COMPACT SPACES AND COMPACTIFICATIONS

AN ALGEBRAIC APPROACH

H. DE VRIES

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AKADEMISCH PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE WISKUNDE EN NATUURWETENSCHAPPEN AAN DE UNIVERSITEIT VAN AMSTERDAM, OP GEZAG VAN DE RECTOR MAGNIFICUS DR. J. KOK, HOOGLERAAR IN DE FACULTEIT DER WISKUNDE EN NATUURWETENSCHAPPEN, IN HET OPENBAAR TE VERDEDIGEN IN DE AULA DER UNIVERSITEIT OP WOENSDAG 13 JUNI 1962 DES NAMIDDAGS OM 4 UUR

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PREFACE

It has become a classical result that there exists a complete duality between the theory of boolean algebras and the theory of zero-dimensional compact Hausdorff spaces (M. H. Stone [32]). In this duality, e.g. the maximal proper filters of a boolean algebra correspond (in our approach) to the points of the corresponding Stone space. In this thesis an exposition is given of a theory which deals with a similar algebraization of the theory of arbitrary compact Hausdorff spaces (all topological spaces considered will be Hausdorff spaces). Though a complete duality has been achieved, it has seemed more practicable not to adhere to such a bare duality theory. The notion which supports the whole theory is that of a so-called compingent (boolean) algebra, i.e. a boolean algebra equipped with an additional relation satisfying a certain set of axioms. A typical example of such a compingent algebra is met in the boolean algebra $\mathrm{B}(C)$ of all regularly open sets of a compact space C, with the compingent relation "$\ll$" defined by: for $a,b \in \mathrm{B}(C)$, $a \ll b \Leftrightarrow \bar{a} \subseteq b$. The possibility of a duality theory as indicated was suggested by J. de Groot; only later the close connection with the theory of proximity spaces became apparent to me.

The theory of compingent algebras can also be considered as a topology without points; this approach to this kind of topology seems more promising than that expounded by K. Menger [23]. However, this side of the theory is not further elaborated here.

The points of the compact space attached to a given compingent algebra are obtained as so-called maximal concordant filters of the compingent algebra. It appeared that essentially the same filters had been used by P. S. Aleksandrov [2] and H. Freudenthal [12] in more concrete cases for their respective compactification theories. A more general theory of such filters in algebraic structures has been developed by J. G. Horne [17], mainly in connection with the theory of rings of continuous functions. In this thesis, it is shown that the compactifications of completely regular spaces can be completely described by means of certain compingent algebras. The practicability of this point of view is illuminated by the proofs, in the author's opinion lucid and simple, of known theorems and generalizations of them. Paradoxically speaking, our method is often more topological than the methods employed in previous proofs.

In the first chapter, the theory of compingent algebras is developed in some detail. For instance, the notion of homomorphism is defined, and its relation to the notion of continuous mapping is studied. Here the relationship with the corresponding Stone theory comes often to the fore.

In the second chapter, the compactification theory of completely regular spaces is developed. The resemblance of the ideas used in this work, and those used by P. S. Aleksandrov [2], H. Freudenthal [12], P. Samuel [26], and J. G. Horne [17], should be noticed. The fourth section gives briefly the connection of our theory with the theory of proximity spaces as developed by V. A. Efremovič [8], Ju. M. Smirnov [30], and Á. Császár [5].

It turns out that compingent algebras are also adequate for the description of the continuous mappings of a topological space into compact spaces; the exposition can be found in section 3 of chapter 2.

The last two chapters deal mainly with applications of the previously developed theory. The principal new results are contained in sections 2 and 4 of the third, and section 3 of the fourth chapter.

In chapter 3, § 1, theorems by C. Kuratowski and H. Freudenthal on quasicomponent spaces are generalized. In chapter 3, § 2, the notion of percompactness is introduced, being a slight generalization of the notion of peripheral compactness (or semi(bi)compactness). In this more general light, known results on the compactifications of peripherally compact spaces are derived in section 3. The last section of the chapter deals with the problem posed by J. de Groot [13] on the characterization of the complements of n-dimensional sets in compacta. General compact spaces are considered. As main results a sufficient condition is presented and it is shown that the weight of the complement need not necessarily be less than the weight of the compact space.

The first two sections of chapter 4 give generalizations of theorems by E. G. Sklyarenko and C. Kuratowski on weight and dimension preserving compactifications. In the final section, the following two results are proved.

Firstly, if given a set Φ of continuous mappings of a completely regular space T into a compact space D, such that the weight of D and the potency of Φ do not exceed the weight of T, then T can be compactified such that the compactification preserves the weight and the dimension of T and the elements of Φ are continuously extendible to the compactification.

Secondly, if given a completely regular space T and a set Φ of continuous mappings of T into itself whose potency does not exceed the weight of T, then there exists a weight and dimension preserving compactification of T which allows continuous extension of the elements of Φ. Various special cases of this theorem have been proved by several authors, e.g. the result without the condition imposed on the dimensions (J. de Groot ([14, 15] and R. H. McDowell [22]).

CONVENTIONS

1. The empty set will be denoted by $\emptyset$. If A and B are sets, then "$A \subseteq B$" will mean that A is a subset B, whereas "$A \subset B$" stands for: A is a *proper* subset of B. If the set A is considered as a subset of a set S, then A^c or $S \setminus A$ will denote the complement of A in S.
   Mappings will be considered as left operators, and written on the left of the argument. If f is a mapping of S into T, and $A \subseteq S$, $B \subseteq T$, then $f[A] = \{f(a) \mid a \in A\}$, $f^{-1}[B] = \{s \in S \mid f(s) \in B\}$. If S' is a subset of S, then $\iota_{S',S}$ will be the injection mapping of S' into S.
2. All topological spaces considered will be Hausdorff spaces. A *neighbourhood* of a point, or a subset, of a topological space will be an open set containing the point, or the subset. The *weight* $\mathrm{w}(T)$ of a topological space T is the minimal potency of a basis for the topology of T.
   The subsets A and B of a topological space T are *functionally separated* if there exists a continuous real function f on T such that:
   $$p \in T \Rightarrow 0 \leq f(p) \leq 1; \quad a \in A \Rightarrow f(a) = 0; \quad b \in B \Rightarrow f(b) = 1.$$
   If A is a subset of a topological space T, then $\bar{A}$ or A^- will denote the *closure* of A in T; the *boundary* of A will be $\mathfrak{R}_T(A) = \mathfrak{R}(A) = \bar{A} \cap (T \setminus A)^-$. A *regularly* open set of a topological space is an open set O of the space such that $O^{-c-c} = O$.
   A topological space is *compact* if every open covering of it has a finite subcovering. A *compactification* of a topological space T will be a pair (α, C), where C is a compact space and α a mapping of T into C, such that $\alpha[T]$ is dense in C, and α induces a topological mapping of T onto $\alpha[T]$.
   If T is a topological space, then $\dim T$ will denote its dimension

defined by means of open coverings, ind T the small inductive dimension of T (defined inductively by means of boundaries of neighbourhoods of points), and Ind T the great inductive dimension of T (defined by means of boundaries of neighbourhoods of closed sets).

For further topological concepts, see e.g. C. Kuratowski [20], W. Hurewicz and H. Wallman [18], and J. L. Kelley [19].

3. A *boolean algebra* is a complemented distributive lattice. The meet of the elements a,b of a boolean algebra B will be denoted by $a \wedge b$, the join by $a \vee b$, and the complement of a by a^o; $a \leq b$ will stand for $a \wedge b = a$, and $a < b$ for: $a \leq b, a \neq b$. If V is a subset of the boolean algebra B, then $\bigwedge V$, respectively $\bigvee V$, will denote the meet of V, respectively the join of V, whenever existent. The minimal element of a boolean algebra will usually be denoted by 0, the maximal element by 1; a boolean algebra will contain at least two elements.

A *boolean subalgebra* of the boolean algebra B will be a non-empty subset of B, containing the elements $a \wedge b$, $a \vee b$ and a^o, whenever it contains a and b; a *subboolean subalgebra* of B will be a subset of B which is a boolean algebra in the partial ordering inherited from B.

For further information on boolean algebras see e.g. G. Birkhoff [4], P. Dwinger [7], and R. Sikorski [27].

CHAPTER I. COMPINGENT BOOLEAN ALGEBRAS

1.1. Definitions.

1.1.1. Definition. A *compingent boolean algebra* B is a boolean algebra in which there is defined a relation $\ll$ satisfying the following conditions:

P1. $0 \ll 0$;

P2. $a \ll b \Rightarrow a \leq b$;

P3. $a \leq a' \ll b \Rightarrow a \ll b$;

P4. $a \ll b, c \ll d \Rightarrow a \wedge c \ll b \wedge d$;

P5. $a \ll b \Rightarrow b^0 \ll a^0$;

P6. $a \ll b \neq 0 \Rightarrow \exists c \neq 0$ such that $a \ll c \ll b$.

The relation "$a \ll b$" should be read as: "a is *surrounded by* b", or "a is *not near to* b^0". The relation $\ll$ which makes the underlying boolean algebra into a compingent boolean algebra is called the *compingent* relation of B. For brevity we shall use the term "compingent algebra" instead of "compingent boolean algebra".

1.1.2. Theorem. Let T be a completely regular space, and $\mathrm{B}(T)$ the boolean algebra of all regularly open sets of T. Then the relation "$\ll$" in $\mathrm{B}(T)$ defined by: for $O_1, O_2 \in \mathrm{B}(T)$,

$$O_1 \ll O_2 \Leftrightarrow O_1 \text{ and } O_2^0 \text{ are functionally separated,}$$

is a compingent relation.

The verification is straightforward, and is left to the reader. In the case where T is *normal* we have by Urysohn's lemma, for $O_1, O_2 \in \mathrm{B}(T)$,

$$O_1 \ll O_2 \Leftrightarrow \bar{O}_1 \subseteq O_2.$$

In the sequel, $\mathrm{B}(T)$ will be considered as a compingent algebra provided with the compingent relation defined in the theorem.

1.1.3. Proposition. Let B be a compingent algebra. Then:

(i) the compingent relation $\ll$ of B is transitive,
(ii) $a,b,b' \in B,\ a \ll b' \leq b \Rightarrow a \ll b,$
(iii) $a,b,c,d \in B,\ a \ll b,\ c \ll d \Rightarrow a \vee c \ll b \vee d,$
(iv) $a \in B \Rightarrow 0 \ll a \ll 1.$

Proof.

(i) Let $a,b,c \in B,\ a \ll b,\ b \ll c$. Then $a \leq b$ by P2, whence $a \ll c$ by P3.
(ii) Let $a,b,b' \in B$, and $a \ll b' \leq b$. Then $b^{\circ} \leq b'^{\circ} \ll a^{\circ}$ by P5; hence $b^{\circ} \ll a^{\circ}$ by P3, and $a \ll b$, again by P5.
(iii) Let $a,b,c,d \in B$, and $a \ll b,\ c \ll d$. By using P5 and P4, we obtain: $b^{\circ} \wedge d^{\circ} \ll a^{\circ} \wedge c^{\circ}$, or, by De Morgan's laws, $(b \vee d)^{\circ} \ll (a \vee c)^{\circ}$. Now the required result follows from P5.
(iv) Let $a \in B$. By P5 it is sufficient to show that $0 \ll a$. This, however, follows from P1 and (ii).

1.1.4. Theorem. Let B be a compingent algebra. Then for $a,b \in B,\quad a \leq b \Leftrightarrow: c \in B,\ b \ll c \Rightarrow a \ll c.$

Proof. The necessity of the condition follows from P3. So let $c \in B,\ b \ll c \Rightarrow a \ll c$. Suppose $a \nleq b$, i.e. $a \wedge b^{\circ} \neq 0$. By P6, there exists a $c \in B$ such that $0 < c \ll a \wedge b^{\circ}$; hence $c \ll b^{\circ}$ and $b \ll c^{\circ}$, by proposition 1.1.3 (ii), and P5. By virtue of the hypothesis, we obtain $a \ll c^{\circ}$, whence $a \leq c^{\circ}$; but this contradicts $0 < c \ll a \wedge b^{\circ} \leq a$.

Remark. The present theorem shows the existence of an axiom system for a compingent algebra involving the compingent relation as the only fundamental notion.

1.1.5. Remark. If T is a connected, completely regular space, then we have for $a \in \mathrm{B}(T)$:

$$a \ll a \Leftrightarrow a = \emptyset \text{ or } a = T.$$

So if we remove the cases $\emptyset \ll \emptyset$ and $T \ll T$ from our relation "$\ll$", we obtain a partial ordering in $\mathrm{B}(T)$ by proposition 1.1.3 (i). It follows from theorem 1.1.4 that the compingent algebra $\mathrm{B}(T)$ is in this case completely determined by a particular partial ordering of the set $\mathrm{B}(T)$.

1.2. Concordant filters.

1.2.1. Definition. Let B be a compingent algebra. A *concordant filter* of B is a non-empty subset $\mathfrak{f}$ of B such that:

F1. $a \in \mathfrak{f},\ a \leq b \Rightarrow b \in \mathfrak{f}$;

F2. $a,b \in \mathfrak{f} \Rightarrow \exists c \in \mathfrak{f}$ such that $c \ll a \wedge b$.

The concordant filter $\mathfrak{f}$ is called *proper* if $\mathfrak{f} \neq B$. A *maximal* concordant filter is a proper concordant filter which is not contained in any other proper concordant filter.

It follows, from Zorn's lemma, that every proper concordant filter is contained in at least one maximal concordant filter since the set of proper concordant filters of a compingent algebra is inductive.

Remark. In the terminology of J. G. Horne [17], our concordant filters are nothing else than $\gg$-ideals. Both notions can be traced in the compactification theory of topological spaces, cf. for instance P. S. Aleksandrov [2] and H. Freudenthal [12].

1.2.2. Theorem. A proper concordant filter $\mathfrak{m}$ of a compingent algebra B is maximal if and only if the following condition is satisfied:

$$a,b \in B,\ a \ll b \Rightarrow b \in \mathfrak{m} \text{ or } a^0 \in \mathfrak{m}.$$

Proof. Firstly, assume the condition is satisfied by the proper concordant filter $\mathfrak{m}$. Let $\mathfrak{m}'$ be a concordant filter of B, such that $\mathfrak{m} \subset \mathfrak{m}'$, and take $a \in \mathfrak{m}$, $b \in \mathfrak{m}' \setminus \mathfrak{m}$. Then there exists a $c \in \mathfrak{m}'$ such that $c \ll a \wedge b$. By the condition we get: $a \wedge b \in \mathfrak{m}$ or $c^0 \in \mathfrak{m}$. But from $a \wedge b \in \mathfrak{m}$ we should obtain: $b \in \mathfrak{m}$, by F1, which is not so; hence $c^0 \in \mathfrak{m}$, which leads to $c^0 \wedge c = 0 \in \mathfrak{m}'$, and $\mathfrak{m}' = B$, by F2 and F1; this proves the maximality of $\mathfrak{m}$.

Secondly, let $\mathfrak{f}$ be a proper concordant filter of B which does not satisfy the condition. We shall see that $\mathfrak{f}$ is not maximal. There exist elements $a,b \in B$ such that $a \ll b$, and neither $b \in \mathfrak{f}$ nor $a^0 \in \mathfrak{f}$. This implies in particular:

$$c \in \mathfrak{f},\ a \ll d \Rightarrow c \wedge d \neq 0.$$

Define: $$\mathfrak{g} = \{c \wedge d \mid c \in \mathfrak{f},\ a \ll d\}.$$

Then $\mathfrak{g} \subset B$, and also $\mathfrak{f} \subset \mathfrak{g}$, since $b \in \mathfrak{g} \setminus \mathfrak{f}$.

It is sufficient to show that $\mathfrak{g}$ is a concordant filter of B. If $c \in \mathfrak{f}$, $a \ll d$ and $c \wedge d \leq e$, then $c \vee e \in \mathfrak{f}$ and $a \ll d \vee e$, which implies: $(c \vee e) \wedge (d \vee e) = e \in \mathfrak{g}$. If $c_1, c_2 \in \mathfrak{f}$, $a \ll d_1$, $a \ll d_2$, then we can choose an element $c \in \mathfrak{f}$ such that $c \ll c_1 \wedge c_2$, and an element $d \in B$ such that $a \ll d \ll d_1 \wedge d_2$ (this by P4 and P6); this shows that $c \wedge d \in \mathfrak{g}$, and also: $c \wedge d \ll (c_1 \wedge c_2) \wedge (d_1 \wedge d_2) = (c_1 \wedge d_1) \wedge (c_2 \wedge d_2)$. So F1 and F2 are both satisfied by $\mathfrak{g}$.

Remark. The theorem is a special case of results by J. G. Horne [17]. Namely, in his terminology, a compingent algebra is also a $\gg$-semiring in 1, with "$\wedge$" as addition, and "$\vee$" as multiplication; then the theorem follows from [17, 4.13 and 4.15].

1.2.3. Theorem. Let B be a compingent algebra, and $a \in B$, $a \neq 0$. Then there exists a maximal concordant filter of B, which contains a.

Proof. By P6 there exists an element $a_1 \in B$, such that $0 < a_1 \ll a$; by induction the existence follows of elements $a_i \in B$ $(i = 1, 2, \ldots)$ such that $a_1 < a$ and $0 < a_{i+1} \ll a_i$ $(i = 1, 2, \ldots)$. Now it is easily seen that

$$\{b \in B \mid \exists a_i \text{ such that } a_i \leq b\}$$

is a proper concordant filter of B which contains a. So there also exists a maximal concordant filter of B which contains a.

1.3. Topological representation.

1.3.1. Notation. Let B be a compingent algebra. Then $\mathfrak{M}_B$ will denote its set of maximal concordant filters. Let ω_B be the mapping defined by:

$$\omega_B(a) = \{\mathfrak{m} \mid a \in \mathfrak{m} \in \mathfrak{M}_B\} \quad (a \in B).$$

1.3.2. Theorem. For any elements a and b of a compingent algebra B, we have:

(i) $a \leq b \Leftrightarrow \omega_B(a) \subseteq \omega_B(b)$,
(ii) $\omega_B(a \wedge b) = \omega_B(a) \cap \omega_B(b)$.
Moreover: (iii) $\omega_B[B]$ is a basis for a topology on $\mathfrak{M}_B$.

Proof. By F1, it is evident that $a \leq b \Rightarrow \omega_B(a) \subseteq \omega_B(b)$. The converse implication follows from theorem 1.2.3: if $a \not\leqq b$, i.e. $a \wedge b^o \neq 0$, and if $a \wedge b^o \in \mathfrak{m} \in \mathfrak{M}_B$, then

$$\mathfrak{m} \in \omega_B(a), \ \mathfrak{m} \notin \omega_B(b).$$

Since for $\mathfrak{m} \in \mathfrak{M}_B$:

$$a \wedge b \in \mathfrak{m} \Rightarrow a \in \mathfrak{m} \text{ and } b \in \mathfrak{m},$$

and
$$a,b \in \mathfrak{m} \Rightarrow a \wedge b \in \mathfrak{m},$$

also (ii) is valid.

It is obvious that $\omega_B(1) = \mathfrak{M}_B$ and $\omega_B(0) = \emptyset$. By (ii), $\omega_B[B]$ is a basis for a topology on $\mathfrak{M}_B$; however, since we consider only Hausdorff spaces, we must show:

$$\mathfrak{m}_1,\mathfrak{m}_2 \in \mathfrak{M}_B, \ \mathfrak{m}_1 \neq \mathfrak{m}_2 \Rightarrow \exists \ a_1 \in \mathfrak{m}_1, a_2 \in \mathfrak{m}_2 \text{ such that } a_1 \wedge a_2 = 0.$$

But if e.g. $a \in \mathfrak{m}_1 \in \mathfrak{M}_B$, $a \notin \mathfrak{m}_2 \in \mathfrak{M}_B$, we can choose $a_1 \in \mathfrak{m}_1$ such that $a_1 \ll a$; then $a_1^o \in \mathfrak{m}_1$ by theorem 1.2.2, and we need only take $a_2 = a_1^o$ to prove the implication.

Remarks. 1. Henceforth $\mathfrak{M}_B$ will be considered as a topological space provided with the topology for which $\omega_B[B]$ is a basis.

2. The topology of $\mathfrak{M}_B$ is the *dual Stone topology* on the set $\mathfrak{M}_B$ of subsets of B (cf. J. G. Horne [17] and M. H. Stone [32]).

1.3.3. Definition. If B is a compingent algebra, and B' a compingent algebra with compingent relation $\ll'$, then B' is called a *subcompingent subalgebra* of B if it is a subboolean subalgebra of B, and if the following conditions are satisfied:

S1. $0 \in B'$,

S2. $a,b \in B' \Rightarrow a \wedge b \in B'$,

S3. $a,b \in B'$, $a \ll' b \Rightarrow a^{o'o} \ll b$, where $a^{o'}$ is the complement of a in B'.

If, moreover, B' is a boolean subalgebra of B, and

$$a,b \in B', \ a \ll b \Rightarrow a \ll' b,$$

then we shall call B' a *compingent subalgebra* of B.

If B' is a subcompingent subalgebra of B and the sets of B and B' coincide, then B' will be said to be a *subcompingent algebra* of B.

For convenience, we introduce the convention that the complementation operation and the join operation of a subcompingent subalgebra will be provided with the same dash, subscript or superscript as the compingent relation of the subcompingent subalgebra.

If a subset B' of a compingent algebra B is a subcompingent subalgebra of B by means of the restriction of the compingent relation of B to B', then B' will be considered as such.

1.3.4. Proposition. Let B' be a subcompingent subalgebra of the compingent algebra B, and $\ll'$ the compingent relation of B'. Then:

$$a,b \in B',\ a \ll' b \Rightarrow a \ll b.$$

Proof. Let $a,b \in B'$, $a \ll' b$. Then $a^{o'o} \ll b$, by S3. Since $0 \in B'$, we have: $a \wedge a^{o'} = 0$, whence $a^{o'} \leq a^{o}$. Hence also: $a \leq a^{o'o}$. But then $a \ll b$, by P3.

Remark. The proposition clearly implies that B' is a compingent subalgebra of B if and only if B' is a boolean subalgebra of B, and for $a,b \in B'$:

$$a \ll' b \Leftrightarrow a \ll b.$$

1.3.5. Lemma. Let B be a compingent algebra. Then:

(i) $\omega_B[B]$ is a boolean subalgebra of $\mathrm{B}(\mathfrak{M}_B)$.

(ii) $a,b \in B,\ a \ll b \Rightarrow (\omega_B(a))^- \subseteq \omega_B(b)$.

Proof. For any subset V of $\mathfrak{M}_B$, we have:

$$\begin{aligned} \overline{V} &= \{\mathfrak{m} \in \mathfrak{M}_B \mid a \in \mathfrak{m} \Rightarrow \exists\ \mathfrak{m}' \in V \cap \omega_B(a)\} \\ &= \{\mathfrak{m} \in \mathfrak{M}_B \mid a \in \mathfrak{m} \Rightarrow a \in \bigcup V\} \\ &= \{\mathfrak{m} \in \mathfrak{M}_B \mid \mathfrak{m} \subseteq \bigcup V\}. \end{aligned}$$

Hence for $a \in B$:

$$\begin{aligned} (\omega_B(a))^- &= \{\mathfrak{m} \in \mathfrak{M}_B \mid \mathfrak{m} \subseteq \bigcup \omega_B(a)\}, \\ (\omega_B(a))^{-c-c} &= \{\mathfrak{n} \in \mathfrak{M}_B \mid \mathfrak{n} \nsubseteq \bigcup \{\mathfrak{m} \in \mathfrak{M}_B \mid \mathfrak{m} \nsubseteq \bigcup \omega_B(a)\}\}. \end{aligned}$$

But for $b \in B$:

$$\begin{aligned} b \in \bigcup \omega_B(a) &\Leftrightarrow \exists\ \mathfrak{m} \in \mathfrak{M}_B \text{ such that } a,b \in \mathfrak{m} \\ &\Leftrightarrow a \wedge b \neq 0 \qquad \text{(by theorem 1.2.3).} \end{aligned}$$

So for $\mathfrak{m} \in \mathfrak{M}_B$:

$$\mathfrak{m} \notin \bigcup \omega_B(a) \Leftrightarrow \exists\, c \in \mathfrak{m} \text{ such that } a \wedge c = 0,$$

whence $\bigcup \{\mathfrak{m} \in \mathfrak{M}_B \mid \mathfrak{m} \notin \bigcup \omega_B(a)\} = \{c \in B \mid c \wedge a^o \neq 0\}$,

using again theorem 1.2.3.

