Notes on Partial Combinatory Algebras

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ACADEMISCH PROEFSCHRIFT

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Promotor: Prof. Dr. A.S. Troelstra

to Henk

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Contents

Chapter 1

Introduction

Originally, *Combinatory logic* and λ-*calculus* were designed to provide a type-free foundation for mathematics. However, the original logical systems of A. Church and H. B. Curry were proved inconsistent and investigation turned to weaker systems.

Currently, **CL** and λ both have the same purpose, namely to describe some of the most primitive and general properties of operators. In doing so, they provide a basis for one of the more recent developments in mathematics, the systematic study of *computations*.

In these theories, functions are regarded as algorithms and not as e.g. in analysis as graphs of mappings. Functions are no longer equivalence classes of algorithms having the same graph but are distinguished in terms of their underlying rules. Thinking of computer science, this is an evident and meaningful point of view: two programs need not necessarily be regarded as being equivalent only because of identical output production. Besides the input-output relation there are other interesting properties such as the complexity of the elementary instructions involved, runtime and storage aspects, etc.

There are two further characteristics of these theories. First of all, the basic operation on functions is not composition $f \circ g$ as in category theory, but *application* $f(g)$. Thinking again of computer science, one can compare this particularity with von Neumann's idea about programs and data: programs and data are given in the same language and are handled equally, that is, programs can operate on and output data that are programs themselves. There is no distinction between objects functioning as operators and objects functioning as arguments. This also means that selfapplication $f(f)$ is meaningful and does not necessarily lead to antinomies as opposed to set theory, where $x \in x$ or $\{x \mid x \notin x\}$ gives rise to Russel's paradox. One is therefore not forced to restrict the formation of new objects or to pursue a typed theory. It is not contradictory to regard every applicative expression built up from functions as a function itself. This is the usual *algebraical completeness.*

Both theories are furthermore based on the principle of *combinatory completeness.* This is the principle that every algebraic function, i.e. every function definable by an applicative expression, is representable. For example, consider the three-place function h, defined by

$$h(x, y, z) = y(z).$$

An instance of combinatory completeness, then, is the requirement that h is representable.

The λ-calculus was invented in the 1930's by A. Church [C]. It is defined by using just variables and the abstraction operator λ. Intuitively, $\lambda x.A(x)$ denotes the function associating x with $A(x)$. The λ-notation can be extended to functions of more than one variable. For example, $\lambda xyz.y(z)$ denotes the three-place function h, defined above. However, one can avoid the need for a special notation for functions of several variables by using functions whose values are again functions. For example, instead of the three-place function h above, consider the one-place function h' denoted by $\lambda x.(\lambda y.(\lambda z.y(z)))$.

This observation was first made and combined with the idea of working without using variables by M. Schönfinkel [S] in the early 1920's and rediscovered and turned into a workable technique by H. Curry [Cu] a couple of years later. This system performs the same tasks as the λ-calculus and avoids technical complications with respect to substitution and congruence. However, for this technical advantage one has to sacrifice the intuitive clarity of the λ-notation.

Schönfinkel and Curry made the observation that combinatory completeness follows already from two of its instances. Requiring the following two functions

$$K(x,y) = x \qquad \text{'Konstantsfunktion' (projection)}$$
$$S(x,y,z) = (x(z))(y(z)) \qquad \text{'Verschmelzungsfunktion' (substitution)}$$

to be representable is sufficient to guarantee combinatory completeness. This means that a structure $(A, *)$ having the following two properties:

(i) $*$ is a binary operation on A (algebraical completeness);

(ii) there are objects $K, S \in A$ such that

(1) $K * a * a' = a$,

(2) $S * a * a' * a'' = a * a'' * (a' * a'')$,

for all $a, a', a'' \in A$;

is combinatory complete. For example, observe that $S * K$ is a representation of the function h defined above:

$$S * K * a * a' * a'' = K * a' * (a * a') * a'' = a' * a''.$$

Such structures are called *combinatory algebras*. But be warned: they are in fact algebraically pathological in as much as they are, except for the trivial one-point structures, never associative, never finite and never commutative.

One can carefully weaken the requirement of algebraical completeness while preserving combinatory completeness. This was first done by S. Feferman [F]. In such a *partial combinatory algebra*, application need not to be defined everywhere. It is even legitimate to consider functions that may be undefined within their 'domain'. This feature corresponds to the situation where a computation initialized with data meeting the input specifications does not terminate.

When I started my research at the Mathematical Institute of the University of Amsterdam my supervisor Anne S. Troelstra confronted me with a puzzle concerning intuitionistic finite type arithmetic - a system which is embedded as a subsystem also in the type-free theories of operators. I could not solve it then and I cannot solve it now. But going to and fro between fruitless attempts to construct a structure meeting the requirements and equally fruitless attempts to show the impossibility of such a phenomenon I came across some new construction methods for (partial) combinatory algebras, some pathologies and some answers to problems I had not thought of before but which I found worth writing down.

This is the result. It is a collection of five loosely connected papers preceded by chapter 2, a whirlwind survey of some of the basic notions for combinatory algebras.

In chapter 3 we present a construction method for extensional combinatory algebras based on probably the simplest known model construction, apart from term models, the graph model D_A. The basic idea for D_A was circulated informally by Plotkin [P] in 1972 and rediscovered by Scott [Sc] and Engeler [E] in 1975 and 1981, respectively. D_A itself is not extensional, but there is a standard procedure for constructing from D_A an extensional combinatory algebra $M(A)$, the *extensional collapse* of D_A. This is shown in the first four pages of chapter 3. The remainder of this chapter is devoted to the proof that the extensional collapse technique does not produce any new models, but that every model constructed in this way is in fact isomorphic with a D_∞-model, the probably most complex known model construction.

Scott's D_∞ was the first 'concrete' model (dating from 1969), and the one whose influence on the semantics of λ and **CL** has been the greatest. Moreover, it has had an impact on abstract lattice theory, having sparked off the study of a new class, the *continuous lattices* (cf. Gierz et al. [G]). The framework in which Scott constructed his nonsyntactical λ-models is that of the *reflexive complete partial orders*. In chapter 4 we modify this approach in order to construct nontotal extensional combinatory algebras. We introduce the notion of a *p-reflexive* cpo and describe a construction method for such structures. Unlike D_∞ this construction method is not a 'projective limit' but is again an extensional collapse technique working on graph models. The final section of chapter 4 comprises some properties of the models constructed in this way. Chapter 3 can be skipped without interfering with the reader's comprehension of chapter 4.

The word 'continuous' suggests topology, and indeed every cpo has a topology called the *Scott topology*, whose (strict) continuous functions are exactly those representable in a (p-)reflexive cpo. Chapter 5 deals with cardinality aspects of such *topological* (partial) combinatory algebras. It turns out that both reflexive and p-reflexive cpo's are uncountable and therefore essentially not effective. Chapter 3 and 4 can be skipped without interfering with the reader's comprehension of chapter 5.

As such, topological (partial) combinatory algebras are probably the most accomplished known models. Engeler and Scott already showed that every applicative structure can be embedded in such a model. In chapter 6 we extend Engeler's proof in order to show that every applicative structure can be embedded into an extensional topological (partial) combinatory algebra. The embedding results rely heavily on the constructions introduced in chapter 3 and 4 and, although we briefly recall the main ingredients involved, the reader is advised to consult the earlier chapters.

Inside a (partial) combinatory algebra, there are 'internal' versions of the finite type structure over ω, which form models of HA^{ω}. One may well wonder which functionals of finite type belong to these structures. We tried to settle this question in chapter 7 for some known models, including D_A, P_ω, T^ω, H_ω and D_∞-models derived from complete lattices. It turns out that the intensional finite type functionals coincide with the extensional finite type functionals in all these models and that, except for H_ω, the type-2 functionals are precisely the *countable* or *continuous* type-2 functionals of Kleene and Kreisel. This is proved in section 3 and 4. The remainder of chapter 7 is devoted to questions concerned with the compatibility and interdependency of extensionality, weak extensionality and finite type extensionality. In particular, we prove that extensionality does not imply finite type extensionality. Except for examples picked from the preceding chapters, chapter 7 is totally selfcontained.

Chapter 2

Preliminaries: Theories, Models and Methods

This chapter briefly reviews the basic definitions of the syntactic and semantic properties of combinatory logic and λ-calculus. They can be found in any text book, e.g. [B], [H,S], and some of them will appear once more in the chapters to follow.

2.1 CL, λ, CL+A_β,CL+EXT and λ+EXT

CL is an equational theory formulated in the following language:

The alphabet of **CL** consists of two individual constants K and S, a fixed countably infinite set $Vars$ of variables, a predicate constant $=$ for equality and the improper symbols (and) for the formation of terms. The set of **CL**-terms, $\mathcal{C}$, is defined in the usual way by

(i) $Vars \subseteq \mathcal{C}$,

(ii) $K, S \in \mathcal{C}$,

(iii) $t, t' \in \mathcal{C} \longrightarrow (tt') \in \mathcal{C}$.

One employs the usual convention of association to the left, i.e. one just writes $t_1t_2...t_n$ instead of $(...((t_1t_2)t_3)...t_n)$.
The formulae of **CL** are equations of the form $t = t'$ with $t, t' \in \mathcal{C}$. Besides the standard axioms and rules for equality, **CL** has only two further axioms, namely those for the combinators:

(i) $Ktt' = t$,

(ii) $Stt't'' = tt''(t't'')$.

The equational theory λ has the following language:

The alphabet of λ consists of a fixed countably infinite set $Vars$ of variables, an abstractor λ, a predicate constant $=$ for equality and parentheses (and) for the formation of terms. The set of λ-terms, Λ, is defined inductively by

(i) $Vars \subseteq \Lambda$,

(ii) $t \in \Lambda \longrightarrow \lambda x.t \in \Lambda$,

(iii) $t, t' \in \Lambda \longrightarrow (tt') \in \Lambda$.

The syntactical conventions for λ are like those for **CL**. Besides the standard axioms and rules for equality, λ has one further axiom, the axiom of β-conversion, and one further rule, the rule ξ:

$$(\lambda x.t)t' = t[x := t'],$$

$$\frac{t = t'}{\lambda x.t = \lambda x.t'}.$$

$t[x := t']$ denotes the result of substituting t' for x in t.

There are standard translations from Λ to $\mathcal{C}$ and back. However, λ and **CL** are not equational equivalent under these translations, i.e. **CL** is essentially weaker. Curry extended **CL** by a finite set $\mathbf{A}_\beta$ of closed equations such that **CL**+$\mathbf{A}_\beta$ is equivalent to λ. We shall list these five axioms below, but only in order to justify the notion of a λ-algebra in the following section. The reader is not supposed to make any detailed sense of these axioms. He may be pleased to hear that we shall not use them in the remainder of this thesis. Now, the axioms $\mathbf{A}_\beta$ are the following:

$K = S(S(KS)(S(KK)K))(K(SKK))$,

$$S = S(S(KS)(S(K(S(KS)))(S(K(S(KK)))S)))(K(K(SKK))),$$

$$S(S(KS)(S(KK)(S(KS)K)))(KK) = S(KK),$$

$$S(KS)(S(KK)) = S(KK)(S(S(KS)(S(KK)(SKK)))(K(SKK))),$$

$$S(K(S(KS)))(S(KS)(S(KS))) =$$

$$S(S(KS)(S(KK)(S(KS)(S(K(S(KS)))S))))(KS).$$

Reverting to the idea about functions as graphs one can extend both theories by extensionality principles. **CL+EXT** can be axiomatized in a way quite similar to **CL+A**$_\beta$. However, for obvious reasons we prefer to extend both the theories by the following rule **EXT**:

$$\frac{tx = t'x}{t = t',}$$

provided $x \notin Vars(tt')$ (in the case of λ, this condition applies to free variables only).
In extending **CL** by **EXT** one actually obtains a theory equivalent to λ+**EXT**. One can thus summarize the proof theoretical strength of the two basic systems and their extensions in the following way:

$$\mathbf{CL} \subseteq \mathbf{CL} + \mathbf{A}_\beta \equiv \lambda \subseteq \lambda + \mathbf{EXT} \equiv \mathbf{CL} + \mathbf{EXT}.$$

2.2 From applicative structures to extensional ca's

Corresponding to the hierarchy of theories above, there are in reverse order not three but four classes of models.

Definition 2.2.1 $M = (A, *)$ is an *applicative structure* if $*$ is a binary operation on A. $*$ is called *application*.

(i) An applicative structure M is *extensional* iff

$$\forall a'' \in A(\, aa'' = a'a''\,) \longrightarrow a = a',$$

for all $a, a' \in A$.

(ii) A *combinatory algebra (ca)* is an applicative structure $M = (A, *, K, S)$ with distinguished elements satisfying

  (1) $Kaa' = a$,

  (2) $Saa'a'' = aa''(a'a'')$,

  for all $a, a', a'' \in A$. □

Again, as in algebra, $a * a'$ is usually written as aa' and $a_1a_2...a_n$ is an abbreviation for $(...((a_1a_2)a_3)...a_n)$.
If M is a ca then $M \models$ **CL**, and if M is an extensional ca then $M \models$ **CL** + **EXT**. One might expect that there is only one major intermediate class of models, models for λ. However, there are two, distinguished in terms of the behaviour of K and S.

Definition 2.2.2 Let M be a ca.

(i) If $M \models \mathbf{A}_\beta$ then M is called a λ-*algebra.*

(ii) Define $I = SKK$ and $1 = S(KI)$. A λ-algebra M is a λ-*model* if

$$(MS) \qquad \forall a'' \in A(\, aa'' = a'a''\,) \longrightarrow 1a = 1a',$$

for all $a, a' \in A$. □

The class of λ-models coincides with the class of *weakly extensional* λ-algebras. The Meyer-Scott condition (MS) is an algebraical formulation of the rather syntactical condition of weak extensionality.
One thus has the following hierarchy of models:

$$\text{extensional ca's} \subseteq \lambda\text{-models} \subseteq \lambda\text{-algebras} \subseteq \text{ca's}.$$

From various examples in the literature it follows that the inclusions are in fact proper.

2.3 Reflexive complete partial orders

Extensional ca's and λ-models can be obtained in a canonical way from reflexive complete partial orders.

Definition 2.3.1 Let $(A, \sqsubseteq)$ be a partially ordered set.

(i) $D \subseteq A$ is *directed* if $D \neq \emptyset$ and

$$\forall a, a' \in D \exists a'' \in D(\, a \sqsubseteq a'' \;\wedge\; a' \sqsubseteq a'' \,).$$

(ii) $(A, \sqsubseteq)$ is a *complete partial order (cpo)* if

(1) there is a $\bot \in A$ such that

$$\forall a \in A(\, \bot \sqsubseteq a \,).$$

$\bot$ is called *bottom*.

(2) $supD$ exists in A, for every directed $D \subseteq A$. □

Every cpo has a topology, the *Scott topology* $\mathcal{O}$, which is given by: $O \in \mathcal{O}$ iff

(i) $a \in O \;\wedge\; a \sqsubseteq a' \longrightarrow a' \in O$,

(ii) $supD \in O \longrightarrow D \cap O \neq \emptyset$, for all directed D.

For cpo's A, A', $[A \to A']$ denotes the continuous function space. Avoiding the Scott topology one can also characterize $[A \to A']$ by: if $f : A \to A'$ then

$$f \in [A \to A'] \longleftrightarrow \forall \text{directed } D \subseteq A(\, f(supD) = sup\{f(a) \mid a \in D\} \,).$$

$[A \to A']$ can be partially ordered pointwise by

$$f \sqsubseteq g \longleftrightarrow \forall a \in A(\, f(a) \sqsubseteq g(a) \,),$$

and if $D \subseteq [A \to A']$ is a directed, then $supD : A \to A'$ defined by

$$(supD)(a) = sup\{f(a) \mid f \in D\}$$

is continuous. $[A \to A']$ is therefore a cpo with bottom $\lambda a.\, \bot$.

Definition 2.3.2 A cpo A is called *reflexive* if $[A \to A]$ is a *retract* of A, i.e. if there are

$$F \in [A \to [A \to A]],\; G \in [[A \to A] \to A]$$

such that $F \circ G = id_{[A \to A]}$. □

Theorem 2.3.3 Let A be a reflexive cpo and define $*$ by

$$a * a' = F(a)(a').$$

Then

(i) $(A, *)$ can be expanded to a λ-model (by choosing K and S);

(ii) $(A, *)$ is extensional iff $G \circ F = id_A$.

PROOF. Cf. e.g. [B], theorem 5.4.4. □

To give an idea of how a reflexive cpo can be constructed, we shall finally describe the graph model P_ω introduced by Plotkin and Scott, independently. The universe of P_ω is the cpo $\mathcal{P}(\omega)$ partially ordered by inclusion with bottom $\emptyset$ and $supD = \cup D$, for directed $D \subseteq \mathcal{P}(\omega)$. We let $(e_n)_{n\in\omega}$ be some standard coding of the finite subsets of ω and (n, m) be some standard coding of pairs of natural numbers. F and G are then defined as follows:

$$F(X)(Y) = \{m \mid \exists e_n \subseteq Y(\,(n, m) \in X\,)\},$$

$$G(f) = \{(n, m) \mid m \in f(e_n)\}.$$

It is easy to see that both F and G are continuous. Moreover, observe that $F(G(f))(X) =$

$= \{m \mid \exists e_n \subseteq X(\,(n, m) \in G(f)\,)\}$

$= \{m \mid \exists e_n \subseteq X(\,m \in f(e_n)\,)\}$

$= \{m \mid m \in \cup\{f(e_n) \mid e_n \subseteq X\}\}$

$= \{m \mid m \in f(X)\}$, since f is continuous and $\{e_n \mid e_n \subseteq X\}$ is directed

$= f(X)$.

Thus $F \circ G = id_{[\mathcal{P}(\omega)\to\mathcal{P}(\omega)]}$.

Chapter 3

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How to construct extensional combinatory algebras

ABSTRACT

We develop a slight modification of Engeler's graph algebras, yielding extensional combinatory algebras. It is shown that by this construction we get precisely the class of Scott's D_∞-models generated by complete atomic Boolean algebras. In section 3 we construct extensional substructures of graph-algebras and $P\omega$-models.

0. INTRODUCTION

0.1 DEFINITION

(i) A *combinatory algebra* (*ca*) is a structure $(A, *, K, S)$ with $*$ a binary operation ('application') on A and two distinguished elements $K, S \in A$ satisfying

$$AK \quad \forall x \in A\, \forall y \in A\;\, Kxy = x$$

$$AS \quad \forall x \in A\, \forall y \in A\, \forall z \in A\;\, Sxyz = xz(yz)$$

where xy is short for $x * y$.

(ii) Moreover, such a structure is *extensional* iff

$$EXT \quad \forall x \in A\, \forall y \in A\; (\forall z \in A\;\, xz = yz \rightarrow x = y) \quad \Box$$

In [E] Engeler introduced the notion of a graph algebra over an arbitrary non-empty set. The construction starts with a non-empty carrier set A. Then $G(A)$ is the least set containing A such that for $b \in G(A)$ and finite $B \subseteq G(A)$ the pair (B, b) is in $G(A)$, assuming that all $a \in A$ are not such pairs, that is

0.2 DEFINITION. Let $A \neq \emptyset$ and $G(A) := \bigcup\{G_n(A) | n \in \omega\}$ where $G_n(A)$ is recursively defined by
(i) $G_0(A) := A$
(ii) $G_{n+1}(A) := G_n(A) \cup \{(B, b) | B \subseteq G_n(A),\ B \text{ finite},\ b \in G_n(A)\}$. □

A binary application operation • on the subsets of $G(A)$ is then defined by

$$X \bullet Y := \{b | \exists B \subseteq Y((B, b) \in X)\}.$$

Engeler showed that the graph algebra $(P(G(A)), \bullet)$ over a non-empty set A can be made into a *ca* by isolating appropriate subsets K and S of $G(A)$. These structures are very elegant, since the notion of application is easy to grasp: the result of applying X to Y depends on the 'elementary instructions' (B, b) of X, which give output b any time the input Y contains B. Since this construction never yields extensional *ca*'s, we shall give:

1. A SLIGHTLY MODIFIED CONSTRUCTION FOR EXTENSIONAL *CA*'S

Again we start with an arbitrary non-empty set A. In the description below we let small letters $a, b, c, \ldots, x, y, z$ range over $G(A)$ and capital letters $B, C, \ldots, X, Y, Z$ denote subsets of $G(A)$. On $P(G(A))$ we define an application operation by

1.1 DEFINITION. $X * Y := \{b | \exists B \leq Y \ (B, b) \in X\} \cup \{a \in A | a \in X\}$ where we put $Z \leq Z' \leftrightarrow \forall x \in Z \exists y \in Z'(x \leq_{G(A)} y)$ and $x \leq_{G(A)} y$ holds if either
(i) $x = y$ or
(ii) $\exists B \exists b \ (x = (B, b) \ \& \ y \in A \ \& \ b \leq_{G(A)} y)$ or
(iii) $\exists b \ (x \in A \ \& \ y = (\emptyset, b) \ \& \ x \leq_{G(A)} b)$ or
(iv) $\exists B_1 \exists B_2 \exists b_1 \exists b_2 \ (x = (B_1, b_1) \ \& \ y = (B_2, b_2) \ \& \ B_2 \leq B_1 \ \& \ b_1 \leq_{G(A)} b_2)$ □

REMARKS. Observe that for all X $A \bullet X = \emptyset = \emptyset \bullet X$, since we have assumed A not to contain pairs of the form (B, b). So, if we want to construct an extensional *ca*, while leaving the application operation unchanged, we are forced to identify A with $\emptyset$, which would have unpleasant consequences. Therefore we consider the elements of A also to be elementary instructions needing no input at all and producing themselves. Moreover, for all X $\{(\emptyset, b)\} \bullet X = \{(\emptyset, b), (B, b)\} \bullet X$. Hence $\{(\emptyset, b)\}$ and $\{(\emptyset, b), (B, b)\}$ represent the same function and should therefore be considered as being equal. On the other hand there is always a subset of $G(A)$ which separates $\{(\emptyset, b)\}$ from $\{(\emptyset, b), (B, b)\}$ if $B \neq \emptyset$: for example, let $D := \{(\{(B, b)\}, b)\}$. Then $D \bullet \{(\emptyset, b)\} = \emptyset$ but $D \bullet \{(\emptyset, b), (B, b)\} = \{b\}$. Therefore we change normal set theoretical inclusion into a relation $Z \leq Z'$ which may be read as "Z' contains at least as strong instructions as Z". y is at least as strong as x ($x \leq_{G(A)} y$) is then defined by 4 clauses:
either x and y denote the same instruction (i)
or y needs no input to produce an output which is at least as strong as the output of x (ii, iii)

or y needs at most as much input as x to produce an output at least as strong as the output of x (iv).

1.2 PROPOSITION

(i) $\forall x \forall y\ (x=y \rightarrow x \leq_{G(A)} y)$
(ii) $\forall X \forall Y\ (X \subseteq Y \rightarrow X \leq Y)$
(iii) $\forall x \in A\ \forall y \in A\ (x=y \leftrightarrow x \leq_{G(A)} y)$
(iv) $\forall X \subseteq A\ \forall Y \subseteq A\ (X \subseteq Y \leftrightarrow X \leq Y)$
(v) $\forall x \exists a \in A\ (x \leq_{G(A)} a)$
(vi) $\forall X\ (\emptyset \leq X \leq A)$

PROOF. Easy. □

$\leq_{G(A)}$ and $\leq$ are transitive relations on $G(A)$ and $P(G(A))$ respectively:

1.3 PROPOSITION

(i) $\forall a \in A\ \forall y\ (a \leq_{G(A)} y \leftrightarrow (\emptyset, a) \leq_{G(A)} y)$
(ii) $\forall a \in A\ \forall y\ (y \leq_{G(A)} a \leftrightarrow y \leq_{G(A)} (\emptyset, a))$
(iii) $\leq_{G(A)}$ is transitive
(iv) $\leq$ is transitive

PROOF. (i) Let $a \in A$ and y be arbitrary. Then $a \leq_{G(A)} y \leftrightarrow$

$a=y$ or $\exists c\ (y=(\emptyset, c)\ \&\ a \leq_{G(A)} y) \leftrightarrow$

$(y \in A\ \&\ a \leq_{G(A)} y)$ or $\exists c\ (y=(\emptyset, c)\ \&\ a \leq_{G(A)} y) \leftrightarrow (\emptyset, a) \leq_{G(A)} y$.

(ii) similar. (iii) We prove with induction on n:

$$\forall n \in \omega\ \forall x \in G_n(A)\ \forall y \in G_n(A)\ \forall z \in G_n(A)\ (x \leq_{G(A)} y\ \&\ y \leq_{G(A)} z \rightarrow x \leq_{G(A)} z).$$

The transitivity of $\leq_{G(A)}$ then follows from the observation that for all $\{x, y, z\} \subseteq G(A)$ there is an $n \in \omega$ with $\{x, y, z\} \subseteq G_n(A)$. If $\{x, y, z\} \subseteq G_0(A)$ are such that $x \leq_{G(A)} y$, $x \leq_{G(A)} y$ then $x=y=z$. Hence $x \leq_{G(A)} z$. Suppose $\{x, y, z\} \subseteq G_{n+1}(A)$ are such that $x \leq_{G(A)} y$, $x \leq_{G(A)} y$. Define

$$S(u) := \begin{cases} (\emptyset, u) \text{ if } u \in A \\ u \quad \text{otherwise} \end{cases}$$

Then from (i), (ii) we get $S(x) \leq_{G(A)} S(y)$, $S(y) \leq_{G(A)} S(z)$. Hence there are finite $B_i \subseteq G_n(A)$, $b_i \in G_n(A)$ for $i=1,2,3$ such that $S(x)=(B_1, b_1)$, $S(y)=(B_2, b_2)$, $S(z)=(B_3, b_3)$, $B_3 \leq B_2 \leq B_1$ and $b_1 \leq_{G(A)} b_2 \leq_{G(A)} b_3$. So from the induction hypothesis it follows that $B_3 \leq B_1$ and $b_1 \leq_{G(A)} b_3$. Thus $S(x) \leq_{G(A)} S(z)$. Then again from (i), (ii) we get $x \leq_{G(A)} z$. (iv) follows immediately from (iii). □

1.4 DEFINITION. Let $A \neq \emptyset$. Define $M(A) := (P(G(A))/\equiv, *, [K], [S])$, where

$$X \equiv Y \leftrightarrow X \leq Y \ \& \ Y \leq X$$

$$[X] := \{Y \mid Y \equiv X\} \text{ and } P(G(A))/\equiv \, := \{[X] \mid X \in P(G(A))\}$$

$$[X] * [Y] := [X * Y]$$

$$K := \{(B,(C,b)) \mid b \in B\}$$

$$S := \{(B,(C,(D,b))) \mid \exists U(\exists D' \leq D \ (D',(U,b)) \in B \ \&$$

$$\& \ \forall u \in U \exists u' \geq_{G(A)} u \exists D' \leq D \ (D',u') \in C)\} \quad \square$$

$\equiv$ is by definition symmetric, reflexive by 1.2(ii) and transitive by 1.3(iii). Hence $\equiv$ is an equivalence relation.

