is *well-foundedness*: M is **well-founded** iff each non-empty set has a minimal element; equivalently, when M is a tree: iff each $m{\downarrow}$ is well-ordered.

Definable well-foundedness, of course, restricts this to *definable* sets.

The proof of theorem 4.1 below can be considered as a paradigm for a method applicable in a variety of situations where the models considered belong to certain types of partial orderings (trees being the simplest example) and the Π^1_1-property involved can be either well-foundedness, *converse* well-foundedness, or, more generally, some kind of completeness as in section 3.3. It did not seem useful however to aim for this greater generality here, as a most general result probably does not exist and the generalizations obtained of the result below all appeared to be rather arbitrary.

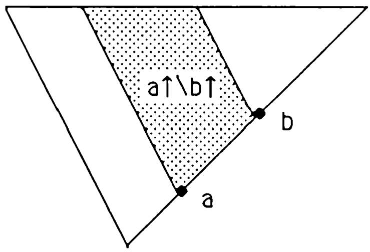
4.1 Theorem. *If M is a tree with finitely many extra unary relations which is definably well-founded, then it has well-founded n-equivalents for all n.*

Before embarking on the proof, we need some surgical terminology on trees and three lemmas.

A **component** of M is a maximal connected subset. An element of M is minimal iff it is the least element of its component. In particular, components are (first-order parametrically) definable.
Therefore, if M is definably well-founded, so are its components. I do not know whether the converse of this holds, though - and this makes for some complications in the formulation of the lemmas below.
If $X \subset M$ is *downward closed* (i.e., if $a < b \in X$ implies $a \in X$) then $M \setminus X$ is *upward closed*, and vice versa.
I shall use the following notations. If $X \subset M$ then (M, X) denotes the expansion of M obtained by adding X as a new unary relation.

Warning. If $a \in M$ then $a\uparrow$ equals $\{c \in M \mid a \leqslant c\}$ in *this* chapter. Thus, $a \in a\uparrow$, but $a \notin a\downarrow$.

4.2 Lemma. *Suppose that the tree* $\boldsymbol{M}$ *is definably well-founded. If* $a<b$ *in* $\boldsymbol{M}$ *then, for each* n, $((a\uparrow)\backslash(b\uparrow), [a,b))$ *has an* n*-equivalent* $(\boldsymbol{N},\beta)$ *such that* β *is well-ordered and all components of* $N\backslash\beta$ *are definably well-founded.*

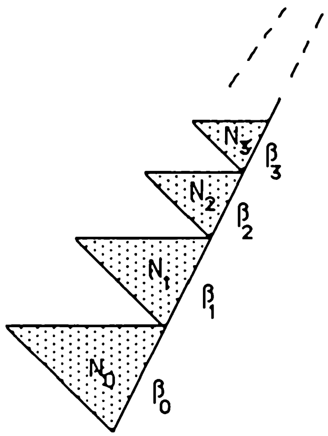


Thus, $\boldsymbol{N}$ can be used as a substitute (within n-equivalence) of the part $(a\uparrow)\backslash(b\uparrow)$ of $\boldsymbol{M}$, thereby exchanging $[a,b)$ for the well-ordered β and preserving definable well-foundedness of the rest - *component-wise*. The other two lemmas are similar in spirit. The proof of 4.1 finally will show how to carry out such substitutions repeatedly, thereby eventually arriving at the desired well-founded n-equivalent. To see that such substitutions actually work, the following remark is needed.

Remark. In what follows, a lot of cutting and pasting of trees has to be performed. To see that in each case n-equivalence is preserved, the Ehrenfeucht-game technique can be applied in much the same way as in the proof of claim 1 in the proof of theorem 2.4.1. The general procedure is as follows. Suppose that $\boldsymbol{M}'$ is obtained from $\boldsymbol{M}$ by exchanging some part $\boldsymbol{N}$ by an n-equivalent $\boldsymbol{N}'$. In all cases occurring, it will be clear how this exchange-process has to be performed since the way $\boldsymbol{N}$ is "attached" to $\boldsymbol{M}\backslash\boldsymbol{N}$ will be particularly simple. Let f be the identity-map on $\boldsymbol{M}\backslash\boldsymbol{N}$. Now, *suppose that, for each partial isomorphism h between **N** and **N'**, the union $f \cup h$ is a partial isomorphism between M and M'.* (In applications it always will be rather obvious that this condition is satisfied.) *Then it will be the case that $M \equiv^n M'$. Proof:* Consider the n-game between M and M'. Player ***II*** wins this if, to answer moves by ***I*** in either ***N*** or ***N'***, he uses a winning strategy for the n-game between these models, and if he copies the moves of player ***II*** on $M \setminus N$.

Proof of 4.2. Let X be the set of $b \in M$ such that for all $a<b$, $(a\uparrow \setminus b\uparrow, [a,b))$ has an n-equivalent of the type desired. The lemma asserts that $X=M$.

Suppose that $X \neq M$. Observe that X is definable: by 1.7.1, there are only finitely many n-characteristics of models (N, β) such that β is well-ordered and all components of $N \setminus \beta$ are definably well-founded; moreover, that $(a\uparrow \setminus b\uparrow, [a,b))$ satisfies a given characteristic is a first-order property of (M,a,b). By definable well-foundedness, $M \setminus X$, assumed to be non-empty, has a minimal element b. Suppose $a<b$ is such that a corresponding n-equivalent of the required type doesn't exist. Obviously, b cannot have an immediate predecessor. Choose $a_0 = a < a_1 < \ldots < a_\xi < \ldots (\xi < \alpha)$ cofinal in $b\downarrow$. By minimality of b, choose $(N_\xi, \beta_\xi) \equiv^n (a_\xi\uparrow \setminus a_{\xi+1}\uparrow, [a_\xi, a_{\xi+1}))$ such that β_ξ is well-ordered and all components of $N_\xi \setminus \beta_\xi$ are definably well-founded for each $\xi < \alpha$.



The model $\Sigma_{\xi<\alpha}(N_\xi, \beta_\xi)$, obtained by glueing the β_ξ one after the other now forms a counter-example to the choice of a and b : to see that this is an n-equivalent of $(a\uparrow\setminus b\uparrow, [a,b))$, apply the remark to $(a\uparrow\setminus b\uparrow, [a,b)) = \bigcup_{\xi<\alpha}(a_\xi\uparrow\setminus a_{\xi+1}\uparrow, [a_\xi, a_{\xi+1}))$ and $\Sigma_{\xi<\alpha}(N_\xi, \beta_\xi)$. ⊠

4.3 Corollary. *Suppose the tree M is definably well-founded and $b \in M$. Then, for each n, (M, b) has an n-equivalent (M', b') such that $b'\downarrow$ is well-ordered and all components of $M'\setminus(b'\downarrow)$ are definably well-founded.*

Proof. If b is minimal in M, then (M, b) itself satisfies the stipulations. Otherwise, let a be the least element of $b\downarrow$. (For a picture: look at the one for 4.2; now, put a in the root of the tree.) By 4.2, $(a\uparrow\setminus b\uparrow, [a,b))$ has an n-equivalent (N, β) with β well-ordered and all components of $N\setminus\beta$ definably well-founded. Replace $(a\uparrow\setminus b\uparrow, [a,b))$ in M by (N, β); the result is M'. In M', $b\downarrow = \beta$. Thus, putting $b' = b$ makes $b'\downarrow$ well-ordered. The components of $M'\setminus(b'\downarrow)$ are the ones of $N\setminus\beta$ plus $b\uparrow$ plus the M-components different from the one containing b (if any); these are all definably well-founded. Finally, $(M', b') \equiv^n (M, b)$ follows from the remark above. ⊠

The next lemma is the version of 4.3 with finitely many b's at the same time:

4.4 Lemma. *Suppose that the tree* $\boldsymbol{M}$ *is definably well-founded and* $A \subset M$ *is finite. Then, for each n,* $(\boldsymbol{M}, a)_{a \in A}$ *has an n-equivalent* $(\boldsymbol{M}', a')_{a \in A}$ *such that each* $a'{\downarrow}$ $(a \in A)$ *is well-ordered and all components of* $M' \setminus \bigcup_{a \in A} a'{\downarrow}$ *are definably well-founded.*

Proof. Induction on the number of elements of A.

To start with, we have 4.3. For the induction-step, choose $a \in A$ and put $B = A \setminus \{a\}$. Apply the inductive hypothesis to $(\boldsymbol{M}, a)$ and B to obtain $(\boldsymbol{M}', a', b')_{b \in B}$ with all $b'{\downarrow}$ $(b \in B)$ well-ordered and $M' \setminus \bigcup_{b \in B} b'{\downarrow}$ definably well-founded - *component-wise.*

(i) Suppose that for some $b \in B$, $a < b$. Then $a' < b'$, $a'{\downarrow}$ is well-ordered, $M' \setminus \bigcup_{a \in A} a'{\downarrow} = M' \setminus \bigcup_{b \in B} b'{\downarrow}$, and we are done.

(ii) If not, let C be the component of $M' \setminus \bigcup_{b \in B} b'{\downarrow}$ containing a'. By 4.3, obtain $(\boldsymbol{C}', a'') \equiv^n (\boldsymbol{C}, a')$ with $a''{\downarrow}$ well-ordered and $C' \setminus (a''){\downarrow}$ definably well-founded, component-wise. Replace $(\boldsymbol{C}, a')$ in $\boldsymbol{M}'$ by $(\boldsymbol{C}', a'')$ to obtain the desired model $(\boldsymbol{M}'', a'', b')_{b \in B}$. ⊠

We are now ready for the

Proof of 4.1.

Define a sequence of models $\boldsymbol{M}_0, \boldsymbol{M}_1, \boldsymbol{M}_2, \ldots$ and sets $T^0, T^1, T^2, \ldots$ such that:

1. $\boldsymbol{M}_0 = \boldsymbol{M}$; $T^0 = \emptyset$;
2. T^i is a well-founded downward-closed part of $\boldsymbol{M}_i$ and every component of $M_i \setminus T^i$ is definably well-founded;

3. T^{i+1} (considered as a submodel of M^{i+1}) is an *end-extension* of T^i (considered as a submodel of M_i) (i.e., $T^i \subset T^{i+1}$, and for $a, b \in T^{i+1}$, if $b \in T^i$ and $a < b$ then $a \in T^i$);
4. $(M_i, t)_{t \in T^i} \equiv^n (M_{i+1}, t)_{t \in T^i}$;
5. for all $a \in M_i \setminus T^i$ there is $b \in T^{i+1}$ such that $(M_i, t, a)_{t \in T^i} \equiv^{n-1} (M_{i+1}, t, b)_{t \in T^i}$.

M_{i+1} and T^{i+1} will be obtained from M_i and T^i by replacing $M_i \setminus T^i$ in M_i by an n-equivalent with a well-founded initial part (namely, $T^{i+1} \setminus T^i$) preserving definable well-foundedness *component-wise*. This will take care of 2-4. However, T^{i+1} must be big enough so as to satisfy 5. This is achieved in the following manner:

Let C be a component of $M_i \setminus T^i$. Choose $A \subset C$ such that for each $c \in C$ there is an $a \in A$ with $(C, a) \equiv^{n-1} (C, c)$ and such that A is finite - this can be done according to 1.7.1. By 4.4, $(C, a)_{a \in A}$ has an n-equivalent $(C', a')_{a \in A}$ with every $a' \downarrow$ well-ordered and $C' \setminus \bigcup_{a \in A} a' \downarrow$ definably well-founded, component-wise.

M_{i+1} is obtained from M_i by exchanging C for C' and making similar replacements for every other component of $M_i \setminus T^i$. T^{i+1} is T^i plus all the $\bigcup_{a \in A} (a' \downarrow \cup \{a'\})$ so encountered.

It is now obvious that 2.-5. are satisfied.

Now, put $N = \bigcup_i T^i$.

By 2-3, N is well-founded. I claim that N is an n-equivalent of M.

Consider the Ehrenfeucht-n-game between these models. Notice that, as $T^i \subset M_i$, in order to win, it suffices for the second player to choose his moves in such a way that after his k-th move sequences $a_0, \ldots, a_{k-1} \in M$ and $t_0, \ldots, t_{k-1} \in N$ have been played such that for all i, if $t_0, \ldots, t_{k-1} \in T^i$ then

$$[*]_i \quad (M, a_0, \ldots, a_{k-1}) \equiv^{n-k} (M_i, t_0, \ldots, t_{k-1}).$$

Notice that $i \leqslant j$ and $[*]_i$ imply $[*]_j$ by condition 4. above.

Now, the second player can keep up with this requirement:

First, if $k=0$ then $[*]_0$ holds since $M_0=M$.

Next, suppose the players have arrived at a position where $[*]_i$ still is satisfied.

(a) Let the first player choose $t_k \in N$, say, $t_k \in T^j$. If $j \leqslant i$ then $[*]_i$ provides the second player with an $a_k \in M$ such that $(M, a_0, \ldots, a_k) \equiv^{n-k-1} (M_i, t_0, \ldots, t_k)$. If $i<j$, simply use $[*]_j$.

(b) Assume the first player chooses $a_k \in M$. By $[*]_i$, there is a $u \in M_i$ such that $(M, a_0, \ldots, a_k) \equiv^{n-k-1} (M_i, t_0, \ldots, t_{k-1}, u)$. If, by a stroke of luck, $u \in T^i$, we are done. If not, by condition 5. there is $t_k \in T^{i+1}$ such that $(M_{i+1}, t, t_k)_{t \in T^i} \equiv^{n-1} (M_i, t, u)_{t \in T^i}$, in particular, $(M_{i+1}, t_0, \ldots, t_k) \equiv^{n-k-1} (M_i, t_0, \ldots, t_{k-1}, u)$. Hence, $(M, a_0, \ldots, a_k) \equiv^{n-k-1} (M_{i+1}, t_0, \ldots, t_k)$; so the second player chooses t_k– thereby ensuring $[*]_{i+1}$ to hold for the resulting sequences. ⊠

III Applications to intensional and intuitionistic logic

Few things are harder to put up with than the annoyance of a good example. - *Pudd'nhead Wilson's Calendar*

5. Fine structure of modal correspondence theory.

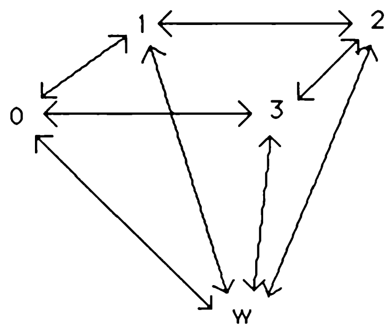
For a given formula φ of modal propositional logic, the condition on frames $\mathbf{W}=(W,R)$ that φ is valid on $\mathbf{W}$ is monadic Π^1_1. (See below for the necessary definitions.) Correspondence theory asks (among other things) whether it can be expressed in first-order terms as well, i.e., whether it is *first-order definable*. This chapter presents some more information about modal formulas known to be not first-order definable.

