$$(\Box(\neg\neg\Box A \to \Box A) \to \Box\Box A)$$

ASPECTS OF DIAGONALIZATION & PROVABILITY

ALBERT VISSER

ASPECTS OF DIAGONALIZATION & PROVABILITY

PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE WIJSBEGEERTE AAN DE RIJKSUNIVERSITEIT TE UTRECHT, OP GEZAG VAN DE RECTOR MAGNIFICUS PROF. DR. M.A. BOUMAN, VOLGENS BESLUIT VAN HET COLLEGE VAN DECANEN IN HET OPENBAAR TE VERDEDIGEN OP MAANDAG 16 NOVEMBER 1981 DES NAMIDDAGS TE 4.15 UUR

DOOR

ALBERT VISSER

GEBOREN OP 19 DECEMBER 1950 TE ZWIJNDRECHT

DRUKKERIJ ELINKWIJK BV – UTRECHT

PROMOTOR: PROF. DR. D. VAN DALEN

voor mijn ouders, mijn broers,
voor Karin en Eva

CONTENTS

ACKNOWLEDGEMENTS

As so often with important things, like the air one breathes, it is only when I pause and reflect that the pervasive influence of Dirk van Dalen becomes clear. As prime mover of the Utrecht Logic Group he created the pleasant atmosphere in which I studied and worked the last six years. I thank him for his guidance in the work on this thesis, his detailed comments on content and style, his attention for matters of organization and strategy.

Henk Barendregt was first my teacher then my colleague. I learned much from him. He helped me especially with part 1 of this thesis, asking good questions, giving relevant information. I am very grateful to him for that.

Jan Willem Klop and Karst Koymans were wonderful roommates. I thank Jan Willem for all our discussions on matters technical, philosophical and otherwise. Karst, I thank, for spotting mistakes in my work with unfailing precision.

Pepe Longo and Michael Beeson deserve special mention. Their enthousiasm and encouragement meant much to me.

I am grateful to professor A.S. Troelstra for stimulating conversations. I thank him and Michael Beeson for their meticulous comments on part 4.

The following people have helped me by conversations or correspondence: J.F.A.K van Benthem, G. Boolos, G. Gargov, D.H.J. de Jongh, M.H. Löb, H. Luckhardt, W. Ruitenburg, D.S. Scott, G. Sundholm, J.F. Zucker.

Perhaps the greatest intellectual debt I owe to Craig Smoryński. Every single paper I wrote is somehow a reaction to an article, a comment, a question of his.

I thank Sophie van Sterkenburg and Karin Stegmann for deciphering my handwriting and carefully typing this thesis.

Part 1 of this thesis is published in: To H.B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism, edited by J.P. Seldin and J.R. Hindly, Academic Press Inc., London, 1980.

Part 2 is published in Indagationes Mathematicae, Volumen 43, (1981).

Part 3 will appear in Studia Logica.

PART 0 INTRODUCTORY PART

1 WHAT THIS THESIS IS ABOUT

Monolithic this thesis is not. It reflects the history of our preoccupation with Gödel's Incompleteness Theorems over the last four years.

In part 1 and 2 we concentrate on generalizations of Gödel's First Incompleteness Theorem like the Gödel-Rosser-Mostowski-Myhill-Kripke Theorem (GRMMK for short). One cannot but be impressed how few specific facts about formal theories are needed to prove this theorem (just consistency and a weak kind of representability of the partial recursive functions). The rest is done by recursion theory; one is tempted to say that here recursion theory plays the role of "general nonsense" with respect to the theory of formal systems. An obvious question in this context is whether the applications of recursion theory to formal systems can be made uniform, whether one can give recursion theoretic results in "the right form" for these applications. In part 1 and 2 tentative answers are considered. In part 1 we adapt Eršov's theory of Numerations for our purpose. A disadvantage of the framework thus obtained is this: when one analyses an application to a formal system it sometimes turns out that there is a direct "theory-free" proof in which less about the system is used than demanded by the framework. For example in certain applications to arithmetic one uses the existence of the Σ_n^0-truthpredicate, to lambda-calculus the existence of the universal constructor E. These facts are not needed for your average recursive inseparability argument. So perhaps still some improvement is possible.. All the same the framework is rather natural for a wide range of applications.

As so many, we were at first victims of that easy, youthful fallacy: to think that Gödel's Second Incompleteness Theorem is "just" a formalization of Gödel's First Incompleteness Theorem and that *therefore* Gödel's Second Incompleteness Theorem does not differ *programmatically* from Gödel's First Incompleteness Theorem. (Clearly this fallacy is just an instance of a far more general fallacy, and is ultimately connected with a wrong appreciation of the very aims of formalization.) In due time we saw the fallacy's true nature, mainly because of Craig Smoryński's benificial influence. The strength of GRMMK is also a weakness, when looked upon as a basis for further work. Because it is so much a recursion theoretic result one is encouraged to forget the intrinsic interest of the concepts of formal provability and concistency.

Consequently part 3,4 and 5 are devoted to the concept of formal provability, or to be more specific to Provability Logic. Provability Logic is of course not the only possible approach to study formal provability (there is e.g. the work of Jeroslow), but we think its results, in particular De Jongh's Theorem and Solovay's Completeness Theorem are at present the deepest ones pertaining specifically to formal provability. Solovay's Theorem seems the natural generalization of Gödel's Second Incompleteness Theorem at least from "a propositional point of view".

Part 4 adds a new element: it is on the provability logic of intuitionistic arithmetic. Maybe this is the proper place to state our view on Intuitionism / Constructivism. We think classical mathematics is just plainly true, e.g. Format's Theorem is either true or false. Even if it is very well possible that it is true but not provable in principle for human beings. In other words that it is simply outside the scope of the human a priori. (Can the mathematical a priori for Martians be stronger at some points than ours?) Still Constructivism is a perfectly legitimate branch of Mathematics like Numerical Analysis, Homological Algebra or Algebraic Geometry. It could be considered as that branch of mathematics that does systematically ("globally") what in other branches is done case for case ("locally"): finding algorithms corresponding to Existence Theorems (this "hackneyed business" of numerical content). A further point is that mathematicians are often interested in theories with many models. Now theories like Classical Arithmetic and Analysis suffer from a lack of (reasonable) models. In the Intuitionistic/ Constructivistic case however there are many models. In current developments in Intuitionistic Model Theory people try to put this aspect to good use. At present it is impossible to judge the merits of topos theory as intuitionistic model theory, but surely there is more than a grain of a good idea behind that approach.

The remaining section of the Introductory Part is an exposition of Solovay's Completeness Theorem, to introduce the reader to parts 3, 4, 5. The thesis is concluded by an Epilogue on the provability logic of Heyting's Arithmetic, both to elaborate on the results of part 4 and to provide some context for them.

2 SOLOVAY'S COMPLETENESS THEOREM, AN INTRODUCTION

This exposition is mainly intended as an introduction to Solovay's way to embed Kripke Models into arithmetic. Consequently we will refer the reader to the literature for most facts about Modal Logics and Kripke Models. Moreover to make this exposition not too long we will freely use notations and definitions from other parts of this thesis.

2.1 Modal Logics and Kripke Models

2.1.1 Modal Logics

Define: $\Box^0\bot := \bot$, $\Box^{n+1}\bot := \Box(\Box^n\bot)$, $\Box^\omega\bot := \top$.

G_a for $a \in \{0,1,\ldots,\omega\}$ is the following theory in the language of Modal Propositional Logic L_{pr}:

- The rules and axioms of Classical Propositional Logic (e.g. in Natural Deduction formulation)
- $\vdash A \Rightarrow\ \vdash \Box A$ (Gödel's Rule)
- $\vdash (\Box A \to \Box\Box A)$
- $\vdash (\Box(A \to B) \to (\Box A \to \Box B))$
- $\vdash (\Box(\Box A \to A) \to \Box A)$ (Löb's Axiom)
- $\vdash \Box^a \bot$

G^ω is also called: G.

G_a^{mon} is $G_a + \vdash p_i \to \Box p_i$.

G* is:

- All theorems of G
- Modus Ponens Rule
- $\vdash (\Box A \to A)$

2.1.2 Kripke Models

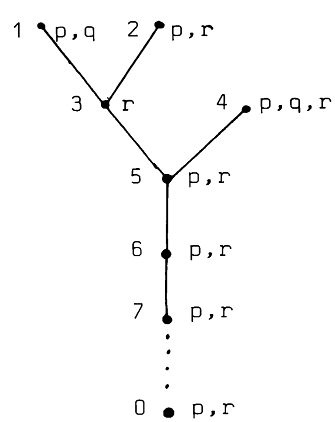
A Kripke model $\langle W, \prec, f\rangle$ is defined as in 6.8 of part 4 of this thesis omitting however the monotonicity condition: $w \prec w' \Rightarrow f(w) \subseteq f(w')$. A model satisfying this condition will be called monotonic. We assume that every model has a bottom. $\models_K$ is defined as in 6.8 of part 4.

We will be interested mainly in finite irreflexive Kripke models. For technical purposes - only to become apperent later - we want to attach an ω-tail to their bottoms. Numerating the elements of W by natural numbers in a suitable way we arrive at the following definition:

A Kripke model $K = \langle\omega, \prec, f\rangle$ is a *tail model* if:

- $\prec$ is irreflexive
- if $m \neq 0$ then $0 \prec m$
- if $n \neq 0$, $n \prec m$, then $n > m$
- for some $N \neq 0$:
  - for every $n, m \geqslant N$ if $n > m$ then $n \prec m$
  - for every $n \geqslant N$ $f(n) = f(N)$
  - $f(0) = f(N)$

Example:



Clearly every finite irreflexive Kripke model is the top part of some tail model (modulo isomorphism) and vice versa: proper top parts of tail models are finite irreflexive Kripke models.

We will call N as in the last clause of the definition a tail element. Note that this is not unique.

In this exposition we will formulate everything in terms of tail models. This causes a slight loss of naturalness. But in this way we avoid the tedious chopping off and tagging on of tails and the accompanying extra burden of notation.

The main lemma on tail models is:

2.1.2.1 Tail Lemma

For every tail model K:

$0 \models_K A$ iff for some M, for all $n \geqslant M$ $\quad n \models_K A$;

$0 \not\models_K A$ iff for some M, for all $n \geqslant M$ $\quad n \not\models_K A$.

Proof: induction on A. □

We need one more definiton. Let K be a tail model. Define the depth of n: $d(n) := \sup \{(d(k)+1) \mid k \succ n\}$

Clearly $d(0) = \omega$ and for topmost k: $d(k) = 0$.

2.1.2.2 Completeness Theorem

i) $G_a \vdash A$ iff for all tail models K, for all n such that $d(n) < a$: $n \models_K A$

ii) $G_a^{mon} \vdash A$ iff for all monotonic tail models K, for all n such that $d(n) < a$ $\quad n \models_K A$

iii) $G^* \vdash A$ iff for all tail models K: $0 \models_K A$

Proof

i) See [Bo][So] and part 5 of this thesis.

ii) This is an easy consequence of *i*).

iii) "$\Rightarrow$" 0 satisfies the theorems of G because K is well-founded upwards. Closure under Modus Ponens is trivial.

Suppose $0 \models_K \Box A$. Then for all $n > 0$, i.e. for all $n \in \omega \setminus \{0\}$: $n \models_K A$. Hence by the Tail Lemma: $0 \models_K A$.

"$\Leftarrow$" Suppose $G^* \nvdash A$. Clearly $G \nvdash ((\bigwedge(\Box B_i \to B_i)) \to A)$, where the B_i are those subformulae of A that have a box in front of them (the boxed subformulae of A).

Hence there is a tail model K such for some $N(\neq 0)$:

$N \not\models_K \bigwedge(\Box B_i \to B_i) \to A$, i.e. $N \models_K \bigwedge(\Box B_i \to B_i)$ and $N \not\models_K A$.

Note that we can arrange that N is a tail element.

That $0 \not\models A$ follows easily from:

Claim for all subformulae C of A:

if $N \models_K C$ then for all $m \leqslant N$ $\; m \models_K C$,

if $N \not\models_K C$ then for all $m \leqslant N$ $\; m \not\models_K C$.

Proof of the claim: the proof is by induction on C. The cases of atoms, $\wedge$, $\vee$, $\neg$ are trivial. Hence assume $C \equiv \Box D$.

If $N \models_K \Box D$ then (because D is a boxed subformula of A: $N \models_K D$. By Induction Hypothesis for all $m \leqslant N$ $\; m \models_K D$. On the other hand for all $m > N$ $\; m \models_K D$. Hence for all k: $k \models_K \Box D$, so certainly for all $m \leqslant N$ $\; m \models_K \Box D$. The second case is again trivial. $\Box$

2.2 Embedding tail models in arithmetic

We want to embed in some sense tail models into PA. Clearly we can consider a Kripke model K as assigning a truth values $[\![A]\!]_K = \{w \in W \mid w \models_K A\}$ to formulae A and consider the logical constants as functions on truth values. The idea is to represent $[\![A]\!]_K$

by a sentence of arithmetic. A formula $A(\vec{p})$ may be considered as a *polynomial* in the truth values of the $\vec{p}$'s. Par abus de langage: $A([\![\vec{p}]\!]_K) = [\![A(\vec{p})]\!]_K$.

Suppose we represent $[\![B]\!]$ by the formula $[B]$ in L, the language of arithmetic; again par abus de langage we want:

$\vdash_{PA} A([\vec{p}]) \leftrightarrow [A(\vec{p})]$ (*)

There is a simple reason why this programme does not work for finite irreflexive K. Clearly in such a K for some n:

$[\![\Box^{k+1}\bot]\!]_K = [\![\Box^k\bot]\!]_K$. * would give (for the case of PA):

$\vdash_{PA} \Box^{k+1}_{PA}\bot \leftrightarrow \Box^k_{PA}\bot$. Quod non.

For the present we fix a tail model K and an RE extension T of PA (T may be inconsistent). A first step is to represent the truth values $[\![A]\!]$ as sets in PA. By the Tail Lemma $[\![A]\!]$ is either finite or cofinite.

Define as formulae of *L*:

in case $[\![A]\!]_K$ is finite: $(x \in [\![A]\!]_K) :\equiv \bigvee\{(x=i) \mid i \models_K A\}$

in case $[\![A]\!]_K$ is cofinite: $(x \in [\![A]\!]_K) :\equiv \bigwedge\{(x \neq i) \mid i \not\models_K A\}$

(By convention the empty disjunction is $\bot$, the empty conjunction T.)

One easily verifies:

$\vdash_{PA} (x \in [\![A]\!]_K \wedge x \in [\![B]\!]_K) \leftrightarrow x \in [\![A \wedge B]\!]_K$

$\vdash_{PA} (x \in [\![A]\!]_K \vee x \in [\![B]\!]_K) \leftrightarrow x \in [\![A \vee B]\!]_K$

$\vdash_{PA} x \notin [\![A]\!]_K \leftrightarrow x \in [\![\neg A]\!]_K$.

We make the following assumptions about the proof predicate $\mathrm{Proof}_T(x,y)$:

$\vdash_{PA} \mathrm{Proof}_T(x,y) \wedge \mathrm{Proof}_T(x,z) \rightarrow y = z$

$\vdash_{PA} \neg\, \mathrm{Proof}(\underline{0},y)$.

Clearly the usual proof predicate satisfies these. (Moreover if $Proof_T(x,y)$ did not satisfy the assumptions $Proof'_T(x,y) :\equiv$ $(\exists z < x \quad x = 2^{z+1} . 3^{y+1} \wedge Proof_T(z,y))$ certainly would.)

Further define for f monotonic in $\prec$:

lim f = s iff for some m f(m) = s and for every p,n f(p) = s and $n > p \Rightarrow f(n) = s$.

The second step is to define a term $\ell :=$ lim h where h is a total recursive function.

Define (by the recursion theorem):

$h(0) := 0$

$$h(k+1) := \begin{cases} n \text{ if for some } n > h(k) \\ \qquad Proof_T(k+1, \ulcorner \ell \neq n \urcorner) \\ h(k) \text{ otherwise.} \end{cases}$$ [1]

(Note that the fact that ℓ occurs in the definition of h is made possible by the recursion theorem.)

We have:

$\vdash_{PA}$ "h is weakly monotonic in $\prec$"

$\vdash_{PA}$ "ℓ exists"

Now (third step) we are in the position to define the representation of $[[A]]_K$ in PA: $[A]_{K,T} :\equiv (\ell \in [[A]]_K)$. Take $f: \mathcal{P} \rightarrow L$ as $f(p_i) :=$ $[p_i]_{K,T}$.

Define $\langle A \rangle_{K,T} := \langle A \rangle^{f,T}$ where $\langle\ \rangle^{f,T}$ is defined as in part 4, 6.4.

---

[1] Because $\vdash_{PA}$ "ℓ exists", scope problems are irrelevant here. We prefer mostly the large scope reading e.g. for $(\ell \neq n)$: $\exists x(\ell = x \wedge x \neq n)$.

We have:

2.2.1 Theorem

$\vdash_{PA} \langle A\rangle_{K,T} \leftrightarrow [A]_{K,T}$

Proof: induction on A. The cases of atoms, $\wedge$, $\vee$, $\neg$ are easy (always using that ℓ provably exists). Now suppose $A \equiv \Box B$.

a) In case $[[\Box B]]_K$ is cofinite, clearly $[[\Box B]]_K = [[B]]_K = \omega$. Hence by Induction Hypothesis: $\vdash_{PA} \langle B\rangle_{K,T} \leftrightarrow \top$, hence $(T \supseteq PA) \vdash_T \langle B\rangle_{K,T}$, so $\vdash_{PA} \Box_T \langle B\rangle_{K,T}$ or $\vdash_{PA} \langle \Box B\rangle_{K,T} \leftrightarrow [\Box B]_{K,T}$.

b) Suppose $[[\Box B]]_K$ is finite. Let $j_0,\ldots,j_s$ be the minimal elements such that $j_k \vDash_K \Box B$ and $j_k \nvDash B$. Note that for each i with $i \nvDash \Box B$ there is a j_k such that $i \prec j_k$.
Clearly by Induction Hypothesis and the fact that $PA \subseteq T$ it is sufficient to prove:

$\vdash_{PA} \Box_T [B]_{K,T} \leftrightarrow [\Box B]_{K,T}$

Argue in PA:

"$\rightarrow$" first:

Suppose $\Box_T [B]_{K,T}$. We have: $\Box_T(\ell \neq j_k)$ by the definition of $[B]_{K,-}$ and the fact that $j_k \nvDash_K B$. Suppose $\mathrm{Proof}_T(p+1, \ulcorner \ell \neq j_k \urcorner)$ and $h(p) = y$. In case $y \prec j_k$ we have $h(p+1) = j_k$, in case $y \not\prec j_k$, $h(p+1) \not\prec j_k$. Hence $\ell \not\prec j_k$. Conclude $\bigwedge\{\ell \not\prec j_k \mid h = 0,\ldots,s\}$, hence by elementary reasoning $\bigvee\{\ell = i \mid i \vDash_K \Box B\}$.

Secondly "$\leftarrow$":

Suppose $\ell = i$ for an $i \vDash \Box B$; by the definition of n and the fact that $i \neq 0$: $\Box_T\, \ell \neq i$ ("How else could h move up to i?"). Moreover from $\exists x\ hx = i$: $\Box_T\ \exists x\ hx = i$. Combining: $\Box_T\, \ell > i$, or $\Box_T \bigvee\{\ell = j \mid j > i\}$. Hence $\Box_T(\ell \in [[B]])$, (for, $j > i \Rightarrow j \vDash_K B$).

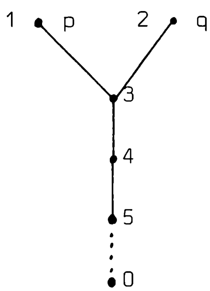
$\Box$

From here on T and K are free again.

2.2.2 <u>Didactical Corollary: Rosser's Theorem</u>

Solovay's proof is a Rosserlike argument, to bring out why this is so, we prove Rosser's Theorem from 2.2.1. Let T be an RE extension of PA.

Consider the simplest countermodel to the disjunction property: $(\Box(p \vee q) \to (\Box p \vee \Box q))$:



We have: $[p] \equiv (\ell = 1)$, $[q] \equiv (\ell = 2)$.

Moreover "$\ell = 1$" and "$\ell = 2$" are provably equivalent to Σ_1-sentences and they are the solutions of Rosserlike equations:

$$\begin{aligned} \vdash_{PA} \ell = 1 &\leftrightarrow \Box_T\, \ell \neq 1 < \Box_T\, \ell \neq 2, \\ \ell = 2 &\leftrightarrow \Box_T\, \ell \neq 2 < \Box_T\, \ell \neq 1. \end{aligned}$$

Here $<$ is defined as in part 1, 4.2.

Further by 2.2.1: $\ell = 1$ is a Rosser sentence for T:

$$\begin{aligned} \vdash_{PA} (\Box_T\, \ell = 1 \vee \Box_T \neg\, \ell = 1) \quad &\equiv \\ \langle \Box p \vee \Box \neg p \rangle_{K,T} \quad &\leftrightarrow \\ [\Box p \vee \Box \neg p]_{K,T} \quad &\equiv \\ [\Box \bot]_{K,T} \quad &\leftrightarrow \\ \langle \Box \bot \rangle_{K,T} \quad &\equiv \\ \Box_T \bot \quad & \end{aligned}$$

□

2.2.3 <u>Definition</u>

$$d(A) := \begin{cases} \text{the least } n \text{ such that there is a tail model } K \text{ and on} \\ m \text{ with } d(m) = n \text{ and } m \models_K A \text{ if there is one.} \\ \omega \text{ otherwise.} \end{cases}$$

2.2.4 <u>Corollary</u>

There is an $f\colon \mathcal{P} \to \mathcal{L}$ such that $\vdash_{PA} \langle A \wedge \Box A\rangle^{f,T} \leftrightarrow \Box_T^{d(A)}\bot$.

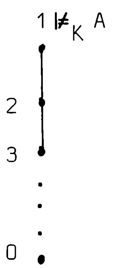
Proof: in case $d(A) = \omega$ we have $G \vdash A$, so any f will work. Suppose $d(A) = M$. Clearly there is a tail model K with tail element N such that $d(N) = M$ and $N \nVdash_K A$. Take $f(p_i) := [p_i]_{K,T}$. By the minimality of $d(A)$ for every $n > N$: $n \Vdash_K (A \wedge \Box A)$. Moreover for $n \leqslant N$ $n \nVdash A \wedge \Box A$ hence:

$$\begin{aligned}\vdash_{PA} \langle A \wedge \Box A\rangle^{f,T} &\leftrightarrow \\ [A \wedge \Box A]_{K,T} &\leftrightarrow \\ [\Box^{d(A)}\bot]_{K,T} &\leftrightarrow \\ \Box_T^{d(A)}\bot &\end{aligned}$$

$\Box$

2.2.5 <u>Example</u>

In case the propositional formula A is not valid we can take the N of 2.2.4: 1.



The interpretations of the atoms come out $\top$ or $\bot$, we find:

$$\begin{aligned}\vdash_{PA} \langle A \wedge \Box A\rangle^{f,T} &\leftrightarrow \\ \Box_T^0\bot &\equiv \\ \bot &\end{aligned}$$

2.2.6 <u>Corollary of 2.2.4</u>

Suppose a is the least element of $\{0,1,\dots,\omega\}$ such that $T \vdash \Box_T^a\bot$. We have:

$\vdash_{G_a} A \Leftrightarrow$ for every $f\colon \mathcal{P} \to \mathcal{L} \vdash_T \langle A\rangle^{f,T}$.

Proof

"$\Rightarrow$" routine.

"$\Leftarrow$" Suppose $G_a \nvdash A$. Then there is a tail model K and a tail element $N \neq 0$ such that $d(N) = d(A)$, $N \nVdash_K A$ and $\vdash_{PA} \langle A \wedge \Box A\rangle_{K,T} \leftrightarrow \Box_T^{d(A)}\bot$.

Suppose $\vdash_T \langle A\rangle_{K,T}$, then $\vdash_T \langle A \wedge \Box A\rangle_{K,T}$, hence $\vdash_T \Box_T^{d(A)}\bot$. But d there are tail models K' and $m \models_{K'} \Box^a \bot$, $m \not\models_{K'} A$. Clearly for such m $d(m) < a$. Hence $d(A) < a$. Contradiction. □

2.2.7 Corollary

Suppose T is a true RE extension of PA. We have:

$\vdash_{G*} A \Leftrightarrow$ for all $f: P \to L$ $\mathbb{N} \models \langle A\rangle^{f,T}$.

Proof

"⇒" routine.

"⇐" Suppose $G' \nvdash A$. Then there is a tail model K with $0 \not\models_K A$. By 2.2.1: $\mathbb{N} \models \langle A\rangle_{K,T} \leftrightarrow [A]_{K,T}$. Moreover $\mathbb{N} \models \ell = 0$. Hence $\mathbb{N} \models \neg \langle A\rangle_{K,T}$. □

2.2.8 Remark

Inspecting the definition of $(\ell = i)$ we see that $[p_i]_{K,T}$ is a Boolean Combination of Σ_1^0-sentences:

$\ell = i :\equiv ((\exists x\ hx = i) \wedge \forall xy((hx = i \wedge y > x) \to hy = i))$.

Hence we can sharpen 2.2.6, 2.2.7 by stating the Completeness Theorems for $f: L_{pr} \to \mathrm{Bool}(\Sigma_1^0\text{-sent})$. When we want to restrict ourselves e.g. to $f: L_{pr} \to (\Sigma_1^0\text{-sent})$ we get:

2.2.9 Corollary

$G_a^{mon} \vdash A \Leftrightarrow$ for all $f: L_{pr} \to \Sigma_1^0$-sent

$\vdash_T \langle A\rangle^{f,T}$,

where T is as in 2.2.6.

Proof: as the proof of 2.2.6; remark that for *monotonic* K and finite $[\![p_j]\!]_K$:

$$\vdash_{PA} W\{\ell = i \mid i \models_K p_j\} \leftrightarrow W\{(\exists x\ hx = i) \mid i \models_K p_j\}$$ □

2.2.10 Remark

As was pointed out by Guaspari the interpretations used in 2.2.6 do not even exhaust Bool(Σ^0_1-sent): there is a Π^0_1-sentence Ω such that for every RE extension T of PA (we need PA $\vdash$ "PA $\subseteq$ T") and every tail model K: $\nvdash_{PA} \Omega \leftrightarrow [p]_{K,T}$.

Proof: By 3.6 of part 1 of this thesis there is a Π^0_1-sentence Ω such that: $PA + \Box_{PA}\bot + \Omega \leftrightarrow A \nvdash_{PA} \bot$ for every Π^0_1-sentence A. By 2.2.1 $PA \vdash W \{\ell = i \mid i \models_K \Box\bot\} \leftrightarrow \Box_T \bot$.

Hence $PA + \Box_{PA}\bot \vdash [p]_{K,T} \leftrightarrow$

$$W\{\ell = i \mid i \models_K (\Box\bot \wedge p)\} \leftrightarrow$$

$$W\{\exists x\ hx = i \mid i \models_K (\Box\bot \wedge p)\}.$$

In other words $PA + \Box_{PA}\bot \vdash [p]_{K,T} \leftrightarrow B$ for some Σ^0_1-sentences B. Clearly $\nvdash_{PA} \Omega \leftrightarrow B$ for $PA + \Box_{PA}\bot + \Omega \leftrightarrow \neg B \nvdash_{PA} \bot$. $\Box$

Next we prove the Uniformination Theorem. This seems to have been proved first by Artyomov using a different method (see [Ar]).

2.2.11 Uniformization Theorem

Let T and a be as in 2.2.6. There are Σ^0_2-sentences $\Omega_0, \Omega_1, \Omega_2 \ldots$ such that for $f: L_{pr} \to L$ with $f(p_i) = \Omega_i$ we have: for every $A \in L_{pr}$: $\vdash_{G_a} A \Rightarrow \vdash_T \langle A\rangle^{f,T}$.

Proof: Let $Y: = \{\langle A\rangle_{K,T} \mid K$ is a tail model such that for some tail element $n \neq 0$ $n \nVdash_K A$ and $d(A) = d(n)\}$.

Clearly Y is recursive and $T \nvdash Y$ (i.e. for every $C \in Y$: $T \nvdash C$).

Define for $B \in \Sigma^0_2(x)$:

$X_B: = \{C \in \Sigma^0_2(x) \mid (\forall x(B \leftrightarrow C) \wedge \Box_T \forall x(B \leftrightarrow C)) \vdash_T Y\}$

Applying part 1, thm. 3.6 we find an $\Omega(x)$ such that $X_{\Omega(x)} = \emptyset$.

Take $\Omega_i := \Omega(i)$. Suppose $G_a \nvdash A(p_0,\ldots,p_n)$ and K is the appropriate counter model. We know $T \nvdash \langle A\rangle_{K,T}$ and we want $T \nvdash \langle A\rangle^{f,T}$, hence it is sufficient to show: $T + (\Omega_0 \leftrightarrow [p_0]_{K,T} \wedge \Box_T(\Omega_0 \leftrightarrow [p_0]_{K,T}))$ $+\ldots+ (\Omega_n \leftrightarrow [p_n]_{K,T} \wedge \Box_T(\Omega_n \leftrightarrow [p_n]_{K,T})) \nvdash \langle A\rangle_{K,T}$.

Take $D(x) := ((x = 0 \wedge [p_0]_{K,T}) \vee \ldots \vee$

$(x = n \wedge [p_n]_{K,T}))$

We have:

$\forall x(\Omega(x) \leftrightarrow D(x)) \wedge \Box_T \forall x(\Omega(x) \leftrightarrow D(x)) \vdash_T$

$\bigwedge(\Omega(i) \leftrightarrow D(i)) \wedge \Box_T \bigwedge(\Omega(i) \leftrightarrow D(i)) \vdash_T$

$\bigwedge(\Omega(i) \leftrightarrow [p_i]_{K,T}) \wedge \Box_T \bigwedge(\Omega(i) \leftrightarrow [p_i]_{K,T})$.

And:

$\forall x(\Omega(x) \leftrightarrow D(x)) \wedge \Box_T \forall x(\Omega(x) \leftrightarrow D(x)) \nvdash_T \langle A_{K,T}\rangle$.

Hence:

$\bigwedge((\Omega_i \leftrightarrow [p_i]_{K,T}) \wedge \Box_T(\Omega_i \leftrightarrow [p_i]_{K,T})) \nvdash_T \langle A\rangle_{K,T}$.

Conclude: $\nvdash_T \langle A\rangle^{f,T}$.

□

2.3 <u>On the metaphysics of Solovay's Theorem</u>

What does Solovay's Theorem tell us? Clearly the question is 'metaphysical', maybe even in the pejorative sense - that dreaded member of the unholy company of 'vague', 'meaningless' and their kin. Nevertheless it imposes itself.

A fist point is that Solovay's Theorem implies Gödel's Second Incompleteness Theorem without using any further information on PA. So Solovay's tells us anything Gödel's does. What then does Gödel's Theorem say? That is a controversial subject. Interpretations are strongly and often strangely dependent on philosophical points of view (examples are [Du], [Sto], for the standard view see [Sm]). We will not pursue that matter here.

So what more does Solovay's Theorem tell us than Gödel's Second Incompleteness Theorem?

Let us split Solovay's Theorem in two: the Soundness and the Completeness half. Soundness is implicit in Löb's 1955 article([Lö]). We will read Soundness as closure of PA under the Löb Conditions and Löb's Rule (even if, strictly speaking, as stated it is a bit weaker). Soundness thus viewed is and is exactly the articulation of the propositional provability principles implicit on the one hand directly in the concept of formal provability, on the other hand via Löb's and De Jongh's theorems in the Arithmetical Fixed Point Theorem or Selfreference Lemma.

But what does Completeness, the fact that these are *all* principles mean? Is Completeness surprising? For Completeness we only need interpretations in Bool (Σ_1^0-sent), doesn't that suggest that we get Completeness just because classical modal propositional logic is too poor to register further provability principles? The case for e.g. modal predicate logic could be far more difficult, possibly using sentences of any complexity, possibly having no RE axiomatization. On the other hand this cannot be the whole truth. Undeniably, interpretations in PA and RE theories extending PA constitute a more specific semantics than say Kripke models, e.g. we cannot get the logic of finite linear irreflexive Kripke models (because of the Rosser-like phenomena in Solovay's Theorem).

As we will see in the epilogue, the provability logic of HA is quite different from that of PA and more complex. Hence Completeness of G for interpretations in PA is in a very essential way bound to classical logic. Inspection of Solovay's proof shows that one needs the fact that existence of limits of recursive functions weakly monotonic in a finite ordering or equivalently decidability of Σ_1^0-sentences is provable *in* the theory. This shows *at least* that Solovay's Theorem is about formal provability in the context of classical logic. It is of course also another sign of the fact that many closure properties of HA are provable in HA. One may however not conclude that Solovay's Theorem shows that the metamathematics of PA is more difficult (in the sense that it needs stronger principles) than the metamathematics of HA: for one thing e.g. the modal predicate logics of HA and PA could very well be equally complex, moreover we cannot even *add* (e.g. using the Selfreference Lemma as in part 4, theorem 5.2) propositional provability principles to PA. The propositional provability logic of the resulting theory always collapses into G or G + $\Box^k\bot$ for some k.

The above remarks are very tentative, they just touch the problem raised, but one point emerges clearly: the relevance of future work on generalizations of propositional provability logic (e.g. to predicate logic, quantified propositional logic, intuitionistic logic), and on interpretations as semantics in general (e.g. the logic of the truth predicates of the complexity classes of sentences, the logic of Δ_2^0-models of PA etc.) to the proper appreciation of Solovay's result.

Let us close with the remark that from a certain point of view

Solovay's theorem *is* surprising. Formal provability is a syntactical, man-made concept, not prima facie a bona fide modality at all. Modal Logic, even if conceived in sin, may have been redeemed by Gödliness, it is also true that the completeness theorem for interpretations is formal provability's Sole Way to salvation from Syntactical Earth into Modal Heaven.

REFERENCES

[Ar] ARTYOMOV S.N., *Arithmetically complete modal theories*, (Russian), Semiotics and Information Sciences, no.14, (Russian), pp 115-133, Akad Nauk SSSR, Vsesojuz. Inst. Naucn. i Tehn. Informackk, Moscow.

[Bo] BOOLOS G., *The unprovability of consistency*, Cambridge University Press, Cambridge, 1979.

[Du] DUMMETT M.A.E., *The Philosophical Significance of Gödel's Theorem*, Ratio, vol. v. (1963), 140-155. Reprinted in: Truth and Other Enigmas, Duckworth, London, 1978, 186-201.

[Mo 1] MONTAGNA F., *On the diagonalizable algebra of Peano Arithmetic*, Bolletino U.M.I. (5) 16-B (1979), 795-812.

[Mo 2] ___________ , *Interpretations of the First-order Theory of Diagonalizable Algebras in Peano Arithmetic*, Studia Logica XXXIX. 4. 1980, 347-354.

[Mo 3] ___________ , *The Undecidability of the First-order Theory of Diagonalizable Algebras*, Studia Logica XXXIX, 4, 1980, 355-359.

[Sm] SMORYŃSKI C., *The incompleteness theorems*, in: Handbook of Mathematical Logic, ed. J. Barwise, North Holland, Amsterdam, 1977.

[So] SOLOVAY R.M, *Provability interpretations of modal logic*, Israel Journal of Mathematics 25, (1976), pp 287-304.

[So Be] SORBI A. & BERNARDI C., *Classifying positive equivalence relations*, Rapporto matematico n.13, 1980, preprint, Instituto di matematica dell' Università di Siena.

[Sto] STOLTENBERG G., *Can an inquiry into the foundations of mathematics tell us anything interesting about mind?* Unpublished manuscript.

PART 1

NUMERATIONS, λ-CALCULUS & ARITHMETIC

Albert Visser

Mathematical Institute, Budapestlaan 6
3508 TA Utrecht, The Netherlands.

Dedicated to H.B. Curry on the occasion of his 80th Birthday

ABSTRACT

Applications of complete and precomplete numerations (as introduced by Eršov) to term models of λ-calculus, structures associated with partial recursive functions and Peano arithmetic. Some results: a version of Gödel's First Incompleteness Theorem for λ-calculus (a consequence is that any countable p.o. can be embedded in the p.o. of RE λ-theories); a topological explanation of the Range Theorem of λ-calculus; representability of the partial recursive functions in any RE λ-theory and in any RE extension of PA.

§0 INTRODUCTION

In this paper I use some of Eršov's concepts to prove results about structures such as term models of λ-calculus. These results and their proofs are "coördinate free" i.e. they use (nearly) no specific properties of e.g. λ-calculus. As a consequence theorems from arithmetic carry over to λ-calculus and vice versa. For example there is a version of the Gödel-Rosser-Mostowski-Myhill-Kripke Theorem for RE λ-theories.

§1 introduces the relevant concepts from the theory of numerations. §2 is a small study of "intensional" phenomena in precomplete numerations by means of topological considerations.

E.g. I bring out the topological content of the well known Range Theorem of λ-calculus. §3 contains some consequences of the Fixed Point Theorem for precomplete numerations. §4 gives a construction which one could call: "How to Kleene a Curry", i.e. a construction to form a complete numeration from a precomplete numeration by identifying certain elements.
To read the paper a general background in recursion theory and λ-calculus (e.g. Rogers (1967) and Barendregt (1978)) should be sufficient. I took some care not to presuppose knowledge of Eršov (1973) and Eršov (1975). Actually, I think the present paper could be considered as an introduction to the idea of precomplete and complete numerations.

§1 INTRODUCTION TO SOME FUNDAMENTAL CONCEPTS

1.1 DEFINITION. A <u>numeration</u> γ is a pair (ν,S), where ν is a surjective function from $\mathbb{N}$ to S.
If $\gamma_1 = (\nu_1,S_1)$ and $\gamma_2 = (\nu_2,S_2)$ are numerations then μ is a <u>morphism</u> from γ_1 to γ_2 if μ is a function from S_1 to S_2 and if there is a recursive ψ s.t.

$$\begin{tikzcd} \mathbb{N} \arrow[r, "\psi"] \arrow[d, "\nu_1"] & \mathbb{N} \arrow[d, "\nu_2"] \\ S_1 \arrow[r, "\mu"] & S_2 \end{tikzcd}$$ commutes.

Numerations with their morphisms form the <u>category of numerations</u>.

1.2 DEFINITION. If $\gamma = (\nu,S)$ is a numeration then we will say $m \sim_\gamma n$ for $\nu(m) = \nu(n)$.

1.3 DEFINITION. We will call a numeration <u>positive</u> if $\sim_\gamma$ is an

RE relation.

1.4 DEFINITION. A numeration $\gamma = (\nu,S)$ is called <u>precomplete</u> if for every partial recursive φ there is a total recursive ψ s.t. for every $n \in \mathrm{Dom}\,\varphi$ $\varphi(n) \sim_\gamma \psi(n)$.
We shall say that ψ <u>makes</u> φ <u>total</u> <u>modulo</u> γ.

1.5 DEFINITION. A numeration $\gamma = (\nu,S)$ is <u>complete</u> if there is an $a \in S$ s.t. for every partial recursive φ there is a total recursive ψ s.t. for every $n \in \mathrm{Dom}\,\varphi$ $\varphi(n) \sim_\gamma \psi(n)$ and for every $n \notin \mathrm{Dom}\,\varphi$ $\nu(\psi(n)) = a$.
We will call "a" a special element of γ.

1.6 EXAMPLES OF NUMERATIONS

1.6.1 TERM MODELS OF $\lambda\beta$-CALCULUS. Let Λ be the set of closed λ-terms. A λ-theory T is a consistent set of identities between elements of Λ closed under the rules of $\lambda\beta$-calculus. Suppose $\ulcorner\ \urcorner$ is an elementary bijective coding of Λ in $\mathbb{N}$ ("elementary" means: easy to give, primitive recursive etc.).
Let for $M \in \Lambda$: $[\![M]\!]_T := \{N \in \Lambda \mid (M=N) \in T\}$ and take $\mathbb{M}_T := \{[\![M]\!]_T \mid M \in \Lambda\}$. Define: $\lambda_T(\ulcorner M \urcorner) := [\![M]\!]_T$. Now the term model $\mathcal{M}_T$ of T is the numeration $(\lambda_T, \mathbb{M}_T)$.

