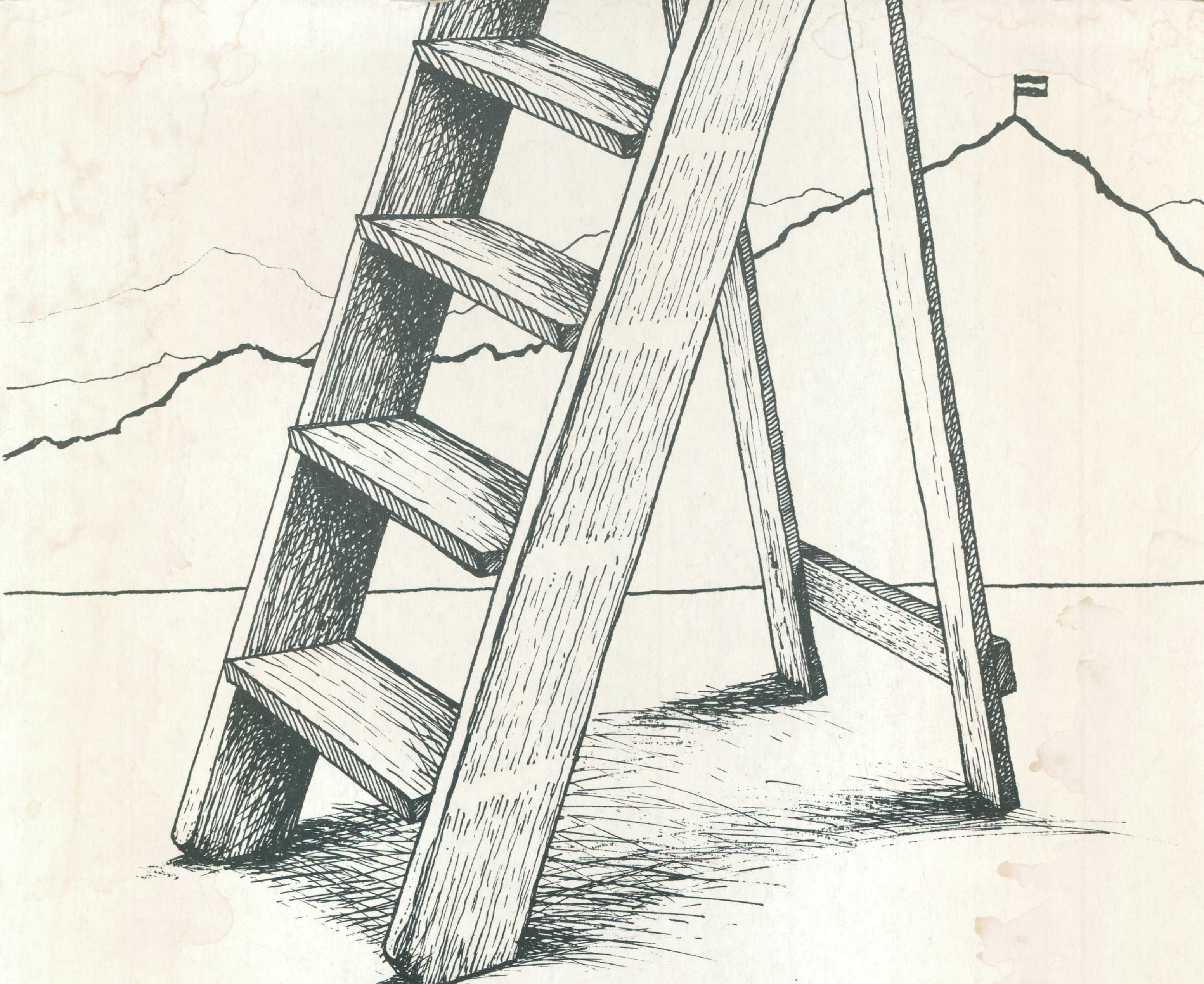

INVESTIGATIONS IN INTUITIONISTIC HIERARCHY THEORY

Wim Veldman

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Promotor: Prof. J.J. de Iongh.

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PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN
DOCTOR IN DE WISKUNDE EN NATUURWETENSCHAPPEN
AAN DE KATHOLIEKE UNIVERSITEIT
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DE RECTOR MAGNIFICUS PROF. DR. P.G.A.B. WIJDEVELD
VOLGENS BESLUIT VAN HET COLLEGE VAN DECANEN
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OP WOENSDAG 20 MEI 1981
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DOOR

WILLEM HENRI MARIA VELDMAN
GEBOREN TE MAASTRICHT

1981
KRIPS REPRO MEPPEL

in een ernstig bestaan
zou hij te gronde gaan
hield niet het zwevende
plezier hem levende.

hans lodeizen.

Hwat nimmen sizze kin yn frij bineamen,
Ik hie in moed en soe it foar jimm' rime?
Unnoazel bern dat seit: ûneindichheit,
As it de stap hat op de fjirde trime!

Obe Postma

De omslag is een ontwerp van mijn kunstbroeder Kees Streefkerk.

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0. INTRODUCTION

This thesis is concerned with constructive reasoning in descriptive set theory.

The venerable subject of descriptive set theory was developed in the early decades of this century, mainly by French and Russian mathematicians.
It started from the following observation:
once the class of continuous real functions has been established, one naturally comes to think of the class of real functions which are limits of everywhere convergent sequences of continuous functions.
This wider class can be extended in its turn, by the same operation of forming limits of everywhere convergent sequences.
This goes on and on, even into the transfinite.
Thus a splendid structure arises, called: Baire's hierarchy.

The same story may be told in terms of sets.
Looking at the subsets of Baire space ${}^{\omega}\omega$ which are forced into existence when we allow for the clopen (= closed-and-open) neighbourhoods and then apply the operations of countable union and intersection again and again, we may wonder once more, because there is no end of it.
One after another, the classes of Borel's hierarchy present themselves, each containing subsets of ${}^{\omega}\omega$ not heard of before.

No Borel class exhausts the possible subsets of ${}^{\omega}\omega$
This can be proved in a few lines: one shows that each class contains a universal element and diagonalizes. (cf. chapter 6, esp. 6.14)

However, the very ease of the proof arouses suspicion.
People like Borel, Baire, Lebesgue, who were the first to raise and answer many questions in this subject, spent much thought on the plausibility of their arguments.
Diagonalizing was felt as cheap reasoning, especially by Baire.
Avoiding the diagonal argument, only relying on methods „from practice", one succeeded in showing up members of the first three or four classes of Baire.

Diagonalizing, of course, was not the worst of all evils. In Lusin's catalogue, to be found on page 55 of Lusin 1930, it comes immediately after „normal constructive argument", before such horrible things as: the use of $\aleph_1$ as a well-defined, completed mathematical set, or, even worse, the essentially incomprehensible argument by which Zermelo established a well-ordering of any set, from the axiom of choice.

Now, for heaven's sake, what might be wrong with the diagonal argument?
From a classical point of view, one cannot bring up much against it.
In fact, as soon as we agree upon the meaning of negation (P and $\neg$P cannot hold together, whatever be the proposition P) we have to accept it.
But in intuitionism we may find an explanation for our uneasiness.

Let us remark that, classically, we may build up the Borel sets in $^\omega\omega$ from the closed-and-open neighbourhoods, using only countable union and intersection. Complementation can be missed as an operation for making new sets out of already existing ones: as the complement of any closed-and-open set is closed-and-open, the complement of any set built from the closed-and-open sets by countable union and intersection, is such a set again.
This certainty is given by such wonderful guardians of classical symmetry as are de Morgan's laws.

De Morgan's laws are not acceptable, intuitionistically, apart from some very simple situations, from which they were derived by a crude generalization.
We cannot explain away complementation, or, more generally, the analogue of logical implication, as methods of constructing sets.
But we might try to do without them.

We will do so in this treatise.
When negation and implication are put aside, the possibility of diagonalizing is taken from our hands, and the hierarchy problem is open again.
A solution is given in chapters 6-9.

There is good reason to consider negation and implication with some caution.
Many unsettled questions in intuitionistic logic are connected with them.
(Compare the discussion in the appendix, chapter 17 We are not able to decide how far the divergence between classical and intuitionistic logic goes.
Also, a curious role is played by negation in the recent discussion of the intuitionistic completeness of intuitionistic predicate logic, cf. de Swart 1976, Veldman 1976).

The intuitionistic hierarchy has a very delicate structure.
The class of the closed subsets of Baire space, for instance, is no longer closed under the operation of finite union. One has to distinguish between closed sets, binary unions of closed sets, ternary unions of closed sets, and so on.
This phenomenon is discussed in chapter 4.
The productive force of disjunction and conjunction is explored further in chapter 11.20-26 and chapter 12.0-7.
Implication, although absent from chapters 6-9, is not completely forgotten, and, we will see, in chapter 5 and chapter 12.8-9 that it shares in some of the properties established for disjunction and conjunction.

Distrust of diagonalization is one of many points on which early descriptive set theorists and intuitionists have similar views.
Their common basic concern might be described as: exploring the constructive continuum.
Brouwer's rejection of classical logic is, of course, a major point of difference.

But one is tempted to ask if not the main theorem of this essay, which establishes the intuitionistic hierarchy (chapter 9, theorems 9.7 and 9.9) might have delivered Baire from his scruples.

Since Addison 1955 it has become customary among logicians to consider descriptive set theory for its connection with recursion theory.
We will bypass this development.
From an intuitionistic point of view, recursion theory is an ambiguous branch on the tree of constructive mathematics.
The deep results of this theory depend on very serious applications of classical logic.
And the classical continuum, which is a rather obscure thing, is accepted without any comment, as a suitable domain of definition for effective operations.

Nevertheless, there is an analogy between recursion theory and the theory to be developed here:
Many paradoxical results of elementary recursion theory are due to the fact that functions and functionals are finite objects, and, therefore, of the same type as natural numbers.
Now, functions from Baire space ${}^\omega\omega$ to ${}^\omega\omega$, being necessarily continuous, are determined by a sequence of neighbourhood functions, and thus may be seen to be themselves members of ${}^\omega\omega$.
Once more, we are in a situation where functions do not differ in type from their arguments and values.

We also have to admit that, if there is any elegance in these pages, it partly is due to modern recursion theory.
For instance, the following concept of many-one reducibility between subsets of ${}^\omega\omega$ is starring

$$A \preceq B := \exists f [\, f \text{ is a continuous function from } {}^\omega\omega \text{ to } {}^\omega\omega \text{ and } \forall\alpha[\alpha \in A \rightleftarrows f(\alpha) \in B]$$

This so-called „Wadge-reducibility" was made the subject of classical study by some students of Addison's (cf. Kechris and Moschovakis 1978, Wadge 198?).
Their methods, however, are very far from constructive.
We introduce this concept in chapter 2, after a short exposition of the principles of intuitionistic analysis.

In the second part of this thesis (chapters 10-14) we turn to analytical sets, and the projective hierarchy. (cf. Note 3 on page 216).
Analytical sets, being close relatives of good old „spreads", get a chapter of their own. It will be seen that the classical duality between analytical and co-analytical sets is severely damaged. (chapter 10).
Some famous results of Souslin's are partly rescued by Brouwer's bar theorem, which we will present here under the name of Brouwer's thesis.

(This expression means to suggest an analogy to Church's thesis in recursive function theory, that all calculable functions from ω to ω are general recursive) (chapter 13).
If we persist in excluding negation and implication, the projective hierarchy does not exceed its second level. (chapter 14).
This is a consequence of the axiom AC_{11} which has been introduced and advocated in chapter 1.

In chapter 11 we study the typically intuitionistic subject of "quantifying over small spreads."
Rather surprisingly, quantifying over the very simple spread σ_{2mon} already leads to sets which are not hyperarithmetical.
Like some sets in chapter 4, these sets turn into more complex ones when they are given a treatment by means of disjunction, conjunction or implication.
In chapter 12 we find many other sets which have similar properties.

The proper place of the last three chapters (15-17) is the margin.

In chapter 15 we ask ourselves what is the domain of validity of the principle of reasoning which we get from the axiom AC_{01}, introduced in chapter 1, by "constructive contraposition".
This principle is vital to many a classical discourse.
It may be seen as a simple case of the axiom of determinacy.
Chapter 16 pursues this line of thought a little further.

In chapter 17 we mention an annoying problem which we could not solve, and some quasi-solutions.

The <u>synopsis</u> is an analytical table of contents.

On the scene of contemporary mathematical logic a family reunion is being held, at which the different branches of the discipline cooperate in seeking for a new understanding of the beautiful problems which occupied our grandfathers.
Recent books like <u>Hinman</u> 1978 and <u>Moschovakis</u> 1980 report about it
Up to now, intuitionism has been absent.
Here it comes, at last, ignoring the question whether it has been missed, or was invited, and raises its voice, somewhat timidly, in the company of so much learning.

1 A SHORT APOLOGY FOR INTUITIONISTIC ANALYSIS

In this chapter, we want to give a sketch of the conception of intuitionistic analysis that guides our thought
As may be expected, our <u>logic</u> will be intuitionistic; indirect arguments are put into their proper place, and are seen to prove less than direct ones; we clearly distinguish $\neg\neg(P \vee Q)$ from $P \vee Q$, and $\neg\neg\exists n[A(n)]$ from $\exists n[A(n)]$
The main objects of our considerations will be: natural numbers and infinite sequences of natural numbers.
Let us take a closer look at them.

1.0 ω is the set of natural numbers, ${}^{\omega}\omega$ is the set of all infinite sequences of natural numbers.
We imagine such sequences to be built up step by step in course of time, there is no necessity for their being completely described at some finite moment
One may restrict the future development of an individual sequence more or less severely, from excluding some possible continuations, up to destroying all freedom – such that the sequence follows a uniquely determined course
This idea, roughly the one Brouwer had in mind, is our point of departure.

In recent expositions of intuitionism, like <u>Troelstra</u> 1977, one sometimes prefers another basic concept: that of sequences growing in complete, never to be restricted freedom.
These objects are supposed to satisfy a very odd set of axioms.
We do not like them.

(Intuitionism is trying to give a precise and reasonable account of the continuum, as it is known by the mathematician.
<u>Lawless sequences</u> are strange things which do not occur in daily life.
Although it is possible to construct something like the continuum from them, one somehow does not like to be told that this is how real numbers really are)

We cling to the older tradition.
We introduce a quartet of <u>axioms of choice and continuity</u> and plead for them.

1.1 AC_{00} Let $A \subseteq {}^{\omega}\omega$

If $\forall n \exists m[A(n,m)]$, then $\exists \alpha \forall n[A(n,\alpha(n))]$

(We use $m, n \ldots$ for members of ω, and $\alpha, \beta, \ldots$ for members of ${}^{\omega}\omega$).

We defend AC_{00} as follows:

Suppose: $\forall n \exists m[A(n,m)]$, we then determine, one after another, first, a natural number n_0 such that $A(0,n_0)$, then a natural number n_1 such that $A(1,n_1)$,... and so on.
This is nothing but creating step-by-step $\alpha \in {}^{\omega}\omega$ such that $\forall n[A(n,\alpha(n))]$. $\boxtimes$

We emphasize that AC_{00} does <u>not</u> say the following:
If $\forall n \exists m [A(n,m)]$, then we can give a <u>finite</u> <u>description</u> of an $\alpha \in {}^{\omega}\omega$ such that $\forall n [A(n, \alpha(n))]$
Sometimes, (cf. <u>Troelstra</u> 1977), it is given this kind of interpretation by intuitionistic mathematicians.
The set A is then subject to the condition that it, too, should admit of a finite description.

1.2 In order to state the next axiom, we need a pairing function on ω.
In view of later developments, we do not go the shortest way.

Let $< > : \bigcup_{k \in \omega} {}^{k}\omega \rightarrowtail\!\!\!\twoheadrightarrow \omega$ be a fixed one-to-one mapping of the set of all finite sequences of natural numbers onto the set of natural numbers.
$< >$ is a <u>coding</u> of the finite sequences.

Every natural number now stands for a finite sequence of natural numbers.
$*: {}^{2}\omega \to \omega$ is the binary function on ω which corresponds to concatenation, i.e. for all $m, n \in \omega$:

$m * n$:= the code number of the finite sequence that one gets by concatenating the finite sequence coded by m and the finite sequence coded by n

We define, for all $m, n \in \omega$

$m \subseteq n$:= the finite sequence coded by n is an initial part of the finite sequence coded by m, i.e.: $\exists p [m = n * p]$

We suppose that our coding fulfils the following condition:

$$\forall m \forall n [m \subseteq n \to n \leq m].$$

Therefore, the empty sequence is coded by the number 0.

For all $\alpha \in {}^{\omega}\omega$ and $n \in \omega$ we define ${}^{n}\alpha$ and α^{n} in ${}^{\omega}\omega$ by:

for all $n \in \omega$: ${}^{n}\alpha(m) := \alpha(n * m)$

for all $n \in \omega$: $\alpha^{n}(m) := \alpha(<n> * m)$

1.3 AC_{01} Let $A \subseteq \omega \times {}^{\omega}\omega$
If $\forall n \exists \alpha [A(n, \alpha)]$, then $\exists \alpha \forall n [A(n, \alpha^{n})]$

We defend AC_{01} as follows:

Suppose: $\forall n \exists \alpha [A(n, \alpha)]$
We first start the creation of an infinite sequence α_0 such that $A(0, \alpha_0)$
This job will ask for our active attention infinitely many times.
This does not prevent our starting a second infinite project in the

meantime: the creation of an infinite sequence α_1 such that $A(1,\alpha_1)$
From time to time we will have to look after the progress of work on α_0, from time to time we will have to look after the progress of work on α_1, but, still, this does not occupy all our mental powers: we can put more kettles on the furnace.
Our program for constructing a sequence α such that $\forall n[A(n,\alpha^n)]$ is as follows:

* Start a project P_0 for creating an infinite sequence α_0 such that $A(0,\alpha_0)$. Continue work on P_0 for one step and define: $\alpha^0(0) := \alpha_0(0)$
* Start a project P_1 for creating an infinite sequence α_1 such that $A(1,\alpha_1)$. Continue work on P_0 for one step and define: $\alpha^0(1) := \alpha_0(1)$ Continue work on P_1 for one step and define: $\alpha^1(0) := \alpha_1(0)$
* Start a project P_2 for creating an infinite sequence α_2 such that $A(2,\alpha_2)$. Continue...

Apparently, we believe in our ability to keep several infinite projects going at the same time. A good memory is useful in these circumstances.
$\boxtimes$

Like AC_{00}, AC_{01} here has a meaning different from the one it has in Troelstra 1977

1.4 The next two axioms usually go under the flag of „principles of continuity" Their introduction requires some more technical conventions.

We define a function $lg: \omega \to \omega$ by:

for all $m \in \omega$: $lg(m) :=$ the length of the finite sequence coded by m.

For all $\alpha \in {}^\omega\omega$ and $n \in \omega$, we define:

$$\bar{\alpha}n := \langle \alpha(0), \ldots, \alpha(n-1) \rangle$$

Remark that, for all $\alpha \in {}^\omega\omega$: $\bar{\alpha}0 = \langle\,\rangle = 0$

We also write, for all $\alpha \in {}^\omega\omega$ and $m \in \omega$:

$$\alpha \in m := \exists n[\bar{\alpha}n = m]$$

(i.e.: the infinite sequence α passes through the finite sequence coded by m)

For all $\gamma \in {}^\omega\omega$, $\alpha \in {}^\omega\omega$, $n \in \omega$, we define:

$$\gamma : \alpha \mapsto n := \exists m[\forall p[p<m \to \gamma(\bar{\alpha}p) = 0] \wedge \gamma(\bar{\alpha}m) = n+1]$$

For all $\gamma \in {}^\omega\omega$, we define:

$$\gamma : {}^\omega\omega \to \omega \quad (\text{or: } fun(\gamma)) := \forall\alpha\exists n[\gamma : \alpha \mapsto n]$$

Let $\gamma \in {}^\omega\omega$ be such that $fun(\gamma)$, and $\alpha \in {}^\omega\omega$. We then write:

$$\gamma(\alpha) := \text{the unique } n \in \omega \text{ such that } \gamma : \alpha \mapsto n$$

1.5 AC_{10} Let $A \subseteq {}^{\omega}\omega \times \omega$

If $\forall\alpha \exists n[A(\alpha,n)]$, then $\exists\gamma[\,fun(\gamma) \wedge \forall\alpha[A(\alpha,\gamma(\alpha))]]$

We defend AC_{10} as follows:

Suppose: $\forall\alpha \exists n[A(\alpha,n)]$
We have to make a sequence γ in ${}^{\omega}\omega$ which fulfils certain conditions, and, as one may guess, we will do so step by step, fixing only one value of γ at a time.
Suppose this work to have proceeded until stage n, i.e.: $\gamma(0), \gamma(1), \ldots$ up to $\gamma(n-1)$ have been determined already.
We now consider the finite sequence of natural numbers which is coded by n, let us say: $n = \langle n_0, n_1, \ldots, n_k \rangle$
This finite sequence may be thought of as being the initial part of an infinite sequence α, which is disclosed to us step by step
While listening to the successively created values of α we are expected to find a natural number p such that $A(\alpha,p)$
We can not wait indefinitely and have to act at some time.
When p eventually is determined, therefore, only a finite part of α will be known to us.
Some finite initial part of α should contain sufficient information, so to say, for p to be calculated.
Looking at n, we may ask: is this finite sequence long enough as an initial part of α so as to enable us to find a natural number p such that $A(\alpha,p)$?
If so, we determine: $\gamma(n) := p+1$, where p is such a number
if not, we put: $\gamma(n) := 0$
In this way the construction of γ is being continued.

Now one may have doubts whether $\forall\alpha \exists n[\gamma(\bar{\alpha}n) \neq 0]$
After all, during the construction of γ only such sequences α are considered, as are growing step by step in freedom, not being subject to any restriction given beforehand, or coming to mind on the way.

This objection may be answered as follows:
Any sequence from ${}^{\omega}\omega$, even a completely determinate one, can be imagined to be the outcome of a step-by-step-creation.
(We do not want to distinguish between sequences α, β which fulfil $\forall n[\alpha(n) = \beta(n)]$, although one may have had different things in mind when making them.
Any sequence is extensionally equal to some sequence growing in complete freedom.
Some modern opinion (cf. Troelstra 1977) holds that this is impossible, as "being equal to some determinate sequence" would conflict with "being created in freedom".
Vexing questions on freedom may be asked now, but they are left to the reader, or any philosopher, to muse upon.)
$\boxtimes$

1.6 We now prepare the way for the last of our four axioms, which is the most debated one.

For all $\gamma \in {}^{\omega}\omega$, $\alpha \in {}^{\omega}\omega$, $\beta \in {}^{\omega}\omega$, we define:

$$\gamma : \alpha \mapsto \beta \quad := \quad \forall n [\, \gamma^n : \alpha \mapsto \beta(n)]$$

For all $\gamma \in {}^{\omega}\omega$, we define:

$$\gamma : {}^{\omega}\omega \to {}^{\omega}\omega \quad (\text{or: } Fun(\gamma)) \quad := \quad \forall n [\, fun(\gamma^n)]$$ (cf. Note 2 on page 216)

Let $\gamma \in {}^{\omega}\omega$ be such that $Fun(\gamma)$, and $\alpha \in {}^{\omega}\omega$. We then write:

$$\gamma | \alpha \quad := \quad \text{the unique } \beta \in {}^{\omega}\omega \text{ such that } \gamma : \alpha \mapsto \beta$$

1.7 AC_{11} Let $A \subseteq {}^{\omega}\omega \times {}^{\omega}\omega$

If $\forall \alpha \exists \beta [A(\alpha, \beta)]$, then $\exists \gamma [Fun(\gamma) \wedge \forall \alpha [A(\alpha, \gamma|\alpha)]]$

AC_{11} will be defended by a rather involved argument, which has features in common with both the argument for AC_{01} and the argument for AC_{10}.

Suppose: $\forall \alpha \exists \beta [\, A(\alpha, \beta)]$
We have to make a sequence γ in ${}^{\omega}\omega$ which satisfies a certain condition.
In fact, this condition on γ is stated in terms of its subsequences $\gamma^0, \gamma^1, \ldots$
We will build up all subsequences $\gamma^0, \gamma^1, \ldots$ step by step, but simultaneously, i.e.: at stage n, all values $\gamma^0(n), \gamma^1(n), \ldots$ will be determined
To be sure, only $\gamma^0, \gamma^1, \ldots$ up to γ^n, properly get into focus at stage n, that is to say: $\forall m \forall n [\, m > n \to \gamma^m(n) = 0]$

Now suppose our work to have progressed so far, that all sequences $\gamma^0, \gamma^1, \ldots$ have their values fixed in all points $0, 1, \ldots$ up to $n-1$.
What about their values in n?
Let us look at the finite sequence of natural numbers coded by n, say: $n = \langle n_0, \ldots, n_k \rangle$
We consider this sequence together with its predecessors: $\langle\,\rangle$, $\langle n_0 \rangle$, $\langle n_0, n_1 \rangle, \ldots \langle n_0, n_1, \ldots, n_{k-1} \rangle$. The values of $\gamma^0, \gamma^1, \ldots$ at these predecessors have been fixed already.
We calculate the smallest number p such that: $\forall m [(n \subseteq m \wedge n \neq m) \to \gamma^p(m) = 0]$
As $\forall m [(n \subseteq m \wedge n \neq m) \to \gamma^n(m) = 0]$, this number may be found.

We now imagine $n = \langle n_0, \ldots, n_k \rangle$ to be the initial part of an infinite sequence α, whose values are given to us one by one, successively.
We should be able to calculate β in ${}^{\omega}\omega$ such that $A(\alpha, \beta)$
We started already a project for creating such a sequence β, as appears from the part of γ which has been completed by now.
The finite sequence n turned out to contain sufficient information for deciding about $\beta(0), \beta(1), \ldots$ up to $\beta(p-1)$

We now continue this same project for creating a suitable partner β to the growing sequence α and ask ourselves: does $n = \langle n_0, n_1, \ldots n_k \rangle$ contain sufficient information for deciding about $\beta(p)$?
If so, we determine a number q which may serve as p-th value of β and say: $\gamma^p(n) := q+1$
If not, we put: $\gamma^p(n) := 0$
All other subsequences of γ are left alone now, so: $\forall \ell [\ell \neq p \to \gamma^\ell(n) = 0]$

In this way the construction of γ is being continued.

Now suppose $\alpha \in {}^\omega\omega$, α being given step by step.
By reflecting upon the construction of γ, one realizes successively:

$$\exists m[\gamma^0(\bar{\alpha}m) \neq 0] \wedge \exists m[\gamma^1(\bar{\alpha}m) \neq 0] \wedge \exists m[\gamma^2(\bar{\alpha}m) \neq 0] \wedge \ldots$$

Hence: $\forall \alpha \forall n \exists m[\gamma^n(\bar{\alpha}m) \neq 0]$, as any sequence α can be thought of as being given step by step, and we see: $\mathrm{Fun}(\gamma)$

In the same way one persuades oneself about: $\forall \alpha [A(\alpha, \gamma|\alpha)]$

$\boxtimes$

1.8 Sometimes, in expositions of intuitionistic analysis, the insight which sustains AC_{10}, is given a less bold formulation, in the following <u>continuity principle</u>:

CP Let $A \subseteq {}^\omega\omega \times \omega$

If $\forall \alpha \exists n [A(\alpha, n)]$, then $\forall \alpha \exists m \exists n \forall \beta [\bar{\beta}m = \bar{\alpha}m \to A(\beta, n)]$

Formally, CP is weaker than AC_{10} (cf. <u>Howard</u> and <u>Kreisel</u> 1966)
As CP easily follows from AC_{10}, we need not defend CP, after all that has been said in favour of AC_{10}

1.9 Let $\alpha \in {}^\omega\omega$ and $\beta \in {}^\omega\omega$. We define:

$$\alpha \in \beta \quad := \quad \forall n [\beta(\bar{\alpha}n) = 0]$$

Let $\beta \in {}^\omega\omega$. β is called a <u>subspread</u> <u>of</u> <u>${}^\omega\omega$</u> if it fulfils the following conditions:

(i) $\beta(\langle\,\rangle) = 0$
(ii) $\forall m [\beta(m) = 0 \rightleftarrows \exists n [\beta(m * \langle n \rangle) = 0]]$

If β is a subspread of ${}^\omega\omega$, we are interested in the set $\{\alpha \mid \alpha \in {}^\omega\omega \mid \alpha \in \beta\}$ which we, at the risk of some confusion, also denote by β, and call a spread.

If β is a subspread of ${}^\omega\omega$, the corresponding subset of ${}^\omega\omega$ may be treated like ${}^\omega\omega$ itself.
It makes sense, therefore, to introduce the following „relativized" concepts:

Let $\beta \in {}^\omega\omega$ be a subspread of ${}^\omega\omega$ and $\gamma \in {}^\omega\omega$

We write: $\gamma : \beta \to \omega$ or: $fun_\beta(\gamma)$ if $\forall\alpha[\alpha\in\beta \to \exists n[\gamma(\bar{\alpha}n) \neq 0]]$
(If $fun_\beta(\gamma)$ and $\alpha\in\beta$, we define:

$\gamma(\alpha)$:= the unique $n\in\omega$ such that $\gamma : \alpha \mapsto n$)

We write: $\gamma : \beta \to {}^\omega\omega$ or: $Fun_\beta(\gamma)$ if $\forall n[\, fun_\beta(\gamma^n)]$
(If $Fun_\beta(\gamma)$ and $\alpha\in\beta$, we define:

$\gamma|\alpha$:= the unique $\beta\in{}^\omega\omega$ such that $\gamma : \alpha \mapsto \beta$)

We are able, now, to enunciate some of our principles of choice and continuity in a more general setting:

GAC_{10} Let $A \subseteq {}^\omega\omega \times \omega$ and $\beta\in{}^\omega\omega$ be a subspread of ${}^\omega\omega$
If $\forall\alpha\exists n[A(\alpha,n)]$, then $\exists\gamma[\, fun_\beta(\gamma) \wedge \forall\alpha\in\beta[A(\alpha,\gamma(\alpha))]]$

GAC_{11} Let $A \subseteq {}^\omega\omega \times {}^\omega\omega$ and $\beta\in{}^\omega\omega$ be a subspread of ${}^\omega\omega$
If $\forall\alpha\exists\delta[A(\alpha,\delta)]$, then $\exists\gamma[\, Fun_\beta(\gamma) \wedge \forall\alpha\in\beta[A(\alpha,\gamma|\alpha)]]$

GCP Let $A \subseteq {}^\omega\omega \times \omega$ and $\beta\in{}^\omega\omega$ be a subspread of ${}^\omega\omega$
If $\forall\alpha\in\beta\,\exists n[A(\alpha,n)]$, then $\forall\alpha\in\beta\,\exists m\,\exists n\,\forall\delta[\,\bar{\delta}m = \bar{\alpha}m \to A(\delta,n)]$

We may argue for these generalized principles in exactly the same way as we did for the ungeneralized ones.
Or, if we prefer so, we may formally derive GAC_{10} from AC_{10}, GAC_{11} from AC_{11} and GCP from CP.
We do not go into details.

1.10 The above presentation of the basic assumptions of intuitionistic analysis owes much, if not all, to many discussions in Nijmegen in which J. J. de Iongh and W. Gielen took the lead (cf. Gielen, de Swart and Veldman 1981, and Gielen 198?)
(This is not to make them responsible for any lack of clarity)
The outcome of our considerations does not differ on any essential point from the axiom system in Kleene and Vesley 1965, commonly known as FIM AC_{11}, for instance, corresponds to *27.2 in Kleene and Vesley 1965
The names we have given to the axioms are new, and differ from the names used in Troelstra 1973, Troelstra 1977
We introduced them in Gielen, de Swart and Veldman 1981

2 AT THE BOTTOM OF THE HIERARCHY. A DISCUSSION OF BROUWER-KRIPKE'S AXIOM

For some time past, it is known, that AC_{11} is inconsistent with a generalized form of Brouwer - Kripke's axiom.
We repeat the simple argument which shows this because the hierarchy theorems that will appear in the following chapters may be viewed as attempts to extend and generalize this fact.
We include a short discussion of the axiom itself.

2.0 <u>Theorem</u>: $\neg\forall\alpha\exists\beta[\ \forall n[\alpha(n)=0] \rightleftarrows \exists n[\beta(n)=0]]$

<u>Proof</u>: Suppose: $\forall\alpha\exists\beta[\forall n[\alpha(n)=0] \rightleftarrows \exists n[\beta(n)=0]]$

Using AC_{11}, determine $\delta\in{}^{\omega}\omega$ such that $\delta: {}^{\omega}\omega \to {}^{\omega}\omega$ and:

$\forall\alpha[\ \forall n[\alpha(n)=0] \rightleftarrows \exists n[(\delta|\alpha)(n)=0]]$

Consider the special element $\underline{0}$ of ${}^{\omega}\omega$ which is defined by: $\forall n[\underline{0}(n)=0]$

We know: $\exists n[(\delta|\underline{0})(n)=0]$ and we determine $m\in\omega$, $n\in\omega$ such that:

$\delta^n(\bar{\underline{0}}m)=1$ and $\forall p[p<m \to \delta^n(\bar{\underline{0}}p)=0]$

Then: $\forall\alpha[\bar{\alpha}m=\bar{\underline{0}}m \to (\delta|\alpha)(n)=0]$

Therefore: $\forall\alpha[\bar{\alpha}m=\bar{\underline{0}}m \to \forall n[\alpha(n)=0]]$

This, of course, is not true.

$\boxtimes$

2.1 BK (Brouwer- Kripke's axiom) Let $\mathfrak{A}$ be a mathematical proposition
Then: $\exists\alpha[\mathfrak{A} \rightleftarrows \exists n[\alpha(n)=0]]$

In order to see the truth of this principle, I have to remember that, essentially, I am alone in this world, doing mathematics.
A theorem is proved only if I myself succeed in making the construction in which its truth consists
(External circumstances (meeting Brouwer, drinking coffee) may have influenced me substantially, but they have no place in a picture of the essence of mathematical truth)

A sequence α from ${}^{\omega}\omega$ may be built up step by step in the course of time, and this may be done without any haste, although, having determined $\alpha(n)$, I have to come with the next value of α, I am not to delay this indefinitely.
But why should not I use the whole of my mathematical future for the construction of α?
Then $\mathfrak{A}$, if true, should be experienced as such during the construction of α.
While numbering the stages of my mathematical life $0, 1, 2, \ldots$ successively, I define $\alpha(n)$ to be 0 if I succeeded in proving $\mathfrak{A}$ at stage n, and to be 1,

if I did not.

(A difficulty is, in our opinion, that, sometimes, we want to perform transfinite constructions. How do we schedule them in a future which is only a countable sequence of stages ?)

BK in full generality conflicts with AC_{11}, as is evident from theorem 2.0 The first published proof of theorem 2.0 is in Myhill 1967.

Theorem 2.0 was a hindrance for people who tried to formalize intuitionistic analysis. Sometimes, they decided to reject AC_{11} in favour of BK. This seemed to be in accordance with Brouwer's own intentions, as, in Brouwer 1949, he used the axiom in the generalized form.

An alternative way out of the conflict was shown by J.J. de Iongh, who suggested to restrict application of BK to determinate propositions $\mathfrak{A}$, i.e. propositions about which all information has been given and which do not depend on objects whose construction has not yet been completed. (We are not thinking of objects whose definition has still to be „worked out", but of objects in whose construction there is some freedom left.)

A more extensive discussion may be found in Gielen, de Swart and Veldman 1981, where BK has been used for giving intuitionistic parallels to classical proofs of the Cantor-Bendixson theorem and its extension by Souslin.

BK does not figure in the following, except that it will sometimes, in a helpful whisper, aid our intuition concerning the truth or falsity of certain propositions. (cf. 4.1).

2.2 Theorem: $\neg \forall \alpha \exists \beta [\, \exists n[\alpha(n)=0] \rightleftarrows \forall n[\beta(n)=0]]$

Proof: Suppose: $\forall \alpha \exists \beta [\, \exists n[\alpha(n)=0] \rightleftarrows \forall n[\beta(n)=0]]$

Using AC_{11}, determine $\delta \in {}^{\omega}\omega$ such that $\delta : {}^{\omega}\omega \to {}^{\omega}\omega$ and:

$\forall \alpha [\, \exists n[\alpha(n)=0] \rightleftarrows \forall n[(\delta|\alpha)(n)=0]]$

Consider the special element $\underline{1}$ of ${}^{\omega}\omega$ which is defined by: $\forall n[\underline{1}(n)=1]$

We claim: $\forall n[(\delta|\underline{1})(n)=0]$

For, suppose: $n \in \omega$ and $(\delta|\underline{1})(n) \neq 0$

We determine $m \in \omega$ such that: $\delta^n(\bar{\underline{1}}m) \neq 0 \wedge \delta^n(\bar{\underline{1}}m) \neq 1 \wedge \forall p[p<m \to \delta^n(\bar{\underline{1}}p)=0]$

Then: $\forall \alpha [\bar{\alpha}m = \bar{\underline{1}}m \to (\delta|\alpha)(n) = (\delta|\underline{1})(n)]$

and: $\forall \alpha [\bar{\alpha}m = \bar{\underline{1}}m \to \neg \forall n[(\delta|\alpha)(n)=0]]$

so: $\forall \alpha [\bar{\alpha}m = \bar{\underline{1}}m \to \neg \exists n[\alpha(n)=0]]$ and this is not so.

Therefore: $\forall n[(\delta|\underline{1})(n)=0]$ and: $\neg \exists n[\underline{1}(n)=0]$

δ's failure is obvious. ⊠

One cannot escape the feeling that δ, the protagonist of this last proof, is being trapped in a base way. One forces him to be careful about $\delta|\underline{1}$ and, later on, this caution is held against him.
In comparison, the play was more fair in theorem 2.0.

2.3 That theorem 2.0 is not an isolated fact and might herald the birth of a new theory, was suggested by J.J. de Iongh.
We now prepare for this more general theory.

Let A, B be subsets of ${}^\omega\omega$. We define:

$A \leq B$ (A is reducible to B) $:= \forall\alpha\exists\beta[A(\alpha) \rightleftarrows B(\beta)]$

Using AC_{11}, we see that: $A \leq B$ if and only if $\exists\delta[Fun(\delta) \wedge \forall\alpha[A(\alpha) \rightleftarrows B(\delta|\alpha)]]$
If we want to avoid the use of AC_{11}, we might define: $A \leq B$ by:
$\exists\delta[Fun(\delta) \wedge \forall\alpha[A(\alpha) \rightleftarrows B(\delta|\alpha)]$ (cf. Note 3 on page 216).

Intuitively, the meaning of „$A \leq B$" might be described as:
We have a method for translating every question whether some element of ${}^\omega\omega$ belongs to A, into a question whether some other element of ${}^\omega\omega$ belongs to B.

This reducibility relation is, obviously, reflexive and transitive.

Classically, this many-one-reducibility-relation is called *Wadge-reducibility*.
(Cf. Kechris and Moschovakis 1978, Moschovakis 1980, Wadge 198?)

We introduce the subsets A_1 and E_1 of ${}^\omega\omega$ by:

for all $\alpha \in {}^\omega\omega$: $A_1(\alpha) := \forall n[\alpha(n) = 0]$

for all $\alpha \in {}^\omega\omega$: $E_1(\alpha) := \exists n[\alpha(n) = 0]$

We have seen, in theorems 2.0 and 2.2 that $\neg(A_1 \leq E_1)$ and $\neg(E_1 \leq A_1)$

We also need the strict reducibility relation:

Let A,B be subsets of ${}^\omega\omega$. We define:

$A \prec B$ (A is strictly reducible to B) $:= A \leq B \wedge \neg(B \leq A)$

3 THE SECOND LEVEL OF THE ARITHMETICAL HIERARCHY

Two theorems will be proved which are a natural extension of the theorems of the previous chapter.
The leading ideas of their proofs will continue to inspire us, up to chapters 7 and 9.

3.0 We consider the subsets A_2 and E_2 of ${}^{\omega}\omega$, which are defined by:

For all $\alpha \in {}^{\omega}\omega$:

$$A_2(\alpha) := \forall m \exists n[\alpha^m(n) = 0]$$

$$E_2(\alpha) := \exists m \forall n[\alpha^m(n) = 0]$$

We leave it to the reader to prove the following easy facts:

$$A_1 \preceq A_2 \quad , \quad E_1 \preceq A_2 \quad , \quad A_1 \preceq E_2 \quad \text{and} \quad E_1 \preceq E_2$$

3.1 The following is an important remark on A_2:

According to AC_{00}: $\forall \alpha [A_2(\alpha) \rightleftarrows \exists \gamma \forall m [\alpha^m(\gamma(m)) = 0]]$

For all $\gamma \in {}^{\omega}\omega$ and $\alpha \in {}^{\omega}\omega$, we define $\gamma \bowtie \alpha$ in ${}^{\omega}\omega$ by:

For all $m \in \omega$, $n \in \omega$:

$$(\gamma \bowtie \alpha)^m(n) := 0 \quad \text{if} \quad n = \gamma(m)$$

$$:= \alpha^m(n) \quad \text{if} \quad n \neq \gamma(m)$$

and: $(\gamma \bowtie \alpha)(0) := 0$

Remark that: $\forall \alpha [A_2(\alpha) \rightleftarrows \exists \gamma [\alpha = \gamma \bowtie \alpha]]$

Let us make $\delta \in {}^{\omega}\omega$ such that $Fun(\delta)$ and $\forall \alpha [\delta | \alpha = \alpha^0 \bowtie \alpha^1]$
Observe that: $\forall \alpha [A_2(\alpha) \rightleftarrows \exists \beta [\alpha = \delta | \beta]]$

Let us define, for all $\delta \in {}^{\omega}\omega$, a subset $Ra(\delta)$ of ${}^{\omega}\omega$ by:

$$Ra(\delta) := \{\alpha | \alpha \in {}^{\omega}\omega \mid \exists \beta [\delta : \beta \mapsto \alpha]\}$$

(„$\mapsto$" has been introduced in 1.6)

We have seen: $\exists \delta [Fun(\delta) \wedge A_2 = Ra(\delta)]$

This is a useful property, which A_2 shares with many other sets.
(cf. 7.0 and 10.7).

3.2 <u>Theorem</u>: $\neg(A_2 \preceq E_2)$

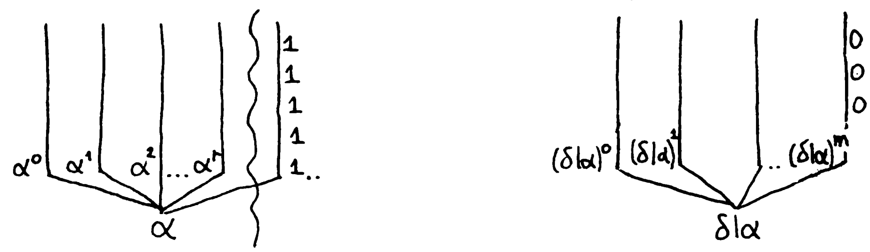
<u>Proof</u>: Suppose: $A_2 \preceq E_2$, i.e.: $\forall\alpha\exists\beta[A_2(\alpha) \rightleftarrows E_2(\beta)]$
Using AC_{11}, determine δ in ${}^\omega\omega$ such that: $Fun(\delta)$ and $\forall\alpha[A_2(\alpha) \rightleftarrows E_2(\delta|\alpha)]$
Consider the intertwining function $\bowtie$, introduced in 3.1, and observe:

$$\forall\gamma\forall\alpha[E_2(\delta|(\gamma\bowtie\alpha))]$$

Consider $\underline{0}$ in ${}^\omega\omega$, the sequence that is defined by: $\forall n[\underline{0}(n)=0]$
Using CP, determine $p\in\omega$, $q\in\omega$, $m\in\omega$ such that:

$$\forall\gamma\forall\alpha[(\bar{\gamma}p = \bar{\underline{0}}p \wedge \bar{\alpha}q = \bar{\underline{0}}q) \to \forall n[(\delta|(\gamma\bowtie\alpha))^m(n) = 0]]$$

Let us pause for a moment and imagine the situation:



We are assuming: $\forall\alpha[A_2(\alpha) \rightleftarrows E_2(\delta|\alpha)]$
We think of "α" in this formula as being built up step by step by a creative subject, whereas $\delta|\alpha$ is being made by a less creative, imitative subject, who does not make a sequence of his own, but transcribes α, using the method coded into δ.
The creative subject is not very fond of the imitative one and plays a trick on him, as follows:
He calculates $r = \max(p,q)$ and defines a sequence α^* in ${}^\omega\omega$ by:

$$\alpha^*(0) = 0 \wedge \forall n[(n \leq r \to (\alpha^*)^n = \underline{0}) \wedge (n > r \to (\alpha^*)^n = \underline{1}]$$

($\underline{1}$ is the sequence in ${}^\omega\omega$ that is defined by: $\forall n[\underline{1}(n)=0]$)
The creative subject will feed the imitative one on α^*, but he does not tell him so.
The imitative subject never sees more than a finite initial part of α^*, and, therefore, he is kept between hope and fear.
His anxiety will grow with the number of 1's, but all the time, he has to reckon with the possibility that things will improve.
Thus, he is forced to make all values of the sequence $(\delta|\alpha^*)^m$ equal to zero.

For, suppose: $k \in \omega$ and $(\delta|\alpha^*)^m(k) \neq 0$

Determine $l \in \omega$ such that $\forall\alpha[\overline{\alpha}l = \overline{\alpha^*}l \to (\delta|\alpha)^m(k) = (\delta|\alpha^*)^m(k)]$

Define a sequence α^+ in ${}^\omega\omega$ by:

$\overline{\alpha^+}l = \overline{\alpha^*}l$ and $\forall n[n \geq l \to \alpha^+(n) = 0]$

We observe that: $A_2(\alpha^+)$ and $\exists\gamma[\alpha^+ = \gamma \ltimes \alpha^+]$

We can say more: as $\forall n[n \leq r \to (\alpha^+)^n = \underline{0}]$, also:

$\exists\gamma[\overline{\gamma}p = \overline{\underline{0}}p \wedge \alpha^+ = \gamma \ltimes \alpha^+] \wedge \overline{\alpha^+}q = \overline{\underline{0}}q$

Therefore: $\forall n[(\delta|\alpha^+)^m(n) = 0]$ and: $(\delta|\alpha^+)^m(k) \neq 0$,

a contradiction.

The imitative subject has no choice and: $\forall n[(\delta|\alpha^*)^m(n) = 0]$

But his caution does not help him

We observe: $\neg A_2(\alpha^*) \wedge E_2(\delta|\alpha^*)$ and: $A_2(\alpha^*) \rightleftarrows E_2(\delta|\alpha^*)$,

a contradiction

$\boxtimes$

3.3 Theorem: $\neg(E_2 \preceq A_2)$

Proof: Suppose: $E_2 \preceq A_2$, i.e.: $\forall\alpha\exists\beta[E_2(\alpha) \rightleftarrows A_2(\beta)]$

Using AC_{11}, determine δ in ${}^\omega\omega$ such that: $Fun(\delta)$ and $\forall\alpha[E_2(\alpha) \rightleftarrows A_2(\delta|\alpha)]$

This time, the creative subject, in order to make the imitative subject fall on his face, uses very foul means from the realm of darkness. He plays the good boy for a while, till the imitative subject, being impressed, cannot refuse him any longer the first of his countably many wishes. As soon as the imitative subject gives in, the creative subject stops playing the good boy.

But not for long. He soon starts to play another good boy and perseveres in it, till the imitative subject loses his firmness again, and grants him the second of his wishes.

Ungratefully, the creative subject breaks off his good conduct, but chooses, after a moment, a third saint to follow, intending to follow him only so far as is required for getting his third wish fulfilled.

And so on.

In the end, the creative subject turns out to be no good boy at all, but he has got all he wanted.

In short, the creative subject makes a sequence α^* such that $\neg E_2(\alpha^*) \wedge A_2(\delta|\alpha^*)$, perplexing the imitative subject, as follows:

First consider $\alpha_0 := \underline{0}$

Remark: $E_2(\alpha_0)$, and determine $p_0 \in \omega$ such that $(\delta|\alpha_0)^0(p_0) = 0$

Determine $n_0 \in \omega$ such that $\forall\alpha[\bar{\alpha}_0 n_0 = \bar{\alpha} n_0 \rightarrow (\delta|\alpha)^0(p_0) = 0]$

Define $\alpha_1 \in {}^\omega\omega$ by:

for all $n \in \omega$, $n \leq n_0$: $\alpha_1(n) := \alpha_0(n)$

$\alpha_1(\langle 0, n_0 \rangle) := 1$

for all $n \in \omega$, $n > n_0$ and $n \neq \langle 0, n_0 \rangle$: $\alpha_1(n) := 0$

Remark: $E_2(\alpha_1)$, and determine $p_1 \in \omega$ such that $(\delta|\alpha_1)^1(p_1) = 0$

Determine $n_1 \in \omega$ such that $n_1 \geq n_0$ and $n_1 \geq \langle 0, n_0 \rangle$ and:

$\forall\alpha[\bar{\alpha}_1 n_1 = \bar{\alpha} n_1 \rightarrow (\delta|\alpha)^1(p_1) = 0]$

Define $\alpha_2 \in {}^\omega\omega$ by:

for all $n \in \omega$, $n \leq n_1$: $\alpha_2(n) := \alpha_1(n)$

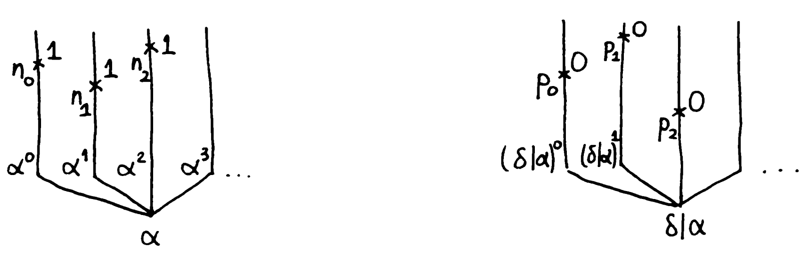
$\alpha_2(\langle 1, n_1 \rangle) := 1$

for all $n \in \omega$, $n > n_1$ and $n \neq \langle 1, n_1 \rangle$: $\alpha_2(n) := 0$

⋮

Continue as before

(One may think of the following picture:



As soon as the imitative subject $\delta|\alpha$ puts 0 in one of his columns, the creative subject answers this move by putting 1 in the corresponding one of his own columns)

In this way one creates successively $\alpha_0, \alpha_1, \alpha_2, \ldots$ in ${}^\omega\omega$ and $p_0, n_0, p_1, n_1, p_2, n_2, \ldots$ in ω such that:

$$n_0 < n_1 < n_2 < \dots$$
$$\forall i \forall j [\, i \le j \to \bar{\alpha}_i n_i = \bar{\alpha}_j n_i \,]$$
$$\forall i \forall \alpha [\, \bar{\alpha} n_i = \bar{\alpha}_i n_i \to (\delta|\alpha)^i (p_i) = 0 \,]$$
$$\forall j [\, (\alpha_{j+1})^j (n_j) = 1 \,]$$
$$\forall i \forall j [\, i \le j \to (\alpha_{j+1})^i (n_i) = 1 \,]$$

Define $\alpha^* \in {}^\omega\omega$ by: $\forall \alpha [\, \overline{\alpha^*} n_i = \bar{\alpha}_i n_i \,]$

We observe: $\neg E_2(\alpha^*) \wedge A_2(\delta|\alpha^*)$ and: $E_2(\alpha^*) \rightleftarrows A_2(\delta|\alpha^*)$,

a contradiction.

$\boxtimes$

3.4 Proofs of more general hierarchy theorems are now within our grasp. We only have to look with some care into the proofs of this chapter.

When we reconsider the proof of theorem 3.2, that $\neg(A_2 \preceq E_2)$, we are struck by its likeness, from a certain moment on, to the proof of theorem 2.2 (whose conclusion reads: $\neg(E_1 \preceq A_1)$).

To be more specific:
Suppose: $\delta \in {}^\omega\omega$ and $Fun(\delta)$ and $\forall \alpha [\, A_2(\alpha) \rightleftarrows E_2(\delta|\alpha) \,]$
Construct numbers m and r, as in the proof of theorem 3.2
Continue by making $\varepsilon \in {}^\omega\omega$ such that $Fun(\varepsilon)$ and:

$$\forall \beta [\, \forall n [\, n \ne r \to (\varepsilon|\beta)^n = \underline{0} \,] \wedge (\varepsilon|\beta)^r = \beta \,]$$

Remark: $\forall \beta [\, E_1(\beta) \rightleftarrows A_2(\varepsilon|\beta) \,]$

and: $\forall \beta [\, A_2(\varepsilon|\beta) \rightleftarrows A_1((\delta|(\varepsilon|\beta))^m) \,]$

Therefore: $\forall \beta [\, E_1(\beta) \rightleftarrows A_1((\delta|(\varepsilon|\beta))^m) \,]$, i.e.: $E_1 \preceq A_1$

Thus, the proof is seen to reduce the supposition: $A_2 \preceq E_2$ to: $E_1 \preceq A_1$

It is not difficult to find a general method for reducing the supposition: $A_{Sn} \preceq E_{Sn}$ to: $E_n \preceq A_n$.
This will be shown in chapter 7, when chapter 6 has given the necessary definitions.

It takes more pains to get a similar conclusion from the converse supposition: $E_{Sn} \preceq A_{Sn}$, but, again, when the work has been done, we see some resemblance to the proof of theorem 3.3, that $\neg(E_2 \preceq A_2)$.

To this proof of theorem 3.3, other useful observations may be made. Perhaps its most memorable feature is, how it pictures the creative subject as a cat bent upon its prey, the imitative subject, moving only in response to moves of its mousy victim.

We understated the conclusion of this proof.
Given a sequence δ in ${}^{\omega}\omega$ such that: $\mathrm{Fun}(\delta) \wedge \forall\alpha[E_2(\alpha) \rightleftarrows A_2(\delta|\alpha)]$, we set ourselves the aim of finding a sequence α^* in ${}^{\omega}\omega$ such that $\neg E_2(\alpha^*) \wedge A_2(\delta|\alpha^*)$.

But the sequence α^* which we constructed, had a more constructive property than: $\neg E_2$, we know that it shows up a number different from zero in each one of its subsequences.
We call this property: A_2^*.

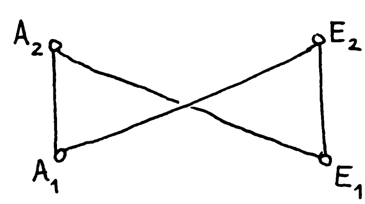
Another important remark on the proof of theorem 3.3 is that we did not use the full strength of the assumption.
Starting from: $\mathrm{Fun}(\delta) \wedge \forall\alpha[E_2(\alpha) \to A_2(\delta|\alpha)]$, we may reach the same conclusion.

A similar thing can be said on the proof of theorem 3.2.

This sharper view of the constructivity of the arguments used will enable us to extend the theorems into the transfinite, in chapter 9.

We decided not to leave out the more clumsy method of chapter 7, although its results are properly contained in those of chapter 9.
This method held us captive for quite a long time, and it deserves of some attention, if only for the sake of comparison.

3.5 We may picture the results of this chapter as follows:



4 SOME ACTIVITIES OF DISJUNCTION AND CONJUNCTION

Both classically and intuitionistically, the intersection of two open subsets of $\mathbb{R}$ is an open subset of $\mathbb{R}$.
However, only by using classical logic, one may infer from this the dual statement: the union of two closed subsets of $\mathbb{R}$ is a closed subset of $\mathbb{R}$.
It need not surprise, therefore, that this statement is not true, if interpreted intuitionistically.
($[0,1] \cup [1,2]$, for example, is not a closed subset of $\mathbb{R}$).

This well-known fact will be confirmed by the theorems of this chapter
We know from chapter 3, that E_2, the subset of ${}^\omega\omega$ which we get from A_1 by an existential projection, is not reducible to A_1 or, for that matter, to A_2.
We will see now that the same holds true for the subset of ${}^\omega\omega$ which we get from A_1 by a disjunctive projection only: D^2A_1.

In the case of A_1, finite disjunction suffices to increase complexity
No wonder, then, that the number of disjuncts is also important: the subset which we get from A_1 by a triple disjunctive projection, is not reducible to the subset we get by a binary disjunctive projection, and so on
Between A_1 and E_2, we may distinguish, in this manner, countably many levels of complexity.

Conjunction, of course, is inactive, if applied to A_1, but it gets productive as soon as we apply it to D^2A_1, for example.
Let us consider the class of all subsets of ${}^\omega\omega$ which originate from A_1, when we apply the operations of finite disjunctive and conjunctive projection again and again.
How does the reducibility relation behave on this countable class?
We partially answer this nice question at the end of this chapter.

4.0 We introduce, for all subsets $P \subseteq {}^\omega\omega$ and $n \in \omega$, a subset D^nP of ${}^\omega\omega$ by:

$$\text{for all } \alpha \in {}^\omega\omega : \quad D^nP(\alpha) := \exists q < n\,[P(\alpha^q)]$$

4.1 <u>Theorem</u>: It is reckless to assume: $D^2A_1 \leq A_1$

<u>Proof</u>: Suppose: $D^2A_1 \leq A_1$, i.e.: $\forall\alpha\exists\beta\,[D^2A_1(\alpha) \rightleftarrows A_1(\beta)]$

Remark: $\forall\beta[\neg\neg\forall n[\beta(n)=0] \rightarrow \forall n[\beta(n)=0]]$

Therefore: $\forall\alpha[\neg\neg D^2A_1(\alpha) \rightarrow D^2A_1(\alpha)]$

This enables us to decide a lot of questions

Let us turn to the decimal development of π which earned itself a reputation in providing counterexamples to all kinds of classically valid but constructively untrue statements.

Construct a sequence α in ${}^{\omega}\omega$ which fulfils the condition:
$\forall n[\ \alpha(n)=0 \rightleftarrows$ At place n in the decimal development of π stands the last 9 of the first block of ninety-nine 9's]

Remark: $\forall m \forall n\,[(\alpha(m)=0 \wedge \alpha(n)=0) \to m=n]$, therefore: $\alpha^0 \neq \underline{0} \to \alpha^1=\underline{0}$ and: $\neg\neg(\alpha^0=\underline{0} \vee \alpha^1=\underline{0})$, i.e.: $\neg\neg D^2A_1(\alpha)$

The conclusion: $D^2A_1(\alpha)$, however, is not empty as a communication on the decimal development of π. We should be able to exclude either all numbers $\langle 0,m\rangle$ or all numbers $\langle 1,m\rangle$ as a possible position of the last 9 in the first block of ninety-nine 9's in π's decimal tail But we are not able to do so.

$\boxtimes$

The axiom of Brouwer and Kripke (cf. chapter 2) increases our doubts concerning: „$D^2A_1 \preceq A_1$"

Let $\mathfrak{A}$ be a determinate (cf. 2.1), as yet undecided mathematical proposition such that: $\neg\neg\mathfrak{A} \to \mathfrak{A}$.

(One might think of Fermat's conjecture, or of any other mathematical proposition which can be brought into the form: $\forall n[F(n)]$, where F is a determinate property of natural numbers, such that: $\forall n[F(n) \vee \neg F(n)]$)

$\mathfrak{A} \vee \neg\mathfrak{A}$ is also a determinate proposition, and, using (BK) and some acrobatics, we determine α in ${}^{\omega}\omega$ such that:

$$\mathfrak{A} \vee \neg\mathfrak{A} \rightleftarrows \exists n[\alpha(n)=0] \quad \text{and:} \quad \mathfrak{A} \rightleftarrows \exists n[\alpha^0(n)=0] \quad \text{and:}$$

$$\neg\mathfrak{A} \rightleftarrows \exists n[\alpha^1(n)=0] \quad \text{and:} \quad \forall m \forall n[(\alpha(m)=0 \wedge \alpha(n)=0) \to m=n]$$

Remark: $\neg\neg D^2A_1(\alpha)$, therefore: $D^2A_1(\alpha)$; i.e.: $\alpha^0=\underline{0} \vee \alpha^1=\underline{0}$, and: $\neg\mathfrak{A} \vee \neg\neg\mathfrak{A}$, therefore: $\neg\mathfrak{A} \vee \mathfrak{A}$

By „$D^2A_1 \preceq A_1$" we are able to decide, in this way, any determinate, stable proposition. (A proposition $\mathfrak{A}$ is called <u>stable</u>, if $\neg\neg\mathfrak{A} \to \mathfrak{A}$)
This is a reckless assumption.

We constructed a „weak" counterexample to: „$D^2A_1 \preceq A_1$". (In Dutch: „een vermetelheidstegenvoorbeeld")
In many such cases, as in this one, we are able to improve on the argument and to derive a contradiction.
This is an art which has been practized much by Wim Gielen.

The axiom of Brouwer and Kripke does not figure in the eventual argument. We also did not use it in proving theorem 4.1.

4.2 We introduced, in 1.2, a coding function $<>: \bigcup_{k\in\omega} {}^k\omega \rightarrowtail\!\!\!\!\twoheadrightarrow \omega$
Thus, every natural number codes a finite sequence of natural numbers.

We also introduced a length function $lg: \omega \to \omega$ such that, for all $n \in \omega$, $lg(n) =$ the length of the finite sequence coded by n.

We now define, for all $n, k \in \omega$ such that $k < lg(n)$:

$n(k) := n_k :=$ the value which the finite sequence coded by n, assumes in k.

Therefore, for each $n \in \omega$: $n = < n(0), n(1), \ldots, n(lg(n) - 1) >$

We define a sequence τ in ${}^\omega\omega$ such that:

$$\forall n [\tau(n) = 0 \rightleftarrows (\forall k [k < lg(n) \to n(k) < 2] \wedge \forall k \forall l [(k < lg(n) \wedge l < lg(n) \wedge n(k) \neq 0 \wedge n(l) \neq 0) \to k = l])]$$

We remark that τ is a subspread of ${}^\omega\omega$ (cf. 1.9) and:

$$\forall \alpha [\alpha \in \tau \rightleftarrows (\forall k [\alpha(k) < 2] \wedge \forall k \forall l [(\alpha(k) \neq 0 \wedge \alpha(l) \neq 0) \to k = l])]$$

The set $\tau = \{\alpha | \alpha \in {}^\omega\omega \mid \forall n [\tau(\bar{\alpha}n) = 0]\}$ consists of those sequences of 0's and 1's which have in at most one point a value different from 0.

The spread τ is very similar to the spread σ_{2mon} which will come to the fore in chapter 11.

4.3 Theorem: $\neg(D^2A_1 \preceq A_1)$

Proof: Suppose: $D^2A_1 \preceq A_1$, i.e. $\forall \alpha \exists \beta [D^2A_1(\alpha) \rightleftarrows A_1(\beta)]$
As in the proof of theorem 4.1, we observe: $\forall \alpha [\neg\neg D^2A_1(\alpha) \to D^2A_1(\alpha)]$
Now: $\forall \alpha \in \tau [\neg\neg D^2A_1(\alpha)]$, where τ is the subspread of ${}^\omega\omega$ which we defined in 4.2
Therefore: $\forall \alpha \in \tau [D^2A_1(\alpha)]$
Remark: $\underline{0} \in \tau$ and, applying to the generalized continuity principle GCP, determine $r \in \omega$ and $k \in \{0,1\}$ such that: $\forall \alpha \in \tau [\bar{\alpha}r = \bar{\underline{0}}r \to \alpha^k = \underline{0}]$
But this is not so, as we may define α_0 in τ such that:
$\bar{\alpha}_0 r = \bar{\underline{0}} r$ and: $(\alpha_0)^k(r) = 1$
$\boxtimes$

We feel content that, in proving this theorem, we did not use AC_{11} or AC_{10}, but GCP only.

4.4 For all $m,n \in \omega$ we define $[n]^m$ to be the code number of the m-th subsequence of the finite sequence coded by n

Therefore, for all $k \in \omega$, $[n]^m(k)$ is defined if and only if $\langle m\rangle * k < lg(n)$ and, in that case: $[n]^m(k) = n(\langle m\rangle * k)$

For every $m \in \omega$ we define a sequence τ_m in ${}^\omega\omega$ such that:
$\forall n [\tau_m(n) = 0 \rightleftarrows \underline{0} \in [n]^m]$

We remark that, for all $m \in \omega$, τ_m is a subspread of ${}^\omega\omega$ (cf. 1.9) and:
$\forall \alpha [\alpha \in \tau_m \rightleftarrows \alpha^m = \underline{0}]$

We also observe: $\forall m \forall \alpha [D^m A_1(\alpha) \rightleftarrows \exists n < m [\alpha \in \tau_n]]$

4.5 Theorem: $\neg(D^3A_1 \preceq D^2A_1)$

Proof: Suppose: $D^3A_1 \preceq D^2A_1$, i.e.: $\forall \alpha \exists \beta [D^3A_1(\alpha) \rightleftarrows D^2A_1(\beta)]$

Using AC_{11}, we find δ in ${}^\omega\omega$ such that:

$Fun(\delta)$ and: $\forall \alpha [D^3A_1(\alpha) \rightleftarrows D^2A_1(\delta|\alpha)]$

We observe: $\forall m<3 \, \forall \alpha [\alpha \in \tau_m \rightarrow (\delta|\alpha \in \tau_0 \vee \delta|\alpha \in \tau_1)]$

and: $\forall m<3 \, [\underline{0} \in \tau_m]$

(The spreads τ_m have been defined in 4.4)

Applying the generalized continuity principle GCP three times, we find natural numbers p_0, p_1, p_2 and k_0, k_1, k_2 such that:

$$\forall m<3 \, [k_m = 0 \vee k_m = 1]$$

and: $\forall m<3 \, \forall \alpha \in \tau_m [\bar{\alpha} p_m = \bar{\underline{0}} p_m \rightarrow \delta|\alpha \in \tau_{k_m}]$

Without loss of generality, we may assume: $k_0 = k_1$

Let $p := max(p_0, p_1, p_2)$

We determine ζ in ${}^\omega\omega$ such that: $Fun(\zeta)$ and, for all $\alpha \in {}^\omega\omega$, $m,n \in \omega$:

$$\begin{aligned}(\zeta|\alpha)^m(n) &:= 0 && \text{if } n<p \vee m>2\\ &:= \alpha^m(n-p) && \text{if } n \geq p \wedge m<2\\ &:= 1 && \text{if } n \geq p \wedge m=2\end{aligned}$$

Now, suppose: $\alpha \in {}^\omega\omega$ and $D^2A_1(\alpha)$, then: $\zeta|\alpha \in \tau_0 \vee \zeta|\alpha \in \tau_1$ and: $\overline{(\zeta|\alpha)}p = \bar{\underline{0}}p$, so: $\delta|(\zeta|\alpha) \in \tau_{k_0}$

Conversely, suppose $\alpha \in {}^\omega\omega$ and $\delta|(\zeta|\alpha) \in \tau_{k_0}$; then $D^2A_1(\delta|(\zeta|\alpha))$ so: $D^3A_1(\zeta|\alpha)$, and: $D^2A_1(\alpha)$

Therefore: $\forall \alpha [D^2A_1(\alpha) \rightleftarrows A_1((\delta|(\zeta|\alpha))^{k_0})]$, i.e.: $D^2A_1 \preceq A_1$

This contradicts theorem 4.3. ⊠

We confess that, in proving theorem 4.5, we did not succeed in avoiding AC_{11}

Without difficulty, we may extend theorem 4.5 to:

4.6 <u>Theorem</u>: $\forall m[\neg(D^{m+1}A_1 \leq D^m A_1)]$

4.7 We introduce, for all subsets $P \subseteq {}^\omega\omega$, a subset $Un(P)$ of ${}^\omega\omega$ by:

for all $\alpha \in {}^\omega\omega$: $(Un(P))(\alpha) := \forall m[P(\alpha^m)]$

We now show that „choosing one-out-of-three" is not to be reduced to „choosing one-out-of-two", even if we are allowed to do the latter infinitely many times.

4.8 <u>Theorem</u>: $\neg(D^3A_1 \leq Un(D^2A_1))$

<u>Proof</u>: Suppose: $D^3A_1 \leq Un(D^2A_1)$, i.e.: $\forall\alpha\exists\beta[D^3A_1(\alpha) \rightleftarrows (Un(D^2A_1))(\beta)]$

Using AC_{11}, we find δ in ${}^\omega\omega$ such that:

$Fun(\delta)$ and $\forall\alpha[D^3A_1(\alpha) \rightleftarrows (Un(D^2A_1))(\delta|\alpha)]$

Let τ be the spread which we introduced in 4.2:

$\forall\alpha[\alpha\in\tau \rightleftarrows (\forall k[\alpha(k)<2] \wedge \forall k\forall l[(\alpha(k)\neq 0 \wedge \alpha(l)\neq 0) \to k=l])]$

We want to show: $\forall\alpha\in\tau\,\forall p[D^2A_1((\delta|\alpha)^p)]$

Let us assume, to this end: $\alpha\in\tau$ and $p\in\omega$

We observe, as in the proof of 4.5:

$$\forall m<3\,\forall\beta\in\tau_m[D^3A_1(\beta)]$$

and: $\forall m<3[\underline{0}\in\tau_m]$

($\tau_0, \tau_1, \tau_2, \ldots$ are the spreads which made their first appearance in 4.4: $\forall m\forall\alpha[\alpha\in\tau_m \rightleftarrows \alpha^m = \underline{0}]$)

By a threefold invocation of the generalized continuity principle GCP we find natural numbers q_0, q_1, q_2 and k_0, k_1, k_2 such that:

$$\forall m<3[k_m=0 \vee k_m=1]$$

and: $\forall m<3\,\forall\beta\in\tau_m[\bar\beta q_m = \bar{\underline{0}}q_m \to (\delta|\beta)^P \in \tau_m]$

Without loss of generality, we may assume: $k_0=k_1$.

Let $q := \max(q_0, q_1, q_2)$.

We distinguish two cases:

Case 1: $\bar\alpha q \neq \bar{\underline{0}}q$

As $\alpha\in\tau$, we may determine, in this case, $m<3$ such that: $\alpha\in\tau_m$, and, thus, we know: $D^2A_1((\delta|\alpha)^p)$

Case 2: $\bar{\alpha}q \neq \bar{\underline{0}}q$

We now turn up our trump card:

$\alpha \in \tau$, therefore: $\neg\neg(\alpha \in \tau_0 \vee \alpha \in \tau_1)$ and $\neg\neg((\delta|\alpha)^P \in \tau_{k_0})$

so: $(\delta|\alpha)^P \in \tau_{k_0}$ and: $D^2A_1((\delta|\alpha)^P)$

In any case: $D^2A_1((\delta|\alpha)^P)$

We have proved now: $\forall\alpha \in \tau\, \forall p[D^2A_1((\delta|\alpha)^p)]$, i.e.:

$\forall\alpha \in \tau[(Un(D^2A_1))(\delta|\alpha)]$, and therefore: $\forall\alpha \in \tau[D^3A_1(\alpha)]$

We observe: $\underline{0} \in \tau$, and, applying to GCP, we determine $r \in \omega$ and $k \in \{0,1,2\}$ such that: $\forall\alpha \in \tau[\bar{\alpha}r = \bar{\underline{0}}r \rightarrow \alpha^k = \underline{0}]$

But this is not so, as we may define α_0 in τ such that:

$\bar{\alpha}_0 r = \bar{\underline{0}}r$ and: $(\alpha_0)^k(r) = 1$

$\boxtimes$

The reader will have remarked that the proof of theorem 4.8 is slightly more economical than the proof of theorem 4.5 and no longer leans on theorem 4.3 In the same way one may prove:

4.9 <u>Theorem</u>: $\forall m[\neg(D^{m+1}A_1 \preceq Un(D^mA_1))]$

We may sharpen the conclusion of theorem 4.3 also in this manner:

4.10 <u>Theorem</u>: $\neg(D^2A_1 \preceq A_2)$

<u>Proof</u>: Suppose: $D^2A_1 \preceq A_2$, i.e.: $\forall\alpha\exists\beta[D^2A_1(\alpha) \rightleftarrows A_2(\beta)]$

Using AC_{11}, we find δ in ${}^\omega\omega$ such that: $Fun(\delta)$ and $\forall\alpha[D^2A_1(\alpha) \rightleftarrows A_2(\delta|\alpha)]$

Let τ be the spread which we introduced in 4.2:

$\forall\alpha[\alpha \in \tau \rightleftarrows (\forall k[\alpha(k) < 2] \wedge \forall k \forall l[(\alpha(k) \neq 0 \wedge \alpha(l) \neq 0) \rightarrow k = l])]$

We want to show: $\forall\alpha \in \tau[A_2(\delta|\alpha)]$

Let us assume, to this end: $\alpha \in \tau$ and $p \in \omega$

We observe: $D^2A_1(\underline{0})$, therefore: $A_2(\delta|\underline{0})$, and: $E_1((\delta|\underline{0})^P)$

We determine $k \in \omega$ such that $(\delta|\underline{0})^P(k) = 0$

And we determine $q \in \omega$ such that: $\forall\beta[\bar{\beta}q = \bar{\underline{0}}q \rightarrow (\delta|\beta)^P(k) = 0]$

We now distinguish two cases:

Case 1: $\bar{\alpha}q \neq \bar{\underline{0}}q$

As $\alpha \in \tau$, we may determine in this case, $m<2$ such that $\alpha^m = \underline{0}$, therefore: $D^2A_1(\alpha)$ and: $A_2(\delta|\alpha)$, esp. $E_1((\delta|\alpha)^P)$

Case 2: $\bar{\alpha}q \neq \bar{\Omega}q$
Then: $E_1((\delta|\alpha)^p)$

In any case: $E_1\ ((\delta|\alpha)^p)$

We have proved, now: $\forall\alpha\in\tau\,\forall p\,[E_1((\delta|\alpha)^p)]$, i.e.: $\forall\alpha\in\tau\,[A_2(\delta|\alpha)]$

Therefore: $\forall\alpha\in\tau\,[D^2A_1(\alpha)]$

This will lead to a contradiction, as in the proof of theorem 4.3

$\boxtimes$

The proofs of the theorems 4.8 and 4.10 are variations upon one theme, the latter being the more simple of the two.
The conclusion of theorem 4.10 marks an improvement upon theorem 3.3, which said: $\neg(E_2 \preceq A_2)$.
In order to see this, one observes, using theorem 4.6: $\forall n[D^nA_1 \prec D^{n+1}A_1 \prec E_2]$
(We defined „$\prec$" in 2.3: $A \prec B \rightleftarrows (A \preceq B \wedge \neg(B \preceq A))$
The reader's task reduces to proving: $\forall n[D^nA_1 \preceq E_2]$).

4.11 We introduce, for all subsets $P \subseteq {}^\omega\omega$ and $n \in \omega$, a subset C^nP of ${}^\omega\omega$ by:
for all $\alpha \in {}^\omega\omega$: $C^nP(\alpha) := \forall q<n\,[P(\alpha^q)]$

4.12 Without difficulty, we establish the following facts: $C^2A_1 \preceq A_1$, $D^2E_1 \preceq E_1$ and $C^2E_1 \preceq E_1$.
First, we determine $\delta \in {}^\omega\omega$ such that: $\mathrm{Fun}(\delta)$ and $\forall\alpha\forall n[(\delta|\alpha)(2n) = \alpha^0(n) \wedge (\delta|\alpha)(2n+1) = \alpha^1(n)]$

Then: $\forall\alpha[C^2A_1(\alpha) \rightleftarrows A_1(\delta|\alpha)]$ and $\forall\alpha[D^2E_1(\alpha) \rightleftarrows E_1(\delta|\alpha)]$

Next, we determine $\delta \in {}^\omega\omega$ such that: $\mathrm{Fun}(\delta)$ and $\forall\alpha\forall n[(\delta|\alpha)(n) = 0 \rightleftarrows (\ell g(n) = 2 \wedge \alpha^0(n(0)) = \alpha^1(n(1)) = 0)]$

Then: $\forall\alpha[C^2E_1(\alpha) \rightleftarrows E_1(\delta|\alpha)]$

This seems to be a good place to mention an important difference between the results of this chapter and the results of chapter 3.
When we set out to prove: $\neg(A_2 \preceq E_2)$, we did not intend to prove as much as we did, eventually.
Starting from a sequence δ, fulfilling only: $\mathrm{Fun}(\delta)$ and: $\forall\alpha[A_2(\alpha) \rightarrow E_2(\delta|\alpha)]$
we were able to point out a sequence α^* such that: $\neg A_2(\alpha^*) \wedge E_2(\delta|\alpha^*)$
When proving: $\neg(E_2 \preceq A_2)$, we also exceeded our own expectations.
(cf. the discussion in 3.4)

There is no hope for a similar reinforcement of a conclusion like: $\neg(D^2A_1 \preceq A_1)$

In order to see this, we consider the subset E_1^* of ${}^\omega\omega$ which is defined by:
for all $\alpha \in {}^\omega\omega$: $E_1^*(\alpha) := \exists n[\alpha(n) = 1]$

We easily find $\delta \in {}^\omega\omega$ such that: $\mathrm{Fun}(\delta)$ and $\forall\alpha[C^2E_1^*(\alpha) \rightleftarrows E_1^*(\delta|\alpha)]$
This same δ also satisfies: $\forall\alpha[\neg C^2E_1^*(\alpha) \rightleftarrows \neg E_1^*(\delta|\alpha)]$ and, therefore:
$\forall\alpha[\neg\neg D^2A_1(\alpha) \rightleftarrows A_1(\delta|\alpha)]$

Remark that $\forall\alpha[D^2A_1(\alpha) \to A_1(\delta|\alpha)]$ and that it is impossible to find $\alpha^* \in {}^\omega\omega$ such that $\neg D^2A_1(\alpha^*) \wedge A_1(\delta|\alpha^*)$

This phenomenon is put into perspective when we recognize that there are classical facts corresponding to the results of chapter 3 whereas, in this chapter, truly intuitionistic idiosyncrasies come to the surface.

4.13 We introduce, for all natural numbers m,n, a finite subset $Exp(m,n)$ of ω by:

$$Exp(m,n) := \{f \mid f \in \omega \mid lg(f) = n \wedge \forall k[k<n \to f(k)<m]\}$$

($Exp(m,n)$ is the set of all functions from n to m, where, following set-theoretical habits, m and n are identified with the sets of their predecessors).

We define, for each $f \in \omega$, a subset A_f of ${}^\omega\omega$ by:

$$\text{for all } \alpha \in {}^\omega\omega: \quad A_f(\alpha) := \forall n[n < lg(f) \to (\alpha^n)^{f(n)} = \underline{0}]$$

We leave it to the reader to verify: $\forall f[A_f \leq A_1]$ and: $\forall f[f \neq \langle\rangle \to A_1 \leq A_f]$
In this last sentence good old A_1 is meant, which we met for the first time in 2.3.

We are guilty of a slight inaccuracy by having introduced, here, namesakes for A_1 and A_2, (cf. 3.0), but it will not harm us.

4.14 Theorem: $\forall n \forall m[C^n(D^m A_1) \leq D^{m^n} A_1]$

Proof: Remark: for all $\alpha \in {}^\omega\omega$:

$$C^n D^m A_1(\alpha) \rightleftarrows \forall k<n \exists l<m[(\alpha^k)^l = \underline{0}]$$
$$\rightleftarrows \exists f[f \in Exp(m,n) \wedge A_f(\alpha)]$$

Also observe that, for all $f \in \omega$, we may define $\delta_f \in {}^\omega\omega$ such that $Fun(\delta_f)$ and $\forall\alpha[A_f(\alpha) \rightleftarrows A_1(\delta_f|\alpha)]$

As $Exp(m,n)$ has m^n members, the construction of a $\delta \in {}^\omega\omega$ such that $Fun(\delta)$ and $\forall\alpha[C^n D^m A_1(\alpha) \rightleftarrows D^{m^n} A_1(\delta|\alpha)]$ is now an easy matter

⊠

4.15 Theorem: $\forall n \forall m \forall q \forall p[C^n D^m A_1 \leq C^q D^p A_1 \to m^n \leq p^q]$

Proof: (The reader has understood, probably, that „$C^n D^m A_1$" stands for: „$C^n(D^m A_1)$".)
Suppose: $m^n > p^q$ and $C^n D^m A_1 \leq C^q D^p A_1$, i.e.: $\forall\alpha\exists\beta[C^n D^m A_1(\alpha) \rightleftarrows C^q D^p A_1(\beta)]$
Using AC_{11}, determine $\delta \in {}^\omega\omega$ such that $Fun(\delta)$ and $\forall\alpha[C^n D^m A_1(\alpha) \rightleftarrows C^q D^p A_1(\delta|\alpha)]$

For every $f \in Exp(m,n)$, consider A_f, as defined in 4.13

Remark: $\forall f \in Exp(m,n)\, \forall\alpha [\, A_f(\alpha) \rightarrow C^n D^m A_1(\alpha)]$

Therefore: $\forall f \in Exp(m,n)\, \forall\alpha [\, A_f(\alpha) \rightarrow C^q D^p A_1(\delta|\alpha)]$

and: $\forall f \in Exp(m,n)\, \forall\alpha [\, A_f(\alpha) \rightarrow \exists h [\, h \in Exp(p,q) \wedge A_h(\delta|\alpha)]]$

Observe that, for every $f \in Exp(m,n)$, $A_f(\underline{0})$, and: A_f is a subspread of ${}^\omega\omega$ (cf. 1.9) so that the generalized continuity principle GCP applies. Applying it for every $f \in Exp(m,n)$ separately and keeping in mind that $m^n > q^p$, one finds $f \in Exp(m,n)$, $g \in Exp(m,n)$, $h \in Exp(p,q)$ and $r \in \omega$ such that: $f \neq g \;\wedge\; \forall\alpha[[\bar\alpha r = \bar{\underline{0}} r \wedge (A_f(\alpha) \vee A_g(\alpha))] \rightarrow A_h(\delta|\alpha)]$

We now again have recourse to τ, the subspread of ${}^\omega\omega$ which we introduced in 4.2 to serve us, in this chapter, as a true sorcerer's apprentice. ($\tau = \{\alpha | \alpha \in {}^\omega\omega \,|\, \forall k[\alpha(k)<2] \wedge \forall k \forall l[(\alpha(k)\neq 0 \wedge \alpha(l) \neq 0) \rightarrow k = l]\}$)

As $f \neq g$ we may determine $k<n$ such that $f(k) \neq g(k)$.

Therefore: $\forall\alpha\in\tau [\, \neg\neg(\, (\alpha^k)^{f(k)} = \underline{0} \;\vee\; (\alpha^k)^{g(k)} = \underline{0})]$

Let us restrict our attention to $\tau^* := \{\alpha | \alpha\in\tau \,|\, \forall l [\, l \neq k \rightarrow \alpha^l = \underline{0}]\}$

τ^* is again a subspread of ${}^\omega\omega$ and: $\forall\alpha\in\tau^* [\neg\neg(A_f(\alpha) \vee A_g(\alpha))]$

Therefore: $\forall\alpha\in\tau^* [\, \bar\alpha r = \bar{\underline{0}} r \rightarrow A_h(\delta|\alpha)]$ and:

$\forall\alpha\in\tau^* [\, \bar\alpha r = \bar{\underline{0}} r \rightarrow C^q D^p A_1(\delta|\alpha)]$, and: $\forall\alpha\in\tau^* [\, \bar\alpha r = \bar{\underline{0}} r \rightarrow C^n D^m A_1(\alpha)]$,

especially: $\forall\alpha\in\tau^* [\, \bar\alpha r = \bar{\underline{0}} r \rightarrow D^m A_1(\alpha^k)]$

We now proceed easily to the contradiction we wanted to reach, following the pattern of the proof of theorem 4.8:

We observe: $\underline{0} \in \tau^*$, and, applying to GCP, determine $s \in \omega$ such that $r \leq s$, and $l \in \omega$ such that: $\forall\alpha\in\tau^* [\, \bar\alpha s = \bar{\underline{0}} s \rightarrow (\alpha^k)^l = \underline{0}]$

This is not so, for we may define α_0 in τ^* such that:

$\bar\alpha_0 s = \bar{\underline{0}} s$ and: $((\alpha_0{}^k)^l)(s) = 1$.

⊠

4.16 <u>Theorem</u>: $\forall m \forall p [\, D^m A_1 \leq Un(D^p A_1) \rightarrow m \leq p]$

<u>Proof</u>: This follows from theorem 4.9.

Assume: $m>p$ and $D^m A_1 \leq Un(D^p A_1)$.

Remark: $D^{p+1} A_1 \leq D^m A_1$, therefore: $D^{p+1} A_1 \leq Un(D^p A_1)$.

This is not so, according to theorem 4.9. ⊠

4.17 <u>Lemma</u>: $\forall n \forall m [\, D^m A_1 \preceq C^{n+1} D^m A_1 \preceq C^{n+2} D^m A_1 \preceq Un(D^m A_1)]$

<u>Proof</u>: Easy. ⊠

4.18 <u>Theorem</u>: $\forall n \forall m \forall q \forall p [\, C^{n+1} D^m A_1 \preceq C^q D^p A_1 \rightarrow m \leq p]$

<u>Proof</u>: Immediate, from 4.16 and 4.17. ⊠

4.19 Many questions are answered by theorems 4.14-18, but some nasty problems remain to be solved.

Conjunctive power demonstrates itself in sequences like the following:

$$D^2A_1 \prec C^2D^2A_1 \prec C^3D^2A_1 \prec \dots$$
$$D^3A_1 \prec C^2D^3A_1 \prec C^3D^3A_1 \prec \dots$$

We know that no set from the second sequence can be reduced to any set from the first sequence.
The converse thing sometimes happens, as $\forall n [C^n D^2 A_1 \preceq C^n D^3 A_1]$
But what about the question if $C^3D^2A_1 \preceq C^2D^3A_1$?
No negative answer may be read off from theorems 4.14-18.
Nevertheless, the answer is negative, as you will suspect after a short walk.
More generally, we may ask, for any set from the first sequence:
what is the first set in the second sequence to which it is reducible?
And: do you know if $C^3D^3A_1 \preceq C^2D^6A_1$, or, if $C^5D^3A_1 \preceq C^4D^4A_1$?

In order to handle these and similar questions we introduce a new notation.

We define, for each $n \in \omega$ a subset $(CD)_n A_1$ of ${}^\omega\omega$ by:

$$\text{for all } \alpha \in {}^\omega\omega: \quad (CD)_n A_1(\alpha) := \forall k [\, k < lg(n) \rightarrow D^{n(k)} A_1(\alpha^k)]$$

$C^3D^2A_1$ reappears as $(CD)_{\langle 2,2,2\rangle} A_1$, and $C^2D^3A_1$ is now called $(CD)_{\langle 3,3\rangle} A_1$
We make a few observations, without striving for completeness:

If the finite sequence coded by n' is a permutation of the finite sequence coded by n, then: $(CD)_n A_1 \preceq (CD)_{n'} A_1$
If $lg(n) = lg(n_1)$ and $\forall k < lg(n) [\, n(k) \leq n'(k)]$, then $(CD)_n A_1 \preceq (CD)_{n'} A_1$
$(CD)_{\langle p,q\rangle} A_1 \preceq D^{p \cdot q} A_1$

More generally, if $n = \langle n_0, n_1, \dots, n_\ell \rangle$, then: $(CD)_{\langle n_0, n_1, \dots n_\ell\rangle} A_1 \preceq (CD)_{\langle n_0 \cdot n_1, n_2, \dots, n_\ell\rangle} A_1$

(The proofs of the last two statements are similar to the proof of theorem 4.14)

The following notion will also be useful:

We define, for all $f, n \in \omega$:

$$f \sqsubset n \;:=\; lg(f) = lg(n) \wedge \forall k[k < lg(n) \rightarrow f(k) < n(k)]$$

If $n = \langle n_0, n_1, \ldots, n_\ell \rangle$, then the number of elements of $\{f | f \in \omega | f \sqsubset n\}$ is $n_0 \cdot n_1 \cdot \ldots \cdot n_\ell$

We use square brackets [ ] to denote the entier-function from $\mathbb{Q}^+$ to ω, which assigns to each positive rational number its integral part.

Sufficiently many preparations have been made now for:

4.20 <u>Theorem</u>: Let m, n be natural numbers, $m = \langle m_0, m_1, \ldots, m_k \rangle$ and $n = \langle n_0, n_1, \ldots, n_\ell \rangle$
Let $m_0 > 0$. Then:

$$(CD)_{\langle m_0, m_1, \ldots, m_k \rangle} A_1 \preceq (CD)_{\langle n_0, n_1, \ldots, n_\ell \rangle} A_1 \text{ if and only if}$$

$$\exists t \leq \ell [m_0 \leq n_t \wedge (CD)_{\langle m_1, \ldots, m_k \rangle} A_1 \preceq (CD)_{\langle n_0, \ldots, [\frac{n_t}{m_0}], \ldots, n_\ell \rangle} A_1]$$

<u>Proof</u>: (I) First suppose: $t \leq \ell \wedge m_0 \leq n_t \wedge (CD)_{\langle m_1, \ldots, m_k \rangle} A_1 \preceq (CD)_{\langle n_0, \ldots, [\frac{n_t}{m_0}], \ldots, n_\ell \rangle} A_1$

A moment's reflection shows:

$$(CD)_m A_1 = (CD)_{\langle m_0, m_1, \ldots, m_k \rangle} A_1 \preceq (CD)_{\langle m_0, n_0, \ldots, [\frac{n_t}{m_0}], \ldots, n_\ell \rangle} A_1 \preceq$$

$$(CD)_{\langle n_0, \ldots, m_0 \cdot [\frac{n_t}{m_0}], \ldots, n_\ell \rangle} A_1 \preceq (CD)_{\langle n_0, \ldots, n_t, \ldots, n_\ell \rangle} A_1 = (CD)_n A_1$$

(II) Now suppose: $(CD)_m A_1 \preceq (CD)_n A_1$, i.e.: $\forall \alpha \exists \beta [(CD)_m A_1(\alpha) \rightleftarrows (CD)_n A_1(\beta)]$
Apply to AC_{11} and determine $\delta \in {}^\omega\omega$ such that: $Fun(\delta)$ and
$\forall \alpha [(CD)_m A_1(\alpha) \rightleftarrows (CD)_n A_1(\delta|\alpha)]$
Observe: $\forall \alpha [(CD)_m A_1(\alpha) \rightleftarrows \exists f \sqsubset m [A_f(\alpha)]]$
(We introduced, in 4.13, for each $f \in \omega$, the set $A_f = \{\alpha | \alpha \in {}^\omega\omega | \forall k < lg(f)[(\alpha^k)^{f(k)} = \underline{0}]\}$)
Call to mind that, for every $f \in \omega$, A_f is a subspread of ${}^\omega\omega$ (cf. 1.9) and: $A_f(\underline{0})$
Remark: $\forall f \sqsubset m \forall \alpha [A_f(\alpha) \rightarrow \exists g \sqsubset n [A_g(\delta|\alpha)]]$.
Invoke the generalized continuity principle GCP and conclude:

$$\forall f \sqsubset m \exists g \sqsubset n \exists s \forall \alpha [(\bar{\alpha}s = \bar{\underline{0}}s \wedge A_f(\alpha)) \rightarrow A_g(\delta|\alpha)].$$

We may construct a function $I: \{f | f \in \omega | f \sqsubset m\} \rightarrow \{g | g \in \omega | g \sqsubset n\}$
and a number $s \in \omega$ such that:

$$\forall f \sqsubset m \forall \alpha [(\bar{\alpha}s = \bar{\underline{0}}s \wedge A_f(\alpha)) \rightarrow A_{I(f)}(\delta|\alpha)]$$

We venture the following
<u>Claim</u>: $\exists t < lg(n) \forall f \sqsubset m \forall h \sqsubset m [f(0) \neq h(0) \rightarrow I(f)(t) \neq I(h)(t)]$
We prove this claim as follows:

Suppose, to the contrary: $\forall t < lg(n)\, \exists f \sqsubset m\, \exists h \sqsubset m\, [f(0) \neq h(0) \wedge (I(f))(t) = (I(h))(t)]$

In this difficult situation, we need our friend from 4.2:

$\tau = \{\alpha \mid \alpha \in {}^\omega\omega \mid \forall k[\alpha(k) < 2] \wedge \forall k \forall l[(\alpha(k) \neq 0 \wedge \alpha(l) \neq 0) \to k = l]\}$

With his help, we define a subset B of ${}^\omega\omega$:

$B := \{\alpha \mid \alpha \in {}^\omega\omega \mid \alpha^0 \in \tau \wedge \forall k > 0[\alpha^k = \underline{0}]$

We remark: $\forall \alpha \in B\, \forall f \sqsubset m\, \forall h \sqsubset m\, [f(0) \neq h(0) \to \neg\neg(A_f(\alpha) \vee A_h(\alpha))]$

therefore: $\forall \alpha \in B\, \forall t < lg(n)\, \exists f \sqsubset m\, [\bar{\alpha}s = \bar{\underline{0}}s \to ((\delta|\alpha)^t)^{(I(f))(t)} = \underline{0}]$

and: $\forall \alpha \in B\, [\bar{\alpha}s = \bar{\underline{0}}s \to (CD)_n A_1(\delta|\alpha)]$

so: $\forall \alpha \in B\, [\bar{\alpha}s = \bar{\underline{0}}s \to (CD)_m A_1(\alpha)]$

Also: $\forall \alpha \in B\, [\bar{\alpha}s \neq \bar{\underline{0}}s \to (CD)_m A_1(\alpha)]$

Therefore: $\forall \alpha \in B[(CD)_m A_1(\alpha)]$, especially: $\forall \alpha \in B[D^{m(0)} A_1(\alpha^0)]$, and: $\forall \alpha \in \tau\, [D^{m(0)} A_1(\alpha)]$. This is not so, as we have seen on several occasions (cf. the end of the proof of 4.15).