To begin with, a couple of examples of van Benthem [1985] is examined. To prove first-order undefinability, use was made there of compactness- and Löwenheim-Skolem theorems. These results are strengthened here by showing non-first-order definability on *finite* frames in all cases. (In one case, we show that even a monadic Σ^1_1-definition on finite frames does not exist.) Of course, on the class of finite frames, other methods have to be used, and so the Ehrenfeucht game has to come to the rescue once more. Secondly, an example is given of a non-first-order definable formula which, however, *is* so definable on countable frames.

For readers unfamiliar with modal logic, let me recall the definitions of the notions involved. The **formulas** of **modal propositional logic** are generated from a set of variables by means of the ordinary connectives plus the (unary) modal operators $\Box$ (**necessity**) and $\Diamond$ (**possibility**). The **Kripke semantics** for these objects relative to **frames**, i.e., models $\mathbf{W}=(W,R)$ where $R\subset W^2$, works as follows. Suppose that V is a **valuation**, that is, a map assigning subsets of W to variables. The triple (W,R,V) is called a (**Kripke**) **model**. The **forcing relation** $\Vdash$ connects elements of W with formulas and is inductively defined by the clauses: (i) $w\Vdash\varphi$ iff $w\in V(\varphi)$ when φ is a variable; (ii) $w\Vdash\neg\varphi$ iff $w\nVdash\varphi$; (iii) $w\Vdash\varphi\wedge\psi$ iff $w\Vdash\varphi$ and $w\Vdash\psi$; (iv) $w\Vdash\Box\varphi$ iff for all v such that wRv: $v\Vdash\varphi$; (v) $w\Vdash\Diamond\varphi$ iff for some v such that wRv: $v\Vdash\varphi$. We say that φ is **valid** on $\mathbf{W}$ iff φ is forced at every $w\in W$ under all possible valuations.

5.1 Theorem. *Validity of $\Box(p\to\Box p)\to(\Diamond p\to\Box p)$ is not first-order on finite frames.*

Proof. This may be seen as a modification of the proof of van Benthem [1985] lemma 10.1. Consider the frame $C_k=(\{0,\dots,k-1\},R)$ where R is defined by: $iRj \equiv |i-j|=1 \pmod k$. C^w_k is obtained from this by the addition of one new element w which is R-connected with every $i<k$. Look at the illustration on the next page.



C^w_4

Claim: $\Box(p\to\Box p)\to(\Diamond p\to\Box p)$ *is valid in every* $C^w{}_k$.

Proof of claim: suppose $\Diamond p$ and $\Box(p\to\Box p)$ are forced at $i<k$. By the first, either (i) $w\Vdash p$ or (ii) $i-1\Vdash p$ or (iii) $i+1\Vdash p$. By the second assumption then, either (i) $w\Vdash\Box p$ or (ii) $i-1\Vdash\Box p$ or (iii) $i+1\Vdash\Box p$. In case (ii) it follows that $w\Vdash p$, hence again $w\Vdash\Box p$, and similarly in case (iii). Thus, $w\Vdash\Box p$ in all cases, whence $i\Vdash\Box p$ follows. Next, suppose $\Diamond p$ and $\Box(p\to\Box p)$ are forced at w. Then $i\Vdash p$ for some $i<k$; therefore, $i-1\Vdash p$, $i+1\Vdash p$, $i-2\Vdash p$, $i+2\Vdash p$ (everything mod k) and so on - therefore, p is forced at every $i<k$; whence $w\Vdash\Box p$. ⊠

Next, let $2C^w{}_k$ be the frame obtained from the disjoint union of two copies of $C^w{}_k$ by identifying the respective nodes w.

Claim: $\Box(p\to\Box p)\to(\Diamond p\to\Box p)$ *is valid in* no $2C^w{}_k$.

For, let $V(p)$ be one of the $C^w{}_k$. Then $\Box(p\to\Box p)$ and $\Diamond p$ are forced at w, but $\Box p$ is not. ⊠

Now, if $\Box(p\to\Box p)\to(\Diamond p\to\Box p)$ had a first-order equivalent, this would have some quantifier rank n. Therefore, the following lemma establishes the result. ⊠

5.2 Lemma. *If* $k\geqslant 2^n$ *then:* $C^w{}_k\equiv^n 2C^w{}_k$.

The lemma is the result of a series of observations 5.3-5. Recall that $\zeta = \omega^* + \omega$.

5.3 Lemma. $\zeta \equiv \zeta + \zeta$.

Proof. By induction on n, check that $\zeta \equiv^n \zeta + \zeta$ for all n, using 1.0.2, 1.0.3(ii) and its consequence, $\omega^* \equiv \zeta + \omega^*$. ⊠

Let φ be a quantifier rank-m formula in the free variables $x_0,\dots,x_{k-1}$. In each frame A, φ defines a relation $\varphi^A = \{ a \in A^k \mid A \models \varphi[a] \} \subset A^k$ which can serve as the interpretation of a new k-ary relation symbol. Now:

5.4 Lemma. *If* $A \equiv^{m+n} B$, *then:* $(A, \varphi^A) \equiv^n (B, \varphi^B)$.

Proof. Replacing the new k-ary relation symbol by its definition φ in a formula raises the quantifier rank of that formula with at most m. ⊠

Now, let η be the formula

$$[x<y \wedge \neg\exists z(x<z \wedge z<y)] \vee [y<x \wedge \neg\exists z(y<z \wedge z<x)].$$

This is a quantifier rank-1 formula defining *immediate neighbourship* in linear orderings. If $A=(A,<)$ is a linear ordering, let $A^\eta = (A, \eta^A)$. By 5.4, in particular, if $A \equiv^{n+1} B$ then $A^\eta \equiv^n B^\eta$.

In the following lemma, $C_m + C_l$ is the disjoint union of C_m and C_l.

5.5 Lemma. *1. If* $k, m \geqslant 2^n$ *then* $C_k \equiv^n C_m$ *;*

2. if $k \geqslant 2^n$ *then* $C_k \equiv^n \zeta^\eta$ *;*

3. if $k, m, l \geqslant 2^n$ *then* $C_k \equiv^n C_m + C_l$.

Proof. Induction on n. For $n=1$, the results are clear. Assume they hold for n and suppose now that $k, m, l \geqslant 2^{n+1}$.

1. A first move $i \in C_k$ by ***I*** is answered with an arbitrary $j \in C_m$ by ***II***. Now, notice the following. $k-1, m-1 \geqslant 2^{n+1}-1$, hence $\mathbf{k-1} \equiv^{n+1} \mathbf{m-1}$ by 1.0.1. Therefore, $(\mathbf{k+1}, 0, k) \equiv^{n+1} (\mathbf{m+1}, 0, m)$ as well and hence $((\mathbf{k+1})^{\eta}, 0, k) \equiv^{n} ((\mathbf{m+1})^{\eta}, 0, m)$. Now, identification of 0 and k in $(\mathbf{k+1})^{\eta}$ produces an isomorph of (C_k, i) and the same goes for 0, m, $(\mathbf{m+1})^{\eta}$ and C_m. Therefore, a trick like the one of 5.4 (identification of 0 and k just means re-interpreting = as: $x=y \vee (x=0 \wedge y=k) \vee (x=k \wedge y=0)$ - and this doesn't need quantifiers) shows that $(C_k, i) \equiv^{n} (C_m, j)$.

2. If $k \geqslant 2^{n+1}$ then $k-1 \geqslant 2^{n+1}-1$, hence $\mathbf{k-1} \equiv^{n+1} \omega+\omega^*$ by 1.0.3(i). Therefore, $(\mathbf{k-1})^{\eta} \equiv^{n} (\omega+\omega^*)^{\eta}$. Adding endpoints, $((\mathbf{k+1})^{\eta}, 0, k) \equiv^{n} ((\omega+\omega^*)^{\eta}, 0, 0^*)$ (where 0^* is the greatest element of ω^*). Identifying 0 and k in $(\mathbf{k+1})^{\eta}$ produces (C_k, i) for any $i<k$; likewise, identifying 0 and 0^* in $(\omega+\omega^*)^{\eta}$ produces $(\zeta^{\eta}, 0)$. Hence, $(C_k, i) \equiv^{n} (\zeta^{\eta}, 0)$, etc.

3. $C_k \equiv^{n} \zeta^{\eta} \equiv^{n} \zeta^{\eta} + \zeta^{\eta} \equiv^{n} C_m + C_l$ by 2. and 5.3. ⊠

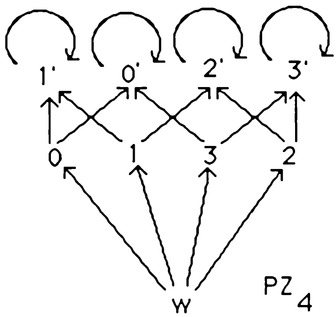
Remark. The bound 2^n in 5.5 is *not* sharp. For instance, if $k, m \geqslant 7$ then $C_k \equiv^{3} C_m$.

Finally, 5.2 now follows from 5.5.3 noting that the addition of w cannot spoil the winning strategy of ***II***.

5.6 Theorem. *The McKinsey formula $\Box\Diamond p \rightarrow \Diamond\Box p$ is not first-order definable on finite frames.*

Proof. (Compare van Benthem [1985] Thm.10.2.) PZ_k is the following frame.

Its universe is $\{w\}\cup\{0,\ldots,k-1\}\cup\{0',\ldots,(k-1)'\}$; its relation R is defined by: aRb iff either $a=w$ and $b\in\{0,\ldots,k-1\}$ or $a\in\{0,\ldots,k-1\}$ and b is either a' or $(a+1)'$ (mod k) or $a=b\in\{0',\ldots,(k-1)'\}$. Truth of $\Box\Diamond p\rightarrow\Diamond\Box p$ is unproblematic at all nodes except w. There, *it depends on whether k is even or odd.* Truth of $\Box\Diamond p$ at w means that for each $i<k$ at least one of i' and $(i+1)'$ (mod k) is in $V(p)$. If k is odd, then for some $i<k$ both i' and $(i+1)'$ must be in $V(p)$, hence $\Box p$ is forced at i and $\Diamond\Box p$ is forced at w.



However, when k is even, set $V(p)=\{\,i' \mid i \text{ even}\}$ and the McKinsey formula clearly is not forced at w. If there is a first-order equivalent, this must have a certain quantifier rank. Therefore, the result is an immediate consequence of the following lemma. ⊠

5.7 Lemma. *If* $k,m\geqslant 2^n$, *then* $\mathrm{PZ}_k\equiv^n\mathrm{PZ}_m$.

Proof. Let C'_k be the asymmetric version of C_k, i.e., it is the frame $(\{0,\ldots,k-1\},R)$ where iRj iff $j=i+1$ (mod k). Just like in 5.5.1, we have $C'_k\equiv^n C'_m$ for $k,m\geqslant 2^n$ (apply 5.4 using the left disjunct of the formula η). Any winning strategy σ for *II* in $G(C'_k,C'_m,n)$ can now be transformed into one for *II* in $G(\mathrm{PZ}_k,\mathrm{PZ}_m,n)$: the move w by *I* in either frame is answered by

w in the other one; when I plays i (resp. i') in either frame, i is fed to σ; if σ prescribes j then II answers j (resp. j'). ⊠

Similar considerations yield the following results. Here, a formula φ is called **locally defined** by $\psi(x)$ if in each frame W, for each $w \in W$: $w \Vdash \varphi$ iff $W \models \psi[w]$. (Definability in 5.1 and 5.6 also is called **global definability**.)

5.8 Theorem. *The following formulas are not locally first-order definable on finite frames:*

1. $\Box(\Box p \vee p) \rightarrow \Diamond(\Box p \wedge p)$ (cf. van Benthem [1985] 10.3)
2. $\Box(p \vee q) \rightarrow \Diamond(\Box p \vee \Box q)$ (l.c. 10.4)
3. $\Diamond(\Box p \vee \Box \neg p)$
4. $\Box(\Box p \vee p) \rightarrow \Diamond(\Diamond p \vee p)$ (l.c.10.6).

Remark. Several results on non-first-order definability are rather unsatisfying in that they show first-order expressibility to be so forbiddingly weak in the poor language given. In such circumstances, it would be quite natural to look for a wider type of definability. A possibility that suggests itself would be definability using an infinitary language. Another one would be higher-order definability. Now, modal formulas are Π^1_1-definable by nature. If such a principle is Σ^1_1 as well, it must be first-order by the interpolation theorem. However, on *non-elementary* model classes, this need not to be so. For instance, which of the previous examples is Δ^1_1 on the finite models? The models used in 5.6 cannot be used to prove non-Δ^1_1-definability here since they can be

$\Delta^1{}_1$-distinguished. The same goes for the models used to prove 5.8: they too employ the even/odd-indistinguishability by first-order means. On the other hand, this is *not* the case in 5.1, as the following theorem shows. Therefore, that result can be strengthened to non-monadic $\Sigma^1{}_1$-definability.