1.6.1.1 THEOREM. $\mathcal{M}_T$ is precomplete.

Proof. Let φ be partially recursive. There is a representation F of φ in $\lambda\beta$-calculus. Let $E \in \Lambda$ be the universal constructor i.e. the λ-term s.t. $\lambda\beta \vdash E\underline{\ulcorner M \urcorner} = M$, for all $M \in \Lambda$. Take $\psi(n) := \ulcorner E(F\underline{n})\urcorner$, then it is easy to see that ψ makes φ total modulo $\mathcal{M}_T$. □

1.6.2 THE RECURSIVE NATURAL NUMBERS. The recursive natural numbers are the 0-place partial recursive functions. Define:

$$\{n\}^0 :\cong \{n\} <> (\equiv \{n\}\, 0\,).$$

Let

$$\mathbb{N}^* := \mathbb{N} \cup \{\uparrow\}.$$

Further take

$$*(n) := \begin{cases} \{n\}^0 & \text{if } \{n\}^0 \downarrow \\ \uparrow & \text{else.} \end{cases}$$

The numeration N^* of recursive natural numbers is $(*, \mathbb{N}^*)$.

1.6.2.1 THEOREM. N^* is complete.

Proof. Let φ be partially recursive. Suppose φ has index p. Take an index p' s.t. $\{p'\}n \cong \{\{p\}n\}^0$ for all n. Construct a primitive recursive S_0^1 with $\{S_0^1(m,n)\}^0 \cong \{m\}n$. Take $\psi(n) \cong S_0^1(p',n)$. It is easy to see that ψ makes φ total modulo N^*. Moreover if $\varphi(n)\uparrow$ we have $*(\psi(n)) = \uparrow$. It is possible to prove that $\uparrow$ is the unique special element of N^*. □

1.6.2.2 REMARK. Define term models of λI-calculus in the obvious way. Call them M_T^I. Take T_0 the closure under the rules of λI of $\{M = N \mid M,N \text{ have no } \lambda I \text{ normal form}\}$. Then T_0 is consistent and $M_{T_0}^I$ is isomorphic (in the sense of the category of numerations) with N^*. No term model of λβ-calculus is isomorphic with N^*.

1.6.3 THE PARTIAL RECURSIVE FUNCTIONS. Let $\mathbb{P}$ be the set of partial recursive functions. Let $K(e) := (\lambda x \in \mathbb{N}.\{e\}x)$. Then $P := (K, \mathbb{P})$ is complete with as unique special element the nowhere defined function. (λ is an informal λ.)

1.6.4 THE RE SETS. Let $\mathbb{R}$ be the set of RE sets. Let $\Pi(e) := W_e$ (W_e is the domain of the partial recursive function with index e). Then $R := (\Pi, \mathbb{R})$ is complete with as unique special element the empty set.

1.6.5 THE 'INTENSIONAL' PARTIAL RECURSIVE FUNCTIONS. Let T be a theory, $T \supseteq PA$. (The language of T may extend the language of PA). Define: $[\![e]\!]^T := \{f \in \mathbb{N} \mid T \vdash \forall x(\{\underline{e}\}x \cong \{\underline{f}\}x)\}$.
Take $K^T(e) := [\![e]\!]^T$ and $\mathbb{P}^T := \{[\![e]\!]^T \mid e \in \mathbb{N}\}$.
Then $P^T = (K^T, \mathbb{P}^T)$ is the numeration of the partial recursive functions intensional in T.

1.6.5.1 THEOREM. P^T is precomplete.

Proof (sketch). Let φ be partially recursive. Suppose p is an index of φ. Define: $\psi(n) := \Lambda k\,.\,\{\{p\}n\}k$, where $(\Lambda k\,.\,\{\{p\}n\}k)$ is an index q s.t. $PA \vdash \forall x\{\underline{q}\}x \cong \{\{\underline{p}\}\underline{n}\}x$.
$(\Lambda k\,.\,\{\{p\}n\}k)$ can be found in a primitive recursive way from p and n. Then ψ makes φ total modulo P^T. □

1.6.6 THE Σ^0_n-FORMULAE WITH FIXED FREE VARIABLES OF A THEORY EXTENDING PA.

Let T be a consistent theory $T \supseteq PA$. Let $\Sigma^0_n(x_0,\ldots,x_{k-1})$ be the class of Σ^0_n formulae (not necessarily in prenex normal form) of the language of PA with free variables among $x_0,\ldots,x_{k-1}$. We shall abbreviate $(x_0,\ldots,x_{k-1})$ to $\vec{x}$. Take $\ulcorner\urcorner$ to be an elementary bijective coding of $\Sigma^0_n(\vec{x})$. Define for $A(\vec{x}) \in \Sigma^0_n(\vec{x})$:

$$[\![A(\vec{x})]\!]^T_{n,\vec{x}} := \{B(\vec{x}) \in \Sigma^0_n(\vec{x}) \mid T \vdash \forall\vec{x}(A(\vec{x}) \longleftrightarrow B(\vec{x}))\}$$

and

$$sig^T_{n,\vec{x}}(\ulcorner A(\vec{x})\urcorner) := [\![A(\vec{x})]\!]^T_{n,\vec{x}}.$$

Put:

$$Sig^T_{n,\vec{x}} := \{[\![A(\vec{x})]\!]^T_{n,\vec{x}} \mid A(\vec{x}) \in \Sigma^0_n(\vec{x})\}.$$

Then

$$\mathcal{S}ig^T_{n,\vec{x}} := (sig^T_{n,\vec{x}}, Sig^T_{n,\vec{x}})$$

is the numeration of $\Sigma^0_n(\vec{x})$ formulae of T.

1.6.6.1 THEOREM. $\mathcal{S}^T_{n,\vec{x}}$ is precomplete.

Proof. Consider φ partially recursive. Let p be an index of φ.

It is well known that there exists a $\Sigma_n^0(y,\vec{x})$ predicate $T_n(y,\vec{x})$ s.t. for all $A(\vec{x}) \in \Sigma_n^0(\vec{x})$: $PA \vdash \forall\vec{x}(A(\vec{x}) \leftrightarrow T_n(\underline{\ulcorner A(\vec{x})\urcorner}, \vec{x}))$.
Now take:

$$\psi(m) :\cong (\ulcorner \exists y\{\underline{p}\}\underline{m} \cong y \ \&\ T_n(y,\vec{x})\urcorner).$$

If $\varphi(m) \cong \ulcorner A(\vec{x})\urcorner$ we have:

$$T \vdash \forall\vec{x}(\exists y\{\underline{p}\}\underline{m} \cong y \ \&\ T_n(y,\vec{x}) \leftrightarrow T_n(\ulcorner A(\vec{x})\urcorner,\vec{x}) \leftrightarrow A(\vec{x})).$$

So ψ makes φ total modulo $Sig^T_{n,\vec{x}}$.

1.7 REMARK. Definition 1.1-1.5 are from Eršov (1973). The expression "making φ total modulo γ" is new. That $M_{\lambda\beta}$ is precomplete was pointed out to me by Henk Barendregt.

§2 A LITTLE EXCURSION INTO TOPOLOGY

2.1 DEFINITION OF $\mathcal{O}_\gamma$. Let $\gamma = (\nu,S)$ be a numeration. Let $\mathcal{B}_\gamma := \{S_0 \subseteq S \mid \nu^{-1}(S_0) \text{ in } \Pi_1^0\}$. Clearly $\mathcal{B}_\gamma$ is a basis for a topology. Call this topology $\mathcal{O}_\gamma$.

2.2 THEOREM. Morphisms are continuous.

Proof. Trivial. □

2.3 REMARK. As far as I know this topology doesn't occur in the literature. In Eršov (1975), Eršov uses a topology based on Σ_1^0 instead of Π_1^0 sets. But doing this one loses Thm. 2.5.

2.4 ILLUSTRATIVE EXAMPLE. The topology $\mathcal{O}_\gamma$ is meant to capture something of the idea of nearness w.r.t. information content. The following example purports to illustrate this.
Let $(T_i)_{i\in\mathbb{N}}$ be a sequence of consistent RE theories containing PA s.t. $T_{i+1} \subseteq T_i$ and $\bigcap_{i\in\mathbb{N}} T_i = PA$. Suppose $(A_i)_{i\in\mathbb{N}}$ is a sequence of Σ_1^0-sentences s.t. $T_i \vdash A_i$. Consider $Sig^{PA}_{1,<>}$ with topology $\mathcal{O}_{Sig^{PA}_{1,<>}}$, then $[\![\underline{0} = \underline{0}]\!]^{PA}_{1,<>}$ is the unique limit of $([\![A_i]\!]^{PA}_{1,<>})_{i\in\mathbb{N}}$.

2.4.1 REMARK. There is a sequence $(T_i)_{i\in\mathbb{N}}$ of RE theories containing PA s.t. $T_{i+1} \subsetneqq T_i$ and $\bigcap_{i\in\mathbb{N}} T_i = PA$.

Proof of remark. It can be shown that for any RE theories U,V with $U \supsetneqq V \supseteq PA$ there is a sentence A s.t. $V \subsetneqq (V+A) \subsetneqq U$ (see §3 of this paper).
Let T be RE, $T \supsetneqq PA$. Let $B_0, B_1, \ldots$ enumerate the sentences of L_T.
Define: $T_0 := T$; B_{i_k} is the first sentence s.t.

$$PA \subsetneqq (PA+B_{i_k}) \subsetneqq T_k;\ T_{k+1} := PA + B_{i_k}.$$

It is easy to see that the construction works and satisfies the desiderata. □

Proof of 2.4. First we show that $[\![\underline{0} = \underline{0}]\!]^{PA}_{1,<>}$ is a limit.
Suppose it is not. Then there is an RE set U closed under $\sim_{Sig^{PA}_{1,<>}}$ s.t. for infinitely many i, $\ulcorner A_i \urcorner \in U$ and $\ulcorner \underline{0} = \underline{0} \urcorner \notin U$.
Let B(x) be a Σ^0_1(x)-formula which represents U in PA.
Let $C :\equiv (\exists x\, B(x) \wedge T_1(x))$. Then: $\forall i \in \mathbb{N}\ T_i \vdash C$, but clearly $PA \nvdash C$ (because PA satisfies the existence property for Σ^0_1-sentences). Contradiction.
For unicity: assume $[\![A]\!]^{PA}_{1,<>}$ is also a limit and $[\![A]\!]^{PA}_{1,<>} \neq [\![\underline{0} = \underline{0}]\!]^{PA}_{1,<>}$. Then there is an N s.t. $T_N \nvdash A$. But $O_A := \{[\![A']\!]^{PA}_{1,<>} \mid T_N \nvdash A'\}$ is open and $[\![A]\!]^{PA}_{1,<>} \in O_A$.
But : $\forall i \geqslant N\ [\![A_i]\!]^{PA}_{1,<>} \notin O_A$ contradiction.

2.5 THEOREM. If γ is precomplete then $\mathcal{O}_\gamma$ is hyperconnected i.e. every two non-empty open sets intersect.

Proof. Clearly it is sufficient to prove the theorem for elements of $\mathcal{B}_\gamma$. So going over to complements, we have to show: for any RE sets U,V closed under $\sim_\gamma$, if $U \cup V = \mathbb{N}$ then $U = \mathbb{N}$ or $V = \mathbb{N}$. Suppose U,V RE, closed under $\sim_\gamma$ and $U \cup V = \mathbb{N}$. We are done if $U \subseteq V$ or $V \subseteq U$. So assume $n_U \in U \setminus V$ and $n_V \in V \setminus U$, in order to derive a contradiction. Let $\Sigma^+, \Sigma^- \subseteq \mathbb{N}$ be two RE, recursively inseparable sets. Define the partial recursive function φ as follows:

$$\varphi(n) :\cong \begin{cases} n_U & \text{if } n \in \Sigma^+ \\ n_V & \text{if } n \in \Sigma^- \\ \uparrow & \text{else} \end{cases}$$

Let ψ make φ total modulo γ.
Consider:

$$W_U := \{m \mid \psi(m) \in U\}$$
$$W_V := \{m \mid \psi(m) \in V\}$$

Then we have:

1. $W_U \cup W_V = \mathbb{N}$, since ψ is total and $U \cup V = \mathbb{N}$.
2. W_U, W_V are RE.
3. $\Sigma^- \cap W_U = \Sigma^+ \cap W_V = \emptyset$.

By familiar arguments we can construct a recursive set which separates Σ^+ and Σ^-, contradiction.

2.6 COROLLARY. Let γ be precomplete. Let $S_0 \subseteq S$ be discrete in $\mathcal{O}_\gamma$. (S_0 is discrete if the induced topology on S_0 is discrete). Let μ be continuous from $S \to S$ and $\mu(S) \subseteq S_0$. Then μ is constant on S.

Proof. This is an elementary topological fact. □

2.7 DEFINITION. Let $\gamma = (\nu, S)$. $S_0 \subseteq S$ is RE without repetitions if *(i)* S_0 is finite and for every $s_0 \in S_0\{n \mid \nu(n) = s_0\}$ is RE.

or

(ii) There is a total recursive ψ s.t. $\nu \circ \psi$ is injective, $\nu \circ \psi(\mathbb{N}) = S_0$ and $m \sim_\gamma \psi(n)$ is an RE relation in m and n.

2.8 FACT. Let $\gamma = (S, \nu)$. If $S_0 \subseteq S$ is RE without repetitions then S_0 is discrete.

Proof. Trivial. □

2.9 FACT. Consider M_T. Define: $\mu_M : M_T \to M_T$ as $\mu_M([\![N]\!])_T = [\![MN]\!]_T$. Then μ_M is a morphism.

Proof. Trivial. □

2.10 APPLICATIONS. Consider $M_{\lambda\beta}$. One easily sees that

(i) All finite $S_0 \subseteq \mathbb{M}_{\lambda\beta}$ are RE without repetitions.

(ii) $\mathbb{N}\,\mathbb{F} = \{[\![M]\!]_{\lambda\beta}$ M is in normal form} is RE without repetitions.

Remember that morphisms are continuous; combining 2.6, 2.8, 2.9, we find:

(i') If there is an $N_1,\ldots,N_k$ s.t. for any $P \in \Lambda$, $MP =_{\lambda\beta} N_i$ for some $1 \leqslant i \leqslant k$ then there is an N s.t. for all P: $MP =_{\lambda\beta} N$.

(ii') Suppose for any P, MP has a normal form, then there is a normal form N s.t. for all P: $MP =_{\lambda\beta} N$.

Of course *(i')* is the range theorem of λ-calculus.

2.11 DEFINITION.

(i) C, the set of <u>contexts</u> in the language $L_{PA} + \Box + p$, is the smallest set s.t.

(a) $A \in L_{PA} \Rightarrow A \in C$ (We allow that A contains free variables)

(b) $p \in C$

(c) $A,B \in C \Rightarrow A \wedge B,\ A \vee B,\ A \to B,\ \neg A \in C$

(d) For any $x_i \in \mathrm{Var} : A \in C \Rightarrow \forall x_i A, \exists x_i A \in C$

(e) $A \in C \Rightarrow \Box A \in C$.

(ii) We define $\mathrm{Sub} : L_{PA} \times C \to L_{PA}$ by:

(a) $A \in L_{PA} \Rightarrow \mathrm{Sub}(D,A) = A$

(b) $\mathrm{Sub}(D,p) = D$

(c) $\mathrm{Sub}(D,A \mathbin{\Delta} B) = \mathrm{Sub}(D,A) \mathbin{\Delta} \mathrm{Sub}(D,B)$, where $\Delta \in \{\wedge,\vee,\to\}$; $\mathrm{Sub}(D,\neg A) = \neg\,\mathrm{Sub}(D,A)$.

(d) $\mathrm{Sub}(D,Qx_i A) = Qx_i\ \mathrm{Sub}(D,A)$. $Q \in \{\forall,\exists\}$.
We allow that x_i is among the free variables of D.

(e) $\mathrm{Sub}(D,\Box A(\vec{x})) = \mathrm{prov}(\ulcorner \mathrm{Sub}(D,A(\vec{x}))\urcorner,\vec{x})$,
where $\vec{x}$ contains all free variables of A in order of first occurrence.

We will write for $C \in \mathcal{C}$: $C[p]$ and for $\mathrm{Sub}(D,C)$: $C[D]$.

2.12 LEMMA. $PA \vdash A \leftrightarrow B \Rightarrow PA \vdash C[A] \leftrightarrow C[B]$.

Proof. Use: $PA \vdash \forall\vec{x}(A(\vec{x}) \leftrightarrow B(\vec{x})) \Rightarrow$
$PA \vdash \forall\vec{x}(\mathrm{Prov}(\ulcorner A(\vec{x})\urcorner,\vec{x}) \leftrightarrow \mathrm{Prov}(\ulcorner B(\vec{x})\urcorner,\vec{x}))$. □

2.13 APPLICATIONS.

2.13.1 Let $C[p], D[p] \in \mathcal{C}$. Suppose that for all $A \in \Sigma^0_1$ $PA \vdash C[A]$ or $PA \vdash D[A]$. Then

for all $A \in \Sigma^0_1$, $PA \vdash C[A]$,

or

for all $A \in \Sigma^0_1$, $PA \vdash D[A]$.

Proof. $\Gamma := \{A \in \Sigma^0_1 \mid PA \vdash C[A]\}$,

$\Delta := \{A \in \Sigma^0_1 \mid PA \vdash D[A]\}$.

Well $\Gamma \cup \Delta = \Sigma^0_1$; Γ,Δ RE; Γ,Δ closed under provable equivalence. Apply hyperconnectedness. □

2.13.2 Suppose $\Box C[p]$, $\Box D[p] \in \mathcal{C}$, where C and D contain no free variables. Then

for all $A \in \Sigma^0_1$ $PA \vdash (\Box C[A] \vee \Box D[A])$

implies

for all $A \in \Sigma^0_1$ $PA \vdash \Box C[A]$

or

for all $A \in \Sigma^0_1$ $PA \vdash \Box D[A]$.

Proof. Because:

$PA \vdash \Box C[A] \vee \Box D[A] \Rightarrow$

$\mathbb{N} \vdash \Box C[A] \vee \Box D[A] \Rightarrow$

$$\mathbb{N} \vdash \Box C[A] \text{ or } \mathbb{N} \vdash \Box D[A] \Rightarrow$$
$$PA \vdash \Box C[A] \text{ or } PA \vdash \Box D[A].$$

(The last step is because $\Box C[A]$, $\Box D[A]$ are Σ^0_1). Now apply 2.13.1. $\Box$

2.13.3 Let $C(x)[p] \in \mathcal{C}$. Suppose that for every $A \in \Sigma^0_1$, there is precisely one n s.t. $PA \vdash C(\underline{n})[A]$ then there is precisely one n_0 s.t. for all $A \in \Sigma^0_1$ $PA \vdash C(\underline{n_0})[A]$.

Proof. $\varphi(\ulcorner A \urcorner) :=$ (the unique n s.t. $PA \vdash C(\underline{n})[A]$) induces a morphism from $Sig^{PA}_{1,<>}$ to $N = \begin{array}{c}\mathbb{N}\\ \downarrow \mathrm{id}\\ \mathbb{N}\end{array}$; and 0_N is discrete.

$\Box$

2.13.4 REMARK. For a different proof of 2.13.2 see Boolos (1979) page 106.

§3 THE FIXED POINT THEOREM

3.1 REMARK. Let γ be precomplete. By definition there is for every partial recursive φ a total recursive ψ which makes φ total modulo γ. We can even show that there is a total recursive $\chi : \mathbb{N}^2 \to \mathbb{N}$ s.t. for every index e $\lambda n\, \chi(e,n)$ makes $\lambda n\{e\}n$ total modulo γ. For consider $\rho(z) := \{z_0\}z_1$. Let ρ^* make ρ total and take $\chi(x,y) := \rho^*(\langle x,y\rangle)$. We shall write $\{x\}^\gamma y$ for $\chi(x,y)$.

3.2 FIXED POINT THEOREM (Eršov). Let γ be precomplete, then for every partial recursive φ we can find an n (effectively from an index of φ) s.t. $\varphi(n)\downarrow \Rightarrow \varphi(n) \sim_\gamma n$.

Proof. (We certainly can afford the space).
Let p be an index of φ. Let q be an index of $\lambda x\,.\,\{p\}(\{x\}^\gamma x)$. Say $\{q\}^\gamma q \cong q^*$. Suppose $\{p\}q^*\downarrow$ then:

$$\{p\}q^* \cong \{p\}(\{q\}^\gamma q) \cong \{q\}q \sim_\gamma \{q\}^\gamma q \cong q^*. \qquad \Box$$

Alternative Proof. Let ψ make $\lambda x\{p\}(\{x\}x)$ total. Let r be an index of ψ. Let r^* be $\{r\}r$. Suppose $\{p\}r^*\downarrow$ then:

$$\{p\}r^* \cong \{p\}(\{r\}r) \sim_\gamma \{r\}r \cong r^*.$$

3.3 DEFINITION. We say that a sequence $(U_i)_{i\in\mathbb{N}}$ of subsets of $\mathbb{N}$ is <u>recursive</u> if there is an RE relation $R(i,k)$ s.t. $k \in U_i \iff R(i,k)$. r is an index of the sequence if r is an index of R.

3.4 DEFINITION. Let $R(x,\vec{y})$ be an RE relation, then we can write R as $\exists z\, R_0(z,x,\vec{y})$, where R_0 is recursive. Define:

$$\varepsilon x R(x,\vec{y}) : \cong (\mu u R_0(u_0,u_1,\vec{y}))_1.$$

So $\varepsilon x.R(x,\vec{y})$ gives an element of $\{x \mid R(x,\vec{y})\}$ is there is one. Note that ε depends on the choice of R_0.

3.5 INDEX AVOIDING THEOREM. Suppose γ is precomplete and $(V_i)_{i\in\mathbb{N}}$ is a recursive sequence s.t.

(i) Each V_i is closed under $\sim_\gamma$
(ii) $i \notin V_i$;

then we can find (effectively from an index of $(V_i)_{i\in\mathbb{N}}$) an i_0 s.t. $V_{i_0} = \emptyset$.

Proof. Take i_0 a fixed point of $\lambda i(\varepsilon n \,.\, n \in V_i)$. □

3.6 THE GÖDEL-ROSSER-MOSTOWSKI-MYHILL-KRIPKE THEOREM. Consider $Sig^{PA}_{n,\vec{x}}$. Let $(T_i)_{i\in\mathbb{N}}$ be a recursive sequence of theories s.t. $T_i \supseteq PA$. Let $(U_i)_{i\in\mathbb{N}}$ be a recursive sequence of codes of formulae (say the coding is " ") s.t. $U_i \subseteq L_{T_i}$ and $T_i \nvdash A$ for all "A" $\in U_i$. (We could say that T_i leaves U_i out). Then there is a $\Sigma^0_n(\vec{x})$ formula A_0 s.t.

$$\forall A \in \Sigma^0_n(\vec{x})\ \forall i \in \mathbb{N}\ \forall \text{"B"} \in U_i\ T_i + \forall \vec{x}(A_0 \leftrightarrow A) \nvdash B.$$

Proof. Apply the Index Avoiding Theorem. Remember $\ulcorner\urcorner$ is a bijective coding of $\Sigma^0_n(\vec{x})$ formulae of L_{PA} in the natural numbers. Take V_i of the theorem as follows:

$$V_{\ulcorner A\urcorner} := \{\ulcorner C\urcorner \mid C \in \Sigma^0_n(\vec{x}) \textit{ and } \exists i \in \mathbb{N}\, \exists \text{"}B\text{"} \in U_i\ \ T_i + \forall\vec{x}(A \leftrightarrow C) \vdash B\}.$$

□

3.7 COROLLARY. Let $T \supseteq PA$ be consistent, RE. Then there is an infinite recursive sequence of Σ^0_1-sentences $B_0, B_1, \ldots$ s.t. for any sequence of Σ^0_1-sentences $C_0, C_1, \ldots$ $T + C_0 \leftrightarrow B_0, C_1 \leftrightarrow B_1, \ldots$ is consistent.

Proof. Apply 3.6 for $Sig^{PA}_{1,x}$ and $T_i = T$ and $U_i = \{\text{"}\underline{0} = \underline{1}\text{"}\}$.
Let A_0 be the formula given by the theorem.
Take $B_0 = A_0(\underline{0})$, $B_1 = A_0(\underline{1}), \ldots$. It is sufficient to prove that for any n: $T + C_0 \leftrightarrow A_0(\underline{0}) + \ldots\ C_n \leftrightarrow A_0(\underline{n})$ is consistent.
Take $D(x) \equiv (x = \underline{0} \wedge C_0) \vee \ldots (x = \underline{n} \wedge C_n)$, then
$T + \forall x(D(x) \leftrightarrow A_0(x))$ is consistent. But
$T + \forall x(D(x) \leftrightarrow A_0(x)) \vdash C_0 \leftrightarrow A_0(\underline{0}), \ldots\ C_n \leftrightarrow A_0(\underline{n})$. □

3.8 THEOREM. Consider $M_{\lambda\beta}$. Let $(T_i)_{i\in\mathbb{N}}$ be a recursive sequence of $\lambda\beta$-theories. Let $(U_i)_{i\in\mathbb{N}}$ be a recursive sequence of non empty sets of codes of identities of closed λ-terms (say coded by " ") s.t.

$$\forall i \in \mathbb{N}\ \ \forall \text{"}P = Q\text{"} \in U_i\ \ T_i \nvdash P = Q.$$

Then there is an $\Omega_0 \in \Lambda$ s.t.

$$\forall M \in \Lambda\ \forall i \in \mathbb{N}\ \ \forall \text{"}P = Q\text{"} \in U_i\ \ T_i + \Omega_0 = M \nvdash P = Q.$$

Proof. As in 3.6. □

3.9 COROLLARY. Let T be an RE λ-theory. Then there are $\Omega_1, \Omega_2, \ldots$ s.t. for every $M_1, M_2, \ldots$ $T + \Omega_1 = M_1 + \Omega_2 = M_2 + \ldots$ is consistent.

Proof. Apply 3.8. Take $T_i = T$, $U_i = \{\text{"}K = I\text{"}\}$. Let Ω_0 be given by the theorem. Take: $\Omega_1 = \Omega_0\,\underline{0}$, $\Omega_2 = \Omega_0\,\underline{1}, \ldots$.
It is sufficient to show that for any n $T + \Omega_0\,\underline{0} = M_0, \ldots \Omega_0\,\underline{n} = M_n$ is consistent. Let φ be recursive s.t. $\varphi(0) = \ulcorner M_0 \urcorner, \ldots \varphi(n) = \ulcorner M_n \urcorner$.

Let F represent φ in $\lambda\beta$. Then $T + \Omega_0 = \lambda x E(Fx)$ is consistent. And $T + \Omega_0 = \lambda x E(Fx) \vdash \Omega_0\ \underline{0} = E(F\underline{0}) = E\ulcorner\underline{M_0}\urcorner = M_0, \ldots \Omega_0\ \underline{n} = M_n$.

3.10 COROLLARY. Let $\mathcal{T}_\infty = (\mathbf{T}_\infty, \subseteq)$ be the partial ordering of $\lambda\beta$-theories or of theories in the language of PA, extending PA, then $P\omega$ can be embedded in $\mathcal{T}_\infty$ i.e. there is an $f : P\omega \to \mathbf{T}_\infty$ s.t.

$$\forall x, y \in P\omega\ (x \subseteq y \Longleftrightarrow f(x) \subseteq f(y)).$$

Proof. Let $x \subseteq \mathbb{N}$. In the case of $\lambda\beta$ take $f(x) = \lambda\beta + \{\Omega_0\ \underline{n} = KI \mid n \in x\}$ where Ω_0 is as in 3.9. In the case of PA take $f(x) = PA + \{A_0(\underline{n}) \mid n \in x\}$. Clearly in both cases $x \subseteq y \Rightarrow f(x) \subseteq f(y)$. By 3.9 and 3.7: $x \neq y \Rightarrow f(x) \neq f(y)$. □

3.11 INTRODUCTION TO 3.12. We are going to prove 3.12 both for RE $\lambda\beta$-theories and for RE theories in the language of PA, extending PA. To avoid unnecessary duplication we need a little dictionary:

NOTATION	MEANING FOR PA	MEANING FOR $\lambda\beta$
A, B, C	Σ^0_1-sentences	Identities between closed λ-terms
Ω_0, X, Y, Z	Σ^0_1-sentences	closed λ-terms
$\sim$	$\leftrightarrow$	$=$
Tr	$\underline{0} = \underline{0}$	KI
Fa	$\underline{0} = \underline{1}$	I
$X \vee A$	$X \vee A$	$XM = XN$ where $A \equiv (M = N)$
$X \wedge A$	$X \wedge A$	$X(K\mathrm{Tr})M = X(K\mathrm{Fa})N$ where $A \equiv (M = N)$
$X \to A$	$X \to A$	$X(K\mathrm{Tr}) \vee A$
$\mathbf{T}$	The set of RE theories, extending PA, in the language of PA	The set of RE $\lambda\beta$-theories

Of the special logical constants for $\lambda\beta$ we really need only Tr, Fa and $\vee$. We use the following fact:

3.11.1 FACT FOR $\lambda\beta$

(i) $\lambda\beta \vdash \mathrm{Tr} \vee A$

(ii) $\lambda\beta, \mathrm{Fa} \vee A \vdash A$

(iii) $\lambda\beta, A \vdash X \vee A$

(iv) $\lambda\beta, X = Y, X \vee A \vdash Y \vee A$

Proof. All the verifications are trivial for example in case *i)* if $A \equiv (M = N)$ then $\lambda\beta \vdash \mathrm{KIM} = \mathrm{I} = \mathrm{KIN}$. □

3.12 THEOREM. Any countable p.o. $S = (S,\leqslant)$ can be embedded in $T = (\mathbf{T},\subseteq)$.
Let $S = (S,\leqslant)$ be a countable p.o.

3.12.1 DEFINITION. Let $P,Q \subseteq S$; $P,Q \neq \emptyset$. We define:
$P \leqslant Q : \iff \exists p \in P,\ q \in Q\ \ p \leqslant q$.

3.12.2 DEFINITION. A function $f : S \to T$ is called faithful to S if: *(i)* $\forall p,q \in S\ \ p \leqslant q \Rightarrow f(p) \subseteq f(q)$

(ii) $(\forall A \bigcap_{p\in P} f(p) \vdash A \Rightarrow \bigcup_{q\in Q} f(q) \vdash A) \Rightarrow P \leqslant Q$, for all $P,Q \neq \emptyset$, $P, Q \subseteq S$.

3.12.3 MINILEMMA. f is faithful $\Rightarrow$ f is an embedding.

Proof. We just have to prove that $f(p_0) \subseteq f(q_0) \Rightarrow p_0 \leqslant q_0$.
Well: $f(p_0) \subseteq f(q_0) \Rightarrow \bigcap_{p\in\{p_0\}} f(p) \subseteq \bigcup_{q\in\{q_0\}} f(q) \Rightarrow$

$$(\forall A \bigcap_{p\in\{p_0\}} f(p) \vdash A \Rightarrow \bigcup_{q\in\{q_0\}} f(q) \vdash A) \Rightarrow$$

$$\{p_0\} \leqslant \{q_0\} \Rightarrow p_0 \leqslant q_0. \qquad \square$$

3.12.4 LEMMA. Let $S = (S,\leqslant)$ be finite with a distinct top $\top$ and bottom $\bot$. Suppose f is faithful to S. Let $S_0 = (S_0,\leqslant_0)$ be a p.o.

s.t. *(i)* $S_0 = S \cup \{s_0\}$, $s_0 \notin S$
(ii) $\leqslant \; = \; \leqslant_0 \upharpoonright S$

then there is an f_0 faithful to $\mathcal{S}_0$ with $f_0 \upharpoonright S = f$.
Let's first prove the theorem from the lemma before proving the lemma.

Proof of theorem 3.12. Let $\mathcal{S} = (S,\leqslant)$ be countable. Add to $\mathcal{S}$ a top and a bottom, $\top$ and $\bot$. ($\top,\bot \notin S$). We get $\mathcal{S}' = \{S \cup \{\top,\bot\},\leqslant')$ where for any $u,v \in S \cup \{\top,\bot\}$:

$$u \leqslant' v \Leftrightarrow ((u,v \in S \text{ and } u \leqslant v) \text{ or } u = \bot \text{ or } v = \top).$$

Suppose $S = \{s_1,s_2,\ldots\}$ $(i \neq j \Rightarrow s_i \neq s_j)$. Define:

$$S_n := \{\top,\bot,s_1,\ldots s_n\} \quad (n = 0,1,\ldots),$$
$$\mathcal{S}_n := (S_n, \leqslant' \upharpoonright S_n).$$

Let T_0,T_1 be any two elements of $\mathcal{T}$ s.t. $T_0 \subseteq T_1$ and there is an A s.t. $T_0 \nvdash A$ and $T_1 \vdash A$. (Note that in the case of PA $A \in \Sigma^0_1$ so what we ask is stronger than $T_0 \subsetneq T_1$). For example for $\lambda\beta$ take $T_0 = \lambda\beta$, $T_1 = \lambda\beta + \Omega = I$ and for PA take $T_0 = PA$ and $T_1 = PA + \neg\,\mathrm{con}(PA)$.
Take: $f_0 : S_0 \to \mathbf{T}$ as $f_0(\bot) = T_0$, $f_0(\top) = T_1$. Clearly f_0 is faithful to $\mathcal{S}_0$. By the lemma we find $f_0 \subseteq f_1 \subseteq \ldots$, where f_i is faithful to $\mathcal{S}_i$ and thus an embedding. Define $g := \bigcup_{i=0}^{\infty} f_i$ and $f = g \upharpoonright S$. It is easy to see that f is an embedding of $\mathcal{S}$ in $\mathcal{T}$. □

3.12.5 REMARK.

(i) We could also do our proof for the case of PA with Π^0_1-sentences instead of Σ^0_1-sentences. This would give the extra result that we could embed any countable p.o. in the <u>true</u> RE theories extending PA in the language of PA.

(ii) We could also do our proof for the case of PA with $\Sigma^0_n(\vec{x})$ or $\Pi^0_n(\vec{x})$ formulae. From this would follow that for any T_0,T_1 with $PA \subseteq T_0 \subset T_1$ (T_0,T_1 RE and in the language of PA) we could

embed any countable p.o. in the RE-theories in the language of PA between T_0 and T_1.
Because if $T_0 \subset T_1$ then there must be an $n, \vec{x}$ s.t. there is a $\Sigma_n^0(\vec{x})$ or a $\Pi_n^0(\vec{x})$ formula F s.t. $T_0 \nvdash F$ and $T_1 \vdash F$.

Now let's do the proof of the Lemma.

Proof of Lemma 3.12.4.

Define: $\hat{P}_0 := \{p \in S \mid p <_0 s_0\}$

$$\check{P}_0 := \{p \in S \mid s_0 <_0 p\}.$$

We have $\bot \in \hat{P}_0, \top \in \check{P}_0$ so $\hat{P}_0, \check{P}_0 \neq \emptyset$.
Let $\hat{P}_0, \hat{P}_1, \ldots \hat{P}_N$ be the subsets of S s.t. $\hat{P}_0 \subseteq \hat{P}_i$ and $\top \notin \hat{P}_i$ $(i = 0,\ldots N)$ (clearly $\top \notin \hat{P}_0$).

Let $\check{P}_0, \ldots \check{P}_{N'}$ be the subsets of S s.t. $\check{P}_0 \subseteq \check{P}_j$ and $\bot \notin \check{P}_j$ $(j = 0,\ldots N')$ (clearly $\bot \notin \check{P}_0$).
Define for $i = 0,\ldots N$:

$$\check{Q}_i := \{q \in S \mid \forall p \in \hat{P}_i \; q \not\leq p\},$$

then: $\top \in \check{Q}_i$ so $\check{Q}_i \neq \emptyset$;

$$\check{Q}_i \not\leq \hat{P}_i;$$

$$Q \not\leq \hat{P}_i \Rightarrow Q \subseteq \check{Q}_i.$$

Define for $j = 0,\ldots N'$:

$$\hat{Q}_j := \{q \in S \mid \forall p \in \check{P}_j \; p \not\leq q\},$$

then: $\bot \in \hat{Q}_j$ so $\hat{Q}_j \neq \emptyset$;

$$\check{P}_j \not\leq \hat{Q}_j;$$

$$\check{P}_j \not\leq Q \Rightarrow Q \subseteq \hat{Q}_j.$$

Let $f(s) = T_s$ for $s \in S$. We will define T_{s_0}. Make $f_0(s) = T_s$ for $s \in S_0$. Because f is faithful to $\mathcal{S}$ we have, for $i = 0,\ldots,N$:

$$\check{Q}_i \not\leq \hat{P}_i,$$

thus there is an A_i s.t. $\bigcap_{q\in\check{Q}_i} T_q \vdash A_i$ and $\bigcup_{p\in\hat{P}_i} T_p \nvdash A_i$.

and for $j = 0,\ldots,N'$:

$$\check{P}_j \not\leqslant \hat{Q}_j$$

and thus there is a B_j s.t. $\bigcap_{p\in\check{P}_j} T_p \vdash B_j$ and $\bigcup_{q\in\hat{Q}_j} T_q \nvdash B_j$.

By theorems 3.8 and 3.6 pick Ω_0 s.t. for all $0 \leqslant i \leqslant N$, $0\leqslant j\leqslant N'$ and for all X

$$\bigcup_{p\in\hat{P}_i} T_p + \Omega_0 \sim X \nvdash A_i$$

$$\bigcup_{q\in\hat{Q}_j} T_q + \Omega_0 \sim X \nvdash B_j$$

Define: $T_{s_0} := \bigcup_{p\in\hat{P}_0} T_p + \Omega_0 \vee B_1 + \ldots\Omega_0 \vee B_{N'}$.

We have to check that f_0 is faithful.

Ad(i) Suppose $u,v \in S_0$ and $u \leqslant_0 v$.

case a) $u,v \in S$, then $T_u \subseteq T_v$.

case b) $u = v = s_0$, then $T_u \subseteq T_v$.

case c) $u \neq s_0$, $v = s_0$. Then $u <_0 s_0$ i.e. $u \in \hat{P}_0$.

So $T_u \subseteq \bigcup_{p\in\hat{P}_0} T_p$. Thus $T_u \subseteq T_{s_0} = T_v$.

case d) $u = s_0$, $v \neq s_0$. Then $s_0 <_0 v$. We have $v \in \check{P}_0$ and thus for any $0 \leqslant j \leqslant N'$ $v \in \check{P}_j$,

so $T_v \supseteq \bigcap_{p\in\check{P}_j} T_p \vdash B_j$. Therefore $T_v \vdash \Omega_0 \vee B_j$.

Moreover because f is faithful and thus an embedding we have:

$$T_v \supseteq \bigcup_{p\in\hat{P}_0} T_p \text{ (for } p \in\hat{P}_0 \Rightarrow p <_0 s_0 <_0 v \Rightarrow p <_0 v \Rightarrow p < v)$$

So

$$T_v \supseteq \bigcup_{p\in\hat{P}_0} T_p + \Omega_0 \vee B_0 + \ldots\Omega_0 \vee B_{N'} = T_{s_0} = T_u.$$

Ad(ii) Suppose $U \not\leq_0 V$ ($u, v \neq \emptyset$); we prove that there is a C s.t. $\bigcap_{u \in U} T_u \vdash C$ and $\bigcup_{v \in V} T_v \nvdash C$.

We distinguish three cases:

case a) $s_0 \notin U$, $s_0 \notin V$. Then we are ready by the faithfulness of f.

case b) $s_0 \in U$, then of course $s_0 \notin V$.

Take $\widetilde{U} := (U \setminus \{s_0\}) \cup \check{P}_0$. Then $\widetilde{U}$ is one of the $\check{P}_j$, for trivially $\bot \notin \widetilde{U}$.

Suppose $\widetilde{U} = \check{P}_{j_0}$. We find $U \not\leq V$ so $V \subseteq \hat{Q}_{j_0}$. Clearly for every $u \in \widetilde{U}$: $T_u \vdash B_{j_0}$ so certainly $T_u \vdash \Omega_0 \vee B_{j_0}$.