Hence for $\mathfrak{n} \in \mathfrak{M}_B$:

$$\begin{aligned} \mathfrak{n} \in (\omega_B(a))^{-c-c} &\Leftrightarrow \exists\, d \in \mathfrak{n} \text{ such that } d \wedge a^o = 0 \\ &\Leftrightarrow \mathfrak{n} \in \omega_B(a); \end{aligned}$$

in other words, $(\omega_B(a))^{-c-c} = \omega_B(a)$; this shows that $\omega_B(a) \in \mathrm{B}(\mathfrak{M}_B)$

Now it follows from theorem 1.3.2 (i), that $\omega_B[B]$ is a subboolean subalgebra of $\mathrm{B}(\mathfrak{M}_B)$. By virtue of theorem 1.3.2 (ii), for the first assertion of the lemma we need only show that $(\omega_B(a))^{-c} = \omega_B(a^o)$ for $a \in B$ (cf. R. Sikorski [27, § 4]), since in $\mathrm{B}(\mathfrak{M}_B)$:

$$O_1 \wedge O_2 = O_1 \cap O_2,\, O^o = O^{-c}.$$

Let $a \in B$; since $\omega_B(0) = \omega_B(a \wedge a^o) = \omega_B(a) \cap \omega_B(a^o)$, we have

$$\omega_B(a^o) \subseteq (\omega_B(a))^{-c};$$

but $\mathfrak{m} \in (\omega_B(a))^{-c} \Rightarrow \exists\, b \in \mathfrak{m}$ such that $a \wedge b = 0 \Rightarrow a^o \in \mathfrak{m}$, which shows the required converse inclusion.

The second assertion of the lemma follows from theorem 1.2.2: let $a,b \in B$, $a \ll b$ and $\mathfrak{m} \in (\omega_B(a))^-$; then $\mathfrak{m} \notin (\omega_B(a))^{-c} = \omega_B(a^o)$, whence $a^o \notin \mathfrak{m}$ and $b \in \mathfrak{m}$; so $(\omega_B(a))^- \subseteq \omega_B(b)$.

1.3.6. Theorem. Let B be a compingent algebra. Then $\mathfrak{M}_B$ is compact.

Remark. The theorem is contained in more general results of J. G. Horne [17, theorem 3.1 and theorem 3.10]; for completeness we present a new proof for the theorem.

Proof. If $\mathfrak{m} \in \mathfrak{M}_B$ and $\mathfrak{m} \in \omega_B(a)$, then there exists an element $b \in \mathfrak{m}$ such that $b \ll a \wedge a = a$; so $\mathfrak{m} \in \omega_B(b) \subseteq (\omega_B(b))^- \subseteq \omega_B(a)$ by lemma 1.3.5. This shows that $\mathfrak{M}_B$ is regular.

Let Σ be a set of closed sets of $\mathfrak{M}_B$ satisfying:

$$\emptyset \notin \Sigma;\; F_1,F_2 \in \Sigma \Rightarrow F_1 \cap F_2 \in \Sigma.$$

To prove the compactness of $\mathfrak{M}_B$, it is necessary and sufficient to show that $\bigcap \Sigma \neq \emptyset$.

Let $\mathfrak{f} = \{a \in B \mid \exists\, F \in \Sigma,\ b \in B \text{ such that } F \subseteq \omega_B(b),\ b \ll a\}$.

Firstly, we shall see that $\mathfrak{f}$ is a proper concordant filter of B. If $a \in \mathfrak{f}$, $a \leq c$, then $c \in \mathfrak{f}$. Now let $a_1,a_2 \in \mathfrak{f}$, and $b_1,b_2 \in B$, $F_1,F_2 \in \Sigma$ such that $b_i \ll a_i$, $F_i \subseteq \omega_B(b_i)$ $(i = 1,2)$; choose $c \in B$ such that $b_1 \wedge b_2 \ll c \ll a_1 \wedge a_2$; since $F_1 \cap F_2 \in \Sigma$ and $F_1 \cap F_2 \subseteq \subseteq \omega_B(b_1) \cap \omega_B(b_2) = \omega_B(b_1 \wedge b_2)$, we obtain that $c \in \mathfrak{f}$; this shows that $\mathfrak{f}$ is a proper concordant filter.

Secondly, we shall show that $\mathfrak{m} \in \bigcap \Sigma$, whence $\bigcap \Sigma \neq \emptyset$, if $\mathfrak{f} \subseteq \mathfrak{m} \in \mathfrak{M}_B$. Indeed, suppose $\mathfrak{m} \notin \bigcap \Sigma$, say $\mathfrak{m} \notin F \in \Sigma$. Since $\mathfrak{M}_B$ is regular and $\omega_B[B]$ a basis for the topology of $\mathfrak{M}_B$, there are elements $a,b \in \mathfrak{m}$ such that:

$$a \ll b,\ (\omega_B(b))^- \cap F = \emptyset.$$

But then $b^o \ll a^o$ and $F \subseteq (\omega_B(b))^{-c} = \omega_B(b^o)$; hence $a^o \in \mathfrak{f} \subseteq \mathfrak{m}$, in contradiction to $a \in \mathfrak{m}$. This proves the theorem.

1.3.7. Definition. A subset B' of a compingent algebra B is called *dense in* B if for every $a,b \in B$ with $a \ll b$ there exists an element $c \in B'$ such that $a \ll c \ll b$.

1.3.8. Definition. The compingent algebras B_1 and B_2 are called *isomorphic* if there exists a boolean isomorphism f of B_1 onto B_2 such that for $a,b \in B$:

$$a \ll_1 b \Leftrightarrow f(a) \ll_2 f(b),$$

where $\ll_i$ is the compingent relation of B_i $(i = 1, 2)$.

1.3.9. Theorem. Let B be a compingent algebra. Then:

(i) $\omega_B[B]$ is a dense compingent subalgebra of $\mathrm{B}(\mathfrak{M}_B)$,

(ii) ω_B is an isomorphism of B onto $\omega_B[B]$.

Proof. By lemma 1.3.5, $\omega_B[B]$ is a boolean subalgebra of $\mathrm{B}(\mathfrak{M}_B)$, and by theorem 1.3.2 (i), ω_B is a boolean isomorphism of B onto $\omega_B[B]$.

Let $O_1,O_2 \in \mathrm{B}(\mathfrak{M}_B)$, $\bar{O}_1 \subseteq O_2$ (by theorem 1.3.6, this is equivalent to $O_1 \ll O_2$). For every $\mathfrak{m} \in \bar{O}_1$, choose $c_\mathfrak{m}$, $d_\mathfrak{m} \in \mathfrak{m}$ such that $c_\mathfrak{m} \ll d_\mathfrak{m}$ and $\omega_B(d_\mathfrak{m}) \subseteq O_2$. Since $\bar{O}_1$ is compact, there exist elements $\mathfrak{m}_1,\mathfrak{m}_2, \ldots, \mathfrak{m}_n \in \mathfrak{M}_B$, such that:

$$\bar{O}_1 \subseteq \bigcup_{i=1}^{n} \omega_B(c_{\mathfrak{m}_i}).$$

Put $c = \bigvee_{i=1}^{n} c_{\mathfrak{m}_i}$, $d = \bigvee_{i=1}^{n} c_{\mathfrak{m}_i}$.

Then $c \ll d$, and $\bar{O}_1 \subseteq \omega_B(c) \subseteq (\omega_B(c))^- \subseteq \omega_B(d) \subseteq O_2$ (using lemma 1.3.5). In particular,

$$O_1 \ll \omega_B(c) \ll O_2;$$

so $\omega_B[B]$ is dense in $\mathrm{B}(\mathfrak{M}_B)$.

If we take $O_1 = \omega_B(a)$, $O_2 = \omega_B(b)$, for $a,b \in B$, then we obtain:

$$a \leq c \ll d \leq b,$$

whence $a \ll b$. This shows that $\omega_B[B]$ is a compingent subalgebra of $\mathrm{B}(\mathfrak{M}_B)$ (use the remark to proposition 1.3.4).

1.3.10. Definition. Let B be a compingent algebra. An element a of B is called *discrete* if $a \ll a$; B is called *discrete* if all its elements are discrete.

1.3.11. Theorem. Let B be a compingent algebra.

Then: (i) $\omega_B[B]$ contains all open-and-closed sets of $\mathfrak{M}_B$,

(ii) for $a \in B$: a discrete $\Leftrightarrow$ $\omega_B(a)$ open and closed,

(iii) B discrete $\Leftrightarrow$ $\omega_B[B]$ consists of all open-and-closed sets of $\mathfrak{M}_B$.

Proof. (i) This assertion is an immediate consequence of theorem 1.3.9 (i).

(ii) Let $a \in B$, $a \ll a$. Then, for any $\mathfrak{m} \in \mathfrak{M}_B$:

$$a \in \mathfrak{m} \text{ or } a^0 \in \mathfrak{m} \qquad \text{(theorem 1.2.2).}$$

So $\mathfrak{M}_B = \omega_B(a) \cup \omega_B(a^0)$, whence $\omega_B(a)$ is open and closed. The converse implication follows from theorem 1.3.9 (i).

The last assertion is a consequence of the other two.

Remark. Theorem (iii) is well-known, since for discrete B, $\mathfrak{M}_B$ turns into the Stone space of the boolean algebra B (cf. M. H. Stone [32]).

1.3.12. Theorem. If B is a compingent algebra, then the lattice of the concordant filters of B is anti-isomorphic to the lattice of the closed sets of $\mathfrak{M}_B$ (both lattices in the set-theoretic partial ordering).

Proof. For a concordant filter $\mathfrak{f}$ of B, let us define:

$$\mathrm{F}(\mathfrak{f}) = \bigcap \{\omega_B(a) \mid a \in \mathfrak{f}\}.$$

Since for every $a \in \mathfrak{f}$ there exists an element $b \in \mathfrak{f}$ such that $b \ll a$, whence $(\omega_B(b))^- \subseteq \omega_B(a)$, it follows that $\mathrm{F}(\mathfrak{f})$ is a closed set of $\mathfrak{M}_B$, being an intersection of closed sets. It is clear that

$$\mathfrak{f} \subseteq \mathfrak{g} \Rightarrow \mathrm{F}(\mathfrak{f}) \supseteq \mathrm{F}(\mathfrak{g}),$$

if $\mathfrak{f}$ and $\mathfrak{g}$ are concordant filters of B.

For a closed set F of $\mathfrak{M}_B$, let us define:

$$\mathfrak{f}(F) = \{a \in B \mid F \subseteq \omega_B(a)\}.$$

It is easy to see that $\mathfrak{f}(F)$ is a concordant filter of B.

Moreover, by conventional techniques, it is readily verified that for any concordant filter $\mathfrak{f}$ of B, and for any closed set F of $\mathfrak{M}_B$:

$$\mathfrak{f}(\mathrm{F}(\mathfrak{f})) = \mathfrak{f}, \text{ and } \mathrm{F}(\mathfrak{f}(F)) = F.$$

This proves the theorem.

1.4. Complete compingent algebras.

1.4.1. Definition. A compingent algebra is called *complete* if its boolean algebra is complete.

1.4.2. Lemma. Let B be a compingent algebra, and A a subset of B. Then:

$$\bigvee A \text{ exists in } B \Rightarrow (\bigcup \omega_B[A])^{-c-c} = \omega_B(\bigvee A).$$

Proof. According to the proof of lemma 1.3.5, we have:

$$(\bigcup \omega_B[A])^{-c-c} = \{\mathfrak{n} \in \mathfrak{M}_B \mid \mathfrak{n} \nsubseteq \bigcup \{\mathfrak{m} \in \mathfrak{M}_B \mid \mathfrak{m} \nsubseteq \bigcup\bigcup \omega_B[A]\}\}.$$

But

$$b \in \bigcup\bigcup \omega_B[A] \Leftrightarrow \exists\, a \in A \text{ such that } b \wedge a \neq 0,$$

whence

$$\mathfrak{m} \notin \bigcup\bigcup\omega_B[A] \Leftrightarrow \exists\, c \in \mathfrak{m} \text{ such that } c \wedge a = 0 \text{ for all } a \in A.$$

If $\bigvee A$ exists, the last condition is equivalent to: $\exists\, c \in \mathfrak{m}$ such that $c \wedge (\bigvee A) = 0$ (cf. R. Sikorski [27, Ch. II, § 19]). The rest of the proof is a repetition of the corresponding part of the proof of lemma 1.3.5.

1.4.3. Theorem. If B is a complete compingent algebra, then ω_B is an isomorphism of B onto $\mathrm{B}(\mathfrak{M}_B)$.

Proof. In view of theorem 1.3.9, it needs only to be shown that ω_B is a mapping onto $\mathrm{B}(\mathfrak{M}_B)$. Let O be a regularly open set of $\mathfrak{M}_B$. Since $\omega_B[B]$ is a basis for the topology of $\mathfrak{M}_B$, there exists a subset A of B such that

$$O = \bigcup \omega_B[A].$$

However, B being complete, by lemma 1.4.2 and the regular openness of O:

$$O = \omega_B(\bigvee A).$$

Hence $O \in \omega_B[B]$.

1.4.4. Theorem. Let C be a compact space, and B a compingent subalgebra of $\mathrm{B}(C)$, such that B is also a basis for the topology of C. Then the mapping μ_B, defined by

$$\mu_B(p) = \{a \in B \mid p \in a\} \quad \text{for } p \in C,$$

is a homeomorphism of C onto $\mathfrak{M}_B$ having the following property:

$$\mu_B[a] = \omega_B(a) \qquad (a \in B).$$

Proof. Let $p \in C$; the verification of the fact that $\mu_B(p)$ is a proper concordant filter of B, is immediate; $\mu_B(p)$ is also maximal since $a,b \in B$, $a \ll b$ implies that $p \in b$ or $p \in a^0$ (because $a \ll b$ means: $\bar{a} \subseteq b$). It is clear that the mapping μ_B is one-to-one into. If $a \in B$, and $p \in a$, then $\mu_B(p) \in \omega_B(a)$; this shows that $\mu_B[C]$ is dense in $\mathfrak{M}_B$. By the compactness of C, for μ_B to be a homeomorphism of C onto $\mathfrak{M}_B$, it suffices to show that μ_B is continuous; this is evident since

$$a \in B \Rightarrow \mu_B^{-1}[\omega_B(a)] = a$$

and $\omega_B[B]$ is a basis for the topology of $\mathfrak{M}_B$.

1.4.5. Theorem.

(i) Every compact space is homeomorphic to a space of the type $\mathfrak{M}_B$, where B is a complete compingent algebra.

(ii) If B_1 and B_2 are complete compingent algebras, then: B_1 isomorphic to $B_2 \Leftrightarrow \mathfrak{M}_{B_1}$ homeomorphic to $\mathfrak{M}_{B_2}$.

(iii) If C_1 and C_2 are compact spaces, then: C_1 homeomorphic to $C_2 \Leftrightarrow \mathrm{B}(C_1)$ isomorphic to $\mathrm{B}(C_2)$.

Proof. The first assertion is contained in theorem 1.4.4. Let B_1 and B_2 be complete compingent algebras. If they are isomorphic, then, a fortiori, $\mathfrak{M}_{B_1}$ and $\mathfrak{M}_{B_2}$ are homeomorphic. Since B_1 and B_2 are complete, B_1 is isomorphic to $\mathrm{B}(\mathfrak{M}_{B_1})$ and B_2 is isomorphic to $\mathrm{B}(\mathfrak{M}_{B_2})$, by theorem 1.4.3. Since the homeomorphy of the compact spaces C_1 and C_2 implies the isomorphy of $\mathrm{B}(C_1)$ and $\mathrm{B}(C_2)$, the rest of the second assertion is obvious.

The third assertion is an immediate consequence of theorem 1.4.3.

1.5. Homomorphisms.

1.5.1. Definition. Let B_1 and B_2 be compingent algebras. A mapping h of B_1 into B_2 is a *homomorphism* if the following three conditions are satisfied for $a,b \in B_1$:

H1. $h(0) = 0$,

H2. $h(a \wedge b) = h(a) \wedge h(b)$,

H3. $a \ll b \Rightarrow (h(a^0))^0 \ll h(b)$.

The homomorphism h is called *chary* if:

H4. $h(a) = \bigvee \{h(c) \mid c \ll a\} \qquad (a \in B_1)$.

The homomorphism h will be called *full* if:

H5. $h(a) \ll h(b) \Rightarrow \exists\ a',b' \in B_1$ such that $a' \ll b'$ and $h(a') = h(a),\ h(b') = h(b) \quad (a,b \in B_1)$.

1.5.2. Proposition. Let h be a homomorphism of the compingent algebra B_1 into the compingent algebra B_2. Then:

(i) $h(1) = 1$,

(ii) $a,b \in B_1,\ a \leq b \Rightarrow h(a) \leq h(b)$,

(iii) $a,b \in B_1,\ a \ll b \Rightarrow h(a) \ll h(b)$.

Proof.

(i) Since $1 \ll 1$, we have $(h(1^o))^o \ll h(1)$, by H3.
Hence $(h(0))^o = 1 \ll h(1)$, and $h(1) = 1$.

(ii) Since $a \leq b \Leftrightarrow a \wedge b = a$, the assertion follows from H2.

(iii) Let $a,b \in B_1$, $a \ll b$. Then

$$0 = h(0) = h(a \wedge a^o) = h(a) \wedge h(a^o) \text{ by H2,}$$

whence $h(a) \leq (h(a^o))^o$.
But also $(h(a^o))^o \ll h(b)$, by H3; hence $h(a) \ll h(b)$.

1.5.3. Theorem. If h is a homomorphism of the compingent algebra B_1 into the complete compingent algebra B_2, then there exists a canonical chary homomorphism h^* of B_1 into B_2, defined by:

$$h^*(a) = \vee \{h(c) \mid c \ll a, c \in B_1\} \qquad (a \in B_1).$$

Proof. It is obvious that $h^*(0) = 0$. Further, let $a,b \in B_1$. It is clear that $h^*(a \wedge b) \leq h^*(a) \wedge h^*(b)$. Suppose $h^*(a \wedge b) < h^*(a) \wedge h^*(b)$, i.e. $x = h^*(a) \wedge h^*(b) \wedge (h^*(a \wedge b))^o \neq 0$. Since $0 < x \leq h^*(a)$, there exists an element $c \in B_1$, such that $c \ll a$ and $x \wedge h(c) \neq 0$. Since $0 < x \wedge h(c) \leq h^*(b)$, there exists an element $d \in B_1$ such that $d \ll b$ and $x \wedge h(c) \wedge h(d) \neq 0$. Hence $c \wedge d$ is such that $c \wedge d \ll a \wedge b$ and $x \wedge h(c \wedge d) \neq 0$; this is in contradiction to the assumption $x \wedge h^*(a \wedge b) = 0$. So axiom H2 is verified.

Now let $a,b \in B_1$, $a \ll b$. Choose $c_1,c_2 \in B_1$ such that $a \ll c_1 \ll c_2 \ll b$. By the definition of h^*, we have:

$$h(c_2) \leq h^*(b) \text{ and } h(c_1^o) \leq h^*(a^o).$$

Hence $\qquad (h^*(a^o))^o \leq (h(c_1^o))^o \ll h(c_2) \leq h^*(b)$, by H3;
so $(h^*(a^o))^o \leq h^*(b)$. Therefore, axiom H3 is satisfied by h^*.

Since for $a \in B_1$:

$$\begin{aligned} h^*(a) &= \vee \{h(c) \mid c \ll a, c \in B_1\} \\ &= \vee \{h(c) \mid c \ll c' \ll a, c,c' \in B_1\} \\ &\leq \vee \{h^*(c') \mid c' \ll a, c' \in B_1\}, \end{aligned}$$

also $h^*(a) = \vee \{h^*(c) \mid c \ll a, c \in B_1\}$; so h^* is chary.

1.5.4. Proposition. Let B_1,B_2 and B_3 be compingent algebras, and h_1 a homomorphism of B_1 into B_2, h_2 a homomorphism of B_2 into B_3. Then $h_2 \circ h_1$ is a homomorphism of B_1 into B_3. If both B_2 and B_3 are complete, then $(h_2 \circ h_1)^* = (h_1^* \circ h_2^*)^*$.

Proof. The conditions H1 and H2 are trivially satisfied by $h_2 \circ h_1$. If $a,b \in B_1$, $a \ll b$, then $(h_1(a^o))^o \ll h_1(b)$; this implies:

$$(h_2(h_1(a^o)))^o \ll h_2(h_1(b));$$

hence $h_2 \circ h_1$ satisfies H3.

Now let $a \in B_1$. Then:

$$(h_2 \circ h_1)^*(a) = \bigvee \{h_2(h_1(c)) \mid c \ll a\},$$

whilst

$$(h_2^* \circ h_1^*)^*(a) = \bigvee \{h_2(b) \mid b \in B_2 \text{ such that } \exists\, d \in B_1 \text{ with } d \ll a \text{ and } b \ll \bigvee \{h_1(e) \mid e \ll d,\ e \in B_1\}\}.$$

It is clear that $(h_2^* \circ h_1^*)^*(a) \leq (h_2 \circ h_1)^*(a)$, since if $d \in B_1$ and $b \in B_2$, as indicated, then $h_2(b) \leq h_2(h_1(d)) \leq (h_2 \circ h_1)^*(a)$.

To show the converse inequality, let $c \in B_1$, $c \ll a$. Choose $e,d \in B_1$, such that $c \ll e \ll d \ll a$. Then

$$h_2(h_1(c)) \leq (h_2^* \circ h_1^*)^*(a);$$

this shows that

$$(h_2 \circ h_1)^*(a) \leq (h_2^* \circ h_1^*)^*(a).$$

1.5.5. Proposition. A homomorphism of a compingent algebra B_1 into a compingent algebra B_2 is an isomorphism if and only if it is one-to-one onto, and full.

Proof. The necessity of the conditions is obvious. So let h satisfy the conditions. From proposition 1.5.2 (iii) and H5, it is clear that for elements $a,b \in B_1$:

$$a \ll b \Leftrightarrow h(a) \ll h(b).$$

Hence, it need only be verified that h is a boolean isomorphism. This follows from theorem 1.1.4.

1.5.6. Proposition. A compingent algebra B', which is also a subset of the compingent algebra B, is a subcompingent subalgebra of B if and only if the injection mapping $\iota_{B',B}$ is a homomorphism of B' into B.

Proof. If B' is a subcompingent subalgebra of B, axioms S1-S3 imply the axioms H1-H3, respectively, for $h = \iota_{B',B}$.

