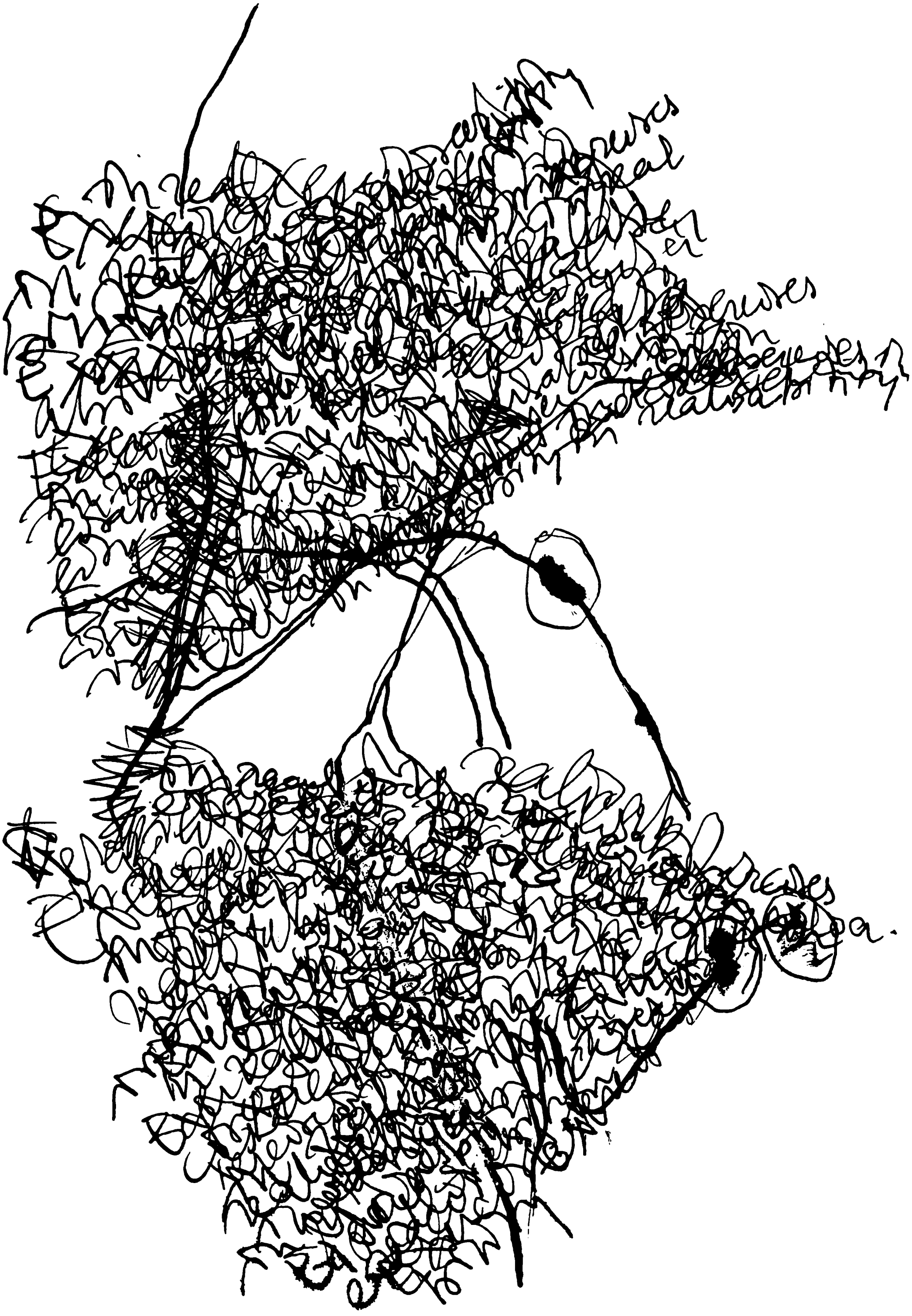
Jaap van Oosten

Exercises in realizability



EXERCISES IN REALIZABILITY

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The cover of this thesis was designed by Tine Blankenstein

I. Realizability: a survey

Since Kleene coined the term "realizability" in 1945 to denote what was, at the time, the first interpretation of intuitionistic arithmetic in a classical context, the word has been used by many authors (including Kleene himself) to denote various modifications of the original definition.

"Realizability" became a *notion* ; by now, there is quite a list of semantical or syntactical interpretations going under this name. It is hard, however, to sift out what all these have in common, or to define what a realizability interpretation ought to be. Let us therefore first have a look at Kleene's definition. Kleene defined a relation between natural numbers and sentences of intuitionistic first-order arithmetic (**HA**), by induction on the logical complexity of the sentences, as follows. Let us write $\langle\rangle$, $(\,)_0$ and $(\,)_1$ for the pairing and unpairing functions, and • for partial recursive application. The relation is written "n realizes A".

n realizes an equation t=s iff it is a true equation and n is the numerical value of t;
n realizes $A \wedge B$ iff $(n)_0$ realizes A and $(n)_1$ realizes B;
n realizes $A \vee B$ iff $((n)_0=0$ and $(n)_1$ realizes A) or $((n)_0 \neq 0$ and $(n)_1$ realizes B);
n realizes $A \to B$ iff for all m, if m realizes A then n•m is defined and realizes B;
n realizes $\exists x\, A(x)$ iff $(n)_1$ realizes $A((\bar{n})_0)$;
n realizes $\forall x\, A(x)$ iff for all m, n•m is defined and realizes $A(\bar{m})$.

By the basic results of recursion theory (most of which were also developed by Kleene), we now have that if **HA** proves a sentence A, then there is a natural number n that realizes A. So, we have some kind of "truth definition" (defining as "true" the realizable sentences) for which **HA** is sound. Since obviously the statement $\bar{0}=\bar{1}$ has no realizer, the consistency of **HA** follows (if you like this kind of argument). One cannot do the same for Peano arithmetic: consider a sentence of the form $\forall x\,(A(x) \vee \neg A(x))$. A realizer of this sentence codes a total recursive function which decides, for all m, whether $A(\bar{m})$ has a realizer or not. It follows that not all such sentences can be realized: Peano arithmetic is not sound for this definition.

The realizability notions I shall consider in this survey share the following features with Kleene's (and I propose them as a loose "definition" of realizability):

Suppose we have a formal system T *and a set* S *which serves as an interpretation of the range of the variables of* T *(or more such sets* S, *if* T *has more than one sort). A relation "d realizes A" is defined between sentences of* T[S] *(i.e.* T *with constants for elements of* S *added) and elements of a certain domain* D, *such that:*

i) if d realizes an implication $A \to B$ *then d codes one or more operations which transform realizers of* A *into realizers of* B*;*

ii) if d realizes an existential statement $\exists x A(x)$ *then d codes information about one or more elements* $s \in S$ *as well as realizers for the sentences* $A(s)$ *for these s;*

iii) if d realizes a universal statement $\forall x A(x)$ *then d codes one or more operations, each transforming elements of* D *which code information about an* $s \in S$ *into realizers of* $A(s)$.

Let us make some remarks. From i) and iii) it follows that elements of D can code partial operations: D ⇀ D. If our realizability is to be sound, then there should be some connection with combinatory logic. Important examples have *partial combinatory algebras* for D, but it would be too strict to restrict the notion of realizability to these. In many cases an element d is said to realize A→B if d codes an operation which, besides transforming realizers of A into realizers of B, satisfies further conditions. For example, one may define an equivalence relation on realizers of A and on realizers of B, and the operation encoded by d is required to respect this relation.

The way elements of D code information about elements of A should, of course, be fixed. There are cases (for instance, analogues of Kleene's realizability for theories about sets) where this is done trivially, in the sense that there is an element d of D which codes information about *every* $a \in A$. In these cases, the dependence on this information is usually suppressed in the presentation.

In this survey, I shall assume that D is nontrivial. This means that I don't consider the various *slash-operations* in existence, as realizabilities. I shall also disregard the Dialectica interpretation and translations where the realizability relation is quantified away in the form "∃d (d realizes A)". My attitude will be that realizability is *semantics* and not a syntactical translation: even a formalized version (see below) will be seen as a model, much in the same way as set theory deals with inner models.

Research on various realizabilities, like with every truth definition, can be naturally divided into four kinds:

1) Straightforward applications of the truth definition, in the sense: T is consistent, T does not prove A, but also, for instance: normalization for a natural deduction-presentation of T (The Curry-Feys isomorphism between natural deduction trees of intuitionistic implicational logic and closed terms of the typed lambda calculus can also be regarded as a realizability interpretation).
2) Investigations of the truth definition itself. What formulas are realizable? What is the logic which is realized?
3) Just like the use of inner models in set theory, *internalization* of realizability is an important tool. Very often, realizability is defined entirely in terms of D. So any formalism which is capable

of describing D, can express realizability. For instance, Kleene's definition can be formalized in **HA** itself, or in **PA**. If we have formalized realizability of T by D in a formalism S, we may ask: does S *prove* the soundness of T for this realizability? For what sentences A does S prove that A has a realizer? Is there an axiom or axiom schema Ψ in the language of T such that T+Ψ proves A iff S proves that A is realizable?
If S and T coincide, one may be interested in questions like: what is the relation (in T) between the sentences A and $\exists d$(d realizes A)? Is there a syntactical criterion by which you can tell whether they are equivalent? What if you iterate the notion: $\exists d$(d realizes $\exists d'$(d' realizes A))? Furthermore one may take a model of S and consider the notion: "d realizes A in the model".
4) The fourth line of research is more mathematical in nature and can be described as the start (as yet) of a "model theory of realizability". Questions that arise are: what is the relation between realizability and other semantics for intuitionistic systems, like Kripke models or sheaf models? How can techniques from the logic of sheaves be applied to realizability? Category theory plays a large role in this research, which has established that many realizabilities fit very well into topos-semantics for higher-order systems. It is quite possible that from this quarter a more rigorous definition emerges of what a realizability interpretation ought to be, and what would constitute a "homomorphism" of realizabilities.

First, I shall concentrate on realizabilities designed for **HA** or systems containing arithmetic; to do this, I present a sample of realizabilities. To avoid pages with inductive definitions, I introduce the following shorthand notation: a realizability notion is presented by specifying a set Σ (to be thought of as the set of possible sets of realizers), together with an implication $\Rightarrow: \Sigma\times\Sigma\rightarrow\Sigma$. These data almost *characterize* a realizability notion, by tripos-theoretic considerations (see chapter 3), but for the moment the reader is not invited to worry about this. For example, in Kleene's definition to each sentence ϕ a set $[\![\phi]\!] = \{n \mid n \text{ realizes } \phi\}$ is associated, such that $[\![\phi\rightarrow\psi]\!] = \{n \mid \forall x\in[\![\phi]\!]\,(n\bullet x\downarrow \ \&\ n\bullet x\in[\![\psi]\!])\}$. This suggests Σ and $\Rightarrow$ in example 1) below.

1) *Kleene's realizability* :

Σ_1 is $P(\mathbb{N})$;

$A\Rightarrow_1 B$ is $\{e \mid \forall a\in A\,(e\bullet a\downarrow \text{ and } e\bullet a\in B)\}$.

2) *Kleene & Vesley's function realizability* :

Σ_2 is $P(\mathbb{N}^{\mathbb{N}})$;

$A\Rightarrow_2 B$ is $\{\varepsilon \mid \forall\alpha\in A\,(\varepsilon|\alpha\downarrow \text{ and } \varepsilon|\alpha\in B)\}$. Here $\varepsilon|\alpha\downarrow$ means that for all x there is an initial segment σ of α such that $\varepsilon(\langle x\rangle * \sigma)\neq 0$; then $(\varepsilon|\alpha)(x) \equiv \varepsilon(\langle x\rangle * \sigma)-1$ for the minimal such σ.

3) *Lifschitz' realizability* :

Let [e] be the finite set $\{x\leq(e)_1 \mid (e)_0\bullet x\uparrow\}$. Let J be $\{e \mid [e]$ is nonempty$\}$ and β be total recursive such that for all e, $[\beta(e)]=\{e\}$. Then

Σ_3 is $\{H\subseteq J \mid \forall e\in J\,(e\in H \Leftrightarrow \forall f\in[e]\,(\beta(f)\in H))\}$;

$H\Rightarrow_3 G$ is $\{e\in J \mid \forall h\in H\forall f\in[e]\,(f\bullet h\downarrow \;\&\; f\bullet h\in H)\}$.

4) *Lifschitz function realizability* :

Let $[\alpha]$ be the compact set $\{\beta\in\mathbb{N}^{\mathbb{N}} \mid \forall n\,(\beta(n)\leq(\alpha(n))_1 \;\&\; \alpha(\bar{\beta}n))_0=0)\}$ where $\bar{\beta}n$ denotes the initial segment of β of length n; let K be $\{\alpha\in\mathbb{N}^{\mathbb{N}} \mid [\alpha]$ is nonempty$\}$, γ such that $\forall\alpha\,([\gamma|\alpha]=\{\alpha\})$.

Then Σ_4 is $\{H\subseteq K \mid \forall\alpha\in K\,(\alpha\in H\Leftrightarrow\forall\beta\in[\alpha]\,(\gamma|\beta\in H))\}$;

$H\Rightarrow_4 G$ is $\{\alpha\in K \mid \forall\beta\in H\forall\gamma\in[\alpha]\,(\gamma|\beta\downarrow \;\&\; \gamma|\beta\in G)\}$.

5) Σ_5 is $\{(A,\approx) \mid A\subseteq\mathbb{N} \;\&\; \approx$ is an equivalence relation on A$\}$;

$(A,\approx)\Rightarrow_5(B,\approx)$ is $(\{e\in A\Rightarrow_1 B \mid \forall a,a'\in A\,(a\approx a'\Rightarrow e\bullet a\approx e\bullet a')\},\doteq)$ where $e\doteq e'$ iff $\forall a\in A\,(e\bullet a\approx e'\bullet a)$.

6) Σ_6 is $\{(A,\approx) \mid A\subseteq\mathbb{N} \;\&\; \approx$ is a partial equivalence relation on A$\}$;

$(A,\approx)\Rightarrow_6(B,\approx)$ is $(A\Rightarrow_1 B,\doteq)$ where $e\doteq e'$ iff $\forall a,a'\in A\,(a\approx a'\Rightarrow e\bullet a\approx e'\bullet a')$.

I call 5) and 6) *extensional realizabilities.*

7) *Kripke models of realizability* :

Suppose $(P,\leq,0)$ is a partial order with bottom element 0. Let for each $p\in P$ a partial combinatory algebra A_p be given and for each $p\leq q$ a map $f_{pq}: A_p\rightarrow A_q$ which preserves K,S and application.Then

Σ_7 is $\{(\alpha_p)_{p\in P} \mid \forall p\,(\alpha_p\subseteq A_p) \;\&\; \forall p\leq q\,(f_{pq}[\alpha_p]\subseteq\alpha_q)\}$;

$((\alpha_p)_p\Rightarrow_7(\beta_p)_p)_p$ is $\{e\in A_p \mid \forall q\geq p\forall a\in\alpha_q\,(e\bullet a\downarrow \;\&\; e\bullet a\in\beta_q)\}$.

8) *Beth models of realizability* :

Let $(P,\leq,0)$, A_p, f_{pq} as in 7); in addition, suppose a *coverage* is defined on P; that is, for every $p\in P$ a set J(p) of *covers* (or *bars*) for p is defined, such that i) $\{p\}$ is a bar for p; ii) if R is a cover for p and $q\geq p$ then $\uparrow(R)$ (the upwards closure of R)$\cap\uparrow(\{q\})$ contains a cover for q, and iii) if $S\subseteq\uparrow(\{p\})$ and for some cover R for p, $\forall r\in R\,(S\cap\uparrow(\{r\})$ contains a cover for r), then S contains a cover for p. Then

Σ_8 is $\{(\alpha_p)_{p\in P}\in\Sigma_7 \mid \forall p\in P\forall a\in A_p\forall R\in J(p)\,(\forall r\in R\, f_{pr}(a)\in\alpha_r\Rightarrow a\in\alpha_p)\}$;

$((\alpha_p)_p\Rightarrow_8(\beta_p)_p)_p$ is $\{e\in A_p \mid \forall q\geq p\forall a\in\alpha_q\exists R\in J(q)\forall r\in R\,(f_{pr}(e)\bullet f_{qr}(a)\downarrow \;\&$
$f_{pr}(e)\bullet f_{qr}(a)\in\beta_r)\}$.

9) *HRO-modified realizability* :

Let e, by the recursion theorem, be such that $\forall x\,(e\bullet x=e)$.

Σ_9 is $\{(p^*,D_p)\in P(\mathbb{N})\times P(\mathbb{N}) \mid p^*\subseteq D_p \;\&\; e\in D_p\}$;

$(p^*,D_p)\Rightarrow_9(q^*,D_q)$ is $((p^*\Rightarrow_1 q^*)\cap(D_p\Rightarrow_1 D_q),D_p\Rightarrow_1 D_q)$.

10) *Modified Lifschitz' realizability* :

Let J and β as in 3) and e such that $\forall x\,(e\bullet x=\beta(e))$.

Σ_{10} is $\{(p^*,D_p)\in\Sigma_3\times\Sigma_3 \mid p^*\subseteq D_p \ \&\ \beta(e)\in D_p\}$;
$(p^*,D_p)\Rightarrow_{10}(q^*,D_q)$ is $((p^*\Rightarrow_3 q^*)\cap(D_p\Rightarrow_3 D_q),D_p\Rightarrow_3 D_q)$.

11) *q-realizability*:
Σ_{11} is $\{(p,x) \mid p\subseteq\mathbb{N},\ x\subseteq\{0\},\ p\neq\emptyset\Rightarrow 0\in x\}$;
$(p,x)\Rightarrow_{11}(q,y)$ is $(\{a\in p\Rightarrow_1 q \mid x\subseteq y\},\{0 \mid x\subseteq y\})$.

Of course this list is only an illustration. There are modified versions of function realizability, q-versions of Lifschitz' realizability and extensional realizability, extensionalizations of modified realizability etc. etc. The notion most thoroughly investigated is of course number 1). In the following discussion I group results about realizability x ($1\leq x\leq 11$) into 4 sections, numbered x.y ($1\leq y\leq 4$), according to the four types of results distinguished before.

1.1. The simplest applications of Kleene's definition are:
- consistency of **HA**+CT_0, where CT_0 is the schema
CT_0 $\quad \forall x\exists y\, A(x,y)\rightarrow\exists z\forall x\exists u\,(T(z,x,u)\wedge A(x,U(u)))$
- consistency of **HA**+$\neg\forall x\,(A(x)\vee\neg A(x))$ for some formula A(x).
In these results, **HA** can be replaced by **HA**+MP, where MP is the schema:
MP $\quad \forall x\,(A(x)\vee\neg A(x))\wedge\neg\neg\exists x\, A(x)\rightarrow\exists x\, A(x)$
It is not hard to see that every instance of MP has a realizer. The initials CT and MP stand for Church's Thesis and Markov's Principle.

1.2. The class of *almost negative formulas* is the class of formulas built up from Σ^0_1-formulas using only $\forall$, $\rightarrow$ and $\wedge$. Let $A(x_1,\ldots,x_n)$ be an almost negative formula with n free variables; then there is a partial recursive function ϕ_A of n arguments, such that for all $k_1,\ldots,k_n$, if $A(\bar{k}_1,\ldots,\bar{k}_n)$ is true then $\phi_A(k_1,\ldots,k_n)$ is defined and realizes $A(\bar{k}_1,\ldots,\bar{k}_n)$; conversely, if an almost negative sentence has a realizer, then it is true. This can be used to show that a strengthening of CT_0 is realizable, the schema ECT_0 (*extended Church's Thesis*):
ECT_0 $\quad \forall x\,(A(x)\rightarrow\exists y B(x,y))\rightarrow\exists z\forall x\,(A(x)\rightarrow\exists u\,(T(z,x,u)\wedge B(x,U(u))))$,
where A(x) must be an almost negative formula.
The "logic of realizability" is stronger than intuitionistic logic, as soon became known after Kleene's definition. In 1953, G.F. Rose gave an example of a propositional schema, not provable in the intuitionistic propositional calculus, yet every arithmetical instance of it being realizable. The relation between realizability and predicate logic was studied very thoroughly by V.E. Plisko (1977, 1978, 1983). He shows that "uniformly realizable" formulas of predicate logic (i.e. formulas of which every arithmetical substitution instance is realizable, uniformly in the Gödel numbers of the substitutions) form a Π^1_1-complete logic.

It should be pointed out that these results depend on classical logic. Gavrilenko (1981) shows that it cannot be intuitionistically provable that the logic of realizability is stronger than intuitionistic propositional logic; for predicate logic, see the remark made in 7-8.1.

1.3. D. Nelson (1947) carried through the internalization of realizability inside **HA**. In a straightforward way, statements "n realises A" are taken as formulas of arithmetic. The soundness theorem can be formalized as: $\mathbf{HA} \vdash A$ implies $\mathbf{HA} \vdash \bar{n}$ realizes A, for some n. Nelson also observed that realizability is *idempotent*, i.e. that $\exists n$ (n realizes A) and $\exists n$ (n realizes $\exists m$ (m realizes A)) are equivalent in **HA**. Several results from 1.2 hold in the internalized version: the equivalence of almost negative formulas with their realizability (Kleene 1960), and the realizability of ECT_0 (Troelstra 1971). Idempotency is a direct consequence of the fact that all formulas of the form "n realizes A" are equivalent to almost negative formulas.
It follows that the realizability of MP is *not* provable in **HA**, for MP implies the almost negative axiom M_{PR}: $\forall e \forall x (\neg\neg\exists z\, T(e,x,z) \rightarrow \exists z\, T(e,x,z))$. It can be shown by modified realizability that M_{PR} is not provable in **HA**. But the realizability of MP is provable in **HA+MP**.
ECT_0 proves to be the key to an axiomatization over **HA** or **HA+MP** of the provably realizable sentences. Troelstra (1971) showed that for any A, $\mathbf{HA}+ECT_0 \vdash A \leftrightarrow \exists x$ (x realizes A) and $\mathbf{HA}+ECT_0 \vdash A$ iff $\mathbf{HA} \vdash \exists x$ (x realizes A). In both of these, **HA** may be replaced by **HA+MP** (in fact, the axiomatization of realizability over **HA+MP** by a schema like ECT_0 is already in Dragalin 1969). ECT_0 can often be applied to show that certain formulas are not realizable. For instance, the schema IP: $(\neg A \rightarrow \exists y B(y)) \rightarrow \exists y (\neg A \rightarrow B(y))$ (y not free in A) was shown by Beeson to be simply inconsistent with ECT_0.

1.4. Well-known semantics for intuitionistic systems like Kripke models, Beth models or topological models, are all special cases of Ω-valued semantics for a complete Heyting algebra Ω. This semantics is described in great detail by Fourman and Scott (1979). Scott felt that realizability should fit in somewhere; the equivalent of Ω should be the set Σ_1. The idea was worked out by J.M.E. Hyland (1982). The theory behind it was developed by A.M. Pitts (1981) and part of it is presented in Hyland, Johnstone, Pitts 1980. We refer to a separate chapter of this thesis for an introduction into this theory.
The "effective topos" generalizes Kleene's realizability in that it provides a uniform extension of it to full higher order intuitionistic arithmetic. This is important not only for the study of intuitionistic systems, but also for understanding the practice of much constructive mathematics, as Hyland showed. Whether influenced by Kleene or not, the Russian school of recursive mathematics (Markov, Shanin, Zaslavskij, Ceitin) used a logic closely related to realizability: "In the overwhelming majority of papers on constructive mathematics, the underlying notion of truth

is essentially equivalent to realizability" (Plisko 1977). This statement acquires a precise meaning in the effective topos: there, Markov's Principle (the "principle of constructive choice" for the Russians) holds, as well as Church's Thesis in the form: all functions from $\mathbb{N}$ to $\mathbb{N}$ are recursive; real numbers are recursive, etc.

The first extension of Kleene's definition to a higher order system was defined by Troelstra (see Troelstra 1973) for second order arithmetic **HAS**. It can be shown that Troelstra's definition coincides with second order arithmetic in the effective topos. As a result, the Uniformity Principle (UP) holds in it: $\forall X \exists y\, A(X,y) \rightarrow \exists y \forall X\, A(X,y)$ (The weaker form UP! is a consequence of CT_0 in higher order logic).

Apart from the construction of the effective topos out of a tripos, there are at least two other presentations of it. One is via the category of "Assemblies" of Carboni, Freyd and Scedrov (1987); the other one shows that the effective topos is obtained by first adjoining recursively indexed non-empty coproducts to the category of sets, and then adding quotients of equivalence relations (Robinson and Rosolini 1990). None of these two constructions seems to be straightforwardly applicable to other realizabilities considered in this thesis, which is why I stick to triposes. However, as regards our phenomenological question "What is realizability?", especially the second one seems promising.

2.1. Function realizability as given here was defined in Kleene & Vesley 1965, but an equivalent formulation in terms of numbers (using "recursive in") was already given in Kleene 1957. The notion was meant to interpret the system of analysis defined there. A simpler description is in Troelstra 1973 where the basic system is called **EL**. **EL** is an extension of **HA** in a language with function variables, a recursion operator and an axiom of quantifier-free choice. The system of Kleene & Vesley can then be rendered as **EL**+AC_{01}+CC+BI_D, where:

AC_{01} $\quad \forall x \exists \alpha\, A(x,\alpha) \rightarrow \exists \alpha \forall x\, A(x, \lambda y.\alpha(\langle x,y \rangle))$

CC $\quad \forall \alpha \exists \beta\, A(\alpha,\beta) \rightarrow \exists \gamma \forall \alpha (\gamma|\alpha\downarrow \wedge A(\alpha, \gamma|\alpha))$

BI_D $\quad [\forall \alpha \exists x\, P(\bar{\alpha}x) \wedge \forall \sigma (P(\sigma) \vee \neg P(\sigma)) \wedge \forall \sigma (P(\sigma) \rightarrow Q(\sigma)) \wedge$
$\forall \sigma (\forall n\, Q(\sigma * \langle n \rangle) \rightarrow Q(\sigma))] \rightarrow Q(\langle\rangle)$

The results obtained by function-realizability are pretty analogous to those obtained by recursive realizability. Of course, the system considered here is not so obviously consistent anymore (For instance, dropping the decidability condition in BI_D makes it inconsistent).

2.2. The definition of almost negative formulas is the same as for **HA**, except that we also allow $\exists \alpha$ directly before quantifier-free formulas. The schema of "Generalized Continuity" GC is realized:

GC $\quad \forall \alpha (A(\alpha) \rightarrow \exists \beta B(\alpha,\beta)) \rightarrow \exists \gamma \forall \alpha (A(\alpha) \rightarrow \gamma|\alpha\downarrow \wedge B(\alpha, \gamma|\alpha))$, for almost negative $A(\alpha)$. This schema is very non-classical, but has no non-classical consequences in the language of **HA**, since by an easy induction one sees that exactly the true arithmetical formulas are realized. So, $\mathbf{EL}+\mathrm{GC}+\mathrm{BI}_D$+(all true arithmetical sentences) is consistent.

2.3. The formalization of function-realizability was carried out in Kleene 1969. All sentences provable in Kleene's system of analysis are provably realized by some *recursive* α. In fact the soundness proof can be carried out in $\mathbf{EL}+\mathrm{BI}_D$, for AC_{01} and CC are realized without assuming them. BI_D is only necessary to prove its own realizability, so we can also regard this realizability over **EL**. Then the role of ECT_0 for recursive realizability is taken over by GC, which axiomatizes realizability over **EL** (Troelstra 1973). It is known that **EL** (even **EL**+CC: Troelstra 1974) is conservative over **HA**, but it is an open conjecture that **EL**+GC is conservative over **HA**.

2.4. Analogous to the effective topos construction, a realizability topos for function realizability can be defined. One can show that function realizability (in a non-formalized version) is equivalent to the theory of the function space $\mathbb{N}^{\mathbb{N}}$ in this topos. True second order arithmetic in this topos can easily be formalized in a second order extension of **EL**, $\mathbf{EL}_2$. $\mathbf{EL}_2$ has variables for sets of functions, and axioms:

Ext $\quad \alpha\in X \wedge \forall x\,(\alpha(x)=\beta(x)) \rightarrow \beta\in X$

CA $\quad \exists X \forall\alpha\,(\alpha\in X \leftrightarrow \phi(\alpha))$, for ϕ not containing X free.

Let $X\rightarrow X^*$ be a 1-1 mapping of the set variables of **HAS** to the set variables of $\mathbf{EL}_2$. Define for formulas ϕ in the language of **HAS** the formula "α realizes ϕ" inductively by:

α realizes ϕ extends the definition on arithmetical formulas;

α realizes $t\in X \equiv \lambda y.\langle t,\alpha(y)\rangle \in X^*$;

α realizes $\forall X A(X) \equiv \forall X^*\,(\alpha$ realizes $A(X))$;

α realizes $\exists X A(X) \equiv \exists X^*\,(\alpha$ realizes $A(X))$.

It is easy to see, that UP holds under this interpretation, so is valid in the function realizability topos. As an immediate corollary, one sees that UP has no non-classical arithmetical consequences, because the function realizability topos satisfies true classical arithmetic.

3.1. Lifschitz' realizability was defined in Lifschitz 1979. Lifschitz wanted to show that Church's Thesis with uniqueness ($\mathrm{CT}_0!$) is strictly weaker than CT_0.

3.2. The following principle is Lifschitz realizable:

$\mathrm{B}\Sigma^0_2$-MP $\quad \neg\neg\exists x{\le}y\,\forall z\,A(x,y,z) \rightarrow \exists x{\le}y\,\forall z\,A(x,y,z)$, for primitive recursive A. M_{PR} is also valid. Define the class of $B\Sigma^0_2$-negative formulas as those built up from Σ^0_1-formulas and formulas of the form $\exists x{\le}y\,\forall z\,A(x,y,z)$ with A primitive recursive, by means of $\wedge$, $\forall$ and $\rightarrow$. Like with the almost negative formulas, $B\Sigma^0_2$-negative formulas are realizable iff true. Also, the following choice principle holds:

ECT_L $\quad\forall x\,(A(x)\rightarrow\exists yB(x,y))\rightarrow\exists z\forall x\,(A(x)\rightarrow z{\bullet}x{\downarrow}\wedge\exists w{\le}(z{\bullet}x)_1\forall u\neg T((z{\bullet}x)_0,w,u)\wedge$
$\quad\wedge\forall w{\le}(z{\bullet}x)_1(\forall u\neg T((z{\bullet}x)_0,w,u)\rightarrow B(x,w)))$,

for $B\Sigma^0_2$-negative formulas A.

3.3. Lifschitz' realizability can be formalized in **HA**+$B\Sigma^0_2$-MP+M_{PR}. This was done in Van Oosten 1990 (also in this thesis). ECT_L axiomatizes Lifschitz' realizability over this theory.

3.4. A "Lifschitz topos" Lif can be constructed which generalizes this realizability. In Lif, MP and CT are valid, but AC_{00} fails. Lif is used in this thesis to show that a certain principle of second order arithmetic:

RP $\quad\forall X\,(\forall x\,(X(x)\vee\neg X(x))\wedge\forall Y\,(\forall y\,(Y(y)\vee\neg Y(y))\rightarrow\forall x\,(X(x)\rightarrow Y(x))\vee\forall x\neg(X(x)\wedge Y(x))$
$\quad\rightarrow\exists n\forall x\,(X(x)\rightarrow x{=}n))$

is not derivable in **HAH** from MP+CT; in fact, the negation of RP is consistent with MP+CT, although RP follows from MP+CT+ECT_0. If the condition $\forall y\,(Y(y)\vee\neg Y(y))$ in RP is dropped, the resulting statement will be true in Lif, because Lif satisfies UP.

4.1. Lifschitz' function realizability is defined in Van Oosten 1990. It is shown that CC!, continuity for functions with uniqueness condition, is consistent with quantifier-free König's Lemma, which is the schema:

QF-KL $\quad\forall n\exists\sigma\,(lth(\sigma){=}n\wedge\forall i{<}n\,(\sigma)_i{\le}1\wedge A(\sigma))\rightarrow\exists\alpha\forall n\,(\alpha(n){\le}1\wedge A(\bar{\alpha}n))$,

for quantifier-free formulas A. It is also shown that QF-KL conflicts in **EL** with Weak Continuity for numbers:

WC-N $\quad\forall\alpha\exists n\,A(a,n)\rightarrow\forall\alpha\exists x\exists y\forall\beta\,(\bar{\beta}y{=}\bar{\alpha}y\rightarrow A(\beta,x))$.

4.2. The class of $B\Sigma^1_2$-negative formulas is analogously defined as in 3.2; built up from formulas of the form $\exists\alpha\,A(a)$ or $\exists\alpha{\le}\beta\,\forall n\,A(\alpha,\beta,n)$ with A quantifier-free, with $\wedge$, $\rightarrow$ and $\forall$. The following choice principle holds:

GC_L $\quad\forall\alpha\,(A(\alpha)\rightarrow\exists\beta B(\alpha,\beta))\rightarrow\exists\gamma\forall\alpha\,(A(\alpha)\rightarrow\gamma|\alpha{\downarrow}\wedge\exists\zeta{\in}[\gamma|\alpha]\wedge\forall\zeta{\in}[\gamma|\alpha]\,B(\alpha,\zeta))$,

for $B\Sigma^1_2$-negative A.

4.3. Lifschitz function realizability can be formalized in **EL**+QF-KL+MP; GC_L axiomatizes it over this theory.

5-6.1. Extensional realizability 5 is presented as realizability topos in Pitts 1981. An erroneous inductive definition is given in Beeson 1985; a better one is given in Renardel de Lavalette 1984, in the context of the theory **APP**. The name "extensional" derives from the analogy with extensional versions of Martin-Löf's type theory. We present an inductive definition for **HA**, which differs from Renardel's in the clauses for the quantifiers, and coincides with the logic of $\mathbb{N}$ in Pitts' topos.

5-6.2. We show that realizability 5 falsifies CT_0; the proof is not entirely trivial (from the point of view of higher order logic it is obvious that 5 falsifies CT since it is easy to see that 5 realizes $AC_{\sigma,\tau}$ for all finite types σ,τ over $\mathbb{N}$).But 5 realizes the following weakening of ECT_0:
$WECT_0$ $\forall x\,(A(x) \to \exists y B(x,y)) \to \neg\neg\exists z \forall x\,(A(x) \to \exists u\,(T(z,x,u) \wedge B(x,U(u))))$
Realizability 6 does not satisfy CT_0 but validates $\neg\neg A$ if A is the closure of an instance of CT_0. This is because the 6-realizable sentences are a subset of the Kleene-realizable ones.

5-6.3. Formalization is straightforward. Our disproof of CT_0 for 5 and 6 has as a corollary that these realizabilities, unlike those presented so far, are not "idempotent" in the sense that the schema $\phi \to \exists x\,(x \text{ realizes } \phi)$ is realizable. Basically, all known axiomatizations of realizabilities rely on the same trick, which presupposes this idempotency; this road is blocked for realizabilities 5-6. To illustrate the difficulty: we cannot have an axiom schema Ψ such that for 5-realizability: $\mathbf{HA}+\Psi \vdash A \leftrightarrow \exists x\,(x \text{ realizes } A)$ for every A. For in that case, the schema $A \to \exists x\,(x \text{ realizes } A)$ would be 5-realizable.

5-6.4. The fact that the 6-realizable sentences of arithmetic are a subset of the Kleene-realizable ones extends to full **HAH**, because the effective topos is an *open subtopos* of the topos generalizing 6-realizability, which I call Ext'; this means that the inverse image functor of the inclusion of Eff into Ext' is a logical functor. So Ext' satisfies $\neg\neg$CT, etc.
Ext, the topos for 5-realizability, refutes the continuity axiom which is, in the presence of choice for finite types, equivalent to WC-N:
Cont $\forall \zeta{:}\,(\mathbb{N}\to\mathbb{N})\to\mathbb{N}\, \forall f{:}\,\mathbb{N}\to\mathbb{N}\, \exists x{:}\,\mathbb{N}\, \forall g{:}\,\mathbb{N}\to\mathbb{N}\, (\bar{g}x=\bar{f}x \to \zeta(f)=\zeta(g))$,
but satisfies the following weakening of it:
WCont $\forall \zeta{:}\,(\mathbb{N}\to\mathbb{N})\to\mathbb{N}\, \forall f{:}\,\mathbb{N}\to\mathbb{N}\, \neg\neg\exists x{:}\,\mathbb{N}\, \forall g{:}\,\mathbb{N}\to\mathbb{N}\, (\bar{g}x=\bar{f}x \to \zeta(f)=\zeta(g))$,
and also
WCT $\forall \alpha{:}\,\mathbb{N}\to\mathbb{N}\, \neg\neg\exists e{:}\,\mathbb{N}\, \forall z{:}\,\mathbb{N}\, \exists u{:}\,\mathbb{N}\, (T(e,z,u) \wedge \alpha(z)=U(u))$

7-8.1. The first Kripke model of realizability was constructed by De Jongh in 1969. His aim was

to prove what is now known as *De Jongh's Theorem* : let $\phi(p_1,...,p_n)$ be a non-provable formula of intuitionistic propositional calculus. Then there are arithmetical sentences $B_1,...,B_n$ such that $\phi(B_1,...,B_n)$ is not provable in **HA**. The proof starts with a Kripke countermodel for ϕ, a system of partial combinatory algebras indexed by the tree of that model such that at every node of the tree, the partial combinatory algebra there contains a decision function for some non-recursive predicate associated with the propositional variable p_i if and only if p_i is forced at that node in the model. De Jongh also proved by this method a weak version of such a theorem concerning predicate logic. A Beth model for realizability was first presented by N. Goodman in 1978; Goodman showed that $\mathbf{HA}^{\omega}$ with decidable equality plus AC is conservative over **HA**.
De Jongh's results have been strengthened by Leivant (1975), and, for the propositional case, by Smorynski (1973); both use proof theory. In this thesis, De Jongh's original method is revived. I construct a sheaf model for **HA** plus a partial application symbol •, together with axioms saying that we have a partial combinatory algebra. In this theory, $\mathbf{HA}^{+}$, we can do realizability with •, and we have the theorem that whenever all arithmetical substitution instances of a predicate logical formula A are, provably in $\mathbf{HA}^{+}$, realizable, then A is provable in the predicate calculus. This gives an indication that the results of Plisko about the predicate logic of realizability depend essentially on the classical metatheory used.

7-8.3. If the system of partial combinatory algebras is arithmetically definable (and **HA** proves their necessary properties) these realizability notions can be formalized in **HA**. The model I give is based on an essentially classical theorem of Kleene & Post, however, and I don't see how it can be constructed inside **HA**.

7-8.4. These models have straightforward extensions to **HAH** (toposes), so we can extend the maximality of predicate logic for **HA** to **HAH**.

9.1. Modified realizability was defined by Kreisel (1959) as interpretation of $\mathbf{HA}^{\omega}$. The version I give results from an interpretation of $\mathbf{HA}^{\omega}$ in the model HRO of "hereditarily recursive operations" and is due to Troelstra (1973); the presentation given above was found by Grayson (1981B). (Think of D_p as a set of "potential realizers" of p, and p* as the set of actual realizers of p). The most outstanding feature of this interpretation is that it falsifies M_{PR}. It does satisfy CT_0 and a schema which is called "independence of premiss":

IP $\quad (\neg A \rightarrow \exists x B) \rightarrow \exists x (\neg A \rightarrow B)$,

x not free in A. Since this schema is inconsistent with ECT_0, we conclude that ECT_0 is stronger than CT_0.
Kleene's "special realizability" (Kleene & Vesley 1965) is an interpretation of **EL** completely

analogous to this notion, but based on partial continuous application (the application in 2)). Moschovakis 1971 is a modification of special realizability, where each set of actual realizers contains only total recursive functions. She shows the consistency of Kleene & Vesley's system + IP (replace $\exists x B$ by $\exists \alpha B$) + CT' where CT' is:

CT' $\quad \exists \alpha A(\alpha) \rightarrow \exists \alpha (GR(\alpha) \wedge A(\alpha))$,

where $GR(\alpha)$ asserts that α is recursive, and $\exists \alpha A(\alpha)$ is closed.

9.2. In contrast to the other realizabilities we have seen, modified realizability validates a purely predicate logical schema (and a propositional one: $(\neg A \rightarrow B \vee C) \rightarrow ((\neg A \rightarrow B) \vee (\neg A \rightarrow C))$) which is not provable in the predicate (propositional) calculus; its logic is therefore stronger than intuitionistic logic, even in an intuitionistic metatheory. Plisko 1990 claims the following result: let M be a model of $\mathbf{HA}^{\omega}$ in which all objects of type $0 \rightarrow 0$ are computable functions. Let $L(T_M)$ denote the modified realizability logic of M, i.e. the set of those formulas of predicate logic for which every arithmetical substitution instance has a modified realizer in M. Then $L(T_M)$ is not arithmetical (I am indebted to Professor V. Shehtman for this reference, as well as for a translation of the abstract).

9.3. Kreisel's notion for $\mathbf{HA}^{\omega}$ is easily axiomatized over a version of $\mathbf{HA}^{\omega}$ with decidable prime formulas (including equality at higher types) by the schemata IP and AC:

AC $\quad \forall x^{\sigma} \exists y^{\tau} A(x,y) \rightarrow \exists z^{\sigma \rightarrow \tau} \forall x^{\sigma} A(x,zx)$

(As observed by E.P. Alward, IP is not realized without decidability of equality at higher types; Eggerz 1986 gives a modification of IP which is always modified realized). The notion we have given above is not easily axiomatizable over **HA**: similar problems as with notions 5) and 6) arise.

9.4. There is a topos for modified realizability; it was constructed by Grayson (1981B). I fill in some gaps in the construction, in this thesis. It will be shown that Troelstra's extension of modified realizability to **HAS** coincides with second order arithmetic in this topos.

10. Modified Lifschitz' realizability was inspired by notion 9). It is defined in this thesis. It satisfies IP but not ECT_L. The higher order treatment is analogous to 9).

11.1. Kleene 1945 introduces a variation on his realizability: in the inductive definition given at the beginning of this chapter, consider the following changes, calling the resulting relation "$\vdash$-realizability":

x $\vdash$-realizes $A \rightarrow B$ iff for all y, if ($\vdash A$ and y $\vdash$-realizes A), then $x \bullet y \downarrow$ and $x \bullet y$ $\vdash$-realizes B;

x $\vdash$-realizes $\exists y A(y)$ iff $(x)_1$ $\vdash$-realizes $A((x)_0)$ and $\vdash A((x)_0)$.
He established soundness for this realizability and used it to prove the "explicit definability for numbers" property (EDN) for **HA**:

EDN If $\mathbf{HA} \vdash \exists x A(x)$, then for some n, $\mathbf{HA} \vdash A(\bar{n})$.

In Kleene 1969, formalized q-realizability for functions is defined. The idea is the same, but $\vdash A$ is replaced by A. Troelstra 1973 works out the similar notion for numbers. What is awkward about this notion is, that although one can prove soundness in the form $\mathbf{HA} \vdash A \Rightarrow \mathbf{HA} \vdash$ n q-realizes A for some n, it is not closed under deduction: it may be that for some A, $\mathbf{HA} \vdash A \rightarrow B$ and $\mathbf{HA} \vdash$ n q-realizes A, without there being an m with $\mathbf{HA} \vdash$ m q-realizes B.
This blemish (from the semantical point of view) was removed by Grayson who changed the definition of Kleene's realizability in the following way:
x q-realizes $A \rightarrow B$ iff $A \rightarrow B$ and for all y, if y q-realizes A then $x \bullet y \downarrow$ and $x \bullet y$ q-realizes B;
the other clauses are the same as for Kleene's original realizability, replacing "realizes" by "q-realizes". The resulting notion suffices for deriving the same proof-theoretical properties of **HA**, besides having the advantage that it is closed under deduction.

11.2. Only true formulas of arithmetic are q-realized. This follows from the fact that one can insert A (in the clause for "x q-realizes A") all along (instead of just doing it for the implication), and get an equivalent definition.

11.3. Important proof-theoretical results are obtained using formalized q-realizability. Apart from the mentioned EDN, one also has the "extended Church's rule":

ECR_0 If $\mathbf{HA} \vdash \forall x (A(x) \rightarrow \exists y B(x,y))$ and A is almost negative, then for some n,
$\mathbf{HA} \vdash \forall x (A(x) \rightarrow \bar{n} \bullet x \downarrow \wedge B(x, \bar{n} \bullet x))$

and as a consequence (in combination with $\mathrm{MR}_{\mathrm{PR}}$), Markov's Rule:

MR If $\mathbf{HA} \vdash \forall x (A(x) \vee \neg A(x)) \wedge \neg\neg \exists x A(x)$, then for some n, $\mathbf{HA} \vdash A(\bar{n})$

But there is a general method behind q-realizability, which can be applied to almost every realizability notion considered so far. In the inductive definitions, it is: carrying truth along. In the model theory, it is: glueing (see 11.4.). Consequently, there are many derived rules for **HA** and systems containing it, that can be proved by q-realizability. q-realizability for **HAS** was defined in Friedman 1977, establishing EDN, ECR_0 and MR for **HAS**. Kleene's q-realizability for functions gives for **EL** the rule of generalized continuity:

GCR If $\mathbf{EL} \vdash \forall \alpha (A(\alpha) \rightarrow \exists \beta B(\alpha, \beta))$ with A almost negative, then for some recursive function Ψ, $\mathbf{EL} \vdash \forall \alpha (A(\alpha) \rightarrow \Psi | \alpha \downarrow \wedge B(\alpha, \Psi | \alpha))$

A q-variant of Lifschitz' realizability gives a derived rule for $\mathbf{HA}+\mathrm{B}\Sigma_1^0\text{-MP}+\mathrm{M}_{\mathrm{PR}}$, analogous to the axiom schema $\mathrm{ECT}_{\mathrm{L}}$, and similar can be done for Lifschitz' function realizability. The

Independence of Premiss Rule:

IPR If $\mathbf{HA} \vdash \forall e(\neg A(e) \rightarrow \exists f B(e,f))$ with f not free in A, then for some number n, $\mathbf{HA} \vdash \forall e(\bar{n} \bullet e\downarrow \wedge (\neg A(e) \rightarrow B(e, \bar{n} \bullet e)))$

can be obtained by applying the q-device to modified realizability. To conclude this enumeration, a q-version of realizability 5) gives the following strengthening of ECR_0:

ECR_e If $\mathbf{HA} \vdash \forall e(\forall x \exists y Bexy \rightarrow \exists z Cez)$ and B is almost negative, then there is a number n such that:

$$\mathbf{HA} \vdash \forall e(n \bullet e\downarrow \wedge \forall f,f'(\forall x(f \bullet x\downarrow \wedge f' \bullet x\downarrow \wedge f \bullet x = f' \bullet x \wedge Bexf \bullet x) \rightarrow (n \bullet e) \bullet f\downarrow \wedge (n \bullet e) \bullet f'\downarrow \wedge (n \bullet e) \bullet f = (n \bullet e) \bullet f' \wedge Ce\,(n \bullet e) \bullet f))$$

(This thesis; chapter 8. To derive ECR_0, take x and y dummies)

11.4. The topos-theoretic approach to q-realizability was found by Grayson (1981A). He showed that the construction underlying it is one that is very familiar to category-theorists, namely glueing of toposes. This is the following: suppose $\mathcal{E}$ and $\mathcal{F}$ are toposes and $F: \mathcal{E} \rightarrow \mathcal{F}$ a left exact functor. Then let Gl(F) be the comma category $(\mathcal{F} \downarrow F)$. Gl(F) is a topos and $\mathcal{E}$ is an open subtopos of it. Now q-realizability corresponds to glueing an appropriate realizability topos along the inclusion from **sets** into it.

It would be nice if a general method for glueing of realizabilities existed; then we could expect derived rules relating different realizabilities to each other. But I have not found it.