1.5 PROPOSITION
(i) $\forall X \forall Y \ (X = Y \rightarrow X \equiv Y)$
(ii) $\forall X \in P(A) \forall Y \in P(A) \ (X = Y \leftrightarrow X \equiv Y)$
(iii) $A \equiv G(A)$

PROOF. Easy. $\square$

Before we show that $M(A)$ is an extensional *ca* we prove

1.6 LEMMA (MONOTONICITY). $*$ is monotone wrt. $\leq$, i.e.
(i) $\forall X \forall Y \forall Z \ (X \leq Y \rightarrow ZX \leq ZY)$
(ii) $\forall X \forall Y \forall Z \ (X \leq Y \rightarrow XZ \leq YZ)$

PROOF. (i) Suppose $X \leq Y$ and let $b \in ZX$. Then if $b \in \{a \in A \mid a \in Z\}$ also $b \in ZY$. If $b \notin \{a \in A \mid a \in Z\}$ then $(B,b) \in Z$ for some $B \leq X$. Thus since $X \leq Y$ also $B \leq Y$. Hence again $b \in ZY$. (ii) Suppose $X \leq Y$ and let $b \in XZ$. If $b \in \{a \in A \mid a \in X\}$ then $b \leq_{G(A)} b'$ for some $b' \in Y$. Hence either $b' \in A$ or $b' = (\emptyset, b'')$ with $b \leq_{G(A)} b''$. So there is a $d \in YZ$ with $b \leq_{G(A)} d$. If $b \notin \{a \in A \mid a \in X\}$ then $(B,b) \in X$ for some $B \leq Z$. Let $b' \in Y$ be such that $(B,b) \leq_{G(A)} b'$. If $b' \in A$ then $b \leq_{G(A)} b'$ and $b' \in YZ$. If $b' \notin A$ then $b' = (D,d)$ with $D \leq B \leq Z$, $b \leq_{G(A)} d$ and $d \in YZ$. $\square$

1.7 LEMMA (EXTENSIONALITY)
(i) $\forall X \forall Y \ (\forall B \text{ finite } (XB \leq YB) \rightarrow X \leq Y)$
(ii) $\forall X \forall Y \ (\forall B \text{ finite } (XB \equiv YB) \rightarrow X \equiv Y)$
(iii) $\forall X \forall Y \ (\forall Z \ (XZ \equiv YZ) \rightarrow X \equiv Y)$

PROOF. (i) Suppose for all finite B $XB \leq YB$ and let $b \in X$. If $b \in A$ then $b \in X\emptyset \leq Y\emptyset$. Let $b' \in Y\emptyset$ be such that $b \leq_{G(A)} b'$. Then either $b' \in \{a \in A \mid a \in Y\}$ or $(\emptyset, b') \in Y$. So $b \leq_{G(A)} b''$ for some $b'' \in Y$. If $b \notin A$ then $b = (C,c)$ and $c \in XC \leq YC$. Hence $c \leq_{G(A)} c'$ for some $c' \in YC$. Then either $c' \in \{a \in A \mid a \in Y\}$

or $(D, c') \in Y$ for some $D \leq C$. Thus again $b \leq_{G(A)} b''$ for some $b'' \in Y$. (ii) and (iii) follow from (i) and (ii) respectively. □

1.8 THEOREM. Let $A \neq \emptyset$. Then $M(A)$ is a *ca* satisfying extensionality.

PROOF. $P(G(A))/\equiv$ is clearly closed under $*$ and from the monotonicity of $*$ it follows that $\forall X \forall Y \forall Z([X]=[Y] \rightarrow [X][Z]=[Y][Z] \ \& \ [Z][X]=[Z][Y])$. So $*$ is a binary operation on $P(G(A))/\equiv$. To prove AK let X, Y be arbitrary. Then

$$KXY = \{(C,b) \mid \exists B \leq X \ b \in B\} Y$$

$$= \{(C,b) \mid \exists b' \in X(b \leq_{G(A)} b')\} Y = \{b \mid \exists b' \in X(b \leq_{G(A)} b')\} \equiv X.$$

Thus $\forall X \forall Y \, [K][X][Y]=[X]$. To prove AS let X, Y, Z be arbitrary and choose $X' \equiv X$ and $Y' \equiv Y$ such that all $x \in X'$ are of the form $(D,(U,b))$ and all $y \in Y'$ of the form (D,b). Then $[S][X][Y][Z]=[S][X'][Y'][Z]$ and $[X][Z]([Y][Z]) = [X'][Z]([Y'][Z])$. We will show $[S][X'][Y'][Z]=[X'][Z]([Y'][Z])$, i.e. $SX'Y'Z \equiv X'Z(Y'Z)$. Now $X'Z = \{(U,b) \mid \exists D \leq Z((D,(U,b)) \in X')\}$ and $Y'Z = \{b \mid \exists D \leq Z((D,b) \in Y')\}$. Hence

$$X'Z(Y'Z) = \{b \mid \exists U \leq Y'Z((U,b) \in X'Z)\} =$$

$$= \{b \mid \exists U \leq Y'Z \exists D \leq Z((D,(U,b)) \in X')\}$$

$$= \{b \mid \exists U(\exists D \leq Z((D,(U,b)) \in X') \ \&$$

$$\& \ \forall u \in U \exists u' \geq u \exists D \leq Z((D,u) \in Y))\}.$$

On the other hand

$$SX'Y'Z = \{b \mid \exists D \leq Z \exists C \leq Y' \exists B \leq X' \exists U(\exists D' \leq D((D',(U,b)) \in B) \ \&$$

$$\& \ \forall u \in U \exists u' \geq_{G(A)} u \exists D' \leq D((D',u) \in C))\} =$$

$$= \{b \mid \exists b' \geq_{G(A)} b \exists U(\exists D \leq Z((D,(U,b')) \in X') \ \&$$

$$\& \ \forall u \in U \exists u' \geq_{G(A)} u \exists D \leq Z((D,u) \in Y))\}.$$

Thus $SX'Y'Z \equiv X'Z(Y'Z)$. Finally, we have to prove $[K] \neq [S]$. As is well known, it suffices to show $[X] \neq [Y]$ for some $[X],[Y] \in P(G(A))/\equiv$. So let $a \in A$ ($A \neq \emptyset$!). Then $[\{a\}], [\{(\{a\},a)\}] \in P(G(A))/\equiv$ and $\neg \ \{a\} \leq \{(\{a\},a)\}$. Hence $[\{a\}] \neq [\{(\{a\},a)\}]$. Thus $M(A)$ is a *ca* and by lemma 1.7(iii) $M(A)$ satisfies extensionality. □

2. THE GLOBAL STRUCTURE OF $M(A)$

Clearly all $M(A)$'s are up to isomorphism uniquely determined by the cardinality of their carrier set A, but we also have the converse, i.e. $\forall A \forall A' \, (M(A) \cong M(A') \rightarrow \text{Card}\,(A) = \text{Card}\,(A))$. Before we prove this fact we will state several properties of $M(A)$. Notice that $(M(A), \leq)$ where

$$[X] \leq [Y] \leftrightarrow X \leq Y$$

is a complete lattice with bottom $[\emptyset]$, top $[A]$ and the supremum $\sup F = [\bigcup\{X \mid [X] \in F\}]$ for arbitrary $F \subseteq P(G(A))/\equiv$. We will show that some of the lattice theoretic properties of $M(A)$ can be expressed in $M(A)$. First we will prove that the binary SUP resp. INF operator is definable (in the language of *ca*'s plus constants) in $M(A)$.

2.1 LEMMA. There is a $\mathrm{SUP}_A \in M(A)$ such that

(i) $\forall X \forall Y\ ([X] \leq \mathrm{SUP}_A[X][Y]\ \&\ [Y] \leq \mathrm{SUP}_A[X][Y])$

(ii) $\forall X \forall Y \forall Z\ ([X] \leq [Z]\ \&\ [Y] \leq [Z] \rightarrow \mathrm{SUP}_A[X][Y] \leq [Z])$

(iii) $\forall Z\ (\forall X \forall Y\ ([Z][X][\emptyset] = [X] = [Z][X][X]\ \&\ [Z][X][Y] = [Z][Y][X]) \rightarrow$
$\rightarrow [Z] = \mathrm{SUP}_A)$

PROOF. Define $\mathrm{SUP}_A := [\{(B,(C,d)) \mid \{d\} \leq B \cup C\}]$.

Then $\mathrm{SUP}_A[X][Y] = [\{d \mid \{d\} \leq X \cup Y\}]$. From this (i) follows immediately. To prove (ii) suppose $[X] \leq [Z]$, $[Y] \leq [Z]$. Then from the monotonicity it follows that $\mathrm{SUP}_A[X][Y] \leq \mathrm{SUP}_A[Z][Z] = [\{d \mid \{d\} \leq Z\}] = [Z]$. Hence $\mathrm{SUP}_A[X][Y] \leq [Z]$. (iii) Let Z be arbitrary such that $\forall X \forall Y\ ([Z][X][\emptyset] = = [X] = [Z][X][X]\ \&\ [Z][X][Y] = [Z][Y][X])$. Then $[X] = [Z][X][\emptyset] \leq \leq [Z][X][Y]$ and $[Y] = [Z][Y][\emptyset] \leq [Z][Y][X] = [Z][X][Y]$. Thus for arbitrary X, Y we get from (ii) $\mathrm{SUP}_A[X][Y] \leq [Z][X][Y]$. Moreover, $[Z][X][Y] \leq \leq [Z](\mathrm{SUP}_A[X][Y])(\mathrm{SUP}_A[X][Y]) = \mathrm{SUP}_A[X][Y]$. Hence for all X, Y $[Z][X][Y] = \mathrm{SUP}_A[X][Y]$. So $[Z] = \mathrm{SUP}_A$ by extensionality. □

2.2 LEMMA. There is a $\mathrm{INF}_A \in M(A)$ such that

(i) $\forall X \forall Y\ (\mathrm{INF}_A[X][Y] \leq [X]\ \&\ \mathrm{INF}_A[X][Y] \leq [Y])$

(ii) $\forall X \forall Y \forall Z\ ([Z] \leq [X]\ \&\ [Z] \leq [Y] \rightarrow [Z] \leq \mathrm{INF}_A[X][Y])$

(iii) $\forall Z\ (\forall X \forall Y\ ([Z][X][A] = [X] = [Z][X][X]\ \&$
$\&\ [Z][X][Y] = [Z][Y][X]) \rightarrow [Z] = \mathrm{INF}_A)$

PROOF. Define $\mathrm{INF}_A = [\{(B,(C,d)) \mid \{d\} \leq B\ \&\ \{d\} \leq C\}]$. □

Next we characterize top and bottom:

2.3 LEMMA.

$$\forall X\ [\exists Y \exists Z\ ([Y] \neq [X]\ \&\ [Z][X] = [X]\ \&\ \forall X' \not\equiv X [Z][X'] = [Y])$$

$$\leftrightarrow ([X] = [\emptyset] \text{ or } ([X] = [A]\ \&\ A \text{ is finite})]$$

PROOF. $\rightarrow$: Let $[Y] \neq [X], [Z]$ be such that $[Z][X] = [X]$ and $\forall X' \not\equiv X$ $[Z][X'] = [Y]$. Suppose $[X] \neq [\emptyset]$ and $[X] \neq [A]$. Then since $[\emptyset] \leq [X] \leq [A]$ we get $[Y] = [Z][\emptyset] \leq [Z][X] = [X] = [Z][X] \leq [Z][A] = [Y]$, i.e. $[X] = [Y]$. Contradiction. Thus $[X] = [\emptyset]$ or $[X] = [A]$. Suppose $[X] \neq [\emptyset]$ and A is infinite. We will show $A \equiv Y$. Clearly $Y \leq A$. To prove $A \leq Y$ let $a \in A$ be arbitrary. Then since $A \equiv ZA$ $a \leq_{G(A)} b$ for some $b \in ZA$. If $b \in \{a \in A \mid a \in Z\}$ then $b \in Z\emptyset \equiv Y$. Hence $a \leq_{G(A)} b \leq_{G(A)} b'$ for some $b' \in Y$. If $b \notin \{a \in A \mid a \in Z\}$ then $(B, b) \in Z$ for some finite $B \leq A$. Now since B is finite $B \not\equiv A$. So $b \in ZB \equiv Y$. Thus again $a \leq_{G(A)} b \leq_{G(A)} b'$ for some $b' \in Y$. So $A \equiv Y$, i.e. $[A] = [Y]$. Contradiction. Hence A is finite. $\leftarrow$: For $[\emptyset]$ choose $a_0 \in A$ and define $[Z] := [\{(B, a_0) | B \neq \emptyset\}]$. Then $[Z][\emptyset] = [\emptyset]$ and $\forall X' \not\equiv \emptyset$ $[Z][X'] = [\{a_0\}] \neq [\emptyset]$. If A is finite then choose $a_0 \in A$ and define $[Z] := [\{(A, a) | a \in A\} \cup \{(\emptyset, (\{a_0\}, a_0))\}]$. Then $[Z][A] = [A \cup \{(\{a_0\}, a_0)\}] = [A]$. Moreover, if $[X'] \neq [A]$ then $\neg A \leq X'$. So $[Z][X'] = [\{(\{a_0\}, a_0)\}] \neq [A]$ for all $[X'] \neq [A]$. □

Observe that $(\{[X] | X \in P(A)\}, \leq)$ is a complete sublattice of $(M(A), \leq)$ which is isomorphic to $(P(A), \subseteq)$ by proposition 1.2(vi). We will finally show that the elements of this sublattice are definable in $M(A)$.

2.4 LEMMA. $\forall Z\ (\forall X\ ([Z][X] = [X]) \leftrightarrow \exists Y \in P(A)\ ([Z] = [Y]))$

PROOF. $\rightarrow$: Suppose $\forall X$ $[Z][X] = [X]$ and define $Y := \{a \in A | \exists X (X \equiv Z \ \&\ a \in X)\}$. Clearly $[Y] \leq [Z]$. To prove $[Z] \leq [Y]$ let $b \in Z$. If $b \in A$ then $b \in Y$. If $b \notin A$ then $b = (B_1, \ldots (B_n, a) \ldots)$ for some $a \in A$ and finite B_i. So $b \leq_{G(A)} a \in ZB_1 \ldots B_n \equiv Z$. Thus $b \leq_{G(A)} a \in Y$. $\leftarrow$: Suppose $[Z] = [Y]$ for some $Y \in P(A)$. Then for all X $[Z][X] = [Y][X] = [YX] = [Y] = [Z]$. □

Now we are ready to prove

2.5 THEOREM. $\forall A\, \forall A'\ (M(A) \cong M(A') \leftrightarrow \text{Card}\ (A) = \text{Card}\ (A'))$

PROOF. $\rightarrow$: Suppose $M(A) \cong M(A')$ by some bijection $\theta : P(G(A_1))/\equiv \rightarrow P(G(A_2))/\equiv$ such that $\forall X \forall Y\ \theta([X][Y]) = \theta([X])\theta([Y])$. Let $B \in P(A)$. Then for all $[X] \in P(G(A))/\equiv$ $[B][X] = [B]$. So for all $[X] \in P(G(A'))/\equiv$ $\theta([B])[X] = \theta([B])$ and thus by lemma 2.4 $\theta([B]) = [C]$ for some $C \in P(A')$. By the same argument we also see that for all $C \in P(A')$ we have $[C] = \theta([B])$ for some $B \in P(A)$. So $\theta[\{[X] | X \in P(A)\}] = \{[X] | X \in P(A')\}$. Hence Card $(\{[X] | X \in P(A)\}) = \text{Card}\ (\{[X] | X \in P(A')\})$ and thus by proposition 1.5(ii) Card $(P(A)) = \text{Card}\ (P(A'))$. Then if A' is finite Card $(A) = \text{Card}\ (A')$. Suppose A' is infinite. We shall prove that $(\{[X] | X \in P(A)\}, \leq) \cong (\{[X] | X \in P(A')\}, \leq)$. By lemma 2.3 there are $[Y] \neq [\emptyset]$ and $[Z] \in P(G(A))/\equiv$ such that $[Z][\emptyset] = [\emptyset]$ and for all $X' \not\equiv \emptyset$ $[Z][X'] = [Y]$. Then again by lemma 2.3 $\theta([\emptyset]) = [\emptyset]$. Now observe that for all $[X], [Y] \in P(G(A))/\equiv$ we have $\text{SUP}_A[X][\emptyset] = [X] = \text{SUP}_A[X][X]$ and $\text{SUP}_A[X][Y] = \text{SUP}_A[Y][X]$. Thus for all $[X], [Y] \in P(G(A'))/\equiv$ $\theta(\text{SUP}_A)[X][\emptyset] = [X] = \theta(\text{SUP}_A)[X][X]$ and $\theta(\text{SUP}_A)[X][Y] = \theta(\text{SUP}_A)[Y][X]$. So by lemma 2.1(iii) $\theta(\text{SUP}_A) = \text{SUP}_{A'}$. Hence for all $B, C \in P(A)$

$$[B] \leq [C] \leftrightarrow \text{SUP}_A[B][C] = [C] \leftrightarrow \text{SUP}_{A'}\theta([B])\theta([C]) = \theta([C]) \leftrightarrow$$

$$\leftrightarrow \theta([B]) \leq \theta([C]).$$

So $(\{[X] | X \in P(A)\}, \leq) \cong (\{[X] | X \in P(A')\}, \leq)$. Hence $(P(A), \subseteq) \cong (P(A'), \subseteq)$ and Card $(A) = \text{Card}\ (A')$ follows immediately. $\leftarrow$: Easy and left to the reader. □

Let $(D_1, \leq_1)$, $(D_2, \leq_2)$ be complete partial orders (cpo), then $[D_1 \to D_2]$ is the set of continuous maps considered as a cpo by pointwise ordering. It is well known that every cpo $(D, \leq')$ with $(D, \leq') \cong [D \to D]$ by some continuous bijection can be made into an extensional *ca*. In [S1], [S2] Scott showed how to construct complete lattices $(D_\infty, \leq_\infty) \cong [D_\infty \to D_\infty]$ starting with an arbitrary complete lattice $(D_0, \leq_0)$. This construction can be done also for cpo's rather than complete lattices. Now, the question arises whether we get different extensional *ca*'s by the graph construction. The answer is no. We shall show that for every $A \neq \emptyset$ and every cpo $(D_0, \leq_0)$ we have

$$M(A) \cong (D_\infty, *_\infty, K_\infty, S_\infty) \leftrightarrow (P(A), \subseteq) \cong (D_0, \leq_0).$$

Hence the graph construction yields up to isomorphism exactly those *ca*'s $(D_\infty, *_\infty, K_\infty, S_\infty)$ with $(D_0, \leq_0)$ a complete atomic Boolean algebra. We shall first give a brief outline of the D_∞-construction and extract the properties of D_∞ we will need in the proofs below. For a very thorough discussion see [B].

Let $(D_0, \leq_0)$ be a cpo and define inductively $(D_{n+1}, \leq_{n+1}) := [D_n \to D_n]$. Then $(D_\infty, \leq_\infty)$ is the cpo with

$D_\infty := \{\langle x_0, x_1, \dots \rangle \mid \forall n \in \omega (x_n \in D_n \;\&\; \varphi_n(x_{n+1}) = x_n\}$ for certain fixed projections $\varphi_n \in [D_{n+1} \to D_n]$

$\langle \vec{x} \rangle \leq_\infty \langle \vec{y} \rangle$ iff $\forall n \in \omega \; x_n \leq_n y_n$

$\sup F := \langle \sup \{x_n \mid x \in F\} \rangle_{n \in \omega}$ for directed $F \subseteq D_\infty$

$\perp_\infty := \langle \perp_n \rangle_{n \in \omega}$, where $\perp_n$ is the bottom of $(D_n, \leq_n)$.

Furthermore, there are projections $P_n \in [D_\infty \to D_\infty]$, such that $(P_n[D_\infty], \leq_\infty)$ is a sub-cpo of $(D_\infty, \leq_\infty)$, and isomorphisms $\text{fun}_n \in [P_n[D_\infty] \to D_n]$. We will abbreviate $P_n(x)$ with x_n. Then the following laws of projection hold in D_∞: $\forall x \in D_\infty \forall n \in \omega \forall m \in \omega$

(L1) $(x_n)_m = x_{\min\{n,m\}}$

(L2) $n \leq m \to x_n \leq_\infty x_m \leq_\infty x$

(L3) $x = \sup \{x_n \mid n \in \omega\}$

(L4) $\perp_n = \perp_\infty$

(L5) $(\sup F)_n = \sup \{x_n \mid x \in F\}$ for arbitrary subsets $F \subseteq D_\infty$ with existing sup's.

Moreover, a binary application operation $*_\infty$ is defined on D_∞ which satisfies the below laws of application: $\forall x \in D_\infty \forall y \in D_\infty \forall n \in \omega \forall m \in \omega$

(L6) $x_{n+1} *_\infty y = x_{n+1} *_\infty y_n = (x *_\infty y_n)_n$

(L7) $x_0 *_\infty y = x_0 = (x *_\infty \perp_\infty)_0$

(L8) $(\sup F) *_\infty x = \sup \{y *_\infty x \mid y \in F\}$ for arbitrary $F \subseteq D_\infty$ with existing sup's

(L9) $x*_\infty \sup F = \sup\{x*_\infty y \mid y \in F\}$ for directed $F \subseteq D_\infty$

(L10) $\forall z\ (y \leq_\infty z \rightarrow x*_\infty y \leq_\infty x*_\infty z \ \&\ y*_\infty x \leq_\infty z*_\infty x)$ (monotonicity)

(L11) $\forall z\ x*_\infty z \leq_\infty y*_\infty z \rightarrow x \leq_\infty y$ } (extensionality)

(L12) $\forall z\ x*_\infty z = y*_\infty z \rightarrow x = y$

(L13) $\mathrm{fun}_n(x_{n+1}*_\infty y_n) = \mathrm{fun}_{n+1}(x_{n+1})(\mathrm{fun}_n(y_n))$.

We turn our attention now to the particular D_∞-models which are generated by algebraic lattices.

2.6 DEFINITION. Let $(D, \leq')$ be a complete lattice. Then
(i) $x \in D$ is *compact* iff for every $X \subseteq D$ one has $x \leq' \sup X \rightarrow x \leq' \sup Y$ for some finite $Y \subseteq X$.
(ii) $C(D) := \{x \in D \mid x \text{ is compact}\}$
(iii) $(D, \leq')$ is *algebraic* iff for all $x \in D$ $x = \sup\{y \mid y \leq' x \ \&\ y \in C(D)\}$ □

The structure of an algebraic lattice is completely determined by the dense subset $C(D)$. Therefore we now characterize $C(D_\infty)$. It is easy to see that if $(D_0, \leq_0)$ is an algebraic lattice, then all $(D_n, \leq_n)$ and $(D_\infty, \leq_\infty)$ are algebraic lattices. In the sequel we assume $(D_0, \leq_0)$ to be an algebraic lattice.

2.7 PROPOSITION
(i) $\forall n \in \omega \forall m \in \omega\ (n \leq m \rightarrow C(P_n[D_\infty]) \subseteq C(P_m[D_\infty])$
(ii) $C(D_\infty) = \bigcup\{C(P_n[D_\infty]) \mid n \in \omega\}$

PROOF. Easy and left to the reader. □

2.8 PROPOSITION. For all $n \in \omega$ and for all $y, y' \in C(P_n[D_\infty])$ there is a unique $x_{y,y'} \in C(P_{n+1}[D_\infty])$ such that for all $z \in D_\infty$

$$x_{y,y'}*_\infty z = \begin{cases} y' \text{ if } y \leq_\infty z \\ \bot_\infty \text{ otherwise} \end{cases} \qquad (\dagger)$$

PROOF. Let $n \in \omega$ and $y, y' \in C(P_n[D_\infty])$ be arbitrary.
Then $\mathrm{fun}_n(y), \mathrm{fun}_n(y') \in C(D_n)$.

Define $f: D_n \rightarrow D_n$ by

$$f(x) := \begin{cases} \mathrm{fun}_n(y') \text{ if } \mathrm{fun}_n(y) \leq_n x \\ \mathrm{fun}_n(\bot_n) \text{ otherwise} \end{cases}$$

Then $f \in [D_n \rightarrow D_n] = D_{n+1}$ and $f \in C(D_{n+1})$:
Suppose $f \leq_{n+1} \sup F$ for some $F \subseteq D_{n+1}$. Then

$$\mathrm{fun}_n(y') = f(\mathrm{fun}_n(y)) \leq_n \sup\{g(\mathrm{fun}_n(y)) \mid g \in F\}.$$

So since $\mathrm{fun}_n(y') \in C(D_n)$ there is a finite $F_0 \subseteq F$ such that

$$\mathrm{fun}_n(y') \leq_n \sup \{g(\mathrm{fun}_n(y)) | g \in F_0\}.$$

Thus for all $z \in D_n$ $f(z) \leq \sup \{g(z) | g \in F_0\}$, i.e. $f \leq_{n+1} \sup F_0$.

Now define $x_{y,y'} := \mathrm{fun}_{n+1}^{-1}(f)$. Then $x_{y,y'} \in C(P_{n+1}[D_\infty])$ and (†) follows from L1, L2, L4, L6 and L13. The uniqueness of $x_{y,y'}$ is due to the extensionality of D_∞. □

2.9 PROPOSITION. For all $n \in \omega$ and all $x \in D_\infty$

$$x \in C(P_{n+1}[D_\infty]) \leftrightarrow \exists m \in \omega \forall i \leq m \exists y_i \in C(P_n[D_\infty]) \exists z_i \in C(P_n[D_\infty])$$

$$x = \sup \{x_{y_i, z_i} | i \leq m\}$$

PROOF. Let $x \in C(P_{n+1}[D_\infty])$ and define $X := \{x_{y,z} | y, z \in C(P_n[D_\infty])$ & & $z \leq_\infty x *_\infty y\}$. Then $X \subseteq P_{n+1}[D_\infty]$ and $x = \sup X$. Hence there is a finite $X_0 \subseteq X$ such that $x = \sup X_0$, i.e. there is a $m \in \omega$ and $y_0, \ldots, y_m, z_0, \ldots, z_m \in C(P_n[D_\infty])$ such that $x = \sup \{x_{y_i, z_i} | i \leq m\}$. □

Before we prove the characterization theorem we shall show

$$(P(G(A)), /\equiv, *) \cong (D_\infty, *_\infty)$$

with $(D_0, \leq_0) = (P(A), \subseteq)$. As a first step in that direction we isolate a certain subset of D_∞ which corresponds to the set of 'elementary instructions' $G(A)$.

2.10 DEFINITION. Let $\mathrm{Elem}(D_\infty) := \bigcup \{\mathrm{Elem}_n(D_\infty) | n \in \omega\}$ where $\mathrm{Elem}_n(D_\infty)$ is recursively defined by

(i) $\mathrm{Elem}_0(D_\infty) := \{\mathrm{fun}_0^{-1}(\{a\}) | a \in A\}$

(ii) $\mathrm{Elem}_{n+1}(D_\infty) := \mathrm{Elem}_n(D_\infty) \cup \{x_{y,z} | \exists$ finite $X \subseteq \mathrm{Elem}_n(D_\infty)$ $y = \sup X$ & & $z \in \mathrm{Elem}_n(D_\infty)\}$ □

2.11 PROPOSITION

(i) $\forall x \in \mathrm{Elem}(D_\infty) \forall X \subseteq \mathrm{Elem}(D_\infty)$ $(x \leq_\infty \sup X \rightarrow \exists x_0 \in X\ x \leq_\infty x_0)$

(ii) $\forall x \in C(D_\infty) \exists$ finite $X \subseteq \mathrm{Elem}(D_\infty)$ $x = \sup X$

(iii) $\forall x \in D_\infty$ $x = \sup \{y \in \mathrm{Elem}(D_\infty) | y \leq_\infty x\}$

PROOF. (i) With induction on $x \in \mathrm{Elem}_n(D_\infty)$. Let $x \in \mathrm{Elem}_0(D_\infty)$ and suppose $x \leq_\infty \sup X$ with $X \subseteq \mathrm{Elem}(D_\infty)$. Then for some $a \in A$

$$x = \mathrm{fun}_0^{-1}(\{a\}) \leq_\infty \sup X.$$

Hence $\mathrm{fun}_0^{-1}(\{a\}) \leq_\infty (\sup X)_0 = \sup \{x_0 | x \in X\}$ by L5. Thus

$$\{a\} \subseteq \bigcup \{\mathrm{fun}_0(x_0) | x \in X\},$$

i.e. $\{a\} \subseteq \text{fun}_0(x_0)$ for some $x \in X$. Hence $\text{fun}_0^{-1}(\{a\}) \leq_\infty x_0 \leq_\infty x$ by L2. Let $x_{y,z} \in \text{Elem}_{n+1}(D_\infty)$ and suppose $x_{y,z} \leq_\infty \sup X$ for $X \subseteq \text{Elem}(D_\infty)$. Then

$$z = x_{y,z} *_\infty y \leq_\infty \sup X *_\infty y = \sup \{x *_\infty y | x \in X\} =$$

$$= \sup \{\text{fun}_0^{-1}(\{a\}), z | \text{fun}_0^{-1}(\{a\}) \in X \text{ or } \exists y' \leq_\infty y \; x_{y',z} \in X\}$$

$$\subseteq \text{Elem}(D_\infty).$$

Thus from the induction hypothesis it follows that $z \leq_\infty \text{fun}_0^{-1}(\{a\}) \in X$ or $z \leq_\infty z'$ for some $x_{y',z'} \in X$ with $y' \leq_\infty y$. Then it follows from the monotonicity of $*_\infty$ that $x_{y,z} \leq_\infty x$ for some $x \in X$. (ii) With induction on $x \in C(P_n[D_\infty])$. If $x \in C(P_0[D_\infty])$ then for some finite $B \subseteq A$ $x = \sup \{\text{fun}_0^{-1}(\{a\}) | a \in B\}$. Let $x \in C(P_{n+1}[D_\infty])$. Then by proposition 2.9 $x = \sup \{x_{y_i, z_i} | i \leq m\}$ for certain $m \in \omega$ and $y_i, z_i \in C(P_n[D_\infty])$. Thus from the induction hypothesis it follows that $y_i = \sup Y_i$, $z_i = \sup Z_i$ with finite $Y_i, Z_i \subseteq \text{Elem}(D_\infty)$. So

$$x = \sup \{x_{y_i, z_i} | i \leq m\} = \sup \{x_{y_i, z} | z \in Z_i \;\&\; i \leq m\}$$

and

$$\{x_{y_i, z} | z \in Z_i \;\&\; i \leq m\} \subseteq \text{Elem}(D_\infty).$$

(iii) follows from (ii) and the algebraic nature of $(D_\infty, \leq_\infty)$. □

2.12 DEFINITION. For $b \in G(A)$ define inductively
(i) $\varphi(a) := \text{fun}_0^{-1}(\{a\})$ if $a \in A$
(ii) $\varphi(B, b) := x_{\sup \{\varphi(b) | b \in B\}, \varphi(b)}$ □

2.13 LEMMA
(i) $\forall b \in G(A) \; \varphi(b) \in \text{Elem}(D_\infty)$
(ii) $\forall b \in G(A) \forall c \in G(A) \; (b \leq_{G(A)} c \leftrightarrow \varphi(b) \leq_{G(A)} \varphi(c))$
(iii) $\forall x \in \text{Elem}(D_\infty) \exists b \in G(A) \; \varphi(b) = x$

PROOF. (i) and (iii) follow immediately from definition 2.12. For (ii) we prove with induction on $n: \forall n \in \omega \forall \{b,c\} \subseteq P(G_n(A))(b \leq_{G(A)} c \leftrightarrow \varphi(b) \leq_\infty \varphi(c))$. For $n = 0$ this is trivial. Let $\{b,c\} \subseteq P(G_{n+1}(A))$. Define

$$S(u) := \begin{cases} (\emptyset, u) \text{ if } u \in A \\ u \quad \text{otherwise} \end{cases}$$

Then if $u \in A$ $\varphi(u) = \text{fun}_0^{-1}(\{u\})$ and $\varphi(S(u)) = x_{\perp_\infty, \text{fun}_0^{-1}(\{u\})}$. So from proposition 2.8 and L7 it follows that $\varphi(u) = \varphi(S(u))$ for all $u \in G(A)$. Suppose $b \leq_{G(A)} c$. Then $S(b) \leq_{G(A)} S(c)$. So there are $D_i \subseteq G_n(A)$ and $d_i \in G_n(A)$ for $i = 1,2$ such that $S(b) = (D_1, d_1)$, $S(c) = (D_2, d_2)$, $D_2 \leq D_1$ and $d_1 \leq_{G(A)} d_2$. Then we get from the induction hypothesis $\sup \{\varphi(d) | d \in D_2\} \leq_\infty \sup \{\varphi(d) | d \in D_1\}$ and $\varphi(d_1) \leq_\infty \varphi(d_2)$. Hence

$$\varphi(b) = \varphi(S(b)) = x_{\sup \{\varphi(d) | d \in D_1\}, \varphi(d_1)} \leq_\infty x_{\sup \{\varphi(d) | d \in D_2\}, \varphi(d_2)} = \varphi(S(c)) = \varphi(c).$$

Suppose $\varphi(b) \leq_\infty \varphi(c)$. Then $\varphi(D_1, d_1) \leq_\infty \varphi(D_2, d_2)$ with $S(b) = (D_1, d_1)$, $S(c) = (D_2, d_2)$, $D_i \subseteq G_n(A)$ and $d_i \in G_n(A)$ for $i = 1, 2$. So

$$\varphi(d_1) = \varphi(S(b)) *_\infty \sup \{\varphi(d) | d \in D_1\} \leq_\infty \varphi(S(c)) *_\infty \sup \{\varphi(d) | d \in D_1\}.$$

Thus $d_1 \leq_\infty d_2$ and $D_2 \leq D_1$ by the induction hypothesis and proposition 2.11(i). □

Next we prove

2.14 THEOREM. $\forall A\ M(A) \cong (D_\infty, *_\infty, K_\infty, S_\infty)$ with $(D_0, \leq_0) = (P(A), \subseteq)$.

PROOF. Define $\theta: P(G(A))/\equiv \to D_\infty$ by $\theta([X]) := \sup \{\varphi(b) | b \in X\}$. Then θ is a monotone bijection by lemma 2.12 and proposition 2.11(iii). Moreover for all $[X], [Y] \in P(G(A))/\equiv$

$$\theta([X][Y]) = \theta([XY]) = \sup \{\varphi(b) | b \in XY\} =$$
$$= \sup \{\varphi(b) | b \in \{b' | \exists B \leq Y (B, b) \in X\} \cup \{a \in A | a \in X\}\} =$$
$$\sup \{\text{fun}_0^{-1}(\{a\}), z | (a \in A \;\&\; a \in X) \text{ or}$$
$$\exists [B] \leq [Y] \exists b\ (\varphi(b) = z \;\&\; (B, b) \in X)\} =$$
$$= \sup \{\text{fun}_0^{-1}(\{a\}), z | (a \in A \;\&\; a \in X) \text{ or}$$
$$\exists y \leq_\infty \theta([Y])\ x_{y,z} \in \{\varphi(b) | b \in X\}\} = \sup \{\varphi(b) *_\infty \theta([Y]) | b \in X\} =$$
$$= \sup \{\varphi(b) | b \in X\} *_\infty \theta([Y]) = \theta([X]) *_\infty \theta([Y]).$$

Finally, since D_∞ is extensional $\theta([K]) = K_\infty$ and $\theta([S]) = S_\infty$. □

Now we are ready to prove

2.15 CHARACTERIZATION THEOREM. For all $A \neq \emptyset$ and for all cpo's $(D_0 \leq_0)$

$$M(A) \cong (D_\infty, *_\infty, K_\infty, S_\infty) \leftrightarrow (P(A), \subseteq) \cong (D_0, \leq_0).$$

PROOF. $\to$: Suppose $M(A) \cong (D_\infty, *_\infty, K_\infty, S_\infty)$ via some bijection

$$\theta: P(G(A))/\equiv \to D_\infty$$

such that for all $[X], [Y] \in P(G(A))/\equiv$ $\theta([X][Y]) = \theta([X]) *_\infty \theta([Y])$. Then by lemma 2.4 for all $X \in P(A)$ and $x \in D_\infty$ $\theta([X]) *_\infty x = \theta([X])$. Since in D_∞ we have

$$\forall x \in D_\infty (x = x_0 \leftrightarrow \forall y \in D_\infty x *_\infty y = x)$$

(cf. [8], 18.4.18) we get $\forall X \in P(A)$ $\theta([X]) = (\theta([X]))_0$. Thus $\theta': P(A) \to D_0$ defined by $\theta'(X) := \text{fun}_0(\theta([X]))$ is a bijection. Now, since $A \neq \emptyset$ $\text{Card}(D_0) \geq 2$. Choose $d_0 \in D_0$ with $d_0 \neq \bot_0$ and define $f: D_0 \to D_0$ by

$$f(x) := \begin{cases} d_0 & \text{if } x \neq \bot_0 \\ \bot_0 & \text{otherwise} \end{cases}$$

Then $f \in D_1$. Hence there is an $y \in D_\infty$ such that for all $z \in D_\infty$ we have

$$y *_\infty z = \begin{cases} \mathrm{fun}_0^{-1}(d_0) & \text{if } z \neq \bot_\infty \\ \bot_\infty & \text{otherwise.} \end{cases}$$

Thus by lemma 2.3 $\theta^{-1}(\bot_\infty) = [\emptyset]$ or $\theta^{-1}(\bot_\infty) = [A]$.