Conversely, let $\iota_{B',B}$ be a homomorphism of B' into B. Then again axioms S1-S3 are implied by the axioms H1-H3, respectively. It remains to be verified that B' is a subboolean subalgebra of B. But if $a,b \in B'$, and $a \wedge' b$ the meet of a and b in B', then $a \wedge' b = = a \wedge b$, by H2 (where $a \wedge b$ is the meet of a and b in B). This implies that B' is a subboolean subalgebra of B.

1.6. Continuous functions and homomorphisms.

1.6.1. Lemma. Let T be a topological space, and O_1 and O_2 open sets of T. Then:

$$(O_1 \cap O_2)^{-c-c} = O_1{}^{-c-c} \cap O^{-c-c}.$$

Proof. Since $(O_1 \cap O_2)^- \subseteq O_1^- \cap O_2^-$, it is clear that

$$(O_1 \cap O_2)^{-c-c} \subseteq O_1^{-c-c} \cap O_2^{-c-c}.$$

However, both open sets $O_1 \cap O_1^{-c-c} \cap O_2^{-c-c}$ and $O_2 \cap O_1^{-c-c} \cap O_2^{-c-c}$ being dense in $O_1^{-c-c} \cap O_2^{-c-c}$, also

$$O_1 \cap O_2 \cap O_1^{-c-c} \cap O_2^{-c-c} \text{ is dense in } O_1^{-c-c} \cap O_2^{-c-c};$$

hence $$(O_1 \cap O_2)^{-c-c} \supseteq O_1^{-c-c} \cap O_2^{-c-c}$$

since $O_1^{-c-c} \cap O_2^{-c-c}$ is regularly open.

1.6.2. Theorem. Let B_1 be a compingent algebra, and B_2 a complete compingent algebra. Then for any continuous mapping φ of $\mathfrak{M}_{B_2}$ into $\mathfrak{M}_{B_1}$ the mapping $\mathrm{b}(\varphi)$ of B_1 into B_2, defined by

$$\omega_{B_2}(\mathrm{b}(\varphi)\,(a)) = \varphi^{-1}[\omega_{B_1}(a)]^{-c-c} \qquad (a \in B_1),$$

is a chary homomorphism of B_1 into B_2.

Proof. Since B_2 is complete, $\mathrm{b}(\varphi)$ is well defined on the whole of B_1, as follows from theorem 1.4.3. Let us check the axioms H1-H4. Clearly, $\mathrm{b}(\varphi)\,(0) = 0$. To show H2, let $a,b \in B_1$. Then:

$$\omega_{B_1}(a) \cap \omega_{B_1}(b) = \omega_{B_1}(a \wedge b) \qquad \text{(theorem 1.3.2.)},$$

whence $\varphi^{-1}[\omega_{B_1}(a)] \cap \varphi^{-1}[\omega_{B_1}(b)] = \varphi^{-1}[\omega_{B_1}(a \wedge b)]$;

so, by lemma 1.6.1,

$$\omega_{B_2}(\mathrm{b}(\varphi)\,(a)) \cap \omega_{B_2}(\mathrm{b}(\varphi)\,(b)) = \omega_{B_2}(\mathrm{b}(\varphi)\,(a \wedge b));$$

and by theorem 1.3.9: $\mathrm{b}(\varphi)\,(a) \wedge \mathrm{b}(\varphi)\,(b) = \mathrm{b}(\varphi)\,(a \wedge b)$.

For the verification of axiom H3, assume $a, b \in B_1$, $a \ll b$.

Since
$$\mathfrak{M}_{B_1} = \omega_{B_1}(a^o) \cup \omega_{B_1}(b), \text{ also}$$
$$\mathfrak{M}_{B_2} = \varphi^{-1}[\omega_{B_1}(a^o)] \cup \varphi^{-1}[\omega_{B_1}(b)];$$

hence
$$\mathfrak{M}_{B_2} = \omega_{B_2}(\mathrm{b}(\varphi)\ (a^o)) \cup \omega_{B_2}(\mathrm{b}(\varphi)\ (b)).$$

This implies:
$$(\mathrm{b}(\varphi)\ (a^o))^o \ll \mathrm{b}(\varphi)\ (b).$$

It remains to show H4. If $a \in B_1$, then
$$\omega_{B_1}(a) = \bigcup \{\omega_{B_1}(b) \mid b \in B_1,\ b \ll a\},$$

and
$$\varphi^{-1}[\omega_{B_1}(a)] = \bigcup \{\varphi^{-1}[\omega_{B_1}(b)] \mid b \in B_1,\ b \ll a\}.$$

This yields
$$\mathrm{b}(\varphi)\ (a) = \bigvee \{\mathrm{b}(\varphi)\ (b) \mid b \in B_1,\ b \ll a\}.$$

1.6.3. Theorem. Let h be a homomorphism of the compingent algebra B_1 into the compingent algebra B_2. Then there exists a canonical continuous mapping $\mathrm{m}(h) = \varphi$ of $\mathfrak{M}_{B_2}$ into $\mathfrak{M}_{B_1}$. If h is chary, then:
$$((\mathrm{m}(h))^{-1}[\omega_{B_1}(a)])^{-c-c} = \omega_{B_2}(h(a)) \qquad (a \in B_1).$$

If B_2 is complete, then:
$$((\mathrm{m}(h))^{-1}[\omega_{B_1}(a)])^{-c-c} = \omega_{B_2}(h^*(a)) \qquad (a \in B_1).$$

Proof. For $\mathfrak{n} \in \mathfrak{M}_{B_2}$, let $\varphi(\mathfrak{n})$ be defined as:
$$\varphi(\mathfrak{n}) = \{a \in B_1 \mid \exists\ b \in B_1 \text{ such that } b \ll a \text{ and } h(b) \in \mathfrak{n}\}.$$

It will be shown that φ meets the requirements. From H1 it follows that $\varphi(\mathfrak{n}) \subset B_1$. If $a \in \varphi(\mathfrak{n})$, and $a \leq c \in B_1$, then also $c \in \varphi(\mathfrak{n})$. Now let $a_1, a_2 \in \varphi(\mathfrak{n})$, $b_1, b_2 \in B_1$ such that $b_1 \ll a_1$, $b_2 \ll a_2$ and $h(b_1), h(b_2) \in \mathfrak{n}$. Then also $h(b_1 \wedge b_2) = h(b_1) \wedge h(b_2) \in \mathfrak{n}$, and $b_1 \wedge b_2 \ll a_1 \wedge a_2$; if we choose $c \in B_1$ such that $b_1 \wedge b_2 \ll c \ll a_1 \wedge a_2$, then $c \in \varphi(\mathfrak{n})$; this proves that $\varphi(\mathfrak{n})$ is a proper concordant filter of B_1. Moreover, let $a, b \in B_1$, $a \ll b$. Choose $c, d \in B_1$, such that $a \ll c \ll d \ll b$. Then $(h(c^o))^o \ll h(d)$ by H3, whence $h(d) \in \mathfrak{n}$ or $h(c^o) \in \mathfrak{n}$, and consequently also: $b \in \varphi(\mathfrak{n})$ or $a^o \in \varphi(\mathfrak{n})$, by the definition of φ. This shows the maximality of $\varphi(\mathfrak{n})$.

The mapping φ is also continuous: if $\mathfrak{n} \in \mathfrak{M}_{B_2}$, and $a \in \varphi(\mathfrak{n})$, then there exists an element $b \in B_1$ such that $b \ll a$ and $h(b) \in \mathfrak{n}$; this implies: $\varphi[\omega_{B_2}(h(b))] \subseteq \omega_{B_1}(a)$; this shows the continuity of φ, since $\omega_{B_1}[\varphi(\mathfrak{n})]$ is a local basis of $\mathfrak{M}_{B_1}$ in $\varphi(\mathfrak{n})$.

Now let $a \in B_1$, and put

$$O = (\bigcup \{\omega_{B_2}(h(b)) \mid b \in B_1, b \ll a\})^{-c-c}.$$

If h is chary, then, by lemma 1.4.2, $O = \omega_{B_2}(h(a))$; if B_2 is complete, then $O = \omega_{B_2}(h^*(a))$.

So we have to show that

$$(\varphi^{-1}[\omega_{B_1}(a)])^{-c-c} = O.$$

It is an immediate consequence of the definitions of φ and O, that $\varphi^{-1}[\omega_{B_1}(a)] \subseteq O$. Since O is a regularly open set, we have also:

$$(\varphi^{-1}[\omega_{B_1}(a)])^{-c-c} \subseteq O.$$

For the converse inclusion it is sufficient to show that $\varphi^{-1}[\omega_{B_1}(a)]$ is dense in O. So let $c \in B_2$, such that $c \neq 0$, and $(\omega_{B_2}(c))^- \subseteq O$. By the definition of O, there exists an element $b \in B_1$, such that $b \ll a$ and $h(b) \wedge c \neq 0$; but if we choose $\mathfrak{n} \in \omega_{B_2}(h(b) \wedge c)$, then $\varphi(\mathfrak{n}) \in \omega_{B_1}(a)$ since $h(b) \wedge c \in \mathfrak{n}$ and hence also $h(b) \in \mathfrak{n}$; so

$$\omega_{B_2}(c) \cap \varphi^{-1}[\omega_{B_1}(a)] \neq \emptyset.$$

This ends the proof of the theorem.

1.6.4. Theorem. Let B_1 be a compingent algebra, and B_2 a complete compingent algebra. Then the mapping m, restricted to the set of all chary homomorphisms of B_1 into B_2, is a one-to-one mapping of this set onto the set of all continuous mappings of $\mathfrak{M}_{B_2}$ into $\mathfrak{M}_{B_1}$; the inverse of this restricted mapping is b.

Proof. To prove the theorem we need only verify:

(i) if h is a chary homomorphism of B_1 into B_2, then $\mathrm{bm}(h) = h$;

(ii) if φ is a continuous mapping of $\mathfrak{M}_{B_2}$ into $\mathfrak{M}_{B_1}$, then $\mathrm{mb}(\varphi) = \varphi$.

The first assertion follows from theorem 1.6.2 and theorem 1.6.3. For the second assertion, let $\mathfrak{n} \in \mathfrak{M}_{B_2}$. Then:

$\mathrm{mb}(\varphi)(\mathfrak{n}) = \{a \in B_1 \mid \exists\, b \in B_1 \text{ such that } b \ll a \text{ and } \mathrm{b}(\varphi)(b) \in \mathfrak{n}\}$

$$= \{a \in B_1 \mid \exists\, b \in B_1 \text{ such that } b \ll a \text{ and}$$
$$\mathfrak{n} \in (\varphi^{-1}[\omega_{B_1}(b)])^{-c-c}\}.$$

But if a and b are as above, then

$$(\varphi^{-1}[\omega_{B_1}(b)])^{-c-c} \subseteq \varphi^{-1}[\omega_{B_1}(a)],$$

whence $\mathfrak{n} \in \varphi^{-1}[\omega_{B_1}(a)]$, and $a \in \varphi(\mathfrak{n})$.

This shows that $\mathrm{mb}(\varphi)\ (\mathfrak{n}) \subseteq \varphi(\mathfrak{n})$; but since both sets are maximal concordant filters of B_1, also $\mathrm{mb}(\varphi)\ (\mathfrak{n}) = \varphi(\mathfrak{n})$.

1.6.5. Corollary. If B_1 is a compingent algebra, B_2 a complete compingent algebra, and h a homomorphism of B_1 into B_2, then $h^* = \mathrm{bm}(h)$, and $\mathrm{m}(h^*) = \mathrm{m}(h)$.

1.6.6. Theorem. Let B_1, B_2 and B_3 be compingent algebras, and h_1 a homomorphism of B_1 into B_2, h_2 a homomorphism of B_2 into B_3. Then:

$$\mathrm{m}(h_2 \circ h_1) = \mathrm{m}(h_1) \circ \mathrm{m}(h_2).$$

Proof. Take $\mathfrak{n} \in \mathfrak{M}_{B_3}$. Then:

$$\mathrm{m}(h_2 \circ h_1)\ (\mathfrak{n}) = \{a \in B_1 \mid \exists\, b \in B_1 \text{ such that } b \ll a \text{ and}$$
$$(h_2 \circ h_1)\ (b) \in \mathfrak{n}\},$$

$$(\mathrm{m}(h_1) \circ \mathrm{m}(h_2))\ (\mathfrak{n}) = \{a \in B_1 \mid \exists\, b \in B_1,\ d \in B_2 \text{ such that}$$
$$b \ll a,\ d \ll h_1(b),\ h_2(d) \in \mathfrak{n}\}.$$

This implies at once:

$$(\mathrm{m}(h_1) \circ \mathrm{m}(h_2))\ (\mathfrak{n}) \subseteq \mathrm{m}(h_2 \circ h_1)\ (\mathfrak{n}).$$

Since both sets are elements of $\mathfrak{M}_{B_1}$, we also have:

$$(\mathrm{m}(h_1) \circ \mathrm{m}(h_9))\ (\mathfrak{n}) = \mathrm{m}(h_2 \circ h_1)\ (\mathfrak{n}).$$

Hence $\mathrm{m}(h_1) \circ \mathrm{m}(h_2) = \mathrm{m}(h_2 \circ h_1)$.

1.6.7. Remark. If $\mathfrak{B}$ is the category of all compingent algebras, with homomorphisms as connecting morphisms, then the pair $(\mathfrak{M}, \mathrm{m})$ is a contravariant functor of $\mathfrak{B}$ into the category $\mathfrak{C}$ of all compact spaces, as follows from theorem 1.6.3 and theorem 1.6.6 (for the notion of category and functor, see S. Eilenberg and S. MacLane [9], and e.g. A. G. Kurosh, A. Kh. Livshits and E. G. Shul'geifer [21]).

If $\mathfrak{B}_1$ is the category of all complete compingent algebras, the restriction of $(\mathfrak{M},\mathrm{m})$ to $\mathfrak{B}_1$ maps non-isomorphic compingent algebras onto non-homeomorphic compact spaces. Moreover, if $\mathfrak{B}_2$ is the category of all complete compingent algebras, with only chary homomorphisms as morphisms and the composition $*$ being defined by: $h * g = (h \circ g)^*$, then, in the terminology of the second paper quoted, the categories $\mathfrak{B}_2$ and $\mathfrak{C}$ are coextensive, as follows from theorem 1.6.4 and corollary 1.6.5.

1.6.8. Example. Here we give an example of a product of two chary homomorphisms which is not chary. Consider the following three compact subspaces of the real line:

$$C_1 = [0,2],\ C_2 = [0,1] \cup [3,4],\ C_3 = [0,2] \cup [3,4].$$

Define $\varphi_1 \colon C_2 \to C_1$ by: $\varphi_1(x) = x \quad (0 \leq x \leq 1)$,
$\varphi_1(x) = x - 2 \quad (3 \leq x \leq 4)$;

$\varphi_2 \colon C_3 \to C_2$ by: $\varphi_2(x) = x \quad (0 \leq x \leq 1 \text{ or } 3 \leq x \leq 4)$,
$\varphi_2(x) = 1 \quad (1 \leq x \leq 2)$.

Then φ_1 and φ_2 are continuous. If we identify the spaces $\mathfrak{M}_{\mathrm{B}(C_i)}$ with C_i $(i = 1, 2, 3)$ in the natural way, we obtain chary homomorphisms $\mathrm{b}(\varphi_1)$ and $\mathrm{b}(\varphi_2)$, such that $\mathrm{b}(\varphi_1 \circ \varphi_2) = (\mathrm{b}(\varphi_2) \circ \mathrm{b}(\varphi_1))^*$. Applying $\mathrm{b}(\varphi_1 \circ \varphi_2)$ and $\mathrm{b}(\varphi_2) \circ \mathrm{b}(\varphi_1)$ to the element $[0,1[\in \mathrm{B}(C_1)$, we see that these two homomorphisms are different; so $\mathrm{b}(\varphi_2) \circ \mathrm{b}(\varphi_1)$ is not chary.

1.7. Some duality theorems.

1.7.1. Theorem. Let h be a homomorphism of the compingent algebra B_1 into the compingent algebra B_2. Then:

$$h \text{ one-to-one into} \Leftrightarrow \mathrm{m}(h) \text{ onto}.$$

Proof.

(i) Let h be one-to-one into. Taking an element $\mathfrak{m} \in \mathfrak{M}_{B_1}$, we find that

$$\mathfrak{f} = \{a \in B_2 \mid \exists\, b \in \mathfrak{m} \text{ such that } h(b) \leq a\}$$

is a proper concordant filter of B_2. If $\mathfrak{f} \subseteq \mathfrak{n} \in \mathfrak{M}_{B_2}$, then it is obvious that $\mathrm{m}(h)\,(\mathfrak{n}) \supseteq \mathfrak{m}$, whence $\mathrm{m}(h)\,(\mathfrak{n}) = \mathfrak{m}$.

(ii) Let $\mathrm{m}(h)$ be onto. Choose $a_1,a_2 \in B_1$ such that $a_1 \neq a_2$, e.g. $a_1^0 \wedge a_2 \in \mathfrak{m} \in \mathfrak{M}_{B_1}$. Assume $\mathfrak{m} = \mathrm{m}(h)\ (\mathfrak{n})$, where $\mathfrak{n}$ is some element of $\mathfrak{M}_{B_2}$. Then, evidently, $h(a_2) \in \mathfrak{n}$ and $h(a_1^0) \in \mathfrak{n}$. This implies $h(a_1) \notin \mathfrak{n}$, since $h(a_1) \wedge h(a_0^1) = 0 \notin \mathfrak{n}$. Hence, in particular, $h(a_1) \neq h(a_2)$.

1.7.2. Corollary. Any compact space C is a continuous image of a zero-dimensional compact space of the same weight.

Proof. Firstly, we notice the following easily verified fact: every infinite compact space C has a basis of potency $\mathrm{w}(C)$ which is a compingent subalgebra of $\mathrm{B}(C)$. So let B be such a basis for the topology of C (evidently, we may assume C to be infinite). If B_0 is the discrete compingent algebra whose boolean algebra coincides with the boolean algebra of B, and if h is the identity mapping of B into B_0, then h is a homomorphism, and $\mu_B^{-1} \circ \mathrm{m}(h)$ a continuous mapping of $\mathfrak{M}_{B_0}$ onto C as required (use theorem 1.7.1, theorem 1.4.4 and theorem 1.3.11 (iii)).

1.7.3. Theorem. Let h be a homomorphism of the compingent algebra B_1 into the compingent algebra B_2. Then:

$$h[B_1] \text{ dense in } B_2, \text{ and } h \text{ full} \Leftrightarrow \mathrm{m}(h) \text{ homeomorphic into.}$$

Proof.

(i) Let $h[B_1]$ be dense in B_2, and h full. Since $\mathfrak{M}_{B_2}$ is compact, it needs only to be shown that $\mathrm{m}(h)$ is one-to-one into. So take $\mathfrak{n}_1,\mathfrak{n}_2 \in \mathfrak{M}_{B_2}$ such that $\mathfrak{n}_1 \neq \mathfrak{n}_2$. Using the hypothesis, we can find elements $a_1,a_2,b_1,b_2 \in B_1$, such that $b_1 \ll a_1$, $b_2 \ll a_2$, $h(a_1)$, $h(b_1) \in \mathfrak{n}_1$, $h(a_2)$, $h(b_2) \in \mathfrak{n}_2$, $h(a_1) \wedge h(a_2) = 0$. Then $a_1 \in \mathrm{m}(h)\ (\mathfrak{n}_1)$, $a_2 \in \mathrm{m}(h)\ (\mathfrak{n}_2)$. Since $h(a_1 \wedge a_2) = h(a_1) \wedge h(a_2) = 0$, it follows that $a_1 \wedge a_2 \notin \mathrm{m}(h)\ (\mathfrak{n}_1)$; so $a_2 \notin \mathrm{m}(h)\ (\mathfrak{n}_1)$, and $\mathrm{m}(h)\ (\mathfrak{n}_1) \neq \mathrm{m}(h)\ (\mathfrak{n}_2)$.

(ii) Let $\mathrm{m}(h)$ be homeomorphic into. Take $\mathfrak{n} \in \mathfrak{M}_{B_2}$, and $d \in \mathfrak{n}$. For every $\mathfrak{n}' \in \mathfrak{M}_{B_2}$ such that $\mathfrak{n}' \neq \mathfrak{n}$, we can choose elements $a_{\mathfrak{n}'}$, $b_{\mathfrak{n}'} \in B_1$ such that $a_{\mathfrak{n}'} \in \mathrm{m}(h)\ (\mathfrak{n})$, $b_{\mathfrak{n}'} \in \mathrm{m}(h)\ (\mathfrak{n}')$, and $a_{\mathfrak{n}'} \ll b_{\mathfrak{n}'}^0$. Since $\mathfrak{M}_{B_2} \setminus \omega_{B_2}(d)$ is compact, there exist an integer n and elements $\mathfrak{n}_i' \in \mathfrak{M}_{B_2}$ ($i = 1, 2, \ldots, n$), such that

$$\mathfrak{M}_{B_2} \setminus \omega_{B_2}(d) \subseteq \bigcup_{i=1}^{n} \omega_{B_2}(h(b_{\mathfrak{n}'_i})).$$

Put $a = \bigwedge_{i=1}^{n} a_{\mathfrak{n}'_i}$, $b = \bigvee_{i=1}^{n} b_{\mathfrak{n}'_i}$.

Then $a \ll b^o$, $a \in \mathrm{m}(h)$ $(\mathfrak{n})$ (whence $h(a) \in \mathfrak{n}$), and

$$d^o \ll \bigvee_{i=1}^{n} h(b_{\mathfrak{n}'_i}) \leq h(b).$$

Using this result, it is easy to show that given $e,d \in B_2$ such that $e \ll d$, there exist elements $a,b \in B_1$ such that

$$a \ll b^o,\ e \ll h(a),\ d^o \ll h(b);$$

then, however, also: $e \ll h(a) \ll d$, since $(h(b))^o \ll d$, and $h(a) \leq (h(b))^o$, because $a \wedge b = 0$. So $h[B_1]$ is dense in B_2. Now assume in addition $e = h(p)$, $d = h(q)$, $p,q \in B_1$. Since $a \ll b^o$, also $p \wedge a \ll q \vee b^o$. Further we have: $h(p \wedge a) =$ $= h(p) \wedge h(a) = e \wedge h(a) = e$.

It is sufficient to show that $h(q \vee b^o) = h(q)$. Suppose $h(q \vee b^o) > h(q)$, e.g. $h(q \vee b^o) \in \mathfrak{n} \in \mathfrak{M}_{B_2}$, $h(q) \notin \mathfrak{n}$. Since $(h(b))^o \ll d = h(q)$, we get $h(b) \in \mathfrak{n}$, and also

$$h(b) \wedge h(q \vee b^o) = h(b \wedge q) \in \mathfrak{n} \text{ and } h(q) \in \mathfrak{n}.$$

This gives the required contradiction, and proves the fullness of h.

1.7.4. Corollary. Let B' be a subcompingent subalgebra of the compingent algebra B. Then $\mathrm{m}(\iota_{B',B})$ is a homeomorphism of $\mathfrak{M}_B$ onto $\mathfrak{M}_{B'}$ if and only if B' is a dense compingent subalgebra of B.

1.7.5. Theorem. Let B be a complete compingent algebra. Then the autohomeomorphism group of $\mathfrak{M}_B$ is isomorphic to the automorphism group of B.