Moreover $T_{s_0} \vdash \Omega_0 \vee B_{j_0}$. So

$\bigcap_{u \in U} T_u \supseteq \bigcap_{u \in \widetilde{U} \cup \{s_0\}} T_u \vdash \Omega_0 \vee B_{j_0}$.

Suppose on the other hand that $\bigcup_{v \in V} T_v \vdash \Omega_0 \vee B_{j_0}$, then certainly $\bigcup_{q \in \hat{Q}_{j_0}} T_q \vdash \Omega_0 \vee B_{j_0}$. But then

$\bigcup_{q \in \hat{Q}_{j_0}} T_q + \Omega_0 \sim Fa \vdash B_{j_0}$.

This contradicts our choice of Ω_0. So we can take $C := \Omega_0 \vee B_{j_0}$.

case c) $s_0 \in V$, then $s_0 \notin U$.

Take $\widetilde{V} := (V \setminus \{s_0\}) \cup \hat{P}_0$. Then we have: $\widetilde{V}$ is one of the $\hat{P}_i$, for trivially $\top \notin \widetilde{V}$.

Suppose $\widetilde{V} = \hat{P}_{i_0}$. We find $U \not\leq \widetilde{V}$ so $U \subseteq \check{Q}_{i_0}$. Then

$\bigcap_{u \in U} T_u \supseteq \bigcap_{q \in \check{Q}_{i_0}} T_q \vdash A_{i_0}$.

Suppose $\bigcup_{v \in V} T_v \vdash A_{i_0}$.

Well: $\bigcup_{v \in V} T_v = \bigcup_{v \in \widetilde{V}} T_v + \Omega_0 \vee B_0 + \ldots \Omega_0 \vee B_{N'}$. It follows that $\bigcup_{v \in \widetilde{V}} T_v + \Omega_0 \sim Tr \vdash A_{i_0}$. But $\widetilde{V} = \hat{P}_{i_0}$, so this contradicts our choice of Ω_0. So take $C := A_{i_0}$. □

3.13 REMARK. Inspection of the proof shows that the theories constructed are all finite over T_0. So using conjunction in case of PA or pairing in case of $\lambda\beta$, we can formulate the theorem for sentences instead of for theories.

3.14 REMARK. The version of the Gödel-Rosser-Mostowski-Myhill-Kripke-Theorem which is closest to mine is that of Kripke in Kripke (1963). Of course we have been a bit too generous using PA: a subsystem like Robinson's Arithmetic would have been sufficient.

§4 THE ANTI DIAGONAL NORMALISATION THEOREM AND SOME OF ITS APPLICATIONS.

4.1 MEDITATION. Consider $M_{\lambda\beta}$. By identifying certain elements, the unsolvables, and closing off under the rules of $\lambda\beta$-calculus we get M_H. M_H is complete. Here we have a natural transformation of a precomplete numeration into a complete one. The elements that we identify first are intuitively the "undefined" elements. By closing off under the rules certain other elements are also identified.
Now if we want to generalize the transition form $\lambda\beta$ to H it would be nice to have some control over the set:

$$A^\gamma := \{\{z_0\}^\gamma z_1 \mid \{z_0\}z_1\uparrow,\ z \in \mathbb{N}\},$$

for arbitrary precomplete γ. The Anti Diagonal Normalisation Theorem seems a step in the right direction; it is not strong enough to get the consistency of the identification of the elements of A^γ in the general case.

4.2 DEFINITION. Suppose $R(\vec{x})$ and $Q(\vec{x})$ are RE-relations. We write them in the form: $\exists y R_0(y,\vec{x})$ and $\exists y Q_0(y,\vec{x})$, where R_0 and Q_0 are recursive. Define:

$$R(\vec{x}) \leqslant Q(\vec{x}) \ :\iff\ \exists y\ \ R_0(y,\vec{x}) \text{ and } \forall z < y \text{ not } Q_0(z,\vec{x}).$$
$$R(\vec{x}) < Q(\vec{x}) \ :\iff\ \exists y\ \ R_0(y,\vec{x}) \text{ and } \forall z \leqslant y \text{ not } Q_0(z,\vec{x}).$$

Then $R(\vec{x}) \leqslant Q(\vec{x})$ and $R(\vec{x}) < Q(\vec{x})$ are RE relations.

4.3 DEFINITION. Let γ be a numeration. A partial recursive Δ is <u>a diagonal function</u> for γ if:

$$\forall x \in \mathrm{Dom}(\Delta) \quad \Delta(x) \not\sim_\gamma x.$$

4.4 THE ANTI DIAGONAL NORMALISATION THEOREM. Given a precomplete numeration γ and a partial recursive diagonal function Δ for γ, there is a total recursive function $\{x\}^{\gamma;\Delta}y$ s.t.

1) $\{x\}y\downarrow \Rightarrow \{x\}y \sim_\gamma \{x\}^{\gamma;\Delta}y$

2) $\{x\}y\uparrow \Rightarrow \{x\}^{\gamma;\Delta}y \notin \mathrm{Dom}(\Delta)$.

Proof. Let δ be an index for Δ. Define:

$$\{d(\delta,e,x)\}y :\cong \begin{cases} \{e\}x \ \mathit{if}\ \{e\}x\downarrow \ \leqslant \{\delta\}(\{y\}^\gamma y)\downarrow & (*) \\ \\ \{\delta\}(\{y\}^\gamma y) \ \mathit{if}\ \{\delta\}(\{y\}^\gamma y)\downarrow < \{e\}x\downarrow & (**) \end{cases}$$

Let us write $d := d(\delta,e,x)$. Suppose: $\{d\}d\downarrow$ because of clause (**), then $\{d\}d \cong \{\delta\}(\{d\}^\gamma d) \not\sim_\gamma \{d\}^\gamma d \sim_\gamma \{d\}d$. Contradiction. So if $\{d\}d\downarrow$ it does so because of (*). Thus we find :
$\{d\}d\downarrow \Rightarrow \{d\}d \cong \{e\}x$.
Suppose $\{d\}d\uparrow$ then we must have $\{\delta\}(\{d\}^\gamma d)\uparrow$, else (**) would give $\{d\}d$ a value. So $\{d\}d\uparrow \Rightarrow \{d\}^\gamma d \notin \mathrm{Dom}(\Delta)$.
So take: $\{x\}^{\gamma;\Delta}y :\cong \{d(\delta,x,y)\}^\gamma d(\delta,x,y)$.

4.5 REMARK. The idea for the Anti Diagonal Theorem occurred to me when I tried to find another proof of the theorem of Smoryński (1978), which is a consequence of the ADNT. Smoryński's forerunners are Shepherdson's fixed point and ultimately the Rosser Sentence.

4.6 COROLLARY. Let γ be precomplete. Let U be RE, non trivial and closed under $\sim_\gamma$, then U is maximal in the RE m-degrees and hence creative. (This corollary can be proved in other ways, see Eršov (1973)).

Proof. Consider an RE set V. Suppose $n_0 \in U$, $n_1 \notin U$. Take:

$$\Delta x :\cong \begin{cases} n_1 & \text{if } x \in U \\ \uparrow & \text{else} \end{cases}$$

and

$$\varphi(x) :\cong \begin{cases} n_0 & \text{if } x \in V \\ \uparrow & \text{else} \end{cases}$$

Clearly Δ is a diagonal function for γ.
Suppose p is an index of φ, then:

$$x \in V \Rightarrow \{p\}x \cong n_0 \Rightarrow \{p\}^{\gamma;\Delta}x \sim_\gamma n_0 \Rightarrow \{p\}^{\gamma;\Delta}x \in U$$
$$x \notin V \Rightarrow \{p\}x \uparrow \Rightarrow \{p\}^{\gamma;\Delta}x \notin \mathrm{Dom}(\Delta) \Rightarrow \{p\}^{\gamma;\Delta}x \notin U. \qquad \square$$

4.7 COROLLARY OF 4.6.

(i) Every non trivial RE set of λ-terms closed under β-convertibility is creative.

(ii) Any RE set U of identities between λ-terms s.t. if $(M = N) \in U$ and $\lambda\beta \vdash M = M'$, $\lambda\beta \vdash N = N'$ then $M' = N' \in U$ is creative or trivial.

(iii) Any RE $\lambda\beta$-theory is creative.

(iv) Let T be any RE $\lambda\beta$-theory s.t. $T \nvdash M = N$ then: $\{(P = Q) \mid T + P = Q \vdash M = N\}$ is creative.

Proof.

(i) routine from 4.6.

(i) $\Rightarrow$ *(ii)* $\Rightarrow$ *(iii)*, *(iv)* routine. $\square$

4.8 THEOREM. Let $(T_i)_{i\in\mathbb{N}}$ be a recursive sequence of $\lambda\beta$-theories. Let $(U_i)_{i\in\mathbb{N}}$ be a recursive sequence of sets of (codes of) identities between λ-terms s.t.

$$\forall i\ \forall \text{"}M = N\text{"} \in U_i\ \ T_i \nvdash M = N.$$

Let

$$I := \{\Omega_0 \in \Lambda \mid \ \forall P \in \Lambda\ \forall i \in \mathbb{N}\ \forall \text{"}M = N\text{"} \in U_i T_i + \Omega_0 = P \nvdash M = N\}.$$

Then there is an $F \in \Lambda$ s.t. for all $i,p,m,n \in \mathbb{N}$:

$$\{p\}m \cong n \iff \lambda\beta \vdash F\,\underline{p}\,\underline{m} = \underline{n} \iff T_i \vdash F\,\underline{p}\,\underline{m} = \underline{n}.$$

$$\{p\}m\uparrow \iff \exists\,\Omega_0 \in I\ \lambda\beta \vdash F\,\underline{p}\,\underline{m} = \Omega_0.$$

Proof. Take:

$$\Delta\ulcorner P\urcorner :\cong \varepsilon\ulcorner Q\urcorner(\exists\, i \in \mathbb{N}\ \exists\ \text{"}M = N\text{"} \in U_i\ \ T_i + P = Q \vdash M = N.$$

Clearly Δ is a diagonal function for $M_{\lambda\beta}$ and $\ulcorner P\urcorner \notin \mathrm{Dom}\,\Delta \iff P \in I$.
Choose p' s.t.

$$\{p'\}m \cong \begin{cases} \ulcorner \underline{n}\urcorner & \text{if } \{p\}m \cong n \\ \uparrow & \text{else} \end{cases}$$

Let p" be an index of $\{p'\}^{M_{\lambda\beta};\Delta}$.
It is clear that we can find p" from p in an effective way. So there is a λ-term Q which represents: $\psi(p,m) :\cong \{p''\}m$ in $\lambda\beta$.
Take $F :\equiv (\lambda xy\,.\,E(Qxy))$. (Remember that E is the universal constructor i.e. $\forall M \in \Lambda\ \ \lambda\beta \vdash E\ulcorner \underline{M}\urcorner = M$).
We have:

(*i*) $\{p\}m \cong n \ \Rightarrow\ \{p'\}m \cong \ulcorner\underline{n}\urcorner \ \Rightarrow\ (\{p''\}m \cong \ulcorner M\urcorner$
where $\ulcorner M\urcorner \sim_{M_{\lambda\beta}} \ulcorner\underline{n}\urcorner) \ \Rightarrow\ \lambda\beta \vdash E(Q\,\underline{p}\,\underline{m}) = E(\ulcorner \underline{M}\urcorner) = M = \underline{n}$
$\Rightarrow\ T_i \vdash E(Q\,\underline{p}\,\underline{m}) = \underline{n}$.

(*ii*) $\{p\}m\uparrow \ \Rightarrow\ \{p'\}m\uparrow \ \Rightarrow\ \{p''\}m = \ulcorner\Omega_0\urcorner$
$\Rightarrow\ \lambda\beta \vdash E(Q\,\underline{p}\,\underline{m}) = E\ulcorner\Omega_0\urcorner = \Omega_0$
$\Rightarrow\ T_i \vdash E(Q\,\underline{p}\,\underline{m}) = \Omega_0$. Where $\Omega_0 \in I$.

Moreover by the properties of elements of I and the fact that numerals cannot be identified consistently with every term it follows $\forall \underline{n}\ T_i \nvdash \Omega_0 = \underline{n}$ for $\Omega_0 \in I$.
A moment's reflection shows that the theorem follows.

4.9 REMARK. Obviously a similar theorem can be given for PA.

4.10 COROLLARY. The partial recursive functions can be represented in any RE λ-theory.

4.11 EXAMPLE OF THE CONSTRUCTION OF A COMPLETE NUMERATION BY MEANS OF $M_{\lambda\beta}$.

Take I as in 4.8 for the constant sequence with $T_i = \lambda\beta$ and $U_i = \{"K = I"\}$. Define:

$$H^- := \lambda\beta + \{(M = N) \mid M,N \in I\}.$$

It is easy to see that if $M \in I$ then M is unsolvable, so H^- is consistent. By 4.8 M_{H^-} is complete.

4.12 REMARKS.

(i) We could do a similar construction for e.g. P^{PA}.

The point where we use "M $\in$ I then M is unsolvable" contains a reference to specific properties of $\lambda\beta$-calculus. It would be nice to eliminate this. As yet I see no way to avoid it. Possibly some natural condition is missing. In the case P^{PA} one uses that the identifications are true.

4.13 REMARK. There are some analogies between λ-calculus and PA that invite further reflection. I will state them without proof. Let for $e,f \in \mathbb{N}$, "$e \sim f$" mean "$\forall x\{e\}x \cong \{f\}x$". Define:

$$T_0 := PA + \{(\underline{e} \sim \underline{f}) \mid \forall n\{e\}n\uparrow,\ \{f\}n\uparrow\};$$

it is easy to see that

$$T_0 = PA + \Sigma_2^0\text{-truth}.$$

Let $T_1 := PA + \{(\underline{e} \sim \underline{f}) \mid e \sim f\}$. $P^{T_1} = P$ up to isomorphism. We have:

1) $\{(M = N) \mid \lambda\beta \vdash M = N\}$ is Σ_1^0-complete
$\{(e \sim f) \mid PA \vdash \underline{e} \sim \underline{f}\}$ is Σ_1^0-complete.

2) H is Σ_2^0-complete (see Barendregt (1978));

$\{(e \sim f) \mid T_0 - \underline{e} \sim \underline{f}\}$ is Σ_2^0. (Σ_2^0 complete?)

3) H^* is Π_2^0-complete and the unique maximal extension of H;

$\{(e \sim f) \mid e \sim f\} = \{(e \sim f) \mid T_1 \vdash \underline{e} \sim \underline{f}\}$ is Π_2^0-complete and is the unique maximal (consistent) extension of $\{(e \sim f) \mid T_0 \vdash \underline{e} \sim \underline{f}\}$ in the sense that $T_0 + \underline{e} \sim \underline{f} \nvdash \underline{0} = \underline{1} \Rightarrow T_1 \vdash \underline{e} \sim \underline{f}$.

4.16 REMARK. For a proof that M_H is complete see Barendregt (1975). The results on many-one degrees of λ-theories mentioned in 4.15 are from Barendregt (1978).

REFERENCES.

Barendregt, H.P., Some extensional term models for combinatory logics and λ-calculi, thesis, Utrecht, 1971.

Barendregt, H.P., Solvability in lambda calculi, Colloque International de Logique, 1975, Editions du Centre National de la Recherche Scientifique, pp. 209-219.

Barendregt, H.P., The Type Free Lambda Calculus, Handbook of Mathematical Logic, 1977, North-Holland, pp. 1091-1132.

Barendregt, H.P., Bergstra J., Klop J.W., Volken, H., Degrees of sensible lambda theories, JSL, volume 43, no. 1, 1978, pp. 45-55.

Boolos, G., The Unprovability of Consistency, 1979, Cambridge University Press.

Eršov, Ju.L., Theorie der Numerierungen I, Zeitschrift für Math. Log., Band 19, Heft 4, 1973, pp. 289-388.

Eršov, Ju.L., Theorie der Numerierungen II, Zeitschrift für Math. Log., Band 21, Heft 6, 1975, pp. 473-584.

Eršov, Ju.L., Model ℂ of Partial Continuous Functionals, Logic Colloquium 1976, North-Holland, 1977, 455-467.

Kripke, S., "Flexible" predicates in formal number theory, Proc. AMS 13, 1963, pp. 647-650.

Rogers, Theory of Recursive Functions and Effective Computability, 1967, McGraw-Hill.

Scott, D., Data types as lattices, Logic Conference, Kiel 1974, Lecture Notes in Mathematics 499, Springer Verlag, 1975, pp. 579-651.

Scott, D., Combinators and classes, λ-Calculus and Computer Science Theory, 1975, Lecture Notes in Computer Science 37, Springer Verlag, pp. 1-26.

Scott D., Some philosophical issues concerning theories of of combinators, λ-calculus & Computer Science Theory, 1975, Lecture Notes in Computer Science 37, Springer Verlag, pp. 346-366.

Smoryński, C., Avoiding Selfreferential Statements, Proceedings of the AMS, volume 70, number 2, July 1978, pp. 181-184.

PART 2

MATHEMATICS Proceedings A 84 (2), June 20, 1981

An incompleteness result for paths through or within O

by Albert Visser

Mathematical Institute, University of Utrecht, the Netherlands

Communicated by Prof. A.S. Troelstra at the meeting of May 31, 1980

ABSTRACT

In this paper an incompleteness result for paths through or within O and related structures will be proved. A consequence of this result will be that no complete Π^0_1 set is partially many-one reducible to any path through or within O. For enumeration reducibility a weak kind of incompleteness is obtained.

1 INTRODUCTION

Roughly there are four kinds of results connected with completeness for paths through or within O:

a) Completeness results for progressions of theories

In [Fe] Feferman produced a path within O and a recursive progression of theories along that path, yielding the true sentences of number theory.

b) Completeness results in the sense of recursion theoretic reducibilities

E.g. it is easy to see that any set of natural numbers is enumeration reducible to a path within O of length ω^2.

c) Incompleteness results for progressions of theories

In [Fe, Sp] Feferman and Spector established a number of incompleteness results for Π^1_1-paths through O. In [Kr] Kreisel gave another incompleteness result for Π^1_1-paths through O. We present here a strengthening of Kreisel's result with a new proof, using the methods of Feferman and Spector. For the definition of and the basic facts about O^*; see [Fe, Sp].

THEOREM: Let π be a Π^1_1-path through O. Let $p \in T_a$ be an RE-relation in p, a s.t. for each $a \in \pi : T_a$ is a set of (codes of) sentences in the language of PA with $\mathbb{N} \models T_a$. Let $\leq$ be an RE-relation s.t. $\leq \upharpoonright O^* = \leq_{O^*}$ and $a \leq b$ and $b \in O^* \Rightarrow a \in O^*$. Define: $T_\pi = \bigcup_{a \in \pi} T_a$. Then there is an RE theory T s.t. $\mathbb{N} \models T$ and $T \supseteq T_\pi$.

PROOF: Without loss of generality we may assume that $a \leq b \Rightarrow T_a \subseteq T_b$. Define $K_a = \{b \mid b \leq a\}$. There is an $a^* \in O^*$ s.t. $K_{a^*} \supseteq \pi$ (see [Fe, Sp]). So $T_{a^*} \supseteq T_\pi$. If T_{a^*} is true, we are done, if not, consider:

$C := \{c \leq a^* \mid T_c$ is not (the set of codes of) a true theory in the language of PA$\}$.

Observe that C is hyperarithmetic, so C has a smallest element w.r.t. $\leq$, say c_0. Pick $c_1 \in O^* \backslash O$ with $c_1 \not\geq c_0$. Put $T := T_{c_1}$. □

d) Incompleteness results for recursion theoretic reducibilities

For example we will show that the halting problem is not partially many-one reducible to any path through or within O.

2 NOTATIONS AND CONVENTIONS

We use $X, Y, \ldots, A, B$ for sets of natural numbers; $x, y, \ldots, a, b$ for natural numbers, and $\varphi, \psi, \varphi_A, \ldots$ for *partial* recursive functions.

$$\Sigma^+ := \{x \mid \{x\}x \cong 0\}$$

$$\Sigma \ := \{x \mid \{x\}x\downarrow \text{ and } \{x\}x \not\cong 0\}$$

$$\Pi^+ := \mathbb{N} \backslash \Sigma^+.$$

Let $R_1(x)$, $R_2(x)$ be RE predicates. There are recursive relations $R'_1(y,x)$ and $R'_2(y,x)$ s.t.:

$$R_1(x) \Leftrightarrow \exists\, y\ R'_1(y,x),$$

$$R_2(x) \Leftrightarrow \exists\, y\ R'_2(y,x).$$

Define:

$$R_1(x) \leq R_2(x) :\leftrightarrow \exists\, y(R'_1(y,x) \text{ and } \forall\, z < y \neg R'_2(z,x))$$

$$R_1(x) < R_2(x) :\leftrightarrow \exists\, y(R'_1(y,x) \text{ and } \forall z \leq y \neg R'_2(z,x))$$

Note that the meaning of $<$, $\leq$ depends on the choice of R'_1, R'_2. For any RE predicate $R(x)$ occurring in the text we will use one fixed $R'(y,x)$ in the whole text. If an RE predicate $Q(x)$ is introduced in the text as (e.g.) $R_1(x) \wedge R_2(x)$, where R_1, R_2 are RE predicates, we take: $Q'(y,x) \equiv (R'_1((y)_0, x) \wedge R'_2((y)_1, x))$. We will often use informal expressions as: "we found $R_1(x)$ before $R_2(x)$" or "we found $R_1(x) \vee R_2(x)$ before $R_3(x)$, because we found $R_1(x)$ first" etc. Clearly on our conventions these expressions have a natural interpretation.

For:

$$\psi(x) :\cong \begin{cases} a_1 & \text{if } R_1(x) \le R_2(x) \\ a_2 & \text{if } R_2(x) < R_1(x) \end{cases}$$ [1]

we use the following shorthand:

$$\psi(x) :\# \begin{cases} a_1 & \text{if } R_1(x) \\ a_2 & \text{if } R_2(x) \end{cases}$$

We will sometimes use abbreviations such as: '$\psi_{\dot{1} \atop 2}(x) \cong y$' for '$\psi_1(x) \cong y$ or $\psi_2(x) \cong y$'.

Scope should be evident from context.

3. THE CONCEPT OF SEPARABILITY AND SOME RELATED HIERARCHIES

The concept of separability introduced here is a generalization of the usual notion of separability in recursion theory. It is inspired by the proof of the Gödel-Rosser-Mostowski-Myhill-Kripke theorem *and* partial many-one reducibility. An analogous concept could be given for enumeration reducibility. However in that case the central result of this paper does not hold.

3.1 DEFINITION

X sep(A,B) (X separates A and B) iff there are φ_A, φ_B s.t.:

a) $\varphi_A^{-1}(X) \cap B = \varphi_B^{-1}(X) \cap A = \emptyset$

b) $\varphi_A^{-1}(X) \cup \varphi_B^{-1}(X) = \mathbb{N}$.

3.2 DEFINITION

$X \le_{pm} Y$ (X is partially many-one reducible to Y) iff there is a φ s.t. $X = \varphi^{-1} Y$.

3.3 REMARK

X sep(A,B) and $X \le_{pm} Y \to Y$ sep(A,B).

3.4 DEFINITIONS

3.4.1 $X \sqsubseteq Y$ iff for every (A,B) : X sep$(A,B) \to Y$ sep(A,B).
3.4.2 $X \Box Y$ iff $X \sqsubseteq Y$ and $Y \sqsubseteq X$.
3.4.3 $(A,B) \le_s (A',B')$ iff for every X : X sep$(A',B') \to X$ sep(A,B).
3.4.4 $(A,B) =_s (A',B')$ iff $(A,B) \le_s (A',B')$ and $(A',B') \le_s (A,B)$.

3.5 EXAMPLE

Let π be a Π_1^1-path through O, then $\pi \Box \{1\}$.

---

[1] I consider the 'else : undefined'-clause as implicit in $\cong$.

PROOF: Let $\leq$ be an RE relation that coincides with $\leq_{O^*}$ in O^* and satisfies: $a \leq b$ and $b \in O^* \Rightarrow a \in O^*$.

Put $K(a) := \{b \mid b \leq a\}$. There is a $p \in O^*$ s.t. $\pi \subseteq K(p)$ (see [Fe, Sp]). Suppose that π *sep*(A,B) via φ_A, φ_B. There are two possibilities:

1. There is an $a^* \in \pi$ s.t. $K(a^*)$ sep(A,B) via φ_A, φ_B. $K(a^*) \leq_{pm} \{1\}$ so then $\{1\}$ sep(A,B) and we are done.
2. There is no such a^*. But in that case: $a \in \pi$ iff ($a \leq p$ and $\varphi_A^{-1}(K(a)) \cup \varphi_B^{-1}(K(a)) \neq \mathbb{N}$). This makes π Σ^0_2. Contradiction. □

The proof works also for enumeration reducibility, thus giving the result of [Kr] p. 313, §3.

3.6 FACT

If φ is a total recursive function then $(A,B) \geq_s (\varphi^{-1}A, \varphi^{-1}B)$.

3.7 FACT

If A,B are RE sets, $A \cap B = \emptyset$, then: $(A,B) \leq_s (\Sigma^+, \Sigma^-)$.

PROOF: Find a total recursive φ s.t.: $\{\varphi(x)\}\varphi(x) \cong \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \in B \end{cases}$
Then: $A = \varphi^{-1}\Sigma^+$, $B = \varphi^{-1}\Sigma^-$. □

3.8 EXAMPLES

i) It is not the case that: $\{1\}$ sep(Σ^+, Σ^-).
ii) Π^+ sep(Σ^+, Σ^+).

PROOF OF ii: Let p^+ be a fixed element of Π^+. Make: $\varphi_{\Sigma^+}(x) :\cong p^+$ if $x \in \Sigma^+$; $\varphi_{\Sigma^-}(x) :\cong x$. □

4. ABOUT DOWNWARD CLOSED AND COMPARABLE SETS

4.1 A set X is Downward Closed and Comparable or DCC if there is an RE relation $\lhd$ s.t.

a) $x \lhd y$ and $y \in X \rightarrow x \in X$.
b) $x, y \in X \rightarrow x \lhd y$ or $y \lhd x$.

4.1.1 *Comment*

Note that every RE set and every path through or within O is DCC.

4.2 THEOREM

No X, which is DCC, separates (Σ^+, Σ^-).

PROOF: Suppose X is DCC and separates Σ^+, Σ^- via φ^+, φ^-. We will construct recursive sets S^+, S^- s.t. $S^+ = (S^-)^c$ and $\Sigma^+ \subseteq S^+$, $\Sigma^- \subseteq S^-$. Thus obtaining a contradiction.

From the definitions of DCC and separability, we see that for every x and y:

$$(*) \qquad \bigvee_{i,j\in\{+,-\}} (\varphi^i(x) \lhd \varphi^j(y)) \quad \text{or} \quad \bigvee_{i,j\in\{+,-\}} (\varphi^i(y) \lhd \varphi^j(x)).$$

(We take $\varphi^i(x) \lhd \varphi^j(y)$ to imply: $\varphi^i(x)\downarrow$, $\varphi^j(y)\downarrow$).

Find e_x (primitive) recursive in x s.t.

$$\{e_x\}y :\# \begin{cases} 0 & \text{if } \varphi^{\pm}(x) \lhd \varphi^+(y) \quad \text{or} \\ & \quad \varphi^-(y) \lhd \varphi^{\pm}(x). \\ 1 & \text{if } \varphi^{\pm}(x) \lhd \varphi^-(y) \quad \text{or} \\ & \quad \varphi^+(y) \lhd \varphi^{\pm}(x). \end{cases}$$

By $(*)$ we see that $\{e_x\}y\downarrow$ for any x, y. We now construct S^+, S^- from the computation of $\{e_x\}e_x$. We will indicate the proof only for the case $\{e_x\}e_x \cong 0$; the case $\{e_x\}e_x \cong 1$ is similar.

Suppose $\{e_x\}e_x \cong 0$, then $e_x \in \Sigma^+$. So $\varphi^+(e_x) \in X$ and ($\varphi^-(e_x)\uparrow$ or $\varphi^-(e_x) \notin X$).

Now there are four cases to consider:
We put $\{e_x\}e_x \cong 0$ because we first found that:

Case 1 $\quad \varphi^+(x) \lhd \varphi^+(e_x)$.

Then $\varphi^+(x) \in X$. Put $x \in S^+$.

Case 2 $\quad \varphi^-(x) \lhd \varphi^+(e_x)$.

Then $\varphi^-(x) \in X$. Put $x \in S^-$.

Case 3 $\quad \varphi^-(e_x) \lhd \varphi^+(x)$.

Because $\varphi^-(e_x) \notin X$, also $\varphi^+(x) \notin X$. Therefore $\varphi^-(x) \in X$. Put $x \in S^-$.

Case 4 $\quad \varphi^-(e_x) \lhd \varphi^-(x)$.

Then $\varphi^+(x) \in X$. Put $x \in S^+$.

That S^+, S^- have the desired properties is immediate from the construction.□

4.4 COROLLARY

For no DCC X we have $X \geq_{pm} \Pi^+$.

4.5 DEFINITION

$X \leq_{en} Y$ iff there is an RE relation R s.t. for every $x : x \in X \leftrightarrow$ there is a y s.t. $(\bigwedge_{i=0}^{\mathrm{lth}(y)\dot{-}1} ((y)_i \in Y)$ and $R(x, y))$.

We say: X is enumeration reducible to Y.

4.6 LEMMA

If Y is DCC:

$X \leq_{en} Y \leftrightarrow$ there is an RE relation Q s.t. for every x:

$$(x \in X \leftrightarrow \text{there is a } y \in Y\ Q(x,y)).$$

PROOF: Let R be as in 4.5. Take Q: $Q(x,y)$ iff there is a z s.t. $\bigwedge_{i=0}^{\mathrm{lth}(z)\dot{-}1} ((z)_i \trianglelefteq y)$ and $R(x,z)$. □

The idea of the lemma is due to Kreisel, see [Kr] p. 313, §2.

4.7 THEOREM

Let Y be DCC, $X \leq_{en} Y$ via Q as in 4.6. Suppose X sep(Σ^+,Σ^-) via φ^+,φ^-. Then there are no ψ^+,ψ^- s.t. for every z:

$$((Q(\varphi^+(z),\ \psi^+(z)) \quad \text{and} \quad \psi^+(z) \in Y)$$

or

$$(Q(\varphi^-(z),\ \psi^-(z)) \quad \text{and} \quad \psi^-(z) \in Y))$$

PROOF: Suppose there were such ψ^+,ψ^-. First remark that $Z = \{\langle x,y\rangle \mid y \in Y$ and $Q(x,y)\}$ in DCC. For suppose that Y is DCC via $\triangleleft$. Put:

$$\langle x,y\rangle \triangleleft' \langle x',y'\rangle \quad \text{iff } y \triangleleft y' \text{ and } Q(x,y) \text{ and } Q(x',y').$$

It is easy to see that Z is DCC via $\triangleleft'$. We find: Z sep(Σ^+,Σ^-) via χ^+,χ^- with:

$$\chi^+(z) :\equiv \langle \varphi^+(z),\ \psi^+(z)\rangle$$

and

$$\chi^-(z) :\equiv \langle \varphi^-(z),\ \psi^-(z)\rangle.$$

Contradiction. □

4.8 COROLLARY

Let Y be DCC. Suppose $q \in T_a$ is an RE relation in q,a s.t.:

a) $a \in Y \rightarrow T_a$ is (a set of codes of) a consistent theory containing PA.
b) For every n there is an a in Y s.t.

$$T_a \vdash \text{“}n \in \Sigma^+\text{”} \quad \text{or} \quad T_a \vdash \text{“}n \notin \Sigma^+\text{”}$$

then there are no ψ^+,ψ^- s.t. for every z:

$$((\psi^+(z) \in Y \quad \text{and} \quad T_{\psi^+(z)} \vdash \text{“}z \in \Sigma^+\text{”})$$

or

$$(\psi^-(z) \in Y \quad \text{and} \quad T_{\psi^-(z)} \vdash \text{“}z \notin \Sigma^+\text{”})).$$

PROOF: By taking in 4.7:

$$X := \bigcup_{a \in Y} T_a,$$

$$\varphi^+(x) := \text{“}x \in \Sigma^+\text{”},$$

$$\varphi^-(x) := \text{“}x \notin \Sigma^+\text{”}. \quad \square$$

4.9 COROLLARY

The Gödel-Rosser-Mostowski-Myhill-Kripke Theorem: Let $(T_a)_{a \in \mathbb{N}}$ be a recursive sequence of consistent theories containing PA. Then there is a b s.t. for every a: $T_a \nvdash$ “$b \in \Sigma^+$” and $T_a \nvdash$ “$b \notin \Sigma^+$”.

PROOF: Take for Y of 4.8 the set $\mathbb{N}$. Clearly condition a) of 4.8 is fulfilled. Suppose b) were fulfilled. It is easy then to give ψ^+, ψ^- s.t.

$$T_{\psi^+(z)} \vdash \text{“}z \in \Sigma^+\text{”}$$

or

$$T_{\psi^-(z)} \vdash \text{“}z \notin \Sigma^+\text{”}.$$

Contradiction. $\square$

REFERENCES

[Fe] Feferman, S. – *Transfinite Recursive Progressions of Axiomatic Theories,* JSL, **27**, 259–316 (1962).

[Fe, Sp] Feferman, S. and C. Spector – *Incompleteness along paths in progressions of theories.* JSL, **27**, 383–390 (1962).

[Kr] Kreisel, G. – *Which Number Theoretic Problems Can be Solved In Recursive Progressions on Π^1_1-Paths Through O?* JSL, **37**, 311–355 (1972).

PART 3

A Propositional Logic with Explicit Fixed Points

Albert Visser

Abstract: This paper studies a propositional logic which is obtained by interpreting implication as formal provability. It is also the logic of finite irreflexive Kripke Models.

A Kripke Model completeness theorem is given and several completeness theorems for interpretations into Provability Logic and Peano Arithmetic.

0.0 The strange tale of the Formalist who lost Modus Ponens

Imagine a formalist. He has become convinced that any philosophy of mathematics worth to be taken seriously, must explain the meaning of the logical constants. Yet he clings to the tenets of formalism. He sets out to produce a hybrid: formal semantics.

Consider the translation * from L, the language of Propositional Logic (PL), to $L_{\Box}$, the language of Modal Propositional Logic (MPL). It is given by:

$$
\begin{aligned}
p_i &\overset{*}{\mapsto} \Box p_i\\
\bot &\overset{*}{\mapsto} \Box \bot\\
(A \wedge B) &\overset{*}{\mapsto} (A^* \wedge B^*)\\
(A \vee B) &\overset{*}{\mapsto} (A^* \vee B^*)\\
(A \to B) &\overset{*}{\mapsto} \Box(A^* \to B^*).
\end{aligned}
$$

Our formalist stipulates: the formalistic meaning of C is to be the classical meaning of C^*, where $\Box$ is interpreted as provability in some fixed formal system. For definiteness he restricts his attention to Peano Arithmetic (PA). An interpretation of L into L_{PA} is just $f \circ *$, where f is some interpretation for Provability Logic (PrL) of $L_\Box$ into L_{PA} (for more details see §5). Formal Validity will be defined as: A is *formally valid* iff for every interpretation g $PA \vdash A^g$.

The formalist proceeds to formalize. He looks for a logic - even before conception he calls it : Formal Propositional Logic (FPL) - satisfying :

$\vdash_{FPL} A$ iff A is formally valid.

By Solovay's Completeness Theorem for Provability Logic (PrL), this is equivalent to : $\vdash_{FPL} A$ iff $\vdash_{PrL} A^*$.

The logic he finds turns out to have Full Explicit Fixed Points, i.e. for any $A(p,\vec{q})$ there is a $B(\vec{q})$ s.t. $B(\vec{q}) \dashv\vdash_{FPL} A(B(\vec{q}),\vec{q})$.

For this luxury however a price has to be paid. The Fixed Point Theorem readily yields an explicit Liar Sentence. Yet $\perp$ should not be derivable. So one of the steps in the usual derivation of $\perp$ must be blocked. It turns out to be Modus Ponens. The reason is that $PA \nvdash Prov\ulcorner \underline{A} \urcorner \to A$ in general, e.g. $PA \nvdash Prov\ulcorner \underline{0 = 1} \urcorner \to \underline{0} = \underline{1}$. This, of course, is part of Gödels Second Incompleteness Theorem.

After some soulsearching our formalist decides to follow the tortoise and repudiate Modus Ponens, for - he argues - what can one expect, to have a formal proof that $A^* \to B^*$, is no evidence whatsoever that if A^* is true, then B^* is true.

Only lately he is having some trouble expressing the thought .... .

0.1 Motivation

A more serious reason than speculation about strange formalists to be interested in logics like FPL is the study of notions like Prov, formal provability as coded in PA, which naturally occur in the metamathematics of formal theories.

0.2 Contents

In the following I will provide a formalization of FPL and prove the associated completeness theorem .

The notion of interpretation described in 0.0 is not the only one which yields FPL and possibly not even the most interesting one. FPL turns out to be also the logic of Σ_1^0 -sentences of PA by interpreting the atoms as Σ_1^0 -sentences, $\perp$ as $0 = 1$ and treating conjunction, disjunction and implication as before.

§ 1 contains the Kripke Model theoretic preliminaries for the development of FPL.

§ 2 gives FPL plus associated Kripke Model completeness theorem.

§ 3 has the basic facts about FPL.

§ 4 studies an extension of FPL which is complete for a certain infinite matrix.

§ 5 finally gives three interpretations into PA plus associated completeness theorems.

0.3 Some partly philosophical remarks

0.3.1 On a certain equation

Let IPL be Intuitionistic Logic. We have :

$$\frac{\text{IPL}}{S_4} = \frac{\text{FPL}}{\text{PrL}} \; .$$

The reason is that * is a Gödel translation for IPL in S_4. We have :

$$\vdash_{IPL} A \quad \text{iff} \quad \vdash_{S_4} A^*$$

as well as

$$\vdash_{FPL} A \quad \text{iff} \quad \vdash_{PrL} A^*.$$

The analogy gains substance when one considers that the axioms of S_4 are valid for true,real or rigid provability. The crucial rule $\frac{\Box A}{A}$, which is blocked in PrL, is justified as follows : suppose we have a proof p of A, then A, *or else p would not be a true proof of A.* This argument suggests that what a proof is, cannot be fixed completely by a set of 'formal' properties; there must be at least some semantical properties.

0.3.2 Liar Paradox and Provability Paradox

With the usual derivation of the Liar Paradox in IPL from the postulated rules $\frac{L}{\neg L}$ (1) and $\frac{\neg L}{L}$ (2) corresponds via * the S_4 derivation of the (true) Provability Paradox from the postulated rules $\frac{G}{\Box\neg G}$ (3) and $\frac{\Box\neg G}{G}$ (4). In FPL there is an explicit sentence L satisfying (1) and (2), likewise in PrL there is an explicit sentence G satisfying (3) and (4). In both cases paradox is blocked by failure of Modus Ponens resp. $\frac{\Box A}{A}$.

0.4 Prerequisites

§ 1-4 are quite selfcontained. The reader only needs a basic understanding of Kripke Models for IPL, see e.g. [4] or [10]. §5 requires an understanding of the main results of PrL see e.g. [2], [9], [7], [8].

0.5 Acknowledgements

The use of the work of the pioneers of Provability Logic should be evident.

For the development of the Kripke Model theory of § 1, [6] has been my been my guide. [4] is related to [3].
The present work seems a companion to [2], [5] where the relation between PrL and S_4 is studied.

1 BASIC PROPOSITIONAL LOGIC

1.1 Language

Let $\mathcal{P} := \{p_0, p_1, \dots\}$ be the set of propositional variables. $\mathcal{L}$ is the smallest set s.t. :

$\mathcal{P} \subseteq \mathcal{L}$; $\bot \in \mathcal{L}$; $A, B \in \mathcal{L} \Rightarrow (A \wedge B), (A \vee B), (A \rightarrow B) \in \mathcal{L}$.