Assume $\theta([A]) = \bot_\infty$. Then from lemma 2.2 it follows that for all $x, y \in D_\infty$ $\theta(\mathrm{INF}_A)x\bot_\infty = x = \theta(\mathrm{INF}_A)xx$ and $\theta(\mathrm{INF}_A)xy = \theta(\mathrm{INF}_A)yx$. Then it is easy to see that for all $x, y \in D_\infty \theta(\mathrm{INF}_A)xy = \sup\{x, y\}$. So for all $[X], [Y] \in P(A)$ we have

$$X \subseteq Y \leftrightarrow \mathrm{INF}_A[X][Y] = [X] \leftrightarrow \theta(\mathrm{INF}_A)\theta([X])\theta([Y]) = \theta([X]) \leftrightarrow$$
$$\leftrightarrow \theta([Y]) \leq_\infty \theta([X]) \leftrightarrow \theta'(Y) \leq_0 \theta'(X).$$

Hence $(P(A), \subseteq) \cong (D_0, \leq_0)$ via $\theta'': P(A) \to D_0$ defined by $\theta''(X) = \theta'(A \setminus X)$.

Assume $\theta(\emptyset) = \bot_\infty$. Then similarly we see that $\theta(\mathrm{SUP}_A)xy = \sup\{x, y\}$. So for all $X, Y \in P(A)$ we have

$$X \subseteq Y \leftrightarrow \mathrm{SUP}_A[X][Y] = [Y] \leftrightarrow \theta(\mathrm{SUP}_A)\theta([X])\theta([Y]) = \theta([Y]) \leftrightarrow$$
$$\leftrightarrow \theta([X]) \leq_\infty \theta([Y]) \leftrightarrow \theta'(X) \leq_0 \theta'(Y).$$

Thus again $(P(A), \subseteq) \cong (D_0, \leq_0)$. $\leftarrow$: If $(P(A), \subseteq) \cong (D_0, \leq_0)$ then clearly $(D'_\infty, *_\infty, K_\infty, S_\infty) \cong (D_\infty, *_\infty, K_\infty, S_\infty)$ with $(D'_0, \leq_0) = (P(A), \subseteq)$. Hence from theorem 2.12 it follows that $M(A) \cong (D_\infty, *_\infty, K_\infty, S_\infty)$. □

3. EXTENSIONAL SUBSTRUCTURES OF $(P(G(A)), \bullet)$ AND $P\omega$

As already mentioned in the introduction Engeler's graph algebra $(P(G(A)), \bullet)$ is never extensional. However, there is always a substructure which can be made into an extensional *ca*, provided $A \neq \emptyset$. We will show this by embedding $(P(G(A))/\equiv, *)$ isomorphically into $(P(G(A)), \bullet)$.

3.1 THEOREM. $\forall A\ (P(G(A))/\equiv, *) \hookrightarrow (P(G(A)), \bullet)$.

PROOF. Define $\theta: P(G(A))/\equiv \to P(G(A))$ by $\theta([X]) := \bigcup\{Z \mid Z \equiv X\}$ and observe that $\bigcup\{Z \mid Z \equiv X\} \equiv X$. Then θ is an injection. Moreover,

$$\theta([X]) \bullet \theta([Y]) = \{b \mid \exists B \subseteq \theta([Y])\ (B, b) \in \theta([X])\} =$$
$$= \{b \mid \exists B \subseteq \bigcup\{Z \mid Z \equiv Y\}\ (B, b) \in \bigcup\{Z \mid Z \equiv X\}\} =$$
$$= \{b \mid \exists B \leq Y\ \{(B, b)\} \leq X\} = \bigcup\{Z \mid Z \equiv XY\} = \theta([XY]) = \theta([X][Y]). \ \square$$

As known from the literature the $P\omega^{c,e}$-models are non-extensional *ca*'s, whose structures depend on the specific codings c of pairs of natural numbers and e of finite subsets of ω used in the construction. Given two bijections $c: \omega^2 \to \omega$, $e: \omega \to \{X \in P(\omega) \mid X \text{ finite}\}$ $P\omega^{c,e}$ is the model $(P(\omega), \blacksquare)$ with the application on $P(\omega)$ defined by

$$X \blacksquare Y := \{m \in \omega \mid \exists n \in \omega (e(n) \subseteq Y \ \&\ c(n, m) \in X)\}.$$

In [S3] Scott presents a very elegant method to construct extensional substructures of the $P\omega^{c,e}$-models. Here we will give a more elementary technique by embedding $(P(G(A_{c,e}))/\equiv, *)$ isomorphically into $P\omega^{c,e}$ for a certain set $A_{c,e}$. However, for rather 'nice' codings only this technique will yield non-trivial extensional substructures.

3.2 DEFINITION. Let $c:\omega^2\to\omega$, $e:\omega\to\{X\in P(\omega)|X \text{ finite}\}$ be arbitrary.
(i) Define $A_{c,e}:=\{n\in\omega|c(e^{-1}(\emptyset),n)=n\}$
(ii) For $b\in G(A_{c,e})$ define inductively

$$\varphi(n):=n \text{ if } n\in A_{c,e}$$

$$\varphi(B,b):=c(e^{-1}(\{\varphi(b)|b\in B\}),\varphi(b))$$

(iii) Define $\theta:P(G(A_{c,e}))/\equiv\to\omega$ by $\theta([X]):=\{\varphi(b)|b\in\bigcup\{Z|Z\equiv X\}\}$ □

3.3 LEMMA. $\forall[X]\in P(G(A_{c,e}))/\equiv\ \forall[Y]\in P(G(A_{c,e}))/\equiv$
(i) $\theta([X])=\theta([Y])\leftrightarrow[X]=[Y]$
(ii) $\theta([X][Y])=\theta([X])\blacksquare\theta([Y])$

PROOF. (i) $\leftarrow$ is trivial. For $\rightarrow$, we prove with induction on n

$$\forall n\in\omega\forall b\in G(A_{c,e})\forall b'\in G(A_{c,e})\ \varphi(b)=\varphi(b')\to b\leq_{G(A_{c,e})}b' \ \&\ b'\leq_{G(A_{c,e})}b.$$

Then if $\theta([X])=\theta([Y])$, we have $X\equiv Y$ and thus $[X]=[Y]$. Clearly, this holds for $n=0$. Let $b,b'\in G_{n+1}(A_{c,e})$ and define

$$S(u):=\begin{cases}(\emptyset,u) & \text{if } u\in A_{c,e}\\ u & \text{otherwise.}\end{cases}$$

Then if $u\in A_{c,e}$ it follows from definition 3.2 that $\varphi(S(u))=\varphi(u)$. Suppose $\varphi(b)=\varphi(b')$. Then also $\varphi(S(b))=\varphi(S(b'))$. Hence $c(e^{-1}(\{\varphi(b)|b\in D_1\}),\varphi(d_1))=$ $=c(e^{-1}(\{\varphi(b)|b\in D_2\}),\varphi(d_2))$ where $S(b)=(D_1,d_1)$ and $S(b')=(D_2,d_2)$. So $\{\varphi(b)|b\in D_1\}=\{\varphi(b)|b\in D_2\}$ and $\varphi(d_1)=\varphi(d_2)$ and from the induction hypothesis it follows that $D_1\leq D_2$ and $d_1\leq_{G(A_{c,e})}d_2, d_2\leq_{G(A_{c,e})}d_1$. Hence

$$S(b)\leq_{G(A_{c',e})}S(b') \text{ and } S(b')\leq_{G(A_{c',e})}S(b).$$

Thus

$$b\leq_{G(A_{c,e})}b' \text{ and } b'\leq_{G(A_{c,e})}b.$$

(ii) By the proof of theorem 3.1 we have

$$\bigcup\{Z|Z\equiv XY\}=\bigcup\{Z|Z\equiv X\}\bullet\bigcup\{Z|Z\equiv Y\}=$$
$$=\{b|\exists B\subseteq\bigcup\{Z|Z\equiv Y\}(B,b)\in\bigcup\{Z|Z\equiv X\}\}.$$

Hence

$$\theta([X][Y]) = \theta([XY]) = \{\varphi(b) | b \in \bigcup\{Z | Z \equiv XY\}\} =$$
$$= \{\varphi(b) | \exists B \subseteq \bigcup\{Z | Z \equiv Y\}(B,b) \in \bigcup\{Z | Z \equiv X\}\} = \{\varphi(b) | \exists \text{ finite}$$
$$B \subseteq P(\omega)(B \subseteq \theta([Y]) \ \& \ c(e^{-1}(B), \varphi(b)) \in \theta([X]))\} =$$
$$= \{m \in \omega | \exists n \in \omega (e(n) \subseteq \theta([Y]) \ \& \ c(n,m) \in \theta([X]))\} = \theta([X]) \blacksquare \theta([Y]).$$

□

3.4 THEOREM. For all bijections $c:\omega^2 \to \omega$, $e:\omega \to \{X \in P(\omega) | X \text{ finite}\}$

$$(P(G(A_{c,e}))/\equiv, *) \hookrightarrow P\omega^{c,e}. \quad \square$$

Thus if $A_{c,e} \neq \emptyset$, i.e. if for some $n \in \omega$ $c(e^{-1}(\emptyset), n) = n$ then $P\omega^{c,e}$ has a substructure which can be made into an extensional *ca*.

EXAMPLES. Let e be the standard coding of finite subsets of ω defined by

$$e(n) = \{k_0, \ldots, k_{m-1}\} \text{ with } k_0 < \ldots < k_{m-1} \leftrightarrow n = 2^{k_0} + \ldots + 2^{k_{m-1}}$$

Then $e(\emptyset) = 0$. Consider the two codings of pairs c and c' given by

$$c(n,m) = \tfrac{1}{2}(n+m)(n+m+1) + m$$
$$c'(n,m) = \tfrac{1}{2}(n+m)(n+m+1) + n$$

Then $A_{c,e} = \{0\}$ and $A_{c',e} = \{0,1\}$. By theorem 2.5 $M(A_{c,e}) \not\cong M(A_{c',e})$. Hence $P\omega^{c,e}$ and $P\omega^{c',e}$ contain non-isomorphic extensional *ca*'s.

Chapter 4

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ON THE EXISTENCE OF EXTENSIONAL PARTIAL COMBINATORY ALGEBRAS

Abstract. The principal aim of this paper is to present a construction method for nontotal extensional combinatory algebras. This is done in §2. In §0 we give definitions of some basic notions for partial combinatory algebras from which the corresponding notions for (total) combinatory algebras are obtained as specializations. In §1 we discuss some properties of nontotal extensional combinatory algebras in general. §2 describes a "partial" variant of reflexive complete partial orders yielding nontotal extensional combinatory algebras. Finally, §3 deals with properties of the models constructed in §2, such as incompletability, having no total submodel and the pathological behaviour with respect to the interpretation of unsolvable λ-terms.

§0. Introduction. Extensional combinatory algebras play an important role in the semantics of the λ-calculus. They form a proper subclass of the so-called *partial applicative structures* (pas). A pas is an untyped structure, where the objects may be thought of as operations which can be applied to each other, though the result of an application need not exist. In such a structure self-application is meaningful, but is not necessarily everywhere defined.

0.1. DEFINITION. (i) $M = (A, *)$ is a *partial applicative structure* iff $*$ is a binary operation on A, called *application*, which may be partial.

(ii) A pas M is *total* iff $a * a'$ is defined in M for all $a, a' \in A$. □

The pas's are exactly the models of the theory **LPT** (logic of partial terms) as described in [Be]. The language of **LPT** consists of a fixed countably infinite set Vars of variables and a single binary operator **AP** for term formation, which however will never appear explicitly: we let $x, y, z, x_0, x_1, \ldots$ range over Vars and instead of $\mathbf{AP}(x, y)$ we just write xy and employ the usual convention of association to the left. Moreover, there is a predicate constant $=$ for equality and the symbol (not a predicate) $\downarrow$ for being defined. The rules for the formation of terms and formulae are as usual, except there is an additional rule: if t is a term, then $t\downarrow$ is an atomic formula.

The propositional axioms and rules of inference are those of first-order predicate calculus. The quantifier rules and axioms are as follows:

$$\frac{\varphi \to \psi}{\varphi \to \forall x\, \psi}, \qquad \frac{\psi \to \varphi}{\exists x\, \psi \to \varphi} \qquad (x \text{ not free in } \varphi);$$

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$$\forall x\,\varphi \;\&\; t\downarrow \to \varphi[x := t],$$
$$\varphi[x := t] \;\&\; t\downarrow \to \exists x\,\varphi.$$

For the equality axioms we introduce the abbreviation $t \simeq s$ for $(t\downarrow \vee s\downarrow \to t = s)$:

$$x = x \;\&\; (x = y \to y = x), \qquad t \simeq s \;\&\; \varphi[x := t] \to \varphi[x := s].$$

In **LPT** = and **AP** are assumed to be strict (STR):

$$t = s \to t\downarrow \;\&\; s\downarrow, \qquad ts\downarrow \to t\downarrow \;\&\; s\downarrow.$$

Finally, variables are assumed to be defined: $x\downarrow$ for every $x \in$ Vars.

Throughout the above list of logical axioms and rules of **LPT**, t and s are arbitrary terms, whereas x and y are variables. Moreover, $\varphi[x := t]$ is the formula obtained from φ by simultaneously replacing every free occurrence of x in φ by t.

The semantics of **LPT** are clearly partial structures M consisting of a set A and a partial binary operation $*$ interpreting **AP**. In formalizing logic with a binary partial operation, i.e. logic permitting the formation of terms which do not necessarily denote anything, one has to be careful. In particular the satisfaction relation $M \models \varphi$ between partial structures on the one hand and formulas φ of **LPT** on the other needs careful attention. In the literature concerning this subject there seems to be both an assumption that the definition is too obvious to need stating and a disagreement about what the definition should be. To avoid misunderstanding we emphasize that throughout this paper we use Definition 0.3.

Before we define satisfaction of a formula φ of **LPT** for such a partial structure M, it is necessary to make clear what it means for the interpretation of a term t under a given assignment ϱ of values to the variables to be defined in M.

0.2. Definition. Let $M = (A, *)$ and ϱ: Vars $\to A$ be arbitrary.

(i) The set of terms over a pas M, denoted $T(M)$, is inductively defined by

$$x \in \text{Vars} \to x \in T(M),$$
$$a \in A \to a \in T(M),$$
$$t, t' \in T(M) \to tt' \in T(M).$$

(ii) For $t \in T(M)$, $[t]^M_\varrho$ is defined inductively by

$$[x]^M_\varrho := \varrho(x) \quad \text{for every } x \in \text{Vars},$$
$$[a]^M_\varrho := a \quad \text{for every } a \in A,$$
$$[tt']^M_\varrho := \begin{cases} [t]^M_\varrho * [t']^M_\varrho & \text{if } [t]^M_\varrho, [t']^M_\varrho \text{ and } [t]^M_\varrho * [t']^M_\varrho \text{ are defined,} \\ \text{undefined} & \text{otherwise.} \quad \square \end{cases}$$

We now define the notion of satisfaction of a formula φ of **LPT**:

0.3. Definition. Let $M = (A, *)$ be arbitrary.

(i) For arbitrary assignments ϱ $M, \varrho \models \varphi$ is defined inductively as follows:

(1) $M, \varrho \models t\downarrow$ iff $[t]^M_\varrho$ is defined in M,

(2) $M, \varrho \models t = t'$ iff both $[t]^M_\varrho$ and $[t']^M_\varrho$ are defined in M and are equal.

The clauses for the connectives and negation are as usual, but bound variables refer to denoting terms only, i.e.

(3) $M, \varrho \models \exists x\, \varphi$ iff there is an $a \in A$ such that $M, \varrho(x := a) \models \varphi$.

(4) $M, \varrho \models \forall x\, \varphi$ iff for all $a \in A, M, \varrho(x := a) \models \varphi$.

Here $\varrho(x := a)$ is the assignment defined by

$$\varrho(x := a)(y) = a \quad \text{if } y \equiv x, \qquad \varrho(x := a)(y) = \varrho(y) \quad \text{if } y \not\equiv x,$$

where $\equiv$ denotes syntactic identity.

(ii) $M \models \varphi$ iff for all assignments ϱ, $M, \varrho \models \varphi$. □

As such, pas's are not our main interest. We shall restrict our attention to a proper subclass. The property that distinguishes the members of this subclass from other pas's is the property of combinatory completeness, which states that every algebraic function, i.e. every function definable by a term, is representable by an element. For a total applicative structure M combinatory completeness is usually defined by

cc′: For every $t \in T(M)$ with variables among $\vec{x}$, $M \models \exists y \forall \vec{x}(y\vec{x} = t)$.

One then has the combinatory completeness theorem, proved by Schönfinkel [S], which states that combinatory completeness follows already from two of its instances, namely the well-known axioms for the combinators K and S. So *M is cc′ iff there are K, $S \in A$ such that $M \models AK'$ & AS'*, where $AK': \forall x \forall y(Kxy = x)$ and $AS': \forall x \forall y \forall z(Sxyz = xz(yz))$. It is tempting to translate cc′ and the axioms AK' and AS' into the partial context by simply replacing $=$ by $\simeq$. However, this translation does not preserve the validity of the above theorem, since there are combinatorial incomplete pas's satisfying $\forall x \forall y(Kxy \simeq x)$ and $\forall x \forall y \forall z(Sxyz \simeq xz(yz))$. We have relegated the proof of this pathology to an appendix. To get the desired algebraic characterization of combinatory completeness one has in fact to strengthen AS' into

$$\forall x \forall y \forall z(Sxy\downarrow \;\&\; Sxyz \simeq xz(yz)).$$

But then cc′ also needs revision in the following way:

0.4. Definition. (i) A pas M is called *combinatory complete* (cc) iff for every sequence $x_0, \ldots, x_n$ and every $t \in T(M)$ with $\mathrm{Vars}(t) \subseteq \{x_0, \ldots, x_n\}$

$$M \models \exists y \forall x_0 \cdots \forall x_n(yx_0 \cdots x_{n-1}\downarrow \;\&\; yx_0 \cdots x_n \simeq t).$$

Here $\mathrm{Vars}(t)$ is the set of variables in t.

(ii) A *partial combinatory algebra* (pca) is a structure $M = (A, *, K, S)$ with $(A, *)$ a pas and K, $S \in A$ such that $M \models AK$ & AS, where

$$AK: \forall x \forall y(Kxy \simeq x),^1 \qquad AS: \forall x \forall y \forall z(Sxy\downarrow \;\&\; Sxyz \simeq xz(yz)). \quad \square$$

Clearly, for total pas's cc and cc′ are equivalent, and so are the notions of combinatory algebra (cf. [B, p. 90]) and partial combinatory algebra.

The strength of a pca is that in it one can simulate the operation of λ-abstraction by using only the two combinators K and S.

[1] We leave it to the reader to verify that AK is in fact equivalent to AK' for pca's.

0.5. Definition. Let M be a pca. Define for all variables x a map $\langle x\rangle: T(M) \to T(M)$ inductively by

$$\langle x\rangle x := SKK,$$
$$\langle x\rangle t := Kt \quad \text{if } t \equiv a \in A \text{ or } t \equiv y \text{ for some variable } y \not\equiv x,$$
$$\langle x\rangle(tt') := S(\langle x\rangle t)(\langle x\rangle t'). \quad \square$$

0.6. Proposition. *Let $M = (A, *, K, S)$ be a pca. Then*
(i) $M \models \forall x(Kx\downarrow \,\&\, Sx\downarrow)$.
Moreover, for all variables x and $t \in T(M)$
(ii) $\text{Vars}(\langle x\rangle t) = \text{Vars}(t)\setminus\{x\}$,
(iii) $M \models \langle x\rangle t\downarrow$,
(iv) $M \models (\langle x\rangle t)x \simeq t$.

Proof. (i) Let $a \in A$. Then $M \models Kaa \simeq a$ by AK and so $M \models Kaa\downarrow$. From AS we get $M \models Saa\downarrow$. Thus STR yields $M \models Ka\downarrow \,\&\, Sa\downarrow$. (ii)–(iv) are proved by simultaneous induction on the complexity of t, using (i), AK and AS. $\square$

In total pca's we usually define $\langle x\rangle t := Kt$, if $x \notin \text{Vars}(t)$. In the partial context, however, we have to make the above modification, since otherwise we cannot prove 0.6(iii). As a consequence $\langle x\rangle$ is less well-behaved with respect to substitution. That is, for arbitrary t, t' with $x \notin \text{Var}(tt')$ we do not have $(\langle x\rangle t)[y := t'] \equiv \langle x\rangle(t[y := t'])$. For example if $t \equiv y$, then $(\langle x\rangle t)[y := t't''] \equiv K(t't'')$, but $\langle x\rangle(t[y := t't'']) \equiv S(\langle x\rangle t')(\langle x\rangle t'')$.

0.7. Theorem. *Let M be a pas. Then M is cc iff M is a pca.*

Proof. $\to$ is trivial.

$\leftarrow$. By induction on the length of the sequence $x_0, \ldots x_n$. For $n = 0$ let t be such that $\text{Vars}(t) \subseteq \{x_0\}$. Then by 0.6(ii) $\text{Vars}(\langle x_0\rangle t) = \varnothing$. Apply both (iii) and (iv) of Proposition 0.6 to x_0, t. Then

$$M \models \langle x_0\rangle t\downarrow \,\&\, \forall x_0((\langle x_0\rangle t)x_0 \simeq t).$$

Hence $M \models \exists y \forall x_0(yx_0 \simeq t)$. Suppose $\text{Vars}(t) \subseteq \{x_0, \ldots, x_{n+1}\}$. By 0.6(ii)

$$\text{Vars}(\langle x_{n+1}\rangle t) \subseteq \{x_0, \ldots, x_n\}.$$

From the induction hypothesis it then follows that

$$M \models \exists y \forall x_0 \cdots \forall x_n(yx_0 \cdots x_n \simeq \langle x_{n+1}\rangle t).$$

Combining 0.6(iii) and (iv) yields

$$M \models \exists y \forall x_0 \cdots \forall x_{n+1}(yx_0 \cdots x_n\downarrow \,\&\, yx_0 \cdots x_{n+1} \simeq t). \quad \square$$

0.8. Definition. A pca is *extensional* iff $M \models \text{EXT}$, where EXT

$$\forall x \forall y(\forall z(xz \simeq yz) \to x = y). \quad \square$$

Recall that by Definition 0.3 we quantify over denoting terms only. So EXT is the principle that every denoting term uniquely represents a function. Clearly for total pca's M, EXT is equivalent to

† $\qquad$ for all $t, t' \in T(M), M \models \forall x(tx \simeq t'x) \to t \simeq t'$.

But this does not hold for nontotal pca's. In fact † is incompatible with nontotality in pca's. Thus suppose M is a pca with † and $M \models \exists x \exists y \neg(xy\downarrow)$. Choose $a, a' \in A$ such that aa' is not defined in M. Then by the combinatory completeness of M there is an $a'' \in A$ with $M \models \forall x(a''x \simeq aa'x)$. So from † it follows that $M \models a'' \simeq aa'$, contradicting the assumption that aa' is not defined in M.

Total extensional pca's are well known. The first model was D_∞ constructed by Scott [S1] as a projective limit of complete lattices. But what about nontotal extensional pca's? Before we show that they do exist, we shall discuss some of their properties.

§1. Some properties of nontotal extensional pca's.

1.1. DEFINITION. A pca M is called *nontrivial* iff $M \models \exists x \exists y(x \neq y)$. □

1.2. PROPOSITION. *Let M be a nontotal pca. Then*

(i) $M \models \exists y \forall x \neg(yx\downarrow)$,

(ii) $M \models S \neq K$, *and*

(iii) *M is nontrivial.*

PROOF. Clearly (iii) follows from (ii).

(i) Let $a, a' \in A$ be such that aa' is not defined in M. By (iii) and (iv) of Proposition 0.6 one has $M \models \langle x\rangle(aa')\downarrow \;\&\; \forall x((\langle x\rangle(aa'))(x) \simeq aa')$. Hence $M \models \exists y \forall x \neg(yx\downarrow)$.

(ii) Suppose $M \models S = K$. Define $I := \langle x\rangle x$ and let $a \in A$ be such that $M \models \forall x \neg(ax\downarrow)$. By AK and AS,

$$M \models a \simeq S(KI)(Ka)a \simeq K(KI)(Ka)a \simeq I.$$

Thus $M \models aa \simeq Ia \simeq a$, i.e. $M \models aa\downarrow$. Contradiction. □

For extensional nontotal pca's M the nowhere-defined function in Proposition 1.2(i) is uniquely represented by an element of A. Henceforth we will denote this element $\perp_M$.

1.3. DEFINITION. (i) A pca $M = (A, *, K, S)$ is *completable* iff there is a total pca $M' = (A', *', K', S')$ and an injection $\varphi: A \to A'$ such that $\varphi(K) = K'$, $\varphi(S) = S'$ and

$$\forall a \in A \forall a' \in A(M \models aa'\downarrow \to M' \models \varphi(aa') = \varphi(a)\varphi(a')).$$

(ii) A pca M has *unique head normal forms* iff for all a, $a' \in A$ the elements K, S, Ka, Sa, Saa' are pairwise distinct and $M \models$ BA, where

BA $$\forall x \forall x' \forall y \forall y'(Sxy = Sx'y' \to x = x' \;\&\; y = y'). \quad \square$$

In [K] Klop showed that having unique head normal forms is a sufficient condition for pca's to be completable. For example Kleene's recursion-theoretic pca, where application is defined by $m * n := \{m\}(n)$, has unique head normal forms and can thus be completed. However, this theorem does not increase our knowledge about the completability of nontotal extensional pca's: Barendregt's axiom BA is incompatible with extensionality in nontotal extensional pca's.

1.4. PROPOSITION. *Let M be a nontotal extensional pca. Then $M \not\models BA$.*

PROOF. $S\perp_M K = SK\perp_M$, since $S\perp_M K$ and $SK\perp_M$ are both nowhere defined. But clearly $\perp_M \neq K$. □

In fact we have

1.5. THEOREM. *Nontotal extensional pca's are not completable.*

PROOF. Let $M = (A, *, K, S)$ be a nontotal extensional pca and suppose $M' = (A', *', K', S')$ is a completion of M via some injective homomorphism φ. Define $a := K(KK)$, $a' := K(KS)$. Then $M \models Sa\bot_M = Sa'\bot_{M'}$, since M is extensional and both $Sa\bot_{M'}$ and $Sa'\bot_M$ are nowhere defined. So $M' \models S'\varphi(a)\varphi(\bot_M) = S'\varphi(a')\varphi(\bot_M)$. Then

$$M' \models K' = S'\varphi(a)\varphi(\bot_M)K' = S'\varphi(a')\varphi(\bot_M)K' = S'.$$

Hence $M \models K = S$, contradicting Proposition 1.2(ii). □

Having seen that nontotal extensional pca's do not arise as submodels of total pca's, we turn our attention to the inside. None of the models we shall construct in the next section possess a total submodel. This is probably merely a characteristic of the construction given in §2 and not a property of nontotal extensional pca's in general. However, one can prove the following:

1.6. PROPOSITION. *No nontotal pca has a total extensional submodel.*

PROOF. Let $M = (A, *, K, S)$ be a nontotal pca and suppose $M' = (A', *, K, S)$ is an extensional total submodel of M. Since M' is total, one has $M' \models \forall x \forall y(SKxy = KIxy)$, where $I := \langle x \rangle x$. Thus from the extensionality of M' it follows that $M' \models SK = KI$, hence $M \models SK = KI$. So $M \models \forall x \forall y(y \simeq KIxy \simeq SKxy \simeq Ky(xy))$, i.e. $M \models \forall x \forall y(Ky(xy)\downarrow)$. Then STR yields $M \models \forall x \forall y(xy\downarrow)$. Contradiction. □

Hence a fortiori no extensional nontotal pca has a total extensional submodel. Total extensional pca's on the other hand can have total extensional submodels. The substructure of D_∞ generated by K and S, for example, is always a total extensional submodel (cf. [B, p. 514, Theorem 20.1.5(ii)]).