5.9 Theorem. *Any monadic $\Sigma^1{}_1$-sentence valid in each C_p holds on some C_q+C_r as well.*

Proof. Suppose that $\exists X_1 \ldots \exists X_k \sigma$ holds on each C_p. Let σ have quantifier-rank n. Choose a finite set Σ of finite linearly ordered models $\alpha=(A,<,X_1,\ldots,X_k)$ such that each such model has an $(n+1)$-equivalent in Σ. By the *finite Ramsey-theorem*, there is a natural number p such that for each p-element set P and each map $h:[P]^2 \to \Sigma$ there exists a 2^{n+1}-element set $Q \subset P$ *homogeneous* for h, i.e., for some $\alpha \in \Sigma$ we have $h(x,y)=\alpha$ for all $x,y \in Q$.
Now, consider a model $(C_p, X_1,\ldots,X_k) \models \sigma$. Cut it open at some place; this produces a linearly ordered p-element model. We may as well assume this to have universe $\mathbf{p}=\{0,\ldots,p-1\}$, where the elements have their natural order. For $i<j<p$, let $h(i,j)$ be an $(n+1)$-equivalent of $[i,j)$ in Σ - here, $[i,j)$ denotes the submodel of $\mathbf{p}$ with universe $\{m<p \mid i \leqslant m<j\}$. By choice of p, there is a 2^{n+1}-element $Q \subset \mathbf{p}$ homogeneous for h; say, $h(i,j)=\alpha$ for all $i<j$ in Q. Put $x=\min Q$ and $y=\max Q$. Then $[x,y) \equiv^{n+1} \alpha$ and, hence, $\mathbf{p} \equiv^{n+1} (\leftarrow,x)+\alpha+[y,\rightarrow)$. Since Q has 2^{n+1} elements, it divides $[x,y)$ into $2^{n+1}-1$ intervals, each $(n+1)$-equivalent with α.
Therefore: $\alpha \equiv^{n+1} \alpha\cdot(2^{n+1}-1)$

$\equiv^{n+1} \alpha\cdot(\omega+\omega^*)$ (for, $2^{n+1}-1 \equiv^{n+1} \omega+\omega^*$ by 1.0.3(i) - use 3.1.8)

$\equiv \alpha\cdot(\omega+\zeta+\omega^*)$ (for, $\omega \equiv \omega+\zeta$ by 1.0.3(ii) - idem);

hence, $\alpha^\eta \equiv^n [\alpha\cdot(\omega+\omega^*)]^\eta+[\alpha\cdot\zeta]^\eta$ (by 5.4 - notice that, after applying η, the ordering between ω, ω^* and ζ doesn't matter any longer);
therefore, $\alpha^\eta \equiv^n \alpha^\eta+[\alpha\cdot\zeta]^\eta$.

It follows that the original model $(C_p, X_1,\dots,X_k)$ is n-equivalent to its disjoint sum with $[\alpha\cdot\zeta]^\eta$. However, for $m=2^n$, $\zeta^\eta \equiv^n C_m$ (by 5.5.2). Hence, $[\alpha\cdot\zeta]^\eta$ is n-equivalent with a model $(C_{am}, Y_1,\dots,Y_k)$ where a is the number of elements of α. Therefore, $\exists X_1\dots\exists X_k\sigma$ holds in C_p+C_{am}. ⊠

The examples given suggest the *pattern*: if a modal formula is not first-order definable, then it is not first-order on *finite* frames already; conversely: if it is first-order on *finite* frames then it is first-order *generally*.
However, this is not true.

5.10 Example. Löb's axiom $\Box(\Box p\to p)\to\Box p$ is first-order on finite models but is not so generally: it is not first-order on countable models. For, the axiom is true in a frame $\mathbf{W}=(W,R)$ iff R is transitive and conversely well-founded (if aRb, bRc but not aRc, put $V(p)=W\setminus\{b,c\}$ and notice that $\Box(\Box p\to p)$ holds at a but $\Box p$ doesn't; if R is transitive, truth of Löb's axiom just means that R is conversely well-founded on each set $\{b \mid aRb\}$ - and so it is well-founded on W) - and this property is known to be not first-order on countable models: a standard consequence of the compactness and downward Löwenheim-Skolem theorems. However, the axiom *is* first-order on *finite* frames since there, (converse) well-foundedness reduces to mere irreflexivity. ⊠

This example triggers the question: if a modal formula is first-order on countable frames, must it be first-order generally?
Again, the answer is negative.

5.11 Example. Consider **Fine's axiom** Φ: $\Diamond\Box(p\vee q)\rightarrow\Diamond(\Box p\vee\Box q)$.
$\psi(x)$ is the first-order condition on frames $(W,\prec)$ that $\forall y\succ x\exists y'\succ x[\forall z\succ y'(z\succ y)\wedge\forall z,z'\succ y'(z=z')]$ (i.e., for each $\prec$-successor y of x there is another one y' which has at most one $\prec$-successor; moreover, each $z\succ y'$ succeeds y as well). By the way, $\forall x\psi(x)$ is the first-order condition with respect to which Fine's axiom is **complete**, i.e., a modal formula is modally derivable from Φ iff it is valid on all frames satisfying $\forall x\psi$ (see van Benthem [1985] lemma 8.10).

Now, notice:

(i). For every frame W and $x\in W$: if $W\models\psi[x]$ then Φ is true at x.

For, suppose that $x\prec y\Vdash\Box(p\vee q)$. Since $W\models\psi[x]$, there is an $y'\succ x$ with at most one successor such that $y'\prec z\Rightarrow y\prec z$. Clearly then, either $y'\Vdash\Box p$ or $y'\Vdash\Box q$, therefore, $y'\Vdash\Box p\vee\Box q$ and $x\Vdash\Diamond(\Box p\vee\Box q)$. ⊠

(ii). For every *countable* W and $x\in W$: if Φ is true at x then $W\models\psi[x]$.

To see this, put $X_y=\{z\in W\mid y\prec z\}$, $U=\{X_y\mid x\prec y\}$ and $X=\bigcup U=\bigcup_{x\prec y}X_y$. Assume $x\Vdash\Phi$; we check that $W\models\psi[x]$: let $x\prec y$. There are two cases.

(a) There is a *finite* $X_{y'}\subset X_y$ in U. Let $X_{y'}$ be one with a minimal number of elements. If it is empty, the rest is trivial. If $X_{y'}=\{z\}$, we are done as well. So it suffices to show that $X_{y'}$ cannot have more than one element. If it has, choose $V(p)$ and $V(q)$ non-empty disjoint such that $X_{y'}=V(p)\cup V(q)$. Now, $y'\Vdash\Box(p\vee q)$, hence $x\Vdash\Diamond\Box(p\vee q)$. As $x\Vdash\Phi$ by assumption, y'' exists such that $x\prec y''$ and $X_{y''}\subset V(p)$ or $X_{y''}\subset V(q)$, contradicting the minimality of $X_{y'}$.

(b) Each $X_{y'}\subset X_y$ in U is infinite. By the countability-assumption, let $X_0,X_1,X_2,\ldots$ be an enumeration of $\{X\in U\mid X\subset X_y\}$. Inductively, choose $a_i,b_i\in X_i$ all different from one another. Put $V(p)=\{a_i\mid i\in\mathbb{N}\}$ and $V(q)=X_y\setminus V(p)$. Then $y\Vdash\Box(p\vee q)$, $x\Vdash\Diamond\Box(p\vee q)$ and hence y' exists such that $x\prec y'$, $y'\Vdash\Box p\vee\Box q$; thus, either $X_{y'}\subset V(p)$ or $X_{y'}\cap V(p)=\emptyset$. For some $i\in\mathbb{N}$, $X_{y'}=X_i$. But, $X_i\subset V(p)$ is impossible as $b_i \in X_i \setminus V(p)$; and neither is $X_i \cap V(p) = \emptyset$ as $a_i \in X_i \cap V(p)$. ⊠

(iii). If Φ would be (locally) first-order, ψ would be a first-order equivalent of Φ: being an equivalent on *countable* frames by (i) and (ii), it must be an equivalent on *all* frames, by the Löwenheim-Skolem theorem. However:

(iv). There are *uncountable* frames $\mathbf{W}$ such that Φ holds at some $x \in W$ but $\mathbf{W} \not\models \psi[x]$. For instance, let U be the collection of all infinite sets of natural numbers; put $W = \{U\} \cup U \cup \mathbb{N}$ and $x \prec y$ iff $y \in x$. It is easy to see that $\mathbf{W} = (W, \prec) \not\models \psi[U]$, because U does not contain singletons or the empty set. Next, choose any valuation $V(p)$, $V(q)$ of p and q. If $U \Vdash \Diamond\Box(p \vee q)$, this means that for some $X \in U$, $X \subset V(p) \cup V(q)$, hence $X = (X \cap V(p)) \cup (X \cap V(q))$. X is infinite, therefore one of $X \cap V(p)$ and $X \cap V(q)$ must be infinite as well. If this is $X \cap V(p)$ then $X \cap V(p) \in U$, $X \cap V(p) \Vdash \Box p$, $X \cap V(p) \Vdash \Box p \vee \Box q$ and $U \Vdash \Diamond(\Box p \vee \Box q)$. ⊠

(v). The above answers the question for *global* first-order definability as well. For clearly, $\forall x \psi$ globally defines Φ on countable models, and the counter-example from (iv) can be transformed into a global one by making $\prec$ reflexive on $\mathbb{N}$. ⊠

(vi). (van Benthem) This example can be transformed into one for strict partial orderings. Let Φ be $\Diamond(\Diamond\top \wedge \Box(p \vee q)) \rightarrow \Diamond(\Diamond\top \wedge (\Box p \vee \Box q))$ and $\psi(x)$: $\forall y \succ x[\exists z(y \prec z) \rightarrow \exists y' \succ x(y'$ has exactly one $\prec$-successor; besides, this successor succeeds y as well)]. Almost the same proofs show (i)-(iii) to be the case for these formulas; the *transitive closure* of the frame of (iv) shows that Φ can hold in an uncountable frame where ψ is false.

5.12 Problem. Determine the least cardinal μ such that: if a modal formula is first-order definable on frames of power $\leqslant \mu$ then it is so generally.

Here are some partial answers on this problem and its variant for tense logic. Restricting to linear (irreflexive) orderings, van Benthem obtained the following

5.13 Theorem. *If a modal formula is first-order on the class of all countable linear orderings then it is first-order on all linear orderings.*

The theorem follows from the

5.14 Lemma. *If the linearly ordered frame F is an elementary submodel of G then each modal formula true on G holds on F as well.*

Proof that 5.14 implies 5.13. Suppose that φ is a modal formula not first-order on all linear orderings. By (a slight modification of) van Benthem [1985] p.91 Theorem 8.6 (with the same proof), there are elementary equivalent linear orderings G_1 and G_2 such that φ holds on G_2 but is false on G_1. Since φ is $\Pi^1{}_1$, G_1 may be taken to be countable, by the downward Löwenheim-Skolem theorem. Let $F \prec G_2$ be countable. If φ is first-order on countable linear orderings, it must be false on F; according to 5.14, this is a contradiction. ⊠

Proof of 5.14. Suppose that $\mathbf{F}=(F,<) \prec \mathbf{G}=(G,<)$ and V is a valuation on F falsifying the modal formula φ. It suffices to construct a φ-falsifying valuation W on G. This is done by *extending* V to G as follows. Let n be the modal rank of φ. The object is to define W such that $(F,V) \prec_n (G,W)$, by which I mean that, for each modal formula ψ of rank $\leqslant n$: (i) for all $a \in F$: a forces ψ in (F,V) iff a forces ψ in (G,W) and (ii) if ψ is forced somewhere on (G,W) then it is forced somewhere on (F,V) as well.

The following needs the material from chapter 6 up to and including 6.4. Define an equivalence $\sim$ on F by $x \sim y$ iff $\exists a \leqslant x,y \exists b \geqslant x,y$: $[\![a]\!]^n = [\![b]\!]^n$. Obviously, $\sim$ is a *condensation* splitting F in a finite number of intervals (since the number of n-characteristics $[\![a]\!]^n$ is finite). Let $u \in G \backslash F$. To decide how we shall define W on u, notice thet there are the following six possibilities as regards the location of u compared to the ondensation $\sim$.

1. u occurs *in* an interval, i.e., there are $a < u$ and $b > u$ with $[\![a]\!]^n = [\![b]\!]^n$.
2. u occurs *between* neighbouring intervals $c_1 < u$ and $c_2 > u$ in $F/\sim$.
   Notice that either c_1 has no maximum or c_2 has no minimum, since $F \prec G$.
   So, we have the subcases 2.1-2.3:
   2.1. c_1 has a maximum; c_2 has no minimum.
   2.2. c_1 has no maximum; c_2 has a minimum.
   2.3. c_1 has no maximum; c_2 has no minimum.
3. u is smaller than all $x \in F$. Notice that F cannot have a least element in this case, since $F \prec G$.
4. Similarly, u is greater that all $x \in F$.

Call $u,v \in G \backslash F$ **equivalent** iff they are not separated by some $x \in F$. There may be infinitely many equivalence classes; I shall extend V step by step to each equivalence class. To do so needs a possibly transfinite iteration which is handled by the following version of the elementary chain lemma:

5.15 Lemma. *Let $\langle A_\xi | \xi < \alpha \rangle$ be a sequence of Kripke models such that $A_\xi \prec_n A_\delta$ whenever $\xi < \delta < \alpha$. Let $A = \bigcup_{\xi < \alpha} A_\xi$; then $A_\xi \prec_n A$ for all $\xi < \alpha$.*

Proof. That for all ψ of rank $\leqslant n$, all $\xi < \alpha$ and all $a \in A_\xi$: $a \Vdash \psi$ (in A_ξ) iff $a \Vdash \psi$ (in A) is proved using induction on the rank of ψ; the rest of the theorem is an immediate consequence of this. ⊠

Now, constuct the sequence $\langle A_\xi|\xi<\alpha\rangle$ as follows. A_0 is the Kripke model (F,V). Take unions at limits - this works by 5.15. $A_{\xi+1}$ is obtained from A_ξ by the addition of one equivalence class of $G\setminus F$ and extending the valuation in the suitable fashion as indicated below, depending on whether we have case 1, 2.1-2.3, 3 or 4.

Case 1. Suppose the equivalence class c of $G\setminus F$ has $a<c$ and $b>c$ in F with $[\![a]\!]^n=[\![b]\!]^n$. Extend the valuation on the $u\in c$ by copying its behaviour on a (which is the same on b). Let $A=A_\xi$ (hence, $[\![a]\!]^n=[\![A,a]\!]^n$) and $B=A\cup c=A_{\xi+1}$ with the valuation as indicated.

Claim 1. If $u\in c$ and $m\leqslant n$ then $[\![B,u]\!]^m=[\![A,a]\!]^m$.
Proof. Induction on m. For $m=0$, this is immediate from the definition of the valuation on B. For $m+1$: (i) if $a'>a$ realizes the m-characteristic σ in A, there must be $b'>b$ realizing σ as well. But, $u<b'$. (ii) if $v>u$ realizes the m-characteristic σ in B and $v\in A$ then, by induction hypothesis, v realizes σ in A as well and $a<v$. And if $v\in c$ then, by induction hypothesis, $[\![A,b]\!]^m=[\![A,a]\!]^m=[\![B,v]\!]^m$, so $b>a$ realizes σ in A. ⊠

Claim 2. If $x\in A$ then $[\![A,x]\!]^n=[\![B,x]\!]^n$.
Proof. Immediate from 1. ⊠

Case 2.1. $c_1<c<c_2$; $c_1,c_2\in F/\sim$ are neighbours; c_1 has a maximum, c_2 has no minimum, c is an equivalence class of $G\setminus F$. Again, let A be the model A_ξ constructed so far, $B=A\cup c=A_{\xi+1}$. Choose an n-characteristic σ which is realized in A by elements occurring initial in c_2; extend the valuation on c by copying from *those* elements.