Our claim has been established, now, and the argument is constructive, although it does not appear so, because we are dealing with finite disjunctions and conjunctions of decidable propositions.

We calculate $t < lg(n)$ such that: $\forall f \sqsubset m\, \forall h \sqsubset m\, [f(0) \neq h(0) \to (I(f))(t) \neq (I(h))(t)]$

Remark that this implies: $m_0 \leq n_t$

We may profit, now, from our training in combinatorics (if we had any): We define a mapping on $\{p \mid p \in \omega \mid p < m_0\}$:

$$p \mapsto \{q \mid q \in \omega \mid \exists f \sqsubset m\, [f(0) = p \wedge (I(f))(t) = q]\}$$

To different numbers, disjoint decidable subsets of ω are associated. We determine p such that: $p < m_0$ and the number of elements of $\{q \mid q \in \omega \mid \exists f \sqsubset m\, [f(0) = p \wedge (I(f))(t) = q]\}$ is at most: $[\frac{n_t}{m_0}]$.

We define a subset E of ${}^\omega\omega$: $E := \{\alpha \mid \alpha \in {}^\omega\omega \mid (\alpha^0)^p = \underline{0}\}$

Without fear, we make a second <u>Claim</u>: we may construct $\zeta \in {}^\omega\omega$ such that: $Fun(\zeta)$ and: $\forall \alpha \in E\, [(CD)_m A_1(\alpha) \rightleftarrows (CD)_{\langle n_0, n_1, \ldots, [\frac{n_t}{m_0}], \ldots n_l \rangle} A_1(\zeta|\alpha)]$

We do not go into a detailed construction of ζ, but it should be clear that ζ may be obtained by a suitable rearrangement of δ.

Finally, we make $\eta \in {}^\omega\omega$ such that $Fun(\eta)$ and $\forall \alpha[(\eta|\alpha)^0 = \underline{0} \wedge \forall j[(\eta|\alpha)^{j+1} = \alpha^j]]$

Then: $\forall \alpha[\, \eta|\alpha \in E \wedge ((CD)_{\langle m_1, \ldots, m_k \rangle} A_1(\alpha) \rightleftarrows (CD)_m A_1(\eta|\alpha))]$

Putting all things together we see

$$\forall\alpha[(CD)_{\langle m_1,\dots,m_k\rangle}A_1(\alpha) \rightleftarrows (CD)_{\langle n_0,n_1,\dots,[\frac{n_t}{m_0}],\dots,n_\ell\rangle}A_1(\zeta|(\eta|\alpha))]$$

i.e.: $(CD)_{\langle m_1,\dots,m_k\rangle}A_1 \preceq CD_{\langle n_0,n_1,\dots,[\frac{n_t}{m_0}],\dots,n_\ell\rangle}A_1$

$\boxtimes$

4.21 Theorem 4.20 delivers us from many problems.
It provides us with an algorithm for the set $\{m \mid m\in\omega \mid lg(m)=2 \wedge (CD)_{m(0)}A_1 \preceq (CD)_{m(1)}A_1\}$
We refrain from a general formulation of this algorithm, and only calculate some special cases:

Suppose $C^3D^2A_1 \preceq C^2D^3A_1$; i.e.: $(CD)_{\langle 2,2,2\rangle}A_1 \preceq (CD)_{\langle 3,3,3\rangle}A_1$
then: $(CD)_{\langle 2,2\rangle}A_1 \preceq (CD)_{\langle 1,3\rangle}A_1$ and $4 \leq 3$: contradiction.

Suppose $C^3D^3A_1 \preceq C^2D^6A_1$; i.e.: $(CD)_{\langle 3,3,3\rangle}A_1 \preceq (CD)_{\langle 6,6\rangle}A_1$
then: $(CD)_{\langle 3,3\rangle}A_1 \preceq (CD)_{\langle 2,6\rangle}A_1$; then: $(CD)_{\langle 3\rangle}A_1 \preceq (CD)_{\langle 2,2\rangle}A_1$: contradiction – there is no entry in $\langle 2,2\rangle$ at least as big as 3.

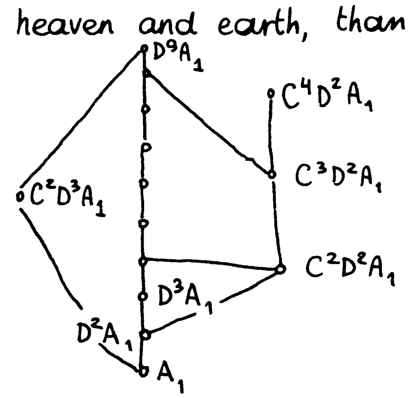
Suppose $C^5D^3A_1 \preceq C^4D^4A_1$; i.e.: $(CD)_{\langle 3,3,3,3,3\rangle}A_1 \preceq (CD)_{\langle 4,4,4\rangle}A_1$
then: $(CD)_{\langle 3,3,3,3\rangle}A_1 \preceq (CD)_{\langle 4,4,4,1\rangle}A_1$, and $81=3^4 \leq 4^3=64$: contradiction.

We may prove, inductively: $\forall m\forall n[C^mD^2A_1 \preceq C^nD^3A_1 \rightleftarrows m\leq n]$.

Theorem 4.20 is a very general statement, which embraces earlier results like theorem 4.18.

We might enter a new field of questions now, by forming „disjunctions" of sets $(CD)_nA_1$, and then again „conjunctions" of these new disjunctions, and so on. We could consider the class of all subsets of ${}^\omega\omega$ which are built from A_1 by a finite tree of disjunctions and conjunctions.

But we are getting tired and prefer to take the bus home.
There is such a choice of playthings here, we cannot go and try them all.
Many problems will be left alone, for, tomorrow, we are visiting another part of the country.
This is a pity, but there are more things in heaven and earth, than are dreamt of in chapter 4.



4.22 Before leaving, however, we buy and send a postcard to our dearest friend:

5. AN ASIDE ON IMPLICATION

We leave the main line of our discourse and look at some subsets of $^\omega\omega$ which are built from A_1 and E_1 by means of implication.
As we announced in the introduction, we do consider implication to be more mysterious and less well understood than disjunction or conjunction, and we try to build a hierarchy of subsets of $^\omega\omega$ without using it.
Someone might be inclined to say to this that logic really starts only when implication comes in.
This chapter offers him some consolation.
We first show how to erect, by repeated use of implication, some towers of subsets of $^\omega\omega$ of ever increasing complexity.
We then shortly discuss the difficult question of how to compare these new subsets with subsets of $^\omega\omega$ which are arithmetical in our restricted sense.

5.0 We define a sequence $I_0, I_1, \dots$ of subsets of $^\omega\omega$ by:

For every $\alpha \in {^\omega\omega}$: $\quad I_0(\alpha) := \quad 1=1$

For every $p \in \omega$:
For every $\alpha \in {^\omega\omega}$: $\quad I_{Sp}(\alpha) := \quad I_p(\alpha) \rightarrow A_1(\alpha^p)$

As usual, S denotes the successor function on ω.
I_4, for example, will turn out to be:

$$I_4(\alpha) := ((\alpha^0 = \underline{0} \rightarrow \alpha^1 = \underline{0}) \rightarrow \alpha^2 = \underline{0}) \rightarrow \alpha^3 = \underline{0}$$

5.1 <u>Theorem</u>: $\forall p [I_p \preceq I_{Sp}]$

<u>Proof</u>: Determine $\delta \in {^\omega\omega}$ such that $\mathrm{Fun}(\delta)$ and $\forall \alpha [(\delta|\alpha)^0 = \underline{0} \wedge \forall p[(\delta|\alpha)^{Sp} = \alpha^p]]$
Then: $\forall p \forall \alpha [\, I_p(\alpha) \rightleftarrows I_{Sp}(\delta|\alpha)]$.
⊠

As the reader may suspect, we are going to prove: $\forall p [\neg (I_{Sp} \preceq I_p)]$
We will do this inductively, and need some auxiliary concepts.

5.2 Let A be a subset of $^\omega\omega$. We define the subset $\mathrm{Neg}(A)$ of $^\omega\omega$ by:

For all $\alpha \in {^\omega\omega}$: $\quad \mathrm{Neg}(A)(\alpha) := \neg A(\alpha)$

Let A be a subset of $^\omega\omega$. A is called a <u>stable</u> subset of $^\omega\omega$ if:

$\mathrm{Neg}(\mathrm{Neg}(A)) = A$, i.e. $\forall \alpha [A(\alpha) \rightleftarrows \neg\neg A(\alpha)]$

5.3 <u>Lemma</u>: (without proof):

For all subsets A, B of $^\omega\omega$: If $A \preceq B$, then $\mathrm{Neg}(A) \preceq \mathrm{Neg}(B)$. And:

For all stable subsets A, B of ${}^\omega\omega$: If $Neg(A) \leq Neg(B)$, then $A \leq B$

5.4 <u>Lemma</u> : $\forall p\,[\,I_p$ is a stable subset of ${}^\omega\omega\,]$

<u>Proof</u> : It is a well-known fact from intuitionistic logic, that A_1 is a stable subset of ${}^\omega\omega$, and that the class of stable subsets of ${}^\omega\omega$ is closed under the operations of (conjunction and) implication.

$\boxtimes$

5.5 <u>Lemma</u> : $\forall p \forall q\,[\,I_{sp} \leq I_{sq} \rightarrow \neg\neg(Neg(I_p) \leq Neg(I_q))\,]$

<u>Proof</u>: Suppose $p, q \in \omega$ and $I_{sp} \leq I_{sq}$, i.e. $\forall\alpha\exists\beta\,[\,I_{sp}(\alpha) \rightleftarrows I_{sq}(\beta)\,]$
Using AC_{11}, determine $\delta \in {}^\omega\omega$ such that $Fun(\delta)$ and: $\forall\alpha\,[\,I_{sp}(\alpha) \rightleftarrows I_{sq}(\delta|\alpha)\,]$
Consider $\alpha_* \in {}^\omega\omega$, where α_* fulfils the conditions:

$$\forall j < p\,[\,\alpha_*^j = \underline{0}\,] \text{ and: } \alpha_*^p = \underline{1}$$

($\underline{1}$ is the sequence in ${}^\omega\omega$ which is defined by: $\forall n\,[\,\underline{1}(n) = 1\,]$)
Remark: $\neg I_{sp}(\alpha^*)$, therefore: $\neg I_{sq}(\delta|\alpha_*)$, and: $(\delta|\alpha_*)^q \neq \underline{0}$
Assume now, for the sake of argument: $\exists n\,[\,(\delta|\alpha_*)^q(n) \neq 0\,]$
Determine $n \in \omega$ such that: $(\delta|\alpha_*)^q(n) \neq 0$
(Both α_* and $\delta|\alpha_*$ now have a „useless" last subsequence, α_*^p, resp. $(\delta|\alpha_*)^q$. Keeping this in mind, one has no difficulty in finding the inductive step:)
Determine $l \in \omega$ such that: $\forall\alpha\,[\,\bar{\alpha}l = \bar{\alpha}_*l \longrightarrow (\delta|\alpha)^q(n) = (\delta|\alpha_*)^q(n)\,]$
(If we have to make α in ${}^\omega\omega$ satisfying: $\bar{\alpha}l = \bar{\alpha}_*l$, our options for the first p subsequences of α are almost open:)
Define $\eta \in {}^\omega\omega$ such that $Fun(\eta)$ and: for all $\alpha \in {}^\omega\omega$:

$$\forall j < p\,[\,(\eta|\alpha)^j = \bar{\underline{0}}l * \alpha^j\,] \text{ and: } (\eta|\alpha)^p = \underline{1}.$$

(For all $m \in \omega$ and $\alpha \in {}^\omega\omega$, $m * \alpha$ denotes the sequence in ${}^\omega\omega$ which one gets by concatenating the finite sequence coded by m and the infinite sequence α)
We have ensured that: $\forall\alpha\,[\,\overline{(\eta|\alpha)}l = \bar{\alpha}_*l\,]$ and: $\forall\alpha\,[\,(\delta|(\eta|\alpha))^q(n) \neq 0\,]$
Moreover, for all $\alpha \in {}^\omega\omega$:

$$\begin{aligned} (Neg(I_p))(\alpha) &\rightleftarrows \neg I_p(\alpha) \\ &\rightleftarrows I_{sp}(\eta|\alpha) \end{aligned}$$

$$\rightleftarrows \; I_{sq}\,(\delta|(\eta|\alpha))$$
$$\rightleftarrows \; \neg I_q\,(\delta|(\eta|\alpha))$$
$$\rightleftarrows \; (Neg(I_q))\,(\delta|(\eta|\alpha))$$

Therefore: $Neg(I_p) \preceq Neg(I_q)$

We reached this conclusion by assuming: $\exists n\,[(\delta|\alpha_*)^q(n) = 0]$

Therefore, from: $\neg\neg\,\exists n\,[(\delta|\alpha_*)^q(n) \neq 0]$ we may come to:

$\neg\neg(Neg(I_p) \preceq Neg(I_q))$.

☒

5.6 <u>Theorem</u>: $\forall p\,[I_p \prec I_{sp}]$

<u>Proof</u>: From 5.1: $\forall p\,[I_p \preceq I_{sp}]$

In order to prove: $\forall p\,[\neg(I_{sp} \preceq I_p)]$, we start from the obvious fact: $\neg(I_1 \preceq I_0)$, and proceed by induction, using lemmas 5.3-5

(Let us prove: $\neg(I_{ssp} \preceq I_{sp})$ from: $\neg(I_{sp} \preceq I_p)$

Suppose: $I_{ssp} \preceq I_{sp}$; then, by 5.5: $\neg\neg(Neg(I_{sp}) \preceq Neg(I_p))$, therefore, by 5.3 and 5.4: $\neg\neg(I_{sp} \preceq I_p)$. Contradiction)

☒

5.7 We define a sequence $J_0, J_1, \ldots$ of subsets of ${}^\omega\omega$ by:

For every $\alpha \in {}^\omega\omega$: $\quad J_0(\alpha) := 1 = 1$

For every $p \in \omega$,
for every $\alpha \in {}^\omega\omega$: $\quad J_{sp}(\alpha) := J_p(\alpha) \rightarrow E_1(\alpha^p)$

5.8 <u>Theorem</u>: $\forall p\,[J_p \preceq J_{ssp}]$

<u>Proof</u>: Like the proof of 5.1. Determine $\delta \in {}^\omega\omega$ such that $Fun(\delta)$ and $\forall \alpha\,[(\delta|\alpha)^0 = \underline{1} \wedge \forall p\,[(\delta|\alpha)^{sp} = \alpha^p]]$. Then: $\forall p\,\forall\alpha\,[J_p(\alpha) \rightleftarrows J_{sp}(\delta|\alpha)]$

☒

We want to prove now: $\forall p\,[\neg(J_{sp} \preceq J_p)]$, and, again, we will do so by induction.

5.9 <u>Lemma</u>: $\forall p\,[J_{ssp} \preceq J_{sp} \rightarrow \neg\neg\,\exists q < sp\,[J_{ssp} \preceq J_q]]$

<u>Proof</u>: Suppose $p \in \omega$ and $J_{ssp} \preceq J_{sp}$, i.e. $\forall\alpha\,\exists\beta\,[J_{ssp}(\alpha) \rightleftarrows J_{sp}(\beta)]$

Using AC_{11}, determine $\delta \in {}^\omega\omega$ such that $\mathrm{Fun}(\delta)$ and $\forall\alpha[J_{SSp}(\alpha) \rightleftarrows J_{Sp}(\delta|\alpha)]$

Observe: $J_{SSp}(\underline{1}) \rightleftarrows \neg J_{Sp}(\underline{1})$ and: $\neg(J_{SSp}(\underline{1}) \rightleftarrows J_{Sp}(\underline{1}))$

Therefore: $\delta|\underline{1} \neq \underline{1}$, and, to be more precise: $\neg\forall t<Sp\,\forall n[(\delta|\underline{1})^t(n) \neq 0]$

Assume now, for the sake of argument: $\exists t<Sp\,\exists n[(\delta|\underline{1})^t(n) = 0]$

Determine $t, n \in \omega$ such that $(\delta|\underline{1})^t(n) = 0$

Determine $\ell \in \omega$ such that: $\forall\alpha[\bar{\alpha}\ell = \bar{\underline{1}}\ell \to (\delta|\alpha)^t(n) = (\delta|\underline{1})^t(n)]$

Define $\eta \in {}^\omega\omega$ such that $\mathrm{Fun}(\eta)$, and, for all $\alpha \in {}^\omega\omega$:

$$\forall j<SSp\,[(\eta|\alpha)^j = \bar{\underline{1}}\ell * \alpha^j]$$

In this way, we ensure: $\forall\alpha[\overline{(\eta|\alpha)}\ell = \bar{\underline{1}}\ell]$ and: $\forall\alpha[(\delta|(\eta|\alpha))^t(n) = 0]$

Moreover, for all $\alpha \in {}^\omega\omega$:

$$\begin{aligned} J_{SSp}(\alpha) &\rightleftarrows J_{SSp}(\eta|\alpha) \\ &\rightleftarrows (J_{Sp}(\delta|(\eta|\alpha)) \wedge (\delta|(\eta|\alpha))^t(n) = 0) \\ &\rightleftarrows (\ldots(E_1((\delta|(\eta|\alpha))^{t+1}) \to E_1((\delta|(\eta|\alpha))^{t+2}) \ldots\ldots \to E_1((\delta|(\eta|\alpha))^p) \end{aligned}$$

Therefore: $J_{SSp} \preceq J_{p-t}$, and: $\exists q<Sp\,[J_{SSp} \preceq J_q]$

We reached this conclusion by assuming: $\exists t<Sp\,[(\delta|\underline{1})^t(n) = 0]$

But: $\neg\forall t<Sp\,\forall n[(\delta|\underline{1})^t(n) \neq 0]$, i.e. $\neg\neg\exists t<Sp\,\exists n[(\delta|\underline{1})^t(n) = 0]$

therefore: $\neg\neg\exists q<Sp\,[J_{SSp} \preceq J_q]$.

☒

5.9 <u>Lemma</u>: $\forall p[J_{SSp} \preceq J_{Sp} \to \neg\neg(J_{Sp} \preceq J_p)]$

<u>Proof</u>: Suppose $p \in \omega$ and $J_{SSp} \preceq J_{Sp}$. By 5.8, we know: $\neg\neg\exists q<Sp[J_{SSp} \preceq J_q]$ Assume, only for a moment: $\exists q<Sp[J_{SSp} \preceq J_q]$ and determine $q<Sp$ such that $J_{SSp} \preceq J_q$. Remark: $J_{Sp} \preceq J_{SSp} \preceq J_q \preceq J_p$, and: $J_{Sp} \preceq J_p$. Therefore, making no additional assumptions, we have: $\neg\neg(J_{Sp} \preceq J_p)$

☒

5.10 <u>Theorem</u>: $\forall p[J_p \prec J_{Sp}]$

<u>Proof</u>: From 5.8, we know: $\forall p[J_p \preceq J_{Sp}]$
In order to prove: $\forall p[\neg(J_{SSp} \preceq J_p)]$, we use induction, starting from the obvious fact: $\neg(J_1 \preceq J_0)$, and applying to 5.9 for the inductive step. The argument is similar to the argument for 5.6 and will not be given in detail. ☒

A classical spectator might guess that all participants in the two processions $I_0, I_1, \dots$ and $J_0, J_1, \dots$ are reducible to both A_2 and E_2.
Let us try and see if this is true.

5.11 Theorem: $J_2 \preceq A_2$

Proof: Note that, for all $\alpha \in {}^\omega\omega$:

$$\begin{aligned} J_2(\alpha) &\rightleftarrows (E_1(\alpha^0) \to E_1(\alpha^1)) \\ &\rightleftarrows (\exists n[\alpha^0(n)=0] \to \exists n[\alpha^1(n)=0]) \\ &\rightleftarrows \forall m[\alpha^0(m)=0 \to \exists n[\alpha^1(n)=0]] \\ &\rightleftarrows \forall m \exists n[\alpha^0(m)=0 \to \alpha^1(n)=0] \end{aligned}$$

Define $\delta \in {}^\omega\omega$ such that $Fun(\delta)$ and:
$\forall\alpha\forall m\forall n[\ (\delta|\alpha)^m(n)=0 \rightleftarrows (\alpha^0(m)=0 \to \alpha^1(n)=0)]$
Then: $\forall\alpha[J_2(\alpha) \rightleftarrows A_2(\delta|\alpha)]$, and: $J_2 \preceq A_2$

⊠

5.12 Theorem: $A_1 \preceq J_2$ and $E_1 \preceq J_2$

Proof: Define $\zeta \in {}^\omega\omega$ such that $Fun(\zeta)$ and $\forall\alpha[\forall n[(\zeta|\alpha)^0(n)=0 \rightleftarrows \alpha(n)\neq 0] \wedge (\zeta|\alpha)^1 = \underline{1}]$
Then: $\forall\alpha[A_1(\alpha) \rightleftarrows J_2(\zeta|\alpha)]$ and: $A_1 \preceq J_2$

Define $\eta \in {}^\omega\omega$ such that $Fun(\eta)$ and: $\forall\alpha[(\eta|\alpha)^0 = \underline{0} \wedge (\eta|\alpha)^1 = \alpha]$
Then: $\forall\alpha[E_1(\alpha) \rightleftarrows J_2(\eta|\alpha)]$ and: $E_1 \preceq J_2$

⊠

5.13 Theorem: $\neg(D^2A_1 \preceq J_2)$

Proof: Suppose: $D^2A_1 \preceq J_2$, i.e.: $\forall\alpha\exists\beta[D^2A_1(\alpha) \rightleftarrows J_2(\beta)]$, and, using AC_{11}, determine $\delta \in {}^\omega\omega$ such that $Fun(\delta)$ and: $\forall\alpha[D^2A_1(\alpha) \rightleftarrows J_2(\delta|\alpha)]$

We now dare to make the following claim:

$$\forall\alpha[D^2A_1(\alpha) \to (\neg E_1((\delta|\alpha)^1)]$$

For, suppose: $\alpha \in {}^\omega\omega$ and $D^2A_1(\alpha)$ and $E_1((\delta|\alpha)^1)$
Determine $n \in \omega$ such that $(\delta|\alpha)^1(n)=0$, and also $\ell \in \omega$ such that $\forall\beta[\bar\beta\ell = \bar\alpha\ell \to (\delta|\beta)^1(n)=0]$
Therefore: $\forall\beta[\bar\beta\ell = \bar\alpha\ell \to E_1((\delta|\beta)^1)]$, and: $\forall\beta[\bar\beta\ell = \bar\alpha\ell \to J_2(\delta|\beta)]$,
and: $\forall\beta[\bar\beta\ell = \bar\alpha\ell \to D^2A_1(\beta)]$.

As there are sequences like $\beta = \bar{\alpha}\ell * \underline{1}$, this is contradictory
We have proved now: $\forall\alpha[D^2A_1(\alpha) \to (\neg E_1((\delta|\alpha)^1))]$, and may conclude: $\forall\alpha[D^2A_1(\alpha) \rightleftarrows \neg E_1((\delta|\alpha)^0)]$, and: $\forall\alpha[D^2A_1(\alpha) \rightleftarrows \forall n[(\delta|\alpha)^0(n) \neq 0]]$
This would mean: $D^2A_1 \preceq A_1$, which we have refuted in theorem 4.3

$\boxtimes$

5.14 Theorem: $\neg(A_2 \preceq J_2)$

Proof: Suppose: $A_2 \preceq J_2$, i.e.: $\forall\alpha\exists\beta[A_2(\alpha) \rightleftarrows J_2(\beta)]$, and, using AC_{11}, determine $\delta \in {}^\omega\omega$ such that $Fun(\delta)$ and: $\forall\alpha[A_2(\alpha) \rightleftarrows J_2(\delta|\alpha)]$.
The proof now proceeds like the proof of theorem 5.13
We first remark that: $\forall\alpha[A_2(\alpha) \to \neg E_1((\delta|\alpha)^1)]$, and then conclude: $A_2 \preceq A_1$, which has been refuted in chapter 3.

$\boxtimes$

As $D^2A_1 \preceq E_2$ and $\neg(D^2A_1 \preceq J_2)$, also: $\neg(E_2 \preceq J_2)$. Actually, E_2 and J_2 are incomparable:

5.15 Theorem: $\neg(J_2 \preceq E_2)$

Proof: This result reinforces theorem 3.2 and is proved in a similar way.
Remark that, for all $\alpha \in {}^\omega\omega$:

$$J_2(\alpha) \rightleftarrows \forall m\exists n[\alpha^0(m)=0 \to \alpha^1(n)=0]$$
$$\rightleftarrows \exists\gamma\forall m[\alpha^0(m)=0 \to (\exists n \le m[\alpha^1(n)=0] \vee \alpha^1(\gamma(m))=0)]$$

For all $\gamma \in {}^\omega\omega$ and $\alpha \in {}^\omega\omega$ we define $\gamma \otimes \alpha$ in ${}^\omega\omega$ by:

For all $n, t \in \omega$:

$$(\gamma\otimes\alpha)^t(n) := \alpha^t(n) \quad \text{if } t \neq 1$$
$$(\gamma\otimes\alpha)^1(n) := 0 \quad \text{if } (\exists m < n[\gamma(m) \le n \wedge \alpha^0(m)=0] \text{ and: } \neg\exists m < n[(\gamma\otimes\alpha)^1(m)=0])$$
$$:= \alpha^1(n) \quad \text{otherwise}$$
$$(\gamma\otimes\alpha)(0) := \alpha(0)$$

(The definition of $(\gamma\otimes\alpha)^1$ apparently goes by induction).

We observe: $\forall\alpha[J_2(\alpha) \rightleftarrows \exists\gamma[\alpha = \gamma\otimes\alpha]]$
Now suppose: $J_2 \preceq E_2$, i.e.: $\forall\alpha\exists\beta[J_2(\alpha) \rightleftarrows E_2(\beta)]$, and,

using AC_{11}, determine $\delta \in {}^\omega\omega$ such that $Fun(\delta)$ and: $\forall\alpha [J_2(\alpha) \rightleftarrows E_2(\delta|\alpha)]$

Remark: $\forall\alpha \forall\gamma [E_2(\delta|\gamma\otimes\alpha)]$, i.e. $\forall\alpha\forall\gamma\exists m [A_1((\delta|\gamma\otimes\alpha)^m)]$.

Using CP, determine $m\in\omega$, $p\in\omega$, $q\in\omega$ such that

$$\forall\gamma\forall\alpha [(\bar\gamma p = \bar{\underline{0}}p \wedge \bar\alpha q = \bar{\underline{1}}q) \rightarrow A_1((\delta|\gamma\otimes\alpha)^m)]$$

(The creative subject, still musing upon his exploits in chapter 3, now has a possibility of reviving his old glories).

Calculate $r := max(p,q)$ and define a special sequence α^* in ${}^\omega\omega$ such that: $(\alpha^*)^0 = \bar{\underline{1}}r * \underline{0}$ and $(\alpha^*)^1 = \underline{1}$ and $\overline{\alpha^*}r = \bar{\underline{1}}r$.

(Not suppressing a sober smile, the creative subject points to the following facts :)

Now: $\neg(J_2(\alpha^*))$ and: $A_1((\delta|\alpha^*)^m)$

> For, suppose $n\in\omega$ and $(\delta|\alpha^*)^m(n)\neq 0$.
> Determine $l\in\omega$ such that: $\forall\alpha[\bar\alpha l = \overline{\alpha^*}l \to (\delta|\alpha)^m(n) = (\delta|\alpha^*)^m(n)]$
> Determine a special sequence β in ${}^\omega\omega$ such that $\bar\beta l = \overline{\alpha^*}l$ and: $\overline{\beta^0}r = \overline{\beta^1}r = \bar\beta r = \bar{\underline{1}}r$ and: $E_1(\beta^0)$ and: $E_1(\beta^1)$
> Remark that: $J_2(\beta)$, and, what is more: $\exists\gamma[\bar\gamma p = \bar{\underline{0}}p \wedge \beta = \gamma\otimes\beta]$
> From this, and: $\bar\beta q = \bar{\underline{1}}q$, we infer: $A_1((\delta|\beta)^m)$, whereas, from: $\bar\beta l = \overline{\alpha^*}l$ we know: $(\delta|\beta)^m(n)\neq 0$. Contradiction.

Therefore: $\neg J_2(\alpha^*)$ and: $E_2(\delta|\alpha^*)$.

(The imitative subject bows his head and goes his way in silence).

$\boxtimes$

This proof tempts us to pause and reflect a little.
It seems that the distinction we proposed to make in 4.12 between „strong" results, which are backed up by solid classical reality, and „weak" results, characteristic of the subtle spirit of intuitionism, is not tenable, since, if the logic were classical, J_2 would be reducible to E_2, and theorem 5.15 refutes this in the strongest possible way.
A second remark is, that it is the same analysis of the true nature of J_2, which, on the one hand, makes one see that it is reducible to A_2, and, on the other hand, that it is not reducible to E_2. One cannot have it both ways.
Thirdly, as a special case of theorem 5.15, we have that the following statement leads to a contradiction:

$$\forall\alpha\forall\beta[(\exists n[\alpha(n)=0] \to \exists n[\beta(n)=0]) \to (\forall n[\alpha(n)\neq 0] \vee \exists n[\beta(n)=0])]$$

This need not surprise, because, if we put $\alpha=\beta$ in this formula, we see that it entails: $\forall\alpha[\forall n[\alpha(n)\neq 0] \vee \exists n[\alpha(n)=0]]$, which, by CP, is obviously untrue

We now turn to the task of comparing I_2, the subset of ${}^\omega\omega$ which we introduced in 5.0, with some other subsets of ${}^\omega\omega$.
Remember that $\forall\alpha[\, I_2(\alpha) \rightleftarrows (\forall n[\alpha^0(n)=0] \rightarrow \forall n[\alpha^1(n)=0])]$
An implication whose antecedens is universal, is less accessible to understanding than an implication whose antecedens is existential.
Whereas we observe at a glance: $A_1 \leq I_1 \leq I_2$, in studying the question of whether E_1 is reducible to I_2, we run up with a deep riddle of intuitionistic analysis.

Consider the statement: $\forall\alpha[\neg\neg\exists n[\alpha(n)=0] \rightarrow \exists n[\alpha(n)=0]]$.
This stands for a very reckless assumption, indeed.
If we should accept it together with the restricted Brouwer-Kripke-axiom, (cf. 2.1), we would be able to decide any determinate proposition, and, probably, would be asked more questions than we are now.

> (Let $\mathfrak{A}$ be a determinate proposition.
> Then: $\mathfrak{A} \vee \neg\mathfrak{A}$ is also a determinate proposition, and we may construct $\alpha \in {}^\omega\omega$ such that: $\mathfrak{A} \vee \neg\mathfrak{A} \rightleftarrows \exists n[\alpha(n)=0]$
> As $\neg\neg(\mathfrak{A} \vee \neg\mathfrak{A})$, also: $\neg\neg\exists n[\alpha(n)=0]$, therefore: $\exists n[\alpha(n)=0]$, and: $\mathfrak{A} \vee \neg\mathfrak{A}$).

Nevertheless, we are not able to prove this statement to be contradictory.
Brouwer himself once stumbled at this stone, using an unrestricted Brouwer-Kripke-axiom in order to get absurdity.

> (It is not difficult to guess how he does this.
> As now any proposition, not only a determinate one, may be assumed to be decidable, we have, for instance: $\forall\gamma[\gamma=\underline{0} \vee \neg(\gamma=\underline{0})]$, which, with help of CP, leads to a contradiction).

In the following we call: $\forall\alpha[\neg\neg\exists n[\alpha(n)=0] \rightarrow \exists n[\alpha(n)=0]]$ an <u>enigma</u>, and we reserve the same title for any proposition which we can prove to be equivalent to it.

5.16 <u>Theorem</u>: „$E_1 \leq Neg(Neg(E_1))$" is an enigma

<u>Proof</u>: Suppose: $\forall\alpha[\neg\neg\exists n[\alpha(n)=0] \rightarrow \exists n[\alpha(n)=0]]$
then: $\forall\alpha[\exists n[\alpha(n)=0] \rightleftarrows \neg\neg\exists n[\alpha(n)=0]]$, and: $E_1 \leq Neg(Neg(E_1))$

Now suppose: $E_1 \leq Neg(Neg(E_1))$, i.e: $\forall\alpha\exists\beta[E_1(\alpha) \rightleftarrows \neg\neg E_1(\beta)]$.
Let $\alpha \in {}^\omega\omega$ and assume $\neg\neg E_1(\alpha)$. Determine $\beta \in {}^\omega\omega$ such that: $E_1(\alpha) \rightleftarrows \neg\neg E_1(\beta)$. Then: $\neg\neg E_1(\beta)$, and: $E_1(\alpha)$
Therefore: $\forall\alpha[\neg\neg\exists n[\alpha(n)=0] \rightarrow \exists n[\alpha(n)=0]]$

$\boxtimes$

Remark that, in this proof, we did not have recourse to AC_{11}.

5.17 <u>Theorem</u>: „ $Neg(Neg(E_1)) \preceq E_1$" is an enigma

<u>Proof</u>: Suppose: $\forall\alpha[\neg\neg\exists n[\alpha(n)=0] \to \exists n[\alpha(n)=0]]$
then: $\forall\alpha[\neg\neg\exists n[\alpha(n)=0] \rightleftarrows \exists n[\alpha(n)=0]]$, and: $Neg(Neg(E_1)) \preceq E_1$

Now suppose: $Neg(Neg(E_1)) \preceq E_1$, i.e.: $\forall\alpha\exists\beta[\neg\neg E_1(\alpha) \rightleftarrows E_1(\beta)]$, and, using AC_{11}, determine $\delta \in {}^\omega\omega$ such that: $Fun(\delta)$ and:
$\forall\alpha[\neg\neg E_1(\alpha) \rightleftarrows E_1(\delta|\alpha)]$

Let $\alpha \in {}^\omega\omega$ and assume $\neg\neg E_1(\alpha)$; then $E_1(\delta|\alpha)$. Calculate $n \in \omega$ such that: $(\delta|\alpha)(n)=0$ and determine $\ell \in \omega$ such that:
$\forall\beta[\bar\beta\ell = \bar\alpha\ell \to (\delta|\beta)(n) = (\delta|\alpha)(n)]$. Consider $\alpha^* := \bar\alpha\ell * \underline{1}$, and remark: $(\delta|\alpha^*)(n)=0$, therefore: $\neg\neg E_1(\alpha^*)$, and: $\exists m<\ell[\alpha^*(m)=\alpha(m)=0]$
i.e.: $E_1(\alpha)$.
We proved: $\forall\alpha[\neg\neg\exists n[\alpha(n)=0] \to \exists n[\alpha(n)=0]]$.

⊠

5.18 <u>Theorem</u>: „ $E_1 \preceq I_2$" is an enigma.

<u>Proof</u>: Define $\zeta \in {}^\omega\omega$ such that $Fun(\zeta)$ and: $\forall\alpha\forall n[(\zeta|\alpha)(n)=0 \rightleftarrows \alpha(n)\neq 0]$
Then: $\forall\alpha[(Neg(E_1))(\alpha) \rightleftarrows A_1(\zeta|\alpha)]$, and: $Neg(E_1) \preceq A_1$
Define $\eta \in {}^\omega\omega$ such that $Fun(\eta)$ and: $\forall\alpha[(\zeta|\alpha)^0=\alpha \wedge (\zeta|\alpha)^1=\underline{1}]$
Then: $\forall\alpha[(Neg(A_1))(\alpha) \rightleftarrows I_2(\eta|\alpha)]$, and: $Neg(A_1) \preceq I_2$
Therefore: $Neg(Neg(E_1)) \preceq Neg(A_1) \preceq I_2$ and: $Neg(Neg(E_1)) \preceq I_2$.

Suppose: $\forall\alpha[\neg\neg\exists n[\alpha(n)=0] \to \exists n[\alpha(n)=0]]$, then, according to theorem 5.16: $E_1 \preceq Neg(Neg(E_1))$ and, consequently: $E_1 \preceq I_2$

Conversely, suppose: $E_1 \preceq I_2$, i.e.: $\forall\alpha\exists\beta[E_1(\alpha) \rightleftarrows I_2(\beta)]$
Let $\alpha \in {}^\omega\omega$ and assume: $\neg\neg E_1(\alpha)$. Determine $\beta \in {}^\omega\omega$ such that: $E_1(\alpha) \rightleftarrows I_2(\beta)$. Then: $\neg\neg I_2(\beta)$, and, as we noted in lemma 5.4: $I_2(\beta)$.
Therefore: $E_1(\alpha)$.
We proved: $\forall\alpha[\neg\neg\exists n[\alpha(n)=0] \to \exists n[\alpha(n)=0]]$.

⊠

AC_{11} has been circumvented once more.

5.19 Theorem: „$J_2 \preceq I_2$" is an enigma

Proof: Suppose: $\forall\alpha[\neg\neg\exists n[\alpha(n)=0] \to \exists n[\alpha(n)=0]]$

Then: $\forall\alpha[(E_1(\alpha^0) \to E_1(\alpha^1)) \rightleftarrows (\neg\neg E_1(\alpha^0) \to \neg\neg E_1(\alpha^1))]$

and: $\forall\alpha[(E_1(\alpha^0) \to E_1(\alpha^1)) \rightleftarrows (\neg E_1(\alpha^1) \to \neg E_1(\alpha^0))]$

As, obviously, $Neg(E_1) \preceq A_1$, we conclude: $J_2 \preceq I_2$

Now, assume: $J_2 \preceq I_2$, remember from 5.12: $E_1 \preceq J_2$, therefore: $E_1 \preceq I_2$, and, according to 5.18: $\forall\alpha[\neg\neg\exists n[\alpha(n)=0] \to \exists n[\alpha(n)=0]]$.

⊠

5.20 Theorem: $\neg(I_2 \preceq E_2)$

Proof. We prove this by re-examining the proof of theorem 5.15

The argument given there may be seen to show the following:

For all $\delta \in {}^{\omega}\omega$, if $Fun(\delta)$ and $\forall\alpha[J_2(\alpha) \to E_2(\delta|\alpha)]$,

there is $\alpha^* \in {}^{\omega}\omega$ such that $\neg J_2(\alpha^*)$ and $E_2(\delta|\alpha^*)$

Now, assume $I_2 \preceq E_2$, i.e.: $\forall\alpha\exists\beta[I_2(\alpha) \rightleftarrows E_2(\beta)]$, and, using AC_{11}, determine $\delta \in {}^{\omega}\omega$ such that $Fun(\delta)$ and: $\forall\alpha[I_2(\alpha) \rightleftarrows E_2(\delta|\alpha)]$

Remark that $\forall\alpha[(\exists n[\alpha^1(n)\neq 0] \to \exists n[\alpha^0(n)\neq 0]) \to (\forall n[\alpha^0(n)=0] \to \forall n[\alpha^1(n)=0])]$

i.e.: $\forall\alpha[(\exists n[\alpha^1(n)\neq 0] \to \exists n[\alpha^0(n)\neq 0]) \to I_2(\alpha)]$

Therefore: $\forall\alpha[(\exists n[\alpha^1(n)\neq 0] \to \exists n[\alpha^0(n)\neq 0]) \to E_2(\delta|\alpha)]$

As in the proof of theorem 5.15, we may construct α^* in ${}^{\omega}\omega$ such that: $\neg(\exists n[(\alpha^*)^1(n)\neq 0] \to \exists n[(\alpha^*)^0(n)\neq 0]$ and: $E_2(\delta|\alpha^*)$

i.e.: $\neg\neg\exists n[(\alpha^*)^0(n)\neq 0] \wedge \forall n[(\alpha^*)^0(n)=0])$ and: $E_2(\delta|\alpha^*)$

and: $\neg I_2(\alpha^*)$ and: $E_2(\delta|\alpha^*)$.

This is the required contradiction.

⊠

We are approaching, now, the limits of our knowledge. Questions like „$I_2 \preceq J_2$" or „$I_2 \preceq A_2$" also have a ring of improbability but seem to belong to a different level of mysteriousness than their predecessors.
We do not pursue this line of research any further.
We do not see a reason why these annoying enigmas are true, and, therefore, we do not want to make axioms of them, although, such things are sometimes done, if only by way of experiment (cf. Troelstra 1973).

We want to conclude this chapter by a short comment on the subsets P and Q of ${}^{\omega}\omega$, which are defined by:

For all $\alpha \in {}^{\omega}\omega$:

$$P(\alpha) \quad := \quad E_1(\alpha^0) \longrightarrow A_1(\alpha^1)$$

$$Q(\alpha) \quad := \quad A_1(\alpha^0) \rightarrow E_1(\alpha^1)$$

We leave it to the reader to verify: $P \preceq A_1$. In contrast to this, we have

5.21 Theorem: „$Q \preceq E_1$" is an enigma

Proof: Suppose first: $\forall\alpha[\neg\neg\exists n[\alpha(n)=0] \rightarrow \exists n[\alpha(n)=0]]$
Under this assumption, for every $\alpha \in {}^{\omega}\omega$ the following holds:

$$\begin{aligned} Q(\alpha) \quad &\rightleftarrows \quad A_1(\alpha^0) \rightarrow E_1(\alpha^1) \\ &\rightleftarrows \quad A_1(\alpha^0) \rightarrow \neg\neg E_1(\alpha^1) \\ &\rightleftarrows \quad \neg\neg(A_1(\alpha^0) \rightarrow E_1(\alpha^1)) \\ &\rightleftarrows \quad \neg\neg\exists n[\alpha^0(n) \neq 0 \vee \alpha^1(n) = 0] \\ &\rightleftarrows \quad \exists n[\alpha^0(n) \neq 0 \vee \alpha^1(n) = 0] \end{aligned}$$

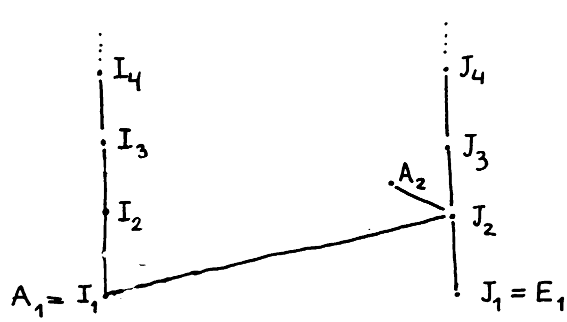
From this, we may conclude: $Q \preceq E_1$

Now assume: $Q \preceq E_1$. By an argument similar to the one given in theorem 5.18: $Neg(Neg(E_1)) \preceq Q$, and, therefore, $Neg(Neg(E_1)) \preceq E_1$. According to theorem 5.17, this implies: $\forall\alpha[\neg\neg\exists n[\alpha(n)=0] \rightarrow \exists n[\alpha(n)=0]]$

$\boxtimes$

We should be careful, in future, not to get entangled in this web of mysteries, but occasionally, and especially in chapter 10 and in the last chapter, we will have to refer to it.

The following picture summarizes the positive results of this chapter:



6. ARITHMETICAL SETS INTRODUCED.

Having plodded heavily through the last pages of chapter 5 where we saw much that we did not really understand, we now enter a glade where simplicity reigns and the sun is shining.
The class of all subsets of ${}^{\omega}\omega$ which are reducible to E_1, is introduced here and baptized Σ^0_1.
Likewise Π^0_1 appears, the class of all subsets of ${}^{\omega}\omega$ which are reducible to A_1.
We verify that these classes behave as one should expect.
Both of them contain a <u>universal element</u>
The other classes of the arithmetical hierarchy, $\Sigma^0_2, \Pi^0_2, \Sigma^0_3, \Pi^0_3, \ldots$ are introduced in a straightforward way, and turn out to behave properly.
A short discussion explains why the diagonal argument does not prove that each of these classes is properly included in one of the following classes.

6.0 We define DEC to be the following class of subsets of ω:

$$\mathrm{DEC} := \{A \mid A \subseteq \omega \mid \forall n[n \in A \vee \neg(n \in A)]\}$$

(Members of DEC are called: <u>decidable</u> subsets of ω).

One might frown at this notion, as we do not have, in intuitionism, a set of all subsets of ω.
But with the help of AC_{00} we can get it into our grasp.
We may remark:

For all subsets A of ω:

If $A \in \mathrm{DEC}$, then $\exists \alpha \forall n[n \in A \rightleftarrows \alpha(n) = 0]$

and: If $\exists \alpha \forall n[n \in A \rightleftarrows \alpha(n) = 0]$, then $A \in \mathrm{DEC}$

We have every reason to recognize DEC, as soon as we accept ${}^{\omega}\omega$, or, for that matter, σ_2 $(:= \{\alpha \mid \alpha \in {}^{\omega}\omega \mid \forall n[\alpha(n) = 0 \vee \alpha(n) = 1]\})$

6.1 We define Σ^0_1 to be the following class of subsets of ${}^{\omega}\omega$:

$$\Sigma^0_1 := \{P \mid P \subseteq {}^{\omega}\omega \mid P \leq E_1\}$$

Once more, one might feel inclined to object. We are very far, indeed from surveying all possible subsets of ${}^{\omega}\omega$.
However, as in the case of DEC, we will be able to reassure ourselves, in a moment.

6.2 <u>Theorem</u>: Let $P \subseteq {}^{\omega}\omega$

$P \in \Sigma^0_1$ if and only if there exists a decidable subset A of ω such that $\forall \alpha[P(\alpha) \rightleftarrows \exists m[\bar{\alpha}m \in A]]$

Proof: (i) Suppose $P \leq E_1$, i.e.: $\forall\alpha\exists\beta[P(\alpha) \rightleftarrows E_1(\beta)]$. Using AC_{11}, determine $\delta \in {}^\omega\omega$ such that: $Fun(\delta)$ and: $\forall\alpha[P(\alpha) \rightleftarrows E_1(\delta|\alpha)]$

Define a decidable subset A of ω by:

For all $b \in \omega$:

$$b \in A \rightleftarrows \exists m[m \leq lg(b) \wedge \exists a[b \subseteq a \wedge \delta^m(a) = 1 \wedge \forall c[a \subseteq c \wedge a \neq c \to \delta^m(c) = 0]]]$$

Now, $\forall\alpha[\ \exists m[(\delta|\alpha)(m) = 0] \rightleftarrows \exists n[\bar{\alpha}n \in A]]$

Therefore, A fulfils the requirements.

(ii) Let A be a decidable subset of ω such that: $\forall\alpha[P(\alpha) \rightleftarrows \exists m[\bar{\alpha}m \in A]]$

Determine $\delta \in {}^\omega\omega$ such that $Fun(\delta)$ and:

For all $\alpha \in {}^\omega\omega$ and $m \in \omega$

$$\begin{aligned} (\delta|\alpha)(m) &:= 0 && \text{if } \bar{\alpha}m \in A \\ &:= 1 && \text{otherwise} \end{aligned}$$

Remark: $\forall\alpha[P(\alpha) \rightleftarrows E_1(\delta|\alpha)]$, therefore $P \leq E_1$

☒

6.3 Theorem: (i) Let P and Q be subsets of ${}^\omega\omega$.
If $P \in \Sigma^0_1$ and $Q \in \Sigma^0_1$, then $P \cap Q \in \Sigma^0_1$.

(ii) Let $P_0, P_1, P_2, \ldots$ be a sequence of subsets of ${}^\omega\omega$
If $\forall n[P_n \in \Sigma^0_1]$, then $\bigcup_{n\in\omega} P_n \in \Sigma^0_1$.

Proof (i) Using the foregoing theorem, determine decidable subsets A and B of ω, such that: $\forall\alpha[P(\alpha) \rightleftarrows \exists m[\bar{\alpha}m \in A]]$ and: $\forall\alpha[Q(\alpha) \rightleftarrows \exists m[\bar{\alpha}m \in B]]$

Define a subset C of ω by:

For all $b \in \omega$:

$$b \in C \rightleftarrows \exists p \exists q[b \subseteq p \wedge b \subseteq q \wedge p \in A \wedge q \in B]$$

Now: $\forall b[b \in C \vee \neg(b \in C)]$ and: $\forall\alpha[(P(\alpha) \wedge Q(\alpha)) \rightleftarrows \exists m[\bar{\alpha}m \in C]]$

Therefore: $P \cap Q \in \Sigma^0_1$.

(ii) Using the foregoing theorem, determine a sequence $A_0, A_1, \ldots$ of decidable subsets of ω, such that: $\forall n\forall\alpha[P_n(\alpha) \rightleftarrows \exists m[\bar{\alpha}m \in A_n]]$

Define a subset A of ω by:

For all $b \in \omega$:

$$b \in A \rightleftarrows \exists n \exists p[n \leq lg(b) \wedge b \subseteq p \wedge p \in A_n]$$

Then: $\forall b[b \in A \vee \neg(b \in A)]$ and: $\forall\alpha[\exists n[P_n(\alpha)] \rightleftarrows \exists m[\bar{\alpha}m \in A]]$

Therefore: $\bigcup_{n\in\omega} P_n \in \Sigma_1^0$.

⊠

We know, from theorem 3.2, that Σ_1^0 is not closed under the operation of countable intersection.

We need a pairing function on ${}^\omega\omega$
In order to spare technical notions, we use our coding of finite sequences of natural numbers (cf. 1.2) and define $\langle\ \rangle : {}^\omega\omega \times {}^\omega\omega \to {}^\omega\omega$ by:

For all $\alpha, \beta \in {}^\omega\omega$:

$$\langle\alpha,\beta\rangle^0 := \alpha \quad \text{and} \quad \langle\alpha,\beta\rangle^1 := \beta \quad \text{and} \quad \forall n[n > 1 \to \langle\alpha,\beta\rangle^n = \underline{0}]$$

$$\text{and} \quad \langle\alpha,\beta\rangle(\langle\,\rangle) := 0$$

This function has the disadvantage of not being surjective, but this will not do any harm.

6.4 Definition: Let $\mathbb{K}$ be a class of subsets of ${}^\omega\omega$ and $\mathcal{U}$ be a member of $\mathbb{K}$.
$\mathcal{U}$ is called a universal element of $\mathbb{K}$, if we are able to prove:

Let $P \subseteq {}^\omega\omega$

If $P \in \mathbb{K}$, then $\exists\beta\forall\alpha[P(\alpha) \rightleftarrows \mathcal{U}(\langle\alpha,\beta\rangle)]$

The careful wording of this definition is to make it apply even in cases where we do not yet know that $\mathbb{K}$ may be viewed as a set.

6.5 Theorem: Σ_1^0 contains a universal element.

Proof: Define the subset $\mathcal{U}$ of ${}^\omega\omega$ by:

For all $\alpha \in {}^\omega\omega$: $\mathcal{U}(\alpha) \rightleftarrows \exists m[\alpha^1(\overline{\alpha^0}m) = 0]$

and note that $\mathcal{U}$ belongs to Σ_1^0

Let $P \subseteq {}^\omega\omega$ and $P \in \Sigma_1^0$

Following theorem 6.2, determine a decidable subset A of ω such that: $\forall\alpha[P(\alpha) \rightleftarrows \exists m[\overline{\alpha}m \in A]]$. Determine $\beta \in {}^\omega\omega$ such that: $\forall n[\beta(n) = 0 \rightleftarrows n \in A]$

Then: $\forall\alpha[P(\alpha) \rightleftarrows \exists m[\beta(\overline{\alpha}m) = 0]]$, i.e.: $\forall\alpha[P(\alpha) \rightleftarrows \mathcal{U}(\langle\alpha,\beta\rangle)]$.

⊠

We are itching to diagonalize.
Consider the subset $\mathcal{U}_0^\#$ of ${}^\omega\omega$ which is defined by:

For all $\alpha \in {}^\omega\omega$: $\mathcal{U}_0^\#(\alpha) := \forall m[\alpha(\overline{\alpha}m) \neq 0]$

One easily verifies, using theorem 6.5.: $\mathcal{U}_0^\# \notin \Sigma_1^0$.

As $\mathcal{U}_0^{\#} \leq A_1$, this confirms theorem 2.0, which said that $\neg(A_1 \leq E_1)$.

6.6 We define Π_1^0 to be the following class of subsets of ${}^\omega\omega$:

$$\Pi_1^0 := \{ P \mid P \subseteq {}^\omega\omega \mid P \leq A_1 \}$$

Like Σ_1^0, this class is manageable:

6.7 Theorem: Let $P \subseteq {}^\omega\omega$
$P \in \Pi_1^0$ if and only if there exists a decidable subset A of ω such that $\forall\alpha[P(\alpha) \rightleftarrows \forall m[\bar{\alpha}m \in A]]$.

Proof: (i) Suppose $P \leq A_1$, i.e.: $\forall\alpha\exists\beta[P(\alpha) \rightleftarrows A_1(\beta)]$. Using AC_{11}, determine $\delta \in {}^\omega\omega$ such that: $Fun(\delta)$ and: $\forall\alpha[P(\alpha) \rightleftarrows A_1(\delta|\alpha)]$

Define a decidable subset A of ω by:

For all $b \in \omega$:

$$b \in A \rightleftarrows \forall m \forall a[(m \leq lg(b) \wedge b \subseteq a \wedge \delta^m(a) \neq 0 \wedge \forall c[(a \subseteq c \wedge a \neq c) \to \delta^m(c) = 0]) \to \delta^m(a) = 1]$$

Now, $\forall\alpha[\ \forall n[(\delta|\alpha)(n) = 0] \rightleftarrows \forall m[\bar{\alpha}m \in A]]$

Therefore, A fulfils the requirements.

(ii) Let A be a decidable subset of ω such that: $\forall\alpha[P(\alpha) \rightleftarrows \forall m[\bar{\alpha}m \in A]]$

Determine $\delta \in {}^\omega\omega$ such that $Fun(\delta)$ and:

For all $\alpha \in {}^\omega\omega$ and $m \in \omega$:

$$\begin{aligned}(\delta|\alpha)(m) &:= 0 \quad \text{if } \bar{\alpha}m \in A \\ &:= 1 \quad \text{if } \bar{\alpha}m \notin A\end{aligned}$$

Remark: $\forall\alpha[P(\alpha) \rightleftarrows A_1(\delta|\alpha)]$, therefore $P \leq A_1$.

$\boxtimes$

6.8 Theorem: Let $P_0, P_1, P_2, \dots$ be a sequence of subsets of ${}^\omega\omega$.
If $\forall n[P_n \in \Pi_1^0]$, then $\bigcap_{n \in \omega} P_n \in \Pi_1^0$.

Proof: Using the foregoing theorem, determine a sequence $A_0, A_1, A_2, \dots$ of decidable subsets of ω, such that: $\forall n \forall\alpha[P_n(\alpha) \rightleftarrows \forall m[\bar{\alpha}m \in A_n]]$

Define a subset A of ω by:

For all $b \in \omega$:

$$b \in A \rightleftarrows \forall m \forall a[b \subseteq a \wedge m \leq lg(b) \to a \in A_m]$$

Then: $\forall b[b \in A \vee \neg(b \in A)]$ and: $\forall\alpha[\forall n[P_n(\alpha)] \rightleftarrows \forall m[\bar{\alpha}m \in A]]$

Therefore: $\bigcap_{n \in \omega} P_n \preceq A_1$.

$\boxtimes$

We know, from theorem 4.3, that $\{\alpha \mid \alpha \in {}^{\omega}\omega \mid \alpha^0 = \underline{0}\} \cup \{\alpha \mid \alpha \in {}^{\omega}\omega \mid \alpha^0 = \underline{1}\}$ does not belong to Π^0_1, and, hence, that it may occur that a union of Π^0_1-sets is not a Π^0_1-set.

6.9 <u>Theorem</u>: Π^0_1 contains a universal element.

<u>Proof</u>: Define the subset $\mathcal{U}$ of ${}^{\omega}\omega$ by:

For all $\alpha \in {}^{\omega}\omega$: $\mathcal{U}(\alpha) \rightleftarrows \forall m[\alpha^1(\bar{\alpha}^0 m) = 0]$

and note that $\mathcal{U}$ belongs to Π^0_1.

Let $P \subseteq {}^{\omega}\omega$ and $P \in \Pi^0_1$.

Following theorem 6.7, determine a decidable subset A of ω such that: $\forall\alpha[P(\alpha) \rightleftarrows \forall m[\bar{\alpha}m \in A]]$. Determine $\beta \in {}^{\omega}\omega$ such that: $\forall n[\beta(n) = 0 \rightleftarrows n \in A]$.

Then: $\forall\alpha[P(\alpha) \rightleftarrows \forall m[\beta(\bar{\alpha}m) = 0]]$, i.e.: $\forall\alpha[P(\alpha) \rightleftarrows \mathcal{U}(\langle\alpha,\beta\rangle)]$.

$\boxtimes$

Let us try and diagonalize once more.
Consider the subset $\mathcal{U}_1^{\#}$ of ${}^{\omega}\omega$ which is defined by:

For all $\alpha \in {}^{\omega}\omega$: $\mathcal{U}_1^{\#}(\alpha) := \exists m[\alpha(\bar{\alpha}m) \neq 0]$

One easily verifies, using theorem 6.9.: $\mathcal{U}_1^{\#} \notin \Pi^0_1$
As $\mathcal{U}_1^{\#} \preceq E_1$, this confirms theorem 2.2, which said that $\neg(E_1 \preceq A_1)$.

6.10 <u>Definition</u>: Let P be a subset of ${}^{\omega}\omega$

We define the subsets $Un(P)$ and $Ex(P)$ of ${}^{\omega}\omega$ by:

For all $\alpha \in {}^{\omega}\omega$:

$$Un(P)(\alpha) := \forall m[P(\alpha^m)]$$

$$Ex(P)(\alpha) := \exists m[P(\alpha^m)]$$

6.11 <u>Definition</u>: We define a sequence $A_1, E_1, A_2, E_2, \ldots$ of subsets of ${}^{\omega}\omega$ by:

(i) For all $\alpha \in {}^{\omega}\omega$: $A_1(\alpha) := \forall n[\alpha(n) = 0]$

$E_1(\alpha) := \exists n[\alpha(n) = 0]$

(ii) For all $n \in \omega$: $A_{Sn} := Un(E_n)$

$E_{Sn} := Ex(A_n)$

We define a sequence $\Pi^0_1, \Sigma^0_1, \Pi^0_2, \Sigma^0_2, \dots$ of classes of subsets of ${}^\omega\omega$ by:

For all $n \in \omega$: $\Pi^0_n := \{P \mid P \subseteq {}^\omega\omega \mid P \preceq A_n\}$

$\Sigma^0_n := \{P \mid P \subseteq {}^\omega\omega \mid P \preceq E_n\}$

6.12 Theorem Let $P \subseteq {}^\omega\omega$ and $n \in \omega$, $n \geq 1$.

$P \in \Pi^0_{Sn}$ if and only if there exists a sequence $Q_0, Q_1, \dots$ of subsets of ${}^\omega\omega$ such that $\forall m[Q_m \in \Sigma^0_n]$ and $P = \bigcap_{m \in \omega} Q_m$.

$P \in \Sigma^0_{Sn}$ if and only if there exists a sequence $Q_0, Q_1, \dots$ of subsets of ${}^\omega\omega$ such that $\forall m[Q_m \in \Pi^0_n]$ and $P = \bigcup_{m \in \omega} Q_m$.

Proof: We prove the first part.

Suppose: $P \in \Pi^0_{Sn}$, and determine $\delta \in {}^\omega\omega$ such that $Fun(\delta)$ and: $\forall\alpha[P(\alpha) \rightleftarrows A_{Sn}(\delta|\alpha)]$. Define, for each $m \in \omega$, a subset Q_m of ${}^\omega\omega$ by: $Q_m := \{\alpha \mid \alpha \in {}^\omega\omega \mid E_n((\delta|\alpha)^m)\}$ and remark:

$\forall m[Q_m \in \Sigma^0_n]$ and: $P = \bigcap_{m \in \omega} Q_m$

Now suppose: $Q_0, Q_1, \dots$ is a sequence of members of Σ^0_n, and, using AC_{11} and AC_{10}, determine $\delta \in {}^\omega\omega$ such that:

$\forall m[Fun(\delta^m) \wedge \forall\alpha[Q_m(\alpha) \rightleftarrows E_n(\delta^m|\alpha)]$.

Determine $\zeta \in {}^\omega\omega$ such that $Fun(\zeta)$ and $\forall\alpha\forall m[(\zeta|\alpha)^m = \delta^m|\alpha]$ and remark: $\forall\alpha[\forall m[Q_m(\alpha)] \rightleftarrows A_{Sn}(\zeta|\alpha)]$, i.e.: $P = \bigcap_{m \in \omega} Q_m \in \Pi^0_{Sn}$.

$\boxtimes$

Like Σ^0_1 and Π^0_1, all classes Σ^0_n, Π^0_n are surveyable:

6.13 Theorem: All classes $\Sigma^0_1, \Pi^0_1, \Sigma^0_2, \Pi^0_2, \dots$ do possess a universal element.

Proof: Use theorems 6.5 and 6.9 and construct a universal element $\mathcal{U}_{11}$ of Σ^0_1 and a universal element $\mathcal{U}_{01}$ of Π^0_1.

We will exhibit universal elements for the other classes by induction. Let $n \in \omega$ and suppose: $\mathcal{U}_{1n}$ and $\mathcal{U}_{0n}$ are universal elements of

Σ_n^0 and Π_n^0, respectively.

Define subsets $\mathcal{U}_{1sn}$ and $\mathcal{U}_{0sn}$ of ${}^\omega\omega$ by:

For all $\alpha \in {}^\omega\omega$,

$$\mathcal{U}_{1sn}(\alpha) := \exists m [\, \mathcal{U}_{0n}(<\alpha^0, (\alpha^1)^m>)\,]$$

$$\mathcal{U}_{0sn}(\alpha) := \forall m [\, \mathcal{U}_{1n}(<\alpha^0, (\alpha^1)^m>)\,]$$

$\mathcal{U}_{1sn}$ and $\mathcal{U}_{0sn}$ do belong to Σ_{sn}^0 and Π_{sn}^0, respectively

We claim that they are universal elements in their classes

Let us prove: $\mathcal{U}_{1sn}$ is a universal element of Σ_{sn}^0.

If P is any member of Σ_{sn}^0, then, using the foregoing theorem and AC_{01}, we may find $\beta \in {}^\omega\omega$ such that:

$\forall\alpha [\, P(\alpha) \rightleftarrows \exists m [\, \mathcal{U}_{0n}(<\alpha, \beta^m>)\,]\,]$, i.e.: $\forall\alpha [\, P(\alpha) \rightleftarrows \mathcal{U}_{1sn}(<\alpha, \beta>)\,]$

$\boxtimes$

Members of $\bigcup_{n\in\omega} \Sigma_n^0$ will be called: arithmetical subsets of ${}^\omega\omega$ (cf. Note 1 on page 216).

An immediate consequence of theorem 6.12 is: $\forall n [\, \Sigma_n^0 \subseteq \Pi_{sn}^0 \wedge \Pi_n^0 \subseteq \Sigma_{sn}^0\,]$

Verifying: $\forall n [\, \Sigma_n^0 \subseteq \Sigma_{sn}^0 \wedge \Pi_n^0 \subseteq \Pi_{sn}^0\,]$ is not difficult.

6.14 Theorems 6.5 and 6.9 gave rebirth to the results of chapter 2.
We may ask, whether theorem 6.13 is also fertile in this sense, and if it may be seen to confirm the conclusions of chapter 3, and, hopefully, to lead us on to new vistas.
It is not, however. Let us try and cut the classical capers in order to find the cause of the trouble.
Consider $\mathcal{U}_{12}$, the universal element of the class Σ_2^0 which has been constructed in the proof of theorem 6.13
Then, for all $\alpha, \beta \in {}^\omega\omega$: $\mathcal{U}_{12}(<\alpha,\beta>) \rightleftarrows \exists m \forall n [\, \beta^m(\bar{\alpha} n) = 0\,]$
Define a subset $\mathcal{U}_{02}^{\#}$ of ${}^\omega\omega$ by:

For all $\alpha \in {}^\omega\omega$: $\mathcal{U}_{02}^{\#}(\alpha) := \forall m \exists n [\, \alpha^m(\bar{\alpha} n) \neq 0\,]$

It is obvious, now, that $\mathcal{U}_{02}^{\#}$ belongs to Π_2^0, but is not so obvious that $\mathcal{U}_{02}^{\#}$ does not belong to Σ_{02}

Suppose. $\mathcal{U}_{02}^{\#} \in \Sigma_2^0$. Determine $\beta \in {}^\omega\omega$ such that: $\forall\alpha [\, \mathcal{U}_{02}^{\#}(\alpha) \rightleftarrows \mathcal{U}_{12}(<\alpha,\beta>)\,]$

Assume: $\mathcal{U}_{02}^{\#}(\beta)$, then $\mathcal{U}_{12}(<\beta,\beta>)$, i.e.: $\forall m \exists n [\, \beta^m(\bar{\beta} n) \neq 0\,]$ and: $\exists m \forall n [\, \beta^m(\bar{\beta} n) = 0\,]$

Contradiction. Therefore: $\neg \mathcal{U}_{02}^{\#}(\beta)$ and: $\neg \mathcal{U}_{12}(<\beta,\beta>)$; i.e.:

$\neg \forall m \exists n [\, \beta^m(\bar{\beta} n) \neq 0\,]$ and: $\neg \exists m \forall n [\, \beta^m(\bar{\beta} n) = 0\,]$

Meeting such a β would be a very memorable event, indeed, but, as matters stand now, we are not able, like classical mathematicians, to exclude the possibility of its existence.

We are reminded of the mysteries which we encountered in chapter 5.
If we assume the enigmatical $\forall\alpha[\neg\neg\exists n[\alpha(n)=0] \to \exists n[\alpha(n)=0]]$, we may carry through the classical argument:

$$\neg\forall m\,\exists n[\beta^m(\bar{\beta}n)\neq 0]\text{, i.e.: }\neg\forall m\neg\neg\exists n[\beta^m(\bar{\beta}n)\neq 0]\text{, i.e.: }\neg\neg\exists m\,\forall n[\beta^m(\bar{\beta}n)=0]$$

The same turn of thought would save us at all future stages of the arithmetical hierarchy.
In chapter 3, we circumvented the mystery, if only for the case of the second level, and gave a truly constructive argument.
We will have no peace till we have extended this to all levels of the hierarchy.

6.15 We could have started the hierarchy with the class of all decidable subsets of ${}^\omega\omega$:

$$\Delta^0_1 := \{P \mid P\subseteq {}^\omega\omega \mid \forall\alpha[P(\alpha)\vee\neg P(\alpha)]\}$$

We may define a special subset D of ${}^\omega\omega$ by:

For all $\alpha\in{}^\omega\omega$: $\quad D(\alpha) := \alpha(0)=0$

and remark: $\Delta^0_1 := \{P \mid P\subseteq{}^\omega\omega \mid P\preceq D\}$ and:

and: $A_1 \preceq Un(D) \preceq A_1$ and: $E_1 \preceq Ex(D)\preceq E_1$

On the other hand, Δ^0_1 does not have a universal element, for, in that case, we would not survive diagonalization.
It is for this reason that we mention Δ^0_1 only now.

In this connection, we are brought to reconsider the classical fact:
$\Pi^0_1\cap\Sigma^0_1 = \Delta^0_1$ (cf. Note 4 on page 216).
This is improbable, in view of the following:
Fermat's last theorem may be written in the form: $\forall n[f(n)=0]$, where f is a primitive-recursive function from ω to $\{0,1\}$
But, using the Brouwer-Kripke-axiom, we may construct β from ω to $\{0,1\}$ such that Fermat's last theorem is equivalent to: $\exists n[\beta(n)=0]$
Consider $C_F := \{\alpha \mid \alpha\in{}^\omega\omega \mid \forall n[f(n)=0]\}$ and assume: $\Pi^0_1\cap\Sigma^0_1 = \Delta^0_1$
Then: C_F is a decidable subset of ${}^\omega\omega$, and Fermat's last theorem has been proved or refuted, a big surprise, indeed.

6.16 A related question, which seems of some interest, refers to the structure $\langle\Sigma^0_1,\preceq\rangle$
Both D and E_1 belong to Σ^0_1 and: $D\prec E_1$
Is it possible to find $P\in\Sigma^0_1$ such that: $D\prec P\prec E_1$?

To be sure, we have no method for deciding, for all $P,Q\in\Sigma^0_1$: $P\preceq Q \vee Q\preceq P$
(Define $P:=C_F$ and $Q:=C_G$, where F, as in 6.15 stands for Fermat's last theorem, and G for some other unsolved proposition, which, as far as we know, has nothing to do with F, i.e. we do not know how to answer: $(F\vee\neg F)\to(G\vee\neg G)$ or $(G\vee\neg G)\to(F\vee\neg F)$)
But we would like to see a P from Σ^0_1 such that the statements

„$P \preceq D$" and „$E_1 \preceq P$" are both contradictory and not but reckless.

The dual problem asks if there exists $P \in \Pi^0_1$ such that $D \prec P \prec A_1$
Like its companion, this problem seems rather inaccessible.

Classically, both questions have to be answered in the negative.

(*) Let us define, for all $\beta \in {}^\omega\omega$: $E_\beta := \{\alpha \mid \alpha \in {}^\omega\omega \mid \exists n[\beta(\bar{\alpha}n)=0]\}$
According to theorem 6.2 and AC_{01}: $\Sigma^0_1 = \{E_\beta \mid \beta \in {}^\omega\omega\}$

Remark that, for all $\beta \in {}^\omega\omega$:

$$E_1 \preceq E_\beta \rightleftarrows \exists \alpha \forall n[\beta(\bar{\alpha}n) \neq 0 \wedge \exists m \leq \bar{\alpha}n[\beta(m)=0]]$$

Suppose: $\neg(E_1 \preceq E_\beta)$ and conclude: $\forall \alpha \exists n[\beta(\bar{\alpha}n)=0 \vee \forall m \leq \bar{\alpha}n[\beta(m) \neq 0]]$
i.e.: $\exists \delta[\mathrm{Fun}(\delta) \wedge \forall \alpha[E_\beta(\alpha) \rightleftarrows D(\delta|\alpha)]]$, i.e.: $E_\beta \preceq D$

(**) Let us define, for all $\beta \in {}^\omega\omega$: $A_\beta := \{\alpha \mid \alpha \in {}^\omega\omega \mid \forall n[\beta(\bar{\alpha}n)=0]\}$
According to theorem 6.7 and AC_{01}: $\Pi^0_1 = \{A_\beta \mid \beta \in {}^\omega\omega\}$
Remark that, for all $\beta \in {}^\omega\omega$

$$A_1 \preceq A_\beta \rightleftarrows \exists \alpha \forall n[\beta(\bar{\alpha}n)=0 \wedge \exists m \leq \bar{\alpha}n[\beta(m) \neq 0]]$$

Suppose: $\neg(A_1 \preceq A_\beta)$ and find: $A_\beta \preceq D$.

We did not succeed in proving similar conclusions by intuitionistic means, and the semi-classical assumption: $\forall \alpha[\neg\neg\exists n[\alpha(n)=0] \to \exists n[\alpha(n)=0]]$ also did not bring any relief.

6.17 We close this chapter by two minor remarks.

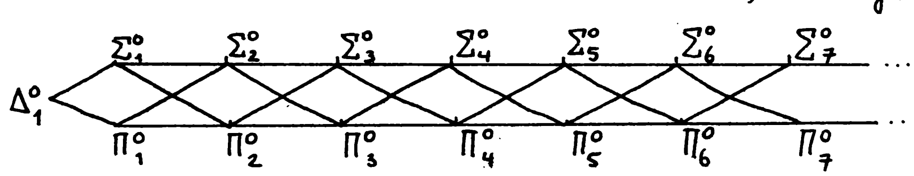
The first one is, that spreads, as they have been introduced in 1.9 do belong to Π^0_1, but that, conversely, not every element of Π^0_1 is a spread.

The second one says, that, in correspondence to chapter 4, we might have introduced a class like:

$$\{P \mid P \subseteq {}^\omega\omega \mid P \preceq D^2 A_1\}$$

and remarked, that a subset of ${}^\omega\omega$ belongs to this class if and only if it is the union of two sets, each belonging to Π^0_1.

We cannot deny, that in 6.14-16, the sky has been clouded slightly.
Our first concern will be to make the arithmetical ladder, now lying down, stand up:



7. THE ARITHMETICAL HIERARCHY ESTABLISHED

We extend the results of chapter 3, in which we learnt that A_2 and E_2 are incomparable, and we prove : $\forall n[\neg(A_n \preceq E_n) \wedge \neg(E_n \preceq A_n)]$.
This conclusion may be framed as follows: $\forall n[\neg(\Pi^0_n \subseteq \Sigma^0_n) \wedge \neg(\Sigma^0_n \subseteq \Pi^0_n)]$
The argument is an inductive one, and develops ideas from chapter 3.

7.0 We will make use of the fact that each one of the sets $A_1, E_1, A_2, E_2, \dots$ is, - as we intend to call it from chapter 10 onwards - : strictly analytical, i.e.:

$$\forall n \exists \delta[\mathrm{Fun}(\delta) \wedge A_n = \mathrm{Ra}(\delta)] \quad \wedge \quad \forall n \exists \delta[\mathrm{Fun}(\delta) \wedge E_n = \mathrm{Ra}(\delta)]$$

In chapter 3, we saw that A_2 has this property.
This is not the full tale.
We indeed construct for each A_n (resp. E_n) a special sequence δ such that $\mathrm{Fun}(\delta)$ and A_n (resp. E_n) $= \mathrm{Ra}(\delta)$.
But the proof of the hierarchy theorem also uses other properties of these sequences δ.

Let us not talk too much and go working.

We first recall and extend some notational conventions which we introduced in the chapters 1 and 4. (Cf. 4.2).

For all $n, k \in \omega$ such that $k < lg(n)$:

$n(k) := n_k :=$ the value which the finite sequence coded by n, assumes in k

Therefore, for each $n \in \omega$: $n = \langle n(0), n(1), \dots, n(lg(n)-1)\rangle$

For all $n, k \in \omega$ such that $k \le lg(n)$

$\bar{n}(k) :=$ the code number of that finite sequence of length k, which is an initial part of the finite sequence, coded by n.

Therefore, for each $n \in \omega$: $\bar{n}(lg(n)) = n$.

Let $\gamma \in {}^\omega\omega$
We introduce two subsets $\Sigma_I(\gamma)$ and $\Sigma_{II}(\gamma)$ of ω by:

$$\Sigma_I(\gamma) := \{n \mid \forall k[2k+1 \le lg(n) \to n(2k) = \gamma(\bar{n}(2k))]\}$$
$$\Sigma_{II}(\gamma) := \{n \mid \forall k[2k+2 \le lg(n) \to n(2k+1) = \gamma(\bar{n}(2k+1))]\}$$

These definitions do need some explanation:
Players I and II are doing a game in which they choose, alternately, a natural number.
Thus finite sequences of natural numbers represent possible positions in one of their plays.

$\Sigma_I(\gamma)$ is the set of positions which may be reached if player I is following the strategy given by γ.
$\Sigma_{II}(\gamma)$ is the set of positions which may be reached if player II is following the strategy given by γ
We remark: $\forall\gamma\forall\delta\exists!\alpha\forall n[\bar{\alpha}n \in \Sigma_I(\gamma) \cap \Sigma_{II}(\delta)]$
(Whenever both player I and player II have decided upon their strategies, there is a unique resulting play).

For all $n\in\omega$, and $\gamma,\alpha \in {}^\omega\omega$ we define $\gamma \mathbin{\rotatebox[origin=c]{90}{$\bowtie$}}_n \alpha$ in ${}^\omega\omega$ by:

For all $p\in\omega$:

$$(\gamma \mathbin{\rotatebox[origin=c]{90}{$\bowtie$}}_n \alpha)(p) := 0 \quad \text{if} \quad p \in \Sigma_I(\gamma) \text{ and } lg(p) = n$$
$$:= \alpha(p) \quad \text{if} \quad p \notin \Sigma_I(\gamma) \text{ or } lg(p) \neq n$$

For all $n\in\omega$, and $\gamma,\alpha\in{}^\omega\omega$ we define $\gamma \bowtie_n \alpha$ in ${}^\omega\omega$ by:

For all $p\in\omega$:

$$(\gamma \bowtie_n \alpha)(p) := 0 \quad \text{if} \quad p\in\Sigma_{II}(\gamma) \text{ and } lg(p) = n$$
$$:= \alpha(p) \quad \text{if} \quad p \notin \Sigma_{II}(\gamma) \text{ or } lg(p)\neq n$$

Appealing repeatedly to AC_{01}, as we did in 3.1, we may verify:

$$\forall n \forall\alpha[E_n(\alpha) \rightleftarrows \exists\gamma[\alpha = \gamma \mathbin{\rotatebox[origin=c]{90}{$\bowtie$}}_n \alpha]]$$

and: $\forall n\forall\alpha[A_n(\alpha) \rightleftarrows \exists\gamma[\alpha = \gamma\bowtie_n\alpha]]$

The intertwining function $\bowtie_2$ is none other than the function $\bowtie$ whose acquaintance we made in 3.1.
To spare the reader and ourselves, we do not go into the trouble of giving a detailed proof of the just mentioned facts, which should go by induction.

For each n, we may make $\delta\in{}^\omega\omega$ such that $Fun(\delta)$ and $\forall\alpha[\delta|\alpha = \alpha^0 \mathbin{\rotatebox[origin=c]{90}{$\bowtie$}}_n \alpha^1]$

We observe: $\forall\alpha[E_n(\alpha) \rightleftarrows \exists\beta[\alpha = \delta|\beta]$

and: $\exists\delta[Fun(\delta) \wedge E_n = Ra(\delta)]$

For each n, we may make $\delta\in{}^\omega\omega$ such that $Fun(\delta)$ and $\forall\alpha[\delta|\alpha = \alpha^0\bowtie_n\alpha^1]$

We observe: $\forall\alpha[A_n(\alpha) \rightleftarrows \exists\beta[\alpha = \delta|\beta]]$

and: $\exists\delta[Fun(\delta) \wedge A_n = Ra(\delta)]$

These remarks vindicate the statement which opened this section, and conclude the preparations we had to make for:

7.1 <u>Lemma</u>: $\forall n{>}0[\text{If } A_{sn} \preceq E_{sn}, \text{ then } E_n \preceq A_n]$

<u>Proof</u>: Suppose $n\in\omega$, $n>0$ and $A_{sn}\preceq E_{sn}$
Using AC_{11}, determine $\delta\in{}^\omega\omega$ such that: $Fun(\delta)$ and: $\forall\alpha[A_{sn}(\alpha) \rightleftarrows E_{sn}(\delta|\alpha)]$

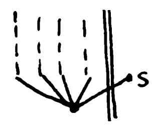
Remark: $\forall\gamma\,\forall\alpha\,[\,A_{sn}(\gamma \bowtie_{sn} \alpha)]$

Therefore: $\forall\gamma\,\forall\alpha\,[\,E_{sn}(\delta|(\gamma \bowtie_{sn}\alpha))]$ and: $\forall\gamma\,\forall\alpha\,\exists m\,[\,A_n((\delta|(\gamma\bowtie_{sn}\alpha))^m)]$

(The camera focuses on the creative subject which is supplying γ and α step-by-step, and then switches to the imitative subject, which is responsible for $\delta|(\gamma\bowtie_{sn}\alpha)$ and has to make a choice about it, notwithstanding the fact that his knowledge about γ and α is, and is to remain, widely insufficient. The creative subject, of course, can not but exploit this state of affairs:)

Using CP, determine $m, p \in \omega$ such that: $\forall\gamma\,\forall\alpha\,[\,\bar\gamma p = \bar{\underline{0}} p = \bar\alpha p \rightarrow A_n((\delta|(\gamma\bowtie_{sn}\alpha))^m)]$

Determine $s\in\omega$ such that $\langle s\rangle > p$.



$\gamma\bowtie_{sn}\alpha$

$\delta|(\gamma\bowtie_{sn}\alpha)$

The creative subject did not place himself under any obligation as regards the sequence α^s; he still may choose anything he likes for it.

Define $\zeta \in {}^\omega\omega$ such that $\mathrm{Fun}(\zeta)$ and: $\forall\beta\,[\,(\zeta|\beta)^s = \beta \wedge \forall\ell[\ell\neq s \rightarrow (\zeta|\beta)^\ell = \underline{0}]]$

Let $\beta \in {}^\omega\omega$ and suppose: $E_n(\beta)$, then: $A_{sn}(\zeta|\beta)$, and, in addition: $\exists\gamma\,\exists\alpha\,[\,\bar\gamma p = \bar\alpha p = \bar{\underline{0}} p \wedge \zeta|\beta = \gamma\bowtie_{sn}\alpha\,]$.

Therefore: $A_n((\delta|(\zeta|\beta))^m)$

Conversely, suppose: $A_n((\delta|(\zeta|\beta))^m)$, then: $E_{sn}(\delta|(\zeta|\beta))$, therefore: $A_{sn}(\zeta|\beta)$, and: $E_n(\beta)$

We have seen: $\forall\beta\,[\,E_n(\beta) \rightleftarrows A_n((\delta|(\zeta|\beta))^m)]$, i.e.: $E_n \preceq A_n$.

⊠

A small refinement of the argument for lemma 7.1 leads to the conclusion: $A_{sn} \preceq A_n$.
(Define $\zeta \in {}^\omega\omega$ such that $\mathrm{Fun}(\zeta)$ and: $\forall\beta\,\forall\ell\,[\,\ell < s \rightarrow (\zeta|\beta)^\ell = \underline{0}) \wedge (\zeta|\beta)^{s+\ell} = \beta^\ell\,]$
This construction brings out that the problem if a given sequence has the property A_{sn}, is not diminished by any knowledge which refers to only finitely many of its subsequences)
But we may do without the stronger conclusion in our inductive scheme. An indispensable element in this scheme is:

7.2 <u>Lemma</u>: $\forall n\,[\,$If $E_{sn} \preceq A_{sn}$, then $A_n \preceq E_n\,]$

<u>Proof</u>: Suppose: $n \in \omega$ and $E_{sn} \leq A_{sn}$

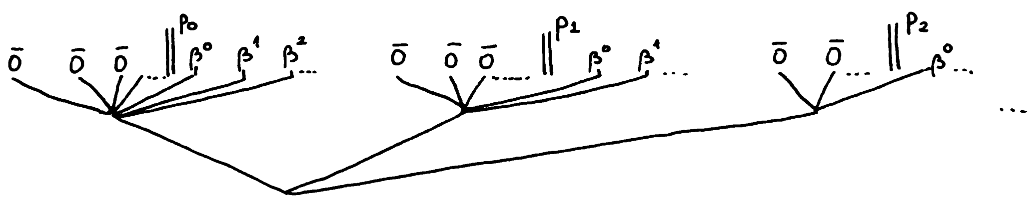
Using AC_{11}, determine $\delta \in {}^{\omega}\omega$ such that: $\mathrm{Fun}(\delta)$ and: $\forall \alpha [E_{sn}(\alpha) \rightleftarrows A_{sn}(\delta|\alpha)]$

We will prove more than the theorem announces, viz. $A_n \leq A_{n-1}$

(We assume: $n > 1$. The cases $n=0$, $n=1$ have been taken care of in theorems 2.2 and 3.3, respectively, and will not be treated here, although, with some precautions, they might be subsumed under this more general theorem).

In order to avoid the sprouting of too many parentheses, we will sometimes write: $\alpha^{m,k}$ in stead of: $(\alpha^m)^k$

We are to construct $\zeta \in {}^{\omega}\omega$ such that $\mathrm{Fun}(\zeta)$, and, for each β, $\zeta|\beta$ looks as follows:



The first-order-subsequences of $\zeta|\beta$ are, all of them, very similar to the sequence β: for each $k \in \omega$, the subsequences of $(\zeta|\beta)^k$ are: finitely many (viz. p_k) times the sequence $\underline{0}$, and, thereafter, the subsequences of β, in due order.

One observes: $\forall \beta [A_n(\beta) \rightleftarrows E_{sn}(\zeta|\beta)]$

The numbers $p_0, p_1, \ldots$ depend on β; for each $k \in \omega$, the choice of p_k will be made such that: $A_n(\beta) \rightarrow E_n((\delta|(\zeta|\beta))^k)$

Moreover, when calculating p_k, we also determine a number m_k such that: $A_n(\beta) \rightarrow A_{n-1}((\delta|(\zeta|\beta))^{k,m_k})$

Carrying out this program will bring us a rich harvest, and we will merrily go round as follows:

$$\begin{tikzcd}
A_n(\beta) \arrow[r] & \forall k [A_{n-1}((\delta|(\zeta|\beta))^{k,m_k})] \arrow[d] \\
 & A_{sn}(\delta|(\zeta|\beta)) \arrow[d] \\
 & E_{sn}(\zeta|\beta) \arrow[uul]
\end{tikzcd}$$

Therefore: $A_n(\beta) \rightleftarrows \forall k [A_{n-1}((\delta|(\zeta|\beta))^{k,m_k}]$

This looks very much like the conclusion we are chasing after.

<u>Construction of ζ</u>

Let $\beta \in {}^{\omega}\omega$, a sequence which is to be held fixed during the rather involved construction of $\zeta|\beta$.

We will make a sequence $\alpha_0, \alpha_1, \alpha_2, \ldots$ of sequences, each depending on β, which converges, in the natural sense of the word. $\zeta|\beta$ is defined as the limit of this sequence.

Let $\gamma_0 := \underline{0}$ and $\alpha_0 := \underline{0}$

<u>First step</u>: Remark: $E_{Sn}(\gamma_0 \boxtimes_{Sn} \alpha_0)$, and, using CP, determine $m_0, p_0 \in \omega$ such that: $\forall \gamma \forall \alpha [(\bar{\gamma} p_0 = \bar{\gamma}_0 p_0 \wedge \bar{\alpha} p_0 = \bar{\alpha}_0 p_0) \rightarrow A_{n-1}((\delta|(\gamma \boxtimes_{Sn} \alpha))^{0,m_0})]$

Now define α_1 as follows:

$$(\alpha_1)^{0,\ell} := \underline{0} \qquad \text{if } \ell < p_0$$
$$(\alpha_1)^{0,p_0+\ell} := \beta^{\ell} \qquad \text{for all } \ell \in \omega$$
$$(\alpha_1)^{m} := \underline{0} \qquad \text{if } m \geq 1$$

Remark: $\bar{\alpha}_1 p_0 = \bar{\alpha}_0 p_0$.

Determine $\gamma_1 \in {}^{\omega}\omega$ such that $\gamma_1(\langle\ \rangle) = 1$ and $\forall t [t \neq \langle\ \rangle \rightarrow \gamma_1(t) = 0]$.

Remark: $\alpha_1 = \gamma_1 \boxtimes_{Sn} \alpha_1$

Suppose: $\alpha \in {}^{\omega}\omega \wedge \alpha^0 = (\alpha_1)^0 \wedge \bar{\alpha} p_0 = \bar{\alpha}_1 p_0 \wedge A_n(\beta)$

Then: $A_n(\alpha^0)$, and, what is more:

$\exists \gamma [\bar{\gamma} p_0 = \bar{\gamma}_0 p_0 \wedge \bar{\alpha} p_0 = \bar{\alpha}_0 p_0 \wedge \alpha = \gamma \boxtimes_{Sn} \alpha]$

Therefore: $A_{n-1}((\delta|\alpha)^{0,m_0})$

We keep this in mind:

$$\forall \alpha [(\alpha^0 = (\alpha_1)^0 \wedge \bar{\alpha} p_0 = \bar{\alpha}_1 p_0 \wedge A_n(\beta)) \rightarrow A_{n-1}((\delta|\alpha)^{0,m_0})]$$

<u>Second step</u> Remark: $E_{Sn}(\gamma_1 \boxtimes_{Sn} \alpha_1)$, and, using CP, determine $m_1, p_1 \in \omega, p_1 \geq p_0$, such that: $\forall \gamma \forall \alpha [(\bar{\gamma} p_1 = \bar{\gamma}_1 p_1 \wedge \bar{\alpha} p_1 = \bar{\alpha}_1 p_1) \rightarrow A_{n-1}((\delta|(\gamma \boxtimes_{Sn} \alpha))^{1,m_1})]$

Now define α_2 as follows:

$$(\alpha_2)^0 := (\alpha_1)^0$$
$$(\alpha_2)^{1,\ell} := \underline{0} \qquad \text{if } \ell < p_1$$
$$(\alpha_2)^{1,p_1+\ell} := \beta^{\ell} \qquad \text{for all } \ell \in \omega$$
$$(\alpha_2)^{m} := \underline{0} \qquad \text{if } m \geq 2$$

Remark: $\bar{\alpha}_2 p_1 = \bar{\alpha}_1 p_1$.

Determine $\gamma_2 \in {}^\omega\omega$ such that $\gamma_2(\langle\rangle) = 2$ and $\forall t[t \neq \langle\rangle \to \gamma_2(t) = 0]$

Remark: $\alpha_2 = \gamma_2 \boxtimes_{sn} \alpha_2$.

Suppose: $\alpha \in {}^\omega\omega \wedge \alpha^0 = (\alpha_2)^0 \wedge \alpha^1 = (\alpha_2)^1 \wedge \bar{\alpha}p_1 = \bar{\alpha}_2 p_1 \wedge A_n(\beta)$

Then: $A_n(\alpha^1)$, and, what is more:

$\exists\gamma[\bar{\gamma}p_1 = \bar{\gamma}_1 p_1 \wedge \bar{\alpha}p_1 = \bar{\alpha}_1 p_1 \wedge \alpha = \gamma \boxtimes_{sn} \alpha]$

Therefore: $A_{n-1}(((\delta|\alpha)^{1,m_1})$.