§2. A construction method for nontotal, extensional pca's. Suitable structures, which under certain circumstances can be made into total extensional pca's, are the complete partial orders (cpo). As is well known, a cpo which is isomorphic to its continuous function space defines in a natural way an extensional λ-model or, equivalently, an extensional total pca. We shall first prove an analogous result for nontotal extensional pca's.

2.1. DEFINITION. (i) Let $(A, \leq_A)$ be a partially ordered set. A $D \subseteq A$ is *directed* iff $D \neq \emptyset$ and for all $d, d' \in D$ there exists a $d'' \in D$ such that $d \leq_A d''$ and $d' \leq_A d''$.

(ii) $A = (A, \leq_A, \bot_A)$ is a *complete partial order* iff $(A, \leq_A)$ is a partially ordered set, $\bot_A \in A$ is the least element of A, and every directed set $D \subseteq A$ has a supremum $\sup_A D \in A$.

(iii) Let A, B be cpo's and $f: A \to B$. f is *continuous* iff for all directed $D \subseteq A$ $f(\sup_A D) = \sup_B\{f(d) \mid d \in D\}$. Moreover, f is called *strict* iff f is continuous and $f(\bot_A) = \bot_B$. □

2.2. LEMMA. *Let A, B be cpo's. Then*:

(i) $[A \to B] := \{f: A \to B \mid f$ *continuous*$\}$ *is a cpo with*

$$f \leq_{[A\to B]} g \text{ iff } \forall a \in A(f(a) \leq_B g(a)),$$

$$\bot_{[A\to B]} = \lambda a \in A.\bot_B,$$

$$\sup_{[A\to B]} D = \lambda a \in A.\sup_B \{f(a) \mid f \in D\}.$$

(ii) $[A \underset{s}{\to} B] := \{f\colon A \to B \mid f \text{ strict}\}$ *is a subcpo of* $[A \to B]$.

(iii) $A \times B$ *is a cpo with*

$$\langle a,b\rangle \leq_{A\times B} \langle a',b'\rangle \quad \textit{iff} \quad a \leq_A a' \text{ and } b \leq_B b',$$
$$\bot_{A\times B} = \langle \bot_a, \bot_B\rangle,$$
$$\sup_{A\times B} D = \left\langle \sup_A D', \sup_B D''\right\rangle,$$

where $D' = \{a \in A \mid \exists b \in B\, \langle a,b\rangle \in D\}$, $D'' = \{b \in B \mid \exists a \in A\, \langle a,b\rangle \in D\}$.

PROOF. Routine. □

One might expect that if one lets $\bot_A$ play the role of the "undefined" in A, $A \cong [A \to A]$ could result in a nontotal extensional pca by taking $A\backslash\{\bot_A\}$ as a model. But $\bot_A$ corresponds under the isomorphism to $\bot_{[A\to A]} = \lambda a \in A.\bot_A$, the nowhere defined function, which by Proposition 1.2(i) has to be present in every nontotal pca. Thus we should rather try $A\backslash\{\bot_A\} \cong [A \to A]$ as a starting point. But this attempt will fail to satisfy extensionality, since two functions in $[A \to A]$ which differ only in $\bot_A$ correspond under the isomorphism with two different operators having identical applicative behaviour on $A\backslash\{\bot_A\}$. We shall show that $A\backslash\{\bot_A\} \cong [A \underset{s}{\to} A]$ is a sufficient condition for cpo's to be successful candidates for nontotal extensional pca's.

2.3. DEFINITION. Let A be a cpo.

(i) A is called *p-reflexive* iff there are $F \in [A \to [A \underset{s}{\to} A]]$, $G \in [[A \underset{s}{\to} A] \to A]$ such that range$(G) \subseteq A\backslash\{\bot_A\}$ and $F \cdot G = \mathrm{id}_{[A \underset{s}{\to} A]}$.

(ii) Let A be p-reflexive via maps F and G. Define a partial application operation * on $A\backslash\{\bot_A\} \times A\backslash\{\bot_A\}$ by

$$a * a' := \begin{cases} F(a)(a') & \text{if } F(a)(a') \neq \bot_A, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

(iii) $\mathrm{PAS}(A) := (A\backslash\{\bot_A\}, *)$. □

2.4. PROPOSITION. *Let A be a p-reflexive cpo via maps F and G. Then* PAS(A) *is a nontotal pas.*

PROOF. Clearly PAS(A) is a pas. Moreover, by Definition 2.3(ii), $G(\lambda a \in A.\bot_A)$ represents the nowhere defined operator in PAS(A). □

We can interpret λ-terms over PAS(A) in PAS(A) in a way very similar to that for the total case, with the obvious modification that this interpretation will now be partial.

2.5. DEFINITION. Let A be a p-reflexive cpo via maps F and G.

(i) The set of λ-terms over PAS(A), denoted $\Lambda(\mathrm{PAS}(A))$, is inductively defined by

$$x \in \mathrm{Vars} \to x \in \Lambda(\mathrm{PAS}(A)),$$
$$a \in A\backslash\{\bot_A\} \to a \in \Lambda(\mathrm{PAS}(A)),$$
$$t, t' \in \Lambda(\mathrm{PAS}(A)) \to tt' \in \Lambda(\mathrm{PAS}(A)),$$
$$t \in \Lambda(\mathrm{PAS}(A)) \to \lambda x.t \in \Lambda(\mathrm{PAS}(A)).$$

(ii) Let $\varrho\colon \text{Vars} \to A\backslash\{\bot_A\}$ be arbitrary. For $t \in \Lambda(\text{PAS}(A))$ define $[t]_\varrho^{\text{PAS}(A)}$ inductively by

$$[x]_\varrho^{\text{PAS}(A)} := \varrho(x) \quad \text{for every } x \in \text{Vars},$$

$$[a]_\varrho^{\text{PAS}(A)} := a \quad \text{for every } a \in A\backslash\{\bot_A\},$$

$$[tt']_\varrho^{\text{PAS}(A)} := \begin{cases} [t]_\varrho^{\text{PAS}(A)} * [t']_\varrho^{\text{PAS}(A)} & \text{if } [t]_\varrho^{\text{PAS}(A)}, [t']_\varrho^{\text{PAS}(A)} \\ & \text{and } [t]_\varrho^{\text{PAS}(A)} * [t']_\varrho^{\text{PAS}(A)} \text{ are defined,} \\ \text{undefined} & \text{otherwise,} \end{cases}$$

$$[\lambda x.t]_\varrho^{\text{PAS}(A)} := G(\lambda a \in A.[t]_{\varrho(x:=a)}^{T,\text{PAS}(A)}),$$

where

$$[t]_{\varrho(x:=a)}^{T,\text{PAS}(A)} := \begin{cases} \bot_A & \text{if } [t]_{\varrho(x:=a)}^{\text{PAS}(A)} \text{ is undefined or } a = \bot_A, \\ [t]_{\varrho(x:=a)}^{\text{PAS}(A)} & \text{otherwise.} \quad \Box \end{cases}$$

Notice that $[\ ]_\varrho^{\text{PAS}(A)}$ behaves well with respect to substitution; that is, for all $t, t' \in \Lambda(\text{PAS}(A))$ if $[t']_\varrho^{\text{PAS}(A)}$ is defined, say $a = [t']_\varrho^{\text{PAS}(A)}$, and one of $[t[x := t']]_\varrho^{\text{PAS}(A)}$ and $[t]_{\varrho(x:=a)}^{\text{PAS}(A)}$ is defined, then so is the other and they are equal.

2.6. Proposition. *$\lambda a \in A.[t]_{\varrho(x:=a)}^{T,\text{PAS}(A)}$ is strict: hence $[\lambda x.t]_\varrho^{\text{PAS}(A)}$ is well-defined.*

Proof. $h := \lambda a \in A.[t]_{\varrho(x:=a)}^{T,\text{PAS}(A)}$ is clearly a total function with $h(\bot_A) = \bot_A$. We prove continuity by induction on the complexity of t. The only nontrivial cases are $t \equiv t_1 t_2$ and $t \equiv \lambda y.t'$.

$t \equiv t_1 t_2$. For $i = 1, 2$ both $f_i := \lambda a \in A.[t_i]_{\varrho(x:=a)}^{T,\text{PAS}(A)}$ are continuous by the induction hypothesis. Hence

$$\langle f_1, f_2 \rangle := \lambda\langle a, a'\rangle \in A \times A.\langle f_1(a), f_2(a')\rangle \in [A \times A \to A \times A].$$

Define $F'\colon A \times A \to A$ by

$$F'(\langle a, a'\rangle) := \begin{cases} \bot_A & \text{if } \bot_A \in \{a, a'\}, \\ F(a)(a') & \text{otherwise.} \end{cases}$$

Then F' is continuous and

$$F'([t_1]_{\varrho(x:=a)}^{T,\text{PAS}(A)}, [t_2]_{\varrho(x:=a)}^{T,\text{PAS}(A)}) = [t_1 t_2]_{\varrho(x:=a)}^{T,\text{PAS}(A)}.$$

So

$$\lambda a \in A.[t_1 t_2]_{\varrho(x:=a)}^{T,\text{PAS}(A)} = F' \cdot \langle f_1, f_2 \rangle \cdot \Delta,$$

where $\Delta = \lambda a \in A.\langle a, a\rangle$. Thus h is continuous.

$t \equiv \lambda y.t'$. Put $f(a, a') := [t']_{\varrho(x:=a)(y:=a')}^{T,\text{PAS}(A)}$. By the induction hypothesis f is continuous in a and a' separately. Hence $f' := \lambda a \in A.\lambda a' \in A.f(a, a') \in [A \to [A \to A]]$. Moreover, $[\lambda y.t']_{\varrho(x:=a)}^{\text{PAS}(A)} = G(f'(a))$ for all $a \in A\backslash\{\bot_A\}$. Thus from the continuity of G and f' it follows that h is continuous. $\Box$

For the definition of the satisfaction of a formula φ in the expansion of the language of **LPT** formed by adding λ-terms we adapt Definition 0.3, using $[\ ]_\varrho^{\text{PAS}(A)}$ instead of $[\ ]_\varrho^M$.

2.7. PROPOSITION. *Let A be a p-reflexive cpo via F and G. Then for all t, t' $\in \Lambda(\mathrm{PAS}(A))$*

(i) $\mathrm{PAS}(A) \models \lambda x.t\downarrow$, *and*

(ii) $\mathrm{PAS}(A) \models \forall x((\lambda x.t)x \simeq t)$.

PROOF. (i) follows immediately from Proposition 2.6.

(ii) For all $a \in A\backslash\{\bot_A\}$

$$F([\lambda x.t]_\varrho^{\mathrm{PAS}(A)})(a) = [t]_{\varrho(x:=a)}^{T,\mathrm{PAS}(A)},$$

since $F \cdot G = \mathrm{id}_{[A \to_s A]}$. So if $[(\lambda x.t)a]_\varrho^{\mathrm{PAS}(A)}$ is defined for $a \in A\backslash\{\bot_A\}$ then

$$\bot_A \neq [(\lambda x.t)a]_\varrho^{\mathrm{PAS}(A)} = [t]_{\varrho(x:=a)}^{T,\mathrm{PAS}(A)}.$$

That is, $[(\lambda x.t)a]_\varrho^{\mathrm{PAS}(A)} = [t]_{\varrho(x:=a)}^{\mathrm{PAS}(A)}$.

If $[(\lambda x.t)a]_\varrho^{\mathrm{PAS}(A)}$ is undefined for $a \in A\backslash\{\bot_A\}$, then

$$\bot_A = F([\lambda x.t]_\varrho^{\mathrm{PAS}(A)})(a) = [t]_{\varrho(x:=a)}^{T,\mathrm{PAS}(A)}.$$

Thus $[t]_{\varrho(x:=a)}^{\mathrm{PAS}(A)}$ is undefined. □

2.8. THEOREM. *Let A be a p-reflexive cpo via F and G, and put* $\mathrm{PCA}(A) := (\mathrm{PAS}(A), K, S)$, *where* $K = [\lambda xy.x]^{\mathrm{PAS}(A)}$, $S = [\lambda xyz.xz(yz)]^{\mathrm{PAS}(A)}$. *Then*

(i) $\mathrm{PCA}(A)$ is a nontotal pca, and

(ii) $\mathrm{PCA}(A)$ *is extensional iff* $G \cdot F = \mathrm{id}_{A\backslash\{\bot_A\}}$.

PROOF. (i) follows immediately from Propositions 2.4 and 2.7.

(ii) Suppose $\mathrm{PCA}(A)$ is extensional. Then, since $F \cdot G = \mathrm{id}_{[A \to_s A]}$, we have for all a, $a' \in A\backslash\{\bot_A\}$

$$F(G(F(a)))(a') = F(a)(a').$$

So $\mathrm{PCA}(A) \models G(F(a)) = a$, i.e. $G \cdot F = \mathrm{id}_{A\backslash\{\bot_A\}}$. Conversely, suppose $G \cdot F = \mathrm{id}_{A\backslash\{\bot_A\}}$. Assume for a, $a' \in A\backslash\{\bot_A\}$

$$\mathrm{PCA}(A) \models \forall x(ax \simeq a'x).$$

Then $F(a)(a'') = F(a')(a'')$ for all $a'' \in A\backslash\{\bot_A\}$, and thus $F(a) = F(a')$. Hence $a = a'$, applying G. □

As an example of how a p-reflexive cpo can be constructed, we shall describe a modification of the free PSE-algebra ([P], [Sc2], [E]), generated by an arbitrary poset A with bottom $\bot_A$. Here, we let $G(A)$ be the least set containing A and all ordered pairs consisting of a *nonempty* finite subset $B \subseteq G(A)$ and an element $b \in G(A)$, assuming that the elements of A are distinquishable from ordered pairs.

2.9. DEFINITION. Let A be a poset with bottom $\bot_A$ and $G(A) := \bigcup\{G_n(A) \mid n \in \omega\}$, where $G_n(A)$ is recursively defined by

(i) $G_0(A) := A$, and

(ii) $G_{n+1}(A) := G_n(A) \cup \{(B, b) \mid B \subseteq G_n(A),\ B \text{ finite},\ B \neq \varnothing,\ b \in G_n(A)\}$. □

Usually in a graph algebra a pair (B, b) corresponds to an elementary instruction giving output b whenever the input contains B. The reason why we exclude pairs of the form $(\varnothing, b)$ is to guarantee the strictness of the application operation which we shall define in Definition 2.12. Another modification is made by forcing also all $a \in A\backslash\{\bot_A\}$ to act as elementary instructions. This is done by defining hereditarily relations $\leq_{G(A)}$ and $\leq_{\mathbf{P}(G(A))}$ on $G(A)$ and $\mathbf{P}(G(A))$ respectively, which will enable us to identify every $a \in A \backslash \{\bot_A\}$ with $(\{\bot_A\}, a)$.

2.10. Definition. For all $x, y \in G(A)$, $x \leq_{G(A)} y$ holds if either

(i) $x = y$, or

(ii) $x = \bot_A$, or

(iii) $x \in A \ \&\ y \in A \ \&\ x \leq_A y$, or

(iv) $x \in A \ \&\ \exists b(y = (\{\bot_A\}, b) \ \&\ x \leq_{G(A)} b)$, or

(v) $y \in A \backslash \{\bot_A\} \ \&\ \exists b \exists B(x = (B, b) \ \&\ b \leq_{G(A)} y)$, or

(vi) $\exists B \exists C \exists b \exists c\, (x = (B, b) \ \&\ y = (C, c) \ \&\ C \leq_{\mathbf{P}(G(A))} B \ \&\ b \leq_{G(A)} c)$

where we put for all $X, Y \in \mathbf{P}(G(A))$

$$X \leq_{\mathbf{P}(G(A))} Y \quad \text{iff} \quad \forall x \in X \exists y \in Y (x \leq_{G(A))} y). \quad \square$$

2.11. Lemma. *For all* $x, y \in G(A)$:

(i) $x \leq_{G(A)} \bot_A \leftrightarrow x = \bot_A$.

(ii) $x \in A \ \&\ y \in A \rightarrow (x \leq_A y \leftrightarrow x \leq_{G(A)} y)$.

(iii) $x \in A \backslash \{\bot_A\} \rightarrow (x \leq_{G(A)} y \leftrightarrow (\{\bot_A\}, x) \leq_{G(A)} y)$.

(iv) $y \in A \backslash \{\bot_A\} \rightarrow (x \leq_{G(A)} y \leftrightarrow x \leq_{G(A)} (\{\bot_A\}, y))$.

(v) $\leq_{G(A)}$ *is transitive.*

(vi) $\leq_{\mathbf{P}(G(A))}$ *is transitive.*

Proof. (i)–(iv) follow immediately from Definition 2.10; (vi) is a consequence of (v). For (v) we prove by induction on n that

$$\forall n \forall x \forall y \forall z (\{x, y, z\} \subseteq G_n(A) \ \&\ x \leq_{G(A)} y \ \&\ y \leq_{G(A)} z \rightarrow x \leq_{G(A)} z).$$

The transitivity of $\leq_{G(A)}$ then follows from the observation that for all $\{x, y, z\} \subseteq G(A)$ there is an $n \in \omega$ with $\{x, y, z\} \subseteq G_n(A)$. For $n = 0$, this follows immediately from (ii). Suppose $\{x, y, z\} \subseteq G_{n+1}(A)$ and $x \leq_{G(A)} y$, $y \leq_{G(A)} z$. If $\bot_A \in \{x, y, z\}$, then by (i) $x = \bot_A$. So $x \leq_{G(A)} z$. Assume $\bot_A \notin \{x, y, z\}$. By (iii) and (iv) we can restrict ourselves to the case where $x = (B, b)$, $y = (C, c)$ and $z = (D, d)$. Then

$$D \leq_{\mathbf{P}(G(A))} C \leq_{\mathbf{P}(G(A))} B \quad \text{and} \quad b \leq_{G(A)} c \leq_{G(A)} d.$$

Notice that, since $\{x, y, z\} \subseteq G_{n+1}(A)$, $B, C, D \in \mathbf{P}(G_n(A))$ and $\{b, c, d\} \subseteq G_n(A)$. So from the induction hypothesis it follows that $D \leq_{\mathbf{P}(G(A))} B$ and $b \leq_{G(A)} d$, i.e $x \leq_{G(A)} z$. $\square$

$\leq_{\mathbf{P}(G(A))}$ is not a partial ordering of $\mathbf{P}(G(A))$, since it fails to be antisymmetric. For example, if $a \in A \backslash \{\bot_A\}$ then

$$\{(\{\bot_A\}, a)\} \leq_{\mathbf{P}(G(A))} \{a\} \quad \text{and} \quad \{a\} \leq_{\mathbf{P}(G(A))} \{(\{\bot_A\}, a)\},$$

but $\{a\} \neq \{(\{\bot_A\}, a)\}$. We therefore define an equivalence relation $\equiv$ on $\mathbf{P}(G(A))$ by

$$X \equiv Y \leftrightarrow X \leq_{\mathbf{P}(G(A))} Y \ \&\ Y \leq_{\mathbf{P}(G(A))} X$$

and take the quotient $\mathbf{P}(G(A))/\equiv$ as our new universe. Notice that $(\mathbf{P}(G(A))/\equiv, \leq)$, where $[X] \leq [Y] \leftrightarrow X \leq_{\mathbf{P}(G(A))} Y$, is a cpo (in fact a complete lattice) with bottom $[\varnothing]$ and the supremum $\sup D = [\bigcup \{X \mid [X] \in D\}]$ for arbitrary subsets D of $\mathbf{P}(G(A))/\equiv$. We shall first prove that $(\mathbf{P}(G(A))/\equiv, \leq)$ is *p*-reflexive.

2.12. Definition. (i) For $[X] \in \mathbf{P}(G(A))/\equiv$ let

$$F([X]) := \lambda [Y] \in \mathbf{P}(G(A))/\equiv . [\{b \mid \exists [B] \leq [Y] [\{(B, b)\}] \leq [X]\}].$$

(ii) Let $f \in [\mathbf{P}(G(A))/\equiv \ \underset{s}{\to} \mathbf{P}(G(A))/\equiv]$. Define

$$G(f) := [\{(C,c) \mid [\{c\}] \le f([C])\} \cup \{\perp_A\}]. \quad \square$$

2.13. LEMMA. (i) $F \in [\mathbf{P}(G(A))/\equiv \ \to [\mathbf{P}(G(A))/\equiv \ \underset{s}{\to} \mathbf{P}(G(A))/\equiv]]$.
(ii) $G \in [[\mathbf{P}(G(A))/\equiv \ \underset{s}{\to} \mathbf{P}(G(A))/\equiv] \to \mathbf{P}(G(A))/\equiv]$.
(iii) $\mathrm{range}(G) \subseteq (\mathbf{P}(G(A))/\equiv)\backslash\{\perp_{\mathbf{P}(G(A))/\equiv}\}$.
(iv) $F \cdot G = \mathrm{id}_{[\mathbf{P}(G(A))/\equiv \ \underset{s}{\to} \mathbf{P}(G(A))/\equiv]}$.

PROOF. Notice that $F([X])([\varnothing]) = [\varnothing]$, since we excluded pairs of the form $(\varnothing, b)$. So

$$F([X])(\perp_{\mathbf{P}(G(A))/\equiv}) = \perp_{\mathbf{P}(G(A))/\equiv} \quad \text{for all } [X] \in \mathbf{P}(G(A))/\equiv.$$

Moreover, by Definition 2.12

$$[\{\perp_A\}] \le G(f) \quad \text{for all } f \in [\mathbf{P}(G(A))/\equiv \ \underset{s}{\to} \mathbf{P}(G(A))/\equiv].$$

Thus $\mathrm{range}(G) \subseteq (\mathbf{P}(G(A))/\equiv)\backslash\{\perp_{\mathbf{P}(G(A))/\equiv}\}$. The continuity properties of F and G are proved straightforwardly.

(iv) Let f and $[X]$ be arbitrary. Then

$$\begin{aligned}
F(G(f))([X]) &= [\{b \mid \exists[B] \le [X][\{(B,b)\}] \le G(f)\}] \\
&= [\{b \mid \exists[B] \le [X][\{(B,b)\}] \le [\{(C,c) \mid [\{c\}] \le f([C])\}]\}] \\
&\hfill \text{by 2.11(i)} \\
&= [\{b \mid \exists C \ne \varnothing \text{ finite } ([C] \le [X] \ \& \ [\{b\}] \le f([C]))\}] \\
&= [\bigcup\{\{b \mid [\{b\}] \le f([C])\} \mid C \ne \varnothing \ \& \ C \text{ finite} \ \& \ [C] \le [X]\}] \\
&= [\bigcup\{Z \mid \exists C \ne \varnothing (C \text{ finite} \ \& \ [C] \le [X] \ \& \ [Z] = f([C]))\}] \\
&= f([X]).
\end{aligned}$$

Here, the last equality follows from the continuity and strictness of f and the observation that, for all $X \ne \varnothing$ the set $D = \{[C] \mid C \ne \varnothing \ \& \ C \text{ finite} \ \& \ [C] \le [X]\}$ is directed and $[X] = \sup D$. $\square$

2.14. THEOREM. *Let A be a poset with bottom $\perp_A$. Then* $\mathrm{PCA}(\mathbf{P}(G(A))/\equiv)$ *is a nontotal extensional pca.*

PROOF. Since $\mathbf{P}(G(A))/\equiv$ is p-reflexive it is sufficient to prove that

$$G \cdot F = \mathrm{id}_{(\mathbf{P}(G(A))/\equiv)\backslash\{\perp_{\mathbf{P}(G(A))/\equiv}\}}$$

(Theorem 2.8). So let $[X] \in \mathbf{P}(G(A))/\equiv$ be such that $X \ne \varnothing$. Then

$$\begin{aligned}
G(F([X])) &= [\{(C,c) \mid [\{c\}] \le [\{b \mid \exists[B] \le [C][\{(B,b)\}] \le [X]\}]\} \cup \{\perp_A\}] \\
&= [\{(C,c) \mid [\{(C,c)\}] \le [X]\} \cup \{\perp_A\}] = [X]. \quad \square
\end{aligned}$$

§3. More properties of $\mathrm{PCA}(\mathbf{P}(G(A))/\equiv)$.

3.1. PROPOSITION. $\perp_{\mathrm{PCA}(\mathbf{P}(G(A))/\equiv)} = [\{\perp_A\}]$.

PROOF. For all X

$$F([\{\perp_A\}])([X]) = [\{b \mid \exists[B] \le [X][\{(B,b)\}] \le [\{\perp_A\}]\}] = [\varnothing].$$

Hence $[\{\perp_A\}]$ represents the unique nowhere-defined function in the model $\mathrm{PCA}(\mathbf{P}(G(A))/\equiv)$. $\square$

First we shall prove that $\mathrm{PCA}(\mathbf{P}(G(A))/\equiv) \models \neg(\Omega\downarrow)$, where Ω is the famous λ-combinator $(\lambda x.xx)(\lambda x.xx)$. To simplify our notation, in this section we shall write interpretations of closed λ-terms in $\mathrm{PCA}(\mathbf{P}(G(A))/\equiv)$ informally, e.g. we shall write $\lambda x.xx$ rather than $[\lambda x.xx]^{\mathrm{PAS}(\mathbf{P}(G(A))/\equiv)}$.

3.2. LEMMA. (i) $\lambda x.xx = [\{(C,c) | [\{c\}] \leq F([C])([C])\} \cup \{\perp_A\}]$.

(ii) $\forall a \in A\backslash\{\perp_A\}\neg([\{a\}] \leq \lambda x.xx)$.

PROOF. (i) follows immediately from Definition 2.5 and 2.12(ii).

(ii) Let $a \in A\backslash\{\perp_A\}$ and suppose $[\{a\}] \leq \lambda x.xx$. By Lemma 2.11(i), $a \leq_{G(A)} (C,c)$ for some (C,c) with $[\{c\}] \leq F([C])([C])$. But by Definition 2.10, $C = \{\perp_A\}$. So

$$[\{c\}] \leq F([\{\perp_A\}])([\{\perp_A\}]) = [\varnothing].$$

Contradiction. □

3.3. PROPOSITION. $\mathrm{PCA}(\mathbf{P}(G(A))/\equiv) \models \neg(\Omega\downarrow)$.

PROOF. Suppose $F(\lambda x.xx)(\lambda x.xx) \neq [\varnothing]$. Then for some $n \in \omega$ there exists a $(B,b) \in G_n(A)$ such that $[B] \leq \lambda x.xx$ and $[\{(B,b)\}] \leq \lambda x.xx$. Let n be minimal. Since $[\{(B,b)\}] \leq \lambda x.xx$ it follows from the monotonicity of F that $[\{b\}] \leq F([B])([B])$. Pick (D,d) such that $[D] \leq [B]$ and $[\{(D,d)\}] \leq [B]$. By 3.2(ii), $(D,d) \leq_{G(A)} (C,c)$ for some $(C,c) \in B$. Then $[C] \leq [D] \leq [B] \leq \lambda x.xx$ and $[\{(C,c)\}] \leq [B] \leq \lambda x.xx$. But $(C,c) \in G_{n-1}(A)$, since $B \subseteq G_n(A)$. Contradiction. □

As a consequence of proposition 3.3 we now have

3.4. THEOREM. $\mathrm{PCA}(\mathbf{P}(G(A))/\equiv)$ *has neither a completion nor a total submodel.*

PROOF. From Theorem 1.5 it follows that $\mathrm{PCA}(\mathbf{P}(G(A))/\equiv)$ is not completable. Suppose $M \subseteq \mathrm{PCA}(\mathbf{P}(G(A))/\equiv)$ is a total submodel. Then $M \models SII(SII)\downarrow$, where $I = SKK$. Hence

$$\mathrm{PCA}(\mathbf{P}(G(A))/\equiv) \models SII(SII)\downarrow.$$

But from the extensionality of $\mathrm{PCA}(\mathbf{P}(G(A))/\equiv)$ it follows that

$$\mathrm{PCA}(\mathbf{P}(G(A))/\equiv) \models \mathrm{SII} = \lambda x.xx.$$

So $\mathrm{PCA}(\mathbf{P}(G(A))/\equiv) \models \Omega\downarrow$. Contradiction. □

The λ-calculus is defined by using just variables and the abstraction operator λ. The basic axiom schema is

$$(\lambda x.M)N = M[x := N].$$

The basic rule is

$$\frac{M = N}{\lambda x.M = \lambda x.N}.$$

The set of λ-terms Λ can be divided into two classes: the solvable and the unsolvable λ-terms.

3.5. DEFINITION. (i) A closed $M \in \Lambda$ is *solvable* iff

$$\exists n \in \omega \exists N_1 \in \Lambda \cdots \exists N_n \in \Lambda(\lambda \vdash MN_1 \cdots N_n = I_\lambda),$$

where $I_\lambda = \lambda x.x$.

(ii) An arbitrary $M \in \Lambda$ is *solvable* iff there is a closed substitution instance of M that is solvable.

(iii) $M \in \Lambda$ is *unsolvable* iff M is not solvable. □

EXAMPLES. Ω and $\lambda x.\Omega$ are both unsolvable λ-terms; this fact however does not follow immediately from the definition but relies e.g. on the Church-Rosser theorem [C-R].

The class of unsolvable λ-terms is often considered as the class of "meaningless" terms in the λ-calculus. And indeed, as is shown in [B], the interpretations of unsolvable λ-terms in Scott's D_∞-models are all identified with the least element $\perp_\infty$. $\mathrm{PCA}(\mathbf{P}(G(A))/\equiv)$, however, behaves quite differently. From Propositions 3.4 and 2.7(i) it follows that

$$\mathrm{PCA}(\mathbf{P}(G(A))/\equiv) \models \neg(\Omega\downarrow) \;\&\; \lambda x.\Omega\downarrow.$$

Hence $\mathrm{PCA}(\mathbf{P}(G(A))/\equiv) \models \neg(\Omega \simeq \lambda x.\Omega)$. In the remainder of this section we shall show that in $\mathrm{PCA}(\mathbf{P}(G(A))/\equiv)$ we even have an ω-chain of denoting unsolvable λ-terms.

3.6. DEFINITION. (i) $T_0 := \lambda x_0.\Omega$.

(ii) $T_{n+1} := \lambda x_{n+1}.T_n$.