1.2 The Theory

The theory Basic Propositional Logic BPL is given by the following groups of ruleschemes :

group I: $\wedge$I $\quad \dfrac{A \quad B}{(A \wedge B)}$ $\quad \wedge$E $\quad \dfrac{(A \wedge B)}{A} \quad \dfrac{(A \wedge B)}{B}$

$\vee$I $\quad \dfrac{A}{(A \vee B)} \quad \dfrac{B}{(A \vee B)}$ $\qquad \bot$E $\quad \dfrac{\bot}{A}$

Tr
(Transitivity) $\quad \dfrac{(A \rightarrow B) \quad (B \rightarrow C)}{(A \rightarrow C)}$

$\wedge$I f
(formalized $\wedge$I) $\quad \dfrac{(A \rightarrow B) \quad (A \rightarrow C)}{(A \rightarrow (B \wedge C))}$

$\vee$E f
(formalized $\vee$E) $\quad \dfrac{(A \rightarrow C) \quad (B \rightarrow C)}{((A \vee B) \rightarrow C)}$

group II: $\rightarrow$I $\quad \dfrac{\begin{array}{c}\cancel{A}\\ \vdots \\ B\end{array}}{(A \rightarrow B)}$ $\qquad \vee$E $\quad \dfrac{(A \vee B) \quad \begin{array}{c}\cancel{A}\\ \vdots \\ C\end{array} \quad \begin{array}{c}\cancel{B}\\ \vdots \\ C\end{array}}{C}$

A ruleschema is considered here as a set of rules. A rule always contains individual elements of $\mathcal{L}$, e.g. $\dfrac{p_0 \quad p_1}{(p_0 \wedge p_1)}$ is a rule and $\dfrac{p_0 \quad p_1}{(p_0 \wedge p_1)} \in \dfrac{A \quad B}{(A \wedge B)}$.

We may add to our system a set of additional rules $\mathcal{R}$ of the form $\dfrac{A_1 \ldots A_k}{B}$. Such rules will be called normal rules.

When $\Gamma \subseteq \mathcal{L}$, $\Gamma \vdash_{\mathcal{R}} A$ means : A is derivable from Γ, with the rules of BPL + $\mathcal{R}$.

We say :

$$\begin{array}{lll}
\vdash_{\mathcal{R}} A & \text{for :} & \emptyset \vdash_{\mathcal{R}} A \\
\Gamma \vdash A & \text{for :} & \Gamma \vdash_{\emptyset} A \\
\vdash A & \text{for :} & \emptyset \vdash_{\emptyset} A \\
A \dashv\vdash_{\Gamma,\mathcal{R}} B & \text{for :} & \Gamma, A \vdash_{\mathcal{R}} B \text{ and } \Gamma, B \vdash_{\mathcal{R}} A \\
A \dashv\vdash_{\mathcal{R}} B & \text{for :} & A \dashv\vdash_{\emptyset,\mathcal{R}} B \\
A \dashv\vdash_{\Gamma} B & \text{for :} & A \dashv\vdash_{\Gamma,\emptyset} B \\
A \dashv\vdash B & \text{for :} & A \dashv\vdash_{\emptyset,\emptyset} B
\end{array}$$

1.3 A few basic facts about BPL

1.3.1 Derived and underived schemes

The following are easily seen to be derived schemes :

$$\wedge E f \quad : \quad \frac{(A \to (B \wedge C))}{(A \to B)} \qquad \frac{(A \to (B \wedge C))}{(A \to C)}$$

$$\vee I f \quad : \quad \frac{(A \to B)}{(A \to (B \vee C))} \qquad \frac{(A \to C)}{(A \to (B \vee C))}$$

$$\to I f \quad : \quad \frac{((A \wedge B) \to C)}{(A \to (B \to C))}$$

The following are not derived ruleschemes as can be seen from the completeness theorem (1.10):

$\to E$ (or Modus Ponens) $\qquad \dfrac{A \quad A \to B}{B}$

$\to E f \qquad \dfrac{(A \to (B \to C))}{((A \wedge B) \to C)}$

1.3.2 Substitution of equivalents

Let a propositional context C [ ] be defined as usual. The following can be proved by easy inductions :

i) S E (Substitution of Equivalents) : $A \dashv\vdash_{\Gamma,R} B \Rightarrow C[A] \dashv\vdash_{\Gamma,R} C[B]$

ii) S E f (formalized S E) : $A \leftrightarrow B \vdash C[A] \leftrightarrow C[B]$

SE justifies us to be careless with brackets, so we will be.

1.4 Models

A (Kripke) Model K is a structure $< W, \lhd, f >$ where W is a set of "worlds"; $\lhd$ a binary transitive relation on W; f a function from W in the subsets of P s.t. $w \lhd w' \Rightarrow f(w) \subseteq f(w')$.

1.4.1 Satisfaction

Let $K = < W, \lhd, f >$ be a Model. $\Vdash_K \subseteq W \times L$ is the smallest relation s.t. :

- $p_i \in f(w) \Rightarrow w \Vdash_K p_i$
- $w \Vdash_K A$ and $w \Vdash_K B \Rightarrow w \Vdash_K A \wedge B$
- $w \Vdash_K A$ or $w \Vdash_K B \Rightarrow w \Vdash_K A \vee B$
- $(\forall w' \rhd w \;\; w' \Vdash_K A \Rightarrow w' \Vdash_K B) \Rightarrow w \Vdash_K (A \rightarrow B)$.

Define further for $\Gamma \subseteq L$:

$\Gamma \Vdash_K A : \Leftrightarrow \forall w \in W \; (w \Vdash_K \Gamma \Rightarrow w \Vdash_K A)$. We write $K \Vdash A$ or $\Vdash_K A$ for $\emptyset \Vdash_K A$.

1.4.2 Fact

For all $A \in L$ and $w, w' \in W$: $w \Vdash_K A$ and $w' \rhd w \Rightarrow w' \Vdash_K A$.

1.4.3 Closure under rules

We say that K is closed under a rule of the form $\frac{A_1 \ldots A_k}{B}$ if $\{A_1, \ldots, A_k\} \Vdash_K B$.

Let $\mathcal{R}$ be a set of normal rules. We say that K is closed under $\mathcal{R}$ if K is closed under every element of $\mathcal{R}$.

Define : $\Gamma \Vdash_{\mathcal{R}} A \; :\Leftrightarrow$ for every $\mathcal{R}$ - closed K $\Gamma \Vdash_K A$.

We have :

1.4.4 Soundness Theorem

$\Gamma \vdash_{\mathcal{R}} A \Rightarrow \Gamma \Vdash_{\mathcal{R}} A$.

Proof : Routine. $\square$

As a preliminary for the completeness theorem we will give a connection between 'formalized' and 'unformalized' .

1.5 Definition

Let $\Gamma \subseteq L$. Define : $\mathcal{R}_\Gamma := \{\frac{A_1 \ldots A_k}{B} \mid \Gamma \vdash_{\mathcal{R}} (A_1 \wedge \ldots A_k) \rightarrow B\}$.

1.5.1 Fact

$\mathcal{R} \subseteq \mathcal{R}_\Gamma$.

Proof : If $\frac{A_1 \ldots A_k}{B} \in \mathcal{R}$ then $\Gamma \vdash_{\mathcal{R}} (A_1 \wedge \ldots A_k) \rightarrow B$. $\square$

1.6 Theorem

For all $\Gamma \subseteq L$; $k \in \mathbb{N}$; $A_1 \ldots A_k, B \in L$:

$\Gamma \vdash_{\mathcal{R}} (A_1 \wedge \ldots A_k) \rightarrow B \Leftrightarrow \Gamma, A_1 \ldots A_k \vdash_{\mathcal{R}_\Gamma} B$.

Proof : "$\Rightarrow$" By definition.

"$\Leftarrow$" By induction on the length of the proof.

Case i) The length of the proof is 0, i.o.w. $B \in \Gamma, A_1 \ldots A_k$.

Trivial.

Case ii) The last rule applied is of group I or of $\mathcal{R}_\Gamma$. Say it is $R = \dfrac{B_1 \ldots B_s}{C}$.

We have :

$$\Gamma, A_1 \ldots A_k \vdash_{\mathcal{R}_\Gamma} B_1, \ldots \Gamma, A_1 \ldots A_k \vdash_{\mathcal{R}_\Gamma} B_s.$$

By Induction Hypothesis and $\wedge$I f :

$$\Gamma \vdash_{\mathcal{R}} (A_1 \wedge \ldots A_k) \to (B_1 \wedge \ldots B_s).$$

When $R \in \mathcal{R}_\Gamma$ we have by definition :

$$\Gamma \vdash_{\mathcal{R}} (B_1 \wedge \ldots B_s) \to C.$$

When R is of group I this follows by $\wedge$E and $\to$I.

By Tr we find :

$$\Gamma \vdash_{\mathcal{R}} (A_1 \wedge \ldots A_k) \to C.$$

Case iii) The last rule applied is $\to$I. We have:

$$\Gamma, A_1 \ldots A_k \vdash_{\mathcal{R}_\Gamma} B \to C \quad \text{from} \quad \Gamma, A_1 \ldots A_k, B \vdash_{\mathcal{R}_\Gamma} C.$$

By Induction Hypothesis :

$$\Gamma \vdash_{\mathcal{R}} (A_1 \wedge \ldots A_k \wedge B) \to C.$$

Using the derived rule $\to$I f we find :

$$\Gamma \vdash_{\mathcal{R}} (A_1 \wedge \ldots A_k) \to (B \to C).$$

Case iv) The last rule applied is vE. We have $\Gamma, A_1 \ldots A_k \vdash_{\mathcal{R}_\Gamma} D$ from

$$\Gamma, A_1 \ldots A_k \vdash_{\mathcal{R}_\Gamma} B \vee C; \quad \Gamma, A_1 \ldots A_k, B \vdash_{\mathcal{R}_\Gamma} D; \quad \Gamma, A_1, \ldots, A_k, C \vdash_{\mathcal{R}_\Gamma} D.$$

By Induction Hypothesis :

$$\Gamma \vdash_{\mathcal{R}} (A_1 \wedge \ldots A_k) \to (B \vee C) \quad \Gamma \vdash_{\mathcal{R}} (A_1 \wedge \ldots A_k \wedge B) \to D \;;$$

$$\Gamma \vdash_{\mathcal{R}} (A_1 \wedge \ldots A_k \wedge C) \to D.$$

By vEf we find :

$$\Gamma \vdash_{\mathcal{R}} ((A_1 \wedge \ldots A_k \wedge B) \vee (A_1 \wedge \ldots A_k \wedge C)) \to D.$$

One easily shows :

$$\Gamma \vdash_{R} ((A_1 \wedge ..A_k) \wedge (B \vee C)) \rightarrow ((A_1 \wedge ..A_k \wedge B) \vee (A_1 \wedge ..A_k \wedge C)).$$

So by Tr :

$$\Gamma \vdash_{R} ((A_1 \wedge ..A_k) \wedge (B \vee C)) \rightarrow D .$$

Using :

$$\frac{(A_1 \wedge ..A_k) \rightarrow (A_1 \wedge ..A_k) \quad (A_1 \wedge ..A_k) \rightarrow B \vee C}{(A_1 \wedge ..A_k) \rightarrow ((A_1 \wedge ..A_k) \wedge (B \vee C))} \wedge I\ f$$

we find using Tr again :

$$\Gamma \vdash_{R} (A_1 \wedge ..A_k) \rightarrow D . \quad \square$$

1.7 Definition

$\Gamma \subseteq L$ is called R-prime if whenever $\Gamma \vdash_R A$ we have $A \in \Gamma$ and whenever $\Gamma \vdash_R A \vee B$ we have $\Gamma \vdash_R A$ or $\Gamma \vdash_R B$. Moreover Γ must be consistent.

1.8 Fact

If $\Gamma \nvdash_R A$ then there is an R-prime $\Gamma' \supseteq \Gamma$ s.t. $\Gamma' \nvdash_R A$.

Proof : Routine. $\square$

1.9 Theorem

If $\Gamma \nvdash_R A$ then there is an R-closed model $K = \langle W, \lhd, f \rangle$ s.t. $K \Vdash \Gamma$ and $K \nVdash A$.

Proof : Take $W := \{ [\Delta] | \Delta \supseteq \Gamma, \Delta$ is R-prime$\}$, where $[\Delta]$ stands for $\langle \emptyset, \Delta \rangle$. We use $[\Delta]$ rather than Δ itself to avoid confusion between $\Delta \Vdash_K A$ as a world and $\Delta \Vdash_K A$ as a theory.

$$[\Delta] \lhd [\Delta'] :\leftrightarrow \Delta' \text{ is } R_\Delta \text{ prime.}$$

$$f([\Delta]) := P \cap \Delta .$$

Note that :

$$[\Delta] \lhd [\Delta'] \Rightarrow \Delta \subseteq \Delta' \Rightarrow f([\Delta]) \subseteq f([\Delta']).$$

It is easy to verify that $\lhd$ is transitive so K is a genuine model.

Claim : $\{B \mid [\Delta] \Vdash_K B\} = \Delta$.

The proof is by induction on the length of B. The only problematic case is $\to$:

Suppose $(D \to E) \in \Delta$. Then $\frac{D}{E} \in \mathcal{R}_\Delta$.

So for every $\mathcal{R}_\Delta$-prime Δ' : $D \in \Delta' \Rightarrow E \in \Delta'$. By Induction Hypothesis: for every $[\Delta'] \trianglerighteq [\Delta]$ ($[\Delta'] \Vdash_K D \Rightarrow [\Delta'] \Vdash_K E$), i.e. $[\Delta] \Vdash_K (D \to E)$.

Now suppose $(D \to E) \notin \Delta$, then by theorem 1.6 $\Delta, D \nvdash_{\mathcal{R}_\Delta} E$. Extend $\Delta \cup \{D\}$ to an $\mathcal{R}_\Delta$-prime Δ' s.t. $\Delta' \nvdash_{\mathcal{R}_\Delta} E$. We have $[\Delta'] \trianglerighteq [\Delta]$.

By Induction Hypothesis :

$$[\Delta'] \Vdash_K D \text{ and } [\Delta'] \nVdash_K E. \text{ So } [\Delta] \nVdash_K (D \to E).$$

It follows easily that :

- K is $\mathcal{R}$-closed
- $K \Vdash \Gamma$, because each $\Delta \supseteq \Gamma$
- $K \nVdash A$, because there is an $\mathcal{R}$-prime Δ s.t. $\Delta \supseteq \Gamma$ and $A \notin \Delta$. □

1.10 Completeness Theorem

$$\Gamma \vdash_{\mathcal{R}} A \Leftrightarrow \Gamma \Vdash_{\mathcal{R}} A .$$

Proof : Combine 1.4.4 and 1.9. □

1.11 Remark

By inspection of the proof, we can see that we have the completeness theorem also for the class of models $K = \langle W, \triangleleft, f \rangle$ s.t. $w \triangleleft w' \triangleleft w \Rightarrow$ $\Rightarrow w = w'$.

1.12 <u>Corollary: Completeness Theorem for Intuitionistic Propostional Logic.</u>

Let Intuitionistic Propositional Logic (IPL) be BPL + $\to$E. Then:
$\Gamma \vdash_{IPL} A \Leftrightarrow$ for all reflexive K : $\Gamma \Vdash_K A \Leftrightarrow$ for all K where $\lhd$ is a weak partial order $\Gamma \Vdash_K A$.

Proof : "$\Rightarrow$" is clear.

"$\Leftarrow$" It is sufficient to show that the model constructed in 1.9 is reflexive when $\mathcal{R} = \to$E. We have to show that any $\to$E-prime $\Delta \supseteq \Gamma$ is $\mathcal{R}_\Delta$-prime. But that is immediate using $\to$E. $\square$

1.13 <u>Lemma</u>

Suppose $\Gamma \nvdash A$, Γ finite. Then there is a finite $K_0 = \langle W_0, \lhd_0, f_0 \rangle$, where $w \lhd_0 w' \lhd_0 w \Rightarrow w = w'$ s.t. $K_0 \Vdash \Gamma$ and $K_0 \nVdash A$.

Proof :Consider the model $K = \langle W, \lhd, f \rangle$ constructed in 1.9 s.t. $K \Vdash \Gamma$ and $K \nVdash A$ (for $\mathcal{R} = \emptyset$). Let Λ be the set of subformulas of the elements of $\Gamma \cup \{A\}$. (Each formula is a subformula of itself).

Define :

- $\tilde{w} := \{B \in \Lambda \mid w \Vdash_K B\}$
- $W_0 := \{\tilde{w} \mid w \in W\}$
- for $a, b \in W_0$: $a \lhd_0 b :\Leftrightarrow a \subseteq b$ and if $(E \to F) \in a$, $E \in b$ then $F \in b$.
- $f_0(a) = a \cap \mathcal{P}$.

It is easy to see that K_0 is a model, that K_0 is finite and that $a \lhd_0 a' \lhd_0 a \Rightarrow a = a'$.

Claim : for every $B \in \Lambda$: $w \Vdash_K B \Leftrightarrow \tilde{w} \Vdash_{K_0} B$. The proof is by induction on the length of B. The only problematic case is $\rightarrow$.
Let $B \equiv (E \rightarrow F) \in \Lambda$.

"$\Rightarrow$" Suppose $w \Vdash_K (E \rightarrow F)$ and $a \rhd_0 \tilde{w}$ and $a \Vdash_{K_0} E$. There is a u such that $a = \tilde{u}$. By Induction Hypothesis: $u \Vdash_K E$, hence $E \in \tilde{u}$ i.e. $E \in a$. Moreover $(E \rightarrow F) \in \tilde{w}$, $a \rhd_0 \tilde{w}$, so $F \in a$. Hence $u \Vdash_K F$. Again by Induction Hypothesis: $\tilde{u} \Vdash_{K_0} F$ i.e. $a \Vdash_{K_0} F$.

"$\Leftarrow$" Suppose $\tilde{w} \Vdash_{K_0} (E \rightarrow F)$, $w' \rhd w$, $w' \Vdash_K E$. Clearly $\tilde{w}' \rhd_0 \tilde{w}$ and by Induction Hypothesis $\tilde{w}' \Vdash_{K_0} E$, hence $\tilde{w}' \Vdash_{K_0} F$. Again by Induction Hypothesis: $w' \Vdash_K F$. $\square$

1.14 <u>Corollary</u>

If Γ is finite we have

i) $\Gamma \vdash A \Leftrightarrow$ for all finite models K : $\Gamma \Vdash_K A$

ii) $\Gamma \vdash_{IPL} A \Leftrightarrow$ for all finite K, where $\lhd$ is a weak partial order $\Gamma \Vdash_K A$.

Proof : From 1.13 Observe that the construction in 1.13 preserves reflexivity. $\square$

2 FORMAL PROPOSITIONAL LOGIC

2.1 <u>The system FPL</u>

Let Löb's Rule L be : $\dfrac{(\top \rightarrow A) \rightarrow A}{\top \rightarrow A}$, where $\top :\equiv (\bot \rightarrow \bot)$.

FPL is BPL + L.

2.2 Completeness Theorem for FPL

Let $\Gamma \subseteq L$, Γ finite, then :

$\Gamma \vdash_{FPL} A \Leftrightarrow$ for every finite irreflexive K $\Gamma \Vdash_K A$.

Proof :

"$\Rightarrow$" We will prove more generally that every K s.t. $\lhd$ is reverse wellfounded (i.e. $\rhd$ is wellfounded) is closed under L.

Consider $w \in W$. Suppose that every $u \rhd w$ is closed under L and suppose $w \Vdash_K (\top \to A) \to A$.

For $u \rhd w$ we have $u \Vdash_K (\top \to A) \to A$ by monotonicity. u is closed under L so $u \Vdash_K \top \to A$. Moreover if $u \Vdash_K \top \to A$, then $u \Vdash_K A$ because $w \Vdash_K (\top \to A) \to A$. So $u \Vdash_K A$ for any $u \rhd w$. So $w \Vdash_K \top \to A$. By transfinite induction we find that K is closed under L.

"$\Leftarrow$" Suppose $\Gamma \nvdash_{FPL} A$. Let K be the model constructed in *1.9* s.t. K is L-closed and $K \Vdash \Gamma$ and $K \nVdash A$. Let K_0 be the model constructed in 1.13. We have $K_0 \Vdash \Gamma$ and $K_0 \nVdash A$. Note that we do not know whether K_0 is closed under L. Define $K_1 = \langle W_0, \lhd_1, f_0 \rangle$, where $a \lhd_1 b :\Leftrightarrow (a \lhd_0 b$ and $a \neq b)$. By the "$\Rightarrow$" part of our proof we see that K_1 is automatically closed under L. Let Λ be as in 1.13. We prove :

For all $B \in \Lambda$, for all $w \in W$: $w \Vdash_K B \Leftrightarrow \tilde{w} \Vdash_{K_0} B \Leftrightarrow \tilde{w} \Vdash_{K_1} B$.

We already have the first equivalence. We prove the second one with induction on the length of B. The only non trivial case is again $\to$.

Suppose $B = (E \to F) \in \Lambda$.

I) If $\tilde{w} \Vdash_{K_0} E \to F$, then for any $\tilde{u} \rhd_0 \tilde{w}$ $(\tilde{u} \Vdash_{K_0} E \Rightarrow \tilde{u} \Vdash_{K_0} F)$.

By Induction Hypothesis : for every $\tilde{u} \rhd_0 \tilde{w}$ $(\tilde{u} \Vdash_{K_1} E \Rightarrow \tilde{u} \Vdash_{K_1} F)$.

So certainly : for any $\tilde{u} \rhd_1 \tilde{w}$ $(\tilde{u} \Vdash_{K_1} E \Rightarrow \tilde{u} \Vdash_{K_1} F)$. Thus :

$\tilde{w} \Vdash_{K_1} (E \to F)$.

II) If $\tilde{w} \not\Vdash_{K_0} E \to F$, then there is an $\tilde{u} \rhd_0 \tilde{w}$ with $(\tilde{u} \Vdash_{K_0} E$ and $\tilde{u} \not\Vdash_{K_0} F)$.

Case i) There is such as $\tilde{u}$ s.t. $\tilde{u} \neq \tilde{w}$. Then $\tilde{u} \rhd_1 \tilde{w}$ and by Induction Hypothesis $\tilde{u} \Vdash_K E$ and $\tilde{u} \not\Vdash_{K_1} F$. So $\tilde{w} \not\Vdash_{K_1} E \to F$.

Case ii) The only such $\tilde{u}$ is $\tilde{w}$ itself. We have :

a) For every $\tilde{u} \rhd_1 \tilde{w}$: $\tilde{u} \Vdash_{K_0} E \Rightarrow \tilde{u} \Vdash_{K_0} F$

b) $\tilde{w} \Vdash_{K_0} E$ and $\tilde{w} \not\Vdash_{K_0} F$ and $\tilde{w} \rhd_0 \tilde{w}$

Consider a $u \rhd w$ with $u \Vdash_K E \to F$. It follows that :

c) $\tilde{u} \rhd_0 \tilde{w}$

d) $\tilde{u} \Vdash_{K_0} (E \to F)$ (because $(E \to F) \in \Lambda$), $\tilde{w} \not\Vdash_{K_0} E \to F$

e) $\tilde{u} \rhd_1 \tilde{w}$ (by (c), (d))

f) $\tilde{u} \Vdash_{K_0} E$ (by (b), (c))

g) $\tilde{u} \Vdash_{K_0} F$ ((a), (e), (f))

h) $u \Vdash_K F$ $(F \in \Lambda)$

$u \rhd w$ with $u \Vdash_K (E \to F)$ was arbitrary , so :

i) $w \Vdash_K (E \to F) \to F$.

By :

$$\dfrac{\dfrac{T}{E \to T} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{T}{E \to T} \qquad T \not\to F}{E \to F}}{(T \to F) \to (E \to F)} \qquad (E \to F) \to F}{(T \to F) \to F}\ L}{T \to F}}{E \to F}$$

we find :

$j)$ $\quad w \Vdash_K E \to F \qquad ((i))$

Thus

$k)$ $\quad \tilde{w} \Vdash_{K_0} E \to F \qquad ((E \to F) \in \Lambda,\ (j))$

Contradiction . $\square$

3 BASIC FACTS ABOUT FPL

3.1 An alternative version of Löb's Rule

There is an alternative version of Löb's Rule L' which is interderivable with L over BPL. Namely :

$$\begin{array}{c} T \not\to A \\ \vdots \\ \dfrac{A}{A}\ L' \end{array}$$

Proof :

$$\dfrac{\dfrac{\begin{array}{c}\cancel{\top \to A}\\ \vdots\ \pi \\ A\end{array}}{(\top \to A) \to A}\to I}{\begin{array}{c}\top \to A\\ \vdots\ \pi \\ A\end{array}}\,L \qquad\qquad \dfrac{\dfrac{\top \to (\cancel{\top} \to A) \quad (\top \to A) \to A}{\top \to A}\,\mathrm{Tr}}{\top \to A}\,L'$$

3.2 Definition

$FC(p_i)$, the set of *Formal Contexts* of p_i is smallest set s.t.

i) $i \neq j \Rightarrow p_j \in FC(p_i)$

ii) $\bot \in FC(p_i)$

iii) $A, B \in L \Rightarrow (A \to B) \in FC(p_i)$

iv) $A, B \in FC(p_i) \Rightarrow (A \wedge B), (A \vee B) \in FC(p_i)$

3.3 Example

$p_0 \wedge (p_1 \to p_0)$ is a formal context of p_1 and not of p_0.

3.4 Theorem: More Substitution of Equivalents

Let $C[p_0]$ be a formal context of p_0.

i) SE^+ : FPL is closed under : $\dfrac{A \leftrightarrow B \quad C[A]}{C[B]}$

ii) SE^+f : FPL is closed under $\dfrac{\top \to (A \leftrightarrow B)}{C[A] \leftrightarrow C[B]}$

Proof : Induction on proof length or use Completeness Theorem. □

3.5 Unicity of Fixed Points in Formal Contexts

Suppose $C[p_0]$ is a formal context of p_0. We have :

i) $(A \dashv\vdash_{\Gamma,FPL} C[A]$ and $B \dashv\vdash_{\Gamma,FPL} C[B]) \Rightarrow A \dashv\vdash_{\Gamma,FPL} B$.

ii) (Formal Version of *i)*) : FPL is closed under :

$$\frac{A \leftrightarrow C[A] \qquad B \leftrightarrow C[B]}{A \leftrightarrow B}$$

Proof :

i)

$$\dfrac{\dfrac{\dfrac{\dfrac{A}{C[A]}\ (\text{ex hypothesi}) \qquad \dfrac{\dfrac{A}{B \to A} \qquad \dfrac{\dfrac{\top}{A \to \top} \qquad \top \cancel{\to} B}{A \to B}}{A \leftrightarrow B}\ SE^{+}}{C[B]}\ (\text{ex hypothesi})}{B}\ L'}{B}$$

ii)

$$\dfrac{\dfrac{\dfrac{A \leftrightarrow C[A] \qquad \dfrac{\top \to (\cancel{A} \leftrightarrow B)}{C[A] \leftrightarrow C[B]}\ SE^{+}f}{A \leftrightarrow C[B]}\ Tr \qquad B \leftrightarrow C[B]}{A \leftrightarrow B}}{A \leftrightarrow B}\ L'. \qquad \square$$

3.6 Fixed Point Theorem

For any $C[p_0]$ we have :

$$C[\top] \dashv\vdash_{FPL} C[\ C[\top]\ ].$$

Before proving 3.6, first we give two lemma's.

3.6.1 Lemma

Let $D[p_0]$ be a formal context of p_0 then :

$$D[\top] \dashv\vdash_{FPL} D[\ D[\top]\ ].$$

Proof : "$\vdash$"

Trivially we have : $D[\top] \dashv\vdash_{D[\top],FPL} \top$,

so by SE : $D[\,D[\top]\,] \dashv\vdash_{D[\top],FPL} D[\top]$.

Thus : $D[\top] \vdash_{FPL} D[\,D[\top]\,]$.

(Note that we did not use that $D[p_0]$ is formal, nor did we use L.)

"$\dashv$"

$$\dfrac{\dfrac{D[\,D[\top]\,] \qquad \dfrac{\top \to \cancel{D}[\top] \qquad \dfrac{\top}{D[\top] \to \top}}{\top \leftrightarrow D[\top]}\;SE^{+}}{D[\top]}\;L'}{D[\top]}$$

□

3.6.2 Definition

$SIC(p_i)$, the set of strictly informal contents of p_i, is the smallest set s.t.

i) $p_i \in SIC(p_i)$

ii) p_i does not occur in $A \Rightarrow A \in SIC(p_i)$

iii) $D,E \in SIC(p_i) \Rightarrow D \wedge E, D \vee E \in SIC(p_i)$

3.6.3 Example

$(p_0 \wedge p_1) \vee (p_1 \to p_2)$ is in $SIC(p_0)$ but not in $SIC(p_1)$.

3.6.4 Lemma

For every $D \in L$, $p_i \in P$ there is a $p_k, p_\ell \in P$, $E[p_k] \langle p_\ell \rangle \in L$ s.t. $D \equiv E[p_i] \langle p_i \rangle$ and $E[p_k] \langle p_\ell \rangle \in FC(p_k)$ and $E[p_k] \langle p_\ell \rangle \in SIC(p_\ell)$.

Proof : routine. □

3.6.5 Example

Let $D \equiv p_0 \vee (p_1 \to p_0)$. For p_0 choose p_2, p_3 and $E[p_2] \langle p_3 \rangle \equiv$
$\equiv p_3 \vee (p_1 \to p_2)$.

3.6.6 Lemma

If $D[p_0] \in SIC(p_0)$, then FPL is closed under $\dfrac{D[A]}{D[\top]}$.

Proof : Induction on $D[p_0]$. (The idea is that the relevant occurences of A are only in positive places.) $\square$

Proof of 3.6

"$\vdash$" By the same reasoning as the "$\vdash$" part of 3.6.1.

"$\dashv$" Write $C[p_0]$ as $D[p_0] < p_0 >$, where $D[p_0] < p_0 >$ is as in 3.6.4.
Apply 3.6.1 to $D[p_0] < \top >$. We find :
$C[\top] \equiv D[\top] < \top > \dashv\vdash_{FPL} D[C\ [\top]\ ] < \top >$.
By 3.6.6:
$C[C\ [\top]\ ] \equiv D[C\ [\top]\ ] < C\ [\top] > \vdash_{FPL} D[C\ [\top]\ ] < \top >$.
So we have: $C[C\ [\top]\ ] \vdash_{FPL} C[\top]$. $\square$

3.7 Remark

Of course we do not have *unicity* of fixed points in general as is seen in the case $C[p_0] \equiv p_0$.
We do get :
$A \dashv\vdash_{\Gamma,FPL} C[A] \Rightarrow A \vdash_{\Gamma,FPL} C[\top]$. Since $A \dashv\vdash_{\Gamma,A,FPL} \top$, we get $C[A] \dashv\vdash_{\Gamma,A,FPL} C[\top]$. So $A, C[A] \vdash_{\Gamma,FPL} C[\top]$. Thus $A \vdash_{\Gamma,FPL} C[\top]$.
So $C[\top]$ is the maximal fixed point w.r.t. $\vdash_{\Gamma,FPL}$.

3.8 Example: "The Liar"

$\top \to \bot$ is the unique A (modulo $\dashv\vdash_{FPL}$) s.t. $A \dashv\vdash_{FPL} \neg A$ ($\equiv A \to \bot$).

4 A "CLASSICAL" VERSION OF FPL

4.1 Definitions

Let $\Gamma, \Delta \subseteq L$.

i) $Sub_\Delta = \{s \mid s : P \to \Delta\}$

ii) - $p_i{}^s = s(p_i)$

- $(A \wedge B)^s = (A^s \wedge B^s)$

- $(A \vee B)^s = (A^s \vee B^s)$

- $(A \to B)^s = A^s \to B^s$

iii) $\Gamma^s = \{B^s \mid B \in \Gamma\}$

iv) $\Gamma \Vdash_{\Delta, R} A :\Leftrightarrow$ for every $s \in Sub_\Delta$ $(\vdash_R \Gamma^s \Rightarrow \vdash_R A^s)$.

4.2 Some Facts

i) Consider $s \in Sub_{\{\perp\}}$. We have for any $A \in L$:

$A^s \dashv\vdash_{IPL} \top$ or $A^s \dashv\vdash_{IPL} \perp$.

ii) Let CPL be Classical Propositional Logic. We have :

$\Gamma \vdash_{CPL} A \Leftrightarrow \Gamma \Vdash_{\{\top, \perp\}, IPL} A$.

iii) $\Gamma \vdash_{CPL} A \Leftrightarrow \Gamma \Vdash_{L, CPL} A$.

Proofs : routine. □

Below we will do something analogous to the "classification" of IPL for FPL.

4.3 Definitions

i) - $\perp_\omega :\equiv \top \; (\equiv \perp \to \perp)$

- $\perp_0 :\equiv \perp$

- $\perp_{n+1} :\equiv \top \to \perp_n$

4.4 Facts

i) $a,b \in \omega+1$ and $a < b \Rightarrow \bot_a \vdash_{FPL} \bot_b$ and $\bot_b \nvdash_{FPL} \bot_a$.

ii) Let $a,b \in \omega+1$, then :

- $\bot_a \wedge \bot_b \dashv\vdash_{FPL} \bot_{\min(a,b)}$
- $\bot_a \vee \bot_b \dashv\vdash_{FPL} \bot_{\max(a,b)}$
- if $a \leqslant b$ $\bot_a \to \bot_b \dashv\vdash_{FPL} \top$

  if $a > b$ $\bot_a \to \bot_b \dashv\vdash_{FPL} \bot_{b+1}$

iii) If $s \in Sub_{\{\bot\}}$, then for any $A \in L$, there is an $a \in \omega+1$ s.t.:
$A^s \dashv\vdash_{FPL} \bot_a$.

Proofs : All the proofs are easy. Let me just prove :

$$a > b \Rightarrow \bot_a \to \bot_b \dashv\vdash_{FPL} \bot_{b+1} .$$

" $\dashv$ "

$$\dfrac{\dfrac{\top}{\bot_a \to \top} \qquad \begin{matrix}\bot_{b+1}\\ \equiv \\ \top \to \bot_b\end{matrix}}{\bot_a \to \bot_b}\ Tr$$

" $\vdash$ "

$$\dfrac{\dfrac{\begin{matrix}\dfrac{\bot_a \to \bot_b \qquad \bot_b \to \dfrac{\dfrac{\cancel{\bot_b}}{\bot_{a-1}}\text{ by (i)}}{}\ \bot_{a-1}}{\bot_a \to \bot_{a-1}} \\ \equiv \\ (\top \to \bot_{a-1}) \to \bot_{a-1}\end{matrix}}{\begin{matrix}\top \to \bot_{a-1} \\ \equiv \\ \dfrac{\bot_a}{\top \to \bot_a}\end{matrix}}\ L \qquad \bot_a \to \bot_b}{\begin{matrix}\top \to \bot_b \\ \equiv \\ \bot_{b+1}\end{matrix}}$$

□

4.5 Definitions

i) $\wedge : (\omega+1)^2 \to (\omega+1)$ with $\wedge(a,b) = \min(a,b)$

$\vee : (\omega+1)^2 \to (\omega+1)$ with $\vee(a,b) = \max(a,b)$

$\to : (\omega+1)^2 \to (\omega+1)$ with $\to(a,b) = \begin{cases} \omega \text{ if } a \leqslant b \\ b+1 \text{ if } a > b \end{cases}$

ii) An assignment f is a function $P \to \omega+1$.

iii) Let f be an assignment. Define :

- $[\![p_i]\!]_f = f(p_i)$
- $[\![\bot]\!]_f = 0$
- $[\![(A \wedge B)]\!]_f = \wedge([\![A]\!]_f, [\![B]\!]_f)$
- $[\![(A \vee B)]\!]_f = \vee([\![A]\!]_f, [\![B]\!]_f)$
- $[\![A \to B]\!]_f = \to([\![A]\!]_f, [\![B]\!]_f)$

iv) Let $\Gamma \cup \{A\} \subseteq L$. Define :

- $\Gamma \models_f^* A :\Leftrightarrow ((\text{for every } B \in \Gamma : [\![B]\!]_f = \omega) \Rightarrow [\![A]\!]_f = \omega)$
- $\Gamma \models^* A$: for every assignment g $\Gamma \models_g A$
- $\Gamma \models_f A :\Leftrightarrow \inf(\{[\![B]\!]_f \mid B \in \Gamma\}) \leqslant [\![A]\!]_f$

  (Note that $\inf(\emptyset) = \omega$)
- $\Gamma \models A :\Leftrightarrow$ for every assignment g: $\Gamma \models_g A$.

4.6 Facts

i) $\models_f^* A \Leftrightarrow \models_f A$

ii) $\models_f^* A \to B \Leftrightarrow \models_f A \to B \Leftrightarrow A \models_f B$

iii) $\Gamma \models_f A \Rightarrow \Gamma \models_f^* A$

iv) $\Gamma \models^* A \Leftrightarrow \Gamma \Vdash_{\{\bot_a \mid a \in \omega+1\},FPL} A$

v) $\Gamma \models^* A \Leftrightarrow (\text{for every } s \in Sub_L : \models^* \Gamma^s \Rightarrow \models^* A^s)$.

Proofs: *i)* - *iv)* are entirely routine. Let us do v).

"$\Rightarrow$" Suppose $\Gamma \models^* A$ and $\models^* \Gamma^s$. Consider an assignment g. We have: $\models^*_g \Gamma^s$. Define an assignment h as: $h(p_i) = [\![s(p_i)]\!]_g$. Then $\models^*_h \Gamma$, so $\models^*_h A$. Thus $\models^*_g A^s$.

"$\Leftarrow$" Suppose for every $s \in \mathrm{Sub}_L$ $\models^* \Gamma^s \Rightarrow \models^* A^s$. Let $\models^*_f \Gamma$. Define s with $s(p_i) = \bot_{f(p_i)}$. Then: $\models^* \Gamma^s$, so $\models^* A^s$, thus $\models^*_f A$. $\square$

4.7 Facts

i) $A, A \to B \models^* B$

ii) $p_0, p_0 \to p_1 \not\models p_1$

iii) $\bot_1 \models^* \bot$, but not $\models^* \bot_1 \to \bot$.

iv) $\Gamma, A \models B \Rightarrow \Gamma \models A \to B$

Proofs: routine. $\square$

Obviously $\models$ looks more like FPL then $\models^*$. Below we will axiomatize $\models$.

4.8 Theorem

Let $\Gamma \cup \{A\} \subseteq L$. We have :

$\Gamma \models A \Leftrightarrow$ for all finite, irreflexive, *linear* K: $\Gamma \Vdash_K A$.

Proof:

"$\Rightarrow$" Suppose $\Gamma \models A$. Let K be finite, irreflexive and linear, and $w \Vdash_K \Gamma$. Without loss of generality we may assume that w is the downmost node of K and that $W = \{1, \ldots N\}$ and that $m \lhd n \Leftrightarrow m > n$.

Define $\alpha_K(A) = \begin{cases} \max\{k \mid 1 \leq k \leq N \text{ and } k \Vdash_K A\} & \text{if there is such a } k \\ 0 & \text{else} \end{cases}$

Define an assignment f_K by: $f_K(p_i) = \alpha_K(p_i)$.

We claim : $\alpha_K(A) = \min([\![A]\!]_{f_K}, N)$.

The proof is by induction on the length of A. Suppose e.g. $A \equiv (B \to C)$. In case $[\![B]\!]_{f_K} \leqslant [\![C]\!]_{f_K}$ we have $\alpha_K(B) \leqslant \alpha_K(C)$ by Induction Hypothesis. So $N \Vdash B \to C$ or $\alpha_K(B \to C) = N$.

When $[\![B]\!]_{f_K} > [\![C]\!]_{f_K}$ we have to consider two possibilities :

- $[\![C]\!]_{f_K} \geqslant N$, then $[\![B \to C]\!]_{f_K} \geqslant N$. By Induction Hypothesis: $N \Vdash_K C$, so $N \Vdash B \to C$.

- $[\![C]\!]_{f_K} < N$. Then by Induction Hypothesis: $\alpha_K(C) < \alpha_K(B)$. So $\alpha_K(B \to C) = \alpha_K(C)+1 = [\![B \to C]\!]_{f_K} = \min([\![B \to C]\!]_{f_K}, N)$.

We assumed $N \Vdash_K \Gamma$. So for $B \in \Gamma$ $\alpha_K(B) = N$ and thus $[\![B]\!]_{f_K} \geqslant N$. Because $\Gamma \models A$, we have $[\![A]\!]_{f_K} \geqslant \min(\{[\![B]\!]_{f_K} | B \in \Gamma\}) \geqslant N$. So $N \Vdash_K A$.