This concludes the discussion of my sample. Two realizability definitions for (extensions of) **EL** that deserve mention, are Scarpellini 1977 and Krol 1983. It seems that Scarpellini combines realizability with the elimination translation for choice sequences, but it is not clear to me what this achieves. Krol's aim is to distinguish various continuity principles. However, in his definition (entirely in terms of the "hardware" of 3-tape Turing machines), evidently some wires have been crossed, so I have been unable to check his proof.

Since this thesis deals primarily with extensions of arithmetic, I shall be more succinct about realizabilities designed for other formalisms. I mentioned the Curry-Feys isomorphism between natural deduction trees for intuitionistic implicational logic and closed terms of the typed λ-calculus. Since second-order propositional logic IPC^2 can be presented with only $\forall$ and $\rightarrow$, there is an obvious isomorphism between IPC^2 and the polymorphic λ-calculus, also known as Girard's system F. System F has also been used for realizability purposes by Martin-Löf, in an unpublished manuscript around 1970, to obtain a characterization of the provably total recursive functions in second-order arithmetic. A realizability interpretation for second-order predicate logic, using the untyped λ-calculus, was given by Tait (1975). The idea in all these realizabilities is the same: one uses normalization of the relevant calculus.

Quite different from the above is Läuchli's intriguing semantics for **IQC** (1970), which he calls an "abstract notion of realizability". It remains a challenging problem to classify this conception among better known semantics. At first sight, it reminds one of modified realizability, but I believe the similarity is only superficial. Maybe there is a connection with J. Medvedev's "calculus of finite problems" (Medvedev 1962,1963,1966) which is an attempt to formalize Kolmogorov's interpretation of the intuitionistic connectives. However, the finite problems are equivalent to some kind of Heyting-valued semantics (Medvedev 1966).

Among theories in first-order predicate logic which have been studied by means of realizability the most prominent, apart from arithmetic, is intuitionistic set theory **IZF**. The first realizability notion for an intuitionistic set theory was given by Tharp (1971); the most influential one is Friedman's (1973). For a survey of Friedman's work on **IZF** the reader is referred to Scedrov 1985. Via realizability for **IZF**, much recursive mathematics can be obtained by using intuitionistic set theoretical arguments; this is carried out in McCarty 1984. A modification of Friedman's definition was given by Khakhayan (1988), who showed that in **IZF**, the uniformity principle is not derivable from Church's Thesis.

A realizability interpretation for analysis in an abstract formalism containing the combinators K and S was first defined by J. Staples (1973). Feferman 1974 describes such a formalism, later called **APP** by Renardel de Lavalette. Feferman does not do realizability, but uses his system to formalize set-theoretic constructions in a realizability-like way (which also resembles Martin-Löf's type theory). Realizability for **APP** itself is done in Renardel de Lavalette 1984. **APP** is an extension of a predicate logical version of combinatory logic, called $(CL)_i$ in Barendregt 1973. Barendregt defines an analogue of Kleene's $\vdash$-realizability for $(CL)_i$ and obtains the usual results (closure under rule of choice; existence and disjunction properties).

The connection between realizability and the formulas-as-types notion is very apparent in the whole structure of Martin-Löf's type theory **ML**. **ML** has a "built-in" realizability via the interpretation of logic in it, and it is called a "calculus of realizability" by Eggerz (1986). Conversely, realizability provides models for **ML** (e.g. Diller & Troelstra 1984; Swaen 1989).

Finally I should mention two realizability notions that have been given for systems based on logics stronger than intuitionistic logic: Lifschitz' theory of "calculable natural numbers" (Lifschitz 1985) and Flagg's realizability for arithmetic based on S4 (Flagg 1985). Lifschitz' formal system is arithmetic based on intuitionistic predicate logic without $\vee$ and with the axiom $\neg\neg\exists xA \rightarrow \exists x\neg\neg A$. There is an extra predicate K(x) which somehow expresses the constructive content of the number x: the K-free fragment is exactly Peano arithmetic, whereas relativizing all quantifiers to K gives a translation of **HA** into this system which Lifschitz conjectures is faithful. A notion of "solvability" is then given whose restriction to the embedded **HA**-formulae is just Kleene's realizability.

Flagg's realizability for Epistemic Arithmetic was obtained by first developing an analogue of Funayama's theorem for the effective tripos. One of his results is that an epistemic form of Church's Thesis is consistent with this arithmetic. An explicit inductive definition for this realizability has been given by Goodman (1986).

I finish this survey with a short philosophical discussion. Kleene explicitly denied that the Heyting-Kolmogorov interpretation was an inspiration for the definition of realizability (Kleene 1973), but he certainly had in mind that realizability should mirror intuitionistic reasoning. It is not quite correct to say that he saw existential statements as "incomplete communications", the realizers of which would provide a completion. Existential statements as incomplete communications was the view of Hilbert & Bernays. Kleene says: "Can we generalize this idea to think of *all* (except, trivially, the simplest) intuitionistic statements as incomplete communications?" (my italics). An implication is as incomplete as an existential statement, and can be completed by giving a recipe for obtaining, out of a completion of the premiss, a completion of the conclusion.

I must admit that I have always failed to understand why $\exists xA$ is more "incomplete" than $\forall xA$, so I am inclined to agree with this point of view. However, implications and universal quantifiers occur in the very definition of realizability (also existential quantifiers, but more innocently). This has led to criticism: "(...) it [i.e.realizability] cannot be said to make the intended meaning of the logical operators more precise. As a "philosophical reduction" of the interpretation of the logical operators it is also moderately successful; e.g. negative formulae are essentially interpreted by themselves" (Troelstra 1973, p.188).

This criticism seems to be valid only for formalized realizability, where the interpreting formula is of the same type as the interpreted.

Nevertheless, it is a pity that non-formalized realizability simply does *not* represent intuitionistic logic faithfully. And the result that it *does* , if you think intuitionistically, is of little help to those who want to understand intuitionism.

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II. Lifschitz' realizability

Abstract. V. Lifschitz defined in 1979 a variant of realizability which validates Church's thesis with uniqueness condition, but not the general form of Church's thesis. In this paper we describe an extension of intuitionistic arithmetic in which the soundness of Lifschitz' realizability can be proved, and we give an axiomatic characterization of the Lifschitz-realizable formulas relative to this extension. By a "q-variant" we obtain a new derived rule. We also show how to extend Lifschitz' realizability to second-order arithmetic. Finally we describe an analogous development for elementary analysis, with partial continuous application replacing partial recursive application.

§0. Introduction. In 1970, the Russian logician A. Dragalin raised the question whether, relative to intuitionistic first-order arithmetic **HA**, the schema

$$\mathrm{CT}_0 \qquad \forall x \exists y A(x,y) \to \exists z \forall x (z \bullet x\downarrow \; \& \; A(x, z \bullet x))$$

is really stronger than the form in which, in the premiss, a unique y is required, i.e.

$$\mathrm{CT}_0! \qquad \forall x \exists ! y A(x,y) \to \exists z \forall x (z \bullet x\downarrow \; \& \; A(x, z \bullet x))$$

(we write $z \bullet x$ for $\{z\}(x)$). The question was answered affirmatively in 1979 by Vladimir Lifschitz, who gave a modification of Kleene's realizability that satisfies $\mathrm{CT}_0!$, but refutes certain instances of CT_0 (Lifschitz [1979]; there is also a good exposition in Dragalin [1979]). This paper is concerned with a further investigation of Lifschitz' realizability.

First an extension **HA**$'$ of **HA** is defined in which Lifschitz' realizability can be formalized. An axiom schema is given which characterizes Lifschitz' realizability over **HA**$'$, much in the same way as ECT_0 characterizes formalized Kleene's realizability over **HA** (Troelstra [1973, 3.2.18]). As an application, a derived rule for **HA**$'$, similar to the extended Church's rule for **HA**, can be given.

It is shown that Lifschitz' realizability has a straightforward extension to **HAS**.

Finally, a Lifschitz analogon to Kleene's realizability for functions (Kleene [1969]) is defined. We show soundness and characterize this realizability over an extension of **EL**. It turns out that this interpretation satisfies general continuity with uniqueness:

$$\mathrm{GC!} \qquad \forall \alpha (A(\alpha) \to \exists ! \beta B(\alpha, \beta)) \to \exists \gamma \forall \alpha (A(\alpha) \to \gamma \mid \alpha\downarrow \; \& \; B(\alpha, \gamma \mid \alpha))$$

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(where $A(\alpha)$ has to satisfy a certain condition), whereas it is incompatible with weak continuity without uniqueness:

WC-N $\qquad \forall\alpha\exists nA(\alpha,n) \rightarrow \forall\alpha\exists n\exists m\forall\beta \in \bar{\alpha}nA(\beta,m)$

The reader will remember that GC (without uniqueness) is shown in Troelstra [1973, (3.3.11)] to characterize Kleene's realizability for functions.

It is also possible to extend Lifschitz' realizability to typed theories like $\mathbf{HA}^\omega$, and develop "modified Lifschitz' realizability". Another interesting aspect of it is that Lifschitz' realizability appears as the internal logic of the natural number object in a subtopos of the effective topos (the latter is described in Hyland [1982]). We hope to expand on these aspects in a later publication.

§5 of this paper formed the core of my master's thesis, which I wrote under Professor A. S. Troelstra. I am very much indebted to him for suggesting the subject to me, reading several successive versions and spotting a lot of mistakes. I am also grateful to the referee for some corrections and suggestions.

§1. Definitions and notation. $n, m, x, y, z, \ldots$ range over numbers: $X, Y, \ldots$ over sets; $\alpha, \beta, \gamma, \ldots$ over functions; $\sigma, \tau, \ldots$ over finite sequences of numbers.

We assume a bijective primitive recursive pairing function $j\colon \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ and inverses j_1 and j_2. The symbol $\bullet$ denotes partial recursive application, T is Kleene's predicate (so that $x \bullet y\downarrow$ iff $\exists zT(x,y,z)$, read *$x \bullet y$ is defined*), and U the result-extracting function. μ is the minimalization operator.

$\bar{\alpha}n$ denotes the initial sequence of α of length n. Recursively, $\bar{\alpha}0 = \langle\ \rangle$ (empty sequence) and $\bar{\alpha}(n+1) = \bar{\alpha}n * \langle\alpha(n+1)\rangle$ ($*$ denotes concatenation of sequences). $\beta(\alpha)\downarrow$ means $\exists x(\beta(\bar{\alpha}x) \neq 0)$ and $\beta(\alpha) = \beta(\bar{\alpha}(\mu z.\beta(\bar{\alpha}z) \neq 0)) - 1$ if it is defined; $\beta\,|\,\alpha\downarrow$ means $\forall x\ \beta(\langle x\rangle * \alpha)\downarrow$, and $\beta\,|\,\alpha = \lambda x.\beta(\langle x\rangle * \alpha)$ if it is defined.

$\sigma \supset \tau$ means that σ is an initial segment of τ; $\alpha \in \sigma$ means that σ is an initial segment of α. $\alpha \leq \beta$ means $\forall i\ \alpha(i) \leq \beta(i)$. $j_i\alpha = \lambda x.j_i(\alpha(x))$ for $i = 1, 2$. $\langle n\rangle^{[m]}$ will stand for a sequence of m n's, and $[n]$ will stand for $\lambda x.n$.

HA is taken to have function symbols and defining axioms for all primitive recursive functions. **HAS** is an extension of **HA** with variables for sets, and as extra axioms:

EXT $\qquad X(t)\ \&\ t = t' \rightarrow X(t'),$

CA $\qquad \exists X\forall y(X(y) \leftrightarrow \phi(y)),$

for every formula ϕ in the extended language (full impredicative comprehension).

EL is an extension of **HA** with variables for functions, abstraction operators $\lambda x.$ for every number variable x and a recursor R, with axioms

λ-CON $\qquad (\lambda x.t)(t') = t[t'/x] \quad (\lambda\text{-conversion}),$

R-ax $\qquad R(t,\phi,0) = t,$

$$R(t,\phi,St') = \phi(R(t,\phi,t'),t'),$$

for numerical terms t, t' and function terms ϕ, and

$\mathrm{QF\text{-}AC}_{00}$ $\qquad \forall x\exists yA(x,y) \rightarrow \exists\alpha\forall xA(x,\alpha(x))$

for quantifier-free formulas A.

Some principles that are used frequently:

M_{PR} $$\neg\neg\exists x A(x) \to \exists x A(x)$$

for A primitive recursive (Markov's principle),

$B\Sigma^0_2$-MP $$\neg\neg\exists x \leq y \forall n A(x,n,e) \to \exists x \leq y \forall n A(x,n,e)$$

for A primitive recursive (Markov's principle for bounded Σ^0_2-formulas).

HA' will stand for **HA** + M_{PR} + $B\Sigma^0_2$-MP.

MP_{QF} is Markov's principle with respect to quantifier-free formulas (in the language of **EL**).

KL_{QF} $$\forall n \exists \sigma(\mathrm{lth}(\sigma) = n \ \&\ \forall i < n(\sigma_i \leq \alpha(i)) \ \&\ R(\sigma)) \\ \to \exists \beta \forall n(\beta(n) \leq \alpha(n) \ \&\ R(\bar{\beta}n)),$$

for quantifier-free R, is quantifier-free König's lemma.

FAN_{QF} $$\forall \beta \leq \alpha \exists n R(\bar{\beta}n) \to \exists z \forall \beta \leq \alpha \exists n \leq z R(\bar{\beta}n)$$

(R quantifier-free) is the quantifier-free fan theorem.

EL' will denote **EL** + MP_{QF} + KL_{QF}.

$V_e = \{x \leq j_2 e \,|\, (j_1 e) \bullet x\uparrow\}$. We will use the abbreviation $V_e \neq \emptyset$ for the formula that says that V_e is *inhabited*, i.e. $\exists y(y \leq j_2 e \ \&\ \forall n \neg\neg T(j_1 e, y, n))$.

$V_\alpha = \{\beta \leq j_2\alpha \,|\, j_1\alpha(\beta)\uparrow\} = \{\beta \leq j_2\alpha \,|\, \forall n j_1\alpha(\bar{\beta}n) = 0\}$, with the same convention about $V_\alpha \neq \emptyset$.

We will freely use the applications $\bullet$ and $\cdot\,|\,\cdot$ as if they were part of the language of **HA** and **EL** respectively; this is justified by the fact that addition of symbols for definable partial functions with the corresponding defining axioms, gives a definitional extension (Troelstra and van Dalen [1988, Chapter 2, §7]). We will adopt the expressions "p-term" and "p-functor" (partially defined numerical term and function term, respectively) from Kleene [1969].

§2. Lifschitz' realizability: formalization, soundness. The crucial idea in proving $CT_0! \nvdash CT_0$ is to find a property $P(e,y)$ such that i) there is an effective procedure which, given that there is a unique y with $P(e,y)$, will find that y (recursively in e), and ii) there is no such procedure if uniqueness is not required.

The property $y \leq j_2 e \ \&\ \forall n \neg\neg T(j_1 e, y, n)$ meets this requirement. For, if there were a code g such that $V_e \neq \emptyset \Rightarrow g \bullet e\downarrow \ \&\ g \bullet e \in V_e$, and W_f and W_h are two disjoint, recursively inseparable r.e. sets, find a recursive function F such that

$$\forall x[F(x) \bullet 0 \simeq f \bullet x \ \&\ F(x) \bullet 1 \simeq h \bullet x].$$

Then always $V_{j(F(x),1)} \neq \emptyset$, so $g \bullet j(F(x),1) \in V_{j(F(x),1)}$ and g serves to construct a recursive separation between W_f and W_h. If V_e is a singleton, however, then one simply waits until $(j_1 e) \bullet x$ has been computed for all $x \leq j_2 e$ save one; the remaining one must be the element of V_e. Note that this same example shows that the principle $B\Sigma^0_2$-MP cannot be Kleene-realizable (since the premiss is equivalent to an almost negative formula, one derives a contradiction with ECT_0).

Lifschitz' realizability reads as follows: define for every formula ϕ a formula $x\underline{r}\phi$ with x not occurring in ϕ and all free variables of $x\underline{r}\phi$ contained in

$\{x\} \cup \{\text{free variables of } \phi\}$:

$$\begin{aligned} x\underline{r}A &\equiv A \quad \text{for } A \text{ atomic};\\ x\underline{r}A \,\&\, B &\equiv j_1 x\underline{r}A \,\&\, j_2 x\underline{r}B;\\ x\underline{r}A \to B &\equiv \forall y(y\underline{r}A \to x \bullet y\downarrow \,\&\, x \bullet y\underline{r}B);\\ x\underline{r}\exists y A(y) &\equiv V_x \neq \varnothing \,\&\, \forall g \in V_x j_2 g\underline{r}A(j_1 g);\\ x\underline{r}\forall y A(y) &\equiv \forall n(x \bullet n\downarrow \,\&\, x \bullet n\underline{r}A(n)). \end{aligned}$$

PROPOSITION 2.1. $\mathbf{B}\Sigma_2^0$-**MP** *is in* **HA** *equivalent to* $\forall e\Psi(e)$, *where*

$$\Psi(e) \quad \forall n\neg\neg[\mathrm{lth}(n) = j_2 e + 1 \,\&\, \forall i \leq j_2 e T(j_1 e, i, (n)_i)] \to \exists i \leq j_2 e \forall n \neg T(j_1 e, i, n).$$

(*This can be read as: if there is no witness for $V_e = \varnothing$, then V_e must contain an element.*)

PROOF. One has to show that

$$\forall n\neg(\mathrm{lth}(n) = j_2 e + 1 \,\&\, \forall i \leq j_2 e T(j_1 e, i, (n)_i)) \leftrightarrow \neg\neg\exists i \leq j_2 e \forall n \neg T(j_1 e, i, n),$$

and use a standard Kleene normal form for Π_1^0-predicates.

Now $\leftarrow$ is trivial because $(\mathrm{lth}(n) = j_2 e + 1 \,\&\, \forall i \leq j_2 e T(j_1 e, i, (n)_i))$ of course implies $\neg\exists i \leq j_2 e \forall n \neg T(j_1 e, i, n)$.

For $\to$, suppose $\neg\exists i \leq j_2 e \forall n \neg T(j_1 e, i, n)$; then $\forall i \leq j_2 e \neg\forall n \neg T(j_1 e, i, n)$, so $\forall i \leq j_2 e \neg\neg\exists n T(j_1 e, i, n)$. And this implies

$$\neg\neg\forall i \leq j_2 e \exists n T(j_1 e, i, n)$$

because of $\vdash \forall i \leq y \neg\neg\exists n T(z, i, n) \to \neg\neg\forall i \leq y \exists n T(z, i, n)$ (induction on y). Now $\neg\neg\forall i \leq j_2 e \exists n T(j_1 e, i, n)$ gives at once $\neg\neg\exists n \forall i \leq j_2 e T(j_1 e, i, (n)_i)$, wherefore $\neg\forall n \neg\forall i \leq j_2 e T(j_1 e, i, (n)_i)$, contradiction. Conclusion:

$$\neg\neg\exists i \leq j_2 e \forall n \neg T(j_1 e, i, n). \quad \boxtimes$$

In the sequel, one or the other of these two equivalent forms will be used whenever convenient. It is easy to show that, with respect to **EL**, $\forall e\Psi(e)$ is a consequence of $\mathrm{KL_{QF}}$ (see §5). The proof that Lifschitz' realizability is sound is a straightforward formalization of Lifschitz' original proof and is given by the following lemmas.

LEMMA 2.2. *There is a total recursive function b such that*

$$\mathbf{HA} \vdash \forall a \forall y(y \in V_{b(a)} \leftrightarrow y = a).$$

LEMMA 2.3. *There is a partial recursive function ϕ such that*

$$\mathbf{HA} + \mathbf{M_{PR}} \vdash \forall e(\exists x \forall y(y \in V_e \leftrightarrow y = x) \to \phi(e)\downarrow \,\&\, \phi(e) \in V_e).$$

The proofs are easy. $\boxtimes$

LEMMA 2.4. *There is a partial recursive function Φ such that*

$$\mathbf{HA}' \vdash \forall e, f\left[\begin{aligned}&\forall g \in V_e(f \bullet g\downarrow) \to \Phi(e,f)\downarrow\\ &\& \ \forall h(h \in V_{\Phi(e,f)} \leftrightarrow \exists g \in V_e(h = f \bullet g))\end{aligned}\right].$$

PROOF. $\exists g \in V_e(h = f \bullet g) \equiv \exists g \leq j_2 e(\forall n \neg T(j_1 e, g, n) \,\&\, \exists m(T(f, g, m) \,\&\, Um = h))$, which is, given that $\forall g \leq j_2 e(\forall n \neg T(j_1 e, g, n) \to \exists m T(f, g, m))$, equivalent to

$$\exists g \leq j_2 e \forall n[\neg T(j_1 e, g, n) \,\&\, (T(f, g, n) \to Un = h)],$$

or $\exists g \leq j_2 e \forall n \neg T(\chi(e, h, f), g, n)$ for a suitable primitive recursive χ; by $\forall e\Psi(e)$,

$\exists g \leq j_2 e \forall n \neg T(\chi(e,h,f),g,n)$ is equivalent to $\forall n \neg(\mathrm{lth}(n) = j_2 e + 1 \ \&\ \forall i \leq j_2 e T(\chi(e,h,f),i,(n)_i))$, or to $\forall n \neg T(\chi'(e,f),h,n)$ for suitable $\chi'(e,f)$; let $\Phi(e,f)$ be $j(\chi'(e,f), \kappa(e,f))$ with $\kappa(e,f) \simeq \max\{Un \mid n = \min_z(T(j_1 e,l,z) \vee T(f,l,z)), l \leq j_2 e\}$. Note that this is defined, by $\mathbf{M_{PR}}$. ☒

LEMMA 2.5. *There is a total recursive function γ such that*

$$\mathbf{HA}' \vdash \forall e \forall h (h \in V_{\gamma(e)} \leftrightarrow \exists g \in V_e(h \in V_g)).$$

In other words, $V_{\gamma(e)} = \bigcup(V_g \mid g \in V_e)$.

PROOF. $\exists g \in V_e(h \in V_g)$ is

$$\exists g \leq j_2 e(\forall n \neg T(j_1 e,g,n) \ \&\ h \leq j_2 g \ \&\ \forall n \neg T(j_1 g,h,n))$$

or $\exists g \leq j_2 e \forall n \neg T(\pi(e,h),g,n)$ for suitable π; which by $\forall e \Psi(e)$ is equivalent to

$$\forall n \neg(\mathrm{lth}(n) = j_2 e + 1 \ \&\ \forall i \leq j_2 e T(\pi(e,h),i,(n)_i))$$

or $\forall n \neg T(\pi'(e),h,n)$ for suitable π'; so if we take $\gamma(e) := j(\pi'(e), \max\{j_2 g \mid g \leq j_2 e\})$, then γ satisfies the lemma. ☒

LEMMA 2.6. *For every formula A in the language of* **HA** *there is a p-term $\chi_A(x)$ (which may contain variables occurring free in A) such that*

$$\mathbf{HA}' \vdash \forall e(V_e \neq \varnothing \ \&\ \forall f \in V_e(f\underline{\mathbf{r}}A) \rightarrow \chi_A(e)\downarrow \ \&\ \chi_A(e)\underline{\mathbf{r}}A).$$

LEMMA 2.7. *For every closed theorem A of* **HA** *there is a number n such that* $\mathbf{HA}' \vdash n\underline{\mathbf{r}}A$.

Lemmas 2.6 and 2.7 are immediate formalizations of Lifschitz' Lemmas 5 and 6. ☒

REMARK. It remains an open problem whether the soundness of Lifschitz' realizability can be proved in **HA**. We have no proof of the impossibility of this, although it seems highly doubtful to us.

§3. Characterization of Lifschitz' realizability. The following lemma gives a more uniform look to Lifschitz' realizability.

LEMMA 3.1. *Define a realizability* $\mathbf{r}'$ *by the following clauses*:

1) $\quad x\mathbf{r}'t = s \equiv V_x \neq \varnothing \ \&\ \forall y \in V_x\, t = s \quad (y \text{ not in } t = s!)$,

2) $\quad x\mathbf{r}'A \ \&\ B \equiv V_x \neq \varnothing \ \&\ \forall y \in V_x\, ((j_1 y)\mathbf{r}'A) \ \&\ ((j_2 y)\mathbf{r}'B)$,

3) $\quad x\mathbf{r}'A \rightarrow B \equiv V_x \neq \varnothing \ \&\ \forall y \in V_x\, \forall w(w\mathbf{r}'A \rightarrow y \bullet w\downarrow \ \&\ y \bullet w\mathbf{r}'B)$,

4) $\quad x\mathbf{r}'\forall z A z \equiv V_x \neq \varnothing \ \&\ \forall y \in V_x\, \forall n(y \bullet n\downarrow \ \&\ y \bullet n\mathbf{r}'A(n))$,

5) $\quad x\mathbf{r}'\exists z A z \equiv V_x \neq \varnothing \ \&\ \forall y \in V_x\, j_2 y\mathbf{r}'A(j_1 y)$.

Then for every formula A in the language of **HA** *there are partial recursive functions ϕ_A and ψ_A (they may contain variables occurring free in A) such that*

$$\mathbf{HA}' \vdash \forall e(e\underline{\mathbf{r}}A \rightarrow \phi_A(e)\downarrow \ \&\ \phi_A(e)\mathbf{r}'A),$$
$$\mathbf{HA}' \vdash \forall e(e\mathbf{r}'A \rightarrow \psi_A(e)\downarrow \ \&\ \psi_A(e)\underline{\mathbf{r}}A).$$

(Note the form of the clauses: apart from a prefix $V_x \neq \varnothing \ \&\ \forall y \in V_x$, it is just the Kleene clauses.)

PROOF. We define ϕ_A and ψ_A and prove the lemma simultaneously by induction on A. The notation is from the lemmas in §2. Following Lifschitz we write g^* for $\Lambda f.\Phi(f,g)$, where Φ is as in Lemma 2.4.

i) $\phi_{t=s}(e) \equiv b(e)$,
$\psi_{t=s}(e) \equiv 0$;

ii) $\phi_{A\,\&\,B}(e) \equiv b(j(\phi_A(j_1 e), \phi_B(j_2 e)))$,
$\psi_{A\,\&\,B}(e) \equiv j(\chi_A((\psi_A \circ j_1)^* \bullet e), \chi_B((\psi_B \circ j_2)^* \bullet e))$;

iii) $\phi_{A\to B}(e) \equiv b(\Lambda h.\phi_B(e \bullet \psi_A(h)))$,
$\psi_{A\to B}(e) \equiv \chi_{A\to B}(g^* \bullet e)$, where $g \equiv \Lambda f.\Lambda a.\psi_B(f \bullet \phi_A(a))$;

iv) $\phi_{\forall x A x}(e) \equiv b(\Lambda n.\phi_A[n/x](e \bullet n))$,
$\psi_{\forall x A x}(e) \equiv \chi_{\forall x A x}(g^* \bullet e)$, where $g \equiv \Lambda f.(\Lambda n.\psi_A[n/x](f \bullet n))$;

v) $\phi_{\exists x A x}(e) \equiv g^* \bullet e$ with $g \equiv \Lambda f.j(j_1 f, \phi_A[j_1 f/x](j_2 f))$,
$\psi_{\exists x A x}(e) \equiv g^* \bullet e$ with $g \equiv \Lambda f.j(j_1 f, \psi_A[j_1 f/x](j_2 f))$.

We trust that the reader will be able to carry out the proof by himself. ⊠

Note that this reformulation makes Lemma 2.6 superfluous; a trivial induction on A shows that if $V_e \neq \varnothing$ & $\forall f \in V_e(f\mathbf{r}'A)$, and γ is as defined in Lemma 2.5, then $\gamma(e)\mathbf{r}'A$.

DEFINITION. Let Γ be the class of formulas inductively generated by the clauses:
1) Σ^0_1-formulas are in Γ;
2) formulas of form $\exists x \leq y\, Ax$, with $A \in \Pi^0_1$, are in Γ;
3) Γ is closed under $\forall$, $\to$ and &.

As Γ will play a role similar to that of the "almost negative" formulas in §3.2 of Troelstra [1973], which could be termed Σ^0_1-negative, let us call Γ-formulas "BΣ^0_2-negative".

LEMMA 3.2 (cf. Troelstra [1973, 3.2.11]). *For every* BΣ^0_2*-negative formula* $A(a)$ *(with free variables* a*) there is a partial recursive function* ψ_A *satisfying*
i) $\mathbf{HA}' \vdash \exists u(u\mathbf{r}'A) \to A$ *and*
ii) $\mathbf{HA}' \vdash A(a) \to \psi_A(a)\downarrow$ & $\psi_A(a)\mathbf{r}'A(a)$.

PROOF. We prove i) and ii) simultaneously by induction on A.

1) Suppose A is $\exists y By$, B prime; then $u\mathbf{r}'A$ is $V_u \neq \varnothing$ & $\forall f \in V_u(V_{j_2 f} \neq \varnothing$ & $\forall h \in V_{j_2 f} B(j_1 f))$ which clearly implies A; for ii) take $\psi_A \equiv b(j(\chi_B, b(0)))$ where $\chi_B \equiv \mu x.Bx$. For then A implies $\chi_B\downarrow$ and $b(0)\mathbf{r}'B(\chi_B)$, so $\psi_A\mathbf{r}'\exists x Bx$. The case of arbitrary Σ^0_1-formulas follows by soundness.

2) Suppose $A \equiv \exists x \leq tBx$, x not in t, B is Π^0_1; say $B \equiv \forall y Cxy$. Then $u\mathbf{r}'\exists x\forall y(x \leq t$ & $Cxy)$ is equivalent to

$$(*)\qquad V_u \neq \varnothing \;\&\; \forall h \in V_u(V_{j_2 h} \neq \varnothing \;\&\; \forall k \in V_{j_2 h}\forall n[k \bullet n\downarrow \;\&\; k \bullet n\mathbf{r}'(j_1 h \leq t \;\&\; Cj_1 hn)])$$

which implies $V_u \neq \varnothing$ & $\forall h \in V_u(V_{j_2 h} \neq \varnothing$ & $\forall n Cj_1 hn)$, which implies A. For ii) let e be such that A is equivalent to $V_e \neq \varnothing$, and let u be such that $V_u = \{j(j_1 h, b(\Lambda n.b(0))) \mid h \in V_e\}$; then $V_e \neq \varnothing$ implies $(*)$ for u.

3) We will only do the case $A \equiv B \to C$; the other cases are left to the reader. $u\mathbf{r}'A$ is $V_u \neq \varnothing$ & $\forall h \in V_u \forall x(x\mathbf{r}'B \to h \bullet x\downarrow$ & $h \bullet x\mathbf{r}'C)$. Now if B then $\psi_B\mathbf{r}'B$, so $\forall h \in V_u$ $(h \bullet \psi_B\downarrow$ & $h \bullet \psi_B\mathbf{r}'C)$; so if χ is such that $V_{\chi(u)} = \{h \bullet \psi_B \mid h \in V_u\}$ then $\gamma(\chi(u))\mathbf{r}'C$ (γ from Lemma 2.5), so C. But if $B \to C$ then $b(\Lambda u.\psi_C)\mathbf{r}'B \to C$, for suppose $u\mathbf{r}'B$, then B, so C, so $\psi_C\mathbf{r}'C$. ⊠

REMARK. Thus the $B\Sigma_2^0$-negative formulas are the "self-realizing" formulas for this realizability. As a quick glance reveals that formulas of form $x\mathbf{r}'A$ are $B\Sigma_2^0$-negative, this realizability is idempotent. Furthermore, since $\forall e\Psi(e)$ is also $B\Sigma_2^0$-negative, as well as $\mathrm{M_{PR}}$, we see that the soundness theorem for **HA** for this realizability can be extended to **HA′**.

We now introduce a principle analogous to ECT_0. Consider

$\mathrm{ECT_L}$ $\forall x(Ax \to \exists yBxy) \to \exists z\forall x(Ax \to z \bullet x\downarrow \;\&\; V_{z\bullet x} \neq \emptyset \;\&\; \forall h \in V_{z\bullet x}Bxh)$,

for A $B\Sigma_2^0$-negative (here "L" stands for "Lifschitz").

LEMMA 3.3 (cf. Troelstra [1973, 3.2.15]). $\mathrm{ECT_L}$ *is* $\mathbf{r}'$-*realizable.*

PROOF. Suppose $u\mathbf{r}'\forall x(Ax \to \exists yBxy)$. This is:

$$V_u \neq \emptyset \;\&\; \forall f \in V_u\forall n(f \bullet n\downarrow \;\&\; V_{f\bullet n} \neq \emptyset \;\&\; \forall h \in V_{f\bullet n}\forall w(w\mathbf{r}'An \to h \bullet w\downarrow \;\&\; V_{h\bullet w} \neq \emptyset \;\&\; \forall k \in V_{h\bullet w}(j_2k\mathbf{r}'Bnj_1k))).$$

Let us simplify a bit. Let u' be such that

$$\forall n(u' \bullet n\downarrow \;\&\; V_{u'\bullet n} = \bigcup(V_{f\bullet n} \mid f \in V_u));$$

then

$$\forall h \in V_{u'\bullet n}\forall w(w\mathbf{r}'An \to h \bullet w\downarrow \;\&\; V_{h\bullet w} \neq \emptyset \;\&\; \forall k \in V_{h\bullet w}(j_2k\mathbf{r}'Bnj_1k)).$$

Put $\beta(h) \equiv h \bullet \psi_A(n)$, and choose u'' such that $V_{u''\bullet n} = \bigcup(V_{\beta(h)} \mid h \in V_{u'\bullet n})$; then

$$\forall w(w\mathbf{r}'An \to u'' \bullet n\downarrow \;\&\; V_{u''\bullet n} \neq \emptyset \;\&\; \forall k \in V_{u''\bullet n}(j_2k\mathbf{r}'Bnj_1k)).$$

It is clear that u'' can be obtained recursively in u.

Now choose z with $\forall x(V_{z\bullet x} = j_1[V_{u''\bullet x}])$, ζ' such that $V_{\zeta'(m)} = \{k \mid j(m,k) \in V_{u''\bullet x}\}$, and ζ'' such that $V_{\zeta''(m)} = \{\Lambda y.\gamma(\zeta'(m))\}$ (γ from Lemma 2.5). Then we have $V_{\zeta''(m)} \neq \emptyset$, and if $g\mathbf{r}'(m \in V_{z\bullet x})$ then $m \in V_{z\bullet x}$ (since this is $B\Sigma_2^0$-negative), so $V_{\zeta'(m)} \neq \emptyset$ & $\forall k \in V_{\zeta'(m)}k\mathbf{r}'Bxm$, so $\gamma(\zeta'(m))\mathbf{r}'Bxm$, by the remark following the proof of Lemma 3.1. Let $\zeta \equiv b(\zeta'')$ (b from Lemma 2.2), then

$$V_\zeta \neq \emptyset \;\&\; \forall l \in V_\zeta\forall m(l \bullet m\downarrow \;\&\; V_{l\bullet m} \neq \emptyset \;\&\; \forall p \in V_{l\bullet m}\forall g(g\mathbf{r}'(m \in V_{z\bullet x}) \to p \bullet g\downarrow \;\&\; p \bullet g\mathbf{r}'Bxm)),$$

which is $\zeta\mathbf{r}'\forall h(h \in V_{z\bullet x} \to Bxh)$. The rest is easy. ⊠

THEOREM 3.4 (cf. Troelstra [1973, 3.2.18]; characterization of $\mathbf{r}'$-realizability).

i) $\mathbf{HA}' + \mathrm{ECT_L} \vdash A \leftrightarrow \exists x(x\mathbf{r}'A)$;

ii) $\mathbf{HA}' \vdash \exists x(x\mathbf{r}'A) \Leftrightarrow \mathbf{HA}' + \mathrm{ECT_L} \vdash A$.

PROOF. i) is proved by induction on A. As usual, the only nontrivial steps are $A \equiv B \to C$ and (similar) $A \equiv \forall yBy$.

Now

$$\begin{aligned}(B \to C) &\leftrightarrow \forall x(x\mathbf{r}'B \to \exists y(y\mathbf{r}'C))\\ &\leftrightarrow \exists z\forall x(x\mathbf{r}'B \to z \bullet x\downarrow \;\&\; V_{z\bullet x} \neq \emptyset \;\&\; \forall y \in V_{z\bullet x}(y\mathbf{r}'C))\\ &\leftrightarrow \exists z\forall x(x\mathbf{r}'B \to z \bullet x\downarrow \;\&\; z \bullet x\mathbf{r}'C) \leftrightarrow \exists x(x\mathbf{r}'(B \to C)).\end{aligned}$$

We leave the other case to the reader.

The proof of ii) (using i)) is completely analogous to 3.2.18 of Troelstra [1973].

REMARKS ON $\mathrm{ECT_L}$. i) $\mathrm{ECT_L}!$ is equivalent to a schema which resembles $\mathrm{ECT_0}!$ except for the condition that A can be taken $\mathrm{B}\Sigma^0_2$-negative. We see that this schema is consistent relative to **HA**, whereas $\mathrm{ECT_0}$ with respect to $\mathrm{B}\Sigma^0_2$-negative formulas is not: if W_e and W_f are disjoint, recursively inseparable r.e. sets, let F be such that $\forall x(F(x) \bullet 0 \simeq e \bullet x \,\&\, F(x) \bullet 1 \simeq f \bullet x)$; then $V_{j(F(x),1)} \neq \emptyset$ for all x, so let $Ax \equiv V_{j(F(x),1)} \neq \emptyset$ ($\mathrm{B}\Sigma^0_2$-negative) and $Bxy \equiv y \in V_{j(F(x),1)}$. Any z as in the conclusion of the schema will give a recursive separation between W_e and W_f.

ii) The example

$$A \equiv \exists yTxxy \vee \neg\exists yTxxy, \qquad B \equiv ((z = 0 \rightarrow \exists yTxxy) \,\&\, (z = 1 \rightarrow \neg Txxy)),$$

given in 3.2.20 of Troelstra [1973], shows that the restriction to $\mathrm{B}\Sigma^0_2$-negative formulas cannot be dropped.

iii) In analogy to Grayson's modification of Kleene's **q**-realizability (see exercises 4.4.7 and 4.4.8 of Troelstra and van Dalen [1988]), we can define a **q**$'$-realizability corresponding to **r**$'$-realizability by:

$$x\mathbf{q}'A \rightarrow B \equiv V_x \neq \emptyset \,\&\, \forall y \in V_x \forall w(w\mathbf{q}'A \rightarrow y \bullet w\downarrow \,\&\, y \bullet w\mathbf{q}'B) \,\&\, A \rightarrow B,$$

the other clauses being the same as for **r**$'$-realizability (with **r**$'$ replaced by **q**$'$).

PROPOSITION 3.5. i) *If A is a closed formula, then* $\mathbf{HA}' \vdash A \Rightarrow \mathbf{HA}' \vdash n\mathbf{q}'A$ *for some n.*

ii) $\mathbf{HA}' \vdash y\mathbf{q}'A \rightarrow A$.

iii) $\mathbf{HA}' \vdash A \rightarrow \psi_A\mathbf{q}'A$ *for ψ_A as in Lemma* 3.2, *if A is* $\mathrm{B}\Sigma^0_2$*-negative.*

PROOF. The first statement is proved by a routine induction on lengths of deductions in **HA**; the reader may wish to consult Theorem 3.2.4 of Troelstra [1973]. The other two statements are proved by induction on A. ☒

COROLLARY 3.6. **HA**$'$ *obeys the following rule*:

$$\vdash \forall x(Ax \rightarrow \exists yBxy) \Rightarrow \exists n \vdash \forall x(Ax \rightarrow \bar{n} \bullet x\downarrow \,\&\, V_{\bar{n}\bullet x} \neq \emptyset \,\&\, \forall h \in V_{\bar{n}\bullet x}Bxh),$$

for A $\mathrm{B}\Sigma^0_2$*-negative. In particular,* **HA**$'$ *obeys the rule*

$$\vdash \forall x(Ax \rightarrow \exists! yBxy) \Rightarrow \exists n \vdash \forall x(Ax \rightarrow \bar{n} \bullet x\downarrow \,\&\, B(x, \bar{n} \bullet x)),$$

for A $\mathrm{B}\Sigma^0_2$*-negative.*

§4. Extension of Lifschitz' realizability to HAS. The extension of Kleene's realizability to **HAS**, described in 3.2.29 of Troelstra [1973], is given by the simple clauses

$$x\mathbf{r}(t_0,\dots,t_{n-1}) \in X \equiv (t_0,\dots,t_{n-1},x) \in X^*,$$
$$x\mathbf{r}\forall XA(X) \equiv \forall X^* x\mathbf{r}A(X),$$
$$x\mathbf{r}\exists XA(X) \equiv \exists X^* x\mathbf{r}A(X),$$

where $X \rightarrow X^*$ is an operation that assigns to each n-ary set variable X an $(n+1)$-ary set variable X^* from a fresh stock of variables.

This extension satisfies the uniformi' rinciple:

UP $$\forall X \exists n A(X, n) \rightarrow \exists n \forall X A(X, n).$$

An extension of Lifschitz' realizability by the same clauses cannot work, because in that case we would have all realizability clauses equal for both interpretations except for the clause for the numerical existential quantifier; but this quantifier can be eliminated in **HAS**, because of the equivalence

$$\exists y A(y) \leftrightarrow \forall X(\forall y(Ay \rightarrow X) \rightarrow X),$$

that holds in second-order logic with full comprehension. So then these two interpretations would be the same, quod non. However, combined with Lemma 3.1, this idea suggests the following extension:

6) $x\mathbf{r}'(t_0,\dots,t_{n-1}) \in X \equiv V_x \neq \varnothing \,\&\, \forall y \in V_x\, (t_0,\dots t_{n-1}, y) \in X^*,$

7) $x\mathbf{r}'\forall X A(X) \equiv V_x \neq \varnothing \,\&\, \forall y \in V_x\, \forall X^* y\mathbf{r}'A(X),$

8) $x\mathbf{r}'\exists X A(X) \equiv V_x \neq \varnothing \,\&\, \forall y \in V_x\, \exists X^* y\mathbf{r}'A(X).$

THEOREM 4.1. $\mathbf{r}'$ *is a sound realizability for* **HAS** $+ \forall e\Psi(e) + \mathrm{M}_{\mathbf{PR}}$.

PROOF. The verification of the rules for second-order predicate logic does not pose any problem. For instance, if $\psi(y)\mathbf{r}'A(y) \rightarrow B$, y not in B, and $x\mathbf{r}'\exists y Ay$, where A and B are arbitrary formulas in the language of **HAS**, then $V_x \neq \varnothing \,\&\, \forall y \in V_x$ $j_2 y\mathbf{r}'A(j_1 y)$, so

$$V_x \neq \varnothing \,\&\, \forall y \in V_x \forall h \in V_{\psi(j_1 y)}(h \bullet (j_2 y)\downarrow \,\&\, h \bullet (j_2 y)\mathbf{r}'B).$$

Let χ be such that $V_\chi = \{h \bullet (j_2 y) \mid h \in V_{\psi(j_1 y)}, y \in V_x\}$; then $\gamma(\chi(x))\mathbf{r}'B$, so $b(\Lambda x.\gamma(\chi(x)))\mathbf{r}'\exists y Ay \rightarrow B$, where b and γ are as defined in Lemmas 2.2 and 2.5.

For the comprehension schema

CA $$\exists X \forall y(y \in X \leftrightarrow Ay),$$

first note that the following holds:

(*) $$V_k \neq \varnothing \,\&\, \forall l \in V_k \exists k'(l \in V_{k'} \,\&\, k'\mathbf{r}'A) \rightarrow k\mathbf{r}'A.$$

(This is trivial from the definition of $\mathbf{r}'$-realizability.) Now $x\mathbf{r}'\exists X \forall y(y \in X \leftrightarrow Ay)$ means

(◦) $$\begin{aligned} &V_x \neq \varnothing \,\&\, \forall f \in V_x \exists X^* \forall y[f \bullet y\downarrow \,\&\, \forall k \\ &\quad (V_k \neq \varnothing \,\&\, \forall l \in V_k (y,l) \in X^* \rightarrow j_1(f \bullet y) \bullet k\downarrow \,\&\, j_1(f \bullet y) \bullet k\mathbf{r}'Ay \\ &\quad \& \, k\mathbf{r}'Ay \rightarrow t = j_2(f \bullet y) \bullet k\downarrow \,\&\, V_t \neq \varnothing \,\&\, \forall l \in V_t (y,l) \in X^*)]. \end{aligned}$$

Now let $V_x = \{f\}$, with f such that $j_1(f \bullet y) \bullet k = j_2(f \bullet y) \bullet k = k$; and if

$$X^* = \{(y,l) \mid \exists k(k\mathbf{r}'Ay \,\&\, l \in V_k)\},$$

then (◦) is easily verified for f, x, and X^*, using (*). The verification of extensionality

EXT $$Ay \,\&\, y = x \rightarrow Ax$$

is completely trivial, which concludes the proof. ⊠

§5. A Lifschitz analogon to realizability for functions. This is based on the following analogy between the sets V_α and the sets V_e: there is no function γ such that for all α, if $V_\alpha \neq \varnothing$ then $\gamma \mid \alpha\downarrow \,\&\, \gamma \mid \alpha \in V_\alpha$. However, if V_α is a singleton, then we can get its element continuously in α.

If we read the principle $\forall e\Psi(e)$ from Proposition 2.1 as: ((there is no witness n for $V_e = \varnothing) \rightarrow V_e \neq \varnothing)$, then the analogous principle in the situation of the sets V_α

would be

$$\forall n \exists \sigma[\mathrm{lth}(\sigma) = n \ \&\ \forall i < n(\sigma_i \leq j_2\alpha(i))) \ \&\ \forall \tau \supset \sigma(j_1\alpha(\tau) = 0)] \to \exists \beta \leq j_2\alpha \forall n j_1\alpha(\bar{\beta}n) = 0,$$

a principle that is an instance of $\mathrm{KL_{QF}}$. Since we will need this principle, as well as $\mathrm{MP_{QF}}$, we will work in the theory $\mathbf{EL}' = \mathbf{EL} + \mathrm{KL_{QF}} + \mathrm{MP_{QF}}$.

Note that $\mathbf{EL}' \vdash \mathrm{FAN_{QF}}$.

PROPOSITION 5.1.

$$\mathbf{EL} + \mathrm{KL_{QF}} \vdash \neg\neg\exists\gamma\forall n(\gamma(n) \leq \alpha(n) \ \&\ A(\beta,\gamma,z,n)) \to \exists\gamma\forall n(\gamma(n) \leq \alpha(n) \ \&\ A(\beta,\gamma,z,n)),$$

for quantifier-free formulas A such that α does not occur in A.

PROOF. Let A' be such that

$$\forall n(\gamma(n) \leq \alpha(n) \ \&\ A(\beta,\gamma,z,n)) \leftrightarrow \forall n(\gamma(n) \leq \alpha(n) \ \&\ A'(\beta,\bar{\gamma}n,z)).$$

If $\neg\neg\exists\gamma\forall n(\gamma(n) \leq \alpha(n) \ \&\ A(\beta,\gamma,z,n))$, then

$$\forall n\neg\neg\exists\sigma(\mathrm{lth}(\sigma) = n \ \&\ \forall i < n(\sigma_i \leq \alpha(i)) \ \&\ A'(\beta,\sigma,z)).$$

The quantifier $\exists\sigma$ is bounded, so

$$\forall n\exists\sigma(\mathrm{lth}(\sigma) = n \ \&\ \forall i < n(\sigma_i \leq \alpha(i)) \ \&\ A'(\beta,\sigma,z)).$$

By $\mathrm{KL_{QF}}$, the conclusion follows. ☒

A counterpart to Proposition 2.1 fails, of course: since $\mathrm{KL_{QF}}$ is false in some models of **EL**, it cannot be equivalent to a schema which is $\neg\neg$-valid.

PROPOSITION 5.2. $\mathbf{EL} + \mathrm{KL_{QF}} \vdash \mathrm{KL}_{\Sigma^0_1}$.