(iii) $T_\infty := WW$, where $W \equiv \lambda xy.xx$. □

Clearly, since Ω is unsolvable, T_n is unsolvable for all $n \in \omega$. Moreover, observe that for $N \in \Lambda$

$$\boldsymbol{\lambda} \vdash T_\infty N = (\lambda xy.xx)WN = (\lambda y.WW)N = WW = T_\infty.$$

Thus T_∞ is also an unsolvable λ-term.

3.7. PROPOSITION. (i) $\forall n \in \omega\ \mathrm{PCA}(\mathbf{P}(G(A))/\equiv) \models T_n\downarrow \;\&\; T_{n+1} = KT_n$.

(ii) $\mathrm{PCA}(\mathbf{P}(G(A))/\equiv) \models T_\infty\downarrow \;\&\; T_\infty = KT_\infty$.

(iii) $\forall i \in \omega \cup \{\infty\} \forall j \in \omega \cup \{\infty\}(i \neq j \rightarrow \mathrm{PCA}(\mathbf{P}(G(A))/\equiv) \models \neg(T_i = T_j))$.

PROOF. (i) By Proposition 2.7(i), $\mathrm{PCA}(\mathbf{P}(G(A))/\equiv) \models T_n\downarrow$ for all $n \in \omega$. Hence, by Proposition 2.7(ii), $\mathrm{PCA}(\mathbf{P}(G(A))/\equiv) \models \forall x(T_{n+1}x = T_n = KT_nx)$. Thus extensionality yields $\mathrm{PCA}(\mathbf{P}(G(A))/\equiv) \models T_{n+1} = KT_n$ for all $n \in \omega$.

(ii) From 2.7(i), (ii) we get

$$\mathrm{PCA}(\mathbf{P}(G(A))/\equiv) \models T_\infty x \simeq (\lambda xy.xx)W \simeq \lambda y.WW.$$

Hence $\mathrm{PCA}(\mathbf{P}(G(A))/\equiv) \models T_\infty\downarrow$. Moreover,

$$\mathrm{PCA}(\mathbf{P}(G(A))/\equiv) \models T_\infty \simeq (\lambda y.WW)x \simeq WW \simeq T_\infty \simeq KT_\infty x.$$

So $\mathrm{PCA}(\mathbf{P}(G(A))/\equiv) \models T_\infty \simeq KT_\infty$ by extensionality.

(iii) follows from the fact that by (i), (ii) and 3.3 we have

$$\mathrm{PCA}(\mathbf{P}(G(A))/\equiv) \models \forall x_0 \cdots \forall x_n(T_n x_0 \cdots x_{n-1}\downarrow$$
$$\& \neg(T_n x_0 \cdots x_n\downarrow) \;\&\; T_\infty x_0 \cdots x_n\downarrow). \quad \square$$

3.8. LEMMA. $T_0 \leq T_1 \leq \cdots \leq T_\infty$.

PROOF. By 3.7(iii), $\mathrm{PCA}(\mathbf{P}(G(A))/\equiv) \models \forall x \neg(T_0 x\downarrow)$. Hence $T_0 = [\{\perp_A\}]$. So $T_0 \leq T_1$ and $T_0 \leq T_\infty$. Suppose $T_n \leq T_{n+1}$ and $T_n \leq T_\infty$. From 3.7(i), (ii) and the monotonicity of $F(K)$ it then follows that

$$T_{n+1} = F(K)(T_n) \leq F(K)(T_{n+1}) = T_{n+2},$$
$$T_{n+1} = F(K)(T_n) \leq F(K)T_\infty = T_\infty. \quad \square$$

Finally we prove that T_∞ is the supremum of $\{T_n | n \in \omega\}$.

3.9. LEMMA. (i) $\forall n \in \omega\ T_{n+1} = [\{(B,b)\}|[\{b\}] \leq T_n\}]$.
(ii) $W = [\{(B,(C,c))|[\{c\}] \leq F([B])([B])\} \cup \{(B,\bot_A)|(B,\bot_A) \in G(A)\} \cup \{\bot_A\}]$.
(iii) $\forall a \in A\backslash\{\bot_A\}(\neg([\{a\}] \leq W))$.
(iv) $\forall n \in \omega \forall (B,b) \in G_n(A)([\{(B,b)\}] \leq W \,\&\, [B] \leq W \rightarrow [\{b\}] \leq T_n)$.
PROOF. (i) Observe that for all $X \neq \varnothing$

$$F([\{(B,b)|[\{b\}] \leq T_n\}])([X]) = T_n = F(T_{n+1})([X]).$$

Hence extensionality yields $[\{(B,b)|[\{b\}] \leq T_n\}] = T_{n+1}$.
(ii) By 2.5(ii) and 2.12(ii) one has

$$\begin{aligned} W &= [\{(B,b)|[\{b\}] \leq [\{(C,c)|[\{c\}] \leq F([B])([B])\} \cup \{\bot_A\}]\} \cup \{\bot_A\}] \\ &= [\{(B,(C,c))|[\{c\}] \leq F([B])([B])\} \cup \{(B,\bot_A)|(B,\bot_A) \in G(A)\} \cup \{\bot_A\}]. \end{aligned}$$

(iii) Let $a \in A\backslash\{\bot_A\}$ be arbitrary and suppose $[\{a\}] \leq W$. Then $[\{a\}] \leq F([\{\bot_A\}])([\{\bot_A\}]) = \varnothing$. Contradiction.
(iv) Trivially this holds for $n = 0$. Let $(B,b) \in G_{n+1}(A)$ and suppose $[\{(B,b)\}] \leq W$ and $[B] \leq W$. If $b = \bot_A$ then clearly $[\{b\}] \leq T_{n+1}$. If $b = c \in A\backslash\{\bot_A\}$ or $b = (C,c)$ then $[\{c\}] \leq F([B])([B])$. Pick D such that $[D] \leq [B]$ and $[\{(D,c)\}] \leq [B]$. By (iii), $(D,c) \leq_{G(A)} (E,e)$ for some $(E,e) \in B$. Then $(E,e) \in G_n(A)$, $[\{(E,e)\}] \leq [B] \leq W$ and $[E] \leq [D] \leq [B] \leq W$. Hence from the induction hypothesis it follows that $[\{c\}] \leq [\{e\}] \leq T_n$. So by (i), $[\{(C,c)\}] \leq T_{n+1}$. □
3.10. PROPOSITION. $T_\infty = \sup\{T_n|n \in \omega\}$.
PROOF. From Proposition 3.8 it follows that $\sup\{T_n|n \in \omega\} \leq T_\infty$. Moreover, by Definition 3.6(iii),

$$T_\infty = F(W)(W) = [\{b|\exists [B] \leq W\,[\{(B,b)\}] \leq W\}].$$

Let b be such that $[\{(B,b)\}] \leq W$ and $[B] \leq W$ for some finite $B \neq \varnothing$. Then it follows from Lemma 3.9 that $[\{b\}] \leq T_n$ for some $n \in \omega$. Thus $T_\infty \leq \sup\{T_n|n \in \omega\}$. □
3.11. THEOREM. *There are unsolvable λ-terms $T_0, T_1, \ldots, T_\infty$ such that*
(i) $T_0 \leq T_1 \leq \cdots \leq T_\infty = \sup\{T_n|n \in \omega\}$ *and*
(ii) $\forall i \in \omega \cup \{\infty\} \forall j \in \omega \cup \{\infty\}(i \neq j \rightarrow \mathrm{PCA}(\mathbf{P}(G(A))/\equiv) \models T_i\downarrow \,\&\, T_j\downarrow \,\&\, \neg(T_i = T_j))$. □

Appendix. $AK'_p + AS'_p$ **does not imply** $\mathbf{cc}'_p$. We shall construct a pas $M = (A, *)$ satisfying

AK'_p	$\forall x \forall y\, Kxy \simeq x$,
AS'_p	$\forall x \forall y \forall z\, Sxyz \simeq xz(yz)$

for some $K, S \in A$ and in addition

P1	$\exists x \exists y \neg(xy\downarrow)$,
P2	$\forall x \exists y\, xy\downarrow$.

Then it is clear that cc'_p:

for every $t \in T(M)$ with variables among $\vec{x}$, $M \models \exists y \forall \vec{x}\, y\vec{x} \simeq t$

cannot hold for M. Thus suppose cc'_p holds for M and let $a, a' \in A$ be such that

$M \models \neg(aa'\downarrow)$(P1). Then, by cc'_p, $M \models \forall x\, a''x \simeq aa'$ for some $a'' \in A$. That is, there is an $a'' \in A$ such that $M \models \forall x \neg(a''x\downarrow)$, contradicting P2.

The construction is again a modification of the free PSE-algebra generated by the one-point set $\{\mathscr{C}\}$, i.e.

A.1. Definition. Define $G := \bigcup\{G_n | n \in \omega\}$ with $G_0 := \{\mathscr{C}\}$ and

$$G_{n+1} := G_n \bigcup \{(B, b) | B \subseteq G_n,\ B \text{ finite},\ b \in G_n\}. \quad \square$$

A.2. Definition. Define $M := (\mathbf{P}(G)\backslash\{\varnothing\}, *, K, S)$, where

$$X * Y := \{b | Y \neq \varnothing \;\&\; \exists B \subseteq Y (B, b) \in X\} \cup \{\mathscr{C} | Y \neq \varnothing \;\&\; \mathscr{C} \in X\},$$
$$K := \{(B, (C, b)) | b \in B\},$$
$$S := \{(B, (C, (D, b))) | b \in B * D * (C * D)\}. \quad \square$$

Clearly, M is a pas.

A.3. Proposition (P1&P2). (i) $M \models \exists x \exists y \neg(xy\downarrow)$.

(ii) $M \models \forall x \exists y\, xy\downarrow$.

Proof. (i) Let $X = Y = \{(\{\mathscr{C}\}, \mathscr{C})\}$. Then $X \neq \varnothing \neq Y$. Moreover, since $\mathscr{C} \notin X$ and $\neg(\{\mathscr{C}\} \subseteq Y)$, $X * Y = \varnothing$. Hence $M \models \exists x \exists y \neg(xy\downarrow)$.

(ii) Let $X \in \mathbf{P}(G)\backslash\{\varnothing\}$. If $\mathscr{C} \in X$, then $\mathscr{C} \in X * \{\mathscr{C}\}$. If $\mathscr{C} \notin X$, then $(B, b) \in X$ for some $(B, b) \in G$. Hence $b \in X * (B \cup \{\mathscr{C}\})$. $\square$

A.4. Proposition. (i) $K, S \in \mathbf{P}(G)\backslash\{\varnothing\}$.

(ii) $M \models AK'_p \;\&\; AS'_p$.

Proof. (i) Observe that $(\{\mathscr{C}\}, (\{\mathscr{C}\}, \mathscr{C})) \in K$ and $(\{\mathscr{C}\}, (\{\mathscr{C}\}, (\{\mathscr{C}\}, \mathscr{C}))) \in S$. So $K, S \in \mathbf{P}(G)\backslash\{\varnothing\}$. (ii) is verified straightforwardly and is left to the reader. $\square$

Chapter 5

Topological PCA's and CA's

5.1 Introduction

Pca's were defined as partial applicative structures in which the representable functions are precisely the algebraic ones. This chapter deals with yet another restriction on the models. Here we shall only study those structures which are combinatory complete and are in addition equipped with a topology, such that the notions of representable and continuous function coincide. Since there are total and nontotal external continuous functions we shall define in fact two classes of structures. The first class consists of those pca's in which every partial continuous function (with open domain) is representable. These structures are nontotal and we shall call them therefore topological pca's. The second class, the class of topological ca's, comprises the total counterparts of topological pca's. The additional condition amounts for these structures merely to the fact that every total continuous function is representable. Topological ca's can therefore not be regarded as a special limit case of topological pca's.

Definition 5.1.1 Let $M = (A, *)$ be a partial applicative structure provided with a topology.
(a) If f is a partial function from A^{n+1} to A, then

(i) f is *algebraic* over M if there is a term $t \in T(M)$ with $Vars(t) \subseteq \{x_1, ..., x_{n+1}\}$ such that for all $a_1, ..., a_{n+1} \in A$

$$f(a_1, ..., a_{n+1}) \simeq t[x_1, ..., x_{n+1} := a_1, ..., a_{n+1}].$$

(ii) f is *representable* in M if there is an $a \in A$ such that for all $a_1, ..., a_{n+1} \in A$

$$a * a_1 * ... * a_n \downarrow \ \wedge \ f(a_1, ..., a_{n+1}) \simeq a * a_1 * ... * a_{n+1}.$$

(iii) f is *continuous with open domain* if $dom f$ is open in A^{n+1} and $f : dom f \to A$ is continuous.

(b) M is a *topological pca* if

$$\{f \mid f \text{ is algebraic over } M\} = \{f \mid f \text{ is representable in } M\} =$$

$$= \{f \mid f \text{ is continuous with open domain }\}.$$

(c) M is a *topological ca* if

$$\{f \mid f \text{ is algebraic over } M\} = \{f \mid f \text{ is representable in } M\} =$$

$$= \{f \mid f \text{ is total continuous }\}.\square$$

Topological pca's and ca's seem to be quite incomparable with respect to their cardinality, since there are 'fewer' representable functions on a topological ca than on a topological pca. In the preceding two chapters we have already encountered representatives of these classes. As is readily checked, reflexive and p-reflexive cpo's give rise to topological ca's and pca's, respectively, when equipped with their Scott topology. It is clear that the set $\mathcal{P}(G(A))$ underlying both our constructions is uncountable. However, one might wonder how the extensional collapse affects the cardinality of $\mathcal{P}(G(A))$. This question will receive a general answer in the next two sections.

5.2 Every topological pca is uncountable

Throughout this section we let $M = (A, *)$ be some fixed topological pca equipped with the topology $\mathcal{O}(A)$. To prove that A is uncountable it is sufficient to prove that $\mathcal{O}(A)$ is uncountable:

Lemma 5.2.1 If $\mathcal{O}(A)$ is uncountable then so is A.

PROOF. For $O \in \mathcal{O}(A)$, define the partial function f_O from A to A by:

$$f_O(a) = \begin{cases} a & \text{if } a \in O, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Then f_O is representable, since f is partial continuous. Hence A is uncountable if $\mathcal{O}(A)$ is so. □

In the remainder of this section we shall therefore focus on proving that $\mathcal{O}(A)$ is uncountable. To this end recall that in any nontrivial pca and hence in M one can develop the rudiments of recursion theory. That is, one can isolate in M a copy N of ω , such that every recursive function is numerically representable in M. We shall not use the recursion-theoretic apparatus explicitly. We shall, however, heavily rely on the existence in M of the copy N of ω together with a numerical definition-by-cases operator D.

Lemma 5.2.2 There is a $D \in A$ such that for all $a, a' \in A$ and all $n, m \in N$

$$D * a * a' * n * n = a' \ \wedge \ (n \neq m \longrightarrow D * a * a' * n * m = a).$$

PROOF. See proposition 7.1.2. □

We shall now distinguish two cases:

CASE I: $card(\mathcal{O}(A)) = 2$.

If $card(\mathcal{O}(A)) = 2$, i.e. $\mathcal{O}(A) = \{\emptyset, A\}$, then *every total* function from A to A is partial continuous and hence representable in M. Since A contains a copy of ω , A is infinite and the set of total functions from A to A is uncountable. Thus M is uncountable.

CASE II: $card(\mathcal{O}(A)) > 2$.

If $card(\mathcal{O}(A)) > 2$ then N equipped with the subspace topology is discrete, i.e.

Lemma 5.2.3 For all $X \subseteq N$ there is an $O \in \mathcal{O}(A)$ such that

$$X \subseteq O \ \wedge \ O \cap N = X.$$

PROOF. It is sufficient to prove that for every $n \in N$ there exists an $O \in \mathcal{O}(A)$ such that $n \in O$ and $O \cap N = \{n\}$. So let $n \in N$ be arbitrary and let $O \in \mathcal{O}(A)$ be distinct from $\emptyset$ and A. Pick $a, a' \in A$ such that $a \in O$ and $a' \notin O$ and define the partial function f from A to A by

$$f(a'') \simeq D * a' * a * n * a''.$$

Then f is algebraic over M and hence continuous. By lemma 5.2.2 $f(n) = a$. Thus $n \in f^{-1}(O) \in \mathcal{O}(A)$. Now let $m \in N$ be such that $m \neq n$ and suppose $m \in f^{-1}(O)$. Then $f(m) \in O$. But $f(m) = a'$, by lemma 5.2.2. Contradiction. Hence $f^{-1}(O) \cap N = \{n\}$.□

Combining lemma 5.2.1 and lemma 5.2.3 one then has

Theorem 5.2.4 M is uncountable. □

5.3 Every countable topological ca is a T_1-space which is not regular

In [B,vM] Barendregt and van Mill raised the question whether countable topological ca's do exist. Although we are not able to answer this question, we shall try to locate countable topological ca's - if they exist at all - according to the axioms of topological separation.

Let $M = (A, *)$ be some fixed topological ca equipped with the topology $\mathcal{O}(A)$. As usual, we shall assume that M is nontrivial. We shall first prove that M is a connected T_0-space. To this end recall that any $a \in A$ has a fixed point, i.e.

$$\forall a \in A \exists a' \in A(a * a' = a').$$

Thus any continuous function $f : A \to A$ has a fixed point, i.e.

$$\forall f : A \to A(f \text{ is continuous} \longrightarrow \exists a \in A(f(a) = a)). \quad (\dagger)$$

Lemma 5.3.1 (i) M is a T_0-space, i.e. for every pair of distinct points of A there exists an open subset containing exactly one of these points. (ii) M is connected, i.e. A cannot be represented as the union of two open, nonempty and disjoint subsets.

PROOF. (i) Let $a, a' \in A$ be such that $a \neq a'$ and define $f : A \to A$ by

$$f(a'') = \begin{cases} a & \text{if } a'' \neq a, \\ a' & \text{otherwise.} \end{cases}$$

Then f has no fixed point and is therefore discontinuous, i.e. there is an $O \in \mathcal{O}(A)$ such that $f^{-1}(O) \notin \mathcal{O}(A)$. This O must clearly contain exactly one of the points a, a'.

(ii) Assume $A = O \cup O'$, where O, O' are two open, nonempty and disjoint subsets of A. Pick $a \in O, a' \in O'$ and define $f : A \to A$ by

$$f(a'') = \begin{cases} a & \text{if } a'' \in O', \\ a' & \text{otherwise.} \end{cases}$$

Then f is continuous, but has no fixed point. Contradiction with (†). □

If M is a proper T_0-space, i.e. a T_0-space which is not T_1, we can repeat the argument used in theorem 5.2.4 in order to show that M is uncountable. To this end recall that for every pair of distinct points $a, a' \in A$ in a T_1-space there exists an $O \in \mathcal{O}(A)$ such that $a \in O$ and $a' \notin O$. So, if M is a proper T_0-space, then there exist two distinct points $a, a' \in A$ such that

$$\forall O \in \mathcal{O}(A)(a \in O \longrightarrow a' \in O). \ (*)$$

Lemma 5.3.2 If M is a proper T_0-space then M is uncountable.

PROOF. Since we have assumed that M is nontrivial we again have the necessary items for a total version of lemma 5.2.3 inside M, viz.

- a copy N of ω and a numerical definition-by-cases operator D;
- two points $a, a' \in A$ and, assuming that M is T_0, an open $O \in \mathcal{O}(A)$ containing exactly one of these points.

Using the same argument as in lemma 5.2.3, one then shows that for all $X \subseteq N$ there is an $O \in \mathcal{O}(A)$ such that $X \subseteq O$ and $O \cap N = X$. Hence $\mathcal{O}(A)$ is uncountable. Now choose $a, a' \in A$ satisfying $(*)$ and define for $O \in \mathcal{O}(A)$ the function $f_O : A \to A$ by

$$f(a'') = \begin{cases} a' & \text{if } a'' \in O, \\ a & \text{otherwise.} \end{cases}$$

Then f_O is continuous and thus representable. Hence A is uncountable. □

Notice that this lemma covers the class of nontrivial reflexive complete partial orders.

Corollary 5.3.3 Every nontrivial reflexive cpo is uncountable.

PROOF. A nontrivial cpo $(D, \sqsubseteq)$ provided with its Scott topology is proper T_0: one can separate two distinct points $d, d' \in D$ by one of the opens $\{d'' \in D \mid \neg\, d'' \sqsubseteq d\}$, $\{d'' \in D \mid \neg\, d'' \sqsubseteq d'\}$; however, the only open containing the bottom element is the space D itself. □

From lemma 5.3.1 and lemma 5.3.2 it now follows that every countable topological ca is a connected T_1-space. However, combining the following two well-known topological facts (cf. e.g. [Ek])

- every connected nontrivial completely regular space is uncountable,
- every countable regular space is normal,

we see that countable topological ca's must be either proper T_1 or proper Hausdorff-spaces. Therefore

Theorem 5.3.4 Every countable topological ca is a T_1-space which is not regular. □

The original question brought forward by Barendregt and van Mill remains open.

Chapter 6

Embedding Theorems

6.1 Introduction

This chapter deals with embeddings of partial applicative structures into structures of increased richness.

Definition 6.1.1 Let $(A, *)$, $(A', *')$ be two partial applicative structures.

(i) A mapping $\Phi : A \to A'$ is a *morphism* if for all $a, a' \in A$

$$a * a' \downarrow \longrightarrow \Phi(a * a') = \Phi(a) *' \Phi(a').$$

(ii) A morphism Φ is an *embedding* if Φ is injective. □

λ-algebras arise as substructures of λ-models, that is, as shown by Barendregt and Koymans [B,K], for every λ-algebra $(A, *, K, S)$ there exist a λ-model $(A', *', K', S')$ together with an embedding Φ such that $\Phi(K) = \Phi(K')$ and $\Phi(S) = \Phi(S')$. Going one step up or down in the hierarchie of models, there is no similar correspondence: neither can λ-models in general be embedded into extensional ca's nor can ca's in general be embedded into λ-algebra's while preserving the combinators (cf. [B,K]). The reason for this incompatibility is that equations between interpretations of closed terms which necessarily hold in structures of the more restricted classes need not to hold in lower class structures. However, dropping the additional requirement that combinators are preserved, the possibilities are less restricted. The following result is due to Engeler.

Theorem 6.1.2 Every pas can be embedded into a topological ca.

PROOF. Let $(A, *)$ be a pas and D_A be Engeler's graph algebra with underlying set A. Define the map Φ of A into $\mathcal{P}(G(A))$ recursively by

$$\Phi_0(a) = \{a\},$$

$$\Phi_{n+1}(a) = \Phi_n(a) \cup \{(\{a'\}, b) \mid a * a' \downarrow \ \wedge\ b \in \Phi_n(a * a')\},$$
$$\Phi(a) = \cup_{n \in \omega} \Phi_n(a).$$

Note that $\Phi(a) \cap A = \{a\}$ $(*)$. Hence $a = a'$, if $\Phi(a) = \Phi(a')$. Thus Φ is injective. It remains to prove that Φ preserves application. For this let $a, a' \in A$ be such that $a * a' \downarrow$ and compute as follows:

$\Phi(a)\Phi(a') = \{b \mid \exists B \subseteq \Phi(a')((B, b) \in \Phi(a))\} =$

$= \{b \mid \exists B \subseteq \Phi(a') \exists n((B, b) \in \Phi_{n+1}(a))\}$, since B is finite

$= \{b \mid \exists n \exists a''(a'' \in \Phi(a') \ \wedge\ a * a'' \downarrow \ \wedge\ b \in \Phi_n(a * a''))\}$,

$= \{b \mid \exists n(b \in \Phi_n(a * a'))\}$, using $(*)$

$= \cup_{n \in \omega} \Phi_n(a * a') = \Phi(a * a')$. □

One can even go one step up in the hierarchie of models and can show that pas's are embeddable into extensional topological ca's. This embedding result is originally due to Scott who showed that every pas is embeddable into a D_∞-model (cf. [B], 18.4.31). However, sticking to graph algebras, we shall follow Engeler and show that every pas is embeddable into an extensional collapse as described in chapter 3.

6.2 Embeddings into extensional topological ca's and pca's

Recall that one can collapse D_A onto the extensional topological ca $M(A)$ as follows: let $\sqsubseteq$ be the reflexive and transitive relation on $\mathcal{P}(G(A))$ as defined in 3.1.1 and put

$$X \equiv Y \longleftrightarrow X \sqsubseteq Y \ \wedge\ Y \sqsubseteq X,$$

$$[X] = \{Y \mid Y \equiv X\},$$
$$\mathcal{P}(G(A))/\equiv \; = \{[X] \mid X \in \mathcal{P}(G(A))\},$$
$$[X] * [Y] = [\{b \mid \exists B \sqsubseteq Y((B,b) \in X)\} \cup (X \cap A)].$$

The extensional collapse of D_A, then, is

$$M(A) = (\mathcal{P}(G(A))/\equiv, *, [K], [S])$$

where K and S are chosen appropriately (cf. 3.1.4).
One can make $\mathcal{P}(G(A))/\equiv$ into a cpo by defining

$$[X] \sqsubseteq [Y] \longleftrightarrow X \sqsubseteq Y.$$

Bottom and *supD* are then given by $[\emptyset]$ and $[\cup\{X \mid [X] \in D\}]$, respectively.

For the Embedding Theorem we now need the following properties of $\sqsubseteq$:

Lemma 6.2.1 Let $a \in A$, $(B,b) \in G(A)$ and $X, Y \subseteq G(A)$. Then

(i) $X \subseteq Y \longrightarrow X \sqsubseteq Y$;

(ii) $\{a\} \sqsubseteq X \longleftrightarrow a \in X \;\vee\; \exists(\emptyset, b) \in X(\{a\} \sqsubseteq \{b\})$;

(iii) $\{(B,b)\} \sqsubseteq X \longleftrightarrow \exists a \in X \cap A(\{b\} \sqsubseteq \{a\}) \;\vee$
$$\exists (C,c) \in X(C \sqsubseteq B \;\wedge\; \{b\} \sqsubseteq \{c\}).$$

PROOF. This follows immediately from definition 3.1.1. □

The embedding will look pretty much the same as the one defined in the previous theorem. However, since elements of A function in the extensional collapse as elementary instructions, one cannot define $\Phi_0(a) = \{a\}$. We therefore modify Φ in the following way:

Definition 6.2.2 Let $(A, *)$ be a pas. Fix $a_0 \in A$ and put for $a \in A$ $\Gamma(a) := (\{a_0\}, a)$. Define the map Φ of A into $\mathcal{P}(G(A))$ recursively by

$$\Phi_0(a) = \{\Gamma(a)\},$$

$$\Phi_{n+1}(a) = \Phi_n(a) \cup \{(\{\Gamma(a')\}, b) \mid a * a' \downarrow \;\wedge\; b \in \Phi_n(a * a')\},$$
$$\Phi(a) = \cup_{n \in \omega} \Phi_n(a). \;\square$$

Then

Lemma 6.2.3 For all $a, a' \in A$ and $n \in \omega$

(i) $\Phi(a) \sqsubseteq \Phi(a') \longrightarrow a = a'$;

(ii) $a * a' \downarrow \longrightarrow [\Phi_{n+1}(a)][\Phi_n(a')] = [\Phi_n(a * a')]$.

PROOF. Notice that for all a firstly, $\Phi(a) \cap A = \emptyset$ and secondly, that $\Phi(a)$ does not contain any pair of the form $(\emptyset, b)$. It thus follows from lemma 6.2.1(ii),(iii) that for all a, a' one has

(1) $\{a\} \not\sqsubseteq \Phi(a')$;

(2) if $\{\Gamma(a)\} \sqsubseteq \Phi(a')$, then

$$\{\Gamma(a)\} \sqsubseteq \{\Gamma(a')\} \;\vee\; \exists a''(\, a' * a'' \downarrow \;\wedge\; \{a\} \sqsubseteq \Phi(a' * a'')\,).$$

(i) Assume $\Phi(a) \sqsubseteq \Phi(a')$. Then $\{\Gamma(a)\} \sqsubseteq \Phi(a')$, since $\{\Gamma(a)\} \sqsubseteq \Phi(a)$. Hence $\{\Gamma(a)\} \sqsubseteq \{\Gamma(a')\}$, by (1) and (2). Thus $a = a'$, by (iii) and (ii) of lemma 6.2.1.
For (ii) compute

$$\{b \mid \exists B \sqsubseteq \Phi_n(a')(\,(B, b) \in \Phi_{n+1}(a)\,)\} =$$

$$= \{b \mid \exists a''(\, a * a'' \downarrow \;\wedge\; \{\Gamma(a'')\} \sqsubseteq \Phi_n(a') \;\wedge\; b \in \Phi_n(a * a'')\,)\} \text{ by (1)}$$
$$= \{b \mid b \in \Phi_n(a * a')\}, \; (*)$$
$$= \Phi_n(a * a')$$

($(*)$: since $\{\Gamma(a'')\} \sqsubseteq \Phi_n(a')$ only if $a'' = a'$). Thus

$$[\Phi_{n+1}(a)][\Phi_n(a')] =$$

$$= [\{b \mid \exists B \sqsubseteq \Phi_n(a')(\,(B, b) \in \Phi_{n+1}(a)\,)\}], \text{ since } \Phi_{n+1}(a) \cap A = \emptyset$$
$$= [\Phi_n(a * a')].\square$$

Now we are ready to prove

Theorem 6.2.4 Every partial applicative structure can be embedded into an extensional topological ca.

PROOF. Define $\Theta : A \to \mathcal{P}(G(A))/\equiv$ by $\Theta(a) = [\Phi(a)]$. Observe that $\Theta(a) = sup\{[\Phi_n(a)] \mid n \in \omega\}$, since $\{[\Phi_n(a)] \mid n \in \omega\}$ is directed by 6.2.1(i).

To prove that Θ is injective assume $\Theta(a) = \Theta(a')$. Then $\Phi(a) \equiv \Phi(a')$. Hence $a = a'$, by 6.2.3(i).

Now let $a, a' \in A$ be such that $a * a' \downarrow$. Then $\Theta(a)\Theta(a') = sup\{[\Phi_{n+1}(a)][\Phi_n(a')] \mid n \in \omega\}$, since application is continuous. Hence $\Theta(a)\Theta(a') = sup\{[\Phi_n(a * a')] \mid n \in \omega\} = \Theta(a * a')$, by lemma 6.2.3 (ii). □

This result may be puzzling at first sight in as much as even nonextensional applicative structures can be embedded into extensional ones. However, it is not puzzling at all once one observes that the embedding is constructed in such a way that distinct elements of A have different graphs under the embedding. Plunging further into the hardware of the extensional collapse construction one will for example discover that $\Theta(a)[\{a_0\}] = sup\{\Theta(a * a_0), [\{a\}]\} \neq sup\{\Theta(a' * a_0), [\{a'\}]\} = \Theta(a')[\{a_0\}]$, provided $a \neq a'$. The puzzle therefore reduces to the less surprising observation that having identical *local* graphs is not enough to ensure that the *global* graphs coincide. We shall come back to the issue of local versus global extensionality in the next chapter.