Claim 1. If $u \in c$ and $m \leqslant n$ then $[\![B,u]\!]^m = \sigma^m$ (by which I mean the m-characteristic of *any* element having n-characteristic σ).

Proof. Induction on m. ⊠

Again, we have a second claim as in case 1 as a consequence. Notice that this argument also works for case 3.

Case 2.2. $c_1 < c < c_2$; c_1, $c_2 \in F/\sim$ are neighbours; c_1 has no maximum but c_2 has a minimum; c is an equivalence class of $G \setminus F$. Let VAR_φ be the set of variables of φ; $a \in A$ has **shape** $X \subset \mathrm{VAR}_\varphi$ when $X = \{x \in \mathrm{VAR}_\varphi | a \in V(x)\}$. Now define W on c such that each shape occurs cofinal in c_1 iff it occurs cofinal in c (since $F \prec G$, c cannot have a maximum). Employing the Ehrenfeucht game characterization, it is clear that we obtain the proper type of extension (i.e., n-characteristics are being preserved and no new n-characteristic is realized).

Case 2.3. $c_1 < c < c_2$; c_1, $c_2 \in F/\sim$ are neighbours; c_1 has no maximum and c_2 has no minimum; c is an equivalence class of $G \setminus F$. If c has no maximum, the method of 2.2 may be employed. However, the method of 2.1 works in all cases.

Case 3. This is handled the same way as 2.1.

Case 4. Use the method of 2.2.

5.16 Remark. (van Benthem) The same procedure can be used for the logic of time - except that there no solution can be obtained for the extension-pattern 2.3 above. But, in some cases, we *know* that pattern cannot occur. For instance, if a tense logical formula is first-order on all countable *discrete* linear orderings, it must be first-order on all discrete orderings.

6. Game theory for intensional logics, exact-universal Kripke models and normal forms.

The Ehrenfeucht game technique can be modified for use in investigations of intensional logics. Here, the case of modal logic is considered only; the modifications needed for tense and intuitionistic logic below are then more or less clear. Using the game characterization, exact-universal models are built which can be used to construct normal forms. This will be used extensively in a comparatively simple case in chapter 9.

6.1. Suppose then that $\mathbf{A}=(A,R)$ $(R\subset A^2)$ is any frame and $V\colon \mathrm{VAR}\rightarrow \mathcal{P}(A)$ is a valuation mapping the set VAR of propositional variables onto subsets of A. The pair $(\mathbf{A},V)$ is called a **Kripke model**. For a modal formula φ and $a\in A$, φ is forced at a ($a\Vdash\varphi$) iff a satisfies the **standard interpretation** $\mathrm{ST}(\varphi)$ of φ in the associated model $(\mathbf{A},V(p))_{p\in\mathrm{VAR}}$, where $\mathrm{ST}(\varphi)$ is defined by the following clauses:

1. $ST(p) = p(v_0)$

(here, $p \in VAR$ on the right-hand side of this equation is used as a unary relation-symbol interpreted by $V(p)$ in $(\mathbf{A}, V(p))_{p \in VAR}$)

2. (i) $ST(\neg\varphi) = \neg ST(\varphi)$

(ii) $ST(\varphi \wedge \psi) = ST(\varphi) \wedge ST(\psi)$

(and similarly for the other connectives if present)

3. (i) $ST(\Box\varphi) = \forall v_1 (R(v_0, v_1) \rightarrow ST(\varphi)^+)$

(ii) $ST(\Diamond\varphi) = \exists v_1 (R(v_0, v_1) \wedge ST(\varphi)^+)$

Here, ψ^+ is obtained from ψ by raising indices of all variables in ψ by 1. Clearly, if the modal formula φ has modal rank n (defined in the obvious way) then $ST(\varphi)$ is an R-restricted first-order formula with v_0 as only free variable which has quantifier rank n.

6.2. The **restricted Ehrenfeucht game of length** n on Kripke models $(\mathbf{A}, V)$ and $(\mathbf{B}, W)$ $(\mathbf{A}=(A,R)$, $\mathbf{B}=(B,S))$ with **initial position** $(a_0, b_0) \in A \times B$ is played by ***I*** and ***II*** as follows. First, ***I*** chooses either $a_1 \in A$ such that $a_0 R a_1$ or $b_1 \in B$ such that $b_0 S b_1$. In the first case, ***II*** answers with some $b_1 \in B$ such that $b_0 S b_1$. In the second, ***II*** chooses $a_1 \in A$ such that $a_0 R a_1$. A position $(a_1, b_1) \in A \times B$ results and the procedure is repeated until each player has had n moves. In so doing, they have set up an n-element sequence $\langle (a_i, b_i) \mid i < n \rangle$ in $A \times B$; and we shall say that ***II* has won the play** iff for each $i < n$ and $p \in VAR$: $a_i \in V(p)$ iff $b_i \in V(p)$ - otherwise ***I* has won**.

Of course, this game has its ordinal-bounded version. But somehow, intensional logic never is considered in the context of an infinitary language. (But see *dynamic logic*.)

6.3. Suppose now that VAR is *finite*.

For $(\mathbf{A}, V)$ a Kripke model, $a \in A$ and $n \in \mathbb{N}$, define the modal formula $[\![a]\!]^n = [\![(\mathbf{A}, V, a)]\!]^n$ as follows:

1. $[\![a]\!]^0 = \bigwedge(\{p \in \mathrm{VAR} \mid a \in V(p)\} \cup \{\neg p \mid p \in \mathrm{VAR} \wedge a \notin V(p)\})$;
2. $[\![a]\!]^{n+1} = [\![a]\!]^0 \wedge \Box \bigvee_{aRa'} [\![a']\!]^n \wedge \bigwedge_{aRa'} \Diamond [\![a']\!]^n$.

Clearly, $[\![a]\!]^n$ is a formula of modal rank n forced at a in $(\mathbf{A}, V)$. A modification of the proofs of 1.5.1/1.6.3 will show that

6.4 Theorem. *For $(\mathbf{B}, W)$ a Kripke model, $b \in B$ and $[\![b]\!]^n = [\![(\mathbf{B}, W, b)]\!]^n$, the following are equivalent:*

1. *II has a winning strategy for the restricted Ehrenfeucht game of length n on $(\mathbf{A}, V)$, $(\mathbf{B}, W)$ with initial position (a, b);*
2. *for each modal formula φ of rank $\leqslant n$: $a \Vdash \varphi$ iff $b \Vdash \varphi$;*
3. *$b \Vdash [\![a]\!]^n$;*
4. *$[\![b]\!]^n = [\![a]\!]^n$.*

Suppose now that K is a class of Kripke models.

6.5 Definition. The Kripke model $(\mathbf{A}, V)$ is called

(i) K-**universal** if for each $(\mathbf{B}, W) \in K$ and $b \in B$ there is an $a \in A$ such that for all $n \in \mathbb{N}$: $[\![a]\!]^n = [\![b]\!]^n$.

(ii) **exact** if for all $a_1, a_2 \in A$: if for all $n \in \mathbb{N}$, $[\![a_1]\!]^n = [\![a_2]\!]^n$, then $a_1 = a_2$.

I discuss one important case only.

6.6 Theorem. *There exists an (obviously: unique within isomorphism) exact Kripke model which is universal with respect to all Kripke models over finite (reflexive) partial orderings.*

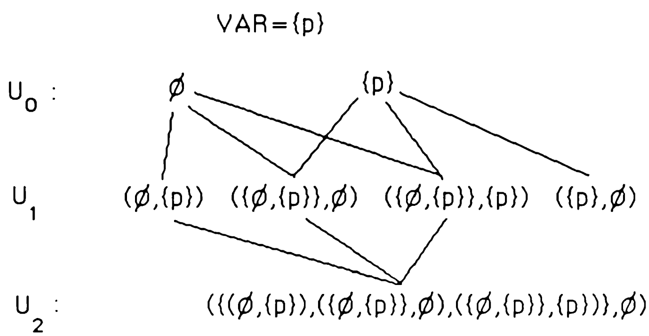
Proof. Let us denote the model to be constructed by $(U, \leqslant, V)$.
It will turn out that $>$ is well-founded. Let ϱk be the **rank** of $k \in U$ relative to $>$ (i.e., $\varrho k = \sup\{\varrho l + 1 \mid l > k\}$). It will turn out that all ranks are finite. Define $U_n = \{k \in U \mid \varrho k = n\}$ $(n \in \omega)$ and $\sigma k = \{p \in \mathrm{VAR} \mid k \in V(p)\}$. It is now easy to build the U_n one after the other explicitly by recursion on n, defining $\leqslant$ and σ along the way, by simply imagining what possibilities may occur in a finite Kripke model over a partially ordered set.

First, $U_0 = \mathcal{P}(\mathrm{VAR})$; $\sigma | U_0$ is defined by: $\sigma k = k$.

Next, assume the U_i constructed up to and including U_n; σ and $\leqslant$ defined on $\bigcup_{i \leqslant n} U_i$. U_{n+1} now consists of the following two types of objects:

1. Each $k \in U_n$ has predecessors $(k, j) \in U_{n+1}$ for each $j \in U_0$ such that $j \neq \sigma k$; σ is defined on these by $\sigma(k, j) = j$.
2. For each *anti-chain* $A \subset \bigcup_{i \leqslant n} U_i$ which intersects U_n and has at least two elements and each $j \in U_0$ there is an element $(A, j) \in U_{n+1}$ which precedes every $k \in A$; σ is defined on these by $\sigma(A, j) = j$.

The picture gives a tiny top-part of the model when VAR$=\{p\}$ is a



singleton. (Sehtman [1978] has a picture of this model with all 30 elements of U_2 shown. The picture which goes with thm.9.1 below represents an important submodel U' of U for VAR$=\{p\}$.) This completes the description of (U, V). That it works is seen below in 6.7/8. ⊠

From now on, K is the class of finite Kripke models over partially ordered sets.

6.7 Theorem. *For each* $(B, W) \in K$, *there is a canonical map* $h: B \to U$ *such that*

(i) $a \leqslant b \Rightarrow ha \leqslant hb$

(ii) $ha \leqslant k \Rightarrow \exists b \geqslant a\,(hb=k)$

(h is a p-morphism*)*

(iii) *for all* $b \in B$, $n \in \mathbb{N}$: $[\![b]\!]^n = [\![hb]\!]^n$.

(hence, (U, V) *is K-universal)*

Proof. hb is defined by recursion on ϱb, the rank of $b \in B$ with respect to the relation $>$, where $B = (B, <)$.

If $\varrho b = 0$, $hb = \sigma b$ ($=\{ p \mid b \in W(p) \}$).

If $\varrho b > 0$, let $C = \{ hb' \mid b < b' \}$; let A be the set of minimal elements of C. Then A is an anti-chain with at least one element of rank $\varrho b - 1$. Distinguish three possibilities.

1. $A = \{k\}$, $\sigma b = \sigma k$. Put $hb = k$.

2. $A = \{k\}$, $\sigma b \neq \sigma k$. Put $hb = (k, \sigma b)$.

3. $|A| \geqslant 2$. Put $hb = (A, \sigma b)$.

Now, (i)-(iii) are clear. ⊠

6.8 Lemma. (U, V) *is exact.*

Proof. Suppose that $k,l \in U$ and for all $n \in \mathbb{N}$, $[\![k]\!]^n = [\![l]\!]^n$. Apply induction on ϱk. Let $A = \{x \in U \mid k < x\}$ and $B = \{x \in U \mid l < x\}$. By induction hypothesis and 6.9 below, it follows that $A = B$. Now, $k = l$ is clear. ⊠

6.9 Lemma. *If a, b are elements of finite Kripke models in K and*

$$[\![a]\!]^{2\varrho a+1} = [\![b]\!]^{2\varrho a+1} \text{ then } [\![a]\!]^n = [\![b]\!]^n \text{ for all } n \in \mathbb{N}.$$

Proof. Induction on ϱa. First, if $\varrho a = 0$, a is maximal and the result is obvious. Next, assume $\varrho a = n > 0$, $[\![a]\!]^{2n+1} = [\![b]\!]^{2n+1}$ and m is minimal such that $[\![a]\!]^m \neq [\![b]\!]^m$. Let I use a winning strategy in the m-game on (a,b) which always picks elements of *minimal* possible rank. If, using this strategy, I starts picking a or b, II answers b resp. a and I "looses a tempo": there are $m-1$ moves left for either player and so II can win by choice of m. If I starts with $a' > a$, II picks $b' \geqslant b$ with $[\![b']\!]^{2n} = [\![a']\!]^{2n}$ and wins by the inductive hypothesis. If I starts with $b' > b$, II picks $a' \geqslant a$ such that $[\![a']\!]^{2n} = [\![b']\!]^{2n}$. There are two cases to distinguish.

(i) $a' \neq a$. Then $\varrho a' < \varrho a$ and II wins by the inductive hypothesis.

(ii) $a' = a$. Consider the second move of I. This cannot be a' or b' for tempo-loss will result. Also, it cannot be $b'' > b'$ since I's strategy picks elements of least rank, so it would have choosen b'' as a *first* move already. Therefore, it will be some $a'' > a'$ and II wins by the inductive hypothesis. ⊠

6.10 Problem. For each $k \in U$, we know by 6.9 that $[\![k]\!]^{2\varrho k+1}$ **defines** k in the sense that k is the only element of U at which the formula is forced. Determine for each $k \in U$ the *least* n such that $[\![k]\!]^n$ defines k (and give a more manageable equivalent of $[\![k]\!]^n$). (The construction of Sehtman [1978] does not seem to satisfy this minimality-requirement.) A special case of this problem has a simple answer, cf. chapter 9 below.

6.11 Lemma. *For all $k \in U$ and $n \in \mathbb{N}$ there is an $l \in U$ such that*

1. $[\![l]\!]^n = [\![k]\!]^n$;
2. $\varrho l \leqslant n$.