Proof. It is clear that to every automorphism f of B, the mapping $\mathrm{n}(f)$ defined by:

$$\mathrm{n}(f)\ (\mathfrak{m}) = f[\mathfrak{m}] \qquad (\mathfrak{m} \in \mathfrak{M}_B)$$

is an autohomeomorphism of $\mathfrak{M}_B$, and that n is a homomorphism of the automorphism group of B into the autohomeomorphism group of $\mathfrak{M}_B$. If f_1 and f_2 are different automorphisms of B, then there exist different elements $a_1, a_2 \in B$, such that $f_1(a_1) = f_2(a_2)$ (consider f_1^{-1} and f_2^{-1}); if e.g. $a_1 \in \mathfrak{m} \in \mathfrak{M}_B$, $a_2 \notin \mathfrak{m}$, then obviously

$$\mathrm{n}(f_1)\,(\mathfrak{m}) \neq \mathrm{n}(f_2)\,(\mathfrak{m}), \text{ whence } \mathrm{n}(f_1) \neq \mathrm{n}(f_2).$$

So n is one-to-one. It is equally easy to prove that n is onto, using the canonical isomorphy of $\mathrm{B}(\mathfrak{M}_B)$ and B.

CHAPTER 2. COMPACTIFICATIONS

2.1. Basic subcompingent subalgebras.

2.1.1. Definition. Let T be a topological space. Then we define $\mathrm{B}^*(T)$ as to be the boolean algebra of all regularly open sets of T, provided with the relation $\ll^*$ defined by:

$$a \ll^* b \Leftrightarrow \bar{a} \subseteq b \qquad (a,b \in \mathrm{B}^*(T)).$$

A *subcompingent subalgebra* B' of $\mathrm{B}^*(T)$ will be a compingent algebra whose boolean algebra is a subboolean subalgebra of $\mathrm{B}^*(T)$, and which satisfies the following conditions:

S*1. $\emptyset \in B'$;

S*2. $a,b \in B' \Rightarrow a \wedge b \in B'$;

S*3. $a,b \in B'$, $a \ll' b \Rightarrow a^{o'o} \ll^* b$, where $\ll'$ is the compingent relation of B'.

The subcompingent subalgebra B' of $\mathrm{B}^*(T)$ will be called *basic* if: $p \in O$, O an open set of $T \Rightarrow \exists\, a,b \in B'$ such that $p \in a \ll' b \subseteq O$.

The subcompingent subalgebra B' of $\mathrm{B}^*(T)$ will be called a *compingent* subalgebra of $\mathrm{B}^*(T)$ whenever, for $a,b \in B'$,

$$a \ll' b \Leftrightarrow a \ll^* b.$$

Remarks. Clearly, $\mathrm{B}^*(T) = \mathrm{B}(T)$ if and only if T is a normal space. A subcompingent subalgebra of $\mathrm{B}(T)$ is also a subcompingent subalgebra of $\mathrm{B}^*(T)$, provided, of course, that T is completely regular.

2.1.2. Theorem. Let T be a topological space, and B' a subcompingent subalgebra of $\mathrm{B}^*(T)$. Then there is a canonical continuous mapping $\mu_{B'}$ of T into $\mathfrak{M}_{B'}$ onto a dense subspace of it. If, moreover, B'' is a subcompingent subalgebra of B', then B'' is a subcompingent subalgebra of $\mathrm{B}^*(T)$, and:

$$\mu_{B''} = \mathrm{m}(\iota_{B'',B'}) \circ \mu_{B'}.$$

Proof. Let $\ll'$ and $\ll''$ be the compingent relations of B' and B'', respectively. We define for $p \in T$:

$$\mu_{B'}(p) = \{a \in B' \mid \exists\, b \in B' \text{ such that } p \in b \ll' a\}.$$

Clearly, if p has been taken from T, $\mu_{B'}(p)$ satisfies condition F1 for a concordant filter. For condition F2, let $a_1,a_2 \in \mu_{B'}(p)$, $b_1,b_2 \in B'$, and $p \in b_1 \ll' a_1, p \in b_2 \ll' a_2$. Choose $c \in B'$ such that $b_1 \wedge b_2 \ll' c \ll' a_1 \wedge a_2$; then $c \in \mu_{B'}(p)$, and condition F2 is satisfied. It is also clear that $\emptyset \notin \mu_{B'}(p)$.

In order to show the maximality of $\mu_{B'}(p)$, assume $a, b \in B', a \ll' b$. Choose $c,d \in B'$, such that $a \ll' c \ll' d \ll' b$. If $p \in d$, then $b \in \mu_{B'}(p)$; if $p \notin d$, then $p \in c^{o'}$, since $d^{o} \ll^{*} c^{o'}$; so, because of $c^{o'} \ll' a^{o'}$, $a^{o'} \in \mu_{B'}(p)$. This shows that $\mu_{B'}(p) \in \mathfrak{M}_{B'}$.

If $a \in \mu_{B'}(p)$, and if $b \in \mu_{B'}(p)$ such that $b \ll' a$, then: $q \in b \Rightarrow a \in \mu_{B'}(q)$; hence $\mu_{B'}[b] \subseteq \omega_{B'}(a)$; this shows the continuity of $\mu_{B'}$.

Since for every $a \in B'$ such that $a \neq \emptyset$, there exists an element $b \in B'$ with $\emptyset < b \ll' a$, whence $\mu_{B'}[b] \subseteq \omega_{B'}(a)$, it follows that $\mu_{B'}[T]$ is dense in $\mathfrak{M}_{B'}$.

To show that B'' is a subcompingent subalgebra of $\mathrm{B}^*(T)$, we need only verify axiom S*3. But if $a,b \in B''$, $a \ll'' b$, then $a^{o''o'} \ll' b$, by axiom S3; hence $(a^{o''o'})^{o'o} \ll^{*} b$, by axiom S*3; in other words, $a^{o''o} \ll^{*} b$, which was to be proved.

For the last assertion, notice that:

$$(\mathrm{m}(\iota_{B'',B'}) \circ \mu_{B'})\,(p) = \{a \in B'' \mid \exists\, b \in B'',\, c \in B' \text{ such that } p \in c \ll' b \ll'' a\},$$

whereas

$$\mu_{B''}(p) = \{a \in B'' \mid \exists\, b \in B'' \text{ such that } p \in b \ll'' a\};$$

this shows that $(\mathrm{m}(\iota_{B'',B'}) \circ \mu_{B'})\,(p) \subseteq \mu_{B''}(p)$; hence also the equality holds.

2.1.3. Proposition. Let T be a completely regular space, and B' a subcompingent subalgebra of $\mathrm{B}^*(T)$. Then B' is a subcompingent subalgebra of $\mathrm{B}(T)$.

Proof. Let $\ll'$ be the compingent relation of B'. Evidently, it is sufficient to prove that:

$$a,b \in B',\ a \ll' b \Rightarrow a^{o'o} \ll b.$$

So assume $a,b \in B'$, $a \ll' b$. Choose $c,d,e,f \in B'$ such that

$$a \ll' c \ll' d \ll' e \ll' f \ll' b.$$

Then:

$$a^{o'o} \ll^* c,\ b^o \ll^* f^{o'},\ \text{by S*3}.$$

Further, it is clear that:

$$c \subseteq \mu_{B'}^{-1}[\omega_{B'}(d)],\ f^{o'} \subseteq \mu_{B'}^{-1}[\omega_{B'}(e^{o'})].$$

In $\mathfrak{M}_{B'}$, $\omega_{B'}(d)$ and $\omega_{B'}(e^{o'})$ are functionally separated, since $d \ll' e$. Hence $\mu_{B'}^{-1}[\omega_{B'}(d)]$ and $\mu_{B'}^{-1}[\omega_{B'}(e^{o'})]$ are functionally separated in T, and then so are $a^{o'o}$ and b^o, q.e.d..

2.1.4. Corollary. If T is a completely regular space, then the subcompingent subalgebras of B*(T) coincide with the subcompingent subalgebras of B(T).

Remark. For this reason, a basic subcompingent subalgebra of B*(T) will be called a basic subcompingent subalgebra of B(T).

2.1.5. Proposition. Let T be a topological space, and B' a basic subcompingent subalgebra of B*(T). Then:

(i) B' is a boolean subalgebra of B*(T);

(ii) if B' is complete, then the sets of B' and B*(T) coincide.

Proof.

(i) We need only show: $a \in B' \Rightarrow a^o \in B'$. But by definition 2.1.1: $a \in B'$, $b \in \text{B}^*(T)$, $0 < b \leq a^o \Rightarrow \exists\, c \in B'$ such that $0 < c \leq b \leq a^o$; this shows that $a^o \in B'$.

(ii) Let B' be complete, and $a \in B^*(T)$. Then, by definition 2.1.1:

$$a = \vee\ \{b \mid b \in B', b \leq a\}.$$

Hence $a' = \vee'\ \{b \mid b \in B',\ b \leq a\} \geq a$, where "$\vee'$" denotes the join operation in B'.
Equally, $a'' = \vee'\ \{c \mid c \in B', c \leq a^o\} \geq a^o$. But $a' \wedge a'' = \emptyset$; hence $a' = a$, and $a \in B'$, q.e.d..

2.1.6. Theorem. Let C be a compact space, and B' a boolean subalgebra of $\mathrm{B}(C)$. Then the following three conditions are equivalent:

(i) B' is a basis for the topology of C;

(ii) B' is a basic compingent subalgebra of $\mathrm{B}(C)$;

(iii) B' is a dense compingent subalgebra of $\mathrm{B}(C)$.

Proof. The easy proof is omitted.

2.2. Compactification.

2.2.1. Theorem. Let T be a topological space, and B' a subcompingent subalgebra of $\mathrm{B}^*(T)$. Then:

$$\mu_{B'} \text{ is homeomorphic into} \Leftrightarrow B' \text{ is basic.}$$

(In other words:

$$(\mu_{B'},\mathfrak{M}_{B'}) \text{ is a compactification of } T \Leftrightarrow B' \text{ is basic.)}$$

Proof. Take $\ll'$ to be the compingent relation of B'.

(i) Assume $\mu_{B'}$ is homeomorphic into, and $p \in O$, where O is an open set of T. Since $\mu_{B'}[O]$ is open in $\mu_{B'}[T]$, and $\omega_{B'}[B']$ a basis for the topology of $\mathfrak{M}_{B'}$, there exists an element $b \in B'$ such that

$$\mu_{B'}(p) \in \mu_{B'}[T] \cap \omega_{B'}(b) \subseteq \mu_{B'}[O];$$

this implies that $b \in \mu_{B'}(p)$. Now, choose $c,d \in \mu_{B'}(p)$, such that $c \ll' d \ll' b$. Then

$$q \in d \Rightarrow b \in \mu_{B'}(q) \Rightarrow \mu_{B'}(q) \in \mu_{B'}[O] \Rightarrow q \in O;$$

hence $p \in c \ll' d \subseteq O$, which shows that B' is basic.

(ii) Let B' be basic. This implies immediately that $\mu_{B'}$ is one-to-one into. Assume $a \in B'$, and $p \in a$. Then we can choose $b \in B'$, such that $p \in b \ll' a$ (by definition 2.1.1). Therefore,

$$\mu_{B'}(p) \in \omega_{B'}(a), \text{ and } \mu_{B'}[a] \subseteq \omega_{B'}(a).$$

It is obvious that $\mu_{B'}[a] \supseteq \mu_{B'}[T] \cap \omega_{B'}(a)$; hence $\mu_{B'}[a] =$ $= \mu_{B'}[T] \cap \omega_{B'}(a)$.

This shows that $\mu_{B'}$ is an open mapping of T onto $\mu_{B'}[T]$ (because B' is basic, it is also a basis for the topology of T). Therefore, $\mu_{B'}$ is homeomorphic into.

2.2.2. Corollary. For a topological space T we have:

T is completely regular $\Leftrightarrow$ $B^*(T)$ has a basic subcompingent subalgebra.

2.2.3. Theorem. Let T_1 and T_2 be completely regular spaces, and φ a continuous mapping of T_1 into T_2. If B_2 is a subcompingent subalgebra of $B(T_2)$, then there is a canonical homomorphism $g = g(\varphi, B_2)$ of B_2 into $B(T_1)$. If, moreover, B_1 is a subcompingent subalgebra of $B(T_1)$ such that g is a homomorphism of B_2 into B_1, then:

$$m(g) \circ \mu_{B1} = \mu_{B2} \circ \varphi$$

(here g considered as a homomorphism of B_2 into B_1).

Proof. The mapping g will be defined by:

$$g(a) = (\varphi^{-1}[a])^{-c-c} \qquad (a \in B_2).$$

It is easily verified that the axioms H1 and H2 are valid for g. Let $\ll_1$ and $\ll_2$ be the compingent relations of B_1 and B_2, respectively.

Assume $a,b \in B_2$, $a \ll_2 b$. Then $a^{o2o} \ll b$, or, in other words, a^{o2o} and b^o are functionally separated. This implies that $(a^{o2o})^-$ and $(b^o)^-$, and also

$$\varphi^{-1}[(a^{o2o})^-] \text{ and } \varphi^{-1}[(b^o)^-],$$

are functionally separated. However, it is easily seen that:

$$(g(a^{o2}))^o \subseteq \varphi^{-1}[(a^{o2o})^-],\ (g(b))^o \subseteq \varphi^{-1}[(b^o)^-];$$

hence $(g(a^{o2}))^o$ and $(g(b))^o$ are functionally separated too, whence

$$(g(a^{o2}))^o \ll g(b).$$

This proves that g is a homomorphism of B_2 into $B(T_1)$.

Further, let $p \in T$. Then:

$$(\mu_{B2} \circ \varphi)\,(p) = \{a \in B_2 \mid \exists\, b \in B_2 \text{ such that } \varphi(p) \in b \ll_2 a\},$$

whereas

$$\begin{aligned}(m(g) \circ \mu_{B1})\,(p) &= \{a \in B_2 \mid \exists\, c \in B_2 \text{ such that } c \ll_2 a \text{ and } g(c) \in \mu_{B1}(p)\}\\ &= \{a \in B_2 \mid \exists\, c \in B_2,\ d \in B_1 \text{ such that } c \ll_2 a \text{ and } p \in d \ll_1 g(c)\}.\end{aligned}$$

However, if a, c, d are as in the latter equality, then we can choose $b \in B_2$ such that $c \ll_2 b \ll_2 a$; since $p \in g(c)$, and $g(c) \subseteq \varphi^{-1}[b]$, it follows that $\varphi(p) \in b$. This implies:

$$(\mathrm{m}(g) \circ \mu_{B_1})(p) \subseteq (\mu_{B_2} \circ \varphi)(p),$$

whence the equality, and $\mathrm{m}(g) \circ \mu_{B_1} = \mu_{B_2} \circ \varphi$.

2.2.4. Theorem. Let T be a completely regular space. Then:

(i) any compactification of T is topologically equivalent to a compactification $(\mu_B, \mathfrak{M}_B)$, where B is a basic subcompingent algebra of $\mathrm{B}(T)$;

(ii) the following equivalence holds for basic subcompingent algebras B' and B'' of $\mathrm{B}(T)$:

B'' is a subcompingent algebra of B' $\Leftrightarrow$

$$\Leftrightarrow (\mu_{B''}, \mathfrak{M}_{B''}) \leq (\mu_{B'}, \mathfrak{M}_{B'}).$$

Proof.

(i) Let (φ, C) be a compactification of T. By theorem 2.2.3, there exists a canonical homomorphism g of $\mathrm{B}(C)$ into $\mathrm{B}(T)$. Since (φ, C) is a compactification of T, g is, in fact, an isomorphism of $\mathrm{B}(C)$ onto a basic subcompingent algebra B_1 of $\mathrm{B}(T)$, as is easily verified. Hence $\mathrm{m}(g)$ is a homeomorphism of $\mathfrak{M}_{B_1}$ onto $\mathfrak{M}_{\mathrm{B}(C)}$. But $\mu_{\mathrm{B}(C)}$ being a homeomorphism of C onto $\mathfrak{M}_{\mathrm{B}(C)}$, it follows that $(\mathrm{m}(g))^{-1} \circ \mu_{\mathrm{B}(C)}$ is a homeomorphism of C onto $\mathfrak{M}_{B_1}$, which, by theorem 2.2.3, satisfies the relation

$$((\mathrm{m}(g))^{-1} \circ \mu_{\mathrm{B}(C)}) \circ \varphi = \mu_{B_1}.$$

This shows that (φ, C) is topologically equivalent to $(\mu_{B_1}, \mathfrak{M}_{B_1})$.

(ii) The implication from the left to the right follows from theorem 2.1.2. For the converse, let ψ be a continuous mapping of $\mathfrak{M}_{B'}$ onto $\mathfrak{M}_{B''}$ such that $\psi \circ \mu_{B'} = \mu_{B''}$. Then $\mathrm{b}(\psi)$ is a one-to-one homomorphism of B'' into B' (theorems 1.7.1 and 1.6.4 (ii)). However, if $a \in \mathrm{B}(T)$, then clearly

$$\psi^{-1}[\omega_{B''}(a)] \subseteq \omega_{B'}(a) \subseteq (\psi^{-1}[\omega_{B''}(a)])^-,$$

by the continuity of ψ, and the fact that $\mu_{B''}[a]$ is dense in $\omega_{B''}(a)$, and $\mu_{B'}[a]$ dense in $\omega_{B'}(a)$. Hence, by the definition of b, $\mathrm{b}(\psi)(a) = a$, so $\mathrm{b}(\psi)$ is the identity mapping of $\mathrm{B}(T)$. This proves the assertion, by proposition 1.5.6.

2.2.5. Corollary. If T is a completely regular space, then $(\mu_{\mathrm{B}(T)},\mathfrak{M}_{\mathrm{B}(T)})$ is a greatest compactification of T, whence topologically equivalent to the Čech-Stone compactification of T.

2.2.6. Corollary. If φ is a continuous mapping of a completely regular space T into a compact space C, then the mapping $\varphi \circ \mu^{-1}_{\mathrm{B}(T)}$ of $\mu_{\mathrm{B}(T)}[T]$ into C can be extended to a continuous mapping ψ of $\mathfrak{M}_{\mathrm{B}(T)}$ into C (here $\mu^{-1}_{\mathrm{B}(T)}$ stands for the mapping of $\mu_{\mathrm{B}(T)}[T]$ onto T defined by: $\mu^{-1}_{\mathrm{B}(T)}(\mu_{\mathrm{B}(T)}(p)) = p \quad (p \in T)$).

Proof. Apply theorem 2.2.3, taking $T_1 = T$, $T_2 = C$, $B_1 = \mathrm{B}(T)$, $B_2 = \mathrm{B}(C)$; then $\psi = \mu^{-1}_{\mathrm{B}(C)} \circ \mathrm{m}(g)$ is the required extension.

2.3. Compaction.

2.3.1. Definition. Let T be a topological space. Generalizing the notion of a compactification of T, we define a *compaction* of T to be a pair (α,C), where C is a compact space, and α a continuous mapping of T into C onto a dense subspace of C.

In a way completely analogous to the case of compactifications, a partial ordering and a topological equivalence relation are introduced in the class of all compactions of T.

2.3.2. Theorem. Let T be a topological space. Then:

(i) for any subcompingent subalgebra B of $\mathrm{B}^*(T)$, $(\mu_B,\mathfrak{M}_B)$ is a compaction of T;

(ii) any compaction (α,C) of T is topologically equivalent to a compaction $(\mu_B,\mathfrak{M}_B)$, where B is a suitable subcompingent subalgebra of $\mathrm{B}^*(T)$.

Proof.

(i) This assertion is contained in theorem 2.1.2.

(ii) We define the mapping g by:

$$g(a) = (\alpha^{-1}[a])^{-\mathrm{c}-\mathrm{c}} \qquad (a \in \mathrm{B}(C)).$$

It is easily verified that g is a one-to-one mapping of $\mathrm{B}(C)$ into $\mathrm{B}^*(T)$, satisfying:

$$g(\emptyset) = \emptyset,\ g(a \wedge b) = g(a) \cap g(b) \qquad (a,b \in \mathrm{B}(C)).$$

Moreover, if $a,b \in \mathrm{B}(C)$ and $a \ll b$, it easily follows that $(g(a^o))^o \ll^* g(b)$. Hence $B' = g[\mathrm{B}(C)]$ is a subcompingent subalgebra of $\mathrm{B}^*(T)$ with compingent relation $\ll'$, if we define:

$$g(a) \ll' g(b) \Leftrightarrow a \ll b \qquad (a,b \in \mathrm{B}(C)).$$

In the same way as in the proof of theorem 2.2.3, it follows that

$$\mathrm{m}(g) \circ \mu_{B'} = \mu_{\mathrm{B}(C)} \circ \alpha,$$

where g is considered as a mapping of $\mathrm{B}(C)$ into B'. In other terms:

$$(\mu_{\mathrm{B}(C)}^{-1} \circ \mathrm{m}(g)) \circ \mu_{B'} = \alpha,$$

where $\mu_{\mathrm{B}(C)}$ and $\mathrm{m}(g)$ are homeomorphisms; this shows that $(\mu_{B'},\mathfrak{M}_{B'})$ is topologically equivalent to (α,C).

2.3.3. Theorem. Let T be a topological space, and B_1 and B_2 subcompingent subalgebras of $\mathrm{B}^*(T)$. Then $(\mu_{B_1},\mathfrak{M}_{B_1}) \leq (\mu_{B_2},\mathfrak{M}_{B_2})$ if and only if the following condition is satisfied:

$$a_1,b_1 \in B_1,\ a_1 \ll_1 b_1 \Rightarrow \exists\ a_2,b_2 \in B_2 \text{ such that } a_1 \leq a_2 \ll_2 b_2 \leq b_1$$

(here $\ll_1$ and $\ll_2$ are the compingent relations of B_1 and B_2, respectively).

Proof.

(i) First we prove the necessity of the condition. So let φ be a continuous mapping of $\mathfrak{M}_{B_2}$ onto $\mathfrak{M}_{B_1}$ such that

$$\varphi \circ \mu_{B_2} = \mu_{B_1}.$$

Take $a_1,b_1 \in B_1$, with $a_1 \ll_1 b_1$. Choose $c_1 \in B_1$ such that $a_1 \ll_1 c_1 \ll_1 \ll_1 b_1$.

Then: $\quad (\varphi^{-1}[\omega_{B_1}(c_1)])^- \subseteq \varphi^{-1}[\omega_{B_1}(b_1)]$;

$$a_1 \subseteq \mu_{B_1}^{-1}[\omega_{B_1}(c_1)] \subseteq c_1 \subseteq \mu_{B_1}^{-1}[\omega_{B_1}(b)] \subseteq b_1.$$

Now we can choose $a_2,b_2,d_2 \in B_2$ such that $a_2 \ll_2 b_2 \ll_2 d_2$ and $(\varphi^{-1}[\omega_{B_1}(c_1)])^- \subseteq \omega_{B_2}(a_2)$, $\omega_{B_2}(d_2) \subseteq \varphi^{-1}[\omega_{B_1}(b_1)]$.

Then, taking the inverse images under μ_{B2}, applying the relation $\varphi \circ \mu_{B2} = \mu_{B1}$, and using the earlier derived inclusions, we obtain

$$a_1 \subseteq \mu_{B_1}^{-1}[\omega_{B1}(c_1)] \subseteq \mu_{B_2}^{-1}[\omega_{B2}(a_2)] \subseteq a_2,$$
$$b_2 \subseteq \mu_{B_2}^{-1}[\omega_{B2}(d_2)] \subseteq \mu_{B_1}^{-1}[\omega_{B1}(b_1)] \subseteq b_1.$$

This shows that $a_1 \leq a_2 \ll_2 b_2 \leq b_1$.