In the case of $(A, *)$ being a nontotal applicative structure, the extensional collapse can reflect its applicative behaviour only to some extent, since application is always defined. We shall therefore discuss now embeddings into extensional topological pca's.

In chapter 4 we introduced a construction method for extensional topological pca's using again essentially Engeler's graph algebra's. Let us briefly recall the necessary ingredients.

This time we start with an arbitrary poset $(A, \sqsubseteq_A)$ with bottom $\bot_A$ and let $G(A)$ be the closure of A under ordered pairs of finite, *nonempty* subsets and elements of $G(A)$. Again we construct a reflexive and transitive relation $\sqsubseteq$ on $\mathcal{P}(G(A))$ such that $(\mathcal{P}(G(A))/\equiv, \sqsubseteq)$ forms a cpo with bottom $[\emptyset]$ and $supD = [\cup\{X \mid [X] \in D\}]$. By defining

$$F([X]) = \lambda[Y].[\{b \mid \exists[B] \sqsubseteq [Y]\ [\{(B, b)\}] \sqsubseteq [X]\}]$$

$$G(f) = [\{(B, b) \mid [\{b\}] \sqsubseteq f([B])\} \cup \{\bot_A\}]$$

$(\mathcal{P}(G(A))/\equiv, \sqsubseteq)$ becomes p-reflexive (cf. 4.2.12). One can therefore define a nontotal application operation on $(\mathcal{P}(G(A))/\equiv)\backslash[\emptyset]$ by

$$[X] * [Y] = \begin{cases} F([X])([Y]) & \text{if } F([X])[Y]) \neq [\emptyset] \\ \text{undefined} & \text{otherwise.} \end{cases}$$

The nontotal extensional collapse, then, is

$$PCA(\mathcal{P}(G(A))/\equiv) = ((\mathcal{P}(G(A))/\equiv)\backslash[\emptyset]\,,\, *\,,\, K\,,\, S)$$

where K and S are again chosen appropriately (cf. 4.2.8).
$PCA(\mathcal{P}(G(A))/\equiv)$ is in fact an extensional topological pca, since F and G satisfy $G \circ F = id_{(\mathcal{P}(G(A))/\equiv)\backslash[\emptyset]}$ (cf. 4.2.14).
Let us first state the properties of $\sqsubseteq$ needed for the Embedding Theorem.

Lemma 6.2.5 Let $a \in A$, $(B, b) \in G(A)$ and $X, Y \subseteq G(A)$. Then

(i) $X \sqsubseteq Y \longleftrightarrow \forall x \in X \exists y \in Y(\{x\} \sqsubseteq \{y\})$;

(ii) $X \subseteq Y \longrightarrow X \sqsubseteq Y$;

(iii) $\{a\} \sqsubseteq X \longleftrightarrow ((a = \perp_A \ \wedge X \neq \emptyset) \vee$

$$\exists a' \in X \cap A(a \sqsubseteq_A a') \ \vee \ \exists(\{\perp_A\}, c) \in X(\{a\} \sqsubseteq \{c\}));$$

(iv) $\{(B, b)\} \sqsubseteq X \longleftrightarrow (\exists a \in X \cap A(a \neq \perp_A \ \wedge \{b\} \sqsubseteq \{a\}) \vee$

$$\exists(C, c) \in X(C \sqsubseteq B \ \wedge \ \{b\} \sqsubseteq \{c\})).$$

PROOF. This follows immediately from definition 4.2.10. □

As often with quotient spaces, one can fiddle around with the definitions such that representatives of equivalence classes appear in a form which is more convenient. We shall first state application in a form which is more suitable for our purpose.

Lemma 6.2.6 For all $X, Y \subseteq G(A)$

$$F([X])([Y]) = [\{b \mid \exists B \sqsubseteq Y((B, b) \in X)\} \cup \{a \in X \cap A \backslash \{\perp_A\} \mid Y \neq \emptyset\}].$$

PROOF. It is sufficient to prove that $Z \equiv Z'$, where

$$Z = \{b \mid \exists B \sqsubseteq Y(\{(B,b)\} \sqsubseteq X)\}$$

and

$$Z' = \{b \mid \exists B \sqsubseteq Y((B,b) \in X)\} \cup \{a \in X \cap A \backslash \{\bot_A\} \mid Y \neq \emptyset\}.$$

If $Y = \emptyset$ then $Z = \emptyset = Z'$, since by 6.2.5(i) $B \sqsubseteq \emptyset$ only if $B = \emptyset$, but $(\emptyset, b) \notin G(A)$. Now suppose $Y \neq \emptyset$. Then

$$Z' = \{b \mid \exists B \sqsubseteq Y((B,b) \in X)\} \cup (X \cap A \backslash \{\bot_A\}).$$

To prove that $Z \sqsubseteq Z'$ let $b \in Z$. Then $\{(B,b)\} \sqsubseteq X$ for some $B \sqsubseteq Y$. Hence there is an $x \in X$ such that either $x \in A \backslash \{\bot_A\}$ and $\{b\} \sqsubseteq \{x\}$, or $x = (C,c)$ and $C \sqsubseteq B$, $\{b\} \sqsubseteq \{c\}$. In the first case one has $x \in Z'$, in the second $c \in Z'$, since $C \sqsubseteq B \sqsubseteq Y$. Thus $Z \sqsubseteq Z'$, by 6.2.5(i).

For the converse we shall prove that $Z' \subseteq Z$. Then also $Z' \sqsubseteq Z$, by 6.2.5(ii). First observe that $\{b \mid \exists B \sqsubseteq Y((B,b) \in X)\} \subseteq Z$, by 6.2.5(ii). Now let $b \in X \cap A \backslash \{\bot_A\}$. Then $\{(\{\bot_A\}, b)\} \sqsubseteq X$, by 6.2.5(iv), and $\{\bot_A\} \sqsubseteq Y$, by 6.2.5(iii). Hence $b \in Z$. □

For practical reasons we shall work with this definition of application rather than the original one. We shall moreover 'restrict' ourselves to *monotone* partial applicative structures.

Definition 6.2.7 An expanded pas $(A, *, \sqsubseteq_A)$ is *monotone* if $(A, \sqsubseteq_A)$ is a poset and $*$ is monotone in both its arguments, i.e. for all $a, a', a'' \in A$

$$a \sqsubseteq_A a' \ \wedge \ a * a'' \downarrow \longrightarrow a' * a'' \downarrow \ \wedge a * a'' \sqsubseteq_A a' * a'',$$

$$a \sqsubseteq_A a' \ \wedge \ a'' * a \downarrow \longrightarrow a'' * a' \downarrow \ \wedge a'' * a \sqsubseteq_A a'' * a'.$$

□

Notice that this does not mean a restriction at all, since *every* partial applicative structure $(A, *)$ trivially becomes monotone when A is regarded to be partially ordered by Δ_A, the diagonal relation on A.

Definition 6.2.8 Let $(A, *, \sqsubseteq_A)$ be a monotone pas.

(i) Extend $(A, \sqsubseteq_A)$ to a poset with bottom by adding a least element, i.e. let $\perp \notin A$ and put

$$A^+ = A \cup \perp,$$

$$\sqsubseteq_{A^+} = \{(\perp, x) \mid x \in A^+\} \cup \sqsubseteq_A .$$

(ii) Fix $a_0 \in A$ and put for $a \in A$ $\Gamma(a) := (\{a_0\}, a)$. Define the map Φ of A into $\mathcal{P}(G(A^+))$ recursively by

$$\Phi_0(a) = \{\Gamma(a)\},$$

$$\Phi_{n+1}(a) = \Phi_n(a) \cup \{(\{\Gamma(a')\}, b) \mid a * a' \downarrow \ \wedge\ b \in \Phi_n(a * a')\},$$

$$\Phi(a) = \cup_{n \in \omega} \Phi_n(a). \ \square$$

Then

Lemma 6.2.9 For all $a, a' \in A$ and $n \in \omega$

(i) $\Phi(a) \sqsubseteq \Phi(a') \longleftrightarrow a \sqsubseteq_A a'$;

(ii) $a * a' \downarrow \longrightarrow [\Phi_{n+1}(a)][\Phi_n(a')] = [\Phi_n(a * a')]$;

(iii) $[\Phi(a)][\Phi(a')] \downarrow \longleftrightarrow a * a' \downarrow$.

PROOF. Notice again that for all $a \in A$ firstly, $\Phi(a) \cap A^+ = \emptyset$ and secondly, that $\Phi(a)$ does not contain any pair of the form $(\{\perp_{A^+}\}, b)$. It thus follows from lemma 6.2.5(iii),(iv) that for all $a, a' \in A$ one has

(1) $\{a\} \not\sqsubseteq \Phi(a')$;

(2) if $\{\Gamma(a)\} \sqsubseteq \Phi(a')$, then

$$\{\Gamma(a)\} \sqsubseteq \{\Gamma(a')\} \ \vee\ \exists a''(\, a' * a'' \downarrow \ \wedge\ \{a\} \sqsubseteq \Phi(a' * a'')\,).$$

(i) Left to right follows again from (1) and (2) applying 6.2.5(iii). For the converse it is sufficient to prove for all n

$$\Phi_n(a) \sqsubseteq \Phi_n(a'),$$

whenever $a \sqsubseteq_A a'$. This is done by induction. The basis case is given by 6.2.5(iv),(iii). Let $a \sqsubseteq_A a'$. By induction hypothesis we can restrict ourselves to showing that

$$\Phi_{n+1}(a)\backslash\Phi_n(a) \sqsubseteq \Phi_{n+1}(a')$$

in order to prove $\Phi_{n+1}(a) \sqsubseteq \Phi_{n+1}(a')$.
So let $(\{\Gamma(a'')\}, b) \in \Phi_{n+1}(a)\backslash\Phi_n(a)$. Then $a * a'' \downarrow$ and $b \in \Phi_n(a * a'')$. Since $(A, *, \sqsubseteq_A)$ is monotone, it follows that $a' * a'' \downarrow$ and by induction hypothesis also $\Phi_n(a*a'') \sqsubseteq \Phi_n(a'*a'')$. Thus by 6.2.5(i) $(\{\Gamma(a'')\}, b') \in \Phi_{n+1}(a')$, for some $\{b'\} \sqsupseteq \{b\}$. But

$$\{(\{\Gamma(a'')\}, b)\} \sqsubseteq \{(\{\Gamma(a'')\}, b')\}$$

by 6.2.5(iv),(ii).

For (ii) compute

$$\{b \mid \exists B \sqsubseteq \Phi_n(a')(\,(B, b) \in \Phi_{n+1}(a)\,)\} =$$

$$= \{b \mid \exists a''(\,a * a'' \downarrow \;\wedge\; \{\Gamma(a'')\} \sqsubseteq \Phi_n(a') \;\wedge\; b \in \Phi_n(a * a'')\,)\} \text{ by (1)}$$

$$\equiv \{b \mid b \in \Phi_n(a * a')\} \; (*)$$

$$= \Phi_n(a * a').$$

($(*)$: $\{\Gamma(a'')\} \sqsubseteq \Phi_n(a')$ only if $a'' \sqsubseteq a'$, but then $\Phi_n(a * a'') \sqsubseteq \Phi_n(a * a')\}$ since $(A, *)$ is monotone.) Thus

$$F([\Phi_{n+1}(a)])([\Phi_n(a')]) =$$

$$= [\{b \mid \exists B \sqsubseteq \Phi_n(a')(\,(B, b) \in \Phi_{n+1}(a)\,)\}], \; (*)$$

$$= [\Phi_n(a * a')] \neq \emptyset$$

($(*)$: apply lemma 6.2.6 and the fact that $\Phi_{n+1} \cap A = \emptyset$). Hence

$$[\Phi_{n+1}(a)][\Phi_n(a')] = [\Phi_n(a * a')].$$

(iii) Right to left follows from (ii), since F is monotone and therefore

$$[\emptyset] \neq [\Phi_n(a * a')] = F([\Phi_{n+1}(a)])([\Phi_n(a)]) \sqsubseteq F([\Phi(a)])([\Phi(a)]).$$

For the converse suppose $F([\Phi(a)])([\Phi(a)]) \neq [\emptyset]$. Then

$$\{b \mid \exists B \sqsubseteq \Phi(a')(\,(B, b) \in \Phi(a)\,)\} \neq \emptyset.$$

Hence $\{\Gamma(a'')\} \sqsubseteq \Phi(a')$ for some $a'' \in A$ such that $a * a'' \downarrow$. But then $a * a' \downarrow$, since $a'' \sqsubseteq a'$ and $(A, *)$ is monotone. □

As an immediate consequence we then have

Theorem 6.2.10 Let $(A, *, \sqsubseteq_A)$ be a monotone partial applicative structure. Then there is a monotone embedding Θ from $(A, *, \sqsubseteq_A)$ into $PCA(\mathcal{P}(G(A^+))/\equiv)$ such that for all $a, a' \in A$

$$a * a' \downarrow \longleftrightarrow \Theta(a)\Theta(a') \downarrow .$$

PROOF. To prove the theorem define $\Theta(a) = [\Phi(a)]$. Then Θ is a monotone injection by 6.2.9(i). For the preservation of application combine the continuity of F and 6.2.9(ii). The last claim follows from 6.2.9(iii) □

Cpo's equipped with a Scott-continuous application operation have been considered as the most natural structures to interpret λ-terms. We shall finally show that every monotone pca possessing a least element arises as a dense (wrt. to the Scott topology) submodel of such a structure.

The closures needed for the embedding theorem can be constructed in a canonical way using theorem 6.2.10. Throughout what follows we let Θ be the embedding of $(A, *', \sqsubseteq_A)$ into $PCA(\mathcal{P}(G(A^+))/\equiv)$ as defined in theorem 6.2.10, $*$ be the partial continuous application operation defined on $PCA(\mathcal{P}(G(A^+))/\equiv)$ and $\sqsubseteq$ be the partial order making $PCA(\mathcal{P}(G(A^+))/\equiv)$ into a cpo.

Definition 6.2.11 Let $M = (A, *', K, S, \bot_A, \sqsubseteq_A)$ be a monotone pca with bottom and define the *closure*, $C(M)$, of M by $C(M) = (C(A), *, \Theta(K), \Theta(S), \Theta(\bot_A), \sqsubseteq)$ where

$$C(A) := \{sup\{\Theta(a) \mid a \in D\} \mid D \subseteq A \text{ directed}\}.\square$$

Then

Lemma 6.2.12 Let $M = (A, *', K, S, \bot_A, \sqsubseteq_A)$ be a monotone pca with bottom. Then

(i) $(C(A), \Theta(\bot_A), \sqsubseteq)$ is a cpo;

(ii) $(C(A), *, \Theta(K), \Theta(S))$ is a pca and $*$ is continuous with respect to the Scott topology.

PROOF. For (ii) observe that if K, S satisfy the axioms of combinatory logic in $(A, *')$, then $\Theta(K), \Theta(S)$ do so in $(C(A), *)$, since Θ preserves application and $*$ is continuous.
(i) Clearly, $(C(A), \sqsubseteq, \Theta(\bot_A))$ is a poset with bottom, since $(A, \sqsubseteq_A, \bot_A)$ is a poset with bottom and Θ is monotone. Before we prove that $C(A)$ is closed under sup's of directed subsets, observe that for $a \in A$, $D \subseteq A$ one has

$$\Theta(a) \sqsubseteq sup\{\Theta(a') \mid a' \in D\} \longrightarrow \exists a' \in D(\, \Theta(a) \sqsubseteq \Theta(a')\,) \;:$$

Assume $\Theta(a) \sqsubseteq sup\{\Theta(a') \mid a' \in D\}$, i.e. $\Phi(a) \sqsubseteq \cup\{\Phi(a') \mid a' \in D\}$. Then also $\{\Gamma(a)\} \sqsubseteq \cup\{\Phi(a') \mid a' \in D\}$. Hence $\{\Gamma(a)\} \sqsubseteq \Phi(a')$, for some $a' \in D$. But this holds only if $a \sqsubseteq_A a'$. Now apply the monotonicity of Θ.

Now let $D \subseteq C(A)$ be directed and define

$$D' := \{a \mid \exists E \subseteq A(\, E \text{ directed} \wedge a \in E \wedge sup\{\Theta(a') \mid a' \in E\} \in D\,)\}.$$

We shall first prove that D' is directed:

Clearly $D' \neq \emptyset$, since $D \neq \emptyset$. Let $a, a' \in D'$. To prove that $a, a' \sqsubseteq a''$ for some $a'' \in D'$ choose directed $E, E' \subseteq A$ such that $sup\{\Theta(a'') \mid a'' \in E\}$,$sup\{\Theta(a'') \mid a'' \in E'\} \in D$ and $a, a' \in E$. Let $d \in D$ be such that $sup\{\Theta(a'') \mid a'' \in E\}$,$sup\{\Theta(a'') \mid a'' \in E'\} \sqsubseteq d$. Then $\Theta(a)$,$\Theta(a') \sqsubseteq d$. But $d = sup\{\Theta(e) \mid e \in E''\}$ for some directed $E'' \subseteq A$. So $\Theta(a) \sqsubseteq \Theta(e)$, $\Theta(a') \sqsubseteq \Theta(e')$ for certain $e, e' \in E''$. Now choose $e'' \in E''$ such that $e, e' \sqsubseteq_A e''$. Then $\Theta(a), \Theta(a') \sqsubseteq \Theta(e'')$. Hence $a, a' \sqsubseteq_A e''$.
Now since D' is directed, $sup\{\Theta(a) \mid a \in D'\} \in C(A)$ and it follows straightforwardly that $sup\{\Theta(a) \mid a \in D'\} = supD$. □

Theorem 6.2.13 Let $M' = (A', *', K', S', \bot_{A'}, \sqsubseteq_{A'})$ be a monotone pca with bottom. Then there is a pca $M = (A, *, K, S, \bot_A, \sqsubseteq_A)$, with $(A, \bot_A, \sqsubseteq_A)$ a cpo and $*$ Scott-continuous, and a monotone embedding Θ such that

(i) $\Theta(K') = K$, $\Theta(S') = S$;

(ii) $\forall a, a' \in A'(\, a *' a' \downarrow \longleftrightarrow \Theta(a) * \Theta(a') \downarrow\,)$;

(iii) for all $x, y \in A$

$$\forall a \in A'(\, x * \Theta(a) \simeq y * \Theta(a)\,) \longrightarrow \forall z \in A(\, x * z \simeq y * z\,).$$

PROOF. Let $M = C(M')$ and apply lemma 6.2.12. The last claim follows from the continuity of $*$. □

In [Sa] Sanchis introduced a Π^1_1-variant of the well-known model P_ω, the so-called *hypergraphmodel* H_ω, by defining on the universe $\mathcal{P}(\omega)$ a total application operation which is monotone with respect to $\subseteq$ but hopelessly discontinuous with respect to the Scott topology on $(\mathcal{P}(\omega), \emptyset, \subseteq)$. By the preceding theorem one can embed H_ω into the ca $C(H_\omega)$ where application is continuous. Notice, however, that continuity is merely half a way to weak extensionality. In [Ko] Koymans proved that H_ω is not a λ-model. It therefore follows from 6.2.13(iii) that $C(H_\omega)$ is not a λ-model either.

Chapter 7

Finite Type Structures within PCA's

7.1 Introduction

The principal aim of this chapter will be to study finite type structures within pca's. In order to do so, we shall define an expansion pca^+ which is enriched with natural numbers, a successor operator, a predecessor operator and a numerical definition-by-cases operator.

Definition 7.1.1 A pca^+, is an expanded partial applicative structure $(A, *, K, S, 0, S_N, P_N, D, N)$ such that

(i) $(A, *, K, S)$ is a pca,

(ii) $0 \in N \ \wedge \ \forall a \in N(S_N a \in N \ \wedge \ P_N(S_N a) = a \ \wedge \ S_N a \neq 0)$,

(iii) $\forall a \in N(a \neq 0 \rightarrow P_N a \in N \ \wedge \ S_N(P_N a) = a)$,

(iv) $\forall a, a' \in N \forall b, b' \in A(Dbb'aa = b' \wedge (a \neq a' \rightarrow Dbb'aa' = b)).\Box$

A common and important feature of nontrivial pca's is, that in them one can define the additional combinators $0, S_N, P_N$ and D with the aid of the combinators K and S. These are standard tricks in combinatory logic, which however require some adaptation to the present situation where application is partial. For the traditional treatment, see for example [B], chap.6, sect.2.

Proposition 7.1.2 Every nontrivial pca $M = (A, *, K, S)$ can be expanded to a pca$^+$.

PROOF. First recall that for each variable x and every term t over M, with variables among $x, x_0, ..., x_n$, there is a term $< x > t$ over M such that for all $a, a_0, ..., a_n \in A$ one has

$$(< x > t)[x_0, ..., x_n := a_0, ..., a_n] \downarrow, \; (1)$$

$$((< x > t)*a)[x_0, ..., x_n := a_0, ..., a_n] \simeq t[x, x_0, ..., x_n := a, a_0, ..., a_n]. \; (2)$$

We now abbreviate (omitting $*$)

$$I := SKK, \; t := K, \; f := KI,$$

$$P :=< x >< y >< z > zxy, \; P_1 :=< x > xK, \; P_2 :=< x > x(KI).$$

Moreover, we let

$$0 := I, \; S_N :=< x > Pfx, \; P_N :=< x > P_1 x 0 (P_2 x)$$

$$N := \{S_N^n 0 \,|\, n \in \omega\}$$

where $S_N^0 := I$, $S_N^{n+1} :=< x > S_N(S_N^n x)$. Using (1) and (2) above it is readily checked that one then has

$$0 \in N \;\wedge\; \forall a \in N(S_N a \in N \;\wedge\; P_N(S_N a) = a \;\wedge\; S_N a \neq 0),$$

$$\forall a \in N(a \neq 0 \rightarrow P_N a \in N \;\wedge\; S_N(P_N a) = a).$$

Note that nontriviality is essential for M in order to satisfy $S_N a \neq 0$ for all $a \in N$.

To prove (iv) we shall use for $t_1 t_2 t_3$ the suggestive notation *if* t_1 *then* t_2 *else* t_3, for if $t_1 \equiv t$ (true) then $t_1 t_2 t_3 \simeq t_2$, and if $t_1 \equiv f$ (false) then $t_1 t_2 t_3 \simeq t_3$. Now observe that by (1), (2) there exists the fixed point operator $FIX \equiv< x > \chi\chi$ with $\chi \equiv< y >< z > x(yy)z$, in M satisfying

$$FIXaa' \simeq a(FIXa)a'$$

for all $a, a' \in A$. If we thus put

$$r :=< u >< v > if \; P_1 v \; then \; Kxv \; else \; < z > yz(u(P_N z))v$$

and $REC :=< x >< y > FIXr$ then REC behaves as a recursor, i.e.

$$RECaa'0 = a,$$

$$a'' \in N \ \wedge \ a'' \neq 0 \rightarrow RECaa'a'' \simeq a'(RECaa'(P_N a''))a''.$$

Hence on the set N of numerals we have explicit definition (via $< x >$) and primitive recursion; $Z := K0$ represents the zero-function and $\Pi_i^n :=< x_1 > ... < x_n > x_i$ a projection. We thus have all primitive recursive functions available and can therefore construct a term t such that $t(S_N^n 0)(S_N^m 0) = S_N^{|n-m|}0$, for all $n, m \in \omega$. The numerical definition-by-cases operator D can then be defined by

$$D :=< x >< y >< u >< v > if\ P_1(tuv)\ then\ x\ else\ y.\Box$$

The reason why we define the expansion separately is that we don't want to restrict ourselves in the choice of models by the special relationship between the natural numbers, successor, etc. and the combinators K, S.

Example 7.1.3 Engeler's D_A is a nontrivial pca and can thus be expanded to a pca$^+$. Consider the special case where $A = \omega$. Here instead of appealing to the combinatorial construction in proposition 7.1.2 we can define N, 0, S_N, P_N and D directly by

$$N := \{\{n\} \,|\, n \in \omega\},$$

$$0 := \{0\},$$

$$S_N = \{(\{n\}, n+1) \,|\, n \in \omega\},$$

$$P_N = \{(\{n+1\}, n) \,|\, n \in \omega\},$$

$$D = \{(B, (C, (\{n\}, (\{m\}, b)))) \in G(\omega)$$

$$n, m \in \omega \ \wedge \ ((n = m \ \wedge \ b \in C) \ \vee \ (n \neq m \ \wedge \ b \in B))\}.$$

We leave the verification of 7.1.1 (ii)-(iv) to the reader. □

Inside a pca$^+$ M there are internal versions of finite type structures over ω, which form models of various systems of finite type arithmetic.

Definition 7.1.4 Let $M = (A, *)$ be a partial applicative structure.
(i) The set of *finite type symbols* $\mathcal{T}$ is inductively generated by the clauses

(a) $0 \in \mathcal{T}$,

(b) $\sigma, \tau \in \mathcal{T} \rightarrow (\sigma)\tau \in \mathcal{T}$ (function types).

(ii) A *finite type structure over* M, $T(M)$, is a collection $< A_\sigma >_{\sigma \in \mathcal{T}}$ such that for all $\sigma, \tau \in \mathcal{T}$

(a) $A_\sigma \subseteq A$,

(b) there are equivalence relations $=_\sigma, =_\tau$ and $=_{(\sigma)\tau}$ on A_σ, A_τ and $A_{(\sigma)\tau}$, respectively, such that $A_{(\sigma)\tau}$ is a collection of total mappings from A_σ to A_τ respecting the equalities, i.e.

$$a =_{(\sigma)\tau} a' \rightarrow \forall b, b'(\, b =_\sigma b' \rightarrow ab =_\tau a'b'\,).$$

(iii) $T(M) =< A_\sigma >_{\sigma \in \mathcal{T}}$ is called *full* iff for all $\sigma, \tau \in \mathcal{T}$

$$\forall a \in A(\,\forall b, b' \in A_\sigma(\, b =_\sigma b' \rightarrow ab =_\tau ab'\,) \rightarrow a \in A_{(\sigma)\tau}\,).$$

(iv) $T(M) =< A_\sigma >_{\sigma \in \mathcal{T}}$ is called *extensional* iff for all $\sigma, \tau \in \mathcal{T}$ and all $a, a' \in A_{(\sigma)\tau}$

$$\forall a'' \in A_\sigma(\, aa'' =_\tau a'a''\,) \rightarrow a =_{(\sigma)\tau} a'.\Box$$

Example 7.1.5 In this chapter we shall only consider the following *standard* finite type structures over a pca$^+$ M:
(a) the *full intensional type structure* $IT(M) =< IT_\sigma >_{\sigma \in \mathcal{T}}$ where

$$IT_0 = N,$$

$$IT_{(\sigma)\tau} = \{a \in A \,|\, \forall a' \in IT_\sigma\, aa' \in IT_\tau\}$$

and $=_\sigma$ is the identity on IT_σ, for all $\sigma \in \mathcal{T}$;
(b) the *full extensional finite type structure* $ET(M) =< ET_\sigma >_{\sigma \in \mathcal{T}}$ where

$$ET_0 = N,$$

$$ET_{(\sigma)\tau} = \{a \in A \,|\, \forall a', a''(\, a' =_\sigma a'' \rightarrow aa' =_\tau aa''\,)\},$$

and

$$a =_0 a' \longleftrightarrow a, a' \in N \;\wedge\; a = a',$$

$$a =_{(\sigma)\tau} a' \longleftrightarrow a, a' \in ET_{(\sigma)\tau} \;\wedge\; \forall a'' \in ET_\sigma(\, aa'' =_\tau a'a''\,).\Box$$

$IT(M)$ and $ET(M)$ are both models of the basic system HA^ω of intuitionistic arithmetic in all finite types. $ET(M)$ forms a model of the extension $HA^\omega + EXT$ of HA^ω (cf. [T], 1.6).

The next proposition is originally due to Zucker [Z] (see also Troelstra [T], 2.4.5). It states that every finite type structure over a partial applicative structure can be collapsed onto an extensional one.

Proposition 7.1.6 Let $M = (A, *)$ be a partial applicative structure. Then there is a standard procedure for constructing from a finite type structure $T(M) =< A_\sigma >_{\sigma \in \mathcal{T}}$ over M an extensional finite type structure $T(M)^E =< A_\sigma^E >_{\sigma \in \mathcal{T}}$, the *extensional collapse* of $T(M)$, such that for all $\sigma \in \mathcal{T}$

(a) $A_\sigma^E \subseteq A_\sigma$,

(b) $\forall a, a' \in A_\sigma^E (a =_\sigma a' \rightarrow a =_\sigma^E a')$.

PROOF. Define

$$a =_0^E a' \longleftrightarrow a =_0 a',$$

$$a =_{(\sigma)\tau}^E a' \longleftrightarrow a, a' \in A_{(\sigma)\tau} \ \wedge \ \forall b, b' (b =_\sigma^E b' \rightarrow ab =_\tau^E a'b').$$

Now put $A_\sigma^E := \{a \in A_\sigma \mid a =_\sigma^E a\}$. Then $=_\sigma^E$ is an equivalence relation on A_σ^E and elements of $A_{(\sigma)\tau}^E$ respect the equalities. Hence $T(M)^E$ is a finite type structure. To prove that $T(M)^E$ is extensional, let $a, a' \in A_{(\sigma)\tau}^E$ be such that $aa'' =_\tau^E a'a''$, for all $a'' \in A_\sigma^E$. Assume $b =_\sigma^E b'$. Then $ab =_\tau^E ab'$, since $a \in A_{(\sigma)\tau}^E$, and $ab' =_\tau^E a'b'$. Thus $ab =_\tau^E a'b'$. Therefore $a =_{(\sigma)\tau}^E a'$.
Clearly $A_\sigma^E \subseteq A_\sigma$. (b) is proved by induction on σ: for the induction step let $a, a' \in A_{(\sigma)\tau}^E$ be such that $a =_{(\sigma)\tau} a'$. Assume $b =_\sigma^E b'$. Then $ab =_\tau a'b$ and $ab, a'b \in A_\tau^E$. Hence $ab =_\tau^E a'b$, by the induction hypothesis. Moreover, $a'b =_\tau^E a'b'$. Thus $ab =_\tau^E a'b'$. Therefore $a =_{(\sigma)\tau}^E a'$. □

Clearly, if a finite type structure is already extensional no more identifications between functionals can be made by the extensional collapse. Conversely, if a type structure coincides with its extensional collapse then it is already extensional.

Proposition 7.1.7 Let M be a pca$^+$ and $T(M) =< A_\sigma >_{\sigma \in \mathcal{T}}$ be a finite type structure over M. Then

$$T(M) \text{ is extensional iff } T(M) = T(M)^E.$$

In particular

$$IT(M) = ET(M) \longleftrightarrow IT(M) = IT(M)^E. \square$$

The coincidence, or lack thereof, of the standard finite type structures over a given pca$^+$ is completely determined by the behaviour of the functionals of type $(\sigma)0$, so that we can concentrate on these types for our further investigations.