Proof. Similar to the one of 6.7. Induction on n. For $n=0$, this is clear. Next, let C be the set of $l \in \bigcup_{i \leqslant n} U_i$ such that for some $k' > k$, $[\![l]\!]^n = [\![k]\!]^n$. Let A be the set of minimal elements of C. The required l is constructed from A and σk. ⊠

We now have the following corollary:

6.12 On normal forms. *Let φ be a formula of modal rank n. On Kripke models in K, φ is equivalent with $\bigvee \{ [\![k]\!]^n \mid \varrho k \leqslant n \wedge k \Vdash \varphi \}$.*

7. Completeness for $\mathbb{Z}$-time.

The theorem of this chapter, asserting completeness of a certain system of tense logic with respect to $\mathbb{Z}$-time, is due to Segerberg [1970]. A different proof is in van Benthem [1983] (cf. II.2.3.15) and another, relatively simple one is in de Jongh et al. [1986]. Our proof is related to the method of chapter 3; however, the relationship is not that exact, due to the fact that the tense logical formalism lacks first-order possibilities such as quantifier relativization. This weakness also is responsible for the fact that the Suslin property of $\mathbb{R}$ has no influence on the theory of $\mathbb{R}$-time; contrast this with 3.3.6/9. Nevertheless, we shall put to good use tense logical versions of n-characteristics.

The logic of time has operators G and F with the same semantics as the modal ones $\Box$ and $\Diamond$. Next to these, there is a dual pair: H ($t \Vdash H\varphi$ iff $\forall t' < t$: $t' \Vdash \varphi$) and P ($t \Vdash P\varphi$ iff $\exists t' < t$: $t' \Vdash \varphi$). Of course, there is the tense logical version of the Ehrenfeucht game: player ***I*** now is allowed to move downward as well as upward in the ordering (which represents the time structure) and ***II*** has to follow ***I*** in this respect. Also, there are the n-characteristics $[\![a]\!]^n$ coding the game-theoretic behaviour of a in the n-game with respect to a finite set of variables.

Theorem. *The tense logical theory of* $\mathbb{Z}$ ***(integer time)*** *is axiomatized by the following principles:*

trans	$\mathrm{G}p \to \mathrm{GG}p$
succ	FT; PT (T *the constant for* true)
r-lin	$\mathrm{F}p \to \mathrm{G}(\mathrm{F}p \vee p \vee \mathrm{P}p)$
l-lin	$\mathrm{P}p \to \mathrm{H}(\mathrm{P}p \vee p \vee \mathrm{F}p)$
modified Löb	$\mathrm{G}(\mathrm{G}p \to p) \to (\mathrm{FG}p \to \mathrm{G}p)$
	$\mathrm{H}(\mathrm{H}p \to p) \to (\mathrm{PH}p \to \mathrm{H}p)$.

For a precise definition of tense logical derivability, cf. van Benthem l.c. pp.167/8.

Proof. Suppose that the formula χ cannot be derived using these principles. I shall show how to construct a valuation V on $\mathbb{Z}$ such that $(\mathbb{Z}, <, V) \models \neg\chi[n]$ for some $n \in \mathbb{Z}$.

As a first step, we need the Henkin construction for tense logic (cf. van Benthem l.c. pp.170-173). This produces a model (M, R, V) such that

1. *the axioms given all hold (universally) in the model;*
2. *for some* $m \in M$, $(M, R, V) \models \neg\chi[m]$.

In fact, M consists of all sets of formulas maximal consistent with the given axioms, and R is defined by

3. xRy *iff for all* φ, *if* $\mathrm{G}\varphi \in x$ *then* $\varphi \in y$.

The basic tense logical axioms now allow one to prove the following *truth lemma*:

4. $(M, R, V) \vDash \varphi[x]$ *iff* $\varphi \in x$

on the basis of the following definition of V

5. $x \in V(p)$ *iff* $p \in x$.

By assumption, $\neg\chi$ will be in some $m \in M$, so 2. follows from 4. Also, 1. follows; as substitution is one of the derivation rules, substitution instances of the axioms are satisfied as well. 6-8 now investigate the effect the axioms **trans** up to **1-lin** have on the structure of (M, R); this is standard procedure.

6. *R is transitive.*

Proof: suppose $xRyRz$. If $G\varphi \in x$ then by 4., x satisfies $G\varphi$ and hence $GG\varphi$ (use **trans**). Hence, $GG\varphi \in x$ by 4. again. Applying 3. twice, this gives $G\varphi \in y$ and $\varphi \in z$. Therefore, xRz by 3. ⊠

7. *M has no R-minimum or R-maximum.*

Proof: immediate from **succ**. ⊠

8. *Every two elements with a common upper bound (resp. lower bound) are comparable.*

Proof: suppose that x, yRz, $x \neq y$, $\neg xRy$, $\neg yRx$. For instance, $\varphi \in x \backslash y$, $G\psi \in x$, $\psi \notin y$, $G\eta \in y$, $\eta \notin x$. Now, z satisfies $P(\varphi \wedge G\psi \wedge \neg\eta)$. Hence, by **1-lin**, it satisfies $H[F(\varphi \wedge G\psi \wedge \neg\eta) \vee (\varphi \wedge G\psi \wedge \neg\eta) \vee P(\varphi \wedge G\psi \wedge \neg\eta)]$ as well. Therefore, y satisfies one of $F(\varphi \wedge G\psi \wedge \neg\eta)$, $\varphi \wedge G\psi \wedge \neg\eta$, $P(\varphi \wedge G\psi \wedge \neg\eta)$. But the first alternative contradicts y satisfying $G\eta$, the second $\varphi \notin y$ and the third $\psi \notin y$. ⊠

Define $\sim$ on M by

$$x \sim y \text{ iff } x = y \text{ or: both } xRy \text{ and } yRx.$$

By 6., $\sim$ is an equivalence.

R induces a partial ordering $R/\sim$ on the set of equivalence clases by

$$|x|\, R/\sim |y| \text{ iff } xRy.$$

Let m be the χ-falsifying element (2.) then restricting to equivalence classes $|x|$ with xRm or $x=m$ or mRx produces a *linear* ordering by 8.; since R is transitive, the model-theoretic properties of (M, R, V) won't change by restricting to such x - hence we may assume xRm or $x=m$ or mRx for all $x \in M$ to begin with.

Let VAR_χ be the (finite) set of variables in χ. $\mathbf{P}(\mathrm{VAR}_\chi)$ is the set of **shapes**; x **has** $S \subset \mathrm{VAR}_\chi$ (S **occurs at** x) iff $S = x \cap \mathrm{VAR}_\chi$.

Let A be an equivalence class of M under $\sim$. If $A=\{a\}$ and $\neg aRa$ then $A^\bullet = A$. In all other cases, $A^\bullet$ denotes a model $(\mathbb{Z}, <, V_A)$ of order-type ζ such that

(i) each shape occurring in it occurs in A;

(ii) if S occurs in A then the set $\{n \in \mathbb{Z} \mid n \text{ has } S\}$ has neither lower nor upper bound.

Now, define $N = \Sigma_{A \in M/\sim} A^\bullet$.

9. *Suppose that $x \in A \in M/\sim$ and $n \in A^\bullet$ have the same shape. Then for all formulas φ over VAR_χ: $M \models \varphi[x]$ iff $N \models \varphi[n]$.*

Proof. Use the Ehrenfeucht game appropriate to tense logic. Notice that *II* can always take care to leave a position (y, m) for *I* for which (i) for all $A \in M/\sim$: $y \in A$ iff $m \in A^\bullet$ and (ii) y and m have the same shape. ⊠

Therefore, we now have a counter-model to χ of an order type which is a sum of ζ's and 1's. To finally transform this into a counter-model of type ζ, I use the **modified Löb**-axioms.

10. *Suppose that φ is a formula over VAR_χ such that $\varphi^N = \{ n \in N \mid N \models \varphi[n] \}$ is non-empty and upward (downward) bounded. Then φ^N has a maximum (minimum).*

Proof. Let $N \models \varphi[n]$ and $m < n$. Then m satisfies $F\varphi$ and $FG\neg\varphi$. Since $F\varphi$ amounts to $\neg G \neg \varphi$, by the first **modified Löb**-axiom (with $\neg\varphi$ substituted for p): $N \models \neg G(G\neg\varphi \rightarrow \neg\varphi)[m]$. Choose $k > m$ such that k satisfies $G\neg\varphi$ and φ; k is the required maximum. ⊠

Let k be the rank of χ. Put $T = \{ [\![x]\!]^k \mid x \in N \}$ ($[\![x]\!]^k$ codes the "behaviour" of x in the game of length k). Define $T^+ = \{ \tau \in T \mid \{ x \in N \mid \tau = [\![x]\!]^k \}$ has an upper bound$\}$ and $T^- = \{ \tau \in T \mid \{ x \in N \mid \tau = [\![x]\!]^k \}$ has a lower bound$\}$. By 10., to each $\tau \in T^+$ there is a maximal $x = x_\tau$ with $[\![x]\!]^k = \tau$ and similarly for T^-. Let A_0 be the set $\{ x_\tau \mid \tau \in T^+ \cup T^- \}$. Choose $A^+ \subset N$ of order type ω such that $A_0 < A^+$ and such that for all $\tau \in T \setminus T^+$, $\{ x \in A^+ \mid [\![x]\!]^k = \tau \}$ is infinite. Similarly, choose $A^- \subset N$ of order type ω^* such that $A^- < A_0$ and each $\{ x \in A^- \mid [\![x]\!]^k = \tau \}$ for $\tau \in T \setminus T^-$ is infinite. Finally, A is the submodel of N obtained by restricting to $A^- \cup A_0 \cup A^+$.
So, A has order type ζ; it suffices to prove

11. *If the formula ψ over VAR_χ has rank $\leqslant k$ and $x \in A$ then $A \models \psi[x]$ iff $N \models \psi[x]$.*

Proof. Induction on ψ. There is but one interesting case. Suppose that $N \models F\psi[x]$. Then $y > x$ exists such that $N \models \psi[y]$. Let $\tau = [\![y]\!]^k$. By construction, there is a $z > x$ in A such that $[\![z]\!]^k = \tau$. But then, $N \models \psi[z]$ as well. By induction hypothesis $A \models \psi[z]$. Hence, $A \models F\psi[x]$. ⊠

8. Rodenburg's tree problem.

8.1 Introduction. By a suggestion of Troelstra, Rodenburg [1982] continued the investigation of intuitionistic correspondence theory which was begun by van Benthem (in a preliminary version of van Benthem [1984]). (For the latest developments, cf. Rodenburg [1986].) Part of this asks for which intuitionistic formulas φ, validity of φ on intuitionistic (i.e., partially ordered) frames - which is a Π^1_1-condition to begin with - can be defined in first-order terms. This question is part of its modal companion: by the Gödel translation, intuitionistic formulas can be translated into modal terms, and this means that the intuitionistic correspondence problem considers rather special modal formulas only. But also, the class of frames is restricted to partial orderings, and in the Π^1_1-condition, the second-order quantifiers range over upward closed sets only. And this makes for the rather different flavor of the subject. In

1983, I solved two problems left open in Rodenburg [1982]. The first, on whether validity of a certain formula B_2 (called SP_2 in Rodenburg [1986]) is first-order definable on the class of (binary) trees, is dealt with in this chapter. The second is the same problem for intuitionistic formulas in one variable and the class of *finite* partial orderings. This is the content of chapter 9.

8.2 Preliminaries. Consider a propositional language over a set VAR of propositional variables with a propositional constant $\bot$ for falsehood and operations $\rightarrow$, $\wedge$ and $\vee$. Negation $\neg$ can either be taken as a primitive or else be explained as $\neg\varphi = (\varphi\rightarrow\bot)$. Let $\mathbf{A}=(A, \leqslant)$ be a partially ordered set and V: VAR$\rightarrow\mathcal{P}(A)$ a valuation; then $(\mathbf{A}, V)$ is the corresponding Kripke model. The **intuitionistic semantics** of the formulas of the above language on such a model can be explained simply in modal terms as follows: for $a\in A$, φ a formula, define $a\Vdash\varphi$ (intuitionistically) iff $a\Vdash\varphi^m$ (modally), where the **modal translation** φ^m of φ is defined as follows:

(i) for $p\in$VAR, $p^m=p$; $\bot^m=\bot$;

(ii) ^{m} preserves $\wedge$ and $\vee$;

(iii) $(\varphi\rightarrow\psi)^m = \Box(\varphi^m\rightarrow\psi^m)$

(hence, $(\neg\varphi)^m = \Box\neg\varphi^m$).

In Kripke models $(\mathbf{A}, V)$ for the intuitionistic language, $\mathbf{A}$ is always a partially ordered set and V is such that each $V(p)$ ($p\in$VAR) is **upward closed**, i.e., $a\in V(p)$ and $a\leqslant b$ imply $b\in V(p)$.

8.3. Rodenburg [1982] considers the intuitionistic formula

B_2: $(\neg\varphi\vee\neg\psi\vee\neg\chi \rightarrow \varphi\vee\psi\vee\chi) \rightarrow \neg\varphi\vee\neg\psi\vee\neg\chi$

where: $\varphi = p \wedge q$

$\psi = p \wedge \neg q$

$\chi = \neg p \wedge q$,

and puts the question whether it is first-order definable on the class of "rooted trees". The binary trees of 2.1 form an elementary subclass of this collection. Therefore, a negative answer would be implied by showing B_2 not to be first-order on binary trees. Now, Rodenburg's arguments show that

8.4 Lemma. *For binary trees T,* B_2 *is valid on T iff T is finite.*

Hence, his problem is answered negatively by corollary 2.2.4 above. Notice, however, that 2.4.1 even shows B_2 not to be $\Sigma^1{}_1$ on binary trees.

Proof of 8.4. First, let T be infinite. By König's lemma, it has an infinite branch α. Suppose $\alpha = \{t_0, t_1, t_2, \ldots\}$ where $t_0 < t_1 < t_2 < \ldots$ and let s_i be the immediate successor of t_i different from t_{i+1}. Define the following valuation on T:

$$V(p) = \{ t \in T \mid \exists i (s_{3i} \leqslant t \vee s_{3i+1} \leqslant t) \}$$

$$V(q) = \{ t \in T \mid \exists i (s_{3i+1} \leqslant t \vee s_{3i+2} \leqslant t) \}.$$

Clearly, $s_{3i} \leqslant t \Rightarrow t \Vdash \psi$

$s_{3i+1} \leqslant t \Rightarrow t \Vdash \varphi$

$s_{3i+2} \leqslant t \Rightarrow t \Vdash \chi$;

hence, $t \in T \setminus \alpha \Rightarrow t \Vdash \varphi \vee \psi \vee \chi$ (1).