We keep this in mind:

$$\forall\alpha[(\alpha^0 = (\alpha_2)^0 \wedge \alpha^1 = (\alpha_2)^1 \wedge \bar{\alpha}p_1 = \bar{\alpha}_2 p_1 \wedge A_n(\beta)) \to A_{n-1}((\delta|\alpha)^{1,m_1})]$$

Sk-th step: Remark: $E_{sn}(\gamma_k \boxtimes_{sn} \alpha_k)$, and, using CP, determine $m_k, p_k \in \omega$, such that $p_k \geq p_{k-1}$ and: $\forall\gamma\forall\alpha[(\bar{\gamma}p_k = \bar{\gamma}_k p_k \wedge \bar{\alpha}p_k = \bar{\alpha}_k p_k) \to A_{n-1}((\delta|(\gamma \boxtimes_{sn} \alpha))^{k,m_k})]$

Now define α_{sk} as follows:

$$(\alpha_{sk})^0 := (\alpha_k)^0 \wedge (\alpha_{sk})^1 = (\alpha_k)^1 \wedge \ldots \wedge (\alpha_{sk})^{k-1} := (\alpha_k)^{k-1}$$

$$(\alpha_{sk})^{k,l} := \underline{0} \qquad \text{if } l < p_k$$

$$(\alpha_{sk})^{k,p_k+l} := \beta^l \qquad \text{for all } l \in \omega$$

$$(\alpha_{sk})^m := \underline{0} \qquad \text{if } m \geq Sk$$

Remark: $\bar{\alpha}_{sk} p_k = \bar{\alpha}_k p_k$.

Determine $\gamma_{sk} \in {}^\omega\omega$ such that $\gamma_{sk}(\langle\rangle) = Sk$ and $\forall t[t \neq \langle\rangle \to \gamma_{sk}(\langle\rangle) = 0]$

Remark: $\alpha_{sk} = \gamma_{sk} \boxtimes_{sn} \alpha_{sk}$.

Suppose: $\alpha \in {}^\omega\omega \wedge \forall l < Sk[\alpha^l = (\alpha_{sk})^l] \wedge \bar{\alpha}p_k = \bar{\alpha}_{sk} p_k \wedge A_n(\beta)$

Then: $A_n(\alpha^k)$, and, what is more:

$\exists\gamma[\bar{\gamma}p_k = \bar{\gamma}_k p_k \wedge \bar{\alpha}p_k = \bar{\alpha}_k p_k \wedge \alpha = \gamma \boxtimes_{sn} \alpha]$

Therefore: $A_{n-1}((\delta|\alpha)^{k,m_k})$

We keep this in mind:

$$\forall\alpha[(\forall l < Sk[\alpha^l = (\alpha_{sk})^l] \wedge \bar{\alpha}p_k = \bar{\alpha}_{sk} p_k \wedge A_n(\beta)) \to A_{n-1}((\delta|\alpha)^{k,m_k})]$$

We conclude the definition of $\zeta|\beta$ by proclaiming:

$$\forall k[(\zeta|\beta)^k := (\alpha_{sk})^k]$$

We make the following observations:

$\forall p \forall k[(\alpha_{sk})^k = (\alpha_{sk+p})^k]$, therefore:

$\forall k[\overline{(\zeta|\beta)}p_k = \bar{\alpha}_{sk} p_k]$ and: $A_n(\beta) \to \forall k[A_{n-1}((\delta|(\zeta|\beta))^{k,m_k})]$

The numbers $m_0, m_1, \ldots$ do depend on β, let us write them as $m_0(\beta), m_1(\beta) \ldots$

We determine $\eta \in {}^{\omega}\omega$ such that $Fun(\eta)$ and:

$$\forall \beta \forall k [(\eta|\beta)^k = (\delta|(\zeta|\beta))^{k, m_k(\beta)}]$$

Remark: $\forall \beta [A_n(\beta) \rightleftarrows \forall k [A_{n-1} ((\eta|\beta)^k)]]$, i.e.:

$$\forall \beta [A_n(\beta) \rightleftarrows (Un(A_{n-1}))(\eta|\beta)], \text{ and: } A_n \preceq Un(A_{n-1})$$

But: $Un(A_{n-1}) \preceq A_{n-1}$, as may be seen from the previous chapter (cf. 6.12), and therefore: $A_n \preceq A_{n-1}$.

$\boxtimes$

In retrospect, lemma 7.1 may be seen to follow from lemma 7.2
For, suppose: $A_{Sn} \preceq E_{Sn}$; then $E_{SSn} \preceq E_{Sn}$, and. $E_{SSn} \preceq A_{SSn}$, therefore: $A_{Sn} \preceq A_n$, and $E_n \preceq A_{Sn} \preceq A_n$.
We maintained lemma 7.1, because its shorter proof might serve to prepare the reader for the proof of lemma 7.2.
And here we find it standing in all its glory:

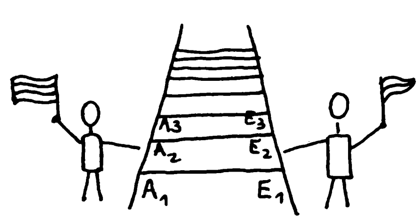
7.3 __Theorem__: (Arithmetical Hierarchy Theorem):

$$\forall n > 0 [\neg(A_n \preceq E_n) \wedge \neg(E_n \preceq A_n)]$$

__Proof__ Theorems 2.1 and 2.2 taught us how to put a first foot on the ladder. (You may choose and start with your left foot or with your right foot).
Lemmas 7.1 and 7.2 taught us how to pass the left foot on to the next higher step, if we lean on the right one, and how to pass the right foot on to the next higher step, if we lean on the left one.
And so we climb, and climb, and still climb.

$\boxtimes$

The following picture visualizes the result of our efforts:



And we dream of higher things ...

8 HYPERARITHMETICAL SETS INTRODUCED

We continue the considerations of the previous chapters, and now enter the domain of the transfinite.
We have to develop something of a theory of countable ordinals
We will identify countable ordinals and their representations as well-ordered <u>stumps</u> in ${}^{\omega}\omega$.
After this, we build hyperarithmetical sets and prove their most obvious properties.

8.0 For every $m \in \omega$ and every subset $A \subseteq \omega$, we define a subset $m * A$ of ω by:

$$m * A := \{ m * p \mid p \in A \}$$

(* has been introduced in 1.2, and denotes concatenation).

We define the set \$ of well-ordered stumps in ${}^{\omega}\omega$ by transfinite induction:

(I) $\emptyset \in \$$

(II) If $A_0, A_1, A_2, \cdots$ is a sequence of elements of \$, then A belongs to \$, where $A := \{<\,>\} \cup \bigcup\limits_{n \in \omega} <n> * A_n$

(III) If any subset A of ω does belong to \$, it does so because of (I) and (II)

It is difficult to judge, if the continually extending stock of well-ordered stumps is a totality which deserves of being called a mathematical set, on a par with ω or ${}^{\omega}\omega$. Some members of the French school of descriptive-set-theorists shrank back from doing so.
Do we survey this totality so well, that propositions, obtained by quantifying over it, are meaningful?
(L.E.J. Brouwer did not unambiguously express himself on this point.(cf. Note 8 on page 217)).
We accept the definition, but keep in mind, that \$, although a set, is very much a set of its own kind, markedly different from both ω and ${}^{\omega}\omega$.

Because of the definition's second clause, members of \$, in general, cannot be assumed to be <u>determinate</u> objects (i.e. objects which admit of a finite description, cf. 2.1.).

Once it has been accepted, \$ may be handled by the method of transfinite induction, i.e.: relations and operations on \$ may be defined, and general statements about all members of \$ may be proved, by "following the definition".

8.1 We will use Greek letters $\sigma, \tau, \ldots$ to vary over \$

Every $\sigma \in \$$ is a decidable subset of ω

Moreover, for all $\sigma \in \$$: $\forall m \forall n[(m \in \sigma \wedge m \subseteq n) \rightarrow n \in \sigma]$

and: $\forall \alpha \exists n [\bar{\alpha} n \notin \sigma]$

We may verify these facts by transfinite induction

For all $\sigma \in \$$ and $n \in \omega$, we define subsets ${}^n\sigma$ and σ^n of ω by:

$${}^n\sigma := \{m \mid n * m \in \sigma\}$$

$$\sigma^n := {}^{\langle n \rangle}\sigma = \{m \mid \langle n \rangle * m \in \sigma\}$$

These definitions conform to the arrangements made in 1.2

One proves easily: for all $\sigma \in \$$ and $n \in \omega$: ${}^n\sigma$ and σ^n do again belong to $\$$

We define a binary predicate $\leq$ on $\$$ by transfinite induction:

(I) For all $\sigma \in \$$: $\sigma \leq \phi \rightleftarrows \sigma = \phi$

(II) For all $\sigma, \tau \in \$, \tau \neq \phi$: $\sigma \leq \tau \rightleftarrows \forall m \exists n [\sigma^m \leq \tau^n]$

We make the following observations:

For all $\sigma \in \$$ $\sigma \leq \sigma$

For all $\sigma, \tau, \varphi \in \$$: $(\sigma \leq \tau \wedge \tau \leq \varphi) \rightarrow \sigma \leq \varphi$

For all $\sigma \in \$, n \in \omega$: $\sigma^n \leq \sigma \wedge {}^n\sigma = (\cdot(\sigma^{n_0})^{n_1 \cdots})^{n(lg(n)-1)} \leq \sigma$

Let A and B be decidable subsets of ω and $\gamma \in {}^\omega\omega$.
We define :

$$\gamma : A \hookrightarrow B := \forall n [lg(\gamma(n)) = lg(n)] \wedge \forall m \forall n [m \subseteq n \rightarrow \gamma(m) \subseteq \gamma(n)] \wedge \forall n [n \in A \rightarrow \gamma(n) \in B]$$

(One should think of γ as an attempt to embed A into B)
We also define:

$$A \leq^* B := \exists \gamma [\gamma : A \hookrightarrow B]$$

8.2 <u>Theorem</u>: For all $\sigma, \tau \in \$$ $\sigma \leq \tau \rightleftarrows \sigma \leq^* \tau$

<u>Proof</u>. Remark: $\forall \sigma [\sigma \leq^* \phi \rightleftarrows \sigma = \phi]$, therefore: $\forall \sigma [\sigma \leq \phi \rightleftarrows \sigma \leq^* \phi]$
Our proof will be by transfinite induction.
Assume, therefore: $\sigma, \tau \in \$$ and $\sigma \leq \tau, \tau \neq \phi$. We have to prove: $\sigma \leq^* \tau$.
We know: $\forall m \exists n [\sigma^m \leq \tau^n]$, and may suppose: $\forall m \exists n [\sigma^m \leq^* \tau^n]$

Using AC_{00} and AC_{01}, we determine $\eta \in {}^\omega\omega$ and for each $m \in \omega$ a sequence $\gamma_m \in {}^\omega\omega$ such that:

$$\forall m [\ \gamma_m : \sigma^m \hookrightarrow \tau^{\eta(m)}\ ]$$

We define a new sequence $\gamma \in {}^\omega\omega$ by:

(i) $\gamma(\langle\rangle) := \langle\rangle$

(ii) for all $m, n \in \omega$: $\gamma(\langle m\rangle * n) := \langle\eta(m)\rangle * \gamma_m(n)$

Then: $\gamma : \sigma \hookrightarrow \tau$, and $\sigma \leq^* \tau$

Now assume: $\sigma \leq^* \tau$ and determine $\gamma \in {}^\omega\omega$ such that $\gamma : \sigma \hookrightarrow \tau$

Let $\delta \in {}^\omega\omega$ be such that $\forall m [\ \gamma(\langle m\rangle) = \langle\delta(m)\rangle\ ]$

Remark: $\forall m [\ \sigma^m \leq^* \tau^{\delta(m)}\ ]$, and use the induction assumption to conclude: $\forall m [\ \sigma^m \leq \tau^{\delta(m)}\ ]$, and: $\sigma \leq \tau$

$\boxtimes$

It is useful to consider the corresponding strict order on \$:

For all $\sigma, \tau \in \$$ $\quad \sigma < \tau := \exists n [\sigma \leq \tau^n]$

We take note of the following:

For all $\sigma, \tau \in \$$ $\quad \sigma < \tau \rightarrow \sigma \leq \tau$

For all $\sigma, \tau, \varphi \in \$$: $\quad (\sigma < \tau \wedge \tau \leq \varphi) \rightarrow \sigma < \varphi$

For all $\sigma, \tau, \varphi \in \$$: $\quad (\sigma \leq \tau \wedge \tau < \varphi) \rightarrow \sigma < \varphi$

For all $\sigma, \tau, \varphi \in \$$: $\quad (\sigma < \tau \wedge \tau < \varphi) \rightarrow \sigma < \varphi$

For all $\sigma \in \$$: $\quad \sigma \neq \phi \rightarrow \neg(\sigma < \sigma)$

One possible way to prove the last-mentioned fact is this one:
Suppose: $\sigma \in \$$ and $\sigma < \sigma$. Determine $n \in \omega$ such that $\sigma \leq \sigma^n$, and, applying to theorem 8.2, determine $\gamma \in {}^\omega\omega$ such that $\gamma : \sigma \hookrightarrow \sigma^n$
Let $\alpha \in {}^\omega\omega$ be such that: $\forall n [\ \bar\alpha(Sn) = \langle n\rangle * \gamma(\bar\alpha n)]$, and assume: $\sigma \neq \phi$
We may establish by induction: $\forall n [\ \bar\alpha n \in \sigma]$, contrary to: $\forall \beta \exists n [\ \bar\beta n \notin \sigma]$.

We seize the opportunity for an explicit statement of the principle of transfinite induction, which, to be sure, has been present for some time already:

8.3 (Principle of transfinite induction)

(I) A first formulation: Let $P \subseteq \$$

If $P(\phi)$ and $\forall\sigma [\forall n [P(\sigma^n)] \rightarrow P(\sigma)]$, then $\forall\sigma[P(\sigma)]$.

(II) A second formulation: Let $P \subseteq \$$

If $P(\emptyset)$ and: $\forall \sigma [\forall \tau [\tau < \sigma \to P(\tau)] \to P(\sigma)]$, then $\forall \sigma [P(\sigma)]$

8.4 We do not want to develop ordinal arithmetic; this stump though inviting subject falls outside the scope of this treatise.
We will profit by introducing a special kind of well-ordered stumps.
Doing so, we have to use a pairing function: $\ll \; \gg : {}^2\omega \rightarrowtail\!\!\twoheadrightarrow \omega$

We define the set HI\$ of hereditarily iterative stumps by transfinite induction:

(I) $\{< >\} \in HI\$$

(II) If $A_0, A_1, A_2, \dots$ is a sequence of elements of HI\$, then A belongs to HI\$, where $A := \{< >\} \cup \bigcup_{n,m \in \omega} <\ll n,m \gg> * A_n$

Hereditarily iterative stumps are quite as nice as ordinary stumps and they enjoy one additional property:

For all $\sigma \in HI\$$ $\quad \forall n \exists m [m > n \wedge \sigma^m = \sigma^n]$

We will write: $① := \{< >\}$

We define, by transfinite induction, for each $\sigma \in HI\$$, a subset A_σ and a subset E_σ of ${}^\omega\omega$:

(I) For all $\alpha \in {}^\omega\omega$:
$$A_{①}(\alpha) := \forall n [\alpha(<n>) = 0]$$
$$E_{①}(\alpha) := \exists n [\alpha(<n>) = 0]$$

(II) For all $\sigma \in HI\$$, such that $\sigma \neq ①$ and all $\alpha \in {}^\omega\omega$:
$$A_\sigma(\alpha) := \forall n [E_{\sigma^n}(\alpha^n)]$$
$$E_\sigma(\alpha) := \exists n [A_{\sigma^n}(\alpha^n)]$$

One might ask why we did not include ϕ into HI\$ and introduce $D := E_\phi := A_\phi$ by: For all $\alpha \in {}^\omega\omega$: $D(\alpha) := \alpha(< >) = 0$, but there are disadvantages to this procedure, as in the case of the arithmetical hierarchy. (Cf. 6.15)

We define, for each $\sigma \in HI\$$, a class Π^0_σ and a class Σ^0_σ of subsets of ${}^\omega\omega$ by:
$$\Pi^0_\sigma := \{ P \mid P \subseteq {}^\omega\omega \mid P \leq A_\sigma \}$$
$$\Sigma^0_\sigma := \{ P \mid P \subseteq {}^\omega\omega \mid P \leq E_\sigma \}$$

Each one of these classes is easy to grasp as a whole.

8.5 <u>Theorem</u>: For all $\sigma \in \mathrm{HI\$}$ Π^0_σ and Σ^0_σ do have a universal element.

<u>Proof</u>: As $\Pi^0_{①} = \Pi^0_1$ and $\Sigma^0_{①} = \Sigma^0_1$, where Π^0_1 and Σ^0_1 are our friends from chapter 6, we know from 6.5 and 6.9 how to construct universal elements for these classes.

We proceed by induction.

Suppose, therefore: $\sigma \in \mathrm{HI\$}$, $\sigma \neq ①$ and let $U_{00}, U_{01}, U_{02}, \dots$ and $U_{10}, U_{11}, U_{12}, \dots$ be two sequences of subsets of ${}^\omega\omega$ such that:
$\forall m[U_{0m}$ is a universal element of $\Pi^0_{\sigma m}$ and U_{1m} is a universal element of $\Sigma^0_{\sigma m}]$

We define subsets U_0 and U_1 of ${}^\omega\omega$ by:

For all $\alpha \in {}^\omega\omega$:

$$U_0(\alpha) := \forall m[U_{1m}(\langle \alpha^0, (\alpha^1)^m \rangle)]$$

$$U_1(\alpha) := \exists m[U_{0m}(\langle \alpha^0, (\alpha^1)^m \rangle)]$$

We claim that U_0 and U_1 are universal elements of Π^0_σ and Σ^0_σ, respectively, and prove only half of this claim, as the other half may be established in a similar way.

Let us first see to it that U_0 does belong to Π^0_σ

Using AC_{01}, we find a sequence $\delta_0, \delta_1, \dots$ of elements of ${}^\omega\omega$ such that:
$\forall m[\mathrm{Fun}(\delta_m)]$ and $\forall m \forall \alpha [U_{1m}(\langle \alpha^0, (\alpha^1)^m \rangle) \rightleftarrows E_{\sigma m}(\delta_m | \alpha)]$

Let $\delta \in {}^\omega\omega$ be such that: $\mathrm{Fun}(\delta)$ and: $\forall m \forall \alpha [(\delta|\alpha)^m = \delta_m | \alpha]$

Remark: $\forall \alpha [U_0(\alpha) \rightleftarrows A_\sigma(\delta|\alpha)]$, i.e.: $U_0 \in \Pi^0_\sigma$

Let us prove now, that U_0 is a universal element of Π^0_σ.

Suppose: $P \subseteq {}^\omega\omega$ and: $P \in \Pi^0_\sigma$. Determine $\delta \in {}^\omega\omega$ such that: $\mathrm{Fun}(\delta)$ and: $\forall \alpha [P(\alpha) \rightleftarrows A_\sigma(\delta|\alpha)]$

Consider, for each $m \in \omega$, the set: $\{\alpha | \alpha \in {}^\omega\omega | E_{\sigma m}((\delta|\alpha)^m)\}$ and remark that this set does belong to $\Sigma^0_{\sigma m}$.

As U_{1m} is a universal element of $\Sigma^0_{\sigma m}$, we may determine $\beta \in {}^\omega\omega$ such that: $\forall \alpha [E_{\sigma m}((\delta|\alpha)^m) \rightleftarrows U_{1m}(\langle \alpha, \beta \rangle)]$

Using AC_{01}, we find $\beta \in {}^\omega\omega$ such that:
$\forall m \forall \alpha [E_{\sigma m}((\delta|\alpha)^m) \rightleftarrows U_{1m}(\langle \alpha, \beta^m \rangle)]$.

Therefore: $\forall \alpha [P(\alpha) \rightleftarrows U_0(\langle \alpha, \beta \rangle)]$.

⊠

The following theorems bring together some nice structural properties of the hyperarithmetical hierarchy.

8.6 Theorem: For all $\sigma, \tau \in HI\$$:

$$\text{If } \sigma \leq \tau, \text{ then: } \Pi^0_\sigma \subseteq \Pi^0_\tau \quad \text{and: } \Sigma^0_\sigma \subseteq \Sigma^0_\tau$$
$$\text{If } \sigma < \tau, \text{ then: } \Pi^0_\sigma \subseteq \Sigma^0_\tau \quad \text{and: } \Sigma^0_\sigma \subseteq \Pi^0_\tau$$

Proof: One may prove the first part by showing:
For all $\sigma, \tau \in HI\$$ if $\sigma \leq \tau$, then: $A_\sigma \preceq A_\tau$ and: $E_\sigma \preceq E_\tau$
This is done by transfinite induction, in conformity with the definition of $\leq$.
For the second part, it suffices to show:
For all $\tau \in HI\$$ and $n \in \omega$: $A_{\tau^n} \preceq E_\tau$ and: $E_{\tau^n} \preceq A_\tau$
Let $n \in \omega$ and $\zeta \in {}^\omega\omega$ such that: $Fun(\zeta)$ and:
$\forall \alpha [(\zeta|\alpha)^n = \alpha \wedge \forall m [m \neq n \rightarrow (\zeta|\alpha)^m = \underline{1}]]$. Then: $\forall \alpha [A_{\tau^n}(\alpha) \rightleftarrows E_\tau(\zeta|\alpha)]$
Let $n \in \omega$ and $\eta \in {}^\omega\omega$ such that: $Fun(\eta)$ and:
$\forall \alpha [(\eta|\alpha)^n = \alpha \wedge \forall m [m \neq n \rightarrow (\eta|\alpha)^m = \underline{0}]]$. Then: $\forall \alpha [E_{\tau^n}(\alpha) \rightleftarrows A_\tau(\eta|\alpha)]$
Therefore: $\forall n \in \omega [A_{\tau^n} \preceq E_\tau \wedge E_{\tau^n} \preceq A_\tau]$.
⊠

8.7 Theorem: Let $P \subseteq {}^\omega\omega$ and $\sigma \in HI\$$, $\sigma \neq ①$

$P \in \Pi^0_\sigma$ if and only if there exists a sequence $Q_0, Q_1, \ldots$ of subsets of ${}^\omega\omega$ such that: $\forall m \exists \tau < \sigma [Q_m \in \Sigma^0_\tau]$ and: $P = \bigcap_{m \in \omega} Q_m$

$P \in \Sigma^0_\sigma$ if and only if there exists a sequence $Q_0, Q_1, \ldots$ of subsets of ${}^\omega\omega$ such that: $\forall m \exists \tau < \sigma [Q_m \in \Pi^0_\tau]$ and: $P = \bigcup_{m \in \omega} Q_m$

Proof: We prove the second part.
Suppose: $P \in \Sigma^0_\sigma$ and determine $\delta \in {}^\omega\omega$ such that: $Fun(\delta)$ and:
$\forall \alpha [P(\alpha) \rightleftarrows E_\sigma(\delta|\alpha)]$. Define, for each $m \in \omega$: $Q_m := \{\alpha \mid A_{\sigma^m}((\delta|\alpha)^m)\}$ and remark: $\forall m [Q_m \in \Pi^0_{\sigma^m} \wedge \sigma^m < \sigma]$ and: $P = \bigcup_{m \in \omega} Q_m$
Now suppose: $Q_0, Q_1, \ldots$ is a sequence of subsets of ${}^\omega\omega$ such that: $\forall m \exists \tau < \sigma [Q_m \in \Pi^0_\tau]$. Using the definition of „<" (cf. 8.2) and theorem 8.6, we infer: $\forall m \exists n [Q_m \in \Pi^0_{\sigma^n}]$

Remembering now, that σ is hereditarily iterative, and using AC_{00}, we find $\zeta \in {}^\omega\omega$ such that: $\zeta(0) < \zeta(1) < \zeta(2) \ldots$ and: $\forall m [Q_m \in \Pi^0_{\sigma\zeta(m)}]$.

We define a sequence $\delta_0, \delta_1, \ldots$ of elements of ${}^\omega\omega$ such that:

$\forall m [\mathrm{Fun}(\delta_m)]$ and: $\forall m \forall \alpha [Q_m(\alpha) \rightleftarrows A_{\sigma\zeta(m)}(\delta_m | \alpha)]$.

Finally, we make a sequence $\delta \in {}^\omega\omega$ such that: $\mathrm{Fun}(\delta)$ and:

$\forall m \forall \alpha [(\delta|\alpha)^{\zeta(m)} = \delta_m | \alpha]$ and: $\forall k [\neg \exists m [k = \zeta(m)] \to \forall \alpha [(\delta|\alpha)^k = \underline{1}]]$

We easily verify: $\forall \alpha [\exists m [Q_m(\alpha)] \rightleftarrows E_\sigma(\delta|\alpha)]$, i.e.: $\bigcup_{m \in \omega} Q_m \in \Sigma^0_\sigma$.

The first part is proved in a similar way.

$\boxtimes$

Let us define, for each $\alpha \in {}^\omega\omega$: $|\alpha| := \{n \mid \alpha(n) = 0\}$.
Thus, $|\alpha|$ is a decidable subset of ω, whose characteristic function is α
We may observe that, for each $\alpha \in {}^\omega\omega$ and each $\sigma \in \$$:

$$|\alpha| \leq^* \sigma \quad \rightleftarrows \quad \forall m \exists n [|\alpha^m| \leq^* \sigma^n]$$

We define, for each $\sigma \in \$$: $K_\sigma := \{\alpha \mid |\alpha| \leq^* \sigma\}$ and remark:
K_σ is hyperarithmetical, that is, it does belong to some class Σ^0_τ, $\tau \in HI\$$.
One would like to calculate from σ the first τ such that $K_\sigma \in \Sigma^0_\tau$.
But we do not study „stump-arithmetic", now, and we have to abandon this question
Another problem arises, when we define a partial ordering $\sqsubseteq$ on $\$$ by:

For all $\sigma, \tau \in \$$: $\quad \sigma \sqsubseteq \tau \ := \ K_\sigma \preceq K_\tau$

and ask for a comparison between $\sqsubseteq$ and $\leq$.
This does not seem to be an easy matter, either, and we leave it alone.

We may define a function $O : \omega \setminus \{0\} \to HI\$$ by:

$\quad (1) := \{\langle\, \rangle\}$

$\quad$ For all $n \in \omega$: $(Sn) := \{m \mid lg(m) \leq n\}$

We observe, without difficulty, that: for all $n \in \omega$: $\Pi^0_{(Sn)} = \Pi^0_{Sn}$ and $\Sigma^0_{(Sn)} = \Sigma^0_n$

Thus, the arithmetical hierarchy is seen to be part of the hyperarithmetical hierarchy.
(Remark: $\forall m \forall n [K_{(Sm)} \preceq K_{(Sn)}]$)

The stage has been set, now, for one of the high-points in our little drama: the resuscitation of the hyperarithmetical hierarchy, which now lies flat and lifeless, although not all warmth has left its feet, as we saw in chapter 7.

9. THE HYPERARITHMETICAL HIERARCHY ESTABLISHED

We want to prove: for every $\sigma \in HI\$$: $\neg(A_\sigma \leq E_\sigma) \wedge \neg(E_\sigma \leq A_\sigma)$
The first thing one thinks of when facing this problem, is some extension of the inductive argument by which the arithmetical hierarchy theorem was proved. But it turned out to be rather difficult to find this extension.
We were brought to make some major changes in the original argument.
First, we replaced the negative statements: $\neg(A_\sigma \leq E_\sigma)$ and: $\neg(E_\sigma \leq A_\sigma)$ by stronger conclusions, in which negation does not figure.
Secondly, the proof of the new theorem is no longer inductive in the sense that it reduces the case σ to all cases τ, $\tau < \sigma$.
Rather, it consists in a schematical construction which has to be carried out from start to finish, for any σ anew.
A minor change is that, henceforth, A_2 and E_2 will be considered as the most simple hyperarithmetical sets, and that A_1 and E_1 will be forgotten.
The germ of the proof is to be found in chapter 9. (Chapter 7 had to make the same acknowledgement).
We have to reveal the true richness of the results of chapter 3 and, for this purpose, we introduce some new technical notions.

9.0 Let $\beta \in {}^\omega\omega$ be a <u>spread</u>, i.e.: β fulfils the condition:

$$\forall a [\beta(a) = 0 \rightleftarrows \exists n [\beta(a * \langle n \rangle) = 0] \quad \wedge \quad \beta(\langle\rangle) = 0.$$

Spreads (subspreads of the universal spread: ${}^\omega\omega$) have been mentioned before in 1.9. Let us recall the following definition:

For all $\alpha, \beta \in {}^\omega\omega$:

$$\alpha \in \beta \quad := \quad \forall n [\beta(\bar{\alpha} n) = 0]$$

When talking about a spread β, we often are thinking of the set $\{\alpha | \alpha \in \beta\}$.

For all $\beta \in {}^\omega\omega$ and $a \in \omega$ we define a decidable subset K_a^β of ω by:

$$K_a^\beta := \{ n \mid n \in \omega \mid \beta(a * \langle n \rangle) = 0 \}$$

If β is a spread, the following holds true:

$$\forall a [\beta(a) = 0 \rightleftarrows \exists n [n \in K_a^\beta]]$$

Members of the spread $\{\alpha | \alpha \in \beta\}$ may be built up step by step, in course of time. When during the construction of such a member we have got so far as the finite sequence a, the „choice set" K_a^β displays the natural numbers by which we may continue the finite sequence a.

In the following we will often meet with spreads β whose members α are thought of as being defined on finite sequences of natural numbers, rather than on natural numbers themselves.

Let $\beta \in {}^\omega\omega$ be a spread and $a \in \omega$.
We want to call the finite sequence a free in β, if for every $\alpha\in\beta$, during the step-by-step-construction of α, we did not receive any restrictive injunction from β, as far as ${}^a\alpha$ was concerned.
(We were left free to determine a value of α at the finite sequence a, and at any continuation of the finite sequence a)
This is the exact definition:

a is free in β :=
$$\forall b \forall c[(\beta(b)=0 \wedge lg(b)=lg(c) \wedge \forall m<lg(b)[b(m)\neq c(m) \rightarrow m\subseteq a]) \rightarrow \beta(c)=0]$$

We remark that a is free in β if and only if:
$$\forall\alpha\forall\gamma[(\alpha\in\beta \quad \wedge \quad \forall m[\alpha(m)\neq\gamma(m) \rightarrow m\subseteq a]) \rightarrow \gamma\in\beta]$$

We observe that, if a is free in β, then:
$$\forall n \forall m[(lg(n)=m \quad \wedge \quad m\subseteq a \quad \wedge \quad \beta(n)=0) \rightarrow K_n^\beta = \omega],$$
The converse of this statement is not true in general.

We define a binary predicate $\perp$ on ω by:
For all $a, b\in\omega$: $\quad a \perp b \ := \ \neg(a\subseteq b) \wedge \neg(b\subseteq a)$

We remark that a is free in β if and only if:
$$\forall\alpha\forall\gamma[(\alpha\in\beta \quad\wedge\quad \forall m[(a\perp m \vee a\subseteq m)\rightarrow \alpha(m)=\gamma(m)]) \rightarrow \gamma\in\beta]$$

We also need the following concept:
Let $\beta\in{}^\omega\omega$ be a spread and $a\in\omega$. Then:
$$a \text{ is almost free in } \beta \ := \ \exists p\forall n[n>p \rightarrow a*n \text{ is free in } \beta]$$

9.1. We will prove a suitable refinement of theorem 3.2.
To this end, we introduce the subsets A_2^* and E_2^* of ${}^\omega\omega$, by the following:
for all $\alpha\in{}^\omega\omega$: $\quad A_2^*(\alpha) := \forall m\exists n[\alpha^m(n)\neq 0]$
for all $\alpha\in{}^\omega\omega$: $\quad E_2^*(\alpha) := \exists m\forall n[\alpha^m(n)\neq 0]$

We observe: $\forall\alpha[\neg(A_2(\alpha)\wedge E_2^*(\alpha)) \wedge \neg(E_2(\alpha)\wedge A_2^*(\alpha))]$

When $\gamma,\beta\in{}^\omega\omega$ are spreads, γ is called a <u>subspread</u> of β if $\forall a[\gamma(a)=0 \rightarrow \beta(a)=0]$, or, equivalently, if $\forall\alpha\in\gamma[\alpha\in\beta]$.
We will write: $\gamma\subseteq\beta$, occasionally

9.2 <u>Theorem</u>: Let $\beta\in{}^\omega\omega$ be a spread, $a,b\in\omega$, $\delta\in{}^\omega\omega$ such that: Fun(δ) and:
(i) a is almost free in β
(ii) $\forall\alpha\in\beta[A_2({}^a\alpha) \rightarrow E_2(\delta|\alpha)]$
(iii) $\beta(b)=0$

We now may construct a subspread β' of β such that:

(I) $\beta'(b) = 0$

(II) $\forall\alpha\in\beta' [\, E_2^*({}^a\alpha) \wedge E_2(\delta|\alpha)]$

(III) $\forall c\, [(c \perp a \wedge c$ is almost free in $\beta) \to (c$ is almost free in $\beta')]$

<u>Proof</u>: We have to relativize the proof of theorem 3.2

We determine $p\in\omega$ such that: $\forall n[n>p \to (a*n$ is free in $\beta]$ and: $p > lg(b)$

We assume our coding of finite sequences of natural numbers (cf. 1.2) to be such that $\forall n[\, n < \langle n\rangle]$

Therefore, also the following holds: $\forall n[n>p \to (a*\langle n\rangle)$ is free in $\beta)]$

We now define $\zeta\in{}^\omega\omega$ such that: $Fun(\zeta)$ and: for all $\gamma,\alpha\in{}^\omega\omega$:

$\zeta(\gamma,\alpha) := \zeta|\langle\gamma,\alpha\rangle$ fulfils these conditions:

For all $m\in\omega$:

$$\zeta(\gamma,\alpha)(m) = \alpha(m) \quad \text{if: } m\perp a \text{ or } a\subseteq m, \text{ or } m<lg(b)$$

For all $n,m\in\omega$:

$$\zeta(\gamma,\alpha)\,(a*\langle n\rangle * m) = 0 \quad \text{if: } \langle n\rangle * m > p \text{ and: } n\le p$$

$$\zeta(\gamma,\alpha)\,(a*\langle n\rangle*\langle m\rangle) = 0 \quad \text{if: } n>p \text{ and } m=\gamma(n)$$

$$\zeta(\gamma,\alpha)\,(a*\langle n\rangle*\langle m\rangle) = \alpha(a*\langle n\rangle*\langle m\rangle) \quad \text{if: } n>p \text{ and } m\ne\gamma(n)$$

We remark: $\forall\gamma\forall\alpha[\alpha\in\beta \to (\zeta(\gamma,\alpha)\in\beta \wedge A_2({}^a\zeta(\gamma,\alpha)))]$

Therefore: $\forall\gamma\,\forall\alpha\in\beta[\, E_2(\delta|\zeta(\gamma,\alpha))]$

We choose some $\alpha^*\in\beta$ such that $\alpha^*\in b$ (i.e. $\overline{\alpha^*}(lg(b)) = b$), and some $\gamma^*\in{}^\omega\omega$

Applying to GCP (cf. 1.9), we determine $q\in\omega$, $m\in\omega$ such that:

$q>p$ and $\forall\gamma\,\forall\alpha\in\beta[(\bar\gamma q = \overline{\gamma^*}q \wedge \bar\alpha q = \overline{\alpha^*}q) \to \forall n[(\delta|\zeta(\gamma,\alpha))^m(n) = 0]]$

We then define a subspread β' of β by saying:

For all $\alpha\in\beta$:

$$\alpha\in\beta' \text{ if and only if: } \alpha\in b \wedge \bar\alpha q = \overline{\zeta(\gamma^*,\alpha^*)}q \wedge {}^{a*\langle q\rangle}\alpha = \underline{1}$$
$$\wedge\ \forall n<q[\,{}^{a*\langle n\rangle}\alpha = {}^{a*\langle n\rangle}\zeta(\gamma^*,\alpha^*)]$$

We have to show that β' does everything we want it to do.

Remark that $\forall\alpha\in\beta'[\,{}^{a*\langle q\rangle}\alpha = \underline{1}]$, therefore: $\forall\alpha\in\beta'[\, E_2^*({}^a\alpha)]$

On the other hand: $\forall\alpha\in\beta'[\,(\delta|\alpha)^m = \underline{0}]$ (and: $\forall\alpha\in\beta'[\, E_2(\delta|\alpha)])$

In order to see this, one should realize:

$\forall a[\beta'(a)=0 \rightarrow \exists\gamma\exists\alpha\in\beta[\bar{\gamma}q=\overline{\gamma^*}q \wedge \bar{\alpha}q=\overline{\alpha^*}q \wedge \zeta(\gamma,\alpha)\in a]]$

Therefore: $\forall a[\beta'(a)=0 \rightarrow \exists\alpha[\alpha\in a \wedge (\delta|\alpha)^m = \underline{0}]]$

Let $\alpha\in\beta'$ and consider $\delta|\alpha$

Remark: $\forall n\,\exists m\,\forall\varepsilon[\bar{\varepsilon}m=\bar{\alpha}m \rightarrow (\delta|\varepsilon)(n)=(\delta|\alpha)(n)]$

and: $\forall\alpha\in\beta'\,\forall n[(\delta|\alpha)^m(n)=0]$

(As we put it in 3.2, it is the conscience-stricken nature of the imitative subject which brings triumph to the creative subject)

The remaining properties of β' are obvious.

$\boxtimes$

We are going to prove a similar counterpart to theorem 3.3
We introduce the subset E_1^* of ${}^\omega\omega$ by:

For all $\alpha\in{}^\omega\omega$:

$$E_1^*(\alpha) := \exists n[\alpha(n)\neq 0]$$

We remind the reader of the conjunctive projection operations which have been mentioned in chapter 4 (cf. 4.11)
Let $P\subseteq{}^\omega\omega$ and $n\in\omega$. The subset C^nP of ${}^\omega\omega$ is defined by

For all $\alpha\in{}^\omega\omega$

$$C^nP(\alpha) := \forall q<n[P(\alpha^q)]$$

9.3 <u>Theorem</u>: Let $\beta\in{}^\omega\omega$ be a spread, $a,b,n\in\omega$, $\delta\in{}^\omega\omega$ such that: $Fun(\delta)$ and:

(I) a is almost free in β

(II) $\forall\alpha\in\beta[E_2({}^a\alpha) \rightarrow A_2(\delta|\alpha)]$

(III) $\beta(b)=0$

We now may construct a subspread β' of β such that:

(I) a is almost free in β'

(II) $\beta'(b)=0$

(III) $\forall\alpha\in\beta'[C^n E_1^*({}^a\alpha) \wedge C^n E_1(\delta|\alpha)]$

(IV) $\forall c[(c\nvdash a \wedge c$ is almost free in $\beta) \rightarrow (c$ is almost free in $\beta')]$

<u>Proof</u>: We use the same method as in the proof of theorem 3.3
The present situation is easier to handle, as we have set ourselves a more modest purpose.

We perform our task in a number of steps

First, determine $q_0 \in \omega$ such that $a * \langle q_0 \rangle$ is free in β and $q_0 > lg(b)$

Determine $\alpha_0 \in \beta$ such that: $\alpha_0 \in b \wedge {}^{a*\langle q_0\rangle}\alpha_0 = \underline{0}$

Remark: $E_2({}^a\alpha_0)$, and determine p_0 such that $(\delta|\alpha_0)^0(p_0) = 0$

Also determine $n_0 \in \omega$ such that: $\forall \alpha \in \beta[\bar{\alpha}_0 n_0 = \bar{\alpha} n_0 \to (\delta|\alpha)^0(p_0) = 0]$

We now construct $m_0 \in \omega$, $q_1 \in \omega$ and $\alpha_1 \in \beta$ such that:

$$\bar{\alpha}_1 n_0 = \bar{\alpha}_0 n_0 \wedge \alpha_1 \in b$$

$$\alpha_1(a * \langle 0 \rangle * \langle m_0 \rangle) \neq 0$$

$$^{a*\langle q_1 \rangle}\alpha_1 = \underline{0}$$

Remark: $E_2({}^a\alpha_1)$, and determine $p_1, n_1 \in \omega$ such that $n_1 > n_0$ and $\forall \alpha \in \beta[\bar{\alpha}_1 n_1 = \bar{\alpha} n_1 \longrightarrow (\delta|\alpha)^1(p_1) = 0]$, and: $n_1 > a * \langle 0 \rangle * \langle m_0 \rangle$

⋮

We continue this process for n steps

In the end, we find a sequence $\alpha_n \in \beta$ and a number $k \in \omega$ such that:

$$\alpha_n \in b \wedge \forall \alpha \in \beta[\bar{\alpha}_n k = \bar{\alpha} k \to (\forall l < n[\alpha(a * \langle l \rangle * \langle m_l \rangle) \neq 0] \wedge \forall l < n[(\delta|\alpha)^l(p_l) = 0])]$$

We define a subspread β' of β by saying:

For all $\alpha \in \beta$:

$$\alpha \in \beta' \text{ if and only if } \alpha \in b \wedge \bar{\alpha}_n k = \bar{\alpha} k.$$

It is not difficult to see that β' fulfils all requirements

⊠

In comparison to theorem 9.2, theorem 9.3 does seem to have a rather weak conclusion. On the other hand, the finite sequence a which figures in theorem 9.3, has been kept almost free during its proof. It will be possible, for this reason, to apply theorem 9.3 several times at the same place.

9.4 We now prepare to attack the hyperarithmetical hierarchy. We made its acquaintance in chapter 8, but we redefine it, because it suits us to have it in a slightly different shape.

For each $\tau \in HI\$$, we define subsets P_τ, Q_τ, P^*_τ, Q^*_τ of $^\omega\omega$

We do this by transfinite induction.
As in chapter 8, we will write ① for $\{<>\}$

For all $\alpha \in {}^{\omega}\omega$:

$$P_{①}(\alpha) := A_2(\alpha) = \forall m \exists n [\alpha^m(n) = 0]$$
$$P^*_{①}(\alpha) := \forall m \exists n [\alpha^m(n) \neq 0]$$
$$Q_{①}(\alpha) := E_2(\alpha) = \exists m \forall n [\alpha^m(n) = 0]$$
$$Q^*_{①}(\alpha) := \exists m \forall n [\alpha^m(n) \neq 0]$$

For all $\tau \in HI\$$, $\tau \neq ①$, for all $\alpha \in {}^{\omega}\omega$:

$$P_\tau(\alpha) := \forall n [Q_{\tau^n}(\alpha^n)] \qquad P^*_\tau(\alpha) := \forall n [Q^*_{\tau^n}(\alpha^n)]$$
$$Q_\tau(\alpha) := \exists n [P_{\tau^n}(\alpha^n)] \qquad Q^*_\tau(\alpha) := \exists n [P^*_{\tau^n}(\alpha^n)]$$

We remark: $\forall \tau \in HI\$ \; \forall \alpha [\neg(P_\tau(\alpha) \wedge Q^*_\tau(\alpha)) \wedge \neg(Q_\tau(\alpha) \wedge P^*_\tau(\alpha))]$

We resume a line of thought which we followed in chapter 7.
We recognized $A_n(\alpha)$ and $E_n(\alpha)$ as boastful announcements of players, who were involved in a game on a tree of uniform height n.
Likewise $P_\tau(\alpha)$ and $Q_\tau(\alpha)$ may be understood to say: „ I ($\forall$ resp. $\exists$) am able to win the quantifier-game determined by α on the well-ordered stump τ, whatever the moves of my opponent!

This idea lies behind the following definition.

Let $\gamma, \alpha \in {}^{\omega}\omega$. For each $\tau \in HI\$$ we will define sequences $\gamma \bowtie_\tau \alpha$ and $\gamma \mathbin{\bowtie\!\!\!\!-}_\tau \alpha$ in ${}^{\omega}\omega$. This is done by transfinite induction:

$$\gamma \bowtie_{①} \alpha := \gamma \bowtie_2 \alpha$$
$$\gamma \mathbin{\bowtie\!\!\!\!-}_{①} \alpha := \gamma \mathbin{\bowtie\!\!\!\!-}_2 \alpha$$

$\bowtie_2$ and $\mathbin{\bowtie\!\!\!\!-}_2$ are the intertwining functions which we defined in 7.0
We know, from 7.0.: $\forall \alpha [P_{①}(\alpha) \rightleftarrows \exists \gamma [\alpha = \gamma \bowtie_2 \alpha]]$
and: $\forall \alpha [Q_{①}(\alpha) \rightleftarrows \exists \gamma [\alpha = \gamma \mathbin{\bowtie\!\!\!\!-}_2 \alpha]]$

Further, for each $\tau \in HI\$$ such that $\tau \neq ①$, we define:

$\gamma \bowtie_\tau \alpha$ by: $\forall n [(\gamma \bowtie_\tau \alpha)^n := \gamma^n \mathbin{\bowtie\!\!\!\!-}_{\tau^n} \alpha^n$
and: $\gamma \bowtie_\tau \alpha (<>) := \alpha(<>)$

and $\gamma \mathbin{\bowtie\!\!\!\!-}_\tau \alpha$ by: $(\gamma \mathbin{\bowtie\!\!\!\!-}_\tau \alpha)^n := \gamma^n \bowtie_{\tau^n} \alpha^n$ if $n = \gamma(0)$
$:= \alpha^n$ if $n \neq \gamma(0)$
and: $\gamma \mathbin{\bowtie\!\!\!\!-}_\tau \alpha (<>) := \alpha(<>)$

One more exercise in transfinite induction will learn:

$$\forall\alpha[\,P_\tau(\alpha) \rightleftarrows \exists\gamma[\alpha = \gamma \bowtie_\tau \alpha]]$$

$$\text{and: } \forall\alpha[\,Q_\tau(\alpha) \rightleftarrows \exists\gamma[\alpha = \gamma \boxtimes_\tau \alpha]]$$

We forge a third weapon for the great battle:

9.5 **Theorem**: Let $\tau \in HI\$$, $\tau \neq ①$

Let $\beta \in {}^\omega\omega$ be a spread, $a, b \in \omega$, $\delta \in {}^\omega\omega$ such that: $Fun(\delta)$ and:

(I) a is almost free in β

(II) $\forall\alpha\in\beta\,[\,P_\tau({}^a\alpha) \to Q_\tau(\delta|\alpha)]$

(III) $\beta(b) = 0$

We may construct a subspread β' of β and $n, m \in \omega$ such that:

(I) $\beta'(b) = 0$

(II) $a * \langle n\rangle$ is free in β'

(III) $\tau^n = \tau^m$

(IV) $\forall\alpha\in\beta'\,[\,Q_{\tau^n}(({}^a\alpha)^n) \to P_{\tau^m}((\delta|\alpha)^m)]$

(V) $\forall c\,[(c \perp a \wedge c \text{ is almost free in } \beta) \to c \text{ is almost free in } \beta']$

Proof: We determine $p \in \omega$ such that $\forall n[\,n > p \to (a*n \text{ is free in } \beta)]$ and: $p > lg(b)$

We define $\zeta \in {}^\omega\omega$ such that: $Fun(\zeta)$ and: for all $\gamma, \alpha \in {}^\omega\omega$ the sequence $\zeta(\gamma,\alpha) := \zeta|\langle\gamma,\alpha\rangle$ fulfils the following conditions:

$$\zeta(\gamma,\alpha) \in b\,, \text{ if } \alpha \in b$$

and, for all $m, n \in \omega$: $\zeta(\gamma,\alpha)(m) := \alpha(m)$ if $m \perp a$ or $a \subseteq m$

$\zeta(\gamma,\alpha)(a * \langle n\rangle * m) := 0$ if $\langle n\rangle * m > p$ and $n \le p$

$\zeta(\gamma,\alpha)(a * \langle n\rangle * m) := (\gamma \bowtie_\tau {}^a\alpha)(\langle n\rangle * m)$, if $n > p$

Remark: $\forall n \le p\, \exists l\, \forall m\,[\,m > l \to \zeta(\gamma,\alpha)(a * \langle n\rangle * m) = 0]$

and: $\forall n > p\,[\,({}^a\zeta(\gamma,\alpha))^n = (\gamma \bowtie_\tau {}^a\alpha)^n\,]$.

Therefore: $\forall\gamma\forall\alpha[\,P_\tau({}^a\zeta(\gamma,\alpha))]$, and: $\forall\gamma\forall\alpha[\,\zeta(\gamma,\alpha)\in\beta \to Q_\tau(\delta|\zeta(\gamma,\alpha))]$

Observe, however, that: $\forall\gamma\,\forall\alpha\,[\alpha\in\beta \to \zeta(\gamma,\alpha)\in\beta]$

We choose some $\alpha^* \in \beta$ such that $\alpha^* \in b$, and some $\gamma^* \in {}^\omega\omega$.

Applying to GCP, we determine $q, m \in \omega$ such that:

$q > lg(b) \wedge \forall\gamma\forall\alpha\in\beta\,[(\bar\gamma q = \overline{\gamma^*}q \wedge \bar\alpha q = \overline{\alpha^*}q) \to P_{\tau^m}((\delta|\zeta(\gamma,\alpha))^m)\,]$

We calculate $n \in \omega$ such that: $n > q$, $n > p$ and: $\tau^n = \tau^m$.

(Here we do need the fact that τ is hereditarily iterative)

We define a subspread β' of β by saying:

For all $\alpha \in \beta$: $\alpha \in \beta'$ if and only if: $\bar{\alpha}q = \overline{\zeta(\gamma^*, \alpha^*)}q$

and: $\forall \ell [\ell \neq n \to ({}^a\alpha)^\ell = ({}^a\zeta(\gamma^*, \alpha^*))^\ell]$

Note that: $\beta'(\beta) = 0$ and: $a * \langle n \rangle$ is free in β'

Moreover: $\forall \alpha \in \beta' \, \forall \ell \neq n [\, Q_{\tau^\ell}(({}^a\alpha)^\ell)]$

Suppose: $\alpha \in \beta'$ and: $Q_{\tau^n}(({}^a\alpha)^n)$. Then: $P_\tau({}^a\alpha)$, but also:

$\exists \gamma \exists \alpha^+ \in \beta [\, \bar{\gamma}q = \overline{\gamma^*}q \wedge \overline{\alpha^+}q = \overline{\alpha^*}q \wedge \alpha = \zeta(\gamma, \alpha^+)]$

Therefore: $P_{\tau^m}((\delta | \zeta(\gamma, \alpha^+))^m)$ and: $P_{\tau^m}((\delta|\alpha)^m)$.

Remark, finally, that a member $\alpha \in \beta$, which has a wish to belong to β', need not restrict seriously any of its subsequences ${}^c\alpha$, where $c \nvdash a$. This shows that β' realizes our great expectations.

⊠

Theorem 9.5 will prove its worth as part of our inductive argument. Like theorem 9.2, it has a (dual) companion, but this is too easy to be formulated as a theorem. If we are in a situation where $Q_\tau({}^a\alpha) \to P_\tau(\delta|\alpha)$, we immediately see: $\forall n \forall m [\, P_{\tau^n}(({}^a\alpha)^n) \to Q_{\tau^m}((\delta|\alpha)^m)]$

9.6 There are still a few technical notions to be mentioned.

Let $a \in \omega$ and $lg(a) > 0$. $Pd(a)$ (predecessor of a) is to be the code number of the finite sequence, which we get by omitting the last number from the finite sequence whose code number is a.

Therefore, for each a such that $lg(a) > 0$: $a = Pd(a) * \langle a_{lg(a)-1} \rangle$

$Pd(\langle\rangle) = Pd(0)$ will be undefined.

Let $\tau \in \$$, and $a \in \tau$. We call a an endpoint of τ if no proper extension of a does belong to τ, i.e. if $\neg \exists n [a * \langle n \rangle \in \tau]$

For any $\tau \in \$$, the collection $\{a \mid a \in \omega \mid a$ is an endpoint of $\tau\}$ is a decidable subset of ω.

One could define the notion of „endpoint of τ" by transfinite induction, as follows:

(We write: $End(\tau)$ for the collection of endpoints of τ)

(I) $End(①) = End(\{\langle\rangle\}) = \{\langle\rangle\}$ and: $End(\phi) = \phi$

(II) If $\tau > ①$: $End(\tau) := \bigcup_{n \in \omega} \langle n \rangle * End(\tau^n)$

This finishes our preparations. We take a long breath and summon up our courage:

9.7 <u>Theorem</u>: (Hyperarithmetical Hierarchy Theorem, First Part)

Let $\tau \in HI\$$ and $\delta \in {}^{\omega}\omega$ such that: $Fun(\delta)$ and: $\forall\alpha[P_\tau(\alpha) \to Q_\tau(\delta|\alpha)]$.
We may construct $\zeta \in {}^{\omega}\omega$ such that: $Q^*_\tau(\zeta)$ and $Q_\tau(\delta|\zeta)$.

<u>Proof</u>: The proof is divided into several paragraphs.
We will spend a lot of words on giving a synopsis of our intentions, before going to work

9.70 We plan to define a decidable subset W of τ such that:

(i) $\langle\,\rangle \in W$

(ii) $\forall a[(a \in W \wedge lg(a)$ is even $\wedge$ a is no endpoint of $\tau) \to \exists! n[a * \langle n\rangle \in W]]$

(iii) $\forall a[(a \in W \wedge lg(a)$ is odd $\wedge$ a is no endpoint of $\tau) \to \forall n[a * \langle n\rangle \in W]]$

The set W represents a strategy for the first player in a quantifier-game on the well-ordered stump τ. It will be the strategy which the statement „$Q^*_\tau(\zeta)$" asserts to exist.

At the same time, we will build a function $H: W \to \tau$, such that:

(i) $H(\langle\,\rangle) = \langle\,\rangle$

(ii) $\forall a \in W\,[\,{}^{a}\tau = {}^{H(a)}\tau\,]$

(iii) $\forall a \in W[(lg(a)$ is even $\wedge$ a is no endpoint of $\tau) \to$
$$\exists n \exists p\,[H(a * \langle n\rangle) = H(a) * \langle p\rangle]$$

(iv) $\forall a \in W[(lg(a)$ is odd $\wedge$ a is no endpoint of $\tau) \to$
$$\forall n\,[H(a * \langle n\rangle) = H(a) * \langle n\rangle]$$

The function H carries positions of τ which belong to W, into structurally equivalent positions of τ. (As τ is hereditarily iterative, there are, at every turn, many such positions).
The range of the function H again represents a first-player-strategy on τ. This strategy will speak for the truth of: $Q_\tau(\delta|\zeta)$.

(We assumed familiarity with the logical convention that „$\exists!$" stands for: „there exists exactly one..").

9.71 In the following we will have to consider all natural numbers, in their natural order, decoding them into finite sequences of natural numbers. (Cf. 1.2).

We assume our coding of finite sequences to be „regular", in the following sense of the word:

(I) $\forall m \forall n \forall p [n<p \rightarrow (m*<n> < m*<p> \wedge <n>*m < <p>*m)]$

(II) $\forall m \forall n [m \leq m*n]$

The latter condition has already been mentioned in 1.2.

9.72 The sequence ζ will be made step-by-step.
We will form a sequence $\beta_0, \beta_1, \ldots$ of subspreads of ${}^\omega\omega$, such that:
${}^\omega\omega = \beta_0 \supseteq \beta_1 \supseteq \cdots$
Each time, having defined β_k, we also determine a next value for ζ, viz. $\zeta(k)$, and ensure: $\beta_k(\bar{\zeta}Sk) = 0$
In the end, we have: $\forall k [\zeta \in \beta_k]$

9.73 The constructions of W, H and $\beta_0, \beta_1, \ldots$ do connect.
They will be made such, that for all $k, a, n \in \omega$:

(I) If $k = <0>*a$ and $a \in W$ and $lg(a)$ is even and a is not an endpoint of τ, then:
$$\forall \alpha \in \beta_k [P_{a_\tau}({}^a\alpha) \rightarrow Q_{a_\tau}({}^{H(a)}(\delta|\alpha))]$$

(II) If $k = <0>*a$ and $a \in W$ and $lg(a)$ is odd and a is not an endpoint of τ, then:
$$\forall \alpha \in \beta_k [Q_{a_\tau}({}^a\alpha) \rightarrow P_{a_\tau}({}^{H(a)}(\delta|\alpha))]$$

(III) If $k = <0>*a$ and $a \in W$ and $lg(a)$ is even and a is an endpoint of τ, then:
$$\forall \alpha \in \beta_k [E_2^*({}^a\alpha) \wedge E_2({}^{H(a)}(\delta|\alpha))]$$

(IV) If $k = <n>*a$ and $a \in W$ and $lg(a)$ is odd and a is an endpoint of τ, then:
$$\forall \alpha \in \beta_k [C^n E_1^*({}^a\alpha) \wedge C^n E_1({}^{H(a)}(\delta|\alpha))]$$

9.74 Once these things come true, we establish:
$$\forall a \in W [(lg(a) \text{ is even} \rightarrow Q^*_{a_\tau}({}^a\zeta) \wedge Q_{a_\tau}({}^{H(a)}(\delta|\zeta))) \wedge (lg(a) \text{ is odd} \rightarrow P^*_{a_\tau}({}^a\zeta) \wedge P_{a_\tau}({}^{H(a)}(\delta|\zeta)))]$$
and therefore, as $<> \in W$: $Q^*_\tau(\zeta) \wedge Q_\tau(\delta|\zeta)$

This is done by transfinite induction.
The principle sustaining this part of the argument, runs as follows:

Let $\tau \in \$$ and $R \subseteq \tau$
If: $\forall a\,[\,a$ is endpoint of $\tau \rightarrow R(a)]$
and: $\forall a\,[\;\forall n\,[R(a * \langle n \rangle)] \rightarrow R(a)]$
then: $\forall a \in \tau\,[R(a)]$, especially : $R(\langle\rangle)$

9.75 The construction of W and H will not be done in advance, at one stroke, but will proceed stepwise, and intertwine with the construction of $\beta_0, \beta_1, \dots$
We should be careful that, for any $a \in \tau$, the decision about a's belonging to W, and, if necessary, the determination of $H(a)$, have been passed before we come to stage $k = \langle 0 \rangle * a$, in which β_k has to be created.
We settle these things, for each $a \in \tau$, if $lg(a)$ is odd, at stage $\langle 0 \rangle * Pd(a)$, and, if $lg(a)$ is even, even earlier, viz. at stage $\langle 0 \rangle * Pd(Pd(a))$

9.76 In our construction, active stages will occur along with inactive ones.
At an inactive stage $k+1$, β_{k+1} is simply put equal to β_k.
At an active stage $k+1$, one of the following cases applies:

(I) $k+1 = \langle 0 \rangle * a$, where $a \in W$, $lg(a)$ is even, and a is not an endpoint of τ.
The formation of β_{k+1} is left to theorem 9.5.

(II) $k+1 = \langle 0 \rangle * a$, where $a \in W$, $lg(a)$ is even, and a is an endpoint of τ
The formation of β_{k+1} is left to theorem 9.2.

(III) $k+1 = \langle n \rangle * a$, where $a \in W$, $lg(a)$ is odd, and a is an endpoint of τ
The formation of β_{k+1} is left to theorem 9.3.

Turn and again, the work is to be done by theorems 9.2, 9.3 and 9.5. They will not object, if only we ensure that, for all $k, a, n \in \omega$: if $k+1 = \langle n \rangle * a$ is an active stage, then a is almost free in β_k.

This necessitates some retrospection. Careful reading of theorems 9.2, 9.3 and 9.5 learns, that a cannot have lost its almost-freedom at any stage $\langle m \rangle * c < k+1$, where $c \neq a$.
In each of the three abovementioned cases we go back to a critical preceding stage:

(I)-(II) $k+1 = \langle 0 \rangle * a$, where $a \in W$ and $lg(a)$ is even.
The critical preceding stage is: $\langle 0 \rangle * Pd(Pd(a))$
We will see that, at this stage, $Pd(a) \in W$ has been chosen such that: $Pd(a)$ is free in $\beta_{\langle 0 \rangle * Pd(Pd(a))}$
Therefore, a itself enjoyed freedom at this stage.
The only possible stage at which a might have lost its almost-liberty, is: $\langle 0 \rangle * Pd(a)$, but there was no activity, then.

(III)′ $k+1 = \langle 0 \rangle * a$, where $a \in W$, $lg(a)$ is odd and a is an endpoint of τ. The critical preceding stage is: $\langle 0 \rangle * Pd(a)$. Going back, we will have to observe: $a \in W$ has been chosen such that: a is free in $\beta_{\langle 0 \rangle * Pd(a)}$
Therefore, a still is almost free in β_k.

(III)″ $k+1 = \langle Sn \rangle * a$, where $n \in \omega$, $a \in W$, $lg(a)$ is odd and a is an endpoint of τ. The critical preceding stage is: $\langle n \rangle * a$. An examination of theorem 9.3 who made the activity at that stage, allays our fears: a is almost-free in $\beta_{\langle n \rangle * a}$, and so it is in β_k.

9.77 We now describe the construction.
At each stage k, β_k and $Z(k)$ will be defined.
Moreover, if $k = \langle 0 \rangle * a$, and $a \in W$ and $lg(a)$ is even, and a is no endpoint of τ, we decide, for all finite sequences c, such that: $c \subseteq a$ and $(lg(c) = lg(a)+1$ or: $lg(c) = lg(a)+2)$, whether c belongs to W, and we define the function H for all finite

sequences which are admitted into W.

Stage 0: We proclaim: $\beta_0 := {}^{\omega}\omega$ and $\zeta(0) := 0$ and $<>\in W$ and $H(<>) := <>$

We know: $\forall\alpha\in\beta_0\,[\,P_{<>_\tau}({}^{<>}\alpha) \to Q_{<>_\tau}({}^{H(<>)}(\delta|\alpha))\,]$

Stage k+1: We distinguish several cases:

(1) $k+1 = <0>*a$, where $a\in W$, $lg(a)$ is even and a is not an endpoint of τ.

We may assume:

(i) a is almost free in β_k

(ii) $\forall\alpha\in\beta_k\,[\,P_{a_\tau}({}^{a}\alpha) \to Q_{a_\tau}({}^{H(a)}(\delta|\alpha))\,]$

(iii) $\beta_k(\,\bar{\zeta}(k+1)) = 0$

Applying theorem 9.5 we construct a subspread β_{k+1} of β_k, and $n, m\in\omega$ such that:

(i) $\beta_{k+1}(\,\bar{\zeta}(k+1)) = 0$

(ii) $a*<n>$ is free in β_{k+1}

(iii) $a*<n>_\tau = a*<m>_\tau$

(iv) $\forall\alpha\in\beta_{k+1}\,[\,Q_{a*<n>_\tau}({}^{a*<n>}\alpha) \to P_{a*<n>_\tau}({}^{H(a)*<m>}(\delta|\alpha))\,]$

(v) $\forall c\,[(c \nvdash a \wedge c$ is almost free in $\beta_k) \to (c$ is almost free in $\beta_{k+1})]$

We extend the definitions of the set W and the function H by:

$$\forall c\,[(c\leq a \wedge lg(c) = lg(a)+1) \to (c\in W \rightleftarrows c = a*<n>)]$$

$$\text{and: } H(a*<n>) := H(a)*<m>$$

If $a*<n>$ is an endpoint of τ, there is no more to be said.
If not, we add:

$$\forall c\,[(c\leq a \wedge lg(c) = lg(a)+2) \to (c\in W \rightleftarrows \exists l\,[c = a*<n>*<l>])]$$

$$\text{and: } H(a*<n>*<l>) := H(a)*<m>*<l>$$

Remark that W may approve of its new members, because, in view of (iv):

$$\forall c\,[\,(c\leq a \wedge lg(c) = lg(a)+2) \to \forall\alpha\in\beta_{k+1}\,[\,P_{c_\tau}({}^{c}\alpha) \to Q_{c_\tau}({}^{H(c)}(\delta|\alpha))]]$$

We finish the activities of this stage by determining $\zeta(k+1)$ such that: $\beta_{k+1}(\,\bar{\zeta}(k+2)) = 0$.

(II) $k+1 = \langle 0 \rangle * a$, where $a \in W$, $lg(a)$ is even and a is an endpoint of τ.

Now: ${}^{a}\tau = {}^{H(a)}\tau = \{\langle\rangle\} = ①$ and: $P_{a_\tau} = A_2$ and: $Q_{a_\tau} = E_2$

We may assume:

(i) a is almost free in β_k.

(ii) $\forall \alpha \in \beta_k\, [\, A_2({}^{a}\alpha) \rightarrow E_2({}^{H(a)}(\delta|\alpha))]$.

(iii) $\beta_k(\bar{\zeta}(k+1)) = 0$.

Applying theorem 9.2 we construct a subspread β_{k+1} of β_k such that:

(i) $\beta_{k+1}(\bar{\zeta}(k+1)) = 0$.

(ii) $\forall \alpha \in \beta_{k+1}\, [\, E_2^*({}^{a}\alpha) \wedge E_2({}^{H(a)}(\delta|\alpha))]$.

(iii) $\forall c[(c \nleq a \wedge c$ is almost free in $\beta_k) \rightarrow (c$ is almost free in $\beta_{k+1})]$.

We finish by determining $\zeta(k+1)$ such that $\beta_{k+1}(\bar{\zeta}(k+2)) = 0$.

(III) $k+1 = \langle n \rangle * a$, where $a \in W$, $lg(a)$ is odd, and a is an endpoint of τ.

We may assume:

(i) a is almost free in β_k.

(ii) $\forall \alpha \in \beta_k\, [\, E_2({}^{a}\alpha) \rightarrow A_2({}^{H(a)}(\delta|\alpha))]$.

(iii) $\beta_k(\bar{\zeta}(k+1)) = 0$.

Applying theorem 9.3 we construct a subspread β_{k+1} of β_k such that:

(i) a is almost free in β_{k+1}

(ii) $\beta_{k+1}(\bar{\zeta}(k+1)) = 0$

(iii) $\forall \alpha \in \beta_{k+1}\, [\, C^n E_1^*({}^{a}\alpha) \wedge C^n E_1({}^{H(a)}(\delta|\alpha))]$

(iv) $\forall c[(c \nleq a \wedge c$ is almost free in $\beta_k) \rightarrow (c$ is almost free in $\beta_{k+1})]$

Our last activity is to determine $\zeta(k+1)$ such that $\beta_{k+1}(\bar{\zeta}(k+2)) = 0$.

(IV) If we are not in case (I)-(II)-(III), stage $k+1$ is an inactive stage. In order not to fall asleep completely, we perform two simple actions: we put $\beta_{k+1} := \beta_k$ and choose $\zeta(k+1)$ such that: $\beta_{k+1}(\bar{\zeta}(k+2)) = 0$.

This concludes the description of our main construction, and ends the proof of theorem 9.7.

⊠

9.8 We do not want to leave theorem 9.7 alone in paradise. It will be but a minor effort to give it a companion.

We remark that, for each $\tau \in HI\$$, the class Π^0_τ is closed under the operation of countable intersection, i.e.: if $Q_0, Q_1, \ldots$ is a sequence of elements of Π^0_τ, then $\bigcap_{m\in\omega} Q_m$ again belongs to Π^0_τ. This follows from theorem 8.7, by a not too difficult argument, based on AC_{01}.
Hence we are able to find, for each $\tau \in HI\$$, $\eta \in {}^\omega\omega$ such that $Fun(\eta)$ and $\forall\alpha[\forall n[P_\tau(\alpha^n)] \rightleftarrows P_\tau(\eta|\alpha)]$.

We introduce a successor-function S on $HI\$$ by:

For all $\tau \in HI\$$ $\quad \forall n[(S\tau)^n = \tau]$.

Referring once more to the previous chapter, esp. theorem 8.6, we observe:
$\tau < S\tau$ and $\Pi^0_\tau \subseteq \Sigma^0_\tau$ and $\Sigma^0_\tau \subseteq \Pi^0_{S\tau}$.

9.9 <u>Theorem</u>: (Hyperarithmetical Hierarchy Theorem, Second Part).

Let $\tau \in HI\$$ and $\delta \in {}^\omega\omega$ such that: $Fun(\delta)$ and: $\forall\alpha[Q_\tau(\alpha) \to P_\tau(\delta|\alpha)]$.
We may construct $\zeta \in {}^\omega\omega$ such that: $P^*_\tau(\zeta)$ and $P_\tau(\delta|\zeta)$.

<u>Proof</u>: Let $\tau \in HI\$$ and $\delta \in {}^\omega\omega$ be such that: $Fun(\delta)$ and: $\forall\alpha[Q_\tau(\alpha) \to P_\tau(\delta|\alpha)]$
Remark: $\forall\alpha[P_{S\tau}(\alpha) \to \forall n[Q_\tau(\alpha^n)]]$, and, therefore:

$$\forall\alpha[P_{S\tau}(\alpha) \to \forall n[P_\tau(\delta|\alpha^n)]].$$

Let $\eta \in {}^\omega\omega$ be such that: $Fun(\eta)$ and: $\forall\alpha[\forall n[P_\tau(\alpha^n)] \rightleftarrows P_\tau(\eta|\alpha)]$.
Let $\delta' \in {}^\omega\omega$ be such that: $Fun(\delta')$ and: $\forall\alpha\forall n[(\delta'|\alpha)^n = \delta|\alpha^n]$.
We observe : $\forall\alpha[P_{S\tau}(\alpha) \to P_\tau(\eta|(\delta'|\alpha))]$.
Let $\varepsilon \in {}^\omega\omega$ be such that: $Fun(\varepsilon)$ and : $\forall\alpha\forall n[(\varepsilon|\alpha)^n = \eta|(\delta'|\alpha)]$
We observe : $\forall\alpha[P_{S\tau}(\alpha) \to Q_{S\tau}(\varepsilon|\alpha)]$.
Applying theorem 9.7 we find $\zeta' \in {}^\omega\omega$ such that : $Q^*_{S\tau}(\zeta')$ and $Q_{S\tau}(\varepsilon|\zeta')$
Determine $m \in \omega$ such that $P^*_\tau((\zeta')^m)$ and remark: $P_\tau(\eta|(\delta'|\zeta'))$, therefore: $\forall n[P_\tau((\delta'|\zeta')^n)]$, and: $\forall n[P_\tau(\delta|(\zeta')^n)]$, especially: $P_\tau(\delta|(\zeta')^m)$.
The sequence $\zeta := (\zeta')^m$ is a good sequence, indeed.

⊠

9.10 Theorems 9.7 and 9.9 do solve many problems.

We may define a function $*: \omega \setminus \{0,1\} \to HI\$$ by:

$$2^* := ① = \{<>\}$$
$$(Sn)^* := S(n^*)$$

We observe: $\forall n > 1 [\ A_n \preceq P_{n^*} \preceq A_n$ and $E_n \preceq Q_{n^*} \preceq E_n\ ]$.

In this way, the arithmetical hierarchy theorem (theorem 7.3) is seen to follow from the hyperarithmetical hierarchy theorem, and proves to admit of a stronger formulation than it has been given in chapter 7.

We may define subsets K, L of ${}^{\omega}\omega$ by:

For all $\alpha \in {}^{\omega}\omega$: $K(\alpha) := \forall n\ [\ A_n(\alpha^n)]$

For all $\alpha \in {}^{\omega}\omega$: $L(\alpha) := \exists n\ [\ A_n(\alpha^n)]$

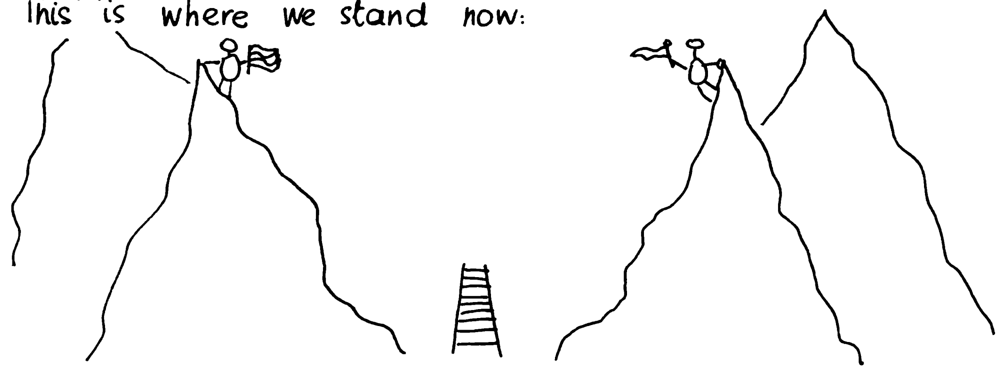
The question whether K and L are reducible to each other, seemed one of the first problems to try one's force on, after the arithmetical hierarchy had been established.
After some reflection, one comes to suspect: $\neg(K \preceq L)$ and $\neg(L \preceq K)$, and, indeed, it is not difficult to see that: $\neg(K \preceq L)$
On the other hand, the proof of: $\neg(L \preceq K)$ took blood, sweat and tears.
Actually, it is a consequence of the hyperarithmetical hierarchy theorem:
Let us define ω^* in $HI\$$ by:

For all $n, m \in \omega$: $\omega^{*\langle n,m\rangle} := n^*$

($\langle\ \rangle$ is the pairing function, introduced in 8.4)
Then: $K \preceq P_{\omega^*} \preceq K$ and: $L \preceq Q_{\omega^*} \preceq L$

As another consequence of the hyperarithmetical hierarchy theorem, we have, that, for each $\tau \in HI\$$, Π^0_τ is <u>not</u> closed under the operation of countable union, and Σ^0_τ is <u>not</u> closed under the operation of countable intersection.

9.11 This is where we stand now:



Perhaps because of breathing deeply the thin air of higher mathematics, we are feeling slightly euphoric ....

10. ANALYTICAL AND CO-ANALYTICAL SETS

We introduce Σ^1_1, the class of analytical sets, and verify that all hyperarithmetical sets are analytical.
We remark that the class of <u>strictly</u> <u>analytical</u> sets, i.e. sets which are the range of a total (and therefore continuous) function on ${}^\omega\omega$, is a proper subclass of Σ^1_1, as not even all hyperarithmetical sets are strictly analytical.
This is a pity, because strictly analytical sets are the things people liked to have of old; indeed, they are none other than Brouwer's dressed spreads.
In the definition of Π^1_1, the class of co-analytical sets, no reference is made to negation.
The symmetry of the classical picture is utterly lost: Σ^0_2 already fails to be included in Π^1_1.
A very annoying question remains, whether Π^1_1 is included in Σ^1_1
We are not able to answer this.

10.0 We define a subset E^1_1 of ${}^\omega\omega$ by:

For all $\alpha\in{}^\omega\omega$:

$$E^1_1(\alpha) := \exists\gamma\forall n[\alpha(\bar\gamma n)=0]$$

We define a class Σ^1_1 of subsets of ${}^\omega\omega$ by:

For every subset P of ${}^\omega\omega$:

$$P\in\Sigma^1_1 \rightleftarrows P\preceq E^1_1$$

This last definition one may feel hesitant to accept, in the absence of a general notion of „subset of ${}^\omega\omega$"
But other characterizations of Σ^1_1 will follow and enable us to survey the whole of its members.
The difficulty then evaporates, like it did in the case of Σ^0_1 and other classes of the (hyper) arithmetical hierarchy. (Cf. 6.0 and 1).

10.1 <u>Theorem</u>: Let $P\subseteq{}^\omega\omega$

$P\in\Sigma^1_1$ if and only if there exists a decidable subset A of ω such that $\forall\alpha[P(\alpha)\rightleftarrows\exists\gamma\forall n[\langle\bar\alpha n,\bar\gamma n\rangle\in A]]$

<u>Proof</u>: (i) Suppose $P\preceq E^1_1$, i.e.: $\forall\alpha\exists\beta[P(\alpha)\rightleftarrows E^1_1(\beta)]$. Using AC_{11}, determine $\delta\in{}^\omega\omega$ such that: $\mathrm{Fun}(\delta)$ and: $\forall\alpha[P(\alpha)\rightleftarrows E^1_1(\delta|\alpha)]$

Define a decidable subset A of ω by:

For all $n\in\omega$:

$$n\in A \rightleftarrows \exists a\exists c[n=\langle a,c\rangle\wedge lg(a)=lg(c)\wedge \forall d\,\forall b[(a\subseteq b\wedge c\subseteq d\wedge \delta^d(b)\neq 0\wedge\forall e[(b\subseteq e\wedge b\neq e)\to\delta^d(e)=0])\to\delta^d(b)=1]]$$

Now, $\forall\alpha[\exists\gamma\forall n[(\delta|\alpha)(\bar{\gamma}n)=0] \rightleftarrows \exists\gamma\forall n[\langle\bar{\alpha}n,\bar{\gamma}n\rangle\in A]]$
Therefore, A fulfils the requirements.

(II) Let A be a decidable subset of ω such that: $\forall\alpha[P(\alpha)\rightleftarrows\exists\gamma\forall n[\langle\bar{\alpha}n,\bar{\gamma}n\rangle\in A]]$
Determine $\delta\in{}^{\omega}\omega$ such that $\mathrm{Fun}(\delta)$ and:

For all $\alpha\in{}^{\omega}\omega$ and $c\in\omega$:

$$(\delta|\alpha)(c)=0 \quad\rightleftarrows\quad \forall n<lg(c)[\langle\bar{\alpha}n,\bar{c}n\rangle\in A]$$

($\bar{c}n$ is the code number of the finite sequence of length n, which is an initial part of the finite sequence coded by c. This notation has been established in 7.0).

Remark: $\forall\alpha[P(\alpha)\rightleftarrows E_1^1(\delta|\alpha)]$, therefore: $P\preceq E_1^1$.

$\boxtimes$

10.2 We again (as in 7.0) extend a notational convention which we introduced in chapter 1, from infinite sequences to finite sequences.

For all $m,c\in\omega$:

c^m := the code number of the m-th subsequence of the finite sequence coded by c.

Therefore, for all $m,c,k\in\omega$, $c^m(k)$ is defined if and only if $\langle m\rangle * k < lg(c)$ and:

$$c^m(k) := c(\langle m\rangle * k) \text{ for all } k \text{ such that } \langle m\rangle * k < lg(c)$$

This notation could give rise to confusion with ordinary exponentiation, but we hope it will not do so, as exponentiation will not occupy us any more (Having figured in chapter 3, it may sink into oblivion).

We remind the reader of another definition which appeared in 7.0:

For all $n,c\in\omega$ such that $n\le lg(c)$

$\bar{c}n = \bar{c}(n)$:= the code number of that finite sequence of length n which is an initial part of the finite sequence coded by c.

10.3 <u>Theorem</u>: Let $P_0, P_1, P_2, \ldots$ be a sequence of subsets of ${}^{\omega}\omega$.

If $\forall m[P_m\in\Sigma_1^1]$, then: $\bigcup_{m\in\omega}P_m\in\Sigma_1^1$ and $\bigcap_{m\in\omega}P_m\in\Sigma_1^1$

<u>Proof</u>: Using theorem 10.1, determine a sequence $A_0, A_1, A_2, \ldots$ of decidable subsets of ω such that: $\forall m \forall \alpha [P_m(\alpha) \rightleftarrows \exists \gamma \forall n [\langle \bar{\alpha} n, \bar{\gamma} n\rangle \in A_m]]$

(I) Define a subset A of ω by:

For all $b \in \omega$:

$b \in A \rightleftarrows \exists m \exists t \exists a \exists c [b = \langle a * \langle t\rangle, \langle m\rangle * c\rangle \wedge lg(a) = lg(c) \wedge \langle a, c\rangle \in A_m]$

Then: $\forall b [b \in A \vee \neg(b \in A)]$ and: $\forall m \forall \alpha [P_m(\alpha) \rightleftarrows \exists \gamma [\gamma(0) = m \wedge \forall n [\langle \bar{\alpha} n, \bar{\gamma} n\rangle \in A]]]$

Therefore: $\forall \alpha [\exists m [P_m(\alpha)] \rightleftarrows \exists \gamma \forall n [\langle \bar{\alpha} n, \bar{\gamma} n\rangle \in A]]$.

and: $\bigcup_{m \in \omega} P_m \in \Sigma^1_1$

(II) Define a subset A of ω by:

For all $b \in \omega$:

$b \in A \rightleftarrows \exists a \exists c [b = \langle a, c\rangle \wedge lg(a) = lg(c) \wedge \forall n \forall m [n < lg(c^m) \rightarrow \langle \bar{a} n, \overline{c^m} n\rangle \in A_m]]$

Then: $\forall b [b \in A \vee \neg(b \in A)]$ and:

$$\forall \alpha \forall \gamma [\forall n [\langle \bar{\alpha} n, \bar{\gamma} n\rangle \in A] \rightleftarrows \forall n \forall m [\langle \bar{\alpha} n, \overline{\gamma^m} n\rangle \in A_m]]$$

Therefore (by AC_{01}): $\forall \alpha [\forall m [P_m(\alpha)] \rightleftarrows \exists \gamma \forall n [\langle \bar{\alpha} n, \bar{\gamma} n\rangle \in A]]$

and: $\bigcap_{m \in \omega} P_m \in \Sigma^1_1$.

$\boxtimes$

10.4 The property of Σ^1_1 which came to light in theorem 10.3 is a beautiful one, and worthy of paraphrase.

Let $P \subseteq {}^\omega\omega$

We define subsets $Ex(P)$ and $Un(P)$ of ${}^\omega\omega$ by:

For all $\alpha \in {}^\omega\omega$: $Ex(P)(\alpha) := \exists m [P(\alpha^m)]$

For all $\alpha \in {}^\omega\omega$: $Un(P)(\alpha) := \forall m [P(\alpha^m)]$.

P is called <u>existentially</u> <u>saturated</u> if: $Ex(P) \preceq P$

P is called <u>universally</u> <u>saturated</u> if: $Un(P) \preceq P$

Theorem 10.3 shows that E^1_1 is both existentially and universally saturated.

We may gather, from theorem 8.7, that, for each $\sigma \in HI\$$, the set A_σ is universally saturated, and the set E_σ is existentially saturated.
Imagine P to be a subset of ${}^\omega\omega$ which is both universally and existentially saturated, such that $A_1 \preceq P$ and $E_1 \preceq P$. Induction shows, that, for every $\sigma \in HI\$$: $A_\sigma \preceq P$ and $E_\sigma \preceq P$.
Thus we learn, from the hyperarithmetical hierarchy theorem (theorems 9.7 and 9.9), that, for each $\sigma \in HI\$$, the set A_σ is <u>not</u> existentially saturated, and

the set E_σ is not universally saturated.
Moreover, as for each $\sigma \in HI\$$, $A_\sigma \preceq E^1_1$ and $E_\sigma \preceq E^1_1$, E^1_1 itself is not reducible to any set A_σ or E_σ, E^1_1 is not hyperarithmetical.
This is another consequence of the hyperarithmetical hierarchy theorem.

10.5 Theorem: Σ^1_1 contains a universal element

Proof: Define the subset $\mathcal{U}$ of ${}^\omega\omega$ by:

For all $\alpha \in {}^\omega\omega$: $\mathcal{U}(\alpha) \rightleftarrows \exists\gamma\forall n [\alpha^1(\langle\bar{\alpha}_0 n, \bar{\gamma}n\rangle) = 0]$

and note that $\mathcal{U}$ belongs to Σ^1_1

Let $P \subseteq {}^\omega\omega$ and $P \in \Sigma^1_1$.

Following theorem 10.1 determine a decidable subset A of ω such that: $\forall\alpha[P(\alpha) \rightleftarrows \exists\gamma\forall n[\langle\bar{\alpha}n,\bar{\gamma}n\rangle \in A]]$. Determine $\beta \in {}^\omega\omega$ such that: $\forall n[\beta(n) = 0 \rightleftarrows n \in A]$. Then: $\forall\alpha[P(\alpha) \rightleftarrows \exists\gamma\forall n[\beta(\langle\bar{\alpha}n,\bar{\gamma}n\rangle) = 0]]$, i.e.: $\forall\alpha[P(\alpha) \rightleftarrows \mathcal{U}(\langle\alpha,\beta\rangle)]$.

$\boxtimes$

It is easy, though not exciting, to exhibit, by diagonalizing, a subset of ${}^\omega\omega$ which does not belong to Σ^1_1.
Our mind is exercised more by the question whether a set outside Σ^1_1 may be found, in whose definition no mention is made of negation.

In 3.1 we defined, for each $\delta \in {}^\omega\omega$, a subset $Ra(\delta)$ of ${}^\omega\omega$ by.

$Ra(\delta) := \{\alpha \mid \alpha \in {}^\omega\omega \mid \exists\beta[\delta : \beta \mapsto \alpha]\}$.

Σ^1_1 may be characterized as follows:

10.6 Theorem: Let $P \subseteq {}^\omega\omega$.

$$P \in \Sigma^1_1 \quad \rightleftarrows \quad \exists\delta[P = Ra(\delta)].$$

Proof: (1) Suppose: $P \in \Sigma^1_1$. Using theorem 10.1, determine a decidable subset A of ω such that: $\forall\alpha[P(\alpha) \rightleftarrows \exists\gamma\forall n[\langle\bar{\alpha}n,\bar{\gamma}n\rangle \in A]]$.

Determine $\delta \in {}^\omega\omega$ such that: $\delta(\langle\rangle) = 0$ and:

For all $n, b \in \omega$:

$$\delta^n(b) := b^0(n)+1 \text{ if } n < lg(b^0) \text{ and } n < lg(b^1) \text{ and } \langle\overline{b^0}(n+1), \overline{b^1}(n+1)\rangle \in A$$
$$:= 0 \quad \text{otherwise.}$$

Remark that: $\forall\alpha\forall\beta[\delta : \beta \mapsto \alpha \rightleftarrows \alpha = \beta^0 \wedge \forall n[\langle\overline{\beta^0}n, \overline{\beta^1}n\rangle \in A]]$

Therefore: $\forall\alpha[P(\alpha) \rightleftarrows \exists\beta[\delta:\beta\mapsto\alpha]$, i.e.: $P = Ra(\delta)$

(II) Let $\delta\in{}^{\omega}\omega$ and consider $P := Ra(\delta)$

Remember, from 1.6, that for all $\beta,\alpha\in{}^{\omega}\omega$:

$\delta:\beta\mapsto\alpha \quad \rightleftarrows \quad \forall n\exists m[\delta^n(\bar\beta m)=\alpha(n)+1 \wedge \forall k<m[\delta^n(\bar\beta k)=0]]$

According to AC_{00}, that is to say, that for all $\beta,\alpha\in{}^{\omega}\omega$:

$\delta:\beta\mapsto\alpha \quad \rightleftarrows \quad \exists\zeta\forall n[\delta^n(\bar\beta(\zeta(n)))=\alpha(n)+1 \wedge \forall k<\zeta(n)[\delta^n(\bar\beta k)=0]]$

Therefore, for all $\alpha\in{}^{\omega}\omega$:

$\alpha\in Ra(\delta) \quad \rightleftarrows \quad \exists\beta\exists\zeta\forall n[\text{\_\_\_\_}]$.

Define a subset A of ω by:

For all $m\in\omega$:

$m\in A \rightleftarrows \exists a\exists c[m=\langle a,c\rangle \wedge lg(a)=lg(c) \wedge \forall n<lg(c)[(c^1(n)$ and $c^0(c^1(n)+1)$ are both defined$) \to (\delta^n(\bar{c}^0(c^1(n))=a(n)+1 \wedge \forall k<c^1(n)[\delta^n(\bar{c}^0 k)=0])]$

Then: $\forall b[b\in A \vee \neg(b\in A)]$.

and: $\forall\alpha\forall\gamma[\forall n[\langle\bar\alpha n,\bar\gamma n\rangle\in A] \to (\delta:\gamma^0\mapsto\alpha)]$.

Conversely, suppose $\delta:\beta\mapsto\alpha$, and determine $\zeta\in{}^{\omega}\omega$ such that $\forall n[-]$.

Defining $\gamma:=\langle\beta,\zeta\rangle$, we observe: $\forall n[\langle\bar\alpha n,\bar\gamma n\rangle\in A]$

Therefore: $\forall\alpha[P(\alpha) \rightleftarrows \exists\gamma\forall n[\langle\bar\alpha n,\bar\gamma n\rangle\in A]]$, and, following theorem 10.1 $P\in\Sigma^1_1$.

⊠

10.7 A subset P of ${}^{\omega}\omega$ will be called <u>analytical</u>, if $P\in\Sigma^1_1$, that is, if $\exists\delta[P=Ra(\delta)]$.
A subset P of ${}^{\omega}\omega$ will be called <u>strictly</u> <u>analytical</u>, if $\exists\delta[Fun(\delta)\wedge P=Ra(\delta)]$ (cf. Note 1 on page 216).
Every strictly analytical set is, trivially, analytical, and the converse is not true, as is shown by the example of the empty set.
The bad habit of reasoning classically arouses the suspicion that this is the only exception.
Indeed, if we assume P to be analytical and „finitely defined", and in possession of at least one element, we may follow John Burgess, and prove, by using Brouwer-Kripke's axiom, that P is strictly analytical. (cf. <u>Burgess</u> 1980, and also: <u>Gielen</u>, <u>de Swart</u> <u>and</u> <u>Veldman</u> 1980)
Restricting oneself to „finitely defined", „determinate" objects, however, is like wearing sunglasses against the dazzling light of constructive truth.
We will see that the supposition that all inhabited Π^0_1-sets are strictly analytical, already leads to a contradiction.

Let us define, for all $\beta \in {}^\omega\omega$: $C_\beta := \{\alpha \mid \forall n[\bar{\alpha}n = \bar{\underline{0}}n \vee \forall m \le n[\beta(\bar{\alpha}m) = 0]]\}$
Remark that, for all $\beta \in {}^\omega\omega$: $C_\beta \in \Pi^0_1$ and $\underline{0} \in C_\beta$.

10.8 Theorem: $\neg \forall\beta \exists\delta[\mathrm{Fun}(\delta) \wedge C_\beta = \mathrm{Ra}(\delta)]$

Proof: Suppose: $\forall\beta\exists\delta[\mathrm{Fun}(\delta) \wedge C_\beta = \mathrm{Ra}(\delta)]$
Using AC_{11}, determine $\zeta \in {}^\omega\omega$ such that:

$$\mathrm{Fun}(\zeta) \wedge \forall\beta[\mathrm{Fun}(\zeta|\beta) \wedge C_\beta = \mathrm{Ra}(\zeta|\beta)]$$

Remark: $C_{\underline{0}} = {}^\omega\omega$, therefore $\underline{1} \in C_{\underline{0}} = \mathrm{Ra}(\zeta|\underline{0})$
Determine $\alpha \in {}^\omega\omega$ such that: $(\zeta|\underline{0})|\alpha = \underline{1}$
Calculate $m \in \omega$ such that: $\forall k < m[(\zeta|\underline{0})^0(\bar{\alpha}k) = 0]$ and:

$$(\zeta|\underline{0})^0(\bar{\alpha}m) = \underline{1}(0) + 1 = 1 + 1 = 2.$$

Determine $n \in \omega$ such that:

$$\forall\beta[\bar{\beta}n = \bar{\underline{0}}n \rightarrow \forall k \le m[(\zeta|\beta)^0(\bar{\alpha}k) = (\zeta|\underline{0})^0(\bar{\alpha}k)]].$$

Then: $\forall\beta[\bar{\beta}n = \bar{\underline{0}}n \rightarrow ((\zeta|\beta)|\alpha \in C_\beta \wedge ((\zeta|\beta)|\alpha)(0) = 1)]$
Therefore: $\forall\beta[\bar{\beta}n = \bar{\underline{0}}n \rightarrow \exists\gamma[\gamma \in C_\beta \wedge \gamma(0) = 1]]$
Bring a blush to your opponent's cheeks by pointing to the sequence $\beta^* \in {}^\omega\omega$ which is defined by:

For all $k \in \omega$: $\beta^*(k) := 0$ if $k < n$
$:= 1$ otherwise.