Lemma 7.1.8 Let M be a pca$^+$. Then
(i) if $IT_\sigma = ET_\sigma, IT_{(\sigma)0} = ET_{(\sigma)0}$ then

$$\forall \tau \in \mathcal{T}(\, IT_\tau = ET_\tau \longrightarrow IT_{(\sigma)\tau} = ET_{(\sigma)\tau}\,);$$

(ii) if for all $\sigma \in \mathcal{T}$

$$IT_\sigma = ET_\sigma \longrightarrow IT_{(\sigma)0} \subseteq ET_{(\sigma)0},$$

then $IT(M) = ET(M)$.

PROOF. (i) We prove this by induction on the complexity of τ. The basis case is given. For $\tau \equiv (\tau')\tau''$, assume $IT_{(\tau')\tau''} = ET_{(\tau')\tau''}$. Then $ET_{(\sigma)(\tau')\tau''} \subseteq IT_{(\sigma)(\tau')\tau''}$. To prove the converse, let $a \in IT_{(\sigma)(\tau')\tau''}$ and $b, b' \in ET_\sigma$ be such that $b =_\sigma b'$. Then $ab, ab' \in ET_{(\tau')\tau''}$, by the assumption. To prove $ab =_{(\tau')\tau''} ab''$ let furthermore $c \in ET_{\tau'}$ be arbitrary. Then $< x > axc \in IT_{(\sigma)\tau''} = ET_{(\sigma)\tau''}$, by the induction hypothesis. Hence $abc =_{\tau''} ab'c$. So $ab =_{(\tau')\tau''} ab'$ and therefore $a \in ET_{(\sigma)(\tau')\tau''}$.

(ii) One proves $IT_\sigma = ET_\sigma$ by induction on the complexity of σ . Clearly, $IT_0 = ET_0$. Let $\sigma \equiv (\rho)\tau$ and assume $IT_\rho = ET_\rho, IT_\tau = ET_\tau$. Then $ET_{(\rho)0} \subseteq IT_{(\rho)0}$ and therefore $IT_{(\rho)0} = ET_{(\rho)0}$ by the assumption. Thus $IT_{(\rho)\tau} = ET_{(\rho)\tau}$ by (i). $\square$

The finite type structures $IT(M)$, $IT(M)^E$ and $ET(M)$ can behave quite differently inside a pca$^+$ M. They may be distinct, isomorphic or even may coincide. Let us mention two examples.

Example 7.1.9 *PRO* ('partial recursive operations', cf. [T,vD]). The objects are the natural numbers and application $n * m$ is defined as $\{n\}(m)$, i.e. the nth partial recursive function applied to m. For N we take ω ; the interpretation of the various constants can be found by the *smn*-theorem. Then $IT(PRO) \neq ET(PRO)$, since the identity function is an element of $IT_{((0)0)0}$; also $IT(PRO)^E \neq ET(PRO)$ (cf. [T] p. 127); but $IT(PRO)^E \cong ET(PRO)$, as proved by Bezem [Bz]. □

Example 7.1.10 In the pca$^+$-expansion of D_ω as described in example 7.1.3 one has $IT(D_\omega) = IT(D_\omega)^E = ET(D_\omega)$. In order to prove this coincidence, observe first that every equivalence class in ET_σ is closed under finite unions, i.e. for all X, X'

$$(*)\ X =_\sigma X' \longrightarrow X \cup X' =_\sigma X :$$

For the induction step let $X, X' \in ET_{(\sigma)\tau}$ be such that $X =_{(\sigma)\tau} X'$. To prove that $X \cup X' \in ET_{(\sigma)\tau}$ let Y, Y' be such that $Y =_\sigma Y'$. Then $XY =_\tau X'Y$. Moreover, from the definition of application in D_ω it follows that $(X \cup X')Y = XY \cup X'Y$. Thus $(X \cup X')Y =_\tau XY$, by the induction hypothesis. Similarly one obtains $(X \cup X')Y' =_\tau XY'$. But $XY =_\tau XY'$. So $(X \cup X')Y =_\tau (X \cup X')Y'$. Hence $X \cup X' \in ET_{(\sigma)\tau}$. $X \cup X' =_{(\sigma)\tau} X$ is proved similarly.
From $(*)$ and the fact that application is monotone it now follows that $IT(D_\omega) = ET(D_\omega)$: By lemma 7.1.8(ii) it is sufficient to prove $IT_{(\sigma)0} \subseteq ET_{(\sigma)0}$, whenever $IT_\sigma = ET_\sigma$. Thus assume $IT_\sigma = ET_\sigma$ and let $X \in IT_{(\sigma)0}$ be arbitrary and Y, Y' be such that $Y =_\sigma Y'$. Then $XY \subseteq X(Y \cup Y')$, $XY' \subseteq X(Y \cup Y')$ and $XY, XY', X(Y \cup Y') \in N = \{\{n\} \mid n \in \omega\}$. Hence $XY = XY'$. Therefore $X \in ET_{(\sigma)0}$.□

Definition 7.1.11 A pca$^+$ M is called *ft-extensional* (extensional on finite types) iff

$$IT(M) = ET(M).\square$$

In the next section we shall present sufficient conditions on pca$^+$'s in order to be ft-extensional.

7.2 FT-Connected pca$^+$'s

The crux of the proof that D_ω is ft-extensional is threefold: firstly, D_ω is monotone; secondly, every pair of equivalent extensional functionals is *connected* by its union which is again an extensional functional; thirdly, the numerals are *consistent,* i.e.

$$\forall X, Y \in N(\, X \subseteq Y \longrightarrow X = Y\,).$$

The latter property, however, is independent of the special choice of N in D_ω and is shared by all monotone pca$^+$'s, i.e. pca$^+$'s which are monotone as partial applicative structures (cf. definition 6.2.7).

Lemma 7.2.1 Let $(M, \sqsubseteq)$ be a monotone pca$^+$. Then M satisfies the following *consistency property*

$$\forall a, a' \in N(\, a \sqsubseteq a' \longrightarrow a = a'\,).$$

PROOF. Let $a, a' \in N$ be such that $a \sqsubseteq a'$. Assume $a \neq a'$. Then it follows from the monotonicity of M that $a' = Daa'aa \sqsubseteq Daa'aa' = a$. Hence $a' \sqsubseteq a$ and therefore $a = a'$, contradiction. □

We now generalize the notion of connectedness in the following way:

Definition 7.2.2 Let $(M, \sqsubseteq)$ be a monotone pca$^+$. Then
(i) $a, a' \in ET_\sigma$ are called *σ-connected* iff there exists a sequence $a_0, ..., a_{n+1}$ in ET_σ such that $a_0 = a, a_{n+1} = a'$ and $a_i \sqsubseteq a_{i+1}$ or $a_i \sqsupseteq a_{i+1}$, for all $0 \leq i \leq n$.
(ii) M is called *ft-connected* iff for all $\sigma \in T$ and all $a, a' \in ET_\sigma$

$$\text{if } a =_\sigma a' \text{ then } a \text{ and } a' \text{ are } \sigma\text{-connected.} \quad \square$$

Applying lemma 7.2.1 one then has

Theorem 7.2.3 Let $(M, \sqsubseteq)$ be a ft-connected pca$^+$. Then M is ft-extensional.

PROOF. We invoke lemma 7.1.8(ii) in order to prove that M is ft-extensional. So assume $IT_\sigma = ET_\sigma$ and let $a \in IT_{(\sigma)0}$ and b, b' be such that $b =_\sigma b'$. Since M is ft-connected there is a sequence $b_0, ..., b_{n+1} \in ET_\sigma$ such that $b = b_0, b' = b_{n+1}$ and $b_i \sqsubseteq b_{i+1}$ or $b_{i+1} \sqsupseteq b_i$, for all $0 \leq i \leq n$. Then $ab_0, ..., ab_{n+1} \in N$ and $ab_i \sqsubseteq ab_{i+1}$ or $ab_{i+1} \sqsupseteq ab_i$, for all $0 \leq i \leq n$. Hence $ab = ab_0 = ... = ab_{n+1} = ab'$, by lemma 7.2.1. Thus $a \in ET_{(\sigma)0}$. □

Having seen that ft-connectedness is a sufficient condition on pca$^+$'s in order to be ft-extensional, we can also ask for sufficient conditions for ft-connectedness. The one we shall give below is again inspired by the algebraic structure of and the behaviour of application in D_ω .

Definition 7.2.4 Let $M = (A, *, \sqsubseteq)$ be a monotone partial applicative structure. M is called *finitely additive in the first argument (fafa)* iff for all $a, a', a'' \in$ A

(i) $sup\{a, a'\}$ exists in $(A, \sqsubseteq)$,

(ii) $aa'' \downarrow \wedge a'a'' \downarrow \longrightarrow (sup\{a, a'\})a'' = sup\{aa'', a'a''\}$.□

Proposition 7.2.5 Let M be fafa and M' be a pca$^+$-expansion of M. Then M' is ft-connected.

PROOF. One proves by induction on the complexity of $\sigma \in \mathcal{T}$ that

$$a =_\sigma a' \longrightarrow sup\{a, a'\} =_\sigma a.$$

For the induction step let a, a' be such that $a =_{(\sigma)\tau} a'$. To prove that $sup\{a, a'\} \in ET_{(\sigma)\tau}$ let b, b' be such that $b =_\sigma b'$. Then $ab =_\tau a'b$. Thus $(sup\{a, a'\})b = sup\{ab, a'b\}$, since M is fafa. Moreover, $sup\{ab, a'b\} =_\tau ab$ by the induction hypothesis. So $(sup\{a, a'\})b =_\tau ab$. Similarly we see that $(sup\{a, a'\})b' =_\tau ab'$. But $ab =_\tau ab'$. Hence $(sup\{a, a'\})b =_\tau (sup\{a, a'\})b'$, i.e. $sup\{a, a'\} \in ET_{(\sigma)\tau}$. $sup\{a, a'\} =_{(\sigma)\tau} a$ is proved similarly. □

Corollary 7.2.6 Let M be fafa and M' be a pca$^+$-expansion of M. Then M' is ft-extensional. □

7.3 Examples.

In this section we shall discuss several examples of ft-extensional pca$^+$'s such as D_A, P_ω, certain D_∞-models, H_ω and T^ω. Note that these examples also show that ft-extensionality does neither imply extensionality nor weak extensionality.

The Graphmodels D_A. Every graphmodel D_A is clearly fafa and thus ft-extensional. Note that the union operator is in fact representable in D_A by the set

$$\cup_A = \{(B,(C,b)) \in G(A) \,|\, b \in B \cup C\}.\Box$$

The Graphmodels P_ω. The structure of these models, as has been shown by Baeten and Boerboom [Ba,Bo], depends heavily on the specific coding used in the construction. Although P_ω-models and D_A-models are never isomorphic as pca's (see Longo [L]), they enjoy the same sufficient properties in order to be ft-extensional: again P_ω is closed under unions and application satisfies

$$(X \cup Y)Z = XZ \cup YZ.\Box$$

Extensional, Reflexive, Complete Lattices. Recall that a complete lattice is a poset $(A, \sqsubseteq)$ where every subset $X \subseteq A$ has a supremum. It is reflexive, if there are continuous maps

$$F : A \to [A \to A],\ G : [A \to A] \to A$$

such that $F \circ G = id_{[A \to A]}$, and extensional, if in addition $G \circ F = id_A$. These structures define in a natural way extensional, continuous λ-models where the total application operation $*$ is given by

$$a * a' = F(a)(a').$$

Now let $M = (A, *, \sqsubseteq)$ be obtained in the canonical way from an extensional, reflexive, complete lattice. To prove that $(sup\{a,a'\})*b = sup\{a*b, a'*b\}$, observe first that $[A \to A]$ is a complete lattice, since A is a reflexive complete lattice. So $sup\{F(a), F(a')\} \in [A \to A]$. But

$$sup\{F(a), F(a')\} = \lambda a''.sup\{F(a)(a''), F(a')(a'')\}.$$

Thus

$$sup\{F(a), F(a')\}(a'') = sup\{F(a)(a''), F(a')(a'')\},$$

for all $a'' \in A$. Since $F(a), F(a') \sqsubseteq sup\{F(a), F(a')\}$ we have

$$a = G(F(a)) \sqsubseteq G(sup\{F(a), F(a')\})$$

and

$$a' = G(F(a')) \sqsubseteq G(sup\{F(a), F(a')\}).$$

Hence $sup\{a, a'\} \sqsubseteq G(sup\{F(a), F(a')\})$. Thus $sup\{a, a'\} * b \sqsubseteq$

$\sqsubseteq G(sup\{F(a), F(a')\}) * b = F(G(sup\{F(a), F(a')\}))(b)$

$= sup\{F(a), F(a')\}(b) = sup\{F(a)(b), F(a')(b)\}.$

Hence $sup\{a, a'\} * b \sqsubseteq sup\{a * b, a' * b\}$. Moreover, it follows from the monotonicity of $*$ that $sup\{a * b, a' * b\} \sqsubseteq sup\{a, a'\} * b$. So M is fafa and therefore ft-extensional. □

In the proof above we only use the fact that $(A, \sqsubseteq)$ is closed under sups of *finite* subsets. Note, however, that the classes of cpo's which are closed under finite sups and complete lattices coincide.

From the discussion above we can conclude that every D_∞-model constructed as a projective limit of complete lattices is ft-extensional. Observe also, that the extensional collapse of Engeler's graphmodels as described in chapter 3 is ft-extensional.

Extensional, P-reflexive, Complete Lattices. Recall that an extensional, p-reflexive, complete lattice is a complete lattice $(A, \sqsubseteq)$ equipped with two continuous maps

$$F : A \to [A \to_s A], \; G : [A \to_s A] \to A$$

such that $range(G) \subseteq A \backslash \{\perp_A\}$, $F \circ G = id_{[A \to_s A]}$ and $G \circ F = id_{A \backslash \{\perp_A\}}$. As we have seen in chapter 4 one can define a nontotal application operation $*$ on $A \setminus \{\perp_A\}$ by

$$a * a' = \begin{cases} F(a)(a') & \text{if } F(a)(a') \neq \perp_A \\ \text{undefined} & \text{otherwise} \end{cases}$$

such that $(A \setminus \{\bot_A\}, *)$ can be expanded to an extensional pca. In a way which is quite similar to the method used in the previous example one can then show that every partial applicative structure $M = (A, *, \sqsubseteq)$ obtained from an extensional, p-reflexive, complete lattice is ft-extensional. □

All the examples we have discussed so far are complete lattices and ft-extensional by virtue of corollary 7.2.6. The situation is slightly more complicated with respect to the last two examples, the hypergraphmodel H_ω and the model T^ω.

The Hypergraphmodel H_ω. Recall that Sanchis' H_ω is the total, monotone applicative structure $(\mathcal{P}(\omega), *, \subseteq)$ where application is defined by

$$X * Y := \{m \mid \forall f \exists p \exists e_n \subseteq Y(< \overline{f}(p), n, m > \in X)\}.$$

Here, $< ., ., . >$ is some bijective coding of triples of natural numbers, $\{e_n \mid n \in \omega\}$ is some enumeration of the finite subsets of ω and if f is a function from ω to ω, then $\overline{f}(p)$ is some code for the sequence $f(0), ..., f(p-1)$. H_ω is a complete lattice but not fafa.

Proposition 7.3.1 H_ω is not fafa.

PROOF. Put (assuming $e_0 = \emptyset$)

$$X := \{< \overline{\lambda x.f(x) + 1}(1), 0, 0 > \mid f : \omega \to \omega\},$$

$$Y := \{< \overline{\lambda x.0}(1), 0, 0 >\}.$$

Then $(X \cup Y)Z = \{0\}$, for every $Z \subseteq \omega$:

Assume $m \in (X \cup Y)Z$, i.e for every f there are $p \in \omega$, $e_n \subseteq Z$ such that $< \overline{f}(p), n, m > \in X \cup Y$. Then $m = 0$. For the converse let $f : \omega \to \omega$. If $f(0) = 0$, then $< \overline{f}(1), 0, 0 > \in Y$. If $f(0) \neq 0$, then $< \overline{\lambda x.f'(x) + 1}(1), 0, 0 > \in X$ where $f' : \omega \to \omega$ is defined by

$$f'(x) = \begin{cases} f(0) - 1 & \text{if } x = 0 \\ f(x) & \text{otherwise.} \end{cases}$$

But $\overline{\lambda x.f'(x) + 1}(1) = \overline{f}(1)$. Hence $< \overline{f}(1), 0, 0 > \in X$. Therefore $0 \in (X \cup Y)Z$.

But $XZ = \emptyset = YZ$: $XZ = \emptyset$, since $< \overline{\lambda x.0}(p), n, m > \notin X$, for every $e_n \subseteq Z$, $p, m \in \omega$. Similarly, $YZ = \emptyset$, since $< \overline{\lambda x.1}(p), n, m > \notin Y$, for every $e_n \subseteq Z$, $p, m \in \omega$. □

In H_ω there exists a sort of saturation operator Σ closing subsets of ω under 'extensions' of triples while preserving its applicative behaviour.

Definition 7.3.2 Define

$$\Sigma := \{< \alpha, n, < \beta, m, l >> \mid \exists \gamma \preceq \beta \exists e_k \subseteq e_m(< \gamma, k, l > \in e_n)\},$$

where we let α, β, γ range over codes of finite sequences and write $\alpha \preceq \beta$ if α codes a sequence that is an initial segment of the sequence coded by β. □

Σ as defined above has the following properties:

Proposition 7.3.3 Let M be a pca$^+$-expansion of H_ω , $X, Y, Z \in \mathcal{P}(\omega)$ and $ET_{(\sigma)\tau} \in ET(M)$. Then

(i) $\Sigma X = \{< \alpha, n, m > \mid \exists \beta \preceq \alpha \exists e_k \subseteq e_n(< \beta, k, m > \in X)\}$,

(ii) $X \subseteq \Sigma X$,

(iii) $\Sigma XY = XY$,

(iv) $X \in ET_{(\sigma)\tau} \longrightarrow \Sigma X =_{(\sigma)\tau} X$,

(v) $(\Sigma X \cap \Sigma Y)Z = XZ \cap YZ$.

PROOF. We leave (i) and (ii) to the reader; (iv) follows from (iii). For (iii) observe $m \in \Sigma XY \longleftrightarrow$

$$\longleftrightarrow \forall f \exists p \exists e_n \subseteq Y(< \overline{f}(p), n, m > \in \Sigma X)$$

$$\longleftrightarrow \forall f \exists p \exists e_n \subseteq Y \exists \beta \preceq \overline{f}(p) \exists e_k \subseteq e_n(< \beta, k, m > \in X)$$

$$\longleftrightarrow \forall f \exists q \exists e_k \subseteq Y(< \overline{f}(q), k, m > \in X)$$

$$\longleftrightarrow m \in XY.$$

(v) is proved in the following way: $m \in XZ \cap YZ \longleftrightarrow$

$$\longleftrightarrow \forall f \exists p, q \exists e_n, e_l \subseteq Z(< \overline{f}(p), n, m > \in X \wedge \ < \overline{f}(q), l, m > \in Y)$$

$$\longleftrightarrow \forall f \exists r \exists e_k \subseteq Z(\ \exists p \leq r \exists e_n \subseteq e_k(< \overline{f}(p), n, m > \in X)$$

$$\wedge \ \exists q \leq r \exists e_l \subseteq e_k(< \overline{f}(p), n, m > \in Y) \ ) \quad (*)$$

$$\longleftrightarrow \forall f \exists r \exists e_k \subseteq Z(< \overline{f}(r), k, m > \in \Sigma X \cap \Sigma Y)$$

$$\longleftrightarrow m \in (\Sigma X \cap \Sigma Y) Z.$$

(*) For $\rightarrow$: take e.g. $r = max(p, q)$ and $e_k = e_n \cup e_l$. □

In the discussion of preceding examples we mentioned the existence of operators within the model connecting equivalent extensional functionals: the operator $\cup_A \in D_A$, for example, connects equivalent extensional functionals X and X' such that $\cup_A X X'$ is an extensional functional of the same type and $X, X' \subseteq \cup_A X X'$.

For H_ω we shall now define operators constructing for equivalent extensional functionals X and X' a finite sequence of extensional functionals of the same type such that

- X and X' are the first and last element of this sequence, respectively,
- every predecessor in the sequence is connected with its successor.

Since 'connected' is a transitive relation it then follows that also X and X' are connected.

These operators will not depend on the choice of the numerals, but can be applied appropriately in every pca$^+$-expansion of H_ω. We shall define them using only Σ and the fact that H_ω is combinatory complete.

Definition 7.3.4 For $n, m \in \omega$ define $\Delta^{n,m} \in H_\omega$ as follows.
(i) If $m \leq n$ then $\Delta^{n,m}$ is defined inductively by

$$\Delta^{n,0} := < x_n > x_n,$$

$$\Delta^{1,1} := < x_1 >< y_1 > x_1 y_1,$$

$$\Delta^{n+2,m+1} := < x_{n+2} >< y_{m+1} > \Sigma(\Delta^{n+1,m}(x_{n+2} y_{m+1})).$$

(ii) If $m > n$ then $\Delta^{n,m} := \Delta^{n,n}$. □

Proposition 7.3.5 Let M be a pca$^+$-expansion of H_ω , $\sigma \in \mathcal{T}$ and $ET_\sigma, ET_{(\sigma)\sigma} \in ET(M)$. If $\sigma \equiv (\sigma_1)...(\sigma_n)0$, then

(i) $\forall i \in \omega\ \Delta^{n,i} \in ET_{(\sigma)\sigma}$.

Moreover, if $0 < n$, then

(ii) $\forall i \in \omega \forall X \in ET_\sigma\ \Sigma(\Delta^{n,i}X) \cap \Sigma(\Delta^{n,i+1}X) =_\sigma X$,

(iii) $\forall X, X' \in ET_\sigma(X =_\sigma X' \longrightarrow \Sigma(\Delta^{n,n}X) \cap \Sigma(\Delta^{n,n}X') =_\sigma X)$.

PROOF. We prove (i),(ii) and (iii) simultaneously by induction on n. If $n = 0$, then $\sigma \equiv 0$. Clearly (i) holds, since $\Delta^{0,i}{=}\Delta^{0,0}{=}< x_0 > x_0$, the identity operator.

If $n = 1$, then $\sigma \equiv (\sigma_1)0$. Clearly $\Delta^{1,i} \in ET_{(\sigma)\sigma}$, since $\Delta^{1,0} = < x_1 > x_1$ and $\Delta^{1,i+1}{=}\Delta^{1,1} =< x_1 >< y_1 > x_1y_1$. To prove (ii) let $X \in ET_\sigma$, $Y \in ET_{\sigma_1}$. Then

$$(\Sigma(\Delta^{1,0}X) \cap \Sigma(\Delta^{1,1}X))Y = \Delta^{1,0}XY \cap \Delta^{1,1}XY = XY \cap XY = XY,$$

by 7.3.3(v), and

$$(\Sigma(\Delta^{1,i+1}X) \cap \Sigma(\Delta^{1,i+2}X))Y = \Delta^{1,i+1}XY \cap \Delta^{1,i+2}XY =$$

$$= \Delta^{1,1}XY \cap \Delta^{1,1}XY = XY,$$

again by 7.3.3(v). Therefore $\Sigma(\Delta^{1,i}X) \cap \Sigma(\Delta^{1,i+1}X) =_\sigma X$, for all $i \in \omega$.

For (iii) let $X =_\sigma X'$ and $Y \in ET_{\sigma_1}$. Then $XY = X'Y$. Hence

$$(\Sigma(\Delta^{1,1}X) \cap \Sigma(\Delta^{1,1}X'))Y = \Delta^{1,1}XY \cap \Delta^{1,1}X'Y =$$

$$= XY \cap X'Y = XY.$$

So $\Sigma(\Delta^{1,1}X) \cap \Sigma(\Delta^{1,1}X') =_\sigma X$.

For $n+2$ observe that $\sigma \equiv (\sigma_1)...(\sigma_{n+2})0$. Then clearly $\Delta^{n+2,0} \in ET_{(\sigma)\sigma}$. To prove (i) for $i = m+1$ let $X =_\sigma X'$ and $Y =_{\sigma_1} Y'$. Then

$$\Delta^{n+2,m+1}XY = \Sigma(\Delta^{n+1,m}(XY)),$$

$$\Delta^{n+2,m+1}X'Y' = \Sigma(\Delta^{n+1,m}(X'Y')),$$

$$XY =_{(\sigma_2)\dots(\sigma_{n+2})0} X'Y'.$$

Thus from the induction hypothesis for (i) it follows that

$$\Delta^{n+1,m}(XY) =_{(\sigma_2)\dots(\sigma_{n+2})0} \Delta^{n+1,m}(X'Y').$$

Hence also

$$\Sigma(\Delta^{n+1,m}(X'Y')) =_{(\sigma_2)\dots(\sigma_{n+2})0} \Sigma(\Delta^{n+1,m}(X'Y'))$$

by 7.3.3(iv). So $\Delta^{n+2,m+1}X =_\sigma \Delta^{n+2,m+1}X'$, i.e. $\Delta^{n+2,m+1} \in ET_{(\sigma)\sigma}$.
To prove (ii) let $X \in ET_\sigma$, $Y \in ET_{\sigma_1}$. Then

$$(\Sigma(\Delta^{n+2,0}X) \cap \Sigma(\Delta^{n+2,1}X))Y = \Delta^{n+2,0}XY \cap \Delta^{n+2,1}XY =$$

$$= XY \cap \Sigma(XY) = XY$$

by 7.3.3(v),(ii) and

$$(\Sigma(\Delta^{n+2,i+1}X) \cap \Sigma(\Delta^{n+2,i+2}X))Y =$$

$$= \Sigma(\Delta^{n+1,i}(XY)) \cap \Sigma(\Delta^{n+1,i+1}(XY)) =_{(\sigma_2)\dots(\sigma_{n+2})0} XY$$

by 7.3.3(v) and the induction hypothesis for (ii). Therefore

$$\Sigma(\Delta^{n+2,i}X) \cap \Sigma(\Delta^{n+2,i+1}X) =_\sigma X,$$

for all $i \in \omega$.
Finally for (iii) let $X =_\sigma X'$ and $Y \in ET_{\sigma_1}$. Then $XY =_{(\sigma_2)\dots(\sigma_{n+2})0} X'Y$. Hence

$$(\Sigma(\Delta^{n+2,n+2}X) \cap \Sigma(\Delta^{n+2,n+2}X'))Y =$$

$$= \Sigma(\Delta^{n+1,n+1}(XY)) \cap \Sigma(\Delta^{n+1,n+1}(X'Y)) =_{(\sigma_2)\dots(\sigma_{n+2})0} XY$$

by 7.3.3(v) and the induction hypothesis for (iii). Thus again

$$\Sigma(\Delta^{n+2,n+2}X) \cap \Sigma(\Delta^{n+2,n+2}X') =_\sigma X.$$

□

From proposition 7.3.5 it now follows that for every pair of equivalent functionals $X, X' \in ET_{(\sigma_1)\dots(\sigma_n)0}$ the sequence

$$X = \Delta^{n,0}X,\ \dots, \Delta^{n,n}X,\ \Delta^{n,n}X',\ \dots, \Delta^{n,0}X' = X'$$

is a connected sequence of extensional functionals of the same type.

Proposition 7.3.6 Let M be a pca$^+$-expansion of H_ω, $\sigma \in \mathcal{T}$, $ET_\sigma \in ET(M)$ and $X, X' \in ET_\sigma$ be such that $X =_\sigma X'$. If $\sigma \equiv (\sigma_1)...(\sigma_n)0$, then

(i) $\Delta^{n,i}X$ and $\Delta^{n,i+1}X$ are σ-connected, for all $i \in \omega$,

(ii) $\Delta^{n,n}X$ and $\Delta^{n,n}X'$ are σ-connected.

PROOF. (i) Define the σ-connection of $\Delta^{n,i}X$ and $\Delta^{n,i+1}X$ by

$$X_0 := \Delta^{n,i}X,\ X_1 := \Sigma(\Delta^{n,i}X),\ X_2 := \Sigma(\Delta^{n,i}X) \cap \Sigma(\Delta^{n,i+1}X)$$

$$X_3 := \Sigma(\Delta^{n,i+1}X),\ X_4 := \Delta^{n,i+1}X.$$

Then $X_0, \dots, X_4 \in ET_\sigma$ by 7.3.3(iv), 7.3.5(i) and 7.3.5(ii). Moreover, $X_0 \subseteq X_1$, $X_2 \subseteq X_1$, $X_2 \subseteq X_3$ and $X_4 \subseteq X_3$.
(ii) Applying 7.3.5(iii) instead of 7.3.5(ii) we also see that

$$X_0 := \Delta^{n,n}X,\ X_1 := \Sigma(\Delta^{n,n}X),\ X_2 := \Sigma(\Delta^{n,n}X) \cap \Sigma(\Delta^{n,n}X'),$$

$$X_3 := \Sigma(\Delta^{n,n}X),\ X_4 := \Delta^{n,n}X'$$

is a σ-connection of $\Delta^{n,n}X$ and $\Delta^{n,n}X'$. □

Theorem 7.3.7 Let M be a pca$^+$-expansion of H_ω. Then M is ft-extensional.

PROOF. We shall prove that M is ft-connected. Equivalent type-0-objects are trivially 0-connected. Let $(\sigma)\tau \equiv (\sigma_1)...(\sigma_{n+1})0$ and X, X' be such that $X =_\sigma X'$. Then every element in the sequence $X = \Delta^{n,0}X, \dots, \Delta^{n,n}X, \Delta^{n,n}X', \dots, \Delta^{n,0}X'$ is an extensional type-$(\sigma)\tau$-object and is $(\sigma)\tau$-connected with its successor in the sequence, by the previous proposition. Thus X and X' are $(\sigma)\tau$-connected. □

The Model T^ω. T^ω was first introduced by Plotkin [P1]. However, here we refer to the description given by Barendregt and Longo in [B,L].

T^ω is a subset of $\mathcal{P}(\omega)^2$ equipped with a very special application operation. The importance of this model lies in the effectiveness properties of its semantics and the way its natural order matches the partial order on B, the λ-model of Böhm-like trees. We shall neither use nor comment on these properties. The only reason for including this model in our list of examples is that it is, as opposed to the first four examples, not fafa but, as opposed to the preceding example, a λ-model. For a thorough investigation of T^ω we refer the reader to [B,L].

The universe of T^ω is $\{< A, B > \mid A, B \in \mathcal{P}(\omega) \ \wedge \ A \cap B = \emptyset\}$. If $a \in T^\omega$ we write $a =< a_-, a_+ >$ and call $a \in T^\omega$ *finite* if $a_- \cup a_+$ is so. We let $\{e_n \mid n \in \omega\}$ be some enumeration of the finite elements of T^ω and $(.,.)$ be some bijective coding of pairs of natural numbers.