PROOF. Suppose $\forall n\exists\sigma[\mathrm{lth}(\sigma) = n \ \&\ \forall i < n(\sigma)_i \leq \alpha(i) \ \&\ \exists m R(\sigma,m)]$ with R quantifier-free, so

$$\forall n\exists m\exists\sigma[\mathrm{lth}(\sigma) = n \ \&\ \forall i < n(\sigma)_i \leq \alpha(i) \ \&\ R(\sigma,m)],$$

so

$$(\mathrm{QF\text{-}AC_{00}}) \qquad \exists\alpha_1\alpha_2\forall n[\mathrm{lth}(\alpha_1(n)) = n \ \&\ \forall i < n(\alpha_1(n))_i \leq \alpha(i) \ \&\ R(\alpha_1(n),\alpha_2(n))],$$

so

$$\exists\alpha_2\forall n\exists\sigma[\mathrm{lth}(\sigma) = n \ \&\ \forall i < n(\sigma)_i \leq \alpha(i) \ \&\ R(\sigma,\alpha_2(n))],$$

which gives, with $\mathrm{KL_{QF}}$,

$$\exists\alpha_2\exists\beta\forall n[\beta(n) \leq \alpha(n) \ \&\ R(\bar{\beta}n,\alpha_2(n))],$$

so

$$\exists\beta\forall n[\beta(n) \leq \alpha(n) \ \&\ \exists m R(\bar{\beta}n,m)]. \quad ☒$$

DEFINITION. We define for every formula A a formula $\alpha \underline{r} A$ with $\alpha \notin \mathrm{FV}(A)$ and $\mathrm{FV}(\alpha \underline{r} A) \subset \{\alpha\} \cup \mathrm{FV}(A)$ as follows:

1) $\alpha \underline{r} A \equiv A$ for A atomic;

2) $\alpha \underline{r} A \,\&\, B \equiv j_1\alpha \underline{r} A \,\&\, j_2\alpha \underline{r} B$;

3) $\alpha \underline{r} A \to B \equiv \forall\beta(\beta \underline{r} A \to \alpha\,|\,\beta\!\downarrow \,\&\, \alpha\,|\,\beta \underline{r} B)$;

4) $\alpha \underline{r} \forall x A x \equiv \forall n(\alpha\,|\,[n]\!\downarrow \,\&\, \alpha\,|\,[n] \underline{r} A n)$;

5) $\alpha \underline{r} \exists x A x \equiv V_\alpha \neq \varnothing \,\&\, \forall\gamma \in V_\alpha(j_2\gamma \underline{r} A(j_1\gamma(0)))$;

6) $\alpha \underline{r} \forall\beta A(\beta) \equiv \forall\beta(\alpha\,|\,\beta\!\downarrow \,\&\, \alpha\,|\,\beta \underline{r} A(\beta))$;

7) $\alpha \underline{r} \exists\beta A(\beta) \equiv V_\alpha \neq \varnothing \,\&\, \forall\gamma \in V_\alpha(j_2\gamma \underline{r} A(j_1\gamma))$.

The proof that **EL** is sound for this realizability goes completely parallel to the proof in §2, and is given in the following lemmas.

LEMMA 5.3. *There is a p-functor β_1 such that*

$$\mathbf{EL}' \vdash \forall\alpha(V_\alpha \textit{ is a singleton} \to \beta_1\,|\,\alpha\!\downarrow \,\&\, \beta_1\,|\,\alpha \in V_\alpha).$$

PROOF. Write $B_\alpha \equiv \{\beta\,|\,\beta \leq j_2\alpha\}$.

If $V_\alpha = \{\beta\}$ then for every n and m such that $m \leq j_2\alpha(n)$ and $m \neq \beta(n)$, a finite computation suffices to show that $j_1\alpha(\gamma)\!\downarrow$, for every γ such that $\gamma \in \bar{\beta}n * \langle m\rangle$ and $\gamma \in B_\alpha$. For, $\{\gamma \in B_\alpha\,|\,\gamma \in \bar{\beta}n * \langle m\rangle\}$ is a finitely branching tree. (Here, of course, we are using FAN.)

Now $\forall\gamma \in B_\alpha(\gamma \in \bar{\beta}n * \langle m\rangle \Rightarrow j_1\alpha(\gamma)\!\downarrow)$ holds for every $m \leq j_2\alpha(n)$ save one; a finite computation shows this and the remaining $m \leq j_2\alpha(n)$ must be equal to $\beta(n)$.

LEMMA 5.4. *There is a p-functor β_2, such that*

$$\mathbf{EL}' \vdash \forall\alpha(\beta_2\,|\,\alpha\!\downarrow \,\&\, V_{\beta_2|\alpha} = \{\alpha\}).$$

PROOF. Let γ be such that $\forall\alpha((\gamma\,|\,\alpha)(\sigma) = 0 \leftrightarrow \alpha \in \sigma)$; take β_2 such that $\forall\alpha(\beta_2\,|\,\alpha = j(\gamma\,|\,\alpha, \alpha))$.

The following sublemma, trivial as it may be, greatly simplifies the proofs of the lemmas hereafter, and will be applied frequently.

SUBLEMMA 5.5. *Let $A(\beta)$ and $C(\beta,\gamma)$ be formulas such that:*

1) *there is a p-functor ψ such that $A(\beta) \vdash \psi\,|\,\beta\!\downarrow \,\&\, \forall\gamma(C(\beta,\gamma) \to \gamma \leq \psi\,|\,\beta)$; and*

2) *$A(\beta) \vdash C(\beta,\gamma) \leftrightarrow \forall n D(\beta,\gamma,n)$, where D is a prime formula.*

Then there is a p-functor Φ such that

$$\mathbf{EL}' \vdash A(\beta) \to \Phi\,|\,\beta\!\downarrow \,\&\, \forall\gamma(\gamma \in V_{\Phi|\beta} \leftrightarrow C(\beta,\gamma)).$$

PROOF. If D is the prime formula from 2), there is a prime formula $D'(\beta,\sigma)$ such that $\forall n D(\beta,\gamma,n)$ is equivalent to $\forall n D'(\beta,\bar{\gamma}n)$. Now let χ be defined as follows: $\chi(\sigma) = 0$ if $D'(\beta,\sigma)$; $\chi(\sigma) = 1$ else. Now put $\Phi := \Lambda\beta.j(\chi,\psi)$, where ψ is the functor from condition 1). ⊠

LEMMA 5.6. *There is a p-functor β_3 such that*

$$\mathbf{EL}' \vdash \forall\alpha\left(\beta_3\,|\,\alpha\!\downarrow \,\&\, V_{\beta_3|\alpha} = \bigcup_{\gamma \in V_\alpha} V_\gamma\right).$$

PROOF. We apply Sublemma 5.5. It is easy to see that $\varepsilon \in \bigcup_{\gamma \in V_\alpha} V_\gamma \to \varepsilon \leq \max\{j_2\gamma\,|\,\gamma \leq j_2\alpha\}$. Furthermore, the formula $\beta \in \bigcup_{\gamma \in V_\alpha} V_\gamma$ is equivalent to

$$\exists\gamma \leq j_2\alpha \forall n(j_1\alpha(\bar{\gamma}n) = 0 \,\&\, \beta(n) \leq j_2\gamma(n) \,\&\, j_1\gamma(\bar{\beta}n) = 0)$$

which is, modulo KL and MP, equivalent to

$$\forall n \exists \sigma[\text{lth}(\sigma) = n \;\&\; j_1\alpha(\sigma) = 0 \;\&\; \forall k < n(\sigma_k \leq j_2\alpha(k) \;\&\; (\bar{\beta}n)_k \leq j_2(\sigma_k) \\ \&\; (\bar{\beta}k < n \rightarrow j_1(\sigma_{\bar{\beta}k}) = 0)],$$

which is a formula of the form required in condition 2) of the sublemma. ☒

LEMMA 5.7. *There is a p-functor Φ such that*

$$\mathbf{EL}' \vdash \forall \phi, \beta[\forall \alpha(\alpha \in V_\beta \rightarrow \phi \,|\, \alpha\!\downarrow) \rightarrow \Phi \,|\, (\phi, \beta)\!\downarrow \\ \&\; \forall \alpha(\alpha \in V_{\Phi|(\phi,\beta)} \leftrightarrow \exists \gamma(\gamma \in V_\beta \;\&\; \alpha = \phi \,|\, \gamma)].$$

In other words: $V_\beta \subseteq \text{dom}(\phi) \rightarrow \phi \,|\, [V_\beta] = V_{\Phi|(\phi,\beta)}$.

In the following, for p-functors ϕ, we will use the abbreviation ϕ^* for the p-functor $\Lambda\beta.\Phi(\phi, \beta)$.

PROOF. Again, we check the conditions of Sublemma 5.5.

1) Suppose $\forall \alpha(\alpha \in V_\beta \rightarrow \phi \,|\, \alpha\!\downarrow)$. Then

$$\forall x \forall \alpha \leq j_2\beta(\forall n(j_1\beta(\bar{\alpha}n) = 0) \rightarrow (\phi \,|\, \alpha)(x)\!\downarrow),$$

which is equivalent to

$$\forall x \forall \alpha \leq j_2\beta \neg\neg \exists n(j_1\beta(\bar{\alpha}n) \neq 0 \vee \phi(\langle x \rangle * \bar{\alpha}n) \neq 0),$$

which by MP is equivalent to

$$\forall x \forall \alpha \leq j_2\beta \exists n(j_1\beta(\bar{\alpha}n) \neq 0 \vee \phi(\langle x \rangle * \bar{\alpha}n) \neq 0),$$

which in turn, by FAN, is equivalent to

$$\forall x \exists n \forall \alpha \leq j_2\beta \exists z \leq n(j_1\beta(\bar{\alpha}z) \neq 0 \vee \phi(\langle x \rangle * \bar{\alpha}z) \neq 0).$$

Note that the part following $\forall x \exists n$ is actually quantifier-free, so we can define ψ by

$$\psi \equiv \lambda x.\mu n.[\forall \alpha \leq j_2\beta \exists z \leq n(j_1\beta(\bar{\alpha}z) \neq 0 \vee \phi(\langle x \rangle * \bar{\alpha}z) \neq 0)].$$

Let $\Phi(x, z)$ be $(\phi \,|\, \alpha)(x)$ if $\phi(\langle x \rangle * \bar{\alpha}z) \neq 0$ (and otherwise, for example, undefined). Now put

$$\eta(x) = \max\{\Phi(x, z) \,|\, z \leq \psi(x) \;\&\; z \text{ witnesses } (\phi \,|\, \alpha)(x)\!\downarrow\},$$

and $\eta(x) = 0$ if this set is empty; then $\chi \equiv \lambda x.\eta(x)$ is the required upper bound.

2) Now $\gamma \in \phi \,|\, [V_\beta]$ is, modulo $V_\beta \subseteq \text{dom}(\phi)$, equivalent to a Π^0_1-formula; in fact, $\exists \delta \in V_\beta(\gamma = \phi \,|\, \delta)$ is equivalent to

$$\exists \delta \forall n(\delta(n) \leq j_2\beta(n) \;\&\; j_1\beta(\bar{\delta}n) = 0 \;\&\; \exists z(\forall k < z\phi(\langle n \rangle * \bar{\delta}k) = 0 \\ \&\; \phi(\langle n \rangle * \bar{\delta}z) = \gamma(n) + 1)),$$

which, modulo $\forall \alpha(\alpha \in V_\beta \rightarrow \phi \,|\, \alpha\!\downarrow)$, is equivalent to

$$\exists \delta \forall n[\delta(n) \leq j_2\beta(n) \;\&\; j_1\beta(\bar{\delta}n) = 0 \;\&\; \forall z((\forall k < z\phi(\langle n \rangle * \bar{\delta}k) = 0 \;\&\; \phi(\langle n \rangle * \bar{\delta}z) > 0) \\ \rightarrow \phi(\langle n \rangle * \bar{\delta}z) = \gamma(n) + 1)],$$

and, in view of the boundedness of δ, this is in $\mathbf{EL}'$ equivalent to a Π^0_1-formula. ☒

LEMMA 5.8. *For every formula A in the language of* **EL** *there is a p-functor χ_A,* *which may contain free variables occurring in A, such that*

$$\mathbf{EL}' \vdash \forall\beta[V_\beta \neq \varnothing \ \&\ \forall\alpha \in V_\beta(\alpha \underline{r} A) \to \chi_A \mid \beta\downarrow \ \&\ \chi_A \mid \beta \underline{r} A].$$

Proof. χ_A is defined by induction on the logical complexity of A:

1) $\chi_A \equiv [1]$ if A and $\chi_A \equiv [0]$ if $\neg A$, for A atomic. Remember that $[0] \mid \alpha\uparrow$ for every α.

2) $\chi_A \equiv \Lambda\beta.j(\chi_B \mid j_1^*\beta, \chi_C \mid j_2^*\beta)$ if $A \equiv B \,\&\, C$. For suppose $V_\beta \neq \varnothing$ and $\forall\alpha \in V_\beta$ $(\alpha \underline{r} B \,\&\, C)$; then $j_1 \mid [V_\beta] = V_{j_1^*\beta}$ (Lemma 5.7) $\neq \varnothing$ and $\forall\alpha \in V_{j_1^*\beta}(\alpha \underline{r} B)$, so $\chi_B \mid j_1^*\beta\downarrow$ and $\chi_B \mid j_1^*\beta \underline{r} B$; analogously for C.

3) $\chi_A \equiv \Lambda\beta.(\Lambda\gamma.(\chi_C \mid (\psi_\gamma^* \mid \beta)))$, where ψ_γ is such that $\forall\alpha.\psi_\gamma \mid \alpha \simeq \alpha \mid \gamma$, if $A \equiv B \to C$. For suppose $V_\beta \neq \varnothing$, $\forall\alpha \in V_\beta(\alpha \underline{r} B \to C)$, and $\gamma \underline{r} B$; then $\psi_\gamma \mid [V_\beta] = V_{\psi_\gamma^* \mid \beta} \neq \varnothing$ and $\forall\delta \in V_{\psi_\gamma^* \mid \beta} \delta \underline{r} C$, so $\chi_C \mid (\psi_\gamma^* \mid \beta)\downarrow$ and $\underline{r} C$.

4) $\chi_A \equiv \Lambda\beta.(\Lambda\gamma.(\chi_{A(x)}[\gamma(0)/x] \mid (\psi_\gamma^* \mid \beta)))$, where ψ_γ is such that $\forall\alpha.\psi_\gamma \mid \alpha \simeq \alpha \mid [\gamma(0)]$, if $A \equiv \forall x A(x)$. For suppose $V_\beta \neq \varnothing$ and $\forall\alpha \in V_\beta(\alpha \underline{r} \forall x A(x))$, γ arbitrary; then $\psi_\gamma \mid [V_\beta] = V_{\psi_\gamma^* \mid \beta} \neq \varnothing$ and $\forall\alpha \in V_{\psi_\gamma^* \mid \beta} \alpha \underline{r} A(x)[\gamma(0)/x]$, so $\chi_{A(x)}[\gamma(0)/x] \mid (\psi_\gamma^* \mid \beta)\downarrow$ and $\underline{r} A(\gamma(0))$.

5) $\chi_A \equiv$ the functor β_3 from Lemma 5.6, if $A \equiv \exists x B(x)$ or $\exists\alpha B(\alpha)$.

6) $\chi_A \equiv \Lambda\beta.(\Lambda\gamma.(\chi_{B(\alpha)}[\gamma/\alpha] \mid (\psi_\gamma^* \mid \beta)))$, where ψ_γ is such that $\forall\alpha.\psi_\gamma \mid \alpha \simeq \alpha \mid \gamma$, if $A \equiv \forall\alpha B\alpha$. For if $V_\beta \neq \varnothing$ and $\forall\delta \in V_\beta(\delta \underline{r} \forall\alpha B\alpha)$, γ arbitrary, then $\psi_\gamma \mid [V_\beta] = V_{\psi_\gamma^* \mid \beta} \neq \varnothing$ and $\forall\delta \in V_{\psi_\gamma^* \mid \beta} \delta \underline{r} B\alpha[\gamma/\alpha]$, so $\chi_{B(\alpha)}[\gamma/\alpha] \mid (\psi_\gamma^* \mid \beta) \underline{r} B\alpha[\gamma/\alpha]$, etc. ☒

LEMMA 5.9. *For every formula A in the language of* **EL** *such that* $\mathbf{EL} \vdash A$ *there is a p-functor* ψ_A *such that* $\mathbf{EL}' \vdash \psi_A\downarrow \,\&\, \psi_A \underline{r} A$; ψ_A *may contain variables occurring free in A.*

PROOF. This goes by induction on proofs in **EL**. Since our realizability differs only in the existential clauses from Kleene's, we only have to check the lemma for those rules and axioms of two-sorted predicate calculus that concern existential formulas, as well as for QF-AC_{00}.

It is clear that

$$\Lambda\alpha.\beta_2 \mid j([t], \alpha) \underline{r} A(t) \to \exists x A(x),$$
$$\Lambda\alpha.\beta_2 \mid j(\phi, \alpha) \underline{r} A(\phi) \to \exists\alpha A(\alpha),$$

for β_2 from Lemma 5.4.

Now suppose $\alpha \underline{r} A(y) \to C$, y possibly in α, not in C. Then $\Lambda\gamma.\chi_C \mid (\psi^* \mid \gamma) \underline{r} \exists x A(x) \to C$, where χ_C from Lemma 5.8 and ψ is such that $\psi \mid \beta \simeq \alpha[j_1\beta(0)/y] \mid j_2\beta$. For suppose $\gamma \underline{r} \exists x A(x)$, so $V_\gamma \neq \varnothing \,\&\, \forall\beta \in V_\gamma(j_2\beta \underline{r} A(j_1\beta(0))$. Then for $\beta \in V_\gamma$ we have that $\psi \mid \beta \underline{r} C$, so $\forall\delta \in \psi \mid [V_\beta] = V_{\psi^* \mid \beta}(\delta \underline{r} C)$, so $\chi_C \mid (\psi^* \mid \gamma) \underline{r} C$.

The argument is completely analogous for $(A(\phi) \to C) \to (\exists\alpha A(\alpha) \to C)$.

The following sublemma will be useful for the proof that QF-AC_{00} is realized. Although we cannot expect, as will be seen further on, to obtain elements of nonempty V_γ's continuously in the γ, we can, if we restrict to a continuous image of $\mathbb{N}$.

SUBLEMMA 5.10. *There is a functor* χ *such that*

$$\mathbf{EL}' \vdash \forall\varepsilon[\forall n(\varepsilon \mid [n]\downarrow \,\&\, V_{\varepsilon \mid [n]} \neq \varnothing) \to \chi \mid \varepsilon\downarrow \,\&\, V_{\chi \mid \varepsilon} \neq \varnothing \,\&\, \forall\gamma \in V_{\chi \mid \varepsilon} \forall n(\gamma \mid [n]\downarrow \,\&\, \gamma \mid [n] \in V_{\varepsilon \mid [n]})].$$

PROOF. To apply Sublemma 5.5, we construct a bounded primitive recursive condition for sequences σ which says that σ is "for the time being" an initial segment of a γ such that $\forall n(\gamma|[n]\downarrow \,\&\, \gamma|[n] \in V_{\varepsilon|[n]})$. Let $\sigma|[n]$ denote the maximal τ such that $\gamma|[n] \in \tau$ for all γ with $\gamma|[n]\downarrow$ and $\gamma \in \sigma$. (This is clearly primitive recursive in n and σ.)

Our condition $A(\varepsilon, \sigma)$ will be the conjunction of the following 4 (we apologise for the somewhat heavy notation):

1) $\forall i < \mathrm{lth}(\sigma)(\sigma_i \leq i)$ (we want $A(\varepsilon,\sigma)$ to be a bounded condition);

2) $\forall n < \sigma \forall i < \mathrm{lth}(\sigma|[n])(i < \mathrm{lth}(\bar{\varepsilon}(\mathrm{lth}(\sigma))|[n]) \to (\sigma|[n])_i \leq j_2((\bar{\varepsilon}(\mathrm{lth}(\sigma))|[n])_i))$ (so if $\gamma \in \sigma$ then for the time being $\gamma|[n] \leq j_2(\varepsilon|[n])$;

3) $\forall n < \sigma \forall i < \mathrm{lth}(\bar{\varepsilon}(\mathrm{lth}(\sigma))|[n])[\exists k < \sigma j_2(((\bar{\varepsilon}(\mathrm{lth}(\sigma))|[n])_i) \leq f(i,n,k) < \mathrm{lth}(\sigma) \to \exists k < \sigma(\sigma_{f(i,n,k)} \neq 0)]$, where $f(i,n,k) = \langle i \rangle * \langle n \rangle^{[k]}$ (we want to force, for every γ such that $\forall n A(\varepsilon, \bar{\gamma}n)$, that $\gamma|[n]$ is defined, for all n; i.e., $\forall i \exists k \gamma(f(i,n,k)) \neq 0$. At the same time, $\gamma(f(i,n,k)) - 1\,(=(\gamma|[n])(i)) \leq j_2((\varepsilon|[n])(i))$ should hold. But we cannot, beforehand, exclude any value $\leq j_2((\varepsilon|[n])(i))$; and since $\sigma_{f(i,n,k)} \leq f(i,n,k)$ by condition 1), we cannot force $\sigma_{f(i,n,k)} \neq 0$ until $f(i,n,k)$ is big enough); and, finally,

4) $\forall n < \sigma \forall \tau \supset \sigma|[n](\tau < \mathrm{lth}(\bar{\varepsilon}(\mathrm{lth}(\sigma))|[n]) \to j_1((\bar{\varepsilon}(\mathrm{lth}(\sigma))|[n])_\tau = 0)$ (so $\gamma \in V_{\varepsilon|[n]}$ if $\forall m A(\varepsilon, \bar{\gamma}m)$).

Now let (Sublemma 5.5) δ be such that $\forall \gamma(\gamma \in V_\delta \leftrightarrow \forall n A(\varepsilon, \bar{\gamma}n))$, and put $\chi \equiv \Lambda\varepsilon.\delta$. Now if $\forall n(\varepsilon|[n]\downarrow \,\&\, V_{\varepsilon|[n]} \neq \emptyset)$, then there are arbitrarily long sequences σ with $A(\varepsilon,\sigma)$; with KL we conclude that $V_{\chi|\varepsilon} \neq \emptyset$.

QF-AC$_{00}$. Let $F \equiv \forall x \exists y Axy \to \exists \alpha \forall x A(x, \alpha x)$ be an instance of QF-AC$_{00}$, and suppose δ realizes the premiss. Then

$$\forall n \delta|[n]\downarrow \,\&\, V_{\delta|[n]} \neq \emptyset \,\&\, \forall \gamma \in V_{\delta|[n]}(j_2\gamma \underline{r} A(n, j_1\gamma(0))).$$

Let ψ be such that $\psi|\gamma \simeq j([j_1\gamma(0)], j_2\gamma)$. Then, for all n, $V_{\psi^*|(\delta|[n])} = \psi|[V_{\delta|[n]}] \neq \emptyset$ (Lemma 5.7), and $\forall \gamma \in V_{\psi^*|(\delta|[n])} j_2\gamma \underline{r} A(n, j_1\gamma(n))$. Apply Sublemma 5.10 to find a χ such that

$$\forall \gamma \in V_{\chi|\delta} \forall n(\gamma|[n]\downarrow \,\&\, \gamma|[n] \in V_{\psi^*|(\delta|[n])}),$$

then this χ realizes F.

This concludes the proof that **EL** is sound for $\underline{r}$. ☒

We now get some lemmas that are analogous to Lemma 3.1 and following.

LEMMA 5.11. *Define a realizability* $\mathbf{r}'$ *by the clauses*:

1) $\alpha\mathbf{r}'A \equiv V_\alpha \neq \emptyset \,\&\, \forall \beta \in V_\alpha\, A$ for A atomic;

2) $\alpha\mathbf{r}'A \,\&\, B \equiv V_\alpha \neq \emptyset \,\&\, \forall \beta \in V_\alpha\, j_1\beta\mathbf{r}'A \,\&\, j_2\beta\mathbf{r}'B$;

3) $\alpha\mathbf{r}'A \to B \equiv V_\alpha \neq \emptyset \,\&\, \forall \beta \in V_\alpha\, \forall \gamma(\gamma\mathbf{r}'A \to \beta|\gamma\downarrow \,\&\, \beta|\gamma\mathbf{r}'B)$;

4) $\alpha\mathbf{r}'\forall x Ax \equiv V_\alpha \neq \emptyset \,\&\, \forall \beta \in V_\alpha\, \forall n(\beta|[n]\downarrow \,\&\, \beta|[n]\mathbf{r}'An)$;

5) $\alpha\mathbf{r}'\exists x Ax \equiv V_\alpha \neq \emptyset \,\&\, \forall \beta \in V_\alpha\, j_2\beta\mathbf{r}'A(j_1\beta(0))$;

6) $\alpha\mathbf{r}'\forall \beta A(\beta) \equiv V_\alpha \neq \emptyset \,\&\, \forall \beta \in V_\alpha\, \forall \gamma(\beta|\gamma\downarrow \,\&\, \beta|\gamma\mathbf{r}'A(\gamma))$;

7) $\alpha\mathbf{r}'\exists \beta A(\beta) \equiv V_\alpha \neq \emptyset \,\&\, \forall \beta \in V_\alpha\, j_2\beta\mathbf{r}'A(j_1\beta)$.

Then for all formulas in the language of **EL** *there are p-functors* ϕ_A *and* ψ_A *such that*

$$\mathbf{EL}' \vdash \forall \alpha(\alpha\underline{r}A \to \phi_A|\alpha\downarrow \,\&\, (\phi_A|\alpha)\mathbf{r}'A),$$

$$\mathbf{EL}' \vdash \forall \alpha(\alpha\mathbf{r}'A \to \psi_A|\alpha\downarrow \,\&\, (\psi_A|\alpha)\underline{r}A).$$

PROOF. For those who are not yet asleep, we give the definitions.

i) $\phi_{t=s} \equiv \lambda\alpha.\beta_2 \,|\, \alpha,$
$\psi_{t=s} \equiv \lambda\alpha.[0];$

ii) $\phi_{A\,\&\,B} \equiv \lambda\alpha.\beta_2 \,|\, j(\phi_A \,|\, j_1\alpha, \phi_B \,|\, j_2\alpha),$
$\psi_{A\,\&\,B} \equiv \lambda\alpha.j(\chi_A \,|\, ((\psi_A \circ j_1)^* \,|\, \alpha), \chi_B \,|\, ((\psi_B \circ j_2)^* \,|\, \alpha));$

iii) $\phi_{A\to B} \equiv \lambda\alpha.\beta_2 \,|\, (\lambda\gamma.\,\phi_B \,|\, (\alpha \,|\, (\psi_A \,|\, \gamma))),$
$\psi_{A\to B} \equiv \lambda\alpha.\chi_{A\to B} \,|\, (\zeta^* \,|\, \alpha),$ where $\zeta \equiv \lambda\beta.\lambda\gamma.\psi_B \,|\, (\beta \,|\, (\phi_A \,|\, \gamma));$

iv) $\phi_{\forall xAx} \equiv \lambda\alpha.\beta_2 \,|\, (\lambda n.\phi_A[n/x] \,|\, (\alpha \,|\, [n])),$
$\psi_{\forall xAx} \equiv \lambda\alpha.\chi_{\forall xAx} \,|\, (\zeta^* \,|\, \alpha),$ where $\zeta \equiv \lambda\beta.(\lambda n.\psi_A[n/x] \,|\, (\beta \,|\, [n]));$

v) $\phi_{\exists xAx} \equiv \lambda\alpha.\zeta^* \,|\, \alpha$ with $\zeta \equiv \lambda\beta.j(j_1\beta, \phi_A[j_1\beta(0)/x] \,|\, (j_2\beta)),$
$\psi_{\exists xAx} \equiv \lambda\alpha.\zeta^* \,|\, \alpha$ with $\zeta \equiv \lambda\beta.j(j_1\beta, \phi_A[j_1\beta(0)/x] \,|\, (j_2\beta));$

vi) $\phi_{\exists\gamma A\gamma} \equiv \lambda\alpha.\zeta^* \,|\, \alpha$ with $\zeta \equiv \lambda\beta.j(j_1\beta, \phi_A[j_1\beta/\gamma] \,|\, (j_2\beta)),$
$\phi_{\exists\gamma A\gamma} \equiv \lambda\alpha.\zeta^* \,|\, \alpha$ with $\zeta \equiv \lambda\beta.j(j_1\beta, \phi_A[j_1\beta/\gamma] \,|\, (j_2\beta));$

vii) $\phi_{\forall\gamma A\gamma} \equiv \lambda\alpha.\beta_2 \,|\, (\lambda\delta.\phi_A[\delta/\gamma] \,|\, (\alpha \,|\, \delta)),$
$\psi_{\forall\gamma A\gamma} \equiv \lambda\alpha.\chi_{\forall\gamma A\gamma}(\zeta^* \,|\, \alpha),$ where $\zeta \equiv \lambda\beta.(\lambda\delta.\psi_A[\delta/\gamma] \,|\, (\beta \,|\, \delta)).$

We hope that it is clear by now how to transpose the rest of §2 to the case of **EL**; therefore we state the following lemmas without proof.

DEFINITION. The class Γ of $\mathrm{B}\Sigma^1_2$-negative formulas is the smallest satisfying the following 3 conditions:

i) Formulas of form $\exists\alpha A(\alpha)$ are in Γ, with A quantifier-free.

ii) Formulas of form $\exists\alpha \le \beta\forall n A(\alpha, n)$ are in Γ, with A quantifier-free.

iii) Γ is closed under $\to$, &, $\forall x$, $\forall\alpha$.

LEMMA 5.12. *For every* $\mathrm{B}\Sigma^1_2$*-negative formula* $A(a)$ *with free variables* a *there is a p-functor* ξ_A *such that*

$$\mathbf{EL}' \vdash \exists\alpha(\alpha\mathbf{r}'A) \to A,$$
$$\mathbf{EL}' \vdash A(a) \to \xi_A \,|\, a\!\downarrow \;\&\; (\xi_A \,|\, a)\mathbf{r}'A(a).$$

COROLLARY 5.13. **EL**′ *is sound for* **r**′.

For, KL and MP are $\mathrm{B}\Sigma^1_2$-negative.

DEFINITION. Let $\mathrm{GC_L}$ be the following schema:

$$\mathrm{GC_L} \qquad \forall\alpha(A\alpha \to \exists\beta B\alpha\beta) \to \exists\gamma\forall\alpha(A\alpha \to \gamma \,|\, \alpha\!\downarrow \;\&\; V_{\gamma|\alpha} \ne \varnothing \;\&\; \forall\zeta \in V_{\gamma|\alpha} B\alpha\zeta),$$

with the restriction that A must be $\mathrm{B}\Sigma^1_2$-negative.

LEMMA 5.14. $\mathrm{GC_L}$ *is* **r**′*-realizable.*

THEOREM 5.15. i) $\mathbf{EL}' + \mathrm{GC_L} \vdash A \leftrightarrow \exists\alpha(\alpha\mathbf{r}'A)$.

ii) $\mathbf{EL}' + \mathrm{GC_L} \vdash \exists\alpha(\alpha\mathbf{r}'A) \Leftrightarrow \mathbf{EL}' \vdash A$.

As a minor application of **r**′-realizability we have that $\mathrm{GC_L}!$, so a fortiori not GC!, is not sufficient to prove GC, the principle of generalized continuity:

$$\mathrm{GC} \qquad \forall\alpha(A\alpha \to \exists\beta B\alpha\beta) \to \exists\gamma\forall\alpha(A\alpha \to \gamma \,|\, \alpha\!\downarrow \;\&\; B\alpha\gamma \,|\, \alpha),$$

which is considered in Troelstra [1973] and is proven there to axiomatize Kleene's realizability based on partial continuous application.

We can do better, for the weakest well-known continuity principle without uniqueness condition in the premiss, the schema

WC-N $$\forall\alpha\exists nA(\alpha,n) \to \forall\alpha\exists n\exists m\forall\beta \in \bar{\alpha}nA(\beta,m)$$

(weak continuity for numbers), is already incompatible with KL:

PROPOSITION 5.16. *WC-N and* KL *are incompatible with respect to* **EL**.

PROOF. Define a functor Γ as follows: $\Gamma(\langle\ \rangle) = 0$, and

$$\Gamma(\langle\sigma\rangle * n) \begin{cases} = 0 & \text{if } \mathrm{lth}(\sigma) > \mathrm{lth}(n), \\ = 1 & \text{if } \mathrm{lth}(\sigma) \le \mathrm{lth}(n)\ \&\ \forall i < \mathrm{lth}(n)(n_i \ne 0\ \&\ n_i \ne 1), \\ = 1 & \text{if } \mathrm{lth}(\sigma) \le \mathrm{lth}(n)\ \&\ \exists i < \mathrm{lth}(n)(n_i = 0\ \&\ \forall j < i n_j \ne 1) \\ & \&\ \forall i < \mathrm{lth}(\sigma)\sigma_i = 0, \\ = 2 & \text{if } \mathrm{lth}(\sigma) \le \mathrm{lth}(n)\ \&\ \exists i < \mathrm{lth}(n)(n_i = 0\ \&\ \forall j < i n_j \ne 1) \\ & \&\ \exists i < \mathrm{lth}(\sigma)\sigma_i \ne 0, \\ = 1 & \text{if } \mathrm{lth}(\sigma) \le \mathrm{lth}(n)\ \&\ \exists i < \mathrm{lth}(n)(n_i = 1\ \&\ \forall j < i n_j \ne 0) \\ & \&\ \forall i < \mathrm{lth}(\sigma)\sigma_i = 1, \\ = 2 & \text{if } \mathrm{lth}(\sigma) \le \mathrm{lth}(n)\ \&\ \exists i < \mathrm{lth}(n)(n_i = 1\ \&\ \forall j < i n_j \ne 0) \\ & \&\ \exists i < \mathrm{lth}(\sigma)\sigma_i \ne 1. \end{cases}$$

Then $(\Gamma\,|\,\alpha)(\sigma) = \Gamma(\langle\sigma\rangle * \bar{\alpha}(\mathrm{lth}(\sigma))) - 1$ is always defined.

Let γ be such that $\forall\alpha\gamma\,|\,\alpha = j(\Gamma\,|\,\alpha, [1])$. Then we have

$$\forall\alpha\forall n\exists\sigma\begin{pmatrix}\forall i < \mathrm{lth}(\sigma)\sigma_i \le 1\ \&\ (\Gamma\,|\,\alpha)(\sigma) = 0\ \&\ \mathrm{lth}(\sigma) = n \\ \&\ \forall\tau \supset \sigma(\Gamma\,|\,\alpha)(\tau) = 0\ \&\ \forall i,j \le \mathrm{lth}(\sigma)\sigma_i = \sigma_j\end{pmatrix},$$

so with KL we conclude that

$$\forall\alpha\exists\beta(\forall n\beta n \le 1\ \&\ \forall n, m\beta n = \beta m\ \&\ \forall n(\Gamma\,|\,\alpha)(\bar{\beta}n) = 0),$$

or, in other words, $\forall\alpha\exists n(n \le 1\ \&\ [n] \in V_{\gamma|\alpha})$. Furthermore,

$$\begin{aligned}\forall\alpha[\forall n\alpha n > 1 &\to \forall\beta(\forall n\beta n \le 1 \to \beta \in V_{\gamma|\alpha})\ \& \\ \exists n(\alpha n = 0\ \&\ \forall m \le n\alpha m \ne 1) &\to V_{\gamma|\alpha} = \{[0]\}\ \& \\ \exists n(\alpha n = 1\ \&\ \forall m \le n\alpha m \ne 0) &\to V_{\gamma|\alpha} = \{[1]\}].\end{aligned}$$

Now we cannot have

(∗) $$\forall\alpha\exists n\exists m\forall\beta \in \bar{\alpha}m(n \le 1\ \&\ [n] \in V_{\gamma|\beta}).$$

For suppose so; let n and m satisfy (∗) for $\alpha = [2]$. Then if $n = 0$ and $\beta = \langle 2\rangle^{[m]} * [1]$ we would have $[0] \in V_{\gamma|\beta}$; if $n = 1$ and $\beta = \langle 2\rangle^{[m]} * [0]$ then $[1] \in V_{\gamma|\beta}$, which is a contradiction in both cases. ☒

§6. Bar induction. Decidable bar induction $\mathbf{BI_D}$ is the following schema: [(1) & (2) & (3) & (4)] → (5), where

(1) $$\forall\alpha\exists xP(\bar{\alpha}x),$$

(2) $$\forall\sigma(P(\sigma) \vee \neg P(\sigma)),$$

(3) $$\forall\sigma(P(\sigma) \to Q(\sigma)),$$

(4) $$\forall\sigma(\forall nQ(\sigma * \langle n\rangle) \to Q(\sigma)),$$

(5) $$Q(\langle\ \rangle).$$

Full bar induction BI is [(1) & (3) & (4)] → (5). The schema $\mathbf{BI_D}$ is often considered when investigating systems based on **EL**, because of its importance in the development of elementary constructive mathematics. The schema **BI**, on the other hand, can be shown to imply ∃-PEM ($\forall\beta(\exists x\beta(x) = 0 \vee \neg\exists x\beta(x) = 0)$; see Exercise 4.8.11 of Troelstra and van Dalen [1988]), and this conflicts already with very mild continuity schemata (for instance, it conflicts with $\mathbf{GC_L}$). In Kleene and Vesley [1965] it is shown that $\mathbf{BI_D}$ is realizable, provably in $\mathbf{EL} + \mathbf{BI_D}$. Decidable bar induction is also consistent with our interpretation:

PROPOSITION 6.1. $\mathbf{BI_D}$ *is* $\underline{r}$*-realizable, provably in* $\mathbf{EL}' + \mathbf{BI_D}$.

PROOF. Let $P'(\sigma) \equiv P(\sigma) \,\&\, \forall\tau \supset \sigma(\tau \neq \sigma \to \neg P(\tau))$; then P' is decidable if P is, and $\vdash (2) \to (\exists x P(\bar{\alpha}x) \to \exists! x P'(\bar{\alpha}x))$. Now it is not difficult to show that for any γ that $\underline{r}$-realizes $\exists x P(\bar{\alpha}x)$ there is a γ', continuous in γ, such that $j_2\gamma' \underline{r} P'(\bar{\alpha}(j_1\gamma'(0)))$. Since nowhere else does an existential quantifier appear in the schema, the formal argument by Kleene and Vesley can now be copied. ☒

Finally, let us remark that $\mathbf{EL}' + \mathbf{GC_L}$ proves FAN with respect to decidable formulas; so bar induction with respect to bounded trees, which is in **EL** equivalent to $\mathrm{FAN_D}$, is $\underline{r}$-realizable without extra assumptions.

III. Tripos-theoretic preliminaries

Part of this thesis deals with extensions of realizabilities to higher order intuitionistic logic. This takes the form of the construction of toposes which generalize the realizability notions we consider, in the following sense: say a realizability notion has been defined for **HA**, then this realizability validates the same first-order sentences of arithmetic, as are valid for the natural number object of the topos.

The vehicle for the construction of these toposes is an abstract framework described in Hyland, Johnstone & Pitts 1980 (HJP 1980 for short) and Pitts 1981: the theory of *triposes* (tripos is an acronym for "topos-representing indexed pre-ordered set"); the most famous example of a topos produced in this way is Hyland's "effective topos" Eff, which is worked out in Hyland 1982. The aim of this chapter is to introduce the reader into this machinery; to provide him with the basic definitions and theorems. We give proofs if they are short, and if they are not easily accessible (it is a great pity that Pitts 1981 has never been published). To give the reader some concrete feel, we develop, as an example, some basic facts about the effective topos as we go along.

1. Definitions and examples

1.1. Definition. A *Heyting pre-algebra* is a preorder $\vdash$ with finite meets $\wedge$, joins $\vee$, top $\top$, bottom $\bot$ and *Heyting implication* $\Rightarrow$, satisfying $a\wedge b\vdash c$ iff $a\vdash b\Rightarrow c$ for all a,b,c. We will write $a \dashv\vdash b$ for the conjuction of $a\vdash b$ and $b\vdash a$. This notation will be extended to isomorphisms of order-preserving maps.

1.2. Definition. Let C be a category with finite products. A C-tripos $\mathcal{P}$ is the following structure.

i) For every object a of C, a Heyting pre-algebra $\mathcal{P}(a)$ is given;
ii) For every morphism f: a$\rightarrow$b in C, an order-preserving map $\mathcal{P}(f)$: $\mathcal{P}(b)\rightarrow\mathcal{P}(a)$ is given, such that:
   a) $\mathcal{P}(f)$ preserves all the Heyting structure;
   b) $\mathcal{P}(id_a)$ is isomorphic to the identity on $\mathcal{P}(a)$, and $\mathcal{P}(g\circ f) \dashv\vdash \mathcal{P}(f)\circ\mathcal{P}(g)$ for a composable pair of morphisms g,f of C;
   c) The maps $\mathcal{P}(f)$ have left and right adjoints $\exists f$ and $\forall f$ respectively, satisfying the *Beck-condition* : if

$$\begin{array}{ccc} \cdot & \xrightarrow{f} & \cdot \\ k\downarrow & & \downarrow g \\ \cdot & \xrightarrow{h} & \cdot \end{array}$$

   is a pullback square ... C, then $\exists f\circ\mathcal{P}(k) \dashv\vdash \mathcal{P}(g)\circ\exists h$ (The dual condition, involving $\forall$, then follows by adjointness);
iii) For every object a of C there is an object [a] of C and an element $\in_a$ of $\mathcal{P}(a\times[a])$ (a *membership predicate* for a) such that for every object b of C and any ϕ in $\mathcal{P}(a\times b)$ there is a

morphism $[\phi]: b \to [a]$ in C with $\mathcal{P}(id_a \times [\phi])(\in_a) \dashv\vdash \phi$ in $\mathcal{P}(a \times b)$ (The morphism $[\phi]$ is *not* required to be unique).

Condition iii) can often be simplified. Consider

iii)' There is an object Σ of C and an element σ of $P(\Sigma)$ (a *generic predicate* for $\mathcal{P}$) such that for every object a of C and every ϕ in $\mathcal{P}(a)$ there is a morphism $\{\phi\}: a \to \Sigma$ in C with $\mathcal{P}(\{\phi\})(\sigma) \dashv\vdash \phi$ in $\mathcal{P}(a)$.

1.3. Proposition. iii) implies iii)'; if C is a cartesian closed category, then the converse holds.

Proof. If iii) let Σ be $[1]$ where 1 is the terminal object of C, and let σ be $\in_1$. Then σ is a generic predicate: if $\phi \in \mathcal{P}(a) \cong \mathcal{P}(1 \times a)$ there is $[\phi]: a \to [1]$ such that $\mathcal{P}(id_1 \times [\phi])(\in_1) \dashv\vdash \phi$ in $\mathcal{P}(a)$.

Conversely if C is cartesian closed and iii)' holds, let $[a]$ be Σ^a and $\in_a$ be $\mathcal{P}(ev)(\sigma) \in \mathcal{P}(a \times [a])$. Now if $\phi \in \mathcal{P}(a \times b)$ there is $\{\phi\}: a \times b \to \Sigma$ in C with $\mathcal{P}(\{\phi\})(\sigma) \dashv\vdash \phi$; let $[\phi]$: $b \to [a]$ be the exponential transpose of $\{\phi\}$. Then $\mathcal{P}(id_a \times [\phi])(\in_a) =$ $\mathcal{P}(id_a \times [\phi])(\mathcal{P}(ev)(\sigma)) \dashv\vdash \mathcal{P}(ev \circ (id_a \times [\phi]))(\sigma) \dashv\vdash \mathcal{P}(\{\phi\})(\sigma) \dashv\vdash \phi$ in $\mathcal{P}(a \times b)$.

In many cases, C is **sets** and $\mathcal{P}$ has a simple form: $\mathcal{P}(a)$ is the set of functions: $a \to \Sigma$ for some set Σ, and $\mathcal{P}(f)$ is just composition with f. The identity on Σ as element of $\mathcal{P}(\Sigma)$, is then clearly a generic predicate ($[a]$ is Σ^a, and the membership predicate in $\mathcal{P}(a \times [a])$ is the evaluation map). Examples of such Σ are:

a) Σ is a complete Heyting algebra. Readers who are familiar with the semantics of "Ω-sets" from Fourman & Scott 1979 or the last chapters of Troelstra & Van Dalen 1988 should keep this example in mind since there is a close analogy between the construction of a topos out of a tripos $\mathcal{P}$, and the topos of Ω-sets.

b) A *partial combinatory algebra* is a set A equipped with a partial binary application • and elements K and S such that for all $a,b,c \in A$:

i) K•a is defined (for short K•a↓) and (K•a)•b↓ and (K•a)•b=a;

ii) S•a↓ and (S•a)•b↓ and ((S•a)•b)•c ≃ (a•c)•(b•c) (≃ means: both or none are defined, and both sides are equal if defined).

If A is a partial combinatory algebra (pca), let Σ be the powerset of A. For sets X, Σ^X is preordered by: $\phi \vdash \psi$ iff there is $a \in A$ such that for all $x \in X$ and all $b \in \phi(x)$, a•b↓ and $a{\bullet}b \in \psi(x)$. There are definable pairing and divorcing functions ‹ ›, $(\ )_0, (\ )_1$ respectively (that is, represented by elements of A) such that $(\langle a,b \rangle)_0 = a$ and $(\langle a,b \rangle)_1 = b$ for all $a,b \in A$. The Heyting structure on Σ^X is as follows: for $\phi, \psi \in \Sigma^X$

$\phi \wedge \psi(x) = \{a | (a)_0 \in \phi(x) \ \& \ (a)_1 \in \psi(x)\}$

$\phi \vee \psi(x) = \{a | (a)_0 = K \ \& \ (a)_1 \in \phi(x), \text{ or } (a)_0 = S \ \& \ (a)_1 \in \psi(x)\}$

$\phi \Rightarrow \psi(x) = \{a \mid \forall b \in \phi(x)\ a \bullet b\downarrow\ \&\ a \bullet b \in \psi(x)\}$
For functions f: Y→X the adjoints $\exists f$ and $\forall f$ are defined by
$\exists f(\phi)(x) = \{a \mid \exists y \in Y(f(y)=x\ \&\ a \in \phi(y))\}$
$\forall f(\phi)(x) = \{a \mid \forall y \in Y(f(y)=x \Rightarrow \forall b \in A(a \bullet b\downarrow\ \&\ a \bullet b \in \phi(y)))\}$ (if f is a surjection, then $\forall f$ may be defined by $\forall f(\phi)(x) = \{a \mid \forall y \in Y(f(y)=x \Rightarrow a \in \phi(y))\}$).
If A is the partial combinatory algebra $\mathbb{N}$ with partial recursive application, we get the "effective tripos" which underlies the effective topos.

c) Suppose $\mathcal{E}$ is a Grothendieck topos and X is an object of $\mathcal{E}$ such that the assignment $Y \dashrightarrow \mathcal{E}(Y,X)$ defines an $\mathcal{E}$-tripos. Then one can take $\Sigma = \mathcal{E}(1,X)$, the set of global elements of X. This follows from proposition 1.4 below. In the chapter "Kripke and Beth models of realizability" we shall describe some triposes of this form.

d) Other examples of such Σ will be given in this thesis; in the chapters on an extension of Lifschitz' realizability, on modified realizability and on "extensional realizability".