$C_{\beta^*} = \{\underline{0}\}$, which is embarassing, in a way.
$\boxtimes$

The gap between strictly analytical and analytical sets is gaping wide and complicates our position seriously.
To be sure, E^1_1 itself, like all the exemplary arithmetical and hyperarithmetical sets from previous chapters: $E_n, A_n, E_\sigma, A_\sigma$, is strictly analytical.
(To see this, define $\delta \in {}^\omega\omega$ such that $\mathrm{Fun}(\delta)$ and:
For all $\alpha \in {}^\omega\omega$ and $b \in \omega$:

$$(\delta|\alpha)(b) := 0 \quad \text{if} \quad \exists m[b = \overline{\alpha^1}m]$$
$$:= \alpha^0(b) \quad \text{otherwise.}$$

Remark: for all $\alpha, \gamma \in {}^\omega\omega$: if $\forall n[\alpha(\bar{\gamma}n) = 0]$, then $\alpha = \delta|\langle\alpha,\gamma\rangle$
Therefore: $E^1_1 = \mathrm{Ra}(\delta)$)

There is no reason whatever for a set which is reducible to a strictly analytical set, to be itself strictly analytical.
This does not add to the reputation of strictly analytical sets.

On the other hand, we should never forget how much, in former endeavours, we leant on the strict analyticity of certain sets. (Cf. chapters 3, 7 and 9).

10.9 We define a subset A^1_1 of ${}^\omega\omega$ by:

For all $\alpha \in {}^\omega\omega$:

$$A^1_1(\alpha) := \forall\gamma\,\exists n\,[\alpha(\bar{\gamma}n) = 0]$$

We define a class Π^1_1 of subsets of ${}^\omega\omega$ by:

For every subset P of ${}^\omega\omega$

$$P \in \Pi^1_1 \rightleftarrows P \leq A^1_1$$

Members of Π^1_1 will be called: <u>co-analytical sets</u>.
Π^1_1 shares in many good properties of Σ^1_1:

10.10 <u>Theorem</u>: Let $P \subseteq {}^\omega\omega$

$P \in \Pi^1_1$ if and only if there exists a decidable subset A of ω such that $\forall\alpha\,[P(\alpha) \rightleftarrows \forall\gamma\,\exists n\,[\langle\bar{\alpha}n, \bar{\gamma}n\rangle \in A]]$

<u>Proof</u>: (i) Suppose $P \leq A^1_1$, i.e.: $\forall\alpha\,\exists\beta\,[P(\alpha) \rightleftarrows A^1_1(\beta)]$. Using AC_{11}, determine $\delta \in {}^\omega\omega$ such that: $Fun(\delta)$ and: $\forall\alpha\,[P(\alpha) \rightleftarrows A^1_1(\delta|\alpha)]$

Define a decidable subset A of ω by:

For all $n \in \omega$:

$$\begin{aligned} n \in A \rightleftarrows\ & \exists a\,\exists c\,[\,n = \langle a, c\rangle \wedge lg(a) = lg(c) \wedge \\ & \exists d\,\exists b\,[\,a \subseteq b \wedge c \subseteq d \wedge \delta^d(b) = 1 \wedge \\ & \forall e\,[(b \subseteq e \wedge b \neq e) \rightarrow \delta^d(e) = 0]]] \end{aligned}$$

Do not shy at all these letters and remark:

$$\forall a\,\forall c\,[\,\langle a,c\rangle \in A \rightarrow \exists d\,[c \subseteq d \wedge \forall\alpha \in a\,[(\delta|\alpha)c = 0]]$$

Be quiet and conclude:

$$\forall\alpha\,[\,\forall\gamma\,\exists n\,[(\delta|\alpha)(\bar{\gamma}n) = 0] \rightleftarrows \forall\gamma\,\exists n\,[\langle\bar{\alpha}n, \bar{\gamma}n\rangle \in A]]$$

Therefore, A fulfils the requirements.

(ii) Let A be a decidable subset of ω such that $\forall\alpha[P(\alpha) \rightleftarrows \forall\gamma\exists n[\langle\bar{\alpha}n, \bar{\gamma}n\rangle \in A]]$
Determine $\delta \in {}^\omega\omega$ such that $Fun(\delta)$ and:

For all $\alpha \in {}^\omega\omega$ and $c \in \omega$

$$(\delta|\alpha)(c) = 0 \rightleftarrows \exists n < lg(c)\,[\langle\bar{\alpha}n, \bar{c}n\rangle \in A]$$

Remark: $\forall\alpha\,[P(\alpha) \rightleftarrows A^1_1(\delta|\alpha)]$, i.e.: $P \leq A^1_1$.

$\boxtimes$

10.11 Theorem: Π^1_1 contains a universal element.

Proof: Define the subset $\mathcal{U}$ of ${}^\omega\omega$ by:

For all $\alpha \in {}^\omega\omega$: $\mathcal{U}(\alpha) \rightleftarrows \forall\gamma\exists n[\alpha^1(<\bar{\alpha}_0 n, \bar{\gamma}n>) = 0]$

and note that $\mathcal{U}$ belongs to Π^1_1.

Let $P \subseteq {}^\omega\omega$ and $P \in \Sigma^1_1$.

Following theorem 10.10 determine a decidable subset A of ω such that: $\forall\alpha[P(\alpha) \rightleftarrows \forall\gamma\exists n[<\bar{\alpha}n, \bar{\gamma}n> \in A]$. Determine $\beta \in {}^\omega\omega$ such that: $\forall n[\beta(n) = 0 \rightleftarrows n \in A]$. Then: $\forall\alpha[P(\alpha) \rightleftarrows \forall\gamma\exists n[\beta(<\bar{\alpha}n, \bar{\gamma}n>) = 0]]$, i.e.: $\forall\alpha[P(\alpha) \rightleftarrows \mathcal{U}(<\alpha, \beta>)]$.

⊠

10.12 Theorem: Let $P_0, P_1, P_2, \ldots$ be a sequence of subsets of ${}^\omega\omega$.
If $\forall m[P_m \in \Pi^1_1]$, then: $\bigcap_{m\in\omega} P_m \in \Pi^1_1$.

Proof: Using theorem 10.10, determine a sequence $A_0, A_1, A_2, \ldots$ of decidable subsets of ω such that: $\forall m\forall\alpha[P_m(\alpha) \rightleftarrows \forall\gamma\exists n[<\bar{\alpha}n, \bar{\gamma}n> \in A_m]]$

Define a subset A of ω by:

For all $b \in \omega$:

$$b \in A \rightleftarrows \exists m\exists t\exists a\exists c[b = <a * <t>, <m> * c> \wedge lg(a) = lg(c) \wedge <a, c> \in A_m].$$

Then: $\forall b[b \in A \vee \neg(b \in A)]$ and: $\forall m\forall\alpha[P_m(\alpha) \rightleftarrows \forall\gamma[\gamma(0) = m \to \exists n[<\bar{\alpha}n, \bar{\gamma}n> \in A]]]$.

Therefore: $\forall\alpha[\forall m[P_m(\alpha)] \rightleftarrows \forall\gamma\exists n[<\bar{\alpha}n, \bar{\gamma}n> \in A]]$.

and: $\bigcap_{m\in\omega} P_m \in \Pi^1_1$.

⊠

In 1.4, we introduced a subset fun of ${}^\omega\omega$ such that: $\forall\delta[\mathrm{fun}(\delta) \rightleftarrows \forall\gamma\exists n[\delta(\bar{\gamma}n) \neq 0]]$
It is easy to see that fun $\in \Pi^1_1$ and it is not difficult to verify that Π^1_1 may be characterized as follows:

Let $P \subseteq {}^\omega\omega$. Then: $P \in \Pi^1_1 \rightleftarrows P \preceq \mathrm{fun}$.

Catching sight of $\mathrm{Fun} := \{\delta \mid \forall n[\mathrm{fun}(\delta^n)]\}$, we observe that it does not do less than its little brother, as fun $\preceq$ Fun $\preceq$ fun.
Fun is funny, for being a natural example of a subset of ${}^\omega\omega$, which is <u>not</u> strictly analytical. (Theorem 10.8 did not provide us with such an example).
One feels a child's joy at arguing this: suppose: Fun is strictly

analytical, and let $\zeta \in {}^\omega\omega$ be such that $Fun(\zeta)$ and $Fun = Ra(\zeta)$
Sitting on grandfather Cantor's knee, we construct $\eta \in {}^\omega\omega$ such that $Fun(\eta)$ and $\forall\alpha[\eta|\alpha \# (\zeta|\alpha)|\alpha]$ (# denotes the well-known apartness relation $\forall\alpha\forall\beta[\alpha \# \beta \rightleftarrows \exists n[\alpha(n) \neq \beta(n)]]$) I go in search of $\beta \in {}^\omega\omega$ such that $\eta = \zeta|\beta$, and, upon finding it, we both start laughing, my grandfather and I.

Observe that this argument does not settle the question whether Fun be analytical.

Are Π^1_1 and Σ^1_1 a pair of identical twins?
In a classical treatment, Π^1_1 could shelter behind Σ^1_1, automatically sharing its reputation, by duality.
But now its weaknesses are exposed.
Doubts concerning Π^1_1 may have been lingering since theorem 10.12, which answered only one half of theorem 10.3.

Recall, from chapter 3: $D^2A_1 := \{\alpha \mid \alpha^0 = \underline{0} \vee \alpha^1 = \underline{0}\}$.

10.13 Theorem: D^2A_1 is not co-analytical.

Proof: The proof does not differ from the proof of theorem 4.10.
Suppose: $D^2A_1 \leq A^1_1$, i.e.: $\forall\alpha\exists\beta[D^2A_1(\alpha) \rightleftarrows A^1_1(\beta)]$.
Using AC_{11}, we find $\delta \in {}^\omega\omega$ such that: $Fun(\delta)$ and: $\forall\alpha[D^2A_1(\alpha) \rightleftarrows A^1_1(\delta|\alpha)]$.
Let τ be the spread which we introduced in 4.2, that is:

$$\tau = \{\alpha \mid \forall k[\alpha(k) < 2] \wedge \forall k \forall l[(\alpha(k) \neq 0 \wedge \alpha(l) \neq 0) \rightarrow k = l]\}$$

We want to show: $\forall\alpha\in\tau[A^1_1(\delta|\alpha)]$.
To this end, let us assume: $\alpha\in\tau$ and $\gamma \in {}^\omega\omega$.
We observe: $D^2A_1(\underline{0})$, therefore: $A^1_1(\delta|\underline{0})$ and $\exists n[(\delta|\underline{0})(\bar{\gamma}n) = 0]$.
We determine $n\in\omega$ such that: $(\delta|\underline{0})(\bar{\gamma}n) = 0$.
And we determine $q \in \omega$ such that: $\forall\beta[\bar{\beta}q = \bar{\underline{0}}q \rightarrow (\delta|\beta)(\bar{\gamma}n) = 0]$.
We now distinguish two cases:

Case 1: $\bar{\alpha}q \neq \bar{\underline{0}}q$.
In this case, we may determine $m<2$ such that $\alpha^m = \underline{0}$.
Therefore, $D^2A_1(\alpha)$, and: $A^1_1(\delta|\alpha)$, esp. $\exists n[(\delta|\alpha)(\bar{\gamma}n) = 0]$.

Case 2: $\bar{\alpha}q = \bar{\underline{0}}q$.
We now immediately see: $\exists n[(\delta|\alpha)(\bar{\gamma}n) = 0]$.

In any case: $\exists n[(\delta|\alpha)(\bar{\gamma}n) = 0]$.
We have proved: $\forall\alpha\in\tau\forall\gamma\exists n[(\delta|\alpha)(\bar{\gamma}n) = 0]$, i.e.: $\forall\alpha\in\tau[A^1_1(\delta|\alpha)]$.
Therefore: $\forall\alpha\in\tau[D^2A_1(\alpha)]$, and this, following 4.3, is contradictory. ⊠

This theorem deals at least two fatal blows to any thought of symmetry between Σ_1^1 and Π_1^1.
As D^2A_1 already is not co-analytical, smiling is the proper answer at the suggestion that all arithmetical, let alone all hyperarithmetical sets belong to Π_1^1
Secondly, as A_1 itself is a plain member of Π_1^1, Π_1^1, obviously, does not make much of closure under the operation of finite union.
And there is more to complain of.
For the sake of contrast, we bring out another comfortable trait of Σ_1^1.

Let $P \subseteq {}^\omega\omega$.
We define subsets $\mathbb{E}(P)$ and $\mathbb{U}(P)$ of ${}^\omega\omega$ by:

For all $\alpha \in {}^\omega\omega$:

$$\mathbb{E}(P)(\alpha) := \exists\gamma [P(\langle\alpha,\gamma\rangle)]$$
$$\mathbb{U}(P)(\alpha) := \forall\gamma [P(\langle\alpha,\gamma\rangle)].$$

$\mathbb{E}$ and $\mathbb{U}$ will be referred to as the operations of <u>existential</u> and <u>universal projection</u>, respectively, and will be studied in chapter 14.
We will see, in that chapter, that Σ_1^1 is closed under the operation of existential projection, and Π_1^1 under the operation of universal projection, as it should be.
As all hyperarithmetical sets are analytical, the existential projection of any hyperarithmetical set is also analytical.
Again, Π_1^1 fails to follow.
A witness to its bad behaviour is the set $Q := \{\alpha \mid D^2A_1(\alpha^0) \wedge A_1^1(\alpha^1)\} = \{\alpha \mid \forall\gamma\exists n[\alpha^1(\bar{\gamma}n)=0 \wedge (\alpha^{00}=\underline{0} \vee \alpha^{01}=\underline{0})]\}$.
Q is the universal projection of an arithmetical set. On the other hand, Q is not co-analytical, as $D^2A_1 \preceq Q$ and $\neg(D^2A_1 \preceq A_1^1)$.

Theorem 10.13 also affords to observe that E_1^1 is not co-analytical, i.e. not reducible to A_1^1 (as $D^2A_1 \preceq E_1^1$).
This is a welcome result, and, in its simplicity, may be the envy of a classical mathematician. In order to set his mind at ease on this point, he would have to resort to diagonalizing.
This is how his argument would run.

Suppose $E_1^1 \in \Pi_1^1$.
Then also: $\{\alpha \mid \exists\gamma\forall n[\alpha(\langle\bar{\alpha}n,\bar{\gamma}n\rangle) \neq 0]\} \in \Pi_1^1$
Using theorem 10.10, and AC_{00}, we find $\beta \in {}^\omega\omega$ such that:

$$\{\alpha \mid \exists\gamma\forall n[\alpha(\langle\bar{\alpha}n,\bar{\gamma}n\rangle)\neq 0]\} = \{\alpha \mid \forall\gamma\exists n[\beta(\langle\bar{\alpha}n,\bar{\gamma}n\rangle)=0]\}.$$

Specializing, we find: $\exists\gamma\forall n[\beta(\langle\bar{\beta}n,\bar{\gamma}n\rangle)\neq 0] \rightleftarrows \forall\gamma\exists n[\beta(\langle\bar{\beta}n,\bar{\gamma}n\rangle=0]$
and, therefore: $\neg\exists\gamma\forall n[\beta(\langle\bar{\beta}n,\bar{\gamma}n\rangle)\neq 0] \wedge \neg\forall\gamma\exists n[\beta(\langle\bar{\beta}n,\bar{\gamma}n\rangle=0]$.

And this sounds like a contradiction, undoubtedly so in the ears of a classical mathematician. An intuitionist, however, may find the sound unpleasant, but he has no easy way of turning it off.
As in 6.14 some solace is offered by the enigmatical assumption

$\forall\alpha[\neg\neg\exists n[\alpha(n)=0] \to \exists n[\alpha(n)=0]]$, which enables us to conclude:
$\forall\gamma\exists n[\beta(<\bar\beta n,\bar\gamma n>)=0]$ from: $\neg\exists\gamma\forall n[\beta(<\bar\beta n,\bar\gamma n>)\neq 0]$.
Eventually, this does not diminish the pain.

We will reformulate the result that E^1_1 is not co-analytical, so as to make it more alike to the hyperarithmetical hierarchy theorem (theorems 9.7 and 9.9) In view of this, we introduce subsets $(E^1_1)^*$ and $(A^1_1)^*$ of ${}^\omega\omega$ by:

For all $\alpha\in{}^\omega\omega$

$$E^{1*}_1(\alpha) := \exists\gamma\forall n[\alpha(\bar\gamma n)\neq 0]$$
$$A^{1*}_1(\alpha) := \forall\gamma\exists n[\alpha(\bar\gamma n)\neq 0]$$

and remark: $A^1_1\cap E^{1*}_1=\phi$ and: $A^{1*}_1\cap E^1_1=\phi$.

10.14 <u>Theorem</u>: Let $\delta\in{}^\omega\omega$ such that: $Fun(\delta)$ and: $\forall\alpha[E^1_1(\alpha)\to A^1_1(\delta|\alpha)]$
We may construct $\zeta\in{}^\omega\omega$ such that: $A^{1*}_1(\zeta)$ and $A^1_1(\delta|\zeta)$.

<u>Proof</u>: Let $\delta\in{}^\omega\omega$ such that: $Fun(\delta)$ and: $\forall\alpha[E^1_1(\alpha)\to A^1_1(\delta|\alpha)]$
Define a sequence $\zeta\in{}^\omega\omega$ by:

For all $c\in\omega$:

$$\zeta(c) := 1 \quad \text{if} \quad \exists d[c\leq d \wedge \exists m<c[\delta^d(\bar\zeta m)=1 \wedge \forall \ell<m[\delta^d(\bar\zeta\ell)=0]]$$
$$:= 0 \quad \text{otherwise.}$$

The following remark springs from some reflection on ζ:
$\forall\gamma[\ \exists n[\zeta(\bar\gamma n)\neq 0] \rightleftarrows \exists n[(\delta|\zeta)(\bar\gamma n)=0]]$.
A classical mathematician probably would leave the proof at this.
But we have to be a bit more careful.
Let us define, for each $\gamma\in{}^\omega\omega$, a sequence $\zeta_\gamma\in{}^\omega\omega$ by:

For all $c\in\omega$

$$\zeta_\gamma(c) := 0 \quad \text{if} \quad \gamma\in c, \text{ i.e.: } \exists n[\bar\gamma n=c]$$
$$:= \zeta(c) \quad \text{otherwise.}$$

Let $\gamma\in{}^\omega\omega$ and consider: ζ_γ
Observe: $\forall n[\ \zeta_\gamma(\bar\gamma n)=0]$, therefore: $E^1_1(\zeta_\gamma)$, and: $A^1_1(\delta|\zeta_\gamma)$
especially: $\exists n[(\delta|\zeta_\gamma)(\bar\gamma n)=0]$.
Determine $k,m\in\omega$ such that: $\delta^{\bar\gamma k}(\overline{\zeta_\gamma}m)=1 \wedge \forall\ell<m[\delta^{\bar\gamma k}(\overline{\zeta_\gamma}\ell)=0]$
and distinguish two cases:

Case (I): $\bar{\zeta}m = \overline{\zeta_\gamma}\, m$.
Then: $(\delta|\zeta)(\bar{\gamma}k) = (\delta|\zeta_\gamma)(\bar{\gamma}k) = 0$ and: $\exists n[(\delta|\zeta)(\bar{\gamma}n)=0]$.

Case (II): $\bar{\zeta}m \neq \overline{\zeta_\gamma}\, m$.
Then: $\exists n[\zeta(\bar{\gamma}n) \neq 0]$, and therefore, by the definition of ζ: $\exists n[(\delta|\zeta)(\bar{\gamma}n)=0]$.

In any case: $\exists n[(\delta|\zeta)(\bar{\gamma}n)]$ and we have to admit: $\forall\gamma\exists n[(\delta|\zeta)(\bar{\gamma}n)=0]$.

Therefore: $A^1_1(\delta|\zeta)$, and, by the construction of ζ: $A^{1*}_1(\zeta)$.

⊠

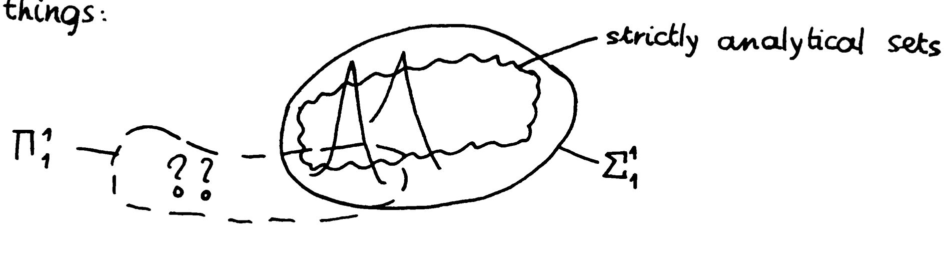
We now miss the looking-glass which once decorated our study, but has been removed on instigation of certain innovators.
Holding up one of the above reasonings against it, a classical mathematician would find an argument establishing that A^1_1 is not analytical, which we, however, know to be lame.
To tell the truth, we did not succeed in finding a constructive argument refuting the analyticity of A^1_1.

A line of thought which seemed to offer some hope, is to parallel the proof of: A_2 is not reducible to E_2. (theorem 3.2).
The creative subject, having at his disposal a great many ways of ensuring, or seeming to ensure: $A^1_1(\alpha)$, might be supposed to be able to delude the imitative subject.
But there is no easy method of surveying „all possible ways of ensuring $A^1_1(\alpha)$", as there was in the case of A_2. This is because A^1_1, like Fun, (cf. the discussion following theorem 10.12) is not strictly analytical.
Paradox is flickering here: as part of the truth is easy (A^1_1 is not strictly analytical), the whole truth (A^1_1 is not analytical) seems unattainable.

A better understanding of A^1_1, which involves a better understanding of the set of well-ordered stumps, $, (cf. chapter 7), as we will see the more clearly, after Brouwer's thesis has entered into discussion (cf. chapter 13), might, eventually, lead to an answer to our problem.

A task which looks more simple, but still is above us, is to refute that A^1_1 be arithmetical, or, better even, hyperarithmetical.

10.15 Gloom and disappointment are upon us, when contemplating the nasty state of things:



11. SOME MEMBERS OF THE ANALYTICAL FAMILY

E^1_1, the subset of ${}^\omega\omega$ which played a leading part in chapter 10, differs from hyperarithmetical subsets of ${}^\omega\omega$ by containing, in its definition, an existential quantifier over ${}^\omega\omega$.
We are going to see some consequences of restricting the range of this existential quantifier to a subspread of the universal spread, ${}^\omega\omega$.
We mainly consider the case of the so-called monotonous fans, $\sigma_{2mon}, \sigma_{3mon}, \ldots$
Thus the first set which offers itself is $S_2 := \{\alpha \mid \exists\gamma\in\sigma_{2mon}\ \forall n[\alpha(\bar{\gamma}n)=0]\}$

We spend a lot of effort to prove the remarkable fact that S_2 is not hyperarithmetical.
According to classical opinion, the fans $\sigma_{2mon}, \sigma_{3mon}, \ldots$ are countable, and the resulting subsets of ${}^\omega\omega$ all belong to Σ^0_2.
Intuitionistically, however, quantifying over a spread, however small it may be, comes to exercising a new art, obeying its own laws, being altogether different from that of quantifying over a countable set, such as ω.

Watching the new sequence: $S_1, S_2, \ldots$ of subsets of ${}^\omega\omega$ and bringing it under the discipline of the reducibility relation, we find another hierarchy. A_1 is a natural leader for this sequence, and certain peculiarities, which we first encountered in chapter 4, when dealing with A_1, reappear.

At the end of the chapter we study, briefly, the case of the binary fan, σ_2.

11.0 We define a sequence $\sigma_{2mon}\in{}^\omega\omega$ by:

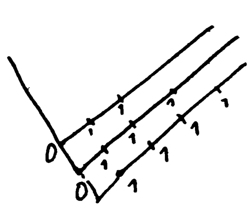
For all $a\in\omega$:

$$\begin{aligned}\sigma_{2mon}(a) &:= 0 \quad \text{if} \quad \forall n[n<lg(a) \to a(n)<2] \\ &\qquad\qquad \text{and: } \forall n[n+1<lg(a) \to a(n)\leq a(n+1)] \\ &:= 1 \quad \text{otherwise.}\end{aligned}$$

It is not difficult to verify that σ_{2mon} is a subspread of ${}^\omega\omega$ (cf. 1.9)
σ_{2mon} will be thought of as the subset of ${}^\omega\omega$ given by:

For all $\gamma\in{}^\omega\omega$:

$$\gamma\in\sigma_{2mon} \rightleftarrows \forall n[\sigma_{2mon}(\bar{\gamma}n)=0]$$



This picture portraits σ_{2mon}.

We define a subset S_2 of ${}^\omega\omega$ by:

For all $\alpha \in {}^\omega\omega$

$$S_2(\alpha) \quad := \quad \exists\gamma\,[\gamma \in \sigma_{2mon} \;\wedge\; \forall n\,[\alpha(\bar{\gamma}n)=0]]$$

(In agreement with 1.9, we sometimes write: „$\gamma \in \alpha$" for: „$\forall n\,[\alpha(\bar{\gamma}n)=0]$"
$\alpha \in {}^\omega\omega$ has the property S_2 if there exists a sequence γ in σ_{2mon}, each of whose initial parts is approved of by α).

The sequence $\underline{0}$ which does belong to σ_{2mon}, is sometimes called the <u>spine</u> of σ_{2mon}.
We fix an enumeration of the other branches of σ_{2mon}, defining a function $*: \omega \to \sigma_{2mon}$ by:

For all $n \in \omega$: $\quad n^* := \bar{\underline{0}}n * \underline{1}$

Therefore: $\quad \forall n \forall k\,[n^*(k)=0 \rightleftarrows k<n]$.

Remark that, classically spoken: $\forall\alpha\,[S_2(\alpha) \rightleftarrows \underline{0}\in\alpha \vee \exists n[n^*\in\alpha]]$, and, therefore: $S_2 \in \Sigma^0_2$.

We remind the reader of definition 4.0 in which we introduced, for each $n\in\omega$ and $P \subseteq {}^\omega\omega$: $\;D^nP := \{\alpha \mid \exists k<n\,[P(\alpha^k)]\}$.

11.1 <u>Theorem</u>: $\forall n\,[D^nA_1 \preceq S_2]$.

<u>Proof</u>: Let $n\in\omega$. Define $\delta \in {}^\omega\omega$ such that $Fun(\delta)$ and such that, for all $\alpha \in {}^\omega\omega$ and $m\in\omega$:

$$\begin{aligned}(\delta|\alpha)(m) &:= 0 \quad \text{if} \quad \exists k \exists p\,[k<n \wedge m=\bar{\underline{0}}k * \bar{\underline{1}}p \wedge \overline{\alpha^k}p=\bar{\underline{0}}p] \\ &:= 1 \quad \text{otherwise}\end{aligned}$$

Make the following observations:
$\forall\gamma\forall\alpha\,[\;\forall p\,[\,(\delta|\alpha)(\bar{\gamma}p)=0] \to \exists k<n\,[\gamma=k^*]]$ and,
$\forall k<n\;\forall\alpha\,[\;k^*\in\delta|\alpha \;\rightleftarrows\; \alpha^k=\underline{0}]$.
Therefore: $\;\forall\alpha\,[D^nA_1(\alpha) \rightleftarrows S_2\,(\delta|\alpha)]$.

⊠

We have seen, in theorem 4.6, that: $\forall n\,[\neg\,(D^{n+1}A_1 \preceq D^nA_1)]$. Therefore:

11.2 <u>Corollary</u>: $\forall n\,[\neg(S_2 \preceq D^nA_1)]$

Corollary 11.2 is actually the first one of a series of theorems which is to culminate in the statement that S_2 is not even hyperarithmetical.

11.3 Theorem: $\neg(S_2 \leq E_1)$

Proof: Suppose: $S_2 \leq E_1$, i.e.: $\forall\alpha\exists\beta[S_2(\alpha) \rightleftarrows E_1(\beta)]$, and, using AC_{11}, determine $\delta \in {}^\omega\omega$ such that: $Fun(\delta)$ and: $\forall\alpha[S_2(\alpha) \rightleftarrows E_1(\delta|\alpha)]$.

Remark: $S_2(\underline{0})$ and determine $p, q \in \omega$ such that $(\delta|\underline{0})(p) = 0$ and $\forall\alpha[\bar{\alpha}q = \bar{\underline{0}}q \rightarrow (\delta|\alpha)(p) = (\delta|\underline{0})(p) = 0]$.

Therefore: $\forall\alpha[\bar{\alpha}q = \bar{\underline{0}}q \rightarrow S_2(\alpha)]$, an absurd conclusion, as is testified by a sequence α^* which satisfies: $\forall n[\alpha^*(n) = 0 \rightleftarrows n < q]$.

$\boxtimes$

We now prepare for proving a converse to theorem 11.3, that E_1 does not reduce to S_2, either.
The analysis of S_2 which we have to make in view of this, will also be useful for other purposes.

For all $\beta \in {}^\omega\omega$ and $a \in \omega$ we define a decidable subset K_a^β of ω by:

$$K_a^\beta := \{ n \mid n \in \omega \mid \beta(a * \langle n \rangle) = 0 \}. \qquad \text{(Cf. 9.0)}$$

If β is a spread, we call it a finitary spread, or a fan, if:

$$\forall a[\beta(a) = 0 \rightarrow K_a^\beta \text{ is finite}].$$

Finitary spreads are remarkable as they are supposed to fulfil the fan theorem:

Let A be a decidable subset of ω, and $\beta \in {}^\omega\omega$ be a fan. If $\forall\alpha\in\beta\exists n[A(\bar{\alpha}n)]$, then $\exists m\forall\alpha\in\beta\exists n[n \leq m \wedge A(\bar{\alpha}n)]$

In the case of a fan like the full binary spread (i.e.: ${}^\omega 2$), this theorem is proved by an appeal to Brouwer's thesis (cf. chapter 13), a rather deep and much debated principle of intuitionistic analysis.
σ_{2mon} however, nimble like all little folk, admits of a more easy treatment:

11.4 Theorem: Let A be a decidable subset of ω.

(i) If $\forall\gamma\in\sigma_{2mon}\exists n[A(\bar{\gamma}n)]$, then $\exists m\forall\gamma\in\sigma_{2mon}\exists n[n \leq m \wedge A(\bar{\gamma}n)]$.

(ii) If $\forall\gamma\in\sigma_{2mon}\neg\neg\exists n[A(\bar{\gamma}n)]$, then $\neg\neg\exists m\forall\gamma\in\sigma_{2mon}\exists n[n \leq m \wedge A(\bar{\gamma}n)]$

Proof: (i) Suppose: A is a decidable subset of ω, and: $\forall\gamma\in\sigma_{2mon}\exists n[A(\bar{\gamma}n)]$

Calculate $n_0 \in \omega$ such that $A(\bar{\underline{0}}n_0)$.

Consider the infinite sequences: $0^*, 1^*, \ldots (n_0 - 1)^*$

Determine natural numbers $k_0, k_1, \ldots k_{n_0-1}$ such that:

$\forall j < n_0 [A(\overline{j^*} k_j)]$

Let $m := \max\{n_0, k_0, \dots k_{n_0-1}\}$. Then: $\forall \gamma \in \sigma_{2mon} \exists n [n \leq m \wedge A(\bar{\gamma} n)]$

(II) Suppose: A is a decidable subset of ω, and $\forall \gamma \in \sigma_{2mon} \neg\neg \exists n [A(\bar{\gamma} n)]$

Assume, for the sake of argument: $\exists n [A(\bar{\underline{0}} n)]$.

Calculate n_0 such that: $A(\bar{\underline{0}} n_0)$.

Remark: $\forall j < n_0 \neg\neg \exists k [A(\overline{j^*} k)]$.

As for all propositions P and Q: $(\neg\neg P \wedge \neg\neg Q) \rightarrow \neg\neg(P \wedge Q)$,

we may conclude: $\neg\neg \forall j < n_0 \exists k [A(\overline{j^*} k)]$, and further,

following the argument in (I): $\neg\neg \exists m \forall \gamma \in \sigma_{2mon} \exists n [n \leq m \wedge A(\bar{\gamma} n)]$.

Therefore: If $\neg \exists m \forall \gamma \in \sigma_{2mon} \exists n [n \leq m \wedge A(\bar{\gamma} n)]$, then: $\neg \exists n [A(\bar{\underline{0}} n)]$,

and: $\neg \forall \gamma \in \sigma_{2mon} \neg\neg \exists n [A(\bar{\gamma} n)]$.

Our conclusion follows by contraposition.

⊠

In 5.2 we defined, to each subset A of ${}^\omega\omega$, a subset $Neg(A)$ of ${}^\omega\omega$ by:
$Neg(A) := \{\alpha \mid \neg A(\alpha)\}$.
Another thing which we may learn from the proof of theorem 11.4, is:

11.5 <u>Corollary</u>: $Neg(Neg(S_1)) \preceq A_1$.

<u>Proof</u>: The proof of theorem 11.4 makes it clear that:

$$\forall \alpha [\neg\neg S_2(\alpha) \rightleftarrows \forall m \exists a [lg(a) \leq m \wedge \sigma_{2mon}(a) = 0 \wedge \alpha(a) = 0]].$$

⊠

The next remark will be made use of in the sequel:

11.6 <u>Lemma</u>: $\neg(Neg(E_1) \preceq Neg(Neg(E_1)))$.

<u>Proof</u>: Suppose: $Neg(E_1) \preceq Neg(Neg(E_1))$, and, using AC_{11}, determine $\delta \in {}^\omega\omega$ such that: $Fun(\delta)$ and: $\forall \alpha [\neg E_1(\alpha) \rightleftarrows \neg\neg E_1(\delta|\alpha)]$.

Remark: $\neg E_1(\underline{1})$, therefore: $\neg\neg \exists n [(\delta|\underline{1})(n) = 0]$.

Assume: $\exists n [(\delta|\underline{1})(n) = 0]$ and determine $n, q \in \omega$ such that:
$(\delta|\underline{1})(n) = 0$ and: $\forall \alpha [\bar{\alpha} q = \bar{\underline{1}} q \rightarrow (\delta|\alpha)(n) = 0]$.

Therefore: $\forall \alpha [\bar{\alpha} q = \bar{\underline{1}} q \rightarrow \neg E_1(\alpha)]$.

This contradiction makes us retire.

We conclude: $\neg \exists n [(\delta|\underline{1})(n) = 0]$, and have another contradiction.

Therefore: $\neg(Neg(E_1) \preceq Neg(Neg(E_1)))$.

$\boxtimes$

11.7 __Theorem:__ $\neg(E_1 \preceq S_2)$

__Proof:__ Suppose: $E_1 \preceq S_2$.

Using lemma 5.3, conclude: $Neg(E_1) \preceq Neg(S_2)$.

As we observed in corollary 11.5: $Neg(Neg(S_2)) \preceq A_1$, and therefore, again by lemma 5.3.: $Neg(S_2) \preceq Neg(A_1)$.

But it is not difficult to see that: $A_1 \preceq Neg(E_1) \preceq A_1$ and: $Neg(A_1) \preceq Neg(Neg(E_1))$.

Taking all things together, we have: $Neg(E_1) \preceq Neg(Neg(E_1))$, and this, according to lemma 11.6, leads to a contradiction.

$\boxtimes$

The fact that E_1 is not reducible to S_2, destroys all hope that A_2, E_2, or any other set to which E_1 itself is reducible, should be so.
We turn to the question, whether S_2 is reducible to A_2.

Like many sets we encountered thus far, S_2 is strictly analytical (cf. 10.7)
In order to see this, we define, for each $\alpha \in {}^\omega\omega$ and $\gamma \in \sigma_{2mon}$, a sequence α_γ in ${}^\omega\omega$ by:

For all $a \in \omega$:

$$\alpha_\gamma(a) := 0 \quad \text{if } \gamma \in a \quad (\text{i.e.: } \bar{\gamma}(lg(a)) = a)$$
$$:= \alpha(a) \quad \text{if } \gamma \notin a$$

We remark: $\forall\alpha[S_2(\alpha) \rightleftarrows \exists\gamma\in\sigma_{2mon}[\alpha = \alpha_\gamma]]$.
(The same construction serves to prove the strict analyticity of E^1_1, cf. the discussion following on theorem 10.8).

We want to mention an important consequence of theorem 11.4:

Let A be a subset of $\sigma_{2mon} \times \omega$
If $\forall\gamma\in\sigma_{2mon}\exists n[A(\gamma, n)]$, then $\exists m\forall\gamma\in\sigma_{2mon}\exists n\leq m[A(\gamma, n)]$

It is not difficult to derive this principle from theorem 11.4 and GCP (cf. 1.9)

We now state a refinement of theorem 11.3:

11.8 __Lemma:__ Suppose: $\delta \in {}^\omega\omega$ and $Fun(\delta)$, and: $\forall\alpha[S_2(\alpha) \to E_1(\delta|\alpha)]$
Then: $\forall\alpha[\neg\neg S_2(\alpha) \to E_1(\delta|\alpha)]$.

Proof: Suppose: $\delta \in {}^\omega\omega$ and: $Fun(\delta)$, and: $\forall\alpha[S_2(\alpha) \to E_1(\delta|\alpha)]$
Let $\alpha \in {}^\omega\omega$ and: $\neg\neg S_2(\alpha)$.
Remark: $\forall\gamma \in \sigma_{2mon}[S_2(\alpha_\gamma)]$, therefore: $\forall\gamma\in\sigma_{2mon}\exists n[(\delta|\alpha_\gamma)(n)=0]$
Also: $\forall\gamma\in\sigma_{2mon}\exists q\exists n\forall\beta[\bar\beta q = \bar\alpha_\gamma q \to (\delta|\beta)(n)=0]$
Using the above-mentioned consequence of theorem 11.4, we calculate $m \in \omega$ such that:

$$\forall\gamma\in\sigma_{2mon}\exists q \leq m\exists n\forall\beta[\bar\beta q = \bar\alpha_\gamma q \to (\delta|\beta)(n)=0].$$

Therefore: $\forall\gamma\in\sigma_{2mon}\exists n\forall\beta[\bar\beta m = \bar\alpha_\gamma m \to (\delta|\beta)(n)=0]$.
And this is useful knowledge.
As $\neg\neg S_2(\alpha)$, we have: $\exists\gamma\in\sigma_{2mon}[\bar\alpha m = \bar\alpha_\gamma m]$, and: $E_1(\delta|\alpha)$.
So we have to admit: $\forall\alpha[\neg\neg S_2(\alpha) \to E_1(\delta|\alpha)]$.
⊠

A further remark is, that S_2 is not a stable subset of ${}^\omega\omega$, i.e.:
$\neg\forall\alpha[\neg\neg S_2(\alpha) \to S_2(\alpha)]$.
For, suppose: $\forall\alpha[\neg\neg S_2(\alpha) \to S_2(\alpha)]$, i.e.: $Neg(Neg(S_2)) = S_2$.
Then, according to corollary 11.5: $S_2 \preceq A_1$, which is refuted by 11.2

Sufficiently many preparations have now been made for:

11.9 Theorem: $\neg(S_2 \preceq A_2)$.

Proof: Suppose: $S_2 \preceq A_2$, and, using AC_{11}, determine $\delta\in{}^\omega\omega$ such that $Fun(\delta)$ and: $\forall\alpha[S_2(\alpha) \rightleftarrows A_2(\delta|\alpha)]$.
Therefore: $\forall\alpha[S_2(\alpha) \to \forall k[E_1((\delta|\alpha)^k)]]$, and, according to lemma 11.8: $\forall\alpha[\neg\neg S_2(\alpha) \to A_2(\delta|\alpha)]$.
But now: $\forall\alpha[\neg\neg S_2(\alpha) \to S_2(\alpha)]$, and this should not be true.
⊠

The next step does not surprise:

11.10 Theorem: $\neg(S_2 \preceq E_3)$.

Proof: Suppose: $S_2 \preceq E_3$, and, using AC_{11}, determine $\delta\in{}^\omega\omega$ such that: $Fun(\delta)$ and: $\forall\alpha[S_2(\alpha) \rightleftarrows E_3(\delta|\alpha)]$.
Remember how we defined, to each $\gamma\in\sigma_{2mon}$ and $\alpha\in{}^\omega\omega$ a sequence α_γ in ${}^\omega\omega$ such that: $\gamma\in\alpha_\gamma$ and: $\gamma\in\alpha \to \alpha = \alpha_\gamma$.

(We did it just before lemma 11.8).

Remark: $\forall\gamma\in\sigma_{2mon}\,\forall\alpha[S_2(\alpha_\gamma)]$, therefore: $\forall\gamma\in\sigma_{2mon}\,\forall\alpha\exists n[A_2((\delta|\alpha_\gamma)^n)]$.

Using GCP, we find $n, q\in\omega$ such that:

$$\forall\gamma\in\sigma_{2mon}\,\forall\alpha[(\bar\gamma q=\bar{\underline{0}}q\wedge\bar\alpha q=\bar{\underline{0}}q)\to A_2((\delta|\alpha_\gamma)^n)].$$

(Again the imitative subject has been forced to a decision, whereas the creative subject did not oblige himself to anything).

Define $\eta\in{}^\omega\omega$ such that: $\mathrm{Fun}(\eta)$ and:

for all $\alpha\in{}^\omega\omega$:

for all $\ell\in\omega$: $\ell\le q\to(\eta|\alpha)(\bar{\underline{0}}\ell)=0$, and:

for all $a\in\omega$: $(\eta|\alpha)(\bar{\underline{0}}q*a):=\alpha(a)$ and:

$\overline{(\eta|\alpha)}q:=\bar{\underline{0}}q$, and:

$\forall j<q\,\exists m[(\eta|\alpha)(\overline{j^*}m)\ne 0]$.

Remark, that for all $\alpha\in{}^\omega\omega$:

$$S_2(\alpha)\rightleftarrows\exists\gamma\in\sigma_{2mon}[\bar\gamma q=\bar{\underline{0}}q\wedge\gamma\in(\eta|\alpha)\wedge\overline{(\eta|\alpha)}q=\bar{\underline{0}}q]$$

$$\rightleftarrows\exists\gamma\in\sigma_{2mon}[\bar\gamma q=\bar{\underline{0}}q\wedge(\eta|\alpha)_\gamma=\eta|\alpha\wedge\overline{(\eta|\alpha)}q=\bar{\underline{0}}q]$$

Therefore: $\forall\alpha[S_2(\alpha)\rightleftarrows A_2((\delta|(\eta|\alpha))^n)]$, i.e.: $S_2\preceq A_2$,

and this is contradictory, according to theorem 11.9.

⊠

In a similar way, we might have obtained the conclusion: $\neg(S_2\preceq E_2)$ from: $\neg(S_2\preceq A_1)$.

This very conclusion also follows from theorem 11.10 itself, as $E_2\preceq E_3$

Looking forward, however, and hoping for absurdity to follow from the assumption: $S_2\preceq A_3$, we want to articulate this truth in a more refined manner.

(The reader should remember how we blew up theorem 11.3 to lemma 11.8, in order to prove theorem 11.9).

We introduce a subset P of ${}^\omega\omega$ by:

For all $\alpha\in{}^\omega\omega$:

$$P(\alpha):=\forall n[\exists j<n[j^*\in\alpha]\vee\neg\neg\exists\gamma\in\sigma_{2mon}[\bar\gamma n=\bar{\underline{0}}n\wedge\gamma\in\alpha]].$$

(As we observed, in corollary 11.5: $\mathrm{Neg}(\mathrm{Neg}(S_2))\preceq A_1$. Therefore, P is an arithmetical set. Actually: $P\preceq A_3$)

Remark: $S_2\subseteq P\subseteq\mathrm{Neg}(\mathrm{Neg}(S_2))$.

Both inclusions are proper, that is to say: either one of the assumptions:

$Neg(Neg(S_2)) \subseteq P$ and: $P \subseteq S_2$, leads to a contradiction.

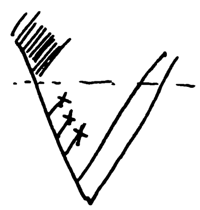
We first prove: $\neg(P \subseteq S_2)$.

We are not making wild accusations but have good reasons for suspecting: $P \subseteq S_2$ of bringing about absurdity:

We think of the decimal development of π, as in 4.1 and in particular of $k := \mu m$ [At place m in the decimal development of π stands the last 9 of a block of ninety-nine 9's].

k is sometimes called: the volatile number of π („het vluchtgetal van π")
It is not a well-defined natural number, of course, but $\{n \mid n \in \omega \mid n < k\}$ is a perfectly clear, decidable subset of ω.

We build a special sequence $\alpha \in {}^{\omega}\omega$, paying exclusive attention to the values it assumes on $\{a \mid a \in \omega \mid \sigma_{2mon}(a) = 0\}$.


k

The picture will help to clarify our wicked project. Up till level k, the only sequences in σ_{2mon} which have a chance of belonging to α, are: all extensions of $\overline{\underline{0}}k$, and the two sequences: $0^* = \underline{1}$ and $1^* = \langle 0 \rangle * \underline{1}$.

If k appears, and turns out to be odd, 0^* will be approved of by α; if k appears, and turns out to be even, 1^* will be the happy one.

In either case we continue α „above" $\overline{\underline{0}}k$ (in the shaded part of the picture) by some sequence β of which it is known that: $\neg\neg S_2(\beta)$, but not known that: $S_2(\beta)$.

Suppose: $S_2(\alpha)$, determine $\gamma \in \sigma_{2mon}$ such that $\gamma \in \alpha$ and consider $\bar{\gamma}2$. We now are able to find out the following alternative:

$$(k \text{ exists} \rightarrow k \text{ is even}) \vee (k \text{ exists} \rightarrow k \text{ is odd}) \vee (k \text{ exists} \rightarrow S_2(\beta))$$

Thus we committed ourselves to a reckless announcement.

Remark, however, that $P(\alpha)$.

The following proof shows that the assumption: $\forall \alpha [P(\alpha) \rightarrow S_2(\alpha)]$ is not but reckless and actually disastrous.

11.11 Theorem: $\neg \forall \alpha [P(\alpha) \rightarrow S_2(\alpha)]$

Proof: Reconsidering corollary 11.5, we find that $Neg(Neg(S_2))$ is not only a member of Π^0_1, but also a subspread of ${}^{\omega}\omega$.

We may define a sequence $\zeta \in {}^{\omega}\omega$ such that:

$$\forall \alpha [\neg\neg S_2(\alpha) \rightleftarrows \forall n [\zeta(\bar{\alpha}n) = 0]].$$

and: ζ is a subspread of ${}^{\omega}\omega$. (cf. 1.9)

We also may define a function $F_0 : {}^{\omega}\omega \to {}^{\omega}\omega$ such that $Neg(Neg(S_2)) = Ra(F_0)$.

In order to do so, we first define a function $f_0 : \omega \to \omega$ by:

$$f_0(\langle\,\rangle) := \langle\,\rangle$$

and, for all $a \in \omega$, $n \in \omega$:

$$\begin{aligned} f_0(a * \langle n \rangle) &:= f_0(a) * \langle n \rangle && \text{if } \zeta(f_0(a) * \langle n \rangle) = 0 \\ &:= f_0(a) * \langle m \rangle && \text{if } \zeta(f_0(a) * \langle n \rangle) \neq 0 \\ &&& \text{and } m = \mu p[\zeta(f_0(a) * \langle p \rangle) = 0] \end{aligned}$$

We then determine F_0 by declaring:

$$\forall \alpha \forall n [F_0(\alpha) \in f_0(\bar{\alpha}n)]$$

We introduce a technical convention:
For all $\alpha \in {}^{\omega}\omega$ and $n \in \omega$, the sequence $\alpha|_n \in {}^{\omega}\omega$ is defined by: $\forall m [\alpha|_n(m) = \alpha(n+m)]$.
(One gets $\alpha|_n$ from α by suppressing the first n values of α).

We are going to define a function $F : {}^{\omega}\omega \to {}^{\omega}\omega$
Let $\beta \in {}^{\omega}\omega$ and: $k := \mu n[\beta(n) \neq 0]$ be the volatile number of β
We define $F(\beta)$ such that:

(i) For all $n \in \omega$, $n \leq k$:

$$F(\beta)(\underline{\bar{0}}n) = F(\beta)(\overline{0^*}n) = F(\beta)(\underline{\bar{1}}^*n) = 0$$
$$F(\beta)(\overline{n^*}(n+1)) = 1 \quad \text{if } n \neq 0 \text{ and } n \neq 1.$$

(ii) For all $n \in \omega$, $n > k$:

$$F(\beta)(\overline{0^*}n) = 0 \quad \rightleftarrows \quad k \text{ is odd}$$
$$F(\beta)(\overline{1^*}n) = 0 \quad \rightleftarrows \quad k \text{ is even}$$

(iii) $\quad {}^{\underline{\bar{0}}k}F(\beta) = F_0(\beta|_{k+1})$ i.e.: for all $a \in \omega$:
$$(F(\beta))(\underline{\bar{0}}k * a) = (F_0(\beta|_{k+1}))(a)$$

We claim that $\forall \beta [P(F(\beta))]$.
For, suppose: $\beta \in {}^{\omega}\omega$ and $n \in \omega$ and $n < k := \mu p[\beta(p) \neq 0]$.

Distinguish two cases:

- If k exists, then: $\forall n \leq k [F(\beta)(\underline{\bar{0}}n) = 0]$ and: $\neg\neg S_2({}^{\underline{\bar{0}}k}F(\beta))$.

Therefore: $\neg\neg S_2(\,^{\underline{\bar{0}}n}F(\beta))$.

- If k does not exist, i.e. $\neg\exists p[\beta(p)\neq 0]$, then: $\underline{0}\in F(\beta)$ and: $S_2(\,^{\underline{\bar{0}}n}F(\beta))$.
- As $\neg\neg(\exists p[\beta(p)\neq 0] \vee \neg\exists p[\beta(p)\neq 0])$, we know: $\neg\neg S_2(\,^{\underline{\bar{0}}n}F(\beta))$.

Now, suppose $n > k := \mu p[\beta(p)\neq 0]$.

If k is odd, then $0^* \in F(\beta)$.

If k is even, then $1^* \in F(\beta)$.

Therefore: $\exists j<n[\, j^*\in F(\beta)]$.

We proved: $\forall\beta\,\forall n[\ \exists j<n[\, j^*\in F(\beta)] \vee \neg\neg S_2(\,^{\underline{\bar{0}}n}F(\beta))$, i.e.: $\forall\beta[P(F(\beta))]$.

We also claim that: $\neg\forall\beta[S_2(F(\beta))]$.

For, suppose: $\forall\beta[S_2(F(\beta))]$.

Then: $\forall\beta\,\exists a\,\exists\gamma[\,lg(a)=2 \wedge \gamma\in a \wedge \gamma\in\sigma_{2mon} \wedge \gamma\in F(\beta)]$.

Using CP, we find $q\in\omega$, $a\in\omega$ such that:

$$lg(a)=2 \quad\wedge\quad \forall\beta[\ \bar\beta q=\bar{\underline{0}}q \to \exists\gamma[\gamma\in a \wedge \gamma\in\sigma_{2mon} \wedge \gamma\in F(\beta)]]$$

We scrutinize a and distinguish three possibilities.

Case (i): $a=\langle 1,1\rangle$.

Then: $\forall\beta[\bar\beta q=\bar{\underline{0}}q \to 0^*\in F(\beta)]$.

Therefore: $\forall\beta[\bar\beta q=\bar{\underline{0}}q \to (\exists n[\beta(n)\neq 0]\to \mu n[\beta(n)\neq 0]$ is odd$)]$.

This is contradictory, as we may define $\beta^*\in{}^\omega\omega$ such that: $2q=\mu n[\beta^*(n)\neq 0]$.

Case (ii): $a=\langle 0,1\rangle$.

Then: $\forall\beta[\bar\beta q=\bar{\underline{0}}q \to 1^*\in F(\beta)]$.

Therefore: $\forall\beta[\bar\beta q=\bar{\underline{0}}q \to (\exists n[\beta(n)\neq 0]\to \mu n[\beta(n)\neq 0]$ is even$)]$.

This is contradictory, as we may define $\beta^*\in{}^\omega\omega$ such that: $2q+1=\mu n[\beta^*(n)\neq 0]$.

Case (iii): $a=\langle 0,0\rangle$.

We claim that, now: $\forall\alpha[S_2(F_0(\alpha))]$.

For, let $\alpha\in{}^\omega\omega$ and consider $\beta^* := \bar{\underline{0}}q * \langle 1\rangle * \alpha$.

We know: $\forall\beta[\bar\beta q=\bar{\underline{0}}q \to \exists\gamma\in\sigma_{2mon}[\bar\gamma 2=\bar{\underline{0}}2 \wedge \gamma\in F(\beta)]]$.

Because of the definition of F, therefore: $S_2(F_0(\alpha))$

But, then: $\forall\alpha[\neg\neg S_2(\alpha)\to S_2(\alpha)]$, and

S_2 is <u>not</u> a stable subset of ${}^\omega\omega$, as we observed just before theorem 11.9

We have seen: $\forall\beta[P(F(\beta))]$ and: $\neg\forall\beta[S_2(F(\beta))]$.

Therefore: $\neg\forall\alpha[P(\alpha)\to S_2(\alpha)]$.

⊠

11.12 <u>Lemma</u>: Suppose: $\delta\in{}^\omega\omega$ and $Fun(\delta)$ and: $\forall\alpha[S_2(\alpha)\to E_2(\delta|\alpha)]$.
Then: $\forall\alpha[P(\alpha)\to E_2(\delta|\alpha)]$.

<u>Proof</u>: Suppose: $\delta\in{}^\omega\omega$ and $Fun(\delta)$ and: $\forall\alpha[S_2(\alpha)\to E_2(\delta|\alpha)]$.
Let $\alpha\in{}^\omega\omega$ and $P(\alpha)$.
Remark: $\forall\gamma\in\sigma_{2mon}[S_2(\alpha_\gamma)]$ and therefore: $\forall\gamma\in\sigma_{2mon}[E_2(\delta|\alpha_\gamma)]$.
(The definition of α_γ has been given just before 11.8).
Observing: $\underline{0}\in\sigma_{2mon}$ and using GCP we find $q, n\in\omega$ such that:

$$\forall\gamma\in\sigma_{2mon}[\bar{\gamma}q=\bar{\underline{0}}q\to(\delta|\alpha_\gamma)^n=\underline{0}]$$

Therefore: $\exists\gamma[\gamma\in\alpha\wedge\bar{\gamma}q=\bar{\underline{0}}q]\to(\delta|\alpha)^n=\underline{0}$.
In view of: $P(\alpha)$, we may distinguish two cases:

Case (i): $\exists j<q[j^*\in\alpha]$.
Now: $S_2(\alpha)$, and therefore: $E_2(\delta|\alpha)$.

Case (ii): $\neg\neg\exists\gamma[\gamma\in\alpha\wedge\bar{\gamma}q=\bar{\underline{0}}q]$.
Then: $\neg\neg((\delta|\alpha)^n=\underline{0})$, therefore: $(\delta|\alpha)^n=\underline{0}$ and: $E_2(\delta|\alpha)$.

In either case: $E_2(\delta|\alpha)$.

We proved: $\forall\alpha[P(\alpha)\to E_2(\delta|\alpha)]$.

⊠

It is not possible to replace the conclusion of lemma 11.12 by: $\forall\alpha[\neg\neg S_2(\alpha)\to E_2(\delta|\alpha)]$
The following example makes this clear:
As one sees easily: $\forall\alpha[S_2(\alpha)\to(0^*\in\alpha\vee\neg\neg\exists\gamma[\gamma(0)=0\wedge\gamma\in\alpha])]$.
The succedens of this implication is indeed Σ^0_2 (cf. 11.5).
Now suppose: $\forall\alpha[\neg\neg S_2(\alpha)\to(0^*\in\alpha\vee\neg\neg\exists\gamma[\gamma(0)=0\wedge\gamma\in\alpha])]$.
Then, in particular: $\forall\alpha[(\alpha(\langle 0,0\rangle)=1\wedge\neg\neg(0^*\in\alpha\vee 1^*\in\alpha))\to(0^*\in\alpha\vee 1^*\in\alpha)]$.
And this, in turn, leads rather straightforwardly to: $\forall\alpha[\neg\neg D^2A_1(\alpha)\to D^2A_1(\alpha)]$.
D^2A_1, however, is not a stable subset of ${}^\omega\omega$. ($Neg(Neg(D^2A_1))$ is reducible to A_1, and D^2A_1 itself is not.)

This also establishes : $\neg\, \forall\alpha\, [\neg\neg S_2(\alpha) \to P(\alpha)]$, a claim which we made at the introduction of P, just after theorem 11.10, but left open until now.

The following, gratifying conclusion is the one we have been striving for:

11.13 <u>Theorem</u>: $\neg\, (S_2 \preceq A_3)$.

<u>Proof</u>: Suppose : $S_2 \preceq A_3$, and, using AC_{11}, determine $\delta \in {}^\omega\omega$ such that $Fun(\delta)$ and: $\forall\alpha\, [S_2(\alpha) \rightleftarrows A_3(\delta|\alpha)]$.

Therefore: $\forall\alpha\, [S_2(\alpha) \to \forall k\, [E_2((\delta|\alpha)^k)]]$, and, according to lemma 11.12 : $\forall\alpha\, [P(\alpha) \to A_3(\delta|\alpha)]$.

But now: $\forall\alpha\, [P(\alpha) \to S_2(\alpha)]$, and this contradicts theorem 11.11

$\boxtimes$

As P itself belongs to Π^0_3, the above proof shows that P is the best possible Π^0_3-approximation to S_2 ; i.e.: $P = \bigcap \{R \mid R \in \Pi^0_3 \mid S_2 \subseteq R\}$.
Similarly, $Neg(Neg(S_2))$, which belongs to Π^0_1, and thus to Π^0_2, is seen to be the best possible Π^0_2-approximation to S_2:
$Neg(Neg(S_2)) = \bigcap \{R \mid R \in \Pi^0_2 \mid S_2 \subseteq R\}$. (Cf. theorem 11.9 and its proof).

11.14 We will generalize the method used in proving : $\neg(S_2 \preceq A_3)$, and prove that S_2 is not hyperarithmetical.

Remark that: $\forall\alpha\, [S_2(\alpha) \rightleftarrows \forall n\, [\exists j\, [j^* \in \alpha] \vee (\forall k \le n [\alpha(\bar{0}k) = 0] \wedge S_2(^{\bar{0}n}\alpha))]]$.

We define $\mathcal{HA}$, a class of hyperarithmetical approximations to S_2 by the following clauses:

(I) $Neg(Neg(S_2))$ belongs to $\mathcal{HA}$.

(II) Whenever $Q_0, Q_1 \ldots$ is a sequence of elements of $\mathcal{HA}$ such that $\forall n\, [Q_{n+1} \subseteq Q_n]$, then Q_ω belongs to $\mathcal{HA}$, where Q_ω is defined by:
For all $\alpha \in {}^\omega\omega$:

$$Q_\omega(\alpha) := \forall n\, [\exists j\, [j^* \in \alpha] \vee (\forall k \le n [\alpha(\bar{0}k) = 0] \wedge Q_n(^{\bar{0}n}\alpha))].$$

(III) Whenever a set Q belongs to $\mathcal{HA}$, it does so because of (I) and (II).

One observes, that for each $Q \in \mathcal{HA}$: $S_2 \subseteq Q$.
We want to show that the converse is not true for any $Q \in \mathcal{HA}$.

11.15 We first remark that all members Q of $\mathcal{H}A$ are <u>proof against procrastination</u>
We will explain what we mean by that.

We want to use the fact that, like S_2, all members Q of $\mathcal{H}A$ have the following property:

$$\forall\alpha\, \forall k\, [\, (Q(\underline{\bar{0}}k\alpha) \wedge \forall n \leq k [\alpha(\underline{\bar{0}}n) = 0]) \to Q(\alpha)]$$

But there is more to it than this.
This „more" is, that we may extend the range of „k" to volatile numbers.
To express ourselves correctly, we have to introduce another new notion.

Let us define a <u>procrastinating function</u> $G: {}^\omega\omega \to {}^\omega\omega$, as follows:
Let $\beta \in {}^\omega\omega$ and $k := \mu n[\beta(n) \neq 0]$ be the volatile number of β.
We define $G(\beta)$ such that:

For all $n \in \omega$, $n \leq k$: $G(\beta)(\underline{\bar{0}}n) = 0$ and $G(\beta)(\overline{n^*}(n+1)) = 1$.

For all $a \in \omega$: $G(\beta)(\underline{\bar{0}}k * a) := \beta|_{k+1}(a)$.

($\beta|_{k+1}$ is the sequence which we get from β by deleting its first $k+1$ values, cf. the proof of theorem 11.11).

We may reformulate the basic properties of G as follows:

$$\forall\beta\, \forall k\, [\, k = \mu n[\beta(n) \neq 0] \to (\forall n < k [n^* \notin G(\beta)] \wedge \forall n \leq k [G(\beta)(\underline{\bar{0}}n) = 0] \wedge {}^{\underline{\bar{0}}k}G(\beta) = \beta|_{k+1})].$$

A subset Q of ${}^\omega\omega$ is called <u>proof against procrastination</u> if:

$$\forall\beta\, [\, \forall k [k = \mu n[\beta(n) \neq 0] \to Q(\beta|_{k+1})] \to Q(G(\beta))].$$

Our first observation is that $Neg(Neg(S_2))$ is proof against procrastination. The proof of this fact has been part of the proof of theorem 11.11, but perhaps it is useful to repeat the argument here.

Suppose: $\beta \in {}^\omega\omega$ and $\forall k\, [\, k = \mu n[\beta(n) \neq 0] \to \neg\neg S_2(\beta|_{k+1})]$.
There are two possibilities:

- $\exists n[\beta(n) \neq 0]$, then calculate $k = \mu n[\beta(n) \neq 0]$ and remark: $\forall n \leq k [G(\beta)(\underline{\bar{0}}n) = 0] \wedge \neg\neg S_2({}^{\underline{\bar{0}}p}G(\beta))$, therefore: $\neg\neg S_2(G(\beta))$.
- $\forall n[\beta(n) = 0]$, then: $\underline{0} \in G(\beta)$ and: $S_2(\beta)$ and: $\neg\neg S_2(\beta)$.

As: $\neg\neg(\exists n[\beta(n) \neq 0] \vee \forall n[\beta(n) = 0])$, we know: $\neg\neg S_2(G(\beta))$.

Now, assume that $Q_0, Q_1, \ldots$ is a sequence of subsets of ${}^\omega\omega$, which are, all of them, proof against procrastination, and such that: $\forall n[Q_{n+1} \subseteq Q_n]$, and consider:

$$Q_\omega = \{\alpha \mid \forall n\, [\exists j [j^* \in \alpha] \vee (\forall k \leq n [\alpha(\underline{\bar{0}}k) = 0] \wedge Q_n({}^{\underline{\bar{0}}n}\alpha))]\}.$$

We first remark that: $\forall n[Q_\omega \subseteq Q_n]$.

For, let $\alpha \in Q_\omega$ and $n \in \omega$.

There are two cases to consider:

(i) $\exists j[j^* \in \alpha]$, then: $S_2(\alpha)$, and: $Q_n(\alpha)$.

(ii) $\forall k \leq n[\alpha(\overline{0}k) = 0] \wedge Q_n(^{\overline{0}n}\alpha)$.

Now, $Q_n(\alpha)$, as Q_n is proof against procrastination.

In either case, therefore: $Q_n(\alpha)$.

Next, we show that Q_ω itself is proof against procrastination.
Suppose: $\beta \in {}^\omega\omega$ and: $\forall k[k = \mu p[\beta(p) \neq 0] \rightarrow Q_\omega(\beta|_{k+1})]$.

Let $n \in \omega$. First suppose: $n < \mu p[\beta(p) \neq 0]$

Then: $\forall k \leq n[G(\beta)(\overline{0}k) = 0]$, and: $^{\overline{0}n}G(\beta) = G(\beta|_n)$.

Remark: $\forall k[k = \mu p[\beta|_n(p) \neq 0] \rightarrow Q_\omega(\beta|_n|_{p+1})]$.

But $Q_\omega \subseteq Q_n$ and Q_n is proof against procrastination.

Therefore: $Q_n(G(\beta|_n))$ and: $Q_n(^{\overline{0}n}G(\beta))$.

Now suppose $n > \mu p[\beta(p) \neq 0]$.

Let $k := \mu p[\beta(p) \neq 0]$ and consider $\beta|_{k+1}$, recalling: $Q_\omega(\beta|_{k+1})$.

There are two cases to distinguish:

(i) $\exists j[j^* \in \beta|_{k+1}]$; calculate $j \in \omega$ such that $j^* \in \beta|_{k+1}$, and remark: $(j+k)^* \in G(\beta)$.

(ii) $Q_n((\beta|_{k+1})^{\overline{0}n})$ and: $\forall l \leq n[\beta|_{k+1}(\overline{0}l) = 0]$

But now: $Q_n(^{\overline{0}n}G(\beta))$ as Q_n is proof against procrastination, and: $\forall l \leq n[G(\beta)(\overline{0}l) = 0]$.

Therefore: $\forall n[\exists j[j^* \in G(\beta)] \vee (\forall k \leq n[G(\beta)(\overline{0}k) = 0] \wedge Q_n(^{\overline{0}n}G(\beta)))]$
i.e.: $Q_\omega(G(\beta))$.

We may trust, now, that all members Q of $\mathcal{H}A$ are proof against procrastination, as $Neg(Neg(S_2))$ has this property, and the property is preserved in the process of making a new element of $\mathcal{H}A$ out of a sequence of earlier-constructed elements.

11.16 We now devote ourselves to the task of proving that no member Q of $\mathcal{H}A$ coincides with S_2. We will define, to each $Q \in \mathcal{H}A$, a function $F: {}^\omega\omega \rightarrow Q$ such that: $\neg\forall\alpha[S_2(F(\alpha))]$.

In the case of $Neg(Neg(S_2))$, this promise is a cheap one.

We have seen that $Neg(Neg(S_2))$, being a spread, is strictly analytical and we constructed a function $F_0: {}^\omega\omega \rightarrow {}^\omega\omega$ such that $Neg(Neg(S_2)) = Ra(F_0)$ in the course of the proof of theorem 11.11.

On the other hand, we know, for some time already, that: $\neg\forall\alpha[\neg\neg S_2(\alpha) \rightarrow S_2(\alpha)]$ (cf. the remark preceding theorem 11.9)

Now, assume $Q_0, Q_1, \ldots$ is a sequence of hyperarithmetical approximations to S_2 such that: $\forall n [Q_{n+1} \subseteq Q_n]$, and $F_0, F_1, \ldots$ is a sequence of functions from ${}^\omega\omega$ to ${}^\omega\omega$ such that:

$$\forall n \forall \alpha [Q_n(F_n(\alpha))] \quad \wedge \quad \forall n \neg \forall \alpha [S_2(F_n(\alpha))]$$

We define a new function $F: {}^\omega\omega \to {}^\omega\omega$ as follows:
Let $\beta \in {}^\omega\omega$ and $k = \mu p[\beta(p) \neq 0]$ be the volatile number of β.
We define $F(\beta)$ such that:

(I) For all $n \in \omega$, $n \leq k$: $\quad F(\beta)(\underline{\bar{0}}n) = F(\beta)(\overline{0^*}n) = F(\beta)(\overline{1^*}n) = 0$.

(II) For all $n \in \omega$, $n \leq k$, $n \neq 0$, $n \neq 1$: $\quad F(\beta)(\overline{n^*}(n+1)) = 1$.

(III) For all $n \in \omega$, $n > k$: $\quad F(\beta)(\overline{0^*}n) = 0 \rightleftarrows k$ is odd

$F(\beta)(\overline{1^*}n) = 0 \rightleftarrows k$ is even

(IV) $\quad {}^{\underline{\bar{0}}k}F(\beta) = F_k(\beta|_{k+1})$.

We claim that: $\forall \beta [Q_\omega(F(\beta))]$.

Let $\beta \in {}^\omega\omega$ and $n \in \omega$.
First suppose: $n < \mu p[\beta(p) \neq 0]$.
Define a sequence $\beta^* \in {}^\omega\omega$ by requiring:

$$\forall k [k = \mu p[\beta(p) \neq 0] \to (\overline{\beta^*}(k+1) = \bar{\beta}(k+1) \wedge \beta^*|_{k+1} = F_k(\beta|_{k+1}))]$$

As $\forall n [Q_{n+1} \subseteq Q_n]$, this implies:

$$\forall k [k = \mu p[\beta^*(p) \neq 0] \to \quad Q_n(\beta^*|_{k+1})],$$

and, since Q_n is proof against procrastination: $Q_n(G(\beta^*))$
Remark, that: $\quad {}^{\underline{\bar{0}}n}F(\beta) = G(\beta^*|_n)$
Almost the same argument proves: $\quad Q_n(G(\beta^*|_n))$.
Therefore: $\forall k \leq n [F(\beta)(\underline{\bar{0}}k) = 0] \wedge Q_n({}^{\underline{\bar{0}}n}F(\beta))$.

Now suppose: $n \geq \mu p[\beta(p) \neq 0]$.
Calculating $k := \mu p[\beta(p) \neq 0]$ and seeing whether it is odd or even, we find: $0^* \in F(\beta) \vee 1^* \in F(\beta)$, therefore: $\exists j [j^* \in F(\beta)]$.

Therefore: $\forall n [\exists j [j^* \in F(\beta)] \vee (\forall k \leq n [F(\beta)\underline{\bar{0}}k = 0] \wedge Q_n({}^{\underline{\bar{0}}n}F(\beta))]$
i.e.: $Q_\omega(F(\beta))$.

We also claim that: $\neg \forall \beta [S_2(F(\beta))]$.

Suppose: $\forall \beta [S_2(F(\beta))]$.
Then: $\forall \beta \exists a \exists \gamma [lg(a) = 2 \wedge \gamma \in a \wedge \gamma \in \sigma_{2mon} \wedge \gamma \in F(\beta)]$
Using CP, we find $q \in \omega$, $a \in \omega$ such that:

$$lg(a) = 2 \wedge \forall \beta [\bar{\beta}q = \underline{\bar{0}}q \to \exists \gamma [\gamma \in a \wedge \gamma \in \sigma_{2mon} \wedge \gamma \in F(\beta)]]$$

We scrutinize a, and distinguish the following cases:

(1) $a = \langle 1, 1 \rangle$ $\quad$ Now: $\forall \beta [\bar{\beta}q = \underline{\bar{0}}q \to 0^* \in F(\beta)]$

Therefore: $\forall\beta[\bar\beta q=\bar{\underline{0}}q \to (\exists n[\beta(n)\neq 0] \to \mu n[\beta(n)\neq 0]$ is odd$)]$
This is contradictory.

(II) $a=\langle 0,1\rangle$. Now: $\forall\beta[\bar\beta q=\bar{\underline{0}}q \to 1^*\in F(\beta)]$
Therefore: $\forall\beta[\bar\beta q=\bar{\underline{0}}q \to (\exists n[\beta(n)\neq 0]\to \mu n[\beta(n)\neq 0]$ is even$)]$
This is contradictory.

(III) $a=\langle 0,0\rangle$. Now: $\forall\beta[\bar\beta q=\bar{\underline{0}}q \to \exists\gamma[\bar\gamma 2=\bar{\underline{0}}2 \wedge \gamma\in\sigma_{2mon} \wedge \gamma\in F(\beta)]$
Therefore: $\forall\beta[(\bar\beta q=\bar{\underline{0}}q \wedge \beta(q)=1) \to S_2({}^{\bar{\underline{0}}q}F(\beta))]$
And: $\forall\beta[(\bar\beta q=\bar{\underline{0}}q \wedge \beta(q)=1) \to S_2(F_q(\beta|_{q+1}))]$
Therefore: $\forall\alpha[S_2(F_q(\alpha))]$.
And this, according to our assumptions, is contradictory.

We put the blame for all these contradictions where it belongs, and conclude: $\neg\forall\beta[S_2(F(\beta))]$.

To any $Q\in\mathcal{H}A$ we may construct, by repeated application of the above, a function $F: {}^{\omega}\omega \to Q$ such that: $\neg\forall\beta[S_2(F(\beta))]$.
Therefore, no member Q of $\mathcal{H}A$ coïncides with S_2.

11.17 Let $Q\in\mathcal{H}A$ and let Q^+ be the set which results when we do apply the generating operation to the sequence $Q, Q, Q, \ldots$
Thus: $Q=\{\alpha \mid \forall n[\exists j[j^*\in\alpha] \vee (\forall k\le n[\alpha(\bar{\underline{0}}k)=0] \wedge Q({}^{\bar{\underline{0}}n}\alpha))]\}$
We have seen, in the previous paragraph, that Q is proof against procrastination, and that $Q^+\subseteq Q$
We observe, now, that $Q^+\neq Q$ and that Q^+ is a proper subset of Q

For, assume $Q^+=Q$
Then: $\forall\alpha[Q(\alpha) \rightleftarrows \forall n[\exists j[j^*\in\alpha] \vee (\forall k\le n[\alpha(\bar{\underline{0}}k)=0] \wedge Q({}^{\bar{\underline{0}}n}\alpha))]]$.
Especially: $\forall\alpha[Q(\alpha) \to (0^*\in\alpha \vee (\alpha(\langle 0\rangle)=0 \wedge Q({}^{\langle 0\rangle}\alpha))]$.
Let $\alpha\in{}^{\omega}\omega$ and $Q(\alpha)$.
We will construct $\gamma\in\sigma_{2mon}$ such that $\gamma\in\alpha$ and we will do so step-by-step.

step (0): We know: $Q(\alpha)$ and distinguish two possibilities:
(I) $0^*\in\alpha$, then $\gamma(0):=1$.
(II) $\alpha(\langle 0\rangle)=0 \wedge Q({}^{\langle 0\rangle}\alpha)$, then $\gamma(0):=0$ (and $Q({}^{\bar\gamma 1}\alpha)$).

step(Sn): $\gamma(0),\ldots\ldots \gamma(n)$ have been defined already.
If $\gamma(n)=1$, we define $\gamma(Sn):=1$.
If $\gamma(n)=0$, we know: $Q({}^{\bar\gamma Sn}\alpha)$, and we distinguish two cases:
(I) $0^*\in{}^{\bar\gamma Sn}\alpha$, then $\gamma(Sn):=1$
(II) ${}^{\bar\gamma Sn}\alpha(\langle 0\rangle)=0 \wedge Q({}^{\bar\gamma Sn*\langle 0\rangle}\alpha)$, then $\gamma(Sn):=0$
Remark, that, in the latter case: $Q({}^{\bar\gamma SSn}\alpha)$.

It is easily verified that: $\forall n[\alpha(\bar{\jmath}n)=0]$
Therefore: $\forall\alpha[Q(\alpha)\to S_2(\alpha)]$, and: $Q\subseteq S_2$, and: $Q=S_2$
But this is impossible, according to 11.16

Remark that, for any $Q\in \mathcal{H}A$, and $m\in\omega$:

$$Q^+(\alpha)\rightleftarrows \forall n>m[\exists j[j^*\in\alpha]\vee(\forall k\leq n[\alpha(\bar{0}k)=0]\wedge Q(\bar{0}^n\alpha))]$$

This is, because Q is proof against procrastination
We use this remark to make the following observation:
If $Q_0, Q_1,\ldots$ is a sequence of hyperarithmetical approximations to S_2, such that: $\forall n[Q_{n+1}\subseteq Q_n]$, and Q_ω is the set which we get by applying the generating operation to this sequence, then:

$$\forall n[Q_\omega\subseteq Q_n^+] \quad \text{and:} \quad \forall n[Q_\omega\neq Q_n].$$

Thus, the process of generating new elements in $\mathcal{H}A$ is endless, a fact which at once surprises and reassures.

A last remark on $\mathcal{H}A$, which we will need in the sequel, is that $\mathcal{H}A$ is closed under the operation of intersection.

We will prove, for all $P\in\mathcal{H}A$, that for all $Q\in\mathcal{H}A$ $P\cap Q\in\mathcal{H}A$, and we will do this inductively.

If $P=Neg(Neg(S_2))$, we remark that for all $Q\in\mathcal{H}A$: $Q\subseteq P$ and: $Q\cap P=Q$.

Now suppose: $P_0, P_1, P_2,\ldots$ is a sequence of elements of $\mathcal{H}A$, such that $\forall n[P_{n+1}\subseteq P_n]$, and such that any intersection of some P_n with any element of $\mathcal{H}A$, belongs to $\mathcal{H}A$ again.
We want to prove that: $P_\omega := \{\alpha\mid\forall n[\exists j[j^*\in\alpha]\vee(\forall k\leq n[\alpha(\bar{0}k)=0]\wedge P_n(\bar{0}^n\alpha))]\}$ has the same property.
To this end, assume $Q\in\mathcal{H}A$, $Q\neq Neg(Neg(S_2))$, and determine a sequence $Q_0, Q_1,\ldots$ of elements from $\mathcal{H}A$, such that

$$Q=Q_\omega := \{\alpha\mid\forall n[\exists j[j^*\in\alpha]\vee(\forall k\leq n[\alpha(\bar{0}k)=0]\wedge Q_n(\bar{0}^n\alpha))]\}.$$

Now consider $A := \{\alpha\mid\forall n[\exists j[j^*\in\alpha]\vee(\forall k\leq n[\alpha(\bar{0}k)=0]\wedge P_n(\bar{0}^n\alpha)\wedge Q_n(\bar{0}^n\alpha))]\}$
We claim that $A=P_\omega\cap Q_\omega$.
The proof is straightforward and may be omitted.
As, by hypothesis, $P_0\cap Q_0$, $P_1\cap Q_1,\ldots$ is a decreasing sequence of members of $\mathcal{H}A$, this shows that $P_\omega\cap Q_\omega$ itself belongs to $\mathcal{H}A$.

The reader may feel anxious about the huge quantifier: „for all $Q\in\mathcal{H}A$" occurring in this proof. But he need not do so We could have been so economical as to avoid it, talking only about those members of $\mathcal{HA}$, which played a role in the construction of P and Q (if we are engaged in proving that the intersection of P and Q belongs to $\mathcal{HA}$).

11.18 The curtain rises for the final act: we prove that S_2 is not hyperarithmetical.

Let Q be a hyperarithmetical approximation to S_2 (i.e.: $Q \in \mathcal{HA}$) and C a hyperarithmetical set.
Q is called a <u>witness against C</u> if

$$\forall\delta[(Fun(\delta) \wedge \forall\alpha[S_2(\alpha) \to C(\delta|\alpha)]) \to \quad \forall\alpha[Q(\alpha) \to C(\delta|\alpha)]]$$

If Q is a witness against C, S_2 cannot be reducible to C, for, in that case, Q and S_2 would coincide, which does not happen, as we saw in 11.6

If Q is a witness against C, Q also witnesses against any set D which is reducible to C.

If Q is a witness against C, the following is also true, for all $m \in \omega$:

$$\forall\delta[(Fun(\delta) \wedge \forall\alpha[S_2({}^m\alpha) \to C(\delta|\alpha)]) \to \quad \forall\alpha[Q({}^m\alpha) \to C(\delta|\alpha)]].$$

Suppose: $\delta \in {}^\omega\omega \wedge Fun(\delta) \wedge m \in \omega \wedge \forall\alpha[S_2({}^m\alpha) \to C(\delta|\alpha)]$
Let $\alpha \in {}^\omega\omega$ and $Q({}^m\alpha)$.
Define a function $\eta: {}^\omega\omega \to {}^\omega\omega$ such that:
$\forall\beta[\ {}^m(\eta|\beta) = \beta \wedge \forall n[\neg(n \le m) \to (\eta|\beta)(n) = \alpha(n)]]$, and consider $\zeta = \delta \circ \eta$.
Remark: $\forall\beta[S_2(\beta) \to C(\zeta|\beta)]$.
Therefore: $\forall\beta[Q(\beta) \to C(\zeta|\beta)]$.
Especially, since $Q({}^m\alpha)$: $C(\zeta|({}^m\alpha))$
But: $\eta|({}^m\alpha) = \alpha$ and $\zeta|({}^m\alpha) = \delta|\alpha$
Therefore: $\forall\alpha[Q({}^m\alpha) \to C(\delta|\alpha)]$.

We have seen, in lemma 11.8 and theorem 11.9 that $Neg(Neg(S_2))$ is a witness against E_1 and A_2, and, therefore, against any set D which belongs to Σ^0_1 or Π^0_1.

Starting from this fact, we may construct a witness against any hyperarithmetical set.

Suppose: $C_0, C_1, C_2, \dots$ is a sequence of hyperarithmetical subsets of ${}^\omega\omega$, and $P_0, P_1, P_2, \dots$ is a sequence of hyperarithmetical approximations to S_2, such that $\forall n[P_n$ is a witness against $C_n]$.

As $\mathcal{HA}$ is closed under intersection, we may assume: $P_0 \supseteq P_1 \supseteq P_2 \dots$
(We may change over to the sequence $P_0, P_0 \cap P_1, P_0 \cap P_1 \cap P_2, \dots$, if we do feel any doubts).

Consider. $P_\omega = \{\alpha \mid \forall n \exists j [j^* \in \alpha \vee (\forall k \le n[\alpha(\bar{\underline{0}}k) = 0] \wedge P_n(^{\bar{\underline{0}}n}\alpha))]\}$.
We claim that P_ω is a witness against $\bigcap_{n\in\omega} C_n$ and also against $\bigcup_{n\in\omega} C_n$.

As $P_\omega \subseteq \bigcap_{n\in\omega} P_n$, we have no difficulty in verifying that P_ω testifies against $\bigcap_{n\in\omega} C_n$.

Now, suppose: $\delta \in {}^\omega\omega$ and $Fun(\delta)$ and: $\forall \alpha [S_2(\alpha) \to \exists n [C_n(\delta|\alpha)]]$
Let $\alpha \in {}^\omega\omega$ and $P_\omega(\alpha)$.
Remark: $\forall \gamma \in \sigma_{2mon}[S_2(\alpha_\gamma)]$, and therefore: $\forall \gamma \in \sigma_{2mon} \exists n [C_n(\delta|\alpha_\gamma)]$.
(The definition of α_γ has been given just before 11.8)
Observing: $\underline{0} \in \sigma_{2mon}$ and using GCP we find $q, n_0 \in \omega$ such that:

$$\forall \gamma \in \sigma_{2mon}[\bar{\gamma}q = \bar{\underline{0}}q \to C_{n_0}(\delta|\alpha_\gamma)].$$

Therefore: $(\forall k \le q [\alpha(\bar{\underline{0}}k) = 0] \wedge S_2(^{\bar{\underline{0}}q}\alpha)) \to C_{n_0}(\delta|\alpha)$.
Let $m := \max(q, n_0)$.
In view of: $P_\omega(\alpha)$ we may distinguish two possibilities:

(I) $\exists j [j^* \in \alpha]$, then: $S_2(\alpha)$, and: $\exists n [C_n(\delta|\alpha)]$

(II) $\forall k \le m [\alpha(\bar{\underline{0}}k) = 0] \wedge P_m(^{\bar{\underline{0}}m}\alpha)$.
Remark, however, that: $(\forall k \le m [\alpha(\bar{\underline{0}}k) = 0] \wedge S_2(^{\bar{\underline{0}}m}\alpha)) \to C_{n_0}(\delta|\alpha)$.
As $P_m (\subseteq P_{n_0})$ witnesses against C_{n_0}, this implies:

$$(\forall k \le m [\alpha(\bar{\underline{0}}k) = 0] \wedge P_m(^{\bar{\underline{0}}m}\alpha) \to C_{n_0}(\delta|\alpha).$$

Therefore: $C_{n_0}(\delta|\alpha)$

Therefore: $\forall \delta [(Fun(\delta) \wedge \forall \alpha [S_2(\alpha) \to \exists n [C_n(\delta|\alpha)]]) \to \forall \alpha [P_\omega(\alpha) \to \exists n [C_n(\delta|\alpha)]]]$
i.e.: P_ω is a witness against $\bigcup_{n\in\omega} C_n$.

We have to abandon every hope that S_2 be hyperarithmetical, as any hyperarithmetical set may be built up from sets which belong to Σ^0_1 and Π^0_1, by repeated use of the operations of countable union and intersection.

We have seen, in 11.16 and 17, that very, very many hyperarithmetical sets are intercalated between S_2 and $Neg(Neg(S_2))$.

The results of this paragraph make us see anew that no hyperarithmetical set can be both existentially and universally saturated, a fact which has been seen to follow from the hyperarithmetical hierarchy theorem. (cf. 9.10).
For, in that case, we would find an element in $\mathcal{HA}$, witnessing against

all hyperarithmetical sets. This is impossible, according to 11.17.

11.19 Let $m \in \omega$, $m > 0$
We define a sequence $\sigma_{m\,mon} \in {}^{\omega}\omega$ by:
For all $a \in \omega$:

$$\sigma_{m\,mon}(a) := 0 \quad \text{if} \quad \forall n[n < lg(a) \to a(n) < m] \text{ and: } \forall n[n+1 < lg(a) \to a(n) \leq a(n+1)]$$

$$:= 1 \quad \text{otherwise.}$$

It is not difficult to verify that $\sigma_{m\,mon}$ is a subspread of ${}^{\omega}\omega$ (cf. 1.9 and 11.0) and that: $\forall m[\sigma_{m\,mon} \subseteq \sigma_{m+1\,mon}]$.

Remark that, for all $\gamma \in {}^{\omega}\omega$:

$$\gamma \in \sigma_{m\,mon} \rightleftarrows \forall n[\sigma_{m\,mon}(\bar{\gamma}n) = 0]$$
$$\rightleftarrows \forall n[\gamma(n) \leq \gamma(n+1) < m].$$

As with σ_{2mon}, we do call $\underline{0}$ the <u>spine</u> of $\sigma_{m\,mon}$.

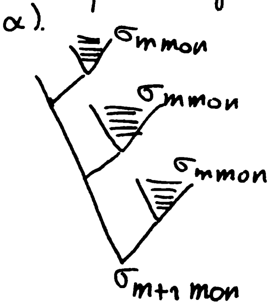
We define a subset S_m of ${}^{\omega}\omega$ by:
For all $\alpha \in {}^{\omega}\omega$:

$$S_m(\alpha) := \exists\gamma[\gamma \in \sigma_{m\,mon} \wedge \forall n[\alpha(\bar{\gamma}n) = 0]].$$

($\alpha \in {}^{\omega}\omega$ has the property S_m if there exists a sequence γ in $\sigma_{m\,mon}$ each of whose initial parts is approved of by α).

Remark that $A_1 \preceq S_1 \preceq A_1$.

Our technical eye also observes the following:



$\sigma_{m+1\,mon}$ is the result of intertwining a whole sequence of copies of $\sigma_{m\,mon}$.

Let us define a function $f^+ : {}^{\omega}\omega \to {}^{\omega}\omega$ by:
For all $\alpha \in {}^{\omega}\omega$:

$$\text{For all } n \in \omega: \quad (f^+(\alpha))(n) := \alpha(n) + 1$$

Remark that, for all $m \in \omega$: $f^+ : \sigma_{m\,mon} \to \sigma_{m+1\,mon}$

We prove a generalization of theorem 11.1 :

11.20 <u>Theorem</u>: $\forall m>0\ \forall n>0\ [\, D^n S_m \precsim S_{m+1} \,]$

<u>Proof</u>: Let $m,n\in\omega$, $m>0$, $n>0$.

Define $\delta\in {}^\omega\omega$ such that: $Fun(\delta)$ and:

(I) For all $\alpha\in {}^\omega\omega$, for all $q<n$: $(\delta|\alpha)(\underline{\bar{0}}q) = 0$

for all $q\geq n$: $(\delta|\alpha)(\underline{\bar{0}}q) = 1$

(II) For all $\alpha\in {}^\omega\omega$, for all $\gamma\in {}^\omega\omega$, for all $q<n$, for all $k\in\omega$:

$$(\delta|\alpha)(\underline{\bar{0}}q * \overline{f^+(\gamma)}\,k) = \alpha^q(\bar{\gamma}k)$$

Remark that:

$$\forall \zeta\in\sigma_{m+1\,mon}\ [\ \zeta\in\delta|\alpha \rightleftarrows \exists q<n\ \exists\gamma\in\sigma_{m\,mon}\,[\zeta = \underline{\bar{0}}q * f^+(\gamma)\ \wedge\ \gamma\in\alpha^q]].$$

Therefore: $\forall\alpha[\ \exists q<n\,[S_m(\alpha^q)] \rightleftarrows S_{m+1}(\delta|\alpha)]$

i.e.: $D^n S_m \precsim S_{m+1}$.

⊠

If this theorem is to bear the same kind of fruit as theorem 11.1, we must prove first:

11.21 <u>Theorem</u>: $\neg(D^2S_2 \precsim S_2)$

<u>Proof</u>: Suppose: $D^2S_2\precsim S_2$ and, using AC_{11}, determine $\delta\in{}^\omega\omega$ such that $Fun(\delta)$ and : $\forall\alpha[\ (S_2(\alpha^0)\ \vee\ S_2(\alpha^1)) \rightleftarrows S_2(\delta|\alpha)]$.

We claim that : $\forall\alpha[\ \underline{0}\in\alpha^0 \rightarrow \underline{0}\in\delta|\alpha]$.

Suppose: $\alpha\in{}^\omega\omega$ and: $\forall n[\alpha^0(\underline{\bar{0}}n)=0]$ and: $\exists n[(\delta|\alpha)(\underline{\bar{0}}n)\neq 0]$.

Calculate $n_0, q\in\omega$ such that:

$$\forall\beta[\ \bar{\beta}q = \bar{\alpha}q \rightarrow (\delta|\beta)(\underline{\bar{0}}\,n_0) = (\delta|\alpha)(\underline{\bar{0}}\,n_0)\neq 0]$$

The imitative subject has severely limited its own possibilities, whereas the creative subject still has all its options open.

Define $\eta\in{}^\omega\omega$ such that: $Fun(\eta)$ and:

for all $\beta\in{}^\omega\omega$:

(I) for all $k\leq q$: $(\eta|\beta)^0(\underline{\bar{0}}k) = 0$

(II) $\overline{(\eta|\beta)}q = \bar{\alpha}q$

(III) for all $k<q$: $k^*\notin(\eta|\beta)^0$

(IV) for all $a\in\omega$: $(\eta|\beta)^0(\underline{\bar{0}}q * a) = \beta(a)$

(V) $\neg S_2((\eta|\beta)^1)$

Let $\zeta \in {}^\omega\omega$ be such that: $Fun(\zeta)$ and: $\forall\beta[\zeta|\beta = \delta|(\eta|\beta)]$
Remark that for all $\beta \in {}^\omega\omega$:

$$\begin{aligned} S_2(\beta) &\rightleftarrows S_2((\eta|\beta)^0) \\ &\rightleftarrows D^2S_2(\eta|\beta) \wedge \overline{(\eta|\beta)}q = \bar{\alpha}q \\ &\rightleftarrows S_2(\delta|(\eta|\beta)) \wedge (\delta|(\eta|\beta))\,\bar{\underline{0}}n_0 \neq 0 \\ &\rightleftarrows S_2(\zeta|\beta) \wedge (\zeta|\beta)\,\bar{\underline{0}}n_0 \neq 0 \\ &\rightleftarrows \exists j < n_0\,[\,j^* \in \zeta|\beta\,] \end{aligned}$$

Therefore: $S_2 \leq D^{n_0}A_1$, and this contradicts corollary 11.2
Therefore: $\forall\alpha[\,\forall n[\alpha^0(\bar{\underline{0}}n) = 0] \rightarrow \forall n[(\delta|\alpha)(\bar{\underline{0}}n) = 0]]$

The same argument also establishes: $\forall\alpha[\,\underline{0} \in \alpha^1 \rightarrow \underline{0} \in \delta|\alpha]$
and, therefore: $\forall\alpha[\,(\underline{0} \in \alpha^0 \vee \underline{0} \in \alpha^1) \rightarrow \underline{0} \in \delta|\alpha]$.
This, however, has an undesirable consequence.