Also, no $t \in \alpha$ forces one of $\neg\varphi$, $\neg\psi$, $\neg\chi$, i.e.,

$$t \in \alpha \Rightarrow t \nVdash \neg\varphi \vee \neg\psi \vee \neg\chi \quad (2).$$

In particular, t_0 does not force $\neg\varphi \vee \neg\psi \vee \neg\chi$. Thus, to see that B_2 is not valid on (T, V), it suffices to see that $t_0 \Vdash \neg\varphi \vee \neg\psi \vee \neg\chi \rightarrow \varphi \vee \psi \vee \chi$. Hence, suppose $t_0 \leqslant u \Vdash \neg\varphi \vee \neg\psi \vee \neg\chi$. By (2), $u \notin \alpha$, therefore, by (1), $u \Vdash \varphi \vee \psi \vee \chi$, and we're done.

Conversely, assume that T is finite, but $T \nVdash B_2$. So, there are a valuation V and $u \in T$ such that $u \nVdash B_2$. For instance, $u \leqslant w_0 \Vdash \neg\varphi \vee \neg\psi \vee \neg\chi \rightarrow \varphi \vee \psi \vee \chi$ but $w_0 \nVdash \neg\varphi \vee \neg\psi \vee \neg\chi$. Since T is finite, we may assume w_0 is a *maximal* element with these properties. Since $w_0 \nVdash \neg\varphi$, there exists $w_1 \geqslant w_0$ with $w_1 \Vdash \varphi$; similarly, there are $w_2, w_3 \geqslant w_0$ such that $w_2 \Vdash \psi$, $w_3 \Vdash \chi$. Now, $\neg\, w_1 \leqslant w_2$, for if not, then $w_2 \Vdash \varphi$ (a formula forced at some place w_1 remains forced at every $w_2 \geqslant w_1$), hence $w_2 \Vdash q$, contradicting $w_2 \Vdash q$.
In the same way, it is seen that no two of w_1, w_2, w_3 are comparable; in particular, they are all $> w_0$. Let w be the greatest lower bound of w_1 and w_2. If $w_0 < w$ then $w \Vdash \neg\varphi \vee \neg\psi \vee \neg\chi \rightarrow \varphi \vee \psi \vee \chi$ (since this is forced by w_0) and $w \Vdash \neg\varphi \vee \neg\psi \vee \neg\chi$ (by maximality of w_0) thus $w \Vdash \varphi \vee \psi \vee \chi$. However, $w \nVdash \varphi$ (if not, then $w_2 \Vdash \varphi$, but $w_2 \Vdash \psi$), $w \nVdash \psi$ (else $w_1 \Vdash \psi$) and $w \nVdash \chi$ (otherwise, $w_1 \Vdash \chi$) - a contradiction. Therefore, $w = w_0$. But a similar argument shows that w is greatest lower bound of each two of w_1, w_2, w_3 - an impossibility in a *binary* tree. ⊠

By contrast, Rodenburg [1982] theorem 6.1.7 says that one-variable formulas all are first-order on (general) trees. Compare this also with the results discussed in the next chapter.

9. Formulas in one variable.

A problem left open by Rodenburg [1982] was to determine which intuitionistic formulas in one variable are first-order definable on *finite* partial orderings. This is solved below. First, however, I construct universal-exact Kripke models for intuitionistic logic and present a logical analysis of the one-variable model using Ehrenfeucht's game. This is applied subsequently to the problem of first-order definability.

In the proof of 6.6, an exact Kripke model (U, V) universal with respect to Kripke models (for modal logic) over finite partial orderings was explicitly constructed. In this model, *no* $V(p)$ is upward closed. (Remember this is required of intuitionistic models.) However, U has a maximal upward closed subset U^i such that $V^i(p) = V(p) \cap U^i$ *is* upward closed, this is

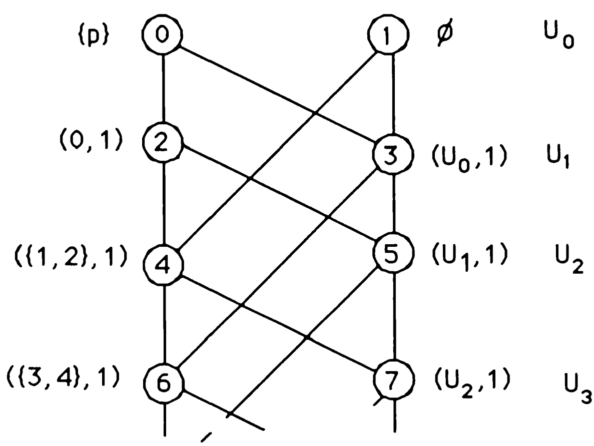
$$U^i = \{\, k \in U \mid \forall p \in \mathrm{VAR}\, \forall m \in U [\, k \leqslant m \wedge k \in V(p) \Rightarrow m \in V(p)\,]\,\}.$$

9.1 Theorem. *1. (U^i, V^i) is (modally) exact;*

2. (U^i, V^i) is universal with respect to finite intuitionistic Kripke models.

Proof. 1. Obvious. 2. Modify the proof of 6.6. ⊠

The structure of (U^i, V^i) for VAR=$\{p\}$ having one element is quite simple and has been known for a long time. Let us have a short look at it.



The elements in the picture are named by their codes from chapter 6 as well as by natural numbers, the reason for which is explained by 9.2 below.

There is but one element in $V(p)$: $0=\{p\}$. Also, there is but one element (k, j) with $k \in U^i$: $2=(0,1)=(0,\emptyset)$. An anti-chain has at most two elements; since $V(p)=\{0\}$, this means that every other non-maximal element has the form $(A, 1)$ with $|A|=2$. Problem 6.10 for this model is solved as follows.

9.2 Lemma. $[\![n]\!]^k = [\![m]\!]^k$ *iff* $n, m > k$ *or* $n = m$.

In particular, n is the least number i such that $[\![n]\!]^i$ *defines n in* (U^i, V^i).

Proof. Let us agree to denote the partial ordering of U^i by $\leqslant^*$; $<$ then can be used for the usual ordering of $\mathbb{N}$ (as is done in 9.2). Hence, $n <^* m$ iff $m+2 \leqslant n$.

If, in the k-game on a position (i, j), player I plays some $l \geqslant^* i, j$, II can win by just copying the moves of I. Therefore, the *optimal* strategy of I always is to pick $i-1 \geqslant^* j$ (if $i<j$) resp. $j-1 \geqslant^* i$ (if $j<i$): he must get at the top of the model as fast as he can without losing, for, in order to win, he has to take care that exactly one of the players has to arrive at 0, the only element in $V(p)$. As long as II cannot copy I, *his* optimal strategy consists in repeating one of the previous moves (namely, i if I has chosen $i-1$ in position (i, j) etc.): this surely is the slowest way to drag him to the top of the model where 0 is lurking. Therefore, if $n<m$, the play will go like this if both players do their utmost; as long as $k \leqslant n$:

I:	$n-1$	$n-2$	$n-3$	$n-k$	
II:	n	$n-1$	$n-2$		$n-(k-1)$.

Obviously then, I wins iff $k \geqslant n$. ⊠

As to defining (*modal*) formulas:

9.3 Lemma. *(i)* $\varphi_0 = p$ *defines* 0;

(ii) $\varphi_1 = \Box\neg p$ *defines* 1;

(iii) $\varphi_2 = \neg p \wedge \Box\Diamond p$ *defines* 2;

(iv) if $\varphi_n, \varphi_{n+1}, \varphi_{n+2}$ *define resp.* $n, n+1, n+2$, *then* $\varphi_{n+3} = \Diamond\varphi_n \wedge \Diamond\varphi_{n+1} \wedge \neg\Diamond\varphi_{n+2}$ *defines* $n+3$ $(n \geqslant 0)$.

Proof. Clear from the picture. ⊠

It is easy to show by induction that for *intuitionistic* formulas φ, the set $\{k \in U^i \mid k \Vdash \varphi\}$ defined by φ always will be upward closed. Now, U^i has the following types of upward closed sets:

1. $\emptyset$;
2. $0\uparrow = \{0\}$; $1\uparrow = \{1\}$; $2\uparrow = \{0,2\}$;
3. $(n+3)\uparrow = \{0,\ldots,n+1,n+3\}$ $(n \geqslant 0)$;
4. $n\uparrow \cup (n+1)\uparrow = \{0,\ldots,n+1\}$ $(n \geqslant 0)$;
5. $U^i = \mathbb{N}$.

9.4 Lemma on normal forms.

1. $\bot$ *defines* $\emptyset$;

2.(i) $\psi_0 = p$ *defines* $0\uparrow$;

(ii) $\psi_1 = \neg p$ *defines* $1\uparrow$;

(iii) $\psi_2 = \neg\neg p$ *defines* $2\uparrow$;

if ψ_n, ψ_{n+1}, ψ_{n+2} *define resp.* $n\uparrow$, $(n+1)\uparrow$ *and* $(n+2)\uparrow$ *then:*

3. $\psi_{n+3} = \psi_{n+2} \to \psi_n \vee \psi_{n+1}$ *defines* $(n+3)\uparrow$, *and*

4. $\psi_n \vee \psi_{n+1}$ *defines* $n\uparrow \cup (n+1)\uparrow$;

5. $p \to p$ *defines* $\mathbb{N}$.

Proof. Notice that *intuitionistic* formulas are intended here. Therefore, a given formula ψ defines $X \subset U^i$ *intuitionistically* iff ψ^m (see 8.2) defines X *modally*, and this is easily checked in all cases. ⊠

Remark. Notice that the φ_n from 9.3 has modal rank n - the least possible rank for an n-defining formula according to 9.2. Also, $\psi_n{}^m$ has rank n. The connection between these two types of formulas is given by the

9.5 Lemma. *On intuitionistic Kripke models, $\psi_{n+1}{}^{m}$ is forced at a node iff $\neg\Diamond\varphi_n$ is.*

Proof. Semantically, this is clear on (U^i, V^i). But a syntactic proof also is possible. ⊠

9.6 Lemma. *Let φ, ψ be intuitionistic one-variable formulas and k the maximum of the modal ranks of φ^m and ψ^m. Then φ and ψ are equivalent (i.e., true on the same frames) iff*

$$\{m \leqslant k+1 \mid m \Vdash \varphi\} = \{m \leqslant k+1 \mid m \Vdash \psi\}.$$

Proof. Suppose this condition holds. By 9.2, $[\![n]\!]^k = [\![n']\!]^k$ when $n' > n > k$. Let a be any element of any model (F, V). Then $[\![a]\!]^k$ must be one of $[\![0]\!]^k, \ldots, [\![k+1]\!]^k$; say, $[\![a]\!]^k = [\![m]\!]^k$ with $m \leqslant k+1$. Hence, $a \Vdash \varphi$ iff $m \Vdash \varphi$ iff $m \Vdash \psi$ iff $a \Vdash \psi$. ⊠

9.7 Remark. It is easy to extend the one-variable model to one that is exact and universal with respect to *all* Kripke models: simply add a *least* element ∞. To see that this suffices, let (A, V) be any Kripke model and $a \in A$. If $a \Vdash \varphi_n$ for some n, then for all k, $[\![a]\!]^k = [\![n]\!]^k$. And, if no φ_n is forced at a then, by induction on n, it follows that $a \Vdash \Diamond\varphi_n$ for all n. But then obviously, for all k, $[\![a]\!]^k = [\![\infty]\!]^k$.

Of course, one can do modal and intuitionistic propositional logic with infinitary connectives $\bigwedge$ and $\bigvee$ and define formulas $[\![a]\!]^\alpha$ for all ordinals α relative to an element a in a Kripke model. The above amended model is still universal with respect to such an extended formalism - by the same proof; this is because the players essentially have one optimal strategy each (cf. 9.2) *not* depending on the bound α of the game. (cf. also de Jongh

[1980]).

I now turn to correspondence theory.
According to 9.4, each intuitionistic one-variable formula is (in the Kripke model semantics) equivalent with a formula of one of the following shapes: $\bot$, $p \to p$, ψ_n, $\psi_n \vee \psi_{n+1}$.
Rodenburg [1982] has a complete classification of these formulas as to first-order definability on arbitrary partial orderings as follows:
$\bot$, $p \to p$, ψ_n for $n \leqslant 4$ and $\psi_n \vee \psi_{n+1}$ for $n \leqslant 3$ are first-order; the rest is not.
He then put the question: what about first-order definability of these formulas on *finite* partially ordered sets?
Of course, only cases $n > 4$ for the ψ_n and $n > 3$ for the $\psi_n \vee \psi_{n+1}$ remain to be investigated.
The answer is given by the

9.8 Theorem. *On finite partial orderings,* ψ_5 *,* ψ_n *for* $n \geqslant 7$ *and* $\psi_n \vee \psi_{n+1}$ *for* $n \geqslant 4$ *are* not *first-order (in fact, they are not* monadically Σ^1_1*); however,* ψ_6 is.

Proof. By 6.7, each finite model is canonically mapped into U^i. h always will denote this map; so $a \Vdash \varphi_n$ iff $ha = n$. A model **realizes** n iff $ha = n$ for some a; n is **realizable** in a frame iff it is realized under a suitable valuation on that frame. As to the ψ_n for $n = 5$ and $n \geqslant 7$, notice that ψ_n does not hold in a given model iff $n-1$ or some $m \geqslant n+1$ is realizable in it and this is true iff $n-1$ is realizable. In particular, ψ_5 does not hold iff 4 is realizable. I shall now show that 4-realizability is not first-order on finite partial orderings; using 5.9, it is easy to modify this into a proof showing 4-realizability not to be Σ^1_1. The argument can be looked at as a "finitisation" of Rodenburg's proof for arbitrary partial orderings.

Consider the frame PZ_k of theorem 5.6. Make it into a partial ordering TPZ_k by replacing the relation of PZ_k by its transitive closure.

Claim 1. *4 is not realizable in any* TPZ_k.

Proof. Suppose 4 is realized under some valuation. If at all, this can happen at the least element w only. There must be top-elements realizing 0 and 1; hence there are $b > w$ and $c, d > b$ with $hc = 0$ and $hd = 1$. But then, $hb = 3$, contradicting $w < b$ and $hw = 4$. ⊠

Next, consider the frame $2TPZ_k$ obtained from two copies of TPZ_k by identifying their least elements.