But also when $\mathcal{P}$ is defined on another category than **sets**, there may be possibilities to look at $\mathcal{P}$ as if it were defined on **sets**:

1.4. Proposition. Suppose C and D are categories with finite products and $\mathcal{P}$ is a C-tripos. Suppose furthermore that F: D→C is a functor which preserves all pullbacks which exist in D, and has a right adjoint G: C→D. Then composition with F defines a D-tripos.
(This is a slight generalization of 3.12 in Pitts 1981)

Proof. Let us write $\mathfrak{R}(a)$ for $\mathcal{P}(Fa)$. The only nontrivial condition is iii). For objects a of D, write Pa for G([Fa]) and δ_a for $\mathcal{P}(1_{Fa}\times\varepsilon_{[Fa]})(\in_a) \in \mathcal{P}(Fa\times FG([Fa])) \cong \mathfrak{R}(a\times Pa)$, where $\varepsilon_{[Fa]}$: FG([Fa])→[Fa] is the counit of $F \dashv G$ at [Fa].
Now if $\phi \in \mathfrak{R}(a\times b) \cong \mathcal{P}(Fa\times Fb)$ then since iii) holds for $\mathcal{P}$ there is $[\phi]$: Fb→[Fa] such that $\mathcal{P}(1_{Fa}\times[\phi])(\in_{Fa}) \dashv\vdash \phi \in \mathcal{P}(Fa\times Fb)$. Let $\{\phi\}$: b→G([Fa]) be the transpose of $[\phi]$ across $F \dashv G$. Then $\mathfrak{R}(1_a\times\{\phi\})(\delta_a) = \mathcal{P}(1_{Fa}\times F(\{\phi\}))(\mathcal{P}(1_{Fa}\times\varepsilon_{[Fa]})(\in_a)) \dashv\vdash \mathcal{P}(1_{Fa}\times(\varepsilon_{[Fa]}\circ F(\{\phi\})))(\in_{Fa}) = \mathcal{P}(1_{Fa}\times[\phi])(\in_{Fa}) \dashv\vdash \phi \in \mathcal{P}(Fa\times Fb) \cong \mathfrak{R}(a\times b)$. So $\delta_a \in \mathfrak{R}(Pa)$ serves as a membership predicate for a.

Proposition 1.4 can be applied, for instance, if $\mathcal{E}$ is a Grothendieck topos, and F: **sets**→$\mathcal{E}$ is the "constant objects" functor which assigns to a set a the sheaf associated to the constant presheaf a.

2. Tripos-semantics

We shall now define how to interpret intuitionistic many-sorted predicate logic without equality in a tripos. This is very basic, since the construction of a topos out of the tripos depends on it. In fact, most calculations about the topos are carried out in the underlying tripos. But, the definition contains no surprises.

Suppose $\mathcal{P}$ a C-tripos and $\mathfrak{L}$ a many-sorted language. An interpretation $[\![\ ]\!]$ of $\mathfrak{L}$ in $\mathcal{P}$ consists of the following:

i) For every sort σ of $\mathfrak{L}$, $[\![\sigma]\!]$ is an object of C;

ii) For every n-ary function symbol f of sort σ, which takes arguments of sorts $\tau_1,...,\tau_n$, $[\![f]\!]$ is a morphism: $[\![\tau_1]\!]\times...\times[\![\tau_n]\!]\rightarrow[\![\sigma]\!]$ in C;

iii) For every n-ary relation symbol R which takes terms of sorts $\tau_1,...,\tau_n$, $[\![R]\!]$ is an element of $\mathcal{P}([\![\tau_1]\!]\times...\times[\![\tau_n]\!])$.

In ii) and iii) the case n=0 is not excluded: constants and propositions.

Given an interpretation $[\![\ ]\!]$ of $\mathfrak{L}$ in $\mathcal{P}$, we can assign to any formula ϕ of $\mathfrak{L}$ with free variables of sorts $\tau_1,...,\tau_n$, an element $[\![\phi]\!]$ of $\mathcal{P}([\![\tau_1]\!]\times...\times[\![\tau_n]\!])$: the basic case is iii) above.

Extend ii) to all terms of $\mathfrak{L}$: if x is a variable of sort σ, $[\![x]\!]$ is the identity on $[\![\sigma]\!]$. Inductively then, $[\![f(t_1,...,t_n)]\!] = [\![f]\!]\circ([\![t_1]\!]\times...\times[\![t_n]\!])$ for function symbols f and $t_1,...,t_n$ of the right sorts. Substitution of a term t for a variable x in a formula is now interpreted by the map $\mathcal{P}([\![t]\!])$: $[\![\phi(^x/_t)]\!] = \mathcal{P}([\![t]\!])([\![\phi]\!])$.

The propositional connectives are taken care of by the Heyting structure. One has to bring the formulas on the same denominator: if a, b and c are respectively the product of the sorts of all distinct variables occurring in ϕ, ψ and $\phi\wedge\psi$, then if π_a: c$\rightarrow$a and π_b: c$\rightarrow$b are the obvious projections, $[\![\phi\wedge\psi]\!] = \mathcal{P}(\pi_a)([\![\phi]\!])\wedge\mathcal{P}(\pi_b)([\![\psi]\!])$ in $\mathcal{P}(c)$.

Quantification is interpreted using the adjoints $\forall\pi$ and $\exists\pi$ for a suitable projection π. The Beck-condition ensures that substitution is well-behaved with respect to this interpretation.

We say that a formula ϕ of $\mathfrak{L}$ is *valid* under the interpretation $[\![\ ]\!]$ in $\mathcal{P}$ (if $[\![\ ]\!]$ is clear, we write $\mathcal{P}\vDash\phi$), iff $[\![\phi]\!]$ is isomorphic to the top element of the Heyting pre-algebra it belongs to.

A trivial, but important theorem (Lemma 2.1 in HJP 1980) asserts that many-sorted intuitionistic predicate logic is sound for this interpretation. A good formulation of this logic to work with, is the one using labelled sequents. These are sequents of the form $\Gamma\vdash_\sigma\phi$, where ϕ is a formula, Γ a finite set of formulas, and σ (the *label* of $\Gamma\vdash_\sigma\phi$) a finite set of variables which contains all the variables occurring free in Γ or ϕ. One has the usual rules of the intuitionistic sequent calculus, taking as label of the conclusion of a binary rule the union of the labels of the premises (including the cut rule). There are only two ways of getting rid of the label: by substitution or by quantifying. The substitution rule says that if b a term substitutable for x in Γ,ϕ then from $\Gamma\vdash_\sigma\phi$ infer $\Gamma[^x/_b]\vdash_{(\sigma\setminus\{x\})\cup FV(b)}\phi[^x/_b]$; the quantifier rules say for instance: from $\Gamma\vdash_\sigma\phi$ infer $\Gamma\vdash_{\sigma\setminus\{x\}}\forall x\phi$, with the obvious conditions. The reason for this fuss with labels is, that one has to cope with the possibility of an empty sort (i.e. a sort σ such that $\mathcal{P}([\![\sigma]\!])$ is a trivial Heyting pre-algebra).

2.1. Example. Let $\mathcal{P}$ be the tripos as in example b) following proposition 1.3; for definiteness we take the partial combinatory algebra $\mathbb{N}$ with partial recursive application, the effective tripos. So, $\mathcal{P}(X)$ is the set of functions $X \to P(\mathbb{N})$ for sets X. The top element $\top$ of $\mathcal{P}(X)$ is the constant function $\lambda x.\mathbb{N}$, and $\phi \vdash \psi$ iff $\exists e \forall x \forall a \in \phi(x)$ ($e \bullet a\downarrow$ and $e \bullet a \in \psi(x)$), so $\phi \dashv\vdash \top$ iff $\cap\{\phi(x) | x \in X\}$ is nonempty. Now if X is nonempty and ϕ and ψ are elements of $\mathcal{P}(X)$ such that $[\![\forall x\,(\phi(x) \vee \psi(x))]\!]$ is isomorphic to the top element of $\mathcal{P}(1)$, this means that, according to the definitions of $\forall$ and $\vee$ given, that there is an element e such that for all x, $(e)_0=0$ and $(e)_1 \in \phi(x)$, or $(e)_0=1$ and $(e)_1 \in \psi(x)$. So, either $[\![\forall x\, \phi(x)]\!]$ or $[\![\forall x\, \psi(x)]\!]$ must be isomorphic to the top element. And since this can be decided recursively in e, $\mathcal{P} \models \forall x\,(\phi(x) \vee \psi(x)) \to \forall x \phi(x) \vee \forall x \psi(x)$.

3. The topos represented by a tripos

3.1. Definition. Let $\mathcal{P}$ be a C-tripos. We define a category $\mathcal{P}$-C as follows:
Objects are pairs (a,=) where a is an object of C and = is an element of $\mathcal{P}(a \times a)$ such that

$\mathcal{P} \models x=x' \to x'=x$ and $\mathcal{P} \models (x=x' \wedge x'=x'') \to x=x''$.

Morphisms : $(a,=) \to (b,\approx)$ are equivalence classes of "functional relations". An element F of $\mathcal{P}(a \times b)$ is a functional relation (with respect to = and $\approx$) iff:

$\mathcal{P} \models (Fxy \wedge x=x' \wedge y \approx y') \to Fx'y'$ (F is a *relation*),

$\mathcal{P} \models Fxy \to x=x \wedge y \approx y$ (F is *strict*),

$\mathcal{P} \models (Fxy \wedge Fxy') \to y \approx y'$ (F is *single-valued*) and

$\mathcal{P} \models x=x \to \exists y Fxy$ (F is *total*).

Two functional relations $F,G \in \mathcal{P}(a \times b)$ are equivalent (represent the same morphism) iff $\mathcal{P} \models Fxy \to Gxy$. This is symmetric (use soundness).
= itself represents a morphism: $(a,=) \to (a,=)$: the identity on (a,=). If F and G represent morphisms: $(a,=) \to (b,\approx)$ and $(b,\approx) \to (c,\doteq)$ respectively, then $\exists y\,(Fxy \wedge Gyz)$ represents the composition of [F] and [G]. Again, the soundness theorem is used to see that $\exists y\,(Fxy \wedge Gyz)$ is a functional relation, that composition so defined does not depend on the representatives F and G, and that it is associative.

3.2. Theorem. $\mathcal{P}$-C is a topos.

This is proven in Pitts 1981 and, for the special case where C is **sets**, in HJP 1980. We give some standard constructions.
Products. The product of (a,=) and $(b,\approx)$ is $(a \times b,\approx)$ with $\langle x,y\rangle \approx \langle x',y'\rangle \equiv x=x' \wedge y \approx y'$.
Pullbacks. The pullback of two morphisms [F]: $(a,=) \to (b,\approx)$ and [G]: $(a',=') \to (b,\approx)$ is the object $(a \times a',\approx)$ where $\approx$ is given by: $\langle x,y\rangle \approx \langle x',y'\rangle$ is $x=x' \wedge y='y' \wedge \exists w\,(Fxw \wedge Gyw)$.
Exponentials. We do this for C cartesian closed. Let (a,=), $(b,\approx)$ be objects of $\mathcal{P}$-C and let $\sigma \in \mathcal{P}(\Sigma)$ be the generic predicate for $\mathcal{P}$. Let Fxy be a predicate (in three variables F,x,y)

interpreted by $\mathcal{P}(ev)(\sigma)$ in $\mathcal{P}(\Sigma^{a\times b}\times a\times b)$. Let $EF\in\mathcal{P}(\Sigma^{a\times b})$ be (the interpretation of) the universal closure of "F is a functional relation". Then $(a,=)^{(b,\approx)}$ is $(\Sigma^{a\times b},\approx)$ with $F\approx G \equiv EF\wedge EG\wedge\forall xy\,(Fxy\leftrightarrow Gxy)$. Let us briefly explain how this works: if $G\in\mathcal{P}(c\times\Sigma^{a\times b})$ represents a morphism $[G]$: $(c,\sim)\to(a,=)^{(b,\approx)}$ then $\tilde{G}\in\mathcal{P}(c\times a\times b)$ given by $\tilde{G}(z,x,y)\equiv \exists F(G(z,F)\wedge Fxy)$ represents its transpose: $(c,\sim)\times(b,\approx)\to(a,=)$. And if $[H]\in\mathcal{P}(c\times b\times a)$ represents $[H]$: $(c,\sim)\times(b,\approx)\to(a,=)$ there is by iii)' a morphism $\{H\}$: $c\times b\times a\to\Sigma$ with $\mathcal{P}(\{H\})(\sigma)\dashv\vdash H$. Let h: $c\to\Sigma^{a\times b}$ be the transpose of $\{H\}$ in C, and define $\bar{H}\in\mathcal{P}(c\times\Sigma^{a\times b})$ by $\bar{H}(z,F)\equiv F\approx h(z)\wedge z\sim z$ (here we use a language with a function symbol h). $\bar{H}$ is a functional relation and represents the transpose of [H]. It is a nice exercise to show that the operations $G\mapsto\tilde{G}$ and $H\mapsto\bar{H}$ are, up to equivalence, inverse to each other.

The *subobject classifier* of $\mathcal{P}$-C is the object $(\Sigma,\leftrightarrow)$.

From these one can, of course, define *power-objects* but it is convenient to know that the power-object of $(a,=)$ is isomorphic to $(\Sigma^a,\approx)$ where $[\![R\approx S]\!]\equiv ER\wedge\forall x\,(R(x)\leftrightarrow S(x))$, where ER is the universal closure of "R is a strict relation".

3.3. Example. We return to the example of the effective tripos $\mathcal{P}$. The topos **$\mathcal{P}$-sets**, the *effective topos* or Eff, has as objects pairs $(X,=)$ with X a set, = a function: $X\times X\to P(\mathbb{N})$ such that there are numbers e and f with the property that for all $x,y,z\in X$: $\forall a\in[\![x=y]\!]$ ($e\bullet a\downarrow$ and $e\bullet a\in[\![y=x]\!]$) and $\forall a\in[\![x=y]\!]\,\forall b\in[\![y=z]\!]$ ($f\bullet\langle a,b\rangle\downarrow$ and $f\bullet\langle a,b\rangle\in[\![x=z]\!]$). Let us look at some special objects: the objects 2_A for $A\subseteq\mathbb{N}$ with A nonempty. 2_A is $(\{0,1\},=_A)$ with $[\![0=_A 0]\!]\equiv A$, $[\![1=_A 1]\!]\equiv\bar{A}$, $[\![0=_A 1]\!]$ is empty. If F: $\{0,1\}^2\to P(\mathbb{N})$ is a functional relation representing a morphism: $2_A\to 2_B$, for every $x\in\{0,1\}$ there is exactly one $y\in\{0,1\}$ with F(x,y) nonempty, so F determines a function f: $\{0,1\}\to\{0,1\}$. Furthermore, since F is total and strict we have a recursive function ϕ with $\forall x\in[\![0=_A 0]\!]$ $(\phi(x)\in[\![f(0)=_B f(0)]\!])$ and $\forall x\in[\![1=_A 1]\!]$ $(\phi(x)\in[\![f(1)=_B f(1)]\!])$. So if f is the identity function, A is many-one reducible to B; and if A is more complex than B, the only morphisms $2_A\to 2_B$ are constant.

The *terminal object* **1** in Eff is the object $(\{0\},=)$ with $[\![0=0]\!]\equiv\mathbb{N}$. A morphism $\mathbf{1}\to(X,=)$ is (equivalent to) an $S\in\mathcal{P}(X)$ such that $\mathcal{P}\vdash\exists x(x=x\wedge\forall y(S(y)\leftrightarrow y=x)))$ (a *singleton* on $(X,=)$).

A *natural numbers object* in a topos is an object N together with morphisms $\bar{0}$: $\mathbf{1}\to N$ and $\bar{S}$: $N\to N$, such that for every object X and morphisms a: $\mathbf{1}\to X$ and b: $X\to X$ there is a unique g: $N\to X$ with $g\circ\bar{0}=a$ and $g\circ\bar{S}=b\circ g$. Consider the object $N\equiv(\mathbb{N},=)$ where $[\![n=n]\!]$ is $\{n\}$ and $[\![n=m]\!]$ is empty if $n\neq m$. There are morphisms $\bar{0}$: $\mathbf{1}\to N$ and $\bar{S}$: $N\to N$ defined by the predicates $[\![n=0]\!]$ and $[\![n=n\wedge m=n+1]\!]$, respectively; let us see that this defines a natural numbers object in Eff. So let $S\in\mathcal{P}(X)$ represent a singleton on $(X,\doteq)$ and $F\in\mathcal{P}(X\times X)$ a functional relation. Define G: $\mathbb{N}\times X\to P(\mathbb{N})$ recursively by:

$G(0,x)\equiv\{\langle 0,a\rangle \mid a\in S(x)\}$

$G(n+1,x)\equiv\{\langle n+1,a\rangle \mid \exists y\in X\,((a)_0\in F(y,x)\ \&\ (a)_1\in G(n,y))\}$.

Then G is obviously strict and relational for =; since S and F are strict for $\doteq$, there are e_0 and e_1

with $e_0 \in S(x) \to \llbracket x \doteq x \rrbracket$ for all x and $e_1 \in Fyx \to \llbracket x \doteq x \rrbracket$ for all y,x; so if e is such that $e \bullet a \simeq e_0 \bullet (a)_1$ if $(a)_0 = 0$, and $e_1 \bullet ((a)_1)_0$ else, then $e \in G(n,x) \to \llbracket x \doteq x \rrbracket$ for all n,x, so G is strict for $\doteq$. Since S and F are relational for $\doteq$ there are f_0 and f_1 with $f_0 \in S(x) \wedge \llbracket x \doteq x' \rrbracket \to S(x')$ and $f_1 \in Fyx \wedge \llbracket x \doteq x' \rrbracket \to Fyx'$ for all x,x',y. Let f be such that $f \bullet a \simeq \langle 0, f_0 \bullet \langle ((a)_0)_1, (a)_1 \rangle\rangle$ if $((a)_0)_0 = 0$, and $\langle ((a)_0)_0, \langle f_1 \bullet \langle (((a)_0)_1)_0, (a)_1 \rangle, (((a)_0)_1)_1 \rangle\rangle$ else; then $f \in G(n,x) \wedge \llbracket x \doteq x' \rrbracket \to G(n,x')$ for all n,x,x', so G is relational for $\doteq$. To prove that G is single-valued, let $h_0 \in S(x) \wedge S(x') \to \llbracket x \doteq x' \rrbracket$ for all x,x' by the fact that S is a singleton, and $h_1 \in Fyx \wedge Fy'x' \wedge \llbracket y \doteq y' \rrbracket \to \llbracket x \doteq x' \rrbracket$ for all y,y',x,x' by the fact that F is a relation and single-valued. Now by the recursion theorem, it is possible to find a code e such that $e \bullet \langle a,b \rangle \simeq h_0 \bullet \langle (a)_1, (b)_1 \rangle$ if $(a)_0 = 0$, and $h_1 \bullet \langle ((a)_1)_0, ((b)_1)_0, e \bullet \langle ((a)_1)_1, ((b)_1)_1 \rangle\rangle$ else. The reader will be able to show by induction on n that $e \in G(n,x) \wedge G(n,x') \to \llbracket x \doteq x' \rrbracket$ for all n,x,x'. The proof that G is total is similar. We leave to the reader the useful exercise of showing that the required commutation relation holds, as well as the uniqueness (up to equivalence) of G.

4. *Topos semantics reduced to tripos semantics*

The interpretation of higher order intuitionistic type theory in toposes can be considered as standard by now; several accounts are available, differing only in the presentation of the calculus, and all these are essentially equivalent. Among others: Johnstone 1977 (Chapter 5), Boileau & Joyal 1981, Lambek & Scott 1986, Bell 1988.

We don't want to redo this; but we shall sketch how, in a topos of the form $\mathcal{P}$-C, the standard interpretation can be reduced to an interpretation of a many-sorted first-order language in the tripos $\mathcal{P}$, as defined in section 2. The topos semantics has two clauses:

i) *types* are interpreted by objects of the topos (respecting, of course, the type formation operations: subobject classifier, exponentials, products);
ii) *terms* t(x) are interpreted by morphisms: $X \to Y$ if X andY interpret the type of x and t, respectively.

Formulas are terms of type Ω, so a formula with free variable x of type interpreted by X is interpreted by a morphism: $X \to \Omega$ (Ω denotes both the type of truth values and the subobject classifier in the topos); equivalently, a subobject of X.

4.1. Proposition. Let (X,=) be an object of $\mathcal{P}$-C and A(X,=) the set of all strict relations on X (for =), considered as sub-preorder of $\mathcal{P}(X)$. Then subobjects of (X,=) are in 1-1 correspondence with isomorphism classes of A(X,=).

Proof. The reader is invited to check that a functional relation $F \in \mathcal{P}(Y \times X)$ represents a monomorphism: $(Y,\approx) \to (X,=)$ in $\mathcal{P}$-C iff $\mathcal{P} \Vdash Fyx \wedge Fy'x \to y \approx y'$. Such a monomorphism induces a strict relation on X given by $\exists y Fyx$, and if $G \in \mathcal{P}(Z \times X)$ represents another mono into (X,=) then [G] factors through [F] iff $\mathcal{P} \Vdash \exists z Gzx \to \exists y Fyx$. Conversely if $R \in \mathcal{P}(X)$ is a strict relation let

$=' \in \mathcal{P}(X\times X)$ be the equality: $x='x' \equiv x=x' \wedge R(x)$. Then $= \in \mathcal{P}(X\times X)$ represents a monomorphism: $(X,=')\to(X,=)$.

The set $A(X,=)$ of strict relations on X for = is a Heyting prealgebra which inherits meets and joins (including $\bot$, but the top element of $A(X,=)$ is the relation x=x) from $\mathcal{P}(X)$, but with Heyting implication $\phi\Rightarrow\psi \equiv x=x \wedge (\phi(x)\to\psi(x))$. Now if $(Y,\approx)$ is another object of $\mathcal{P}$-C and $=\wedge\approx$ is the product equality on X×Y, there is a map: $A(X,=)\to A(X\times Y,=\wedge\approx)$ which sends ϕ to $\phi(x)\wedge y\approx y$ (this corresponds to pulling back the subobject of (X,=) represented by ϕ along the projection: $(X,=)\times(Y,\approx)\to(X,=)$). This map preserves all Heyting structure and has a left and a right adjoint Σ and Π respectively, given by:

$\Sigma(\phi) \equiv \exists y\phi$

$\Pi(\phi) \equiv x=x \wedge \forall y(y\approx y \to \phi)$.

This discussion suggests the following translation. Let $(\phi)^+$ be defined by:

$(\phi)^+ \equiv \phi$ for atomic ϕ;

$(\ )^+$ commutes with $\wedge$, $\vee$ and $\exists$;

$(\phi\to\psi)^+ \equiv x=x \wedge ((\phi)^+\to(\psi)^+)$;

$(\forall y\phi)^+ \equiv x=x \wedge \forall y(y\approx y \to (\phi)^+)$.

In the last two clauses, x=x is meant to be the conjunction of the existence of all free variables occurring in $\phi\to\psi$ and $\forall y\phi$. Now if an interpretation of a language of intuitionistic type theory in the topos $\mathcal{P}$-C has been given, assigning to atomic formulas subobjects of the object interpreting the product of the types of all free variables, we can replace a *type* interpreted by (X,=) by a *sort* interpreted by X, and apply the translation $(\ )^+$, interpreting = as the equality of the corresponding type. We have:

4.2. Proposition. ϕ holds in $\mathcal{P}$-C iff $\mathcal{P}\models x=x\to(\phi)^+$, where x=x stands for the conjunction of the existence of all variables occurring free in $(\phi)^+$.

The translation $(\ )^+$ can be given a more familiar look.

4.3. Proposition. Let $(\ )^-$ be given by:

$(\phi)^- \equiv \phi$ for atomic ϕ;

$(\ )^-$ commutes with propositional connectives;

$(\exists y\phi)^- \equiv \exists y(y\approx y \wedge (\phi)^-)$;

$(\forall y\phi)^- \equiv \forall y(y\approx y \to (\phi)^-)$.

Then if all atomic formulas are interpreted by strict relations and x denotes all free variables occurring in ϕ, $x=x \wedge (\phi)^- \dashv\vdash (\phi)^+$.

This is proven by induction on ϕ. So: $\mathcal{P}\models x=x \to (\phi)^+$ iff $\mathcal{P}\models x=x \to (\phi)^-$. We should add a word about substitution. Again, if we have interpreted function symbols as morphisms, we can extend this to all terms.

The topos semantics interprets substitutions as pullbacks: if a term t(x) is interpreted by a morphism [T]: $(Y,\approx)\to(X,=)$ and ϕ a formula with free variable x of type $(X,=)$, then $\phi[^x/_t]$ is the subobject of $(Y,\approx)$ obtained by pulling $[\![\phi]\!]$ back along [T]. By the characterization of pullbacks in $\mathcal{P}$-C following Theorem 3.2, if R(x) is a strict relation on X interpreting ϕ and $T\in\mathcal{P}(Y\times X)$ a functional relation representing [T], then the pullback of $[\![\phi]\!]$ along [T] is the strict relation $\exists x\,(Tyx\wedge Rx)$ on Y.

Sometimes T has a special form: if there is a morphism t: $Y\to X$ in C such that $y\approx y\wedge t(y)=x$ in $\mathcal{P}(Y\times X)$ is a functional relation and represents [T], the pullback of R along [T] can be given as $y\approx y\wedge R(t(y))$.

4.4 Remark. Since, by the above, in the case of a topos induced by a tripos, the topos semantics can be reduced to tripos semantics, it would be perfectly possible to give a direct interpretation of higher order intuitionistic logic in a tripos, without mentioning the topos at all. Of course, the soundness proof for such an interpretation would contain a proof of Theorem 3.2, and vice versa. I believe that such an interpretation would look rather ad hoc, however; it would be difficult to justify the need for translations like $()^{-}$.

4.5. Example. We pursue the example of the effective topos. We want to show that the interpretation of second order arithmetic (**HAS**) in Eff coincides with an informal reading of Troelstra's extension of Kleene's realizability to **HAS**. This fact is certainly known to several people, but we have not found a decent proof in the literature.

Troelstra's extension has the three clauses:

n **r** $x\in X$ $\equiv \langle n,x\rangle\in X^*$

n **r** $\forall XA$ $\equiv \forall X^*(n$ **r** $A)$

n **r** $\exists XA$ $\equiv \exists X^*(n$ **r** $A)$,

where $X \dashrightarrow X^*$ is a 1-1 operation assigning a fresh variable to any set variable X. Let $(\mathbb{N},=)$ denote the natural numbers object of Eff, defined in example 3.3. It is easy to see that for every primitive recursive function t: $\mathbb{N}\to\mathbb{N}$ the predicate $[\![n=n\wedge m=t(n)]\!]$ represents a functional relation: $(\mathbb{N},=)\to(\mathbb{N},=)$ in Eff.

In the effective topos, the power object of $(\mathbb{N},=)$ is $(P(\mathbb{N})^{\mathbb{N}},\approx)$ with $[\![R\approx S]\!]\equiv[\![\forall n(R(n)\to n=n)\wedge \forall nm(R(n)\wedge n=m\to R(m)\wedge\forall n(R(n)\leftrightarrow S(n))]\!]$. But since $\Lambda x.(x)_0$ is an element of $[\![\forall nm(R(n)\wedge n=m\to R(m)]\!]$ for every $R\in P(\mathbb{N})^{\mathbb{N}}$, it is not hard to show that this object is isomorphic to $(X,\approx)$ where X is $\{\phi\in P(\mathbb{N})^{\mathbb{N}}\mid\forall n,x\,(x\in\phi(n)\to(x)_1=n)\}$ and $R\approx S$ is $\forall n.(R(n)\leftrightarrow S(n))$. The element relation: $\in\ \rightarrowtail(\mathbb{N},=)\times(X,\approx)$ is represented by the restriction of the evaluation map to $X\times\mathbb{N}$ (note, that this is a strict relation).

Now $(X,\approx)$ has the property that $\langle\Lambda x.x,\Lambda x.x\rangle$ is an element of $[\![R\approx R]\!]$ for every $R\in X$; this means

that for any predicate ϕ with a variable R of sort X the predicates $\forall R\,(R{\approx}R\rightarrow\phi(R))$ and $\forall R\,\phi(R)$ are isomorphic, as well as $\exists R\,(R{\approx}R\wedge\phi(R))$ and $\exists R\,\phi(R)$. We may forget the existence of variables of sort X. So if we define a translation $(\ )^-$ from the language of **HAS** to a language with two sorts $\mathbb{N}$ and X, symbols for all primitive recursive relations and functions, and an element relation, such that $(\ )^-$ is the identity on prime formulas, commutes with propositional connectives and second-order quantifiers, and $(\forall x\phi)^- = \forall x(x{=}x\rightarrow(\phi)^-)$, $(\exists x\phi)^- = \exists x(x{=}x\wedge(\phi)^-)$, then ϕ holds in Eff iff $\mathbf{x{=}x}\rightarrow(\phi)^-$ holds in the tripos underlying Eff, where $\mathbf{x{=}x}$ abbreviates the existence of all free number-variables in ϕ.
We use the bijection $X\rightarrow P(\mathbb{N})$ given by $\tilde{R} \equiv \{y\in\mathbb{N} \mid y\in R((y)_1)\}$ for $R\in X$, with inverse $\bar{A}(n) = \{x\in A \mid (x)_1=n\}$ for $A\subseteq\mathbb{N}$.
For ϕ in the language of **HAS** with free set variable X, and $A\subseteq\mathbb{N}$ we denote by $\phi[^X/_A]$ the interpretation of ϕ in the standard model, A interpreting X.
Now we have:

4.5.1. Proposition. For every formula ϕ in the language of **HAS** with free variables $x_1,\ldots,x_k,X_1,\ldots,X_j$ there are two k-ary primitive recursive functions s_ϕ and t_ϕ such that for any k-tuple $n_1,\ldots,n_k$ from $\mathbb{N}$ and j-tuples $A_1,\ldots,A_j$ from $P(\mathbb{N})$ and $R_1,\ldots,R_j$ from X:
i) $e\in[\![(\phi)^-(n_1,\ldots,n_k,R_1,\ldots,R_j)]\!] \Rightarrow s_\phi(n_1,\ldots,n_k)\bullet e\downarrow$ and $(s_\phi(n_1,\ldots,n_k)\bullet e\ \mathbf{r}\ \phi(n_1,\ldots,n_k,X_1,\ldots,X_j))[^{X_1*}/_{\tilde{R}_1},\ldots,{}^{X_j*}/_{\tilde{R}_j}]$;
ii) $(e\ \mathbf{r}\ \phi(n_1,\ldots,n_k,X_1,\ldots,X_j))[^{X_1*}/_{A_1},\ldots,{}^{X_j*}/_{A_j}] \Rightarrow t_\phi(n_1,\ldots,n_k)\bullet e\downarrow$ and $t_\phi(n_1,\ldots,n_k)\bullet e \in [\![(\phi)^-(n_1,\ldots,n_k,\bar{A}_1,\ldots,\bar{A}_j)]\!]$.

Proof. Let us do this for $\phi \equiv x\in X$. Define $s_{x\in X}(n) \equiv \Lambda e.(e)_0$; $t_{x\in X}(n) \equiv \Lambda e.\langle e,n\rangle$. Then $e\in R(n) \Rightarrow (e)_1=n$ and $e\in\tilde{R}$, so $e=\langle(e)_0,n\rangle$ and $((e)_0\ \mathbf{r}\ n\in X)[^{X*}/_{\tilde{R}}]$; conversely if $(e\ \mathbf{r}\ n\in X)[^{X*}/_A]$ then $\langle e,n\rangle\in A$ so $\langle e,n\rangle\in\bar{A}(n)$.
The other functions are now defined by induction on ϕ.

5. Relations between triposes and their toposes

5.1. Definition. Let $\mathcal{P}$ and $\mathfrak{R}$ be two C-triposes. A *geometric morphism* $\Phi:\mathcal{P}\rightarrow\mathfrak{R}$ consists of a system of order-preserving maps $\Phi_+(a):\mathcal{P}(a)\rightarrow\mathfrak{R}(a)$ and $\Phi^+(a):\mathfrak{R}(a)\rightarrow\mathcal{P}(a)$ for every object a of C, satisfying the following conditions:
i) $\Phi^+(a)$ is left adjoint to $\Phi_+(a)$ and preserves finite meets;
ii) For every morphism f: b→a in C we have $\Phi_+(a)\circ\mathcal{P}(f) \dashv\vdash \mathfrak{R}(f)\circ\Phi_+(b)$ and $\Phi^+(a)\circ\mathfrak{R}(f) \dashv\vdash \mathcal{P}(f)\circ\Phi^+(b)$ (One says that Φ_+ and Φ^+ are *C-indexed functors*).

5.2. Definition. $\Delta_{\mathcal{P}}$: C→$\mathcal{P}$-C denotes the following functor:
On objects a, $\Delta_{\mathcal{P}}(a)$ is $(a,\exists\delta(\top))$ where δ: a→a×a is the diagonal embedding and $\top$ the top

element in $\mathcal{P}(a\times a)$; on morphisms f: a→b, $\Delta_{\mathcal{P}}(f)$ is represented by $\exists(id_a\times f)(\top)\in\mathcal{P}(a\times b)$.

$\Delta_{\mathcal{P}}$ is a kind of "constant objects" functor. Indeed, if for instance $\mathcal{P}$ is defined on **sets** by $\mathcal{P}(a)=\Omega^a$ for some locale Ω, the topos $\mathcal{P}$-**sets** is equivalent to the topos of sheaves on Ω and $\Delta_{\mathcal{P}}(a)$ corresponds under this equivalence to the sheaf generated by the constant presheaf a. $\Delta_{\mathcal{P}}$ then has a right adjoint, and is inverse image of a geometric morphism. But when $\mathcal{P}$ is the effective tripos defined on **sets**, $\Delta_{\mathcal{P}}$ does not have a right adjoint. In this case $\Delta_{\mathcal{P}}$ is direct image of a geometric morphism. There are also examples where $\Delta_{\mathcal{P}}$ is not part of any geometric morphism.

5.3. Example. Again, let $\mathcal{P}$ be the effective tripos. The objects $\Delta_{\mathcal{P}}(X)$ are (X,=) where $[\![x=y]\!]$ is $\mathbb{N}$ if x=y, and $\emptyset$ else. Note, that every $R\in\mathcal{P}(X)$ is a strict relation with respect to this equality, so the preorder reflection of $\mathcal{P}(X)$ is isomorphic to the lattice of subobjects of $\Delta_{\mathcal{P}}(X)$. This holds for any tripos. Also, for any object (X,=) of $\mathcal{P}$-C, there is a morphism: $\Delta_{\mathcal{P}}(X)\to(X,=)$ iff $\mathcal{P}\models\forall x(x=x)$. In the effective topos, one has that (X,=) is isomorphic to a subobject of some $\Delta_{\mathcal{P}}(Y)$ iff $\mathcal{P}\models\forall xy(x=x\wedge y=y\wedge\neg\neg x=y\to x=y)$ (One direction is clear; for the other, let Y be the set $\{x\in X \mid [\![x=x]\!] \text{ is nonempty}\}/\approx$ where $x\approx x'$ iff $[\![x=x']\!]$ is nonempty. Then the relation F(x,[y]) $=\cup\{[\![x=y']\!] \mid y'\in[y]\}$ defines always a morphism $(X,=)\to\Delta_{\mathcal{P}}(Y)$. This is a monomorphism iff the given condition holds).

5.4. Theorem. Let $\mathcal{P}$ and $\mathfrak{R}$ be C-triposes and Φ: $\mathcal{P}\to\mathfrak{R}$ a geometric morphism of triposes. Then there is a geometric morphism of toposes (Φ_*,Φ^*): $\mathcal{P}$-C→$\mathfrak{R}$-C. The inverse image part Φ^* is given by: $\Phi^*(a,=)$ is $(a,\Phi^+(a\times a)(=))$ and $\Phi^*([F])$ is $[\Phi^+(a\times b)(F)]$: $\Phi^*(a,=)\to\Phi^*(b,\approx)$. Moreover, $\Phi^*\circ\Delta_{\mathcal{P}}$ is naturally isomorphic to $\Delta_{\mathfrak{R}}$.

This is proven in Pitts 1981 (let us explain the terminology: a geometric morphism f: $\mathcal{E}\to\mathcal{F}$ of toposes is a pair of functors f_*: $\mathcal{E}\to\mathcal{F}$ and f^*: $\mathcal{F}\to\mathcal{E}$ such that $f^*\dashv f_*$ and f^* preserves finite limits; f^* is called the *inverse image part of* f, f_* the *direct image*) and, for the case C = **sets**, in HJP 1980. Since we shall be interested in interpretations of arithmetic, an important fact for us is, that inverse image functors preserve natural numbers objects (use $f^*\dashv f_*$ and the definition of the natural numbers object in example 3.3).

A special kind of geometric morphisms are *inclusions* , which are those geometric morphisms for which the counit ε: $f^*f_*\to id$ of the adjunction $f^*\dashv f_*$ is an isomorphism (equivalently: f_* is full and faithful, or: f_* preserves exponentials). $\mathcal{E}$ is then called a *subtopos* of $\mathcal{F}$. There is also a notion of inclusion of triposes: say Φ: $\mathcal{P}\to\mathfrak{R}$ is an inclusion of triposes if for any object a of C, $\Phi^+(a)\circ\Phi_+(a)$ is isomorphic to the identity on $\mathcal{P}(a)$ (this is equivalent to: $\Phi_+(a)$ preserves the Heyting implication, or: $\Phi_+(a)$ is full and faithful). $\mathcal{P}$ is a *subtripos* of $\mathfrak{R}$.

5.5. Proposition. If Φ: $\mathcal{P}\rightarrow\mathfrak{R}$ is an inclusion of C-triposes, then the corresponding geometric morphism (Φ_*,Φ^*): $\mathcal{P}$-C$\rightarrow\mathfrak{R}$-C is an inclusion of toposes.

Proof. This is a consequence of the way Φ_* is constructed and is implicit in 3.3-3.5 of HJP 1980. There is no difficulty in extending this to the general case.

A special kind of tripos inclusions is defined by the following lemma, which is analogous to an easy exercise in locale theory: suppose A is a locale and $B\subseteq A$ is closed under arbitrary meets and satisfies: $b\in B$ implies $a\rightarrow b\in B$, for every $a\in A$ ($\rightarrow$ being the Heyting implication in A). Then B is a sublocale of A (Johnstone 1982, p. 50).

5.6. Lemma. Let Σ be a set such that the collection of sets Σ^X and maps Σ^f forms a **sets**-tripos $\mathcal{P}$. Let $\Sigma'\subseteq\Sigma$ be a subset such that:

i) whenever $q\in\Sigma'$, $p\Rightarrow q\in\Sigma'$;

ii) If $\phi\in\mathcal{P}(X)$ maps X into Σ' and $f:X\rightarrow Y$ is a function, then $\forall f(\phi)$ maps Y into Σ'.

Then the collection of sets Σ'^X and maps Σ'^f forms a **sets**-subtripos $\mathcal{P}'$ of $\mathcal{P}$ (Σ'^X inherits the preorder of Σ^X).

Proof. In Theorem 1.4 of HJP 1980 it is shown that conditions, under which a system of sets and maps is a tripos, can be given entirely in terms of $\vdash$, $\Rightarrow$, the maps $\forall f$, and the generic element. The generic element plays no role in these conditions except that there should be one. Since $\mathcal{P}'$ inherits $\vdash$, $\Rightarrow$ and $\forall f$ from $\mathcal{P}$ and $\mathcal{P}$ is a tripos, it follows immediately that $\mathcal{P}'$ is a tripos. Let I: $\mathcal{P}'\rightarrow\mathcal{P}$ be the **sets**-indexed functor induced by the inclusion $\Sigma'\subseteq\Sigma$. I show that I has a **sets**-indexed left adjoint, which preserves finite meets.

Consider $\phi:\Sigma\times\Sigma'\rightarrow\Sigma$, given by $\phi(p,q)=(p\Rightarrow q)\Rightarrow q$, as element of $\mathcal{P}(\Sigma\times\Sigma')$. Then ϕ maps $\Sigma\times\Sigma'$ into Σ' by i); so by ii), $\forall q\,\phi(p,q)$ maps Σ into Σ'. Let J: $\Sigma\rightarrow\Sigma'$ denote this map. Then the **sets**-indexed functor induced by J is left adjoint to I: if $q\in\Sigma'$, $p\vdash q$, then $\vdash p\Rightarrow q$, so $\forall q'(p\Rightarrow q')\Rightarrow q'\vdash_{p,q}(p\Rightarrow q)\Rightarrow q\vdash q$. Conversely if $\forall q'(p\Rightarrow q')\Rightarrow q'\vdash_{p,q}q$ then $p\vdash\forall q'(p\Rightarrow q')\Rightarrow q'\vdash_{p,q}q$. The reader will verify that J preserves finite meets.

Another trivial analogue of a situation for locales says that existential quantification in a subtripos of a tripos P can be derived from that of P (If $B\subseteq A$ is a sublocale belonging to the j-operator $j:A\rightarrow A$) and $B'\subseteq B$, then $\bigvee_B B'=j(\bigvee_A B')$).

5.7. Lemma. Let (Φ_+,Φ^+): $\mathfrak{R}\rightarrow\mathcal{P}$ be an inclusion of triposes and let $\exists f$ denote existential quantification in $\mathcal{P}$, so $\exists f$: $\mathcal{P}(X)\rightarrow\mathcal{P}(Y)$ is left adjoint to $\mathcal{P}(f)$. Then existential quantification in $\mathfrak{R}$ along f is isomorphic to $\Phi^+(Y)(\exists f(\Phi_+(X)(-)))$.

Proof. For $\phi \in \mathfrak{R}(X)$, $\psi \in \mathfrak{R}(Y)$: $\Phi^+(Y)(\exists f(\Phi_+(X)(\phi))) \vdash \psi$ iff $\exists f(\Phi_+(X)(\phi)) \vdash \Phi_+(Y)(\psi)$ iff $\Phi_-(X)(\phi) \vdash \mathcal{P}(f)(\Phi_+(Y)(\psi))$ iff $\Phi_+(X)(\phi) \vdash \Phi_+(X)(\mathfrak{R}(f)(\psi))$ (since Φ_+ is an indexed functor) iff $\phi \vdash \mathfrak{R}(f)(\psi)$ (since Φ_+ is full and faithful).

We conclude this chapter with a result of Pitts' on *iteration* . Given a C-tripos $\mathcal{P}$ and a ($\mathcal{P}$-C)-tripos $\mathfrak{R}$, what does the topos $\mathfrak{R}$-($\mathcal{P}$-C) look like?

5.8. Theorem (Pitts 1981, 6.2). If $\mathfrak{R}$ is such that $\Delta_{\mathfrak{R}}$: $\mathcal{P}$-C$\rightarrow\mathfrak{R}$-($\mathcal{P}$-C) preserves epimorphisms, then composing $\mathfrak{R}$ with $\Delta_{\mathcal{P}}$ gives a C-tripos $\mathcal{K}$. Moreover, there is an equivalence k: $\mathfrak{R}$-($\mathcal{P}$-C) $\rightarrow \mathcal{K}$-C such that $k \circ \Delta_{\mathfrak{R}} \circ \Delta_{\mathcal{P}} = \Delta_{\mathcal{K}}$.

This theorem is particularly useful when we have to consider triposes on toposes other than **sets**: if we can recognise such a topos as $\mathcal{P}$-**sets** for some **sets**-tripos $\mathcal{P}$, then we can, equivalently, work over **sets**, which often considerably simplifies computations.

IV. A topos for Lifschitz' realizability

§0. Introduction

In this chapter a topos Lif is constructed which generalizes Lifschitz' realizability: a sentence of first-order arithmetic is valid in this topos iff it is realized in Lifschitz' sense. The construction is analogous to that of the effective topos in Hyland 1982.
It is shown that Lif is a subtopos of Eff. However, it is not one of the well-known subtoposes of Eff of the form 'recursive in A' (studied e.g. in Phoa 1990) and it cannot be equivalent to a realizability topos based on a partial combinatory algebra, since the axiom of countable choice fails in it.
Nevertheless, Lif shares some features of Eff: it satisfies Church's Thesis, Markov's Principle and the Uniformity Principle; and the (Dedekind) reals are the recursive real numbers.
Lif and Eff are neatly separated by a classically valid principle of second-order arithmetic, which I call Richman's Principle because F. Richman raised the question whether it is intuitionistically derivable from Church's Thesis and Markov's Principle. I show that the principle is valid in Eff, but refuted in Lif.
The chapter ends with some trivialities concerning the internal topology k on Eff for which Lif is equivalent to k-sheaves in Eff. When we look at Lif from the point of view of Eff, we retrieve the original realizability definition of Lifschitz 1979.

Some notation. From now on I write [e] for Lifschitz' V_e: $[e] \equiv \{x \leq (e)_1 \mid (e)_0 \bullet x\uparrow\}$. The letter β is reserved for a primitive recursive function such that $[\beta(e)]=\{e\}$ for all e, and δ is a partial recursive function such that if [e] is a singleton, then $\delta(e)\downarrow$ and $\delta(e)\in[e]$ (lemma 2.3, chapter 2). The functions F and γ from lemmas 2.4 and 2.5 of chapter 2 will not be used: I simply write things like $\Lambda yw.\bigcup\{[g\bullet h] \mid g\in[y], h\in[w]\}$ by which is meant the partial recursive function assigning a standard code for this set to y,w. I find this set notation more suggestive and I trust the reader with the ability to construct such partial recursive functions with the help of lemmas 2.2-2.5 in chapter 2.

§1. Construction of Lif

1.1. Definition.

1) Put $J \equiv \{e\in\mathbb{N} \mid [e]\neq\emptyset\}$. Let Σ consist of those $H\subseteq J$ that satisfy:
   $\forall e\in J\,(e\in H \Leftrightarrow \forall f\in[e]\,(\beta(f)\in H))$.
2) Define an implication $\Rightarrow: \Sigma\times\Sigma\to\Sigma$ by
   $G\Rightarrow H \equiv \{e\in J \mid \forall f\in[e]\forall g\in G\,(f\bullet g\downarrow \;\&\; f\bullet g\in H)\}$
3) For any set X, define a binary relation ⊢ on Σ^X by
   $\phi\vdash\psi$ iff $\bigcap\{\phi(x)\Rightarrow\psi(x) \mid x\in X\}$ is nonempty.

Note, that $H \in \Sigma$ iff for some $A \subseteq \mathbb{N}$, $H = \{e \in J \mid [e] \subseteq A\}$. In particular, if $H \in \Sigma$ and $e \in J$ is such that $f \in H$ for all $f \in [e]$, then $\bigcup\{[f] \mid f \in [e]\} \in H$.

1.2. Proposition.

i) $\vdash$ is a preorder on Σ^X.

ii) There are operations $\wedge$, $\vee$, $\top$, $\bot$ on Σ that, together with $\Rightarrow$, make Σ^X (with the pointwise operations) a Heyting pre-algebra (see chapter 3).

Proof. i) $\beta(\Lambda x.x) \in \bigcap\{\phi(x) \Rightarrow \phi(x) \mid x \in X\}$, so $\vdash$ is reflexive. If $e \in \bigcap\{\phi(x) \Rightarrow \psi(x) \mid x \in X\}$ and $f \in \bigcap\{\psi(x) \Rightarrow \chi(x) \mid x \in X\}$ then $\beta(\Lambda a.\bigcup\{[h\bullet(g\bullet a)] \mid g \in [e], h \in [f]\})$ is an element of $\bigcap\{\phi(x) \Rightarrow \chi(x) \mid x \in X\}$, so $\vdash$ is transitive.

ii) Let $\top \equiv J$, $\bot \equiv \emptyset$, $G \wedge H \equiv \{e \in J \mid \forall f \in [e]\,((f)_0 \in G \;\&\; (f)_1 \in H)\}$, $G \vee H \equiv \{e \in J \mid \forall f \in [e]\,((f)_0 = 0 \text{ and } (f)_1 \in G, \text{ or } (f)_0 \neq 0 \text{ and } (f)_1 \in H)\}$. The calculations showing that Σ^X is a Heyting pre-algebra are all straightforward. For instance, $\beta(\Lambda x.\bigcup\{[(a)_0] \mid a \in [x]\})$ is an element of $\bigcap\{G \wedge H \Rightarrow G \mid G, H \in \Sigma\}$. The rest is left to the reader.

1.3. Proposition. For f: $X \to Y$ in **sets**, the map Σ^f: $\Sigma^Y \to \Sigma^X$ defined by composition with f, preserves all the Heyting structure and has both a left adjoint $\exists f$ and a right adjoint $\forall f$, which satisfy the Beck condition.

Proof. It is immediate that Σ^f preserves the Heyting structure since it is defined pointwise.