(ii) Assume the condition in the theorem is satisfied. Then we define the mapping φ in the following manner:

$$\varphi(\mathfrak{n}) = \{a_1 \in B_1 \mid \exists\, a_1' \in B_1,\, a_2 \in \mathfrak{n} \text{ such that } a_2 \leq a_1' \ll_1 a_1\} \quad (\mathfrak{n} \in \mathfrak{M}_{B2}).$$

It is clear that $\emptyset \notin \varphi(\mathfrak{n})$, and:

$a_1 \in \varphi(\mathfrak{n})$, $a_1 \leq b_1 \in B_1 \Rightarrow b_1 \in \varphi(\mathfrak{n})$ (having taken $\mathfrak{n}$ from $\mathfrak{M}_{B2}$).

If $a_2 \leq a_1' \ll_1 a_1$, $b_2 \leq b_1' \ll_1 b_1$, where $a_2, b_2 \in \mathfrak{n}$, then we choose $c_1 \in B_1$ such that $a_1' \wedge b_1' \ll c_1 \ll_1 a_1 \wedge b_1$, which shows that $a_2 \wedge b_2 \leq a_1' \wedge b_1' \ll_1 c_1$ and $c_1 \ll_1 a_1 \wedge b_1$, whilst $a_2 \wedge b_2 \in \mathfrak{n}$; hence $c_1 \in \varphi(\mathfrak{n})$, and $\varphi(\mathfrak{n})$ is a proper concordant filter of B_1.

We proceed to prove the maximality of $\varphi(\mathfrak{n})$. So let $a_1, b_1 \in B_1$, $a_1 \ll_1 b_1$. Choose $c_1, d_1, e_1 \in B_1$ such that

$$a_1 \ll_1 c_1 \ll_1 d_1 \ll_1 e_1 \ll_1 b_1.$$

Because of our condition, there are elements $d_2, e_2 \in B_2$ such that $d_1 \leq d_2 \ll_2 e_2 \leq e_1$. Now $e_2 \in \mathfrak{n}$ or $d_2^{\circ_2} \in \mathfrak{n}$. If $e_2 \in \mathfrak{n}$, then $b_1 \in \varphi(\mathfrak{n})$; so assume $d_2^{\circ_2} \in \mathfrak{n}$. Since

$$d_2^{\circ_2} \leq d_2^{\circ} \leq d_1^{\circ} \ll^* c_1^{\circ_1} \ll_1 a_1^{\circ_1},$$

whence

$$d_2^{\circ_2} \leq c_1^{\circ_1} \ll_1 a_1^{\circ_1},$$

it follows that $a_1^{\circ} \in \varphi(\mathfrak{n})$, which had to be proved.

If $\mathfrak{n} \in \mathfrak{M}_{B2}$, $a_1 \in \varphi(\mathfrak{n})$, and $a_1' \in B_1$, $a_2 \in B_2$ such that $a_2 \leq a_1' \ll_1 a_1$, then it is clear that

$$\varphi[\omega_{B2}(a_2)] \subseteq \omega_{B1}(a_1);$$

this shows the continuity of φ.

Lastly, we should verify that $\varphi \circ \mu_{B2} = \mu_{B1}$.

So let $p \in T$. Then:

$$(\varphi \circ \mu_{B_2})(p) = \{a_1 \in B_1 \mid \exists\, a_1' \in B_1,\ a_2 \in B_2,\ b_2 \in B_2 \text{ such that } p \in b_2 \ll_2 a_2 \leq a_1' \ll_1 a_1\},$$

whereas

$$\mu_{B_1}(p) = \{a_1 \in B_1 \mid \exists\, a_1' \in B_1 \text{ such that } p \in a_1' \ll_1 a_1\}.$$

Clearly, $(\varphi \circ \mu_{B_2})(p) \subseteq \mu_{B_1}(p)$, whence $(\varphi \circ \mu_{B_2})(p) = \mu_{B_1}(p)$, q.e.d..

2.4. Connection with proximity spaces.

2.4.1. Definition. According to P. S. Aleksandrov and V. I. Ponomarëv [3], a *proximity space* can be defined as a topological space T together with a *proximity* relation $\Subset$ in the set of its subsets, which satisfies the following conditions:

E1. $A \Subset B \Rightarrow T \setminus B \Subset T \setminus A$;

E2. $A \Subset B \Rightarrow A \subseteq B$;

E3. $A_1 \subseteq A \Subset B \subseteq B_1 \Rightarrow A_1 \Subset B_1$;

E4. $A_1 \Subset B_1, A_2 \Subset B_2 \Rightarrow A_1 \cup A_2 \Subset B_1 \cup B_2, A_1 \cap A_2 \Subset B_1 \cap B_2$;

E5. $A \Subset B \Rightarrow \exists\, C \subseteq T$ such that $A \Subset C \Subset B$;

E6. $\emptyset \Subset \emptyset$;

E7. If $p \in O$, and O an open set of T, then $\{p\} \Subset O$; if $p \in T$ and $\{p\} \Subset A$, then there exists a neighbourhood O of p such that $O \Subset A$.

The subsets A and B of T are called *near* if $A \Subset T \setminus B$ does not hold.

2.4.2. Theorem. Let T be a topological space. Then there is a canonical one-to-one correspondence between the proximity relations on T and the basic subcompingent algebras of $\mathrm{B}^*(T)$.

Proof. Indeed, given a proximity relation $\Subset$ on T, we define, for $a,b \in \mathrm{B}^*(T)$:

$$a \ll' b \Leftrightarrow a \Subset b.$$

If A and B are subsets of T such that $A \Subset B$, and $p \in T$, $p \notin B$, then it follows from the consecutive application of axioms E1 and E3, that $\{p\} \Subset T \setminus A$; hence, by axioms E7 and E2, $p \notin \bar{A}$; so $\bar{A} \subseteq B$.

Using this result, the verification of the facts that $\ll'$ is a compingent relation and the compingent algebra with the relation $\ll'$ is a basic subcompingent algebra of B*(T), is immediate.

On the other hand, given a basic subcompingent algebra B' of B*(T), with compingent relation $\ll'$, a proximity relation $\Subset$ on T is obtained by the definition: for subsets A,B of T,

$$A \Subset B \Leftrightarrow \exists\, a,b \in \mathrm{B}^*(T) \text{ such that } A \subseteq a \ll' b \subseteq B.$$

The verification of axioms E1-E7 is quite easy. Moreover, it is obvious that the two operations defined above are each other's inverses. This proves the theorem.

2.4.3. Corollary. A topological space which admits a proximity relation, is completely regular.

Proof. The corollary follows from theorem 2.4.2 and corollary 2.2.2.

2.4.4. Remark. It is well-known that there exists a natural one-to-one correspondence between the proximity relations on a completely regular space and the equivalence classes of topologically equivalent compactifications of that space (Ju. M. Smirnov [30]; cf. also P. Samuel [26] and Á. Császár [5]). Another proof of the same result is obtained by combining theorem 2.2.4 and theorem 2.4.2.

CHAPTER 3. QUASICOMPONENT SPACES AND PERIPHERAL COMPACTNESS

3.1. Quasicomponent spaces.

3.1.1. Definition. Let T be a topological space. Then we define:

(i) $Q(T)$ as the set of all quasicomponents of T, provided with the quotient topology;

(ii) $K(T)$ as the set of all quasicomponents of T, provided with the topology for which a basis is obtained by taking all those sets of quasicomponents which are contained in an open-and-closed set of T.

The following proposition is obvious.

3.1.2. Proposition. If T is a topological space, then the identity mapping of the set of all quasicomponents of T is a one-to-one continuous mapping of $Q(T)$ onto $K(T)$. Moreover, if T is compact or $Q(T)$ is compact, then this mapping is a homeomorphism, and $K(T)$ is compact too.

3.1.3. Definition. Let B be a compingent algebra. Then we define:

$$B^{\mathfrak{q}} = \{a \in B \mid a \ll a\}.$$

Clearly, $B^{\mathfrak{q}}$ is a discrete compingent subalgebra of B.

3.1.4. Theorem. If B is a compingent algebra, then $\mathfrak{M}_{B^{\mathfrak{q}}}$ is homeomorphic to $K(\mathfrak{M}_B)$, by means of a canonical homeomorphism.

Proof. By theorem 1.3.11, $\omega_B[B^{\mathfrak{q}}]$ consists of all open-and-closed sets of $\mathfrak{M}_B$. Now, it follows from the definitions that the mapping

$$\mathfrak{m} \to \bigcap \{\omega_B(a) \mid a \in \mathfrak{m}\} \qquad (\mathfrak{m} \in \mathfrak{M}_{B^{\mathfrak{q}}}),$$

is a homeomorphism of $\mathfrak{M}_{B^{\mathfrak{q}}}$ onto $K(\mathfrak{M}_B)$.

3.1.5. Theorem. Let T be a topological space, and B a subcompingent subalgebra of $\mathrm{B}^*(T)$. Then the mapping $\varkappa_B$, defined by

$$\varkappa_B(Q) = \{a \in B^{\mathfrak{q}} \mid Q \subseteq a\} \qquad (Q \in \mathrm{K}(T)),$$

is a continuous mapping of $\mathrm{K}(T)$ into $\mathfrak{M}_{B^{\mathfrak{q}}}$, yielding a dense subspace of $\mathfrak{M}_{B^{\mathfrak{q}}}$ as its image. As is obvious, $\varkappa_B$ is one-to-one into if and only if to every two quasicomponents of T, there exists an element of $B^{\mathfrak{q}}$ which contains exactly one of them.

Proof. First, we make the useful observation that a discrete subcompingent subalgebra of $\mathrm{B}^*(T)$ is necessarily a (discrete) compingent subalgebra of $\mathrm{B}^*(T)$. Hence, given any quasicomponent Q and any element a of $B^{\mathfrak{q}}$, it follows that either $Q \subseteq a$ or $Q \subseteq a^{\mathrm{o}}$; this proves that indeed $\varkappa_B(Q)$ is an element of $\mathfrak{M}_{B^{\mathfrak{q}}}$, and $\varkappa_B$ a mapping of $\mathrm{K}(T)$ into $\mathfrak{M}_{B^{\mathfrak{q}}}$. If $a \in B^{\mathfrak{q}}$, then

$$\varkappa_B^{-1}[\omega_{B^{\mathfrak{q}}}(a)] = \{Q \in \mathrm{K}(T) \mid Q \subseteq a\}$$

is an open set of $\mathrm{K}(T)$. This shows that $\varkappa_B$ is continuous. If $\emptyset \neq a \in B^{\mathfrak{q}}$, then $\exists\ Q \in \mathrm{K}(T)$ such that $Q \subseteq a$, and then $\varkappa_B(Q) \in \omega_{B^{\mathfrak{q}}}(a)$, which proves that $\varkappa_B[\mathrm{K}(T)]$ is dense in $\mathfrak{M}_{B^{\mathfrak{q}}}$.

3.1.6. Lemma. Let B be a compingent algebra and S an infinite subset of B. Then there exists a compingent subalgebra B' of B such that $|B'| = |S|$.

Proof. Inductively we define a monotonously non-shrinking sequence $(S_n)_{n=1}^{\infty}$ of subsets of B such that $S_1 = S$. If S_n has been already defined, then we define S_{n+1} as follows.

To every $a,b \in S_n$ with $a \ll b \neq 0$, choose $c(a,b) \in B$ such that $c(a,b) \neq 0$, and $a \ll c(a,b) \ll b$. Now put

$$S_{n+1} = \{a \wedge b,\ a \vee b,\ a^{\mathrm{o}},\ c(a,b) \mid a,b \in S_n\}.$$

Then it is easily verified that $|S_n| = |S|$ $(n = 1, 2, \ldots)$, and that

$$B' = \bigcup_{n=1}^{\infty} S_n$$

satisfies the lemma.

3.1.7. Lemma. If the topological space T has weight $\mathrm{w}(T) = \tau$, then there exists a set S of open-and-closed sets of T such that $|S| \leq \tau$ and to every two quasicomponents of T, there exists an element of S which contains exactly one of them.

Proof. For finite spaces, the lemma is obvious. So let $\mathrm{w}(T) \geq \aleph_0$. Let A be a basis for the topology of T such that $|A| = \tau$. To every pair $(O_1,O_2) \in A \times A$, choose an open-and-closed set F of T such that $O_1 \subseteq F \subseteq T \setminus O_2$, whenever this is possible. Define S as to be the set of all such obtained open-and-closed sets; then $|S \leq \tau$.

Now, let Q_1 and Q_2 be two quasicomponents of T. Then Q_1 and Q_2 are separated by the empty set, i.e. there are open-and-closed disjoint subspaces T_1 and T_2 of T, such that

$$T = T_1 \cup T_2, \;\; Q_1 \subseteq T_1, \;\; Q_2 \subseteq T_2.$$

Since A is a basis, there are elements $O_1, O_2 \in A$, such that $O_i \subseteq T_i$, $Q_i \cap O_i \neq \emptyset$ $(i = 1, 2)$.

By the construction of S, there exists an $F \in S$ such that

$$O_1 \subseteq F \subseteq T \setminus O_2;$$

but then also $Q_1 \subseteq F$, $Q_2 \cap F = \emptyset$.

3.1.8. Theorem. If T is a completely regular space of weight $\tau \geq \aleph_0$, then there exists a basic compingent subalgebra B of $\mathrm{B}(T)$ such that $\varkappa_B$ is one-to-one into and $|B| = \tau$.

Hence, $(\mu_B, \mathfrak{M}_B)$ is a compactification of T such that $\mathrm{w}(\mathfrak{M}_B) = \tau$ and the canonical mapping of $\mathrm{K}(T)$ into $\mathrm{K}(\mathfrak{M}_B)$ is one-to-one into (cf. theorem 3.1.4).

Proof. Let A be a basis for the topology of T, consisting of regularly open sets, such that $|A| = \tau$; moreover, let S be as in lemma 3.1.7. By lemma 3.1.6, there exists a compingent subalgebra B of $\mathrm{B}(T)$ such that:

$$A \cup S \subseteq B, \; |B| = \tau.$$

Since a compingent subalgebra of $\mathrm{B}(T)$ which also is a basis, is necessarily basic, the theorem follows from theorem 3.1.5.

Remark. For $\tau = \aleph_0$, the topological consequence of the theorem is proved in C. Kuratowski [20, II, § 41.V.4].

3.1.9. Corollary. If T is a completely regular space of weight τ, then there exists a continuous mapping ψ of T into the Cantor space D_τ such that:

$$\psi^{-1}(p) = \emptyset \text{ or } \psi^{-1}(p) \in \mathrm{K}(T) \qquad (p \in \mathrm{D}_\tau).$$

Proof. Let B be as in theorem 3.1.8. By R. Sikorski [27, 14.4], there exists a homomorphism h of $\mathrm{B}^{\mathfrak{q}}(\mathrm{D}_\tau)$ onto $B^{\mathfrak{q}}$. If ν is the canonical mapping of T onto $\mathrm{K}(T)$, we need only take

$$\psi = \mu^{-1}_{\mathrm{B}^{\mathfrak{q}}(\mathrm{D}_\tau)} \circ \mathrm{m}(h) \circ \varkappa_B \circ \nu.$$

(Notice that $\mathrm{m}(h)$ is one-to-one into, by theorem 1.7.3, or by [27, § 10].)

Remark. The present corollary generalizes II, § 41.V.3 in [20].

3.1.10. Theorem. A completely regular space T has a compactification (α, C) such that $\mathrm{K}(T)$ is homeomorphic to $\mathrm{K}(C)$ under the canonical mapping, if and only if $\mathrm{K}(T)$ is compact.

Proof. By proposition 3.1.2, the condition is necessary. Conversely, if $\mathrm{K}(T)$ is compact, and B as in theorem 3.1.8, then

$$\varkappa_B[\mathrm{K}(T)] = \mathfrak{M}_{B^{\mathfrak{q}}}, \text{ by theorem 3.1.5,}$$

and $\varkappa_B$ a homeomorphism of $\mathrm{K}(T)$ onto $\mathfrak{M}_{B^{\mathfrak{q}}}$. By theorem 3.1.4, the present theorem follows.

Remark. For the purpose of the theorem, we could have taken $(\mu_{\mathrm{B}(T)}, \mathfrak{M}_{\mathrm{B}(T)})$ as the required compactification. But the proof shows the following corollary.

3.1.11. Corollary. If T is a completely regular space such that $\mathrm{K}(T)$ is compact, then there exists a compactification (α, C) of T such that $\mathrm{w}(C) = \mathrm{w}(T)$ and $\mathrm{K}(T)$ is homeomorphic to $\mathrm{K}(C)$ in a natural way.

3.1.12. Theorem. Let T be a completely regular space such that $\mathrm{w}(T) \geq \aleph_0$ and $\mathrm{K}(T)$ is compact. Then the potency of the set of all open-and-closed sets of T is at most $\mathrm{w}(T)$.

Proof. Let B be as in theorem 3.1.8. Then, in particular, $|B^{\mathfrak{q}}| \leq \mathrm{w}(T)$. By theorem 1.3.11 (iii), $\omega_{B^{\mathfrak{q}}}[B^{\mathfrak{q}}]$ consists of all open-and-closed sets of $\mathfrak{M}_{B^{\mathfrak{q}}}$. However, $\mathfrak{M}_{B^{\mathfrak{q}}}$ is homeomorphic to $\mathrm{K}(T)$, by the proof of theorem 3.1.10. Since there exists a canonical one-to-one correspondence between the open-and-closed sets of $\mathrm{K}(T)$ and those of T, the theorem follows.

Remark. The theorem is a generalization of the case in which $\mathrm{w}(T) = \aleph_0$, proved by H. Freudenthal [12].

3.1.13. Theorem. Let T be a topological space such that $\mathrm{K}(T)$ is compact. Then, for any closed set F of T with compact boundary, $\mathrm{K}(F)$ is compact.

Proof. In accordance with definition 3.1.3, $\mathrm{B}^{\mathfrak{q}}(T)$ is the discrete compingent algebra of all open-and-closed sets of T. It is clear that for any open set U of $\mathrm{K}(T)$, $\varkappa_{\mathrm{B}^{\mathfrak{q}}(T)}[U]$ is an open set of $\varkappa_{\mathrm{B}^{\mathfrak{q}}(T)}[\mathrm{K}(T)]$, for *any* space T.

Applying this to F, we need only show that $\varkappa_{\mathrm{B}^{\mathfrak{q}}(F)}$ is not only one-to-one, which it is by theorem 3.1.5, but also onto $\mathfrak{M}_{\mathrm{B}^{\mathfrak{q}}(F)}$. In other words, given $\mathfrak{m} \in \mathfrak{M}_{\mathrm{B}^{\mathfrak{q}}(F)}$, it should be shown that $\bigcap\mathfrak{m} \neq \emptyset$.

If $a \cap \mathfrak{R}(F) \neq \emptyset$, for all $a \in \mathfrak{m}$, then, by the compactness of $\mathfrak{R}(F)$,

$$(\bigcap\mathfrak{m}) \cap \mathfrak{R}(F) \neq \emptyset, \text{ whence } \bigcap\mathfrak{m} \neq \emptyset.$$

Now assume there exists an element $a \in \mathfrak{m}$ such that $a \cap \mathfrak{R}(F)=\emptyset$ Then:

$$\bigcap\mathfrak{m} = \bigcap \{b \in \mathfrak{m} \mid b \leq a\}.$$

But clearly $b \in \mathrm{B}^{\mathfrak{q}}(T)$ for every $b \in \mathfrak{m}$ with $b \leq a$. Hence,

$$\mathfrak{f} = \{c \in \mathrm{B}^{\mathfrak{q}}(T) \mid \exists\, b \in \mathfrak{m} \text{ with } b \leq a \wedge c\}$$

is a proper concordant filter of $\mathrm{B}^{\mathfrak{q}}(T)$. So, if $\mathfrak{f} \subseteq \mathfrak{n} \in \mathfrak{M}_{\mathrm{B}^{\mathfrak{q}}(T)}$,

$$\bigcap \mathfrak{m} = \bigcap \mathfrak{f} \supseteq \bigcap \mathfrak{n} \neq \emptyset, \text{ since } \mathrm{K}(T) \text{ is compact.}$$

This proves the theorem.

Remark. Though H. Freudenthal [12] proved the theorem with the acceptance of the second countability axiom, his proof is not essentially different from ours and applies equally well to the general case.

3.2. Percompactness.

3.2.1. Definition. A topological space is called *percompact* if every two points of it are separated by a compact set.

The notion of percompactness is introduced as a slight generalization of the well-known concept of peripheral compactness, which will be defined in the next section.

3.2.2. Theorem. Let T be a percompact topological space, and C and D disjoint closed sets of T with compact boundaries. Then C and D have disjoint neighbourhoods with compact boundaries (which neighbourhoods can, of course, be chosen to be regularly open).

Proof. Let $p \in \mathfrak{R}(C)$. By definition 3.2.1, for every $q \in \mathfrak{R}(D)$ we can choose disjoint open sets O_q and U_q, containing p and q respectively, such that $\mathfrak{R}(O_q)$ and $\mathfrak{R}(U_q)$ are compact. Since $\mathfrak{R}(D)$ is compact, there exist an integer n and elements $q_1, q_2, \ldots, q_n \in \mathfrak{R}(D)$, such that:

$$\mathfrak{R}(D) \subseteq \bigcup_{i=1}^{n} U_{q_i}.$$

Put

$$V_p = (\bigcap_{i=1}^{n} O_{q_i}) \cap D^c, \quad W_p = (\bigcup_{i=1}^{n} U_{q_i}) \cup D.$$

It is easily seen that V_p is a neighbourhood of p, W_p a neighbourhood of D, $V_p \cap W_p = \emptyset$, and $\mathfrak{R}(V_p)$ and $\mathfrak{R}(W_p)$ are compact.

Take such sets V_p and W_p for every $p \in \mathfrak{R}(C)$. Since $\mathfrak{R}(C)$ is compact, there exist an integer m and elements $p_1, p_2, \ldots, p_m \in \mathfrak{R}(C)$ such that:

$$\mathfrak{R}(C) \subseteq \bigcup_{j=1}^{m} V_{p_j}.$$

Now

$$V = (\bigcup_{j=1}^{m} V_{p_j}) \cup C \text{ and } W = (\bigcap_{j=1}^{m} W_{p_j}) \cap C^c$$

are neighbourhoods of C and D respectively, as sought.

3.2.3. Theorem. Let T be a percompact topological space. Then

$$\mathrm{B_c}(T) = \{a \in \mathrm{B}^*(T) \mid \mathfrak{R}(a) \text{ compact}\}$$

is a compingent subalgebra of $\mathrm{B}^*(T)$.

Proof. If $a,b \in \mathrm{B_c}(T)$, then $\mathfrak{R}(a \wedge b)$, $\mathfrak{R}(a \vee b)$ and $\mathfrak{R}(a^o)$ are closed subsets of the compact set $\mathfrak{R}(a) \cup \mathfrak{R}(b)$, whence $a \wedge b$, $a \vee b$, $a^o \in \mathrm{B_c}(T)$. So $\mathrm{B_c}(T)$ is a boolean subalgebra of $\mathrm{B}^*(T)$.

Let $a,b \in \mathrm{B_c}(T)$, and $a \ll^* b$ (i.e. $\bar{a} \subseteq b$), $b \neq \emptyset$, e.g. $p \in b$. By theorem 3.2.2, there exists an element $c \in \mathrm{B_c}(T)$ such that

$\bar{a} \cup \{p\} \subseteq c \subseteq \bar{c} \subseteq b$. So $a \ll^* c \ll^* b$, and $c \neq \emptyset$. This shows the theorem.

3.2.4. Definition. Let S be a subspace of a completely regular space T. Then S is called (≤ 0)-*Inductionally embedded* in T if any two functionally separated sets of T are separated by a set disjoint to S. (Then, in particular, ind $S \leq 0$).

This definition is included in a more general definition which will be given in the fourth section.