"$\Leftarrow$" Suppose $\Gamma \not\models A$, then there is an f s.t. $\min(\{[\![B]\!]_f | B \in \Gamma\}) > [\![A]\!]_f$. Suppose $\min(\{[\![B]\!]_f | B \in \Gamma\}) = M$. We construct $K_f = \langle W, \lhd, g \rangle$. Let $W = \{1,\ldots, M\}$ and for $m,n \in W$: $m \lhd n :\Leftrightarrow n < m$. Take $p_i \in g(m) :\Leftrightarrow f(p_i) \geqslant m$.

By the usual induction on the length of A we prove for $1 \leqslant m \leqslant M : m \Vdash_{K_f} A \Leftrightarrow [\![A]\!]_f \geqslant m$. It follows that $M \Vdash_{K_f} \Gamma$, but $M \nVdash_{K_f} A$. □

4.9 Definitions

We shall consider an axiom as a rule with empty premiss.

i) BPLL is BPL + $((A \to B) \vee ((A \to B) \to A))$

ii) FPL^{CL} is FPL + $((A \to B) \vee ((A \to B) \to A))$

or

BPLL + L

4.10 Completeness for BPLL and FPL^{CL}

i) Let $\Gamma \cup \{A\} \subseteq L$:

$\Gamma \vdash_{BPLL} A \Leftrightarrow$ for all linear* Kripke models K : $\Gamma \Vdash_K A$.

ii) Let $\Gamma \subseteq L$ be *finite*, $A \in L$, then:

$\Gamma \vdash_{FPL^{CL}} A \Leftrightarrow$ for all finite, irreflexive, linear K $\Gamma \Vdash_K A$

$\Leftrightarrow \Gamma \models A$.

Proof:

i) "$\Rightarrow$" The validity of $(A \to B) \vee (A \to B) \to A$ is routine.

"$\Leftarrow$" Suppose $\Gamma \nvdash_{BPLL} A$. Consider the model $K = \langle W, \lhd, f \rangle$ constructed in 1.9 Let $\Delta \supseteq \Gamma$ be BPLL-prime with $A \notin \Delta$.
Let $K_1 := \langle W_1, \lhd_1, f_1 \rangle$ where $W_1 = \{[\Delta'] \in W \mid [\Delta'] \unrhd [\Delta]\}$ and $\lhd_1 = \lhd \upharpoonright W_1 \times W_1$ and $f_1 = f \upharpoonright W_1$.
Clearly $\Gamma \nVdash_{K_1} A$, so it is sufficient to show that K_1 is linear.
Consider $[\Delta_0], [\Delta_1] \in W_1$. The case that $\Delta_0 = \Delta$ or $\Delta_1 = \Delta$ is trivial, so assume $\Delta_0, \Delta_1 \neq \Delta$. We have :

$[\Delta] \lhd_1 [\Delta_0]$

$[\Delta] \lhd_1 [\Delta_1]$.

The case that $\Delta_0 = \Delta_1$ is trivial, so assume that for some C :

$[\Delta_0] \nVdash_{K_1} C$,

$[\Delta_1] \Vdash_{K_1} C$.

Suppose :

$[\Delta_0] \Vdash_{K_1} A \to B$ and $[\Delta_1] \Vdash_{K_1} A$.

We distinguish:

case i) $[\Delta_0] \nVdash_{K_1} A$, then $[\Delta] \nVdash_{K_1} (A \to B) \to A$, so $[\Delta] \Vdash_{K_1} A \to B$.
So $[\Delta_1] \Vdash_{K_1} B$.

*(Here linear means: for any $w, w' \in W$ $w \lhd w'$ or $w = w'$ or $w' \lhd w$)

case ii) $[\Delta_0] \Vdash_{K_1} A$, then $[\Delta_0] \Vdash_{K_1} \top \to B$.

Well: $[\Delta] \Vdash_{K_1} (C \to B) \vee ((C \to B) \to C)$. Because $(\top \to B) \Vdash_{K_1} (C \to B)$ we have :

$[\Delta_0] \Vdash_{K_1} (C \to B)$ and $[\Delta_0] \nVdash_{K_1} C$. So $[\Delta] \nVdash_{K_1} (C \to B) \to C$.

So $[\Delta] \Vdash_{K_1} (C \to B)$. But $[\Delta_1] \Vdash_{K_1} C$. So $[\Delta_1] \Vdash_{K_1} B$.

So Δ_1 is R_{Δ_0}-prime and $[\Delta_1] \rhd_1 [\Delta_0]$ □

ii) "⇒" as in (i).

"⇐" use (i) and the fact that the constructions in *1.13* and 2.2 preserve linearity. □

4.11 Fact

Let

$$DM_1: \frac{\neg(A \wedge B)}{\neg A \vee \neg B} \qquad DM_2: \frac{\neg A \vee \neg B}{\neg(A \wedge B)}$$

$$DM_3: \frac{\neg(A \vee B)}{\neg A \wedge \neg B} \qquad DM_4: \frac{\neg A \wedge \neg B}{\neg(A \vee B)}.$$

i) DM_2, DM_3, DM_4 are derived ruleschemes for BPL.

ii) DM_1 is derived for BPLL.

Proof: i) routine.

ii)

$$\begin{array}{c}\dfrac{(B \to \bot) \vee (B \to \bot) \to B \qquad \dfrac{B \cancel{\to} \bot}{(A \to \bot) \vee (B \to \bot)} \qquad \dfrac{\dfrac{\dfrac{A \to A \qquad \dfrac{\dfrac{\begin{array}{c}\neg(A \wedge B)\\ \vert\vert\vert\\ (A \wedge B) \to \bot\end{array}}{A \to (B \to \bot)}\to\mathrm{If} \qquad (B \to \cancel{\bot)}) \to B}{A \to B}}{A \to A \wedge B}\wedge\mathrm{If} \qquad A \wedge B \to \bot}{A \to \bot}}{(A \to \bot) \vee (B \to \bot)}}{(A \to \bot) \vee (B \to \bot)}\\ \vert\vert\vert \\ \neg A \vee \neg B\end{array}$$

or via the completeness theorem. □

5 INTERPRETRATIONS

We will interprete FPL in Peano Arithmetic (PA) via PrL and directly.

5.1 The Language of Modal Propositional Logic (MPL)

Let $\mathcal{P}$, $\wedge$, $\vee$, $\rightarrow$, $\perp$ be as in the case of L. $\Box$ is an additional logical constant. $L_\Box$, the language of MPL, is the smallest set s.t.

- $\mathcal{P} \subseteq L_\Box$, $\perp \in L_\Box$
- $A, B \in L_\Box \Rightarrow (A \wedge B), (A \vee B), (A \rightarrow B), (\Box A) \in L_\Box$.

5.2 Kripke Models for MPL

i) A (Kripke) Model for MPL is a structure $M = \langle W, \lhd, f \rangle$, where W is a set (of worlds), $\lhd$ a transitive binary relation on W and f a function $W \rightarrow \mathcal{P}$.

ii) $\models_M \subseteq W \times L_\Box$ is the smallest relation s.t. :

- $p_i \in f(w) \Rightarrow w \models_M p_i$
- $(w \models_M A \text{ and } w \models_M B) \Rightarrow w \models_M (A \wedge B)$
- $(w \models_M A \text{ or } w \models_M B) \Rightarrow w \models_M (A \vee B)$
- $(w \models_M A \Rightarrow w \models_M B) \Rightarrow w \models_M (A \rightarrow B)$
- $(\forall w' \rhd w \;\; w' \models_M A) \Rightarrow w \models_M (\Box A)$

iii) Define for $\Gamma \subseteq L_\Box$:

- $w \models_M \Gamma \;:\Leftrightarrow\; \forall B \in \Gamma \;\; w \models_M B$
- $\Gamma \models_M A \;:\Leftrightarrow\; (\forall w \in W \;\; w \models_M \Gamma \Rightarrow w \models_M A)$
- $M \models A \;:\Leftrightarrow\; \models_M A \;:\Leftrightarrow\; \emptyset \models_M A$

iv) - K is the class of all finite irreflexive Kripke Models for BPL

- M is the class of all finite irreflexive Kripke Models for MPL

- $\Gamma \models_{PrL} A$ means: for every $M \in \mathbf{M}$ $\Gamma \models_M A$.

5.3 Two Gödel Translations

We define :

i) $0 : L \to L_\Box$ by :

$(p_i)^0 := \Box p_i$

$(\bot)^0 := (\Box \bot)$

$(A \wedge B)^0 := (A^0 \wedge B^0)$

$(A \vee B)^0 := (A^0 \vee B^0)$

$(A \to B)^0 : \Box(A^0 \to B^0)$

ii) $1 : L \to L_\Box$ by :

$(p_i)^1 := (p_i \wedge \Box p_i)$

$(\bot)^1 := \bot$

$(A \wedge B)^1 := (A^1 \wedge B^1)$

$(A \vee B)^1 := (A^1 \vee B^1)$

$(A \to B)^1 := \Box(A^1 \to B^1)$

We have :

5.4 Theorem

For finite $\Gamma \subseteq L$, $A \in L$:

i) $\Gamma \vdash_{FPL} A \Leftrightarrow \Gamma^0 = \{B^0 | B \in \Gamma\} \vdash_{PrL} A^0$

ii) $\Gamma \vdash_{FPL} A \Leftrightarrow \Gamma^1 \vdash_{PrL} A^1$

Proof: We know :

for $\Gamma \subseteq L$ finite, $A \in L$: $\Gamma \vdash_{FPL} A \leftrightarrow \Gamma \Vdash_{FPL} A$. And for $\Gamma' \subseteq L_\Box$ finite, $A' \in L_\Box$: $\Gamma' \vdash_{PrL} A' \leftrightarrow \Gamma' \models_{PrL} A'$ (see [1], [9]). So it is sufficient to show for $\Gamma \subseteq L$ finite, $A \in L$:

i) $\Gamma \Vdash_{FPL} A \leftrightarrow \Gamma^0 \models_{PrL} A^0$

ii) $\Gamma \Vdash_{FPL} A \leftrightarrow \Gamma^1 \models_{PrL} A^1$

Case *ii*) is rather easy, so let us do case *i*) :

It is sufficient to provide $\Phi : \mathcal{M} \to \mathcal{K}$ and $\Psi : \mathcal{K} \to \mathcal{M}$ s.t.:

$$\Phi(M) \Vdash A \Leftrightarrow M \models A^0$$

$$K \Vdash A \Leftrightarrow \Psi(K) \models A^0$$

For assume $\Gamma \Vdash_{FPL} A$ and $M \models \Gamma^0$. Then $\Phi(M) \Vdash \Gamma$. So $\Phi(M) \Vdash A$. Conclude: $M \models A^0$.(Clearly $\Delta \models_{PrL} B \Leftrightarrow \forall M \;\; M \models \Delta \Rightarrow M \models B$.) The other direction is similar.

Now let us construct Φ, Ψ .

Consider $M = \langle W, \lhd, f \rangle \in \mathcal{M}$. Take $\Phi(M) = M^\Phi = \langle W^\Phi, \lhd^\Phi, f^\Phi \rangle$, where :

- $W^\Phi := \{w \in W \mid \exists w' \;\; w \lhd w'\}$
- for $w, w' \in W^\Phi$: $w \lhd^\Phi w' :\Leftrightarrow w \lhd w'$
- for $w \in W^\Phi$: $f^\Phi(w) = \bigcap_{\substack{w' \rhd w \\ w' \in W}} f(w')$

Consider on the other hand $K = \langle W, \lhd, f \rangle \in \mathcal{K}$. Take $\Psi(K) = K^\Psi = \langle W^\Psi, \lhd^\Psi, f^\Psi \rangle$ where :

- $W^\Psi := W \cup (W \times \{W\})$
- for $w, w' \in W^\Psi$: $w \lhd^\Psi w' :\Leftrightarrow ((w, w' \in W$ and $w \lhd w')$ or $(w = \langle w'', W \rangle$ and $(w \lhd w''$ or $w = w'')))$.
- for $w \in W^\Psi$: $f^\Psi(w) = \bigcup_{\substack{w' \lhd^\Psi w \\ w' \in W}} f(w')$

Now it is easy to see that Φ, Ψ have the derived properties (by induction on the length of A; note that $K^{\Psi\Phi} = K$). $\Box$

5.5 <u>Definition</u>

Let $f : P \to L_{PA}$. Define :

i) $(p_i)^f := f(p_i)$

$(\bot)^f := (0 = 1)$

$(A \wedge B)^f := (A^f \wedge B^f)$

$(A \vee B)^f := (A^f \vee B^f)$

$(A \to B)^f := \Box(A^f \to B^f)$ $(\equiv \mathrm{Prov}(\ulcorner A^f \to B^f \urcorner))$

ii) $(p_i)^{0f} := \Box f(p_i)$

$(\bot)^{0f} := \Box(0 = 1)$

$(A \wedge B)^{0f} := (A^{0f} \wedge B^{0f})$

$(A \vee B)^{0f} := (A^{0f} \vee B^{0f})$

$(A \to B)^{0f} := \Box(A^{0f} \to B^{0f})$

iii) Let $g(p_i) := (f(p_i) \wedge \Box f(p_i))$, then $A^{1f} := A^g$

5.6 Theorem

Let $\Gamma \subseteq L$ be finite and $A \in L$, then :

i) $\Gamma \vdash_{FPL} A \Leftrightarrow \forall f : P \to \Sigma_1^0\ PA + \Gamma^f \vdash A^f$

ii) $\Gamma \vdash_{FPL} A \Leftrightarrow \forall f : P \to L_{PA}\ PA + \Gamma^{0f} \vdash A^{0f}$

iii) $\Gamma \vdash_{FPL} A \Leftrightarrow \forall f : P \to L_{PA}\ PA + \Gamma^{1f} \vdash A^{1f}$.

Proof: The proofs of *ii*) and *iii*) are by combining 5.4 with Solovay's Completeness Theorem [9]. Note that : $\Gamma \vdash_{FPL} A \Leftrightarrow \vdash_{FPL} \bigwedge\!\!\!\bigwedge \Gamma \to A$, by an easy Kripke Model proof, and $PA + \Gamma^{if} \vdash A^{if} \Leftrightarrow PA \vdash \Box(\bigwedge\!\!\!\bigwedge \Gamma^{if} \to A^{if})$ $(i = 0,1)$. Let us turn to the proof of *i*)

There are standard ways of reducing certain sentences to provably (in PA) equivalent Σ_1^0 sentences. It will be convenient to forget to mention these reductions. Alternatively one can read: '$\{A \in L_{PA} \mid \exists B \in \Sigma_1^0\ PA \vdash A \leftrightarrow B\}$' for '$\Sigma_1^0$' in the statement of the theorem.

"⇒" By induction on the length of the proof.
L uses Löb's Rule for PA. $\rightarrow$I uses the fact that for $A \in \Sigma^0_1$: $PA \vdash A \rightarrow \Box A$.

"⇐" Suppose $\Gamma \nvdash_{FPL} A$. Let K be the finite irreflexive model of 2.2 such that $K \Vdash \Gamma$ and $K \nVdash A$. Without loss of generality we may assume that $W = \{1,\ldots,N\}$ and that $1 \lhd 2,\ldots,1 \lhd N$. '$m \lhd n$' can be birepresented in PA in the obvious way.

We now turn to the proof of Solovay's Completeness Theorem (see [9]). Solovay provides a (primitive) recursive function h s.t. :

i) $PA \vdash (h : \mathbb{N} \rightarrow \{0,1,\ldots N\})$

ii) $PA \vdash (h(m) \neq 0 \rightarrow h(m+1) \trianglerighteq h(m))$

Let $\ell := \lim_{m\to\infty} h(m)$. (By i), ii) ℓ exists)

iii) $(PA + \ell = i)$ is consistent for $i = 0,\ldots N$.

iv) $PA \vdash (\ell = i \rightarrow \Box(\ell \rhd i))$ for $i = 1,\ldots N$

v) $PA \vdash (\ell = i \rightarrow \neg\Box\neg\, \ell = j)$, for $j \rhd i$, for $i = 1,\ldots N$.

Clearly $i \trianglelefteq \ell$ is provably equivalent to $\exists m\ i \trianglelefteq h(m)$, a Σ^0_1 sentence.

Define :

$$g(p_j) = \begin{cases} W\{"i \trianglelefteq \ell" \mid i \Vdash_K p_j\} & \text{if there is such an } i \\ "0 = 1" & \text{else.} \end{cases}$$

Clearly $g(p_j)$ is Σ^0_1. We claim :

$$\mathrm{I} : i \Vdash_K A \Rightarrow PA \vdash (\ell = i \rightarrow A^g)$$

$$\mathrm{II} : i \nVdash_K A \Rightarrow PA \vdash (\ell = i \rightarrow \neg A^g)$$

The proof is by induction on A, simultaneously over I, II.

Suppose $i \Vdash_K A$:

a) $A \equiv p_i$. Clearly $PA \vdash \ell = i \rightarrow i \trianglelefteq \ell$

$$\rightarrow g(p_i)$$

b) Then $\wedge, \vee$ case is trivial.

c) $A \equiv (B \to C)$. We have:

$\forall j \rhd i \; j \Vdash_K B \Rightarrow j \Vdash_K C$.

Using the induction hypothesis and the fact that there are only finitely many $j \rhd i$, we find :

$PA \vdash (\forall j \rhd i \quad \ell = j \to (B^g \to C^g))$

(Note that we use the induction hypothesis on II for B.)

By iv) : $PA \vdash (\ell = i \to \Box\, \ell \rhd i)$. So $PA \vdash (\ell = i \to \Box (B^g \to C^g))$ i.e.

$PA \vdash (\ell = i \to A^g)$.

Suppose $i \nVdash A$:

a') Suppose $A \equiv p_j$. Consider i' s.t. $i' \Vdash_K p_j$, then $i' \not\rhd\; i$.

Thus $PA \vdash (i = \ell \to i' \not\rhd \ell)$. So $PA \vdash i = \ell \to \neg\, g(p_j)$.

(If there is no $i' \Vdash_K p_j$ the case is trivial).

b') The $\wedge$, $\vee$, $\bot$ cases are trivial. In the $\bot$ case we use iii).

c') Suppose $A \equiv (B \to C)$. So there is an $i' \rhd i$ with $i' \Vdash_K B$ and $i' \nVdash_K C$. By Induction Hypothesis and propositional logic :

$PA \vdash \ell = i' \to \neg\,(B^g \to C^g)$.

By v) : $PA \vdash (\ell = i \to \neg\, \Box\, \neg\, \ell = i')$.

So : $PA \vdash (\ell = i \to \neg\, \Box\, (B^g \to C^g))$

i.e. : $PA \vdash \ell = i \to \neg\, A^g$.

We find : $PA \vdash \ell = 1 \to \bigwedge \Gamma^g$

and : $PA \vdash \ell = 1 \to \neg\, A^g$.

By *iii*) : $(PA + \Gamma^g + \neg\, A^g)$ is consistent.

So $PA + \Gamma^g \nvdash A^g$. $\Box$

5.7 Remark

Let for $f : P \to L_{PA}$, $(A)^{*f}$ denote the usual interpretation of $L_{\Box}$ in L_{PA}. It is easily seen that the proof of **5.6**.i can be adapted to give :

$$(\vdash_{PrL+\{p_i \to \Box p_i \mid i \in \mathbb{N}\}} A) \Leftrightarrow (\forall f : P \to \Sigma_1^0 \; PA \vdash A^{*f}).$$

5.8 Corollary

Let $C = \{"0 = 0"\} \cup \{"\neg\, con^n\,(PA)" \mid n \in \mathbb{N}\}$. Then for finite $\Gamma \subseteq L$, $A \in L$:

$$\Gamma \vdash_{FPL^{CL}} A \Leftrightarrow \forall f : P \to C \; PA + \Gamma^f \vdash A^f.$$

Proof: Note that $(\perp_n)^f$ is provably equivalent with $\neg\, con^n\,(PA)$ and $(\perp_\omega)^f$ with $0 = 0$. $\Box$

REFERENCES

[1] BOOLOS G., *The unprovability of consistency*, Cambridge University Press, Cambridge 1979.

[2] __________, *Provability, Truth and Modal Logic*, Journal of Philosophical Logic (1980), pp 1-7.

[3] DUMMETT M.A., *A Propositional Calculus with a Denumerable Matrix*. JSL 24 (1959), pp 97-106.

[4] ____________, *Elements of Intuitionism*, Clarendon Press, Oxford, 1977.

[5] GOLDBLATT R., *Arithmetical necessity, provability and intuitionistic logic*, Theoria vol 44,2 (1978), pp 38-46.

[6] SMORYŃSKI C., *Investigations of Intuitionistic Formal Systems by Means of Kripke Models*, thesis.

[7] _____________, *Calculating Self-Referential Statements I: Explicit Calculations*, Studia Logica 38 (1979), pp 17-36.

[8] _____________, *Calculating Self-Referential Statements II: Non-Explicit Calculations*, Fund.Math., to appear.

[9] SOLOVAY R.M., *Provability Interpretations of Modal Logic*, Israel Journal of Mathematics, vol 25, (1976), pp 287-304.

[10] TROELSTRA A.S. (ed), *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis*, Lecture Notes in Math. 344, Springer Verlag, Berlin, 1973.

PART 4

On the Completeness Principle

A study of provability in Heyting's Arithmetic and extensions.

Albert Visser

Abstract: In this paper extensions of HA are studied that prove their own completeness, i.e. they prove $A \to \Box A$, where $\Box$ is interpreted as provability in the theory itself. Motivation is threefold: firstly these theories are thought to have some intrinsic interest, secondly they are a tool for producing and studying provability principles, thirdly they can be used to prove independence results. Work done in the paper connected with these motivations is respectively:

i) a characterization is given of theories proving their own completeness, including an appropriate conservation result.

ii) some new provability principles are produced. The provability logic of HA is not a sublogic of that of PA. A provability logic plus completeness theorem is given for a certain intuitionistic extension of HA. De Jongh's Theorem for propositional logic is a corollary.

iii) FP-realizability in Beeson's proof that $\nvdash_{HA}$ KLS is replaced by theories proving their own completeness. New consequences are $\nvdash_{HA+\neg M_{PR}}$ KLS, $\nvdash_{HA+DNS}$ KLS.

1 INTRODUCTION

1.1 What is the Completeness Principle?

Theories are in this paper recursively enumerable extensions of Heyting's Arithmetic (HA) in the language of Arithmetic. By *the Completeness Principle for a theory* T we mean:

(CP_T) $\qquad A \rightarrow \Box_T A.$

Here the scheme is interpreted as the set of universal closures of formulae in the language of HA of the displayed form.

1.2 Excurs: prima facie facts about CP

An air of paradoxality lingers around the Completeness Principle. The reader may well wonder: doesn't Gödel's Theorem refute the principle, or what? Let us see what happens.

By the Fixed Point Theorem (also known as Self Reference or Diagonal Lemma, see [Bo]) we can avail ourselves of a sentence G such that: $\vdash_{HA} G \leftrightarrow \neg \Box_T G$. Hence $\vdash_{HA+CP_T} \neg G$. From this we have on the one hand: $\vdash_{HA+CP_T} \neg\neg \Box_T G$, on the other: $\vdash_{HA+CP_T} \Box_T \neg G$. Hence by logic and the closure of $\Box_T$ under Modus Ponens: $\vdash_{HA+CP_T} \neg\neg\Box_T \bot$. In the paper we show that we cannot go further and derive $\vdash_{HA+CP_T} \Box_T \bot$. We will for example exhibit a theory HA* such that $HA^* = HA+CP_{HA^*}$ which is conservative over HA with respect to a wide class of formulae including all formulae of the form: $\forall x_1 \exists y_1 \forall x_2 \exists y_2 \ldots \forall x_n \exists y_n A$, where A contains only bounded quantifiers.

Clearly the principle becomes trivial over Peano Arithmetic (PA) for we have: $(PA+CP_T) = (PA+\Box_T \bot)$.

1.3 Motivation

Given a theory U, we call V a *selfcompletion* of U if $V = U+CP_V$. This paper studies CP and selfcompletions for (mainly) the following reasons:

i) The principle CP has a certain *intrinsic interest*.

(*a*) CP is the natural counterpart of the Reflexion Principle.

(*b*) Selfcompletions bring intuitively the concept of truth closer to that of formal provability. There is a definite feeling that for example $\neg\neg\Box_T\bot$ which is provable in a selfcompletion T *expresses* the familiar provability theoretic fact that a consistent formal theory can never prove its own consistency or 'inconsistency cannot be excluded'. These intuitions are supported by the translation of section 4, which could be considered as an interpretation depending on an Intuitionistic Semantics given in advance.

ii) There are a number of *technical applications*.

(*a*) We derive some new provability principles for e.g. HA.

(*b*) We prove certain independence results for e.g. HA.

1.4 Contents

Section 2 is devoted to preliminaries and elementary facts. Section 3 proves the equivalence of CP with the Strong Löb Principle $(\Box A \to A) \to A$. Section 4 developes most of the basic technical apparatus needed in the remaining sections and provides some examples and minor applications. Section 5 studies Selfcompletions. Section 6 is on Provability Logic. We prove for example that $\Box(\neg\neg\Box A \to \Box A) \to \Box\Box A$ is a provability principle of HA. This shows that the provability logic of HA is not a sublogic of that of PA. Moreover we treat the provability logic of a certain extension of HA: PA*. For this logic we prove a Completeness Theorem. A corollary is De Jongh's Theorem for propositional logic. In section 7 we give an alternative proof of the independence of the Kreisel-Lacombe - Shoenfield Theorem (KLS) for HA and certain extensions. This is done by replacing FP-realizability in Beeson's proof by selfcompletions. As new results we find:

i) $\nvdash_{HA+\neg M_{PR}}$ KLS, where M_{PR} is primitive recursive Markov's Principle i.e:

(M_{PR}) $\quad \forall e\, \forall x(\neg\neg\exists n\ T(e,x,n) \rightarrow \exists n\ T(e,x,n))$.

$T(e,x,n)$ is Kleene's T-predicate.

ii) $\nvdash_{HA+DNS}$ KLS, where DNS is the scheme Double Negation Shift i.e.

(DNS) $\quad (\forall x\ \neg\neg A(x) \rightarrow \neg\neg \forall x\ A(x))$.

A consequence is:

$\nvdash_{HA} \neg\neg \mathrm{KLS} \rightarrow \mathrm{KLS}$.

1.5 Prerequisites

Some familiarity with HA and elementary facts about provability, is sufficient for most sections. At some points realizability, Kripke Models and work on Provability Logic like Solovay's Completeness Theorem are used, however any of these sections could be skipped. All basic facts can be found in ([Tr] ∪ [Bo]).

1.6 Acknowledgements

I wish to thank professors D. van Dalen and A.S. Troelstra for stimulating discussions. Special thanks must go to Michael Beeson for his helpful comments and questions.

2 CONVENTIONS, NOTIONS, ELEMENTARY FACTS

2.1 Language

L is the language of Heyting's Arithmetic (HA), considered as a set of *formulae*. We choose as logical constants $\bot$, $\wedge$, $\vee$, $\rightarrow$, $\forall$, $\exists$. $(\neg A)$ is defined as $(A \rightarrow \bot)$. It makes no difference for our treatment whether we choose the language with just $\underline{0}$, $(\ )'$, $+$, $\cdot$, or with symbols for all primitive recursive functions. The last choice is Troelstra's in [Tr]; it has the nice property that for any $A \in \Delta$ (see 2.2.1) $\vdash_{HA}(A \leftrightarrow s = t)$ for some terms s and t.

2.2 Special classes of formulae

2.2.1 Δ is the smallest class of formulae such that

i) $(s = t) \in \Delta$ for any terms s,t. $\bot \in \Delta$.

ii) Δ is closed under $\wedge$, $\vee$, $\rightarrow$.

iii) $A \in \Delta$, t a term $\Rightarrow (\forall x < tA) \in \Delta$, $(\exists x < tA) \in \Delta$.

2.2.2 $\Sigma^0_1 := \{\exists x A \mid A \in \Delta\}$

2.2.3 $\Pi^0_2 := \{\forall y A \mid A \in \Sigma^0_1\}$

2.2.4 Σ is the smallest class of formulae s.t.

i) $(s=t)$, $\bot \in \Sigma$

ii) Σ is closed under $\wedge$, $\vee$, $\exists$

iii) $A \in \Delta$, $B \in \Sigma \Rightarrow (A \rightarrow B) \in \Sigma$

iv) $A \in \Sigma$, t a term $\Rightarrow (\forall x < tA) \in \Sigma$

We use Σ instead of Σ^0_1, because Σ has nicer closure properties. This enables us to avoid saying things like 'A is a Σ^0_1-formula modulo provable equivalence' all the time.

2.2.5 A is the smallest class of formulae s.t.

i) $(s=t), \perp \in \mathcal{A}$

ii) $\mathcal{A}$ is closed under $\wedge, \vee, \forall, \exists$.

iii) $A \in \Sigma, B \in \mathcal{A} \Rightarrow (A \to B) \in \mathcal{A}$.

Roughly speaking $\mathcal{A}$ is the set of formulae with only Σ-formulae before $\to$. $\mathcal{A}$ is the class with respect to which we will obtain various conservation results.

2.2.6 $\mathcal{B}$ is the smallest set of formulae s.t.

i) $A \in \mathcal{L} \Rightarrow (\neg A) \in \mathcal{B}$

ii) $A \in \Sigma \Rightarrow A \in \mathcal{B}$

iii) $\mathcal{B}$ is closed under $\wedge, \forall$.

iv) $A \in \mathcal{A}, B \in \mathcal{B} \Rightarrow (A \to B) \in \mathcal{B}$

Various nice properties of HA are preserved when we add axioms from $\mathcal{B}$ to HA.

2.3 Convention

"$A(x)$" will sometimes stand for "$A(x_1,\dots,x_n)$". This introduces an ambiguity which is certainly harmless, given the coding possibilities available in HA.

2.4 Some Facts

i) $A(x) \in \Delta \Rightarrow \vdash_{HA} \forall x(A(x) \vee \neg A(x))$

ii) $A \in \Sigma \Rightarrow$ there is a $B \in \Sigma_1^0 \vdash_{HA} A \leftrightarrow B$.

2.5 Logical System

We will work in Natural Deduction (see [Tr]). We distinguish axioms from assumptions: *an axiom is a rule with empty premiss set.* The difference is that free variables in axioms may be universally generalized, but not in assumptions. In the expression "$\Gamma \vdash_T A$", "Γ"

denotes a set of assumptions, while "T" denotes a set of axioms and rules. E.g. we have $\vdash_{A(x)} \forall x A(x)$, but not in general $A(x) \vdash \forall x A(x)$.

2.6 Theories

In this paper theories are always RE theories in L extending HA. Strictly speaking a theory T is a pair $\langle A(x), X\rangle$ where $A(x)$ is a one place Σ-formula satisfied by the Gödel numbers of the axioms of T *additional* to those of HA and X is the set of theorems of T. We will often write $T(x)$ for $A(x)$. When we speak about e.g. HA+DNS, we will always mean this theory *as given by a natural formula*. Of course it is very difficult to state what a natural formula is, but in specific cases natural formulae are easy to give or recognize.

2.7 The Provability Predicate

"$\Box_T A(x_1,\ldots,x_n)$" will mean:

$\mathrm{Prov}_T(\ulcorner A(\dot{x}_1,\ldots,\dot{x}_n)\urcorner)$, where:

i) Prov_T is the standard provability predicate for T built up from $(Ax_{HA}(x) \vee T(x))$. $(Ax_{HA}(x))$ is a standard formula satisfied by the Gödel numbers of the axioms of HA.

ii) The free variables of $A(x_1,\ldots,x_n)$ are precisely $x_1,\ldots,x_n$.

iii) $\ulcorner A(\dot{x}_1,\ldots,\dot{x}_n)\urcorner$ is the 'Gödel term' for $A(x_1,\ldots,x_n)$ as defined in [Fe 2], [Sm]. Troelstra writes this as $\ulcorner A(\bar{x}_1,\ldots,\bar{x}_n)\urcorner$ (see [Tr] pp 25,26). The variables $x_1,\ldots,x_n$ are free in $\ulcorner A(\dot{x}_1,\ldots,\dot{x}_n)\urcorner$ and we have:

$[\underline{m}_1|x_1] \,..\, [\underline{m}_n|x_n] \ulcorner A(\dot{x}_1,\ldots,\dot{x}_n)\urcorner = GN(A(\underline{m}_1,\ldots,\underline{m}_n))$, where $[\,|\,]$ is the usual substitution function and GN a standard Gödel numbering.

2.8 Some relations between theories

Define: $T \subseteq U :\Leftrightarrow$ (for all $A(\vdash_T A \Rightarrow \vdash_U A)$)

$T = U :\Leftrightarrow T \subseteq U$ and $U \subseteq T$

$T \leqslant U :\Leftrightarrow T \subseteq U$ and $\vdash_{HA} \forall x(\mathrm{Prov}_T(x) \to \mathrm{Prov}_U(x))$

$T \equiv U :\Leftrightarrow T \leqslant U$ and $U \leqslant T$.

Note that under our conventions we have: $HA \leqslant T$ for each T.

2.9 Theorem

$A \in \Sigma \Rightarrow \vdash_{HA}(A \to \Box_{HA} A)$

Proof: We can copy the proof for Peano Arithmetic verbatim. See for a sketch [Sm]. For term identities it is given in [Tr] pp 37,38. One can also derive the theorem from the theorem for PA. Let $A \in \Sigma$. First we remark: $\vdash_{PA} A \Rightarrow \vdash_{HA} A$ and $\vdash_{HA} \Box_{PA} A \to \Box_{HA} A$, by the $\neg\neg$ translation and verifiable closure of HA under Markov's Rule (see 2.15). We know $\vdash_{PA} A \to \Box_{PA} A$, so by the second half of the remark: $\vdash_{PA} (A \to \Box_{HA} A)$. $(A \to \Box_{HA} A)$ is provably (in HA) equivalent to $\forall x\, C(x)$ for some $C(x) \in \Sigma$. So $\vdash_{PA} C(x)$. By the first part of the remark: $\vdash_{HA} C(x)$. So $\vdash_{HA} A \to \Box_{HA} A$.

$\Box$

2.10 Corollary

$\vdash_{HA} ([s|x]\, \Box_T A(x) \leftrightarrow \Box_T A(s))$.

Proof: Suppose x is not free in s. In HA:

$$\begin{aligned} s = x \to\ & \Box_T\, s = x \\ \to\ & \Box_T(A(s) \leftrightarrow A(x)) \\ \to\ & ((\Box_T A(s)) \leftrightarrow (\Box_T A(x))) \end{aligned}$$

So

$$\begin{aligned} s = s &\to [s|x]((\Box_T A(s)) \leftrightarrow (\Box_T A(x))) \\ &= (\Box_T A(s) \leftrightarrow [s|x]\, \Box_T A(x)) \end{aligned}$$

If x is free in s, first prove the corollary for A(y), y not free in s and then substitute $([s|x]\Box_T A(x))$ for $([s|y]\Box_T A(y))$.

$\Box$

2.11 The Fixed Point Theorem

For every $A(x,y) \in L$ there is a $B(y) \in L$ such that:

$$\vdash_{HA} B(y) \leftrightarrow A(\underline{GN(B(y))},y)$$

Proof: see e.g. [Bo] pp 49,50.

$\Box$

2.12 Definitions

i) $\bot_T^0 = \bot$

$\bot_T^{n+1} = \Box_T \bot_T^n$

ii) T is n-inconsistent if $\vdash_T \bot_T^n$.

2.13 The Löb Conditions

We have:

i) $\vdash_T A \Rightarrow \vdash_{HA} \Box_T A$

ii) $\vdash_{HA} (\Box_T A \rightarrow \Box_T \Box_T A)$

iii) $\vdash_{HA} \Box_T (A \rightarrow B) \rightarrow (\Box_T A \rightarrow \Box_T B)$

2.14 Löb's Theorem

$$\vdash_{HA} \Box_T (\Box_T A \rightarrow A) \rightarrow \Box_T A$$

2.15 The Friedman Translation and Markov's Rule

H. Friedman defines (see [Fr]) the following translation for $A \in L$:

$(\ )^A : L \rightarrow L$ is given by:

i) $(s = t)^A := ((s = t) \vee A)$

$(\perp)^A \quad := A$

ii) $(\ )^A$ commutes with all logical constants. (Only free variables in A may not be bound in $(B)^A$. When this threatens to occur we have first to rename the bound variables of B).

One can prove by induction verifiably in HA:

a) $B \in \Sigma \Rightarrow\ \vdash_{HA} (B)^A \leftrightarrow (B \vee A)$

b) $B \in \mathcal{A} \Rightarrow\ \vdash_{HA} B \to (B)^A$

c) $\Gamma^A \vdash_{HA} B^A$ as a relation between Γ and B is closed under the axioms and rules of HA.

We can give two applications:

Application 1:

Suppose $\Gamma \subseteq \mathcal{A}$, $A \in \Sigma$ and $\Gamma \vdash_{HA} \neg\neg A$.

We claim: $\Gamma \vdash_{HA} A$.

Proof: $\Gamma \vdash_{HA} \neg\neg A$, so $\Gamma^A \vdash_{HA} (\neg\neg A)^A$.

Hence: $\Gamma \vdash_{HA} \Gamma^A$

$\vdash_{HA} ((A \vee A) \to A) \to A$

$\vdash_{HA} A$.

$\square$

Application 2:

Suppose $\Gamma \subseteq \mathcal{A}$, $A \in \Sigma$ and $\Gamma \vdash_{HA} \neg\neg A \to A$.

We claim: $\Gamma \vdash_{HA} A \vee \neg A$.

Proof: $\Gamma \vdash_{HA} \neg\neg A \to A$, so $\Gamma^{\neg A} \vdash_{HA} (\neg\neg A \to A)^{\neg A}$.

Hence: $\Gamma \vdash_{HA} \Gamma^{\neg A}$

$\vdash_{HA} (((A \vee \neg A) \to \neg A) \to \neg A) \to (A \vee \neg A)$

$\vdash_{HA} (A \vee \neg A)$.

$\square$

We can formalize this proof in HA, so we find for $A \in \Sigma$:

$$\vdash_{HA} \Box_{HA}(\neg\neg A \to A) \to \Box_{HA}(A \vee \neg A).$$

3 THE COMPLETENESS PRINCIPLE AND THE STRONG LÖB PRINCIPLE

3.1 Definition

The *Strong Löb Principle for T* is the axiom scheme:

(SLP_T) $\qquad ((\Box_T A \to A) \to A)$

Because we work in Natural Deduction there are the following two variants:

(SLP'_T) $$\frac{\Box_T A \to A}{A}$$

and

(SLP''_T) $$\frac{\begin{array}{c}[\Box_T A] \\ \vdots \\ A\end{array}}{A}$$

3.2 Theorem

CP_T is interderivable with SLP_T over HA.

Proof:

i) $\vdash_{HA+CP_T} SLP_T$

$$\dfrac{\dfrac{\dfrac{[\Box_T A \to A]^{(1)}}{\Box_T(\Box_T A \to A)}\,CP_T}{\Box_T A}\,\text{Löb's Theorem} \qquad [\Box_T A \to A]^{(1)}}{\dfrac{A}{(\Box_T A \to A) \to A}\,\to I\ (1)}$$

ii) $\vdash_{HA+SLP_T} CP_T$.

$$\dfrac{\dfrac{\dfrac{\dfrac{[A]^{(2)} \qquad \dfrac{[\Box_T\ (A \wedge \Box_T\ A)]^{(1)}}{\Box_T\ A}}{A \wedge \Box_T\ A}}{A \wedge \Box_T\ A}\ (1)\ SLP''_T}{\Box_T\ A}}{A \to \Box_T\ A}\ (2) \to I$$

□

4 THE T - TRANSLATION

This section introduces the necessary technical apparatus: the T-translation and U^V.

4.1 Definition: the T-translation

Let T be a theory. We define the translation $(\ )^T : L \to L$ as:

i) $(s = t)^T :\, = (s = t)$, $(\bot)^T :\, = \bot$

ii) $(\ )^T$ commutes with $\wedge, \vee, \exists$.

iii) $(A \to B)^T :\, = ((A^T \to B^T) \wedge \Box_T(A^T \to B^T))$

iv) $(\forall x\ A)^T :\, = ((\forall x\ A^T) \wedge \Box_T(\forall x\ A^T))$

4.2 Substitution Lemma

$\vdash_{HA} ([s|x]\ (A(x))^T) \leftrightarrow (A(s))^T$.