Define $\forall f(\phi)(y) \equiv \{e \in J \mid \forall x \in X \forall g \in J\,(f(x) = y \Rightarrow \forall h \in [e]\,(h\bullet g\downarrow \;\&\; h\bullet g \in \phi(x)))\}$

$\exists f(\phi)(y) \equiv \{e \in J \mid \forall h \in [e] \exists x \in X\,(f(x) = y \;\&\; h \in \phi(x))\}$

By way of example, we show for $\phi \in \Sigma^X$ and $\psi \in \Sigma^Y$: $\Sigma^f(\psi) \vdash \phi$ in Σ^X iff $\psi \vdash \forall f(\phi)$ in Σ^Y (In particular, this will show that $\forall f$ is order-preserving). So let $e \in \bigcap\{\psi(f(x)) \Rightarrow \phi(x) \mid x \in X\}$ and $y \in Y$, $a \in \psi(y)$. If $f(x) = y$ and $h \in J$, then $\bigcup\{[g\bullet a] \mid g \in [e]\} \in \phi(x)$ so for $e' = \beta(\Lambda a.\beta(\Lambda h.\bigcup\{[g\bullet a] \mid g \in [e]\}))$ we have that $e' \in \bigcap\{\psi(y) \Rightarrow \forall f(\phi)(y) \mid y \in Y\}$; conversely, if $e' \in \bigcap\{\psi(y) \Rightarrow \forall f(\phi)(y) \mid y \in Y\}$ and $x \in X$, $a \in \psi(f(x))$, then $\bigcup\{[f\bullet a] \mid f \in [e']\} \in \forall f(\phi)(f(x))$ so $\bigcup\{[(f\bullet a)\bullet\beta(0)] \mid f \in [e']\} \in \phi(x)$.

The Beck condition is trivial.

It is now immediate that the structure of the Heyting pre-algebras Σ^X and the maps Σ^f forms a tripos. We call the topos represented by it, Lif (the "Liffective topos").

1.4. Proposition. There is an inclusion of toposes: Lif$\to$Eff.

Proof. First we define a geometric morphism of triposes (def. 5.1 in chapter 3). Let $\Psi_+(X)$: $\Sigma^X \to P(\mathbb{N})^X$ be defined by composition with the inclusion: $\Sigma \subseteq P(\mathbb{N})$. Let

$\Psi^+(X)$: $P(\mathbb{N})^X \to \Sigma^X$ be defined by composition with the map: $P(\mathbb{N}) \to \Sigma$ which sends A to $\{e \in J \mid [e] \subseteq A\}$. It is obvious that $\Psi_+(X)$ is order-preserving. We show that $\Psi^+(X)$ is left adjoint to $\Psi_+(X)$.
Suppose $\phi \in P(\mathbb{N})^X$, $\psi \in \Sigma^X$ and $\phi \vdash \Psi_+(X)(\psi)$ in $P(\mathbb{N})^X$, say for all x and all $a \in \phi(x)$, $e{\bullet}a\downarrow$ & $e{\bullet}a \in \Psi_+(X)(\psi)(x) = \psi(x)$. Then if $x \in X$ and $b \in \Psi^+(X)(\phi)(x)$, so $[b] \subseteq \phi(x)$, then $\bigcup\{[e{\bullet}h] \mid h \in [b]\} \in \psi(x)$. So for all x, $\beta(\Lambda b.\bigcup\{[e{\bullet}h] \mid h \in [b]\}) \in \Psi^+(X)(\phi)(x) \Rightarrow \psi(x)$, so $\Psi^+(X)(\phi) \vdash \psi$ in Σ^X. Conversely, if for all x, $e \in \Psi^+(X)(\phi)(x) \Rightarrow \psi(x)$, then [e] is nonempty and for every $g \in [e]$, for every $x \in X$ and $a \in \phi(x)$, $g{\bullet}\beta(a)$ is defined and in $\Psi_+(X)(\psi)(x)$. So $\phi \vdash \Psi_+(X)(\psi)$.
Now $\Psi^+(X)$ obviously preserves $\top$. Furthermore, if $A,B \in P(\mathbb{N})$ and $a \in \Psi^+(A \wedge B)$ then $\forall e \in [a]\,((e)_0 \in A$ & $(e)_1 \in B)$ so $\{\langle \beta((e)_0), \beta((e)_1)\rangle \mid e \in [a]\} \in \Psi^+(A) \wedge \Psi^+(B)$; if $a' \in \Psi^+(A) \wedge \Psi^+(B)$ then $\{\langle g,h\rangle \mid g \in [(e)_0], h \in [(f)_1], e,f \in [a']\} \in \Psi^+(A \wedge B)$, and codes for these elements can be obtained recursively in a and a'. So $\Psi^+(X)$ preserves finite meets.
Moreover, if $G \in \Sigma$ and $a \in G$ then $\beta(a) \in \Psi^+\Psi_+(G)$ so $\Psi^+(X) \circ \Psi_+(X)$: $\Sigma^X \to \Sigma^X$ is isomorphic to the identity map. So (Ψ_+, Ψ^+) determines an inclusion of triposes and by proposition 5.5 of chapter 3 an inclusion of toposes.

1.5. Corollary. The natural numbers object (NNO) in Lif is up to isomorphism given by the pair $(\mathbb{N},=)$ where $[\![n=m]\!] \equiv \{e \in J \mid [e] \subseteq \{n\} \cap \{m\}\}$.

Proof. This follows from the characterization of the NNO in Eff (example 3.3 in chapter 3), theorem 5.4 (chapter 3) and the fact that natural numbers objects are preserved by inverse image functors.

§2. Some logical properties of Lif

2.1. Theorem. An arithmetical sentence is valid in Lif (interpreting the variables as ranging over the NNO) iff it is Lifschitz realizable.

Proof. Use the definition of Lifschitz' realizability given in lemma 3.1 of chapter 2 and the translation $()^-$ of proposition 4.3 in chapter 3.
One defines, by an induction on the complexity of formulas ϕ with free variables $x_1,\dots,x_k$ primitive recursive functions t_ϕ and s_ϕ of k arguments, such that for all $e, m_1,\dots,m_k$:
i) if e **r** $\phi(m_1,\dots,m_k)$ then $t_\phi(m_1,\dots,m_k){\bullet}e$ is defined and is an element of $[\![(\phi)^-]\!](m_1,\dots,m_k)$;
ii) if $e \in [\![(\phi)^-]\!](m_1,\dots,m_k)$ then $s_\phi(m_1,\dots,m_k){\bullet}e$ is defined and **r** $\phi(m_1,\dots,m_k)$.
If ϕ is a prime formula t=s, then $[\![(\phi)^-]\!](m_1,\dots,m_k) = \{e \in J \mid [e] = \{t\}$ and $t=s\}$ so we can put:
$t_\phi \equiv \Lambda m_1,\dots,m_k.\Lambda e.\beta(t)$; $s_\phi \equiv \Lambda m_1,\dots,m_k.\Lambda e.\beta(0)$.
The induction steps for the propositional connectives are trivial. If ϕ is $\forall x\psi$, then

$[\![(\phi)^-]\!](m_1,...,m_k) = \{e\in J \mid \forall f\in [e]\, \forall n\in \mathbb{N}\, \forall h\in J\, (f\bullet h\downarrow\ \&\ \forall g\in [f\bullet h] \forall w\, ([w]=\{n\} \Rightarrow g\bullet w\downarrow\ \&$ $g\bullet w\in [\![(\psi)^-]\!](m_1,...,m_k,n)))\}$. So if $e\in [\![(\phi)^-]\!](m_1,...,m_k)$ then $\forall f\in [e]\, \forall n\in \mathbb{N}\, (f\bullet\beta(0)\downarrow$ $\&\forall g\in [f\bullet\beta(0)]\, (g\bullet\beta(n)\downarrow\ \&\ s_\psi(m_1,...,m_k,n)\bullet(g\bullet\beta(n))\ \mathbf{r}\ \psi(m_1,...,m_k,n)))$. So if $e'=\beta(\Lambda n.\bigcup\{s_\psi(m_1,...,m_k,n)\bullet(g\bullet\beta(n)) \mid g\in [f\bullet\beta(0)], f\in [e]\})$, then $e'\ \mathbf{r}\ \forall x\psi$. Conversely, if $e\ \mathbf{r}\ \forall x\psi(m_1,...,m_k)$ then $[e]\neq\emptyset\ \&\ \forall f\in [e] \forall n\, (f\bullet n\downarrow\ \&$ $t_\psi(m_1,...,m_k,n)\bullet(f\bullet n)\in [\![(\psi)^-]\!](m_1,...,m_k,n))$. So if e' is such that $[e'] = \{\Lambda h.\beta(\Lambda w.$ $t_\psi(m_1,...,m_k,\delta(w))\bullet(f\bullet\delta(w)) \mid f\in [e]\}$, then $e'\in [\![(\phi)^-]\!](m_1,...,m_k)$. In both cases, e' can evidently be obtained recursively in e.
The induction step for the existential quantifier, equally tedious, is left to the reader.

2.2. Proposition. CT and MP are valid in Lif.

Proof. According to the characterization of exponentials given after Theorem 3.2 in chapter 3, the function space $\mathbb{N}^{\mathbb{N}}$ has as underlying set $\Sigma^{\mathbb{N}\times\mathbb{N}}$. The equality is given by: $[\![F=G]\!]$ is the interpretation of $E(F)\wedge E(G)\wedge\forall xy(Fxy\leftrightarrow Gxy)$ where E(F) is the universal closure of formula $(F(x,y)\rightarrow x=x\wedge y='y') \wedge (F(x,y)\wedge x=x'\wedge y='y'\rightarrow F(x',y')) \wedge (F(x,y)\wedge F(x,y')\rightarrow y='y') \wedge$ $(x=x\rightarrow\exists yF(x,y))$. Here of course = is interpreted as the equality on $\mathbb{N}$ given in 1.5. Now if $e\in [\![E(F)]\!]$ then we can find, recursively in e, $f\in [\![\forall xy\ (F(x,y)\wedge F(x,y')\rightarrow y='y')]\!]$, $g\in [\![\forall x(x=x\rightarrow\exists yF(x,y))]\!]$ and $h\in [\![\forall xy\ (F(x,y)\rightarrow x=x\wedge y='y)]\!]$. This means $\forall x(g'\bullet x\downarrow\ \&\ [g'\bullet x]\neq\emptyset$ $\&\ \forall n\in [g'\bullet x]\exists y(n\in Fxy))$ for some g. Using h, we can find a z such that $\forall x(z\bullet x\downarrow\ \&\ [z\bullet x]\neq\emptyset\ \&$ $\forall n\in [z\bullet x]\exists y(n\in Fxy\wedge y=y))$. But then, using f, we know that for every x, $\{(n)_1 \mid n\in [z\bullet x]\}$ must be a singleton. So, by lemma 2.3 of chapter 2, there is a w such that $\forall x(w\bullet x\downarrow\ \&\ Fx(w\bullet x)$ is nonempty)). This w then codes a total recursive function, w can be found recursively in e, and, in Lif, F is the function coded by w.
Markov's Principle is easier: note that if $e\in [\![A\vee\neg A]\!]$ then $\{(a)_0 \mid a\in [e]\}$ must be a singleton.

Now it is easy to see that AC_{00} cannot hold in Lif, because AC_{00}+CT implies CT_0 which by theorem 2.1 and chapter 2 is not valid in Lif; the counterexample mentioned in chapter 2 is also a counterexample to AC_{00} in Lif.
Note also that proposition 2.2 implies that Lif is not equivalent to a subtopos of Eff of the form "recursive in A" (see Phoa 1990): in such toposes, the maps $N\rightarrow N$ are exactly the A-recursive functions.

I shall now show that true second-order arithmetic in Lif is formalized by the extension of Lifschitz' realizability to **HAS** defined in §4 of chapter 2. The whole treatment is analogous to example 4.5 in chapter 3. By considerations similar to those in that example, we may identify the power-object of N in Lif up to isomorphism as the pair $(X,\approx)$, where

$X \equiv \{\phi: \mathbb{N}\rightarrow\Sigma \mid \forall n,x\, (x\in\phi(n) \rightarrow \forall y\in [x]\, ((y)_0=n))\}$, and

$R \approx S \equiv [\![\forall n\,(R(n) \leftrightarrow S(n))]\!]$.

And again, just as in the example, we may forget about the existence of variables of sort X since $\beta(\langle\beta(\Lambda x.x),\beta(\Lambda x.x)\rangle)$ is an element of $[\![R\approx R]\!]$ for every $R\in X$. So we use the translation $(\,)^-$ from the example, as well as the following bijection from X to P(ℕ): for $R\in X$ put

$\breve{R} \equiv \{y\in \mathbb{N} \mid \beta(y)\in R((y)_0)\}$

and inversely for $A\subseteq\mathbb{N}$ put

$\bar{A}(n) \equiv \{x\in J \mid \forall y\in [x]\,(y\in A \,\&\, (y)_0=n)\}$.

Then

2.3. Proposition. For every formula ϕ in the language of **HAS** with free variables $x_1,\ldots,x_k,X_1,\ldots,X_j$ there are two k-ary primitive recursive functions s_ϕ and t_ϕ such that for any k-tuple $n_1,\ldots,n_k$ from ℕ and j-tuples $A_1,\ldots,A_j$ from P(ℕ) and $R_1,\ldots,R_j$ from X:

i) $e\in [\![(\phi)^-(n_1,\ldots,n_k,R_1,\ldots,R_j)]\!] \Rightarrow s_\phi(n_1,\ldots,n_k)\bullet e\downarrow$ and $(s_\phi(n_1,\ldots,n_k)\bullet e\ \mathbf{r}\ \phi(n_1,\ldots,n_k,X_1,\ldots,X_j))[{X_1}^*/\breve{R}_1,\ldots,{X_j}^*/\breve{R}_j]$;

ii) $(e\ \mathbf{r}\ \phi(n_1,\ldots,n_k,X_1,\ldots,X_j))[{X_1}^*/A_1,\ldots,{X_j}^*/A_j] \Rightarrow t_\phi(n_1,\ldots,n_k)\bullet e\downarrow$ and $t_\phi(n_1,\ldots,n_k)\bullet e \in [\![(\phi)^-(n_1,\ldots,n_k,\bar{A}_1,\ldots,\bar{A}_j)]\!]$.

Proof. We only have to do this for $\phi \equiv x\in X$; the second-order quantifier clauses are trivial (because we have a bijection from X onto P(ℕ)), and the induction steps for the first-order connectives are as in the proof of 2.1.

Let $s_\phi(n) \equiv \Lambda e.\{(f)_1 \mid f\in [e]\}$

$t_\phi(n) \equiv \Lambda g.\{\langle n,h\rangle \mid h\in [g]\}$

For if $e\in[\![R(n)]\!]$ then $\forall f\in [e]\,((f)_0=n$ and $\beta(f)\in[\![R(n)]\!])$, (by definitions of X and Σ), so $\forall f\in [e]\ (\langle n,(f)_1\rangle\in \breve{R})$, so $\forall g\in \{(f)_1 \mid f\in [e]\}\ (\langle n,g\rangle\in \breve{R})$ which means $(\{(f)_1 \mid f\in [e]\}\ \mathbf{r}\ n\in X)[X^*/\breve{R}]$. The other one is left to the reader.

2.4. Corollary. The Uniformity Principle holds in Lif.

Proof. Easy, with the definition in §4 of chapter 2.

Now I want to show that the notions of Cauchy real and Dedekind real coincide in Lif. Generally, this is a consequence of AC_{00}, but we do not have that.

2.5. Proposition. Reals in Lif are recursive real numbers.

Proof. By CT, the Cauchy reals in Lif are the recursive reals. A *Dedekind cut* is a pair of predicates (L,U) on ℚ, such that:

1) $\forall q\,(L(q)\leftrightarrow\exists q'\,(L(q')\wedge q<q'))$

2) $\forall q\,(U(q) \leftrightarrow \exists q'\,(U(q') \wedge q'<q))$
3) $\forall q,q'\,(L(q) \wedge U(q') \rightarrow q<q')$
4) $\forall n \exists q,q'\,(L(q) \wedge U(q') \wedge n(q'-q)<1)$.

Call (L,U) a *strong real* if:

(*) $\exists \alpha: \mathbb{Q}^2 \rightarrow \mathbb{N}\, \forall q,q'\,(q<q' \rightarrow ((\alpha(q,q')=0 \wedge L(q)) \vee (\alpha(q,q')\neq 0 \wedge U(q'))))$

By proposition 5.5.10 of Troelstra & Van Dalen 1988, the object of strong reals is (order-) isomorphic to the object of Cauchy reals (the definitions there are different, but equivalent); so I show that every Dedekind cut is a strong real in Lif.

Suppose e realizes 4) which means:

$\forall g \in [e]\, \forall n\, (g{\bullet}n\!\downarrow \,\&\, [g{\bullet}n]\neq\emptyset \,\&\, \forall f \in [g{\bullet}n] \exists q,q'\,(f \in [\![q=q \wedge q'=q' \wedge L(q) \wedge U(q') \wedge n(q'-q)<1]\!]))$.

Let r_k be such that $[r_k] \equiv \{\langle a,b\rangle \mid a \in [(f)_1],\, b \in [(f)_3],\, f \in [g{\bullet}k],\, g \in [e]\}$;

let u_k be such that $[u_k] \equiv \{\langle c,d\rangle \mid c \in [(f)_2],\, d \in [(f)_4],\, f \in [g{\bullet}k],\, g \in [e]\}$.

Then if $q<q'$, either $\exists k \forall p \in [r_k]\,(q \leq (p)_1 - 2^{-k})$ or $\exists k \forall p \in [u_k]\,((p)_1 + 2^{-k} \leq q')$. Since both properties are Σ^0_1 in q,q' and from e.g. k such that $\forall p \in [r_k]\,(q \leq (p)_1 - 2^{-k})$ we can, by 1), find a realizer for L(q), it is now easy to construct α as in (*).

Now I want to discuss a principle of second-order arithmetic which separates Lif and Eff. Consider the statement:

RP $\quad \forall^d X\, (\forall^d Y\, (X \subseteq Y \vee X \cap Y = \emptyset) \rightarrow \exists n \forall x\, (x \in X \rightarrow x=n))$

where the quantifiers $\forall^d X$ and $\forall^d Y$ mean that X and Y range over decidable subsets of N. This principle is discussed in Blass & Scedrov 1986. F. Richman had raised the question whether it is a valid principle of intuitionistic higher order logic, because he needed it for the construction of divisible hulls of countable abelian groups; which is why I call it RP from Richman's Principle.

Blass & Scedrov exhibit a topological model and a sheaf model in which RP is not valid; however, $\neg\neg$RP is valid in their models. Moreover they write: "Our models do not satisfy further conditions imposed by Richman, namely Church's Thesis and Markov's Principle, so the full conjecture remains an open problem". I show that in Lif, RP is false, whereas it is true in Eff.

Dropping the decidability condition on Y gives a weakening of RP which still does not hold in the models of Blass & Scedrov. But in the presence of UP, if $\forall Y\, (X \subseteq Y \vee X \cap Y = \emptyset)$ then of course $\forall Y\, (X \subseteq Y) \vee \forall Y\, (X \cap Y = \emptyset)$, and X must be the empty set.

In the presence of CT we can reduce RP to a first-order statement. For, decidable subsets of N are sets of zeroes of functions: N→N, and these are recursive. So, RP is equivalent to

$RP_0 \quad \forall e\, [\forall x \exists y Texy \wedge \forall f (\forall x \exists y Tfxy \rightarrow \forall x\, (e{\bullet}x=0 \rightarrow f{\bullet}x=0) \vee \forall x \neg(e{\bullet}x=0 \wedge f{\bullet}x=0))$
$\rightarrow \exists n \forall x\, (e{\bullet}x=0 \rightarrow x=n)]$

Recall from chapter 2 that Lifschitz' realizability is axiomatized by the schema:

$ECT_L \quad \forall x\, (Ax \rightarrow \exists y Bxy) \rightarrow \exists z \forall x\, (Ax \rightarrow z{\bullet}x\!\downarrow \wedge \exists w (w \in [z{\bullet}x]) \wedge \forall w \in [z{\bullet}x]\, Bxw)$, where A must be $B\Sigma^0_2$-negative (definition before lemma 3.2 in chapter 2).

2.6. Proposition. **HA+RP_0+ECT_L+MP** is inconsistent.

Proof. Assuming $\forall x\exists yTexy$, $\forall f(\forall x\exists yTfxy \rightarrow \forall x(e{\bullet}x=0\rightarrow f{\bullet}x=0) \vee \forall x\neg(e{\bullet}x=0 \wedge f{\bullet}x=0))$ is equivalent to:

$$C(e) \qquad \forall f(\forall x\exists yTfxy \rightarrow \forall xyz(Texy \wedge Tfxz \wedge Uy=0 \rightarrow Uz=0) \vee \forall xyz\neg(Texy \wedge Uy=0 \wedge Tfxz \wedge Uz=0)),$$

which is equivalent to a $B\Sigma^0_2$-negative formula; we may apply ECT_L to RP_0 which would give a z such that:

(1) $\forall e[\forall x\exists yTexy \wedge C(e) \rightarrow z{\bullet}e\downarrow \wedge \exists w(w\in[z{\bullet}e]) \wedge \forall w\in[z{\bullet}e]\forall x(e{\bullet}x=0\rightarrow x=w)]$,

which means the existence of a z such that:

(2) $\forall e[\forall x\exists yTexy \wedge C(e) \rightarrow z{\bullet}e\downarrow \wedge \exists w\leq z{\bullet}e\forall x(e{\bullet}x=0\rightarrow x=w)]$,

and this is contradictory: suppose z as in (2). Let, by the recursion theorem, e be such that:

$$e{\bullet}x \simeq \begin{cases} 0 \text{ if } Tzex \\ 1 \text{ else.} \end{cases}$$

Then $z{\bullet}e$ is defined. For if not, then $\forall x(e{\bullet}x=1)$ and C(e) clearly holds, so $z{\bullet}e\downarrow$, contradiction; so $\neg\neg(z{\bullet}e\downarrow)$; apply MP. Furthermore, C(e) holds, for if f codes a total function, we only have to look at $f{\bullet}(\mu x.Tzex)$ to decide which of the two possibilities holds. But $\exists w\leq z{\bullet}e\forall x(e{\bullet}x=0\rightarrow x=w)$ is obviously false, since if Tzex, then $z{\bullet}e<x$ (for any standard coding).

2.7. Proposition. **HA+ECT_0** $\vdash$ RP_0

Proof. We argue in **HA+ECT_0**. Suppose $\forall x\exists yTexy \wedge \forall f(\forall x\exists yTfxy \rightarrow \forall x(e{\bullet}x=0\rightarrow f{\bullet}x=0) \vee \forall x\neg(e{\bullet}x=0 \wedge f{\bullet}x=0))$. By ECT_0, there is a z such that, for all f, if $\forall x\exists yTfxy$ then $z{\bullet}f\downarrow$ and:

i) $z{\bullet}f=0 \rightarrow \forall x(e{\bullet}x=0\rightarrow f{\bullet}x=0)$;

ii) $z{\bullet}f\neq 0 \rightarrow \forall x\neg(e{\bullet}x=0 \wedge f{\bullet}x=0)$.

Moreover:

iii) $\neg\exists xy\,(x\neq y \wedge e{\bullet}x=0 \wedge e{\bullet}y=0)$.

Use the recursion theorem to find a code f such that:

$$f{\bullet}x \simeq \begin{cases} 1 \text{ if } \forall y\leq x\neg T(z,f,y) \\ 0 \text{ if } T(z,f,x) \wedge Ux=0 \\ 1 \text{ if } \exists y<x\,(T(z,f,y) \wedge Uy=0)) \\ 0 \text{ if } \exists y\leq x\,(T(z,f,y) \wedge Uy\neq 0)). \end{cases}$$

Then f codes a total function, so $z{\bullet}f\downarrow$. Say T(z,f,x). Two possibilities:

a) Ux=0. Then by i), $\forall y(e{\bullet}y=0\rightarrow f{\bullet}y=0)$. But the only zero of f is x. So $\forall y(e{\bullet}y=0\rightarrow y=x)$.

b) $Ux\neq 0$. Then by ii), $\forall y\neg(e{\bullet}y=0 \wedge f{\bullet}y=0)$. But $\forall y\geq x\,(f{\bullet}y=0)$. So $\forall y(e{\bullet}y=0\rightarrow y<x)$.

In both cases, $\exists n\forall y\,(e{\bullet}y=0\rightarrow y=n)$ (in case b, check $e{\bullet}y$ for all $y<x$. Use iii)).

§3. Lif as k-sheaves of Eff

This section presumes some basic knowledge of topos theory. Since the results are far from spectacular, I don't think it is worthwile to spell out the definitions; the reader may safely skip this section.
Since Lif is a subtopos of Eff, Lif is equivalent to the topos of sheaves for an internal topology k in Eff. Let us write U(G) for $\{a\in J \mid [a]\subseteq G\}$, and (Ψ_*,Ψ^*) for the inclusion: Lif→Eff defined in 1.4.

3.1. Proposition. k is represented by K: $P(\mathbb{N})\times P(\mathbb{N})\rightarrow P(\mathbb{N})$ defined by
$K(G,H) \equiv [\![H\leftrightarrow U(G)]\!]$.

Proof. Let f: $(X,=)\rightarrow(Y,\approx)$ be a morphism in Eff, represented by F: $X\times Y\rightarrow P(\mathbb{N})$. Then $\Psi^*(f)$ is an epimorphism in Lif iff there is a number e such that:
$\forall y\in Y\forall f\in [\![y\approx y]\!]\,(e\bullet f\downarrow \ \&\ [e\bullet f]\neq\emptyset\ \&\ \forall h\in [e\bullet f]\,\exists x\in X\,(h\in [\![x=x\wedge F(x,y)]\!]))$,
but this is the same as saying that $\forall y\in Y.U(\exists x\in X.F(x,y))$ is valid in Eff.
Now the function U: $P(\mathbb{N})\rightarrow P(\mathbb{N})$ is a strict relation for ↔; let L denote the subobject of Ω (Ω is the subobject classifier in Eff) determined by U.
Then $\Psi^*(f)$ is an epimorphism iff $\forall y\in Y.U(\exists x\in X.F(x,y))$ is valid in Eff, but this is equivalent to: the classifying map of the image of f factors through L. It follows that L is the generic k-closed subobject of Ω in Eff, and k classifies L.

3.2. Example. Let $S(i,x,y) \equiv T(x,x,y)\wedge U(y)=i$ for i=0,1, and $R\subseteq N\times 2$ be the subobject defined by the relation
$R(n,z) \equiv [\![z=0\rightarrow\neg\exists yS(0,n,y)\wedge z=1\rightarrow\neg\exists yS(1,n,y)]\!]$. Then the first projection: R→N is k-almost epi in Eff (i.e. the Ψ^*-image of it is epi in Lif), but not epi.

3.3. Proposition. N is a k-sheaf.

Proof. The reader is invited to check that morphisms (X,=)→N in Eff are in 1-1 correspondence with partial recursive functions ϕ: $\cup\{Ex \mid x\in X\}\rightarrow\mathbb{N}$, such that:
(*) $\forall x,x'([\![x=x']\!]\neq\emptyset \Rightarrow \forall a\in Ex\forall b\in Ex'(\phi(a)=\phi(b)))$
The same is true for morphisms: (X,≈)→N in Lif.
Now if ϕ: $\cup\{Ex \mid x\in X\}\rightarrow\mathbb{N}$ satisfies (*) for (X,=) in Eff and x∈X, [a]⊆Ex, then $\{\phi(a') \mid a'\in [a]\}$ is a singleton, so $\Lambda a.\delta(\{\phi(a') \mid a'\in [a]\})$: $\cup\{U(Ex) \mid x\in X\}\rightarrow\mathbb{N}$ satisfies (*) and so defines a morphism: $\Psi^*((X,=))\rightarrow$N in Lif.
Conversely if ϕ: $\cup\{U(Ex) \mid x\in X\}\rightarrow\mathbb{N}$ satisfies (*) then $\phi\circ\beta$: $\cup\{Ex \mid x\in X\}\rightarrow\mathbb{N}$ satisfies (*).

So we have established a natural isomorphism: Eff((X,=),N) $\xrightarrow{\sim}$ Lif(Ψ*((X,=)),N); which means that N is isomorphic to $\Psi_*\Psi^*$(N).

In a topos, an internal topology k can be used to interpret a modal operator ⊠ defined on the internal language, with the additional axioms:

1) $p \rightarrow ⊠(p)$
2) $⊠⊠(p) \rightarrow ⊠(p)$
3) $⊠(p \wedge q) \leftrightarrow ⊠(p) \wedge ⊠(q)$.

If N is a k-sheaf, validity of an arithmetical sentence ϕ in the topos of k-sheaves is equivalent to validity of $(\phi)^{⊠}$ in the original topos, where $(-)^{⊠}$ is the translation which is the identity on atomic formulas, commutes with all the negative connectives and puts a ⊠ before $\exists$ and $\vee$. Writing this out for the topology k on Eff, one gets an inductive definition of realizability which is exactly Lifschitz' original definition (the definition before proposition 2.1 in chapter 2).

V. Modified realizability and modified Lifschitz realizability

§1. Introduction

In this chapter, triposes are presented that represent toposes for HRO-modified realizability and a Lifschitz analogue of it. A tripos-theoretic account of HRO-modified realizability was given in Grayson 1981B; I fill in a gap in this construction.

For a definition of abstract modified realizability and the system $\mathbf{HA}^{\omega}$ I refer to Troelstra 1973. There, the following inductive definition of HRO-modified realizability for **HA** is given:

1.1. Definition. Assign to every formula ϕ two predicates D_ϕ and $\underline{mr}\,\phi$, as follows:

$D_\phi(x) \equiv x{=}x$; $\quad x\,\underline{mr}\,\phi \equiv \phi \quad$ if ϕ is a prime formula;

$D_{\phi\wedge\psi}(x) \equiv D_\phi((x)_0) \wedge D_\psi((x)_1)$; $x\,\underline{mr}\,\phi\wedge\psi \equiv (x)_0\,\underline{mr}\,\phi \wedge (x)_1\,\underline{mr}\,\psi$;

$D_{\phi\to\psi}(x) \equiv \forall y\,(D_\phi(y) \to x{\bullet}y\downarrow \wedge D_\psi(x{\bullet}y))$; $x\,\underline{mr}\,\phi\to\psi \equiv D_{\phi\to\psi}(x) \wedge \forall y\,(y\,\underline{mr}\,\phi \to x{\bullet}y\downarrow \wedge$
$x{\bullet}y\,\underline{mr}\,\psi)$;

$D_{\exists y\phi}(x) \equiv D_\phi((x)_1)$; $x\,\underline{mr}\,\exists y\phi \equiv (x)_1\,\underline{mr}\,\phi[{}^{y}/{}_{(x)_0}]$;

$D_{\forall y\phi}(x) \equiv \forall y\,(x{\bullet}y\downarrow \wedge D_\phi(x{\bullet}y))$; $x\,\underline{mr}\,\forall y\phi \equiv \forall y\,(x{\bullet}y\downarrow \wedge x{\bullet}y\,\underline{mr}\,\phi[{}^{x}/{}_{y}])$.

The predicates D_ϕ depend on the logical structure of ϕ only (this, however, is an inessential feature). This inductive definition results from expressing in **HA** the interpretation of modified realizability in the model HRO (more precisely: interpret the finite type functionals in abstract modified realizability in HRO).

1.2. Proposition. Let, by the recursion theorem, **e** be such that for all x, $\mathbf{e}{\bullet}x = \mathbf{e}$. Suppose that in the definition of D_ϕ and $\underline{mr}\,\phi$ above, $(x)_0$ and $(x)_1$ refer to a primitive recursive pairing function which satisfies $\langle\mathbf{e},\mathbf{e}\rangle = \mathbf{e}$. Then:

i) $\forall x\,(x\,\underline{mr}\,\phi \to D_\phi(x))$

ii) $D_\phi(\mathbf{e})$.

Proof. Trivial.

§2. Tripos-theoretic treatment of modified realizability

As is apparent from definition 1.1., modified realizability assigns two sets of "realizers" to each formula, such that one is included in the other, and the largest one contains a fixed element **e** (proposition 1.2).

In order to generalize this to a tripos, let us first have a look at the situation without the special element **e**.

2.1. Notational convention. Write, for $A,B \subseteq \mathbb{N}$:

$A \to B \equiv \{n \in \mathbb{N} \mid \forall a \in A\,(n \bullet a\downarrow \wedge n \bullet a \in B)\}$
$A \& B \equiv \{n \in \mathbb{N} \mid (n)_0 \in A \text{ and } (n)_1 \in B\}$

2.2. Definition. Let Δ be the set of all pairs $p=(p_1,p_2) \in P(\mathbb{N}) \times P(\mathbb{N})$ such that $p_1 \subseteq p_2$. Define $\Rightarrow: \Delta \times \Delta \to \Delta$ by:
$(p \Rightarrow q)_1 \equiv p_1 \to q_1 \cap p_2 \to q_2$
$(p \Rightarrow q)_2 \equiv p_2 \to q_2$.
For every set X, define a preorder on Δ^X by putting
$\phi \vdash \psi$ iff $\bigcap\{(\phi(x) \Rightarrow \psi(x))_1 \mid x \in X\}$ is nonempty.

2.3. Proposition.
1) Let $\mathcal{P}(X) \equiv (\Delta^X,\vdash)$; for f: X→Y let $\mathcal{P}(f)$ be Δ^f. Then $\mathcal{P}$ is a tripos, with the following logical structure:

$$
\begin{array}{lcl}
\phi\wedge\psi(x) & \equiv & (\phi(x)_1\&\psi(x)_1,\phi(x)_2\&\psi(x)_2) \\
\phi\vee\psi(x) & \equiv & (\{0\}\&\phi(x)_1\cup\{1\}\&\psi(x)_1,\{0\}\&\phi(x)_2\cup\{1\}\&\psi(x)_2) \\
\phi\Rightarrow\psi(x) & \equiv & \phi(x)\Rightarrow\psi(x) \\
\top(x) & \equiv & (\mathbb{N},\mathbb{N}) \\
\bot(x) & \equiv & (\emptyset,\emptyset) \\
\exists f(\phi)(y) & \equiv & (\bigcup\{\phi(x)_1 \mid f(x)=y\},\bigcup\{\phi(x)_2 \mid f(x)=y\}) \\
\forall f(\phi)(y) & \equiv & (\bigcap\{\mathbb{N}\to\phi(x)_1 \mid f(x)=y\},\bigcap\{\mathbb{N}\to\phi(x)_2 \mid f(x)=y\})
\end{array}
$$

(if f is surjective one may take $(\bigcap\{\phi(x)_1 \mid f(x)=y\},\bigcap\{\phi(x)_2 \mid f(x)=y\})$ for $\forall f(\phi)(y)$);
2) There is a geometric morphism of triposes (Φ_+,Φ^+): $\mathcal{P}\to\mathcal{E}$, where $\mathcal{E}$ denotes the effective tripos. This is induced by the first projection: $\Delta \to P(\mathbb{N})$ and the diagonal embedding: $P(\mathbb{N}) \to \Delta$.

Proof. 1) is a special case of "Kripke models of realizability", to be treated in chapter 6, but the reader may enjoy a direct verification which meets no difficulties. 2) is a not too difficult verification.

2.4. Corollary. 1) The natural numbers in **$\mathcal{P}$-sets** are up to isomorphism given by $(\mathbb{N},=)$ where $[\![n=m]\!] \equiv (\{n\}\cap\{m\},\{n\}\cap\{m\})$;
2) **$\mathcal{P}$-sets** satisfies the same sentences of arithmetic as Eff.

Proof. 1) is immediate from 2.3; 2) follows from the observation that the functor Φ^+: $\mathcal{E}\to\mathcal{P}$ preserves all first-order structure.

From Corollary 2.4 we see that $\mathcal{P}$ is not the right generalization of modified realizability, although the implication looks the same. We need to use the special element **e**.
For the rest of this section I assume that primitive recursive pairing satisfies $\langle \mathbf{e},\mathbf{e}\rangle=\mathbf{e}$.

2.5. Definition. Let Σ be the set of all pairs $p=(p^*,E_p)\in P(\mathbb{N})\times P(\mathbb{N})$ such that $p^*\subseteq E_p$ and $\mathbf{e}\in E_p$. Define $\Rightarrow$ as in 2.2:
$E_{p\Rightarrow q} \equiv E_p \to E_q$
$(p\Rightarrow q)^* \equiv E_{p\Rightarrow q} \cap (p^*\to q^*)$.
Note, that $\mathbf{e}\in E_q$ implies $\mathbf{e}\in E_{p\Rightarrow q}$; so $\Rightarrow$ is well-defined.
For every set X, a preorder $\vdash$ on Σ^X is defined exactly as on Δ^X in 2.2:
$\phi\vdash\psi$ iff $\bigcap\{(\phi(x)\Rightarrow\psi(x))^* \mid x\in X\}$ is nonempty.

2.6. Proposition. Let $\mathfrak{R}(X)$ be $(\Sigma^X,\vdash)$; $\mathfrak{R}(f)$ is Σ^f. Then $\mathfrak{R}$ is a tripos, and the logical structure on $\mathfrak{R}$ is the same as on $\mathcal{P}$, except for $\bot$, $\vee$ and $\exists f$.

Proof. One can verify directly that $\vdash$ defines a preorder and that $\wedge$, $\Rightarrow$, $\top$ and $\forall f$ have the right properties.
Of course $(\emptyset,\emptyset)$ is not an element of Σ; the bottom element $\bot$ of $\mathfrak{R}(X)$ is $\lambda x.(\emptyset,\{\mathbf{e}\})$ (or, isomorphically, $\lambda x.(\emptyset,\mathbb{N})$).
For the join $\vee$ of two elements ϕ and ψ of $\mathfrak{R}(X)$ we may take
$\phi\vee\psi(x) \equiv (\{\mathbf{e}\}\&\phi(x)^* \cup \{\mathbf{e}+1\}\&\psi(x)^*, \{\mathbf{e}\}\&E_{\phi(x)} \cup \{\mathbf{e}+1\}\&E_{\psi(x)})$,
using that $\langle \mathbf{e},\mathbf{e}\rangle=\mathbf{e}$.
The definition of existential quantification however, is not immediate. Grayson (1981B) defined directly:
$\exists f(\phi)(y) \equiv (\bigcup\{\phi(x)^* \mid f(x)=y\}, \{\mathbf{e}\}\cup\bigcup\{E_{\phi(x)} \mid f(x)=y\})$.
This is wrong, as the following example shows: let $X=\{0,1\}$, $f:X\to X$ be $\lambda x.1$ and $\phi,\psi\in\mathfrak{R}(X)$ be given by $\phi(0)=\phi(1)=\psi(0)=(\{\mathbf{e}\},\{\mathbf{e}\})$, $\psi(1)=(\{\mathbf{e}+1\},\{\mathbf{e},\mathbf{e}+1\})$. Then $\phi\vdash\mathfrak{R}(f)(\psi)$ via $\Lambda x.x+1$, but any element of $(\exists f(\phi)(0)\Rightarrow\psi(0))^* \cap (\exists f(\phi)(1)\Rightarrow\psi(1))^*$ would send $\mathbf{e}$ at once to $\mathbf{e}$ and to $\mathbf{e}+1$, which is impossible; so $\exists f(\phi)\nvdash\psi$ and $\exists f$ fails to have the required adjointness property. We have to be slightly more sophisticated.
The solution is clear once we realize that Σ is a subset of Δ, satisfying:
i) $p\in\Delta$, $q\in\Sigma$ implies $p\Rightarrow q\in\Sigma$;
ii) If $\phi\in\mathcal{P}(X)$ maps X into Σ then for every function $f:X\to Y$, $\forall f(\phi)$ maps Y into Σ.
These are the conditions of Lemma 5.6 of chapter 3; so by the lemma, $\mathfrak{R}$ is a subtripos of $\mathcal{P}$; and by lemma 5.7 of chapter 3, we can derive existential quantification in $\mathfrak{R}$ from that in $\mathcal{P}$.
Following lemma 5.6 we have to use the function $J: \Delta\to\Sigma$ which is the interpretation of $\forall q\,(p\Rightarrow q)\Rightarrow q$, where q runs over Σ. Now if $p\in\Delta$, $p=(p_1,p_2)$, then
$E_{J(p)} \equiv \bigcap\{(p_2\to E_q)\to E_q \mid q\in\Sigma\}$ and
$J(p)^* \equiv E_{J(p)}\cap\bigcap\{(p_1\to q^*)\to q^* \mid q\in\Sigma\}$.
More explicitly, define maps $V,W: P(\mathbb{N})\to P(\mathbb{N})$ by
$V(B) \equiv \{f\in\mathbb{N} \mid \forall c\,(\forall b\in B\,(c\bullet b\downarrow)\Rightarrow f\bullet c\downarrow \text{ and } (f\bullet c=\mathbf{e} \text{ or } \exists b\in B\,(f\bullet c=c\bullet b)))\}$
$W(A) \equiv \{f\in\mathbb{N} \mid \forall c\,(\forall a\in A\,(c\bullet a\downarrow)\Rightarrow f\bullet c\downarrow \text{ and } \exists a\in A\,(f\bullet c=c\bullet a))\}$.
Then the assiduous reader may like to show that

$J(p) = (W(p_1)\cap V(p_2), V(p_2))$.
Accordingly, by lemma 5.7 in chapter 3, existential quantification in $\mathfrak{R}$ can be defined by:
$\exists f(\phi)(y) \equiv (W(p_1)\cap V(p_2), V(p_2))$, where $p_1 = \cup\{\phi(x)^* \mid f(x)=y\}$ and $p_2 = \cup\{E_{\phi(x)} \mid f(x)=y\}$.
This still looks rather complicated; but if f is a surjection, $\exists f$ is (up to isomorphism) given by:
$\exists f(\phi)(y) = (\cup\{\phi(x)^* \mid f(x)=y\}, \cup\{E_{\phi(x)} \mid f(x)=y\})$.
This latter expression is all we need (concerning existential quantification) since we shall only deal with projections from nonempty sets; however, the maps V and W will be used to characterize the natural numbers in **$\mathfrak{R}$-sets**. From now on I call this topos Mod.

2.7. Corollary (of the proof of 2.6). In Mod, the natural numbers object N is given by $(\mathbb{N},=)$ where $[\![n=m]\!] \equiv (W(\{n\}\cap\{m\}), V(\{n\}\cap\{m\}))$; V and W being as defined in the proof of 2.6.

Proof. Immediate from the inclusion: $\mathfrak{R}\to\mathfrak{P}$ in the proof of 2.6 and the characterization of N in **$\mathfrak{P}$-sets**.

2.8. Remark. As we have seen in the proof of 2.6, existential quantification for numbers in $\mathfrak{R}$ does not differ from that in $\mathfrak{P}$, since the projections: $\mathbb{N}^{k+1}\to\mathbb{N}^k$ are all surjective. Nor is there any difference in the other logical clauses (of course, it does not really matter whether you use 0 and 1 in the definition of $\vee$, or **e** and **e**+1; besides, $\vee$ and $\bot$ are defined notions in first-order arithmetic). This means that if you write out the truth clauses for arithmetic for both toposes, you get two inductive realizability definitions that only differ in the clause for atomic formulas and in the quantifier clauses.

2.9. Proposition. Validity in Mod of sentences in the language of arithmetic is equivalent to HRO-modified realizability.

Proof. Using the translation $()^-$ defined in proposition 4.3. of chapter 3, one constructs for every formula ϕ with free variables $x_1,\dots,x_k$ two primitive recursive functions t_ϕ and s_ϕ of k arguments, such that for all $n_1,\dots,n_k,a\in\mathbb{N}$:
i) $a\in D_{\phi(n_1,\dots,n_k)} \Rightarrow t_\phi(n_1,\dots,n_k)\bullet a\downarrow$ and $t_\phi(n_1,\dots,n_k)\bullet a\in E_{[\![(\phi)^-(n_1,\dots,n_k)]\!]}$;
ii) $a\,\underline{mr}\,\phi(n_1,\dots,n_k) \Rightarrow t_\phi(n_1,\dots,n_k)\bullet a\downarrow$ and $t_\phi(n_1,\dots,n_k)\bullet a\in [\![(\phi)^-(n_1,\dots,n_k)]\!]^*$;
iii) $a\in E_{[\![(\phi)^-(n_1,\dots,n_k)]\!]} \Rightarrow s_\phi(n_1,\dots,n_k)\bullet a\downarrow$ and $s_\phi(n_1,\dots,n_k)\bullet a\in D_{\phi(n_1,\dots,n_k)}$;
iv) $a\in [\![(\phi)^-(n_1,\dots,n_k)]\!]^* \Rightarrow s_\phi(n_1,\dots,n_k)\bullet a\downarrow$ and $s_\phi(n_1,\dots,n_k)\bullet a\,\underline{mr}\,\phi(n_1,\dots,n_k)$.
The functions t_ϕ and s_ϕ are defined by induction on ϕ.
For $\phi \equiv u=v(x_1,\dots,x_k)$ put $t_\phi(n_1,\dots,n_k) \equiv \Lambda g.[\Lambda f.f\bullet u(n_1,\dots,n_k)$ if $u=v(n_1,\dots,n_k)$; **e** else];
$s_\phi(n_1,\dots,n_k) \equiv \Lambda g.\Lambda f.f$.
The conjunction case is trivial.
If $\phi \equiv \psi\to\chi\ (x_1,\dots,x_k)$ put $t_\phi(n_1,\dots,n_k) \equiv \Lambda e.\Lambda a.t_\chi(n_1,\dots,n_k)\bullet(e\bullet(s_\psi(n_1,\dots,n_k)\bullet a))$;
$s_\phi(n_1,\dots,n_k) \equiv \Lambda e.\Lambda a.s_\chi(n_1,\dots,n_k)\bullet(e\bullet(t_\psi(n_1,\dots,n_k)\bullet a))$;

The quantifier step is slightly less obvious. Let f be a code for $\lambda w.w+\mathbf{e}+1$.
If $\phi \equiv \forall x\psi$ define:

$$t_\phi(n_1,...,n_k) \equiv \Lambda g.\Lambda a.[\mathbf{e} \text{ if } a\bullet f=\mathbf{e};\; t_\psi(w,n_1,...,n_k)\bullet(g\bullet w) \text{ if } a\bullet f=w+\mathbf{e}+1;\; \text{undefined else}];$$

$$s_\phi(n_1,...,n_k) \equiv \Lambda g.\Lambda n.\ s_\psi(n,n_1,...,n_k)\bullet(g\bullet(\Lambda u.u\bullet n)).$$

Recall that $E_{[\![(\forall n\psi)^-(n_1,...,n_k)]\!]}$ is $\bigcap\{V(\{n\})\to E_{[\![(\psi)^-(n,n_1,...,n_k)]\!]} \mid n\in\mathbb{N}\}$ and $[\![(\forall n\psi)^-(n_1,...,n_k)]\!]^*$ is $\bigcap\{ V(\{n\})\to E_{[\![(\psi)^-(n,n_1,...,n_k)]\!]} \cap W(\{n\})\to[\![(\psi)^-(n,n_1,...,n_k)]\!]^* \mid n\in\mathbb{N}\}$.
Now suppose $g\in D_{\forall n\psi(n,n_1,...,n_k)}$, so $\forall n\,(g\bullet n\in D_{\psi(n,n_1,...,n_k)})$. Let $n\in\mathbb{N}$, $a\in V(\{n\})$. Since $f\bullet n\!\downarrow$, $a\bullet f\!\downarrow$ and ($a\bullet f=\mathbf{e}$ or $a\bullet f=f\bullet n=\mathbf{e}+n+1$). If $a\bullet f=\mathbf{e}$, then $(t_\phi(n_1,...,n_k)\bullet g)\bullet a=\mathbf{e}$ so $\in E_{[\![(\psi)^-(n,n_1,...,n_k)]\!]}$; and if $a\bullet f=\mathbf{e}+n+1$ then $g\bullet n\in D_{\psi(n,n_1,...,n_k)}$, so $(t_\phi(n_1,...,n_k)\bullet g)\bullet a=\mathbf{e}$ so $\in E_{[\![(\psi)^-(n,n_1,...,n_k)]\!]}$.
So $t_\phi(n_1,...,n_k)\bullet g$ is always an element of $\bigcap\{V(\{n\})\to E_{[\![(\psi)^-(n,n_1,...,n_k)]\!]} \mid n\in\mathbb{N}\}$ whenever $g\in D_{\forall n\psi(n,n_1,...,n_k)}$. Similarly, $t_\phi(n_1,...,n_k)\bullet g\in W(\{n\})\to[\![(\psi)^-(n,n_1,...,n_k)]\!]^*$ whenever $g\ \underline{mr}\ \forall n\psi(n,n_1,...,n_k)$ (if $a\in W(\{n\})$ then $a\bullet f=\mathbf{e}+n+1$).
The verification that s_ϕ works is left to the reader (Note, that $\Lambda u.u\bullet n$ is an element of $W(\{n\})$).
If $\phi \equiv \exists n\psi$ define:

$$t_\phi(n_1,...,n_k) \equiv \Lambda g.\langle\Lambda f.f\bullet(g)_0, t_\psi((g)_0,n_1,...,n_k)\rangle$$

$$s_\phi(n_1,...,n_k) \equiv \Lambda g.[\mathbf{e} \text{ if } (g)_0\bullet f = \mathbf{e};\ (f \text{ is } \Lambda w.w+\mathbf{e}+1)\;\; s_\psi(n,n_1,...,n_k)\bullet(g)_1 \text{ if } (g)_0\bullet f=\mathbf{e}+n+1;\;\; \text{undefined else}]$$

Recall that $E_{[\![(\exists n\psi)^-(n_1,...,n_k)]\!]} = \bigcup\{V(\{n\})\& E_{[\![(\psi)^-(n,n_1,...,n_k)]\!]} \mid n\in\mathbb{N}\}$ and that $[\![(\exists n\psi)^-(n_1,...,n_k)]\!]^*$ is $\bigcup\{W(\{n\})\&[\![(\psi)^-(n,n_1,...,n_k)]\!]^* \mid n\in\mathbb{N}\}$. The proof is left to the reader.