Define $\eta \in {}^\omega\omega$ such that: $Fun(\eta)$ and:
for all $\beta \in {}^\omega\omega$:

(i) $\forall n[\,(\eta|\beta)^0(\bar{\underline{0}}n) = \beta^0(n) \wedge (\eta|\beta)^1(\bar{\underline{0}}n) = \beta^1(n)]$

(ii) $\forall n[\,n^* \notin (\eta|\beta)^0 \wedge n^* \notin (\eta|\beta)^1]$.

Let $\zeta \in {}^\omega\omega$ be such that: $Fun(\zeta)$ and: $\forall\beta[\zeta|\beta = \delta|(\eta|\beta)]$
Remark that for all $\beta \in {}^\omega\omega$:

$$\begin{aligned} D^2A_1(\beta) &\rightleftarrows \beta^0 = \underline{0} \vee \beta^1 = \underline{0} \\ &\rightleftarrows \underline{0} \in (\eta|\beta)^0 \vee \underline{0} \in (\eta|\beta)^1 \\ &\rightleftarrows \underline{0} \in \zeta|\beta \\ &\rightleftarrows \forall n[(\zeta|\beta)(\bar{\underline{0}}n) = 0] \end{aligned}$$

Therefore: $D^2A_1 \leq A_1$, and this contradicts theorem 4.3.

We better admit: $\neg(D^2S_2 \leq S_2)$.

⊠

We thirst for more wisdom and are eager to know whether $D^3S_2 \leq D^2S_2$.

Let us introduce, for each $\alpha \in {}^\omega\omega$, $\gamma \in {}^\omega\omega$, $m \in \omega$, a sequence $\alpha_{\gamma,m} \in {}^\omega\omega$ by:

(i) $\alpha_{\gamma,m}(\langle\,\rangle) = 0$

(ii) for all $q \in \omega$, $q \neq m$ $\quad (\alpha_{\gamma,m})^q = \alpha^q$

(iii) for all $q \in \omega$: $\quad (\alpha_{\gamma,m})^m(q) := 0$ if $\bar{\gamma}(lg(q)) = q$
$:= \alpha^m(q)$ otherwise

(The reader will be reminded of the definition which has been given immediately after theorem 11.7).

We remark: $\forall m \forall\alpha [S_2(\alpha^m) \rightleftarrows \exists\gamma\in\sigma_{2mon} [\alpha = \alpha_{\gamma,m}]]$.

11.22 Theorem: $\forall n>0 [\neg (D^{n+1}S_2 \preceq D^n S_2)]$.

Proof: Suppose: $n\in\omega$, $n>1$, and: $D^{n+1}S_2 \preceq D^n S_2$, i.e.: $\forall\alpha\exists\beta[D^{n+1}S_2(\alpha)\rightleftarrows D^n S_2(\beta)]$

Using AC_{11}, determine $\delta\in{}^\omega\omega$ such that: $Fun(\delta)$ and:

$$\forall\alpha [D^{n+1}S_2(\alpha) \rightleftarrows D^n S_2(\delta|\alpha)].$$

Remark: $\forall m<n+1 \ \forall\alpha \ \forall\gamma\in\sigma_{2mon} [D^{n+1} S_2 (\alpha_{\gamma,m})]$.

Therefore: $\forall m<n+1 \ \forall\alpha \ \forall\gamma\in\sigma_{2mon} [D^n S_2 (\delta|\alpha_{\gamma,m})]$

i.e.: $\forall m<n+1 \ \forall\alpha \ \forall\gamma\in\sigma_{2mon} \exists p<n [S_2((\delta|\alpha_{\gamma,m})^p)]$.

Observing: $\underline{0}\in\sigma_{2mon}$ and using GCP we determine natural numbers $q_0, r_0, p_0, q_1, r_1, p_1, \ldots, q_n, r_n, p_n$ such that:

$$\forall m<n+1 \ \forall\alpha \ \forall\gamma\in\sigma_{2mon} [(\bar{\alpha}r_m = \bar{\underline{0}}r_m \wedge \bar{\gamma}q_m = \bar{\underline{0}}q_m) \rightarrow S_2((\delta|\alpha_{\gamma,m})^{p_m})]$$

Therefore: $\forall m<n+1 \ \forall\alpha [(\bar{\alpha}r_m = \bar{\underline{0}}r_m \wedge \exists\gamma\in\sigma_{2mon}[\bar{\gamma}q_m = \bar{\underline{0}}q_m \wedge \gamma\in\alpha^m]) \rightarrow S_2((\delta|\alpha)^{p_m})]$

As each of the numbers $p_0, p_1, \ldots p_n$ belongs to $\{0, 1, \ldots, n-1\}$, we may assume, without loss of generality: $p_0 = p_1 = 0$ and we perceive, putting $q := \max(q_0, q_1)$ and $r := \max(r_0, r_1)$:

$$\forall\alpha[(\bar{\alpha}r = \bar{\underline{0}}r \wedge \exists\gamma\in\sigma_{2mon}[\bar{\gamma}q = \bar{\underline{0}}q \wedge (\gamma\in\alpha^0 \vee \gamma\in\alpha^1)]) \rightarrow S_2((\delta|\alpha)^0)].$$

Once more, we have eaten too much from the tree of knowledge:

Let $s := \max(q, r)$ and define $\eta\in{}^\omega\omega$ such that: $Fun(\eta)$ and for all $\beta\in{}^\omega\omega$:

(I) $\overline{(\eta|\beta)}s = \bar{\underline{0}}s$

(II) $\forall k\le s [(\eta|\beta)^0(\bar{\underline{0}}k) = (\eta|\beta)^1(\bar{\underline{0}}k) = 0]$

(III) for all $a\in\omega$: $(\eta|\beta)^0(\bar{\underline{0}}s * a) = \beta^0(a)$ and $(\eta|\beta)^1(\bar{\underline{0}}s * a) = \beta^1(a)$

(IV) $\forall k<s [k^* \notin (\eta|\beta)^0 \wedge k^* \notin (\eta|\beta)^1]$

(V) $\forall m>1 [\neg S_2((\eta|\beta)^m)]$.

Define $\zeta\in{}^\omega\omega$ such that: $Fun(\zeta)$ and: $\forall\beta[\zeta|\beta = (\delta|(\eta|\beta))^0]$.

Remark, that for all $\beta\in{}^\omega\omega$:

$$D^2 S_2(\beta) \rightleftarrows \exists\gamma\in\sigma_{2mon}[\bar{\gamma}s = \bar{\underline{0}}s \wedge (\gamma\in(\eta|\beta)^0 \vee \gamma\in(\eta|\beta)^1)] \wedge \overline{(\eta|\beta)}s = \bar{\underline{0}}s$$
$$\rightleftarrows S_2(\zeta|\beta).$$

Therefore: $D^2S_2 \preceq S_2$, and this leads to absurdity (cf. 11.21)

We better leave paradise and keep in mind: $\forall n>1 [\neg D^{n+1}S_2 \preceq D^n S_2]$. ⊠

It is not difficult to establish: $\forall n>0[D^n S_2 \preceq D^{n+1}S_2]$, and, therefore: $\forall n>0[D^n S_2 \prec D^{n+1}S_2]$.
Combining this with theorem 11.20, we find: $\forall n>0[D^n S_2 \prec D^{n+1}S_2]$.
Now, the world starts to move again.
Looking into the proof of theorem 11.21, we see that it made us jump from: $\forall n>0\,[D^n A_1 \prec D^{n+1}A_1 \prec S_2]$ to: $S_2 \prec D^2 S_2$.
Nothing prevents a similar jump from: $\forall n>0[D^n S_2 \prec D^{n+1}S_2 \prec S_3]$ to: $S_3 \prec D^2 S_3$
Theorem 11.22 taught us how to conclude: $\forall n>0\,[D^n S_2 \prec D^{n+1}S_2]$ from: $S_2 \prec D^2 S_2$
Leaning on this experience, we trust that: $\forall n>0\,[D^n S_3 \prec D^{n+1}S_3]$ will follow from: $S_3 \prec D^2 S_3$.

Gradually, the following picture unfolds itself:

$$A_1 \prec D^2A_1 \prec D^3A_1 \ldots\; S_2 \prec D^2S_2 \prec D^3S_2 \ldots\; S_3 \prec D^2S_3 \prec D^3S_3 \ldots\; S_4 \prec D^2S_4 \ldots$$

Or, to put the same into a learned formula:

$$\forall m>0\,\forall n>0\,\forall p>0\,\forall q>0\,[\,D^m S_n \preceq D^p S_q \rightleftarrows (\,n<q \vee (n=q \wedge m\le p))]$$

It comes somewhat as a surprise, that much of this game may be played also with conjunction. We remind the reader of the easy fact that: $C^2A_1 \preceq A_1$.
(In definition 4.11 we introduced, for each $n\in\omega$ and $P\subseteq {}^\omega\omega$:
$C^n P := \{\alpha \mid \forall k<n\,[\,P(\alpha^k)]\}$).
In contrast to this, we have:

11.23 <u>Theorem</u>: $\neg\,(C^2S_2 \preceq S_2)$

<u>Proof</u>: The proof is a charming variation upon the proof of theorem 11.21
Suppose: $C^2S_2 \preceq S_2$ and, using AC_{11}, determine $\delta\in{}^\omega\omega$ such that: $Fun(\delta)$ and: $\forall\alpha[\,C^2S_2(\alpha) \rightleftarrows S_2(\delta|\alpha)]$, i.e.: $\forall\alpha[\,(S_2(\alpha^0)\wedge S_2(\alpha^1)) \rightleftarrows S_2(\delta|\alpha)]$
We claim that: $\forall\alpha[\,(\underline{0}\in\alpha^0 \wedge S_2(\alpha^1)) \to \underline{0}\in\delta|\alpha]$.

Suppose: $\alpha\in{}^\omega\omega$ and: $\forall n[\alpha^0(\bar{\underline{0}}n)=0]$ and: $S_2(\alpha^1)$ and: $\exists n\,[\,(\delta|\alpha)(\bar{\underline{0}}n)\neq 0]$.
Calculate $n_0, q\in\omega$ such that:

$$\forall\beta\,[\,\bar\beta q = \bar\alpha q \to (\delta|\beta)(\bar{\underline{0}}n_0) = (\delta|\alpha)(\bar{\underline{0}}n_0)\neq 0]$$

With a sigh, we point out to the imitative subject that it should not have made this overhasty step:
Define $\eta\in{}^\omega\omega$ such that: $Fun(\eta)$ and:
for all $\beta\in{}^\omega\omega$:
(i) $(\eta|\beta)^1 = \alpha^1$
(ii) $\overline{(\eta|\beta)}q = \bar\alpha q$

(iii) for all $k \leq q$: $(\eta|\beta)^0(\bar{\underline{0}}k) = 0$

(iv) for all $k < q$: $k^* \notin (\eta|\beta)^0$

(v) for all $a \in \omega$: $(\eta|\beta)^0(\bar{\underline{0}}q * a) = \beta(a)$.

Let $\zeta \in {}^\omega\omega$ be such that: $Fun(\zeta)$ and: $\forall\beta[\zeta|\beta = \delta|(\eta|\beta)]$

Remark that for all $\beta \in {}^\omega\omega$:

$$\begin{aligned} S_2(\beta) &\rightleftarrows S_2((\eta|\beta)^0) \\ &\rightleftarrows C^2S_2(\eta|\beta) \wedge \overline{(\eta|\beta)}q = \bar{\alpha}q \\ &\rightleftarrows S_2(\delta|(\eta|\beta)) \wedge (\delta|(\eta|\beta))\bar{\underline{0}}n_0 \neq 0 \\ &\rightleftarrows S_2(\zeta|\beta) \wedge (\zeta|\beta)\bar{\underline{0}}n_0 \neq 0 \\ &\rightleftarrows \exists j < n_0[j^* \in \zeta|\beta] \end{aligned}$$

Therefore: $S_2 \preceq D^{n_0}A_1$, and this contradicts corollary 11.2

We retire and conclude:

$$\forall\alpha[(\forall n[\alpha^0(\bar{\underline{0}}n) = 0] \wedge S_2(\alpha^1)) \rightarrow \forall n[(\delta|\alpha)(\bar{\underline{0}}n) = 0]].$$

Now that our claim has been established, it remains to see how it gets us into a further mess. But it does so rather quickly.

Define $\eta \in {}^\omega\omega$ such that: $Fun(\eta)$ and, for all $\beta \in {}^\omega\omega$:

(i) $(\eta|\beta)^0 = \underline{0}$

(ii) $(\eta|\beta)^1 = \beta$.

Let $\zeta \in {}^\omega\omega$ be such that: $Fun(\zeta)$ and: $\forall\beta[\zeta|\beta = \delta|(\eta|\beta)]$.

Remark that for all $\beta \in {}^\omega\omega$:

$$\begin{aligned} S_2(\beta) &\rightleftarrows (\underline{0} \in (\eta|\beta)^0 \wedge S_2((\eta|\beta)^1) \\ &\rightleftarrows \underline{0} \in \delta|(\eta|\beta) \\ &\rightleftarrows \forall n[(\zeta|\beta)(\bar{\underline{0}}n) = 0]. \end{aligned}$$

Therefore: $S_2 \preceq A_1$, and this contradicts corollary 11.2

We have to bow our head: $\neg(C^2S_2 \preceq S_2)$.

⊠

Pondering this last proof, we come to reflect that for all $m \in \omega$, $m > 0$, $n \in \omega$, for all $\alpha \in {}^\omega\omega$: $S_{m+1}(\alpha) \wedge \alpha(\bar{\underline{0}}n) \neq 0 \rightleftarrows \exists j < n\, \exists \gamma \in \sigma_m[\bar{\underline{0}}j * f^+(\gamma) \in \alpha]$

We may construct therefore, $\zeta \in {}^\omega\omega$ such that: $Fun(\zeta)$ and:

$$\forall\alpha \in {}^\omega\omega[\alpha(\bar{\underline{0}}n) \neq 0 \rightarrow (S_{m+1}(\alpha) \rightleftarrows D^nS_m(\zeta|\alpha)].$$

Thus, mimicking the proof of theorem 11.23 and using the fact that: $\forall n [D^n S_m \prec D^{n+1} S_m \prec S_{m+1}]$ we find that: $S_{m+1} \prec C^2 S_{m+1}$.
Remark that, in doing so, we take advantage of previously acquired knowledge on disjunction, rather than conjunction.

And now, dear and patient reader, we would like you to join us and climb the conjunctive towers which are based on $S_2, S_3, \ldots$
We have to warn you that the steps are high and many, but the view is a nice one...
An indication of conjunctive vitality is given by:

11.24 <u>Theorem</u>: $\forall m > 1\ \forall n > 1 [\neg (C^n S_m \preceq S_{m+n-2})]$.

<u>Proof</u>: Let $m > 1$.

We prove: $\forall n > 1 [\neg (C^n S_m \preceq S_{m+n-2})]$ by induction on n
The case: $n = 2$ has been disposed of in theorem 11.23 and the subsequent discussion.

Suppose: $n > 1$ and: $\neg (C^n S_m \preceq S_{m+n-2})$.
In order to take the next step, assume: $C^{n+1} S_m \preceq S_{m+n-1}$
Using AC_{11}, determine $\delta \in {}^\omega\omega$ such that: $Fun(\delta)$ and:

$$\forall \alpha [C^{n+1} S_m (\alpha) \rightleftarrows S_{m+n-1} (\delta | \alpha)]$$

We claim that: $\forall \alpha [(\forall p < n [\underline{0} \in \alpha^p] \wedge S_m (\alpha^n)) \to \underline{0} \in \delta | \alpha]$.

Suppose: $\alpha \in {}^\omega\omega$ and: $\forall p < n [\underline{0} \in \alpha^p]$ and: $S_m (\alpha^n)$ and: $n_0 \in \omega$ and: $(\delta | \alpha)(\overline{\underline{0}} n_0) \neq 0$.
(The next argument, which looks rather technical, is to bring out that now, by skilful grafting, one may reduce $C^n S_m$ to S_{m+n-2}).
Let us define a subset $\mathcal{T}$ of ${}^\omega\omega$ by:
$\mathcal{T} := \{ \beta | \beta \in {}^\omega\omega | \forall p < n+1 [\beta^{2p} \in \sigma_{mmon}] \}$.
We observe that $\mathcal{T}$ is a subspread of ${}^\omega\omega$.
Let us define, as we did on earlier occasions, to each $\beta; \gamma \in {}^\omega\omega$ a sequence $\beta_\gamma \in {}^\omega\omega$ by:

$$\text{for all } a \in \omega: \quad \beta_\gamma(a) := 0 \quad \text{if } \gamma \in a$$
$$:= \beta(a) \quad \text{otherwise.}$$

Let us define a function $F: {}^\omega\omega \to {}^\omega\omega$ such that:

$$\forall \beta \in {}^\omega\omega\ \forall p < n+1 [(F(\beta))^p = (\beta^{2p+1})_{\beta^{2p}}].$$

We observe that: $\forall \beta \in \mathcal{T} [C^{n+1} S_m (F(\beta))]$ and

we determine $\beta_* \in T$ such that: $F(\beta_*) = \alpha$ and: $\forall p<n\,[(\beta_*)^{2p} = \underline{0}]$.
Now: $\forall \beta \in T\,[S_{m+n-1}(\delta | F(\beta))]$.
Especially: $\forall \beta \in T\, \exists a\,[lg(a) = n_0 \wedge \exists \gamma [\gamma \in a \wedge \gamma \in \sigma_{m+n-1mon} \wedge \gamma \in \delta | F(\beta)]]$.
Applying GCP, we find $a \in \omega$, $q \in \omega$ such that:

$$lg(a) = n_0 \wedge \forall \beta \in T[\bar{\beta} q = \bar{\beta}_* q \rightarrow \exists \gamma[\gamma \in a \wedge \gamma \in \sigma_{m+n-1mon} \wedge \gamma \in \delta | F(\beta)]].$$

We define $\eta \in {}^\omega\omega$ such that: $Fun(\eta)$ and:
for all $\zeta \in {}^\omega\omega$:

(I) for all $p<n$: for all $a \in \omega$: $(\eta|\zeta)^p(\underline{\bar{0}}q * a) = \zeta^p(a)$
and: $\forall b\,[lg(b) < q \rightarrow (\eta|\zeta)^p(b) = \alpha^p(b)]$
and: $\forall b\,[(lg(b) = q \wedge b \neq \underline{\bar{0}}q) \rightarrow (\eta|\zeta)^p(b) \neq 0]$
and: $(\eta|\zeta)^p(\underline{\bar{0}}q) = 0$.

(II) $(\eta|\zeta)^n = \alpha^n$ and: $\forall m>n\,[(\eta|\zeta)^m = \alpha^m]$
and: $(\eta|\zeta)(\langle\rangle) = \alpha(\langle\rangle)$.

η might be called: the grafting function.

Remark that, for all $\zeta \in {}^\omega\omega$:

$$\begin{aligned} C^n S_m(\zeta) &\rightleftarrows C^{n+1} S_m(\eta|\zeta) \\ &\rightleftarrows \exists \beta \in T[\bar{\beta} q = \bar{\beta}_* q \wedge \eta|\zeta = F(\beta)] \\ &\rightleftarrows \exists \gamma \in \sigma_{m+n-1mon}[\gamma \in a \wedge \gamma \in \delta|(\eta|\zeta)] \end{aligned}$$

Looking back, we realize that: $a \neq \underline{\bar{0}} n_0$.
(As: $(\delta|\alpha)(\underline{\bar{0}} n_0) \neq 0$ and: $\exists \gamma \in \sigma_{m+n-1mon}[\gamma \in a \wedge \gamma \in \delta|\alpha]$).
Suppose that: $a(n_0 - 1) = 1$.
Define a function $f^+ : {}^\omega\omega \rightarrow {}^\omega\omega$ such that $\forall \alpha \forall n[(f^+(\alpha))(n) = \alpha(n) + 1]$.
Remark that, for all $\zeta \in {}^\omega\omega$:

$$C^n S_m(\zeta) \quad \rightleftarrows \quad \exists \gamma \in \sigma_{m+n-2mon}[a * f^+(\gamma) \in \delta|(\eta|\zeta)]$$

Therefore: $C^n S_m \preceq S_{m+n-2}$, and this leads to a contradiction, according to the induction hypothesis.
If $a(n_0 - 1) > 1$, we also find ourselves in an impossible situation, by a similar reasoning.
Therefore: $\forall \alpha\,[(\forall p<n\,[\underline{\dot{0}} \in \alpha^p] \wedge S_m(\alpha^n)) \rightarrow \forall j[(\delta|\alpha)(\underline{\bar{0}} j) = 0]]$

It is now an easy matter to bring the proof to its conclusion. (The more so, if we do remember the last bars of the proof of thm. II.23).

Define $\eta \in {}^\omega\omega$ such that: $Fun(\eta)$ and, for all $\beta \in {}^\omega\omega$:
$\forall p<n\,[(\eta|\beta)^p = \underline{0}]$ and: $(\eta|\beta)^n = \beta$.

Let $\zeta \in {}^\omega\omega$ be such that: $Fun(\zeta)$ and: $\forall\beta[\zeta|\beta = \delta|(\eta|\beta)]$
Remark that for all $\beta \in {}^\omega\omega$:

$$S_m(\beta) \rightleftarrows (\forall p<n[\underline{0} \in (\eta|\beta)^p] \wedge S_m((\eta|\beta)^n))$$
$$\rightleftarrows \underline{0} \in \delta|(\eta|\beta)$$
$$\rightleftarrows \forall j[(\zeta|\beta)(\bar{\underline{0}}j) = 0].$$

Therefore: $S_m \preceq A_1$, and, as $S_2 \preceq S_m$ (cf. the discussion after theorem 11.22), this contradicts corollary 11.2.

Admitting: $\neg(C^{n+1}S_m \preceq S_{m+n-1})$, we complete the induction step and, thereby, the proof of the theorem.

⊠

One of the consequences of this theorem is that: $\neg(C^{17}S_2 \preceq S_{17})$
As m increases, the complexity of C^mS_2 outgrows the complexity of any given member of the sequence $S_2, S_3, \ldots$
In retrospect, disjunction did not behave half as wildly as conjunction.

Let us introduce, for all subsets $P \subseteq {}^\omega\omega$, $Q \subseteq {}^\omega\omega$, a subset $C(P,Q)$ of ${}^\omega\omega$ by:
For all $\alpha \in {}^\omega\omega$: $C(P,Q)(\alpha) := P(\alpha^0) \wedge Q(\alpha^1)$.

11.25 <u>Theorem</u>: $\forall p>1\, \forall q>1\, [C(S_p, S_q) \preceq S_{p+q-1}]$.

<u>Proof</u>: Let us define a function $\pi: {}^\omega\omega \times {}^\omega\omega \to {}^\omega\omega$ such that:
for all $\alpha \in {}^\omega\omega$, $\beta \in {}^\omega\omega$: $\pi(\alpha,\beta) = \langle \alpha(0), \alpha(0)+\beta(0), \alpha(1)+\beta(0), \alpha(1)+\beta(1), \ldots$
i.e.: $\pi(\alpha,\beta)(0) := \alpha(0)$ and: $\forall n[\pi(\alpha,\beta)(2n+1) = \alpha(n)+\beta(n) \wedge \pi(\alpha,\beta)(2n+2) = \alpha(n+1)+\beta(n)]$.

(As usual, $m \dot{-} n := m-n$ if $m \geq n$, and $m \dot{-} n := 0$ if $m \leq n$).

Let us define a function $\lambda: {}^\omega\omega \to {}^\omega\omega$ such that:
for all $\alpha \in {}^\omega\omega$: $\lambda(\alpha) = \langle \alpha(0), \alpha(2) \dot{-} \alpha(1), \alpha(4) \dot{-} \alpha(3), \ldots$
i.e.: $\lambda(\alpha)(0) := \alpha(0)$ and: $\forall n[\lambda(\alpha)(n+1) := \alpha(2n+2) \dot{-} \alpha(2n+1)]$.

Let us define a function $\rho: {}^\omega\omega \to {}^\omega\omega$ such that
for all $\alpha \in {}^\omega\omega$: $\rho(\alpha) = \langle \alpha(1) \dot{-} \alpha(0), \alpha(3) \dot{-} \alpha(2), \alpha(5) \dot{-} \alpha(4), \ldots$
i.e.: $\forall n[\rho(\alpha)(n) := \alpha(2n+1) \dot{-} \alpha(2n)]$.

Remark that: $\forall\alpha\forall\beta[\lambda(\pi(\alpha,\beta)) = \alpha \wedge \rho(\pi(\alpha,\beta)) = \beta]$.

We also want a function $L: \omega \to \omega$ such that:
for all $a \in \omega$: $lg(L(a)) := \mu p[2p \geq lg(a)]$ and:
$L(a)(0) := a(0)$ and: $\forall n[2n+2 < lg(a) \to L(a)(n+1) := a(2n+2) \dot{-} a(2n+1)]$

Thus, L does to finite sequences what λ does to infinite sequences.
Similarly, we introduce a function $R: \omega \to \omega$ such that:
for all $a \in \omega$: $lg(R(a)) := \mu p[2p+1 \geq lg(a)]$ and:
$\forall n[2n+1 < lg(a) \to R(a)(n) := a(2n+1) \dot{-} a(2n)]$

Remark that: $\forall\alpha \in Ra(\pi)\, \forall n[\overline{\lambda(\alpha)}n = L(\bar{\alpha}2n) \wedge \overline{\rho(\alpha)}n = R(\bar{\alpha}2n)]$.

Let $p \in \omega$, $q \in \omega$, $p > 1$, $q > 1$
Remark that: $\forall \gamma \forall \delta [(\gamma \in \sigma_{p\,mon} \wedge \delta \in \sigma_{q\,mon}) \rightarrow \pi(\gamma, \delta) \in \sigma_{p+q-1\,mon}]$

Let us define $\eta \in {}^\omega\omega$ such that: $Fun(\eta)$ and:
for all $\alpha \in {}^\omega\omega$, for all $a \in \omega$:

$$(\eta|\alpha)(a) \quad := 0 \quad \text{if: } \alpha^0(L(a)) = 0 \text{ and } \sigma_{p\,mon}(L(a)) = 0 \text{ and } \alpha^1(R(a)) = 0 \text{ and } \sigma_{q\,mon}(R(a)) = 0$$
$$:= 1 \quad \text{otherwise}$$

We make two observations:

$$\forall \alpha \forall \gamma \forall \delta [(\gamma \in \sigma_{p\,mon} \wedge \gamma \in \alpha^0 \wedge \delta \in \sigma_{q\,mon} \wedge \delta \in \alpha^1) \rightarrow (\pi(\gamma, \delta) \in \sigma_{p+q-1\,mon} \wedge \pi(\gamma, \delta) \in \eta|\alpha)]$$

$$\forall \alpha \forall \gamma [(\gamma \in \sigma_{p+q-1\,mon} \wedge \gamma \in \eta|\alpha) \rightarrow (\lambda(\gamma) \in \sigma_{p\,mon} \wedge \lambda(\gamma) \in \alpha^0 \wedge \rho(\gamma) \in \sigma_{q\,mon} \wedge \rho(\gamma) \in \alpha^1)]$$

Therefore: $\forall \alpha [(S_p(\alpha^0) \wedge S_q(\alpha^1)) \rightleftarrows S_{p+q-1}(\eta|\alpha)]$

i.e.: $C(S_p, S_q) \preceq S_{p+q-1}$

⊠

When making theorems 11.24 and 11.25 join hands, we find a result which is worth remembering:

11.26 <u>Theorem</u>: $\forall m > 1\ \forall n > 0\ [\neg(C^{n+1}S_m \preceq C^n S_m)]$.

<u>Proof</u>: Suppose: $m \in \omega$, $m > 1$ and: $n \in \omega$ and: $C^{n+1}S_m \preceq C^n S_m$.
Now: $C^{n+2}S_m \preceq C(S_m, C^{n+1}S_m) \preceq C(S_m, C^n S_m) \preceq C^{n+1}S_m \preceq C^n S_m$.
In this way, we come to see: $\forall p \geq n\ [C^p S_m \preceq C^n S_m]$.
On the other hand, we may derive from theorem 11.25, that:
$C^n S_m \preceq S_{n \cdot m - n + 1}$.
Therefore: $\forall p \geq n\ [C^p S_m \preceq S_{n \cdot m - n + 1}]$.
This calls for a protest by theorem 11.24, which says that, if we choose p large enough: $\neg(C^p S_m \preceq S_{n \cdot m - n + 1})$
Therefore: $\neg(C^{n+1}S_m \preceq C^n S_m)$.

⊠

As we have no difficulty in seeing that: $\forall m > 0\ \forall n > 0\ [C^n S_m \preceq C^{n+1} S_m]$, we quiet down and relish the sight of the following towers:

$S_2 \prec C^2 S_2 \prec C^3 S_2 \prec \ldots$

$S_3 \prec C^2 S_3 \prec C^3 S_3 \prec \ldots$

Unlike the disjunctive ones, these towers have no easy upper bounds, and are very much entangled into each other.

11.27 We define a sequence $\sigma_2 \in {}^\omega\omega$ by:

For all $a \in \omega$:

$$\sigma_2(a) := 0 \quad \text{if } \forall n[\, n < lg(a) \rightarrow a(n) < 2]$$
$$:= 1 \quad \text{otherwise.}$$

σ_2 is a well-known example of a subspread of ${}^\omega\omega$
σ_2, or the set $\{\alpha \mid \forall n[\sigma_2(\bar{\alpha}n) = 0]\}$ is called: the binary fan.

We define a subset S of ${}^\omega\omega$ by:

For all $\alpha \in {}^\omega\omega$:

$$S(\alpha) \;:=\; \exists \gamma[\, \gamma \in \sigma_2 \wedge \forall n[\alpha(\bar{\gamma}n) = 0]].$$

We introduce a class $\mathcal{C}$ of subsets of ${}^\omega\omega$ by:

For every subset P of ${}^\omega\omega$:

$$P \in \mathcal{C} \quad \rightleftarrows \quad P \preceq S.$$

Other definitions of $\mathcal{C}$ may be given, which avoid quantifying over all subsets of ${}^\omega\omega$. (Cf. 10.0). Perhaps the most easy solution, here and now, is to restrict oneself to members P of Σ^1_1.

We remark that $\mathcal{C}$ is closed under the operations of finite union and countable intersection.

Suppose: P and Q are subsets of ${}^\omega\omega$ and: $P \in \mathcal{C}$ and $Q \in \mathcal{C}$
Determine $\delta_0 \in {}^\omega\omega$ such that: $Fun(\delta_0)$ and: $\forall \alpha[\, P(\alpha) \rightleftarrows S(\delta_0|\alpha)]$
Determine $\delta_1 \in {}^\omega\omega$ such that: $Fun(\delta_1)$ and: $\forall \alpha[\, Q(\alpha) \rightleftarrows S(\delta_1|\alpha)]$
Define $\delta \in {}^\omega\omega$ such that: $Fun(\delta)$ and: for all $\alpha \in {}^\omega\omega$, for all $a \in \omega$:

$$(\delta|\alpha)(\langle 0\rangle * a) = (\delta_0|\alpha)(a)$$

and: $(\delta|\alpha)(\langle 1\rangle * a) = (\delta_1|\alpha)(a)$ and: $(\delta|\alpha)(\langle\,\rangle) = 0$.
One has to allow that: $\forall \alpha[(S(\delta_0|\alpha) \vee S(\delta_1|\alpha)) \rightleftarrows S(\delta|\alpha)]$,
i.e.: $\forall \alpha[(P(\alpha) \vee Q(\alpha)) \rightleftarrows S(\delta|\alpha)]$, and: $P \cup Q \in \mathcal{C}$.

Suppose: $P_0, P_1, P_2, \ldots$ is a sequence of subsets of ${}^\omega\omega$ such that: $\forall m[\, P_m \in \mathcal{C}]$.
Determine a sequence $\delta_0, \delta_1, \delta_2, \ldots$ of elements of ${}^\omega\omega$ such that:
$\forall m[\, Fun(\delta_m) \wedge \forall \alpha[\, P_m(\alpha) \rightleftarrows S(\delta_m|\alpha)]$
Define $\delta \in {}^\omega\omega$ such that: $Fun(\delta)$ and: for all $\alpha \in {}^\omega\omega$, for all $a \in \omega$:

$$(\delta|\alpha)(a) := 0 \quad \text{if } \forall m < lg(a)\, \forall k[\, k < lg(a^m) \rightarrow (\delta_m|\alpha)(\overline{a^m}k) = 0]$$
$$:= 1 \quad \text{otherwise}$$

(The notations a^m and $\overline{a^m}$ have been mentioned in 10.2).
We observe that: $\forall \alpha\, \forall \gamma \in \sigma_2[\, \gamma \in \delta|\alpha \;\rightleftarrows\; \forall m[\, \gamma^m \in \delta_m|\alpha]]$
Therefore: $\forall \alpha[\, \forall m[\, S(\delta_m|\alpha)] \rightleftarrows S(\delta|\alpha)]$,
and: $\forall \alpha[\, \forall m[\, P_m(\alpha)] \rightleftarrows S(\delta|\alpha)]$, i.e.: $\bigcap_{m \in \omega} P_m \in \mathcal{C}$.

Recall that $\beta \in {}^\omega\omega$ is called a subfan of ${}^\omega\omega$ if β is a subspread of ${}^\omega\omega$ and: $\forall a [\beta(a) = 0 \rightarrow (K_a^\beta = \{n \mid \beta(a * \langle n \rangle) = 0\}$ is finite$)\}$.
(Cf. 9.0 and the discussion following on theorem 11.3).

To any subspread β of ${}^\omega\omega$ we may consider a corresponding subset S_β of ${}^\omega\omega$ which, in analogy to S_2 and S, is defined by:

For all $\alpha \in {}^\omega\omega$:

$$S_\beta(\alpha) \quad := \quad \exists \gamma \forall n [\beta(\bar{\gamma}n) = 0 \ \wedge \ \alpha(\bar{\gamma}n) = 0]$$

We remark that, for every subfan β of ${}^\omega\omega$, S_β belongs to $\mathcal{C}$.

> A proof of this fact is readily found, if one realizes that any subfan β of ${}^\omega\omega$ may be embedded into σ_2.

Therefore, $\mathcal{C}$ is a quite complicated class of subsets of ${}^\omega\omega$.

> Many subsets of ${}^\omega\omega$ which have been mentioned in this chapter, do belong to $\mathcal{C}$; like $S_2, S_3, \ldots$ and all sets which we get from them by applications of the operations of finite union and countable intersection, for instance: $C^2D^{17}S_3$.

We remark that S is not hyperarithmetical, as $S_2 \preceq S$ and S_2 already is not hyperarithmetical. (Cf. 11.18)
Also: $\forall n [S_n \prec S]$, as $\forall n [S_n \prec S_{n+1} \preceq S]$ (Cf. theorem 11.22 and the ensuing discussion.)

We define a subset T of ${}^\omega\omega$ by:

For all $\alpha \in {}^\omega\omega$:

$$T(\alpha) \quad := \quad \forall \gamma \in \sigma_2 \ \exists n [\alpha(\bar{\gamma}n) = 0].$$

We introduce a class $\mathcal{D}$ of subsets of ${}^\omega\omega$ by:

For every subset P of ${}^\omega\omega$:

$$P \in \mathcal{D} \quad \rightleftarrows \quad P \preceq T.$$

(In this definition, we may restrict our attention to members P of Π_1^1).

According to the fan theorem, which we mentioned already after theorem 11.3, for every subfan β of ${}^\omega\omega$, and all $\alpha \in {}^\omega\omega$:

$$\forall \gamma \in \beta \ \exists n [\alpha(\bar{\gamma}n) = 0] \quad \rightleftarrows \quad \exists m \forall \gamma \in \beta \ \exists n \leq m [\alpha(\bar{\gamma}n) = 0]$$

Thus, we find that $\mathcal{D} \subseteq \Sigma_1^0$, and, actually, that $\mathcal{D} = \Sigma_1^0$.

As $S_2 \notin \Sigma_1^0$ (cf. theorem 11.3), and $S_2 \in \mathcal{C}$, also: $\neg(\mathcal{C} \subseteq \mathcal{D})$

There are different ways of establishing this last truth.
(The use of the fan theorem, which is a difficult principle of

intuitionistic analysis, should be avoided as much as possible).

We may remark, that, according to theorem 11.1, $D^2A_1 \in \mathcal{C}$, and, according to theorem 10.11, $D^2A_1 \notin \Pi^1_1$, whereas $\mathcal{D} \subseteq \Pi^1_1$
Neither one of these results depends on the fan theorem.

Or, our memory may go back to theorem 10.12
We may cite its proof almost literally to obtain the following conclusion:

Let $\delta \in {}^\omega\omega$ such that: $\mathrm{Fun}(\delta)$ and: $\forall\alpha [\, S(\alpha) \to T(\delta|\alpha)]$
We may construct $\zeta \in {}^\omega\omega$ such that:

$$\forall\gamma \in \sigma_2\ \exists n\, [\, \alpha(\bar{\gamma}n) \neq 0] \text{ and: } \forall\gamma \in \sigma_2\, \exists n\, [\, (\delta|\alpha)(\bar{\gamma}n) = 0].$$

Like Σ^1_1 and Π^1_1, $\mathcal{C}$ an $\mathcal{D}$ do form a mysterious couple.
One is tempted to compare the two.

The reader will remember how we deplored, at the end of chapter 10, not to be able to prove that: $\neg(\Pi^1_1 \subseteq \Sigma^1_1)$

There is much more that we do not know.

(i) Is $\mathcal{D} \subseteq \mathcal{C}$? Is $\Sigma^0_1 \subseteq \mathcal{C}$?

(At the assumption of the fan theorem, these two questions are equivalent.
Remark that the proof of: $\neg(E_1 \leq S_2)$ (theorem 11.7) depended on theorem 11.4 (ii)
It is not known whether σ_2 has this property).

(ii) Is $\mathcal{C}$ closed under the operation of countable union?

(If so, all hyperarithmetical sets belong to $\mathcal{C}$).

(iii) Is $\Sigma^1_1 = \mathcal{C}$?

(Remark that, on the other hand, $\Pi^1_1 \neq \mathcal{D}$, as $\Pi^0_1 \subseteq \Pi^1_1$ and, at the assumption of the fan theorem, $\neg\, \Pi^0_1 \subseteq \mathcal{D} = \Sigma^0_1$.
Is there a proof this fact, which avoids the use of the fan theorem?)

One would like to understand why these questions are giving so much trouble. A positive answer to any one of them would be very surprising, fooling classical opinion which holds, for duality reasons, that Π^0_1 and $\mathcal{C}$ coïncide.

While this new cloud of unknowing descends upon us, we feel that it is time to end the chapter.

12 AN OUTBURST OF DISJUNCTIVE, CONJUNCTIVE AND IMPLICATIVE PRODUCTIVITY.

We still are under the spell of the theme which captivated us in the second half of the previous chapter.
We have seen, there, that S_2 is an upper bound to the increasing sequence $A_1, D^2A_1, \dots$ and, as such, rivals E_2, although the two do not admit of a comparison.
Trying to understand why S_2 should be so rude as to disturb the peace of the hyperarithmetical hierarchy, we might think of the fact that S_2, itself, is not a hyperarithmetical set.
It turns out, however, that agitators may be found under our own roof: S_2 has some hyperarithmetical relatives that are equal to similar mischief, being superior to all sets $A_1, D^2A_1, \dots$ and, nevertheless, incomparable to E_2.
Like S_2, these sets also support disjunctive and conjunctive towers.

A subset P of ${}^\omega\omega$, such as A_1, or S_2, for which $P < D^2P < D^3P \dots$ will be called disjunctively productive.
General methods will be indicated, to assign to any disjunctively productive set P a disjunctively productive subset Q of ${}^\omega\omega$ such that $\forall m[D^mP \leq Q]$.
Fortunately, these methods assign to a set P which is hyperarithmetical, a set Q which is hyperarithmetical as well.
We will find, in this way, that, for instance between A_1 and A_3, uncountably many levels of complexity have to be distinguished.

A similar game may be played with conjunction.

Also, notions of implicative productivity will be around, carrying along, in their development, a generalization of some theorems of chapter 5.

12.0 Consider, as an example of the type of constructions which will occupy us, the set:

$$R := \{\alpha \mid \forall n[n = \mu p[\alpha^0(p) \neq 0] \rightarrow D^nA_1(\alpha^{Sn})]\}.$$

Remark, that $R \in \Pi^0_3$, and so is arithmetical, as $R = \{\alpha \mid \forall n[n \neq \mu p[\alpha^0(p) \neq 0] \vee D^nA_1(\alpha^{Sn})]\}$.

Remark, that $\forall n[D^nA_1 \leq R]$.

> Let $n \in \omega$. Define $\delta \in {}^\omega\omega$ such that: $Fun(\delta)$ and: $\forall\alpha[n = \mu p[(\delta|\alpha)^0(p) \neq 0] \wedge (\delta|\alpha)^{Sn} = \alpha]$
> Then: $\forall\alpha[D^nA_1(\alpha) \rightleftarrows R(\delta|\alpha)]$, i.e.: $D^nA_1 \leq R$.

Also observe, that: $\neg(S_2 \leq R)$.

> We may appeal to 11.18 where it is proved that S_2 is not hyperarithmetical, or even to theorem 11.13 which says only that S_2 is not Π^0_3.
> The following argument resulted from an attempt at a direct proof:
> Suppose: $S_2 \leq R$, and, using AC_{11}, determine $\delta \in {}^\omega\omega$ such that: $Fun(\delta)$ and: $\forall\alpha[S_2(\alpha) \rightleftarrows R(\delta|\alpha)]$, i.e.: $\forall\alpha[\exists\gamma[\gamma \in \sigma_{2mon} \wedge \gamma \in \alpha] \rightleftarrows R(\delta|\alpha)]$.

Consider: $\mathcal{T} = \{\alpha \mid \forall n [\, \forall m \le n [\alpha(\underline{\bar{0}} m) = 0] \vee \forall m \le n [\alpha(\underline{\bar{1}} m) = 0]]\}$.

We claim that: $\forall \alpha \in \mathcal{T} [R(\delta|\alpha)]$.

Let $\alpha \in \mathcal{T}$ and $n \in \omega$ and: $n = \mu p\, [(\delta|\alpha)^{\circ}(p) \neq 0]$
Determine $q \in \omega$ such that: $\forall \beta [\bar{\beta} q = \bar{\alpha} q \to \overline{\delta|\beta}\,^{\circ}(n+1) = \overline{(\delta|\alpha)}^{\circ}(n+1)]$.
We now claim that: $\exists r \le q [\alpha(\underline{\bar{0}} r) \neq 0]$.

Suppose: $\forall r \le q [\alpha(\underline{\bar{0}} r) = 0]$.
Define $\eta \in {}^{\omega}\omega$ such that Fun(η) and, for all $\zeta \in {}^{\omega}\omega$:
(I) $\overline{(\eta|\zeta)}\, q = \bar{\alpha} q$
(II) for all $a \in \omega$: $(\eta|\zeta)(\underline{\bar{0}} q * a) = \zeta(a)$
(III) $\forall r < q \,[\, r^* \notin \eta|\zeta \,]$.
Then, for all $\zeta \in {}^{\omega}\omega$:

$$S_2(\zeta) \rightleftarrows S_2(\eta|\zeta) \wedge \overline{(\eta|\zeta)}\, q = \bar{\alpha} q$$
$$\rightleftarrows D^n A_1((\eta|\zeta)^{Sn})$$

Therefore: $S_2 \le D^n A_1$, which contradicts theorem 11.2.

Therefore: $\exists r \le q [\alpha(\underline{\bar{0}} r) \neq 0]$, and, as $\alpha \in \mathcal{T}$: $\forall m [\alpha(\underline{\bar{1}} m) = 0]$
i.e.: $\underline{1} \in \alpha$, and: $S_2(\alpha)$
Therefore: $R(\delta|\alpha)$, and: $D^n A_1((\delta|\alpha)^{Sn})$.

We proved: $\forall n [n = \mu p [(\delta|\alpha)^{\circ}(p) \neq 0] \to D^n A_1((\delta|\alpha)^{Sn})]$.
i.e.: $R(\delta|\alpha)$.

Therefore: $\forall \alpha \in \mathcal{T} [R(\delta|\alpha)]$, and: $\forall \alpha \in \mathcal{T} [S_2(\alpha)]$.

Consider the following subset of $\mathcal{T}$:

$$\mathcal{T}^* := \{\alpha \mid \alpha \in \mathcal{T} \mid \forall n > 0\, \exists m\, [\alpha(\overline{n^*} m) \neq 0]\}.$$

Now: $\forall \alpha \in \mathcal{T}^* [S_2(\alpha)]$ and so: $\forall \alpha \in \mathcal{T}^* [\, \forall m [\alpha(\underline{\bar{0}} m) = 0] \vee \forall m [\alpha(\underline{\bar{1}} m) = 0]]$.
From this, it may be proved that: $\forall \alpha [\neg\neg D^2 A_1(\alpha) \to D^2 A_1(\alpha)]$.
which, as we know, is not true. (Cf. theorem 4.3 and its proof).

Finally, remark that: $R \preceq S_2$.

Define $\delta \in {}^{\omega}\omega$ such that: Fun(δ) and, for all $\alpha \in {}^{\omega}\omega$:

(I) $(\delta|\alpha)(\underline{\bar{0}} n) := 0$ if $n < 2 \cdot \mu p [\alpha^{\circ}(p) \neq 0]$
$(\delta|\alpha)(\underline{\bar{0}} n) := 1$ if $n = 2 \cdot \mu p [\alpha^{\circ}(p) \neq 0]$.

(II) $(\delta|\alpha)(\overline{n^*}(n+1)) := 1$ if $n < \mu p [\alpha^{\circ}(p) \neq 0]$.

(III) for all $n \in \omega$ such that: $n = \mu p [\alpha^{\circ}(p) \neq 0]$, for all $k < n$, for all $l \in \omega$:
$(\delta|\alpha)(n+k)^*(n+l) = \alpha^{Sn,k}(l)$.

Then, for all $\alpha \in {}^\omega\omega$:

$$\begin{aligned} R(\alpha) &\rightleftarrows \forall n\,[\,n = \mu p[\alpha^0(p) \neq 0] \rightarrow D^n A_1(\alpha^{Sn})] \\ &\rightleftarrows \forall n\,[\,2n = \mu p\,[\,(\delta|\alpha)(\underline{\bar{0}}\,p) = 0] \rightarrow \exists k[n \leq k < 2n \wedge k^* \in \delta|\alpha]] \\ &\rightleftarrows S_2(\delta|\alpha). \end{aligned}$$

Therefore: $R \preceq S_2$.

Apparently, R is a smaller upper bound to the sequence $A_1, D^2A_1, \ldots$ than is S_2 and it has the advantage of being arithmetical.

12.1 We generalize the construction that has been sketched in 12.0 and discover nice properties of the sets which are produced by it.

Let $P_0, P_1, \ldots$ be a sequence of subsets of ${}^\omega\omega$ which fulfils the condition: $\forall m \exists n\,[P_m \prec P_n]$.

Now define $Q := \{\alpha \mid \forall n\,[\,n = \mu p\,[\,\alpha^0(p) \neq 0] \rightarrow P_n(\alpha^{Sn})]\}$.

Remark that, by this definition, α^0 has to play the role of a signalling sequence, and, as such, may be compared to $\alpha \circ \underline{0}$, i.e. α, as behaving on the spine of σ_{2mon}, if we are studying whether α has the property S_2.

Remark that: $\forall n\,[\,P_n \prec Q]$.

Let $n \in \omega$. Define $\delta \in {}^\omega\omega$ such that: $Fun(\delta)$ and: $\forall \alpha[n = \mu p[(\delta|\alpha)^0(p) \neq 0] \wedge (\delta|\alpha)^{Sn} = \alpha]$
Then: $\forall \alpha\,[\,P_n(\alpha) \rightleftarrows R(\delta|\alpha)]$, i.e.: $P_n \preceq Q$.

We make a minor assumption on the sequence $P_0, P_1, \ldots$ namely, that $\forall n \exists \alpha[\neg P_n(\alpha)]$, and prove: $\neg\,(D^2Q \preceq Q)$.

The proof is similar to the proof of theorem 11.21 which stated that: $\neg\,(D^2S_2 \preceq S_2)$.

Suppose: $D^2Q \preceq Q$ and, using AC_{11}, determine $\delta \in {}^\omega\omega$ such that: $Fun(\delta)$ and: $\forall \alpha\,[\,D^2Q(\alpha) \rightleftarrows Q(\delta|\alpha)]$.

Determine a sequence $\beta_0, \beta_1, \ldots$ of members of ${}^\omega\omega$, such that: $\forall n[\neg P_n(\beta_n)]$.
Consider $T := \{\alpha \mid \forall n\,[\,\forall m \leq n\,[\,\alpha^{0,0}(m) = 0] \vee \forall m \leq n\,[\alpha^{1,0}(m) = 0] \wedge \forall n[\alpha^{0,Sn} = \alpha^{1,Sn} = \beta^n]\}$
We claim that: $\forall \alpha \in T\,[\,(\delta|\alpha)^0 = \underline{0}]$.

Suppose: $\alpha \in T$ and $n \in \omega$ and $n = \mu p\,[\,(\delta|\alpha)^0(p) \neq 0]$.
Determine $q \in \omega$ such that: $\forall \zeta[\,\bar{\zeta}q = \bar{\alpha}q \rightarrow \overline{(\delta|\zeta)^0}(n+1) = \overline{(\delta|\alpha)^0}(n+1)]$
Determine $r > q$ such that: $P_n \prec P_r$
Either: $\overline{\alpha^{0,0}}r = \bar{\underline{0}}r$ or: $\overline{\alpha^{1,0}}r = \bar{\underline{0}}r$ and it does no harm to assume that: $\overline{\alpha^{0,0}}r = \bar{\underline{0}}r$.

Define $\eta \in {}^\omega\omega$ such that: $\mathrm{Fun}(\eta)$ and, for all $\zeta \in {}^\omega\omega$:

(i) $\overline{(\eta|\zeta)}q = \bar{\alpha}q$

(ii) $r = \mu p[(\eta|\zeta)^{0,0}(p) \neq 0]$ and: $(\eta|\zeta)^{0,Sr} = \zeta$

(iii) $r \geq \ell = \mu p[(\eta|\zeta)^{1,0}(p) \neq 0]$ and: $(\eta|\zeta)^{1,S\ell} = \beta_\ell$.

Then, for all $\zeta \in {}^\omega\omega$

$$\begin{aligned} P_r(\zeta) &\rightleftarrows Q((\eta|\zeta)^0) \wedge \overline{(\eta|\zeta)}q = \bar{\alpha}q \\ &\rightleftarrows D^2Q(\eta|\zeta) \wedge \overline{(\eta|\zeta)}q = \bar{\alpha}q \\ &\rightleftarrows P_n((\delta|(\eta|\zeta))^{Sn}). \end{aligned}$$

Therefore: $P_r \leq P_n$, which contradicts: $P_n \prec P_r$.

Therefore: $\forall \alpha \in \mathcal{T}[(\delta|\alpha)^0 = \underline{0}]$ and: $\forall \alpha \in \mathcal{T}[Q(\delta|\alpha)]$ and: $\forall \alpha \in \mathcal{T}[D^2Q(\alpha)]$ and: $\forall \alpha \in \mathcal{T}[\forall m[\alpha^{0,0}(m) = 0] \vee \forall m[\alpha^{1,0}(m) = 0]]$
We are almost in the same position as in 12.0, and may conclude, as we did there: $\forall \alpha[\neg\neg D^2A_1(\alpha) \to D^2A_1(\alpha)]$ which still is contradictory.

We make another minor assumption on the sequence $P_0, P_1, \ldots$ namely, that $\neg(Q \leq A_1)$, and prove: $\neg(C^2Q \leq Q)$.

The proof is similar to the proof of theorem 11.23 which stated that: $\neg(C^2S_2 \leq S_2)$.

Suppose: $C^2Q \leq Q$ and, using AC_{11}, determine $\delta \in {}^\omega\omega$ such that: $\mathrm{Fun}(\delta)$ and: $\forall \alpha[C^2Q(\alpha) \rightleftarrows Q(\delta|\alpha)]$.

We claim that: $\forall \alpha[(\alpha^{0,0} = \underline{0} \wedge Q(\alpha^1)) \to (\delta|\alpha)^0 = \underline{0}]$.

Let $\alpha \in {}^\omega\omega$ be such that: $\alpha^{0,0} = \underline{0}$ and $Q(\alpha^1)$, and $n \in \omega$, $n = \mu p[(\delta|\alpha)^0(p) \neq 0]$.
Determine $q \in \omega$ such that $\forall \zeta[\bar{\zeta}q = \bar{\alpha}q \to \overline{(\delta|\zeta)^0}(n+1) = \overline{(\delta|\alpha)^0}(n+1)]$
Determine $r > q$ such that $P_n \prec P_r$.
Define $\eta \in {}^\omega\omega$ such that: $\mathrm{Fun}(\eta)$ and, for all $\zeta \in {}^\omega\omega$:

(i) $\overline{(\eta|\zeta)}q = \bar{\alpha}q$

(ii) $r = \mu p[(\eta|\zeta)^{0,0}(p) \neq 0]$ and: $(\eta|\zeta)^{0,Sr} = \zeta$

(iii) $(\eta|\zeta)^1 = \alpha^1$.

Then, for all $\zeta \in {}^\omega\omega$:

$$\begin{aligned} P_r(\zeta) &\rightleftarrows C^2Q(\eta|\zeta) \wedge \overline{(\eta|\zeta)}q = \bar{\alpha}q \\ &\rightleftarrows P_n((\delta|(\eta|\zeta))^{Sn}). \end{aligned}$$

Therefore: $P_r \leq P_n$, which contradicts: $P_n \prec P_r$.

Therefore: $\forall \alpha[(\alpha^{0,0} = \underline{0} \wedge Q(\alpha^1)) \to (\delta|\alpha)^0 = \underline{0}]$ and...

we fall into an abyss, as follows:

Define $\eta \in {}^\omega\omega$ such that: $Fun(\eta)$ and, for all $\zeta \in {}^\omega\omega$:
$(\eta|\zeta)^0 = \underline{0}$ and: $(\eta|\zeta)^1 = \zeta$

Then, for all $\zeta \in {}^\omega\omega$:
$Q(\zeta) \rightleftarrows (\eta|\zeta)^{0,0} = \underline{0} \wedge Q((\eta|\zeta)^1)$

$\rightleftarrows \forall m [(\delta|(\eta|\zeta))^0(m) = 0]$

Therefore: $Q \leq A_1$, which, by our rather weak assumption, is not true.

Remark that the results of this paragraph apply to the set R, which we defined in 12.0, so that R, indeed, seems to do very well as a substitute for S_2: $\neg(D^2R \leq R)$ and: $\neg(C^2R \leq R)$.

12.2 Once more, let $P_0, P_1, \ldots$ be a sequence of subsets of ${}^\omega\omega$ which fulfils the condition: $\forall m \exists n [P_m \prec P_n]$.

Define $Q^* := \{\alpha \mid \exists n [n = \mu p [\alpha^0(p) \neq 0] \wedge P_n(\alpha^{Sn})]\}$.

Q^* challenges Q, as defined in 12.1, probably deserving as good a record.

Remark that: $\forall n [P_n \prec Q^*]$.

Let $n \in \omega$. Define $\delta \in {}^\omega\omega$ such that: $Fun(\delta)$ and: $\forall \alpha [n = \mu p [(\delta|\alpha)^0(p) \neq 0 \wedge (\delta|\alpha)^{Sn} = \alpha]$
Then: $\forall \alpha [P_n(\alpha) \rightleftarrows Q^*(\delta|\alpha)]$, i.e.: $P_n \leq Q^*$

We make an assumption on the sequence $P_0, P_1, \ldots$ namely that: $\forall a \exists \alpha [\alpha \in a \wedge \neg Q^*(\alpha)]$
(i.e.: $Neg(Q^*)$ is dense in ${}^\omega\omega$).
Observe that this holds, for instance, if $\forall n \forall a \exists \alpha [\alpha \in a \wedge \neg P_n(\alpha)]$.
We also assume that $\exists \alpha [P_0(\alpha)]$ and prove: $\neg(D^2Q^* \leq Q^*)$.

Suppose: $D^2Q^* \leq Q^*$ and, using AC_{11}, determine $\delta \in {}^\omega\omega$ such that: $Fun(\delta)$ and: $\forall \alpha [D^2Q^*(\alpha) \rightleftarrows Q^*(\delta|\alpha)]$.

Determine $\alpha \in {}^\omega\omega$ such that: $\alpha^{0,0} = \underline{0}$ and: $\alpha^{1,0}(0) \neq 0$ and: $P_0(\alpha^{1,1})$.
Remark: $Q^*(\alpha^1)$, therefore: $D^2Q^*(\alpha)$ and: $Q^*(\delta|\alpha)$.
Determine $n = \mu p [(\delta|\alpha)^0(p) \neq 0]$.
Determine $q \in \omega$ such that: $\forall \zeta [\bar{\zeta}q = \bar{\alpha}q \rightarrow \overline{(\delta|\zeta)^0}(n+1) = \overline{(\delta|\alpha)^0}(n+1)]$.
Determine $\beta \in {}^\omega\omega$ such that: $\bar{\beta}q = \overline{\alpha^1}q$ and: $\neg Q^*(\beta)$.
Determine $r > q$ such that: $P_n \prec P_r$.

Now define $\eta \in {}^\omega\omega$ such that: $Fun(\eta)$ and, for all $\zeta \in {}^\omega\omega$:
(i) $\overline{(\eta|\zeta)}q = \bar{\alpha}q$
(ii) $r = \mu p [(\eta|\zeta)^{0,0}(p) \neq 0]$ and: $(\eta|\zeta)^{0,Sr} = \zeta$
(iii) $(\eta|\zeta)^1 = \beta$.

Then, for all $\zeta \in {}^\omega\omega$:

$$\begin{aligned} P_r(\zeta) &\rightleftarrows Q^*((\eta|\zeta)^0) \wedge \overline{(\eta|\zeta)}q = \bar\alpha q \\ &\rightleftarrows D^2Q^*(\eta|\zeta) \wedge \overline{(\eta|\zeta)}q = \bar\alpha q \\ &\rightleftarrows Q^*(\delta|(\eta|\zeta)) \wedge n = \mu p[(\delta|(\eta|\zeta))^0(p) \neq 0] \\ &\rightleftarrows P_n((\delta|(\eta|\zeta))^{Sn}). \end{aligned}$$

Therefore: $P_r \leq P_n$ and this contradicts: $P_n \prec P_r$

Specializing this construction, we introduce, as a rival to the set R from 12.0, a subset R^* of ${}^\omega\omega$ by:

$$R^* := \{\alpha \mid \exists n[n = \mu p[\alpha^0(p) \neq 0] \wedge D^nA_1(\alpha^{Sn})]\}.$$

The general argument which we outlined a moment ago, applies to R^* and shows that: $\neg(D^2R^* \leq R^*)$.
On the other hand, it is true that: $C^2R^* \leq R^*$.

Let $\langle\ \rangle: {}^\omega\omega \times {}^\omega\omega \to {}^\omega\omega$ be a pairing function on ${}^\omega\omega$.
(We mentioned such a function just before definition 6.4).
It is an easy matter - and we leave it to the reader - to define for each $m \in \omega, n \in \omega$ a sequence $\zeta_{m,n} \in {}^\omega\omega$ such that:
$Fun(\zeta_{m,n})$ and: $\forall\alpha\forall\beta[(D^mA_1(\alpha) \wedge D^nA_1(\beta)) \rightleftarrows D^{m\cdot n}A_1(\zeta_{m,n}|\langle\alpha,\beta\rangle)]$
Now define $\delta \in {}^\omega\omega$ such that: $Fun(\delta)$ and, for all $\alpha \in {}^\omega\omega$:

(i) $\exists p[(\delta|\alpha)^0(p) \neq 0] \rightleftarrows (\exists p[\alpha^{0,0}(p) \neq 0] \wedge \exists p[\alpha^{1,0}(p) \neq 0])$

(ii) For all $m \in \omega, n \in \omega$:
If: $m = \mu p[\alpha^{0,0}(p) \neq 0] \wedge n = \mu p[\alpha^{1,0}(p) \neq 0]$
then: $m\cdot n = \mu p[(\delta|\alpha)^0(p) \neq 0] \wedge (\delta|\alpha)^{S(m\cdot n)} = \zeta_{m,n}|\langle\alpha^{0,Sm},\alpha^{1,Sn}\rangle$

One soon realizes that: $\forall\alpha[C^2R^*(\alpha) \rightleftarrows R^*(\delta|\alpha)]$.
Therefore: $C^2R^* \leq R^*$.

We compare R and R^* and establish that: $\neg(R \leq R^*)$.

Suppose: $R \leq R^*$ and, using AC_{11}, determine $\delta \in {}^\omega\omega$ such that:
$Fun(\delta)$ and: $\forall\alpha[R(\alpha) \rightleftarrows R^*(\delta|\alpha)]$.

Let $\alpha \in {}^\omega\omega$ and $\alpha^0 = \underline{0}$.
Remark: $R(\alpha)$, and: $R^*(\delta|\alpha)$ and: $\exists n[(\delta|\alpha)^0(n) \neq 0]$.
Determine $n =\mu p[(\delta|\alpha)^0(p) \neq 0]$.
Determine $q \in \omega$ such that: $\forall\zeta[\bar\zeta q = \bar\alpha q \to \overline{(\delta|\zeta)^0}(n+1) = \overline{(\delta|\alpha)^0}(n+1)]$
Determine $r > q$ such that $P_n \prec P_r$.
Define $\eta \in {}^\omega\omega$ such that: $Fun(\eta)$ and, for all $\zeta \in {}^\omega\omega$:

(i) $\overline{(\eta|\zeta)}q = \bar\alpha q$

(ii) $r = \mu p[(\eta|\zeta)^0(p) \neq 0] \wedge (\eta|\zeta)^{Sr} = \zeta$.

Then, for all $\zeta \in {}^\omega\omega$:

$$P_r(\zeta) \rightleftarrows R(\eta|\zeta) \wedge \overline{(\eta|\zeta)}q = \bar{\alpha}q$$
$$\rightleftarrows R^*(\delta|(\eta|\zeta)) \wedge n = \mu p[(\delta|(\eta|\zeta))^0(p) \neq 0]$$
$$\rightleftarrows P_n((\delta|(\eta|\zeta))^{Sn}).$$

Therefore: $P_r \preceq P_n$ and this contradicts: $P_n \prec P_r$
We have to admit: $\neg(R \preceq R^*)$.

It is seen at a glance that: $E_1 \preceq R^*$
Therefore: $\neg(R^* \preceq S_2)$ because, according to theorem 11.7: $\neg(E_1 \preceq S_2)$
Also: $\neg(R^* \preceq R)$, as, according to 12.0: $R \preceq S_2$.

After all, R seems nearer to S_2 than R^*.
But we destroyed any claims that R might put forward, to be a least upper bound to the sequence $A_1, D^2A_1, D^3A_1, \dots$

(We did not seriously consider the question of least upper bounds with respect to the reducibility relation $\preceq$. It does not seem easy to find a nice example. The reader may try his wits on finding a least upper bound for A_1 and E_1).

12.3 Let us return to the construction which we studied in 12.1

Let $P_0, P_1, \dots$ be a sequence of subsets of ${}^\omega\omega$ which fulfils the condition: $\forall m \exists n[P_m \prec P_n]$, and define $Q := \{\alpha | \forall n[n = \mu p[\alpha^0(p) \neq 0] \rightarrow P_n(\alpha^{Sn})]\}$.

We have seen, in 12.1, that: if $\forall n \exists \alpha[\neg P_n(\alpha)]$, then $\neg(D^2Q \preceq Q)$.
We would like to prove the stronger statement: $\forall n[\neg(D^{Sn}Q \preceq D^nQ)]$.

We will do so, in two different ways, but, each time, we have to extend our assumptions concerning the sequence $P_0, P_1, \dots$

We observe, that, if the sets $P_0, P_1, \dots$ are, all of them, strictly analytical, then the resulting set Q is also strictly analytical.
(Strictly analytical sets have been discussed in 10.7).

Suppose: $\forall n[P_n$ is strictly analytical]
Determine a sequence $\delta_0, \delta_1, \dots$ of elements of ${}^\omega\omega$ such that $\forall n[\mathrm{Fun}(\delta_n) \wedge P_n = \mathrm{Ra}(\delta_n)]$.
Define $\delta \in {}^\omega\omega$ such that: $\mathrm{Fun}(\delta)$ and, for all $\alpha \in {}^\omega\omega$:

(i) $(\delta|\alpha)^0 := \alpha^0$

(ii) For all $n \in \omega$:

$$(\delta|\alpha)^{Sn} := \alpha^{Sn} \quad \text{if } n \neq \mu p[\alpha^0(p) \neq 0]$$
$$:= \delta_n|\alpha^{Sn} \quad \text{if } n = \mu p[\alpha^0(p) \neq 0].$$

A moment's thought will convince you, that $Q = \mathrm{Ra}(\delta)$.

In addition, δ has the following two properties:

(i) $\delta|\underline{0} = \underline{0}$

(ii) $\forall q \forall \alpha [(\bar{\alpha}q = \bar{\underline{0}}q \wedge Q(\alpha)) \to \exists \beta [\bar{\beta}q = \bar{\underline{0}}q \wedge \alpha = \delta|\beta]]$

More or less imitating the proof of 11.22, we find:

12.3.0 <u>Theorem</u>: Let $P_0, P_1, \ldots$ be a sequence of strictly analytical subsets of ${}^\omega\omega$, such that: $\forall m \exists n [P_m \prec P_n]$ and: $\forall n \exists \alpha [\neg P_n(\alpha)]$.
Let $Q := \{\alpha \mid \forall n [n = \mu p[\alpha^0(p) \neq 0] \to P_n(\alpha^{Sn})]\}$.
Then: Q is strictly analytical and: $\forall n > 0 [D^n Q \prec D^{n+1} Q]$.

<u>Proof</u>: It is easy to see that: $\exists \alpha [\neg Q(\alpha)]$, and, therefore, that $\forall n > 0 [D^n Q \leq D^{n+1} Q]$.
We also know, from the discussion in 12.1, that: $\neg(D^2 Q \leq Q)$.
We build a sequence $\delta_0, \delta_1, \ldots$ of elements of ${}^\omega\omega$ such that:
$\forall n [\mathrm{Fun}(\delta_n) \wedge P_n = \mathrm{Ra}(\delta_n)]$, and, from it, an element $\delta \in {}^\omega\omega$ such that:
$\mathrm{Fun}(\delta) \wedge Q = \mathrm{Ra}(\delta)$, like we did it just before embarking upon this proof.

Now, let $n \in \omega$, $n > 0$ and suppose: $D^{n+1} Q \leq D^n Q$.
Using AC_{11}, we determine $\eta \in {}^\omega\omega$ such that: $\mathrm{Fun}(\eta)$ and $\forall \alpha [D^{n+1} Q(\alpha) \rightleftarrows D^n Q(\eta|\alpha)]$.
We also define, for each $m \in \omega$, an element $\varepsilon_m \in {}^\omega\omega$ such that:
$\mathrm{Fun}(\varepsilon_m)$ and, for all $\alpha \in {}^\omega\omega$:

(i) $(\varepsilon_m|\alpha)(\langle\,\rangle) = \alpha(\langle\,\rangle)$

(ii) $(\varepsilon_m|\alpha)^m := \delta|\alpha^m$

(iii) For all $n \in \omega$, $n \neq m$: $(\varepsilon_m|\alpha)^n = \alpha^n$.

We observe: $\forall m < n+1 \forall \alpha [Q((\varepsilon_m|\alpha)^m)]$, therefore:
$\forall m < n+1 \forall \alpha [D^{n+1} Q(\varepsilon_m|\alpha)]$ and: $\forall m < n+1 \forall \alpha [D^n Q(\eta|(\varepsilon_m|\alpha))]$.
Using CP, we determine natural numbers $q_0, p_0, \ldots q_n, p_n$ such that:
$\forall m < n+1 \forall \alpha [\bar{\alpha}q_m = \bar{\underline{0}}q_m \to Q((\eta|(\varepsilon_m|\alpha))^{p_m})]$.
Ruminating the last remark which preceded this theorem, we conclude:
$\forall m < n+1 \forall \alpha [(\bar{\alpha}q_m = \bar{\underline{0}}q_m \wedge Q(\alpha^m)) \to Q((\eta|\alpha)^{p_m})]$

For, let $m \in \omega$, $m < n+1$ and $\alpha \in {}^\omega\omega$ and $\bar{\alpha}q_m = \bar{\underline{0}}q_m$.
Determine $\beta \in {}^\omega\omega$ such that: $\varepsilon_m|\beta = \alpha$ and: $\bar{\beta}q_m = \bar{\underline{0}}q_m$.
We then see: $Q((\eta|(\varepsilon_m|\beta))^{p_m})$, i.e.: $Q((\eta|\alpha)^{p_m})$.

As $\forall m < n+1 [p_m < n]$, we may assume, without loss of generality, that $p_0 = p_1$. Let $q := \max(q_0, q_1)$.

The reader will sense how this is to end: we are able, now, by skilful grafting, to reduce D^2Q to Q.

First, we define $\gamma \in {}^\omega\omega$ such that $\forall n[q < \gamma(n) < \gamma(n+1) \wedge P_n \leq P_{\gamma(n)}]$

Then, we define a sequence $f_0, f_1, f_2, \ldots$ of elements of ${}^\omega\omega$ such that $\forall n[\mathrm{Fun}(f_n) \wedge \forall \alpha[P_n(\alpha) \rightleftarrows P_{\gamma(n)}(f_n|\alpha)]$

Finally, we define $\zeta \in {}^\omega\omega$ such that: $\mathrm{Fun}(\zeta)$ and, for all $\alpha \in {}^\omega\omega$:

(i) $\overline{(\zeta|\alpha)}q = \bar{\underline{0}}q$

(ii) $\forall n[(\zeta|\alpha)^{0,0}(n) = 0 \rightleftarrows \exists m[\alpha^{0,0}(m) = 0 \wedge n = \gamma(m)]$ and: $\forall n[(\zeta|\alpha)^{0,\gamma(n)} = f_n|\alpha^{0,n}]$

$\forall n[(\zeta|\alpha)^{1,0}(n) = 0 \rightleftarrows \exists m[\alpha^{1,0}(m) = 0 \wedge n = \gamma(m)]$ and: $\forall n[(\zeta|\alpha)^{1,\gamma(n)} = f_n|\alpha^{1,n}]$

(iii) $\forall n > 1[\neg Q((\zeta|\alpha)^n)]$.

Then, for all $\alpha \in {}^\omega\omega$:

$$D^2Q(\alpha) \rightleftarrows D^2Q(\zeta|\alpha) \wedge \overline{(\zeta|\alpha)}q = \bar{\underline{0}}q$$
$$\rightleftarrows Q((\eta|(\zeta|\alpha))^{Pm}).$$

Therefore: $D^2Q \leq Q$, and, as we know, this is not true.

Therefore: $\forall n[\neg(D^{n+1}Q \leq D^nQ)$ and: $\forall n[D^nQ \prec D^{n+1}Q]$.

$\boxtimes$

Our heart is flooded with joy at this result.
To our regret, the underlying method did not help us to prove the same thing about Q^* (as defined in 12.3), or to set up the conjunctive tower on the base Q.
The reader will remember that, in connection with S_2, we treated disjunction and conjunction rather differently. (Cf. theorems 11.21 and 11.26).
Rethinking theorem 12.3.0, we come to prove it anew, on slightly other conditions, thus paving the way for a similar handling of conjunction.

12.4 Let us introduce, for all subsets $P \subseteq {}^\omega\omega$, $Q \subseteq {}^\omega\omega$, a subset $D(P,Q)$ of ${}^\omega\omega$ by:

For all $\alpha \in {}^\omega\omega$: $D(P,Q)(\alpha) := P(\alpha^0) \vee Q(\alpha^1)$.

Let $P_0, P_1, P_2, \ldots$ be a sequence of subsets of ${}^\omega\omega$.
We call this sequence <u>disjunctively closed</u> if $\forall m \forall n \exists k[D(P_m, P_n) \leq P_k]$.

12.4.0 <u>Theorem</u>: Let $P_0, P_1, P_2, \ldots$ be a disjunctively closed sequence of subsets of ${}^\omega\omega$ such that: $\forall m \exists n[P_m \prec P_n]$ and: $\forall n \exists \alpha[\neg P_n(\alpha)]$.

Let $Q := \{\alpha \mid \forall n[n = \mu p[\alpha^0(p) \neq 0] \rightarrow P_n(\alpha^{Sn})]\}$.

Then: $\forall n > 0[D^nQ \prec D^{n+1}Q]$.

<u>Proof</u>: It is easy to see that: $\exists \alpha[\neg Q(\alpha)]$ and, therefore, that $\forall n > 0[D^nQ \leq D^{n+1}Q]$.

This, of course, is a cheap observation.

In view of the large work at hand, we send for our old friend
$\tau := \{\alpha \mid \forall m \forall n [(\alpha(m) \neq 0 \wedge \alpha(n) \neq 0) \to m = n]\}$. (cf. 4.2)

Observe that: $\neg \forall \alpha \in \tau [E_2(\alpha)]$.

> Suppose: $\forall \alpha \in \tau [E_2(\alpha)]$.
> As $\underline{0} \in \tau$ and τ is a subspread of ${}^\omega\omega$ (cf. 4.2), we apply GCP and calculate $n \in \omega$, $q \in \omega$ such that:
> $\forall \alpha \in \tau [\bar{\alpha}q = \bar{\underline{0}}q \to \alpha^n = \underline{0}]$.
> This is not true, as we may define $\alpha^* \in \tau$ such that $\overline{\alpha^*}q = \bar{\underline{0}}q$ and $(\alpha^*)^n(q) \neq 0$.
> Therefore: $\neg \forall \alpha \in \tau [E_2(\alpha)]$.

We have, at the same time, that: $\forall n > 1 \neg \forall \alpha \in \tau [D^n A_1(\alpha)]$ and that:
$\forall n \forall \alpha \in \tau \forall k [\# \{m \mid m < n+1 \mid \overline{\alpha^m}k = \bar{\underline{0}}k\} \geq n]$.

(The symbol $\#$ has usual function of denoting the cardinal number of a finite set).

Determine a sequence $\beta_0, \beta_1, \ldots$ of members of ${}^\omega\omega$ such that: $\forall n [\neg P_n(\beta_n)]$
Define a subset T of ${}^\omega\omega$ by: $T := \{\alpha \mid \forall n \forall m [\alpha^{n,Sm} = \beta_m]\}$.
Remark that: $\forall \alpha \in T \forall n > 0 [D^n Q(\alpha) \rightleftarrows \exists m < n [\alpha^{m,0} = \underline{0}]]$.

Suppose: $n \in \omega$, $n > 0$ and: $D^{n+1}Q \preceq D^n Q$.
Using AC_{11}, determine $\delta \in {}^\omega\omega$ such that: Fun(δ) and: $\forall \alpha [D^{n+1}Q(\alpha) \rightleftarrows D^n Q(\delta|\alpha)]$.
Let us define, for each $\alpha \in {}^\omega\omega$ and $k \in \omega$, natural numbers $c_\alpha(k)$ and $d_\alpha(k)$ by:

$$c_\alpha(k) := \# \{m \mid m < n+1 \mid \overline{\alpha^{m,0}}(k) = \bar{\underline{0}}k\}$$

(we pronounce: the critical number of α at stage k)

$$d_\alpha(k) := \# \{m \mid m < n \mid \overline{(\delta|\alpha)^{m,0}}(k) = \bar{\underline{0}}k\}.$$

(The number $c_\alpha(k)$ represents, so to say, the number of alternatives that α has left open, up till stage k).

We claim that: $\forall p < n \forall \alpha \in T [\forall k [c_\alpha(k) > p] \to \forall k [d_\alpha(k) > p]]$

> We prove this by induction, and start with the case $p = 0$.
> Suppose, therefore, that $\alpha \in T$ and: $\forall k [c_\alpha(k) > 0]$ and: $\exists k [d_\alpha(k) = 0]$.
> Calculate, for each $m < n$: $\ell_m := \mu k [(\delta|\alpha)^{m,0}(k) \neq 0]$.

Calculate $q \in \omega$ such that:
$\forall \gamma [\ \bar{\gamma} q = \bar{\alpha} q \ \rightarrow \ \forall m < n [\ \overline{(\delta|\gamma)^{m,0}}(\ell_m + 1) = \overline{(\delta|\alpha)^{m,0}}(\ell_m+1)]]$.
Finally, remember that the sequence $P_0, P_1, \ldots$ is disjunctively closed and calculate $N \in \omega$ such that:

$N > q$ and: $D(P_{\ell_0}, P_{\ell_1}, \ldots, P_{\ell_{n-1}}) \prec P_N$
(We write: $D(P_{\ell_0}, P_{\ell_1}, \ldots P_{\ell_{n-1}})$ for: $D(\ldots(D(P_{\ell_0}, P_{\ell_1}) \cdots), P_{\ell_{n-1}}))$
We may assume, without loss of generality, that $\overline{\alpha^{0,0}} N = \bar{\underline{0}} N$
We define $\eta \in {}^\omega\omega$ such that: $\mathrm{Fun}(\eta)$, and, for all $\gamma \in {}^\omega\omega$:

(I) $\overline{(\eta|\gamma)}\, q = \bar{\alpha} q$

(II) $N = \mu k [\ (\eta|\gamma)^{0,0}(k) \neq 0]$ and: $(\eta|\gamma)^{0,SN} = \gamma$

(III) For all $j \in \omega$, $j > 0$:
$N > \ell := \mu k [\ (\eta|\gamma)^{j,0}(k) \neq 0]$ and: $(\eta|\gamma)^{j,S\ell} = \beta_\ell$

(This last „β_ℓ" is the fixed sequence which fulfils: $\neg P_\ell(\beta_\ell)$.
The third clause is to ensure that $\forall \gamma \forall j > 0 [\neg Q((\eta|\gamma)^j)])$

Then, for all $\gamma \in {}^\omega\omega$:

$$P_N(\gamma) \rightleftarrows D^{n+1} Q(\eta|\gamma) \wedge \overline{(\eta|\gamma)}\, q = \bar{\alpha} q$$
$$\rightleftarrows P_{\ell_0}((\delta|(\eta|\gamma))^{0,S\ell_0} \vee \ldots \vee P_{\ell_{n-1}}((\delta|(\eta|\gamma))^{n-1,S\ell_{n-1}})$$

Therefore: $P_N \leq D(P_{\ell_0}, \ldots, P_{\ell_{n-1}})$ and this conflicts with the choice of N.
This contradiction shows that: $\forall \alpha \in T [\ \forall k [c_\alpha(k) > 0] \rightarrow \forall k [d_\alpha(k) > 0]]$.

Suppose, now, that $p \in \omega$, $p < n-1$ and:
$\forall \alpha \in T [\ \forall k [\ c_\alpha(k) > p] \ \rightarrow \ \forall k [\ d_\alpha(k) > p]$
We wish to prove that: $\forall \alpha \in T [\ \forall k [c_\alpha(k) > p+1] \rightarrow \forall k [d_\alpha(k) > p+1]]$.
Assume, therefore: $\alpha \in T$ and: $\forall k [\ c_\alpha(k) > p+1]$ and:
$\exists k [\ d_\alpha(k) = p+1]$.
Calculate $k_0 \in \omega$ such that $d_\alpha(k_0) = p+1$.
Calculate $q \in \omega$ such that $\forall \gamma [\bar{\gamma} q = \bar{\alpha} q \rightarrow \forall m < n [\overline{(\delta|\gamma)^{m,0}}\, k_0 = \overline{(\delta|\alpha)^{m,0}}\, k_0]$.
We may assume, without loss of generality

$\forall m < p+1 [\ \overline{\alpha^{m,0}}\, q = \bar{\underline{0}} q]$ and: $\overline{(\delta|\alpha)^{0,0}}\, k_0 = \bar{\underline{0}} k_0$.
We define $\zeta \in {}^\omega\omega$ such that: $\mathrm{Fun}(\zeta)$ and, for all $\gamma \in {}^\omega\omega$:

(I) $\overline{(\zeta|\gamma)}\, q = \bar{\alpha} q$

(II) For all $m\in\omega$, $m<p+1$: $(\zeta|\gamma)^{m,0} = \underline{\bar{0}}q * \gamma^m$

(III) For all $m\in\omega$, $n\in\omega$: $(\zeta|\gamma)^{m,Sn} = \beta_n$

(IV) For all $m\in\omega$, $m\geq p+1$: $\neg Q((\eta|\gamma)^m)$.

Remark that: $\forall\gamma[\zeta|\gamma\in\mathcal{T}]$.
Also observe that: $\forall\gamma\in\tau\,\forall k[c_{\zeta|\gamma}(k)>p]$.
Therefore, taking into account what we proved at the previous stage: $\forall\gamma\in\tau\,\forall k[d_{\zeta|\gamma}(k)>\dot{p}]$.
However, as: $\forall\gamma\in\tau[\overline{(\zeta|\gamma)}q=\bar{\alpha}q]$, also:
$\forall\gamma\in\tau[d_{\zeta|\gamma}(k_0)=d_\alpha(k_0)=p+1]$, and:
$\forall\gamma\in\tau[\overline{(\delta|(\zeta|\gamma))^{0,0}}k_0=\overline{(\delta|\alpha)^{0,0}}k_0=\underline{\bar{0}}k_0]$.
Therefore: $\forall\gamma\in\tau\,\forall k[\overline{(\delta|(\zeta|\gamma))^{0,0}}k=\underline{\bar{0}}k]$, and: $\forall\gamma\in\tau[Q((\delta|(\zeta|\gamma))^0)]$,
and: $\forall\gamma\in\tau[D^nQ(\delta|(\zeta|\gamma))]$, and: $\forall\gamma\in\tau[D^{n+1}Q(\zeta|\gamma)]$
and: $\forall\gamma\in\tau[D^{n+1}A_1(\gamma)]$.
And this is not true, as we have seen at the beginning of this proof.
This contradiction shows: $\forall\alpha\in\mathcal{T}[\forall k[c_\alpha(k)>p+1]\rightarrow\forall k[d_\alpha(k)>p+1]]$.

This establishes our claim: $\forall p<n\,\forall\alpha\in\mathcal{T}[\forall k[c_\alpha(k)>p]\rightarrow\forall k[d_\alpha(k)>p]]$.

Thus, we know that: $\forall\alpha\in\mathcal{T}[\forall k[c_\alpha(k)\geq n]\rightarrow\forall k[d_\alpha(k)\geq n]]$
Victory cannot escape us any more.
We define $\zeta\in{}^\omega\omega$ such that: $Fun(\zeta)$ and: for all $\gamma\in{}^\omega\omega$:

(I) for all $m\in\omega$: $(\zeta|\gamma)^{m,0}=\gamma^m$

(II) for all $m\in\omega$, $n\in\omega$: $(\zeta|\gamma)^{m,Sn}=\beta_n$

Remark that: $\forall\gamma[\zeta|\gamma\in\mathcal{T}]$
Also observe that: $\forall\gamma\in\tau[\forall k[c_{\zeta|\gamma}(k)\geq n]]$
And finish as follows, holding up your arms in triumph:
$\forall\gamma\in\tau\,\forall k[d_{\zeta|\gamma}(k)\geq n]$, therefore: $\forall\gamma\in\tau\,\forall m<n[(\delta|(\zeta|\gamma))^{m,0}=\underline{0}]$
and: $\forall\gamma\in\tau\,\forall m<n[Q((\delta|(\zeta|\gamma))^m]$, and: $\forall\gamma\in\tau\,D^nQ(\delta|(\zeta|\gamma))$,
and so: $\forall\gamma\in\tau[D^{n+1}Q(\zeta|\gamma)]$, and: $\forall\gamma\in\tau[D^{n+1}A_1(\gamma)]$,
a flat contradiction, as we saw before.
Looking for a culprit, we conclude: $\forall n>0[\neg(D^{n+1}Q\leq D^nQ]$
and: $\forall n>0[D^nQ<D^{n+1}Q]$. ⊠

12.5 Conjunction, anxious to fly at least as high as disjunction, now attracts our attention.

We introduced, just before theorem 11.25, for all subsets $P \subseteq {}^\omega\omega$, $Q \subseteq {}^\omega\omega$, a subset $C(P,Q)$ of ${}^\omega\omega$ by:

For all $\alpha \in {}^\omega\omega$: $C(P,Q)(\alpha) := P(\alpha^0) \wedge Q(\alpha^1)$.

Let $P_0, P_1, \ldots$ be a sequence of subsets of ${}^\omega\omega$.
We call this sequence <u>conjunctively</u> <u>closed</u> if: $\forall m \forall n \exists k [C(P_m, P_n) \preceq P_k]$.

12.5.0 <u>Theorem</u>: Let $P_0, P_1, \ldots$ be a conjunctively closed sequence of subsets of ${}^\omega\omega$ such that: $\forall m \exists n [P_m \prec P_n]$ and: $\exists n [A_1 \preceq P_n]$.
Let $Q := \{\alpha \mid \forall n [n = \mu p[\alpha^0(p) \neq 0] \rightarrow P_n(\alpha^{Sn})]\}$.
Then: $\forall n > 0 [C^n Q \prec C^{n+1} Q]$.

<u>Proof</u>: It is easy to see that $\exists \alpha [Q(\alpha)]$ and, therefore, that $\forall n > 0 [C^n Q \preceq C^{n+1} Q]$.
This remark serves to loose our tongue.

Suppose: $n \in \omega$, $n > 0$ and: $C^{n+1} Q \preceq C^n Q$.
Using AC_{11}, determine $\delta \in {}^\omega\omega$ such that: $Fun(\delta)$ and: $\forall \alpha [C^{n+1}Q(\alpha) \rightleftarrows C^n Q(\delta|\alpha)]$.
As in the proof of theorem 12.4.0, we define, for each $\alpha \in {}^\omega\omega$ and $k \in \omega$, so-called critical numbers $c_\alpha(k)$ and $d_\alpha(k)$ by:

$$c_\alpha(k) := \# \{m \mid m < n+1 \mid \overline{\alpha^{m,0}} k = \overline{0} k\}$$

$$d_\alpha(k) := \# \{m \mid m < n \mid \overline{(\delta|\alpha)^{m,0}} k = \overline{0} k\}.$$

We claim that: $\forall p < n \forall \alpha [(C^{n+1} Q(\alpha) \quad \forall k [c_\alpha(k) > p]) \rightarrow \forall k [d_\alpha(k) > p]]$.

We prove this by induction and start with the case: $p = 0$.
Suppose, therefore: $\alpha \in {}^\omega\omega$ and $C^{n+1}Q(\alpha)$ and $\forall k [c_\alpha(k) > 0]$ and: $\exists k [d_\alpha(k) = 0]$.
Calculate, for each $m < n$: $\ell_m := \mu k [(\delta|\alpha)^{m,0}(k) \neq 0]$.
Calculate $q \in \omega$ such that:
$\forall \gamma [\overline{\gamma} q = \overline{\alpha} q \rightarrow \forall m < n [\overline{(\delta|\gamma)^{m,0}} (\ell_m + 1) = \overline{(\delta|\alpha)^{m,0}} (\ell_m + 1)]$
Remember that the sequence $P_0, P_1, \ldots$ is conjunctively closed and calculate $N \in \omega$ such that:
$N > q$ and: $C(P_{\ell_0}, P_{\ell_1}, \ldots P_{\ell_{n-1}}) \prec P_N$
(We write: $C(P_{\ell_0}, P_{\ell_1}, \ldots, P_{\ell_{n-1}})$ for: $C(\ldots(C(P_{\ell_0}, P_{\ell_1})\ldots), P_{\ell_{n-1}}))$

We may assume, without loss of generality, that $\overline{\alpha^{0,0}}N = \overline{\underline{0}}N$.
We define $\eta \in {}^{\omega}\omega$ such that: $\mathrm{Fun}(\eta)$ and, for all $\gamma \in {}^{\omega}\omega$:

(i) $\overline{(\eta|\gamma)}q = \bar{\alpha}q$

(ii) $N = \mu k[(\eta|\gamma)^{0,0}(k) = 0]$ and $(\eta|\gamma)^{0,SN} = \gamma$

(iii) For all $j \in \omega$, $j > 0$: $(\eta|\gamma)^{j} = \alpha^{j}$.

Then, for all $\gamma \in {}^{\omega}\omega$:

$$P_N(\gamma) \rightleftarrows C^{n+1}Q(\eta|\gamma) \wedge \overline{(\eta|\gamma)}q = \bar{\alpha}q$$
$$\rightleftarrows P_{\ell_0}((\delta|(\eta|\gamma))^{0,S\ell_0}) \wedge \ldots \wedge P_{\ell_{n-1}}((\delta|(\eta|\gamma))^{n-1,S\ell_{n-1}})$$

Therefore: $P_N \preceq C(P_{\ell_0}, \ldots, P_{\ell_{n-1}})$ and this conflicts with the choice of N.

This contradiction shows that:

$$\forall\alpha[(C^{n+1}Q(\alpha) \wedge \forall k[c_\alpha(k) > 0]) \rightarrow \forall k[d_\alpha(k) > 0]].$$

Suppose, now, that $p \in \omega$, $p < n-1$ and:

$$\forall\alpha[(C^{n+1}Q(\alpha) \wedge \forall k[c_\alpha(k) > p]) \rightarrow \forall k[d_\alpha(k) > p]]$$

We wish to prove that: $\forall\alpha[(C^{n+1}Q(\alpha) \wedge \forall k[c_\alpha(k) > p+1]) \rightarrow \forall k[d_\alpha(k) > p+1]]$

Assume therefore: $\alpha \in {}^{\omega}\omega$ and $C^{n+1}Q(\alpha)$ and $\forall k[c_\alpha(k) > p+1]$ and $\exists k[d_\alpha(k) = p+1]$.

Calculate $k_0 \in \omega$ such that $d_\alpha(k_0) = p+1$.