Proof: Induction on $A(x)$. Use 2.10.

□

4.3 Definitions

i) For $U \leqslant T$ we define:

$U^T :\, \equiv HA + \{A \mid \vdash_U A^T\}$

ii) $U^* :\, \equiv U^U$

4.4 Definition

Let GN: $\mathcal{L} \to \mathbb{N}$ be the standard Gödel numbering. Let h_T be the natural primitive recursive function such that $h_T(GN(A)) = GN((A)^T)$. Let (Sent(x)) be a formula of $\mathcal{L}$ that expresses 'x is the Gödelnumber of a *sentence* of $\mathcal{L}$'.

U is *a base* if for every T with $U \leqslant T$:

i) $\vdash_U A \Rightarrow \vdash_U A^T$ and

ii) $\vdash_{HA} \forall x((Sent(x) \wedge Prov_U(x)) \to Prov_U(h_T(x)))$

4.5 Definition: the Weak Completeness Principle

By the Weak Completeness Principle for T we mean:

(WCP_T) $\qquad A \to \Box_T A^T$

4.6 Technical Lemma

i) $U \leqslant V \leqslant W \Rightarrow U^W \leqslant V^W$

ii) $\vdash_{HA} A^T \to \Box_T A^T$

iii) $A \in \Sigma \Rightarrow \vdash_{HA} (A^T \leftrightarrow A)$

iv) $A \in \mathcal{A} \Rightarrow \vdash_{HA} (A^T \to A)$

v) $A \in \mathcal{B} \Rightarrow \vdash_{HA} ((A \wedge \Box_T A) \to A^T)$

vi) $\vdash_{HA+CP_T} A \leftrightarrow A^T$

vii) $\vdash_{HA+WCP_T} A \leftrightarrow A^T$

viii) $\vdash_{HA+CP_T} WCP_T$

ix) $\vdash_{HA+\Box_T\perp} CP_T$

x) $\vdash_{HA+WCP_T+\Box_T\Box_T\perp} CP_T$

Proof:

i) trivial.

ii) Induction on A, e.g. the case of $A = (B \to C)$:

$$\dfrac{\dfrac{\begin{matrix}(B \to C)^T \\ \| \\ (B^T \to C^T) \wedge \Box_T (B^T \to C^T)\end{matrix}}{\Box_T (B^T \to C^T)}}{\begin{matrix}\Box_T ((B^T \to C^T) \wedge \Box_T (B^T \to C^T)) \\ \| \\ \Box_T (B \to C)^T\end{matrix}} \text{ By the Löb Conditions}$$

iii) Induction on A using 2.9.

iv) Induction on A using *iii)*.

v) Induction on A. In the "$\to$" case use *ii)* and *iv)*. we treat the case that $A = (\neg B)$, $B \in L$:

First note that $\Box_T \perp$ 'blows up the T-translation' i.e. $\Box_T \perp \vdash_{HA} (B \leftrightarrow B^T)$, this of course because it makes the second conjuncts in clause *iii)* and *iv)* of the definition of $(\ )^T$ true.

We have: $\Box_T \perp \vdash_{HA} (B \leftrightarrow B^T)$, hence $\neg B, B^T \vdash_{HA} \Box_T \perp \to \perp$ (*).

So: $\Box_T \neg B, B^T \vdash_{HA} \Box_T \neg B, \Box_T B^T$ (by *ii)*)

$\vdash_{HA} \Box_T (\Box_T \perp \to \perp)$ (by * and the Löb Conditions)

$\vdash_{HA} \Box_T \perp$ (by Löb's Theorem)

We find, combining this with (*):

$\neg B, \Box_T \neg B, B^T \vdash_{HA} \perp$, or:

$(\neg B \wedge \Box_T \neg B) \vdash_{HA} \neg B^T$.

Thus, using the Löb Conditions again:

$(\neg B \wedge \Box_T \neg B) \vdash_{HA} (\neg B)^T$.

$\Box$

vi)
Induction on A.
vii)

viii) $\vdash_{HA+CP_T} A \leftrightarrow A^T$, so by CP_T:

$\vdash_{HA+CP_T} \Box_T(A \leftrightarrow A^T)$. Using the Löb Conditions we find:

$\vdash_{HA+CP_T} \Box_T A \leftrightarrow \Box_T A^T$. So:

$\vdash_{HA+CP_T} A \rightarrow \Box_T A^T$.

ix) trivial

x) Clearly:

$$\vdash_{HA+\Box_T\Box_T\perp} \Box_T A \leftrightarrow \Box_T A^T .$$

$\Box$

4.7 <u>Characterization of U^V</u>

Let $U \leqslant V$, then:

i) $\Gamma \vdash_{U^V} A \Leftrightarrow \Gamma^V \vdash_U A^V$

ii) $\vdash_{HA} \Box_{U^V} A \leftrightarrow \Box_U A^V$

iii) $\vdash_{U^V} WCP_V$

Proof:

i) "$\Leftarrow$" Suppose $\Gamma^V \vdash_U A^V$. Then there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0^V \vdash_U A^V$. Let C be the conjunction of the elements of Γ_0. Clearly $\vdash_U C^V \rightarrow A^V$. $U \leqslant V$, hence $\vdash_U((C^V \rightarrow A^V) \wedge \Box_V(C^V \rightarrow A^V))$. Or: $\vdash_U(C \rightarrow A)^V$. Then by definition: $\vdash_{U^V} C \rightarrow A$. Conclude: $\Gamma \vdash_{U^V} A$.

"$\Rightarrow$" By induction on the length of the proof of $\Gamma \vdash_{U^V} A$.

The length of proof is 0

In this case A is an axiom of HA or $\vdash_U A^V$. We treat for example the Induction Axiom:

We have: $\vdash_{HA}(B^V o \wedge \forall x(B^V x \rightarrow B^V x')) \rightarrow \forall x B^V x$

Since $\vdash_{HA} B^V o \rightarrow \Box_V B^V o$:

$\vdash_{HA}(B^V o \wedge \Box_V \forall x(B^V x \rightarrow B^V x')) \rightarrow \Box_V \forall x B^V x$

Combining:

$$\vdash_{HA} (B^V o \wedge \forall x((B^V x \to B^V x') \wedge \ldots) \wedge \Box_V \ \forall x((B^V x \to B^V x') \wedge \ldots)) \to$$
$$(\forall x B^V x \wedge \Box \ \ \forall x B^V x)$$

Where ... is the (for our reasoning spurious) $\Box_V (B^V x \to B^V x')$.

The length of the proof is n+1

We treat for example the case that the last step in the proof was:

$$\frac{\Gamma, B \vdash_{U^V} C}{\Gamma \vdash_{U^V} B \to C.}$$

By the Induction Hypothesis we have:

$$\frac{\dfrac{\Gamma^V \vdash_U \Box_V \Gamma^V \qquad \dfrac{\dfrac{\Gamma^V,\ B^V \vdash_U C^V}{\Gamma^V \vdash_U (B^V \to C^V)}}{\Box_V \Gamma^V \vdash_U \Box_V (B^V \to C^V)}}{\Gamma^V \vdash_U \Box_V (B^V \to C^V)} \qquad \dfrac{\Gamma^V, B^V \vdash_U C^V}{\Gamma^V \vdash_U B^V \to C^V}}{\Gamma^V \vdash_U (B \to C)^V}$$

(By 4.6 *ii*)   By the Löb Conditions and $U \leqslant V$

ii) By formalizing the proof of *i*). Remark:

a) We must formalize the proof of 4.6 *ii*.

b) We really need $U \leqslant V$ rather than $U \subseteq V$ here.

c) What we prove is even the stronger 'quantified' version of *ii*).

iii) By 4.6 *ii*):

$\vdash_{HA} (A^V \to \Box_V \ A^V) \Rightarrow$ (4.6 *iii*))

$\vdash_{HA} (A^V \to (\Box_V \ A^V)^V) \Rightarrow$ ($HA \leqslant V$)

$\vdash_{HA} (A \to \Box_V \ A^V)^V \Rightarrow$ ($HA \leqslant U$)

$\vdash_U (A \to \Box_V \ A^V)^V \Rightarrow$

$\vdash_{U^V} \ A \to \Box_V \ A^V .$

$\Box$

4.8 Theorem

Let $U \leqslant V$, then:

i) $A \in \mathcal{A}$ and $\vdash_{U^V} A \Rightarrow\ \vdash_U A$

ii) $A \in \mathcal{B}$ and $\vdash_U A \Rightarrow\ \vdash_{U^V} A$

Proof:

i) Suppose $A \in \mathcal{A}$:

$\vdash_{U^V} A \Rightarrow$ (4.7 *i*)

$\vdash_U A^V \Rightarrow$ ($A \in \mathcal{A}$).

$\vdash_U A$ □

ii) Suppose $A \in \mathcal{B}$:

$\vdash_U A \Rightarrow$ ($U \leqslant V$)

$\vdash_U A \wedge \Box_V A \Rightarrow$ ($A \in \mathcal{B}$)

$\vdash_U A^V \Rightarrow$ (4.7 *i*)

$\vdash_{U^V} A$ □

4.9 Definitions

i) Let $\Gamma \subseteq L$, T a theory.

Define: T is Γ-*sound* if for every sentence $A \in \Gamma$:

$(\vdash_T A \Rightarrow$ (A is (classically) true)).

ii) A theory T satisfies the *Disjunction Property* (DP) if for every sentence $(A \vee B)$: $\vdash_T A \vee B \Rightarrow (\vdash_T A$ or $\vdash_T B)$.

iii) A theory T satisfies the *Numerical Existence Property* if for every sentence $(\exists x A(x))$: $\vdash_T \exists x A x \Rightarrow (\vdash_T A\underline{n}$ for some $n \in \mathbb{N}$).

4.10 Theorem

i) $\vdash_{U*} CP_{U*}$

ii) U* is a base

iii) If U is Σ-sound then U* satisfies DP and EP.

Proof:

i) By 4.7 *iii)* $\vdash_{U*} A \to \Box_U A^U$ and by 4.7 *ii)* $\vdash_{HA} \Box_U A^U \leftrightarrow \Box_{U*} A$.

ii) Suppose $U* \leqslant T$. We have $\vdash_{U*} CP_{U*}$, hence $\vdash_{U*} CP_T$. By 4.6 *vi*: $\vdash_{U*} A \leftrightarrow A^T$. Conclude $\vdash_{U*} A \Rightarrow \vdash_{U*} A^T$. To prove the verifiability clause we must formalize this argument.

iii) Suppose U is Σ-sound. We treat EP. Let $(\exists x A(x))$ be a sentence. We have:

$\vdash_{U*} \exists x A(x) \Rightarrow$	(4.10 *i*)
$\vdash_{U*} \exists x \Box_{U*} A(x) \Rightarrow$	(4.8 *i*)
$\vdash_{U} \exists x \Box_{U*} A(x) \Rightarrow$	(U is Σ-sound)
$\vdash_{U*} A(\underline{n})$ for some $n \in \mathbb{N}$.	

□

4.11 Example: PA*

In the next sections we will mostly be interested in theories U such that $U \leqslant U*$. Bases have this property (see 4.12). A nice example of a theory without this property is PA: clearly $PA \not\leqslant PA*$, for PA* satisfies DP and any extension of PA that satisfies DP is either not RE or inconsistent. In section 6 we will return to PA* to charactarize its provability logic.

By the Trace Principle we mean:

(TP) $\qquad (\Box_{PA*} \forall x(Ax \to Bx)) \to ((\exists x Ax) \vee \forall x(Ax \to Bx))$

It is called Trace Principle because it is the trace left by Excluded Third in PA*.

We have:

i) $\vdash_{PA*}$ CP_{PA*}

ii) $\vdash_{PA*}$ TP

iii) $\vdash_{PA*}$ DNS, where DNS is the scheme $(\forall x(\neg\neg A(x)) \rightarrow \neg\neg(\forall x Ax))$

Proof:

ii) We have:

$$\vdash_{HA} (\Box_{PA*} \forall x(Ax \rightarrow Bx)) \leftrightarrow$$
$$\Box_{PA} (\forall x(Ax \rightarrow Bx))^{PA} \leftrightarrow$$
$$\Box_{PA} \forall x((Ax)^{PA} \rightarrow (Bx)^{PA})$$

and

$$\vdash_{HA} (\forall x(Ax \rightarrow Bx))^{PA} \leftrightarrow$$
$$((\forall x(Ax)^{PA} \rightarrow (Bx)^{PA}) \wedge \Box_{PA} \forall x((Ax)^{PA} \rightarrow (Bx)^{PA}))$$

Moreover:

$$\vdash_{PA} (\exists x(Ax)^{PA} \vee \forall x((Ax)^{PA} \rightarrow (Bx)^{PA}))$$

So $\vdash_{PA} (TP)^{PA}$

iii) We can prove $\vdash_{PA*}$ DNS in two ways. The first is by remarking that $HA + \{\neg A \mid \vdash_{PA} \neg A\} \equiv HA+DNS$. Moreover $(\neg A) \in \mathcal{B}$, hence: $\vdash_{PA} \neg A \Rightarrow \vdash_{PA*} \neg A$.

The second is by deriving DNS in $HA+CP_{PA*}+TP$. First note that from TP a slightly weakened form, the Propositional Trace Principle, follows:

(PTP): $\quad \Box_{PA*}(A \rightarrow B) \rightarrow A \vee (A \rightarrow B)$.

We have in HA+TP:

$$\frac{\dfrac{\dfrac{\dfrac{[\Box_{PA*}\perp]}{\Box_{PA*}(\neg Ax)}}{Ax \vee \neg Ax}}{\forall x(Ax \vee \neg Ax)}}{\Box_{PA*}\perp \to \forall x(Ax \vee \neg Ax)}$$

Moreover: $\vdash_{HA+CP_{PA*}} \neg\neg\Box_{PA*}\perp$, so

$\vdash_{HA+CP_{PA*}+TP} \neg\neg\forall x\,(Ax \vee \neg Ax)$

From this it is easy to derive:

$\vdash_{HA+CP_{PA*}+TP}$ DNS.

□

Open problem: Is $PA* \equiv HA+CP_{PA*}+TP$?

We now turn to bases.

4.12 Theorem

Let U be a base, $U \leqslant V$, then:

i) $U^V \equiv U+WCP_V$

ii) $U* \equiv U+CP_{U*}$

Proof:

i) We prove: $U^V = U+WCP_V$. $\equiv$ follows by formalizing this proof.

$U+WCP_V \subseteq U^V$

By 4.7 (*iii*) $\vdash_{U^V} WCP_V$

Further $\vdash_U A \Rightarrow$ (U is a base)

$\vdash_U A^V \Rightarrow$ (4.7 *i*)

$\vdash_{U^V} A$

$U^V \subseteq U{+}WCP_V$

$$\vdash_{U^V} A \quad \Rightarrow \qquad (4.7\ i)$$
$$\vdash_U A^V \quad \Rightarrow \qquad (4.6\ vii)$$
$$\vdash_{U+WCP_V} A$$

ii) By *i)*: $U^* \equiv U{+}WCP_U$. By 4.7 *ii)*: $\vdash_{HA} \Box_U A^U \leftrightarrow \Box_{U^*} A$ so $U{+}WCP_U = U{+}CP_{U^*}$. Formalization of this argument gives: $U{+}WCP_U \equiv U{+}CP_{U^*}$.

□

4.13 Theorem

Let U be a base, $U \leqslant V$, then $A \in \mathcal{A} \Rightarrow (\vdash_U A \Leftrightarrow \vdash_{U^V} A)$

Proof: by 4.8 *i)* and the fact that $U \leqslant U^V$.

□

4.14 Application

i) HA is a base

ii) $\vdash_{HA+CP_V} A \Rightarrow \vdash_{HA^V+\Box_V \Box_V \bot} A$

iii) $A \in \mathcal{A} \Rightarrow (\vdash_{HA+CP_V} A \Rightarrow \vdash_{HA+\Box_V \Box_V \bot} A)$

iv) $\nvdash_{HA+CP_V} \Box_V \bot$, if V is consistent.

v) $HA{+}CP_{HA} \equiv HA^* + \Box_{HA} \Box_{HA} \bot$

vi) $A \in \mathcal{A} \Rightarrow (\vdash_{HA+CP_{HA}} A \Leftrightarrow \Box_{HA}\Box_{HA} \bot \vdash_{HA} A)$

Proof:

i) The proof is implicit in the proof of 4.7 *i)*, *ii)*.

ii) We have: $\vdash_{HA^V} WCP_V$ (by 4.7 *iii)*), so $\vdash_{HA^V+\Box_V \Box_V \bot} CP_V$ (by 4.6 *x)*).

iii) Suppose $A \in \mathcal{A}$. Then $(\Box_V \Box_V \bot \to A) \in \mathcal{A}$.

We have:

$A \vdash_{HA+CP_V} A \Rightarrow$ (i)

$\vdash_{HA^V+\Box_V\Box_V\bot} A \Rightarrow$

$\vdash_{HA^V} (\Box_V\Box_V\bot \to A) \Rightarrow$ (4.8 i)

$\vdash_{HA} (\Box_V\Box_V\bot \to A) \Rightarrow$

$\vdash_{HA+\Box_V\Box_V\bot} A.$

iv) If $\vdash_{HA+CP_V} \Box_V\bot$, then by iii): $\vdash_{HA} \Box_V\Box_V\bot \to \Box_V\bot$, so by $HA \leqslant V$, Löb's Theorem and logic: $\vdash_{HA}\Box_V\bot$. HA is Σ-sound, hence $\vdash_V \bot$.

v) By ii) we have $HA+CP_{HA} \subseteq HA^*+\Box_{HA}\Box_{HA}\bot$.

On the other hand:

$\vdash_{HA^*} A \Rightarrow$ (4.7 i)

$\vdash_{HA} A^{HA} \Rightarrow$ (4.6 vi)

$\vdash_{HA+CP_{HA}} A$

and (by 3.2)

$\vdash_{HA+CP_{HA}} \neg\neg\Box_{HA}\bot \Rightarrow$ (CP_{HA})

$\vdash_{HA+CP_{HA}} \Box_{HA}\neg\neg\Box_{HA}\bot \Rightarrow$ (2.15)

$\vdash_{HA+CP_{HA}} \Box_{HA}\Box_{HA}\bot.$

So $HA+CP_{HA} = HA^*+\Box_{HA}\Box_{HA}\bot$.

We find $\equiv$ by formalizing this argument.

vi) Suppose $A \in \mathcal{A}$, we have:

$\vdash_{HA+CP_{HA}} A \Leftrightarrow$ (v)

$\vdash_{HA^*} (\Box_{HA}\Box_{HA}\bot \to A) \Leftrightarrow$ (4.13)

$\vdash_{HA} (\Box_{HA}\Box_{HA}\bot \to A)$

$\Box$

4.15 Examples of Bases

Our result in section 7 on the independence of KLS will hold for Σ-sound bases. Hence we will note when an example is Σ-sound.

i) HA is a Σ-sound base.

Proof: that HA is a base, we remarked in 4.14 *i*). Σ-soundness follows from the truth of HA. □

ii) Let $\mathcal{B}(x)$ be a natural formula of L such that the x satisfying it are precisely the Gödelnumbers of the elements of $\mathcal{B}$. Let $\Gamma(x)$ be a Σ-formula with just x free. Suppose that: $\vdash_{HA} \forall x(\Gamma(x) \to \mathcal{B}(x))$. Then $U :\equiv HA + \Gamma$ is a base.

Proof: Suppose $A \in \Gamma$, $\vdash_U A$, $U \leqslant T$. We have:

$\vdash_{HA} (\bigwedge\!\!\!\bigwedge \Gamma_0 \to A)$, where $\Gamma_0 \subseteq \Gamma$, Γ_0 finite. Hence by *i*:

$\vdash_{HA} \bigwedge\!\!\!\bigwedge \Gamma_0^T \to A^T$. By 4.8*ii*: $\vdash_U \bigwedge\!\!\!\bigwedge \Gamma_0^T$. Conclude $\vdash_U A^T$.

To prove the verifiability condition we have to formalize this proof. □

iii) Let $U :\equiv HA + DNS$, where DNS is the scheme: $(\forall x(\neg\neg Ax) \to \neg\neg(\forall x Ax))$. Then U is a Σ-sound base.

Proof: Let $\neg\neg$DNS be the scheme:

$(\neg\neg DNS)\ \neg\neg(\forall x(\neg\neg Ax) \to \neg\neg(\forall x Ax))$.

By simple logic $U \equiv HA + \neg\neg DNS$. Apply *ii*).

The Σ-soundness of U follows from the fact that HA+DNS is classically true. □

iv) Let Primitive Recursive Markov's Principle be:

$(M_{PR})\qquad \forall x \forall y(\neg\neg\exists z Txyz \to \exists z Txyz)$

where Txyz is Kleene's T-predicate. Then $U :\equiv HA + \neg M_{PR}$ is a Σ-sound base.

Proof: U is a base by *ii*). It is Σ-sound by a result of Kreisel, see [Tr] 3.8.3 page 264.

v) The Reflexion Principle for HA is the axiomscheme:

$(RP_{HA}) \qquad \Box_{HA} A \to A.$

Let U: ≡ HA+RP_{HA}. Then U is a Σ-sound base.

Proof: U is true, hence Σ-sound. Let T ⩾ U, we have:

$$\begin{array}{ll} \vdash_U (\Box_{HA} A)^T \to & (4.6\ iii) \\ \quad \Box_{HA} A \to & (\text{HA is a base}) \\ \quad \Box_{HA} A^T \to & (RP_{HA}) \\ \quad A^T. & \end{array}$$

Hence: $\vdash_U (\Box_{HA} A \to A)^T$.

To prove the verifiability condition we have to formalize this proof.

vi) Let $<$ be a well founded RE relation, represented in *L* by the formula $(x < y)$. Let $TI_<$ be the scheme:

$(TI_<) \ ((\forall y((\forall x < y\ Ax) \to Ay)) \to \forall z Az)$

Let U: ≡ HA+$TI_<$. Then U is a Σ-sound base.

Proof: The Σ-soundness follows from truth.

Let T ⩾ U. Clearly it is sufficient to prove:

$\vdash_U (\forall y((\forall x < y\ Ax) \to Ay))^T \to (\forall z Az)^T$

(and formalize this proof in HA).

Assume:

a) $\Box_T (\forall z Az)^T$

b) $(\forall y((\forall x < y\ Ax) \to Ay))^T$

We first prove: $\forall z(\forall x < z\ Ax)^T$, by $TI_<$ in $(\forall x < z\ Ax)^T$:

Suppose we have $\forall u < z\ (\forall x < u\ Ax)^T$.

By *b*): $\forall u < z\ (Au)^T$.

By *a*): $\Box_T\ \forall u < z\ (Au)^T$.

Combining: $(\forall u < z\ Au)^T$.

By $TI_<$ conclude: $\forall z(\forall u < z\ Au)^T$, hence by *b*): $\forall z(Az)^T$.

From *a*): $(\forall zAz)^T$.

So we have :

(*) $b \vdash_U \Box_T (\forall zAz)^T \to (\forall zAz)^T$.

We have $U \leqslant T$ and $b \vdash_U \Box_T b$ (according to 4.6 *ii*)) hence:

$b \vdash_U \Box_T (\Box_T(\forall zAz)^T \to (\forall zAz)^T)$.

By Löb's Theorem:

(**) $b \vdash_U \Box_T (\forall zAz)^T$.

Combining (*) and (**):

$$b \vdash_U (\forall zAz)^T .$$

□

4.16 Theorem

Let $T(y,x)$ be a Σ-formula with just y,x free. Let T_i given by the formula $(T(\underline{i},x))$ be a theory. Suppose $\vdash_{HA}$ "$\forall i\ T_i$ is a base". Let $U \equiv \bigcup_{i \in \omega} T_i$, i.e. the theory given by $\exists y\ T(y,x)$. Then U is a base.

Proof: Suppose $\vdash_U A$ and $U \leqslant V$. Then there are $B_1,\ldots,B_k$ such that $\vdash_{T_{i_\ell}} B_\ell$ for $\ell = 1,\ldots,k$, and $B_1,\ldots,B_k \vdash_{HA} A$. For every $\ell \in \{1,\ldots,k\}$: $T_{i_\ell} \leqslant U \leqslant V$, so $\vdash_{T_{i_\ell}} (B_\ell)^V$. Moreover $B_1^V,\ldots,B_k^V \vdash_{HA} A^V$. So $\vdash_U A^V$.

This proof can be formalized in HA. □

5 ON SELFCOMPLETIONS

This section is about selfcompletions. We show how to construct them and give the appropriate conservation results. Selfcompletions over a fixed theory are not in general unique but selfcompletions of a base, for which the fact that they are selfcompletions of that base is provable in HA, are unique.

5.1 Definition

Let T, U be theories.

i) T is a *selfcompletion of U* if $T = U + CP_T$.

ii) T is a *strong selfcompletion of U* if $T \equiv U + CP_T$.

5.2 Theorem

Every theory U has a strong selfcompletion.

Proof: Clearly we can find a primitive recursive h such that:

$h(GN((T(x))), GN(A)) = GN((A \to \Box_T A))$

Let (Form(z)) be a formula just satisfied by the Gödelnumbers of formulae and define:

$(B(y,x)) := (U(x) \vee \exists z(Form(z) \wedge x = h(y,z)))$

By the Fixed Point Theorem (2.11) we find a formula (T(x)) such that:

$\vdash_{HA} (T(x) \leftrightarrow B(\underline{GN((T(x)))},x))$ □

5.3 Theorem

Let U be a base.

i) Suppose T is a selfcompletion of U, then $T = U^T$.

ii) Suppose T is a strong selfcompletion of U, then $T \equiv U^T$.

Proof:

i) By 4.12 $U^T \equiv U + WCP_T$. Moreover $\vdash_T A \leftrightarrow A^T$ (by 4.6 *vi*), so $\vdash_{HA} \Box_T A \leftrightarrow \Box_T A^T$. Hence WCP_T and CP_T are interderivable over HA. Thus $T = U + CP_T = U + WCP_T \equiv U^T$. □

ii) By formalizing the proof of *i)*. □

5.4 Theorem

Let U be a base. Then any selfcompletion T of U is conservative over U w.r.t. $\mathcal{A}$.

Proof: by 4.13 and 5.3 *i*). $\square$

5.5 Theorem

Let T be a selfcompletion of U. Then $\vdash_T \neg A \Leftrightarrow \neg\neg\Box_T \bot \vdash_U \neg A$.

Proof:

"$\Rightarrow$" $T = U + CP_T \subseteq U + \Box_T \bot$ (4.6 *ix*), so $\vdash_T \neg A \Rightarrow \Box_T \bot \vdash_U \neg A$

$\Rightarrow \neg\neg\Box_T \bot \vdash_U \neg A$.

"$\Leftarrow$" $T \supseteq U + \neg\neg\Box_T \bot$ by 3.2 $\square$

5.6 Theorem

Let U be a Σ-sound base, T a selfcompletion of U. Then T satisfies EP and DP.

Proof: We treat EP: Suppose $(\exists x A x)$ is a sentence, and $\vdash_T (\exists x A x)$.

We have:

$\vdash_T (\exists x A x) \Rightarrow$ (CP_T)

$\vdash_T \exists x \Box_T Ax \Rightarrow$ (5.4)

$\vdash_U \exists x \Box_T Ax \Rightarrow$ (U is Σ-sound)

$\vdash_T A\underline{n}$ for some n. $\square$

5.7 Uniqueness Theorem

Let U be a base. Then U* is the unique strong selfcompletion of U. (unique in the sense of $\equiv$).

Proof: By 4.12 *ii*) U* is a strong selfcompletion of U. Now suppose V and W are strong selfcompletions of U. We claim: $V \equiv W$.

By 5.3 *ii*) $V \equiv U^V$, $W \equiv U^W$, hence it is sufficient to show $U^V \equiv U^W$. We treat $U^V = U^W$. To get $\equiv$ one must as usual formalize the argument. Clearly one needs only to prove: $\vdash_{HA}(A^V \leftrightarrow A^W)$. We proceed by induction on A; we treat the case that $A = (B \to C)$:

$$
\begin{array}{lll}
\vdash_{HA} ((B^V \to C^V) \wedge \Box_V(B^V \to C^V)) & \leftrightarrow & (4.6\ vi) \\
\quad ((B^V \to C^V) \wedge \Box_V(B \to C)) & \leftrightarrow & (5.3 \text{ and the Löb Conditions}) \\
\quad ((B^V \to C^V) \wedge \Box_U(B^V \to C^V)) & \leftrightarrow & (IH) \\
\quad ((B^W \to C^W) \wedge \Box_U(B^W \to C^W)) & \leftrightarrow & (\text{by the symmetrical argument}) \\
\quad ((B^W \to C^W) \wedge \Box_W(B^W \to C^W)) & &
\end{array}
$$

□

5.8 Non Uniqueness Theorem

There are different selfcompletions of HA, what is more: for every false Σ-sentence B there is a selfcompletion T of HA s.t. $T \subseteq HA+B$. Moreover if U and V are selfcompletions of HA and $U \neq V$ then there are D, E in L s.t. $\vdash_U D, \nvdash_V D$, $\nvdash_U E$, $\vdash_V E$.

Proof: Let B be a false Σ-sentence. Find using the Fixed Point Theorem for HA a $(T(x))$ such that:

$\vdash_{HA} (T(x) \leftrightarrow (``\exists A \in L\ x = \ulcorner A \to \Box_T A \urcorner" \vee B))$.

Then T is clearly a selfcompletion of HA, because B is false. We have: $B \vdash_{HA} \Box_T \bot$, so $HA+B \supseteq HA+\Box_T\bot \supseteq T$. (Note that we do not get $\geqslant$, unless $\vdash_{HA} \neg B$).

Now suppose U and V are distinct selfcompletions of HA. Suppose e.g. there is a D such that $\vdash_U D$, $\nvdash_V D$. Take $E := (D \to \Box_V D)$. Surely $\vdash_V E$. Suppose $\vdash_U E$, then $\vdash_U \Box_V D$. Hence (by 5.4) $\vdash_{HA} \Box_V D$, so $\vdash_V D$. Contradiction. Conclude: $\nvdash_U E$.

□

5.9 Example

We provide two concrete distinct selfcompletions of HA: take T as in the first half of the proof of 5.8 with $B \equiv \Box_{HA}\Box_{HA}\bot$. Compare T with HA*.

We have: $\vdash_{HA*} \neg\neg\Box_{HA*}\bot$, so $\vdash_{HA*} \neg\neg\Box_{HA}\bot$ (by 4.7 *ii* and the fact that $(\bot)^{HA} = \bot$). Suppose that $\vdash_T \neg\neg\Box_{HA}\bot$, then:
$\Box_{HA}\Box_{HA}\bot \vdash_{HA} \neg\neg\Box_{HA}\bot$. By 2.15 application 1: $\Box_{HA}\Box_{HA}\bot \vdash_{HA} \Box_{HA}\bot$.
Applying Löb's Theorem: $\vdash_{HA}\Box_{HA}\bot$, quod non. Conclude: $\nvdash_T \neg\neg\Box_{HA}\bot$.

5.10 Remark

In the introduction we have seen that any selfcompletion T of PA coincides with PA$+\Box_T\bot$. We have also seen that no selfcompletion of HA is equal to HA$+\Box_T\bot$. The following theorem will dispel the fear that there is a selfcompletion T of HA such that $T = $ HA$+\neg\neg\Box_T\bot$.

5.11 Theorem

Every selfcompletion T of HA is a proper extension of HA$+\neg\neg\Box_T\bot$.

Proof: Clearly $T \supseteq$ HA$+\neg\neg\Box_T\bot$. Suppose $T = $ HA$+\neg\neg\Box_T\bot$. Let M be a classical model of PA$+\Box_T\bot$. Consider the Kripke model:

$$\begin{array}{l} \bullet\ M \\ | \\ \bullet\ \omega \end{array}$$

This is a model of HA (see Smoryński's contribution to [Tr], pp 340, 341.) We have: $\omega \Vdash \neg\neg\Box_T\bot$.

By an easy induction on $A(x)$ in Σ (all free variables shown) we see that:

$$\omega \Vdash A(m) \Leftrightarrow \omega \models A(m) \qquad \text{(for all m).}$$

We have for sentences B of L:

$\vdash_T B \Rightarrow$ (ex hypothesi: $\omega \Vdash T$)

$\omega \Vdash B \Rightarrow$ ($\omega \Vdash T$)

$\omega \Vdash \Box_T B \Rightarrow$ ($\Box_T\ B \in \Sigma$)

$\vdash_T B$.

So $\omega \Vdash B \Leftrightarrow \vdash_T B$.

Moreover: $M \models B \Leftrightarrow \omega \Vdash \neg\neg B$.

So finally:

$$M \models B \Leftrightarrow \vdash_T \neg\neg B.$$

This makes the true sentences of M RE. Contradiction. (The simplest way to see this, is by considering two recursively inseparable RE sets).

5.12 Open problems

i) Is HA* finitely axiomatizable over HA?

ii) Is HA* axiomatizable over HA by a set of formulae of bounded complexity? (The reader may consider his favourite measure of complexity.)

Conjecture: no, to both questions.

5.13 On HA+ECT_0

Our presentation here, leans heavily on that in [Tr] pp 188-205. For Extended Church's Thesis (ECT_0) see [Tr] page 195.

We have not been able to show that HA+ECT_0 is a base:

Open Problem

Is HA+ECT_0 a base?

Yet we can get a conservation result as in 5.4 for selfcompletions of HA+ECT_0.

Suppose T is a selfcompletion of $HA+ECT_0$. Define: A^r, as $(\exists x\ xrA)$, where r is a formalized version of Kleene's relizability (see [Tr] page 189, 3.2.3 A). We have:

i) $\vdash_T A \Rightarrow \vdash_{HA+CP_T} A^r$

The proof of *i*) is along the lines of the proof of 3.2.18, [Tr] page 196, if necessary adapted to a language with just ( )', +, . . In that case we need e.g. that for B(x) in Σ we can find a partial recursive term $\Psi_B(x)$ s.t.:

$\vdash_{HA} B(x) \leftrightarrow \Psi_B(x) r B(x)$

Moreover we must show that CP_T is realized in $HA+CP_T$. We have (in $HA+CP_T$):

$$\dfrac{\dfrac{\dfrac{\dfrac{yrA(x)}{(A(x))^r}}{\Box_T\,(A(x))^r}}{\Box_T\,A(x)}}{\Psi_{\Box_T A}(x) r \,\Box_T\, A(x).}$$

(because $\vdash_T ECT_0$ and $\vdash_{HA+ECT_0} B \leftrightarrow B^r$)

Take $\chi_A(x) := (\Lambda y\ \Psi_{\Box_T A}(\dot{x}))$ i.e. a certain (standard) primitive recursive term such that for all y $\{\chi_A(x)\}\,y \cong \Psi_{\Box_T A}(x)$. Then:

$\vdash_{HA+CP_T}\ \chi_A(x) r (A(x) \to \Box_T\, A(x))$

ii) $\vdash_{HA+CP_T} A^r \Rightarrow \vdash_T A$.

For we have $\vdash_{HA+ECT_0} A \leftrightarrow A^r$.

iii) $\vdash_{HA} B^T \Leftrightarrow \vdash_{HA+CP_T} B$.

By 4.7 *i*), 4.12 *i*), and the fact that $HA+CP_T = HA+WCP_T$, because T is a selfcompletion.

iv) $\vdash_{HA} A^{rT} \Leftrightarrow \vdash_T A$.

Combining *i)*, *ii)*, *iii)*.

We claim: for $A \in \mathcal{A}$: $\vdash_{HA} A^{rT} \rightarrow A$.
Troelstra in [Tr] page 250 gives a class of formulae Γ_0 such that for $A \in \Gamma_0$: $\vdash_{HA} A^r \rightarrow A$. It is easily shown that $\mathcal{A} \subseteq \Gamma_0$. We find for $A \in \mathcal{A}$: $\vdash_{HA} A^r \rightarrow A$, so $\vdash_{HA} A^{rT} \rightarrow A^T$ (HA is a base).
Hence: $\vdash_{HA} A^{rT} \rightarrow A$ $(A \in \mathcal{A})$.

So for $A \in \mathcal{A}$:

$\vdash_T A \Rightarrow$ (*iv)*)

$\vdash_{HA} A^{rT} \Rightarrow$ $(A \in \mathcal{A})$

$\vdash_{HA} A \Rightarrow$

$\vdash_T A$.

$\square$

6 REMARKS ON THE PROVABILITY LOGIC OF HA AND EXTENSIONS

This section is devoted to the provability logic of HA and extensions. First we give certain principles for the provability logic of HA and extensions informally. Then the apparatus necessary to state these principles *as* principles of provability logic is introduced. The last part characterizes PA*. A corollary is de Jongh's Theorem for Intuitionistic Propositional Logic.

The next theorem is folklore:

6.1 <u>Thecrem</u>

Let $HA \leqslant T \leqslant PA$ and $A \in \Sigma$, then :

i) $\vdash_T \neg\neg A \Rightarrow \vdash_T A$

ii) $\vdash_{HA} \Box_T \neg\neg A \rightarrow \Box_T A$

Proof:

i) $\vdash_T \neg\neg A \Rightarrow$

$\vdash_{PA} A \Rightarrow$ $\neg\neg$ translation, $A \in \Sigma$

$\vdash_{HA} \neg\neg A \Rightarrow$ (2.15)

$\vdash_{HA} A \Rightarrow$

$\vdash_T A$

ii) Formalize the proof of *i)*

□

6.2 Theorem

$\vdash_{HA} \Box_{HA}(\neg\neg A \to A) \to \Box_{HA}(A \vee \neg A)$

Proof: This is application 2 of 2.15.

□

6.3 Theorem

Let $A, B_1, \ldots, B_n \in \mathcal{A}$, then:

i) if $\vdash_{HA} (\bigwedge_{i=1}^{n} (B_i \to \Box_{HA} B_i)) \to A$ then $\vdash_{HA} A$.

ii) $\vdash_{HA} (\Box_{HA}(\bigwedge_{i=1}^{n} (B_i \to \Box_{HA} B_i) \to A) \to \Box_{HA} A$.

Proof:

i) Suppose $A, B_1, \ldots, B_n \in \mathcal{A}$ and $\vdash_{HA} (\bigwedge_{i=1}^{n} (B_i \to \Box_{HA} B_i) \to A)$.

We have $\vdash_{HA} \Box_{HA*} B_i \leftrightarrow \Box_{HA} B_i$ (by formalizing theorem 4.13), hence: $\vdash_{HA*} A$. Conclude $\vdash_{HA} A$ (4.13).

ii) Formalize the proof of *i)*.

□

Clearly we could formulate a similar principle for SLP.

6.4 Definitions

i) $\mathcal{P} := \{p_0, p_1, p_2, \dots\}$

ii) $\mathcal{L}_p$ is the closure of $\mathcal{P} \cup \{\top, \bot\}$ under $\wedge, \vee, \rightarrow$.

iii) $\mathcal{L}_{pr}$ is the closure of $\mathcal{P} \cup \{\top, \bot\}$ under $\wedge, \vee, \rightarrow, \Box$.

iv) Σ_{pr} is the closure of $\{\top, \bot\} \cup \{\Box A \mid A \in \mathcal{L}_{pr}\}$ under $\wedge, \vee$.

v) $\mathcal{A}_{pr}$ is the closure of $\{\top, \bot\} \cup \{\Box A \mid A \in \mathcal{L}_{pr}\}$ under $\wedge, \vee$ and the rule: $(A \in \Sigma_{pr}, B \in \mathcal{A}_{pr} \Rightarrow (A \rightarrow B) \in \mathcal{A}_{pr}.)$

vi) Let $f : \mathcal{P} \rightarrow L$. Define: $< \,>^{f,T} : \mathcal{L}_{pr} \rightarrow \mathcal{L}$ as:

- $< \top >^{f,T} := (\underline{0} = \underline{0})$
- $< \bot >^{f,T} := \bot$
- $< p_i >^{f,T} := f(p_i)$
- $< \,>^{f,T}$ commutes with $\wedge, \vee, \rightarrow$
- $< \Box A >^{f,T} := \Box_T < A >^{f,T}$.

vii) Let $\Gamma \subseteq \mathcal{L}$. We will say that $A \in \mathcal{L}_p$ is Γ,T-*valid* or $\models_{\Gamma,T} A$ if: for every $f : \mathcal{P} \rightarrow \Gamma \;\vdash_T < A >^{f,T}$. When $\Gamma = \mathcal{L}$ we will speak simply about T-valid and $\models_T$.