2.10. Proposition. i) For every subobject A of $\mathbb{N}\times\mathbb{N}$ in Mod, we have

$$\vdash \forall x{:}\mathbb{N}\,\exists y{:}\mathbb{N}\, A(x,y) \to \exists z{:}\mathbb{N}\,\forall x{:}\mathbb{N}\,\exists y{:}\mathbb{N}\,(T(z,x,y)\wedge A(x,Uy))$$

ii) For every two objects X and Y, subobjects A of X and B of X×Y in Mod:

$$\vdash \forall x{:}X\,[(\neg A(x)\to\exists y{:}Y\,B(x,y)) \to \exists y{:}Y\,(\neg A(x)\to B(x,y))]$$

Proof. Straightforward.

Now we turn to second-order arithmetic in Mod. Consider the following extension of definition 1.1. to the language of **HAS**:

2.11. Definition. To every set variable X of **HAS** we suppose two set variables $\underline{X}^*$ and $D_{\underline{X}}$ are associated, such that for different X,Y, the variables $\underline{X}$, $\underline{Y}$, $D_{\underline{X}}$, $D_{\underline{Y}}$ are all different. Now we define predicates D_ϕ and $\underline{mr}\,\phi$ for formulas ϕ of the extended language as follows:

$$D_{n\in X}(a) \equiv \forall y\, D_{\underline{X}}(\langle\mathbf{e},y\rangle) \to D_{\underline{X}}(\langle a,n\rangle)$$

$$a\ \underline{mr}\ n\in X \equiv \underline{X}^*\subseteq D_{\underline{X}} \wedge \underline{X}^*(\langle a,n\rangle)$$

$$D_{\forall X\phi} \equiv \forall D_{\underline{X}}(\forall y\, D_{\underline{X}}(\langle\mathbf{e},y\rangle) \to D_\phi(a))$$

$$a\ \underline{mr}\ \forall X\phi \equiv \forall \underline{X}^*\forall D_{\underline{X}}(\underline{X}^*\subseteq D_{\underline{X}} \wedge \forall y\, D_{\underline{X}}(\langle e,y\rangle) \to a\ \underline{mr}\ \phi)$$
$$D_{\exists X\phi} \equiv \exists D_{\underline{X}}(\forall y\, D_{\underline{X}}(\langle e,y\rangle) \wedge D_\phi(a))$$
$$a\ \underline{mr}\ \exists X\phi \equiv \exists \underline{X}^*\exists D_{\underline{X}}(\underline{X}^*\subseteq D_{\underline{X}} \wedge \forall y\, D_{\underline{X}}(\langle e,y\rangle) \wedge a\ \underline{mr}\ \phi)$$

The proof that 2.11 defines a realizability for which **HAS** is provably sound, is left to the reader. I show that this definition formalizes true second-order arithmetic in Mod; the proof is similar to the proof in example 4.5 of chapter 3. The power object of $\mathbb{N}$ in Mod is the pair $(\Sigma^{\mathbb{N}},\approx)$, where $R\approx S$ is the interpretation of $\forall n(R(n)\to n=n) \wedge \forall nm(R(n)\wedge n=m \to R(m)) \wedge \forall n(R(n)\leftrightarrow S(n))$.

2.12. Proposition. $(\Sigma^{\mathbb{N}},\approx)$ is isomorphic to $(X,\approx)$ where X is the set of all $\phi\in\Sigma^{\mathbb{N}}$ which are of the form $\phi(n) \equiv \psi(n)\wedge n=n$, and $\phi\approx\psi$ is the interpretation of $\forall n(\phi(n)\leftrightarrow\psi(n))$.

Proof. There is a uniform realizer for $\forall nm(R(n)\wedge n=m\to R(m))$ for every $R\in\Sigma^{\mathbb{N}}$, i.e. an a such that for all $R\in\Sigma^{\mathbb{N}}$,and all $n,m\in\mathbb{N}$: $a\in[\![R(n)\wedge n=m\to R(m)]\!]^*$. For, let f be, again, a code for $\lambda x.e+x+1$. Let $\langle c,b\rangle\in[\![R(n)\wedge n=m]\!]$. Then $b\bullet f\downarrow$. Send $\langle c,b\rangle$ to **e** if $b\bullet f$=**e**, and to c otherwise.

2.13. Definition. Define a 1-1 correspondence between the set X as defined in proposition 2.12 and the set of all pairs (A,B) which satisfy $A\subseteq B$ and $\forall n(\langle e,n\rangle\in B)$, as follows:
for $\phi\in X$ let $G(\phi) \equiv (\{y\in\mathbb{N} \mid \langle (y)_0,\Lambda f.f\bullet(y)_1\rangle\in\phi((y)_1)^*\},\{y\in\mathbb{N} \mid \langle (y)_0,\Lambda f.f\bullet(y)_1\rangle\in E_{\phi((y)_1)}\})$;
for (A,B) as above let $H((A,B))(n)^* \equiv \{\langle (x)_0,y\rangle \mid x\in A,\ (x)_1=n,\ y\in[\![n=n]\!]^*\}$ and
$$E_{H((A,B))(n)} \equiv \{\langle (x)_0,y\rangle \mid x\in B,\ (x)_1=n,\ y\in E_{[\![n=n]\!]}\}$$

2.14. Proposition. Let $()^-$ be the translation as in 4.5 of chapter 3. Then for any formula ϕ in the language of **HAS** with free number variables $x_1,\dots,x_n$ and set variables $X_1,\dots,X_k$ there are primitive recursive functions s_ϕ and t_ϕ such that for all $m_1,\dots,m_n\in\mathbb{N}$, $\alpha_1,\dots,\alpha_k\in X$ and $(A_1,B_1),\dots,(A_k,B_k)\in P(\mathbb{N})\times P(\mathbb{N})$ satisfying $A_i\subseteq B_i$ and $\forall n(\langle e,n\rangle\in B_i)$ for $1\le i\le k$:
i) $(e\in D_\phi)(^{m_1,\dots,m_n}/_{x_1,\dots,x_n};\ ^{(A_1,B_1)}/(\underline{X}^*_1,D_{\underline{X}_1}),\dots,^{(A_k,B_k)}/(\underline{X}^*_k,D_{\underline{X}_k}))$ implies $s_\phi(m_1,\dots,m_n)\bullet e\downarrow$ and $s_\phi(m_1,\dots,m_n)\bullet e\in E_{[\![(\phi)^-(m_1,\dots,m_n,H(A_1,B_1),\dots,H(A_k,B_k))]\!]}$;
ii) $(e\ \underline{mr}\ \phi)(^{m_1,\dots,m_n}/_{x_1,\dots,x_n};\ ^{(A_1,B_1)}/(\underline{X}^*_1,D_{\underline{X}_1}),\dots,^{(A_k,B_k)}/(\underline{X}^*_k,D_{\underline{X}_k}))$ implies $s_\phi(m_1,\dots,m_n)\bullet e\downarrow$ and $s_\phi(m_1,\dots,m_n)\bullet e\in[\![(\phi)^-(m_1,\dots,m_n,H(A_1,B_1),\dots,H(A_k,B_k))]\!]^*$;
iii) $e\in E_{[\![(\phi)^-(m_1,\dots,m_n,\alpha_1,\dots,\alpha_k)]\!]}$ implies $t_\phi(m_1,\dots,m_n)\bullet e\downarrow$ and $(t_\phi(m_1,\dots,m_n)\bullet e\in D_\phi)(^{m_1,\dots,m_n}/_{x_1,\dots,x_n};\ ^{G(\alpha_1)}/(\underline{X}^*_1,D_{\underline{X}_1}),\dots,^{G(\alpha_k)}/(\underline{X}^*_k,D_{\underline{X}_k}))$;
iv) $e\in[\![(\phi)^-(m_1,\dots,m_n,\alpha_1,\dots,\alpha_k)]\!]^*$ implies $t_\phi(m_1,\dots,m_n)\bullet e\downarrow$ and $(t_\phi(m_1,\dots,m_n)\bullet e\ \underline{mr}\ \phi)(^{m_1,\dots,m_n}/_{x_1,\dots,x_n};\ ^{G(\alpha_1)}/(\underline{X}^*_1,D_{\underline{X}_1}),\dots,^{G(\alpha_k)}/(\underline{X}^*_k,D_{\underline{X}_k}))$;

Proof. We only have to do this for formulas of the form $x\in X$; the first order induction steps are

as in the proof of proposition 2.9, and the steps for the second order quantifiers are trivial from the definition (e.g.$s_{\forall X\phi}(m_1,\dots,m_k) = s_\phi(m_1,\dots,m_k)$).
Put $s_{x\in X}(n) \equiv \Lambda a.\langle a, \Lambda f.f\bullet n\rangle$. If $(a\in D_{n\in X})[(A,B)]$ then $\langle a,n\rangle\in B$, so $\langle a,\Lambda f.f\bullet n\rangle\in E_{H(A,B)(n)}$; similarly, if $(a\ \underline{mr}\ n\in X)[(A,B)]$ then $\langle a,n\rangle\in A$ and $\langle a,\Lambda f.f\bullet n\rangle\in H(A,B)(n)^*$.
Put $t_{x\in X}(n) \equiv \Lambda a.(a)_0$. If $a\in E_{\alpha(n)}$ then because α is of the form $\lambda n.[\![\beta(n)\wedge n=n]\!]$, $\langle (a)_0,\Lambda f.f\bullet n\rangle\in E_{\alpha(n)}$, so $\langle (a)_0,n\rangle\in G(\alpha)_2$ (the second component of $G(\alpha)$), which means $((a)_0\in D_{x\in X})[n,G(\alpha)]$. And similar for the case $a\in\alpha(n)^*$.

2.15. Remark. Let (f_*,f^*): $\mathcal{P}$-**sets**$\to$Eff be the geometric morphism induced by the geometric morphism of triposes defined in 2.3. Then the constant objects functor $\Delta_{\mathcal{P}}$: **sets**$\to\mathcal{P}$-**sets** factors through f^* (Theorem 5.4 in chapter 3). Now it follows from general tripos theory that for the geometric morphism (g_*,g^*): Mod$\to$Eff, obtained by composing (f_*,f^*) with the inclusion Mod$\to\mathcal{P}$-**sets**, every object of Mod is a subquotient of some $g^*(X)$, X an object of Eff. In topos-theoretical language, (g_*,g^*) is *bounded* and by the Giraud-Mitchell-Diaconescu theorem (see Johnstone 1977, theorem 4.46), Mod is a subtopos of the topos of internal presheaves on an internal category in Eff. So it should be possible to express HRO-modified realizability as forcing over a site in the effective topos. I do not have an easy description of such a site; it might be interesting to have one, though.

2.16. Remark. A tripos for q-modified realizability can be defined, analogous to Grayson (see example 11 of the list of realizabilities in chapter 1): truth-values are pairs (p,x) with $p\in\Sigma$, $x\subseteq\{0\}$ satisfying $\exists n\,(n\in p^*)\Rightarrow 0\in x$; $(p,x)\to(q,y)$ is defined as the pair $(r,\{0\mid x\subseteq y\})$ where r is $(\{a\in(p\Rightarrow q)^*\mid x\subseteq y\},D_{p\Rightarrow q})$ (Glueing of **sets** and Mod). **Sets** is an open subtopos of the topos represented by this tripos, say q-Mod. Performing the construction of Mod and q-Mod over the free topos yields another proof of the Independence of Premiss Rule for intuitionistic type theory (cf. Lambek & Scott 1986, §21 of part II).

§3. Modified Lifschitz realizability

The treatment of modified Lifschitz realizability is quite analogous to that of §2. As in chapter 4, let J be $\{e\mid [e]\neq\emptyset\}$ and let Σ consist of those $H\subseteq J$ that satisfy:

$e\in H$ iff for all $f\in[e]$, $\beta(f)\in H$,

where β is a primitive recursive function satisfying $[\beta(e)]=\{e\}$ for all e. Let, by the recursion theorem, **e** be such that for all x, $\mathbf{e}\bullet x=\beta(\mathbf{e})$. We assume a primitive recursive pairing function $\langle\rangle$ satisfying $\langle\beta(\mathbf{e}),\beta(\mathbf{e})\rangle=\mathbf{e}$. We write $\dashrightarrow$ for the implication in Σ:
$G\dashrightarrow H = \{a\in J\mid \forall g\in G\forall b\in[a]\ (b\bullet g\downarrow\ \&\ b\bullet g\in H)\}$.
T is the set $\{p=(p_0,p_1)\in\Sigma\times\Sigma\mid p_0\subseteq p_1\}$ with implication $p\Rightarrow q=(p_0\dashrightarrow q_0\cap p_1\dashrightarrow q_1, p_1\dashrightarrow q_1)$ and $\mathcal{P}'$ is the tripos based on T.
Θ is the set $\{p=(p_0,p_1)\in\Sigma\times\Sigma\mid p_0\subseteq p_1$ and $\beta(\mathbf{e})\in p_1\}$ and $\mathfrak{R}'$ is the tripos based on Θ. Just as in

the proof of proposition 2.6, there is an inclusion $\mathfrak{R}' \to \mathcal{P}'$ given by j: $T \to \Theta$ defined by $j(q) = \forall p \in \Theta\,[(q \Rightarrow p) \Rightarrow p]$. Just as in §2, existential quantification in $\mathfrak{R}'$ is defined by: first compute it in $\mathcal{P}'$, then apply j.
We denote the topos represented by $\mathfrak{R}'$ Modlif.

3.1. Proposition. j is isomorphic (as element of $\mathcal{P}'(T)$) to the map which sends q to $(W(q_0,q_1),V(q_1))$, where V and W are defined as follows:

$$V(q_1) \equiv \{a \in J \mid \forall e\,(\forall b \in q_1\, e\bullet b\downarrow \Rightarrow \forall a' \in [a]\ (a'\bullet e\downarrow \ \&\ a'\bullet e \in J\ \&\ [a'\bullet e] \subseteq \{e\} \cup \{e\bullet b \mid b \in q_1\}))\}$$

$$W(q_0,q_1) \equiv V(q_0) \cap \{a \in J \mid \forall e\,(\forall b \in q_0 e\bullet b\downarrow \Rightarrow \forall a' \in [a]\ (a'\bullet e\downarrow \ \&\ a'\bullet e \in J\ \&\ [a'\bullet e] \subseteq \{e\bullet b \mid b \in q_0\}).$$

Proof. Straightforward.

Let us write $[n=m]_L$ for $\{a \in J \mid [a]=\{n\} \text{ and } n=m\}$; then by proposition 3.1, a natural numbers object in Modlif is given by $(\mathbb{N},=)$ where $[\![n=m]\!] \equiv (W([n=m]_L,[n=m]_L),V([n=m]_L))$.

3.2. Definition. (Modified Lifschitz realizability) Assign to every formula ϕ in the language of arithmetic two predicates E_ϕ and $\underline{mr}_L\,\phi$, by induction on ϕ, as follows:

1) $E_{t=s}(x) \equiv [x] \neq \emptyset$;
   $x\ \underline{mr}_L\ t=s \equiv [x] \neq \emptyset$ and $t=s$ is true;
2) $E_{\phi\wedge\psi}(x) \equiv [x] \neq \emptyset$ and $\forall y \in [x]\,(E_\phi((y)_0)$ and $E_\psi((y)_1)$;
   $x\ \underline{mr}_L\ \phi\wedge\psi \equiv [x] \neq \emptyset$ and $\forall y \in [x]\,((y)_0\ \underline{mr}_L\ \phi$ and $(y)_1\ \underline{mr}_L\ \psi)$;
3) $E_{\phi\to\psi}(x) \equiv [x] \neq \emptyset$ and $\forall y \in [x]\,\forall w\,(E_\phi(w) \Rightarrow y\bullet w\downarrow$ and $E_\psi(y\bullet w))$;
   $x\ \underline{mr}_L\ \phi\to\psi \equiv\ E_{\phi\to\psi}(x)$ and $\forall y \in [x]\,\forall w\,(w\ \underline{mr}_L\ \phi \Rightarrow y\bullet w\downarrow$ and $y\bullet w\ \underline{mr}_L\ \psi)$;
4) $E_{\forall y\phi(y)}(x) \equiv [x] \neq \emptyset$ and $\forall y \in [x]\,\forall w\,(y\bullet w\downarrow$ and $E_{\phi(w)}(y\bullet w))$;
   $x\ \underline{mr}_L\ \forall y\phi(y) \equiv [x] \neq \emptyset$ and $\forall y \in [x]\,\forall w\,(y\bullet w\downarrow$ and $y\bullet w\ \underline{mr}_L\ \phi(w))$;
5) $E_{\exists y\phi(y)}(x) \equiv [x] \neq \emptyset$ and $\forall y \in [x]\,(E_{\phi((y)_0)}((y)_1)$;
   $x\ \underline{mr}_L\ \exists y\phi(y) \equiv [x] \neq \emptyset$ and $\forall y \in [x]\,((y)_1\ \underline{mr}_L\ \phi((y)_0))$.

3.3. Proposition. Validity of arithmetical sentences in Modlif is equivalent to $\underline{mr}_L$-realizability.

Proof. Let $[\![\phi]\!] \equiv ([\![\phi]\!]_0,[\![\phi]\!]_1)$ denote the canonical interpretation of ϕ in the tripos $\mathfrak{R}'$, and $()^-$ the usual translation. Define primitive recursive functions s_ϕ and t_ϕ of k arguments for each formula ϕ with k free variables, such that for all $n_1,\dots,n_k$ and all e:

i) $E_{\phi(n_1,\dots,n_k)}(e) \Rightarrow s_\phi(n_1,\dots,n_k)\bullet e\downarrow$ and $s_\phi(n_1,\dots,n_k)\bullet e \in [\![(\phi)^-(n_1,\dots,n_k)]\!]_1$;
ii) $e\ \underline{mr}_L\ \phi(n_1,\dots,n_k) \Rightarrow s_\phi(n_1,\dots,n_k)\bullet e\downarrow$ and $s_\phi(n_1,\dots,n_k)\bullet e \in [\![(\phi)^-(n_1,\dots,n_k)]\!]_0$;
iii) $e \in [\![(\phi)^-(n_1,\dots,n_k)]\!]_1 \Rightarrow t_\phi(n_1,\dots,n_k)\bullet e\downarrow$ and $E_{\phi(n_1,\dots,n_k)}\,(t_\phi(n_1,\dots,n_k)\bullet e)$;
iv) $e \in [\![(\phi)^-(n_1,\dots,n_k)]\!]_0 \Rightarrow t_\phi(n_1,\dots,n_k)\bullet e\downarrow$ and $t_\phi(n_1,\dots,n_k)\bullet e\ \underline{mr}_L\ \phi(n_1,\dots,n_k)$.

The definition of s_ϕ and t_ϕ and the proof of i)-iv) is done simultaneously by induction on ϕ.
1) If $\phi \equiv t=s(x_1,...,x_k)$ let $s_\phi(n_1,...,n_k) \equiv \Lambda x.\beta(\Lambda e.F(e))$, where F(e) is $\beta(e\bullet\beta(t(n_1,...,n_k)))$ if $t=s(n_1,...,n_k)$ and $\beta(\mathbf{e})$ else. Now $[\![(\phi)^-(n_1,...,n_k)]\!]_1$ is $V([t=s]_L)$ and $[\![(\phi)^-(n_1,...,n_k)]\!]_0$ is $W([t=s]_L,[t=s]_L)$; suppose $E_{\phi(n_1,...,n_k)}(x)$. Then $s_\phi(n_1,...,n_k)\bullet x$ is defined. Suppose e is such that $e\bullet b\downarrow$ for all $b\in[t=s]_L$. Now either $t=s(n_1,...,n_k)$ or not; in both cases $F(e)\downarrow$ and $F(e)\in J$ and $[F(e)]\subseteq\{\mathbf{e}\}\cup\{e\bullet b \mid b\in[t=s]_L\}$, so $\beta(\Lambda e.F(e))\in V([t=s]_L)$. The proof of ii) is similar.
Let $t_\phi(n_1,...,n_k) \equiv \Lambda x.\beta(\mathbf{e})$. iii) and iv) are obvious.
2) Let $s_{\phi\wedge\psi}(n_1,...,n_k) \equiv \Lambda e.\beta(\langle k,m\rangle)$ where
$k \equiv \bigcup\{[s_\phi(n_1,...,n_k)\bullet(f)_0] \mid f\in[e]\}$ and
$l \equiv \bigcup\{[s_\psi(n_1,...,n_k)\bullet(f)_1] \mid f\in[e]\}$; i) and ii) are clear. The definition of $t_{\phi\wedge\psi}(n_1,...,n_k)$ is exactly the same (replacing s by t).
3) $s_{\phi\to\psi}(n_1,...,n_k) \equiv \Lambda e.\beta(\Lambda f.s_\psi(n_1,...,n_k)\bullet\bigcup\{[f'\bullet(t_\phi(n_1,...,n_k)\bullet f)] \mid f'\in[e]\})$ and t results from this by interchanging s and t. i)-iv) are left to the reader.
4) $s_{\forall n\psi}(n_1,...,n_k) \equiv \Lambda x.\beta(\Lambda a.\bigcup\{[\zeta_x(g)] \mid g\in[a'\bullet\mathbf{f}],a'\in[a]\})$, where
$\mathbf{f} \equiv \Lambda b.e+\delta(b)+1$ (here δ is such that if [b] is a singleton, then $\delta(b)\in[b]$), and
$\zeta_x(g) \equiv$ undefined if $g<e$; $b(\mathbf{e})$ if $g=\mathbf{e}$; and $s_\psi(n,n_1,...,n_k)\bullet(x\bullet n)$ if $g=e+n+1$.
Proof of i)-ii): suppose $E_{\forall n\psi}(x)$, $a\in V([n=n]_L)$. Then since $\mathbf{f}\bullet b\downarrow$ for all $b\in[n=n]_L$, we have $a'\bullet\mathbf{f}\downarrow$ for all $a'\in[a]$, and $\zeta_x(g)\downarrow$ for all $g\in[a'\bullet\mathbf{f}]$, and moreover $\zeta_x(g)\in[\![(\psi)^-(n,n_1,...,n_k)]\!]_1$. Etcetera.
Note that if x $\underline{mr}_L$ $\forall n\,\psi$, $a\in W([n=n]_L)$, $a'\in[a]$, $g\in[a'\bullet\mathbf{f}]$, always $g=e+n+1$.
Put $t_{\forall n\psi}(n_1,...,n_k) \equiv \Lambda x.\Lambda n.t_\psi(n,n_1,...,n_k)\bullet\bigcup\{[y\bullet\beta(\Lambda f.\beta(f\bullet\beta(n)))] \mid y\in[x]\}$. iii)-iv) are left to the reader.
5) Let $s_{\exists n\psi(n)}(n_1,...,n_k) \equiv \Lambda x.\{\langle\beta(\Lambda f.\beta(f\bullet\beta((y)_0))),s_\psi((y)_0,n_1,...,n_k)\bullet(y)_1\rangle \mid y\in[x]\}$.
i)-ii) left to the reader.
Put $t_{\exists n\psi(n)}(n_1,...,n_k) \equiv \Lambda x.\bigcup\{\bigcup\{[\zeta_y(a'')] \mid a''\in[a'\bullet\mathbf{f}],a'\in[(y)_0]\} \mid y\in[x]\}$ where $\mathbf{f}$, again, is $\Lambda b.e+\delta(b)+1$, and
$\zeta_y(a'') \equiv$ undefined if $a''<e$; $\beta(\mathbf{e})$ if $a''=\mathbf{e}$; $\beta(\langle n,t_\psi(n,n_1,...,n_k)\bullet(y)_1\rangle)$ if $a''=\mathbf{e}+n+1$.
Now if $x\in[\![(\exists n\psi)^-]\!]_1$ then $\forall y\in[x]\exists n\,((y)_0\in V([n=n]_L$ and $(y)_1\in[\![(\psi)^-]\!]_1)$. Let $y\in[x]$. Then for all a' in $[(y)_0]$, $a'\bullet\mathbf{f}$ is defined and $[a'\bullet\mathbf{f}]\neq\emptyset$ and $\forall a''\in[a'\bullet\mathbf{f}]$ ($a''\geq e$, and if $a''=e+n+1$, then $(y)_1\in[\![(\psi)^-(n,n_1,...,n_k)]\!]_1$). So for all $a''\in[a'\bullet\mathbf{f}]$, $\zeta_y(a'')\downarrow$ and $[\zeta_y(a'')]\neq\emptyset$, and for all $v\in[\zeta_y(a'')]$, $E_{\psi((v)_0,n_1,...n_k)}((v)_1)$. So the same holds for $\bigcup\{[\zeta_y(a'')] \mid a''\in[a'\bullet\mathbf{f}],a'\in[(y)_0]\}$ which is recursively obtained in y, etc.

I close this chapter with some results analogous to those about modified realizability.

3.4. Proposition. Independence of premiss holds in Modlif.

Proof. (For the arithmetical case). $\beta(\Lambda z.z')$ $\underline{mr}_L$ $(\neg A\to\exists xB(x))\to\exists x(\neg A\to B(x))$, where z' such that $[z'] = \{\langle(h')_0,\beta(\Lambda x.(h')_1\rangle \mid h'\in[h\bullet\beta(e)],h\in[z]\}$.

3.5. Proposition. For every negative formula A of arithmetic with free variables $x_1,\ldots,x_n$ there is a term t_A such that:
i) $E_{\forall x_1\ldots\forall x_n A}(t_A)$;
ii) For all $k_1,\ldots,k_n$: if $A(k_1,\ldots,k_n)$ is true then $\forall h\in[t_A]\,(h\bullet\langle k_1,\ldots,k_n\rangle\ \underline{mr}_L\ A(k_1,\ldots,k_n))$;
iii) For all $k_1,\ldots,k_n$: if for some x, $x\ \underline{mr}_L\ A(k_1,\ldots,k_n)$, then $A(k_1,\ldots,k_n)$ is true.

Proof. 1) Put $t_A\equiv\beta(e)$ if A is a prime formula.
2) $t_{A_1\wedge A_2}\equiv\beta(\Lambda e.\{\langle h\bullet e,h'\bullet e\rangle \mid h\in[t_{A_1}],h'\in[t_{A_2}]\})$;
3) $t_{A_1\to A_2}\equiv\beta(\Lambda e.\beta(\Lambda x.\bigcup\{[h\bullet e]\mid h\in[t_{A_2}]\}))$;
4) $t_{\forall nA}\equiv\beta(\Lambda x_1,\ldots,\Lambda x_k.\beta(\Lambda u.\bigcup\{[h\bullet\langle x_1,\ldots,x_n,u\rangle]\mid h\in[t_A]\}))$. The proofs of i)-iii) are left to the reader.

3.6. Proposition. M_{PR} is refuted in Modlif.

Proof. Suppose $x\ \underline{mr}_L\ \forall e(\neg\neg\exists yT(e,e,y)\to\exists yT(e,e,y))$. Then $[x]\neq\emptyset$; let $y\in[x]$. Then for all e, $y\bullet e\downarrow$ and $y\bullet e\ \underline{mr}_L\ \neg\neg\exists yT(e,e,y)\to\exists yT(e,e,y)$. Let $g\equiv\bigcup\{[v\bullet\beta(e)]\mid v\in[y\bullet e]\}$ and F be the primitive recursive function which sends h to 0 if $\neg T(e,e,(h)_0)$, and to 1 otherwise. Now if $\exists zT(e,e,z)$ then $\beta(e)\ \underline{mr}_L\ \neg\neg\exists zT(e,e,z)$ so $\forall h\in[g]\ T(e,e,(h)_0)$; if $\neg\exists zT(e,e,z)$ then certainly $\forall h\in[g]\ \neg T(e,e,(h)_0)$. So $\neg\neg(\{F(h)\mid h\in[g]\}$ is a singleton), etc.

3.7. Conjecture. $B\Sigma^0_2$-MP is refuted in Modlif.

Motivation. Suppose $a\ \underline{mr}_L\ \forall e(\neg\neg[e]\neq\emptyset\to[e]\neq\emptyset)$ (I use $[e]\neq\emptyset$ as abbreviation for $\exists x\leq(e)_1\forall n\neg T((e)_0,x,n)$). Let F be total recursive such that $[F(e)]=\{(h)_0\mid h\in\bigcup\{[b\bullet\beta(e)]\mid b\in[a'\bullet e],a'\in[a]\}\}$. Then $[F(e)]$ is nonempty since $\beta(e)\in E_{\neg\neg[e]\neq\emptyset}$. Suppose $[e]$ is nonempty; then $b(e)\ \underline{mr}_L\ \neg\neg[e]\neq\emptyset$, so for all $a'\in[a]$ and all $h\in\bigcup\{[b\bullet\beta(e)]\mid b\in[a'\bullet e]\}$, $(h)_1\ \underline{mr}_L\ ((h)_0\in[e])$. Since $(h)_0\in[e]$ is a negative formula, by proposition 3.5 it follows that $(h)_0\in[e]$ is true. Summarizing: F is a total recursive function which satisfies: $\forall e([F(e)]$ is nonempty and $([e]$ is nonempty $\Rightarrow[F(e)]\subseteq[e]))$. I believe that this is impossible, although I do not have a proof at the moment.

3.8. Proposition. ECT_L and IP are incompatible w.r.t. **HA**.

Proof. Let:

$$A(e)\equiv\neg\neg\exists zTeez\to\exists zTeez$$

$$B(e,y)\equiv\neg\neg\exists zTeez\to Teey$$

Then $A(e)\to\exists yB(e,y)$ is an instance of IP; by ECT_L (since A is $B\Sigma^0_2$-negative) we have a z such that

$$\forall e(A(e)\to z\bullet e\downarrow\wedge[z\bullet e]\neq\emptyset\wedge\forall w\in[z\bullet e]\ B(e,w))$$

Suppose $z \bullet z\downarrow$. Then $A(z)$, so $[z \bullet z] \neq \emptyset \wedge \forall w \in [z \bullet z]\, T(z,z,w)$. But this means: if $T(z,z,w)$ then $w \leq (U(w))_1$. This is contradictory for a standard coding, so $\neg(z \bullet z\downarrow)$. But then $\neg A(z)$, which is also contradictory.

Proposition 3.8 is an analogue for the situation with ordinary modified realizability, which satisfies CT_0 but not ECT_0. This raises the question whether CT_L is $\underline{mr}_L$-realized. It is rather unsatisfactory that I do not know the answer. The problem with a direct verification is, that the part $\forall w \in [z \bullet e]\, B(e,w)$ contains an implication.

VI. Kripke and Beth models of realizability

§0. Introduction. In the next chapter I give a semantical proof of De Jongh's Theorem. This is done via the construction of a "Beth model of realizability". In order to get some more insight into what is going on there, as well as to substantiate certain claims that are made in that chapter (in particular, that Beth realizability is part of a topos), in this chapter some generalities are developed about these models and the relations between them.
In chapter 3, the example of a **sets**-tripos based on a partial combinatory algebra was given. Now this can be carried out in any topos, provided we know what an "internal partial combinatory algebra" in a topos is.
The reader is referred to Troelstra & Van Dalen 1988 (Chapter 9, section 3) for an exposition of the theory **APP**.

0.1. Definition. Let $\mathcal{E}$ be a topos. An internal partial combinatory algebra is a model of **APP** in $\mathcal{E}$.

0.2. Remark. The theory **APP** has, besides the axioms for application, K and S, also a unary predicate for natural numbers as well as constants for successor, predecessor and definition by numerical cases. Although these can be explicitly defined in any partial combinatory algebra, it is convenient to have them because for realizability one seldom wishes to use these "Church-numerals".

0.3. Linguistic intermezzo. Now the theory **APP** (being a theory of *partial* application) is formulated in E^+-logic (see Troelstra & Van Dalen 1988, chapter 2, section 2 for a treatment), so a few words about interpreting E^+-logic in a topos are in order.
APP has one partial binary application symbol *. This should be interpreted as a partial map: $A\times A \rightharpoonup A$ in $\mathcal{E}$, or a 3-ary predicate Ap, satisfying $Ap(a,b,c)\wedge Ap(a,b,c')\rightarrow c=c'$ in the internal logic of $\mathcal{E}$. In a topos, partial maps to A are equivalent to maps into $\tilde{A}$, where $\tilde{A}$ (in the internal language of $\mathcal{E}$) is $\{\alpha\subseteq A \mid \forall x,y\in\alpha\,(x=y)\}$ ($\tilde{A}$ is the *partial map classifier* of A). Now we can extend a partial map Ap: $A\times A \rightharpoonup A$ to a map $\tilde{A}p$: $\tilde{A}\times\tilde{A}\rightarrow\tilde{A}$ by putting
$\tilde{A}p(\alpha,\beta)\equiv\{c\in A \mid \exists a\in\alpha\exists b\in\beta\, Ap(a,b,c)\}$.
The *existence predicate* E is intepreted by the collection of inhabited $\alpha\in\tilde{A}$: i.e.
$E(\alpha)\equiv\exists x(x\in\alpha)$. Note, that $\tilde{A}p$ is *strict* for this interpretation: $E(\tilde{A}p(\alpha,\beta))\rightarrow E(\alpha)\wedge E(\beta)$.
Equality $\equiv$ is interpreted as: $\alpha\equiv\beta$ is $\exists x\,(x\in\alpha\wedge x\in\beta)$. Then $\alpha\equiv\alpha\leftrightarrow E(\alpha)$;
Directed equality $\simeq$ is interpreted as equality in $\tilde{A}$. Then $\alpha\simeq\beta\leftrightarrow(E(\alpha)\vee E(\beta)\rightarrow\alpha\equiv\beta)$.
Let η: $A\rightarrow\tilde{A}$ be defined by $\eta(x)=\{x\}$. Then always $E(\eta(x))$ (this is the interpretation of: *variables exist*).
Now interpret every formula of the language of **APP** into the internal language of $\mathcal{E}$, such that all terms become terms of type $\tilde{A}$ (variables x are translated as $\eta(x)$, K and S as $\eta(K)$, $\eta(S)$),

whereas the quantifiers run only over A. So $\forall x(Sx\downarrow)$ translates into $\forall x{:}A.(E(\breve{A}p(\eta(S),\eta(x))))$. Let us see that E^+-logic is sound for this translation. The axioms Ex translate into $E(\eta(x))$ which are true; the rules $\forall$I and $\exists$E are immediate; let us look at $\exists I^E$: $A[^x/_t],E(t)\Rightarrow\exists xA$. Suppose α interprets t, then $E(\alpha)$ implies $\exists y(\alpha=\eta(y))$ (in the internal language), so $\exists yA[^x/_{\eta(y)}]$ which is the interpretation of $\exists yA$. Similarly for $\forall E^E$: $\forall xA,E(t)\Rightarrow A[^x/_t]$: $\forall xA$ translates into $\forall xA[^x/_{\eta(x)}]$ and if α interprets t, then $\exists y(\alpha=\eta(y))$ as before, so $A[^x/_\alpha]$ which interprets $A[^x/_t]$. Now we have interpreted E^+-logic into the internal language of $\mathcal{E}$, we may use the internal language without appealing to this translation every time: I write in the language of **APP**, which I find more convenient.

0.4. Proposition. Let A be an internal partial combinatory algebra in a topos $\mathcal{E}$. Let P(A) denote the power-object of A in $\mathcal{E}$. Then the assignment $\mathcal{P}(X)\equiv\mathcal{E}(X,P(A))$, and for morphisms $f{:}X\to Y$ in $\mathcal{E}$, $\mathcal{P}(f)\equiv\mathcal{E}(f,P(A))$: $\mathcal{E}(Y,P(A))\to\mathcal{E}(X,P(A))$ defines an $\mathcal{E}$-tripos.

Proof. All the definitions are the same as in example b) following proposition 1.3, chapter 3; provided they are read in the internal language of $\mathcal{E}$.

0.5. Definition. A *Kripke model* of realizability is a tripos of the form given in 0.4, where $\mathcal{E}$ is the topos $\mathbf{sets}^P$, P being a partial order with bottom element.

0.6. Proposition. Let $\mathcal{E}$ be a topos, $j{:}\Omega\to\Omega$ an internal topology in $\mathcal{E}$, A an internal partial combinatory algebra. Let $P_j(A)$ be the object of j-closed subsets of A, i.e. $P_j(A)=\{B\subseteq A \mid \forall x\in A(j(x\in B)\to x\in B)\}$. Then the assignment $\mathcal{P}(X)\equiv\mathcal{E}(X,P_j(A))$ defines an $\mathcal{E}$-tripos.

We shall see this in section 2.

0.7. Definition. A *Beth model* of realizability is a tripos of the form given in 0.6, $\mathcal{E}$ again of the form $\mathbf{sets}^P$.

§1. Kripke models of realizability

It is easily verified that a model of the theory **APP** in the topos $\mathbf{sets}^P$, where P is a partial order with bottom element p_0, is given by a *P-indexed system of partial combinatory algebras*, that is:

i) for every $p\in P$ a partial combinatory algebra A_p;

ii) for every $p\leq p'$ a map $f_{pp'}$: $A_p\to A_{p'}$, satisfying:

a) f_{pp} is the identity on A_p, and $f_{p'p''}\circ f_{pp'}=f_{pp''}$ for every $p\leq p'\leq p''$;

b) if $a\bullet b\downarrow$ in A_p, then $f_{pp'}(a)\bullet f_{pp'}(b)\downarrow$ in $A_{p'}$, and $f_{pp'}(a)\bullet f_{pp'}(b)=f_{pp'}(a\bullet b)$;

c) the maps $f_{pp'}$ preserve the combinators K and S, the specified copy N of $\mathbb{N}$, and the combinators for successor, predecessor and definition by numerical cases.

Such a system $\langle(A_p)_{p\in P},(f_{pp'})_{p\leq p'}\rangle$ will be denoted A. It is clear that such A are just functors: P→**pca** for a suitably defined category **pca** of partial combinatory algebras.

The power object of A in the topos $\mathbf{sets}^P$, denoted by P(A), is the P-indexed system of sets:

$$(P(A))_p \equiv \{(\alpha_{p'})_{p'\geq p} \mid \alpha_{p'}\subseteq A_{p'} \text{ and } f_{p'p''}[\alpha_{p'}]\subseteq\alpha_{p''} \text{ for all } p'\leq p''\}.$$

Now one can work out the logical structure of the $\mathbf{sets}^P$-tripos based on A (I write $\mathcal{P}_A$ for this tripos), from the internal language of $\mathbf{sets}^P$. Let X be object of $\mathbf{sets}^P$, $\phi,\psi\in\mathcal{P}_A(X)$.

Then $\phi\vdash\psi$ iff $\exists a\in A_{p_0}\forall p\in P\forall x\in X_p\forall b\in f_p(x)\,(f_{p_0p}(a)\bullet b\downarrow$ and $f_{p_0p}(a)\bullet b\in\psi_p(x))$.

There is a map ⇒: P(A)×P(A)→P(A) in $\mathbf{sets}^P$ defined by:

$(\Rightarrow_p((\alpha_{p'})_{p'\geq p},(\beta_{p'})_{p'\geq p}))_q = \{a\in A_q \mid \forall r\geq q\forall b\in\alpha_r\,(f_{qr}(a)\bullet b\downarrow$ and $f_{qr}(a)\bullet b\in\beta_r)\}$.

If f: X→P(Y) is a morphism from X into a power-object, the intersection of the image of f, $\bigcap\{f(x)\mid x\in X\}$, is given by:

$(\bigcap\{f(x)\mid x\in X\})_p = \{y\in Y_p \mid \forall q\geq p\forall x\in X_q(y\in(f_q(x))_q)\}$.

So for ϕ,ψ: X→P(A) we see that $\phi\vdash\psi$ iff $\bigcap\{\phi(x)\Rightarrow\psi(x)\mid x\in X\}$ has a global element (i.e., an element at p_0, in this case). I leave it to the reader to define the rest of the logical structure belonging to $\mathcal{P}_A$ explicitly.

Let (Γ,Δ) be the unique geometric morphism: $\mathbf{sets}^P$→**sets** (Γ sends the presheaf X to X_{p_0}, and Δ sends a set X to the constant presheaf with value X). Applying proposition 1.4 of chapter 3 to the tripos $\mathcal{P}_A$, we get a **sets**-tripos $\Gamma(\mathcal{P}_A)$. For a set X, $\Gamma(\mathcal{P}_A)(X) = \mathcal{P}_A(\Delta(X)) = \mathbf{sets}^P(\Delta(X),P(A)) \cong \mathbf{sets}(X,\Gamma(P(A))) = \mathbf{sets}(X,P(A)_{p_0})$. Now $P(A)_{p_0}$ is the set $\{(\alpha_p)_{p\in P}\mid\alpha_p\subseteq A_p$ and $f_{pq}[\alpha_p]\subseteq\alpha_q\}$, and $\mathbf{sets}(X,P(A)_{p_0})$ is preordered by: $\phi\vdash\psi$ iff for some $a\in A_{p_0}$: $\forall x\in X\forall p\in P\forall b\in\phi(x)_p\,(f_{p_0p}(a)\bullet b\downarrow$ and $f_{p_0p}(a)\bullet b\in\psi(x)_p)$.

1.1. Proposition. Let 1 be the trivial system of partial combinatory algebras in $\mathbf{sets}^P$: $1_p = \{*\}$ for all $p\in P$. Then $\Gamma(\mathcal{P}_1)$-**sets** is equivalent to $\mathbf{sets}^P$ by an equivalence k such that $k\circ\Delta_{\Gamma(\mathcal{P}_1)}$ is equal to Δ (here $\Delta_{\Gamma(\mathcal{P}_1)}$ is the functor: **sets**→$\Gamma(\mathcal{P}_1)$-**sets** defined in 5.2 of chapter 3).

Proof. Let X a set. $\Gamma(\mathcal{P}_1)(X) = \mathbf{sets}(X,\Gamma(P(1))) \cong \mathbf{sets}(X,\{\alpha\subseteq P\mid\alpha$ is upwards closed$\})$, and $\phi\vdash\psi$ iff $\forall x\in X(\phi(x)\subseteq\psi(x))$. So a $\Gamma(\mathcal{P}_1)$-set is a pair $(X,\approx)$ where X is a set and $\approx$ is a P-indexed set of partial equivalence relations $(\approx_p)_{p\in P}$ on X, i.e. $x\approx_p y$ and $p\leq q$ imply $x\approx_q y$. This defines an obvious presheaf $k((X,\approx))$.

A $\Gamma(\mathcal{P}_1)$-set in the image of $\Delta_{\Gamma(\mathcal{P}_1)}$ is a pair $(X,\approx)$ where all $\approx_p$ are the equality on X. So $k(\Delta_{\Gamma(\mathcal{P}_1)}(X))$ is $\Delta(X)$. And k is an equivalence, for define, for every presheaf $X = \langle(X_p)_{p\in P},(f_{pq})_{p\leq q}\rangle$, l(X) as $(Y,\approx)$ where $Y = \{(x,p)\mid x\in X_p\}$ and $(x,p)\approx_q(y,r)$ iff $p,r\leq q$ and $f_{pq}(x)=f_{rq}(y)$. Then lk and kl are both naturally isomorphic to the identity.

The following proposition says that instead of the tripos $\mathcal{P}_A$ on $\mathbf{sets}^P$ we may equivalently study the tripos $\Gamma(\mathcal{P}_A)$ on **sets**:

1.2. Proposition. Let $\mathcal{P}$ be a tripos defined on $\mathbf{sets}^P$. Then $\mathcal{P}$-$\mathbf{sets}^P$ is equivalent to $\Gamma(\mathcal{P})$-**sets**.

Proof. This is an application of Pitts' *iteration lemma* (Theorem 5.8 in chapter 3). By proposition 1.1, $\mathbf{sets}^P$ is equivalent to $\Gamma(\mathcal{P}_1)$-**sets**, so $\mathcal{P}_A$-$\mathbf{sets}^P$ is equivalent to $\mathcal{P}_A$-($\Gamma(\mathcal{P}_1)$-**sets**); by the iteration lemma, $\mathcal{P}_A$-($\Gamma(\mathcal{P}_1)$-**sets**) is equivalent to $\mathcal{P}_A \circ \Delta^{op}$-**sets** which is $\Gamma(\mathcal{P}_1)$-**sets**.

1.3. Remark. It is worth to note that proposition 1.2 holds only for posets P; if C is an arbitrary small category and (Γ,Δ) the geometric morphism: $\mathbf{sets}^C \to \mathbf{sets}$, and 1 the trivial partial combinatory algebra in $\mathbf{sets}^C$, then $\mathcal{P}_1$-$\mathbf{sets}^C$ is of course equivalent to $\mathbf{sets}^C$ but $\Gamma(\mathcal{P}_1)$-**sets** is equivalent to $\mathbf{sets}^Q$ where Q is the partial order reflection of C.

1.4. Proposition. There is an inclusion of toposes: $\mathbf{sets}^P \to \Gamma(\mathcal{P}_A)$-**sets**.

Proof. This is, in view of proposition 1.2, analogous to the inclusion: **sets**→Eff, discussed in Hyland 1982.
Define Z: $\Gamma(\mathcal{P}_A)$-**sets**→ $\mathbf{sets}^P$ by $Z((X,=))_p = \{x \in X \mid [\![x=x]\!]_p \text{ is nonempty}\}/\approx$ where $x \approx x'$ iff $[\![x=x']\!]_p$ is nonempty. Transition maps are induced by the identity. For morphisms $f:(X,=)\to(Y,=')$ represented by $F:X\times Y \to \Gamma(P(A))$, define Z(f) by $Z(f)_p([x]) = [y]$ iff $F(x,y)_p$ is nonempty.
Conversely define a functor E: $\mathbf{sets}^P \to \Gamma(\mathcal{P}_A)$-**sets** by $E(G) = (\coprod_{p\in P} G_p, =)$ with $[\![x_q = x'_r]\!]_p = A_p$ if $q,r \leq p$ and $x\restriction p = x'\restriction p$, and $\emptyset$ otherwise. For morphisms $\mu:G\to H$ in $\mathbf{sets}^P$ let $E(\mu)$ be represented by M: $\coprod_{p\in P} G_p \times \coprod_{p\in P} H_p \to \Gamma(P(A))$ defined as follows: $M(x_q,y_r)_p = A_p$ if $q,r\leq p$ and $\mu_p(x\restriction p) = \mu_q(x)\restriction p = y\restriction p$, and $\emptyset$ else.
Z is left adjoint to E: if μ: $Z((X,=))\to G$ in $\mathbf{sets}^P$ then its transpose $\tilde{\mu}$ is represented by $\tilde{M}$: $X \times \coprod_{p\in P} G_p \to \Gamma(P(A))$ given by $\tilde{M}(x,y_q)_p = [\![x=x \wedge \mu_p([x]_p) = y\restriction p]\!]_p$ if $q \leq p$, and $\emptyset$ else. The other direction is straightforward.
It is easy to see that Z preserves finite limits. Now $ZE(G)_p = \{x_q \in \coprod_{p\in P} G_p \mid q \leq p\}/\approx$ with $x_q \approx x'_{q'}$ iff $x\restriction p = x'\restriction p$. Clearly, this is G_p. So (Z,E): $\mathbf{sets}^P \to \Gamma(\mathcal{P}_A)$-**sets** is an inclusion of toposes.⊠

On the other hand, **sets** in general is *not* a subtopos of $\Gamma(\mathcal{P}_A)$-**sets**. The constants object functor Δ has a left adjoint $\varinjlim \circ Z$, explicitly: $\varinjlim \circ Z((X,=)) = X'/\approx$ where $X' = \{x \in X \mid \exists p.[\![x=x]\!]_p$ is nonempty$\}$ and $\approx$ is generated by: $x \approx x'$ if $\exists p.[\![x=x']\!]_p$ is nonempty. However, this functor does not preserve finite products; this will only be the case when every p and q in P have an upper bound in P.