Claim 2. *4 is realizable in each* $2TPZ_k$.

Proof. Just let $V(p)$ be the set of top-elements of one of the TPZ_k-copies in $2TPZ_k$. ⊠

Therefore, in order to see that 4-realizability is not first-order on finite frames, we need

Claim 3. *If* $k \geqslant 2^n$ *then* $TPZ_k \equiv^n 2TPZ_k$.

Proof. Compare the proof of 5.2. ⊠

Showing that ψ_6 is first-order definable on finite frames amounts to showing that 5-realizability is so definable. This is the content of the following

Claim 4. 5 *is realizable in a finite frame A iff there are* x, x_0, y_0, x_1, x_2 *and* x_3 *such that:*

(1) $x < x_2 < x_0$; $x < x_3 < y_0, x_1$

(2) $\neg x_2 \leqslant x_3$

(3) $\forall y(x < y \,\mathbf{comp}\, x_2 \rightarrow y \bigvee x_1)$

(4) $\forall y(x_3 < y \,\mathbf{comp}\, y_0 \rightarrow y \bigvee x_1)$.

Here, $x \,\mathbf{comp}\, y$ means: $x \leqslant y \vee y \leqslant x$; and $x \bigvee y$ stands for $\neg\exists z(x \leqslant z \wedge y \leqslant z)$.

Proof. To see this, first suppose that $V(p) \subset A$ and $x \in A$ are such that $hx=5$. We may assume that x is a maximal element with this property. Choose $x_3 > x$ maximal with $hx_3=3$ Finally, choose x_0, y_0, x_1, x_2 with h-values 0, 0, 1 and 2 respectively such that (1) holds. Now, (2) holds as well. As to (3): if $x < y \,\mathbf{comp}\, x_2$ then, by maximality of x, $hy=2$ or $=0$; hence $y \bigvee x_1$. As to (4): if $x_3 < y \,\mathbf{comp}\, y_0$ then, by maximality of x_3, $hy=0$; hence $y \bigvee x_1$.

Conversely, assume that x, x_0, y_0, x_1, x_2, x_3 satisfy (1)-(4).

Define $V(p) = \{a \in A \mid \neg a \leqslant x_2 \wedge \neg a \leqslant x_3 \wedge a \bigvee x_1\}$.

Notice that $V(p)$ is upward closed. Now:

$hx_0=0$: $\neg x_0 \leqslant x_2$; $\neg x_2 \leqslant x_3$ and $x_2 < x_0$ hence $\neg x_0 \leqslant x_3$; by (3), $x_0 \bigvee x_1$. So $x_0 \in V(p)$, i.e., $hx_0=0$.

$hy_0=0$: $\neg y_0 \leqslant x_3$; $\neg x_3 \leqslant x_2$ and $x_3 < y_0$, hence $\neg y_0 \leqslant x_2$; by (4), $y_0 \bigvee x_1$. So, $y_0 \in V(p)$.

$hx_1=1$: Suppose $x_1 \leqslant z$. Obviously, $\neg z \bigvee x_1$. Thus, $z \notin V(p)$.

$hx_2=2$: $x_2 \notin V(p)$ is clear. Suppose $x_2 \leqslant y$. If $y=x_2$ then $y < x_0 \in V(p)$. If $x_2 < y$ then by (3), $y \bigvee x_1$; furthermore, $\neg y \leqslant x_3$, since if $y \leqslant x_3$ then $x_2 \leqslant x_3$. Hence, $y \in V(p)$.

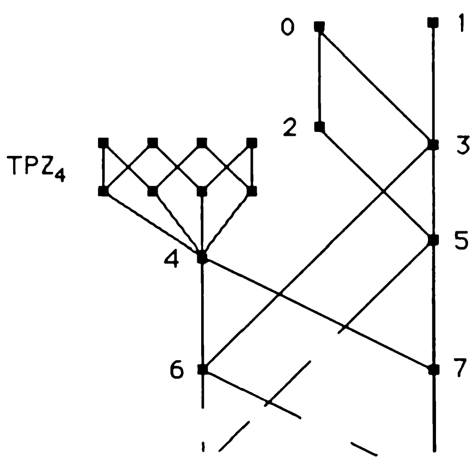
$hx_3=3$: First, $x_3 < y_0, x_1$. Second, suppose $x_3 < z$ and $hz=2$. Then $z \notin V(p)$, $\neg z \leqslant x_3$, $\neg z \leqslant x_2$, thus by definition of $V(p)$, $\neg z \bigvee x_1$. But then, $hz=2$ is impossible.

Finally, it suffices to show that $hx=5$; and for this, it suffices to show that no $z>x$ has $hz=4$. Suppose not. Then $\neg z \leqslant x_3$. Suppose $z \leqslant x_2$. Then zVx_1 by (3). Choose $z_1>z$ with $hz_1=1$. Then $\neg z_1 \leqslant x_2$, x_3 and $z_1 \notin V(p)$, hence $\neg z_1 V x_1$ by definition of $V(p)$; therefore $\neg zVx_1$ - a contradiction. Hence, $\neg z \leqslant x_2$. Choose $z_2>z$ with $hz_2=2$. Then $\neg z_2 \leqslant x_2$ and $\neg z_2 \leqslant x_3$. Since $z_2 \notin V(p)$, we have that $\neg z_2 V x_1$ by definition of $V(p)$ - contradicting $hz_2=2$ and $hx_1=1$. ⊠

For the rest of the proof, I need a

Trivial remark. *Suppose that V is any valuation on $\boldsymbol{U}^i$ different from V^i. Then under it, no $a \in U^i$ obtains a canonical value >3 in $(\boldsymbol{U}^i, V^i)$.*

Now, modify the frame $\boldsymbol{U}^i$ by disconnecting the element 4 from its successors 0, 1 and 2. Consider the frames obtained from this by identifying the now successor-less element 4 with the least node w of a model TPZ_k or $2TPZ_k$; cf. the picture below where TPZ_4 is attached at the $\boldsymbol{U}^i$-node nr. 4.



By the trivial remark and by what we know about the (2)TPZ_k, for $n \geqslant 6$, the resulting frame can realize the canonical value n iff it realizes this value at n iff a model 2TPZ_k is attached at 4. Therefore, final segments of these models (from n upward) can be used to show that ψ_n is not first-order definable on finite partial orderings for $n \geqslant 7$.

As to the $\psi_n \vee \psi_{n+1}$, finally, a frame can be made into a counter-example to this formula iff under a suitable valuation a canonical value $\geqslant n+2$ is realized, and this happens iff one of $n+2$, $n+3$ is realized. To see this is not a first-order condition when $n \geqslant 4$, the model classes constructed for ψ_{n+4} can be used.

For instance, take the case $n=4$.
Look at the partial ordering used to construct the ψ_8-model classes. Whatever the valuation on such a model, $h5=4$ is impossible by the trivial remark. Therefore, the model realizes a value $n \geqslant 6$ iff it does so at element 7 iff $h4=4$. (In fact, if $h4=4$ then $h7=7$ iff $h5=5$ and $h7=6$ iff $h5=3$.) But we know this to be not first-order on the given partial orderings. This finishes the proof of 9.8. ⊠

To contrast the $\psi_n \vee \psi_{n+1}$-case with that of the ψ_n, let us call $A \subset \mathbb{N}$ **realizable** on a model iff some element obtains a value in A under a suitable valuation. Then notice that by Rodenburg [1982], realizability for {4, 5} and {5, 6} *is* first-order definable while this is not the case for 4, 5 and 6 separately (and it is so for 5 only if we restrict attention to *finite* partial orderings).

Appendix A: can time be directional?

Van Benthem [1983] contains a list of properties which the time-structure $(T, \leqslant)$ may have:

SYM : $(T, \leqslant) \cong (T, \geqslant)$.

SYM*: *each* $t \in T$ *splits* T *into order-isomorphic halves* $(\leftarrow, t) = \{ t' \in T \mid t' < t \}$ *and* $(t, \rightarrow) = \{ t' \in T \mid t < t' \}$.

HOM : *for all* $t, t' \in T$ *there is an automorphism moving* t *to* t'.

HOM*: *for all* $t_1 < t_2$, $t_3 < t_4$, *if* $|(t_1, t_2)| = |(t_3, t_4)|$ *then some automorphism maps* (t_1, t_2) *onto* (t_3, t_4).

REF : *each interval* (t_1, t_2) *is isomorphic with* $(T, \leqslant)$.

On p.45, the question is posed as to whether **REF** implies **SYM**. Below, I answer this negatively by constructing, for each uncountable power, a linear ordering of that power which has *every* property listed above - except the first one.

Another solution to this problem is given by Droste [1987].

Given an uncountable initial number, let Ω be a well-ordering of that type. Define the increasing sequence of linear orderings $A_0 \subset A_1 \subset A_2 \subset \ldots$ as follows:

(i) $A_0 = \Omega$;

(ii) A_1 is obtained from A_0 by inserting, for each $a \in A_0$, a copy $\Omega(a)$ of Ω between a and its predecessors in A_0;

(iii) A_{i+2} is obtained from A_{i+1} by inserting a copy $\Omega(a)$ of Ω between a and its predecessors for each $a \in A_{i+1} \setminus A_i$ ($i \geqslant 0$).

$\mathbf{\Omega}_\infty$ is the union $\bigcup_n A_n$.

Clearly, $\mathbf{\Omega}_\infty$ has the same power as Ω.

By induction on n it follows that each A_n is well-ordered (it has the order type Ω^{n+1}). Suppose $X \subset \mathbf{\Omega}_\infty$ has order type Ω^* (the *reversal* of Ω). Then each set $X \cap A_n$, being both well-ordered and conversely well-ordered, must be finite. Therefore, $X = \bigcup_n (X \cap A_n)$ is at most countable - a contradiction. Since $\Omega \subset \mathbf{\Omega}_\infty$, $\mathbf{\Omega}_\infty$ does not satisfy **SYM**.

We are going to prove the

Lemma. *If $a \in \mathbf{\Omega}_\infty$ then $(\leftarrow, a)$ and $(a, \rightarrow)$ are both isomorphic with $\mathbf{\Omega}_\infty$.*

From this, **SYM*** and **HOM** easily follow. As to **REF**, if $t_1 < t_2$ then $(t_1, \rightarrow)$ is isomorphic with $\mathbf{\Omega}_\infty$ by the lemma. Since $\mathbf{\Omega}_\infty$ is isomorphic with each of its initials $(\leftarrow, a)$, so is $(t_1, \rightarrow)$. In particular then, $(t_1, t_2) \cong (t_1, \rightarrow) \cong \mathbf{\Omega}_\infty$. **HOM*** now is a trivial consequence.

Proof of lemma.

I use the following notations. $\Omega_n = A_n \setminus \bigcup_{i<n} A_i$. $a \leftarrow b$ means that a is an element of the copy $\Omega(b)$ of Ω inserted immediately in front of b. For $X \subset \Omega_\infty$, $C(X) = X \cup \bigcup_{x \in X} \Omega(x)$. C^n is the n-th iterate of C. Hence, $A_n = C^n(\Omega)$.

Now, suppose that $a \in \Omega_\infty$. Say, $a \in \Omega_n$.

For $i \leqslant n$, let a_i be the least element of $(a, \rightarrow) \cap \Omega_i$.

Then $a_n \leftarrow a_{n-1} \leftarrow \ldots \leftarrow a_0$ and $(a, a_i) \cap \Omega_i = \emptyset$; if $n > 0$ then $a \leftarrow a_{n-1}$ as well.

Dealing with the second half of the lemma first: since Ω has initial type, $[a_0, \rightarrow) \cap \Omega \cong \Omega$. Let h be the isomorphism.

h extends to an isomorphism h_1: $[a_1, \rightarrow) \cap C(\Omega) \rightarrow C(\Omega)$ which maps $[a_1, \rightarrow) \cap \Omega(a_0)$ onto $\Omega(ha_0)$ and, for $a_0 < b$, $\Omega(b)$ onto $\Omega(hb)$.

Repeating this extension-procedure, ultimately an isomorphism h_n between $[a_n, \rightarrow) \cap C^n(\Omega)$ and $C^n(\Omega)$ is obtained.

Going further one more step but this time using $\Omega(a_n) \cong \Omega(b)$ (b least element of $C^n(\Omega)$), we get $(a, \rightarrow) \cap C^{n+1}(\Omega) \cong C^{n+1}(\Omega)$.

Proceeding, we finally find $(a, \rightarrow) \cong \Omega_\infty$, as desired.

Turning to $(\leftarrow, a)$, define

$$X_0 = (\leftarrow, a) \cap \Omega_0$$
$$X_{i+1} = (\leftarrow, a) \cap \Omega(a_i) \qquad (i < n)$$

Then $X_i \subset \Omega_i$ and $X_0 < X_1 < X_2 < \ldots < X_n < \Omega(a) < a$.

Situation for $a \in \Omega_2$:

Ω_0:	X_0	[	empty	]	a_0	
Ω_1:	$\Omega(a_0)$: X_1	[	empty	]	$a_1 \ldots$	
Ω_2:	$\Omega(a_1)$:	X_2	[empty]	a	$a_2 \ldots$	
Ω_3:			$\Omega(a)$			

Because X_0 is bounded by a in Ω and X_{i+1} is bounded by a_{i+1} in $\Omega(a_i)$ $(i<n)$, these well-ordered sets all have cardinal $<\Omega$. Therefore, $\bigcup_{i\leqslant n} X_i$ has cardinal $<\Omega$; hence $X = \bigcup_{i\leqslant n} X_i \cup \Omega(a) \cong \Omega$ by some isomorphism h. Extend h to an isomorphism: $C(X) \to C(\Omega)$, using $\Omega(x) \cong \Omega(hx)$ for each $x\in X$, and repeat this procedure; eventually, obtain $\bigcup_n C^n(X) \cong \Omega_\infty$.

It remains to show that $(\leftarrow, a) = \bigcup_n C^n(X)$.

One inclusion is clear. So, assume $b < a$. Suppose that $b = b_m \leftarrow b_{m-1} \leftarrow \ldots \leftarrow b_0 \in \Omega_0$.