3.2.5. Theorem. Let T be a percompact topological space, and

$$N = \mathfrak{M}_{\mathrm{B_c}(T)} \setminus \mu_{\mathrm{B_c}(T)}[T].$$

Then:

(i) N is (≤ 0)-Inductionally embedded in $\mathfrak{M}_{\mathrm{B_c}(T)}$,

(ii) $\mu_{\mathrm{B_c}(T)}$ is one-to-one into.

Proof. Let us put $B = \mathrm{B_c}(T)$ for short.

(i) Let F be a closed set of $\mathfrak{M}_B$, and U a neighbourhood of F. By theorem 2.1.6, and the normality of $\mathfrak{M}_B$, there exists an element $a \in B$ such that

$$F \subseteq \omega_B(a) \subseteq (\omega_B(a))^- \subseteq U.$$

So it suffices to show that $\mathfrak{R}(\omega_B(a)) \cap N = \emptyset$; for this it is sufficient to show that $\mu_B[\mathfrak{R}(a)] = \mathfrak{R}(\omega_B(a))$. Now let $\mathfrak{m} \in \mathfrak{R}(\omega_B(a))$; this means:

$$b \in \mathfrak{m} \Rightarrow b \wedge a \neq \emptyset,\ b \wedge a^0 \neq \emptyset.$$

If $b \cap \mathfrak{R}(a) \neq \emptyset$ for every $b \in \mathfrak{R}$, then, by the compactness of $\mathrm{R}(a)$, there exists an element

$$p \in (\bigcap \mathfrak{m}) \cap \mathfrak{R}(a).$$

Then, however, $\mu_B(p) = \mathfrak{m}$. So suppose there exists an element $b \in \mathfrak{m}$ such that $b \cap \mathfrak{R}(a) = \emptyset$. Choose $c \in \mathfrak{m}$ such that $c \ll^* b$, i.e. $\bar{c} \subseteq b$. Then it follows that

$$\mathfrak{R}(c \wedge a) \subseteq \mathfrak{R}(c) \cap a,$$

whence $c \wedge a \ll^* a$.

So $a \in \mathfrak{m}$ or $(c \wedge a)^o \in \mathfrak{m}$; but the first possibility contradicts the implication above, whilst the second possibility is equally absurd, since it would imply that $(c \wedge a)^o \wedge c = c \wedge a^o \in \mathfrak{m}$, and $a^o \in \mathfrak{m}$.

(ii) Take $p,q \in T$, such that $p \neq q$. Since T is percompact, there exists an element $a \in B$ such that $p \in a$, $q \notin a$. By theorem 3.2.2, there exists an element $b \in B$ such that $p \in b \subseteq \bar{b} \subseteq a$. This shows that $a \in \mu_B(p)$. Since evidently $a \notin \mu_B(q)$, we obtain:

$$\mu_B(p) \neq \mu_B(q).$$

3.2.6. Theorem. Let T be a percompact topological space. Then $(\mu_{B_c(T)}, \mathfrak{M}_{B_c(T)})$ is a greatest compaction amongst those compactions (α, C) of T which have the following property: given any disjoint closed sets S_1 and S_2 of C, there exists an open set O of T which separates $\alpha^{-1}[S_1]$ and $\alpha^{-1}[S_2]$, whilst $\mathfrak{R}(O)$ is compact.

Proof. Put $B_c(T) = B$. First, we show that $(\mu_B, \mathfrak{M}_B)$ has the property mentioned. Let S_1 and S_2 be disjoint closed sets of $\mathfrak{M}_B$. Then there are elements $a,b \in B$ such that

$$S_1 \subseteq \omega_B(a),\ S_2 \subseteq \omega_B(b^o),\ a \ll^* b$$

(remember that B is a compingent subalgebra of $B^*(T)$). Then

$$\mu_B^{-1}[\omega_B(a)] \subseteq a,\ \mu_B^{-1}[\omega_B(b^o)] \subseteq b^o;$$

it follows that $a^o \wedge b$ is a separating set as required.

To prove the theorem, by theorem 2.3.2 we need only consider an arbitrary compaction of T of the form $(\mu_{B_1}, \mathfrak{M}_{B_1})$ having the property of the theorem, where B_1 is a subcompingent subalgebra of $B^*(T)$. We shall apply theorem 2.3.3. So let $a_1, b_1 \in B_1$, with $a_1 \ll_1 b_1$, where $\ll_1$ is the compingent relation of B_1. Choose $c_1, d_1, e_1 \in B_1$ such that $a_1 \ll_1 c_1 \ll_1 d_1 \ll_1 e_1 \ll_1 b_1$. Then:

$$a_1 \subseteq \mu_{B_1}^{-1}[\omega_{B_1}(c_1)] \subseteq c_1,\ b_1^o \subseteq e_1^{o_1} \subseteq \mu_{B_1}^{-1}[\omega_{B_1}(d_1^{o_1})];$$

by our hypothesis, there exists an open set c of T separating $\mu_{B_1}^{-1}[(\omega_{B_1}(c_1))^-]$ and $\mu_{B_1}^{-1}[\omega_{B_1}(d_1^{o_1}))^-]$; evidently, we may assume $c \in B$. So there exists a disjoint union

$$T = T_1 \cup c \cup T_2, \text{ with } T_1 \cap T_2 \subseteq c, \text{ and}$$
$$\mu_{B_1}^{-1}[(\omega_{B_1}(c_1))^-] \subseteq T_1,\ \mu_{B_1}^{-1}[(\omega_{B_1}(d_1^{o_1}))^-] \subseteq T_2.$$

Now it is easy to show that the sets $a = T_1^{\mathrm{c}-\mathrm{c}}$ and $b = T_1 \cup c$ are regularly open, and that $a_1 \leq a \ll^* b \leq b_1$.

3.2.7. Theorem. Let T be a percompact topological space such that $\mathrm{K}(T)$ is compact. Then

$$\mathrm{w}(\mathfrak{M}_{\mathrm{B_c}(T)}) \leq \mathrm{w}(T).$$

Proof. Assume $\mathrm{w}(T) \geq \aleph_0$. Take a basis A for the topology of T, with minimal potency; we may assume that to any two elements of A, A contains their union. To every pair $(O_1,O_2) \in A \times A$, choose an element $a \in \mathrm{B_c}(T)$ such that $O_1 \subseteq a \subseteq \bar{a} \subseteq T \setminus O_2$, whenever this is possible. Let C be the boolean subalgebra of $\mathrm{B_c}(T)$ generated by these elements of $\mathrm{B_c}(T)$. Then $|C| \leq \mathrm{w}(T)$. Take D as to be the boolean subalgebra of $\mathrm{B_c}(T)$, generated by the interiors in T of the open-and-closed sets of the closures of the elements of C; using theorem 3.1.12 and theorem 3.1.13, we obtain: $|D| \leq \mathrm{w}(T)$. It suffices to show that D is dense in $\mathrm{B_c}(T)$ (since then $\omega_{\mathrm{B_c}(T)}[C]$ is a basis for the topology of $\mathfrak{M}_{\mathrm{B_c}(T)}$). So take $a,b \in \mathrm{B_c}(T)$, such that $a \ll^* b$. By theorem 3.2.2, $\mathfrak{R}(a)$ and $\mathfrak{R}(b)$ have disjoint neighbourhoods a' and b' respectively, with $a',b' \in \mathrm{B_c}(T)$. By the compactness of $\mathfrak{R}(a)$ and $\mathfrak{R}(b)$, and the hypotheses on A, there are elements $O_a,O_b \in A$, such that $\mathfrak{R}(a) \subseteq O_a \subseteq a'$, $\mathfrak{R}(b) \subseteq O_b \subseteq b'$. Now, by the construction of C, there exists an element $c \in C$ such that

$$O_a \subseteq c \subseteq \bar{c} \subseteq T \setminus O_b \text{ (e.g. } a' \text{ might be in } C).$$

Then, in particular, $\mathfrak{R}(a) \subseteq c \subseteq \bar{c} \subseteq T \setminus \mathfrak{R}(b)$.

Since $\mathfrak{R}(b) \cap \mathfrak{R}(c) = \emptyset$, $\bar{c} \cap b$ is an open-and-closed set of $\bar{c}$, with interior $c \cap b$, and since $\mathfrak{R}(a) \cap \mathfrak{R}(c) = \emptyset$, $c^{\mathrm{c}} \cap a$ is an open-and-closed set of $c^{\mathrm{o}-}$, with interior $c^{\mathrm{o}} \cap a$. Hence

$$(c \wedge b) \vee (c^{\mathrm{o}} \wedge a) \in D,$$

and

$$a \ll^* (c \wedge b) \vee (c^{\mathrm{o}} \wedge a) \ll^* b.$$

This proves the theorem.

Remark. The proof is an adaptation to the greater generality of the theorem and to the application of our theory, of the analogous proof by H. Freudenthal [12].

3.2.8. Problem. If T is as in theorem 3.2.7, can it ever happen that $\mathrm{w}(\mathfrak{M}_{\mathrm{B_c}(T)}) < \mathrm{w}(T)$? For $\mathrm{w}(T) = \aleph_0$, it is not difficult to show that $\mathrm{w}(\mathfrak{M}_{\mathrm{B_c}(T)}) = \mathrm{w}(T)$.

3.3. Peripheral compactness.

3.3.1. Definition. A topological space is *peripherally compact* if each point of it has arbitrarily small neighbourhoods with compact boundaries.

3.3.2. Theorem. (K. Morita [24]). A peripherally compact space is a percompact completely regular space.

Proof. Let T be a peripherally compact space. The percompactness of T follows from the definitions (remember that only Hausdorff spaces are being considered). Let $p \in O$, where O is an open set of T. Then there exists a neighbourhood a of p with $a \subseteq O$, and $a \in \mathrm{B_c}(T)$. By theorem 3.2.2, there exists a neighbourhood b of p with $p \in b \ll^* a$. This shows that $\mathrm{B_c}(T)$ is a basic compingent subalgebra of $\mathrm{B}^*(T)$; hence, T is completely regular by corollary 2.2.2.

3.3.3. Theorem. Let T be a percompact topological space. Then the following three conditions are equivalent.

(i) T is peripherally compact;

(ii) $\mathrm{B_c}(T)$ is basic;

(iii) $(\mu_{\mathrm{B_c}(T)}, \mathfrak{M}_{\mathrm{B_c}(T)})$ is a compactification of T.

Proof. The equivalence of (ii) and (iii) is contained in theorem 2.2.1, whereas the equivalence of (i) and (ii) follows from the proof of theorem 3.3.2, and the definitions.

3.3.4. Theorem. Let T be a completely regular space, and (α, C) a compactification of T such that $C \setminus \alpha[T]$ is (≤ 0)-Inductionally embedded in C. Then

(i) T is peripherally compact;

(ii) $(\alpha, C) \leq (\mu_{\mathrm{B_c}(T)}, \mathfrak{M}_{\mathrm{B_c}(T)})$ (i.e. $(\mu_{\mathrm{B_c}(T)}, \mathfrak{M}_{\mathrm{B_c}(T)})$ is a greatest compactification of T by means of an (≤ 0) -Inductionally embedded set).

Proof.

(i) Let $p \in T$, and U be a neighbourhood of p; then there exists a neighbourhood V of $\alpha(p)$ such that $V \cap \alpha[\mathrm{T}] = \alpha[\mathrm{U}]$. By the hypothesis, there is a neighbourhood W of $\alpha(p)$ such that $W \subseteq V$ and $\mathfrak{R}(W) \subseteq \alpha[T]$. Then, evidently, $p \in \alpha^{-1}[W] \subseteq U$ and $\mathfrak{R}(\alpha^{-1}[W])$ is compact.

(ii) The assertion follows from (i) and a simple application of theorem 3.2.6.

3.3.5. Theorem. (P. S. Aleksandrov - V. I. Ponomarëv [3]). The peripherally compact spaces are the complements of the (≤ 0)-Inductionally embedded sets in compact spaces.

Proof. Indeed, by theorem 3.3.3 and theorem 3.2.5, every peripherally compact space is homeomorphically embedded in $\mathfrak{M}_{\mathrm{B_c}(T)}$ with an (≤ 0)-Inductionally embedded complement; the converse is proved as theorem 3.3.4 (i).

Remark. For the case in which the spaces considered satisfy the second countability axiom, the theorem reduces to a theorem proved by J. de Groot [13].

3.3.6. Theorem. (K. Morita [24]). To every peripherally compact space T there exists a greatest amongst the compactifications of T by means of an (≤ 0)-Inductionally embedded set.

Proof. The theorem follows from theorem 3.3.5 and theorem 3.3.4 (ii).

3.3.7. Theorem (E. G. Sklyarenko [28]). If T is a peripherally compact space, then there is a compactification (α, C) of T by means of an (≤ 0)-Inductionally embedded set, such that $\mathrm{w}(C) = \mathrm{w}(T)$.

Proof. Let $\mathrm{w}(T) \geq \aleph_0$. Since $\mathrm{B_c}(T)$ is a basis for the topology of T, there exists a subset S of $\mathrm{B_c}(T)$ such that S is a basis for the topology of T and $|S| = \mathrm{w}(T)$. (This useful potency lemma has been communicated to me by J. de Groot.) By lemma 3.1.6, there exists a compingent subalgebra B of $\mathrm{B_c}(T)$ such that $|B| = \mathrm{w}(T)$ and $S \subseteq B$. Then B is a basic compingent subalgebra of $\mathrm{B}(T)$; so $(\mu_B, \mathfrak{M}_B)$ is a compactification of T, and since $\omega_B[B]$ is a basis for the topology of $\mathfrak{M}_B$, $\mathrm{w}(\mathfrak{M}_B) = \mathrm{w}(T)$. The proof that the complement of $\mu_B[T]$ in $\mathfrak{M}_B$ is (≤ 0)-Inductionally embedded in $\mathfrak{M}_B$ is not different from the proof of theorem 3.2.5 (i).

3.3.8. **Theorem.** If T is a peripherally compact space such that $\mathrm{K}(T)$ is compact, then

$$\mathrm{w}(\mathfrak{M}_{\mathrm{B_c}(T)}) = \mathrm{w}(T).$$

Proof. The proof follows from theorem 3.2.7 and the observation that $\mathrm{w}(\mathfrak{M}_{\mathrm{B_c}(T)}) \geq \mathrm{w}(T)$, since $\mathfrak{M}_{\mathrm{B_c}(T)}$ contains a homeomorphic image of T.

Remark. For the case where $\mathrm{w}(T) = \aleph_0$ the theorem is due to H. Freudenthal [12].

3.3.9. **Remark.** We have noticed that for an (≤ 0)-Inductionally embedded set S we necessarily have: $\mathrm{ind}\ S \leq 0$. However, the converse is not true, as is shown by an example of Ju. M. Smirnov [31] of a non-peripherally compact space W such that $\beta W \setminus W = 0$, using theorem 3.3.5. It is not known to the author whether there exists an example of a subspace S of a completely regular space T such that $\mathrm{Ind}\ S = 0$, but with S not being (≤ 0)-Inductionally embedded in T. However, C. H. Dowker [6] has given an example of a normal space N containing a subspace M such that $\mathrm{Ind}\ N = 0$, $\mathrm{Ind}\ M = 1$. Then, a fortiori, M is an example of an (≤ 0)-Inductionally (in N) embedded set with positive Inductive dimension (the ambient space can be made even compact by taking its Čech-Stone compactification).

3.4. Compactness deficiency.

3.4.1. **Definition.** Let T be a completely regular space, and T' a subspace of T. Then T' is (-1)-*Inductionally embedded in* T if $T' = \emptyset$ (notation: $\mathrm{Ind}_T T' = -1$); T' is called $(\leq n)$-*Inductionally embedded* *in* T (notation: $\mathrm{Ind}_T T' \leq n$) for a non-negative integer n if any two functionally separated sets of T are separated by a set C such that $\mathrm{Ind}_T(T' \cap C) \leq n-1$.

If $\mathrm{Ind}_T T' \leq n$, but not $\mathrm{Ind}_T T' \leq n-1$, then we say that $\mathrm{Ind}_T T' = n$; if there is no integer n such that $\mathrm{Ind}_T T' \leq n$, then we put $\mathrm{Ind}_T T' = \infty$.

3.4.2. **Proposition.** If T is a completely regular space, and T' and T'' subsets of T, then:

(i) $T'' \subseteq T' \Rightarrow \mathrm{Ind}_T T'' \leq \mathrm{Ind}_T T'$,

(ii) $\mathrm{ind}\, T' \leq \mathrm{Ind}_T T'$,

(iii) $\mathrm{Ind}\, T = \mathrm{Ind}_T T$ if T is normal.

Proof. All three assertions follow directly from the definitions.

3.4.3. **Proposition.** Let T be a normal space, and T' a subspace of T. Then, for any closed set F of T:

$$\mathrm{Ind}_F\, T' \cap F \leq \mathrm{Ind}_T\, T'.$$

Proof. Let F_1 and F_2 be two functionally separated sets of F. Then, by the Tietze extension theorem, F_1 and F_2 are also functionally separated in T. So, if $\mathrm{Ind}_T T' = n$, there exists a set C of T, separating F_1 and F_2 in T, such that $\mathrm{Ind}_T\, C \cap T' \leq n-1$ (we may assume that $n \geq 0$). Using induction, we find that $\mathrm{Ind}_F\, F \cap C \cap T' \leq n-1$. Since $F \cap C$ separates F_1 and F_2 in F, the proposition follows from the definition.

3.4.4. **Proposition.** Let T be a compact space, $T' \subseteq T$ and n a non-negative integer. Then:

$\mathrm{Ind}_T\, T' \leq n \Leftrightarrow \exists$ a basic compingent subalgebra of $\mathrm{B}(T)$ in which the set of elements c with $\mathrm{Ind}_T\, \mathfrak{R}(c) \cap T' \leq n-1$ is dense.

Proof. The verification of both implications is straightforward.

3.4.5. **Definition.** Let T be a completely regular space. Then the *compactness deficiency* comp def T of T is defined as follows:

comp def $T \leq n \Leftrightarrow T$ has a compactification (α, C) such that

$$\operatorname{Ind}_C C \setminus \alpha[T] \leq n.$$

Obviously, we define:

comp def $T = n \Leftrightarrow$ comp def $T \leq n$ and comp def $T \nleq n - 1$.

3.4.6. Lemma. Let C be a compact space, N a subset of C and B_0 a subset of $\mathrm{B}(C)$ such that $|B_0| = \tau \geq \aleph_0$. Then there exists a compingent subalgebra B of $\mathrm{B}(C)$ such that:

$$B_0 \subseteq B,\ |B| = \tau,\ \operatorname{Ind}_{\mathfrak{M}_B} \mathfrak{M}_B \setminus \mu_B[C \setminus N] \leq \operatorname{Ind}_C N.$$

Proof. We may assume that $\operatorname{Ind}_C N = n < \infty$. By means of induction we shall define a monotonously non-shrinking sequence $(B_i)_{i=0}^{\infty}$ of subsets of $\mathrm{B}(C)$.

Assume $i \geq 0$, and B_i already defined.

(i) If $0 \leq k \leq n$, and $c_1, c_2, \ldots, c_k \in B_i$ such that:

$$\operatorname{Ind}_C \bigcap_{j=1}^{k} \mathfrak{R}(c_j) \cap N \leq n - k,$$

then choose for every $a,b \in B_i$ with $a \ll b$ an element $d \in \mathrm{B}(C)$ such that $a \ll d \ll b$ and

$$\operatorname{Ind}_C \mathfrak{R}(d) \cap \bigcap_{j=1}^{k} \mathfrak{R}(c_j) \cap N \leq n - k - 1;$$

this is possible by definition 3.4.1 and proposition 3.4.2 (i).

(ii) If $2 \leq k \leq n + 1$, $c_1, c_2, \ldots, c_k \in B_i$ such that

$$\operatorname{Ind}_C \bigcap_{j=1}^{k} \mathfrak{R}(c_j) \cap N \leq n - k,$$

and $a,b \in B_i$ such that $a \ll b$ and $b \cap \bigcap_{j=1}^{k} \mathfrak{R}(c_j) = \emptyset$, then choose $e,f \in \mathrm{B}(C)$ such that:

$$e \ll f,\ \bar{a} \cap \bigcap_{j=1}^{k-1} \mathfrak{R}(c_j) \subseteq e,\ f \cap \mathfrak{R}(c_k) = \emptyset;$$

this is possible by the normality of C.

(iii) Let B_i' be a compingent subalgebra of $\mathrm{B}(C)$ containing B_i, and such that $|B_i'| = \tau$; B_i' exists by lemma 3.1.6. Now we define B_{i+1} as the union of B_i' and the set consisting of all elements d,e,f chosen under (i) and (ii).

Put $B = \bigcup_{i=1}^{\infty} B_i$. It is clear that B is a compingent subalgebra of B(C), with $B_0 \subseteq B$ and $|B| = \tau$. There remains the last assertion of the lemma to be shown. The rest of the proof will be divided into two steps.

(a) Let $1 \leq k \leq n+1$, and $c_1, c_2, \ldots, c_k \in B$ be such that:

$$\mathrm{Ind}_C \bigcap_{j=1}^{l} \mathfrak{N}(c_j) \cap N \leq n - l \qquad (l = 1, 2, \ldots, k).$$

It will be shown that this implies:

$$\mu_B[\bigcap_{j=1}^{l} \mathfrak{N}(c_j)] = \bigcap_{j=1}^{l} \mathfrak{N}(\omega_B(c_j)) \qquad (l = 1, 2, \ldots, k).$$

Let us use induction with respect to k. For $k = 1$ the assertion follows from the compactness of $\mathfrak{N}(c_1)$ (cf. the proof of theorem 3.2.5 (i)).

Now assume $1 < k \leq n+1$, and $c_1, c_2, \ldots, c_k \in B$ as in the hypotheses of the assertion. Using the induction hypothesis, it is sufficient to show that:

$$\mu_B[\bigcap_{j=1}^{k} \mathfrak{N}(c_j)] \supseteq \mu_B[\bigcap_{j=1}^{k-1} \mathfrak{N}(c_j)] \cap \mu_B[\mathfrak{N}(c_k)],$$

since the converse inclusion is obvious. So, assume

$$\mathfrak{m} \in \mathfrak{M}_B, \qquad \mathfrak{m} \notin \mu_B [\bigcap_{j=1}^{k} \mathfrak{N}(c_j)].$$

By the compactness of $\bigcap_{j=1}^{k} \mathfrak{N}(c_j)$, it easily follows that there are elements $a,b \in \mathfrak{m}$ such that $a \ll b$ and $b \cap \bigcap_{j=1}^{k} \mathfrak{N}(c_j) = \emptyset$.

Choose i such that $a,b \in B_i$, $c_1, c_2, \ldots, c_k \in B_i$. By (ii), there exist elements $e,f \in B$ such that $e \ll f$ and

$$\bar{a} \cap \bigcap_{j=1}^{k-1} \mathfrak{N}(c_j) \subseteq e, \; f \cap \mathfrak{N}(c_k) = \emptyset.$$

Since $a \wedge e \ll b \wedge f$, we have: $b \wedge f \in \mathfrak{m}$ or $(a \wedge e)^0 \in \mathfrak{m}$. Now notice that if F is a set of C, and $\mathfrak{n}$ a maximal concordant filter of B containing an element disjoint to F, then $\mathfrak{n} \notin \mu_B[F]$. So, if $b \wedge f \in \mathfrak{m}$, then, because of the relation $(b \wedge f) \cap \mathfrak{N}(c_k) = \emptyset$, $\mathfrak{m} \notin \mu_B[\mathfrak{N}(c_k)]$. In the other case, $(a \wedge e)^0 \in \mathfrak{m}$, whence $a \wedge (a \wedge e)^0 = a \wedge e^0 \in \mathfrak{m}$.