Combining propositions 1.2 and 1.4 there is an internal topology j in the topos $\mathcal{P}_A$-**sets**^{P}, the sheaves for which are **sets**^{P}. An object (X,=) of $\mathcal{P}_A$-**sets**^{P} is j-separated iff:

(∗) $\forall p\in P\ \forall xy\in X_p\ ([\![x=_p y]\!]_p$ is nonempty $\Rightarrow x=y)$

holds, completely analogous to the characterization of the $\neg\neg$-separated objects of Eff in Hyland 1982. In **sets**^{P} one can read (∗) as $\forall xy\in X\ ([\![x=y]\!]$ is inhabited $\Rightarrow x=y)$. If furthermore we require: $\forall xy\in X\ ([\![x=y]\!] = Ex\cap Ey)$ (Ex abbreviates $[\![x=x]\!]$) and $\forall x\in X$ (Ex is inhabited) we get something like Hyland's "effective objects", or "modest functors:P→**sets**".

In the particular case when A is a constant pca $\Delta(U)$ one might wonder about the relation between $\Gamma(\mathcal{P}_A)$-**sets** and $(\mathrm{Eff}_A)^P$ (when P is finite, this is a topos). There is an obvious functor G: $\Gamma(\mathcal{P}_A)$-**sets** $\rightarrow (\mathrm{Eff}_A)^P$ given by $G((X,=))_p = (X,=_p)$ where $[\![x=_p y]\!] = [\![x=y]\!]_p$. Transition maps $G((X,=))_p \rightarrow G((X,=))_q$ are represented by $H(x,y) = [\![x=_p x \wedge x=_q y]\!]$. If $F: X\times Y\rightarrow \Gamma(P(\Delta(U)))$ represents a morphism f in $\Gamma(\mathcal{P}_A)$-**sets** define a natural transformation G(f), represented at level p by $G(f)_p(x,y) = F(x,y)_p$. I do not know whether G is faithful.

§2. Beth models of realizability

2.1. Definition. A *Grothendieck topology* on a poset P is an operation J that assigns to every $p\in P$ a family J(p) of upwards closed subsets of $\uparrow(p)$ (called *covers* of p; I write $\uparrow(p)$ for $\{q \mid q\geq p\}$), such that:

i) $\uparrow(p)\in J(p)$;

ii) if $R\in J(p)$ and $S\subseteq\uparrow(p)$ is upwards closed and $\forall r\in R\ (S\cap\uparrow(r)\in J(r))$, then $S\in J(p)$;

iii) if $R\in J(p)$ and $q\geq p$ then $R\cap\uparrow(q)\in J(q)$.

2.2. Example. Let P be a tree and
$J(p)=\{S\subseteq\uparrow(p) \mid S$ is upwards closed and contains a bar for $p\}$; then J is a Grothendieck topology on P (I hope this example is helpful to the reader who is used to traditional presentations of Beth forcing).

The subobject classifier in **sets**^{P}, denoted by Ω, is the P-indexed system of sets

$\Omega_p \equiv \{\alpha\subseteq\uparrow(p) \mid a$ is upwards closed$\}$; restriction: $\Omega_p\rightarrow\Omega_{p'}$ is intersection with $\uparrow(p')$.

Let J be a Grothendieck topology on P. Then J determines the following morphism j: $\Omega\rightarrow\Omega$ in **sets**^{P}: $j_p(\alpha) = \{q\geq p \mid \alpha$ contains a cover of $q\}$. The following proposition is standard.

2.3. Proposition.

1) The following hold in **sets**^{P}:

i) $\alpha\rightarrow j(\alpha)$

ii) $jj(\alpha)\rightarrow j(\alpha)$

iii) $j(\alpha\wedge\beta)\leftrightarrow j(\alpha)\wedge j(\beta)$

2) Let j: $\Omega\rightarrow\Omega$ be a morphism such that i)-iii) of 1) hold in $\mathbf{sets}^P$. Then j is determined by a unique Grothendieck topology on P.
(In any topos, a morphism j: $\Omega\rightarrow\Omega$ satisfying i)-iii) is called an *internal topology*)

2.4. Proposition. Let A be an internal partial combinatory algebra in $\mathbf{sets}^P$. Let $\vdash_j$ be the following binary relation on $\mathbf{sets}^P(X,P(A))$: $\phi\vdash_j\psi$ iff (in $\mathbf{sets}^P$):
$\exists a\in A\forall x\in X\forall b\in\phi(x)\,j(a\bullet b\downarrow\ \&\ a\bullet b\in\psi(x))$.
Then the assignment $\mathcal{P}_{A,j}(X)\equiv\langle\mathbf{sets}^P(X,P(A)),\vdash_j\rangle$ defines a $\mathbf{sets}^P$-tripos.

Proof. This is fairly straightforward: reason in $\mathbf{sets}^P$ and use the properties i)-iii) of 2.3 (1). Let I be $\Lambda x.x$ in A. Then $\forall x\in X\forall b\in\phi(x)\,j(I\bullet b\downarrow\ \&\ I\bullet b\in\phi(x))$ (use i)) so $\phi\vdash_j\phi$.
Suppose $\phi\vdash_j\psi$ via a, and $\psi\vdash_j\chi$ via c. Let d be $\Lambda u.c\bullet(a\bullet u)$.
Then $\forall x\in X\forall b\in\phi(x)\,j(a\bullet b\downarrow\ \&\ a\bullet b\in\psi(x))$, so
$\forall x\in X\forall b\in\phi(x)\,j(a\bullet b\downarrow\ \&\ j(c\bullet(a\bullet b)\downarrow\ \&\ c\bullet(a\bullet b)\in\chi(x)))$ from which easily (with i)-iii)):
$\forall x\in X\forall b\in\phi(x)\,j(d\bullet b\downarrow\ \&\ d\bullet b\in\chi(x))$, so $\phi\vdash_j\chi$. So $\vdash_j$ is a preorder.
Define $\Rightarrow$: $P(A)\times P(A)\rightarrow P(A)$ in $\mathbf{sets}^P$ by: $\alpha\Rightarrow\beta\equiv\{c\in A\mid\forall a\in\alpha\,j(c\bullet a\downarrow\ \&\ c\bullet a\in\beta)\}$. Then $\phi\vdash_j\psi$ in $\mathcal{P}_{A,j}(X)$ iff in $\mathbf{sets}^P$: $\exists a\forall x\in X\,(a\in\phi(x)\Rightarrow\psi(x))$.
If f: $X\rightarrow Y$ is a morphism in $\mathbf{sets}^P$, $\mathcal{P}_{A,j}(f)$ is defined by composition with f, and its adjoints $\exists f$ and $\forall f$ are defined by:
$\exists f(\phi)(y)\equiv\{a\in A\mid j(\exists x\in X\,(f(x)=y\ \&\ a\in\phi(x)))\}$
$\forall f(\phi)(y)\equiv\{a\in A\mid\forall x\in X\forall b\in A\,(f(x)=y\rightarrow j(a\bullet b\downarrow\ \&\ a\bullet b\in\phi(x)))\}$

2.5. Definition. Let J be a Grothendieck topology on P and j: $\Omega\rightarrow\Omega$ in $\mathbf{sets}^P$ the associated internal topology. If $X\rightarrowtail Y$ is a subobject, X is called *closed* w.r.t. j iff $\forall y\in Y\,(j(y\in X)\rightarrow y\in X)$ holds. By extension, I call a morphism f: $X\rightarrow P(Y)$ from X into a power-object closed, iff $\forall x\in X\forall y\in Y\,(j(y\in f(x))\rightarrow y\in f(x))$ holds. For arbitrary f: $X\rightarrow P(Y)$ define its *closure* j(f): $X\rightarrow P(Y)$ by $j(f)(x)=\{y\in Y\mid j(y\in f(x))\}$.

2.6. Proposition.
i) $\Rightarrow$: $P(A)\times P(A)\rightarrow P(A)$ is closed;
ii) if $\phi\in\mathcal{P}_{A,j}(X)$ and f: $X\rightarrow Y$ then $\forall f(\phi)$: $Y\rightarrow P(A)$ is closed;
iii) every $\phi\in\mathcal{P}_{A,j}(X)$ is isomorphic (in $\mathcal{P}_{A,j}(X)$) to its closure.

Proof. i) If $j(\forall b\in\alpha\,(a\bullet b\downarrow\ \&\ a\bullet b\in\beta))$ then $\forall b\in\alpha\,j(a\bullet b\downarrow\ \&\ a\bullet b\in\beta)$. ii) is similar. For iii), if $a\in j(\phi)(x)$ then $j(a\in\phi(x))$ so $j(I\bullet a\downarrow\ \&\ I\bullet a\in\phi(x))$ (I is $\Lambda x.x$), so $j(\phi)\vdash_j\phi$ via I. The converse inequality is likewise trivial.

Now let $\mathcal{P}'_{A,j}(X)$ be the subset of $\mathcal{P}_{A,j}(X)$ consisting of the j-closed ϕ, with the restricted preorder. Then it follows from proposition 2.6 that closure w.r.t. j: $\mathcal{P}_{A,j}(X) \to \mathcal{P}'_{A,j}(X)$ defines a $\mathbf{sets}^P$-indexed equivalence of preorders; in particular, $\mathcal{P}'_{A,j}$ is a $\mathbf{sets}^P$-tripos. Since $\mathcal{P}'_{A,j}(X)$ is simply the set of maps $X \to P_j(A)$ as defined in 0.6, this proves proposition 0.6.

§3. Beth realizability and sheaves

We reserve the notation $\tilde{X}$ for the partial map classifier of X. Let j be an internal topology in $\mathbf{sets}^P$. A *sheaf* for j is an object X of $\mathbf{sets}^P$ such that:

i) $\forall \alpha \in \tilde{X}\,(j(\exists x.x \in \alpha) \to \exists x.x \in \alpha)$ and

ii) $\forall x,y \in X\,(j(x=y) \to x=y)$

both hold.

I recall the following facts about sheaves, which can be found in any textbook:

1) Call Sh(P,j) the full subcategory of $\mathbf{sets}^P$ generated by the j-sheaves. Then the inclusion: $Sh(P,j) \to \mathbf{sets}^P$ has a left adjoint L (called *sheafification*), which is left exact. Define X^+ as $\bar{\tilde{X}}/{\approx}$, where $\bar{\tilde{X}} \equiv \{\alpha \in \tilde{X} \mid j(\exists x.x \in \alpha)\}$ and $\alpha \approx \alpha'$ iff $j(\alpha = \alpha')$. Then $(-)^+$ is the object part of a functor, and L results from applying $(-)^+$ twice.

2) Sh(P,j) is a topos; the subobject classifier of Sh(P,j) is the object $\Omega_j \equiv \{\alpha \in \Omega \mid j(\alpha) = \alpha\}$.

3) If X is a sheaf, then X^Y is a sheaf for any object Y of $\mathbf{sets}^P$.

Now let A be an internal partial combinatory algebra in $\mathbf{sets}^P$. Since $P_j(A) = (\Omega_j)^A$, $P_j(A)$ is a sheaf, which means that the restriction of $\mathcal{P}'_{A,j}$ to Sh(P,j) forms a Sh(P,j)-tripos. Let us write $\mathfrak{R}$ for this tripos. Let us show that $\mathfrak{R}$-Sh(P,j) is equivalent to $\mathcal{P}'_{A,j}$-$\mathbf{sets}^P$.

3.1. Proposition.

i) $\mathfrak{R} \circ L^{op}$ and $\mathcal{P}'_{A,j}$ are equivalent $\mathbf{sets}^P$-triposes;

ii) $\mathfrak{R} \circ L^{op}$-$\mathbf{sets}^P$ is equivalent to $\mathfrak{R}$-Sh(P,j).

Proof. i) There is an order-preserving isomorphism: $\mathbf{sets}^P(X,P_j(A)) \xrightarrow{\sim} \mathbf{sets}^P(X^+,P_j(A))$ as follows: for $\phi \in \mathbf{sets}^P(X,P_j(A))$ let ϕ^+ be induced by $\psi(\alpha) \equiv \{a \in A \mid j(\exists x \in \alpha.a \in \phi(x))\}$. Then ψ respects $\approx$ so ϕ^+ is well-defined. Conversely for $\phi \in \mathbf{sets}^P(X^+,P_j(A))$ let $\phi^-(x) \equiv \phi([\{x\}])$. Then $\phi^{+-}(x) = \phi^+([\{x\}]) = \{a \in A \mid j(\exists y \in \{x\}.a \in \phi(y))\} = \{a \in A \mid j(a \in \phi(x))\} = \phi(x)$, and $\phi^{-+}([a]) = \{a \in A \mid j(\exists x \in \alpha.a \in \phi^-(x))\} = \{a \in A \mid j(\exists x \in \alpha.a \in \phi(\{x\}))\} = \{a \in A \mid j(a \in \phi(\alpha))\} = \phi([\alpha])$. Let us show that this isomorphism is order-preserving: if $\phi \vdash_j \psi$ via a, then $b \in \phi^+([\alpha])$ implies $j(\exists x \in \alpha.b \in \phi(x))$ so $j(\exists x \in \alpha.j(a \bullet b\downarrow \wedge a \bullet b \in \psi(x)))$ so $j(\exists x \in \alpha.j(a \bullet b\downarrow \wedge a \bullet b \in \psi^+([\alpha])))$ so $j(a \bullet b\downarrow \wedge a \bullet b \in \psi^+([\alpha]))$. So $\phi^+ \vdash_j \psi^+$ via a. The other direction is easier.

ii) Let $\mathcal{P}$ be the $\mathbf{sets}^P$-tripos defined by $\mathcal{P}(X) = \mathbf{sets}^P(X,\Omega_j)$. Then there is an equivalence k: $\mathcal{P}$-$\mathbf{sets}^P \to Sh(P,j)$ such that $k \circ \Delta_{\mathcal{P}}$ is L: $\mathbf{sets}^P \to Sh(P,j)$. Now applying the iteration lemma twice gives: $\mathfrak{R}$-Sh(P,j) is equivalent to $\mathfrak{R} \circ k^{op}$-($\mathcal{P}$-$\mathbf{sets}^P$) which is equivalent to $\mathfrak{R} \circ k^{op} \circ (\Delta_{\mathcal{P}})^{op}$-$\mathbf{sets}^P$ which is $\mathfrak{R} \circ L^{op}$-$\mathbf{sets}^P$.

From proposition 3.1 we may conclude that the transition from $\mathcal{P}_A$ to $\mathcal{P}_{Aj}$ entails sheafification on the index category $\mathbf{sets}^P$. But it does not entail sheafification of A. There is, of course, a natural isomorphism of *sets* : $\mathbf{sets}^P(X,P_j(A)) \to \mathbf{sets}^P(L(X),P_j(L(A)))$ but this isomorphism does not reflect the order, as from the definition of $\vdash_j$ follows: it may be that
$j(\exists a\in A \forall x\in X \forall b\in \phi(x)\, j(a\bullet b\downarrow \wedge a\bullet b\in \psi(x)))$ but not
$\exists a\in A \forall x\in X \forall b\in \phi(x)\, j(a\bullet b\downarrow \wedge a\bullet b\in \psi(x))$; in which case $\phi \vdash_j \psi$ in $\mathbf{sets}^P(L(X),P_j(L(A)))$, but not in $\mathbf{sets}^P(X,P_j(A))$.
I now turn to some conditions on A from which some relations can be deduced between the various triposes considered so far. Let:
(1) $\forall a,b\in A\, (j(a\bullet b\downarrow) \to a\bullet b\downarrow)$
(2) $\forall \alpha\in \tilde{A}\, (j(\exists x.x\in\alpha) \to \exists x.x\in\alpha)$

3.2. Proposition. If A satisfies (1), then $\mathcal{P}'_{Aj}$ is a subtripos of $\mathcal{P}_A$.

Proof. Let F: $\mathcal{P}_A \to \mathcal{P}'_{A,j}$ be closure w.r.t. j. I show that F is left adjoint to the inclusion H: $\mathcal{P}_{A,j} \to \mathcal{P}_A$ (it is evident that F preserves finite meets). It is clear that if ψ is j-closed then $\phi \vdash \psi$ implies $j(\phi) \vdash_j \psi$; conversely if $j(\phi) \vdash_j \psi$ via a, then $\forall x\in X \forall b\in \phi(x)\, j(a\bullet b\downarrow \wedge a\bullet b\in \psi(x))$ (because $\phi(x) \subseteq j(\phi(x))$), so because of (1), $\forall x\in X \forall b\in \phi(x)\, (a\bullet b\downarrow \wedge j(a\bullet b\in \psi(x)))$, so $\phi \vdash \psi$ via a if ψ is j-closed.

Now if A is an internal partial combinatory algebra, then A^+ can be made into a partial combinatory algebra by putting $\alpha\bullet\beta \simeq \{x\bullet y \mid x\in\alpha, y\in\beta, x\bullet y\downarrow \text{ in } A\}$ (This is defined iff $j(\exists x,y.x\in\alpha\ \&\ y\in\beta\ \&\ x\bullet y\downarrow)$). It is immediate that A^+ satisfies (1), so $\mathcal{P}'_{A^+,j}$ is a subtripos of $\mathcal{P}_{A^+}$ by proposition 3.2.

3.3. Proposition. If A satisfies (2), then $\mathcal{P}_{Aj}$ is equivalent to $\mathcal{P}_{A^+j}$.

Proof. Define:

Φ: $\mathcal{P}_{A,j} \to \mathcal{P}_{A^+,j}$ by $\Phi(X)(\phi) \equiv \lambda x.\{a\in A^+ \mid a \subseteq \phi(x)\}$

Ξ: $\mathcal{P}_{A^+,j} \to \mathcal{P}_{A,j}$ by $\Xi(X)(\psi) \equiv \lambda x.\{a\in A \mid \{a\}\in \psi(x)\}$.

The proof that these are functors and define an equivalence is a similar exercise in logic to the propositions before.

3.4. Example. In Goodman 1978, P is a subtree of the tree of all partial functions r: $\mathbb{N} \to \mathbb{N}$, ordered by inclusion, and J is the $\neg\neg$-topology. A is the internal partial combinatory algebra with $A_r=\mathbb{N}$ and application on A_r is partial recursive application in r. It is easy to see that this A does not satisfy (1), for if x is such that $\{x\}^{P}(y)\simeq p(0)$, and $0\notin \mathrm{dom}(r)$, then not $x\bullet y\downarrow$ at r, yet $\neg\neg(x\bullet y\downarrow)$. So it is not immediately clear that Goodman's model is: realizability over an internal partial combinatory algebra in the topos of $\neg\neg$-sheaves over P, as Pitts asserts (Pitts 1981,p. 74).

To close this chapter I want to show that Sh(P,j) is a subtopos of $\mathcal{P}'_{A,j}$-**sets**^{P}. To do this I present a purely topos-theoretical lemma which is doubtless known among topos-theorists although I have not been able to find a reference for it. First, two definitions.

3.5. Definition. Let $\mathcal{E}$ be a topos, j an internal topology in $\mathcal{E}$.

i) An object X of $\mathcal{E}$ is called *separated* if $\forall x,y\in X.j(x=y)\to x=y$ holds;

ii) A monomorphism $\sigma:X\rightarrowtail Y$ is *dense* if $\forall y\in Y.j(\exists x\in X\,(\sigma(x)=y))$ holds.

3.6. Lemma. Let $\mathcal{E}$ be a topos, j an internal topology in $\mathcal{E}$. Then

i) a category $\mathcal{E}$-j can be defined, which has as objects the j-separated objects of $\mathcal{E}$, and as morphisms $X\to Y$: j-closed subobjects $A\rightarrowtail X\times Y$ such that the composition $\pi_1 a:A\to X$ is a j-dense monomorphism;

ii) $\mathcal{E}$-j is equivalent to $Sh_j(\mathcal{E})$.

Proof. i) We have to define composition of morphisms. So suppose $a:A\rightarrowtail F\times G$ and $b:B\rightarrowtail G\times H$ represent morphisms $F\to G$ and $G\to H$. Form the pullback:

$$\begin{array}{ccc} W & \xrightarrow{v_1} & A \\ {\scriptstyle v_2}\downarrow & & \downarrow{\scriptstyle \pi_2\circ a} \\ B & \xrightarrow{\pi_1\circ b} & G \end{array}$$

and consider $\langle\pi_1 av_1,\pi_2 bv_2\rangle: W\to F\times H$.
This is a monomorphism, for suppose it coequalises $f,g:U\to W$; then $\pi_1 a\, v_1 f= \pi_1 av_1 g$, $\pi_2 bv_2 f= \pi_2 bv_2 g$. Since $\pi_1 a$ is mono, $v_1 f= v_1 g$, and $\pi_1 bv_2 f=\pi_2 av_1 f=\pi_2 av_1 g=\pi_1 bv_2 g$ so ($\pi_1 b$ is mono) $v_2 f=v_2 g$. By the pullback property, f=g. Furthermore, $W\to F\times H\to F$ is a j-dense mono because this is $\pi_1 a\, v_1$, a composition of j-dense monos. Define the composition ba as the closure of W in $F\times H$. Then $\bar{W}\to F$ is still mono because of the following fact: if $\sigma:U\to U'$ is a j-dense mono into a separated object and $\tau:U'\to V$ is such that $\tau\sigma$ is mono, then τ is mono (by internal logic: if $\tau(x)=\tau(y)$ then $j(\exists zz'\in U.\sigma(z)=x \wedge \sigma(z')=y \wedge \tau(x)=\tau(y))$ so $j(x=y)$ because $\tau\sigma$ is mono, so x=y because U' is separated).
Checking associativity is left to the reader. The diagonal $X\rightarrowtail X\times X$ (which is closed if X is separated) acts as identity.

ii) L': $\mathcal{E}\text{-j}\to Sh_j(\mathcal{E})$ is given on objects by sheafification L and on morphisms $a:A\to F\times G$ by

$L'(a)=L(\pi_2 a)\circ(L(\pi_1 a))^{-1}$. It is easy to check that L' commutes with composition, so L' is a functor.
$I':Sh_j(\mathcal{E})\to\mathcal{E}$-j is given by the identity on objects and on morphisms by $I'(f)=\langle id_X,f\rangle:X\rightarrowtail X\times Y$. Clearly, L'I' is the identity in $Sh_j(E)$. Moreover, if F is separated then the universal map η_F: $F\to L(F)$ is a j-dense mono and $\langle id_F,\eta_F\rangle$: $F\rightarrowtail F\times L(F)$ is closed because it is $\eta_F{}^*(\delta:L(F)\rightarrowtail L(F)\times L(F))$, so represents a morphism $F\to I'L'(F)$ in $\mathcal{E}$-j. This is an isomorphism, because $\langle\eta_F,id_F\rangle$: $F\rightarrowtail L(F)\times F$ is an inverse for it.

3.7. Proposition. Sh(P,j) is a subtopos of $\mathcal{P}'_{A,j}$-**sets**^{P}.

Proof. With 3.6 it is enough to show an inclusion: **sets**^{P}-j$\to\mathcal{P}'_{A,j}$-**sets**^{P}. Everything is very similar to 1.4.
Define E: **sets**^{P}-j$\to\mathcal{P}'_{A,j}$-**sets**^{P} on objects by E(X) = (X,=) where $[\![x=y]\!] = \{a\in A \mid x=y\}$.
Since X is separated, $[\![.=.]\!]$ is a map: $X\times X\to P_j(A)$.
If $f\rightarrowtail X\times Y$ is a morphism in **sets**^{P}-j, E(f): $X\times Y\to P_j(A)$ defined by $E(f)(x,y) = \{a\in A \mid (x,y)\in f\}$ is well-defined because f is a closed subobject of X×Y, and represents a morphism: E(X)→E(Y) in $\mathcal{P}'_{A,j}$-**sets**^{P}.
Conversely, define a functor Z: $\mathcal{P}'_{A,j}$-**sets**$^P\to$**sets**^{P}-j as follows.
$Z((X,=)) = \{x\in X \mid j(\exists a\in[\![x=x]\!])\}/\approx$, where $x\approx x'$ iff $j(\exists a\in[\![x=x']\!])$. Then $\approx$ is an equivalence relation on $\{x\in X \mid j(\exists a\in[\![x=x]\!])\}$, and since it is closed, the quotient is separated.
On morphisms $(X,=)\to(Y,\simeq)$ represented by F: $X\times Y\to P_j(A)$, Z is defined as the following subobject of $Z((X,=))\times Z((Y,\simeq))$: $Z(F) = \{([x],[y]) \mid j(\exists a\in F(x,y))\}$. One should check that this expression is well-defined, that it does not depend on the particular representative F, that it defines a closed subobject, etc.
Now if F: $X\times Y\to P_j(A)$ represents a morphism f: (X,=)→E(Y) in $\mathcal{P}'_{A,j}$-**sets**^{P}, its transpose $\bar{f}$: $Z((X,=))\to Y$ in **sets**^{P}-j is defined by $\bar{f} = \{([x],y) \mid j(\exists a\in F(x,y))\}$.
Conversely if $f\rightarrowtail Z((X,=))\times Y$ represents a morphism in **sets**^{P}-j its transpose $\bar{f}$: (X,=)→E(Y) in $\mathcal{P}'_{A,j}$-**sets**^{P} is represented by the map $\bar{F}(x,y) = \{\langle a,a'\rangle \mid a\in[\![x=x]\!]\ \&\ j(([x],y)\in f)\}$.
I leave it to the reader to verify that these transpositions are inverse to each other, that Z preserves finite limits, and that ZE(X) is naturally isomorphic to X in **sets**^{P}-j.

VII. A semantical proof of De Jongh's Theorem

(This is the text of a paper accepted by the Archives for Mathematical Logic)

0. Introduction

In 1969, Dick de Jongh proved an interesting theorem. In order to state it, let us introduce the following notation.
If A is a formula of intuitionistic predicate calculus **IQC**, and P a unary predicate symbol not occurring in A, let $A^{(P)}$ be A with all quantifiers relativised to P (i.e. replace $\forall x$ by $\forall x(P(x)\to\ldots)$ and $\exists x$ by $\exists x(P(x)\wedge\ldots)$), and $A' \equiv \exists x P(x)\to A^{(P)}$. **HA** denotes, as usual, intuitionistic first order arithmetic.

Theorem 0.1. *If* **HA** *proves every arithmetical substitution instance of* A'*, then* A' *is provable in* **IQC**.

The proof was an ingenious combination of Kripke semantics and realisability. However, De Jongh never published it and his method remained unknown until N. Goodman [1978] presented a very similar semantics, for different purposes (A theorem similar to Theorem 0.1, concerning **HA** and *propositional* logic, was also proved by De Jongh by the same method. This theorem is given by Smorynski in Troelstra [1973] with a proof that uses only Kripke models and some proof-theoretic facts).
By purely proof-theoretic means, D. Leivant was able to strengthen Theorem 0.1 considerably (Leivant [1975]):

Theorem 0.2. *There are* Π^0_2*-predicates* $\{A_{ij}\}_{i,j<\omega}$*, such that* A_{ij} *has j free variables and for any formula* F *of* **IQC** *with* n_j*-ary predicate letters* $P_{i_j n_j}$*,* $j=1,\ldots,k$*, if* **HA** $\vdash F[A_{i_1 n_1},\ldots,A_{i_k n_k}]$ *then* **IQC** $\vdash F[P_{i_1 n_1},\ldots,P_{i_k n_k}]$.

The aim of this paper is to give a semantical proof of a slightly weaker version of Theorem 0.2. Throughout the rest of this paper, we assume that languages contain relation symbols only, and furthermore, that they admit an enumeration $(A_i)_{i\in\mathbb{N}}$ of their predicate symbols such that the arity of the A_i is a primitive recursive function of i.

Theorem 0.3. *Let* T *be a recursively enumerable theory, formulated in a language* $\mathcal{L}$ *in* **IQC**. *Then for every j-place predicate letter* A_{ij} *of* $\mathcal{L}$ *there is a j-place number-theoretic predicate* B_{ij}*, resulting in a translation (by substitution)* $(-)^*: \mathcal{L}\to\mathcal{L}(\mathbf{HA})$ *such that for every sentence* F *of* $\mathcal{L}$: $T\vdash F$ *if and only if* $\mathbf{HA}+(T)^*\vdash F^*$.

Note that Theorem 0.3 is contained in Theorem 0.2, so we do not claim a new result. We believe, however, that our proof, which is a refinement of De Jongh's original one, has some interest of its own, besides being much shorter than Leivant's.

The proof consists of the construction of a realisability model that "matches" the truth in an appropriate Beth model: we will be using a "universal Beth model" for T.

We could, of course, have formulated Theorem 0.3 the same way as Theorem 0.2, without reference to T (let T be the empty theory in a universal language); however, we would like to point out that there is a mass of realisability models obtained in this way, one for each T, and this is not immediately clear if one restricts attention to just the empty theory (if this paper has any interest, it is the *method* , not the result).

The reader will have noted that we didn't mention the complexity of our substitutions in the statement of Theorem 0.3. We cannot have Π^0_2-substitutions since our models will satisfy exactly the true Π^0_2-sentences, but classically they will be in Π^0_4.

It is possible to replace **HA** in Theorem 0.3 by certain extensions of **HA**. These extensions will be easy corollaries of our proof and will be discussed in section 3. Section 1 gives preliminaries; the actual construction of the model will take up section 2.

Section 3 also contains a corollary of the proof of Theorem 0.3 that is, we think, a new result. Consider an expansion of **HA** in a language that contains, besides the function symbols of **HA**, a partial binary operation symbol • and constants K and S, as well as axioms saying that ($\mathbb{N}$,•,K,S) is a partial combinatory algebra (this can be done in a logic with partial terms). Call this expansion **HA⁺**. Just as Kleene-realisability, one can define realisability w.r.t. •, a kind of "abstract realisability" over **HA**. Then if for a predicate formula A all its arithmetical substitution instances are, provably in **HA⁺**, realisable in this sense, A is a provable formula of the intuitionistic predicate calculus. This result is proposition 3.2 and it can be compared to results, most notably Plisko's (see references) about the relationship between predicate logic and realisability.

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§1. Beth models and realisability

Definition 1.1. A (fallible) Beth model for a language $\mathfrak{L}$ in **IQC** consists of the following:

i) a tree P and a P-indexed collection of sets (this is, for every $p\in P$ a set X_p as well as a collection of functions $(f_{pp'}:X_p\rightarrow X_{p'})_{p,p'\in P,\, p\leq p'}$ such that f_{pp} is the identity and $f_{p'p''}\circ f_{pp'}=f_{pp''}$ whenever $p\leq p'\leq p''$);

ii) a specified upwards closed subset U of P such that for any $p\in P$, if every path through p meets U somewhere, then already $p\in U$;

iii) for every n-ary relation symbol A of $\mathfrak{L}$ an interpretation $A^*=(A^*_p)_{p\in P}$ with $A^*_p\subseteq(X_p)^n$ such that:

a) $(d_1,\dots,d_n)\in A^*_p$ and $p\leq p'$ implies $(f_{pp'}(d_1),\dots,f_{pp'}(d_n))\in A^*_{p'}$;

b) If $(d_1,...,d_n)\in (X_p)^n$ is such that on every path through p there is a p' with $(f_{pp'}(d_1),...,f_{pp'}(d_n))\in A^*_{p'}$, then $(d_1,...,d_n)\in A^*_p$;

c) $A^*_p=(X_p)^n$ for $p\in U$.

Let us call a set R that is such that every path through p meets R eventually, a *bar* for p. Given a fallible Beth model we can interpret, in any $p\in P$, sentences of $\mathfrak{L}(X_p)$ (constants for elements of X_p added) as follows:

$p\Vdash A(d_1,...,d_n)$ iff $(d_1,...,d_n)\in A^*_p$;

$p\Vdash \phi\wedge\psi$ iff $p\Vdash\phi$ and $p\Vdash\psi$;

$p\Vdash \phi\vee\psi$ iff there is a bar R for p with $\forall r\in R\ (r\Vdash\phi$ or $r\Vdash\psi)$;

$p\Vdash \phi\rightarrow\psi$ iff for every $p'\geq p$, if $p'\Vdash\phi$ then $p'\Vdash\psi$;

$p\Vdash \exists x\phi(x)$ iff there is a bar R for p with $\forall r\in R\,\exists d\in X_r(r\Vdash\phi(d))$;

$p\Vdash \forall x\phi(x)$ iff for every $p'\geq p$ and for all $d\in X_{p'}$, $p'\Vdash\phi(d)$.

Here, if $\phi\equiv\phi(d_1,...,d_n)$ with $d_1,...,d_n\in X_p$ and $p\leq p'$, $p'\Vdash\phi$ is read as $p'\Vdash\phi(f_{pp'}(d_1),...,f_{pp'}(d_n))$. From the definition it follows immediately that if $p\in U$, $p\Vdash\phi$ for any formula ϕ (we take the absurdity as a 0-place predicate); this is why these models are called *fallible*. A fallible Beth model is said to have a *constant domain* if all X_p are equal and the maps $f_{pp'}$ are identities.

The main result about fallible Beth models is the following.

Theorem 1.2. *Let* T *be a recursively enumerable theory in a language* $\mathfrak{L}$ *in* **IQC**. *Then there is a fallible Beth model* $\mathfrak{B}$ *with constant domain* $\mathbb{N}$ *and as underlying poset the binary tree* P *(i.e. the tree of all finite 01-sequences), such that for every sentence* A *in the language of* $\mathfrak{L}$: $\langle\rangle\Vdash A$ *iff* $T\vdash A$ *(* $\mathfrak{B}$ *is called a* universal Beth model *for* T*). Moreover, there is an enumeration* $(A_i)_i$ *of* $\mathfrak{L}$ *such that the relation* $p\Vdash A_i(n_1,...,n_{k_i})$ *is* Σ^0_1 *in* p,i,n.

This result can be found in Troelstra & Van Dalen [1988], chapter 13. It is an adaptation by the authors of a proof by Friedman.

Definition 1.3. A *partial combinatory algebra* (pca) consists of a set A and a partial binary operation • on A, as well as elements K and S of A, for which hold:

i) For every $x,y\in A$, K•x and (K•x)•y are defined and (K•x)•y is equal to x;

ii) For $x,y,z\in A$, S•x and (S•x)•y are defined, and ((S•x)•y)•z is defined whenever (x•z)•(y•z) is, and equal to it in that case.

The reader is referred to Barendregt [1981] for proofs of the following facts:

i) λ-abstraction can be defined in A;

ii) A contains a definable system of natural numbers $\{\bar{n}\mid n\in\mathbb{N}\}$, such that for every partial recursive function f there is a definable element $\bar{f}$ of A which satisfies: f(n) is defined and equal

to m $\Leftrightarrow$ $\bar{f}\bullet\bar{n}$ is defined and equal to $\bar{m}$, for n,m$\in\mathbb{N}$. Moreover, the smn- and recursion theorems are satisfied in A by definable elements.
("definable" means roughly: in terms of K,S and •. More precisely, pca's are models of a theory formulated in a logic with partial terms. See Beeson 1985 for details.)

Now suppose we have a tree P and a specified upwards closed subset U as in definition 1.1. Consider a P-indexed system of pca's: that is, a pca A_p is attached to every p$\in$P, and functions $f_{pp'}:A_p\to A_{p'}$ are given for each inequality p$\leq$p' satisfying the same conditions as in definition 1.1, and furthermore:

i) the $f_{pp'}$ preserve the combinators K and S, and

ii) application: if a•b is defined in A_p, then $f_{pp'}(a)\bullet f_{pp'}(b)$ is defined in $A_{p'}$ and equal to $f_{pp'}(a\bullet b)$

(This ensures that every closed λ-term retains its meaning under the $f_{pp'}$).
Furthermore we fix a λ-definable choice of natural numbers, denoted $\{\bar{n} \mid n\in\mathbb{N}\}$, as well as λ-definable pairing and unpairing operators j, j_1, j_2. We will now define, for sentences A of arithmetic, elements p of P, and a of A_p, what it means that a "p-realises A", by induction on A. Let us call a set R such that R$\cup$U is a bar for p, a *U-bar* for p.

1) a p-realises t=s iff there is a U-bar R for p with $\forall r\in R$ (t=s is true and $f_{pr}(a)=\bar{t}$);

2) a p-realises A$\wedge$B iff j_1a p-realises A and j_2a p-realises B;

3) a p-realises A$\vee$B iff there is a U-bar R for p with $\forall r\in R$ ($j_1(f_{pr}(a))=\bar{0}$ and $j_2(f_{pr}(a))$ r-realises A, or $j_1(f_{pr}(a))=\bar{1}$ and $j_2(f_{pr}(a))$ r-realises B);

4) a p-realises A$\to$B iff for every p'$\geq$p and for every b$\in A_{p'}$, if b p'-realises A then there is a U-bar R for p' such that $\forall r\in R$ ($f_{pr}(a)\bullet f_{p'r}(b)$ is defined and r-realises B);

5) a p-realises $\exists$xA(x) iff there is a U-bar R for p with $\forall r\in R\,\exists n\in\mathbb{N}$ ($j_1(f_{pr}(a))=\bar{n}$ and $j_2(f_{pr}(a))$ r-realises A(n));

6) a p-realises $\forall$xA(x) iff for every n there is a U-bar R for p with $\forall r\in R$ ($f_{pr}(a)\bullet\bar{n}$ is defined and r-realises A(n)).

When talking about $\bar{t}$ and $\bar{n}$ we mean, of course, their interpretations in the appropriate pca; but since these are stable in the sense that, for p$\leq$p', $f_{pp'}((\bar{t})_p)=(\bar{t})_{p'}$ for every term t of $\mathfrak{L}$(**HA**), we suppress the reference to p.
We say that a sentence A is *p-realisable* iff there is an a$\in A_p$ that p-realises A. We say that A is *realisable* iff A is ⊥-realisable, where ⊥ denotes the bottom element of the tree P. A trivial induction on A shows that:

i) A is always p-realisable when p$\in$U;

ii) if a p-realises A then $f_{pp'}(a)$ p'-realises A, for p$\leq$p';

iii) if a$\in A_p$ and R is a U-bar for p such that for every r$\in$R, $f_{pr}(a)$ r-realises A, then a p-realises A.

Theorem 1.4. *All axioms and rules of* **HA** *are p-realisable, for every* $p\in P$.

The reader is referred to Goodman [1978] for a proof (some obvious modifications have to be made); people familiar with topos theory may be satisfied with the remark that we have just defined the internal logic of the natural numbers object in an appropriate realisability topos defined over the topos of sheaves on a closed subset of Cantor space. Finally, one may note that a P-indexed system of pca's is just a Kripke model of an intuitionistic theory of pca's, and that the normal soundness theorem is entirely constructive (note, however, that there is a difference from the constructivist's point of view between working with U-bars and simply cutting U out).

Definition 1.5. Let a system of pca's and functions be given as above. We say that this system is a *sheaf* iff the following two conditions are satisfied:

i) For every p and every minimal U-bar R for p (meaning that no proper subset of R is a U-bar for p), for every family $(a_r\in A_r)_{r\in R}$ there is a *unique* $a\in A_p$ with $\forall r\in R\,(f_{pr}(a)=a_r)$;

ii) For every p, every $a,b\in A_p$, if there is a U-bar R for p with $\forall r\in R\,(f_{pr}(a)\bullet f_{pr}(b)$ is defined), then $a\bullet b$ is defined.

The notion of sheaf defined here depends on the j-operator on the complete Heyting algebra of upwards closed subsets of P given by $j(A)=\{p\mid A$ contains a U-bar for $p\}$. Goodman's realisability was defined using the $\neg\neg$-operator: $\neg\neg(A)=\{p\mid \forall q\geq p\exists r\geq q(r\in A)\}$, but his system of pca's is not a sheaf. For more information about j-operators, sheaves and sheafification (used below) the reader is referred to Fourman and Scott 1979.

Suppose the system of pca's given in the definition of realisability is a sheaf. Then the clauses for implication and universal quantification in the realisability definition can be simplified into:

4') a p-realises $A\to B$ iff for all $p'\geq p$ and all $b\in A_{p'}$, if b p'-realises A then $f_{pp'}(a)\bullet b$ is defined and p'-realises B;

6') a p-realises $\forall xA(x)$ iff for all $n\in\mathbb{N}$, $a\bullet\bar{n}$ is defined and p-realises A(n).

Furthermore, an induction on A shows that in this case, A has a p-realiser iff there is a U-bar R for p with $\forall r\in R$(A has an r-realiser).

Since a similar property holds for fallible Beth models ($p\Vdash A$ iff there is a U-bar R for p with $\forall r\in R(r\Vdash A)$), and we are we are steering towards realisabilities that match the truth in certain Beth models, it is clear that we need sheafs of pca's.

De Jongh's proof, which used Kripke models and (in a hidden way) a corresponding system of pca's, suffered from the fact that Kripke models with constant domain are not complete (this explains the need to restrict to formulas of the form A' in Theorem 0.1). Using sheaves, one can work with fallible Beth models which are better in this respect. Furthermore, our system of pca's was inspired by Goodman's.

2. Construction of the model

The structure of the proof of Theorem 0.3 will be the following. Given a recursively enumerable theory T, we have a universal Beth model for T (i.e. the model given by theorem 1.2); this model will be used to define a sheaf of pca's, as well as substitutions for the predicates of $\mathfrak{L}$, such that the following will hold: for any formula A in the language $\mathfrak{L}$ with, say, n free variables, for any $p \in P$ and for any n-tuple $y_1,...,y_n \in \mathbb{N}$, $A^*(y_1,...,y_n)$ has a p-realiser if and only if $p \Vdash A(y_1,...,y_n)$.

We start with a P-indexed system of pca's of the following form. Consider an acceptable Gödel-numbering (i.e., satisfying enumeration and smn-theorem, see Odifreddi[1989]) of Turing machines that are enriched with two types of standard instructions, namely ask for values of F and G at a certain argument, where F and G are abstract partial oracle functions. A pca will be obtained by providing interpretations for F,G, i.e. concrete partial functions f and g: $\mathbb{N} \rightharpoonup \mathbb{N}$. The interpretations f,g will vary with $p \in P$ and since we will declare a computation to diverge whenever a value of F (or G) is asked at an argument not in the domain of f (resp. g), in order to satisfy the conditions for a P-indexed system of pca's we must have $f(p) \subseteq f(p')$ and $g(p) \subseteq g(p')$ whenever $p \leq p'$.

Let F_p be the pca $(\mathbb{N}, \{\cdot\}^{f(p),g(p)}(\cdot))$ where $\{x\}^{f(p),g(p)}(y)$ will denote the outcome (if there is any) of a computation of machine x with input y, and f(p) and g(p) interpreting F and G. Transition maps:$F_p \to F_{p'}$ are identities. This gives a system of pca's which is not a sheaf; therefore we let the system $(A_p)_{p \in P}$ be the sheafification of it : A_p consists of equivalence classes of partial functions $\alpha: \uparrow(p) \rightharpoonup \cup_{q \geq p} F_q$ that satisfy:

i) $q \in dom(\alpha) \Rightarrow \alpha(q) \in F_q$;

ii) $q \in dom(\alpha)$, $q' \geq q \Rightarrow q' \in dom(\alpha)$ and $\alpha(q') = f_{qq'}(\alpha(q))$;

iii) there is a U-bar R for p such that $R \subseteq dom(\alpha)$).

Two such functions are equivalent iff there is a U-bar for p at which they are both defined and equal. In A_p an application is defined by: $[\alpha] \bullet [\beta]$ is defined iff there is a U-bar R for p with $\forall r \in R\, (\{\alpha(r)\}^{f(r),g(r)}(\beta(r))$ is defined in F_r), and in that case $[\alpha] \bullet [\beta]$ is the equivalence class of the function that assigns $\{\alpha(r)\}^{f(r),g(r)}(\beta(r))$ to r (note, that this does not depend on the choice of representatives).

The idea behind the oracle functions is: the functions g(p) will provide "independent enough" information, assuring that certain formulas can only be p-realised if the relevant information is in g(p). The functions f(p) will be partial recursive and code some information about the Beth model for T.

Now for the choice of the functions f(p) and g(p) we need a recursion-theoretic fact.

Theorem 2.1. *Let u be a numerical function in 0', i.e. u is the characteristic function of some non-recursive Σ^0_1-predicate. Then there is a 2-place number-theoretic predicate* $D(x,y) \in 0''$ *such that (putting* $D_n(x) \equiv D(x,n)$, $D^m(x,n) \equiv D(x, n+sg(n+1-m))$*), D_n is not recursive in u, D^n (the* *sequence D_n is called* recursively independent).

This is Theorem 2 of Kleene & Post [1951]. We owe the use of this theorem to De Jongh[1969].

Suppose $\mathfrak{B}$ is a universal Beth model for T as given by theorem 1.2. Let $(A_i| i=0,1,...)$ be an enumeration of $\mathfrak{L}$, such that for some primitive recursive b and #, $R_j=A_{b(j)}$ and A_i has exactly #(i) free variables. Furthermore, we suppose that the enumeration $(A_i| i=0,1,...)$ is such that, for instance, $A_i \wedge A_j = A_{g(i,j)}$ for primitive recursive g, etc. Then the function u defined by:

$u(p,i,y)=1$ if $y=\langle y_1,...,y_{\#(i)}\rangle$ and $p \Vdash A_i(y_1,...,y_{\#(i)})$, and 0 otherwise,

is in 0' by theorem 1.2. Let D be a 2-place predicate as given by theorem 2.1. For $p \in P$ define the predicate $D^{(p)}$ by: $D^{(p)}(x,y)$ iff $y=\langle i,y_1,...,y_{\#(i)}\rangle$ and $u(p,i,y)=1$ and $D(x,y)$. Then $D^{(p)}$ is obviously recursive in u,D; and if $u(p,i,\langle w_1,...,w_{\#(i)}\rangle)=0$ then $D^{(p)}$ is recursive in u, $D^{\langle i,w_1,...,w_{\#(i)}\rangle}$. So D_y is recursive in $D^{(p)}$ iff $y=\langle i,y_1,...,y_{\#(i)}\rangle$ and $u(p,i,\langle y_1,...,y_{\#(i)}\rangle)=1$; for if not ($y=\langle i,y_1,...,y_{\#(i)}\rangle$ and $u(p,i,\langle y_1,...,y_{\#(i)}\rangle)=1$), then $D^{(p)}$ is recursive in u,D^y, and D_y is not.

We are now ready to define the partial functions f(p),g(p) and the substitutions ϕ_j for the predicates R_j.