Now, $a_0 < b_0$ is impossible since then $a_0 < b_i$ for all i, in particular, $a_0 < b$. If $b_0 < a_0$ then, since $(a, a_0) \cap \Omega_0 = \emptyset$, $b_0 \in X_0$, and we are done. The remaining possibility is $b_0 = a_0$. But then, $m > 0$. Consider b_1. $a_1 < b_1$ is impossible and $b_1 < a_1$ leads to $b_1 < a$ and $b_1 \in X_1$ since $(a, a_1) \cap \Omega_1 = \emptyset$. So, assume $b_1 = a_1$. Then $m > 1$. Repeating this argument, we arrive at the case $b_n = a_n$ and $m > n$. But then, $a < b_{n+1}$, hence $a < b$ - a contradiction. ⊠

Appendix B:
reduction of higher-order logic.

In 1975, S.K.Thomason gave a reduction of second-order logic to modal logic. Van Benthem [1985] pp.23-24 added interest to this by explaining how all of higher-order logic could then be reduced as well. For the reduction of higher-order to second-order logic, he referred to an old unpublished note of mine. However, it appeared that such a reduction was given by at least two people already, viz. Hintikka [1955] and Montague [1965].

These authors use rather different methods. Hintikka employed a rather straightforward direct translation into monadic second-order logic, of which a modern presentation occurs in van Benthem and Doets [1983]. Montague's paper was based on a translation into the language of set theory, as was my note.
For completeness' sake, this is presented below.

(Montague goes on to transfinite type theory and has applications to spectra besides.)

B.1 Definition. The (ZF) model $A=(A,E)$ is called **absolute for powers** if for each $a\in A$ and $X\subset\{x\in A\mid xEa\}$ there exists $b\in A$ such that $X=\{x\in A\mid xEb\}$.

Power absoluteness obviously is a Π^1_1-property which is expressed by the Π^1_1-form of Zermelo's *Aussonderung-axiom*. (Notice however that it is *first-order* on *finite* models.)

Let $V_\alpha=\bigcup_{\beta<\alpha}\mathcal{P}(V_\beta)$ be the α-th level of the *cumulative hierarchy*. *Natural models* have the form $(V_\alpha,\in)$. Clearly, these are power absolute. Conversely:

B.2 Lemma. *If $A=(A,E)$ is a power absolute model of* ZF*, then E is well-founded and the isomorph $(B,\in)$ of A where B is transitive is a natural model.*

Proof. Suppose that $X\subset A$ is a non-empty set without E-minimal element. Since ZF proves the existence of transitive closures, there is an $a\in A$ such that $X\cap\{x\in A\mid xEa\}$ is non-empty and has no E-minimal element (simply let a be the "transitive closure" of an element of X). By power absoluteness, choose $b\in A$ such that $\{x\in A\mid xEb\}=X\cap\{x\in A\mid xEa\}$. Now, b contradicts the regularity axiom of ZF. Next, let $\mathbf{B}=(B,\in)$ be the isomorph of $\mathbf{A}$ with B transitive. Suppose that α is the set of ordinals in B. By transfinite induction on $\beta\in\alpha$, it follows that $V_\beta{}^{\mathbf{B}}=\mathbf{B}\cap V_\beta=V_\beta$, using power absoluteness. Hence, $B=\bigcup_{\beta<\alpha}V_\beta{}^{\mathbf{B}}=\bigcup_{\beta<\alpha}V_\beta=V_\alpha$. ⊠

Suppose that L is a (finite) language; L^ω is the set of formulas of finite type theory over L. Without loss of generality we may suppose that $L^\omega \subset V_\omega$. Let $\ulcorner\ \urcorner$ be an effective operation mapping L^ω into the set of ZF-terms such that for each transitive ZF-model $\boldsymbol{B}=(B, \in)$ and each $\varphi \in L^\omega$: $\ulcorner \varphi \urcorner^{\boldsymbol{B}} = \varphi$. (Such an operation exists if the identification of formulas with hereditarily finite sets is a reasonable one.) Let $T(x,y)$ be a set-theoretic formula obtained as a straightforward translation of the definition of "φ is true in $\boldsymbol{A}$" ($\varphi \in L^\omega$, $\boldsymbol{A}$ any L-model) in set-theoretic terms.

B.3 Lemma. *For any natural* ZF*-model* $(V_\alpha, \in)$, *if* $M \in V_\alpha$ *is an L-model and* $\varphi \in L^\omega$, *then:* $(V_\alpha, \in) \models T(\ulcorner \varphi \urcorner, M)$ *iff* $M \models \varphi$.

Proof. Being a ZF-model, V_α must contain the ("full") type-structure M^ω over the model M as soon as it contains M. Now T can be read as expressing ordinary (many-sorted) *first-order truth with respect to* M^ω. Any advanced set theory book (e.g., Drake [1974]) explains how this is done in a way absolute for transitive models. ⊠

The above holds for power absolute transitive models of Zermelo set theory as well. If $\boldsymbol{B}=(B, \in)$ is such a model and $M \in B$, then M^ω need not be in B of course; however, any particular φ refers only to types below a certain level $n=n_\varphi$ and the type-structure on M up to and including n will be in B.

Now, let $L\text{-}mod(y)$ be a set-theoretic formula expressing being an L-model which is absolute for transitive models. The reduction is given now by the following

B.4 Theorem. *For* $\varphi \in L^\omega$: φ *is valid iff for each power absolute* ZF*-model* $\boldsymbol{A}$: $\boldsymbol{A} \models \forall y[L\text{-}mod(y) \rightarrow T(\ulcorner \varphi \urcorner, y)]$.

Proof. ($\Leftarrow$) If M is an L-model, let $\mathbf{A}$ be a natural model containing M and use B.3. ($\Rightarrow$) Let $\mathbf{A}=(A,E)$ be power absolute. By B.2, we may suppose it is *natural*. Let $M\in A$ be such that $\mathbf{A}\models L\text{-}mod(M)$. Hence, M actually *is* an L-model, whence $M\models\varphi$ by assumption. Therefore, $\mathbf{A}\models T(\ulcorner\varphi\urcorner,M)$ by B.3. ⊠

Lastly, since power absoluteness is a monadic $\Pi^1{}_1$-condition, B.4 reduces higher-order truth to truth for monadic $\Sigma^1{}_1$-sentences in a language with one binary relation (ϵ).

B.5. Power absoluteness of the McKinsey axiom.

Let A be any set, $\Gamma\subset\mathbf{P}(A)$, $W=\{\Gamma\}\cup\Gamma\cup A\cup(A\times\{0,1\})$, $\prec$ is the relation on W given by van Benthem [1984] p.197 in somewhat different terms:

(i) $\Gamma\prec B$ for all $B\in\Gamma$

(ii) $B\prec(a,i)$ iff $a\in B$ and $i=0$, or: $a\notin B$ and $i=1$

(iii) $\Gamma\prec a$ for all $a\in A$

(iv) $a\prec(a,i)$ for all $a\in A$, $i<2$

(v) $(a,i)\prec(a,i)$ for all $a\in A$, $i<2$.

Van Benthem l.c. shows that, for $\Gamma=\mathbf{P}(A)$, the McKinsey axiom $\Box\Diamond p\rightarrow\Diamond\Box p$ holds in $(W,\prec)$ but (if A is infinite) it does not hold in any proper elementary submodel. Now actually, he proves something better: for general $\Gamma\subset\mathbf{P}(A)$, the axiom holds in the corresponding model *iff* $\Gamma=\mathbf{P}(A)$. Therefore, it can be used to express power absoluteness, and the previous theorem indicates what this may mean.
Van Benthem [1985] has more examples of this phenomenon.

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Samenvatting

Dit proefschrift bevat resultaten over eerste- en tweede-orde logica (delen I en II) en modale- tijds- en intuïtionistische (propositie-) logica (deel III).

Een verbindend thema is Ehrenfeucht's spel en enkele varianten daarvan.

Hoofdstuk 1 is een inleiding in Ehrenfeucht-speltheorie en zijn relatie met (quantorrang-) α-equivalentie in (infinitaire) logica. Sectie 1.0 bedoelt een smaakmaker voor het eindige spel te zijn.

In hoofdstuk 2 wordt het spel gespeeld op binaire bomen. Er wordt een karakterisering verkregen van alle bomen die n-equivalent zijn met de binaire boom B_m waarvan alle takken lengte m hebben. In het bizonder volgt dat B_m *oneindige* n-equivalenten heeft als $m \geqslant 2^n - 1$. Dit werd toegepast door Rodenburg [1986] bij het oplossen van een probleem uit de intuïtionistische correspondentie-theorie; het verhaal staat in hoofdstuk 8.

Deel II laat zien hoe sommige monadische $\Pi^1{}_1$-theorieën kunnen worden geaxiomatizeerd (de meesten gaan over gefundeerdheid). Hoofdstuk 3 heeft betrekking op lineaire ordeningen. Sectie 3.3 isoleert het effect van de Suslin-eigenschap op de monadische $\Pi^1{}_1$-theorie van $\mathbb{R}$. Hoofdstuk 4 generaliseert de methode van 3 naar het geval van bomen.

In deel III bespreekt hoofdstuk 5 Löwenheim-Skolem-type problemen in modale correspondentie-theorie. Aangetoond wordt dat de meeste voorbeelden van niet-eerste-orde-definieerbare formules al niet eerste-orde zijn op eindige strukturen. Aan de andere kant wordt een voorbeeld gegeven van een niet-eerste-orde definieerbare formule die wèl eerste-orde is op alle aftelbare strukturen.

Hoofdstuk 6 modificeert het Ehrenfeucht-spel voor gebruik in intensionele logica; exacte Kripke-modellen worden geconstrueerd die universeel zijn met betrekking tot eindige partieel geordende Kripke-modellen.

Hoofdstuk 7 geeft onze versie van het bewijs van de $\mathbb{Z}$-volledigheidsstelling.

In hoofdstuk 9 worden spelen en het universeel-exacte Kripke-model adekwaat voor één-variabele-intuïtionistische formules toegepast bij het oplossen van enkele problemen in intuïtionistische correspondentietheorie die open bleven in Rodenburg [1982].

Appendix A construeert asymmetrische lineaire ordeningen met veel homogeniteitseigenschappen in ieder overaftelbaar kardinaalgetal.

Appendix B reduceert hogere-orde logica tot monadische tweede-orde logica - dit geeft een indruk van de expressie-mogelijkheden van modale logica onder de Kripke-semantiek.

STELLINGEN
behorend bij het proefschrift
"Completeness and Definability"
van Kees Doets.

1

Er is een uniforme bewijsmethode voor resultaten zoals het ordenings-extensie-principe en de boolese priemideaalstelling (die zeggen dat een model een expansie heeft die zekere universele eerste-orde condities vervult) zonder gebruik van logica. (H.C.Doets: *A theorem on the existence of expansions*. Bull. de l'Ac.Pol. des Sci. XIX (No.1) 1971, pp.1-3.)

2

Het verzamelingstheoretisch reflectie-principe is zeer eenvoudig te bewijzen in de volgende gedaante: laat Σ een eindige verzameling van formules zijn die gesloten is onder subformules. Dan is in ZF afleidbaar: als $A=\bigcup_{\alpha\in OR} A(\alpha)$ een cumulatieve, continue hierarchie van verzamelingen is dan is de collectie

$$\{\alpha\in OR \mid \bigwedge_{\Phi\in\Sigma} \forall a\in A(\alpha)\,[\Phi^{A(\alpha)}(a) \leftrightarrow \Phi^{A}(a)]\}$$

gesloten en onbegrensd.

3

ZF bewijst een klasse-vorm van de kleinste-dekpuntstelling voor monotone operatoren H die voldoen aan de conditie: $H(X)\subset\bigcup\{H(x) \mid x\subset X \wedge x \text{ is een verzameling}\}$; bovendien bestaat een "algebraïsch" bewijs hiervoor (m.b.v. het collectie-principe) dat ordinalen en transfiniete recursie vermijdt.

4

"Simpele" consistentie-bewijzen voor verzamelingentheorieën (m.b.v. "natuurlijke" en andere super-transitieve modellen) onderscheiden zich van hun niet-simpele soortgenoten (via OD, L, forcing) daardoor dat ze tevens voor de corresponderende *tweede-orde* theorie werken.

5

Verzamelingstheoretisch forceren gaat even gemakkelijk met het infinitaire toelaatbare fragment dat door het basismodel wordt bepaald.

6

Voor de basisresultaten van infinitaire logica zijn "consistency-properties" niet nodig. (H.C.Doets: **Notes on admissible model theory.** Rapport 83-10, Universiteit van Amsterdam 1983.)

7

De constructie van de construeerbare hierarchie boven een model (J.Barwise: **Admissible Sets and Structures**, Springer 1975) kan aanzienlijk worden vereenvoudigd door het gebruik van een niet-standaard definitie van *n*-tupel van Scott. (Doets, **Notes on admissible model theory**, l.c.)

8

Het gebruik van recursief verzadigde modellen maakt een kort bewijs mogelijk van Lindström's karakterisering van eerste-orde logica in termen van compactheid en de neerwaartse Löwenheim-Skolem eigenschap.

9

$\omega + \zeta$ is een model van de monadische $\Sigma^1{}_1$-theorie van ω (maar er bestaan verzamelingen $X \subset \omega$ zódat geen expansie van $\omega + \zeta$ elementair equivalent is met (ω, X)).

10

Kampeerders worden in ons land gediscrimineerd ten opzichte van andere gebruikers van verplaatsbare recreatie-accomodaties.

Errata, remarks

page 11

Replace the last two sentences on lines -5 — -3 by:

Suppose that II counters with $\beta < \omega^{n+1}$. Then $(2n+2, \Omega, \omega^{n+1})$ is a winning move for I: II has to answer this move with some $\alpha < \beta$; by (*) we have that $\alpha \not\equiv^{2n+2} \omega^{n+1}$ and I wins.

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In the proof of Theorem 1.8.1, replace part 2⇒3 by:

The required set consists of all positions that can be reached in a play where II uses his winning strategy.

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Theorem 3.3.9 was obtained earlier by John P. Burgess and Yuri Gurevich, cf. **The decision problem for linear temporal logic**. *Notre Dame J. of Formal Logic* **26** (2) 1985 pp 115—128.

page 59

Lines –9 — –8 hints at the question wether component-wise definable well-foundedness implies definable well-foundedness. The answer is affirmative. (P. Rodenburg, private communication, march 1987.)

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In line +7, replace 'II' by 'I'.

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To Theorem 5.8.1—4, add

5. $\Diamond\Box(\Box p \rightarrow p)$ (l.c. 10.5).

(P. Rodenburg, private communication.)

page 73

Theorem 5.9 was obtained earlier by R. Fagin: **Monadic Generalized Spectra.** *Zeitschrift für Math. Logik u. Gr.d.M.* **21** (1975) pp 89—96.