6.5 Theorem

We have the following provability principles:

i) Let $HA \leqslant T \leqslant PA$ and $A \in \Sigma_{pr}$ then: $\models_T \Box \neg\neg A \rightarrow \Box A$

ii) Let $A \in \Sigma_{pr}$, then: $\models_{HA} (\Box(\neg\neg A \rightarrow A) \rightarrow \Box(A \vee \neg A))$.

iii) Let $A, B_1, \dots, B_n \in \mathcal{A}_{pr}$, then: $\models_{HA} \Box(\bigwedge(B_i \rightarrow \Box B_i) \rightarrow A) \rightarrow \Box A$.

Proof: by 6.1, 6.2, 6.3 and the fact that for $A \in \Sigma_{pr}$: $< A >^{f,T} \in \Sigma$, etc.

$\Box$

6.6 Corollary

i) Let $A \in \mathcal{A}_{pr}$ then: $\models_{HA} \Box(\neg\neg \Box\bot \rightarrow A) \rightarrow \Box A$

ii) $\models_{HA} \Box(\neg\neg\Box A \to \Box A) \to \Box\Box A$

Proof:

i) Use 6.5 *iii*) and:

$$\models_{HA} ((\Box\bot \to \bot) \to \Box(\Box\bot \to \bot)) \to$$
$$((\Box\bot \to \bot) \to \Box\bot) \to$$
$$\neg\neg\Box\bot$$

plus the fact that $(\Box\bot \to \bot) \in A_{pr}$.

ii) This is a consequence of *i*).

$\Box$

6.7 Comments

Let $A \in L_p$, $T \subseteq U$. Clearly we have $\models_T A \Rightarrow \models_U A$. Or: the propositional logic of theories is monotone in theories. 6.6 *ii*) shows that this does not extend to L_{pr}, for:

$\not\models_{PA} \Box(\neg\neg\Box\bot \to \Box\bot) \to \Box\Box\bot$.

If it did we would have: $\models_{PA} \Box\Box\bot$.

Professor Löb asked whether this last observation also holds for the $(\to, \Box)$-fragment of L_{pr} i.e. the closure of P under $\to$, $\Box$. The answer is yes. From 6.5 we find e.g.:

$\models_{HA} (\Box(((\Box\Box p_0 \to \Box p_0) \to \Box(\Box\Box p_0 \to \Box p_0)) \to \Box\Box p_0) \to \Box\Box\Box p_0)$.

Hence by Löb's Theorem:

$\models_{HA} (\Box(((\Box\Box p_0 \to \Box p_0) \to \Box\Box p_0) \to \Box\Box p_0) \to \Box\Box\Box p_0)$ (*)

But if $\models_{PA}$ (*), then we would have by Peirce's Law: $\models_{PA} \Box\Box\Box p_0$. Quod non.

We now turn to the characterization of the provability logic of PA*.

6.8 Definitions

i) Σ^c is the set of closed Σ-sentences

ii) $G(\Sigma^c)$ is the following theory in L_{pr}:

- Classical Propositional Logic (in Natural Deduction formulation)
- $\vdash A \Rightarrow \vdash \Box A$ (Gödel's Rule)
- $\vdash \Box A \rightarrow \Box \Box A$
- $\vdash \Box (A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
- $\vdash \Box (\Box A \rightarrow A) \rightarrow \Box A$
- $\vdash p_i \rightarrow \Box p_i$ (for all $p_i \in P$)

Thus $G(\Sigma^c)$ is $G + (p_i \rightarrow \Box p_i)$ (all $p_i \in P$). Note that $G(\Sigma^c)$ does not satisfy the usual substitution property: $\vdash A(p_i) \Rightarrow \vdash A(B)$, for $\vdash B \rightarrow \Box B$ does not hold in general.

iii) H is the following theory in L_{pr}:

- Intuitionistic Propositional Logic in Natural Deduction formulation.
- $\vdash A \rightarrow \Box A$ (CP)
- $\vdash \Box (A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
- $\vdash (\Box A \rightarrow A) \rightarrow A$ (SLP)
- $\vdash \Box (A \rightarrow B) \rightarrow (A \vee (A \rightarrow B))$ (the Propositional Trace Principle)

iv) $(\ )^{\Box} : L_{pr} \rightarrow L_{pr}$ is the following translation:

- $(p_i)^{\Box} := p_i$
- $(\ )^{\Box}$ commutes with $\top, \bot, \wedge, \vee, \Box$.
- $(A \rightarrow B)^{\Box} := ((A^{\Box} \rightarrow B^{\Box}) \wedge \Box(A^{\Box} \rightarrow B^{\Box}))$

v) A Kripke Model K is a structure $< W, <, f >$ where:

- W is a set

- $<$ is a transitive binary relation on W, such that $w < w' < w \Rightarrow w = w'$.
- $f: W \to$ the powerset of $\mathcal{P}$, such that $w < w' \Rightarrow f(w) \subseteq f(w')$.

The monotonicity condition on f is usual for Kripke Models for Intuitionistic Propositional Logic, unusual for Kripke Models for Modal Propositional Logic. We have it in this paper because we are as far as Modal Logic is concerned just interested in Logics with the principle $(p_i \to \Box p_i)$ (for all $p_i \in \mathcal{P}$).

vi) Let $K = < W, <, f >$ be a Kripke Model. We define $\models_K$, *the satisfaction relation on K,* as follows: for $w \in W$:

- $w \models_K \top$, $w \not\models_K \bot$
- $w \models_K p_i :\Leftrightarrow p_i \in f(w)$
- $w \models_K (A \wedge B) :\Leftrightarrow (w \models A$ and $w \models B)$,
  and similarly for $\vee$, $\to$.
- $w \models_K \Box A :\Leftrightarrow$ for all $w' > w$ $w' \models_K A$.

Clearly $\models$ is a satisfaction relation for *classical* modal logic, with a little modification.

vii) Let $K = <W, <, f>$, $w \in W$. We define $\Vdash_K$ *the forcing relation on K* as follows: for $w \in W$:

- $w \Vdash_K \top$, $w \nVdash_K \bot$
- $w \Vdash_K p_i :\Leftrightarrow p_i \in f(w)$
- $w \Vdash_K (A \wedge B) :\Leftrightarrow (w \Vdash_K A$ and $w \Vdash_K B)$,
  and similarly for $\vee$.
- $w \Vdash_K (A \to B) :\Leftrightarrow$ (for every $w' \geq w$ $(w' \Vdash_K A \Rightarrow w' \Vdash_K B))$
- $w \Vdash_K \Box A :\Leftrightarrow$ (for every $w' > w$ $w' \Vdash_K A)$.

Clearly $\Vdash$ is a forcing relation for a kind of intuitionistic modal logic. When we restrict it to L_p we get a forcing relation for intuitionistic propositional logic.

viii) Let $\mathcal{K}$ be a class of Kripke Models. Define:

$\models^{\mathcal{K}} A :\Leftrightarrow$ for all $K = \langle W, <, f \rangle$ in $\mathcal{K}$, for all $w \in W : w \models_K A$

$\vdash^{\mathcal{K}} A :\Leftrightarrow$ for all $K = \langle W, <, f \rangle$ in $\mathcal{K}$, for all $w \in W : w \Vdash_K A$.

ix) $\mathcal{FI}$ is the class of all finite irreflexive Kripke Models.

6.9 <u>Theorem</u>

$\vdash_H A \Leftrightarrow \models_{PA*} A$

Proof:

"$\Rightarrow$" Induction on the length of the proof using 4.11.

"$\Leftarrow$" We give the proof modulo a number of subsequent lemma's:

$\models_{PA*} A \Rightarrow$	(specialization)
$\models_{\Sigma^c, PA*} A \Rightarrow$	(Lemma 6.10)
$\models_{\Sigma^c, PA} (A)^{\Box} \Rightarrow$	(Lemma 6.11)
$\vdash_{G(\Sigma^c)} (A)^{\Box} \Rightarrow$	(Lemma 6.12)
$\models^{\mathcal{FI}} (A)^{\Box} \Rightarrow$	(Lemma 6.13)
$\Vdash^{\mathcal{FI}} A \Rightarrow$	(Lemma 6.14)
$\vdash_H A$	

$\Box$

6.10 <u>Lemma</u>

i) Let $f : P \to \Sigma^c$. We have:

$\vdash_{HA} (\langle A \rangle^{f,T*})^T \leftrightarrow \langle (A)^{\Box} \rangle^{f,T}$

ii) $\models_{\Sigma^c, T*} A \Leftrightarrow \models_{\Sigma^c, T} (A)^{\Box}$.

Proof:

i) Induction on A, for example:

- $(<p_i>^{f,T*})^T = (f(p_i))^T$, $(<(p_i)^{\Box}>^{f,T}) = f(p_i)$, and by the fact that $f(p_i) \in \Sigma$ and 4.6 *iii)*: $\vdash_{HA} ((f(p_i))^T \leftrightarrow f(p_i))$.

- Suppose $A = \Box B$. We have:

$$(<\Box B>^{f,T*})^T = (\Box_{T*} <B>^{f,T*})^T \quad \text{and:}$$

$$<(\Box B)^{\Box}>^{f,T} = (\Box_T(<(B)^{\Box}>^{f,T})).$$

Moreover:

$$\begin{array}{lll} \vdash_{HA} (\Box_{T*} <B>^{f,T*})^T & \leftrightarrow & (4.6\ \textit{iii}) \\ (\Box_{T*} <B>^{f,T*}) & \leftrightarrow & (4.7\ \textit{ii}) \\ (\Box_T\ (<B>^{f,T*})^T) & \leftrightarrow & \text{(Induction Hypothesis)} \\ (\Box_T\ (<(B)^{\Box}>^{f,T})) & & \end{array}$$

□

6.11 <u>Lemma</u>

$\vdash_{G(\Sigma^c)} A \Leftrightarrow \models_{\Sigma^c, PA} A$

Proof: This is a simple corollary of *the proof* of Solovay's Completeness Theorem. See [Vi].

□

6.12 <u>Lemma</u>

$\vdash_{G(\Sigma^c)} A \Leftrightarrow \models^{FI} A$

Proof: This is a simple corollary of the usual Kripke Model Completeness Theorem for G. (see [Bo] , [So]).

□

6.13 <u>Lemma</u>

Let $K = <W, <, f>$ be a Kripke Model, $w \in W$. We have: $w \Vdash_K A \Leftrightarrow w \models_K A^{\Box}$

Proof: a simple induction on A.

□

6.14 <u>Lemma: Completeness Theorem for H</u>

$\vdash_H A \Leftrightarrow \Vdash^{FI} A$.

Proof:

"$\Rightarrow$" routine.

"$\Leftarrow$"

Step 1 We define $K_H = < W_H, <_H, f_H >$ as follows:

call a set $\Gamma \subseteq L_{pr}$ H-saturated if Γ is closed under $\vdash_H$ and $((A \vee B) \in \Gamma \Rightarrow A \in \Gamma$ or $B \in \Gamma)$.

- $W_H := \{\Gamma | \Gamma$ is H-saturated$\}$.
- $\Gamma <_H \Delta :\Leftrightarrow (\{A \mid (\Box A) \in \Gamma\} \subseteq \Delta)$
- $f_H(\Gamma) := (\Gamma \cap P)$

Claim: $\Gamma \Vdash_{K_H} A \Leftrightarrow \Gamma \vdash_H A$.

The proof is by induction on A. Let us just treat the case of $\rightarrow$:

suppose $A = (B \rightarrow C)$.

"$\Leftarrow$" Suppose $(B \rightarrow C) \in \Gamma$, $\Delta \geqslant \Gamma$, $\Delta \Vdash_K B$. By the Induction Hypothesis $\Delta \vdash_H B$. Clearly because of CP: $\Gamma \subseteq \Delta$, so $(B \rightarrow C) \in \Delta$, hence $\Delta \vdash_H C$. By the Induction Hypothesis: $\Delta \Vdash_{K_H} C$.

"$\Rightarrow$" Suppose $(B \rightarrow C) \notin \Gamma$.

- If $(\Box(B \rightarrow C)) \in \Gamma$ then by PTP: $(B \vee (B \rightarrow C)) \in \Gamma$. Γ is saturated, hence $B \in \Gamma$ or $(B \rightarrow C) \in \Gamma$. By our assumption $(B \rightarrow C) \notin \Gamma$, so $B \in \Gamma$. By IH: $\Gamma \Vdash_{K_H} B$ and $\Gamma \nVdash_{K_H} C$. Hence $\Gamma \nVdash_{K_H} B \rightarrow C$.

- If $\Box(B \to C) \notin \Gamma$, let $\Delta_0 := \{D \mid \Box D \in \Gamma\}$. A moment's reflexion shows that: $\Delta_0, B \nvdash_H C$. Using a standard argument we can find a saturated $\Delta \supseteq \Delta_0$ such that $B \in \Delta$ and $C \notin \Delta$. We have $\Delta \geqslant \Gamma$ and by IH: $\Delta \Vdash_{K_H} B$, $\Delta \nVdash_{K_H} C$.

Step 2 Let $K = \langle W, <, f \rangle$ be any Kripke Model such that $\Vdash_K A$ for all $w \in W$. Sub(A) is the set of subformulae of A. Let:

$w^A := \{B \in \mathrm{Sub}(A) \mid w \Vdash_K B\}$.

We define $K^A = \langle W^A, <^A, f^A \rangle$ as follows:

- $W^A := \{w^A \mid w \in W\}$
- for $a, b \in W^A$:

  $a <^A b :\Leftrightarrow (a \cup \{C \mid (\Box C) \in a\} \subseteq b$ and $a \neq b)$.
- $f^A(a) := a \cap P$.

One easily verifies that K^A is a finite irreflexive Kripke Model.

Claim: for every $B \in \mathrm{Sub}(A)$: $(w \Vdash_K B \Leftrightarrow w^A \Vdash_{K^A} B)$.

The proof is by induction on B. We treat the case $B = (\Box C)$. Suppose $(\Box C) \in \mathrm{Sub}(A)$.

"$\Rightarrow$" Suppose: $w \Vdash_K \Box C$ and $a >_{K^A} w^A$. $a = v^A$ for some v. We have: $v^A >_{K^A} w^A$ and $(\Box C) \in w^A$, hence $C \in v^A$. Conclude $v \Vdash_K C$. By the IH: $a = v^A \Vdash_{K^A} C$.

"$\Leftarrow$" Suppose: $w^A \Vdash_{K^A} C$ and $v > w$.

We have two cases:

Case I $v^A \neq w^A$. Then clearly $v^A >^A w^A$, hence $v^A \Vdash_{K^A} C$ and by IH: $v \Vdash_K C$.

Case II $v^A = w^A$. Suppose $w \nVdash_K \Box C$.

We claim: $w \Vdash_K \Box C \to C$. For consider $x \geqslant w$.

If $x^A \neq w^A$, we have $x \Vdash_K C$ by case I. If $x^A = w^A$, we have $(\Box C) \notin x^A$, hence: $x \nVdash_K \Box C$.

By SLP $w \Vdash_K C$, hence $w \Vdash_K \Box C$. (CP). Contradiction.

We conclude: $w \Vdash_K \Box C$.

Step 3 Suppose $\nvdash_H A$. Then there is a saturated Δ such that $\Delta \nvdash_H A$. (In fact we can take $\Delta = \emptyset$, because H has the disjunction property). By step 1: $\Delta \nVdash_{K_H} A$. By step 2: $\Delta^A \nVdash_{K_H^A} A$, and K_H^A is a finite irreflexive Kripke Model.

□

6.15 <u>Corollary: De Jongh's Theorem for Σ^c-substitution instances</u>

Let $A \in L_p$, then:

$\vdash_{IPL} A \Leftrightarrow \models_{\Sigma^c, HA} A$, where IPL is Intuitionistic Propositional Logic.

Proof:

"⇒" trivial.

"⇐" Suppose $A \in L_p$, we have:

$\models_{\Sigma^c, HA} A \Rightarrow$

$\models_{\Sigma^c, PA*} A \Rightarrow$ (6.10 - 13)

$\Vdash^{FI} A \Rightarrow$ (Completeness Theorem for IPL)

$\vdash_{IPL} A$ □

7 ON KLS

In this section we adapt Beeson's proof of the independence of KLS from HA, to the framework of selfcompletions. As a result we find the independece of KLS from any Σ-sound base.

7.1 Definitions

We define the necessary concepts in L. Let Txyz be Kleene's T-predicate, U the result extracting function.

i) y total: $\Leftrightarrow \forall u \exists v Tyuv$

ii) $y \sim z$: $\Leftrightarrow \forall u \forall v \forall w((Tyuv \wedge Tzuw) \rightarrow Uv = Uw)$

iii) z is an effective operation: $\Leftrightarrow$

$E_2(z)$: $\Leftrightarrow$

$(\forall y(y \text{ total} \rightarrow \exists u Tzyu) \wedge \forall y \forall w \forall u \forall v((y \text{ total} \wedge w \text{ total} \wedge$

$y \sim w \wedge Tzyu \wedge Tzwv) \rightarrow Uu = Uv)))$

iv) n is a modulus for z at y: $\Leftrightarrow$

$Mod(n,z,y)$: $\Leftrightarrow$

$\forall u \forall v \forall w((w \text{ total} \wedge Tzwu \wedge Tzyv \wedge \forall x \leqslant n\ \forall b \forall c((Twxb \wedge Tyxc) \rightarrow$

$Ub = Uc)) \rightarrow Uu = Uv)$

v) KLS: $\Leftrightarrow \forall z \forall y((E_2(z) \wedge y \text{ total}) \rightarrow \exists u\ Mod(u,z,y))$

7.2 Theorem

Let U be a Σ-sound base, then $\nvdash_U$ KLS. In particular:

- $\nvdash_{HA}$ KLS
- $\nvdash_{HA+TI(<)}$ KLS, where < is a wellfounded RE-relation
- $\nvdash_{HA+RP_{HA}}$ KLS
- $\nvdash_{HA+DNS}$ KLS
- $\nvdash_{HA+\neg M_{PR}}$ KLS

(Moreover inspection of the proof combined with our remarks in 5.13 shows:

- $\nvdash_{HA+ECT_0}$ KLS)

For Beeson's original proof, see [Be] or [Tr] pp 267-273. Our result that $\nvdash_{HA+\neg M_{PR}}$ KLS answers a question of Kreisel, which was not answered by FP-realizability.

We prove 7.2 from a lemma:

7.3 Lemma

Let T be any RE theory, $T \geqslant HA$, s.t. $\vdash_T CP_T$. We have:

$\vdash_T KLS \rightarrow \Box_T \bot$.

Proof of 7.2 from 7.3

Let U be a Σ-sound base. Suppose $\vdash_U$ KLS. Then $\vdash_{U*}$ KLS. Hence by the lemma: $\vdash_{U*} \Box_{U*}\bot$. $(\Box_{U*} \bot) \in \Sigma$, so $\vdash_U \Box_{U*} \bot$. By Σ-soundness $\vdash_{U*} \bot$. $\bot \in \Sigma$, so $\vdash_U \bot$. By Σ-soundness: $\bot$.

$\Box$

Proof of 7.3

Let T be a theory such that $\vdash_T CP_T$.

We will produce an index e such that $\vdash_T E_2(\underline{e})$ and $\vdash_T (\exists n\, Mod(n,\underline{e}, \underline{\Lambda x.0}) \rightarrow \Box_T \bot)$, where $(\Lambda x.0)$ is a canonical index of the identically zero function.

First we give the computation of $\{e\}y$. Let s_y be the smallest number s such that:

$$(\{y\}s \downarrow \wedge((\{y\}s = 0 \wedge Proof_T(s,\ulcorner \dot{y} \text{ total}\urcorner)) \vee \{y\}s \neq 0)).$$

Here $\{y\}s\downarrow$ abbreviates $(\exists uTysu)$. To compute $\{e\}y$ we first compute

s_y by first considering $\{y\}0$, in case $\{y\}0 \cong 0$, checking whether $\mathrm{Proof}_T(0,\ulcorner\dot{y}\text{ total}\urcorner)$ etc. Of course this procedure need not terminate. When we have found s_y there are two cases:

i) $\{y\}s_y = 0$, then put $\{e\}y :\cong 0$

ii) $\{y\}s_y \neq 0$. Check whether:

$((\forall w < s_y\ \forall p < s_y\ \mathrm{Proof}_T(p,\ulcorner\dot{w}\text{ total}\urcorner)) \to \exists k < \{y\}s_y\ T(w,s_y,k))$.

If so put $\{e\}y :\cong 1$, else put $\{e\}y \cong 0$.

We have:

Claim 1 $\vdash_T$ y total $\to \{\underline{e}\}y\downarrow$

<u>Reason in T</u>: Suppose y total, then by CP_T: $\exists p\ \mathrm{Proof}_T\ (p,\ulcorner\dot{y}\text{ total}\urcorner)$. Clearly s_y is bounded by p. y is total, hence the computation of s_y will terminate. Moreover whenever the computation of s_y terminates, we have $\{e\}y\downarrow$.

Claim 2 $\vdash_T ((y \text{ total} \wedge y' \text{ total} \wedge y \sim y') \to \{\underline{e}\}y \cong \{\underline{e}\}y')$

<u>Reason in T</u>: Suppose y total, y' total, $y \sim y'$. We know from claim 1: $\{e\}y\downarrow$, $\{e\}y'\downarrow$. Suppose $\{e\}y = 0$. $\{e\}y' = 1$. Let $s := s_y$, $s' := s_{y'}$.

Assume $\{y\}s = 0$, so $\mathrm{Proof}_T(s,\ulcorner\dot{y}\text{ total}\urcorner)$. We have $y < s$, because in a standard Gödel numbering the proof is longer than the theorem. Moreover $s < s'$ because $y \sim y'$ and $\{y'\}s' \neq 0$. Hence $\exists k < \{y'\}s' = \{y\}s'\ T(y,s',k)$. But this is impossible, for in a standard Gödel numbering the computation k must be longer than the result $\{y\}s'$.

Hence $\{y\}s \neq 0$. We have $s = s'$, $\{y\}s = \{y\}s'$. Conclude $\{e\}y = \{e\}y'$. Contradiction.

Hence, because = is decidable: $\{e\}y = \{e\}y'$.

Taking claim 1 and 2 together we have $\vdash_T E_2(\underline{e})$.

<u>Reason in T</u>: Suppose KLS, then $\exists n\, \mathrm{Mod}(n,e,(\Lambda x.0))$ and hence by CP_T: $\exists n\, \Box_T\, \mathrm{Mod}(n,e,(\Lambda x.0))$. We have:

$(\forall w < n+1\ \forall p < n+1(\mathrm{Proof}_T(p,\ulcorner \dot{w} \text{ total}\urcorner) \to \Box_T(\exists k\, Tw\,(n+1)k)))$.

Consider $w < n+1$, $p < n+1$. We have

$$\begin{aligned}\mathrm{Proof}_T(p,\ulcorner \dot{w} \text{ total}\urcorner) &\to \Box_T\ \exists k\ Tw(n+1)k \\ &\to \Box_T(\mathrm{Proof}_T(p,\ulcorner \dot{w} \text{ total}\urcorner) \to \exists k\, Tw\,(n+1)k)\end{aligned}$$

and

$$\begin{aligned}\neg\, \mathrm{Proof}_T(p,\ulcorner \dot{w} \text{ total}\urcorner) &\to \Box_T\ \neg\, \mathrm{Proof}_T(p,\ulcorner \dot{w} \text{ total}\urcorner) \\ &\to \Box_T(\mathrm{Proof}_T(p,\ulcorner \dot{w} \text{ total}\urcorner) \to \exists k\, Tw\,(n+1)k)\end{aligned}$$

Hence by the decidability of Proof_T:

$(\forall w < n+1\ \forall p < n+1\ \ \Box_T(\mathrm{Proof}_T(p,\ulcorner \dot{w} \text{ total}\urcorner) \to (\exists k\, Tw\,(n+1)k)))$

By a familiar induction argument:

$\Box_T(\forall w < n+1\ \forall p < n+1(\mathrm{Proof}_T(p,\ulcorner \dot{w} \text{ total}\urcorner) \to (\exists k\, Tw\,(n+1)k)))$

<u>Reason in $\Box_T$</u>: We have

$\forall w < n+1\ \ \forall p < n+1(\mathrm{Proof}_T(p,\ulcorner \dot{w} \text{ total}\urcorner) \to (\exists k\, Tw\,(n+1)k)))$.

Hence clearly there is an m such that:

$$\forall w < n+1\ \forall p < n+1(\mathrm{Proof}_T(p,\ulcorner \dot{w} \text{ total}\urcorner) \to \exists k < m(Tw(n+1)k). \qquad (*)$$

Let f be a canonical index such that:

$$\{f\}x :\cong \begin{cases} 0 \text{ if } x \neq n+1 \\ m \text{ if } x = n+1 \end{cases}$$

We have $\mathrm{Mod}(n,e,\ \Lambda x.0)$, hence: $\{e\}f \cong 0$.

On the other hand: if $\{f\}s_f$ were 0, we had $f < s_f$ and because f is a canonical index: $n+1 < f$. Hence $n+1 < s_f$. This is impossible. So $\{f\}s_f = m$ and $s_f = n+1$. But then by $(*)$: $\{e\}f \cong 1$. Contradiction.

Hence we conclude in T: $\Box_T \bot$.

Hence $\vdash_T$ KLS $\rightarrow \Box_T \bot$.

$\Box$

7.4 Remark

Clearly we cannot establish the consistency of HA+ $\neg$ KLS with our methods, for HA+ $\neg\neg$ KLS is a Σ-sound base. Thus:

Open problem: do we have $\nvdash_{HA} \neg\neg$ KLS ?

7.5 Remark

Of course other independence proofs using CP and bases are possible. E.g. we can prove that for Σ-sound bases U: $\nvdash_U M_{PR}$ and $\nvdash_U$ MS, where MS is Myhill-Shepherdson's Theorem. The proof of the last fact is a simple adaptation of the proof in [Be].

REFERENCES

[Be] BEESON M., *The non-derivability in intuitionistic formal systems of theorems on the continuity of effective operations*, JSL 40 (1975), 321-346.

[Bo] BOOLOS G., *The unprovability of consistency*, Cambridge University Press, Cambridge, 1979.

[Fe 1] FEFERMAN S., *Arithmetization of metamathematics in a general setting*, Fundamenta Mathematicae 49 (1960), 35-92.

[Fe 2] __________, *Transfinite recursive progressions of axiomatic theories* JSL 27 (1962), 259-316.

[Fr] FRIEDMAN H., *Classically and Intuitionistically Provably Recursive Functions*, in: Higher Set Theory, ed. G.H.Müller and D.S. Scott, Springer Lecture Notes in Mathematics 669, Berlin, 1977, pp 21-27.

[Sm] SMORYŃSKI C., *The incompleteness theorems*, in: Handbook of Mathematical Logic, ed. by J. Barwise, North Holland, Amsterdam, 1977.

[So] SOLOVAY R.M., *Provability interpretations of modal logic*, Israel Journal of Mathematics, vol.25, 1976, 287-304.

[Tr] TROELSTRA A.S., *Metamathematical investigation of intuitionistic arithmetic and analysis*, Springer Lecture Notes in Mathematics 344, Berlin, 1973.

[Vi] VISSER A., A *propositional logic with explicit fixed points*, University of Utrecht, Preprint Series nr. 141, 1980. To appear: Studia Logica.

PART 5

On the provability logic of any recursively enumerable extension of Peano Arithmetic

or

Another look at Solovay's proof

Albert Visser

1) In [So] Solovay shows that for any recursively axiomatizable theory T such that T is Σ_2^0-sound and P is relatively interpretable in T, we have: a modal formula χ is T-valid iff χ is a theorem of G.

Actually a rather lazy inspection of his proof yields a characterization of the provability logic of any RE theory T such that P is relatively interpretable in T.

In this paper we will follow [So] rather closely. Not only do we adopt most of Solovay's notations, but we will also refer freely to subproofs and observations in [So]. Hence the reader is advised to consult [So].

2) We will state our main result for RE extensions T of P. The case of relative interpretability of P in T is an easy consequence.

Define: $\Box^0\bot := \bot$, $\Box^{n+1}\bot := \Box(\Box^n\bot)$, $\Box^\omega\bot = \top$. $\Box_T^a\bot$ for $0 \leqslant a \leqslant \omega$ is similarly defined. Here $\Box_T$ stands for the standard arithmetical provability predicate for T.

Let $G_a := G + \Box^a\bot$, $(0 \leqslant a \leqslant \omega)$.

Theorem

Let T be an RE theory extending P. Then there is an a, $0 \leqslant a \leqslant \omega$ such that: a modal formula χ is T-valid iff χ is a theorem of G_a.

3) Before proving our theorem we first turn to the avowedly unexciting Kripke frame completeness theorems for the G_a.
Let $F = \langle x; \prec \rangle$ be a finite, irreflexive, transitive Kripke frame. (Remember that Solovay's trees grow downwards).
Define the maximal downward pathlength $m: X \to \omega$ as:

$$m(x): = \sup\{m(x') \mid x' \prec x\} + 1.$$

Clearly for bottom nodes x: $m(x) = 1$.
Let $m(F): = \max\{m(x) \mid x \in X\}$.

Theorem

Let χ be a modal formula, then:
$G_a \vdash \chi$ iff χ is valid in every finite, irreflexive transitive Kripke frame F, with $m(F) \leqslant a$.
Proof: The only if side is routine. For the if side: suppose $G_a \nvdash \chi$. Then $G \nvdash \Box^a \bot \to \chi$. By the Kripke frame completeness theorem for G (see [So] 3.6, 3.7) there is a Kripke model $K = \langle X; \succ, e \rangle$ such that for some $x_0 \in X$ $e((\Box^a \bot \to \chi), x_0) = 0$. Without loss of generality we may assume that x_0 is the top node of $\langle X; \succ \rangle$. By an easy induction one may show: for all $x \in X$, all b with $0 \leqslant b \leqslant \omega$: $e(\Box^b \bot, x) = 1$ iff $m(x) \leqslant b$. Clearly $e(\Box^a \bot, x_0) = 1$, hence $m(x_0) \leqslant a$. So $m(F) = m(x_0) \leqslant a$. Moreover $e(\chi, x_0) = 0$. $\Box$

4) We turn to the proof of the Theorem of 2). Let a be the smallest b, $0 \leqslant b \leqslant \omega$ such that $\Box_T^b \bot$ is a theorem of T. The case that $a = 0$ is trivial, hence we will assume $a \neq 0$.

The if side of the theorem is routine.

For the only if side, assume $G_a \nvdash \chi$. Then by 3) there is a finite, transitive, irreflexive Kripke model $K = \langle X; \succ, e \rangle$ with top node x_0 such that $m(x_0) \leqslant a$ and $e(\chi, x_0) = 0$. We identify x_0 with 1, X with $\{1, \ldots, n\}$.

Define ℓ and h for K as in [So] 4.3, substituting T for P.

We have:

i) $0 \leqslant \ell \leqslant n$ and $T \vdash 0 \leqslant \ell \leqslant n$

ii) if $0 \leqslant i \leqslant n$ and $j \in S_i$ then:

- if $\ell = i$ then $T + \ell = j$ is consistent
- $T \vdash (\ell = i \rightarrow \neg \Box_T\, \ell \neq j)$

iii) if $0 < i \leqslant n$ and $j \notin S_i$ then:

- if $\ell = i$ then $T \vdash \ell \neq j$
- $T \vdash (\ell = i \rightarrow \Box_T\, \ell \neq j)$

iv) if $0 < i \leqslant n$ then:

- if $\ell = i$ then $T \vdash \ell \in S_i$
- $T \vdash \ell = i \rightarrow \Box_T\, \ell \in S_i$

v) if $0 < i \leqslant n$ then:

- if $\ell = i$ then $\vdash_T \Box_T^{m(i)-1} \bot$
- $\vdash_T \ell = i \rightarrow \Box_T^{m(i)} \bot$

Proof of $i)$-$v)$: The proofs of i)-iv) are simple consequences of the considerations in [So] 4.4.

The proof of $v)$ is by induction on m(i).

<u>m(i) = 1</u>: We have $S_i = \emptyset$, $T \vdash (S_i = \emptyset)$, $T \vdash \dot{\Box}_T(S_i = \emptyset)$, hence by $iv)$:

- if $\ell = i$ then $T \vdash \perp$.

- $T \vdash \ell = i \to \Box_T \perp$.

<u>m(i) > 1</u>: Suppose $\ell = i$, then by $iv)$:

$T \vdash \ell \in S_i$ i.e.

$T \vdash \bigvee_{j \in S_i} \ell = j$. By Induction Hypothesis:

$T \vdash \bigvee_{j \in S_i} \Box^{m(j)} \perp$. Hence $T \vdash \Box_T^{\max\{m(j) \mid j \in S_i\}} \perp$ i.e.

$T \vdash \Box_T^{m(i)-1} \perp$. The second case is by formalizing this argument. □

Suppose $\ell \neq 0$. Then $\ell = i$, $0 < i \leqslant n$.

By $v)$: $\vdash_T \Box_T^{m(i)-1} \perp$. But $m(i)-1 < m(i) \leqslant m(1) \leqslant a$ and a was the least b with $0 \leqslant b \leqslant \omega$ s.t. $\vdash_T \Box^a \perp$. Contradiction.

Hence $\ell = 0$. By $ii)$: $T + \ell = 1$ is consistent.

Let $(\ )^*$ be as in [So] 4.7. We have $\nvdash_T (\chi)^*$, for the obvious adaptation of [So] 4.7 just uses $i)$-$iv)$ and to repeat [So] 4.8 we just need that $T + \ell = 1$ is consistent. □

5) We have shown that the provability logics of RE extensions of P are among the $G_a (0 \leqslant a \leqslant \omega)$. Conversely for any G_a there is an RE extension T of P such that G_a is the provability logic of T.

Proof: Take $T := P + \Box_P^a \bot$. Clearly $T \vdash \Box_P \Box_P^a \bot$ hence $T \vdash$ "P = T". We conclude $T \vdash \Box_T^b \bot \leftrightarrow \Box_P^b \bot$ for any b with $0 \leqslant b \leqslant \omega$. Hence $T \vdash \Box_T^a \bot$, but not (in case $a \neq 0, \omega$) $T \vdash \Box_T^{a-1} \bot$ for otherwise $P \vdash \Box_P^a \bot \rightarrow \Box_P^{a-1} \bot$ and by Löb's Theorem $P \vdash \Box_P^{a-1} \bot$. Quod non. The cases $a = 0$, $a = \omega$ are trivial. $\Box$

Reference

[So] SOLOVAY R.M., *Provability interpretations of modal logic*, Israel Journal of Mathematics, vol 25, 1976 pp 287-304.

PART 6 EPILOGUE

EPILOGUE

Remarks on the Provability Logic of HA and extensions

In the epilogue we will make full use of the notations of part 4, section 6 and of the introductory part. Define further:

$L(T) := \{A \in \mathcal{L}_{pr} \mid \models_T A\}$ (T is an RE theory extending HA)

$L(*) := \{A \in \mathcal{L}_{pr} \mid \text{for all RE } T \text{ extending HA} \models_T A\}$;

G^I is defined as G, only with intuitionistic instead of classical logic.

We are interested in questions like the following (all open):

- What is the provability logic of HA?
- What are the provability logics of interesting extensions of HA such as $HA+ECT_0$, $HA+M_{PR}$, HA+KLS, HA+IP etc.?
- What is $L(*)$? Conjecture: $L(*) = G^I$.
- Is there an RE T extending HA such that $L(*) = L(T)$? One can show: $T \not\leqslant PA$. (Any $T \leqslant PA$ satisfies $\Box\neg\neg\Box A \rightarrow \Box\Box A$ and e.g. HA* doesn't.)
- What are the possible complexities (our favorite measure is many-one reducibility) of the L(T)'s? L(PA) is (primitive) recursively decidable; the L(T)'s can be at most complete Π^0_2.

The questions above are good questions, not only because of their intrinsic interest, but also because they promise to be technically fruitful. In particular the first two may be hard.

1 On naively applying Solovay's method

G, of course, is not sound for HA, but one might hope to prove: $L(HA) \subseteq G$ (here we confuse conveniently theories with the sets of theorems provable in those theories), by repeating the completeness side of Solovay's proof for HA.

Consider A in L_{pr}, such that $G \nvdash A$. There is a tail model K and a tail element m with d(m) = d(A). We would like to prove: $HA \vdash (<A \wedge \Box A>_{K,HA} \leftrightarrow \Box_{HA}^{d(A)}\bot)$, in order to get completeness. Clearly for completeness: $HA + \Box_{HA}^{d(A)+1}\bot \vdash (<A \wedge \Box A>_{K,HA} \leftrightarrow \Box_{HA}^{d(A)}\bot)$ is sufficient. When we try to repeat the arguments of part 0 2.2, it turns out that the only non intuitionistic step is the assumption that $HA + \Box_{HA}^{d(A)+1}\bot \vdash$ "lim h exists". Clearly $HA + \Box_{HA}^{d(A)+1}\bot \vdash \exists x\ hx \succcurlyeq m$, for some $m \neq 0$, hence we need only the existence of the limit in a given finite upper part of $\prec$. So the relevant natural missing principle is given by the following scheme:

$(*)_{e,k}$: (" $\underline{k}$ codes a finite sequence of ordered pairs that determines a partial irreflexive ordering $\prec$"

$\wedge\ (\forall x\ \forall y\quad x \prec y \rightarrow \exists u\ \exists v\ T\underline{e}xu \wedge T\underline{e}yv \wedge Uu \preccurlyeq Uv) \rightarrow$

"$\lim\{\underline{e}\}$ exists").

Let:

(**): for Σ_1^0-*sentences*: $A \vee \neg A$.

We show:

1.1 <u>Theorem</u>

(*) and (**) are interderivable over HA.

Proof:

"(*) $\Rightarrow$ (**)".

Consider $A \equiv \exists x\ A_0(x)$, $A_0(x)$ primitive recursive. Reason in HA+ (*)

Define:

$$\{e\}x :\cong \begin{cases} 0 \text{ if } \forall y \leqslant x\ \neg\ A_0(x) \\ 1 \text{ if } \exists y \leqslant x\ A_0(x) \end{cases}$$

Then $\{e\}$ is total and monotonic in $\prec$ with $(0 \prec 1)$. Hence $\lim\{e\}$ exists. If $\lim\{e\} = 0$ we have $\neg A$, if $\lim\{e\} = 1$: A.

"$(**) \Rightarrow (*)$"

Argue in HA+$(**)$:

Assume the hypothesis of $(*)_{e,k}$. Let the domain of $\prec$ given by k be $\{1,\ldots,N\}$. We have by decidability of Σ_1^0-sentences and propositional logic that one of:

$\bigwedge\{\pm\exists x\{e\}x \cong i \mid i \in \{1,\ldots,N\}\}$ is true. Pick any such possibility. If it is inconsistent with the totality and monotonicity of $\{e\}$, we have a contradiction, hence $\lim\{e\}$ exists. Otherwise there is a j such that $\exists x\{e\}x \cong j$ and $\bigwedge\{\neg\exists x\{e\}x \cong s \mid s \succ j\}$. Conclude $\lim\{e\} = j$, hence $\lim\{e\}$ exists.

□

The theorem indicates that for a naive extension of the completeness part of Solovay's Theorem, one must consider extensions of HA+$(**)$. We think these are hardly interesting.

2 Solovay's Theorem and the Completeness Principle.

We can execute something like 1's plan for theories $T \geqslant$ HA, with $T \vdash CP_T$ (see part 4, p.2). Hence suppose $T \vdash CP_T$.

Let K be a *monotonic* tail model. Let $m \Vdash_K A$ be defined as in part 4, 6.8.

Define:

$h(0) := 0$

$$h(k+1) := \begin{cases} n & \text{if for some } n \succ h(k)\ \mathrm{Proof}_T(k+1, \ulcorner(\exists x\ hx \not\preccurlyeq n)\urcorner) \\ h(k) & \text{otherwise} \end{cases}$$

Clearly: $\vdash_{HA}$ "h is weakly monotonic in $\prec$".