We may assume, without loss of generality, that:
$\forall m < p+1[\overline{(\delta|\alpha)^{m,0}}k_0 = \overline{\underline{0}}k_0]$ and: $\forall m[p+1 \leq m < n \rightarrow \overline{(\delta|\alpha)^{m,0}}k_0 \neq \overline{\underline{0}}k_0]$.
Calculate, for each $m \in \omega$ such that $p+1 \leq m < n$: $\ell_m := \mu k[(\delta|\alpha)^{m,0}(k) \neq 0]$.
Calculate $q \in \omega$ such that: $\forall\gamma[\bar{\gamma}q = \bar{\alpha}q \rightarrow \forall m < n[\overline{(\delta|\gamma)^{m,0}}k_0 = \overline{(\delta|\alpha)^{m,0}}k_0]]$.
Remember, that the sequence $P_0, P_1, \ldots$ is conjunctively closed and that $\exists n[A_1 \preceq P_n]$ and calculate $N \in \omega$ such that $N > q$ and: $C(A_1, P_{\ell_{p+1}}, P_{\ell_{p+2}}, \ldots, P_{\ell_{n-1}}) \prec P_N$.
We again need not fear to endanger the generality of the argument when assuming: $\forall m \leq p+1[\overline{\alpha^{m,0}}N = \overline{\underline{0}}N]$.

We define $\zeta \in {}^{\omega}\omega$ such that: $\mathrm{Fun}(\zeta)$ and, for all $\gamma \in {}^{\omega}\omega$:

(i) $\overline{(\zeta|\gamma)}q = \bar{\alpha}q$

(ii) $N = \mu k[(\zeta|\gamma)^{0,0}(k) \neq 0]$ and $(\zeta|\gamma)^{0,SN} = \gamma$

(iii) For all $j \in \omega$ such that: $0 < j \leq p+1$: $(\zeta|\gamma)^{j,0} = \underline{0}$

(iv) For all $j \in \omega$ such that $p+1 < j < n+1$: $(\zeta|\gamma)^j = \alpha^j$.

The most pleasing property of this function ζ is, that: $\forall\gamma\,\forall k\,[\,c_{\zeta|\gamma}(k) > p\,]$, which, in view of what we proved before the break, has the further consequence that: $\forall\gamma\,\forall k\,[\,d_{\zeta|\gamma}(k) > p\,]$.

Therefore, for each $\gamma \in {}^\omega\omega$:

$$\begin{aligned} P_N(\gamma) &\rightleftarrows C^{n+1}Q(\zeta|\gamma) \wedge \overline{(\zeta|\gamma)}q = \overline{\alpha}q \wedge \forall k\,[c_{\zeta|\gamma}(k) > p] \\ &\rightleftarrows C^nQ(\delta|(\zeta|\gamma)) \wedge \forall k > k_0\,[d_{\zeta|\gamma}(k) = d_{\zeta|\gamma}(k_0)] \\ &\rightleftarrows \big(\forall m < p+1\,[(\delta|(\zeta|\gamma))^{m,0} = \underline{0}] \wedge \\ &\qquad \wedge \forall m\,[\,p+1 \le m < n \to P_{\ell_m}((\delta|(\zeta|\gamma))^{m,S\ell_m})]\big) \end{aligned}$$

It is beyond doubt, now, that $P_N \preceq C(A_1, P_{\ell_{p+1}}, \ldots, P_{\ell_{n-1}})$ and this conflicts with the choice of N.

This contradiction shows that:

$$\forall\alpha\,[\,(C^{n+1}Q(\alpha) \wedge \forall k\,[c_\alpha(k) > p+1]) \to \forall k\,[d_\alpha(k) > p+1]\,].$$

Our claim obviously has been saved from all insinuations and: $\forall p < n\,\forall\alpha\,[\,(C^{n+1}Q(\alpha) \wedge \forall k\,[c_\alpha(k) > p]) \to \forall k\,[d_\alpha(k) > p]\,]$.

Thus, we know that: $\forall\alpha\,[\,(C^{n+1}Q(\alpha) \wedge \forall k\,[c_\alpha(k) \geq n]) \to \forall k\,[\,d_\alpha(k) \geq n]\,]$.
And this knowledge clears the way for a swift and joyful conclusion.
Remark that: $\forall\alpha\,[\,(\forall m < n\,[\alpha^{m,0} = \underline{0}\,] \wedge Q(\alpha^n)) \to \forall m < n\,[(\delta|\alpha)^{m,0} = \underline{0}]$
We define $\zeta \in {}^\omega\omega$ such that: $\mathrm{Fun}(\zeta)$ and, for all $\gamma \in {}^\omega\omega$:

$$(\zeta|\gamma)^n := \gamma \text{ and, for all } m \in \omega \text{ such that } m \neq n\text{: } (\zeta|\gamma)^m = \underline{0}$$

Then, for all $\gamma \in {}^\omega\omega$:

$$\begin{aligned} Q(\gamma) &\rightleftarrows C^{n+1}Q(\zeta|\gamma) \wedge \forall k\,[\,c_{\zeta|\gamma}(k) \geq n] \\ &\rightleftarrows \forall m < n\,\forall k\,[\,(\delta|(\zeta|\gamma))^{m,0}(k) = 0] \end{aligned}$$

Therefore: $Q \preceq A_1$, and this is not true, as: $\exists n\,[\,A_1 \preceq P_n]$ and: $\forall n\,[\,P_n \prec Q]$

Tired as we may be, we write down, out of love of truth:
$\forall n > 0\,[\,\neg(C^{n+1}Q \preceq C^nQ)]$ and: $\forall n > 0\,[\,C^nQ \prec C^{n+1}Q]$.

⊠

12.6 We apologize, but we long for the disjunctive ascension of the set Q^*, whose acquaintance we made in 12.2, and are going to sing our magic song a third time.

12.6.0 <u>Theorem</u>: Let $P_0, P_1, \dots$ be a disjunctively closed sequence of subsets of ${}^\omega\omega$ such that: $\forall m \exists n [P_m < P_n]$.
Let $Q^* := \{\alpha \mid \exists n [n = \mu p[\alpha^0(p) \neq 0] \wedge P_n(\alpha^{Sn})]\}$ and assume:
$\forall a \exists \alpha [\alpha \in a \wedge \neg Q^*(\alpha)]$.
Then: $\forall n > 0 [D^n Q^* < D^{n+1} Q^*]$.

<u>Proof</u>: It is easy to see that: $\exists \alpha [\neg Q^*(\alpha)]$ and, therefore, that $\forall n > 0 [D^n Q^* \leq D^{n+1} Q^*]$
That is not where the shoe pinches.

Suppose: $n \in \omega$, $n > 0$ and: $D^{n+1} Q^* \leq D^n Q^*$
Using AC_{11}, determine $\delta \in {}^\omega\omega$ such that: $Fun(\delta)$ and: $\forall \alpha [D^{n+1} Q^*(\alpha) \rightleftarrows D^n Q^*(\delta|\alpha)]$
We define, for each $\alpha \in {}^\omega\omega$ and $k \in \omega$, critical numbers $c_\alpha(k)$ and $d_\alpha(k)$ by:

$$c_\alpha(k) := \{m \mid m < n+1 \mid \overline{\alpha^{m,0}} k = \overline{\underline{0}} k\}$$

$$d_\alpha(k) := \{m \mid m < n \mid \overline{(\delta|\alpha)^{m,0}} k = \overline{\underline{0}} k\}$$

We claim that: $\forall p < n \, \forall \alpha [\, \forall k [c_\alpha(k) > p] \rightarrow \forall k [d_\alpha(k) > p]]$.

We prove this by induction and start with the case: $p = 0$.
Suppose, therefore: $\alpha \in {}^\omega\omega$ and: $\forall k [c_\alpha(k) > 0]$ and $\exists k [d_\alpha(k) = 0]$.
Calculate, for each $m < n$: $\ell_m := \mu k [(\delta|\alpha)^{m,0}(k) \neq 0]$.
Calculate $q \in \omega$ such that:
$\forall \gamma [\bar{\gamma} q = \bar{\alpha} q \rightarrow \forall m < n [(\delta|\gamma)^{m,0}(\overline{\ell_m + 1}) = (\delta|\alpha)^{m,0}(\overline{\ell_m + 1})]$
Remember, that the sequence $P_0, P_1, \dots$ is disjunctively closed and calculate $N \in \omega$ such that: $N > q$ and $D(P_{\ell_0}, \dots, P_{\ell_{n-1}}) < P_N$
We may assume, without loss of generality, that $\overline{\alpha^{0,0}} N = \overline{\underline{0}} N$
We define $\eta \in {}^\omega\omega$ such that: $Fun(\eta)$, and for all $\gamma \in {}^\omega\omega$:

(I) $\overline{(\eta|\gamma)} q = \bar{\alpha} q$

(II) $N = \mu k [(\eta|\gamma)^{0,0}(k) = 0]$ and $(\eta|\gamma)^{0,SN} = \gamma$

(III) For all $j \in \omega$, $0 < j < n+1$: $\neg Q^*((\eta|\gamma)^j)$.

Then, for all $\gamma \in {}^\omega\omega$:

$$\begin{aligned} P_N(\gamma) &\rightleftarrows D^{n+1} Q^*(\eta|\gamma) \wedge \overline{(\eta|\gamma)} q = \bar{\alpha} q \\ &\rightleftarrows P_{\ell_0}((\delta|(\eta|\gamma))^{0,S\ell_0}) \vee \dots \vee P_{\ell_{n-1}}((\delta|(\eta|\gamma))^{n-1,S\ell_{n-1}}) \end{aligned}$$

Therefore: $P_N \preceq D(P_{\ell_0}, \dots, P_{\ell_{n-1}})$, and this conflicts with the choice of N.

We may trust, now, that $\forall\alpha[\forall k[c_\alpha(k)>0] \to \forall k[d_\alpha(k)>0]]$.

Suppose, now, that $p\in\omega$, $p<n-1$ and: $\forall\alpha[\forall k[c_\alpha(k)>p] \to \forall k[d_\alpha(k)>p]]$.
We wish to prove that: $\forall\alpha[\forall k[c_\alpha(k)>p+1] \to \forall k[d_\alpha(k)>p+1]]$.
Assume, therefore: $\alpha\in{}^\omega\omega$ and $\forall k[c_\alpha(k)>p+1]$ and $\exists k[d_\alpha(k)=p+1]$.
Calculate $k_0\in\omega$ such that $d_\alpha(k_0)=p+1$.
We assume, and do not damage, thereby, the generality of the argument, that: $\forall m<n[\overline{(\delta|\alpha)^{m,0}}k_0 = \bar{\underline{0}}k_0 \rightleftarrows m<p+1]$.
Calculate, for each $m\in\omega$ such that $p+1\le m<n$: $\ell_m := \mu k[(\delta|\alpha)^{m,0}(k)\neq 0]$.
Calculate $q\in\omega$ such that: $\forall\gamma[\bar{\gamma}q=\bar{\alpha}q \to \forall m<n[\overline{(\delta|\gamma)^{m,0}}k_0 = \overline{(\delta|\alpha)^{m,0}}k_0]]$.
Remember, that the sequence $P_0, P_1, \dots$ is disjunctively closed and calculate $N\in\omega$ such that:
$N>q$ and $D(P_{\ell_{p+1}}, P_{\ell_{p+2}}, \dots, P_{\ell_{n-1}}) \prec P_N$.
Again, we do not expect to be accused of dirty tricks, when assuming: $\forall m\le p+1[\overline{\alpha^{m,0}}N = \bar{\underline{0}}N]$.

We define $\zeta\in{}^\omega\omega$ such that: $\mathrm{Fun}(\zeta)$ and, for all $\gamma\in{}^\omega\omega$

(i) $\overline{(\zeta|\gamma)}q = \bar{\alpha}q$

(ii) $N = \mu k[(\zeta|\gamma)^{0,0}(k)\neq 0]$ and $(\zeta|\gamma)^{0,SN} = \gamma$

(iii) For all $j\in\omega$ such that: $0<j\le p+1$: $(\zeta|\gamma)^{j,0} = \underline{0}$

(iv) For all $j\in\omega$ such that: $p+1<j<n+1$: $\neg Q^*((\zeta|\gamma)^j)$.

Remark that: $\forall\gamma\forall k[c_{\zeta|\gamma}(k)>p]$ and, therefore: $\forall\gamma\forall k[d_{\zeta|\gamma}(k)>p]$.
Therefore, for each $\gamma\in{}^\omega\omega$:

$$\begin{aligned} P_N(\gamma) &\rightleftarrows D^{n+1}Q^*(\zeta|\gamma) \wedge \overline{(\zeta|\gamma)}q = \bar{\alpha}q \\ &\rightleftarrows D^n Q^*(\delta|(\zeta|\gamma)) \wedge \forall m<p+1[(\delta|(\zeta|\gamma))^{m,0} = \underline{0}] \\ &\rightleftarrows P_{\ell_{p+1}}((\delta|(\zeta|\gamma))^{p+1,S\ell_{p+1}}) \vee \dots \vee P_{\ell_{n-1}}((\delta|(\zeta|\gamma))^{n-1,S\ell_{n-1}}) \end{aligned}$$

For this reason: $P_N \prec D(P_{\ell_{p+1}}, \dots P_{\ell_{n-1}})$ and this conflicts with the choice of N.

We are forced to conclude: $\forall\alpha[\forall k[c_\alpha(k)>p+1] \to \forall k[d_\alpha(k)>p+1]]$.

This establishes our claim: $\forall p<n\,\forall\alpha[\forall k[c_\alpha(k)>p] \to \forall k[d_\alpha(k)>p]]$.

Thus, we know that: $\forall\alpha\,[\,\forall k[c_\alpha(k) \geq n] \to \forall k[d_\alpha(k) \geq n]]$.
Faster than ever, we are to receive the palm of honour.
We observe that: $\forall\alpha\,[\,\forall m<n\,[\alpha^{m,0} = \underline{0}] \to \forall m<n\,[\,(\delta|\alpha)^{m,0} = \underline{0}]]$.
Therefore: $\forall\alpha\,[\,\forall m<n\,[\alpha^{m,0} = \underline{0}] \to \neg D^n Q^*(\delta|\alpha)]$ and:
$\forall\alpha\,[\,\forall m<n\,[\alpha^{m,0} = \underline{0}] \to \neg D^{n+1} Q^*(\alpha)]$.
This is contradictory, because, as $\exists\alpha\,[P_0(\alpha)]$, also: $\exists\alpha\,[Q^*(\alpha)]$, and we may define $\alpha^* \in {}^\omega\omega$ such that: $\forall m<n\,[(\alpha^*)^{m,0} = \underline{0}]$ and: $Q^*((\alpha^*)^n)$, therefore $D^{n+1} Q^*(\alpha^*)$.
A new grain of wisdom may be added to our treasury:
$\forall n>0\,[\,\neg(D^{n+1}Q^* \leq D^n Q^*)]$ and: $\forall n>0\,[\,D^n Q^* \prec D^{n+1} Q^*]$

⊠

The method underlying the proofs of theorems 12.4.0, 12.5.0 and 12.6.0 is a general one, admitting of application under not too restrictive and varying circumstances.
The proof of the last theorem, which *stated* that: $\forall n>0\,[D^nQ^* \prec D^{n+1}Q^*]$. shows more likeness to the proof of the conjunctive ascension of Q (i.e.: $\forall n>0\,[\,C^nQ \prec C^{n+1}Q\,]$, theorem 12.5.0) than to the proof of the disjunctive ascension of Q (i.e.: $\forall n>0\,[\,D^nQ \prec D^{n+1}Q]$, theorem 12.4.0)
Some understanding of why this should be so, is gained, when, one realizes, that the set $Q^* := \{\alpha \mid \exists n\,[\,n = \mu p\,[\,\alpha^0(p) \neq 0] \wedge P_n(\alpha^{Sn})]\}$ is, classically, parented to the set: $\{\alpha \mid \neg\,\forall n\,[\,n = \mu p[\alpha^0(p) \neq 0] \to \neg P_n(\alpha^{Sn})]\}$ i.e. the complement of the set which results from letting loose the operation which generated Q, on the sequence: $Neg(P_0)$, $Neg(P_1)$, ...

12.7 Let us rest ourselves a little, and philosophize.

Let us call a subset P of ${}^\omega\omega$, <u>disjunctively productive</u>, if $\forall n>0\,[\,D^nP \prec D^{n+1}P]$

We know, from theorems 4.6 and 11.22 respectively, that there are disjunctively productive subsets of ${}^\omega\omega$, for instance A_1 and S_2.

And, now, theorem 12.4.0, (or, for that matter, theorem 12.3.0) enables us to find many more of them.

Starting with $A_1 = R_0$, and applying the generating operation to the sequence $A_1 \prec D^2 A_1 \prec \ldots$, we find R_1, and, thereafter, applying the same operation to the sequence $R_1 \prec D^2 R_1 \prec \ldots$, we find R_2, and, continuing in this way, successively: $R_0 \prec R_1 \prec R_2 \prec \ldots$
But this sequence itself is also an increasing (in the sense of the reducibility relation $\leq$) and disjunctively closed sequence. (We also may refer to the fact that all its members are strictly analytical).
Therefore, another application of the generating operation gives birth to

a disjunctively productive set R_ω such that $\forall n[R_n < R_\omega]$.
Continuing, we find an uncountable multitude of disjunctively productive sets.

The hyperarithmetical hierarchy theorem (theorem 9.7) showed us a very different way to the truth that, with respect to the reducibility relation $\leq$ uncountably many levels of complexity have to be distinguished.

Here, we are facing a phenomenon of a more local nature.
This is even more apparent from the conjunctive story.

Let us call a subset P of ${}^\omega\omega$ conjunctively productive, if $\forall n>0[C^nP < C^{n+1}P]$.

We already met with some conjunctively productive subsets of ${}^\omega\omega$, for example D^2A_1 (cf. theorem 4.15) and S_2 (cf. theorem 11.26).

We also know, from theorem 4.14, that the sequence $A_1, D^2A_1, \ldots$ is conjunctively closed. According to theorem 12.5.0, then, $R = W_0 = \{\alpha \mid \forall n[n = \mu p[\alpha^0(p)\neq 0] \to D^nA_1(\alpha^{Sn})\}$ is a conjunctively productive subset of ${}^\omega\omega$.
The sequence $W_0, C^2W_0, C^3W_0, \ldots$ is, obviously, increasing and conjunctively closed, and theorem 12.5.0 crowns it with a conjunctively productive set W_1.
As in the disjunctive case, a whole sequence $W_0 < W_1 < W_2 < \ldots$ is, successively, called up, and after it, applying the generating operation to this sequence, we find a conjunctively productive set W_ω such that $\forall n[W_n < W_\omega]$
This process will never end.

Reflecting, now, that each one of the sets $W_0, W_1, \ldots$ and W_ω, and the whole of their yet unborn offspring (under the same generating operation) do belong to Π^0_3, we lose ourselves in wonder: Π^0_3 seems to be rather complex.

The foregoing statement rests on two observations:
(I) Π^0_3 is closed under the operation of countable intersection (cf. theorem 6.8).
(II) For all subsets A, D of ${}^\omega\omega$:
If $A \in \Pi^0_3$ and D is a decidable subset of ${}^\omega\omega$, then $A \cup D \in \Pi^0_3$
(The same is true if we replace "Π^0_3" by "Π^0_n" or "Σ^0_n")

We also remark that the sets $W_0, W_1, \ldots, W_\omega$, and their following, and the sets $R_0, R_1, \ldots R_\omega$ and their following, are, all of them, reducible to $S = \{\alpha \mid \exists\gamma[\gamma\in\sigma_2 \wedge \forall n[\alpha(\bar{\gamma}n)=0]]\}$ and, thus belong to the class $\mathcal{C}$, which we discussed in 11.27. This is, because $\mathcal{C}$, as we have seen, is closed under the operations of finite union and countable intersection.
This is some new evidence for the complexity of $\mathcal{C}$.

Still in our pensive mood, we turn to theorem 12.6.0. This theorem gives occasion to similar considerations. We remark that, if we start again with the sequence $A_1, D^2A_1, \ldots$, repeated application of the operation advertized by this theorem, keeps us within the bounds of Σ^0_2. The complexity of Σ^0_2, like that of Π^0_3, is almost beyond imagination.

We mention only some of the many questions that remain to be asked.

Are all universal representatives from the hyperarithmetical hierarchy, i.e.: the sets A_σ, as they have been introduced in 8.4, disjunctively productive?

We know, from the hyperarithmetical hierarchy theorem (theorem 9.7), that these sets are „existentially productive", i.e.: $\forall \sigma \in HI\$ [A_\sigma \prec Ex(A_\sigma)]$ (cf. 10.4)

We have proved, in theorem 4.6, that A_1 is disjunctively productive, and are prepared, on payment, to do the same for A_2 and A_3.

We conjecture, that all sets A_σ are disjunctively productive, but miss a general argument.

Is there any subset A of ${}^\omega\omega$ which is both „disjunctively saturated" and „existentially productive" i.e.: $D^2A \leq A$ and: $A \prec Ex(A)$?

(Remark that E_1 is an example of a set which is „conjunctively saturated" and „universally productive": $C^2E_1 \leq E_1 \prec A_2$)

If so, we would be surprised, but we do not know.
A candidate is $S := \{\alpha \mid \exists \gamma [\gamma \in \sigma_2 \wedge \forall n[\alpha(\bar\gamma n) = 0]\}$.

This lands us into the quicksands of 11.27. We have seen, there, that S is disjunctively saturated, and have stressed, that we do not know how to prove that S is existentially productive, although we would like to do so.

12.8 Implication, like an impatient little brother, has been watching the performances of disjunction and conjunction, eager to show its own abilities.

Negation plays an important part in the implicational show:
Recall, how we defined, in 5.2, to each subset P of ${}^\omega\omega$, a subset $Neg(P)$ of ${}^\omega\omega$, by: $Neg(P) := \{\alpha \mid \neg P(\alpha)\}$. A subset P of ${}^\omega\omega$ is called <u>stable</u>, if $Neg(Neg(P)) = P$.

12.8.0 <u>Lemma</u>: Let $P_0, P_1, \ldots$ be a sequence of stable subsets of ${}^\omega\omega$ such that:
$\forall m \exists n [P_m \prec P_n]$ and $\forall m \exists n [Neg(P_m) \prec P_n]$.
Let $Q := \{\alpha \mid \forall n [n = \mu p[\alpha^0(p) \neq 0] \rightarrow P_n(\alpha^{Sn})$.
Then: $\neg(Neg(Q) \leq Q)$.

<u>Proof</u>: Suppose: $Neg(Q) \leq Q$, and, using AC_{11}, determine $\delta \in {}^\omega\omega$ such that:
$Fun(\delta)$ and $\forall \alpha [\neg Q(\alpha) \rightleftarrows Q(\delta|\alpha)]$.
Remark: $Q(\underline{0})$, therefore: $\neg Q(\delta|\underline{0})$ and $\neg\neg \exists p[(\delta|\underline{0})^0(p) \neq 0]$.
Assume, for the sake of argument: $\exists p[(\delta|\underline{0})^0(p) \neq 0]$ and determine $n_0 := \mu p[(\delta|\underline{0})^0(p) \neq 0]$.
Calculate $q \in \omega$ such that: $\forall \alpha [\bar\alpha q = \bar{\underline{0}} q \rightarrow \overline{(\delta|\alpha)^0}(n_0+1) = \overline{(\delta|\underline{0})^0}(n_0+1)]$.
Calculate $N \in \omega$ such that: $N > q$ and: $Neg(P_{n_0}) \prec P_N$.

Finally, determine $\zeta \in {}^\omega\omega$ such that: $Fun(\zeta)$ and, for all $\gamma \in {}^\omega\omega$:

(i) $\overline{(\zeta|\gamma)}\, q = \overline{\underline{0}}q$

(ii) $N = \mu p [\, (\zeta|\gamma)^0(p) \neq 0]$ and $(\zeta|\gamma)^{SN} = \gamma$.

Then, for all $\gamma \in {}^\omega\omega$:

$$\neg P_N(\gamma) \rightleftarrows \neg Q(\zeta|\gamma) \wedge \overline{(\zeta|\gamma)}\, q = \overline{\underline{0}}q$$
$$\rightleftarrows Q(\delta|(\zeta|\gamma)) \wedge n_0 = \mu p[(\delta|(\zeta|\gamma))^0(p) \neq 0]$$
$$\rightleftarrows P_{n_0}((\delta|(\zeta|\gamma))^{Sn_0}).$$

Therefore: $Neg(P_N) \preceq P_{n_0}$ and, as P_N and P_{n_0} are stable subsets of ${}^\omega\omega$, also: $P_N \preceq Neg(P_{n_0})$ and this conflicts with the choice of N.

This contradiction shows that: $\neg \exists p [(\delta|\underline{0})^0(p) \neq 0]$.
And thus, the assumption: $Neg(Q) \preceq Q$ is seen to lead us to absurdity.

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This lemma is a worthy sequel to lemma 11.6 which stated that: $\neg(A_1 \preceq Neg(A_1))$
To be sure, we never did encounter a subset A of ${}^\omega\omega$ such that: $A \preceq Neg(A)$, and if anybody sees one, he should warn us.

Let R be a subset of ${}^\omega\omega$. We define a sequence $I_0R, I_1R, \ldots$ of subsets of ${}^\omega\omega$ by:

(i) For all $\alpha \in {}^\omega\omega$: $I_0R(\alpha) := R(\alpha^0)$

(ii) For all $p \in \omega$, for all $\alpha \in {}^\omega\omega$: $I_{Sp}R(\alpha) := I_pR(\alpha) \to A_1(\alpha^{Sp})$

Remark: $Neg(R) \preceq I_1R$
Using the technique of lemma 12.8.0, we prove a further result:

12.8.1 <u>Lemma</u>: Let $P_0, P_1, \ldots$ be a sequence of stable subsets of ${}^\omega\omega$ such that: $\forall m \exists n [P_m \prec P_n]$.
Let $Q := \{\alpha \mid \forall n [\, n = \mu p[\alpha^0(p) \neq 0] \to P_n(\alpha^{Sn})]\}$.
Then: $\neg(I_1Q \preceq Neg(Q))$.

<u>Proof</u>: Suppose: $I_1Q \preceq Neg(Q)$ and, using AC_{11}, determine $\delta \in {}^\omega\omega$ such that: $Fun(\delta)$ and $\forall \alpha [\, (Q(\alpha^0) \to \alpha^1 = \underline{0}) \rightleftarrows \neg Q(\delta|\alpha)]$
We claim that: $\forall p [\, (\delta|\underline{0})^0(p) = 0]$.

Suppose: $\exists p [\, (\delta|\underline{0})^0(p) \neq 0]$ and calculate $n_0 := \mu p[\, (\delta|\underline{0})^0(p) \neq 0]$.
Calculate $q \in \omega$ such that: $\forall \alpha [\, \bar{\alpha}q = \overline{\underline{0}}q \to \overline{(\delta|\alpha)^0}(n_0+1) = \overline{(\delta|\underline{0})^0}(n_0+1)]$.

Calculate $N \in \omega$ such that: $N > q$ and: $P_{n_0} \prec P_N$.
Finally, determine $\zeta \in {}^\omega\omega$ such that: $\mathrm{Fun}(\zeta)$ and, for all $\gamma \in {}^\omega\omega$:

(i) $\overline{(\zeta|\gamma)}q = \overline{\underline{0}}q$

(ii) $N = \mu p[(\zeta|\gamma)^{0,0}(p) \neq 0]$ and $(\zeta|\gamma)^{0,SN} = \gamma$

(iii) $(\zeta|\gamma)^1(N) \neq 0$.

Then, for all $\gamma \in {}^\omega\omega$:

$$\begin{aligned}
\neg P_N(\gamma) \quad &\rightleftarrows \quad (P_N((\zeta|\gamma)^{0,SN}) \to A_1((\zeta|\gamma)^1)) \wedge \neg A_1((\zeta|\gamma)^1))\\
&\rightleftarrows \quad (Q((\zeta|\gamma)^0) \to A_1((\zeta|\gamma)^1)) \wedge \neg A_1((\zeta|\gamma)^1))\\
&\rightleftarrows \quad I_1 Q(\zeta|\gamma) \wedge \overline{(\zeta|\gamma)}q = \overline{\underline{0}}q\\
&\rightleftarrows \quad \neg Q(\delta|(\zeta|\gamma)) \wedge n_0 = \mu p[(\delta|(\zeta|\gamma))^0(p) \neq 0]\\
&\rightleftarrows \quad \neg P_{n_0}((\delta|(\zeta|\gamma))^{Sn_0}).
\end{aligned}$$

Therefore: $\mathrm{Neg}(P_N) \leq \mathrm{Neg}(P_{n_0})$ and, since P_N and P_{n_0} are stable subsets of ${}^\omega\omega$: $P_N \leq P_{n_0}$ and this conflicts with the choice of N.

This contradiction shows that $\forall p[(\delta|\underline{0})^0(p) = 0]$.

As $(\delta|\underline{0})^0 = \underline{0}$, we have: $Q(\delta|\underline{0})$.
We are in an impossible situation, because, just as well: $I_1 Q(\underline{0})$.
Let us be wise and give up the assumption: $I_1 Q \leq \mathrm{Neg}(Q)$.

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Let R be a subset of ${}^\omega\omega$. We say that R is <u>wavering in $\underline{0}$</u> if:
$$\forall n \exists \zeta[\mathrm{Fun}(\zeta) \wedge \forall \alpha[\overline{(\zeta|\alpha)}n = \overline{\underline{0}}n] \wedge \forall \alpha[R(\alpha) \rightleftarrows R(\zeta|\alpha)].$$

This means, more or less, that for each $n \in \omega$, $R \cap \overline{\underline{0}}n$ is as complicated as R itself. (We might say: $R \leq R \cap \overline{\underline{0}}n$).

If you come to think upon it, very many sets are wavering in $\underline{0}$.

We take the last preparations before launching implication, and try to follow a line of argument which has been successful in the past (cf. lemma 5.5).

12.8.2 <u>Lemma</u>: Let R be a subset of ${}^\omega\omega$, which is wavering in $\underline{0}$, and such that: $R(\underline{0})$
Then: $\forall p \forall q[I_{Sp}R \leq I_{Sq}R \to \neg\neg(\mathrm{Neg}(I_p R) \leq \mathrm{Neg}(I_q R))]$

<u>Proof</u>: Suppose: $p, q \in \omega$ and $I_{Sp}R \leq I_{Sq}R$.
Using AC_{11}, determine $\delta \in {}^\omega\omega$ such that: $\mathrm{Fun}(\delta)$ and
$\forall \alpha[I_{Sp}R(\alpha) \rightleftarrows I_{Sq}R(\delta|\alpha)]$.

Consider a special sequence α_* in ${}^\omega\omega$ which fulfils the conditions: $\forall j \le p [(\alpha_*)^j = \underline{0}]$ and $(\alpha_*)^{Sp}(0) = 1$.

Remark: $\neg I_{Sp} R(\alpha_*)$, therefore $\neg I_{Sq} R(\delta|\alpha_*)$ and: $(\delta|\alpha_*)^{Sq} \ne \underline{0}$.

Assume now, for the sake of argument: $\exists n [(\delta|\alpha_*)^{Sq}(n) \ne 0]$, and determine $n_0 \in \omega$ such that: $(\delta|\alpha_*)^{Sq}(n_0) \ne 0$.

Also determine $\ell \in \omega$ such that $\forall \alpha [\bar\alpha\ell = \bar\alpha_*\ell \to (\delta|\alpha)^{Sq}(n_0) = (\delta|\alpha_*)^{Sq}(n_0)]$

Remember, that R is wavering in $\underline{0}$, and determine $\zeta \in {}^\omega\omega$ such that: $\mathrm{Fun}(\zeta)$ and: $\forall \alpha [\overline{(\zeta|\alpha)}\ell = \bar{\underline{0}}\ell]$ and: $\forall \alpha [R(\alpha) \rightleftarrows R(\zeta|\alpha)]$

Finally, determine $\eta \in {}^\omega\omega$ such that: $\mathrm{Fun}(\eta)$, and for all $\gamma \in {}^\omega\omega$:

(i) $\overline{(\eta|\gamma)}\ell = \bar\alpha_*\ell$ and: $(\eta|\gamma)^{Sp}(0) = 1$

(ii) $(\eta|\gamma)^0 = \zeta|\gamma^0$

(iii) $\forall j [0 < j \le p \to (\eta|\gamma)^j = \bar{\underline{0}}\ell * \gamma^j]$.

Then, for all $\gamma \in {}^\omega\omega$:

$$\begin{aligned} \neg I_p R(\gamma) &\rightleftarrows I_{Sp} R(\eta|\gamma) \wedge \overline{(\eta|\gamma)}\ell = \bar\alpha_*\ell \\ &\rightleftarrows I_{Sq} R(\delta|(\eta|\gamma)) \wedge (\delta|(\eta|\gamma))^{Sq}(n_0) \ne 0 \\ &\rightleftarrows \neg I_q R(\delta|(\eta|\gamma)). \end{aligned}$$

Therefore: $\mathrm{Neg}(I_p R) \preceq \mathrm{Neg}(I_q R)$.

We have proved, now, that: $\exists n [(\delta|\alpha_*)^{Sq}(n) \ne 0] \to (\mathrm{Neg}(I_p R) \preceq \mathrm{Neg}(I_q R))$.

And we know that: $\neg\neg \exists n [(\delta|\alpha_*)^{Sq}(n) \ne 0]$.

Therefore: $\neg\neg(\mathrm{Neg}(I_p R) \preceq \mathrm{Neg}(I_q R))$.

$\boxtimes$

Lemma 5.5 is a special case of lemma 12.8.2: consider $R := A_1$

Implication now fulfils its promises and, really, goes far:

12.8.3 <u>Theorem</u>: Let $P_0, P_1, \ldots$ be a sequence of stable subsets of ${}^\omega\omega$ such that:
$\forall m \exists n [P_m \prec P_n]$
Let $Q := \{\alpha \mid \forall n [n = \mu p[\alpha^0(p) \ne 0] \to P_n(\alpha^{Sn})]\}$
Then: $\forall n [I_n Q \prec I_{n+2} Q]$.

<u>Proof</u>: Let us first remark that Q, and likewise all sets $I_1Q, I_2Q, \ldots$ are stable subsets of ${}^\omega\omega$, as they are built from the sets A_1, $P_0, P_1, \ldots$ by means of operations (countable intersection, implication)

which preserve stability.

It is easily seen that, for each subset R of ${}^{\omega}\omega$: $Neg(Neg(R)) \preceq I_2 R$.

And: $\forall n\, [\, I_n Q = Neg(Neg(I_n Q))]$, therefore: $\forall n [I_n Q \preceq I_{n+2} Q]$.

Suppose, now: $I_2 Q \leq Q$. Then: $Neg(I_1 Q) \preceq I_2 Q \preceq Q$, and, as we have to do with stable subsets of ${}^{\omega}\omega$: $I_1 Q \preceq Neg(Q)$

This, however, has been refuted in lemma 12.8.1

Therefore: $\neg (I_2 Q \preceq Q)$.

Remark that: Q is wavering in $\underline{0}$ and: $Q(\underline{0})$

Let $n \in \omega$ and define $\zeta \in {}^{\omega}\omega$ such that: $Fun(\zeta)$, and, for all $\gamma \in {}^{\omega}\omega$:

(i) $\overline{(\zeta|\gamma)}\, n = \overline{\underline{0}} n$

(ii) $(\zeta|\gamma)^0 = \overline{\underline{0}} n * \gamma^0$

(iii) For all $j \in \omega$: $(\zeta|\gamma)^{n+Sj} = \gamma^{Sj}$

Then: $\forall \gamma [\, Q(\gamma) \rightleftarrows Q(\zeta|\gamma)]$.

Therefore: lemma 12.8.2 applies and, observing first that, again because of stability: $\forall p \forall q\, [\,(Neg(I_p Q) \preceq Neg(I_q Q)) \rightleftarrows (I_p Q \preceq I_q Q)]$, we establish, successively: $\neg (I_3 Q \preceq I_1 Q)$, $\neg (I_4 Q \preceq I_2 Q)$, ...

i.e.: $\forall n [\, \neg (I_{n+2} Q \leq I_n Q)]$ and: $\forall n [\, I_n Q \prec I_{n+2} Q]$.

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Thus, we get an increasing sequence $Q \prec I_2 Q \prec I_4 Q \prec \ldots$

We better leave out: $I_1 Q$, $I_3 Q$, ...

It is an easy consequence of theorem 12.8.3 that: $\neg (I_1 Q \preceq Q)$
(For: if $I_1 Q \preceq Q$, then $I_2 Q \leq Q$).
On somewhat stricter conditions, the same conclusion follows from lemma 12.8.0.

It is doubtful, on the other hand, whether $Q \preceq I_1 Q$.
We observed earlier, just before theorem 5.21, that $I_1 E_1 \preceq A_1$, therefore: $\neg (E_1 \preceq I_1 E_1)$
We admit that this is not a very convincing example, as E_1 is not a stable subset of ${}^{\omega}\omega$. Therefore: $\forall n [\, \neg (E_1 \preceq I_n E_1)]$.

But we need not trouble ourselves with these questions, if we concentrate upon the ascension of implication.

It is clear, already, that, like its disjunctive and conjunctive predecessors 12.4.0 and 12.5.0, theorem 12.8.3 is capable of repeated application.

First, consider the sequence $I_1\ (:= A_1)$, $I_2\ (:= I_1A_1)$, $I_3\ (:= I_2A_1), \dots$ which we introduced in 5.0 and, using 12.8.3, build a set $\mathcal{U}_0$.
Remark: $\forall n > 0\,[\,I_n \prec I_{n+1} \prec \mathcal{U}\,]$.
Then, consider the sequence: $I_0\mathcal{U}_0,\ I_2\mathcal{U}_0,\ I_4\mathcal{U}_0, \dots$ and, using 12.8.3 again build a set $\mathcal{U}_1$. Remark: $\forall n\,[\,I_{2n}\mathcal{U}_0 \prec I_{2n+2}\mathcal{U}_0 \prec \mathcal{U}_1\,]$.
Similarly, from $\mathcal{U}_1$ build $\mathcal{U}_2$, from $\mathcal{U}_2$ build $\mathcal{U}_3, \dots$
Then, consider the sequence: $\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2, \dots$ and, using 12.8.3 again, build a set $\mathcal{U}_\omega$
And so on.

12.9 Also the second construction of chapter 5, which led to theorem 5.10, may be generalized.

Let R be a subset of ${}^\omega\omega$. We define a sequence $J_0R, J_1R, \dots$ of subsets of ${}^\omega\omega$ by:

(i) For all $\alpha \in {}^\omega\omega$: $J_0R(\alpha) := R(\alpha^0)$

(ii) For all $p \in \omega$, for all $\alpha \in {}^\omega\omega$: $J_{Sp}R(\alpha) := J_pR(\alpha) \rightarrow E_1(\alpha^{Sp})$.

Remark that, if $R := \{\alpha \mid \alpha(0) = 0\}$, the sequence $J_0, J_1, \dots$ which caught our attention in 5.7, reappears.

This time, we do without long preparations and we take the truth by surprise:

12.9.0 __Theorem__: Let $P_0, P_1, \dots$ be a sequence of subsets of ${}^\omega\omega$ such that:
$$\forall \ell\, \forall p\, \forall q\, \forall n\, \exists N\,[\,N > \ell \wedge \neg(J_pP_N \leq J_qP_n)\,].$$
Let $Q^* := \{\alpha \mid \exists n\,[\,n = \mu p[\alpha^0(p) \neq 0] \wedge P_n(\alpha^{Sn})\,]\}$.
Then: $\forall p\, \forall q\,[\,(p+q \text{ is odd}) \rightarrow \neg(J_pQ^* \leq J_qQ^*)\,]$.

__Proof__: Suppose: $p \in \omega$, $q \in \omega$, $p+q$ is odd and: $J_pQ^* \leq J_qQ^*$.
Using AC_{11}, determine $\delta \in {}^\omega\omega$ such that: $Fun(\delta)$ and: $\forall\alpha[\,J_pQ^*(\alpha) \rightleftarrows J_qQ^*(\delta|\alpha)\,]$.

We call a sequence $\alpha \in {}^\omega\omega$ __negativist__ if: $\alpha^{0,0} = \underline{0}$ and: $\forall j > 0\,[\,\alpha^j = \underline{1}\,]$.
Observe, that for all negativist $\alpha \in {}^\omega\omega$: $\neg Q^*(\alpha^0)$ and: $\forall j > 0\,[\neg E_1(\alpha^j)]$.
Therefore, for all negativist $\alpha \in {}^\omega\omega$, for all $n \in \omega$:

if n is odd, then $J_nQ^*(\alpha)$, and: if n is even, then $\neg J_nQ^*(\alpha)$.

We see now that, as $p+q$ is odd: $\forall\alpha[\,\alpha \text{ is negativist} \rightarrow \delta|\alpha \text{ is not negativist}\,]$
More precisely: $\forall\alpha\,[\,\alpha \text{ is negativist} \rightarrow \neg\neg\exists n[(\delta|\alpha)^{0,0}(n) \neq 0 \vee \exists j[0 < j \leq q \wedge (\delta|\alpha)^j(n) = 0]]\,]$.

All the same, we announce, boldly, that: $\forall\alpha[\,\alpha \text{ is negativist} \rightarrow (\delta|\alpha)^{0,0} = \underline{0}\,]$.

Suppose: $\alpha \in {}^\omega\omega$, α is negativist, and: $\exists n\,[(\delta|\alpha)^{0,0}(n) \neq 0]$.
Calculate $n_0 := \mu n\,[(\delta|\alpha)^{0,0}(n) \neq 0]$.
Calculate $\ell \in \omega$ such that: $\forall\beta[\,\bar\beta\ell = \bar\alpha\ell \rightarrow \overline{(\delta|\beta)^{0,0}}(n_0+1) = \overline{(\delta|\alpha)^{0,0}}(n_0+1)\,]$.

Determine $N \in \omega$ such that: $N > \ell$ and $\neg(J_p P_N \preceq J_q P_{n_0})$.
Finally, determine $\zeta \in {}^\omega\omega$ such that: $Fun(\zeta)$ and, for all $\gamma \in {}^\omega\omega$:

(i) $\overline{(\zeta|\gamma)}\ell = \bar{\alpha}\ell$

(ii) $N = \mu n[(\zeta|\gamma)^{0,0}(n) \neq 0]$ and $(\zeta|\gamma)^{0,SN} = \gamma^0$

(iii) for all $j \in \omega$, $0 < j \leq p$: $(\zeta|\gamma)^j = \underline{1}\ell * \gamma^j$

Observe that, for all $\gamma \in {}^\omega\omega$:

$$J_p P_N(\gamma) \rightleftarrows J_p Q^*(\zeta|\gamma) \wedge \overline{(\zeta|\gamma)}\ell = \bar{\alpha}\ell$$
$$\rightleftarrows J_q Q^*(\delta|(\zeta|\gamma)) \wedge n_0 = \mu n[(\delta|(\zeta|\gamma))^{0,0}(n) \neq 0].$$

Therefore: $J_p P_N \preceq J_q P_{n_0}$ and this conflicts with the choice of N.
This contradiction shows that: $\forall n[(\delta|\alpha)^{0,0}(n) = 0]$.

Going one step further, we assert: $\forall\alpha[\alpha \text{ is negativist} \to \forall j[0<j\leq q \to \forall n[(\delta|\alpha)^j(n) \neq 0]]]$.

Suppose: $j_0 \in \omega$, $0 < j_0 \leq q$ and: $n_0 \in \omega$, $(\delta|\alpha)^{j_0}(n_0) = 0$.
Calculate $\ell \in \omega$ such that: $\forall\beta[\bar{\beta}\ell = \bar{\alpha}\ell \to (\delta|\beta)^{j_0}(n_0) = (\delta|\alpha)^{j_0}(n_0)]$.
Calculate $N \in \omega$ such that: $N > \ell$ and: $\neg(J_p P_N \preceq J_{Sq} P_0)$.

Remark that: $J_{q-1} \preceq J_{Sq} P_0$.
(Define $\eta \in {}^\omega\omega$ such that: $Fun(\eta)$ and, for all $\gamma \in {}^\omega\omega$:
$(\eta|\gamma)^1 = \underline{0} \wedge \forall j \leq q-1\,[(\eta|\gamma)^{SSj} = \gamma^j]$.
Then: $\forall\gamma[J_{q-1}(\gamma) \rightleftarrows J_{Sq} P_0(\eta|\gamma)]$).

As in the previous paragraph, define, from α, ℓ, N, a sequence $\zeta \in {}^\omega\omega$ such that: $Fun(\zeta)$ and ...
Observe that for all $\gamma \in {}^\omega\omega$:

$$J_p P_N(\gamma) \rightleftarrows J_q Q^*(\delta|(\zeta|\gamma)) \wedge (\delta|(\zeta|\gamma))^{j_0}(n_0) = 0.$$

Therefore: $J_p P_N \preceq J_{q-j} \preceq J_{q-1} \preceq J_{Sq} P_0$, (cf. theorem 5.8), and this conflicts with the choice of N.
This contradiction shows that: $\forall j[0<j\leq q \to \forall n[(\delta|\alpha)^j(n) \neq 0]]$.

The quarreling conclusions that we reached will only cease to annoy us, if we accept: $\forall p \forall q[(p+q \text{ is odd}) \to \neg(J_p Q^* \preceq J_q Q^*)]$
We do so.

⊠

This theorem enables us, once more, to scrape the sky:

Using theorem 5.10, we start with the sequence: $J_0, J_1, J_2, \dots$ and, applying 12.9.0, find V_0.

Then, considering the sequence: $J_0 V_0,\ J_1 V_0,\ J_2 V_0, \dots$ we see that 12.9.0 applies again, and we find V_1.

In the same way, from V_1 we find V_2, from V_2 we find V_3, and so on.

We then consider the sequence: $V_0, V_1, V_2, \dots$ and we observe

For all $p, q, n \in \omega$:

$$\exists N\,[\neg(J_q \cdot J_N V_n \preceq J_p V_n)] \quad \text{and:} \quad \forall N\,[J_N V_n \preceq V_{n+1}]$$

$$\text{therefore:} \quad \neg(J_q V_{n+1} \preceq J_p V_n)$$

Therefore: 12.9.0 applies again, and we welcome the new set V_ω.

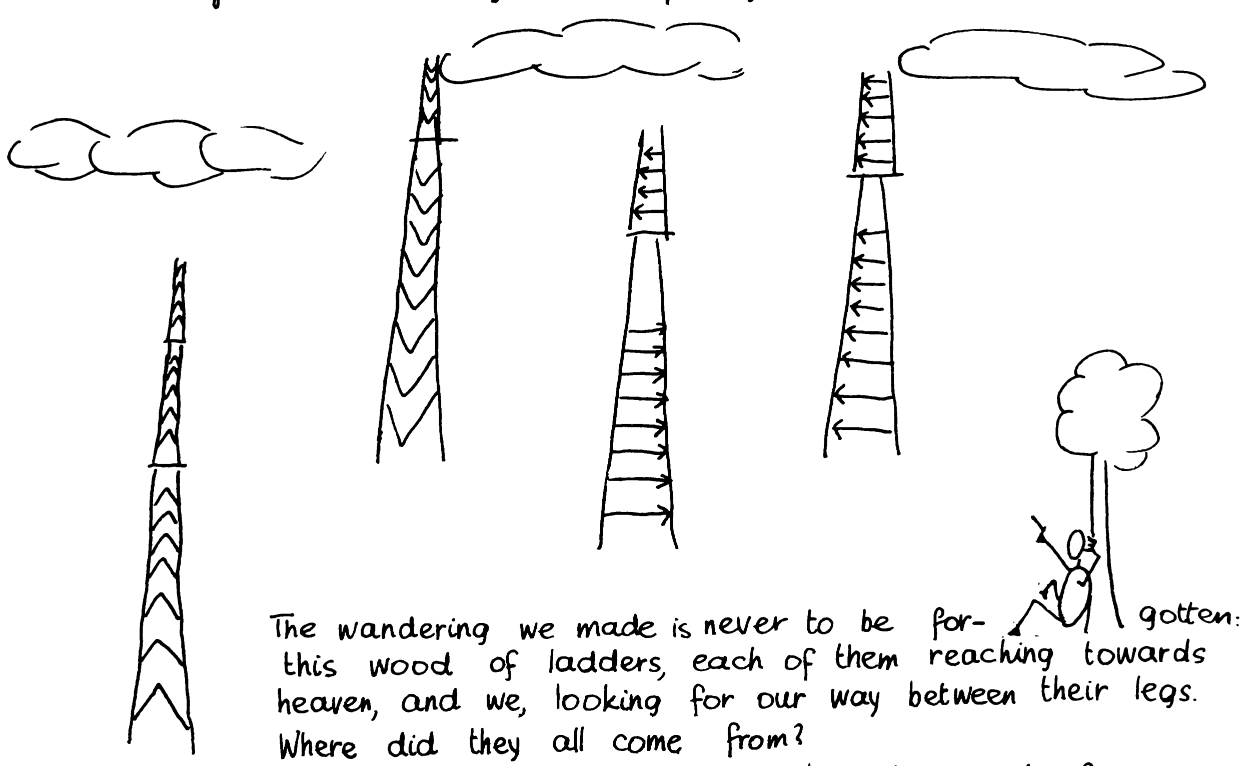
We may continue in this way for quite a long time.

A strange property of this construction is that we do not see, how to prove, or to refute: $\exists n > 0\,[V_0 \preceq J_n V_0]$.

In 12.8, we established an increasing line in $U_0, I_2 U_0, I_4 U_0, \dots$ „by means of stability".

Here, the sets $V_0, J V_0, J_2 V_0 \dots$ are like an unorderable crowd, which we only use to go up from V_0 to V_1.

Remark, before leaving this chapter, that we did our implicational clambering without raising the complexity of the succedens.



The wandering we made is never to be for- gotten:
this wood of ladders, each of them reaching towards heaven, and we, looking for our way between their legs.
Where did they all come from?
Is Saint Peter asking us to clean his window?

13 BROUWER'S THESIS, AND SOME OF ITS CONSEQUENCES

Having made, in the chapters 11 and 12, an excursion into typically intuitionistic phenomena, we now come to some more classical questions, which it is natural to ask in connection with chapter 10, but which we did not yet mention.
One of the famous, beautiful theorems that Souslin proved for classical descriptive set theory, during its infancy, says that the class of all sets which are both analytical and co-analytical, coincides with the class of all hyperarithmetical sets.
One half of this theorem has gone lost in 10.13 already, where it was shown that it is rather exceptional, for a hyperarithmetical set, to be co-analytical.
We now turn to the other half, and prove, in this chapter, that every set, which is both <u>strictly</u> analytical and co-analytical, is hyperarithmetical, indeed.
Souslin was not completely wrong, therefore, and we should perhaps be kind to him and not make too much of the difference between analytical and strictly analytical sets. (cf. 10.7-8).
In defending Souslin, we appeal to the bar theorem, a fundamental tenet of intuitionistic analysis, and, probably, the most questionable one.
We put this theorem into a formulation, which slightly differs from the usual ones, and refer to it as „Brouwer's thesis."
Brouwer's thesis deserves our sympathy, for creating, in the midst of the waste land into which the classical paradise has withered, under the blaze of his harsh criticisms, some things of beauty.
We will see that it also secures a separation theorem for strictly analytical sets, and a corollary thereof, saying that the range of a (strongly) injective function on ${}^\omega\omega$, is hyperarithmetical.
We hope for the truth of Brouwer's thesis, really, and we first try to get clear in what way Brouwer conquered his own doubts.

13.0 We recall, from 8.0, that the set $\$$ of well-ordered stumps in ${}^\omega\omega$ has been defined by transfinite induction, as follows:

(i) $\phi \in \$$

(ii) If $A_0, A_1, A_2, \ldots$ is a sequence of elements of $\$$, then A belongs to $\$$, where $A := \{\langle\,\rangle\} \cup \bigcup_{n\in\omega} \langle n\rangle * A_n$

(iii) If any subset A of ω does belong to $\$$, it does so because of (i) and (ii).

We have observed, in 8.1, that every $\sigma \in \$$ is a decidable subset of ω, and that for all $\sigma \in \$$: $\forall m \forall n[(m\in\sigma \wedge m \subseteq n) \to n \in \sigma]$ and: $\forall\alpha \exists n[\bar{\alpha}n \notin \sigma]$.

We now introduce:

Brouwer's Thesis, General Version:

Let $R \subseteq \omega$ and: $\forall\gamma \exists n [R(\bar{\gamma}n)]$.

Then: $\exists\sigma \in \$ \; \forall a [a \notin \sigma \rightarrow \exists b [a \subseteq b \wedge R(b)]]$.

(In words, which go back to Brouwer's discussion:
the finite sequences which do not belong to \$, have to be past-secured with respect to R) (cf. Note 5 on page 216).

To justify his thesis, Brouwer used a metamathematical argument, saying that, if we have some way of proving: $\forall\gamma \exists n [R(\bar{\gamma}n)]$, we also have a standardized way of proving it.

We should start to break down: $\forall\gamma \exists n [R(\bar{\gamma}n)]$ into:
$\forall\gamma [\gamma(0) = 0 \rightarrow \exists n [R(\bar{\gamma}n)]] \wedge \forall\gamma [\gamma(0) = 1 \rightarrow \exists n [R(\bar{\gamma}n)]] \wedge \forall\gamma [\gamma(0) = 2 \rightarrow \exists n [R(\bar{\gamma}n)]] \wedge \ldots$
and then do the same with each of the countably many propositions which we have before us, now, and continue this process, again and again.

Sometimes, we will strike upon an <u>elementary</u> <u>fact</u>, i.e. a statement of the form: $\forall\gamma [\gamma \in a \rightarrow \exists n [R(\bar{\gamma}n)]]$ which is obviously true, for the reason that $\exists b [a \subseteq b \wedge R(b)]$ and which, therefore, needs no further breaking down.
Brouwer says that this will happen quite often.
He claims that, if $\forall\gamma \exists n [R(\bar{\gamma}n)]$, then the truth of: $\forall\gamma \exists n [R(\bar{\gamma}n)]$ should admit of reconstruction, by a straightforward organization of elementary facts.

The structure of this new proof is isomorphic to the stump σ, which Brouwer's thesis asserts to exist.

This short sketch of the argument should suffice, as we, in any case, are not able to speak the last word upon it.

We will not exploit the full strength of Brouwer's thesis.

Let us introduce, for each $\alpha \in {}^{\omega}\omega$, a subset $|\alpha|^*$ of ω by:

$$|\alpha|^* := \{ a \mid \forall b [a \subseteq b \rightarrow \alpha(b) \neq 0]\}$$

We now present:

Brouwer's Thesis, Special Version

Let $\alpha \in {}^{\omega}\omega$ and: $\forall\gamma \exists n [\alpha(\bar{\gamma}n) = 0]$.

Then: $\exists\sigma \in \$ \; [|\alpha|^* \subseteq \sigma]$.

($|\alpha|^*$ is a decidable subset of ω, which consists of those finite sequences of natural numbers, that are unsecured with respect to α).

Thus, Brouwer's thesis has an important thing to say about Π^1_1.

13.1 Let P be a one-to-one function from $\omega\times\omega$ onto ω, i.e. a pairing function on ω. Let ℓ and r be functions from ω to ω which are left- resp. right-inverse to P, i.e.: $\forall m\,[\,P(\ell(m), r(m)) = m\,]$.

Using P, we introduce a new pairing function on ${}^{\omega}\omega$, and forget all earlier remarks on pairing functions:

Let us define, for all $\alpha\in{}^{\omega}\omega$, $\beta\in{}^{\omega}\omega$, a sequence $\langle\alpha,\beta\rangle$ in ${}^{\omega}\omega$ by:
For all $n\in\omega$: $\quad\langle\alpha,\beta\rangle(n) := P(\alpha(n), \beta(n))$.

Obviously, $\langle\ \rangle$ is a one-to-one function from ${}^{\omega}\omega\times{}^{\omega}\omega$ onto ${}^{\omega}\omega$, i.e. a pairing function on ${}^{\omega}\omega$.
Its left- and right-inverses are called λ, resp. ρ so that $\forall\alpha\,[\,\langle\lambda|\alpha, \rho|\alpha\rangle = \alpha\,]$.

Finally, we introduce a corresponding function from $\{\langle a,b\rangle \mid lg(a) = lg(b)\}$ to ω, as follows:

Let $a\in\omega$, $b\in\omega$ and $lg(a) = lg(b)$.
We define $\langle a,b\rangle$ in ω such that $lg(\langle a,b\rangle) = lg(a)$ and,
for all $n < lg(a)$: $\quad\langle a,b\rangle(n) := P(a(n), b(n))$

We observe, that for each $a\in\omega$, there exist exactly one $x\in\omega$ and exactly one $y\in\omega$ such that $a = \langle x,y\rangle$ and call these numbers $L(a)$, resp. $R(a)$.
Therefore: $\forall a\,[\,\langle L(a), R(a)\rangle = a\,]$.

Remark that: $\forall\alpha\forall\beta\forall n\,[\,\overline{\langle\alpha,\beta\rangle}n = \langle\bar{\alpha}n, \bar{\beta}n\rangle\,]$.

13.2 We defined, in 8.1, a binary predicate $\leq$ on $\$$ by transfinite induction, as follows:

(i) $\phi\leq\phi$

(ii) For all $\sigma,\tau\in\$$: $\quad\sigma\leq\tau := \forall m\exists n\,[\,\sigma^m\leq\tau^n\,]$.

We also defined, for all decidable subsets A, B of ω:

$$A\leq^* B := \exists\gamma\,[\,\forall n\,[\,lg(\gamma(n)) = lg(n)\,]\wedge\forall m\forall n\,[\,m\subseteq n\to\gamma(m)\subseteq\gamma(n)\,]\wedge\forall n\,[\,n\in A\to\gamma(n)\in B\,]\,]$$

And we established, in 8.2 that for all $\sigma,\tau\in\$$: $\quad\sigma\leq\tau \;\rightleftarrows\; \sigma\leq^*\tau$.

This completes our equipment for the next step: piling the wood which will be kindled by Brouwer's thesis:

13.2.0 <u>Lemma</u> (Boundedness lemma)

Let $\delta\in{}^{\omega}\omega$ such that: $Fun(\delta)$ and: $\forall\alpha\,[\,A^1_1(\delta|\alpha)\,]$
Then $\exists\beta\,[\,A^1_1(\beta)\wedge\forall\alpha\,[\,|\delta|\alpha|^*\leq^*|\beta|^*\,]\,]$.

<u>Proof</u>: (The idea of this proof is quite simple: we know that $\forall\alpha\forall\gamma\exists n[(\delta|\alpha)(\bar{\gamma}n)]=0$, and, therefore have to do with a bar in ${}^{\omega}\omega\times{}^{\omega}\omega$. β will be the product of translating this bar into a bar in ${}^{\omega}\omega$. A bar, of course, is nothing but a member of A^1_1)

We define a sequence β in ${}^{\omega}\omega$ such that, for all $a\in\omega$:

$$\beta(a) := 0 \quad \text{if} \quad \exists\ell\exists m[\ell\le lg(a)\wedge m\le lg(a)\wedge \delta^{\overline{R(a)}m}(\overline{L(a)}\ell)=1\wedge$$
$$\wedge\ \forall t[t<\ell\rightarrow \delta^{\overline{R(a)}m}(\overline{L(a)}t)=0]]$$
$$:= 1 \quad \text{otherwise.}$$

Then, for all $a\in\omega$: If $\beta(a)=0$, then $\exists m<lg(a)\,\forall\alpha\in L(a)[(\delta|\alpha)\,\overline{R(a)}m=0]$

We claim that: $A^1_1(\beta)$.

Suppose: $\gamma\in{}^{\omega}\omega$

We write: $\gamma_0:=\lambda|\gamma$ and $\gamma_1:=\rho|\gamma$, therefore: $\gamma=\langle\gamma_0,\gamma_1\rangle$

Determine $m\in\omega$ such that $(\delta|\gamma_0)(\bar{\gamma}_1 m)=0$.

Determine $\ell\in\omega$ such that $\delta^{\bar{\gamma}_1 m}(\bar{\gamma}_0\ell)=1\wedge\forall t<\ell[\delta^{\bar{\gamma}_1 m}(\bar{\gamma}_0 t)=0]$.

Let $n:=\max(m,\ell)$ and remark: $\beta(\bar{\gamma}n)=\beta(\langle\bar{\gamma}_0 n,\bar{\gamma}_1 n\rangle)=0$

We understand, now: $\forall\gamma\exists n[\beta(\bar{\gamma}n)=0]$, i.e.: $A^1_1(\beta)$.

In addition, we claim that: $\forall\alpha[\,|\delta|\alpha|^*\le^*|\beta|^*]$.

Let $\alpha\in{}^{\omega}\omega$.

Define a sequence γ in ${}^{\omega}\omega$ such that, for all $c\in\omega$:

$$\gamma(c) := \langle\bar{\alpha}\,lg(c),\ c\rangle.$$

We observe, without difficulty, that $\forall c[lg(\gamma(c))=lg(c)]$ and:

$\forall c\forall d[c\subseteq d\rightarrow\gamma(c)\subseteq\gamma(d)]$ and $\forall c[\forall t\le lg(c)[(\delta|\alpha)\bar{c}t\ne 0]\rightarrow\forall t\le lg(c)[\beta(\overline{\gamma(c)}t)\ne 0]]$

i.e.: $\forall c[\,c\in|\delta|\alpha|^*\rightarrow\gamma(c)\in|\beta|^*]$.

Therefore: $|\delta|\alpha|^*\le^*|\beta|^*$.

We kept our word.

⊠

13.2.1 <u>Lemma</u>: (i) $\forall\sigma\in\$\ \forall\alpha[\,|\alpha|^*\subseteq\sigma\rightarrow A^1_1(\alpha)]$.

(ii) $\forall\alpha\forall\beta[(A^1_1(\beta)\wedge|\alpha|^*\le^*|\beta|^*)\rightarrow A^1_1(\alpha)]$.

Proof: We prove only the second part, as the first part is easy. Suppose: $\alpha \in {}^{\omega}\omega$, $\beta \in {}^{\omega}\omega$, $A^1_1(\beta)$ and: $|\alpha|^* \leq |\beta|^*$ and determine $\zeta \in {}^{\omega}\omega$ such that: $\forall c[\mathrm{lg}(\zeta(c)) = \mathrm{lg}(c)]$ and: $\forall c \forall d[c \subseteq d \to \zeta(c) \subseteq \zeta(d)]$ and $\forall c\,[\,\forall t \leq \mathrm{lg}(c)[\alpha(\bar{c}t) \neq 0] \to \forall t \leq \mathrm{lg}(c)\,[\,\beta(\overline{\zeta(c)}t) \neq 0]$.
Determine $\eta \in {}^{\omega}\omega$ such that: $\mathrm{Fun}(\eta)$ and: $\forall \gamma \forall n\,[\,\overline{(\eta|\gamma)}n = \zeta(\bar{\gamma}n)]$.
Let $\gamma \in {}^{\omega}\omega$ and determine $n_0 \in \omega$ such that: $\beta(\overline{(\eta|\gamma)}n_0) = 0$.
Then: $\beta(\zeta(\bar{\gamma}n_0)) = 0$, and, therefore: $\exists t < n_0\,[\alpha(\bar{\gamma}t) = 0]$.
We see now, that: $\forall \gamma \exists t[\alpha(\bar{\gamma}t) = 0]$, i.e.: $A^1_1(\alpha)$.

⊠

13.2.2 Theorem: (Souslin - Brouwer) (cf. Note 6 on page 217)

Let P be a subset of ${}^{\omega}\omega$ which is co-analytical and strictly analytical. Then P is hyperarithmetical.

Proof: Determine $\delta \in {}^{\omega}\omega$ such that: $\mathrm{Fun}(\delta)$ and: $\forall \alpha[P(\alpha) \rightleftarrows A^1_1(\delta|\alpha)]$.
Determine $\zeta \in {}^{\omega}\omega$ such that: $\mathrm{Fun}(\zeta)$ and $P = \mathrm{Ra}(\zeta)$, i.e.:
$\forall \alpha[\,P(\alpha) \rightleftarrows \exists \gamma\,[\alpha = \zeta|\gamma]]$.
Remark that: $\forall \gamma\,[\,A^1_1\,(\,\delta|(\zeta|\gamma))]$, and, applying the boundedness lemma (13.2.0), determine $\beta \in {}^{\omega}\omega$ such that $A^1_1(\beta)$ and: $\forall \gamma[\,|\delta|(\zeta|\gamma)|^* \leq^* |\beta|^*\,]$
Now, Brouwer's thesis (13.0) steps forward and finds us a $\sigma \in \$$ such that $\beta^* \subseteq \sigma$
We claim that: $\forall \alpha\,[\,P(\alpha) \rightleftarrows |\delta|\alpha|^* \leq^* \sigma\,]$.

First, suppose: $\alpha \in {}^{\omega}\omega$ and $P(\alpha)$
Determine $\gamma \in {}^{\omega}\omega$ such that $\alpha = \zeta|\gamma$ and remark:
$|\delta|\alpha|^* = |\delta|(\zeta|\gamma)|^* \leq^* |\beta|^* \leq \sigma$
Therefore: $|\delta|\alpha|^* \leq^* \sigma$.

Conversely, suppose: $\alpha \in {}^{\omega}\omega$ and $|\delta|\alpha|^* \leq^* \sigma$.
Then, according to lemma 13.2.1: $A^1_1(\delta|\alpha)$, and, therefore, by choice of δ, $P(\alpha)$.

This establishes our claim.

We observed, at the end of chapter 8, that the set $K_\sigma := \{\alpha \,|\, |\alpha| \leq^* \sigma\}$ is hyperarithmetical, and, as $P \preceq K_\sigma$, P is hyperarithmetical as well.

⊠

It follows from the boundedness lemma (13.2.0), in the proof of which Brouwer's thesis did not yet figure, that A_1^1, itself, is not strictly analytical:

Suppose: $\delta \in {}^\omega\omega$ and: $Fun(\delta)$ and: $\forall\alpha[A_1^1(\delta|\alpha)]$
Using 13.2.0, determine $\beta \in {}^\omega\omega$ such that: $A_1^1(\beta)$ and: $\forall\alpha[\,|\delta|\alpha|^* \leq^* \beta^*]$
Define a sequence $S\beta \in {}^\omega\omega$ such that: $S\beta(\langle\,\rangle) = 1$ and $\forall n[(S\beta)^n = \beta]$
(Cf. the definition of $S\sigma$, for $\sigma \in \$$, in 9.8)

Then: $\forall\alpha[\,|\delta|\alpha|^* \leq^* |(S\beta)^0|^*]$ and: $A_1^1(S\beta)$
Therefore: $\forall\alpha[\delta|\alpha \neq S\beta]$.

For, suppose $\alpha \in {}^\omega\omega$ and $\zeta := \delta|\alpha = S\beta$
Then: $A_1^1(\zeta)$ and: $|\zeta|^* \leq^* |\zeta^0|^*$.

Let $\gamma \in {}^\omega\omega$ be such that: $\forall c[lg(c) = lg(\gamma(c))]$ and: $\forall c \forall d[c \subseteq d \to \gamma(c) \subseteq \gamma(d)]$ and: $\forall c[\forall t < lg(c)[\gamma(\bar{c}t) \neq 0] \to \forall t < lg(c)[\zeta^0(\overline{\gamma(c)}t) \neq 0]]$.

Consider the following sequence:
$d_0 := \langle\,\rangle$, $d_1 := \langle 0 \rangle$, $d_2 := \langle 0 \rangle * \gamma(d_1)$ .... $d_{Sn} := \langle 0 \rangle * \gamma(d_n)$,...

Remark that: $\forall n[lg(d_n) = n \wedge d_{Sn} \subseteq d_n]$.
Determine the unique $\eta \in {}^\omega\omega$ such that: $\forall n[\eta \in d_n]$.

Also observe, using induction, that $\forall n \forall t \leq n[\zeta(\overline{d_n}t) \neq 0]$.
Therefore: $\neg\exists n[\zeta(\bar{\eta}n) = 0]$ and this contradicts: $A_1^1(\zeta)$

Therefore: $\delta|\alpha \neq S\beta$.

Slightly adapting this proof, we may use it to find, effectively, $m \in \omega$, such that: $(\delta|\alpha)(m) \neq S\beta(m)$.

Let η be defined as above, and determine $p \in \omega$ such that $S\beta(\bar{\eta}p) = 0$. Then $\exists t < p[(\delta|\alpha)(\bar{\eta}t) \neq S\beta(\bar{\eta}t)]$.

In any case: $A_1^1(S\beta)$ and: $\forall\alpha[\delta|\alpha \neq S\beta]$.

We have seen, now:

$$\forall\delta[(Fun(\delta) \wedge \forall\alpha[A_1^1(\delta|\alpha)]) \to \exists\beta[A_1^1(\beta) \wedge \forall\alpha[\delta|\alpha \neq \beta]]]$$

Therefore: A_1^1 is not strictly analytical.

To appease our surprise, our thoughts go back to the short discussion following upon theorem 10.12, where we saw that Fun is not strictly analytical.

The two arguments are worth of comparison, leading to similar conclusions along, at least at first sight, rather different ways.

13.3 Let $\sigma \in \$$.
A well-ordered stump, like σ, may be used as a skeleton for mathematical proofs.

We may verify, by transfinite induction, the following

Principle of stump induction.

Let $\sigma \in \$$
Let $Q \subseteq \omega$ and suppose:

(I) $\forall a [a \notin \sigma \to Q(a)]$
(II) $\forall a [\forall n [Q(a * \langle n \rangle)] \to Q(a)]$.

Then: $\forall a [Q(a)]$ and, especially, $Q(\langle\ \rangle)$.

Combining this principle with Brouwer's thesis (13.0), we are led to a

Principle of bar induction.

Let $\beta \in {}^\omega\omega$ be such that: $\forall \gamma \exists n [\beta(\bar{\gamma} n) = 0]$.
Let $Q \subseteq \omega$ and suppose:

(I) $\forall a [\exists b [a \subseteq b \wedge \beta(b) = 0] \to Q(a)]$
(II) $\forall a [\forall n [Q(a * \langle n \rangle)] \to Q(a)]$.

Then: $\forall a [Q(a)]$ and, especially, $Q(\langle\ \rangle)$.

13.4 As is well-known, intuitionists like to consider, besides the negatively defined inequality relation, a constructive <u>apartness</u> relation on ${}^\omega\omega$, which is denoted by $\#$ and defined by:

For all $\alpha \in {}^\omega\omega$, $\beta \in {}^\omega\omega$:

$$\alpha \# \beta \quad := \exists n [\alpha(n) \neq \beta(n)].$$

We are not going to recite the litany of good properties of $\#$ and only mention that: $\forall \alpha \forall \beta \forall \gamma [\alpha \# \beta \to (\alpha \# \gamma \vee \gamma \# \beta)]$.

Let P and Q be subsets of ${}^\omega\omega$.
We say that P is <u>separate</u> from Q, and write: $Sep(P, Q)$ if:

$$\forall \alpha \forall \beta [(P(\alpha) \wedge Q(\beta)) \to \alpha \# \beta].$$

Let P, Q, S and T be subsets of ${}^\omega\omega$
We say that the pair $\langle S, T \rangle$ separates the pair $\langle P, Q \rangle$ if:

$$P \subseteq S \wedge Q \subseteq T \text{ and } Sep(S, T).$$

Let P and Q be subsets of ${}^\omega\omega$
We say that the pair $\langle P, Q \rangle$ is <u>hyperarithmetically</u> <u>separable</u> (or: Borel-separable) if there exists a pair $\langle S, T \rangle$ of hyperarithmetical sets, which separates the pair $\langle P, Q \rangle$.

We are going to prove that any pair of separate, strictly analytical sets is hyperarithmetically separable, and have to make some preparations:

13.4.0. Lemma: Let $A_0, A_1, A_2, \ldots$ and $B_0, B_1, B_2, \ldots$ be two sequences of subsets of ${}^\omega\omega$ such that: $\forall m \forall n [<A_m, B_n> $ is hyperarithmetically separable$]$
Then: $< \bigcup_{n\in\omega} A_n, \bigcup_{n\in\omega} B_n >$ is hyperarithmetically separable.

Proof: Using countable choice, determine for each $m\in\omega$, $n\in\omega$ hyperarithmetical sets $E_{m,n}$ and $F_{m,n}$ such that:

$$A_m \subseteq E_{m,n} \wedge B_n \subseteq F_{m,n} \quad \text{and} \quad Sep(E_{m,n}, F_{m,n})$$

Consider the sets $E := \bigcup_{n\in\omega} \bigcap_{m\in\omega} E_{n,m}$ and $F := \bigcup_{n\in\omega} \bigcap_{m\in\omega} F_{m,n}$ and remark that both E and F are hyperarithmetical and that $\bigcup_{n\in\omega} A_n \subseteq E$ and $\bigcup_{n\in\omega} B_n \subseteq F$.

Finally, we show that: $Sep(E, F)$

Suppose: $\alpha \in {}^\omega\omega$ and $E(\alpha)$, and: $\beta \in {}^\omega\omega$ and $F(\beta)$
Determine $n_0 \in \omega$ such that: $\alpha \in \bigcap_{m\in\omega} E_{n_0,m}$
Determine $n_1 \in \omega$ such that: $\beta \in \bigcap_{m\in\omega} F_{m,n_1}$
Remark: $\alpha \in E_{n_0,n_1}$ and: $\beta \in F_{n_0,n_1}$ and $Sep(E_{n_0,n_1}, F_{n_0,n_1})$
Therefore: $\alpha \# \beta$
We see, now, that: $\forall\alpha \forall\beta [(E(\alpha) \wedge F(\beta)) \to \alpha\#\beta]$, i.e.: $Sep(E,F)$.

Therefore: $\bigcup_{n\in\omega} A_n \subseteq E$ and: $\bigcup_{n\in\omega} B_n \subseteq F$ and: $Sep(E,F)$, i.e.: the pair: $< \bigcup_{n\in\omega} A_n, \bigcup_{n\in\omega} B_n >$ is hyperarithmetically separable.

$\boxtimes$

We introduce another notational convention.
Let $\delta \in {}^\omega\omega$ be such that: $Fun(\delta)$, and let $a \in \omega$
Then:

$$\delta``a := \{ \beta \mid \exists \alpha \in a [\delta|\alpha = \beta] \}$$

$\delta``a$ is the image of the set $a := \{\alpha \mid \alpha \in a\} = \{\alpha \mid \bar{\alpha} lg(a) = a\}$ under the function δ.

Remark that $Ra(\delta) = \delta``< >$ (Cf. 3.1).

13.4.1 <u>Theorem</u>: (Separation theorem of Lusin - Brouwer). (cf. Note 6 on page 217).

Let $\langle P, Q\rangle$ be a pair of separate, strictly analytical subsets of ${}^\omega\omega$
Then: $\langle P, Q\rangle$ is hyperarithmetically separable.

<u>Proof</u>: Determine $\delta \in {}^\omega\omega$ such that: $\mathrm{Fun}(\delta)$ and: $\forall\alpha[P(\alpha) \rightleftarrows \exists\gamma[\alpha = \delta|\gamma]]$
Determine $\zeta \in {}^\omega\omega$ such that: $\mathrm{Fun}(\zeta)$ and: $\forall\alpha[Q(\alpha) \rightleftarrows \exists\gamma[\alpha = \zeta|\gamma]]$
We then know: $\forall\alpha\forall\beta[\delta|\alpha \# \zeta|\beta]$ and, therefore:

$$\forall\gamma\exists n[(\delta|(\lambda|\gamma))(n) \neq (\zeta|(\rho|\gamma))(n)]$$

(Here, λ and ρ are the inverse functions of the pairing function $\langle\!\langle\ \rangle\!\rangle$, as they were defined in 13.1).

Thus, we are offered a bar in ${}^\omega\omega$, and we will reach our goal by an application of the principle of bar induction (cf. 13.3).

First, define a sequence β in ${}^\omega\omega$ such that, for all $a \in \omega$:

$$\begin{aligned}\beta(a) := 0 \quad \text{if} \quad & \exists n < lg(a)\,\exists p < lg(a)\,\exists q < lg(a)\,[\,\delta^n(\overline{L(a)}p) \neq 0 \;\wedge \\ & \wedge\, \forall t < p\,[\delta^n(\overline{L(a)}t) = 0] \wedge \zeta^n(\overline{R(a)}q) \neq 0 \;\wedge \\ & \wedge\, \forall t < q\,[\zeta^n(\overline{R(a)}t) = 0] \wedge \delta^n(\overline{L(a)}p) \neq \zeta^n(\overline{R(a)}q) \;\wedge \\ & \wedge\, \forall m < n\,\exists t < lg(a)\,[\delta^m(\overline{L(a)}t) \neq 0] \;\wedge \\ & \wedge\, \forall m < n\,\exists t < lg(a)\,[\zeta^m(\overline{R(a)}t) \neq 0]\,] \\ := 1 \quad & \text{otherwise.}\end{aligned}$$

(Here, $L(a)$ and $R(a)$ are finite sequences of the same length as the finite sequence a, which result from cutting a into two, as in 13.1).

Remark that, for all $a \in \omega$, if $\beta(a) = 0$, then:

$$\begin{aligned}&\exists b\,\exists c\,[\,b \neq c \wedge lg(b) = lg(c) \leq lg(a) \;\wedge \\ &\qquad \wedge\, \forall\alpha \in L(a)\,[\delta|\alpha \in b] \wedge \forall\beta \in R(a)\,[\zeta|\beta \in c]\,].\end{aligned}$$

Remark also that: $\forall\gamma\exists n[\beta(\bar{\gamma}n) = 0]$

Let $\gamma \in {}^\omega\omega$ and determine $n_0 \in \omega$ such that: $(\delta|(\lambda|\gamma))(n_0) \neq (\zeta|(\rho|\gamma))(n_0)$.
Determine $p_0 \in \omega$ such that $\forall m \leq n_0\,\exists t < p_0\,[\delta^m(\overline{(\lambda|\gamma)}t) \neq 0]$.
Determine $q_0 \in \omega$ such that $\forall m \leq n_0\,\exists t < q_0\,[\zeta^m(\overline{(\rho|\gamma)}t) \neq 0]$.
Let $n := \max(n_0, p_0, q_0)$ and observe: $\beta(\bar{\gamma}n) = 0$.
This justifies the remark.

Next, we define a subset Q of ω by:

For all $a \in \omega$:

$$Q(a) := <\delta``L(a), \zeta``R(a)> \text{ is hyperarithmetically separable.}$$

We claim that: $\forall a [\beta(a) = 0 \rightarrow Q(a)]$

Suppose: $a \in \omega$ and $\beta(a) = 0$

Determine $b \in \omega$, $c \in \omega$ such that: $lg(b) = lg(c) \leq lg(a)$ and:

$b \neq c$ and: $\forall \alpha \in L(a) [\delta|\alpha \in b]$ and: $\forall \beta \in R(a) [\zeta|\beta \in c]$

$<b, c>$ is, obviously, a pair of hyperarithmetical subsets of ${}^\omega\omega$, which separates $<\delta``L(a), \zeta``R(a)>$

It is easily seen, now, that: $\forall a [\exists b [a \leq b \wedge \beta(b) = 0] \rightarrow Q(a)]$

We also claim that: $\forall a [\forall n [Q(a * <n>)] \rightarrow Q(a)]$.

Suppose: $a \in \omega$ and: $\forall n [Q(a * <n>)]$

Then: $\forall m \forall n [<\delta``(L(a) * <m>), \zeta``(R(a) * <n>)>$ is hyperarithmetically separable$]$.

Using lemma 13.4.0, we conclude that:

$$<\bigcup_{n \in \omega} \delta``(L(a) * <n>), \bigcup_{n \in \omega} \zeta``(R(a) * <n>)> = <\delta``L(a), \zeta``R(a)>$$

is hyperarithmetically separable, i.e.: $Q(a)$

This establishes our claim.

The principle of bar induction (13.3) now tells us: $Q(<>)$, i.e.: the pair $<\delta``L(<>), \zeta``R(<>)> = <\delta``<>, \zeta``<>> = <Ra(\delta), Ra(\zeta)> = <P, Q>$ is hyperarithmetically separable.

And this is the conclusion we sought for.

$\boxtimes$

In the classical theory, this grand separation theorem is foreshadowed in more modest statements, for which, however, there is no obvious constructive equivalent.

For example, it is not true that any pair of separate members of Σ^0_1 is separable by a pair of decidable subsets of ${}^\omega\omega$.

Let $\gamma \in {}^\omega\omega$ and $k := \mu n [\gamma(n) = 0]$ be the volatile number of γ (cf. 11.10)

Let $P := \{\alpha \mid (\alpha(0) = 0 \wedge \exists n [n = k \wedge 2|n]) \vee (\alpha(0) \neq 0 \wedge \exists n [n = k \wedge \neg(2|n)])\}$

Let $Q := \{\alpha \mid (\alpha(0) = 0 \wedge \exists n [n = k \wedge \neg(2|n)]) \vee (\alpha(0) \neq 0 \wedge \exists n [n = k \wedge 2|n])\}$

Remark that P and Q belong to Σ^0_1 and that:

$\forall\alpha\forall\beta[(P(\alpha)\wedge Q(\beta))\rightarrow\alpha(0)\neq\beta(0)]$, i.e.: P is separate from Q
Suppose, now, that $\langle S,T\rangle$ is a pair of separate, decidable subsets of ${}^{\omega}\omega$ and that: $P\subseteq S$ and $Q\subseteq T$.
Consider the question whether $\gamma\in S$:

If $\gamma\in S$, then $\forall n[n=k\rightarrow 2|n]$

If $\gamma\notin S$, then $\forall n[n=k\rightarrow\neg(2|n)]$.

Both answers are reckless, and a general method to answer this question, for each $\gamma\in{}^{\omega}\omega$, does not exist.

In 6.15, we have seen other symptoms, that, at the lowest level of the arithmetical hierarchy, disappointment may be waiting for us.

Another feature of the classical theory is that, therein, theorem 13.2.2 (Souslin - Brouwer) may be derived from the separation theorem 13.4.1 (Lusin - Brouwer).
We can not go this way, for two reasons: we do not identify analytical and strictly analytical sets and we distinguish between co-analytical sets and sets whose complement is analytical.

One succulent fruit, however, is still hanging there, and does not seem to be affected by the sickness of unconstructivity.
Let us try and eat it.

13.5.0 Lemma: Let $A_0, A_1, A_2, \ldots$ be a sequence of subsets of ${}^{\omega}\omega$ such that $\forall m\forall n[m\neq n\rightarrow\langle A_m,A_n\rangle$ is hyperarithmetically separable$]$.
Then there exists a sequence $B_0, B_1, B_2, \ldots$ of hyperarithmetical subsets of ${}^{\omega}\omega$ such that:

(i) $\forall n[A_n\subseteq B_n]$

(ii) $\forall m\forall n[m\neq n\rightarrow B_m$ is separate from $B_n]$.

Proof: Using countable choice, determine, for each $m\in\omega$, $n\in\omega$ such that $m\neq n$, hyperarithmetical sets $E_{m,n}$ and $F_{m,n}$ such that:

$$A_m\subseteq E_{m,n}\ \wedge\ A_n\subseteq F_{m,n}\quad\text{and:}\quad Sep(E_{m,n}, F_{m,n})$$

Define, for each $n\in\omega$, a subset B_n of ${}^{\omega}\omega$ by:

$$B_n := \bigcap_{m\neq n} E_{n,m}\ \cap\ \bigcap_{m\neq n} F_{m,n}$$

It is easily verified that the sequence $B_0, B_1, B_2, \ldots$ fulfils our promises.

⊠

Let $\delta \in {}^\omega\omega$ be such that: $\mathrm{Fun}(\delta)$
We say that δ is <u>strongly injective</u> if: $\forall\alpha\forall\beta[\alpha \# \beta \to \delta|\alpha \# \delta|\beta]$

13.5.1 <u>Theorem</u>: Let $\delta \in {}^\omega\omega$ be such that: $\mathrm{Fun}(\delta)$ and δ is strongly injective.
Then: $Ra(\delta)$ is a hyperarithmetical subset of ${}^\omega\omega$.

<u>Proof</u>: Let $n \in \omega$ and consider $S_n := \{a \mid lg(a) = n\}$.
Remark that: $\forall a \in S_n \forall b \in S_n [a \neq b \to Sep(\delta``a, \delta``b)]$.
Therefore, according to theorem 13.4.1

$\forall a \in S_n \forall b \in S_n [a \neq b \to \langle\delta``a, \delta``b\rangle$ is hyperarithmetically separable$]$.

And, according to lemma 13.5.0, we may define a system $(B_a)_{a\in S_n}$ of hyperarithmetical subsets of ${}^\omega\omega$ such that

$\forall a \in S_n [\delta``a \subseteq B_a]$ and $\forall a \in S_n \forall b \in S_n [a \neq b \to Sep(B_a, B_b)]$.

Doing this for each $n \in \omega$, we assign, to each $a \in \omega$, a hyperarithmetical set B_a.
Next, we define, for each $a \in \omega$, a hyperarithmetical set C_a by:

$$C_a := \bigcap_{a \subseteq b} B_b = B_{\langle\rangle} \cap B_{\bar{a}1} \cap B_{\bar{a}2} \cap \dots \cap B_a.$$

We observe that: $\forall a \forall b [\neg(a \subseteq b \vee b \subseteq a) \rightleftarrows C_a \cap C_b = \phi]$.
We claim that: $\forall\alpha[\forall n \exists a \in S_n [C_a(\alpha)] \to \exists\gamma\forall n [C_{\bar{\gamma}n}(\alpha)]]$

Suppose: $\alpha \in {}^\omega\omega$ and: $\forall n \exists a \in S_n [C_a(\alpha)]$.
Using AC_{00}, we determine a sequence $a_0, a_1, a_2, \dots$ of natural numbers such that: $\forall n [lg(a_n) = n \wedge C_{a_n}(\alpha)]$
Remark that: $\forall n \forall m [\alpha \in C_{a_n} \cap C_{a_m} \neq \Phi]$ and, therefore: there exists exactly one $\gamma \in {}^\omega\omega$ such that $\forall n [\gamma \in a_n]$ and: $\forall n [\bar{\gamma}n = a_n]$.
Thus, our claim is established.

Finally, we observe that: $\forall\gamma\forall\alpha[\forall n [C_{\bar{\gamma}n}(\alpha)] \to \alpha = \delta|\gamma]$.
Therefore, for all $\alpha \in {}^\omega\omega$:

$$\alpha \in Ra(\delta) \rightleftarrows \exists\gamma[\alpha = \delta|\gamma]$$
$$\rightleftarrows \forall n \exists a \in S_n [C_a(\alpha)].$$

And: $Ra(\delta) = \bigcap_{n\in\omega} \bigcup_{a \in S_n} C_a$ is hyperarithmetical indeed. ⊠.

The classical converse of 13.5.1, does not survive constructive criticism.

14 THE COLLAPSE OF THE PROJECTIVE HIERARCHY

Classically, A^1_1 and E^1_1, the sets we studied in chapter 10, walk at the head of a long procession of subsets of ${}^\omega\omega$: $A^1_1, E^1_1, A^1_2, E^1_2, A^1_3, E^1_3, \ldots$

The members of this procession are defined rather straightforwardly, by repeated use of the operations of existential and universal projection with respect to ${}^\omega\omega$.

In perfect analogy to the arithmetical hierarchy, one finds that:
$\forall n>0 [A^1_n \prec E^1_{Sn} \wedge E^1_n \prec A^1_{Sn}]$.

Intuitionistically, however, the axiom AC_{11} disturbs this dream, making, as we will see in this chapter, that $A^1_2 \leq E^1_2$ and $A^1_3 \leq E^1_2$.

This is a serious application of AC_{11}.
(Many other applications in this treatise could have been avoided by a change in the definition of the reducibility relation (cf. 2.3), but not this one).

Thus, the projective hierarchy breaks off at E^1_2.
This only happens by our refusal to recognize complementation as a blameless method of building new subsets of ${}^\omega\omega$. Complementation immediately enables one to make subsets of ${}^\omega\omega$ which are not reducible to E^1_2, by diagonalizing.

At the end of the chapter we again have to face some nasty questions, which resisted our attempts to answer them, such as, whether $E^1_2 \leq A^1_2$.

14.0 We want to use, in this chapter, the pairing functions on ω and ${}^\omega\omega$ which have been introduced in 13.1.

$\ll\ \gg$ is a pairing function on ${}^\omega\omega$ such that, for all $\alpha\in{}^\omega\omega$, $\beta\in{}^\omega\omega$, $n\in\omega$, the value of the sequence $\ll\alpha,\beta\gg$ at n is produced by glueing together $\alpha(n)$ and $\beta(n)$.
The left- and right- inverses of this pairing function are called λ and ρ.

$\ll\ \gg$ also denotes a function which pairs finite sequences of equal length into a finite sequence of the same length, employing the same method that his namesake uses in pairing infinite sequences.
Remark that the domain of this function is not the whole of $\omega\times\omega$, but only $\{ \langle a,b\rangle \mid \langle a,b\rangle \in {}^\omega\omega \mid lg(a) = lg(b)\}$

Its left and right - inverses are total functions, and are called L and R.

The pairing function $\ll\ \gg$ on ${}^\omega\omega$ is different from the one we introduced in chapter 6, just before definition 6.4, where we learned what it means, if $\mathcal{B}$ is a class of subsets of ${}^\omega\omega$, and P belongs to $\mathcal{B}$, that P is a universal element of $\mathcal{B}$.
This notion depends on the pairing function that we use, but in a rather innocent way:

Let us assume that the class $\mathbb{K}$ is closed under reducibility:
i.e.: for all subsets P and Q of ${}^\omega\omega$: if $P \in \mathbb{K}$, and $Q \preceq P$, then $Q \in \mathbb{K}$.

In general, this is a difficult notion, because of the huge quantifier: „for all subsets P and Q of ${}^\omega\omega$".
In practice, however, this quantifier may be tamed often (cf. similar remarks in 6.1, 6.6, 8.4, 10.0) and we observe, easily, that all classes of the hyperarithmetical hierarchy, and also Σ^1_1 and Π^1_1, fulfil the condition.

Suppose, now, that $\mathcal{U} \in \mathbb{K}$ and $\mathcal{U}$ is a universal element of $\mathbb{K}$ with respect to the pairing function $\langle\ \rangle$.
Define $\mathcal{U}^* := \{\alpha \mid \mathcal{U}(\langle\lambda|\alpha, \rho|\alpha\rangle)\}$ and observe: $\mathcal{U}^*$ is a universal element of $\mathbb{K}$ with respect to the pairing function $\langle\!\langle\ \rangle\!\rangle$.
Conversely, suppose that $\mathcal{U} \in \mathbb{K}$ and $\mathcal{U}$ is a universal element of $\mathbb{K}$ with respect to the pairing function $\langle\!\langle\ \rangle\!\rangle$.
Define $\mathcal{U}^\circ := \{\alpha \mid \mathcal{U}(\langle\!\langle\alpha^0, \alpha^1\rangle\!\rangle)\}$ and observe: $\mathcal{U}^\circ$ is a universal element of $\mathbb{K}$ with respect to the pairing function $\langle\ \rangle$.

We may be convinced, now, that the new pairing function is, to all purposes, quite as good as the old one, and we will see that it is technically superior.

We remind the reader of 10.0, where we defined a subset E^1_1 of ${}^\omega\omega$ by: $E^1_1 := \{\alpha \mid \exists\gamma\forall n[\alpha(\bar\gamma n) = 0]\}$, and introduced the class Σ^1_1 of all subsets of ${}^\omega\omega$ that are reducible to E^1_1.