But $(a \wedge e^o) \cap \bigcap_{j=1}^{k-1} \mathfrak{R}(c_j) = \emptyset$, so then

$$\mathfrak{m} \notin \mu_B[\bigcap_{j=1}^{k-1} \mathfrak{R}(c_j)].$$

Hence in either case:

$$\mathfrak{m} \notin \mu_B[\bigcap_{j=1}^{k-1} \mathfrak{R}(c_j)] \cap \mu_B[\mathfrak{R}(c_k)].$$

This shows the required inclusion.

(b) Let $1 \leq k \leq n+1$, and $c_1, c_2, \ldots, c_k \in B$ be such that:

$$\mathrm{Ind}_C \bigcap_{j=1}^{l} \mathfrak{R}(c_j) \cap N \leq n - l \quad (l = 1, 2, \ldots, k);$$

we shall see that this implies:

$$\mathrm{Ind}_{\mathfrak{M}_B} \bigcap_{j=1}^{k} \mathfrak{R}(\omega_B(c_j)) \cap M \leq n - k,$$

where $M = \mathfrak{M}_B \setminus \mu_B[C \setminus N]$. We shall use induction with respect to $n - k$.

Let $k = n + 1$. Then by (a) we obtain at once:

$$\bigcap_{j=1}^{n+1} \mathfrak{R}(\omega_B(c_j)) \cap M = \emptyset, \text{ q.e.d..}$$

Now assume the assertion true for "k" replaced by "$k+1$", where $1 \leq k < n+1$, and assume $c_1, c_2 \ldots, c_k \in B$ as above. If $a \ll b$, $a,b \in B$, then by (i) there is an element $d \in B$ such that $a \ll d \ll b$, and

$$\mathrm{Ind}_C \mathfrak{R}(d) \cap \bigcap_{j=1}^{k} \mathfrak{R}(c_j) \cap N \leq n - k - 1.$$

By the induction hypothesis,

$$\mathrm{Ind}_{\mathfrak{M}_B} \mathfrak{R}(\omega_B(d)) \cap \bigcap_{j=1}^{k} \mathfrak{R}(\omega_B(c_j)) \cap M \leq n - k - 1.$$

This shows that $\mathrm{Ind}_{\mathfrak{M}_B} \bigcap_{j=1}^{k} \mathfrak{R}(\omega_B(c_j)) \cap M \leq n - k$, using proposition 3.4.4, q.e.d..

Lastly, applying (b) for the case $k = 1$, we obtain:

$$d \in B, \mathrm{Ind}_C \mathfrak{R}(d) \cap N \leq n - 1 \Rightarrow \mathrm{Ind}_{\mathfrak{M}_B} \mathfrak{R}(\omega_B(d)) \cap M \leq n - 1.$$

Since by (i) the elements d satisfying the hypothesis of the implication form a dense subset of B, the lemma follows from proposition 3.4.4 (applied to the space $\mathfrak{M}_B$).

3.4.7. Theorem. Let T be a completely regular space. Then there exists a compactification (δ, D) of T such that:

(i) $\operatorname{Ind}_D D \setminus \delta[T] = \operatorname{comp\,def} T$,

(ii) $\mathrm{w}(D) = \mathrm{w}(T)$.

Proof. We may assume that $\mathrm{w}(T) = \tau \geq \aleph_0$, and that comp def $T = n < \infty$. By definition 3.4.5, and theorem 2.2.4, there exists a basic subcompingent algebra B' of $\mathrm{B}(T)$ such that:

$$\operatorname{Ind}_{\mathfrak{M}_{B'}} N = n, \text{ where } N = \mathfrak{M}_{B'} \setminus \mu_{B'}[T].$$

Let B_0 be a basis for the topology of T such that $|B_0| = \tau$ and $B_0 \subseteq \mathrm{B}(T)$. By lemma 3.4.6, there exists a compingent subalgebra B of B' such that $\omega_{B'}[B_0] \subseteq \omega_{B'}[B]$, $|B| = \tau$ and

$$\operatorname{Ind}_{\mathfrak{M}_{B^*}} M \leq n, \text{ where } M = \mathfrak{M}_{B^*} \setminus \mu_{B^*}[\mathfrak{M}_{B'} \setminus N],$$

and where B^* is the isomorphic image of B under the mapping $\omega_{B'}$ restricted to B; this isomorphism will be denoted by e. It is easily verified that B is a basic subcompingent subalgebra of $\mathrm{B}(T)$ and that:

$$\mu_{B^*} \circ \mu_{B'} = \mathrm{m}(e^{-1}) \circ \mu_B.$$

Hence $(\mu_{B^*} \circ \mu_{B'}, \mathfrak{M}_{B^*})$ is a compactification of T (topologically equivalent to $(\mu_B, \mathfrak{M}_B)$), which satisfies all requirements of the theorem.

3.4.8. Remark. The property of completely regular spaces of having a certain given compactness deficiency, might be called an external property since its definition involves the consideration of relationships of the set space to other spaces. It follows from theorem 3.3.5 that the property "comp def $T \leq 0$" is equivalent to the internal property "T is peripherally compact". For separable metrizable spaces, the compactness deficiency coincides with the *compactification degree* as defined by J. de Groot [13], if only metrizable compactifications are admitted. By J. de Groot, l.c., the problem has been posed to find internal characterizations of separable metrizable spaces with given compactification degrees. In our more general setting, his conjecture becomes: for $n \geq 0$, the completely regular spaces with compactness deficiency $\leq n$ are those completely regular spaces in which every point has arbitrarily small neighbourhoods whose boundaries have compactness deficiencies $\leq n-1$. The restricted conjecture, and so the more the general conjecture, remains as yet unproved. In the rest of this section we shall give a sufficient condition for a space to have a compactness deficiency $\leq n$.

3.4.9. **Definition.** Let T be a normal space. We shall say that T has property P_{-1} if T is compact. Inductively, we define for a non-negative integer n, that T has property P_n if every closed set of T has arbitrarily small neighbourhoods of which the boundaries have property P_{n-1}.

3.4.10. **Lemma.** If T is a normal space having property P_n and F is a closed set of T, then F has property P_n.

Proof. The lemma will be proved by induction. For $n = -1$, the lemma is obvious. So assume $n > -1$. Let A be a closed set of F, and U a neighbourhood in F of A. Then there exists a neighbourhood U' in T of A such that $U' \cap F = U$. Since T has property P_n and $n > -1$, there exists a neighbourhood V in T of A such that $V \subseteq U'$ and $\mathfrak{R}(V)$ has property P_{n-1}. Since $\mathfrak{R}_F(V \cap F)$ is a closed subset of $\mathfrak{R}(V)$, and $V \cap F$ is a neighbourhood in F of A contained in U, the lemma follows from the induction hypothesis.

3.4.11. **Lemma.** Let T be a normal space, and $c_1, c_2, \ldots, c_k \in \mathrm{B}(T)$. Then:

$$(\mu_{\mathrm{B}(T)}[\bigcap_{j=1}^{k} \mathfrak{R}(c_j)])^{-} = \bigcap_{j=1}^{k} (\mu_{\mathrm{B}(T)}[\mathfrak{R}(c_j)])^{-} = \bigcap_{j=1}^{k} \mathfrak{R}(\omega_{\mathrm{B}(T)}(c_j)).$$

Proof. Let us call the consecutive closed sets in the lemma F, G and H. Then it is obvious that $F \subseteq G \subseteq H$ (notice that for a set A of T:

$$\mathfrak{m} \in (\mu_{\mathrm{B}(T)}[A])^{-} \Leftrightarrow a \cap A \neq \emptyset \text{ (all } a \in \mathfrak{m})).$$

So we need only prove: $F \supseteq H$. This will be done by induction.

Let $k = 1$, and $\mathfrak{m} \notin F$, i.e. $\exists\, a \in \mathfrak{m}$ such that $a \cap \mathfrak{R}(c_1) = \emptyset$.

Choose $b \in \mathfrak{m}$ such that $b \ll a$. Then $b \wedge c_1 \ll c_1$, whence $c_1 \in \mathfrak{m}$ or $(b \wedge c_1)^o \in \mathfrak{m}$ (and also $b \wedge (b \wedge c_1)^o = b \wedge c_1^o \in \mathfrak{m}$, and $c_1^o \in \mathfrak{m}$); in either case $\mathfrak{m} \notin \mathfrak{R}(\omega_{\mathrm{B}(T)}(c_1))$. Now assume $k > 1$, and $\mathfrak{m} \notin F$, e.g. $a \in \mathfrak{m}$ such that $a \cap \bigcap_{j=1}^{k} \mathfrak{R}(c_j) = \varnothing$. Choose $b \in \mathfrak{m}$ such that $b \ll a$. By the normality of T, we can choose $e,f \in \mathrm{B}(T)$ such that:

$$f \ll e,\ \bar{b} \cap \mathfrak{R}(c_k) \subseteq f,\ e \cap \bigcap_{j=1}^{k-1} \mathfrak{R}(c_j) = \varnothing.$$

Because of $b \wedge f \ll a \wedge e$, we have: $a \wedge e \in \mathfrak{m}$ or $(b \wedge f)^o \in \mathfrak{m}$. In the first case $\mathfrak{m} \notin (\mu_{\mathrm{B}(T)}[\bigcap_{j=1}^{k-1} \mathfrak{R}(c_j)])^-$, and in the second case $\mathfrak{m} \notin (\mu_{\mathrm{B}(T)}[\mathfrak{R}(c_k)])^-$. So

$$\mathfrak{m} \notin (\mu_{\mathrm{B}(T)}[\bigcap_{j=1}^{k-1} \mathfrak{R}(c_j)])^- \cap (\mu_{\mathrm{B}(T)}[\mathfrak{R}(c_k)])^- =$$
$$= \bigcap_{j=1}^{k-1} \mathfrak{R}(\omega_{\mathrm{B}(T)}(c_j)) \cap \mathfrak{R}(\omega_{\mathrm{B}(T)}(c_j)),$$

by the induction hypothesis and the case $k = 1$. This proves the lemma.

3.4.12. Theorem. Let T be a completely normal space with property P_n. Then:

$$\mathrm{Ind}_{\mathfrak{M}_{\mathrm{B}(T)}} \mathfrak{M}_{\mathrm{B}(T)} \setminus \mu_{\mathrm{B}(T)}[T] \leq n;$$

so, in particular, comp def $T \leq n$.

Proof. If $0 \leq k \leq n$, and $c_1, c_2, \ldots, c_k \in \mathrm{B}(T)$ such that

$$\bigcap_{j=1}^{k} \mathfrak{R}(c_j) \text{ has property } P_{n-k},$$

then to every $a,b \in \mathrm{B}(T)$ with $a \ll b$ there exists an element $d \in \mathrm{B}(T)$ such that $a \ll d \ll b$, and

$$\mathfrak{R}(d) \cap \bigcap_{j=1}^{k} \mathfrak{R}(c_j) \text{ has property } P_{n-k-1}.$$

Indeed, put $F = \bigcap_{j=1}^{k} \mathfrak{R}(c_j)$. Then $\bar{a} \cap F$ has a neighbourhood U in F such that $\bar{U} \subseteq b$ and

$$\mathfrak{R}_F(U) \text{ has property } P_{n-k-1}.$$

Since T is completely normal, $\bar{a} \cup U$ has a neighbourhood $d \in \mathrm{B}(T)$ in T such that

$$\bar{d} \cap ((F \setminus \bar{U}) \cup b^{\mathrm{c}}) = \emptyset.$$

So $a \ll d \ll b$, and $\mathfrak{R}(d) \cap F \subseteq \mathfrak{R}_F(U)$; hence, $\mathfrak{R}(d) \cap F$ also has property P_{n-k-1}, by lemma 3.4.10.

Now let $1 \leq k \leq n + 1$, and $c_1, c_2, \ldots, c_k \in \mathrm{B}(T)$ be such that $\bigcap_{j=1}^{l} \mathfrak{R}(c_j)$ has property P_{n-l} $(l = 1, 2, \ldots, k)$. We shall show that this implies:

$$\mathrm{Ind}_{\mathfrak{M}_{\mathrm{B}(T)}} \bigcap_{j=1}^{k} \mathfrak{R}(\omega_{\mathrm{B}(T)}(c_j)) \cap M \leq n - k,$$

where $M = \mathfrak{M}_{\mathrm{B}(T)} \setminus \mu_{\mathrm{B}(T)}[T]$. We shall use induction with respect to $n - k$.

For $k = n + 1$, the assertion follows from the compactness of $\bigcap_{j=1}^{n+1} \mathfrak{R}(c_j)$ and lemma 3.4.11. Now assume the assertion true for "k" replaced by "$k + 1$", where $1 \leq k < n + 1$, and assume $c_1, c_2, \ldots, c_k \in \mathrm{B}(T)$ as above. If $a, b \in \mathrm{B}(T)$, and $a \ll b$, then by what we proved first, there is an element $d \in \mathrm{B}(T)$ such that $a \ll d \ll b$ and

$$\mathfrak{R}(d) \cap \bigcap_{j=1}^{k} \mathfrak{R}(c_j) \text{ has property } P_{n-k-1}.$$

Hence, by the induction hypothesis,

$$\mathrm{Ind}_{\mathfrak{M}_{\mathrm{B}(T)}} \mathfrak{R}(\omega_{\mathrm{B}(T)}(d)) \cap \bigcap_{j=1}^{k} \mathfrak{R}(\omega_{\mathrm{B}(T)}(c_j)) \cap M \leq n - k - 1.$$

This shows that

$$\mathrm{Ind}_{\mathfrak{M}_{\mathrm{B}(T)}} \bigcap_{j=1}^{k} \mathfrak{R}(\omega_{\mathrm{B}(T)}(c_j)) \cap M \leq n - k.$$

Lastly, applying this for the case $k = 1$, we obtain: $d \in \mathrm{B}(T)$, $\mathfrak{R}(d)$ has property $P_{n-1} \Rightarrow \mathrm{Ind}_{\mathfrak{M}_{\mathrm{B}(T)}} \mathfrak{R}(\omega_{\mathrm{B}(T)}(d)) \cap M \leq n - 1$. Since these elements d form a dense subset of $\mathrm{B}(T)$, the theorem follows from proposition 3.4.4.

CHAPTER 4. COMPACTIFICATIONS, PRESERVING DIMENSION AND CONTINUOUS MAPPINGS

4.1. On a theorem of E. G. Sklyarenko.

4.1.1. Lemma. Let B be a compingent algebra, and $a_1, a_2, \ldots, a_k \in B$. Then:

$$\mathfrak{M}_B = \bigcup_{i=1}^{k} \omega_B(a_i) \Leftrightarrow \exists\, b_i \in B \text{ such that } b_i \ll a_i \ (i = 1, 2, \ldots, k)$$
$$\text{and } 1 = \bigvee_{i=1}^{k} b_i.$$

Proof. To prove the sufficiency, let b_i $(i = 1, 2, \ldots, k)$ be as in the lemma. Let $\mathfrak{m} \in \mathfrak{M}_B$, and suppose $a_i \notin \mathfrak{m}$ $(i = 1, 2, \ldots, k)$. Then $b_i^{\mathrm{o}} \in \mathfrak{m}$ $(i = 1, 2, \ldots, k)$, whence

$$\bigwedge_{i=1}^{k} b_i^{\mathrm{o}} = (\bigvee_{i=1}^{k} b_i)^{\mathrm{o}} = 1^{\mathrm{o}} = 0 \in \mathfrak{m},$$

which is a contradiction.

To prove the necessity, we shall define b_l $(l = 1, 2, \ldots, k)$ by induction, such that the following conditions are satisfied:

$$b_i \ll a_i \ (i = 1, 2, \ldots, l), \mathfrak{M}_B = (\bigcup_{i=1}^{l} \omega_B(b_i)) \cup (\bigcup_{i=l+1}^{k} \omega_B(a_i)).$$

Let $l < k$, and b_l be defined. Put

$$F = (\bigcup_{i=1}^{l} \omega_B(b_i))^{\mathrm{c}} \cap (\bigcup_{i=l+2}^{k} \omega_B(a_i))^{\mathrm{c}}.$$

It is clear that F is a closed set contained in $\omega_B(a_{l+1})$. Then there exists an element $b_{l+1} \in B$ such that $b_{l+1} \ll a_{l+1}$ and $F \subseteq \omega_B(b_{l+1})$. Then b_{l+1} is as required. This proves the lemma.

4.1.2. Definition. Let u be a finite open covering of a topological space T. A *strong* refinement of u is an open covering v of T, such that there exists a mapping f of u onto v with:

$$O \in u \Rightarrow \overline{f(O)} \subseteq O.$$

4.1.3. Lemma. Let T be a normal space and u a finite open covering of T. Then there exists a regularly open, strong refinement of u.

The easy proof of this lemma and the next is left to the reader (cf. P. Alexandroff und H. Hopf [1, I, § 6.8]).

4.1.4. Lemma. Let T be a compact space, B a basic compingent subalgebra of T, and u a finite open covering of T. Then there exists a strong refinement of u which consists of elements of B.

4.1.5. Theorem. Let T be a normal space of weight τ, and Σ a set of closed sets of T such that $|\Sigma| \leq \tau$. Then there exists a compactification (α, C) of T such that:

(i) $\mathrm{w}(C) = \tau$;

(ii) $\dim\, (\alpha[F])^- = \dim F$ (all $F \in \Sigma$).

Proof. Assume $\tau \geq \aleph_0$. Take $F \in \Sigma$. If $\dim F < \infty$, then we choose a finite open covering $\{U_i\}_{i=1}^{k}$ of the subspace F which does not admit an open refinement of order $\dim F$. Then $\{U_i \cup F^c\}_{i=1}^{k}$ is a finite open covering of T. By lemma 4.1.3, there exists a regularly open, strong refinement $v = v_F$ of $\{U_i \cup F^c\}_{i=1}^{k}$. It is clear that $\{a \cap F \mid a \in v\}$ is a finite open covering of F which does not admit an open refinement of order $\dim F$. If $\dim F = \infty$, then, in a similar way, we can find finite regularly open coverings $v^{(m)} = v_F^{(m)}$ of T such that $\{a \cap F \mid a \in v^{(m)}\}$ is a finite open covering of F which does not admit an open refinement of order m $(m = 1, 2, \ldots)$. Let $w = w_F$, or $w^{(m)} = w_F^{(m)}$, be a regularly open, strong refinement of v, or $v^{(m)}$ $(m = 1, 2, \ldots)$, according to whether $\dim F < \infty$ or $\dim F = \infty$. Put

$$S = \bigcup \{v_F, w_F \mid F \in \Sigma,\ \dim F < \infty\} \cup \bigcup_{m=1}^{\infty} \bigcup \{v_F^{(m)}, w_F^{(m)} \mid F \in \Sigma,\ \dim F = \infty\}.$$

Then, obviously, $|S| \leq \tau$. Let B_0 be a basis for the topology of T, such that:

$$S \subseteq B_0 \subseteq \mathrm{B}(T).$$

We shall define a monotonously non-shrinking sequence $(B_n)_{n=0}^{\infty}$ of subsets of $\mathrm{B}(T)$.

Assume $n \geq 0$, and B_n already defined. Then B_{n+1} will be determined in two steps.

(i) Let $F \in \Sigma$, and $\dim F < \infty$. Let u be any finite open covering of T by elements of B_n. Then $\{a \cap F \mid a \in u\}$ is a finite open covering of the subspace F, which has a finite open refinement $\{V_i\}_{i=1}^{k}$ of order $\leq \dim F + 1$. We may assume: $i \neq j \Rightarrow V_i \neq V_j$. To every $i = 1, 2, \ldots, k$, choose $a_i \in u$ such that $V_i \subseteq a_i$. Then $V_i \cup (a_i \cap F^c)$ is an open set of T $(i = 1, 2, \ldots, k)$. Now

$$u' = u'_F = \{V_i \cup (a_i \cap F^c)\}_{i=1}^{k} \cup \{a \cap F^c \mid a \in u\}$$

is a finite open refinement of u. Let $u^* = u_F^*$ be a regularly open, strong refinement of u' (which exists by lemma 4.1.3). Notice that $\{a \cap F \mid a \in u^*\}$ is a finite open covering of F of order $\leq \dim F + 1$, and u^* is a refinement of u. In its turn, u^* has a regularly open, strong refinement $u^{**} = u_F^{**}$. Put

$$B'_n = \bigcup \{u_F^*, u_F^{**} \mid u \text{ a finite covering of } T \text{ by elements of } B_n,\ F \in \Sigma,\ \dim F < \infty\}.$$

Clearly, $|B'_n| \leq \tau$, under the assumption $|B_n| = \tau$.

(ii) Let B''_n be a compingent subalgebra of $\mathrm{B}(T)$, such that:

$$B_n \subseteq B''_n,\ |B''_n| = \tau;$$

B''_n exists by lemma 3.1.6.

Now we define: $B_{n+1} = B'_n \cup B''_n$, and $B = \bigcup\limits_{n=0}^{\infty} B_n$. It is clear that B is a basic compingent subalgebra of $\mathrm{B}(T)$ of potency τ; hence $\mathrm{w}(\mathfrak{M}_B) = \tau$. We shall show that $(\mu_B, \mathfrak{M}_B)$ also satisfies the second assertion of the theorem.

Let $F \in \Sigma$, $\dim F < \infty$. Let $z = \{W_i\}_{i=1}^{s}$ be a finite open covering of the subspace $(\mu_B[F])^-$. Then:

$$z' = \{W_i \cup (\mu_B[F])^{-c}\}_{i=1}^{s}$$

is a finite open covering of $\mathfrak{M}_B$. By lemma 4.1.4, z' admits a strong refinement by elements of $\omega_B[B]$, say

$$\{\omega_B(b_i)\}_{i=1}^{s}, \text{ where } b_i \in B \ (i = 1, 2, \ldots, s).$$

Obviously, $u = \{b_i\}_{i=1}^{s}$ is a covering of T.
Let n be chosen in such a way that $b_1, b_2, \ldots, b_s \in B_n$. By the construction (i), we have found u^* and u^{**}. If $u^* = \{c_i\}_{i=1}^{t}$, then, by lemma 4.1.1, $\{\omega_B(c_i)\}_{i=1}^{t}$ is a covering of $\mathfrak{M}_B$. We may assume: $i \neq j \Rightarrow c_i \neq c_j$. By the construction of u^*, we have:

$$\bigcap_{k=1}^{\dim F+2} c_{i_k} \cap F = \emptyset \text{ if } k \neq l \Rightarrow i_k \neq i_l.$$

Hence $\displaystyle\bigcap_{k=1}^{\dim F+2} \omega_B(c_{i_k}) \cap \mu_B[F] = \emptyset$, under the same condition;

and then $\displaystyle\bigcap_{k=1}^{\dim F+2} \omega_B(c_{i_k}) \cap (\mu_B[F])^- = \emptyset.$

So the order of $y = \{\omega_B(c_i) \cup (\mu_B[F])^-\}_{i=1}^{t}$ is at most $\dim F + 1$. Since y is a refinement of z, it follows that

$$\dim (\mu_B[F])^- \leq \dim F.$$

However, if $F \in \Sigma$, by the construction of B_0 we have: $\{\omega_B(a) \cap (\mu_B[F])^- \mid a \in v\}$, or $\{\omega_B(a) \cap (\mu_B[F])^- \mid a \in v^{(m)}\}$, according to whether $\dim F < \infty$, or $\dim F = \infty$, is a finite open covering of $(\mu_B[F])^-$, which does not admit an open refinement of order $\dim F + 1$, or m $(m = 1, 2, \ldots)$, respectively.
This shows that $\dim F = \dim (\mu_B[F])^- \quad (F \in \Sigma)$.