For the definition of f(p) let W_e be $\{\langle p,i,y\rangle| y=\langle y_1,...,y_{\#(i)}\rangle \ \& \ p\Vdash A_i(y_1,...,y_{\#(i)})\}$. Now f(p) is the partial recursive function given by the following instructions:

f(p)(i,y) = undefined if A_i is not an existential formula or a disjunction, or if y is not of the form $\langle y_1,...,y_{\#(i)}\rangle$;

If A_i is $\exists x A_j(y_1,...,y_{\#(i)},x)$, let w be the least z with $T(e,(z)_0,(z)_1)$ & $(z)_0=\langle q,j,v\rangle$ for some initial sequence q of p and v of form $\langle y_1,...,y_{\#(i)},n\rangle$ for some n; undefined if w does not exist;

If w does exist, check whether there is a $z \le w$ with $T(e,(z)_0,(z)_1)$ & $(z)_0=\langle q,j,v\rangle$ for some extension q of p and v of form $\langle y_1,...,y_{\#(i)},n\rangle$ and n is different from the corresponding n in w; undefined if such a z exists; this n from w else.

If A_i is $A_j \vee A_k$ let y|j, y|k be the subsequences of y that occur in A_j and A_k, respectively; let w be the least z such that $T(e,(z)_0,(z)_1)$ and either $(z)_0$ is $\langle q,j,y|j\rangle$ for some initial sequence q of p, or $(z)_0$ is $\langle q,k,y|k\rangle$ for some initial sequence q of p; undefined if w does not exist;

If w does exist, again check if there is $z \le w$ doing this for some extension q of p and such that $((z)_0)_1 \neq ((w)_0)_1$; undefined if such a z exists; else: 0 if $((w)_0)_1=j$; 1 if $((w)_0)_1=k$.

Put g(p)(y,x) = undefined if y is not of form $\langle i,y_1,...,y_{\#(i)}\rangle$ or $y=\langle i,y_1,...,y_{\#(i)}\rangle$ and $p \Vdash A_i(y_1,...,y_{\#(i)})$;

= 1 if $y=\langle i,y_1,...,y_{\#(i)}\rangle$, $p\Vdash A_i(y_1,...,y_{\#(i)})$ and D(x,y);

$= \quad 0$ if $y=\langle i,y_1,\ldots,y_{\#(i)}\rangle$, $p\Vdash A_i(y_1,\ldots,y_{\#(i)})$ and not $D(x,y)$.

The reader is invited to check that $f(p)\sqsubseteq f(q)$ whenever p is an initial segment of q; also, that if A_i is a disjunction or an existential formula and $p\Vdash A_i(y_1,\ldots,y_{\#(i)})$ then there is a U-bar R for p such that for every $r\in R$, $f(r)(i,\langle y_1,\ldots,y_{\#(i)}\rangle)$ is defined.

For j=1,... let $C_j(x,y_1,\ldots,y_{\#(b(j))})$ be a negative formula, expressing $D(x,\langle b(j),y_1,\ldots,y_{\#(b(j))}\rangle)$, and put $\phi_j(y_1,\ldots,y_{\#(b(j))})\equiv\forall x(C_j(x,y_1,\ldots,y_{\#(b(j))})\vee\neg C_j(x,y_1,\ldots,y_{\#(b(j))}))$.

By a *partial term* we mean something that is built up from: free variables, primitive recursive functions, λ-abstraction, and $\{\cdot\}^{F,G}(\cdot)$. If t is a partial term we denote by t_p its (possibly undefined) meaning in F_p, interpreting F,G by f(p),g(p) respectively. t represents an element of A_p if t is defined on a U-bar for p. We express this by "$t\in A_p$".

Lemma 2.2. For every negative formula $C(x_1,\ldots,x_k)$ of $\mathfrak{L}$(**HA**) there is a partial term t(C), whose free variables are contained in $\{x_1,\ldots,x_k\}$, such that for all $p\in P$ and all $n_1,\ldots,n_k$:
i) $C(n_1,\ldots,n_k)$ is true in $\mathbb{N} \Rightarrow (t(C)(\bar{n}_1,\ldots,\bar{n}_k))_p\in A_p$ and $(t(C)(\bar{n}_1,\ldots,\bar{n}_k))_p$ p-realises $C(n_1,\ldots,n_k)$;
ii) $C(n_1,\ldots,n_k)$ has a p-realiser and $p\notin U \Rightarrow C(n_1,\ldots,n_k)$ is true in $\mathbb{N}$.

Proof. Standard. ⊠

The translation $(-)^*$: $\mathfrak{L}\rightarrow\mathfrak{L}$(**HA**) is given by substituting ϕ_j for R_j.Theorem 0.3 will now follow from the following lemma:

Lemma 2.3. For every formula A of $\mathfrak{L}$ there is a partial term t_A with the same number k of free variables, such that the following holds: for every p and all $y_1,\ldots,y_k\in\mathbb{N}$,

i) $p\Vdash A(y_1,\ldots,y_k)\Rightarrow t_A(y_1,\ldots,y_k)_p\in A_p$ & $t_A(y_1,\ldots,y_k)_p$ p-realises $A^*(y_1,\ldots,y_k)$;

ii) $A^*(y_1,\ldots,y_k)$ has a p-realiser $\Rightarrow p\Vdash A(y_1,\ldots,y_k)$.

Proof. By induction on A. We define t_A and prove i) and ii) simultaneously. The main step is the one for prime formulas.

If $A\equiv R_j$ let $t_A(y_1,\ldots,y_{\#(b(j))})$ be $\lambda x.\begin{cases} j(0,t(C_j)(x,y_1,\ldots,y_{\#(b(j))})) \text{ if } G(\langle b(j), y_1,\ldots,y_{\#(b(j))}\rangle,x)=1 \\ j(1,t(\neg C_j)(x,y_1,\ldots,y_{\#(b(j))})) \text{ if} G(\langle b(j),y_1,\ldots,y_{\#(b(j))}\rangle,x)=0\end{cases}$

here the expressions $t(C_j)$ and $t(\neg C_j)$ are as defined in lemma 2.2.
Then i) is immediate; for ii), suppose $[\alpha]$ p-realises

$\forall x\,(C_j(x,y_1,...,y_{\#(b(j))})\vee\neg C_j(x,y_1,...,y_{\#(b(j))}))$ and $p\nVdash R_j(y_1,...,y_{\#(b(j))})$. There is a U-bar R for p such that $R\subseteq dom(\alpha)$ and for at least one $r\in R$, $r\nVdash R_j(y_1,...,y_{\#(b(j))})$, so we may as well assume $p\in dom(\alpha)$. Then for all n, $[\alpha]\bullet n$ is defined and $[\alpha]\bullet\bar{n}$ p-realises $C_j(n,y_1,...,y_{\#(b(j))})\vee\neg C_j(n,y_1,...,y_{\#(b(j))})$, so for all n there is a U-bar R_n for p with $\forall r\in R_n$ $(j_1(\alpha\bullet\bar{n})(r)=\bar{0}$ & $j_2(\alpha\bullet\bar{n})(r)$ r-realises $C_j(n,y_1,...,y_{\#(b(j))})$ or $j_1(\alpha\bullet\bar{n})(r)=\bar{1}$ & $j_2(\alpha\bullet\bar{n})(r)$ r-realises $\neg C_j(n,y_1,...,y_{\#(b(j))}))$.
But since C_j is negative and $p\notin U$ (because $p\nVdash R_j(y_1,...,y_{\#(b(j))})$), exactly one of $C_j,\neg C_j$ is realised at p, according to whether C_j is true or not. So if $\beta\equiv\lambda n.j_1(\alpha\bullet\bar{n})$, then β is a decision function for $D_{\langle b(j),y_1,...,y\#b(j)\rangle}$. But if β needs $G(\langle b(j),y_1,...,y_{\#b(j)}\rangle,n)$ for some n then $\beta\bullet\bar{n}$ can never be defined (because the interpretation g(p) of the oracle function G at p is undefined at this argument, since $p\nVdash R_j(y_1,...,y_{\#(b(j))})$). So $D_{\langle b(j),y_1,...,y\#b(j)\rangle}$ is recursive in $D^{\langle b(j),y_1,...,y_{\#b(j)}\rangle}$, contradiction.
2) If $A\equiv B_1\wedge B_2$ put $t_A\equiv j(t_{B_1},t_{B_2})$.
3) If $A\equiv B_1\rightarrow B_2$ put $t_A\equiv\lambda x.t_{B_2}$.
4) If $A=A_i$ is a disjunction, say A is $A_n\vee A_k$, let $t_A(y_1,...,y_{\#(i)})$ be $j(0,t_{A_n}(y|n))$ if $F(i,\langle y_1,...,y_{\#(i)}\rangle)=0$ and $j(1,t_{A_k}(y|k))$ if $F(i,\langle y_1,...,y_{\#(i)}\rangle)=1$ (Here y|n denotes the subsequence of $y=\langle y_1,...,y_{\#(i)}\rangle$ that occurs in A_n, and similar for y|k). (i) follows by the definition of f(p). For ii) suppose α p-realises $A_k^*\vee A_l^*$. Pick a U-bar R for p such that $\forall r\in R\ (\alpha(r)\downarrow$ & $(j_1(\alpha(r))=\bar{0}\rightarrow j_2\alpha$ r-realises $A_k^*)$ & $(j_1(\alpha(r))=\bar{1}\rightarrow j_2\alpha$ r-realises $A_l^*))$. Then $\forall r\in R$ (A_k^* has a r-realiser or A_l^* has a r-realiser), so by induction hypothesis $p\Vdash A_k\vee A_l$.
5) $A\equiv\forall xB(x)$. Similar to 3).
6) If $A=A_i$ is $\exists xA_j(x)$ let $t_A(y_1,...,y_{\#(i)})$ be $j(F(i,\langle y_1,...,y_{\#(i)}\rangle),t_{A_j}(\langle y_1,...,y_{\#(i)}\rangle,F(i,\langle y_1,...,y_{\#(i)}\rangle)))$. Again, (i) and (ii) follow from the construction of f(p) and the induction hypothesis. ⊠

To conclude the proof of the theorem: $\Rightarrow$ is obvious. Suppose $\mathbf{HA}+(T)^*\vdash A^*$, then A^* has a $\langle\rangle$-realiser, so by lemma 2.3 $\langle\rangle\Vdash A$, which means $T\vdash A$ by the property of a universal Beth model.⊠

§3. Extensions of ***HA****; some corollaries*

A casual glance at the model will convince the reader that it satisfies all true Π^0_2-sentences; moreover, we have remarked that our model is part of a topos (this has not been explained, but since this is a general phenomenon we prefer to leave this for a separate treatment). So it is immediate that **HA**, in theorem 0.3, can be replaced by **HAH**+all true Π^0_2-sentences, where **HAH** is Higher Order Heyting Arithmetic.

We now want to show that transfinite induction over all primitive recursive well-orderings holds in our model. Let $\mathbf{HA}^+$ be the expansion of **HA** in a language that contains an extra partial function symbol $\bullet$, and with additional axioms asserting that $(\mathbb{N},\bullet)$ is a partial combinatory algebra. Since the sheaf of pca's constructed in the model has the sheafification of ℕ as underlying sheaf, it is an ordinary sheaf model of $\mathbf{HA}^+$. Moreover, the realisability definition in our model is the sheaf model interpretation of Kleene realisability with •. So if F is some arithmetical principle or schema that holds in the model, and we have, for every instance A of F, a proof in **HA**+F that A is Kleene-realisable such that the proof doesn't use any particular property of the pca of partial recursive application, then the proof can be carried out in $\mathbf{HA}^+$, doing realisability with •, and consequently the principle will be realised in our model, if it is valid in it.

Let us apply this to the transfinite induction schema $\mathrm{TI}_{\prec}$, which is:

$\forall u\,(\forall v{<}u\,A(v) \to A(u)) \to \forall u A(u)$,

where < is a primitive recursive well-ordering. It is easy to convince oneself that this schema is valid in a sheaf model, so what remains to prove is the following:

Proposition 3.1. *For every instance* F *of* $\mathrm{TI}_{\prec}$, $\mathbf{HA}^+ + \mathrm{TI}_{\prec} \vdash \exists n\,(n\,\mathbf{r}\,F)$, *where* **r** *means realisability with* •.

Proof. This is a slight adaptation of the proof given in Troelstra [1973], 3.2.23. Let F be $\forall u\,(\forall v{<}u\,A(v) \to A(u)) \to \forall u A(u)$ for some formula A, and suppose w realises the premiss. This means:

$(\oplus)\ \forall u \forall w'\,(\forall v\,(v{<}u \to \forall k\,(w'\bullet v)\bullet k\,\mathbf{r}\,A(v)) \to (w\bullet u)\bullet w'\,\mathbf{r}\,A(u))$.

We want a g that realises $\forall u A(u)$ or $\forall u\,(g\bullet u\ \mathbf{r}\ A(u))$ or, with $\mathrm{TI}_{\prec}$,

$\forall u\,(\forall v{<}u\ g\bullet v\ \mathbf{r}\ A(v)\ \to g\bullet u\ \mathbf{r}\ A(u))$.

Take a number G such that for all g,u:

$G\bullet\langle g,u\rangle \simeq (w\bullet u)\bullet(\Lambda v.\ \Lambda k.\ g\bullet v)$,

and find with the recursion theorem for •, a number g such that for all u:

$g\bullet u \simeq G\bullet\langle g,u\rangle$.

Now $\forall v{<}u\ g\bullet v\ \mathbf{r}\ A(v)$ implies $\forall v{<}u\,\forall k\,((\Lambda v.\ \Lambda k.\ g\bullet v)\bullet v)\bullet k\ \mathbf{r}\ A(v)$, so with $(\oplus)$: $(w\bullet u)\bullet(\Lambda v.\ \Lambda k.\ g\bullet v)\ \mathbf{r}\ A(u)$, which is $g\bullet u\ \mathbf{r}\ A(u)$. Note, that $\mathbf{HA}^+$ need not prove anything about < ! ⊠

Proposition 3.2. *Let A be a formula of* **IQC** *such that for all arithmetical substitution instances A* of A,* $\mathbf{HA}^+ \vdash \exists n\,(n\ \mathbf{r}\ A^*)$. *Then* $\mathbf{IQC} \vdash A$.

Proposition 3.2 follows immediately from the considerations preceding proposition 3.1 and the proof of theorem 0.3. This corollary is interesting in view of the research done, mainly by V.E. Plisko (see Plisko 1978 and 1984), on the logic of realisability: i.e. those formulas of predicate logic all of whose arithmetical substitution instances are realisable. Plisko shows that this logic is quite complicated: it is Π^1_1-complete. Proposition 3.2 shows that this feature depends on the metamathematics used.

VIII. Two versions of "extensional realizability"

Jaap van Oosten

In this chapter I shall consider two realizability interpretations for arithmetic that are "extensionalisations" of Kleene's 1945-realizability, much in the same way as the models **HRO**^{E} and **HEO** are extensionalisations of **HRO** (See Troelstra 1973). These realizabilities will be denoted e̲ and e̲'. e̲-Realizability was erroneously defined in Beeson 1985; another definition can be found in Renardel de Lavalette 1984, in an abstract setting. The topos corresponding to e̲-realizability, discussed in section 3, was already defined in Pitts 1981; some of its internal logic was explained by Hyland in a talk in 1982 (I am indebted to Professor A.S. Troelstra for notes of this talk). I thought it worth-wile to present the matter in some more detail and to complete some of the arguments.

§1. Definition and some basic properties

1.1. Definition.

1) Define, inductively, for any formula A of arithmetic, a partial equivalence relation on the set of all Kleene-realizers (r̲) of A, as follows:

$$x \sim_A x' \equiv x=x' \wedge D_A(x) \text{ for A atomic}$$
$$x \sim_{A\wedge B} x' \equiv (x)_0 \sim_A (x')_0 \wedge (x)_1 \sim_B (x)_1$$
$$x \sim_{A\to B} x' \equiv x \,\underline{r}\, A\to B \wedge x' \,\underline{r}\, A\to B \wedge \forall yy'(y \sim_A y' \Rightarrow x\bullet y \sim_B x'\bullet y')$$
$$x \sim_{\forall yAy} x' \equiv \forall n\,(x\bullet n\downarrow \wedge x'\bullet n\downarrow \wedge x\bullet n \sim_{A(n)} x'\bullet n)$$
$$x \sim_{\exists yAy} x' \equiv (x)_0=(x')_0 \wedge (x)_1 \sim_{A((x)_0)} (x')_1$$

2) Define predicates E_A and $=_A$ simultaneously by induction on A:

$$E_A(x) \equiv x=x \wedge A \text{ for A atomic;} \quad x=_A x' \equiv x=x' \wedge A$$
$$E_{A\wedge B}(x) \equiv E_A((x)_0) \wedge E_B((x)_1); \quad x=_{A\wedge B} x' \equiv (x)_0=_A(x')_0 \wedge (x)_1=_B(x')_1$$
$$E_{A\to B}(x) \equiv \forall yy'(y=_A y' \Rightarrow x\bullet y\downarrow \wedge x\bullet y'\downarrow \wedge x\bullet y=_B x\bullet y');$$
$$x=_{A\to B} x' \equiv E_{A\to B}(x) \wedge E_{A\to B}(x') \wedge \forall y(E_A(y) \Rightarrow x\bullet y=_B x'\bullet y)$$
$$E_{\forall yAy}(x) \equiv \forall n\,(x\bullet n\downarrow \wedge E_{A(n)}(x\bullet n)); \quad x=_{\forall yAy} x' \equiv \forall n\,(x\bullet n=_{A(n)} x'\bullet n)$$
$$E_{\exists yAy}(x) \equiv E_{A((x)_0)}((x)_1); \quad x=_{\exists yAy} x' \equiv (x)_0=(x')_0 \wedge (x)_1=_{A((x)_0)}(x')_1$$

Note, that $x \sim_A x$ implies x r̲ A, and that $\sim_A$ is symmetric and transitive. We say that x e̲' A iff $x \sim_A x$.

$E_A(x)$ is equivalent to $x=_A x$ and $=_A$ is symmetric and transitive. We say that x e̲ A iff $x=_A x$.

1.2. Proposition. **HA** ⊢ A ⇒ for some n,m, **HA** ⊢ n e̲ A ∧ m e̲'A

Proof. A routine induction on **HA** ⊢ A.

The difference between these two notions of realizability that presents itself immediately is in the implication (and consequently the negation) clauses. For $\underline{e}$-realizability it is evident that, with classical logic, $A \vee \neg A$ is realizable for sentences A; for $\underline{e}'$-realizability this is not the case, but one sees from the remarks following definition 1.1 that classically $A \vee \neg A \vee \neg\neg A$ is $\underline{e}'$-realizable for sentences A: if A and $\neg A$ are not $\underline{e}'$-realizable then A must be Kleene-realizable, so $\sim_{\neg A}$ must be the empty relation and any number $\underline{e}'$-realizes $\neg\neg A$. Let us see that all three possibilities occur. First a trivial remark:

1.3. Proposition. Let, for almost negative formulas A, ψ_A be the p-term from Troelstra 1973, 3.2.11, i.e. satisfying $\vdash A(x) \rightarrow \psi_A(x)\ \underline{r}\ A(x)$. Then:

$\vdash A(x) \rightarrow \psi_A(x)\ \underline{e}\ A(x) \wedge \psi_A(x)\ \underline{e}'\ A(x)$

$\vdash \exists y(y\ \underline{e}\ A \vee y\ \underline{e}'\ A) \rightarrow A$

Proof. Trivial. Note, that the formulas $x\ \underline{e}\ A$ and $x\ \underline{e}'\ A$ are equivalent to almost negative formulas.

1.4. Proposition. The following instance of the open schema CT_0 is not $\underline{e}$-realizable or $\underline{e}'$-realizable:

A $\quad \forall e[\forall x \exists y(\neg\neg\exists z Texz \rightarrow Texy) \rightarrow \exists v \forall x \exists u(Tvxu \wedge (\neg\neg\exists y Texy \rightarrow TexUu))]$

Proof. We reason informally; the proof can be formalized in $\mathbf{HA}+\mathbf{M}_{PR}$. Since the proofs are similar, we give it for $\underline{e}$-realizability. Suppose w $\underline{e}$-realizes A (we will derive a contradiction). Some remarks:

i) If e codes the empty function, then $\Lambda f.((w \bullet e) \bullet f)_0$ is an effective operation of type 2, for every total function will realize $\forall x \exists y(\neg\neg\exists z Texz \rightarrow Texy)$, and equal functions are equivalent realizers.

ii) If k realizes $\forall x \exists y(\neg\neg\exists z Texz \rightarrow Texy)$, then $((w \bullet e) \bullet k)_1$ realizes $\forall x \exists u(T\ ((w \bullet e) \bullet k)_0\ xu \wedge (\neg\neg\exists y Texy \rightarrow TexUu))$. This is equivalent to an almost negative formula, so we always have: $\forall x[((w \bullet e) \bullet k)_0 \bullet x$ is defined and $(\neg\neg\exists y Texy \rightarrow Tex\ ((w \bullet e) \bullet k)_0 \bullet x))]$

Using the recursion theorem, we pick a code e for a partial recursive function of three variables such that:

$$e \bullet (k,n,x) \simeq \begin{cases} \text{undefined if not } Tnnx \\ \text{if } Tnnx\text{:} \\ \text{undefined if } ((w \bullet S^2{}_1(e,k,n)) \bullet \Lambda x.0)_0 \bullet x \text{ is undefined;} \\ 0 \text{ if } ((w \bullet S^2{}_1(e,k,n)) \bullet \Lambda x.0)_0 \bullet x \text{ is defined and not} \\ T(S^2{}_1(e,k,n),x, ((w \bullet S^2{}_1(e,k,n)) \bullet \Lambda x.0)_0 \bullet x); \\ U[((w \bullet S^2{}_1(e,k,n)) \bullet \Lambda x.0)_0 \bullet x)]+1 \text{ else.} \end{cases}$$

Some remarks:

iii) If Tnnx, then $((w\bullet S^2_1(e,k,n))\bullet\Lambda x.0)_0\bullet x$ is always defined. Otherwise $S^2_1(e,k,n)$ would code the empty function, and see i)-ii).
iv) If Tnnx, then never $T(S^2_1(e,k,n),x,((w\bullet S^2_1(e,k,n))\bullet\Lambda x.0)_0\bullet x)$. For then we would have $S^2_1(e,k,n)\bullet x = U[((w\bullet S^2_1(e,k,n))\bullet\Lambda x.0)_0\bullet x)] \neq U[((w\bullet S^2_1(e,k,n))\bullet\Lambda x.0)_0\bullet x)]+1 = e\bullet(k,n,x)$; contradiction.

Again using the recursion theorem, with e as just defined, we take a code k for a partial recursive function of two variables, such that:

$$k\bullet(n,x) \simeq \begin{cases} 0 \text{ if not Tnnx;} \\ \langle \mu z.T(S^2_1(e,S^1_1(k,n),n),x,z), \Lambda x.0\rangle \text{ else.}\end{cases}$$

Then $S^1_1(k,n)$ always realizes
$\forall x\exists y(\neg\neg\exists zT(S^2_1(e,S^1_1(k,n),n),x,z)\to T(S^2_1(e,S^1_1(k,n),n),x,y))$.
Furthermore:
If $n\bullet n$ is undefined then $S^1_1(k,n)$ codes $\Lambda x.0$ and $S^2_1(e,S^1_1(k,n),n)$ the empty function, so $((w\bullet S^2_1(e,S^1_1(k,n),n))\bullet S^1_1(k,n))_0 = ((w\bullet S^2_1(e,S^1_1(k,n),n))\bullet\Lambda x.0)_0$.
If $n\bullet n$ is defined, say Tnnx, then (see remark ii) $((w\bullet S^2_1(e,S^1_1(k,n),n))\bullet S^1_1(k,n))_0\bullet x$ is defined and $T(S^2_1(e,S^1_1(k,n),n),x,((w\bullet S^2_1(e,S^1_1(k,n),n))\bullet S^1_1(k,n))_0\bullet x)$. By remarks iii)-iv) we have that $((w\bullet S^2_1(e,S^1_1(k,n),n))\bullet\Lambda x.0)_0\bullet x$ is defined and not $T(S^2_1(e,S^1_1(k,n),n),x,((w\bullet S^2_1(e,S^1_1(k,n),n))\bullet\Lambda x.0)_0\bullet x)$. So then:
$((w\bullet S^2_1(e,S^1_1(k,n),n))\bullet S^1_1(k,n))_0 \neq ((w\bullet S^2_1(e,S^1_1(k,n),n))\bullet\Lambda x.0)_0$.
(Note, that both sides are always defined!)
The conclusion is that $n\bullet n$ defined? is decidable, contradiction.

1.5. Corollary. $\underline{e}$- and $\underline{e}'$-realizability are not equivalent.

Proof. For, since A (from 1.4) is Kleene-realizable, $\neg\neg A$ must be $\underline{e}'$-realizable.

1.6. Corollary. The open schema $A\to\exists x(x\ \underline{e}\ A)$ is not $\underline{e}$-realizable.

Proof. Take for A the formula $\forall x\exists y(\neg\neg\exists zTexz \to Texy)$. Then $\exists x(x\ \underline{e}\ A)$ is equivalent to $\exists v\forall x\exists u(Tvxu \wedge (\neg\neg\exists zTexz\to TexUu))$, and apply proposition 1.4.

1.7. Proposition. Let $WECT_0$ (Weak Extended Church's Thesis) denote the schema:
$\forall x(Ax\to\exists yBxy)\to\neg\neg\exists z\forall x(Ax\to\exists u(Tzxu\wedge BxUu))$ with A almost negative. Then $WECT_0$ is $\underline{e}$-realizable, provably in **HA**.

Proof. Note that for $\underline{e}$-realizability two realizers of a negation are always equivalent. So $\Lambda x.0$ realizes every instance of the schema.

§2. A q-variant of $\underline{e}$-realizability; strengthening of ECR_0

2.1. Definition. Define predicates Q_A, $=_A$ as follows:

$Q_A(x) \equiv x{=}x \wedge A$ for A atomic; $x{=}_A x' \equiv x{=}x' \wedge A$

$Q_{A\wedge B}(x) \equiv Q_A((x)_0) \wedge Q_B((x)_1)$; $x{=}_{A\wedge B}x' \equiv (x)_0{=}_A(x')_0 \wedge (x)_1{=}_B(x')_1$

$Q_{A\to B}(x) \equiv \forall yy'(y{=}_A y' \Rightarrow x{\bullet}y{\downarrow} \wedge x{\bullet}y'{\downarrow} \wedge x{\bullet}y{=}_B x{\bullet}y') \wedge A{\to}B$;

$x{=}_{A\to B}x' \equiv Q_{A\to B}(x) \wedge Q_{A\to B}(x') \wedge \forall y(Q_A(y) \Rightarrow x{\bullet}y{=}_B x'{\bullet}y)$

$Q_{\forall yAy}(x) \equiv \forall n\,(x{\bullet}n{\downarrow} \wedge Q_{A(n)}(x{\bullet}n)$; $x{=}_{\forall yAy}x' \equiv \forall n\,(x{\bullet}n{=}_{A(n)}x'{\bullet}n)$

$Q_{\exists yAy}(x) \equiv Q_{A((x)_0)}((x)_1)$; $x{=}_{\exists yAy}x' \equiv (x)_0{=}(x')_0 \wedge (x)_1{=}_{A((x)_0)}(x')_1$

2.2. Proposition.
i) $\vdash x{=}_A x' \to Q_A(x) \wedge Q_A(x')$
ii) $\vdash Q_A(x) \to x{=}_A x$
iii) $\vdash Q_A(x) \to A$
iv) $\vdash A \Rightarrow\ \vdash Q_A(n)$ for some n

Proof. Straightforward.

2.3. Proposition. Let ψ_A be as in proposition 1.3, for almost negative A. Then $\vdash Q_A(\psi_A)$.

Proof. Trivial.

2.4. Proposition. Suppose **HA** $\vdash \forall e\,(\forall x\exists yBexy \to \exists zCez)$ and B is almost negative. Then there is a number n such that:
HA $\vdash \forall e\,(n{\bullet}e{\downarrow} \wedge \forall f,f'(\forall x(f{\bullet}x{\downarrow} \wedge f'{\bullet}x{\downarrow} \wedge f{\bullet}x{=}f'{\bullet}x \wedge Bexf{\bullet}x) \to (n{\bullet}e){\bullet}f{\downarrow} \wedge (n{\bullet}e){\bullet}f'{\downarrow} \wedge$
$(n{\bullet}e){\bullet}f{=}(n{\bullet}e){\bullet}f' \wedge Ce\,(n{\bullet}e){\bullet}f))$.
In particular, **HA** satisfies ECR_0 (take x and y dummy variables).

Proof. Let A be $\forall e\,(\forall x\exists yBexy \to \exists zCez)$; suppose **HA** $\vdash$ A. Let m such that **HA** $\vdash Q_A(m)$. Then **HA** $\vdash \forall e(m{\bullet}e{\downarrow}) \wedge \forall f,f'(f{=}_{\forall x\exists yBexy}f' \to (m{\bullet}e){\bullet}f{=}_{\exists zCez}(m{\bullet}e){\bullet}f')$. If $\forall x(f{\bullet}x{=}f'{\bullet}x \wedge Bef{\bullet}x)$ then, since B is almost negative, $\Lambda x.\langle f{\bullet}x,\psi_B(e,x,f{\bullet}x)\rangle{=}_{\forall x\exists yBexy}\Lambda x.\langle f'{\bullet}x,\psi_B(e,x,f'{\bullet}x)\rangle$, so $y \equiv ([(m{\bullet}e){\bullet}\Lambda x.\langle f{\bullet}x,\psi_B(e,x,f{\bullet}x)\rangle])_0{=}\ ([(m{\bullet}e){\bullet}\Lambda x.\langle f'{\bullet}x,\psi_B(e,x,f'{\bullet}x)\rangle])_0$ and $Q_{Cey}(([(m{\bullet}e){\bullet}\Lambda x.\langle f{\bullet}x,\psi_B(e,x,f{\bullet}x)\rangle])_1)$. By 2.2 iii), Cey. So $n \equiv \Lambda f.y$ satisfies the proposition.

§3. Higher-order extension of e- and e'-realizabilities

The following, up to 3.6, can also be found in Pitts 1981. We define a tripos $\mathfrak{R}$ as follows. Let Σ' be the set of all pairs $(p,\sim)$ with $p \subseteq \mathbb{N}$ and $\sim$ an equivalence relation on p. A binary operation $\to$ on Σ' is defined by $(p,\sim)\to(q,\sim') = (\{n \mid \forall a,b \in p\ (n{\bullet}a{\downarrow}\ \&\ n{\bullet}b{\downarrow}\ \&\ (a{\sim}b \Rightarrow n{\bullet}a{\sim'}n{\bullet}b))\},\approx)$, with $n{\approx}m$ iff $\forall a \in p\ (n{\bullet}a{\sim'}m{\bullet}a)$. For sets X, $\mathfrak{R}(X)$ is the collection of all maps: $X \to \Sigma'$, preordered by putting $\phi \vdash \psi$ iff there is an e such that for all $x \in X$, e is in the underlying set of

$\phi(x)\to\psi(x)$. We define logical operators in $\mathfrak{R}(X)$:

3.1. Definition. i) If ϕ, $\psi\in\mathfrak{R}(X)$, $\phi(x)=(p_x,\sim_x)$, $\psi(x)=(q_x,\sim'_x)$:

$\phi\wedge\psi(x) \equiv (\{\langle a,b\rangle | a\in p_x, b\in q_x\},\approx)$ with $\langle a,b\rangle\approx\langle a',b'\rangle$ iff $a\sim_x a'$ and $b\sim'_x b'$

$\phi\to\psi(x) \equiv \phi(x)\to\psi(x)$

$\phi\vee\psi(x) \equiv (\{\langle 0,c\rangle,\langle 1,d\rangle | c\in p_x \;\&\; d\in q_x\},\approx)$ with $\langle n,c\rangle\approx\langle m,d\rangle$ iff ($n=m=0$ and $c\sim_x d$) or ($n=m=1$ & $c\sim'_x d$)

ii) If $f:X\to Y$ is function, then:

$\forall f\phi(y) \equiv (\{c | \forall x\in X(f(x)=y \Rightarrow \forall n\in\mathbb{N}(c\bullet n\downarrow \;\&\; c\bullet n\in p_x))\},\approx)$ with $c\approx c'$ iff $\forall x\in X(f(x)=y \Rightarrow \forall n\in\mathbb{N}(c\bullet n\sim_x c'\bullet n))$

$\exists f\phi(y) \equiv (\{c | \exists x\in X(f(x)=y \;\&\; c\in p_x)\},\approx)$ where $c\approx c'$ is the transitive closure of: $\exists x\in X(f(x)=y \;\&\; c\sim_x c')$

3.2. Proposition. $\mathfrak{R}$ is a tripos, with the operations defined in 3.1.

Let $\wp$ be the tripos underlying the effective topos.

3.3. Definition. i) $\Phi_+(X)$: $\wp(X)\to\mathfrak{R}(X)$ is the order-preserving map given by:

$\Phi_+(X)(\phi) \equiv \Lambda x.(\phi(x),\top)$ where $\top$ is the maximal equivalence relation on $\phi(x)$

ii) $\Phi^+(X)$: $\mathfrak{R}(X)\to\wp(X)$ is the order-preserving map given by:

$\Phi^+(X)(\phi) \equiv \Lambda x.p_x$, if $\phi(x)=(p_x,\sim_x)$.

3.4. Proposition. The pair (Φ_+,Φ^+) constitutes a geometric morphism: $\wp\to\mathfrak{R}$, which is an inclusion of triposes.

3.5. Definition. i) $\Psi_+(X)$: $\mathfrak{R}(X)\to\wp(X)$ is $\Phi^+(X)$

ii) $\Psi^+(X)$: $\wp(X)\to\mathfrak{R}(X)$ is given by

$\Psi^+(X)(\phi) \equiv \Lambda x.(\phi(x),\Delta)$ where Δ is the minimal equivalence relation on $\phi(x)$.

3.6. Proposition. The pair (Ψ_+,Ψ^+) constitutes a geometric morphism: $\mathfrak{R}\to\wp$, which is a right inverse to (Φ_+,Φ^+).

Let us call the topos represented by $\mathfrak{R}$, Ext. Propositions 3.5 and 3.6 show that there is an inclusion (Φ_*,Φ^*): Eff$\to$Ext and a geometric morphism (Ψ_*,Ψ^*): Ext$\to$Eff, making Eff a retract of Ext. I shall use this to compute the finite-type structure (i.e. the structure generated from $\mathbb{N}$ by exponentials and products) in Ext. The natural number object in Ext is the set $\mathbb{N}$ with equality $[\![n=m]\!]=(\{n | n=m\},\sim)$, $\sim$ the unique equivalence relation. From this:

3.7. Proposition. The internal logic of $\mathbb{N}$ in Ext coincides with $\underline{e}$-realizability.

Proof. Use definition 3.1.

Recall (Hyland 1982) that an object (X,=) of Eff is canonically separated iff $[\![x=x']\!]$ is nonempty implies x=x'.
In general, Φ_* is not given by: $\Phi_*((X,=)) = (X,\Phi_+(=))$. But this holds when (X,=) is canonically separated:

3.8. Proposition. Let (X,=) be a canonically separated object of Eff. Then $\Phi_*((X,=))$ is isomorphic to $(X,\Phi_+(=))$.

Proof. First observe that for sets X,Y, a function $f:X\to Y$ and $\phi\in\wp(X)$, $\Phi_+(Y)(\exists f(\phi))$ is isomorphic to $\exists f(\Phi_+(X)(\phi))$ if for all $x,x'\in X$, $y\in Y$, $n,m\in\mathbb{N}$: if $n\in\phi(x)$, $m\in\phi(x')$ and $f(x)=f(x')=y$ then there are $x=x_1,\ldots,x_{k+1}=x',n_1,\ldots,n_k$ with $f(x_1)=\ldots=f(x_{k+1})$ and $n_i\in\phi(x_i)\cap\phi(x_{i+1})$ for i=1,...,k. Clearly, this condition holds if f is a projection: $Y\times X\to X$, (X,=) is a canonically separated object and $\phi\in\wp(Y\times X)$ represents a functional relation for (Y,=') and (X,=). So if $F\in\wp(Y\times X)$ represents a morphism in Eff into a canonically separated object (X,=), $\Phi_+(Y\times X)(F)$ represents a morphism in Ext: $(Y,\Phi_+(='))\to(X,\Phi_+(=))$.
Now there is a natural isomorphism K: $Eff((Z,\Phi^+(='),(X,=))\to Ext((Z,='),(X,\Phi_+(=))$ for canonically separated (X,=), given by $K(F)=\Phi_+(F)\wedge Ez$, with inverse L given by $L(G)=\Phi^+(G)$: $LK(F) = \Phi^+\Phi_+(F)\wedge\Phi^+(Ez) \dashv\vdash F\wedge\Phi^+(Ez) \dashv\vdash F$, since $\Phi^+\Phi_+$ is the identity and F is functional for $\Phi^+(Ez)$; furthermore $G\vdash KL(G)$ and both are functional relations, so they must be isomorphic.

3.9. Proposition. The finite type structure in Ext is given by: the object of type σ has as underlying set the hereditarily effective operations of type σ, and as equality $[\![\alpha=\alpha']\!] = (\{n|n \text{ codes } \alpha\},\top)$ if $\alpha=\alpha'$, and $(\emptyset,\emptyset)$ else.

Proof. This follows directly from proposition 3.8, taking into account that the finite type objects in Eff are canonically separated and that Φ_*, being direct image of an inclusion, preserves exponents.

3.10. Corollary. (Hyland) $Ext\models AC_{\sigma,\tau}$ and $Ext\models\neg CT$.

Proof. Immediate.

3.11. Corollary. Define:

WCT $\quad\forall f:\mathbb{N}\to\mathbb{N}\,\neg\neg\exists z:\mathbb{N}\,\forall x:\mathbb{N}\,\exists u:\mathbb{N}\,(Tzxu\wedge Uu=f(x))$

WC-N $\quad\forall f:\mathbb{N}\to\mathbb{N}\,\exists x:\mathbb{N}\,A(f,x) \to \forall f:\mathbb{N}\to\mathbb{N}\,\exists x,y:\mathbb{N}\,\forall g:\mathbb{N}\to\mathbb{N}\,(\bar{f}y=\bar{g}y\to A(x,g))$

Then $Ext\models WCT$ and $Ext\not\models WC\text{-}N$.

Proof. The first statement is analogous to proposition 1.7; as to the second, it can be proven in $\mathbf{E\text{-}HA}^{\omega}$ that WC-N is incompatible with $AC_{2,0}$ (see Troelstra 1977). Informally: there is no effective operation of type 3 sending an object of type 2 to its modulus of continuity at $\Lambda x.0$.

In the presence of AC, the schema WC-N is equivalent to the continuity axiom:

Cont $\quad \forall\zeta{:}(\mathbb{N}\to\mathbb{N})\to\mathbb{N}\,\forall f{:}\mathbb{N}\to\mathbb{N}\,\exists x{:}\mathbb{N}\,\forall g{:}\mathbb{N}\to\mathbb{N}\,(\bar{g}x=\bar{f}x\to\zeta(f)=\zeta(g))$

The following weakening of this axiom is valid in Ext:

WCont $\quad \forall\zeta{:}(\mathbb{N}\to\mathbb{N})\to\mathbb{N}\,\forall f{:}\mathbb{N}\to\mathbb{N}\,\neg\neg\exists x{:}\mathbb{N}\,\forall g{:}\mathbb{N}\to\mathbb{N}\,(\bar{g}x=\bar{f}x\to\zeta(f)=\zeta(g))$

which follows from the Kreisel-Lacombe-Shoenfield theorem. I conclude:

3.12. Proposition. $\mathbf{E\text{-}HA}^{\omega}$+AC+WCT+¬¬CT+WCont is consistent.

I now sketch an analogous treatment for $\underline{e}'$-realizability. Let Σ'' be the set of all pairs (p,~) with $p\subseteq\mathbb{N}$ and ~ a *partial* equivalence relation on p. An implication is defined by $(p,\sim)\to(q,\sim') = \{a \mid \forall x\in p\,(a\bullet x\downarrow \,\&\, a\bullet x\in q)\}$ with partial equivalence relation $a\approx a'$ iff $\forall x,x'\in p\,(x\sim x'\Rightarrow a\bullet x\sim' a'\bullet x')$. Let $\mathcal{K}(X)$ denote the set of maps $X\to\Sigma''$, preordered by: if $\phi,\psi\in\mathcal{K}(X)$, then $\phi\vdash\psi$ iff there is an a such that for all $x\in X$, $a\approx a$ in $\phi(x)\to\psi(x)$. The proof that $\mathcal{K}$ is a tripos is very similar to the case of $\mathfrak{R}$ and will be omitted. The topos represented by $\mathcal{K}$ will be denoted Ext'.

3.13. Proposition. The inclusion (Φ_*,Φ^*): Eff→Ext factors through an open inclusion: Eff→Ext'.

Proof. Define $V_+(X)$: $\wp(X)\to\mathcal{K}(X)$ by $V_+(X)(\phi)(x)=(\phi(x),\top)$; $V^+(X)$ is defined by forgetting the partial equivalence relation. Let H_+: $\mathcal{K}\to\mathfrak{R}$ be induced by $(A,\sim)\to(\{a\in A\mid a\sim a\},\sim)$ and $H^+{:}\mathfrak{R}\to\mathcal{K}$ by the inclusion: $\Sigma'\subseteq\Sigma''$. Then $\Phi^+=V^+\circ H^+$.

Now Eff is clearly equivalent to the full subcategory of Ext' whose objects have equalities of the form $[\![x=x']\!] = (p,\emptyset)$. But this is an open subtopos of Ext' because it is equivalent to the slice category Ext'/U, where $U\rightarrowtail 1$ is $(\{*\},[\![*=*]\!]=(\mathbb{N},\emptyset))$. And "forgetting": Ext'→Eff factors through the pullback functor U*.

The effect of proposition 3.13 is that the statement "A is $\underline{e}'$-realizable implies that it is Kleene-realizable" extends to sentences of full **HAH**, since inverse image parts of open geometric morphisms are logical functors.

3.14. Corollary. Ext' ⊩ ¬¬CT ∧ ¬¬Cont, but Ext' refutes instances of WC-N and AC.

§4. Direct treatment of ***HAS***.

$\underline{e}$-Realizability for **HAS** can be derived from the considerations in section 3. However, power objects in realizability toposes are clumsy to work with, so a direct definition (formalizable in the language of **HAS**) seems desirable. The definition is quite straightforward.

For sets X and Y let Eq(Y,X) be the formula expressing that Y is an equivalence relation on X. Assign to every set variable X two set variables X^s and X^e such that for different X,Y the variables X^s, X^e, Y^s, Y^e, X and Y are all different. For convenience, we work in a version of **HAS** with only unary set variables. We extend the relation $=_A$ to formulas in the language of **HAS** as follows:

$$x =_{X(t)} x' \equiv X^s(\langle x,t\rangle) \wedge X^s(\langle x',t\rangle) \wedge (Eq(X^e,X^s) \to X^e(\langle\langle x,t\rangle,\langle x',t\rangle\rangle))$$
$$x =_{\forall X A(X)} x' \equiv \forall X^s \forall X^e (x =_{A(X)} x')$$
$$x =_{\exists X A(X)} x' \equiv \exists \sigma (lth(\sigma) \geq 2 \wedge (\sigma)_0 = x \wedge (s)_{lth(\sigma)-1} = x' \wedge \forall i \leq lth(\sigma)-2\, \exists X^s X^e ((\sigma)_i =_{A(X)} (\sigma)_{i+1}))$$

The reader sees that the relation $=_A$ is symmetric and transitive for every formula A in the extended language. Again, we say that $x\,\underline{e}\,A$ if $x =_A x$.

4.1. Proposition. **HAS** is sound for the given interpretation.

Proof. We check some rules of the second-order predicate calculus and the comprehension axiom. Suppose $x\,\underline{e}\,\forall X(A(X)\to B)$ with X not free in B, and $y =_{\exists X A(X)} y'$. Then by induction on $lth(\sigma)$ $x \bullet y =_B x \bullet y'$ so $x\,\underline{e}\,\exists X A(X) \to B$. Conversely if $x\,\underline{e}\,\exists X A(X)\to B$ and $y =_{A(X)} y'$ then $y =_{\exists X A(X)} y'$ and $x \bullet y =_B x \bullet y'$, so $x\,\underline{e}\,\forall X(A(X)\to B)$. Analogously for the equivalence $\forall X(A \to B(X))$ with $A \to \forall X B(X)$.

The axiom $X(t) \wedge t=s \to X(s)$ is trivially realized. For the comprehension scheme $\exists X \forall y (X(y) \leftrightarrow A(y))$, take $X^s = \{\langle w,y\rangle \mid w\,\underline{e}\,A(y)\}$ and $X^e = \{\langle\langle w,y\rangle,\langle w',y\rangle\rangle \mid w =_{A(y)} w'\}$, let $x = \Lambda y.\langle \Lambda w.w, \Lambda w.w\rangle$. Then $x\,\underline{e}\,\forall y (X(y) \leftrightarrow A(y))$ for the given X^s, X^e, so $x\,\underline{e}\,\exists X \forall y (X(y) \leftrightarrow A(y))$.

4.2. Proposition. The Uniformity Principle is $\underline{e}$-realizable.

Proof. The identity realizes every instance of it.

4.3. Corollary. CT is independent (in **HAH**) of the Uniformity Principle.

Index of notions

(only those notions are included of which the reader can find a definition on the indicated page)

Index of systems, axioms and rules

Barendregt, H.	1973	Combinatory logic and the axiom of choice	Indagationes Mathematicae 35, 203-221	
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Samenvatting

"Realiseerbaarheid" is een verzamelnaam voor verschillende interpretaties van intuïtionistische formalismen (in dit proefschrift beschouw ik alleen uitbreidingen van de rekenkunde); in al deze interpretaties staat het begrip *operatie* centraal. Men definieert inductief een relatie "d realiseert ϕ" tussen elementen d van een domein D en zinnen ϕ van een taal L, en een zin heet realiseerbaar als er een $d \in D$ is die hem realiseert. Het domein D zit zo in elkaar dat elementen ervan een of meer partiële functies van D naar D coderen, en anders dan in de gewone modeltheorie, wordt een implicatie $\phi \rightarrow \psi$ door d gerealiseerd, als alle partiële functies door d gecodeerd, alle realisatoren van ϕ naar realisatoren van ψ sturen.
Het oervoorbeeld van zo'n waarheidsdefinitie is Kleene's realiseerbaarheid uit 1945; in wezen zijn alle andere interpretaties varianten op dit idee.
Aanvankelijk was het nut van realiseerbaarheid beperkt tot het leveren van consistentie- en onafhankelijkheidsbewijzen; van betrekkelijk recente datum dateert het onderzoek van realiseerbaarheid vanuit modeltheoretisch oogpunt. Men zoekt een goede kategorie van realiseerbaarheids"modellen" en morfismen tussen deze. Hyland heeft in 1979 ontdekt dat de realiseerbaarheid van Kleene beschreven kan worden als de interpretatie van rekenkunde in een elementaire topos, de zogenaamde "effectieve topos".
In dit proefschrift wordt een aantal realiseerbaarheidsdefinities onderzocht, waarbij ook dit topos-theoretische gezichtspunt aan de orde komt: topossen worden geconstrueerd die de betreffende realiseerbaarheidsbegrippen generaliseren. Tevens worden deze interpretaties syntactisch behandeld, en worden bewijstheoretische eigenschappen van (uitbreidingen van) de rekenkunde afgeleid. In hoofdstuk 7 wordt een "abstracte" notie van realiseerbaarheid, geformaliseerd in een rekenkundige theorie, gedefinieerd waarvoor de intuïtionistische predicatenlogica *maximaal* is: een predicaatlogische formule is bewijsbaar dan en slechts dan als al zijn rekenkundige substitutie-instanties, bewijsbaar in deze theorie, een realisator hebben. Dit suggereert dat realiseerbaarheid een correcte interpretatie van de intuïtionistische logische connectieven geeft, zoals ook Kleene's oorspronkelijke bedoeling was; mits men intuïtionistisch redeneert (Klassiek redenerend kan men aantonen, dat dit *niet* het geval is).
Al met al hoop ik dat dit proefschrift een redelijk beeld geeft van wat met behulp van realiseerbaarheid kan worden bereikt.

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