One easily shows: $[\![A]\!]_K := \{w \mid w \Vdash_K A\}$ is finite or $\mathbb{N}$.

Define: $[A]_{K,T} := W\{\exists x\; hx = i \mid i \in [\![A]\!]_K\}$ if $[\![A]\!]_K$ is finite

$[A]_{K,T} := (0 = 0)$ if $[\![A]\!]_K = \mathbb{N}$.

We will conveniently identify $W\{\exists x\; hx = i \mid i \in \mathbb{N}\}$ with $(0 = 0)$.

Take $f\colon P \to L$ as: $f(p_i) = [A]_{K,T}$ and define: $\langle A\rangle_{K,T} := \langle A\rangle^{f,T}$.

2.1 <u>Theorem</u>

$\vdash_T \langle A\rangle_{K,T} \leftrightarrow [A]_{K,T}$.

Proof: induction on A.

i) $A \equiv p_i$ or $A \equiv \bot$. By definition.

ii) $A \equiv (B \vee C)$. Trivial.

iii) $A \equiv (B \wedge C)$.

We have to show:

$\vdash_T ([B]_{K,T} \wedge [C]_{K,T}) \leftrightarrow [B \wedge C]_{K,T}$.

Reason in T:

"$\leftarrow$" trivial.

"$\rightarrow$" Assume $W\{\exists x\; hx = i \mid i \Vdash_K B\}$, $W\{\exists x\; hx = j \mid j \Vdash_K C\}$.

Consider some $\exists x\; hx = i$ of the first disjunction, some $\exists x\; hx = j$ of the second. In case i and j are incomparable w.r.t. $\preccurlyeq$, we have $\bot$, hence $[B \wedge C]_{K,T}$; otherwise $i \preccurlyeq j$ or $j \preccurlyeq i$, so e.g. if $i \preccurlyeq j$, we have $\exists x\; hx = j$ and $j \Vdash B \wedge C$, hence $[B \wedge C]_{K,T}$.

iv) $A \equiv (B \rightarrow C)$.

We have to show:

$\vdash_T ([B]_{K,T} \rightarrow [C]_{K,T}) \leftrightarrow [B \rightarrow C]_{K,T}$.

In case $[\![B \rightarrow C]\!]_K = \mathbb{N}$ this is easy. Hence assume $[\![B \rightarrow C]\!]_K$ is finite. Let $j_0,\ldots,j_s$ be the maximal elements such that $j_k \nVdash_K (B \rightarrow C)$. Note that $j_k \Vdash_K B$ and $j_k \nVdash_K C$.

Reason in T:

First "$\rightarrow$"

Suppose $([B]_{K,T} \rightarrow [C]_{K,T})$ and $\Box_T [B \rightarrow C]_{K,T}$. We have $\Box_T(\exists x\ hx \not\prec j_k)$. Suppose $\mathrm{Proof}_T(p+1, \ulcorner(\exists x\ hx \not\prec j_k)\urcorner)$ and $h(p) = y$. In case $y \prec j_k$, $h(p+1) = j_k$ and so $[B]_{K,T}$, hence $[C]_{K,T}$. From $h(p+1) = j_k$ and $[C]_{K,T}$ we have: $\exists x\ hx > j_k$ (for $j_k \notin [\![C]\!]_K$). In case $y \not\prec j_k$, $h(p+1) \not\prec j_k$. Conclude: $\exists x\ hx \not\prec j_k$. Hence $\bigwedge\{\exists x\ hx \not\prec j_k \mid k = 0,\ldots,s\}$, so by the monotonicity of h: $\exists x \bigwedge\{hx \not\prec j_k \mid k = 0,\ldots,s\}$. From this we have: $\bigvee\{\exists x\ hx = i \mid i \Vdash_K (B \rightarrow C)\}$, i.e. $[B \rightarrow C]_{K,T}$.

By the SLP (see part 4, 3) we may conclude $[B \rightarrow C]_{K,T}$ without assuming $\Box_T[B \rightarrow C]_{K,T}$.

Secondly "$\leftarrow$"

Suppose $[B \rightarrow C]_{K,T}$ and $[B]_{K,T}$. Hence by the reasoning of the "$\rightarrow$" case of $\wedge$: $\bigvee\{\exists x\ hx = i \mid i \Vdash_K ((B \rightarrow C) \wedge B)\}$. So $[C]_{K,T}$.

v) $A \equiv \Box B$

The case that $[\![A]\!]_K = \mathbb{N}$ is easy, hence assume $[\![A]\!]_K$ is finite.

Let $j_0,\ldots,j_s$ be all elements such that $j_k \Vdash_K \Box B$, $j_k \nVdash_K B$. Note that for each i s.t. $i \nVdash_K \Box B$ there is a j_k with $i \prec j_k$.

We have to show: $\vdash_T \Box_T[B]_{K,T} \leftrightarrow [\Box B]_{K,T}$.

Argue in T:

"$\rightarrow$" first:

Suppose $\Box_T[B]_{K,T}$. We have: $\Box_T(\exists x\ hx \not\prec j_k)$ by the definition of $[B]_{K,T}$ and the fact that $j_k \nVdash_K B$. Suppose

$\mathrm{Proof}_T(p+1, \ulcorner(\exists x\ hx \not\prec j_k)\urcorner)$ and $h(p) = y$. In case $y \prec j_k$ we have $h(p+1) = j_k$, in case $y \not\prec j_k$ $h(p+1) \not\prec j_k$. Hence $\exists x\ hx \not\prec j_k$.

Conclude $\bigwedge\{(\exists x\ hx \not\prec j_k) \mid k = 0,\ldots,s\}$, hence by the monotonicity of h: $\exists x \bigwedge\{(hx \not\prec j_k) \mid k = 0,\ldots,s\}$. So by elementary reasoning: $\bigvee\{\exists x\ hx = i \mid i \Vdash_K \Box B\}$.

Secondly "$\leftarrow$"

Suppose $\exists x\ hx = i$ for an $i \Vdash_K \Box B$, by the definition of h and the fact that $i \neq 0$: $\Box_T\ \exists x\ hx \not\prec i$. ("How else could h move up to i".) Moreover from $\exists x\ hx = i$: $\Box_T(\exists x\ hx = i)$. Combining: $\Box_T\ \exists x\ hx \succ i$ or $\Box_T \bigvee\{\ell = j \mid j \succ i\}$. Hence $\Box_T\ [B]_{K,T}$ (because $j \succ i \Rightarrow j \models_K B$).

$\Box$

2.2 Corollary

Let d(A) be as in part 0,2.3.3 but now defined for $\Vdash_K$. Suppose $T \vdash CP_T$. There is an $f: P \to L$ such that: $\vdash_T \langle A\rangle^{f,T} \leftrightarrow \Box_T^{d(A)}\bot$.

Proof: as in part 0, 2.3.3.

$\Box$

2.3 Corollary

Suppose $T \vdash CP_T$ and a is the least element of $\{0,1,\ldots,\omega\}$ such that $T \vdash \Box_T^a\bot$. Then $L(T) \subseteq H + \Box^a\bot$, where H is as in part 4, 6.8.

Proof: as in part 0, 2.2.5, using lemma 6.14 of part 4.

$\Box$

2.4 Examples and Applications

i) L(PA*) = H. Because PA* is sound for H.

ii) $L(T) \subseteq H$ for e.g. the following T: $T \equiv HA^*$, $T \equiv HA+ECT_0+CP_T$, $T \equiv (HA+DNS)^*$.

iii) We have De Jongh's Theorem for HA, HA+ECT_0, HA+DNS.
From *ii*).

iv) $L(HA) \subseteq H+\Box\Box\bot$

Proof: Consider $T \equiv HA+CP_{HA}$. By 2.3 and part 4, 4.14 $L(T) \subseteq H+\Box\Box\bot$. Moreover $L(HA) \subseteq L(T)$, for $HA \leqslant T$ and $T \vdash \Box_{HA} A \leftrightarrow \Box_T A$ (because $T \vdash \Box_{HA} CP_{HA}$). $\Box$

Of course this is a rather silly upper bound because $H+\Box\Box\bot \vdash \Box(A \vee \neg A)$; but it is the only one we know.

3 Translations and Principles

3.1 Independence of Premiss, The Friedman Translation and The De Jongh Slash.

The De Jongh Slash was introduced by De Jongh [DJ] in 1973. It is given by:

- $(E \underset{\sim}{|} P) := (E \to P)$ for atoms P of L.
- $(E \underset{\sim}{|} .)$ commutes with $\wedge$, $\forall$.
- $(E \underset{\sim}{|} (A \vee B)) := ((E \underset{\sim}{|} A \wedge (E \to A)) \vee (E \underset{\sim}{|} B \wedge (E \to B)))$
- $(E \underset{\sim}{|} (A \to B)) := ((E \underset{\sim}{|} A \wedge (E \to A)) \to E \underset{\sim}{|} B)$
- $(E \underset{\sim}{|} \exists x\, A) := (\exists x (E \underset{\sim}{|} A \wedge (E \to A)))$

The Friedman Translation was introduced by Friedman [Fr] in 1977. It is given by:

- $(P)^A := (P \vee A)$ for atoms P of L.
- $(.)^A$ commutes with $\wedge$, $\vee$, $\to$, $\forall$, $\exists$.

A typical application of the De Jongh Slash is a proof of closure of HA under the Independence of Premiss Rule (IPR), a typical application of the Friedman Translation is a proof of closure of HA under Markov's Rule.

Closer inspection of the De Jongh Slash gives us:

Lemma

The De Jongh Slash is a special case of Friedmans translation.

To be specific: $\vdash_{HA} E \underset{\sim}{|} A \leftrightarrow (A)^{\neg E}$.

Proof: it is clearly sufficient to show that $(.)^{\neg E}$ satisfies, modulo provable equivalence, the inductive clauses of the definition of $(E \underset{\sim}{|} .)$:

- $\vdash_{HA} (P \vee \neg E) \leftrightarrow (E \rightarrow P)$, because atoms are decidable in HA.
- The other clauses follow easily by noting that $\vdash_{HA} E \rightarrow ((A)^{\neg E} \leftrightarrow A)$, hence $\vdash_{HA} (A)^{\neg E} \rightarrow (E \rightarrow A)$. So e.g.:

$$\begin{aligned} \vdash_{HA} (B \vee C)^{\neg E} &\leftrightarrow (B^{\neg E} \vee C^{\neg E}) \\ &\leftrightarrow ((B^{\neg E} \wedge E \rightarrow B) \vee (C^{\neg E} \wedge E \rightarrow C)) \\ &\leftrightarrow ((E \underset{\sim}{|} B \wedge E \rightarrow B) \vee (E \underset{\sim}{|} C \wedge E \rightarrow C)) \end{aligned}$$

$\Box$

The crucial lemmas on Friedman's Translation are treated in part 4 of this thesis, 2.15.

We show that IPR is a derived rule of HA.

Suppose $\vdash_{HA} \neg A \rightarrow \exists x\, B(x)$, then $\vdash_{HA} (\neg A)^{\neg\neg A} \rightarrow (\exists x\, B(x))^{\neg\neg A}$, hence $\vdash_{HA} (\neg A)^{\neg\neg A} \rightarrow \exists x(\neg A \rightarrow B(x))$.

$$\begin{aligned} \text{But } \vdash_{HA} ((A)^{\neg\neg A} &\rightarrow (\neg A \rightarrow A) \\ &\rightarrow \neg\neg A) \end{aligned}$$

and $(\neg A)^{\neg\neg A} = (A^{\neg\neg A} \rightarrow \neg\neg A)$, so $\vdash_{HA} (\neg A)^{\neg\neg A}$.

Clearly this proof can be formalized in HA, so

$\vdash_{HA} \Box_{HA}(\neg A \rightarrow \exists x\, B(x)) \rightarrow \Box_{HA}\, \exists x(\neg A \rightarrow B(x))$.

Consequences visible in L_{pr} are e.g.:

$(\Box(\neg A \rightarrow B \vee C) \rightarrow \Box((\neg A \rightarrow B) \vee (\neg A \rightarrow C)))$ is in L(HA). And for $B \in \Sigma_{pr}$: $(\Box(\neg A \rightarrow B) \rightarrow \Box(\neg\neg A \vee B))$ is a principle of L(HA).

To show this principle; suppose $\vdash_{HA} \neg A \rightarrow \exists x\, B_0(x)$, where $B_0(x)$ is primitive recursive. We have by IPR: $\vdash_{HA} \exists x(\neg A \to B_0(x))$.

Argue in HA:

Pick an x such that $\neg A \to B_0(x)$. In case $B_0(x)$ we have B and so $\neg\neg A \vee B$. In case $B_0(x)$, we have $\neg\neg A$ and so $\neg\neg A \vee B$. $\square$

3.2 Leivants Principle, q-realizability, $(\ )^{qHA}$.

In "Stelling" 1 accompanying his thesis of 1975, Leivant states that: $(\square(A \vee B) \to \square(\square A \vee \square B)) \in L(HA)$.

When writing part 4 of this thesis we totally overlooked this. With this principle we think Leivant was the first to state an HA valid provability principle that together with G yields some degree of falsity. We give two proofs of a slightly stronger version of Leivant's Principle:

Theorem

$L(HA) \vdash \square(A \vee B) \to \square(A \vee \square B)$

First proof: Argue in HA:

Suppose $\square_{HA}(A \vee B)$ then for some recursive term t: $\square_{HA}(\exists x\ t \cong x \wedge xq(A \vee B))$, where q is q-realizability (see [Tr] pp 188-203). Hence by Gödel's Rule: $\square_{HA}\square_{HA}(\exists x\ t \cong x \wedge xq\ A \vee B)$.

Argue in $\square_{HA}$:

Let $t \cong x$ and $xq(A \vee B)$. In case $(x)_0 = 0$ we have $(x)_1qA$ and hence A. Moreover we have: $\square_{HA}\ t \cong x$(for $(t \cong x)$ is Σ^0_1) and $\square_{HA}(\exists x\ t \cong x \wedge xq(A \vee B))$, so $\square_{HA}\ xq\,(A \vee B)$, hence in case $(x)_0 = 1$: $\square_{HA}(x)_1qB$. Conclude $\square_{HA}B$. $\square$

Second proof:

Define $(\ )^{qHA}$ as follows:

- $(P)^{qHA} := P$ for atoms P of L.

- $(.)^{qHA}$ ccmmutes with $\wedge$, $\vee$, $\exists$.
- $(A \to B)^{qHA} := ((A^{qHA} \to B^{qHA}) \wedge (A \to B) \wedge \Box_{HA}(A \to B))$
- $(\forall x\ A)^{qHA} := ((\forall x\ A^{qHA}) \wedge \Box_{HA}(\forall x\ A))$

One easily shows:

a) $\Gamma \vdash_{HA} A \Rightarrow \Gamma^{qHA} \vdash_{HA} A^{qHA}$

b) $A^{qHA} \vdash_{HA} A \wedge \Box_{HA} A$.

(One does not have: $\Gamma^{qHA} \vdash_{HA} A^{qHA} \Rightarrow \Gamma \vdash_{HA} A$, e.g. $(\neg\neg\Box_{HA}\bot)^{qHA} \vdash_{HA} (\Box_{HA}\Box_{HA}\bot)^{qHA}$.)

Moreover the proofs of *a* and *b* can be formalized in HA. We have, in HA:

Suppose $\Box_{HA}(A \vee B)$, then $\Box_{HA}(A \vee B)^{qHA}$. So $\Box_{HA}(A^{qHA} \vee B^{qHA})$, hence $\Box_{HA}(A \vee \Box_{HA} B)$.

$\Box$

The principle $(\Box(A \vee B) \to \Box(A \vee \Box B))$ is 2-inconsistent with G, for in G plus this principle:

$\Box(\Box\bot \vee \neg\Box\bot)$, so $\Box(\Box\bot \vee (\Box\neg\Box\bot))$. By Löb's Axiom:
$\Box(\Box\bot \vee \Box\bot)$, or $\Box\Box\bot$.

4 General insights on the L(T)

We end our remarks on the provability logics of HA and extensions with two gereral results.

The first is due to Gargov:

4.1 Theorem

Suppose $T \geqslant HA$, T has the Disjunction Property, then L(T) has the Disjunctior Property.

Proof: Suppose for all $f: P \to L: \to \vdash_T \langle A \vee B\rangle^{f,T}$. We prove: for all $f: P \to L: \vdash_T \langle A\rangle^{f,T}$ or $\vdash_T \langle B\rangle^{f,T}$.

Say $f(p_i) = E_i$, $g(p_i) = D_i$. Define $E_i(n) :\,= ((\{\underline{n}\}\ \underline{n} \cong 0 \to C_i) \wedge ((\{\underline{n}\}\ \underline{n}\downarrow \wedge \{\underline{n}\}\ \underline{n} \not\cong 0) \to D_i))$ and $h_n(p_i) :\,= E_i(n)$.

$$\text{Let } \{d\}n :\cong \begin{cases} 0 \text{ if } (\vdash_T \langle A\rangle^{h_n,T}) \leqslant (\vdash_T \langle B\rangle^{h_n,T}) \\ 1 \text{ if } (\vdash_T \langle B\rangle^{h_n,T}) < (\vdash_T \langle A\rangle^{h_n,T}) \end{cases}$$

Clearly $\{d\}$ is total recursive, since just the restriction of h_n to the atoms of A **and** B is relevant for finding $\langle A\rangle^{h_n,T}$ and $\langle B\rangle^{h_n,T}$.

Suppose $\{d\}\ d \cong 0$ then $\vdash_T \{d\}\ \underline{d} \cong 0$ and so: $\vdash_T C_i \leftrightarrow E_i(d)$. Hence by an easy induction $\vdash_T \langle A\rangle^{h_n,T} \leftrightarrow \langle A\rangle^{f,T}$. Moreover by the definition of $\{d\}$: $\vdash_T \langle A\rangle^{h_n,T}$. Conclude $\vdash_T \langle A\rangle^{f,T}$.

Suppose $\{d\}\ d \cong 1$. By similar reasoning as above: $\vdash_T \langle B\rangle^{g,T}$. Conclude: $\forall f\ \forall g(\vdash_T \langle A\rangle^{f,T}$ or $\vdash_T \langle B\rangle^{g,T})$. Hence by *classical* logic: $(\forall f \vdash_T \langle A\rangle^{f,T})$ or $(\forall g \vdash_T \langle B\rangle^{g,T})$.

$\square$

Remark: the proof of 4.1 can be formalized in HA (replacing reference to functions by reference to numbers as codes of finite functions) up to the last step. ("Hence bv *classical* logic ...".) What we can get is:

$\exists f \nvdash_T \langle A\rangle^{f,T} \Rightarrow \forall g \vdash_T \langle B\rangle^{g,T}$.

Or another variant:

$-\ \forall f \vdash_T \langle A\rangle^{f,T} \Rightarrow \forall g\ \neg\neg \vdash_T \langle B\rangle^{g,T}$.

Hence in HA+MP we can prove: T has DP, $(A \vee B) \in L(T)$ and $A \notin L(T) \Rightarrow B \in L(T)$.

Another question of interest is this: as we have seen: $T \leqslant U \not\Rightarrow L(T) \subseteq L(U)$, e.g. $L(HA) \not\subseteq L(PA)$, but is it possible to find for any T and U a V such that $L(V) \subseteq L(T) \cap L(U)$?

The next theorem provides a partial answer:

4.2 <u>Theorem</u>

Let $(T_i)_{i \in \mathbb{N}}$ be a recursive sequence of theories, (extending HA), i.e. $T_i \vdash A$ is an RE relation in A and i. Let (X_i) be a recursive sequence of non empty subsets of L (i.e. $A \in X_i$ is an RE relation in A and i) such that X_i is a counterexample set for T_i; we say X_i is a counterexample set for T_i if each element of T_i is of the form $\langle B \rangle^{f,T_i}$ for some f and some $B \in L_{pr}$ and $T_i \nvdash \langle B \rangle^{f,T_i}$.

We have: there is a $T = HA$ such that for each i, for each $\langle B \rangle^{f,T_i} \in X_i$: $T \nvdash \langle B \rangle^{f,T}$.

Proof: By the methods of part 1 of this Thesis one can find an index e such that for each n and i and each $A \in X_i$:

$T_i + \{\underline{e}\}\ \underline{0} \cong \underline{n} \nvdash A$.

Define $T :\equiv HA + \{B \mid \exists x\ \{\underline{e}\}\ \underline{0} \cong x \wedge T_x \vdash B\}$.

Clearly $T = HA$ (but $T \neq HA$). Consider T_i and $\langle B_i \rangle^{f,T_i} \in X_i$. We have $T_i + \{\underline{e}\}\ \underline{0} \cong \underline{i} \nvdash \langle B_i \rangle^{f,T_i}$, so certainly $HA + \{\underline{e}\}\ \underline{0} \cong \underline{i} \nvdash \langle B_i \rangle^{f,T_i}$. But $HA + \{\underline{e}\}\ \underline{0} \cong \underline{i} \vdash$ '$T \equiv T_i$', so $HA + \{\underline{e}\}\ \underline{0} \cong \underline{i} \vdash \langle B_i \rangle^{f,T_i} \leftrightarrow \langle B_i \rangle^{f,T}$. Hence $HA + \{\underline{e}\}\ \underline{0} \cong \underline{i} \nvdash \langle B_i \rangle^{f,T}$, so $HA \nvdash \langle B_i \rangle^{f,T}$. Conclude $T \vdash \langle B_i \rangle^{f,T}$.

4.3 <u>Application</u>

There is an RE T, $T = HA$, such that $L(T) \subseteq H \cap G$.

Proof: Let $T_0 := PA$, $T_i := PA^*$, $i \neq 0$. Inspecting the proofs of the completeness theorems for resp. G and H, we find recursive sets X_0 and X_i $(i \neq 0)$ such that for each j:

$A \notin L(T_j) \Leftrightarrow$ for some f $\langle A \rangle^{f,T_j} \in X_j$.

Take T as in 4.2. □

REFERENCES

[DJ] DE JONGH D.H.J., *Formulas of one propositional variable in intuitionistic arithmetic*, Report 73-103, 1973, Department of Mathematics, University of Amsterdam.

[Fr] FRIEDMAN H., *Classically and Intuitionistically Provably Recursive Functions*, in: Higher Set Theory, ed. G.H. Müller and D.S. Scott, Springer Lecture Notes in Mathematics 669, Berlin, 1977, pp 21-27.

[Ga] GARGOV G.K., *A Note on the provability logics of certain extensions of Heyting's Arithmetic*. Unpublished manuscript.

[Tr] TROELSTRA A.S., *Metamathematical investigation of intuitionistic arithmetic and analysis*, Springer Lecture Notes in Mathematics 344, Berlin, 1973.

[Ur 1] URSINI A., *Intuitionistic Diagonalizable Algebras*, Algebra Universalis 9 (1979). pp 229-237.

[Ur 2] _________, *A Modal Calculus Analogous to K4W, Based on Intuitionistic Propositional Logic*, I^{o}, Studia Logica XXXVIII, 3, (1979), pp 297-311.

SAMENVATTING

Dit proefschrift gaat (voornamelijk) over generalisaties van de eerste en tweede onvolledigheidsstelling van Gödel en over begrippen die in het bewijs van deze stellingen voorkomen.

In Gödels eerste stelling wordt een zin G in de taal van de rekenkunde geconstrueerd zodat de rekenkunde (bijvoorbeeld zoals geformaliseerd door Dedekind / Peano, met inductie*schema*) deze zin noch zijn negatie impliceert. Dit resultaat is op spectaculaire manier te generalizeren met behulp van een zogenaamd recursief onscheidbaarheidsargument. Om dit argument toe te kunnen passen is nodig dat de partiëel recursieve functies semirepresenteerbaar zijn in de beschouwde (consistente) theorieën. De formalizering van het begrip bewijsbaarheid en het begrip consistentie is uit het bewijs geëlimineerd. Deel 1 en 2 van dit proefschrift gaan in wezen over recursieve onscheidbaarheidsargumenten met het oog op toepassing op formele theorieën. (De bestudeerde methoden kunnen natuurlijk ook elders toegepast worden.)

In zijn tweede stelling liet Gödel zien dat bovengenoemde zin G bewijsbaar equivalent is met (de formalizering van) 'de rekenkunde is consistent'. Löb heeft in 1955 drie principes geïsoleerd die om zo te zeggen aan het bewijs van deze equivalentie ten grondslag liggen. Natuurlijk is 'ten grondslag liggen aan' vaag, en men kan dan ook heel goed andere principes formuleren die 'ten grondslag' zouden 'liggen' aan het bewijs. De principes van Löb hebben echter de prachtige eigenschap dat ze op te vatten zijn als principes van een *modale propositielogica*. Bovendien is het bewijs van voornoemde equivalentie, uit Löbs principes plus de 'definitievergelijking' van G geheel in modale propositielogica te geven. Löbs werk heeft aanleiding gegeven tot een nieuwe tak van onderzoek: bewijsbaarheidslogica (zie bijvoorbeeld G. Boolos, *The unprovability of consistency*, Cambridge University Press, Cambridge, 1979). Belangrijke resultaten op dit gebied zijn de stellingen van D.H.J de Jongh en van R.M. Solovay. Deel 3, 4 en 5 van dit proefschrift zijn bijdragen tot de bewijsbaarheidslogica.

CURRICULUM VITAE

Op 19 december 1950 ben ik te Zwijndrecht geboren. Van 1963 tot en met 1969 bezocht ik het Johannes Calvijn Lyceum te Kampen. In 1969 ging ik aan de Technische Hogeschool Twente studeren, waar ik in 1974 mijn Baccalaureaatsexamen in de toegepaste wiskunde behaalde. Intussen was mijn belangstelling verschoven naar zuivere wiskunde en filosofie. De Rijksuniversiteit te Utrecht accepteerde mijn Baccalaureaatsdiploma als grond van toelating tot de doctoraalstudie in de wiskunde. Van 1974 tot en met 1976 studeerde ik hier wiskunde met als bijvak filosofie. Mijn specialisatie in de afstudeerfase was metamathematica onder leiding van Van Dalen / Barendregt. Het onderwerp van de afstudeerscriptie - geschreven samen met J.S. Lub - was predicatieve analyse. Van 1976 tot 1981 was ik als promotiemedewerker verbonden aan het Mathematisch Instituut in Utrecht. Mijn eerste artikel 'Numerations, λ-Calculus & Arithmetic' kwam voort uit een stafseminarium over Eršovs werk.

Vanaf januari 1981 ben ik werkzaam als medewerker bij de Centrale Interfaculteit te Utrecht.

Albert Visser
Centrale Interfaculteit
Heidelberglaan 2

3508 TC Utrecht

STELLINGEN

Bij het proefschrift *'Aspects of Diagonalization & Provability'* van Albert Visser.

1. Om mee te kunnen praten moet je weten waar je het over hebt. Om "Julius Caesar is dood" te begrijpen moet je weten naar wie "Julius Caesar" refereert. Anders dan D. Bell (zie BELL D. *Frege's Theory of Judgement*, Clarendon Press, Oxford, 1979, pp 60-62) houd ik dat voor onbetwijfelbaar. De problemen die men daarin gezien heeft zijn het gevolg van het geloof in naïve thesen over kennis, zoals:

   a) kennis zit in het hoofd (of wellicht in de individuele geest).

   b) kennis is zeker.

   De eerste these lijkt plausibel als men meent dat het toeschrijven van kennis aan iemand het toeschrijven van een *eigenschap* is, en dat iemand een eigenschap kan hebben los van de rest van de wereld. Het is beter kennis te zien als een *relatie* tussen een persoon, een propositie en de wereld. Geloof in de tweede these is het gevolg van eeuwenlange filosofische preoccupatie met sceptische argumenten.

2. Iemand weet in voldoende mate naar wie "Julius Caesar" refereert om hem begrip van "Julius Caesar is dood" toe te schrijven precies dan als hij in de positie is om met "Julius Caesar = Julius Caesar" een waarheid over Julius Caesar uit te drukken.

3. In tegenstelling tot wat bijna iedereen vindt, gelooft iemand die denkt dat Hesperus = Phosphorus, daarmee simpelweg dat Venus = Venus. De reden dat zovelen anders geloven is dat men onbewust "Hesperus = Phosphorus" leest als *elliptisch* voor " "Hesperus" refereert naar Phosphorus en "Phosphorus" refereert naar Hesperus". Zo ongeveer zoals men "Jongens zijn jongens" leest alsof er stond "Jongens halen nu eenmaal typisch jongensachtige streken uit".

4. De eenheid van interpretatie is niet de zin, maar, laten we zeggen, het discourse: een groep op zekere manier samenhangende zinnen. Een voorwaarde om een aantal door Piet geuite zinnen tot één discourse te rekenen is dat Piet van de coreferentiële namen uit die zinnen weet dat ze coreferentiëel zijn, zoals het een voorwaarde is voor een zin om überhaupt tot een discourse van Piet te horen dat Piet weet waar de in die zin voorkomende namen naar refereren (even voorbijgaande aan zinnen als "Sinterklaas bestaat niet").

   Stel Piet beweert oprecht (1) "Hesperus is een grote planeet" en (2) "Phosphorus is een kleine planeet". We kunnen Piet nu, aannemende dat hij z'n Nederlands beheerst en weet waar "Hesperus" resp. "Phosphorus" naar refereert,

zowel het geloof dat Venus een grote planeet is, als het geloof dat Venus een kleine planeet is toeschrijven. Het volgt echter niet dat Piet gelooft dat Venus zowel een grote als een kleine planeet is. Immers (1) en (2) kunnen niet tot één discourse gerekend worden.
Geloof is niet gesloten onder conjunctie.
Stel Piet beweert oprecht (3) "Hesperus ≠ Phosphorus". Nu volgt niet dat Piet gelooft dat Venus ≠ Venus. (3) behoort tot geen enkel discourse en drukt dus geen propositie uit.

5. Stel Piet weet niet dat "Hesperus" en "Phosphorus" naar dezelfde planeet refereren, maar Piet is wel in de positie met "Hesperus" en met "Phosphorus" naar Venus te refereren. (Piet weet dat "Hesperus" naar Venus refereert en Piet weet dat "Phosphorus" naar Venus refereert, maar Piet weet niet dat "Hesperus" en "Phosphorus" beide naar Venus refereren).
Stel nu Jan zegt tegen Piet: "Hesperus = Phosphorus" en dat Piet weet dat Jan altijd de waarheid spreekt. Nu gebeurt er iets heel merkwaardigs. Piet kan "Hesperus = Phosphorus" tot nog toe niet interpreteren, aangezien hij niet weet dat beide namen coreferentiëel zijn. Echter uit het feit dat Jan zegt: "Hesperus = Phosphorus" kan Piet concluderen dat "Hesperus" en "Phosphorus" beide naar Venus refereren. En daarna kan hij ook begrijpen wát Jan gezegd heeft: dat Venus = Venus.
Het bovenstaande verklaart waarom "Hesperus = Phosphorus" bijna altijd als elliptisch voor " "Hesperus" refereert naar Phosphorus en "Phosphorus" refereert naar Hesperus" begrepen wordt. Men stelt zich immers, wanneer men zich afvraagt wat een zin betekent, bijna altijd een *communicatiesituatie* voor. Tevens laat het bovenstaande zien dat de *informatie* die Jans *uiting* "Hesperus = Phosphorus" Piet geeft iets heel anders is dan *wat* Jan zegt. De taal is als een bril waarvan men altijd *ook* de glazen ziet.

6. Fields claim (zie FIELD H.H., *Mental Representation*, Erkenntnis, vol.13, no 1, july 1978, pp 9-61; de relevante opmerking staat op p 49) dat als zinnen proposities betekenen dit een algemeen synonymie begrip oplevert, is juist. Echter niet juist is zijn verdere claim dat dit zowel intra- als interlinguistisch van toepassing is.

7. Wittgensteins opmerking: "*Von zwei Dingen zu sagen, sie seien identisch, ist ein Unsinn, und von Einem zu sagen, es sei identisch mit sich selbst, sagt gar nichts.*" is indien welwillend geïnterpreteerd, juist. (Zie WITTGENSTEIN L. *Tractatus Logico Philosophicus*, Routledge & Kegan Paul, 1969, 5.5303.) Opgemerkt dient echter te worden:

i) Wittgensteins opmerking zegt meer over wat het is iets over iets te zeggen dan over de identiteit. Identiteit is een doodgewone binaire relatie.

ii) Wittgensteins opmerking zou aantonen dat het hebben van een teken voor identiteit in de taal overbodig was, als we in een zin alleen naar een relatie zouden verwijzen om van zekere objecten te zeggen dat die relatie tussen hen bestaat. Dat laatste is echter niet zo.

8. Als ik iets geloof, dan geloof ik dat dat zo is. Echter ik geloof niet, dat als ik iets geloof, dat dat dan zo is. Sterker nog, ik weet dat ik iets geloof dat niet zo is, maar ik kan onmogelijk weten wat.

9. Wanneer we Kripkes Fixed Points (zie KRIPKE S.A., *Outline of a theory of truth*, Journal of Philosophy, Nov.6, 1975, pp 690-716) beschouwen met als basis model de natuurlijke getallen met plus en maal (of meer algemeen: een 'acceptable structure') dan blijkt:

   i) Elke verzameling elementen van het domein van het basis model is representeerbaar in een zeker Fixed Point.

   ii) Elke inductieve verzameling elementen van het basis model is representeerbaar in *elk* inductief Fixed Point.

   iii) Er is geen maximaal inductief Fixed Point.

10. Definiëer voor Σ^0_1-zinnen A,B:

   i) $A \lesssim B :\Leftrightarrow PA \vdash B \rightarrow A$

   ii) $A \sim B :\Leftrightarrow A \lesssim B$ en $B \lesssim A$.

   iii) $[A] := \{C \in \Sigma^0_1\text{-zinnen} \mid C \sim A\}$

   iv) Laat 'a', 'b', 'c' over equivalentieklassen als in *iii)* lopen.
   $a \lesssim b :\Leftrightarrow$ Er is een $A \in a$, $B \in b$ met $A \lesssim B$.

   v) Voor $a \lnsim b$: $[a,b] := \{c \mid a \lesssim c \lesssim b\}$

   Dan geldt: elke aftelbare partiële ordening is in te bedden in $<[a,b],\lesssim>$ voor willekeurige a, b met $a \lnsim b$. (Dit is een direct gevolg van *het bewijs* van stelling 3.12 van deel 1 van dit proefschrift.)

11. Volgens een stelling van H.P. Barendregt en M. Dezani is iedere *recursieve* applicatieve structuur isomorf inbedbaar in een willekeurige combinatorische algebra.
   Voor elke aftelbare combinatorische algebra is er echter een aftelbare applicatieve structuur die daarin niet isomorf ingebed kan worden.

12. Een S-term is een term van de Combinatorische Logica geheel opgebouwd uit S-en en variabelen (en haakjes). Zij A een S-term, X,Y,Z variabelen en U een variabele die niet in A voorkomt. Dan geldt: elk reduct van AXYZU is van de vorm BU, waar U niet in B voorkomt. In slogan: S-termen hebben een beperkte eetlust, tenzij het genuttigde voedsel zelf actief wordt.

13. Zij C[ ] een context van de λβ-calculus met tenminste één gat. De termen M, die in een β-stap reduceren naar C[M], modulo α-reducties, (waar M geen subterm is van C[ ] modulo α-reducties), zijn precies de termen:

$$M \equiv \underbrace{C[C[\ldots C}_{n\ C's}[D[((\lambda x.E[D[(xx)]])(\lambda x.E[D[(xx)]]))]\ldots]$$

waar n = 0 als er meer dan één gat in C[ ] zit, n ⩾ 0 wanneer er één gat in C[ ] is, en waar D[ ] en E[ ] contexten zijn zodat C[ ] ≡ D[E[ ]] en waar er precies één gat in D[ ] zit. (Bedenk dat [ ] ook een context is.)

14. De eerste fysicus moet een wijdse armbeweging gemaakt hebben met de woorden: ik wil weten wat hier achter zit. (Wat hier achter zit = φυσιϲ.) De verdere ontwikkeling van de fysica moet niet slechts gezien worden als het geven van steeds betere antwoorden op déze vraag, maar evenzeer als een steeds nader interpreteren van die vraag, in het byzonder als een verdere precisering van dat 'hier'.

15. P. Aczel heeft in verband met het formalizeren van constructivistische theorieën het begrip 'taboe' ingevoerd. Een taboe is een test voor theorieën: een theorie dient een taboe te impliceren noch te weerleggen, tenzij nieuwe substantiële argumenten pro of contra worden aangevoerd. (Dit betekent constructivistisch gezien niet dat de negatie van een taboe weer een taboe is.)

Het verdient aanbeveling de methodologie van taboes ook in de wetenschapsfilosofie toe te passen. Een wetenschapsfilosofische theorie dient een taboe niet te impliceren of te weerleggen, tenzij:

a) op grond van nieuwe substantiële argumenten

b) op grond van expliciet gemaakte metafysische vooronderstellingen.

Bewering: de volgende uitspraken dienen taboe te zijn:

- Computer simulatie van wetenschappelijk onderzoek is mogelijk.
- Onze fysica en die van de Marsmannetjes zullen op den duur convergeren.
- Op een gegeven moment is de fundamentele fysica klaar.

16. Over W. Veldmans argument voor het axiom AC_{10} (zie VELDMAN W., *Investigations in Intuitionistic Hierarchy Theory*, proefschrift, Katholieke Universiteit Nijmegen, 1981, p 8) valt het volgende op te merken:

Laat 'α' over stap voor stap gecreëerde functies van $\mathbb{N} \to \mathbb{N}$ lopen en 'f' over wetmatige functies van $\mathbb{N} \to \mathbb{N}$. W. Veldman hanteert het principe
$\forall f \, \exists \alpha \, \forall n \; \alpha(n) = f(n)$.   (*)

i) volgens Veldmans argument is dit een niet wiskundig, want intensioneel, principe, en daarmee is het argument zelf niet wiskundig.

ii) Aangezien we voor gegeven α en f per definitie nooit een bewijs kunnen hebben dat ze voor alle n overeen komen, zou volgens de gebruikelijke uitleg van het constructivistisch redeneren (*) moeten fungeren als falsum. Daar Veldman zijn argument duidelijk niet beschouwt als instantie van ex falso sequitur quodlibet, moet hij een andere (wellicht 'intuitionistische' in tegenstelling tot 'constructivistische') uitleg hebben van de betekenis van de logische voegtekens. Het is te hopen dat hij deze ooit nader toe wil lichten.

17. Zij K een Kripke model van HA. Zij A een zin uit de taal van HA. We maken K' uit K door alle knopen van K die A forceren weg te laten. Dan is K' weer een Kripke model van HA.
Bovenstaande simpele stelling vat de betekenis van Friedmans vertaling voor HA samen in termen van Kripke modellen. (Zie FRIEDMAN H., *Classically and Intuitionistically Provably Recursive Functions*, in: Higher Set Theory, ed. G.H. Müller and D.S. Scott, Springer Lecture Notes in Mathematics 669, Berlin, 1977, pp 21-27.)

18. De intuitionistische eerste-orde theorie van paring en identiteit plus het axioma $(\exists x \, \exists y \; x \neq y)$ is niet conservatief over de theorie van de identiteit plus de axiomas $(\exists x_1 .. \exists x_n \bigwedge \{(x_i \neq x_j) \mid 1 \leqslant i < j \leqslant n\})$.

19. Het is heel goed mogelijk dat een ethisch principe een handeling voorschrijft, wat ook de gevolgen van die handeling zijn. Dat betekent echter niet dat men ontslagen is van de morele plicht na te denken over die gevolgen. Immers wat een individuele handeling ethisch rechtvaardigt is niet het feit dat ze uit een ethisch principe volgt, maar dat men gegeven dat ethische principe de verantwoordelijkheid voor de verwachte gevolgen op zich neemt.

Het is bijvoorbeeld mogelijk 'Gij zult niet doden' zo op te vatten, dat daaruit volgt dat we eenzijdig moeten ontwapenen ook al zou dit een kernoorlog eerder bespoedigen dan verhinderen. Wie op grond van dat principe voor eenzijdige ontwapening is, moet toch over de mogelijke reakties van verschillende naties op die stap nadenken.

16 november 1981   Albert Visser