Remark. E. G. Sklyarenko [29] proved the theorem for the case in which $|\Sigma| \leq \aleph_0$.

4.1.6. Corollary. If T is a normal space, then there exists a compactification (α, C) of T such that:

(i) $\mathrm{w}(C) = \mathrm{w}(T)$;

(ii) $\dim C = \dim T$.

4.1.7. Remark. One might wonder whether a peripherally compact normal space can always be compactified by means of an (≤ 0)-

Inductionally embedded set, such that the dimension of the space is not raised. However, it follows from connected separable metric examples of L. Zippin [33] and T. Nishiura [25], that this is not always possible.

4.2. On a theorem of C. Kuratowski.

4.2.1. Theorem. Let T be a normal space of weight τ, and Σ a set of closed sets of T with $|\Sigma| \leq \tau$. Then there exists a compactification (α, C) of T such that:

(i) $\mathrm{w}(C) = \tau$;

(ii) $F_i \in \Sigma$ $(i = 1, 2, \ldots, m)$ $\Rightarrow$

$$(\alpha[\bigcap_{i=1}^{m} F_i])^- = \bigcap_{i=1}^{m} (\alpha[F_i])^- \ (m = 1, 2, \ldots).$$

Proof. Assume $\tau \geq \aleph_0$, and:

$$F_i \in \Sigma \ (i = 1, 2, \ldots, m) \Rightarrow \bigcap_{i=1}^{m} F_i \in \Sigma \ (m = 1, 2, \ldots).$$

Let B_0 be a basis for the topology of T which consists of regularly open sets, and is such that $|B_0| = \tau$.

We shall define a monotonously non-shrinking sequence $(B_n)_{n=0}^{\infty}$ of subsets of $\mathrm{B}(T)$.

Assume $n \geq 0$, and B_n already defined. Then B_{n+1} will be determined in two steps.

(i) Let $a,b \in B_n$, $F_1,F_2 \in \Sigma$, and $a \ll b$, $b \cap F_1 \cap F_2 = \emptyset$. Then, by the normality of T, we can choose elements $c,d \in \mathrm{B}(T)$ such that $\bar{a} \cap F_1 \subseteq c \ll d$ and $d \cap F_2 = \emptyset$. Let B_n' be the set of all elements $c,d \in \mathrm{B}(T)$ so chosen. Clearly, $|B_n'| \leq \tau$, under the assumption $|B_n| = \tau$.

(ii) Let B_n'' be a compingent subalgebra of $\mathrm{B}(T)$, such that:

$$B_n \subseteq B_n'', \ |B_n| = \tau.$$

Now, put $B_{n+1} = B_n' \cup B_n$; and $B = \bigcup_{i=1}^{\infty} B_n$.

It is obvious that B is a basic compingent subalgebra of $\mathrm{B}(T)$ of potency τ; hence $\mathrm{w}(\mathfrak{M}_B) = \tau$. Let us verify that $(\mu_B, \mathfrak{M}_B)$ also satisfies the second assertion of the theorem.

Because of the condition we have imposed on Σ, we need only take $m = 2$. The proof that for $F_1, F_2 \in \Sigma$:

$$\mathfrak{m} \notin (\mu_B[F_1 \cap F_2])^- \Rightarrow \mathfrak{m} \notin (\mu_B[F_1])^- \cap (\mu_B[F_2])^-,$$

follows the same pattern as the corresponding part of the proof of lemma 3.4.11 or lemma 3.4.6.

4.2.2. Theorem. Let T be a normal space of weight τ, and Σ a set of closed sets of T with $|\Sigma| \leq \tau$. Then there exists a compactification (α, C) of T such that:

(i) $\mathrm{w}(C) = \tau$;

(ii) $F_i \in \Sigma \; (i = 1, 2, \ldots, m) \Rightarrow$

$$\dim \bigcap_{i=1}^{m} (\alpha[F_i])^- = \dim \bigcap_{i=1}^{m} F_i \; (m = 1, 2, \ldots).$$

Proof. Assume $\tau \geq \aleph_0$, and:

$$F_i \in \Sigma \; (i = 1, 2, \ldots, m) \Rightarrow \bigcap_{i=1}^{m} F_i \in \Sigma \; (m = 1, 2, \ldots).$$

By combining the constructions of the proofs of theorem 4.1.5 and theorem 4.2.1, a basic compingent subalgebra B of $\mathrm{B}(T)$ is obtained such that $(\mu_B, \mathfrak{M}_B)$ is a compactification of T, possessing the properties of both theorem 4.1.5 and theorem 4.2.1. Then, obviously, $(\mu_B, \mathfrak{M}_B)$ has the properties required in the present theorem too.

Remark. For the case $\tau = \aleph_0$ the result can be found in C. Kuratowski [20, § 40, VII.5].

4.3. The extension of continuous mappings to compactifications.

4.3.1. Lemma. Let T be a normal space of weight $\tau \geq \aleph_0$, and S a subset of $\mathrm{B}(T)$ such that $|S| \leq \tau$. Then there exists a basic compingent subalgebra B of $\mathrm{B}(T)$ such that:

$$S \subseteq B, \; |B| = \tau, \; \dim \mathfrak{M}_B = \dim T.$$

Proof. Taking $\Sigma = \{T\}$ in theorem 4.1.5, we merely need take care that in the proof of theorem 4.1.5, $S \subseteq B_0$. Then the compingent subalgebra B of $\mathrm{B}(T)$, obtained in the proof of that theorem, satisfies the lemma.

4.3.2. Theorem. Let T be a normal space of weight τ, and Φ a set of continuous mappings of T into a compact space D, where $|\Phi| \leq \tau$ and $\mathrm{w}(D) \leq \tau$. Then there exists a compactification (α, C) of T such that:

(i) $\mathrm{w}(C) = \tau$,

(ii) $\dim C = \dim T$,

(iii) $\varphi \circ \alpha^{-1}$ can be extended to a continuous mapping of C into D (all $\varphi \in \Phi$).
(Here α^{-1} stands for the mapping of $\alpha[T]$ onto T defined by: $\alpha^{-1}(\alpha(p)) = p$ $(p \in T)$.)

Proof. Assume $\tau \geq \aleph_0$. Let B_2 be a basic compingent subalgebra of $\mathrm{B}(D)$ of potency at most τ. By theorem 2.2.3, there exists a canonical homomorphism $\mathrm{g}(\varphi) = \mathrm{g}(\varphi, B_2)$ of B_2 into $\mathrm{B}(T)$ $(\varphi \in \Phi)$. Put

$$S = \cup \{\mathrm{g}(\varphi)[B_2] \mid \varphi \in \Phi\};$$

clearly, $|S| \leq \tau$. By lemma 4.3.1, there exists a basic compingent subalgebra B_1 of $\mathrm{B}(T)$ such that

$$S \subseteq B_1,\ |B_1| = \tau,\ \dim \mathfrak{M}_{B_1} = \dim T.$$

Then, by theorem 2.2.3,

$$\mathrm{m}(\mathrm{g}(\varphi)) \circ \mu_{B_1} = \mu_{B_2} \circ \varphi \quad (\varphi \in \Phi).$$

Since B_2 is basic and D compact, μ_{B_2} is a homeomorphism of D onto $\mathfrak{M}_{B_2}$. So

$$\mu_{B_2}^{-1} \circ \mathrm{m}(\mathrm{g}(\varphi)) \circ \mu_{B_1} = \varphi,$$

and $\mu_{B_2}^{-1} \circ \mathrm{m}(\mathrm{g}(\varphi))$ is a continuous extension of $\varphi \circ \mu_{B_1}^{-1}$ $(\varphi \in \Phi)$, which shows that $(\mu_{B_1}, \mathfrak{M}_{B_1})$ is a compactification of T as required.

Remark. A. B. Forge [11] proved the specialization of the theorem which is obtained by ~~omitting condition (ii)~~, taking $\tau = \aleph_0$, and D equal to the closed unit interval.

4.3.3. Theorem. Let B_n be subcompingent subalgebras of a compingent algebra B, such that their boolean algebras are boolean subalgebras of B, whilst B_n is a subcompingent subalgebra of B_{n-1} $(n = 1, 2, \ldots)$. Then $B_\infty = \bigcup_{n=1}^{\infty} B_n$ is made into a subcompingent subalgebra of B, containing every B_n as a subcompingent subalgebra, by the following definition of its compingent relation $\ll_\infty$: for $a,b \in B_\infty$,

$$a \ll_\infty b \Leftrightarrow \exists\, n \text{ such that } a,b \in B_n,\ a \ll_n b$$

(here "$\ll_n$" denotes the compingent relation of B_n, $n = 1, 2, \ldots$). We shall call B_∞ the *unification* of $(B_n)_{n=1}^\infty$.

Proof. As a union of a monotonously non-shrinking sequence of boolean subalgebras of B, B_∞ is a boolean subalgebra of B. The further verification of the fact that B_∞ is a compingent algebra is straightforward, and the same applies to the remaining assertions of the theorem.

4.3.4. **Theorem.** Let B, B_n $(n = 1, 2, \ldots)$, B_∞ be as in theorem 4.3.3. Then:

$$\dim \mathfrak{M}_{B_\infty} \leq \liminf_{n\to\infty} \dim \mathfrak{M}_{B_n}.$$

Proof. If $\liminf\limits_{n\to\infty} \dim \mathfrak{M}_{B_n} = \infty$, nothing has to be proved. So assume $k = \liminf\limits_{n\to\infty} \dim \mathfrak{M}_{B_n} < \infty$.

Let u be a finite open covering of $\mathfrak{M}_{B_\infty}$. By virtue of lemma 4.1.4, there exists a finite refinement $\{\omega_{B_\infty}(a_i)\}_{i=1}^s$ of u, where $a_i \in B_\infty$ $(i = 1, 2, \ldots, s)$. By lemma 4.1.1, there are elements $b_i \in B_\infty$ such that:

$$1 = \bigvee_{i=1}^{s} b_i \text{ and } b_i \ll_\infty a_i \quad (i = 1, 2, \ldots, s).$$

Now there exists a positive integer n such that:

$$a_i,b_i \in B_n, \qquad b_i \ll_n a_i \quad (i = 1, 2, \ldots, s).$$

Choose $m \geq n$ such that $k = \dim \mathfrak{M}_{B_m}$. Since B_n is a subcompingent subalgebra of B_m, we also have:

$$a_i,b_i \in B_m, \qquad b_i \ll_m a_i \quad (i = 1, 2, \ldots, s).$$

Hence, by lemma 4.1.1, $\{\omega_{B_m}(a_i)\}_{i=1}^s$ is a covering of $\mathfrak{M}_{B_m}$.

Using the definition of dimension, and lemma 4.1.4, we find that there exists a finite refinement

$$v = \{\omega_{B_m}(c_j)\}_{j=1}^t \text{ of } \{\omega_{B_m}(a_i)\}_{i=1}^s, \text{ where } c_1, c_2\ \ldots, c_j \in B_m,$$

and where the order of v is at most $k + 1$.

Again by lemma 4.1.1, there are elements $d_1, d_2, \ldots, d_t \in B_m$ such that

$$1 = \bigvee_{j=1}^{t} d_i \text{ and } d_j \ll_m c_j \quad (j = 1, 2, \ldots, t).$$

Then, the more, $c_j, d_j \in B_\infty$ and $d_j \ll_\infty c_j$ $(j = 1, 2, \ldots, t)$. Hence by the same lemma 4.1.1, $w = \{\omega_{B_\infty}(c_j)\}_{j=1}^{t}$ is a covering of $\mathfrak{M}_{B_\infty}$. It is obvious that w is a refinement of u. Since the order of w is equal to the order of v, it is at most $k + 1$; this proves that $\dim \mathfrak{M}_{B_\infty} \leq k$, q.e.d..

4.3.5. **Theorem.** Let T be a normal space of weight τ, and Φ a set of continuous mappings of T into T such that $|\Phi| \leq \tau$. Then there exists a compactification (α, C) of T such that:

(i) $\mathrm{w}(C) = \mathrm{w}(T)$,

(ii) $\dim C = \dim T$,

(iii) $\alpha \circ \varphi \circ \alpha^{-1}$ can be extended to a continuous mapping $\bar{\varphi}$ of C into C (for every $\varphi \in \Phi$),

(iv) if $\varphi \in \Phi$, and φ an autohomeomorphism of T, then $\bar{\varphi}$ is an autohomeomorphism of C.

Proof. Assume $\tau \geq \aleph_0$. We may also assume that Φ contains φ^{-1} whenever φ is an autohomeomorphism of T.

Let B_0 be a basis for the topology of T, as determined in the proof of theorem 4.1.5 for the case $\Sigma = \{T\}$. As shown in the proof of that theorem, this entails

$$\dim \mathfrak{M}_B \geq \dim T,$$

for any basic compingent subalgebra B of $\mathrm{B}(T)$ containing B_0.

We shall construct a monotonously non-shrinking sequence $(B_n)_{n=1}^{\infty}$ of basic compingent subalgebras of $\mathrm{B}(T)$ such that, in addition to other properties, $B_0 \subseteq B_1$, $|B_n| = \tau$, $\dim \mathfrak{M}_{B_n} = \dim T$ $(n = 1, 2, \ldots)$.

B_1 is defined such that the conditions mentioned are satisfied (B_1 exists by lemma 4.3.1). Now assume $n \geq 1$, and B_n already defined. Then B_{n+1} will be determined as follows.

According to theorem 2.2.3, every $\varphi \in \Phi$ induces a canonical homomorphism $\mathrm{g}(\varphi, B_n)$ of B_n into $\mathrm{B}(T)$. It is clear that $|S_n| = \tau$ if

$$S_n = B_n \cup \bigcup \{\mathrm{g}(\varphi, B_n)[B_n] \mid \varphi \in \Phi\}.$$

By lemma 4.3.1, we can choose B_{n+1} to be a compingent subalgebra of $\mathrm{B}(T)$ containing S_n.

We shall see that the unification B_∞ of $(B_n)_{n=1}^\infty$ (which in this case is merely the union of the sequence with compingent relation inherited from $\mathrm{B}(T)$), is such that $(\mu_{B_\infty}, \mathfrak{M}_{B_\infty})$ is a compactification of T as required.

It is clear that $|B_\infty| = \tau$, whence $\mathrm{w}(\mathfrak{M}_{B_\infty}) = \tau$. By theorem 4.3.4, and the condition imposed on B_0, we also have:

$$\dim \mathfrak{M}_{B_\infty} = \dim T.$$

Let $\varphi \in \Phi$. From the construction of B_∞, it is clear that φ induces a homomorphism $\mathrm{g}(\varphi) = \mathrm{g}(\varphi, B_\infty)$ of B_∞ into B_∞. Hence by theorem 2.2.3,

$$\mathrm{m}(\mathrm{g}(\varphi)) \circ \mu_{B_\infty} = \mu_{B_\infty} \circ \varphi,$$

and

$$\mathrm{m}(\mathrm{g}(\varphi)) \mid \mu_{B_\infty}[\mathrm{T}] = \mu_{B_\infty} \circ \varphi \circ \mu_{B_\infty}^{-1}.$$

This shows that $\mathrm{m}(\mathrm{g}(\varphi))$ is the required extension $\bar{\varphi}$ of φ. If, moreover, φ is an autohomeomorphism, then $\overline{\varphi^{-1}} \circ \bar{\varphi}$ is the identity mapping of $\mathfrak{M}_{B_\infty}$ since it induces the identity mapping of its dense subspace $\mu_{B_\infty}[T]$. Hence $\overline{\varphi^{-1}} = (\bar{\varphi})^{-1}$, and $\bar{\varphi}$ is an autohomeomorphism of $\mathfrak{M}_{B_\infty}$.

Remark. J. de Groot and R. H. McDowell [16], omitting condition (ii), except for the case dim T = 0, proved the theorem under the assumption: $\tau = \aleph_0$. Later R. Engelking [10] gave a proof of the theorem for the case $\tau = \aleph_0$, whilst R. H. McDowell (cf. [22]) and J. de Groot [14, 15] proved the theorem omitting condition (ii).

4.3.6. Remark. It is easily verified that the proofs of theorem 4.1.5 and theorem 4.3.5 can be combined in order to obtain a compactification possessing the properties mentioned in either theorem, if, of course, the hypotheses of both theorems are taken together.

SAMENVATTING

Er bestaat een volledige dualiteit tussen de theorie der boolealgebra's en de theorie van de nuldimensionale compacte hausdorffruimten (M. H. Stone [32]). In dit proefschrift wordt o.a. een soortgelijke algebraïzering van de theorie van willekeurige compacte ruimten uitgevoerd (alleen hausdorffruimten worden beschouwd). Het begrip, dat de hele theorie ten grondslag ligt, is dat van een zogenaamde compingente algebra. Een karakteristiek voorbeeld van zo'n compingente algebra wordt gevonden in de boolealgebra B(C) van alle regulair open verzamelingen van een compacte ruimte C, voorzien van de relatie „$\ll$" gedefinieerd door: voor $a,b \in$ B(C), $a \ll b \Leftrightarrow \bar{a} \subseteq b$. Een volledige dualiteit wordt verkregen door slechts dergelijke, d.z. volledige, compingente algebra's in aanmerking te nemen. De mogelijkheid van zo'n dualiteit werd gesuggereerd door J. de Groot. Compingente algebra's kunnen bijvoorbeeld ook gebruikt worden om de compactificaties van volledig reguliere ruimten te beschrijven.

Ofschoon de theorie der compingente algebra's ook opgevat kan worden als een puntloze topologie (vgl. K. Menger [23]), wordt dit aspect hier niet verder bekeken.

In het eerste hoofdstuk worden de compingente algebra's als zodanig bestudeerd, terwijl ook het verband met de bijbehorende compacte ruimten opgehelderd wordt.

In het tweede hoofdstuk wordt de compactificatietheorie van volledig reguliere ruimten behandeld, waarbij op de gelijkenis met bestaande compactificatiemethoden gewezen wordt. Ook wordt de verwantschap met de theorie der nabijheidsruimten, vooral als ontwikkeld door J. M. Smirnow [30], aangegeven.

De laatste twee hoofdstukken handelen over de toepassingen van de voordien ontwikkelde theorie. Eerst worden o.a. stellingen van C. Kuratowski en H. Freudenthal over quasicomponentenruimten en eindpuntcompactificaties enigszins veralgemeend, waarna een tweetal stellingen bewezen wordt over het vraagstuk van de karakterizering van de complementen van n-dimensionale ruimten in compacte ruimten, dat afkomstig is van J. de Groot [13]. De eerste twee paragrafen van hoofdstuk 4 geven veralgemeningen van stellingen van E. G. Sklyarenko en C. Kuratowski over gewicht- en dimensiebewarende compactificaties. Ten slotte worden stellingen bewezen over het bestaan van compactificaties die niet alleen het gewicht en de dimensie bewaren, maar ook de voortzetting van gegeven continue afbeeldingen toestaan. Hierbij wordt aangesloten bij resultaten die verkregen zijn door J. de Groot, R. H. McDowell en R. Engelking.

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STELLINGEN

I

Er zijn torsievrije abelse groepen van rang 2, waarvan de automorfismengroep een cyclische groep van de orde 4 is.

Lit. H. de Vries and A. B. de Miranda, Math. Zeitschr. **68** (1958), 450-464.

II

Voor iedere eindige groep G is er een eindige primaire groep P waarvan de nilpotentieklasse 2 is, terwijl

$$\mathrm{A}(P) \,/\, Z \cong G,$$

als $\mathrm{A}(P)$ de automorfismengroep van P is en Z bestaat uit de centrale automorfismen van P die de centrumelementen invariant laten.

III

Zij G een eindige groep, die voor elke priemdeler p van zijn orde precies $p + 1$ p-sylowondergroepen heeft. Dan geldt:

$p \mid | G | \Rightarrow p = 2$ of p is een priemgetal van Mersenne.

IV

Zij G een eindige groep, die voor elke priemdeler p van zijn orde precies $p + 1$ p-sylowondergroepen heeft.

Zij verder $\mathfrak{S}_n$, opvolgend $\mathfrak{A}_n$, de symmetrische, opvolgend alternerende, groep van n objecten, en L_p een splijtuitbreiding van een elementair abelse groep van orde $p + 1$ als normaaldeler, met behulp van een automorfisme van orde p, waarbij p een priemgetal van Mersenne is. Stel

$$A = \times \{L_p \mid p \mid |G|, p > 3\}.$$

Dan geldt: G heeft een zodanige nilpotente normaaldeler N, dat:

$$G/N \cong \mathfrak{S}_4 \times A \text{ of } G/N \cong \mathfrak{A}_4 \times \mathfrak{S}_3 \times A.$$

V

Er zijn meta-abelse groepen met exponent 6 en continue machtigheid, waarvan alle sylowondergroepen aftelbaar zijn.

Lit. L. G. Kovács, B. H. Neumann F.R.S. and H. de Vries, Proc. Roy. Soc. A, **260** (1961), 304-316.

VI

Zij G een abelse topologische groep, die elementen van oneindige orde bevat. Dan is de topologische-automorfismengroep van G niet isomorf met een diëdergroep waarvan de orde ten minste 6 is.

VII

Iedere oneindige, lokaal compacte, periodieke, topologische groep heeft oneindig veel topologische automorfismen.

VIII

Iedere compact voortgebrachte topologische groep, waarvan de conjugatieklassen eindig zijn en waarin de verzameling der periodieke elementen dicht ligt, is compact.

IX

Als V een verzameling van continue machtigheid is en f een eenduidige afbeelding van V op zichzelf, dan bestaat er, onder aanname van de continuümhypothese, een topologie voor V, waardoor V tot een compactum wordt en f tot een autohomeomorfisme van dat compactum.

Lit. H. de Vries, Bull. Acad. Polon. Sci. Cl. III, **5** (1957), 943-945.

X

Als H een hausdorffruimte is, dan geldt, de notatie van het proefschrift gebruikende:

$$\mathrm{Q}(H) = \mathrm{K}(H) \Leftrightarrow \mathrm{Q}(H) \text{ perifeer compact.}$$

XI

Als S een semiconvexe deelverzameling is van de n-dimensionale projectieve ruimte en S geen lijn omvat, dan omvat het complement van S een $(n - 1)$-dimensionale deelruimte $(n \geq 1)$.

Lit. J. de Groot and H. de Vries,
Comp. Math. **13** (1957), 113-118.

XII

In het bewijs, dat C. Lech geeft van een uitbreiding van stellingen van K. Mahler en Th. Skolem over het op de duur periodiek voorkomen van de nulwaarden van de taylorcoëfficiënten van een rationale functie, wordt een niet geheel gerechtvaardigd beroep gedaan op resultaten van K. Hensel.

Lit. C. Lech, Arkiv för Math. **2** (1954), 417-421.
K. Hensel, Theorie der algebraischen Zahlen I, Leipzig und Berlin, 1908.

XIII

Uit velerlei overwegingen blijkt het gewenst en mogelijk, dat een gelijkvormig schrift voor de talen der wereld ingevoerd wordt, en dat tevens spellingen gebruikt worden die bij voldoende benadering fonetisch zijn.

XIV

Bij de inrichting van het onderwijs zoals deze in de zogenaamde „mammoetwet" wordt voorgesteld, is het ongewenst de cursus h.b.s.-b zesjarig te maken, tenzij hieraan een wezenlijke verdieping van het programma gepaard gaat.

Stellingen behorende bij H. de Vries
Compact spaces and compactifications.
Amsterdam 1962

C V C
GOEDE VOL COURAGIE