We also introduced, in the discussion following upon theorem 10.13, for each subset P of ${}^\omega\omega$, a subset $\mathbb{E}(P)$ of ${}^\omega\omega$ by:

$$\mathbb{E}(P) := \{\alpha \mid \exists\gamma[P(\langle\alpha,\gamma\rangle)]\}.$$

We now consider: $\mathbb{E}^*(P) := \{\alpha \mid \exists\gamma[P(\langle\!\langle\alpha,\gamma\rangle\!\rangle)]\}$ and prove:

14.1 <u>Theorem</u>: Let P be a subset of ${}^\omega\omega$ such that $P \in \Sigma^1_1$.
Then: $\mathbb{E}^*(P) \in \Sigma^1_1$

<u>Proof</u>: Using theorem 10.1, determine a decidable subset A of ω such that:
$\forall\alpha[P(\alpha) \rightleftarrows \exists\beta\forall n[\langle\bar\alpha n, \bar\beta n\rangle \in A]]$.
Remark that: $\forall\alpha[\mathbb{E}^*(P)(\alpha) \rightleftarrows \exists\gamma\exists\beta\forall n[\langle\overline{\langle\!\langle\alpha,\gamma\rangle\!\rangle}n, \bar\beta n\rangle \in A]]$
Define a subset A^* of ω by:
For all $n \in \omega$:

$$n \in A^* \rightleftarrows \exists a\exists b[n = \langle a,b\rangle \wedge lg(a) = lg(b) \wedge \langle\langle\!\langle a, L(b)\rangle\!\rangle, R(b)\rangle \in A].$$

Observe that A^* is a decidable subset of ω, and that:
$\forall\beta\forall\alpha[\forall n[\langle\bar\alpha n, \bar\beta n\rangle \in A^*] \rightleftarrows \forall n[\langle\overline{\langle\!\langle\alpha, \lambda|\beta\rangle\!\rangle}n, \overline{(\rho|\beta)}n\rangle \in A]]$.
We claim that $\forall\alpha[\mathbb{E}^*(P)(\alpha) \rightleftarrows \exists\beta\forall n[\langle\bar\alpha n, \bar\beta n\rangle \in A^*]]$.

Suppose: $\alpha \in {}^\omega\omega$ and $\mathbb{E}^*(P)(\alpha)$.
Determine $\gamma \in {}^\omega\omega$, $\zeta \in {}^\omega\omega$ such that: $\forall n [< \overline{\lessdot \alpha, \gamma \gtrdot} n, \bar{\zeta} n > \in A]$.
Define $\beta := \lessdot \gamma, \zeta \gtrdot$ and remark: $\forall n [< \bar{\alpha} n, \bar{\beta} n > \in A^*]$.

Now, suppose: $\alpha \in {}^\omega\omega$ and $\beta \in {}^\omega\omega$ and: $\forall n [< \bar{\alpha} n, \bar{\beta} n > \in A^*]$.
Define: $\gamma := \lambda | \beta$ and $\zeta := \rho | \beta$ and remark:
$\forall n [< \overline{\lessdot \alpha, \gamma \gtrdot} n, \bar{\zeta} n > \in A]$.
Therefore: $\mathbb{E}^*(P)(\alpha)$.

Using theorem 10.1 again, we conclude: $P \in \Sigma^1_1$.

⊠

It follows from theorem 14.1 that, for each subset P of ${}^\omega\omega$: if $P \in \Sigma^1_1$, then $\mathbb{E}(P) \in \Sigma^1_1$
(It suffices to call up $P^* := \{ \alpha \mid P(< \lambda | \alpha, \rho | \alpha >) \}$).

The operation $\mathbb{E}$ did not come alone.
We introduced, in the discussion following upon theorem 10.13, for each subset P of ${}^\omega\omega$, a subset $\mathbb{U}(P)$ of ${}^\omega\omega$:

$$\mathbb{U}(P) := \{ \alpha \mid \forall \gamma [P(< \alpha, \gamma >)] \}$$

We now prefer to consider $\mathbb{U}^*(P) := \{ \alpha \mid \forall \gamma [P(\lessdot \alpha, \gamma \gtrdot)] \}$.

We define a subset A^1_2 of ${}^\omega\omega$ by:

$$A^1_2 := \{ \alpha \mid \forall \gamma \exists \beta \forall n [\alpha (\overline{\lessdot \beta, \gamma \gtrdot} n) = 0] \}$$

Remark that we did not define: $A^1_2 := \mathbb{U}^*(E^1_1)$, because we did not think this definition to be the most convenient one.

We define a class Π^1_2 of subsets of ${}^\omega\omega$ by:
For every subset P of ${}^\omega\omega$: $P \in \Pi^1_2 \rightleftarrows P \preceq A^1_2$

Like Σ^1_1, Π^1_2 has many nice properties.
We introduce a notational convention which is to help us in proving this:

Let $\delta \in {}^\omega\omega$ and $a \in \omega$
We write $\delta | a$ for the unique $p \in \omega$ such that:

$$lg(p) \le lg(a) \wedge \forall t < lg(p) \exists n < lg(a) [\delta^t(\bar{a} n) = p(t) + 1 \wedge \forall m < n [\delta^t(\bar{a} m) = 0]]$$
$$\wedge (lg(p) < lg(a) \to \forall n < lg(a) [\delta^{lg(p)+1}(\bar{a} n) = 0])$$

Remark that, if $Fun(\delta)$, then $\forall a \forall \alpha [\alpha \in a \to \delta | \alpha \in \delta | a]$ and:
$\forall \alpha \forall m \exists n [lg(\delta | \bar{\alpha} n) \ge m]$

14.2 Theorem: Let $P \subseteq {}^\omega\omega$

$P \in \Pi^1_2$ if and only if there exists a decidable subset A of ω such that: $\forall\alpha[P(\alpha) \rightleftarrows \forall\gamma\exists\beta\forall n[<\bar\alpha n, \bar\beta n, \bar\gamma n> \in A]]$

Proof: (I) Suppose: $P \in \Pi^1_2$ and, using AC_{11}, determine $\delta \in {}^\omega\omega$ such that: $Fun(\delta)$ and: $\forall\alpha[P(\alpha) \rightleftarrows A^1_2(\delta|\alpha)]$, i.e.: $\forall\alpha[P(\alpha) \rightleftarrows \forall\gamma\exists\beta\forall n[(\delta|\alpha)(\overline{\ll\beta,\gamma\gg} n) = 0]]$

Define a decidable subset A of ω by:

For all $n \in \omega$:

$$n \in A \rightleftarrows \exists a\exists b\exists c[lg(a) = lg(b) = lg(c) \wedge n = <a,b,c> \wedge \forall t < lg(a)[\overline{\ll b,c\gg} t < lg(\delta|a) \to (\delta|a)(\overline{\ll b,c\gg} t) = 0]].$$

Remark that: $\forall\alpha\forall\beta\forall\gamma[\forall n[(\delta|\alpha)(\overline{\ll\beta,\gamma\gg} n) = 0] \rightleftarrows \forall n[<\bar\alpha n, \bar\beta n, \bar\gamma n> \in A]]$

Therefore: $\forall\alpha[P(\alpha) \rightleftarrows \forall\gamma\exists\beta\forall n[<\bar\alpha n, \bar\beta n, \bar\gamma n> \in A]]$.

(II) Let A be a decidable subset of ω such that:

$$\forall\alpha[P(\alpha) \rightleftarrows \forall\gamma\exists\beta\forall n[<\bar\alpha n, \bar\beta n, \bar\gamma n> \in A]].$$

Determine $\delta \in {}^\omega\omega$ such that $Fun(\delta)$ and:

For all $\alpha \in {}^\omega\omega$ and $b \in \omega$, $c \in \omega$ such that: $lg(b) = lg(c)$

$$(\delta|\alpha)(\ll b,c\gg) = 0 \rightleftarrows <\bar\alpha lg(b), b, c> \in A.$$

Remark that: $\forall\alpha\forall\beta\forall\gamma[\forall n[<\bar\alpha n, \bar\beta n, \bar\gamma n> \in A] \rightleftarrows \forall n[(\delta|\alpha)(\overline{\ll\beta,\gamma\gg} n) = 0]]$

Therefore: $\forall\alpha[P(\alpha) \rightleftarrows \forall\gamma\exists\beta\forall n[(\delta|\alpha)(\overline{\ll\beta,\gamma\gg} n) = 0]]$,

i.e.: $\forall\alpha[P(\alpha) \rightleftarrows A^1_2(\delta|\alpha)]$ and: $P \in \Pi^1_2$.

$\boxtimes$

14.3 Theorem: Let $P \subseteq {}^\omega\omega$.

$P \in \Pi^1_2$ if and only if there exists a subset Q of ${}^\omega\omega$ such that $Q \in \Sigma^1_1$ and $P = \mathcal{U}^*(Q)$

Proof: (I) First, suppose: $P \in \Pi^1_2$, and, using theorem 14.2, determine a decidable subset A of ω such that: $\forall\alpha[P(\alpha) \rightleftarrows \forall\gamma\exists\beta\forall n[<\bar\alpha n, \bar\beta n, \bar\gamma n> \in A]]$

Determine a decidable subset A^* of ω such that:

For all $a \in \omega$, $b \in \omega$

$$<a,b> \in A^* \rightleftarrows (lg(a) = lg(b) \wedge <L(a), b, R(a)> \in A).$$

Define $Q := \{\alpha \mid \exists\beta\forall n[<\bar\alpha n, \bar\beta n> \in A^*]\}$, and, using theorem 10.1, remark that: $Q \in \Sigma^1_1$.

Also observe that, for all $\alpha \in {}^\omega\omega$, $\gamma \in {}^\omega\omega$:

$$Q(\langle\!\langle \alpha,\gamma \rangle\!\rangle) \rightleftarrows \exists\beta \forall n [\langle \overline{\langle\!\langle \alpha,\gamma\rangle\!\rangle} n, \bar\beta n\rangle \in A^*]$$
$$\rightleftarrows \exists\beta\forall n [\langle \bar\alpha n, \bar\beta n, \bar\gamma n\rangle \in A].$$

Therefore, for all $\alpha \in {}^\omega\omega$:

$$\forall\gamma [Q(\langle\!\langle \alpha,\gamma\rangle\!\rangle)] \rightleftarrows \forall\gamma\exists\beta\forall n [\langle \bar\alpha n, \bar\beta n, \bar\gamma n\rangle \in A]$$
$$\rightleftarrows P(\alpha).$$

i.e.: $P = \mathcal{U}^*(Q)$.

(ii) Conversely, suppose: $Q \in \Sigma^1_1$, and, using theorem 10.1, determine a decidable subset A of ω such that $\forall\alpha[Q(\alpha) \rightleftarrows \exists\beta\forall n[\langle \bar\alpha n, \bar\beta n\rangle \in A]]$

Then: $\forall\alpha[\mathcal{U}^*(Q)(\alpha) \rightleftarrows \forall\gamma\exists\beta\forall n[\langle \overline{\langle\!\langle \alpha,\gamma\rangle\!\rangle} n, \bar\beta n\rangle \in A]]$

Determine a decidable subset A^* of ω such that:

For all $a \in \omega$, $b \in \omega$ $c \in \omega$:

$$\langle a,b,c\rangle \in A^* \rightleftarrows (lg(a) = lg(b) = lg(c) \wedge \langle\langle\!\langle a,c\rangle\!\rangle, b\rangle \in A)$$

Remark that: $\forall\alpha[\mathcal{U}^*(Q)(\alpha) \rightleftarrows \forall\gamma\exists\beta\forall n[\langle \bar\alpha n, \bar\beta n, \bar\gamma n\rangle \in A^*]]$.

and therefore, according to theorem 14.2: $P = \mathcal{U}^*(Q) \in \Pi^1_2$.

$\boxtimes$

14.4 <u>Theorem</u>: Π^1_2 contains a universal element.

<u>Proof</u>: Define the subset $\mathcal{U}$ of ${}^\omega\omega$ by:

For all $\alpha \in {}^\omega\omega$: $\mathcal{U}(\alpha) \rightleftarrows \forall\gamma\exists\beta\forall n [(\rho|\alpha)(\langle \overline{(\lambda|\alpha)} n, \bar\beta n, \bar\gamma n\rangle) = 0]]$

and note that $\mathcal{U}$ belongs to Π^1_2.

Let $P \subseteq {}^\omega\omega$ and $P \in \Pi^1_2$.

Following theorem 14.2, determine a decidable subset A of ω such that: $\forall\alpha[P(\alpha) \rightleftarrows \forall\gamma\exists\beta\forall n[\langle \bar\alpha n, \bar\beta n, \bar\gamma n\rangle \in A]]$. Determine $\delta \in {}^\omega\omega$ such that: $\forall n[\delta(n) = 0 \rightleftarrows n \in A]$. Then: $\forall\alpha[P(\alpha) \rightleftarrows \forall\gamma\exists\beta\forall n[\delta(\langle \bar\alpha n, \bar\beta n, \bar\gamma n\rangle) = 0]]$,

i.e.: $\forall\alpha[P(\alpha) \rightleftarrows \mathcal{U}(\langle\!\langle \alpha,\delta\rangle\!\rangle)]$.

$\boxtimes$

A very minor change in this argument would have given a universal element with respect to any other pairing function.

Like Σ^1_1, Π^1_2 is one of a pair of twins.
The time has come, now, to consider its brother Σ^1_2.
Our speculations on Π^1_2 will be mirrored.

We remind the reader of 10.9, where we defined a subset A^1_1 of ${}^\omega\omega$ by:
$A^1_1 := \{\alpha \mid \forall\gamma \exists n[\alpha(\bar\gamma n) = 0]\}$, and introduced the class Π^1_1 of all subsets of ${}^\omega\omega$ that are reducible to A^1_1.

14.5 Theorem: Let P be a subset of ${}^\omega\omega$ such that $P \in \Pi^1_1$.
Then: $\mathfrak{U}^*(P) \in \Pi^1_1$.

Proof: Using theorem 10.10, determine a decidable subset A of ω such that:
$\forall\alpha[P(\alpha) \rightleftarrows \forall\beta\exists n[\langle\bar\alpha n, \bar\beta n\rangle \in A]]$.

Remark that: $\forall\alpha[\mathfrak{U}^*(P)(\alpha) \rightleftarrows \forall\gamma\forall\beta\exists n[\langle\overline{\langle\!\langle\alpha,\gamma\rangle\!\rangle}n, \bar\beta n\rangle \in A]]$.

Define a subset A^* of ω by:

For all $n \in \omega$:

$$n \in A^* \rightleftarrows \exists a \exists b[n = \langle a, b\rangle \wedge lg(a) = lg(b) \wedge \langle\langle\!\langle a, L(b)\rangle\!\rangle, R(b)\rangle \in A]$$

Observe that A^* is a decidable subset of ω and that:

$\forall\beta\forall\alpha[\exists n[\langle\bar\alpha n, \bar\beta n\rangle \in A^* \rightleftarrows \exists n[\langle\overline{\langle\!\langle\alpha, \lambda|\beta\rangle\!\rangle}n, \overline{(\rho|\beta)}n\rangle \in A]]$

We claim that: $\forall\alpha[\mathfrak{U}^*(P)(\alpha) \rightleftarrows \forall\beta\exists n[\langle\bar\alpha n, \bar\beta n\rangle \in A^*]]$.

Suppose: $\alpha \in {}^\omega\omega$ and $\mathfrak{U}^*(P)(\alpha)$.
Let $\beta \in {}^\omega\omega$ and determine $n \in \omega$ such that:
$\langle\overline{\langle\!\langle\alpha, \lambda|\beta\rangle\!\rangle}n, \overline{(\rho|\beta)}n\rangle \in A$, and, therefore: $\langle\bar\alpha n, \bar\beta n\rangle \in A^*$.
We see, now, that: $\forall\beta\exists n[\langle\bar\alpha n, \bar\beta n\rangle \in A^*]$.

Now, suppose: $\alpha \in {}^\omega\omega$ and: $\forall\beta\exists n[\langle\bar\alpha n, \bar\beta n\rangle \in A^*]$.
Let $\gamma \in {}^\omega\omega$ and $\zeta \in {}^\omega\omega$ and determine $n \in \omega$ such that:
$\langle\bar\alpha n, \overline{\langle\!\langle\gamma, \zeta\rangle\!\rangle}n\rangle \in A^*$, and, therefore: $\langle\overline{\langle\!\langle\alpha,\gamma\rangle\!\rangle}n, \bar\zeta n\rangle \in A$.
We see, now, that: $\forall\gamma\forall\zeta\exists n[\langle\overline{\langle\!\langle\alpha,\gamma\rangle\!\rangle}n, \bar\zeta n\rangle \in A]$,
i.e.: $\mathfrak{U}^*(P)(\alpha)$.

$\boxtimes$

We define a subset E^1_2 of ${}^\omega\omega$ by:

$$E^1_2 := \{\alpha \mid \exists\gamma\forall\beta\exists n[\alpha(\overline{\langle\!\langle\beta,\gamma\rangle\!\rangle}n) = 0]\}.$$

This definition parallels exactly the definition of A^1_2.

We define a class Σ^1_2 of subsets of ${}^\omega\omega$ by:

For every subset P of ${}^\omega\omega$: $P \in \Sigma^1_2 \rightleftarrows P \leq E^1_2$

When it comes to pleasant properties, Σ^1_2 does not yield to Π^1_2:

14.6 Theorem: Let $P \subseteq {}^\omega\omega$.

$P \in \Sigma^1_2$ if and only if there exists a decidable subset A of ω such that $\forall\alpha[P(\alpha) \rightleftarrows \exists\gamma\forall\beta\exists n[<\bar{\alpha}n, \bar{\beta}n, \bar{\gamma}n> \in A]]$.

Proof: (i) Suppose: $P \in \Sigma^1_2$ and, using AC_{11}, determine $\delta \in {}^\omega\omega$ such that: $Fun(\delta)$.
and: $\forall\alpha[P(\alpha) \rightleftarrows E^1_2(\delta|\alpha)]$, i.e.: $\forall\alpha[P(\alpha) \rightleftarrows \exists\gamma\forall\beta\exists n[(\delta|\alpha)(\overline{\ll\beta,\gamma\gg}n)=0]]$
Define a decidable subset A of ω by:
For all $n\in\omega$:

$$n \in A \rightleftarrows \exists a \exists b \exists c[\, lg(a) = lg(b) = lg(c) \wedge n = <a,b,c> \wedge \exists t < lg(a)[\overline{\ll b,c\gg}t < lg(\delta|a) \wedge (\delta|a)(\overline{\ll b,c\gg}t) = 0].$$

(The notation „$\delta|a$" has been introduced just before theorem 14.2).
Remark that: $\forall\alpha\forall\beta\forall\gamma[\exists n[(\delta|\alpha)(\overline{\ll\beta,\gamma\gg}n)=0] \rightleftarrows \exists n[<\bar{\alpha}n,\bar{\beta}n,\bar{\gamma}n>\in A]]$
Therefore: $\forall\alpha[P(\alpha) \rightleftarrows \exists\gamma\forall\beta\exists n[<\bar{\alpha}n,\bar{\beta}n,\bar{\gamma}n>\in A]$.

(ii) Let A be a decidable subset of ω such that:

$$\forall\alpha[P(\alpha) \rightleftarrows \exists\gamma\forall\beta\exists n[<\bar{\alpha}n,\bar{\beta}n,\bar{\gamma}n>\in A]]$$

Determine $\delta \in {}^\omega\omega$ such that $Fun(\delta)$ and:

For all $\alpha \in {}^\omega\omega$ and $b \in \omega, c \in \omega$ such that $lg(b) = lg(c)$

$$(\delta|\alpha)(\ll b, c\gg) = 0 \rightleftarrows <\bar{\alpha}lg(b), b, c> \in A.$$

Remark that: $\forall\alpha\forall\beta\forall\gamma[\exists n[<\bar{\alpha}n,\bar{\beta}n,\bar{\gamma}n>\in A] \rightleftarrows \exists n[(\delta|\alpha)(\overline{\ll\beta,\gamma\gg}n)=0]]$
Therefore: $\forall\alpha[P(\alpha) \rightleftarrows \exists\gamma\forall\beta\forall n[(\delta|\alpha)(\overline{\ll\beta,\gamma\gg}n)=0]$.
i.e.: $\forall\alpha[P(\alpha) \rightleftarrows E^1_2(\delta|\alpha)]$ and: $P \in \Sigma^1_2$.

$\boxtimes$

14.7 Theorem: Let $P \subseteq {}^\omega\omega$.

$P \in \Sigma^1_2$ if and only if there exists a subset Q of ${}^\omega\omega$ such that $Q \in \Pi^1_1$ and $P = \mathbb{E}^*(Q)$.

Proof: (i) First, suppose: $P \in \Sigma^1_2$ and, using theorem 14.6, determine a decidable subset A of ω such that: $\forall\alpha[P(\alpha) \rightleftarrows \exists\gamma\forall\beta\exists n[<\bar{\alpha}n,\bar{\beta}n,\bar{\gamma}n>\in A]$.
Determine a decidable subset A^* of ω such that:
For all $a\in\omega$, $b\in\omega$:

$$<a,b> \in A^* \rightleftarrows (lg(a) = lg(b) \wedge <L(a), b, R(a)> \in A)$$

Define $Q := \{\alpha \mid \forall\beta\exists n[<\bar{\alpha}n,\bar{\beta}n>\in A^*]\}$, and, using theorem 10.10, remark that $Q \in \Pi^1_1$

Also observe that, for all $\alpha \in {}^\omega\omega$, $\gamma \in {}^\omega\omega$:

$$Q(\langle\!\langle \alpha,\gamma \rangle\!\rangle) \rightleftarrows \forall\beta\,\exists n\,[\,\langle \overline{\langle\!\langle \alpha,\gamma\rangle\!\rangle} n, \bar\beta n\rangle \in A^*\,]$$
$$\rightleftarrows \forall\beta\,\exists n\,[\,\langle \bar\alpha n, \bar\beta n, \bar\gamma n\rangle \in A\,]$$

Therefore, for all $\alpha \in {}^\omega\omega$:

$$\exists\gamma\,[\,Q(\langle\!\langle \alpha,\gamma\rangle\!\rangle)\,] \rightleftarrows \exists\gamma\,\forall\beta\,\exists n\,[\,\langle \bar\alpha n, \bar\beta n, \bar\gamma n\rangle \in A\,]$$
$$\rightleftarrows P(\alpha)$$

i.e.: $P = \mathbb{E}^*(Q)$.

(ii) Conversely, suppose: $Q \in \Pi^1_1$ and, using theorem 10.10, determine a decidable subset A of ω such that: $\forall\alpha\,[\,Q(\alpha) \rightleftarrows \forall\beta\,\exists n\,[\langle\bar\alpha n, \bar\beta n\rangle \in A]\,]$

Then: $\forall\alpha\,[\,\mathbb{E}^*(Q)(\alpha) \rightleftarrows \exists\gamma\,\forall\beta\,\exists n\,[\,\langle \overline{\langle\!\langle\alpha,\gamma\rangle\!\rangle} n, \bar\beta n\rangle \in A\,]\,]$

Determine a decidable subset A^* of ω such that:

For all $a \in \omega$, $b \in \omega$, $c \in \omega$:

$$\langle a, b, c\rangle \in A^* \rightleftarrows (\,lg(a) = lg(b) = lg(c) \wedge \langle\langle\!\langle a,c\rangle\!\rangle, b\rangle \in A\,).$$

Remark that: $\forall\alpha\,[\,\mathbb{E}^*(Q)(\alpha) \rightleftarrows \exists\gamma\,\forall\beta\,\exists n\,[\,\langle\bar\alpha n, \bar\beta n, \bar\gamma n\rangle \in A^*\,]\,]$.

and therefore, according to theorem 14.6: $P = \mathbb{E}^*(Q) \in \Sigma^1_2$.

$\boxtimes$

14.8 <u>Theorem</u>: Σ^1_2 contains a universal element.

<u>Proof</u>: Define the subset $\mathcal{U}$ of ${}^\omega\omega$ by:

For all $\alpha \in {}^\omega\omega$: $\mathcal{U}(\alpha) \rightleftarrows \exists\gamma\,\forall\beta\,\exists n\,[\,(\rho|\alpha)(\langle \overline{(\lambda|\alpha)} n, \bar\beta n, \bar\gamma n\rangle) = 0\,]$

and note that $\mathcal{U}$ belongs to Σ^1_2

Let $P \subseteq {}^\omega\omega$ and $P \in \Sigma^1_2$

Following theorem 14.6, determine a decidable subset A of ω such that: $\forall\alpha\,[\,P(\alpha) \rightleftarrows \exists\gamma\,\forall\beta\,\exists n\,[\,\langle\bar\alpha n, \bar\beta n, \bar\gamma n\rangle \in A\,]\,]$. Determine $\delta \in {}^\omega\omega$ such that: $\forall n\,[\,\delta(n) = 0 \rightleftarrows n \in A\,]$. Then: $\forall\alpha\,[\,P(\alpha) \rightleftarrows \exists\gamma\,\forall\beta\,\exists n\,[\,\delta(\langle\bar\alpha n, \bar\beta n, \bar\gamma n\rangle) = 0\,]\,]$

i.e.: $\forall\alpha\,[\,P(\alpha) \rightleftarrows \mathcal{U}(\langle\!\langle\alpha,\delta\rangle\!\rangle)\,]$.

$\boxtimes$

In this last proof, any pairing function, other than $\langle\!\langle\ \rangle\!\rangle$, would do as well.

Until now, our narrative has been straightforward, and almost boring.

But the following, simple remark is surprising:

14.9 **Theorem:** $\Pi^1_2 \subseteq \Sigma^1_2$.

Proof: It is sufficient to show that A^1_2 belongs to Σ^1_2
Using AC_{11}, observe, that for all $\alpha \in {}^\omega\omega$:

$$\begin{aligned} A^1_2(\alpha) &\rightleftarrows \forall\gamma \exists\beta \forall n [\alpha(\overline{\langle\beta,\gamma\rangle} n) = 0] \\ &\rightleftarrows \exists\delta [\mathrm{Fun}(\delta) \wedge \forall\gamma \forall n [\alpha(\overline{\langle \delta|\gamma,\gamma\rangle} n) = 0]] \\ &\rightleftarrows \exists\delta [\mathrm{Fun}(\delta) \wedge \forall c [\alpha(\langle \delta|c, \bar{c}\, lg(\delta|c)\rangle) = 0]] \end{aligned}$$

(The notation „$\delta|c$" has been established just before theorem 14.2)
Recall, from chapter 10, that $\mathrm{Fun} \in \Pi^1_1$, and remark that
$\{\langle\alpha,\delta\rangle \mid \forall c [\alpha(\langle\delta|c, \bar{c}\, lg(\delta|c)\rangle) = 0]\}$ belongs to $\Pi^0_1 \subseteq \Pi^1_1$.
As Π^1_1 is closed under the operation of finite intersection (cf. theorem 10.12) we may conclude, using theorem 14.7, that A^1_2 belongs to Σ^1_2.

⊠

We now prepare to deal a final blow to any remaining hope of a projective hierarchy.

We define a subset A^1_3 of ${}^\omega\omega$ by:

$$A^1_3 := \{\alpha \mid \forall\delta \exists\gamma \forall\beta \exists n [\alpha(\overline{\langle\langle\beta,\gamma\rangle,\delta\rangle} n) = 0]\}$$

We define a class Π^1_3 of subsets of ${}^\omega\omega$ by:

For every subset P of ${}^\omega\omega$: $P \in \Pi^1_3 \rightleftarrows P \preceq A^1_3$

The reader may trust, or else, for one time, go for himself into the treadmill of patient calculation, that:

For every subset P of ${}^\omega\omega$:

$P \in \Pi^1_3 \rightleftarrows$ there exists a decidable subset A of ${}^\omega\omega$ such that
$\forall\alpha [P(\alpha) \rightleftarrows \forall\delta \exists\gamma \forall\beta \exists n [\langle \bar\alpha n, \bar\beta n, \bar\gamma n, \bar\delta n\rangle \in A]]$

and: $P \in \Pi^1_3 \rightleftarrows$ there exists a subset Q of ${}^\omega\omega$ such that:
$Q \in \Sigma^1_2$ and $P = \mathbb{U}^*(Q)$.

14.10 **Theorem:** $\Pi^1_3 = \Sigma^1_2$.

Proof: We leave it for the reader to prove that $\Sigma^1_2 \subseteq \Pi^1_3$.
As to the converse, it is sufficient to show that A^1_3 belongs to Σ^1_2.

Using AC_{11}, observe, that for all $\alpha \in {}^{\omega}\omega$:

$$A_3'(\alpha) \rightleftarrows \forall\delta\exists\gamma\forall\beta\exists n[\alpha(\overline{\langle\!\langle\langle\!\langle\beta,\gamma\rangle\!\rangle,\delta\rangle\!\rangle}n)=0]$$

$$\rightleftarrows \exists\zeta[\mathrm{Fun}(\zeta) \wedge \forall\delta\forall\beta\exists n[\alpha(\overline{\langle\!\langle\langle\!\langle\beta,\zeta|\delta\rangle\!\rangle,\delta\rangle\!\rangle}n)=0]$$

$$\rightleftarrows \exists\zeta[\mathrm{Fun}(\zeta) \wedge \forall\delta\forall\beta\exists n\exists a[\delta\in a \wedge lg(\zeta|a)=n \wedge \alpha(\langle\!\langle\langle\!\langle\bar\beta n,\zeta|a\rangle\!\rangle,\bar\delta n\rangle\!\rangle)=0]].$$

(The notation „$\zeta|a$" has been established just before theorem 14.2.
In the last line, $\langle\!\langle\ \rangle\!\rangle$ denotes a function which pairs finite sequences of equal length, cf. 14.0)

Recall, from chapter 10, that $\mathrm{Fun}\in\Pi_1^1$, and remark that

$\{\langle\!\langle\alpha,\zeta\rangle\!\rangle \mid \forall\delta\forall\beta\exists n\exists a[\delta\in a \wedge lg(\zeta|a)=n \wedge \alpha(\langle\!\langle\langle\!\langle\bar\beta n,\zeta|a\rangle\!\rangle,\bar\delta n\rangle\!\rangle)=0]\}$

belongs to Π_1^1.

As Π_1^1 is closed under the operation of finite intersection (cf. 10.12), we may conclude, using theorem 14.7, that A_3' belongs to Σ_2^1.

$\boxtimes$

Putting together theorems 14.7 and 14.10, we see, that for all subsets P of ${}^{\omega}\omega$:
If $P\in\Sigma_2^1$, then both $\mathbb{E}^*(P)$ and $\mathbb{U}^*(P)$ belong to Σ_2^1.

It is not difficult to verify that the operations of countable union and intersection are but special cases of $\mathbb{E}^*$, resp. $\mathbb{U}^*$

It is impossible, therefore, to go beyond Σ_2^1 by any one of these methods.

If we are so obstinate as not to use negation, or implication, and so dull as not to invent different methods of building subsets of ${}^{\omega}\omega$, Σ_2^1 is the end.

From a classical point of view, theorems 14.9 and 14.10 are strange, indeed.
We still may learn something from attempting the good old diagonal argument:

Let us consider $D := \{\alpha \mid \forall\gamma\exists\beta\forall n[\alpha(\langle\bar\alpha n,\bar\beta n,\bar\gamma n\rangle)\neq 0]\}$

D is easily seen to be a member of Π_2^1, and may be called: the diagonal member of Π_2^1.

According to theorem 14.9, D also belongs to Σ_2^1, and, using theorem 14.6, we determine a decidable subset A of ω such that:

$$\forall\alpha[D(\alpha) \rightleftarrows \exists\gamma\forall\beta\exists n[\langle\bar\alpha n,\bar\beta n,\bar\gamma n\rangle\in A]].$$

Using AC_{01}, we find $\delta\in{}^{\omega}\omega$ such that:

$$\forall\alpha[D(\alpha) \rightleftarrows \exists\gamma\forall\beta\exists n[\delta(\langle\bar\alpha n,\bar\beta n,\bar\gamma n\rangle)=0]]$$

We observe, now, that

$$D(\delta) \rightleftarrows \forall\gamma\exists\beta\forall n[\delta(\bar\delta n,\bar\beta n,\bar\gamma n)\neq 0]$$

$$\rightleftarrows \exists\gamma\forall\beta\exists n[\delta(\bar\delta n,\bar\beta n,\bar\gamma n)=0].$$

Therefore: $\neg D(\delta)$, i.e.:

$$\neg \forall\gamma \exists\beta \forall n [\delta(\bar{\delta}n, \bar{\beta}n, \bar{\gamma}n) \neq 0] \wedge \neg \exists\gamma \forall\beta \exists n [\delta(\bar{\delta}n, \bar{\beta}n, \bar{\gamma}n) = 0]$$

Such a δ is worth a prize: it embodies the nonsense of classical logic. Looking for a place where to lodge it in our zoo, we choose a cage next to this animal:

We claim that: $\neg \forall\alpha \exists n \forall m [\alpha(n)=0 \to \alpha(m)=0] \wedge \neg \exists\alpha \forall n \exists m [\alpha(n)=0 \wedge \alpha(m) \neq 0]$

First, suppose: $\forall\alpha \exists n \forall m [\alpha(n)=0 \to \alpha(m)=0]$.

Using CP (cf. 1.8), we determine $n \in \omega$, $q \in \omega$ such that:

$\forall\beta [\bar{\beta}q = \bar{\underline{0}}q \to \forall m [\beta(n)=0 \to \beta(m)=0]]$.

Let $N := \max(q, n+1)$

Then: $\forall\beta [\bar{\beta}N = \bar{\underline{0}}N \to \beta = \underline{0}]$, and this is not so.

Therefore: $\neg \forall\alpha \exists n \forall m [\alpha(n)=0 \to \alpha(m)=0]$

Next, suppose: $\exists\alpha \forall n \exists m [\alpha(n)=0 \wedge \alpha(m) \neq 0]$.

Choose such an α, an observe: $\alpha = \underline{0} \quad \wedge \neg(\alpha = \underline{0})$!

Therefore: $\neg \exists\alpha \forall n \exists m [\alpha(n)=0 \wedge \alpha(m) \neq 0]$.

This harmless creature seems to be the most simple representative of its species which perhaps might be called: the species of de Morgan's nightmares.
(We do not know if there are any de Morgan's nightmares around, that cause panic about the quantifier-combination: „$\forall\alpha\exists n$")

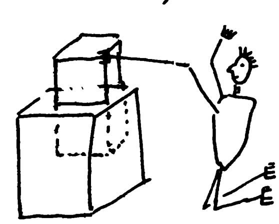
We cannot conceal our ignorance concerning some important points any longer.

At the end of chapter 10, we mentioned our inability to settle the question whether $A_1^1 \preceq E_1^1$, or, equivalently, $\Pi_1^1 \subseteq \Sigma_1^1$.

If it should be so that $A_1^1 \preceq E_1^1$, nothing remains of the, once, proud projective hierarchy, as $\Sigma_2^1 \subseteq \Sigma_1^1$.

Otherwise, if not: $A_1^1 \preceq E_1^1$, then also not: $\Sigma_2^1 \subseteq \Sigma_1^1$, as $\Pi_1^1 \subseteq \Sigma_2^1$.

In this case there is another problem to haunt us, namely, whether $E_2^1 \preceq A_2^1$



dear parents,
the blocks are very nice.
but if I try to build a
tower from them, the one sinks into the other.
this I deplore. your son.

15. A CONTRAPOSITION OF COUNTABLE CHOICE

This chapter presupposes some love of the fan theorem
A fan is used to bring distraction and a moderate breeze, during the unimportant chatterings which may occur when the heat of the day is over.

Leaning on the axiom AC_{01}, we were able to prove, in chapter 10, that all hyperarithmetical sets are analytical.
We are not able to prove that all hyperarithmetical sets are co-analytical, for, as we have seen, the arithmetical set D^2A_1 is not co-analytical.

In this respect we fall behind a classical mathematician, who will stand on his head and then, making the movements required for analyticity, soothe his conscience.

To carry through the classical argument, we need a constructive contraposition of AC_{01}, the second of the two principles of countable choice that we admitted (cf. 1.3)
The resulting principle of reasoning, therefore, cannot be valid in full generality.

Once, watching the classical circus in the company of some good friends, we discussed the question, what is the range of validity of AC_{01}- turned-upside-down.
This question, though not too serious in itself, could be given a simple and elegant answer, which will be the subject of this chapter.

Contraposition might be another method of constructing hierarchical structures of (neo-)classical beauty.
We mention this possibility at the end of the chapter, but are not elaborating it.

The following lines are dedicated, in friendship, to Jo Gielen and Mervyn Jansen. (cf. Note 9 on page 217).

15.0 We remind the reader of the axiom AC_{01}, that has been introduced and defended in 1.3.:

AC_{01} Let $A \subseteq \omega \times {}^{\omega}\omega$.
If $\forall n \exists \alpha [A(n,\alpha)]$, then $\exists \alpha \forall n [A(n,\alpha^n)]$.

Dancing to the piping of A. de Morgan, we are led on to the following crazy principle:

CRP Let $A \subseteq \omega \times {}^{\omega}\omega$.
If $\forall \alpha \exists n [A(n,\alpha^n)]$, then $\exists n \forall \alpha [A(n,\alpha)]$.

As we are entertaining already some grave suspicions against CRP, it seems wise to consider also a relativized version of it.
For each subspread σ of ${}^{\omega}\omega$ which fulfils the condition:
$\forall \alpha [\alpha \in \sigma \rightleftarrows \forall n [\alpha^n \in \sigma]]$, it makes sense to study:

CRP_σ Let $A \subseteq \omega \times {}^\omega\omega$.

If $\forall \alpha \in \sigma \exists n [A(n, \overline{\alpha}^n)]$, then $\exists n \forall \alpha \in \sigma [A(n, \alpha)]$.

We remark that the above-mentioned condition is met by the binary fan σ_2, whose acquaintance we made in 11.27.
More generally, we may introduce, for each $p \in \omega$, the p-ary fan σ_p, by:

For all $a \in \omega$:

$$\sigma_p(a) := 0 \quad \text{if } \forall n [n < lg(a) \to a(n) < p]$$
$$:= 1 \quad \text{otherwise.}$$

We remark that, for each $p \in \omega$, σ_p meets the above-mentioned condition.

15.1 The arguments given in the preface to this chapter may have convinced the reader that CRP leads to a contradiction.
Perhaps because of a morbid trait in our character, we follow it once more on its way to absurdity.
We first introduce a consequence of it, which, at the sight of it, is somewhat less disturbing:

CRP^* Let $A \subseteq \omega \times \omega$.

If $\forall \alpha \exists n [A(n, \alpha(n))]$, then $\exists n \forall m [A(n, m)]$.

(The attentive reader may observe that CRP^* is AC_{00}-turned-upside-down, just as CRP is nothing but AC_{01}-turned-upside-down).

We claim that CRP implies CRP^*.

Let $A \subseteq \omega \times \omega$ be such that $\forall \alpha \exists n [A(n, \alpha(n))]$.

Define $A^* \subseteq \omega \times {}^\omega\omega$ by:

For all $n \in \omega$, $\alpha \in {}^\omega\omega$

$$A^*(n, \alpha) := A(n, \alpha(0))$$

We claim that: $\forall \alpha \exists n [A^*(n, \alpha^n)]$.

Let $\alpha \in {}^\omega\omega$ and determine $\alpha^* \in {}^\omega\omega$ such that $\forall n [\alpha^*(n) = \alpha^n(0)]$
Determine $n_0 \in \omega$ such that: $A(n_0, \alpha^*(n_0))$ and observe that $A^*(n_0, \alpha^{n_0})$.

Applying CRP we find $n_1 \in \omega$ such that $\forall \alpha [A^*(n_1, \alpha)]$.
Therefore: $\forall m [A(n_1, m)]$, and thus, our claim proves harmless.

The danger of CRP* glimmers through the following consideration:

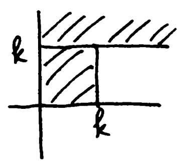
Suppose: $\gamma \in {}^\omega\omega$ and let $k := \mu n[\gamma(n) \neq 0]$ be the volatile number of γ (we discussed this notion just after theorem 11.10)

Let us define a subset A of $\omega \times \omega$ by:

For all $n \in \omega$, $m \in \omega$:

$$A(n,m) := (n \leq k \wedge m \leq k) \vee n \geq k$$

A is pretty close to having the property mentioned in the conclusion of CRP*, for, if k exists, then $\forall m[A(k,m)]$, and, if not, then $\forall m[A(0,m)]$



The reader will see for himself that still, in some cases, it may be reckless to assert that $\exists n \forall m[A(n,m)]$

And he will observe that, on the other hand, the premiss of CRP* goes through.

And now, a fat contradiction, unable to hide itself any longer, creeps from the bushes behind CRP*:

Let us define, for each $\gamma \in {}^\omega\omega$, a subset A_γ of $\omega \times \omega$ by:

For all $n \in \omega$, $m \in \omega$:

$$A_\gamma(n,m) := \forall \ell \leq m[\gamma(\ell) = 0] \vee \exists \ell \leq n[\gamma(\ell) \neq 0]$$

We claim that: $\forall \gamma \forall \alpha \exists n[A_\gamma(n, \alpha(n))]$.

Suppose $\gamma \in {}^\omega\omega$, $\alpha \in {}^\omega\omega$ and consider $\alpha(0)$

We distinguish two cases:

- If $\forall \ell \leq \alpha(0)[\gamma(\ell) = 0]$, then: $A_\gamma(0, \alpha(0))$
- If $\exists \ell \leq \alpha(0)[\gamma(\ell) \neq 0]$, then: $A_\gamma(\alpha(0), \alpha(\alpha(0)))$

In either case, therefore: $\exists n[A_\gamma(n, \alpha(n))]$

Applying CRP*, we find that: $\forall \gamma \exists n \forall m[A_\gamma(n,\alpha)]$.

Therefore: $\forall \gamma \exists n[\gamma(n) = 0 \rightarrow \forall m[\gamma(m) = 0]]$.

And this is easily seen to be contradictory:

Using CP, the principle of continuity mentioned in 1.8, we determine $q \in \omega$, $n \in \omega$ such that:

$$\forall \gamma[(\bar{\gamma}q = \bar{\underline{0}}q \wedge \gamma(n) = 0) \rightarrow \forall m[\gamma(m) = 0]]$$

What about a member γ^* of ${}^\omega\omega$ such that

$$\max(n, q) < \mu p[\gamma^*(p) \neq 0] \quad ?$$

$\boxtimes$

15.2 We mentioned the fan theorem just before theorem 11.4 and repeat it, now:

Let A be a decidable subset of ω and $\beta \in {}^{\omega}\omega$ be a fan.
If $\forall \gamma \in \beta \exists n [A(\bar{\gamma}n)]$, then $\exists m \forall \gamma \in \beta \exists n [n \leq m \wedge A(\bar{\gamma}n)]$.

Recall that $\beta \in {}^{\omega}\omega$ is a fan if the set of finite sequences determined by it is, everywhere, finitely-splitting, i.e.:

$$\forall a [\beta(a) = 0 \rightarrow K_a^{\beta} = \{n \mid n \in \omega \mid \beta(a * \langle n \rangle) = 0\} \text{ is finite}].$$

The fan theorem is a most famous consequence of Brouwer's thesis, which has been presented in 13.0 and, in its special version, reads as follows:

Let $\alpha \in {}^{\omega}\omega$ and $\forall \gamma \exists n [\alpha(\bar{\gamma}n) = 0]$
Then: $\exists \sigma \in \$ [\,|\alpha|^* \leq \sigma]$.

The valuable set $\$$ is the set of well-ordered stumps in ${}^{\omega}\omega$, as we know it from 13.0 and 8.0.

The proof of the fan theorem goes by showing, by transfinite induction, that for each $\beta \in {}^{\omega}\omega$ and each $\sigma \in \$$:

If β is a fan, then $\{a \mid a \in \sigma \wedge \beta(a) = 0\}$ is finite

Once this observation has been made, we quickly enter the promised land:

Let A be a decidable subset of ω and $\beta \in {}^{\omega}\omega$ be a fan, such that $\forall \gamma \in \beta \exists n [A(\bar{\gamma}n)]$.
Determine $\alpha \in {}^{\omega}\omega$ such that $\forall a [\alpha(a) = 0 \rightleftarrows (a \in A \vee \beta(a) \neq 0)]$
Observe that $\forall \gamma \exists n [\alpha(\bar{\gamma}n) = 0]$ (cf. Note 7 on page 217).
Using Brouwer's thesis, we determine $\sigma \in \$$ such that:
$|\alpha|^* := \{a \mid \forall b [a \sqsubseteq b \rightarrow \alpha(b) \neq 0]\} \leq \sigma$
We remark that: $\{a \mid a \in \sigma \wedge \beta(a) = 0\}$ is finite, and calculate $m \in \omega$ such that $\forall a [(a \in \sigma \wedge \beta(a) = 0) \rightarrow lg(a) \leq m]$
We finish by noticing that: $\forall \gamma \in \beta \exists n [n \leq m+1 \wedge A(\bar{\gamma}n)]$.

Combining the axiom AC_{10} (cf. 1.5) and the fan theorem, we find the following principle of reasoning, which we want to apply freely in the sequel:

15.2.0 Let $A \subseteq {}^{\omega}\omega \times \omega$ and $\beta \in {}^{\omega}\omega$ be a fan
If $\forall \gamma \in \beta \exists n [A(\gamma, n)]$, then $\exists m \forall \gamma \in \beta \exists n [n \leq m \wedge A(\gamma, n)]$

The contents of this section will not surprise someone who is acquainted with an introduction to intuitionistic analysis, for instance, Kleene and Vesley 1965.

15.3 We will prove, for each $\tau \in {}^\omega\omega$, which is a fan and fulfils the condition: $\forall\alpha[\alpha\in\tau \rightleftarrows \forall n[\alpha^n\in\tau]]$, that CRP_τ.

We first make a simple observation:

15.3.0 Lemma: Let $A\subseteq \omega\times{}^\omega\omega$ and $\tau\in{}^\omega\omega$ be a fan such that $\forall\alpha[\alpha\in\tau \rightleftarrows \forall n[\alpha^n\in\tau]]$, and $\forall\alpha\in\tau\,\exists n[A(n,\alpha^n)]$.
Then: $\forall\alpha\in\tau\,\exists n[A(n,\alpha)]$.

Proof: Let $\alpha\in\tau$ and determine $\beta\in\tau$ such that $\forall n[\beta^n=\alpha]$.
Determine $n_0\in\omega$ such that $A(n_0,\beta^{n_0})$ and conclude: $A(n_0,\alpha)$.
$\boxtimes$

The next observation is more than twice as difficult:

15.3.1 Lemma: Let $A\subseteq\omega\times{}^\omega\omega$ and $\tau\in{}^\omega\omega$ be a fan such that $\forall\alpha[\alpha\in\tau\rightleftarrows\forall n[\alpha^n\in\tau]]$, and $\forall\alpha\in\tau\,\exists n[A(n,\alpha^n)]$
Then: $\forall\alpha\in\tau\,\forall\beta\in\tau\,\exists n[A(n,\alpha)\wedge A(n,\beta)]$.

Proof: Let $\alpha\in\tau$, $\beta\in\tau$.
We need the assistance of the binary fan $\sigma_2 := \{\gamma \mid \forall n[\gamma(n)<2]\}$ (cf. 15.0 and 11.27).
We determine $\zeta\in{}^\omega\omega$ such that: $Fun(\zeta)$ and, for all $\gamma\in\sigma_2$, for all $n\in\omega$:
- if $\gamma(n)=0$, then $(\zeta|\gamma)^n=\alpha$
- if $\gamma(n)=1$, then $(\zeta|\gamma)^n=\beta$.

Thus, we have a mapping from σ_2 onto the set of all members of τ whose only subsequences are α and β.
We know: $\forall\gamma\in\sigma_2\,\exists n[A(n,(\zeta|\gamma)^n)]$, and, applying 15.2.0, we calculate $M\in\omega$ such that:

$$\forall\gamma\in\sigma_2\,\exists n[n\le M \wedge A(n,(\zeta|\gamma)^n)]$$

Let us assume, for a moment only, that $M=2$.
We then know, how to find, for each $\gamma\in\sigma_2$, a natural number n, such that: $n\le 2 \wedge A(n,(\zeta|\gamma)^n)$, i.e.: $A(0,(\zeta|\gamma)^0)\vee A(1,(\zeta|\gamma)^1)\vee A(2,(\zeta|\gamma)^2)$

In determining the triple $(\zeta|\gamma)^0$, $(\zeta|\gamma)^1$, $(\zeta|\gamma)^2$ we have to choose one out of eight possibilities, from α,α,α ... up to β,β,β.
Thus, we are offered eight pieces of truth, to wit:

$$A(0,\alpha) \vee A(1,\alpha) \vee A(2,\alpha)$$

and: $A(0,\alpha) \vee A(1,\alpha) \vee A(2,\beta)$

$\vdots$

and: $A(0,\beta) \vee A(1,\beta) \vee A(2,\beta)$.

Each of these eight statements produces at least one true fact of the form: $A(i,\delta)$, where $i \in \{0,1,2\}$ and $\delta \in \{\alpha,\beta\}$.
Now, either: $A(0,\alpha)$ and $A(0,\beta)$ are both among these true facts, or: $A(1,\alpha)$ and $A(1,\beta)$ are both among these true facts, or: $A(2,\alpha)$ and $A(2,\beta)$ are both among these true facts.
For, if, for instance $A(0,\beta)$, $A(1,\alpha)$ and $A(2,\beta)$ are, all three of them, not among these true facts, this conflicts with our having found true: $A(0,\beta) \vee A(1,\alpha) \vee A(2,\beta)$.

Therefore: $\exists n[A(n,\alpha) \wedge A(n,\beta)]$

This wordy argument has been necessary, as we do not know that A is a decidable subset of $\omega \times {}^{\omega}\omega$, a subtlety which eludes the classical mathematician.

We close the proof by expressing our confidence that, should M have been some other number than 2, we could have played a similar game.

$\boxtimes$

Lemma 15.3.1 has an obvious generalization:

15.3.2 <u>Lemma</u>: Let $A \subseteq \omega \times {}^{\omega}\omega$ and $\tau \in {}^{\omega}\omega$ be a fan such that $\forall\alpha[\alpha\in\tau \rightleftarrows \forall n[\alpha^n\in\tau]]$, and $\forall\alpha\in\tau\,\exists n[A(n,\alpha^n)]$.
Let $p \in \omega$, $p > 0$.
Then: $\forall\alpha_0\in\tau\,\forall\alpha_1\in\tau \dots \forall\alpha_p\in\tau\,\exists n[A(n,\alpha_0)\wedge A(n,\alpha_1)\wedge\dots\wedge A(n,\alpha_p)]$.

<u>Proof</u>: Let $\alpha_0\in\tau, \alpha_1\in\tau, \dots, \alpha_p\in\tau$
We need help from the p-ary fan $\sigma_p := \{\gamma \mid \forall n[\gamma(n)<p]\}$. (cf. 15.0)

We determine $\zeta \in {}^\omega\omega$ such that: Fun(ζ), and:
for all $\gamma \in \sigma_p$, for all $n \in \omega$, for all $m \in \omega$, $m < p$:

- if $\gamma(n) = m$, then $(\zeta|\gamma)^n = \alpha_m$

Thus, we have a mapping from σ_p onto the set of all members of τ, all whose subsequences are chosen from $\{\alpha_0, \alpha_1, \ldots \alpha_p\}$.

The rest of the proof is also quite similar to the proof of lemma 15.3.1 and will be omitted.

$\boxtimes$

Without further delay, we close our eyes, and jump:

15.3.3 Theorem: Let $A \subseteq \omega \times {}^\omega\omega$ and $\tau \in {}^\omega\omega$ be a fan such that $\forall\alpha[\alpha\in\tau \rightleftarrows \forall n[\alpha^n \in \tau]]$, and $\forall\alpha\in\tau\, \exists n[A(n,\alpha^n)]$.
Then: $\exists n\, \forall\alpha\in\tau\, [A(n,\alpha)]$.

Proof: The water is colder than we thought. But never mind.

Suppose: $\forall\alpha\in\tau\, \exists n[A(n,\alpha^n)]$.
Then, according to GCP (cf. 1.9):

$$\forall\alpha\in\tau\, \exists n\, \exists m\, \forall\beta\in\tau\, [\bar\beta m = \bar\alpha m \to A(n,\beta^n)].$$

Therefore: $\forall\alpha\in\tau\, \exists n\, \exists m\, \forall\beta\in\tau\, [\overline{\beta^n} m = \overline{\alpha^n} m \to A(n,\beta^n)]$.

We define a subset A^* of $\omega\times\omega$ by:

For all $n\in\omega$, $a\in\omega$:

$$A^*(n,a) \;:=\; \forall\alpha\in a\, [\alpha\in\tau \to A(n,\alpha)]$$

Observe that: $\forall\alpha\in\tau\, \exists n\, \exists m [A^*(n, \overline{\alpha^n} m)]$
Using 15.2.0, we determine $M\in\omega$ such that:

$$\forall\alpha\in\tau\, \exists n\, \exists m \le M\, [A^*(n, \overline{\alpha^n} m)],$$

and we remark that, now: $\forall\alpha\in\tau\, \exists n\, [A^*(n, \overline{\alpha^n} M)]$

We define a subset A^{**} of $\omega\times{}^\omega\omega$ by:

For all $n\in\omega$, $\alpha\in{}^\omega\omega$

$$A^{**}(n,\alpha) \;:=\; A^*(n, \bar\alpha M)$$

Observe that: $\forall\alpha\in\tau\, \exists n[A^{**}(n,\alpha^n)]$
We now consider $S_M := \{a \mid lg(a) = M \wedge \tau(a) = 0\}$

As τ is a fan, S_M is a finite set.
To each $a \in S_M$ we determine a sequence $\alpha_a \in {}^\omega\omega$ such that:

$$\overline{\alpha}_a M = a \quad \wedge \quad \alpha_a \in \tau.$$

We apply lemma 15.3.2 and find $n \in \omega$ such that

$$\forall a \in S_M [\ A^{**}(n, \alpha_a)].$$

Retranslating, we see that:

$$\forall a \in S_M [\ A^*(n, \overline{\alpha}_a M)], \quad \text{i.e.:} \ \forall a \in S_M [A^*(n,a)]$$

Therefore: $\forall a \in S_M [\ \forall \alpha \in a [\alpha \in \tau \to A(n,\alpha)]]$
and: $\forall \alpha \in \tau [\ A(n,\alpha)]$.

⊠

15.4 We will prove a converse to theorem 15.3.3.

We first treat the reader to a small technicality.

Let $\tau \in {}^\omega\omega$ be a spread, which fulfils the condition: $\forall \alpha [\alpha \in \tau \rightleftarrows \forall n [\alpha^n \in \tau]]$
Let us define, as in 9.0, for each $a \in \omega$, a decidable subset K_a^τ of ω by:

$$K_a^\tau := \{ n \mid n \in \omega \mid \tau(a * \langle n \rangle) = 0 \}$$

We claim that $\forall a [\ \tau(a) = 0 \to K_a^\tau = K_{\langle\rangle}^\tau\ ]$.

To justify this claim, we reflect on the coding function (cf. 1.2)
Remark that: $\forall a [\ a > 0 \to \exists n \exists b [\ b < a \wedge a = \langle n \rangle * b]]$
Therefore: $\forall a [(a > 0 \wedge \tau(a) = 0) \to \exists b [\ b < a \wedge \tau(b) = 0 \wedge K_a^\tau = K_b^\tau\ ]]$.

Henceforth, if $\tau \in {}^\omega\omega$ is a spread such that: $\forall \alpha [\alpha \in \tau \rightleftarrows \forall n [\alpha^n \in \tau]]$,
we write $K^\tau := K_{\langle\rangle}^\tau$
τ may be thought of as the set ${}^\omega(K^\tau)$
τ is a fan if and only if K^τ is a finite set of natural numbers.

Next, we take a look of something which <u>almost</u> is a fan:
consider a spread $\tau \in {}^\omega\omega$ such that: $\forall \alpha [\alpha \in \tau \rightleftarrows \forall n [\alpha^n \in \tau]]$ and
$K_\tau = \{0, k\}$ where k is a volatile number (cf. the discussion following on theorem 11.10), and a far one.

It is reckless to assert that τ is a fan, this comes down to: $\exists n [n = k]$

It also is dangerous to claim that τ fulfils the fan theorem (cf. 15.2), for, as $\forall \alpha \in \tau \exists m [\ \alpha(0) = m]$, the fan theorem would imply
$\exists n \forall \alpha \in \tau [\ \alpha(0) \le n\ ]$, i.e.: $\exists n [\ k \le n]$.

Finally, we advise the reader against preaching that τ fulfils

the crazy principle CRP_τ.
We have our reasons for doing so:
We define a subset A of $\omega \times {}^\omega\omega$ by:

For all $n \in \omega$, $\alpha \in {}^\omega\omega$:

$$A(n,\alpha) := \alpha(0)=0 \vee n = k.$$

We claim that: $\forall\alpha\in\tau\,\exists n\,[\,A(n,\alpha^n)]$.

Let $\alpha\in\tau$ and consider $\alpha^0(0)$.

- If $\alpha^0(0)=0$, then: $A(0,\alpha^0)$.
- If $\alpha^0(0)\neq 0$, then: $\exists n[n=k]$ and: $\exists n[n=k \wedge A(n,\alpha^n)]$.

In any case, therefore: $\exists n[A(n,\alpha^n)]$.

Applying CRP_τ, we would find: $\exists n\,\forall\alpha\in\tau\,[\,A(n,\alpha)]$.

If $\forall\alpha\in\tau\,[\,A(0,\alpha)]$, then k is a hallucination.
If $\exists n[\,n\neq 0 \wedge \forall\alpha\in\tau[A(n,\alpha)]]$, then $\exists n\,[\,n=k]$, i.e.:
k has been caught.

Both assertions are overhasty.

It seems wise, therefore, not to claim: CRP_τ.

Taking to heart the lesson that this example forces upon us, we find:

15.4.0 <u>Lemma</u>: Let $\tau\in{}^\omega\omega$ be a spread such that $\forall\alpha[\alpha\in\tau \rightleftarrows \forall n[\alpha^n\in\tau]]$, and CRP_τ
Then: $\forall m\,[\;\forall l\,[\;\tau(\langle l\rangle)=0 \to l\leq m]\;\vee\;\exists l\,[\;l>m \wedge \tau(\langle l\rangle)=0]]$

(That is to say: the „choice set" $K^\tau = K^\tau_{\langle\rangle}$ is, in a sense, perspicuous. We may find out, for each $m\in\omega$, if there is a member of K^τ, greater than m, or not).

<u>Proof</u>: Let $m\in\omega$.
We define a subset A of $\omega\times{}^\omega\omega$ by:

For all $n\in\omega$, $\alpha\in{}^\omega\omega$:

$$A(n,\alpha) := \alpha(0)\leq m \vee (n>m \wedge \tau(\langle n\rangle)=0)$$

We claim that: $\forall\alpha\in\tau\,\exists n[\,A(n,\alpha^n)]$.

Let $\alpha\in\tau$ and consider $\alpha^0(0)$.

- If $\alpha^\circ(0) \le m$, then: $A(0,\alpha^\circ)$
- If $\alpha^\circ(0) > m$, then: $\tau(<\alpha^\circ(0)>) = 0$,
  therefore, putting $n := \alpha^\circ(0)$: $A(n,\alpha^n)$

In any case, therefore: $\exists n[A(n,\alpha^n)]$.

Applying CRP_τ, we calculate new such that $\forall\alpha\in\tau[A(n,\alpha)]$.
We then distinguish two cases.

(I) $n \le m$, then: $\forall\alpha\in\tau[\alpha(0) \le m]$ and:
$\forall\ell[\tau(<\ell>) = 0 \to \ell \le m]$

(II) $n > m$, then $\tau(<n>) = 0$ and:
$\exists\ell[\ell > m \wedge \tau(<\ell>) = 0]$.

⊠

Now that we have placed the ladder, we have no hesitation to pick the apple, and eat it:

15.4.1 <u>Theorem:</u> Let $\tau \in {}^\omega\omega$ be a spread such that $\forall\alpha[\alpha\in\tau \rightleftarrows \forall n[\alpha^n\in\tau]]$, and CRP_τ.
Then: $\exists m\,\forall\ell[\tau(<\ell>) = 0 \to \ell \le m]$, and, therefore: τ is a fan.

<u>Proof</u>: We define a subset A of $\omega \times {}^\omega\omega$ by:

For all $\alpha \in {}^\omega\omega$:
$$A(0,\alpha) := \exists\ell[\ell > \alpha(0) \wedge \tau(<\ell>) = 0]$$
For all $\alpha\in{}^\omega\omega$, for all $n \in {}^\omega\omega$, $n > 0$:
$$A(n,\alpha) := \forall\ell[\tau(<\ell>) = 0 \to \ell \le n].$$

We claim that: $\forall\alpha\in\tau\,\exists n[A(n,\alpha^n)]$.

Let $\alpha\in\tau$ and consider $\alpha^\circ(0)$
Applying lemma 15.4.0, we distinguish two cases:

(I) $\exists\ell[\ell > \alpha^\circ(0) \wedge \tau(<\ell>) = 0]$
Then: $A(0,\alpha^\circ)$

(II) $\forall\ell[\tau(<\ell>) = 0 \to \ell \le \alpha^\circ(0)]$
Then, putting $n := \alpha^\circ(0) + 1$, $A(n,\alpha^n)$.

Using CRP_τ, we calculate new such that $\forall\alpha\in\tau[A(n,\alpha)]$.
Again, there are two possibilities:

(I) $n = 0$

Then: $\forall \alpha \in \tau \, \exists \ell [\, \ell > \alpha(0) \;\wedge\; \tau(\langle \ell \rangle) = 0 \,]$

Therefore: $\forall n \in K^{\tau} \, \exists \ell [\, \ell > n \;\wedge\; n \in K^{\tau} \,]$.

As $\exists n [\, n \in K^{\tau} \,]$, this shows that K^{τ} is an infinite and decidable subset of ω.

Thus, there is no important difference between τ and ${}^{\omega}\omega$

According to 15.1, then, τ does not fulfil CRP_{τ}.

This case has to be excluded, and we are led to:

(II) $n > 0$

Then: $\forall \ell [\, \tau(\langle \ell \rangle) = 0 \rightarrow \ell \leq n \,]$

Therefore: K^{τ} is a finite set and τ is a fan.

We reached our goal.

$\boxtimes$

The theorems 15.3.3 and 15.4.1 complement each other and characterize the fans among the spreads τ that fulfil the condition $\forall \alpha [\alpha \in \tau \rightleftarrows \forall n [\alpha^n \in \tau]]$ as those spreads τ that obey the crazy principle CRP_{τ}.

This is a new occasion to throw the ranks of the classical army into disorder.

For, upon classical reading of the quantifiers, CRP_{τ} is valid for all spreads τ satisfying the above-mentioned condition, especially for ${}^{\omega}\omega$ itself, and the fan theorem is not.

15.5 In conclusion of this chapter we invite the reader for an exercise in the difficult art of counterpoint.

Do not the sweet melodies of chapter 7 and 9 deserve of a counterpart?

Consider $A_2 := \{\alpha \mid \forall n \exists m [\alpha^n(m) = 0]\}$, write $A_2 = \{\alpha \mid \exists \gamma \forall n [\alpha^n(\gamma(n)) = 0]\}$

and try $P_2 := \{\alpha \mid \forall \gamma \exists n [\alpha^n(\gamma(n)) = 0]\}$.

Consider $A_4 := \{\alpha \mid \forall n \exists m [A_2(\alpha^{n,m})]\}$, write $A_4 = \{\alpha \mid \exists \gamma \forall n [A_2(\alpha^{n,\gamma(n)})]\}$

and try $P_4 := \{\alpha \mid \forall \gamma \exists n [P_2(\alpha^{n,\gamma(n)})]\}$.

Or, write: $A_4 = \{\alpha \mid \exists \gamma \exists \delta \forall n \forall p [\alpha^{n,\gamma(n),p}(\delta(\langle n,p \rangle)) = 0]\}$ and

try $Q_4 := \{\alpha \mid \forall \gamma \forall \delta \exists n \exists p [\alpha^{n,\gamma(n),p}(\delta(\langle n,p \rangle)) = 0]\}$.

What is there to say on the behaviour of this kind of sets under the reducibility relation $\preceq$?

We did not explore this question and it does not look easy.

A^1_1 probably would like it to have some more sets to boss.

16 THE TRUTH ABOUT DETERMINACY

The axiom of determinacy is playing first fiddle in recent discussions on the subject of descriptive set theory. (cf. Moschovakis 1980)
At the next audition, we want to hear if it is able to play a constructive tune.
Our expectations are low.
Its style of playing is that of A. de Morgan, and of the two notes he produced, only one was right.

Music aside, it is clear that we do not have a method to decide which one of two players is to win a one-move-game, if the number of alternatives at this one move is infinite.
The axiom of determinacy makes this claim and ventures to extend it to games where there are infinitely many moves.

It seems that the statement of the axiom of determinacy : under such-and-such circumstances, either player I is bound to have a winning strategy, or player II is, expresses an idle hope.

We improve its chances by not taking it on its disjunctive face value, and testing instead the following hypothesis:

> Suppose player I has an answer to each strategy player II might follow.
> Then player I has a winning strategy.

Observe that, when the game is being played, player I does not know which strategy player II is following.
In calculating his moves, he has to reckon with all possibilities.

This formulation of the determinacy problem, is reminiscent of situations in daily life, like playing chess with a clever uncle.
Suppose that player I is able to win the game, if, at each move, he is allowed access to any finite information on the answers which player II will give, whatever be the outcome of this information. Then he should be able to find the right moves without asking questions as well.

The device of robbing classical statements of their disjunctive structure (and, thereby, of blatant falsity) by making the constructive contraposition of one of the two disjuncts into a condition from which the other one should follow, has been successfull in other cases.
The continuum hypothesis, to mention only one example, comes true, by this treatment. (cf. Gielen, de Swart and Veldman 1981).

Having made this first and sensible step, we have to face another disappointment: two-move-games still need not be determined.
To be sure, we proved that in the previous chapter.
We have seen, in 15.1, that the following step, in general, is not permitted:

CRP* Let $A \subseteq \omega \times \omega$
If $\forall\alpha \exists n [A(n, \alpha(n))]$, then $\exists n \forall m [A(n,m)]$

Here, α should be interpreted as a possible strategy for the second player

The first player, though having an answer to each possible strategy of his opponent, may not know how to move.

Thus, we are forced back to a more restricted situation, where, at each move, a player faces a finite choice. As the number of moves is still infinite, counting does not suffice and we have to think.

Now, the fan theorem comes to our aid and saves the honour of determinacy. The story of it will be told in this chapter.

We first reconsider the determinacy of finite games as we cannot trust A. de Morgan with this task.
We then go on to some not too difficult infinitary games which are enacted in the monotonous fans that we know from chapter 11.

Finally, we solve the problem for fans in general.

The conclusion of this chapter is, therefore, that, from a constructive point of view, determinacy is a compactness phenomenon.
(cf. Note 10 on page 217)

16.0 We first have to coin some terms

Let $\tau \in {}^\omega\omega$ be a spread, and S be a subset of ${}^\omega\omega$

Together, τ and S determine the following game $G(\tau, S)$

> Players I and II co-operate in producing some $\alpha \in \tau$
> Player I chooses $\alpha(0)$, then player II chooses $\alpha(1)$, then player I chooses $\alpha(2)$, etc.
> These choices are restricted by the condition that $\forall n[\tau(\bar{\alpha}n)=0]$
> (τ being a spread, the game is not frustrated at any finite stage).
> Player I wins the game if $\alpha \in S$.
> Player II may be said to win if $\alpha \notin S$: his interest is in preventing player II from winning.

We already had an occasion to use game-theoretic terminology, viz. in chapter 7, and we will build on what we have laid down there.

Let $\gamma \in {}^\omega\omega$. γ may be interpreted as a function defined on finite sequences of natural numbers, and therefore, as a strategy for either one of the two players I and II, which says him, at each possible position, how he has to move.
We introduce two subsets $\Sigma_I(\gamma)$ and $\Sigma_{II}(\gamma)$ of ω by:

$$\Sigma_I(\gamma) := \{a \mid \forall k\,[2k+1 \le lg(a)] \rightarrow a(2k) = \gamma(\bar{a}(2k))]$$

$$\Sigma_{II}(\gamma) := \{a \mid \forall k\,[2k+2 \le lg(a)] \rightarrow a(2k+1) = \gamma(\bar{a}(2k+1))]$$

$\Sigma_I(\gamma)$ is the set of positions which may be reached if player I keeps to the strategy given by γ.
$\Sigma_{II}(\gamma)$ is the set of positions which may be reached if player II keeps to the strategy given by γ.

Let $\tau \in {}^\omega\omega$ be a spread. We define $Strat_I(\tau)$, the set of strategies for player I which keep him within the spread τ, provided that his opponent does not leave it, either, by:

$$Strat_I(\tau) := \{\gamma \mid \forall a\,[((a \in \Sigma_I(\gamma) \wedge lg(a) \text{ is even} \wedge \tau(a)=0) \to \tau(a*\langle\gamma(a)\rangle)=0)$$
$$\wedge ((a \notin \Sigma_I(\gamma) \vee lg(a) \text{ is odd} \vee \tau(a) \neq 0) \to \gamma(a)=0)]\}.$$

The corresponding set $Strat_{II}(\tau)$ is defined by:

$$Strat_{II}(\tau) := \{\gamma \mid \forall a\,[((a \in \Sigma_{II}(\gamma) \wedge lg(a) \text{ is odd} \wedge \tau(a)=0) \to \tau(a*\langle\gamma(a)\rangle)=0)$$
$$\wedge ((a \notin \Sigma_{II}(\gamma) \vee lg(a) \text{ is even} \vee \tau(a) \neq 0) \to \gamma(a)=0)]\}.$$

It is easy to see that $Strat_I(\tau)$ and $Strat_{II}(\tau)$ themselves are spreads.

We also introduce the notion of „obeying to a strategy"
For all $\alpha \in {}^\omega\omega$, $\gamma \in {}^\omega\omega$ we define:

$$\alpha\, E_I\, \gamma \quad := \quad \forall n\,[\,\bar{\alpha}n \in \Sigma_I(\gamma)]$$

(i.e.: the sequence α is the result of some play, in which player I obeys to the strategy given by γ).

$$\alpha\, E_{II}\, \gamma \quad := \quad \forall n\,[\,\bar{\alpha}n \in \Sigma_{II}(\gamma)]$$

(i.e.: the sequence α is the result of some play in which player II obeys to the strategy given by γ).

The following property is to be the object of our investigations:

Let $\tau \in {}^\omega\omega$ be a spread and S be a subset of ${}^\omega\omega$.
We define: $Det(\tau, S)$ (i.e.: S is determined in τ), by:

$$Det(\tau, S) \;:=\; \forall\gamma \in Strat_{II}(\tau)\, \exists\alpha\,[\alpha E_{II}\gamma \wedge S(\alpha)] \to$$
$$\exists\gamma \in Strat_I(\tau)\, \forall\alpha\,[\,\alpha E_I \gamma \to S(\alpha)].$$

In the introduction to this chapter we have given some explanation, as to why we prefer this formulation above other possible ones.

16.1 Before losing ourselves in infinite games, we have a careful look at finite ones. We will treat them along similar, but shorter lines.

Let T be a finite subset of ω such that $\forall a \forall b\,[(a \in T \wedge a \subseteq b) \to b \in T]$
Let S be a subset of ω.
Together, T and S determine the following game $G(T,S)$:

Players I and II co-operate in producing some $a \in T$
Player I chooses $a(0)$, then player II chooses $a(1)$, then player I chooses $a(2)$, etc.
These choices are subject to the condition that, at each stage, the finite sequence produced until then, belongs to T.
The play ends, and ends only, if there is no continuation of the finite sequence within T, and player I wins, if the final finite sequence belongs to S; otherwise, player II, whose interest is in preventing player I from winning, may be said to win.

$a \in T$ is called T-complete if $\neg \exists n [a * \langle n \rangle \in T]$

Let $c \in \omega$. c may be interpreted as a finite sequence, and also as a function whose domain is a finite set of finite sequences, and therefore, as a strategy for either one of the two players in some game $G(T,S)$.
A natural number c is called a strategy for player I in T if:

$$lg(c) = \max(T) + 1 \ \wedge\ \forall a [(a \in T \wedge lg(a) \text{ is even} \wedge \exists n[a * \langle n \rangle \in T]) \to a * \langle c(a) \rangle \in T].$$

The set of all strategies for player I in T is a finite subset of ω, which is called: $Strat_I(T)$
Likewise, a natural number c is called a strategy for player II in T if:

$$lg(c) = \max(T) + 1 \ \wedge\ \forall a [(a \in T \wedge lg(a) \text{ is odd} \wedge \exists n[a * \langle n \rangle \in T]) \to a * \langle c(a) \rangle \in T].$$

The set of all strategies for player II in T is a finite subset of ω, which is called: $Strat_{II}(T)$.

Finally, we introduce the notion of „obeying to a strategy"
For all $a \in \omega$, $c \in \omega$, we define:

$$a \in_I c \ := \ \forall b [(a \subseteq b \wedge a \neq b \wedge lg(b) \text{ is even}) \to (b < lg(c) \wedge a \subseteq b * \langle c(b) \rangle)]$$

(i.e.: the finite sequence a is the result of some finite play, in which player I obeys to the strategy given by c).

$$a \in_{II} c \ := \ \forall b [(a \subseteq b \wedge a \neq b \wedge lg(b) \text{ is odd}) \to (b < lg(c) \wedge a \subseteq b * \langle c(b) \rangle)]$$

(i.e.: the finite sequence a is the result of some finite play, in which player II obeys to the strategy given by c).

We had to go through all these definitions for the sake of the following simple truth:

16.1.0 __Theorem:__ (Determinacy of finite games)

Let T be a finite subset of ω such that $\forall a \forall b\, [(a \in T \wedge a \subseteq b) \rightarrow b \in T]$

Let S be a subset of ω.

Suppose: $\forall c \in Strat_{II}(T)\, \exists a\, [\, a$ is T-complete $\wedge\, a \,\varepsilon_{II}\, c \,\wedge\, S(a)]$.

Then: $\exists c \in Strat_{I}(T)\, \forall a\, [(a$ is T-complete $\wedge\, a \,\varepsilon_{I}\, c) \rightarrow S(a)]$.

__Proof:__ The proof goes by induction on $\max\{\, lg(a) \mid a \in T\}$.

Determine $f \in \omega$ such that $lg(f) = \max(Strat_{II}(T)) + 1$ and:

$\forall c \in Strat_{II}(T)\, [\, f(c)$ is T-complete $\wedge\, f(c) \,\varepsilon_{II}\, c \,\wedge\, S(f(c))]$

A strategy c for player II may be divided into different parts, each part answering one of the possible first moves of the first player.

Let us consider the finite set $K^{T}_{\langle\rangle} := \{\, i \mid \langle i \rangle \in T\}$

As in 10.2, we define, for each $c \in \omega$ and $i \in \omega$, such that $i < lg(c)$,

c^{i} := the code number of the i-th subsequence of the finite sequence, coded by c.

We claim that: $\exists i \in K^{T}_{\langle\rangle}\, \forall c \in Strat_{II}(T)\, \exists d \in Strat_{II}(T)\, [\, c^{i} = d^{i} \wedge (f(d))(\langle\rangle) = i]$

(i.e.: there is a subtree of T such that, whatever player II is scheming on this subtree, player I knows how to answer him)

For, suppose not

(Remark that the statement which we want to prove is a decidable one)

We now determine, for each $i \in K^{T}_{\langle\rangle}$, $c_i \in Strat_{II}(T)$ such that: $\forall i \in K^{T}_{\langle\rangle}\, \forall d \in Strat_{II}(T)\, [\, c^{i} = d^{i} \rightarrow (f(d))(\langle\rangle) \neq i]$

It is clear that, building $c \in Strat_{II}(T)$ such that: $\forall i \in K^{T}_{\langle\rangle}\, [\, c^{i} = (c_i)^{i}]$, we find: $\forall i \in K^{T}_{\langle\rangle}\, [(f(c))(\langle\rangle) \neq i]$,

i.e.: a contradiction.

We determine $i_0 \in K^{T}_{\langle\rangle}$ such that $\forall c \in Strat_{II}(T)\, \exists d \in Strat_{II}(T) [c^{i_0} = d^{i_0} \wedge (f(d))(\langle\rangle) = i_0]$

i_0 is a safe first move for player I.

Let us consider, for each $j \in K^{T}_{\langle i_0 \rangle} := \{\, j \mid \langle i_0, j \rangle \in T\}$ the game

$G(T^*, S^*)$, where $T^* := {}^{\langle i_0, j\rangle}T := \{a \mid \langle i_0, j\rangle * a \in T\}$
and $S^* := {}^{\langle i_0, j\rangle}S := \{a \mid \langle i_0, j\rangle * a \in S\}$
(We relativize the game $G(T,S)$ to the position $\langle i_0, j\rangle$)
By our choice of i_0, we know that

$$\forall c \in Strat_{II}(T^*) \exists a [\, a \text{ is } T^*\text{-complete} \wedge a \in_{II} c \wedge S^*(a)]$$

and, as $\max\{lg(a) \mid a \in T^*\} < \max\{lg(a) \mid a \in T\}$, we may determine $c_j \in Strat_I(T^*)$ such that $\forall a[(a \text{ is } T^*\text{-complete} \wedge a \in_I c) \rightarrow S^*(a)]$
Putting all these things together, we find $c \in Strat_I(T)$ such that

$$c(\langle\rangle) := i_0$$

and, for all $j \in K^T_{\langle i_0\rangle}$, for all $a \in {}^{\langle i_0, j\rangle}T$:

$$c(\langle i_0, j\rangle * a) := c_j(a)$$

We easily observe that $\forall a[(a \text{ is } T\text{-complete} \wedge a \in_I c) \rightarrow S(a)]$

We should complete this proof by treating, separately, the case: $\max\{lg(a) \mid a \in T\} < 2$
But this will be left to the reader.

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Remark that, in theorem 16.1.0, we did not impose any condition on the set S. In case S is a decidable subset of ω, we may of course prove the theorem by classical joggling with quantifiers.
This is an easy method, but not very promising for the kind of problems we are studying.

16.2 A simple example of a spread is σ_{2mon} (cf. 11.0)

It is in this spread that we want to play our first infinite games. Strong nerves will help you, when playing in σ_{2mon}.
There is one decisive move, viz. mentioning the first 1 in the sequence α which the players I and II are working upon.
This move may be done by either one of the two players, as long as his opponent has not yet made it.
It is possible, in case both players like suspense, that nothing happens. We then are witnessing an endlessly protracted cold war, in which the first strike is, necessarily, the last one.

We will prove that, in σ_{2mon}, every game is determined.

Before doing so, we reflect, for a moment, on $Strat_{II}(\sigma_{2mon})$
This is another simple spread, not very different from σ_{2mon} itself: remark that: $\forall \gamma \in Strat_{II}(\sigma_{2mon}) \forall m \forall n[(\gamma(m) \neq 0 \wedge \gamma(n) \neq 0) \rightarrow m = n]$.

For elementary reasons, therefore, $Strat_{II}(\sigma_{2mon})$ satisfies the conclusion of the fan theorem (cf. 11.4)

We want to use the following corollary of the fan theorem (cf. 15.2):

Let $\tau \in {}^\omega\omega$ be a fan and $\delta \in {}^\omega\omega$ be such that: $\delta : \tau \to \omega$ (cf. 1.9)
Then: $A := \{ n \mid n \in \omega \mid \exists \alpha \in \tau [\, \delta(\alpha) = n]\}$ is a finite subset of ω,
especially: $\forall n [\, n \in A \vee \neg (n \in A)]$

We now redeem our word.

16.2.0 <u>Theorem</u>: Let S be a subset of ${}^\omega\omega$.
Then: $Det(\sigma_{2mon}, S)$.

<u>Proof</u>: Suppose: $\forall \gamma \in Strat_{II}(\sigma_{2mon}) \exists \alpha [\, \alpha E_{II} \gamma \wedge S(\alpha)]$
Using GAC_{11} (cf. 1.9), determine $\delta \in {}^\omega\omega$ such that $\delta : Strat_{II}(\sigma_{2mon}) \to {}^\omega\omega$ and: $\forall \gamma \in Strat_{II}(\sigma_{2mon}) [\, \delta | \gamma \, E_{II} \, \gamma \wedge S(\delta | \gamma)]$

We now describe a strategy ζ for player I.
What will be his first move?

He considers $A := \{ (\delta|\gamma)(0) \mid \gamma \in Strat_{II}(\sigma_{2mon})\}$
As we remarked just before theorem 16.2.0, this is a decidable subset of ω.
Player I distinguishes two possibilities:

If $1 \in A$, then $\zeta(\langle\rangle) := 1$
If $1 \notin A$, then $\zeta(\langle\rangle) := 0$

Now suppose that the game has been played, for some time, and players I and II have reached, in co-operation, the position $\bar{\underline{0}}2n = \langle 0, 0, \ldots 0 \rangle$ (2n times)
Player I still has a choice.

He considers $A := \{ \overline{(\delta|\gamma)}(2n+1) \mid \gamma \in Strat_{II}(\sigma_{2mon})\}$
This is, again, a decidable subset of ω
Player I discerns two possibilities:

If $\bar{\underline{0}}2n * \langle 1 \rangle \in A$, then $\zeta(\bar{\underline{0}}2n) := 1$
If $\bar{\underline{0}}2n * \langle 1 \rangle \notin A$, then $\zeta(\bar{\underline{0}}2n) := 0$

This completes the description of a strategy for player I, as, in all other cases, he has no choice.

We have to show that this strategy ζ which we described, is a winning strategy for player I, i.e.: that $\forall\alpha\in\sigma_{2mon}[\alpha E_I \zeta \to S(\alpha)]$

We will do this by proving: $\forall\alpha\in\sigma_{2mon}[\alpha E_I\zeta \to \exists\gamma\in Strat_{II}(\sigma_{2mon})[\alpha = \delta|\gamma]]$

(We have to reason in this careful way, as we do not know how complicated S is as a subset of ${}^\omega\omega$).

Let $\alpha\in\sigma_{2mon}$ and $\alpha E_I \zeta$.

First, we establish that:

$$\forall n[n=\mu p[\alpha(p)\neq 0] \to \exists\gamma\in Strat_{II}(\sigma_{2mon})[\forall m<n[\gamma(\overline{0}m)=0]\wedge \delta|\gamma=\alpha]].$$

Suppose $n\in\omega$ and $n=\mu p[\alpha(p)\neq 0]$.

We distinguish two possibilities:

Case (i): n is odd. Player II has made the decisive move.

As player I has been following the strategy ζ, we know that: $\forall\gamma\in Strat_{II}(\sigma_{2mon})[\overline{(\delta|\gamma)}n = \overline{0}n]$.

Let $\gamma_0\in Strat_{II}(\sigma_{2mon})$ be such that: $\alpha E_{II}\gamma_0$.

Remark: $\overline{(\delta|\gamma_0)}n = \overline{0}n$ and: $\delta|\gamma_0 E_{II}\gamma_0$.

Therefore: $(\delta|\gamma_0)(n)=\alpha(n)=1$ and: $\delta|\gamma_0=\alpha$.

Observe that: $\gamma_0(\overline{0}n)=1$ and: $\forall m<n[\gamma_0(\overline{0}m)=0]$.

Case (ii): n is even. Player I has made the decisive move.

As he is following the strategy ζ, he has done so for the reason that:

$$\exists\gamma\in Strat_{II}(\sigma_{2mon})[\overline{(\delta|\gamma)}(n+1) = \overline{0}n * \langle 1\rangle = \overline{\alpha}(n+1)]$$

We now determine $\gamma\in Strat_{II}(\sigma_{2mon})$ such that $\delta|\gamma=\alpha$, and observe that, as $\overline{\alpha}n=\overline{0}n$ and $\alpha E_{II}\gamma$, also: $\forall m<n[\gamma(\overline{0}m)=0]$

We now describe how to find, step-by-step, $\gamma\in Strat_{II}(\sigma_{2mon})$ such that $\alpha=\delta|\gamma$.

For all $n\in\omega$, we say:

- if $n<\mu p[\alpha(p)\neq 0]$, then: $\forall m<n[\gamma(\overline{0}m)=0]$
- if $n=\mu p[\alpha(p)\neq 0]$, then γ may be determined completely such that $\forall m<n[\gamma(\overline{0}m)=0]\wedge\delta|\gamma=\alpha$.

Observe that, for this γ: $\forall n[\overline{(\delta|\gamma)}n=\overline{\alpha}n]$, i.e.: $\delta|\gamma=\alpha$.

⊠

16.3 The reader may suspect that theorem 16.2.0 generalizes to the other monotonous spreads $\sigma_{3mon}, \sigma_{4mon}, \ldots$ (cf. 11.19) and it does so, indeed. Before proving it, we first establish a lemma which is also useful for other purposes.

Recall, how we defined, in 9.0, for all $\tau \in {}^\omega\omega$ and $a \in \omega$, a decidable subset K_a^τ of ω by: $K_a^\tau := \{n \mid n \in \omega \mid \tau(a * \langle n \rangle) = 0\}$

If $\tau \in {}^\omega\omega$ is a spread and $\tau(a) = 0$, the set K_a^τ is the set of natural numbers by which the finite sequence a may be continued within τ.

16.3.0 Lemma: Let $\tau \in {}^\omega\omega$ be a fan and S be a subset of ${}^\omega\omega$ such that:

$$\forall \gamma \in Strat_{II}(\tau)\ \exists \alpha \in \tau\ [\ \alpha \, E_{II}\, \gamma \ \wedge\ S(\alpha)]$$

Then: $\exists i \in K_{\langle\rangle}^\tau\ \forall \gamma \in Strat_{II}(\tau)\ \exists \alpha \in \tau\ [\ \alpha \, E_{II}\, \gamma \ \wedge\ S(\alpha) \ \wedge\ \alpha(0) = i\ ]$.

Proof: Using GAC_{11}, determine $\delta \in {}^\omega\omega$ such that $\delta: Strat_{II}(\tau) \to {}^\omega\omega$ and: $\forall \gamma \in Strat_{II}(\tau)\ [\ \delta|\gamma \in \tau \ \wedge\ \delta|\gamma \, E_{II}\, \gamma \ \wedge\ S(\delta|\gamma)]$.

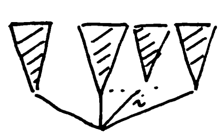
Remark that, as τ is a fan, $Strat_{II}(\tau)$ is also a fan.

Using the fan theorem (cf. 15.2), we calculate $m \in \omega$ such that:

$$\forall \gamma \in Strat_{II}(\tau)\ \forall \zeta \in Strat_{II}(\tau)\ [\ \bar{\gamma}m = \bar{\zeta}m \to (\delta|\gamma)(0) = (\delta|\zeta)(0)]$$

Let $\gamma \in Strat_{II}(\tau)$. γ is a strategy for player II in τ and naturally falls apart into different parts γ^i, $i \in K_{\langle\rangle}^\tau$, each part answering one of the first moves that are open to player I.

As $\bar{\gamma}m$ already is sufficient to decide about $(\delta|\gamma)(0)$, we may reason as in the proof of the determinacy of finite games, (theorem 16.1.0) and we claim that:



$$\exists i \in K_{\langle\rangle}^\tau\ \forall \gamma \in Strat_{II}(\tau)\ \exists \zeta \in Strat_{II}(\tau)\ [\ \overline{\gamma^i}m = \overline{\zeta^i}m \ \wedge (\delta|\zeta)(0) = i]$$

(i.e.: one of the first-level subfans of τ has the property that, whatever player II plans in this subfan, player I knows some answer to it.)

For, suppose not.

(Remark that the statement which we want to prove is a decidable one; we have to examine only $\{\bar{\gamma}m \mid \gamma \in Strat_{II}(\tau)\}$)

We now determine, for each $i \in K_{\langle\rangle}^\tau$, $\gamma_i \in Strat_{II}(\tau)$ such that: $\forall i \in K_{\langle\rangle}^\tau\ \forall \zeta \in Strat_{II}(\tau)\ [\ \overline{\gamma_i^i}m = \overline{\zeta^i}m \to (\delta|\zeta)(0) \neq i]$

It is clear that, building $\gamma \in Strat_{II}(\tau)$ such that:

$\forall i \in K^{\tau}_{<>}\ [\ \gamma^i = (\gamma_i)^i]$, we find: $\forall i \in K^{\tau}_{<>}\ [\ (\delta|\gamma)(0) \neq i]$
i.e.: a contradiction

We determine $i_0 \in K^{\tau}_{<>}$ such that $\forall \gamma \in Strat_{II}(\tau) \exists \zeta \in Strat_{II}(\tau)[\overline{\gamma^{i_0}}m = \overline{\zeta^{i_0}}m \wedge (\delta|\zeta)(0) = i_0]$
We observe that $\forall \gamma \in Strat_{II}(\tau)\ \exists \alpha \in \tau [\alpha E_{II} \gamma \ \wedge\ S(\alpha) \wedge \alpha(0) = i_0]$
and sigh our relief.

⊠

Let $\tau \in {}^{\omega}\omega$ be a spread and $a \in \omega$ be such that $\tau(a) = 0$
Then ${}^{a}\tau$ (cf. 1.2) is also a spread, consisting of those infinite sequences α for which $a * \alpha \in \tau$.
($a * \alpha$ is the infinite sequence which we get by concatenating the finite sequence a and the infinite sequence α, i.e.: $a * \alpha \in a\ \wedge\ \forall n [a * \alpha(lg(a) + n) = \alpha(n)]$)
The spread ${}^{a}\tau$ is the result of relativizing the spread τ to the position a.
Suppose, in addition, that S is a subset of ${}^{\omega}\omega$ such that:
$\forall \gamma \in Strat_{II}(\tau)\ \exists \alpha \in \tau [\ \alpha E_{II} \gamma\ \wedge\ S(\alpha)]$.
Let us call $a \in \omega$ such that $\tau(a) = 0$ a position which is S-safe-for-player-I
if: $lg(a)$ is even $\wedge\ \forall \gamma \in Strat_{II}({}^{a}\tau)\ \exists \alpha \in {}^{a}\tau [\ \alpha E_{II} \gamma\ \wedge\ S(a * \alpha)]$.
We have seen, in lemma 16.3.0, that:
$\exists i \in K^{\tau}_{<>}\ \forall j \in K^{\tau}_{<i>}\ [\ <i,j>$ is S-safe-for-player-I].
We easily generalize this to the following conclusion:
$\forall a [(\tau(a) = 0 \wedge a$ is S-safe-for-player-I$] \longrightarrow$
$\exists i \in K^{\tau}_{a}\ \forall j \in K^{\tau}_{a*<i>} [\ a*<i,j>$ is S-safe-for-player-I]].

16.4 <u>Theorem</u>: Let S be a subset of ${}^{\omega}\omega$ and $m \in \omega$, $m \geq 2$.
Then: $Det(\sigma_{m\,mon}, S)$.

<u>Proof</u>: Suppose: $\forall \gamma \in Strat_{II}(\sigma_{m\,mon})\ \exists \alpha \in \sigma_{m\,mon} [\ \alpha E_{II} \gamma\ \wedge\ S(\alpha)]$
Using GAC_{11} (cf. 1.9), determine $\delta \in {}^{\omega}\omega$ such that $\delta: Strat_{II}(\sigma_{m\,mon}) \rightarrow {}^{\omega}\omega$
and: $\forall \gamma \in Strat_{II}(\sigma_{m\,mon}) [\ \delta|\gamma \in \sigma_{m\,mon} \wedge \delta|\gamma E_{II} \gamma\ \wedge\ S(\delta|\gamma)]$.

Let $S^* := \{\ \alpha\ |\ \alpha \in \sigma_{m\,mon}\ |\ \exists \gamma \in Strat_{II}(\sigma_{m\,mon}) [\ \alpha = \delta|\gamma]\}$
Remark that $S^* \subseteq S$.
We advise player I to go, each time, to the rightmost S^*-safe-position, but we will refine this advice in a moment.
The proof that such a strategy will bring victory to player I, is by induction to m.

Suppose, therefore, that $m>2$ and that the theorem has been proved for all m', $m'<m$

(The case $m=2$ has been taken care of in theorem 16.2.0).

Let us make a start with the description of the strategy ζ, which we want to commend to player I.

Using the fan theorem (which is an elementary theorem, in the case of these monotonous fans, cf. theorem 11.4) we find $k\in\omega$ such that: $\forall\gamma\in Strat_{II}(\sigma_{mmon})\forall\zeta\in Strat_{II}(\sigma_{mmon})[\bar\gamma k=\bar\zeta k\to(\delta|\gamma)(0)=(\delta|\zeta)(0)]$

Following the proof of lemma 16.3.0, we distinguish two cases:

Case (i): $\exists i>0\,\forall\gamma\in Strat_{II}(\sigma_{mmon})\exists\zeta\in Strat_{II}(\sigma_{mmon})[\overline{\gamma^i}k=\overline{\zeta^i}k\wedge(\delta|\zeta)(0)=i]$

In this case, we choose such a number, say i_0, and we determine: $\zeta(<>):=i_0$

Remark that, after the answering move by player II, we reach a position $<i_0,j>$ such that

$$\forall\gamma\in Strat_{II}(^{<i_0,j>}\sigma_{mmon})\exists\alpha\in{}^{<i_0,j>}\sigma_{mmon}[\alpha E_{II}\gamma\wedge S(<i_0,j>*\alpha)]$$

Observe that $^{<i_0,j>}\sigma_{mmon}$ is isomorphic to some $\sigma_{m'mon}$, $m'<m$

Applying the induction hypothesis we know how to complete the construction of ζ as a winning strategy for player I.

Case (ii) $\neg$ Case (i)

Now, it seems that player I need not hesitate very long: we determine: $\zeta(<>):=0$

From lemma 16.3.0 we know that:

$$\forall\gamma\in Strat_{II}(\sigma_{mmon})\exists\zeta\in Strat_{II}(\sigma_{mmon})[\overline{\gamma^0}k=\overline{\zeta^0}k\wedge(\delta|\zeta)(0)=0]$$

It is clear that player I has made a sensible first move. But he does something more.

He is a very human being and he wants to know what player II would have done, should his (player I's) first move have been different.

Not catching player I's intentions, player II does not want to tell, estimating that, in any case, a bit of mystery will add to his reputation

Player I, therefore, has to make a conjecture

Chewing on the proof of lemma 16.3.0, he finds μ_0 in $Strat_{II}(\sigma_{mmon})$ such that:

$$\forall i>0\ \forall \gamma\in Strat_{II}(\sigma_{mmon})[\gamma^i=(\mu_0)^i \rightarrow (\delta|\gamma)(0)\neq i]$$

and, therefore:

$$\forall \gamma\in Strat_{II}(\sigma_{mmon})[\forall i>0[\gamma^i=(\mu_0)^i] \rightarrow (\delta|\gamma)(0)=0]$$

Player I now suspects that player II would have answered a possible move to $\langle i\rangle$, $i>0$, by following the strategy $(\mu_0)^i$.

In reality, however, his first move is to $\langle 0\rangle$.

After the answering move by player II we reach a position $\langle 0,j\rangle$.

> Let $\lambda\in Strat_{II}({}^{\langle 0,j\rangle}\sigma_{mmon})$
>
> Determine $\gamma\in Strat_{II}(\sigma_{mmon})$ such that:
>
> $\forall i>0[\gamma^i=(\mu_0)^i]\wedge \gamma(\langle\rangle)=j \wedge {}^{\langle 0,j\rangle}\gamma=\lambda$
>
> and remark: $(\delta|\gamma)\,E_{II}\,\gamma \wedge \delta|\gamma \in \langle 0,j\rangle$

Let us define:

$$S_{\langle 0,j\rangle} := \{\alpha \mid \alpha\in {}^{\langle 0,j\rangle}\sigma_{mmon} \mid \exists\gamma\in Strat_{II}(\sigma_{mmon})[$$
$$[\gamma(\langle\rangle)=j \wedge \forall i>0[\gamma^i=(\mu_0)^i] \wedge \delta|\gamma=\langle 0,j\rangle * \alpha]\}$$

We observe that:

$$\forall\lambda\in Strat_{II}({}^{\langle 0,j\rangle}\sigma_{mmon})\,\exists\alpha\in{}^{\langle 0,j\rangle}\sigma_{mmon}[\alpha E_{II}\lambda\wedge S_{\langle 0,j\rangle}(\alpha)]$$

We, of course, do understand what player I is aiming at, as we witnessed his wrestling in theorem 16.2.0.

He wants to ensure that, if α is a game played according to his strategy Z, we are able to find $\mu\in Strat_{II}(\sigma_{mmon})$, such that: $\alpha=\delta|\mu$.

He will be successful if he continues his strategy in the way he has begun it.

While making Z, he chooses, for each $n\in\omega$ such that $\overline{0}(2n+1)$ belongs to $\Sigma_I(Z)$, a strategy μ_n from $\Sigma_{II}(\sigma_{mmon})$, such that:

(i) $\overline{\underline{0}}(2n+1) \in \Sigma_{II}(\mu_n)$

(ii) As a conjecture about the strategy used by player II, μ_n extends μ_{n-1}, i.e.:
$\forall a[(\sigma_{mmon}(a)=0 \wedge lg(a)<2n \wedge \underline{0} \notin a) \to {}^a\mu_n = {}^a\mu_{n-1}]$

(iii) $\forall \gamma \in Strat_{II}(\sigma_{mmon})[\ \forall a[(\sigma_{mmon}(a)=0 \wedge lg(a) \le 2n+1 \wedge \underline{0} \notin a) \to$
$\to {}^a\gamma = {}^a\mu_n] \longrightarrow \overline{(\delta|\gamma)}(2n+1) = \overline{\underline{0}}(2n+1)]$

(Player I has made the move from $\overline{\underline{0}}2n$ to $\overline{\underline{0}}(2n+1)$, only because he had no other safe possibility. This is why he may determine μ_n such that (iii) holds.)

We define, for each $n \in \omega$, $j \in \omega$ such that: $\overline{\underline{0}}(2n+1) * \langle j \rangle \in \Sigma_I(\zeta)$:

$$S_{\overline{\underline{0}}(2n+1)*\langle j\rangle} := \{\alpha \mid \alpha \in {}^{\overline{\underline{0}}(2n+1)*\langle j\rangle}\sigma_{mmon} \mid \exists \gamma \in Strat_{II}(\sigma_{mmon})[$$
$$[\forall a[(\sigma_{mmon}(a)=0 \wedge lg(a) \le 2n+1 \wedge \underline{0} \notin a) \to {}^a\gamma = {}^a\mu_n]$$
$$\wedge\ \delta|\gamma = \overline{\underline{0}}(2n+1) * \langle j \rangle * \alpha]\}.$$

And we observe that:
$$\forall \lambda \in Strat_{II}({}^{\overline{\underline{0}}(2n+1)*\langle j\rangle}\sigma_{mmon})\,\exists \alpha \in {}^{\overline{\underline{0}}(2n+1)*\langle j\rangle}\sigma_{mmon}\,[\alpha E_{II} \lambda \wedge S_{\overline{\underline{0}}(2n+1)*\langle j\rangle}(\alpha)]$$

We now see how player I is going to win.
He is trying to leave the spine of σ_{mmon} as soon as possible.
While building, in co-operation with player II, a sequence $\alpha \in \sigma_{mmon}$, he conjectures more and more about the strategy $\mu \in Strat_{II}(\sigma_{mmon})$ which is to fulfil: $\alpha = \delta|\mu$.
When arriving at $\overline{\alpha}2n$ he conjectures the value of μ on at least all positions of length $\le 2n$.
As soon as the play, either by his own choice or by the command of player II, leaves the spine of σ_{mmon}, player I knows, using the induction hypothesis, how to complete α and μ.
In any case, both α and μ are growing, step-by-step, and, observing them, we establish: $\forall n[\ \overline{\alpha}n = \overline{(\delta|\mu)}n]$, i.e.: $\alpha = \delta|\mu$.

We abstain from a formal definition of ζ, and we guess that the reader will not deplore this decision.

$\boxtimes$

16.5 Player I, having lost his fear of player II, is brooding on tactics to be used in fans other than the monotonous fans, which, now that he has seen through them, do not attract him any more.

It does not seem easy to generalize the proof of theorem 16.4

Player I was successful in the monotonous fans, because, while playing a run α in such a fan, he was able to guess large parts of the strategy which player II appeared to follow.

As he only conjectured on the possible behaviour of player II in parts of the monotonous fan that they could not enter any more, during the present play, his dreams would never be disturbed by reality.

In general, however, he has to base his moves on a supposition concerning the future doings of player II, also at some positions which they still might come to pass, in the further course of the game.

Player I might be mistaken, therefore, in his assumptions regarding player II, the more so, as player II will try to thwart his expectations.

Thus, we have to go a new way.
Happily, we learnt a lesson from the classical adventures of the axiom of determinacy.

We first try to prove it, in case the payoff-set S is rather simple (in the sense of the hyperarithmetical hierarchy).

In this section, we will come ahead with Gale and Stewart 1953, and prove, for fans in general, the determinacy of open and of closed sets.

16.5.0 Theorem: Let $\tau \in {}^\omega\omega$ be a fan and S be a subset of ${}^\omega\omega$ such that $S \in \Sigma^0_1$.
Then: $Det(\tau, S)$.

Proof: Using theorem 6.2, determine a decidable subset A of ω such that:

$\forall\alpha[S(\alpha) \rightleftarrows \exists m[\bar{\alpha}m \in A]]$.

Suppose that: $\forall\gamma \in Strat_{II}(\tau)\,\exists\alpha\in\tau[\alpha E_{II}\gamma \wedge S(\alpha)]$.

Then: $\forall\gamma\in Strat_{II}(\tau)\,\exists\alpha\in\tau\,\exists m[\alpha E_{II}\gamma \wedge \bar{\alpha}m \in A]$:

Remark, as in the proof of lemma 16.3.0, that, as τ is a fan, $Strat_{II}(\tau)$ is also a fan.

Using the fan theorem (cf. 15.2.0) we calculate $M \in \omega$ such that:

$$\forall\gamma\in strat_{II}(\tau)\,\exists\alpha\in\tau\,\exists m[\alpha E_{II}\gamma \wedge \bar{\alpha}m \in A \wedge m \le M]$$

We define: $T := \{a \mid \tau(a) = 0 \wedge lg(a) \le M\}$ and:

$$A^* := \{a \mid \exists b[a \subseteq b \wedge b \in A]\}$$

We observe: $\forall c \in Strat_{II}(T)\,\exists a[a \text{ is } T\text{-complete} \wedge a E_{II} c \wedge a \in A^*]$

The finite game $G(T, A^*)$ is determined, according to theorem 16.1.0, and we calculate $c \in Strat_I(T)$ such that

$\forall a[(a \text{ is } T\text{-complete} \wedge a \mathrel{E_I} c) \rightarrow a \in A^*]$

Remark that for all $\gamma \in Strat_I(\tau)$ which agree with c on T:

$\forall \alpha \in \tau[\alpha \mathrel{E_I} \gamma \rightarrow \bar{\alpha}m \in A^*]$, and: $\forall \alpha \in \tau[\alpha \mathrel{E_I} \gamma \rightarrow S(\alpha)]$.

⊠

16.5.1 Theorem: Let $\tau \in {}^{\omega}\omega$ be a fan and S be a subset of ${}^{\omega}\omega$ such that $S \in \Pi^0_1$. Then: $Det(\tau, S)$.

Proof: Using theorem 6.7, determine a decidable subset A of ω such that

$\forall \alpha[S(\alpha) \rightleftarrows \forall m[\bar{\alpha}m \in A]]$.

Suppose that: $\forall \gamma \in Strat_{II}(\tau)\, \exists \alpha \in \tau[\alpha \mathrel{E_{II}} \gamma \wedge S(\alpha)]$

Let us call $a \in \omega$ such that $\tau(a) = 0$ a position which is S-safe-for-player I, if: $lg(a)$ is even $\wedge\ \forall \gamma \in Strat_{II}({}^{a}\tau)\, \exists \alpha \in {}^{a}\tau[\alpha \mathrel{E_{II}} \gamma \wedge S(a * \alpha)]$.

Using lemma 16.3.0 and the subsequent discussion, we find $\gamma \in Strat_I(\tau)$ such that $\forall a[\tau(a) = 0 \wedge lg(a)$ is even $\wedge\ a \in \Sigma_I(\gamma)) \rightarrow a$ is S-safe-for-player-I$]$

Remark that: $\forall a[(\tau(a) = 0 \wedge a$ is S-safe-for-player-I$) \rightarrow \forall b[a \subseteq b \rightarrow b \in A]]$

and, therefore: $\forall \alpha \in \tau[\alpha \mathrel{E_I} \gamma \rightarrow \forall m[\bar{\alpha}m \in A]]$ and:

$\forall \alpha \in \tau[\alpha \mathrel{E_I} \gamma \rightarrow S(\alpha)]$.

⊠

16.6 The gods are smiling upon us, at our next undertaking. The determinacy of Π^0_2- and Σ^0_2-sets has to be conquered, now.

16.6.0 Theorem: Let $\tau \in {}^{\omega}\omega$ be a fan and S be a subset of ${}^{\omega}\omega$ such that $S \in \Pi^0_2$. Then: $Det(\tau, S)$.

Proof: Using theorems 6.12 and 6.2, we determine a sequence $A_0, A_1, \ldots$ of decidable subsets of ω such that: $\forall \alpha[S(\alpha) \rightleftarrows \forall n \exists m[\bar{\alpha}m \in A_n]]$

Suppose that: $\forall \gamma \in Strat_{II}(\tau)\, \exists \alpha \in \tau[\alpha \mathrel{E_{II}} \gamma \wedge S(\alpha)]$.

Using GAC_{11}, we determine $\delta \in {}^{\omega}\omega$ such that: $\delta: Strat_{II}(\tau) \rightarrow {}^{\omega}\omega$ and $\forall \gamma \in Strat_{II}(\tau)[\delta|\gamma \in \tau \wedge \delta|\gamma \mathrel{E_{II}} \gamma \wedge S(\delta|\gamma)]$.

As τ is a fan, $Strat_{II}(\tau)$ is also a fan, and, applying the fan theorem (cf. 15.2) we determine a sequence $m_0, m_1, \ldots$ of natural

numbers such that $\forall n \forall \gamma \in Strat_{II}(\tau) \exists m [m \le m_n \wedge \overline{(\delta|\gamma)}m \in A_n]$
Next, we define a subset S^* of ${}^{\omega}\omega$ by:

$$S^* := \{\alpha \mid \forall n \exists m [m \le m_n \wedge \bar{\alpha}m \in A_n]\}.$$

We observe that $S^* \subseteq S$ and that. $\forall \gamma \in Strat_{II}(\tau) \exists \alpha \in \tau [\alpha E_{II} \gamma \wedge S^*(\alpha)]$.

As in the proof of theorem 16.5.1 we find $\gamma \in Strat_I(\tau)$ such that $\forall a [(\tau(a) = 0 \wedge lg(a)$ is even $\wedge\ a \in \Sigma_I(\gamma)) \to a$ is S^*-safe-for-player-I$]$.
Recall that $a \in \omega$ such that $\tau(a)=0$ is called S^*-safe-for-player-I if: $lg(a)$ is even $\wedge\ \forall \gamma \in Strat_{II}({}^{a}\tau) \exists \alpha \in {}^{a}\tau [\alpha E_{II} \gamma \wedge S^*(a * \alpha)]$.
Remark that:
$\forall n [\forall a [(\tau(a) = 0 \wedge a$ is S^*-safe-for-player-I $\wedge\ lg(a) \ge m_n) \to \exists b [a \le b \wedge b \in A_n]]]$
Therefore: $\forall \alpha \in \tau [\alpha E_I \gamma \to \forall n \exists m [m \le m_n \wedge \bar{\alpha}m \in A_n]]$.
and: $\forall \alpha \in \tau [\alpha E_I \gamma \to S(\alpha)]$.

We might have concluded the proof also by perceiving that $S^* \in \Pi^0_1$. and then referring to theorem 16.5.1.

⊠

The proof of the determinacy of Σ^0_2-sets will be in two steps.
First, we make a remark which improves on lemma 16.3.0.

Let $\tau \in {}^{\omega}\omega$ be a spread and S be a subset of ${}^{\omega}\omega$.
We define a subset $W_\tau(S)$ of ω by:

$$W_\tau(S) := \{a \mid \tau(a) = 0 \wedge lg(a) \text{ is even} \wedge \forall \gamma \in Strat_{II}({}^{a}\tau) \exists \alpha \in {}^{a}\tau [\alpha E_{II} \gamma \wedge S(a * \alpha)]\}$$

$W_\tau(S)$ is the set of all positions in τ which are of even length and S-safe-for-player-I.

16.6.1 <u>Lemma</u>: Let $\tau \in {}^{\omega}\omega$ be a fan and $S_0, S_1, \ldots$ be a sequence of subsets of ${}^{\omega}\omega$ such that: $\forall \gamma \in Strat_{II}(\tau) \exists \alpha \in \tau [\alpha E_{II} \gamma \wedge \exists n [S_n(\alpha)]]$.
Then: $\forall \gamma \in Strat_{II}(\tau) \exists a [\tau(a) = 0 \wedge a \in \Sigma_{II}(\gamma) \wedge \exists n [a \in W_\tau(S_n)]]$.

<u>Proof</u>: Using GAC_{11}, determine $\delta \in {}^{\omega}\omega$ such that $\delta: Strat_{II}(\tau) \to {}^{\omega}\omega$ and $\forall \gamma \in Strat_{II}(\tau) [\delta|\gamma \in \tau \wedge \delta|\gamma E_{II} \gamma \wedge \exists n [S_n(\delta|\gamma)]]$
Remark that, as τ is a fan, $Strat_{II}(\tau)$ is also a fan.
Let $\gamma \in Strat_{II}(\tau)$.
Using the fan theorem, (cf. 15.2) we calculate $m \in \omega$, $n \in \omega$ such that:

$$\forall \zeta \in Strat_{II}(\tau) [\bar{\gamma}m = \bar{\zeta}m \to S_n(\delta|\zeta)]$$

We consider $\bar{\gamma}m$.

$\bar{\gamma}m$ is a finite initial part of a strategy for player II.

We define a subset B of ω by:

$$B := \{ a \mid \tau(a)=0 \wedge lg(a) \text{ is even } \wedge m \leq a$$
$$\wedge \forall b[(a \subseteq b \wedge a \neq b \wedge lg(b) \text{ is even}) \rightarrow (b<m \wedge a \subseteq b * \langle \gamma(b) \rangle)]\}.$$

(When arriving at a position in B, player II has to make up his mind, because, from now on, his moves are not determined any more by $\bar{\gamma}m$).

When we choose, for any member a of the finite set B, a strategy ζ_a in $Strat_{II}({}^a\tau)$, there exists exactly one strategy ζ in $Strat_{II}(\tau)$ such that $\bar{\zeta}m = \bar{\gamma}m \wedge \forall a \in B[{}^a\zeta = \zeta_a]$.

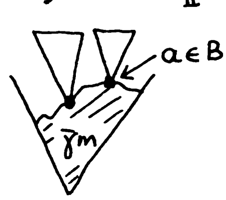
Remark that: $\forall \zeta \in Strat_{II}(\tau)[\bar{\zeta}m = \bar{\gamma}m \rightarrow \exists a \in B[\delta|\zeta \in a]]$.

Using the fan theorem, we calculate $p \in \omega$ such that $p > m$ and:

$\forall \zeta \in Strat_{II}(\tau) \forall \eta \in Strat_{II}(\tau)[(\bar{\zeta}m = \bar{\eta}m = \bar{\gamma}m \wedge \bar{\zeta}p = \bar{\gamma}p) \rightarrow \forall a \in B[\delta|\zeta \in a \rightleftarrows \delta|\eta \in a]]$

(i.e.: for any $\zeta \in Strat_{II}(\tau)$ such that $\bar{\zeta}m = \bar{\gamma}m$, it is sufficient to know $\bar{\zeta}p$, in order to decide which member of B $\delta|\zeta$ will pass through).

Let $\zeta \in Strat_{II}(\tau)$ such that $\bar{\zeta}m = \bar{\gamma}m$



ζ naturally falls apart into different parts ${}^a\zeta$, $a \in B$, each one representing a continuation of $\bar{\gamma}m$ from the position in B to which player I likes to go.

Reasoning exactly as in the proof of lemma 16.3.0, we conclude:

$$\exists a \in B \, \forall \zeta \in Strat_{II}(\tau)[\bar{\zeta}m = \bar{\gamma}m \rightarrow \exists \eta \in Strat_{II}(\tau)[\bar{\eta}m = \bar{\gamma}m \wedge$$
$$\wedge \overline{{}^a\eta}p = \overline{{}^a\zeta}p \wedge \delta|\eta \in a].$$

Calculating such a number, a, we observe:

$$\forall \zeta \in Strat_{II}({}^a\tau) \cdot \exists \alpha \in {}^a\tau[\alpha \in_{II} \zeta \wedge S(a * \alpha)]$$

i.e.: $a \in W_\tau(S_n)$.

⊠

16.6.2 <u>Theorem</u>: Let $\tau \in {}^{\omega}\omega$ be a fan and S be a subset of ${}^{\omega}\omega$ such that $S \in \Sigma^0_2$.
Then: $Det(\tau, S)$.

<u>Proof</u>: Using theorem 6.12 we determine a sequence $S_0, S_1, \dots$ of subsets of ${}^{\omega}\omega$ such that $\forall n [S_n \in \Pi^0_1]$ and $S = \bigcup_{n \in \omega} S_n$

Suppose that: $\forall \gamma \in Strat_{II}(\tau) \exists \alpha \in \tau [\alpha E_{II} \gamma \wedge S(\alpha)]$

i.e.: $\forall \gamma \in Strat_{II}(\tau) \exists \alpha \in \tau [\alpha E_{II} \gamma \wedge \exists n [S_n(\alpha)]]$.

Using lemma 16.6.1, we observe that:

$$\forall \gamma \in Strat_{II}(\tau) \exists a [\tau(a) = 0 \wedge a \in \Sigma_{II}(\gamma) \wedge \exists n [a \in W_\tau(S_n)]]$$

Reading through the proof of theorem 16.5.0 we see that we may use it to find $\gamma \in Strat_I(\tau)$ such that:

$$\forall \alpha \in \tau [\alpha E_I(\gamma) \rightarrow \exists m \exists n [\bar{\alpha}m \in W_\tau(S_n)]]$$

(We never used the fact that the subset A of ω, which occurs in that proof, is a decidable subset of ω)

Assuming the grateful role of player I, we obey to this strategy γ, and call the play that now develops: α.
Quietly, we make our moves, but when we come up to a position $\bar{\alpha}m$, such that $\exists n [\bar{\alpha}m \in W_\tau(S_n)]$, we ask some time for reflection.
We calculate $n \in \omega$ such that $\bar{\alpha}m \in W_\tau(S_n)$ and we observe that $\forall \gamma \in Strat_{II}({}^{\bar{\alpha}m}\tau) \exists \beta \in {}^{\bar{\alpha}m}\tau [\beta E_{II} \gamma \wedge S_n(\bar{\alpha}m * \beta)]$.
As $S_n \in \Pi^0_1$, we recall theorem 16.5.1 and find a strategy $\zeta \in Strat_I({}^{\bar{\alpha}m}\tau)$ such that $\forall \beta \in {}^{\bar{\alpha}m}\tau [\beta E_I \zeta \rightarrow S_n(\bar{\alpha}m * \beta)]$
It seems wise to be obedient, from now on, to this strategy ζ, and we do so.
Continuing the play, we are sure that α will belong to $S_n \subseteq S$, and this is happiness.

$\boxtimes$

16.7 The reader who is classically educated, will expect a long series of further adventures in determinacy.
But he will be disappointed.
A slight extension of the method used in theorem 16.6.2, solves the problem once and for all.

16.7.0 __Theorem__: Let $\tau \in {}^\omega\omega$ be a fan and S be a subset of ${}^\omega\omega$ such that $S \in \Sigma^1_1$.
Then: $\mathrm{Det}(\tau, S)$.

__Proof__: Using theorem 10.1 we determine a decidable subset A of ω such that $\forall\alpha[\, S(\alpha) \rightleftarrows \exists\beta\forall m[\langle\bar{\alpha}m, \bar{\beta}m\rangle \in A]]$.

Suppose that: $\forall\gamma \in \mathrm{Strat}_{\mathrm{II}}(\tau)\, \exists\alpha \in \tau\, [\alpha\, E_{\mathrm{II}}\, \gamma \wedge S(\alpha)]$.

We again imagine ourselves to be player I, for we do not like games that we do not win.

We define, for each $n \in \omega$, a subset S_n of ${}^\omega\omega$ by:

$$S_n := \{\alpha \mid \exists\beta\,\forall m\, [\,\langle\bar{\alpha}m, \bar{\beta}m\rangle \in A \;\wedge\; \beta(0) = n]\}.$$

and we observe that $S = \bigcup_{n\in\omega} S_n$.

Applying lemma 16.6.1, we remark that:

$$\forall\gamma \in \mathrm{Strat}_{\mathrm{II}}(\tau)\, \exists a\, [\tau(a) = 0 \;\wedge\; a \in \Sigma_{\mathrm{II}}(\gamma) \;\wedge\; \exists n[\, a \in W_\tau(S_n)]].$$

and, using the method of the proof of theorem 16.5.0, we find a strategy $\gamma_0 \in \mathrm{Strat}_{\mathrm{I}}(\tau)$ such that:

$$\forall\alpha \in \tau\, [\,\alpha\, E_{\mathrm{I}}\, \gamma_0 \;\rightarrow\; \exists m\, \exists n\, [\,\bar{\alpha}m \in W_\tau(S_n)]].$$

We now start the game, producing, in co-operation with player II, a play α, while keeping to the strategy γ_0.
When we come up to a position $\bar{\alpha}m_0$ such that $\exists n[\,\bar{\alpha}m_0 \in W_\tau(S_n)]$ we ask for a break.

> Remark that we may assume that $m_0 > 0$.
> This follows by a short reflection on the proof of lemma 16.6.1
> We may slightly modify the definition of the set B, mentioned there, to ensure that all numbers in B have a positive length.

We calculate n_0 such that $\bar{\alpha}m_0 \in W_\tau(S_{n_0})$ and we observe that

$$\tau(\bar{\alpha}m) = 0 \;\wedge\; m_0 \text{ is even} \;\wedge\; \forall\gamma \in \mathrm{Strat}_{\mathrm{II}}({}^{\bar{\alpha}m_0}\tau)\, \exists\zeta \in {}^{\bar{\alpha}m_0}\tau\, [\,\zeta\, E_{\mathrm{II}}\, \gamma \wedge S(\bar{\alpha}m_0 * \zeta)]$$

Therefore: $\forall\gamma \in \mathrm{Strat}_{\mathrm{II}}({}^{\bar{\alpha}m_0}\tau)\, \exists\zeta \in {}^{\bar{\alpha}m_0}\tau\, \exists\beta[\,\zeta\, E_{\mathrm{II}}\, \gamma \;\wedge$
$\wedge\, \forall k[\,\langle \overline{\bar{\alpha}m_0 * \zeta}\, k, \bar{\beta}k\rangle \in A] \;\wedge\; \beta(0) = n_0].$

Especially: $\langle\langle\alpha(0)\rangle, \langle n_0\rangle\rangle \in A$.

We get it into our head, to produce, while α develops, a sequence $\beta \in {}^\omega\omega$ such that: $\forall m[\,\langle\bar{\alpha}m, \bar{\beta}m\rangle \in A]$.

We start this project by putting: $\beta(0) := n_0$.

We define, for each $b \in \omega$, a subset $S(b)$ of ${}^\omega\omega$ by:

$$S(b) = \{ \alpha \mid \exists\beta \forall m [<\bar{\alpha}m, \bar{\beta}m> \in A \wedge \beta \in b>] \}$$

Remark that: $\forall n [S(<n>) = S_n]$ and: $\forall b [S(b) = \bigcup_{n\in\omega} S(b * <n>)]$

It will be clear that we have to repeat ourselves.

Arguing like we did before we started the play α, we find a strategy $\gamma_1 \in Strat_I(^{\bar{\alpha}m_0}\tau)$ such that:

$$\forall \zeta \in {}^{\bar{\alpha}m_0}\tau [\zeta \in_I \gamma_1 \rightarrow \exists m \exists n [\bar{\alpha}m_0 * \bar{\zeta}m \in W_\tau(S(<n_0> * <n>))]]$$

We continue the play α, keeping ourselves to this strategy γ_1, till we reach, in co-operation with player II, a position $\bar{\alpha}m_1$ such that $m_1 > m_0$ and: $\exists n [\bar{\alpha}m_1 \in W_\tau(S(<n_0> * <n>))]$

We calculate n_1 such that $\bar{\alpha}m_1 \in W_\tau(S(<n_0> * <n_1>))$, observe that $< <\alpha(0), \alpha(1)>, <\beta(0), n_1> > \in A$ and, confidently, put $\beta(1) := n_1$

And thus we go on.

While playing α, we find a sequence $m_0, b_0, m_1, b_1, \ldots$ of natural numbers such that:

(i) $\forall k [m_{k+1} > m_k \wedge lg(b_k) = k+1 \wedge b_{k+1} \subseteq b_k]$

(ii) $\forall k [\bar{\alpha}m_k \in W_\tau(S(b_k))]$

In order to move from $\bar{\alpha}m_k$ to $\bar{\alpha}m_{k+1}$, we use a strategy γ_{k+1} which we find by an application of lemma 16.6.1

Finally, we consider the sequence $\beta \in {}^\omega\omega$ that fulfils: $\forall k [\bar{\beta}(k+1) = b_k]$ and we observe that: $\forall n [<\bar{\alpha}n, \bar{\beta}n> \in A]$, i.e.: $S(\alpha)$

We hold a small reception, to celebrate our victory.

⊠

Actually, we have nothing left to wish for:

16.7.1 <u>Theorem</u>: (Determinacy of games in finitary spreads)

Let $\tau \in {}^\omega\omega$ be a fan and S be a subset of ${}^\omega\omega$.
Then: $Det(\tau, S)$.

Proof: Suppose that $\forall \gamma \in Strat_{II}(\tau)\, \exists \alpha \in \tau\, [\alpha\, E_{II}\, \gamma \wedge S(\alpha)]$.
Using GAC_{11}, determine $\delta \in {}^\omega\omega$ such that $\delta: Strat_{II}(\tau) \to {}^\omega\omega$ and $\forall \gamma \in Strat_{II}(\tau)\, [\, \delta|\gamma \in \tau \wedge \delta|\gamma\, E_{II}\, \gamma \wedge S(\delta|\gamma)]$.
Define a subset S^* of ${}^\omega\omega$ by:

$$S^* := \{\alpha \mid \exists \gamma \in Strat_{II}(\tau)\, [\, \delta|\gamma = \alpha]\}.$$

Observe that $S^* \subseteq S$ and: $\forall \gamma \in Strat_{II}(\tau)\, \exists \alpha \in \tau\, [\alpha\, E_{II}\, \tau \wedge S^*(\alpha)]$.
Moreover, S^* is strictly analytical and, therefore, belongs to Σ^1_1 (cf. 10.7)
Applying theorem 16.7.0 we find $\gamma \in Strat_I(\tau)$ such that

$$\forall \alpha \in \tau\, [\, \alpha\, E_I\, \gamma \to S^*(\alpha)].$$

This satisfies us.

$\boxtimes$

16.8 Theorem 16.7.1 admits of a minor extension.

Suppose that $\tau \in {}^\omega\omega$ is a spread which fulfils the condition:
$\forall a\, [\, (\tau(a) = 0 \wedge lg(a) \text{ is odd}) \to K^\tau_a \text{ is finite}]$.

τ need not be a fan.
When a game $G(\tau, S)$ is enacted in the spread τ, a move by player II is always the result of a choice among finitely many possibilities, whereas player I may be offered, now and then, an infinite list of alternatives to choose from.

It is easy to see that $Strat_{II}(\tau)$ is a finitary spread.

Assume that S is a subset of ${}^\omega\omega$ such that $\forall \gamma \in Strat_{II}(\tau)\, \exists \alpha \in \tau\, [\alpha\, E_{II}\, \gamma \wedge S(\alpha)]$
As usual, we determine, with the help of GAC_{11}, $\delta \in {}^\omega\omega$ such that:
$\delta: Strat_{II}(\tau) \to {}^\omega\omega$ and $\forall \gamma \in Strat_{II}(\tau)\, [\, \delta|\gamma \in \tau \wedge \delta|\gamma\, E_{II}\, \gamma \wedge S(\delta|\gamma)]$
Using the fan theorem, we observe that, for each $n \in \omega$:
$\{\, \overline{(\delta|\gamma)}n \mid \gamma \in Strat_{II}(\tau)\}$ is a finite set.
Therefore, the range of the function δ is but a limited part of the spread τ.
Working steadily, we find $\tau^* \in {}^\omega\omega$ such that:

(i) τ^* is a finitary spread.

(ii) τ^* is a subspread of τ

(iii) $\forall a\, [\, (\tau^*(a) = 0 \wedge lg(a) \text{ is odd}) \to K^{\tau^*}_a = K^\tau_a\,]$

(iv) $\forall \gamma \in Strat_{II}(\tau)\, [\, \delta|\gamma \in \tau^*]$

Player I is able to ensure that any play in τ is actually in τ^*, and, of course, he resolves to do so.

This restraint pays itself, because, now, theorem 16.7.1 applies, and player I will steer by the winning strategy that this theorem finds him.

Finally, we ask ourselves if the above-mentioned condition is necessary:

> Suppose that $\tau \in {}^{\omega}\omega$ is a spread such that: for all subsets S of ${}^{\omega}\omega$: $\mathrm{Det}(\tau, S)$
>
> > (If the huge quantifier „for all subsets S of ${}^{\omega}\omega$" worries you, you may safely replace it by: „for all $S \in \Sigma^1_1$" (cf. 16.7))
>
> Are we allowed to infer that $\forall a[(\tau(a)=0 \wedge lg(a) \text{ is odd}) \rightarrow K^{\tau}_{a} \text{ is finite}]$?

We are not, for example, in the extreme case that player I never has a choice, i.e.: if $\forall a[(\tau(a)=0 \wedge lg(a) \text{ is even}) \rightarrow K^{\tau}_{a} \text{ has exactly one element}]$.

But it seems reasonable to require from τ that:
$\forall a[\tau(a)=0 \rightarrow K^{\tau}_{a} \text{ has at least two elements}]$, so that there are no compulsory moves in τ.

Now, the conclusion in question may be justified, as follows:

We treat an exemplary case:

Suppose that: $0 \in K^{\tau}_{<>}$, $1 \in K^{\tau}_{<>}$
We prove that $K^{\tau}_{<>}$ is a finite set.

Let $\gamma \in \sigma_{2mon}$. We define a subset S_γ of ${}^{\omega}\omega$ by:

$S_\gamma := \{\alpha \mid (\alpha(0)=0 \wedge \forall p \leq \alpha(1)[\gamma(p)=0]) \vee (\alpha(0)=1 \wedge \exists p[\gamma(p) \neq 0])\}$

Remark that: $\forall \eta \in \mathrm{Strat}_{II}(\tau)\, \exists \alpha \in \tau\, [\alpha E_{II} \gamma \wedge S(\alpha)]$:

> Let $\eta \in \mathrm{Strat}_{II}(\tau)$
>
> Determine: $\alpha(0) := 0$ if $\forall p \leq \eta(<0>)[\gamma(p)=0]$
> $:= 1$ if not

Therefore, we may find $\zeta \in \mathrm{Strat}_I(\tau)$ such that: $\forall \alpha \in \tau[\alpha E_I \zeta \rightarrow S(\alpha)]$

Consider $\zeta(<>)$, and distinguish two cases:

(i) $\zeta(<>)=0$, then: $\forall q \in K^{\tau}_{<0>} \forall p[p \leq q \rightarrow \gamma(p)=0]$

(ii) $\zeta(<>)=1$, then: $\exists p[\gamma(p) \neq 0]$

We are able to make this decision for every $\gamma \in \sigma_{2mon}$.
Using GCP, we calculate $N \in \omega$ such that:

$\forall \gamma \in \sigma_{2mon}[\bar{\gamma}N = \bar{\underline{0}}N \rightarrow \forall q \in K^{\tau}_{<0>} \forall p[p \leq q \rightarrow \gamma(p)=0]$

(The alternative possibility immediately leads to a contradiction.)
We remark that: $\forall q \in K^{\tau}_{<0>}[q \leq N]$, and: $K^{\tau}_{<0>}$ is a finite set.

Observe that the sets S_γ, which occurred in this proof, are only Σ^0_2

The results of this section improve upon our refutation of CRP^* in 15.1 and may be related to the discussion in 15.4.

17 APPENDIX: STRANGE LIGHTS IN A DARK ALLEY.

We met with Ignorance, during our long travel, on more than one occasion. Living in the modern age, we should ask ourselves, if our failure to fight it down is not explained by the poorness of our equipment.
It might be that the axioms of intuitionistic analysis, as we paraded them in chapter 1, do not decide some of the questions that keep us awake. We should jump into metamathematics.

Much work has been done on the metamathematics of intuitionistic analysis, but, mostly, classical interpretations of intuitionistic formal systems were looked for, and found.
Therefore, the results of this discipline have to be welcomed with caution, approximately, like the findings of a Japanese professor in Netherlandic studies, on reading closely a Dutch poem.

Intuitionism should develop its own metamathematics, but, until now, perhaps because of its famous distrust of logic, it has done so only with great reluctance, and very partially.

Great things will not be done in this chapter.
We meditate, briefly, on an agonizing problem that we are carrying with us since chapter 10, viz., whether $A^1_1 \preceq E^1_1$
The classical devil is prepared to sell us a NO to this question if we only give up some very tiny part of our soul, it does not seem to matter which one.
We try not to listen to him.

The light that comes from adding semi-classical assumptions to the axioms of intuitionistic analysis, is artificial light, and personally, we prefer to stumble under the twinkling of the stars, although there are but few of them.

17.0 We first consider a generalized form of Markov's principle:

$$\text{GMP} \qquad \forall\alpha[\neg\neg\exists n[\alpha(n)=0] \rightarrow \exists n[\alpha(n)=0]].$$

We discussed GMP already, just after theorem 5.15, and have seen, in theorems 5.16-21 other possible formulations of it.

GMP also occurs as the last formula of Kleene and Vesley 1965 and is the subject of that book's section IV. 18.2

Many (weaker) versions of it, and their relation to intuitionistic arithmetic have been studied, mainly by classical methods, cf. Troelstra 1973.

We remarked, in 6.15, that acceptance of GMP would have saved us the trouble of establishing the arithmetical hierarchy the way we did it in chapter 7.

Even the proud hyperarithmetical hierarchy shrivels – when touched by GMP – into a rather obvious phenomenon.

17.0.0 <u>Remark</u>: If $A^1_1 \preceq E^1_1$, then $\neg$ GMP

<u>Proof</u>: Suppose $A^1_1 \preceq E^1_1$, i.e.: $A^1_1 \in \Sigma^1_1$

Then also: $\{\alpha \mid \forall\gamma \exists n [\alpha(<\bar{\alpha}n, \bar{\gamma}n>) \neq 0]\} \in \Sigma^1_1$

Using theorem 10.1 and AC_{00} we find $\beta \in {}^\omega\omega$ such that:

$$\{\alpha \mid \forall\gamma \exists n [\alpha(<\bar{\alpha}n, \bar{\gamma}n>) \neq 0]\} = \{\alpha \mid \exists\gamma \forall n [\beta(<\bar{\alpha}n, \bar{\gamma}n>) = 0]\}$$

Specializing, we find: $\forall\gamma \exists n [\beta(<\bar{\beta}n, \bar{\gamma}n>) \neq 0] \rightleftarrows \exists\gamma \forall n [\beta(<\bar{\beta}n, \bar{\gamma}n>) = 0]$

and, therefore: $\neg \forall\gamma \exists n [\beta(<\bar{\beta}n, \bar{\gamma}n>) \neq 0] \wedge \neg \exists\gamma \forall n [\beta(<\bar{\beta}n, \bar{\gamma}n>) = 0]$

Using GMP, we observe: $\neg\forall\gamma \neg\neg \exists n [\beta(<\bar{\beta}n, \bar{\gamma}n>) \neq 0]$, and:

$\neg\neg \exists\gamma \forall n [\beta(<\bar{\beta}n, \bar{\gamma}n>) = 0]$.

This is a contradiction.

$\boxtimes$

17.1 Another fancy, which may attract some half-hearted intuitionists, is the following scheme, proposed by <u>Kuroda</u> 1951. (cf. Note 11 on page 217)

KUR Let $P \subseteq \omega$.

If $\forall n [\neg\neg P(n)]$, then $\neg\neg \forall n [P(n)]$.

An immediate consequence of KUR is, that for every subset P of ω:
$\neg\neg \forall n [P(n) \vee \neg P(n)]$

17.1.0 <u>Remark</u>: If $A^1_1 \preceq E^1_1$, then $\neg$ KUR

<u>Proof</u>: Suppose: $A^1_1 \preceq E^1_1$.

Using theorem 10.1 we determine a decidable subset A of ω such that: $\forall\alpha [A^1_1(\alpha) \rightleftarrows \exists\gamma \forall n [<\bar{\alpha}n, \bar{\gamma}n> \in A]]$.

Now, we define a subset A^* of ω by:

For all $n \in \omega$:

$$n \in A^* \rightleftarrows \exists a \exists c [n = \langle\!\langle a, c \rangle\!\rangle \wedge lg(a) = lg(c) \wedge \exists\alpha \exists\gamma [\alpha \in a \wedge \gamma \in c \wedge \forall p [<\bar{\alpha}p, \bar{\gamma}p> \in A]]]$$

(Here, $\langle\!\langle\ \rangle\!\rangle$ is the function, introduced in 13.1, which fuses two finite sequences of equal length into one finite sequence of the same length, operating like its namesake, a pairing function on ${}^\omega\omega$).

Using Kuroda's scheme KUR, we observe that: $\neg\neg\forall n[n \in A^* \vee \neg(n \in A^*)]$

Let us assume, for the sake of argument, that: $\forall n[n \in A^* \vee \neg(n \in A^*)]$.

Remark that: $\forall n[n \in A^* \rightleftarrows \exists p[n * \langle p \rangle \in A^*]]$.

The set: $\{\alpha \mid \forall n[\bar{\alpha}n \in A^*]\}$ is, therefore, a subspread of ${}^{\omega}\omega$, and, as such, a strictly analytical subset of ${}^{\omega}\omega$. (cf. 10.7).

(Let $\beta \in {}^{\omega}\omega$ be a subspread of ${}^{\omega}\omega$ (cf. 1.9), i.e.: $\beta(\langle\rangle) = 0$ and: $\forall n[\beta(n) = 0 \rightleftarrows \exists p[\beta(n * \langle p \rangle) = 0]]$

Define $\delta \in {}^{\omega}\omega$ such that: $\mathrm{Fun}(\delta)$ and, for all $\alpha \in {}^{\omega}\omega$, for all $n \in \omega$:

$$(\delta|\alpha)(n) := \alpha(n) \quad \text{if } \beta(\overline{(\delta|\alpha)}n * \langle \alpha(n) \rangle) = 0.$$
$$:= \mu p[\beta(\overline{(\delta|\alpha)}n * \langle p \rangle) = 0], \quad \text{if not.}$$

Observe that $\beta = (\{\alpha \mid \forall n[\beta(\bar{\alpha}n) = 0]\} =) \mathrm{Ra}(\delta) (= \{\alpha \mid \exists\gamma[\alpha = \delta|\gamma]\})$ and that, therefore β is strictly analytical).

Observe, that, for all $\alpha \in {}^{\omega}\omega$:

$$A^1_1(\alpha) \rightleftarrows \exists\gamma\forall n[\langle \bar{\alpha}n, \bar{\gamma}n \rangle \in A]$$
$$\rightleftarrows \exists\gamma\forall n[\overline{\langle\!\langle \alpha, \gamma \rangle\!\rangle}n \in A^*]$$
$$\rightleftarrows \alpha \in \{\lambda|\zeta \mid \forall n[\bar{\zeta}n \in A^*]\}$$

(λ is the left-inverse of the pairing function $\langle\!\langle\ \rangle\!\rangle$ on ${}^{\omega}\omega$, cf. 14.0).

Therefore, A^1_1 is a strictly analytical subset of ${}^{\omega}\omega$, and we have seen that this is not true, in the discussion following after theorem 13.22 (cf. also: the remarks concerning Fun, just after theorem 10.12).

We conclude: $\neg\forall n[n \in A^* \vee \neg(n \in A^*)]$ and, thereby, bring shame upon Kuroda's schema KUR.

$\boxtimes$

Our thoughts go back to theorem 10.8, where we have seen that the assumption that all analytical subsets of ${}^{\omega}\omega$ are strictly analytical, leads to a contradiction.
If we assume that A^1_1 is analytical, we may add A^1_1, being an example of an analytical subset of ${}^{\omega}\omega$ which, surely, is not strictly analytical, to our collection of curiosities.

17.2 In 10.7 we mentioned that one may prove, using the restricted principle of Brouwer and Kripke, introduced in chapter 2, that every finitely defined,

analytical subset of ${}^{\omega}\omega$ is strictly analytical, indeed.
We need not be surprised, therefore, by the following

17.2.0 <u>Remark</u>: If $A^1_1 \preceq E^1_1$, then $\neg BK$. (?)

<u>Proof(?)</u>: There is (at least) one questionable step in this proof.

Suppose $A^1_1 \preceq E^1_1$

Using theorem 10.1 we determine a decidable subset A of ω such that $\forall\alpha [A^1_1(\alpha) \rightleftarrows \exists\gamma \forall n [\langle \bar{\alpha}n, \bar{\gamma}n \rangle \in A]]$

We like to assume, now, and this is the moot point, that A is a determinate subset of ω, arguing this, if urged, by saying that A^1_1 itself is a determinate subset of ${}^{\omega}\omega$.

Does not it sound reasonable that an object which is created to fulfil certain needs of other determinate objects, may be constructed in such a way that it is itself determinate?

As in the proof of remark 17.1.0 we define a subset A^* of ω by:

$A^* := \{n \mid \exists a \exists c [n = \langle\langle a, c \rangle\rangle \wedge lg(a) = lg(c) \wedge \exists\alpha\exists\gamma[\alpha \in a \wedge \gamma \in c \wedge \forall p[\langle \bar{\alpha}p, \bar{\gamma}p \rangle \in A]]]\}$

Like A itself, A^* is a determinate subset of ω, at least, we hope so.

Using BK and AC_{01}, we determine $\beta \in {}^{\omega}\omega$ such that:

$$\forall n [n \in A^* \rightleftarrows \exists m [\beta^n(m) = 0]]$$

We claim that the set $\{\alpha \mid \forall n [\bar{\alpha}n \in A^*]\}$ is a strictly analytical subset of ${}^{\omega}\omega$.

(Remark that $\forall n [n \in A^* \rightleftarrows \exists p [n * \langle p \rangle \in A^*]]$:

Define $\delta \in {}^{\omega}\omega$ such that: $Fun(\delta)$ and, for all $\alpha \in {}^{\omega}\omega$, for all $n \in \omega$:

$$\begin{aligned} (\delta|\alpha)(n) &:= (\lambda|\alpha)(n) && \text{if } \beta^{\overline{(\delta|\alpha)}n * \langle(\lambda|\alpha)(n)\rangle}((\rho|\alpha)(n)) = 0 \\ &:= p && \text{where fulfils: } \overline{(\delta|\alpha)}(n) * \langle p \rangle \in A^*, \\ & && \text{if not} \end{aligned}$$

Observe that $\forall\alpha [\forall n [\overline{(\delta|\alpha)}n \in A^*]]$

On the other hand: suppose $\alpha \in {}^{\omega}\omega$ and: $\forall n [\bar{\alpha}n \in A^*]$.

Determine a sequence $\gamma \in {}^{\omega}\omega$ such that $\forall n [\beta^{\bar{\alpha}(n+1)}(\gamma(n)) = 0]$

and remark: $\alpha = \delta | \langle\langle \alpha, \gamma \rangle\rangle$

We conclude, as in the proof of 17.1.0 that A^1_1 itself is a strictly analytical subset of ${}^{\omega}\omega$, and, as we know, it is not.

⊠

The argument in this proof, showing that, on the assumption of BK, every (finitely defined) analytical subset of ${}^\omega\omega$ is a strictly analytical subset of ${}^\omega\omega$, is due to John Burgess. (cf. Burgess 1980).

He used the axiom of Brouwer and Kripke in a more general form and did not restrict himself to finitely defined analytical subsets of ${}^\omega\omega$.

To be honest, we deny support to the conjecture, made in the course of this proof, that a construction made in behalf of determinate objects, may be expected to yield a determinate object.

This conjecture would extend to the sequences themselves which are claimed to exist by the axiom of Brouwer and Kripke.

But, given a determinate proposition $\mathfrak{A}$, the making of a sequence $\alpha \in {}^\omega\omega$ such that $\mathfrak{A} \rightleftarrows \exists n[\alpha(n)=0]$ requires an unbounded stretch of creative attention.

A similar remark has been made in Gielen, de Swart and Veldman 1981, section 3.3.

17.3 We remind the reader of the set S, introduced in 11.27: $S = \{\alpha \mid \exists \gamma[\gamma \in \sigma_2 \wedge \forall n[\alpha(\bar{\gamma}n)=0]]\}$
One of the problems we have in connection with S is the question whether $E_1 \preceq S$.

17.3.0 Remark: If $E_1 \preceq S$, then $\neg$ GMP

Proof: We may indulge in some sweet memories from chapter 11.

the fan theorem. (cf. the discussion after 11.3, and 15.2)

Assuming GMP we find, that for every decidable subset A of ω:

if $\forall \gamma \in \sigma_2 \neg\neg \exists n[A(\bar{\gamma}n)]$, then $\neg\neg \exists m \forall \gamma \in \sigma_2 \exists n[n \le m \wedge A(\bar{\gamma}n)]$

Repeating the argument, set forth in 11.5-7, we conclude that: $\neg(E_1 \preceq S)$.

$\boxtimes$

17.4 Probably, other theorems, of the same kind as 17.0.0 – 17.3.0, may be formulated and proved.
We are not interested in them.
In our ears, they sound like as many stanzas in an old ballad on lost and faraway classical truth.
If we surrender ourselves to these distressful thoughts, we may overlook theorems like those of chapter 7 and 9.

The axiom of Brouwer and Kripke keeps bad company, in this chapter. Sometimes, also this axiom seems the invention of a nasty child, wanting to make life easier than it is.

NOTES.

[1] (cf. pages 3, 51, 88)
We have to warn the reader: our terminology is somewhat confusing. In recursion theory, the „effective versions" of Borel-sets-of-finite-order, general Borel sets, and projective sets, are called arithmetical, hyperarithmetical and analytical sets, respectively. (It is not difficult to understand how these effective notions are made: for instance: a subset P of ${}^{\omega}\omega$ is effectively open if there exists a recursive function $\beta: \omega \rightarrow \omega$ such that $\forall\alpha[P(\alpha) \rightleftarrows \exists m[\beta(\bar{\alpha}m) = 0]]$, (cf. theorem 6.2 on page 45))
„Analytic sets" is the classical name for members of Σ^1_1. (cf. Moschovakis 1980, page 157 and notice the distinction between „analytic" and „analytical")
Our notions are not effective in the recursion-theoretic sense, and, perhaps, we would have done better in using the classical terminology.
On the other hand, our notions are not to be identified with the classical ones, either.

[2] (cf. page 9)
Remark that: $\forall\gamma[\gamma(\langle\rangle) = 0 \rightarrow (\mathrm{Fun}(\gamma) \rightleftarrows \mathrm{fun}(\gamma))]$

[3] (cf. page 14, section 2.3, page 89, theorem 10.8, page 175, theorem 14.9)
The axiom AC_{11} plays an important part only in theorems 10.8 and 14.9, and in many theorems of chapter 16.
By a change in the definition of „$\mathrm{Det}(\tau, S)$" in section 16.0 on page 191, similar to the one proposed for: „$A \leq B$" in section 2.3, we may reduce its role still further.

[4] (cf. page 52)
Remark that this constructive formulation of „Post's theorem" does not use negation.
We might also consider the question, if, for all subsets $P \subseteq {}^{\omega}\omega$, if $P \in \Sigma^0_1$ and $\mathrm{Neg}(P) \in \Sigma^0_1$, then $P \in \Sigma^0_1$.
This is a stronger statement than ours, and it is easily seen to be an enigma, i.e. equivalent to the generalized Markov principle GMP: $\forall\alpha[\neg\neg\exists n[\alpha(n) = 0] \rightarrow \exists n[\alpha(n) = 0]]$.
(cf. Luckhardt 1976).

[5] (cf. page 156)
This version of Brouwer's thesis avoids a difficulty which is touched upon in Kleene and Vesley 1965, sections 6.8 and 7.14
There is no apparent intuitive reason, why, in the bar theorem as it is formulated there, only effective predicates should be

considered, and, for this reason, our version might be preferable.

[6] (cf. page 159, theorem 13.2.2, and page 163, theorem 13.4.1)

The names „Souslin-Brouwer-theorem" and „Lusin-Brouwer-theorem" may be misleading.
Brouwer never proved these theorems, but the classical arguments are „rescued" by his bar theorem.
Souslin's theorem has been announced in Souslin 1917, and proofs may be found in Lusin and Sierpinski, 1918 and 1923.

The bar theorem may be found in Brouwer 1927, and is a central topic in the intuitionistic literature (cf. note [5])

[7] (cf. page 181)

We are reasoning rather quickly, at this place.
First, build $\delta \in {}^{\omega}\omega$ such that $\mathrm{Fun}(\delta)$ and $\forall \alpha [\delta | \alpha \in \beta]$ and $\forall \alpha \in \beta [\delta | \alpha = \alpha]$
This may be done by defining, for each $\alpha \in {}^{\omega}\omega$ and $n \in \omega$:

$$\begin{aligned} (\delta|\alpha)(n) &:= \alpha(n) \quad \text{if: } \beta(\overline{(\delta|\alpha)}(n) * \langle \alpha(n) \rangle) = 0 \\ &:= \mu m [\beta(\overline{(\delta|\alpha)}n * \langle m \rangle) = 0], \quad \text{otherwise} \end{aligned}$$

Remark: $\forall \alpha \exists n [A(\overline{(\delta|\alpha)}n)]$ and: $\forall \alpha \forall n [A(\overline{(\delta|\alpha)}n) \rightleftarrows (A(\bar{\alpha}n) \vee \beta(\bar{\alpha}n) \neq 0)]$.

[8] (cf. page 61)

Brouwer's ambivalent attitude towards $\aleph_1$ appears from Brouwer 1975, page 133, which, however, seems to contradict loc. cit., page 388, where he mentions: „die Spezies O der Ordinalzahlen."

[9] (cf. page 178)

In Kleene 1955, Kleene admits that he is, sometimes, standing on his head.

[10] (cf. page 190)

As a foundational problem, determinacy made its appearance in Mycielski 1964. Further references may be found in Moschovakis 1980.

[11] (cf. page 212).

A similar question has been discussed in van Dantzig 1942.

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SYNOPSIS

4 Some activities of disjunction and conjunction. 21

We introduce, for every subset P of ${}^{\omega}\omega$ and $n \in \omega$, subsets D^nP and C^nP, $Ex(P)$ and $Un(P)$ of ${}^{\omega}\omega$ by:

$$D^nP := \{\alpha \mid \exists q{<}n\,[P(\alpha^q)]\} \qquad Ex(P) := \{\alpha \mid \exists q\,[P(\alpha^q)]\}$$
$$C^nP := \{\alpha \mid \forall q{<}n\,[P(\alpha^q)]\} \qquad Un(P) := \{\alpha \mid \forall q\,[P(\alpha^q)]\}$$

We define, for all subsets P, Q of ${}^{\omega}\omega$: $P \prec Q := P \preceq Q \wedge \neg(Q \preceq P)$

We prove: $\forall n\,[D^nA_1 \prec D^{n+1}A_1]$ (theorem 4.6), $\neg(D^3A_1 \preceq Un(D^2A_1))$ (theorem 4.8), $\neg(D^2A_1) \preceq Un(E_1))$ (theorem 4.10) and:
$\forall n \forall m \forall p \forall q\,[C^{n+1}D^mA_1 \preceq C^qD^pA_1 \rightarrow m^{n+1} \leq p^q \wedge m \leq p]$
(theorems 4.15 and 4.18)

Theorem 4.20 provides us with an algorithm to decide which quadruples $\langle n+1, m, q, p\rangle$ satisfy: $C^{n+1}D^mA_1 \preceq C^qD^pA_1$

In order to solve this problem, we consider a wider class of subsets of ${}^{\omega}\omega$, viz. for each $m \in \omega$, reading m as a finite sequence of natural numbers:

$$(CD)_m A_1 := (CD)_{\langle m_0,\dots,m_t\rangle} A_1 := \{\alpha \mid D^{m_0}A_1(\alpha^0) \wedge \dots \wedge D^{m_t}A_1(\alpha^t)\}$$

In hindsight, some of the earlier theorems may be seen to follow from theorem 4.20

5 An aside on implication. 34

We introduce a sequence $I_0, I_1, \dots$ of subsets of ${}^{\omega}\omega$ by:
$I_0 := {}^{\omega}\omega$ and, for each $p \in \omega$: $I_{Sp} := \{\alpha \mid I_p(\alpha) \rightarrow A_1(\alpha^p)\}$

We prove: $\forall p\,[I_p \prec I_{Sp}]$ (theorem 5.6)

We introduce a sequence $J_0, J_1, \dots$ of subsets of ${}^{\omega}\omega$ by:
$J_0 := {}^{\omega}\omega$ and, for each $p \in \omega$. $J_{Sp} := \{\alpha \mid J_p(\alpha) \rightarrow E_1(\alpha^p)\}$

We prove: $\forall p\,[J_p \prec J_{Sp}]$ (theorem 5.10)

Some minor results (5.11-15) are added which try to locate subsets of ${}^{\omega}\omega$, built by means of implication, with respect to other ones.

Theorems 5.16-20 collect a number of so-called enigmas, i.e. statements equivalent to the generalized Markov principle:
$\forall \alpha\,[\neg\neg E_1(\alpha) \rightarrow E_1(\alpha)]$

6 Arithmetical sets introduced. 45

Starting from A_1 and E_1, we define a sequence $A_2, E_2, A_3, E_3, \dots$ of subsets of ${}^{\omega}\omega$ by: for all $n \in \omega$: $A_{Sn} = Un(E_n)$ and $E_{Sn} = Ex(A_n)$

We introduce classes Π^0_1, Σ^0_1, Π^0_2, Σ^0_2, ... of subsets of ${}^\omega\omega$ by:
for all $n \in \omega$, $n > 0$: $\Pi^0_n := \{P \mid P \subseteq {}^\omega\omega \mid P \preceq A_n\}$ and $\Sigma^0_n := \{P \mid P \subseteq {}^\omega\omega \mid P \preceq E_n\}$

These classes behave as one would expect; for instance:
Π^0_n (Σ^0_n) is closed under the operation of countable intersection (union).
Π^0_n (and similarly Σ^0_n) possesses a universal element; i.e. there exists a member $\mathcal{U}$ of Π^0_n such that $\Pi^0_n = \{\mathcal{U}_\beta \mid \beta \in {}^\omega\omega\}$ where $\mathcal{U}_\beta := \{\alpha \mid \mathcal{U}(\langle\alpha,\beta\rangle)\}$
($\langle\ \rangle$ denotes a suitable pairing function on ${}^\omega\omega$)

We easily find, by diagonalizing, a subset of ${}^\omega\omega$ which does not belong to Π^0_n, but this set cannot be said to belong to Σ^0_n (cf. 6.14)
Most of the results of this chapter conform with the results of classical descriptive set theory

We introduce $D := \{\alpha \mid \alpha(0) = 0\}$ and shortly discuss two questions:
in 6.15 if for all subsets P of ${}^\omega\omega$: $(P \preceq E_1 \wedge P \preceq A_1) \to P \preceq D$
in 6.16: do there exist subsets P of ${}^\omega\omega$ such that
$D \prec P \prec E_1$ or $D \prec P \prec A_1$?

7 The arithmetical hierarchy established. 54

We prove: $\forall n > 0\,[\text{ If } A_{Sn} \preceq E_{Sn}, \text{ then } E_n \preceq A_n]$ (lemma 7.1)
and: $\forall n > 0\,[\text{ If } E_{Sn} \preceq A_{Sn}, \text{ then } A_n \preceq E_n]$ (lemma 7.2)
The proofs extend the methods of chapter 3.

The arithmetical hierarchy theorem (theorem 7.3) follows easily:
$$\forall n > 0\,[\neg(A_n \preceq E_n) \wedge \neg(E_n \preceq A_n)]$$

8 Hyperarithmetical sets introduced. 61

We define the set HI\$ of hereditarily iterative stumps by transfinite induction: (every element of HI\$ is a (decidable) subset of ω and ω is identified with the set of finite sequences of natural numbers)

(i) $\{\langle\,\rangle\} \in \mathrm{HI\$}$

(ii) If $A_0, A_1, A_2, \ldots$ is a sequence of elements of HI\$, then A belongs to HI\$ where $A := \{\langle\,\rangle\} \cup \bigcup_{n,m \in \omega} \langle\langle\!\langle n,m\rangle\!\rangle\rangle * A_n$

(* denotes the operation of concatenation of finite sequences.
If $A \subseteq \omega$, then $n * A := \{n * m \mid m \in A\}$.
$\langle\!\langle\ \rangle\!\rangle$ denotes some pairing function on ω.
If $\sigma \in \mathrm{HI\$}$ and $n \in \omega$, then $\sigma^n := \{m \mid \langle n\rangle * m \in \sigma\}$)

We define, by transfinite induction, for each $\sigma \in \mathrm{HI\$}$, subsets A_σ and E_σ of ${}^\omega\omega$ by:
$$A_{\{\emptyset\}} := \{\alpha \mid \forall n[\alpha(\langle n\rangle) = 0]\} \qquad A_\sigma := \{\alpha \mid \forall n[E_{\sigma^n}(\alpha^n)]\}$$
$$E_{\{\emptyset\}} := \{\alpha \mid \exists n[\alpha(\langle n\rangle) = 0]\} \qquad E_\sigma := \{\alpha \mid \exists n[A_{\sigma^n}(\alpha^n)]\}$$

We introduce, for each $\sigma \in HI\$$, classes Π^0_σ and Σ^0_σ of subsets of ${}^\omega\omega$ by:
$\Pi^0_\sigma := \{ P \mid P \subseteq {}^\omega\omega \mid P \preceq A_\sigma \}$ and $\Sigma^0_\sigma := \{ P \mid P \subseteq {}^\omega\omega \mid P \preceq E_\sigma \}$
We introduce a strict and a reflexive ordering relation, $<, \leq$, respectively, on $HI\$$ (which is a subclass of the class $\$$ of stumps, presented in 8.0) such that: for all $\sigma, \tau \in HI\$$: $\sigma \leq \tau \rightleftarrows \forall m[\sigma^m < \tau]$ and: $\sigma < \tau \rightleftarrows \exists n[\sigma \leq \tau^n]$

We prove, in theorem 8.7, that, for each $\sigma \in HI\$$, and $P \subseteq {}^\omega\omega$:
$P \in \Pi^0_\sigma$ if and if there exists a sequence $Q_0, Q_1, \ldots$ of subsets of ${}^\omega\omega$ such that $\forall m \exists \tau < \sigma [Q_m \in \Sigma^0_\tau]$ and $P = \bigcap_{m \in \omega} Q_m$
An analogous result holds for Σ^0_σ. Furthermore, Π^0_σ and Σ^0_σ do possess universal elements and remarks, similar to those in chapter 6, apply

9 The hyperarithmetical hierarchy established. 68

We introduce subsets A^*_2 and E^*_2 of ${}^\omega\omega$ by:
$A^*_2 := \{ \alpha \mid \forall m \exists n [\alpha^m(n) \neq 0]\}$ and $E^*_2 := \{ \alpha \mid \exists m \forall n [\alpha^m(n) \neq 0]\}$
We introduce, by transfinite induction, for each $\sigma \in HI\$$, subsets P_σ, Q_σ, P^*_σ and Q^*_σ of ${}^\omega\omega$ by:

$$P_{\{\langle\rangle\}} := A_2 \quad Q_{\{\langle\rangle\}} := E_2 \quad P^*_{\{\langle\rangle\}} = A^*_2 \quad Q^*_{\{\langle\rangle\}} = E^*_2$$
$$P_\sigma := \{\alpha \mid \forall n [Q_{\sigma^n}(\alpha^n)]\} \qquad Q_\sigma := \{\alpha \mid \exists n [P_{\sigma^n}(\alpha^n)]\}$$
$$P^*_\sigma := \{\alpha \mid \forall n [Q^*_{\sigma^n}(\alpha^n)]\} \qquad Q^*_\sigma := \{\alpha \mid \exists n [P^*_{\sigma^n}(\alpha^n)]\}$$

We observe that, for each $\tau \in HI\$$: $P_\tau \cap Q^*_\tau = P^*_\tau \cap Q_\tau = \phi$
We prove the hyperarithmetical hierarchy theorem (theorem 9.7):
Let $\tau \in HI\$$ and $\delta \in {}^\omega\omega$ such that: $Fun(\delta)$ and $\forall \alpha [P_\tau(\alpha) \rightarrow Q_\tau(\delta|\alpha)]$
We may construct, now, $\zeta \in {}^\omega\omega$ such that $Q^*_\tau(\zeta)$ and $Q_\tau(\delta|\zeta)$

(This result is complemented by its corollary, theorem 9.8)
The formulation of the theorem shows that we had to reason more carefully than in the case of the arithmetical hierarchy theorem in chapter 7.
We first strengthen the results of chapter 3, concerning A_2 and E_2 (lemmas 9.2 and 9.3).
Theorem 9.5 is a basic tool in the inductive construction

10 Analytical and co-analytical sets. 84

We introduce a subset E^1_1 of ${}^\omega\omega$ by: $E^1_1 := \{\alpha \mid \exists \gamma \forall n [\alpha(\bar{\gamma}n) = 0]\}$
We introduce Σ^1_1, the class of all <u>analytical</u> subsets of ${}^\omega\omega$, by:
$\Sigma^1_1 := \{ P \mid P \subseteq {}^\omega\omega \}$
We verify, in theorem 10.3, that Σ^1_1 is closed under the operations of countable union and intersection and, therefore, contains all

hyperarithmetical sets. Σ^1_1 also has a universal element (theorem 10.5).
We call a subset P of ${}^\omega\omega$ <u>strictly</u> <u>analytical</u> if $\exists\delta[\mathrm{Fun}(\delta) \wedge P = \mathrm{Ra}(\delta)]$ (i.e.: P is the range of a total (and therefore continuous) function on ${}^\omega\omega$.
We show that the supposition that all analytical inhabited (i.e.: constructively non-empty) subsets of ${}^\omega\omega$ are strictly analytical, is contradictory (theorem 10.8).

We introduce a subset A^1_1 of ${}^\omega\omega$ by: $A^1_1 := \{\alpha \mid \forall\gamma\exists n[\alpha(\bar{\gamma}n)=0]\}$
We introduce Π^1_1, the class of all co-analytical subsets of ${}^\omega\omega$, by: $\Pi^1_1 := \{P \mid P \subseteq {}^\omega\omega \mid P \preceq A^1_1\}$.
Π^1_1 is closed under the operation of countable intersection, but D^2A_1 is not co-analytical (theorem 10.13), and, therefore Π^1_1 is not closed under the operation of countable union.
We give a constructive version of the result that E^1_1 is not co-analytical (theorem 10.14) and have to admit that we do not know whether A^1_1 is analytical. It is easy to prove that Fun and A^1_1 are not strictly analytical.

We study the effect of restricting the range of the existential quantifier which occurs in the definition of E^1_1, to some subspread of ${}^\omega\omega$.
First, we consider $\sigma_{2mon} := \{\alpha \mid \forall n[\alpha(n) \leq \alpha(n+1) \leq 1]\}$ and introduce

$S_2 := \{\alpha \mid \exists\gamma\in\sigma_{2mon}\forall n[\alpha(\bar{\gamma}n)=0]\}$
We establish the following: $\forall n[D^nA_1 \prec S_2]$ (in 11.2), $\neg(S_2 \preceq E_1)$ (in 11.3) and $\neg(E_1 \preceq S_2)$ (in 11.7)
Refining the proofs of these facts, we find that $\neg(S_2 \preceq A_2)$ (in 11.9), $\neg(S_2 \preceq E_3)$ (in 11.10) and, after some effort: $\neg(S_2 \preceq A_3)$ (in 11.13)
We go further, now, and prove that S_2 is not hyperarithmetical. This is a big task which engages us up to 11.18.
While performing it, we observe that uncountably many hyperarithmetical sets may be intercalated between S_2 and the arithmetical set $\mathrm{Neg}(\mathrm{Neg}(S_2)) := \{\alpha \mid \neg\neg S_2(\alpha)\}$ (cf. 11.16).

We introduce, in 11.19, for each $m\in\omega$: $\sigma_{m\,mon} := \{\alpha \mid \forall n[\alpha(n) \leq \alpha(n+1) < m]\}$ and $S_m := \{\alpha \mid \exists\gamma\in\sigma_{m\,mon}\forall n[\alpha(\bar{\gamma}n)=0]\}$
We find that $\forall n[D^nS_2 \prec D^{n+1}S_2 \prec S_3]$ (theorems 11.20, 11.22). and remark that it is easy to generalize this to:
$\forall n\forall m[D^nS_m \prec D^{n+1}S_m \prec S_{m+1}]$
Trying to do similar things for conjunction, we have to work harder but find: $\forall m>1\ \forall n>0[C^nS_m \prec C^{n+1}S_m]$ (theorem 11.26)
Remark, however, that, for instance: $\neg(C^2S_2 \preceq S_3)$ (cf. theorem 11.24)

In 11.27 we consider the binary fan $\sigma_2 := \{\alpha \mid \forall n[\alpha(n) \leq 1]\}$ and introduce $S := \{\alpha \mid \exists\gamma\in\sigma_2\forall n[\alpha(\bar{\gamma}n)=0]\}$.
We make some observations on the class $\mathcal{C} := \{P \mid P \subseteq {}^\omega\omega \mid P \preceq S\}$ and formulate difficult questions.

We introduce a subset R of ${}^\omega\omega$ by: $R := \{\alpha \mid \forall n[n = \mu p[\alpha^0(p) \neq 0] \rightarrow D^n A_1(\alpha^{Sn})]\}$ and prove, in 12.0, that $\forall n[D^n A_1 < R]$ and $R < S_2$ and $R \in \Pi^0_3$.

Let $P_0, P_1, \ldots$ be a sequence of subsets of ${}^\omega\omega$ such that $\forall m \exists n[P_m < P_n]$
We define: $Q := \{\alpha \mid \forall n[n = \mu p[\alpha^0(p) \neq 0] \rightarrow P_n(\alpha^{Sn})]\}$

Using methods from chapter 11 we prove, in 12.1, that:
if $\forall n \exists \alpha[\neg P_n(\alpha)]$, then $\neg(D^2Q \preceq Q)$, and: if $\neg(Q \preceq A_1)$, then $\neg(C^2Q \preceq Q)$

Starting from the same sequence $P_0, P_1, \ldots$, we define:
$Q^* := \{\alpha \mid \exists n[n = \mu p[\alpha^0(p) \neq 0] \wedge P_n(\alpha^{Sn})]\}$ and we prove, in 12.2, that:
if Q^* is dense in ${}^\omega\omega$, then $\neg(D^2Q^* \preceq Q^*)$

We introduce a subset R^* of ${}^\omega\omega$ by $R^* := \{\alpha \mid \exists n[n = \mu p[\alpha^0(p) \neq 0] \wedge D^n A_1(\alpha^{Sn})]\}$ and we observe that: $\neg(D^2R^* \preceq R^*)$ but, on the other hand: $C^2R^* \preceq R^*$

We prove, in two different ways, that: $\forall n[D^nQ < D^{n+1}Q]$
The first time, in 12.3, we require that each one of the sets $P_0, P_1, \ldots$ is strictly analytical.

We define, for all subsets P and Q of ${}^\omega\omega$, a subset $D(P,Q)$ of ${}^\omega\omega$ by:
$D(P,Q) := \{\alpha \mid P(\alpha^0) \vee Q(\alpha^1)\}$
We call the sequence $P_0, P_1, \ldots$ disjunctively closed if $\forall m \forall n \exists k[D(P_m, P_n) \preceq P_k]$.
We call a subset P of ${}^\omega\omega$ disjunctively productive if $\forall n[D^nP < D^{n+1}P]$.

We prove, in 12.4: if the sequence $P_0, P_1, \ldots$ is disjunctively closed and $\forall n \exists \alpha[\neg P_n(\alpha)]$, then Q is disjunctively productive.

Similarly, we prove, in 12.5, having made the obvious definitions:
if the sequence $P_0, P_1, \ldots$ is conjunctively closed and $\exists n[A_1 \preceq P_n]$, then Q is conjunctively productive.

Thirdly, we prove, in 12.6: if the sequence $P_0, P_1, \ldots$ is disjunctively closed, and Q^* is dense in ${}^\omega\omega$, then Q^* is disjunctively productive.

These results imply that uncountably many levels of complexity may be distinguished in Π^0_3, and even in Σ^0_2 (cf. the discussion in 12.7).

Let R be a subset of ${}^\omega\omega$: We introduce a sequence $I_0R, I_1R, \ldots$ of subsets of ${}^\omega\omega$ by: $I_0R := \{\alpha \mid R(\alpha^0)\}$ and, for each $p \in \omega$:
$I_{Sp}R := \{\alpha \mid I_pR(\alpha) \rightarrow A_1(\alpha^{Sp})\}$.
We prove, in 12.8, for the very set Q we introduced in 12.1, that:
$\forall n[I_nQ < I_{n+2}Q]$.

Let R be a subset of ${}^\omega\omega$. We introduce a sequence $J_0R, J_1R, \ldots$ of subsets of ${}^\omega\omega$ by: $J_0R := \{\alpha \mid R(\alpha^0)\}$ and, for each $p \in \omega$:
$J_{Sp}R := \{\alpha \mid J_pR(\alpha) \rightarrow E_1(\alpha^{Sp})\}$.
We prove, in 12.9, that, if the sequence $P_0, P_1, \ldots$ fulfils the condition:
$\forall \ell \forall p \forall q \forall n \exists N[N > \ell \wedge \neg(J_pP_N \preceq J_qP_n)]$ and, as in 12.2,
$Q^* := \{\alpha \mid \exists n[n = \mu p[\alpha^0(p) \neq 0] \wedge P_n(\alpha^{Sn})\}$, then:
$\forall p \forall q[(p+q \text{ is odd}) \rightarrow \neg(J_pQ^* \preceq J_qQ^*)]$.

13 Brouwer's thesis, and some of its consequences 155

We discuss, briefly, Brouwer's thesis, and formulate it in a way which suits our purposes.
We introduce, for each $\alpha \in {}^{\omega}\omega$, a subset $|\alpha|^*$ of ω by:
$|\alpha|^* := \{a \mid \forall b\,[a \subseteq b \rightarrow \alpha(b) \neq 0]\}$.
("$a \subseteq b$" means that the finite sequence of natural numbers a extends the finite sequence b).
We define, for all decidable subsets A, B of ω:
$A \leq^* B := \exists\gamma \forall n[\,lg(\gamma(n)) = lg(n) \wedge \forall m \forall n[m \subseteq n \rightarrow \gamma(m) \subseteq \gamma(n)] \wedge \forall n[n \in A \rightarrow \gamma(n) \in B]]$
($lg(m)$ denotes the length of the finite sequence m)
and we observe that for all stumps σ, τ: $\sigma \leq \tau \rightleftarrows \sigma \leq^* \tau$

We prove the boundedness lemma 13.2.2:
Let $\delta \in {}^{\omega}\omega$ be such that: $Fun(\delta)$ and $\forall\alpha[A^1_1(\delta|\alpha)]$
Then: $\exists\beta[A^1_1(\beta) \wedge \forall\alpha[\,|\delta|\alpha|^* \leq^* |\beta|^*]$

We prove the Souslin-Brouwer theorem 13.2.2:
A subset of ${}^{\omega}\omega$ which is both co-analytical and strictly analytical, is hyperarithmetical.

Let P and Q be subsets of ${}^{\omega}\omega$. We say that $<P, Q>$ is a separate pair of subsets of ${}^{\omega}\omega$ if: $\forall\alpha\forall\beta[\,P(\alpha) \wedge Q(\beta) \rightarrow \alpha \# \beta]$
($\#$ denotes the usual apartness relation on ${}^{\omega}\omega$).
We say that $<P, Q>$ is hyperarithmetically separable if there are hyperarithmetical sets S, T such that: $P \subseteq S$ and $Q \subseteq T$ and $Sep(S, T)$.

We prove the separation theorem of Lusin and Brouwer 13.4.1:
A separate pair of strictly analytical subsets of ${}^{\omega}\omega$, is hyperarithmetically separable.

Let $\delta \in {}^{\omega}\omega$ be such that: $Fun(\delta)$ We call δ strongly injective if:
$\forall\alpha\forall\beta[\alpha \# \beta \rightarrow \delta|\alpha \# \delta|\beta]$

We prove, in theorem 13.5.1, that the range of a strongly injective and everywhere defined function from ${}^{\omega}\omega$ to ${}^{\omega}\omega$, is hyperarithmetical.

14 The collapse of the projective hierarchy. 167

We introduce, for each subset P of ${}^{\omega}\omega$, subsets $\mathbb{E}(P)$ and $\mathbb{U}(P)$ of ${}^{\omega}\omega$ by:
$\mathbb{E}(P) := \{\alpha \mid \exists\gamma[P(<\alpha, \gamma>)]\}$ and $\mathbb{U}(P) := \{\alpha \mid \forall\gamma[P(<\alpha, \gamma>)]\}$
($< >$ denotes a pairing function on ${}^{\omega}\omega$)
We prove, in theorem 14.1, that Σ^1_1 is closed under the operation $\mathbb{E}$
We define a subset A^1_2 of ${}^{\omega}\omega$ by: $A^1_2 := \{\alpha \mid \forall\gamma\exists\beta\forall n[\alpha(\overline{\ll\beta,\gamma\gg} n) = 0]\}$
($\ll \gg$ denotes a pairing function on ${}^{\omega}\omega$.
For all $\alpha \in {}^{\omega}\omega$ and $n \in \omega$: $\overline{\alpha}n := <\alpha(0), \ldots, \alpha(n-1)>$)
We introduce a class Π^1_2 of subsets of ${}^{\omega}\omega$ by: $\Pi^1_2 := \{P \mid P \subseteq {}^{\omega}\omega \mid P \leq A^1_2\}$
We prove, in theorem 14.3, that $\Pi^1_2 := \{\mathbb{U}(P) \mid P \in \Sigma^1_1\}$ and, in theorem 14.4 that Π^1_2 has a universal element.

We repeat the story, defining E^1_2 and Σ^1_2 in the obvious way and proving their (by now) obvious properties. (cf. 14.6-8)
We then prove that, by intervention of AC_{11} : $\Pi^1_2 \subseteq \Sigma^1_2$ and $\Pi^1_3 = \Sigma^1_2$ (theorems 14.9-10)
These are strange results, from a classical point of view, and they fascinate us.

We consider the following crazy principle, for any subspread σ of ${}^\omega\omega$:
CRP_σ: Let $A \subseteq {}^\omega\omega$
If $\forall\alpha\in\sigma\,\exists n[A(n,\overline{\alpha}n)]$, then $\exists n\,\forall\alpha\in\sigma[A(n,\alpha)]$
(The intuitionistic notions of „subspread of ${}^\omega\omega$" and "fan" have been mentioned in section 1.9 and just before theorem 11.4, respectively The fan theorem is recalled in 15.2).

Using the fan theorem, we prove, in theorem 15.3.3, that CRP_σ holds, for any subfan σ of ${}^\omega\omega$ which fulfils the condition: $\forall\alpha[\alpha\in\sigma \rightleftarrows \forall n[\overline{\alpha}n\in\sigma]]$
We also prove, in theorem 15.4.1, that every subspread σ of ${}^\omega\omega$ such that CRP_σ holds and $\forall\alpha[\alpha\in\sigma \rightleftarrows \forall n[\overline{\alpha}n\in\sigma]]$ is a fan.
The proof of this theorem develops a line of thought from section 15.1, where we made sure that CRP_σ is not true if $\sigma = {}^\omega\omega$, the universal spread.

For any subspread τ of ${}^\omega\omega$ and any subset S of ${}^\omega\omega$, we introduce the usual infinite game for players I and II, and we define $Strat_I(\tau)$, $Strat_{II}(\tau)$, the set of strategies in τ for players I and II, respectively. These two sets are spreads.
We say that the game associated with τ and S is determined, and write: $Det(\tau,S)$ if:
$\forall\gamma\in Strat_{II}(\tau)\exists\alpha\in\tau[\alpha \text{ obeys to } \gamma \wedge S(\alpha)] \to \exists\gamma\in Strat_I(\tau)\forall\alpha\in\tau[\alpha \text{ obeys to } \gamma \to S(\alpha)]$
Adapting these definitions to the case of finite games, we first prove, in 16.1, that every finite game is determined.
We then prove, in theorem 16.2.0, that, for any subset S of ${}^\omega\omega$: $Det(\sigma_{2mon},S)$
We extend this result and prove, in theorem 16.4.0, that, for all $m\in\omega$, for all subsets S of ${}^\omega\omega$: $Det(\sigma_{mmon}, S)$.

We leave the domain of the monotonous fans and prove, in theorems 16.5.0-1, that, for all subfans τ of ${}^\omega\omega$, and all subsets S of ${}^\omega\omega$ which belong to Σ^0_1 or Π^0_1 : $Det(\tau,S)$
In section 16.6 we extend this result to subsets S which belong to Σ^0_2 or Π^0_2
In section 16.7, we conclude, to our own surprise, that, for all subfans τ of ${}^\omega\omega$ and all subsets S of ${}^\omega\omega$ which belong to Σ^1_1: $Det(\tau,S)$

Therefore, by AC_{11}, for all subfans τ of ${}^{\omega}\omega$, for all subsets S of ${}^{\omega}\omega$: $Det(\tau, S)$ (theorem 16.7.1)

Actually, theorem 16.7.1 embraces the earlier results on monotonous fans, but we left those on their own, in order not to deny the reader the fun of discovery.

In 16.8 we remark that the result 16.7.1 may be extended to subspreads τ of ${}^{\omega}\omega$ which offer only finitely many alternatives at any move by player II, but, possibly, infinitely many at some moves by player I. Conversely, if a subspread τ of ${}^{\omega}\omega$ is such that for all subsets S of ${}^{\omega}\omega$: $Det(\tau, S)$ and it offers, at each move by either player I or player II, at least two alternatives, then τ offers only finitely many alternatives, at any move by player II.

17. Appendix: strange lights in a dark alley. 211

We could not answer the question whether $A^1_1 \preceq E^1_1$, in chapter 13. We observe that, assuming $A^1_1 \preceq E^1_1$, we would have to abandon various schemes which have been proposed as additions to the axioms of intuitionistic analysis, such as: the generalized Markov principle GMP, saying that: $\forall\alpha[\neg\neg\exists n[\alpha(n)=0] \rightarrow \exists n[\alpha(n)=0]]$, or Kuroda's scheme KUR, saying that, for all subsets P of ω: if $\forall n[\neg\neg P(n)]$, then $\neg\neg\forall n[P(n)]$.
A somewhat dubious argument which forces us, on the assumption of: $A^1_1 \preceq E^1_1$, to give up the restricted principle of Brouwer and Kripke BK, is also given.

SUBJECT INDEX

References below are given as follows: first, the number of the section, and then, between parentheses, the number of the page.
Thus, 2.1(12) means: in section 2.1, on page 12

INDEX OF SYMBOLS

References are given as in the subject index: first, the number of the section, and then, between parentheses, the number of the page.
$m, n, \ldots$ are used for natural numbers; $\alpha, \beta, \ldots$ for members of $^\omega\omega$.

CURRICULUM VITAE

De schrijver van dit proefschrift werd op 7 oktober 1947 te Maastricht geboren.
Hij volgde het gymnasium aan het Sint Ignatius college te Amsterdam.
Hij studeerde wiskunde te Nijmegen van 1965 tot 1970
Tot zijn leermeesters behoorden J.H. de Boer, J.J. de Iongh, A.H.M. Levelt, A.C.M. van Rooij, H.O. Varma en H. de Vries.
Sinds 1970 is hij medewerker aan het mathematisch instituut van de Katholieke Universiteit te Nijmegen.

STELLINGEN

behorende bij het proefschrift:

Investigations in intuitionistic hierarchy theory.

1

De opmerkingen die K. Menger in 1928 maakte over de gelijkenis tussen sommige intuïtionistische begrippen en begrippen uit de klassieke beschrijvende verzamelingsleer hebben, tot nu toe, niet de aandacht gekregen die zij verdienden.
Hij redeneerde wel klassiek, en dus, voor een intuïtionist, niet zorgvuldig genoeg: ofschoon nauw verwant, moeten spreidingen en analytische verzamelingen toch van elkaar onderscheiden worden.

Vgl.: Karl Menger
Selected Papers in Logic and Foundations, Didactics, Economics
D. Reidel Publ. Co., Dordrecht 1979
i.h.b. blz. 79-87, blz. 246

dit proefschrift, hoofdstuk 10

2

E. Bishop en P. Martin-Löf bespreken beiden de vraag, hoe de Borel-verzamelingen in de constructieve wiskunde moeten worden ingevoerd.
Onafhankelijk van elkaar, komen beiden er toe, de betekenis van het begrip "complement" zo te veranderen dat hun bouwwerken klassieke symmetrie vertonen.
Ze gaan voorbij aan het eigenlijke hiërarchie-probleem: of deze bouwwerken nu ook bewoond zijn
De oplossing die in dit proefschrift wordt geboden, berust op een typisch intuïtionistisch continuïteitsbeginsel, en is voor hen vermoedelijk niet aanvaardbaar.

Vgl.: Errett Bishop
Foundations of Constructive Analysis
Mc. Graw Hill, New York 1967
i.h.b. blz. 66-69

Per Martin-Löf
Notes on Constructive Mathematics
Almqvist & Wiksell, Stockholm 1970
i.h.b. blz. 79-84

dit proefschrift, hoofdstuk 9

3

In de intuïtionistische reële analyse kan de continuïteit van (overal op $\mathbb{R}$ gedefiniëerde) reële functies bewezen worden met behulp van alleen het zwakke continuïteitsbeginsel. (In 1.9 van dit proefschrift wordt dit beginsel CP genoemd).
Gebruik makend van het sterke continuïteitsbeginsel (AC_{10} in 1.9 van dit proefschrift), kan men bewijzen:

(*) Als $f: \mathbb{R} \to \mathbb{R}$ en $m \in \mathbb{N}$, dan bestaat er een functie $g: \mathbb{R} \to \mathbb{R}$ zodat: $\forall x \in \mathbb{R}\, [\, g(x) > 0 \,]$ en: $\forall x \in \mathbb{R}\, \forall y \in \mathbb{R}\, [\, |x-y| < g(x) \to |f(x)-f(y)| < 2^{-m} \,]$

Ook hierbij is de waaierstelling niet nodig.
Het vermoeden, uitgesproken door Charles Parsons, in zijn inleiding bij de heruitgave van Brouwer's artikel: „Ueber Definitionsbereiche von Funktionen", is dus niet juist.
De bewering die aan dit vermoeden voorafgaat, dat (*) gelijkwaardig zou zijn met: f is locaal uniform continu, is onwaar.

Vgl.: Jean van Heijenoort
From Frege to Gödel
Harvard University Press, Cambridge, Mass. 1967
i.h.b. blz. 448, voetnoot, laatste zin.

4

Bij de intuïtionistische behandeling van de volledigheid van de predikatenrekening, behoeven geen byzondere structuren als Beth- en Kripke-modellen ter sprake te komen.
Alleen voor het verkrijgen van een klassieke volledigheidsstelling voor de intuïtionistische predikatenrekening – een oud, maar wonderlijk verlangen – is het nodig een andere dan de voor de hand liggende interpretatie van het begrip "geldigheid" te bedenken.

5

Het volgende speciale geval van het lemma van Teichmüller en Tukey is constructief bewijsbaar:

Zij τ een waaier en B een decidabele deelverzameling van de collectie van de eindige deelverzamelingen van τ
Dan bestaat er een deelwaaier van τ, die maximaal is onder de deelverzamelingen van τ, waarvan alle eindige deelverzamelingen tot B behoren.

6

De beide functies, die Brouwer ziet voor de taal in verband met het wiskundig denken: het vasthouden van wiskundige constructies in het geheugen, en het suggereren van wiskundige constructies aan anderen, vertonen gelijkenis: de denker is een leraar die zijn eigen leerling is.

Vgl.: Gilbert Ryle,
Thinking and Self-teaching
Rice University Studies 58, no. 3, 1972

ook in: Gilbert Ryle, On thinking
Blackwell, Oxford 1979
blz. 65-78

7

Verschillende axioma's van de verzamelingsleer, kunnen niet, in de termen van B. Nieuwentijt (1654-1718), gekenschetst worden als:

„algemene Bekentenisse, aanstonts klaar aan ymant die de woorden verstaat"

Eerder zijn het „hypotheses of onderstellingen", dit is: „door ondervinding bekomen denkbeelden."
Zijn de verzamelingstheoretici geen „suyvere Wiskundigen, die Waarheden soeken en bewijsen, omtrent hare enkele of blote Denkbeelden" en bestuderen zij „Saken, die buiten haar verstant en Denkbeelden wesentlyk bestaan"?

Vgl. Bernard Nieuwentijt
Gronden van Zekerheid of de regte betoogwijse der wiskundigen
Johannes Pauli, Amsterdam 1739
i.h.b. blz. 11, blz. 1, blz. 27

Kurt Gödel,
What is Cantor's continuum problem?
Amer. Math. Monthly 54(1947) 515-525

Yiannis Moschovakis
Descriptive Set Theory
North Holland Publ. Co., Amsterdam 1980
i.h.b. blz. 604-611

8

De Nederlandse man van wetenschap zou zich moeten uitdrukken

> „in plat Neerduytsch sonder vermenging van quade Barbarische woorden, die hy in syns moederstael beter heeft."

Het is jammer dat de Nederlandse waarschijnlijkheidsrekenaars het woord „stochastiek" meer gebruiken dan het woord „giskunde".

Vgl.: Simon Stevin, de Sterckenbouwing,
Leiden 1594, i.h.b. blz. 91

ook in: The principal works of Simon Stevin, Vol IV,
Swets & Zeitlinger, Amsterdam 1964, i.h.b. blz. 230

Nijmegen, 20 mei 1981
Wim Veldman.