On T^ω one can define a partial order by

$$a \sqsubseteq b \longleftrightarrow a_- \subseteq b_- \text{ and } a_+ \subseteq b_+.$$

It is readily checked that $(T^\omega, \sqsubseteq)$ forms a cpo with bottom $< \emptyset, \emptyset >$ and $supD =< \cup\{d_- \mid d \in D\}, \cup\{d_+ \mid d \in D\} >$, for directed $D \subseteq T^\omega$.

T^ω belongs to the class of reflexive cpo's and defines therefore a λ-model. In order to define the appropriate retraction map, Barendregt and Longo introduce the following notations: for $n, m \in \omega$, put

$$n \uparrow m \longleftrightarrow \exists a \in T^\omega(e_n \sqsubseteq a \ \wedge \ e_m \sqsubseteq a),$$

$$D_{(n,2m+1)} := \{(n', 2m) \mid n' \uparrow n \ \wedge \ (n', 2m) \leq (n, 2m+1)\},$$

$$D_{(n,2m)} := \{(n', 2m+1) \mid n' \uparrow n \ \wedge \ (n', 2m+1) \leq (n, 2m)\}.$$

To prevent any misgivings as to the relationship between the sets D_n and the numerical definition-by-cases operator D, let us stress that there is none. We just keep close to the notations introduced in [B,L].

Definition 7.3.8 For $a, b \in T^\omega$ and $f \in [T^\omega \to T^\omega]$, define

$$(F(a)(b))_- := \{m \mid \exists e_n \sqsubseteq b((n, 2m) \in a_- \ \wedge \ D_{(n,2m)} \subseteq a_+)\},$$

$$(F(a)(b))_+ := \{m \mid \exists e_n \sqsubseteq b((n, 2m+1) \in a_- \ \wedge \ D_{(n,2m+1)} \subseteq a_+)\},$$

$$(G(f))_- := \{(n, 2m) \mid m \in (f(e_n))_-\} \cup \{(n, 2m+1) \mid m \in (f(e_n))_+\},$$

$$(G(f))_+ := \{(n, 2m) \mid \exists l(e_n \sqsubseteq e_l \ \wedge \ m \in (f(e_l))_+)\}$$

$$\cup\{(n, 2m+1) \mid \exists l(e_n \sqsubseteq e_l \ \wedge \ m \in (f(e_l))_-)\}. \ \square$$

Theorem 7.3.9 T^ω is reflexive via F and G.

PROOF. cf. [B,L], 1.5 to 1.7, section 1.

As usual one defines the total application operation $*$ on T^ω by $a * b := F(a)(b)$, that is

$(a * b)_- = \{m \,|\, \exists e_n \sqsubseteq b(\,(n, 2m) \in a_- \ \wedge \ D_{(n,2m)} \subseteq a_+\,)\}$,

$(a * b)_+ = \{m \,|\, \exists e_n \sqsubseteq b(\,(n, 2m+1) \in a_- \ \wedge \ D_{(n,2m+1)} \subseteq a_+\,)\}$.

$*$ is then continuous with respect to the Scott topology induced by $\sqsubseteq$. However, it is not fafa. First of all, T^ω is not closed under finite sup's: e.g. $< \emptyset, \{0\} >, < \{0\}, \emptyset > \in T^\omega$, but if $< \emptyset, \{0\} >, < \{0\}, \emptyset > \sqsubseteq a$, then $0 \in a_- \cap a_+$. But even if $sup\{a, b\}$ does exist it does not necessarily satisfy $(sup\{a, b\})c = sup\{ac, bc\}$. Observe, however, that T^ω is closed under inf's of nonempty sets: for all $\emptyset \neq X \subseteq T^\omega$

$$infX =< \cap\{x_- \,|\, x \in X\}, \cap\{x_+ \,|\, x \in X\} > \in T^\omega.$$

But inf does not in general satisfy $(inf\{a, b\})c = inf\{ac, bc\}$ either, so that the whole enterprise is not merely a matter of reversing the order. In order to increase the familiarity with respect to the definitions involved, we shall give an example confirming the remarks on sup and inf made above and already indicating, how we intend to connect equivalent extensional functionals.

Example 7.3.10 We shall construct $a, b, c \in T^\omega$ such that $sup\{a, b\} \in T^\omega$ and

$$(inf\{a, b\})c \sqsubset inf\{ac, bc\} \sqsubset sup\{ac, bc\} \sqsubset (sup\{a, b\})c. \ (\dagger)$$

Observe that $< \{0\}, \emptyset >$ and $< \emptyset, \{1\} >$ are finite elements of T^ω, say $< \{0\}, \emptyset >= e_m$ and $< \emptyset, \{1\} >= e_n$. Put $c :=< \{0\}, \{1\} >$. Then $c \in T^\omega$ and $e_n, e_m \sqsubseteq c$. Now define a, b by

$a_- := \{(n, 0), (n, 4)\}$,

$a_+ := D_{(n,0)} = \{(n', 1) \,|\, n' \uparrow n \ \wedge \ (n', 1) \leq (n, 0)\}$,

$$b_- := \{(m,0),(m,3)\},$$

$$b_+ := D_{(m,0)} \cup D_{(m,3)} \cup D_{(n,4)} = \{(m',1) \,|\, m' \uparrow m \ \wedge \ (m',1) \leq (m,0)\}$$
$$\cup\{(m',2) \,|\, m' \uparrow m \ \wedge \ (m',2) \leq (m,3)\}$$
$$\cup\{(n',5) \,|\, n' \uparrow n \ \wedge \ (n',5) \leq (n,0)\}.$$

Observe that $a_- \cap a_+ = \emptyset = b_- \cap b_+$, so that $a, b \in T^\omega$. Observe furthermore that
(1) $inf\{a,b\} =< a_- \cap b_-, a_+ \cap b_+ >=< \emptyset, a_+ \cap b_+ >$. Hence

$$(inf\{a,b\})c =< \emptyset, \emptyset >,$$

since $a_- = \emptyset$.
(2) $ac =< \{0\}, \emptyset >$ and $bc =< \{0\}, \{1\} >$. Therefore

$$inf\{ac, bc\} =< (ac)_- \cap (bc)_-, (ac)_+ \cap bc_+ >=< \{0\}, \emptyset >,$$

$$sup\{ac, bc\} =< (ac)_- \cup (bc)_-, (ac)_+ \cup bc_+ >=< \{0\}, \{1\} > .$$

(3) $(a_- \cup b_-) \cap (a_+ \cup b_+) = \emptyset$, since every element of $a_- \cup b_-$ is a pair of the form $(l,0), (l,3)$ or $(l,4)$, but $a_+ \cup b_+$ contains only pairs of the form $(l,1), (l,2)$ or $(l,5)$. Thus $sup\{a,b\} \in T^\omega$. Finally, since $sup\{a,b\} =$

$$=< \{(n,0),(n,4),(m,0),(m,3)\}, D_{(n,0)} \cup D_{(n,4)} \cup D_{(m,0)} \cup D_{(m,3)} >,$$

it follows that $(sup\{a,b\})c =< \{0,2\}, \{1\} >$.
Combining (1), (2) and (3) yields (†).
a is a representation of the external function

$$f_a(x) = \begin{cases} < \{0\}, \emptyset > & \text{if } e_n \sqsubseteq x \\ < \emptyset, \emptyset > & \text{otherwise.} \end{cases}$$

However, a is certainly not the only way to represent f_a in T^ω. To envisage other representations of f_a observe that the pair $(n,4)$ does not contribute to a's applicative behaviour, since $\neg D_{(n,4)} \subseteq a_+$. Hence

$$a_e :=< \{(n,0)\}, D_{(n,0)} >$$

is a second representation of f_a. a_e represents f_a in a minimal way and embodies a's essence with respect to its applicative behaviour. A third representation emerges once one observes that, without any modification of the intended applicative behaviour, one can close $(a_e)_-$ under pairs of the form $(n', 0)$ satisfying $e_n \sqsubseteq e_{n'}$ and, moreover, can drop the boundary constraint on elements in $(a_e)_+$. That is,

$$a_s := < \{(n', 0) \,|\, e_n \sqsubseteq e_{n'}\}, \{(n', 1) \,|\, n' \uparrow n\} >$$

is another representation of f_a and is a saturated version of the essence of a. Finally observe that a_e connects a with a_s, i.e.

$$a \sqsupseteq a_e \sqsubseteq a_s. \; \square$$

In T^ω there exist two operators: the first, which we shall call the essence operator, prunes those parts of an element of T^ω which do not contribute to its applicative behaviour; the second, which is again a sort of saturation operator, closes pruned elements under 'extensions' of pairs. Both operators preserve again applicative behaviour.

Definition 7.3.11 For $a \in T^\omega$, define

$$f_E(a) := < \{m \,|\, m \in a_- \;\wedge\; D_m \subseteq a_+\}, \bigcup\{D_m \,|\, m \in a_- \;\wedge\; D_m \subseteq a_+\} >,$$

$$f_\Sigma(a) := < \{(n, m) \,|\, \exists e_l \sqsubseteq e_n(\,(l, m) \in a_- \;\wedge\; D_{(l,m)} \subseteq a_+\,)\},$$

$$\bigcup\{K_m \,|\, m \in a_- \;\wedge\; D_m \subseteq a_+\} >,$$

where K_m is the unbounded version of D_m, i.e.

$$K_{(n,2m)} := \{(n', 2m+1) \,|\, n' \uparrow n\}$$

$$K_{(n,2m+1)} := \{(n', 2m) \,|\, n' \uparrow n\}. \square$$

Then

Lemma 7.3.12 f_E, $f_\Sigma \in [T^\omega \to T^\omega]$.

PROOF. We shall only prove that f_E and f_Σ are well-defined. The continuity of these functions is proved straightforwardly using the finiteness of the sets D_m. Let $a \in T^\omega$. Then $a_- \cap a_+ = \emptyset$ (†).
For f_E, observe that $f_E(a) \sqsubseteq a$. Thus $(f_E(a))_- \cap (f_E(a))_+ = \emptyset$. Therefore $f_E(a) \in T^\omega$.

To prove that f_Σ is well-defined, observe first that

(1) $(n,m) \in K_k \;\wedge\; e_l \sqsubseteq e_n \longrightarrow (l,m) \in K_k$,

(2) $(l,m) \in K_k \setminus D_k \longrightarrow k \in D_{(l,m)}$:

Say, $k = (k_0, k_1)$. (1) follows from the fact that if $n \uparrow k_0$ and $e_l \sqsubseteq e_n$, then also $l \uparrow k_0$. For (2) assume $(l,m) \in K_k \setminus D_k$. Then

$$(\, m = 2n+1 \longrightarrow k_1 = 2n \,) \;\wedge\; (\, m = 2n \longrightarrow k_1 = 2n+1 \,)$$

since $(l,m) \in K_k$, and

$$l \uparrow k_0 \;\wedge\; \neg(l,m) \leq k,$$

since $(l,m) \notin D_k$, but $(l,m) \in K_k$. Thus $k_0 \uparrow l$ and $k < (l,m)$. So $k \in D_{(l,m)}$.
Now suppose that $(n,m) \in (f_\Sigma(a))_- \cap (f_\Sigma(a))_+$. Then there are l, k such that

(3) $e_l \sqsubseteq e_n \;\wedge\; (l,m) \in a_- \;\wedge\; D_{(l,m)} \subseteq a_+$,

(4) $(n,m) \in K_k \;\wedge\; k \in a_- \;\wedge\; D_k \subseteq a_+$.

Since $e_l \sqsubseteq e_n$, it follows from (1) and (4) that $(l,m) \in K_k$. So

(5) $(l,m) \in a_- \;\wedge\; D_{(l,m)} \subseteq a_+$,

(6) $(l,m) \in K_k \;\wedge\; k \in a_- \;\wedge\; D_k \subseteq a_+$.

We now can conclude that $(l,m) \notin D_k$, since otherwise

$$(l,m) \in a_- \cap a_+.$$

Thus $(l,m) \in K_k \setminus D_k$. Therefore $k \in D_{(l,m)}$, by (2). Hence

$$k \in D_{(l,m)} \subseteq a_+ \;\wedge k \in a_-,$$

by (5) and (6). Contradiction with (†). Hence $(f_\Sigma(a))_- \cap (f_\Sigma(a))_+ = \emptyset$.
□

Lemma 7.3.13 In T^ω there exist operators E, Σ satisfying for all $a, b, c \in T^\omega$

(i) $Ea \sqsubseteq a \;\wedge\; Ea \sqsubseteq \Sigma a$,

(ii) $Eab = ab \ \wedge \ \Sigma ab = ab$,

(iii) $(inf\{\Sigma a, \Sigma b\})c = inf\{ac, bc\}$.

Moreover, if M is a pca$^+$-expansion of T^ω and $ET_{(\sigma)\tau} \in ET(M)$ then

(iv) $a \in ET_{(\sigma)\tau} \longrightarrow Ea =_{(\sigma)\tau} a =_{(\sigma)\tau} \Sigma a$.

PROOF. Put $E := G(f_E)$ and $\Sigma := G(f_\Sigma)$. Then $Ea = f_E(a)$ and $\Sigma a = f_\Sigma(a)$. (i) is left to the reader and (iv) follows from (ii). For (ii) and (iii), observe first that

$$e_l \sqsubseteq e_n \longrightarrow D_{(n,m)} \subseteq K_{(l,m)} \ (\dagger) :$$

Suppose $e_l \sqsubseteq e_n$ and let $(n', m') \in D_{(n,m)}$. Then m' is appropriate with respect to m and $n' \uparrow n$. Then also $n' \uparrow l$, since $e_l \sqsubseteq e_n$. Hence $(n', m') \in K_{(l,m)}$.

(ii) Clearly $Eab = ab$, since Ea contains precisely those elements which contribute to a's applicative behaviour. To prove $\Sigma ab = ab$ observe that $m \in (\Sigma ab)_- \longleftrightarrow$

$$\longleftrightarrow \exists e_n \sqsubseteq b(\ (n, 2m) \in (\Sigma a)_- \ \wedge \ D_{(n,2m)} \subseteq (\Sigma a)_+\ )$$

$$\longleftrightarrow \exists e_n \sqsubseteq b \exists e_l \sqsubseteq e_n(\ (l, 2m) \in a_- \ \wedge \ D_{(n,2m)} \subseteq a_+\ )\ (*)$$

$$\longleftrightarrow \exists e_l \sqsubseteq b(\ (l, 2m) \in a_- \ \wedge \ D_{(n,2m)} \subseteq a_+\ ) \longleftrightarrow m \in (ab)_-.$$

((*): for $\longleftarrow$, observe that if $(l, 2m) \in a_-$ and $D_{(l,2m)} \subseteq a_+$, then $K_{(l,2m)} \subseteq (\Sigma a)_+$. Hence by ($\dagger$) $D_{(n,2m)} \subseteq (\Sigma a)_+$, since $e_l \sqsubseteq e_n$.)
$(\Sigma ab)_+ = (ab)_+$ is proved similarly.

(iii) Since application is monotone it follows from (ii) that

$$(inf\{\Sigma a, \Sigma b\})c \sqsubseteq inf\{\Sigma ac, \Sigma bc\} = inf\{ac, bc\}.$$

For the converse, observe that $m \in (inf\{ac, bc\})_- \longrightarrow$

$$\longrightarrow m \in (ac)_- \cap (bc)_-$$

$$\longrightarrow \exists e_n, e_l \sqsubseteq c(\ (n, 2m) \in a_- \ \wedge \ D_{(n,2m)} \subseteq a_+$$

$$\wedge\ (l, 2m) \in b_- \ \wedge D_{(l,2m)} \subseteq b_+\ )$$

$$\longrightarrow_1 \exists e_n, e_l, e_k \sqsubseteq c(\,(k, 2m) \in (\Sigma a)_- \cap (\Sigma b)_- \ \wedge e_n, e_l \sqsubseteq e_k$$

$$\wedge\ K_{(n,2m)} \cap K_{(l,2m)} \subseteq (\Sigma a)_+ \cap (\Sigma b)_+\ )$$

$$\longrightarrow_2 \exists e_k \sqsubseteq c(\,(k, 2m) \in (inf\{\Sigma a, \Sigma b\})_- \wedge D_{(k,2m)} \subseteq (inf\{\Sigma a, \Sigma b\})_+\ )$$

$$\longrightarrow m \in (inf\{\Sigma a, \Sigma b\}c)_-.$$

For $\rightarrow_1$, put $e_k :=< (e_n)_- \cup (e_l)_-, (e_n)_+ \cup (e_l)_+ >$. For $\rightarrow_2$, apply (†). $(inf\{ac, bc\})_+ \subseteq ((inf\{\Sigma a, \Sigma b\})c)_+$ is proved similarly. □

We shall now proceed as in the preceding example and define operators in T^ω connecting equivalent extensional functionals by a finite sequence of connected extensional functionals of the same type.

Definition 7.3.14 For $n, m \in \omega$, define $\Delta^{n,m} \in T^\omega$ as follows.
(i) If $m \leq n$ then $\Delta^{n,m}$ is defined inductively by

$$\Delta^{n,0} :=< x_n > x_n,$$

$$\Delta^{1,1} :=< x_1 >< y_1 > x_1 y_1,$$

$$\Delta^{n+2,m+1} :=< x_{n+2} >< y_{m+1} > \Sigma(\Delta^{n+1,m}(x_{n+2} y_{m+1})).$$

(ii) If $m > n$ then $\Delta^{n,m} := \Delta^{n,n}$. □

Then

Proposition 7.3.15 Let M be a pca$^+$-expansion of T^ω, $\sigma \in \mathcal{T}$ and $ET_\sigma, ET_{(\sigma)\sigma} \in ET(M)$. If $\sigma \equiv (\sigma_1)...(\sigma_n)0$, then

(i) $\forall i \in \omega\ \Delta^{n,i} \in ET_{(\sigma)\sigma}$.

Moreover, if $0 < n$, then

(ii) $\forall i \in \omega \forall a \in ET_\sigma\ inf\{\Sigma(\Delta^{n,i} a), \Sigma(\Delta^{n,i+1} a)\} =_\sigma a$,

(iii) $\forall a, a' \in ET_\sigma(\ a =_\sigma a' \longrightarrow inf\{\Sigma(\Delta^{n,n} a), \Sigma(\Delta^{n,n} a')\} =_\sigma a\ )$.

PROOF. (i) and (iii) are proved as in proposition 7.3.5 applying 7.3.13(iii) instead of 7.3.3(v) and 7.3.13(iv) instead of 7.3.3(iv). For (ii), however, we have to make a slight modification, since the proof of 7.3.5(ii) involves 7.3.3(ii), a fact for which there is no analogous result in the present situation. The proof is again carried out by induction on n.

If $n = 1$, then $\sigma \equiv (\sigma_1)0$. Let $X \in ET_\sigma$, $Y \in ET_{\sigma_1}$. Then

$$(inf\{\Sigma(\Delta^{1,0}X), \Sigma(\Delta^{1,1}X)\})Y = inf\{\Delta^{1,0}XY, \Delta^{1,1}XY\} =$$

$$= inf\{XY, XY\} = XY,$$

by 7.3.13(iii), and

$$(inf\{\Sigma(\Delta^{1,i+1}X), \Sigma(\Delta^{1,i+2}X)\})Y = inf\{\Delta^{1,i+1}XY, \Delta^{1,i+2}XY\} =$$

$$= inf\{\Delta^{1,1}XY, \Delta^{1,1}XY\} = XY,$$

again by 7.3.13(iii). Therefore $inf\{\Sigma(\Delta^{1,i}X), \Sigma(\Delta^{1,i+1}X)\} =_\sigma X$, for all $i \in \omega$.

For $n+2$ observe that $\sigma \equiv (\sigma_1)...(\sigma_{n+2})0$. Let $X \in ET_\sigma$ and $Y \in ET_{\sigma_1}$. Then

$$(inf\{\Sigma(\Delta^{n+2,0}X), \Sigma(\Delta^{n+2,1}X)\})Y = inf\{\Delta^{n+2,0}XY, \Delta^{n+2,1}XY\} =$$

$$= inf\{XY, \Sigma(XY)\}$$

by 7.3.13(iii). To prove

$$inf\{XY, \Sigma(XY)\} =_{(\sigma_2)...(\sigma_{n+2})0} XY$$

it is clearly sufficient to prove

$$(inf\{XY, \Sigma(XY)\})Z =_{(\sigma_3)...(\sigma_{n+2})0} XYZ,$$

for all $Z \in ET_{\sigma_2}$. To this end observe that by monotonicity and 7.3.13(i),(ii) one has

$$XYZ = E(XY)Z \sqsubseteq (inf\{XY, \Sigma(XY)\})Z \sqsubseteq XYZ.$$

Hence $(inf\{XY, \Sigma(XY)\})Z = XYZ$. So

$$inf\{\Sigma(\Delta^{n+2,0}X), \Sigma(\Delta^{n+2,1}X)\} =_\sigma X.$$

Finally,

$$(inf\{\Sigma(\Delta^{n+2,i+1}X), \Sigma(\Delta^{n+2,i+2}X)\})Y =$$
$$= inf\{\Sigma(\Delta^{n+1,i}(XY)), \cap\Sigma(\Delta^{n+1,i+1}(XY))\} =_{(\sigma_2)...(\sigma_{n+2})0} XY$$

by 7.3.13(iii) and the induction hypothesis. Therefore also

$$inf\{\Sigma(\Delta^{n+2,i+1}X), \Sigma(\Delta^{n+2,i+2}X) =_\sigma X.\Box$$

Again it follows from proposition 7.3.15 that for every pair of equivalent functionals $a, a' \in ET_{(\sigma_1)...(\sigma_n)0}$ the sequence

$$a = \Delta^{n,0}a, \ldots, \Delta^{n,n}a, \Delta^{n,n}a', \ldots, \Delta^{n,0}a' = a'$$

is a connected sequence of extensional functionals of the same type.

Proposition 7.3.16 Let M be a pca$^+$-expansion of T^ω, $\sigma \in \mathcal{T}$, $ET_\sigma \in ET(M)$ and $a, a' \in ET_\sigma$ be such that $a =_\sigma a'$. If $\sigma \equiv (\sigma_1)...(\sigma_n)0$, then

(i) $\Delta^{n,i}a$ and $\Delta^{n,i+1}a$ are σ-connected, for all $i \in \omega$,

(ii) $\Delta^{n,n}a$ and $\Delta^{n,n}a'$ are σ-connected.

PROOF. (i) Define the σ-connection of $\Delta^{n,i}a$ and $\Delta^{n,i+1}a$ by

$$a_0 := \Delta^{n,i}a, \; a_1 := E(\Delta^{n,i}a), \; a_2 := \Sigma(\Delta^{n,i}a)$$
$$a_3 := inf\{\Sigma(\Delta^{n,i}a), \Sigma(\Delta^{n,i+1}a)\}$$
$$a_4 := \Sigma(\Delta^{n,i+1}a), \; a_5 := E(\Delta^{n,i+1}a), \; a_6 := \Delta^{n,i+1}a.$$

Then $a_0, \ldots, a_6 \in ET_\sigma$ by 7.3.15(i), 7.3.13(iv), and 7.3.15(ii). Moreover, $a_1 \sqsubseteq a_0$, $a_1 \sqsubseteq a_2$, $a_3 \sqsubseteq a_2$, $a_3 \sqsubseteq a_4$, $a_5 \sqsubseteq a_4$ and $a_5 \sqsubseteq a_6$, by 7.3.13(i).

(ii) Applying 7.3.15(iii) instead of 7.3.15(ii) we also see that

$$a_0 := \Delta^{n,n}a, \; a_1 := E(\Delta^{n,n}a), \; a_2 := \Sigma(\Delta^{n,n}a)$$
$$a_3 := inf\{\Sigma(\Delta^{n,n}a), \Sigma(\Delta^{n,n}a')\}$$
$$a_4 := \Sigma(\Delta^{n,n}a'), \; a_5 := E(\Delta^{n,n}a'), \; a_6 := \Delta^{n,n}a'$$

is a σ-connection of $\Delta^{n,n}a$ and $\Delta^{n,n}a'$. $\Box$

Theorem 7.3.17 Let M be a pca$^+$-expansion of T^ω. Then M is ft-extensional. $\Box$

7.4 The Countable Functionals CF.

The *pure finite types* (denoted by natural numbers) are 0 and with n also $n+1 := (n)0$. We have so far seen for various pca's that, no matter how these models are expanded to pca$^+$'s, the intensional and extensional finite type structures always coincide. We shall now identify the functionals of type 2 belonging to certain pca$^+$-expansions.

$CF =< CF_n >_{n\in\omega}$ is known as the *countable* functionals [Kl] or *continuous* functionals [Kr], and has been extensively investigated. It is a finite type structure in which each functional is globally described by a countable amount of information, coded in a type-1 object, and is locally determined by a finite amount of information about its argument. We shall first give Kleene's definition via associates:
Define

(i) $CF_0 = \omega$ and $Ass(n) = \{n\}$, for all $n \in \omega$.

(ii) $CF_1 = \omega^\omega$ and $Ass(f) = \{f\}$, for all $f \in \omega^\omega$.

(iii) Let $\Phi \in \omega^{CF_{n+1}}$.
$f \in Ass(\Phi)$ iff $f \in \omega^\omega$ and for every $\Psi \in CF_{n+1}$, $g \in Ass(\Psi)$

(1) $\exists m(f(\overline{g}(m)) > 0)$,

(2) $\forall m(f(\overline{g}(m)) > 0 \longrightarrow f(\overline{g}(m)) = \Phi(\Psi) + 1)$.

$\Phi \in CF_{n+2}$ iff $Ass(\Phi) \neq \emptyset$.

Then $CF =< CF_n >_{n\in\omega}$. As usual, $\overline{g}(m)$ is some code for the sequence $g(0), ..., g(m-1)$.

In order to interpret this definition within pca's, we shall impose a restriction on pca$^+$-expansions. In this section we shall only consider pca$^+$-expansions which have *standard integers*, i.e. we require N together with the successor operator S_N to be isomorphic to the usual structure of ω and successor. Observe e.g. that both the combinatorial pca$^+$-expansion as described in proposition 7.1.2 and the pca$^+$-expansion of D_ω as described in example 7.1.3 have standard integers. For such a pca$^+$-expansion we shall write

- $\overline{n}$ for the element of N corresponding to the natural number n;
- $\overline{n} \leq \overline{m}$ (or $\overline{n} < \overline{m}$) if $n \leq m$ (or $n < m$);
- if $a \in N$, then $a + 1$ for $S_N a$;
- if $a \in IT_1$, then $< a\overline{0}, ..., a\overline{m} >$ or $\overline{a}(\overline{m+1})$ for the element of N corresponding to $< g(0), ..., g(m) >$, where $g \in \omega^\omega$ is such that $a\overline{n} = \overline{g(n)}$, for all $n \in \omega$.

One can then interpret Kleene's original definition in the following way:

Definition 7.4.1 Let M be a pca$^+$ having standard integers. Then $CF(M) =< CF_n >_{n\in\omega}$ is the following finite type structure over M:

(i) $CF_0 = N$ and $Ass(a) = \{a\}$, for all $a \in N$.

(ii) $CF_1 = IT_1$ and $Ass(a) = \{a\}$, for all $a \in IT_1$.

(iii) Let a be such that $aa' \in N$, for all $a' \in CF_{n+1}$.
$b \in Ass(a)$ iff $b \in IT_1$ and for every $a' \in CF_{n+1}$, $b' \in Ass(a')$

(1) $\exists \overline{m}(\, b(\overline{b'}(\overline{m})) > \overline{0}\,)$,

(2) $\forall \overline{m}(\, b(\overline{b'}(\overline{m})) > \overline{0} \longrightarrow b(\overline{b'}(\overline{m})) = (aa') + 1\,)$.

$a \in CF_{n+2}$ iff $Ass(a) \neq \emptyset$.

Equivalence is identity for all pure finite types. □

We shall now show for certain pca$^+$-expansions M of D_A, P_ω, T^ω and extensional, (p)-reflexive, complete lattices that CF_2 and IT_2 coincide. In addition to having standard integers we shall require the numerals to have some *finite* character. This constraint, as will become clearer later on, ensures that the type-2 functionals in $IT(M)$ are locally determined by a finite amount of information about their argument. Let us first make the notion of pca$^+$'s as decribed above more precise.

Definition 7.4.2 $M = (A, *, K, S, 0, S_N, P_N, D, N, \sqsubseteq, \perp)$ is a *CF-pca$^+$* iff

(i) $(A, *, K, S, 0, S_N, P_N, D, N)$ is an ft-extensional pca$^+$,

(ii) $(A, \sqsubseteq, \perp)$ is (p)-reflexive via F, G and $*$ is the canonical application operation based on F,

(iii) $(A, \sqsubseteq, \perp)$ is a cpo such that for every $A' \subseteq A$

$$\exists a \forall a' \in A'(\, a' \sqsubseteq a\,) \longrightarrow supA' \in A,$$

(iv) $(A, *, K, S, 0, S_N, P_N, D, N)$ has standard integers,

(v) a is *compact*, for all $a \in N$, i.e. for every $a \in N$ and every directed $A' \subseteq A$ one has

$$a \sqsubseteq supA' \longrightarrow a \sqsubseteq a_0$$

for some $a_0 \in A'$. □

This is a long catalogue, but observe that any pca$^+$-expansion of the five model types mentioned above satisfies (i) to (iii): Firstly, any pca$^+$-expansion of these models is ft-extensional. Secondly, every model is a (p)-reflexive cpo and comes along with its canonical application. Thirdly, every model, except for T^ω, is a complete lattice and therefore closed under arbitrary sup's. T^ω, however, is closed under sup's of sets having an upper bound. Observe also that compactness comes down to finite sets, in the case of D_A and P_ω, and to finite elements, in the case of T^ω. The pca$^+$-expansion of D_ω as described in example 7.1.3 is therefore a CF-pca$^+$.

We shall now prove for arbitrary CF-pca$^+$'s that every $a \in IT_2$ is indeed locally determined by a finite amount of information about its argument. To these ends we shall define canonical approximations of type-1 functionals.

Definition 7.4.3 Let M be a CF-pca$^+$. For $a \in IT_1$, finite $X \subseteq N$ and $a' \in A$, define

$$f_{a,X}(a') = sup\{ax \mid x \in X \ \wedge \ x \sqsubseteq a'\}. \ \square$$

Lemma 7.4.4 Let M be a CF-pca$^+$ and $a \in IT_1$. Then

(i) $\forall$finite $X \subseteq N(\, f_{a,X} \in [A \to A]\,)$,

(ii) $sup\{f_{a,X} \mid X \subseteq N \ \wedge \ X \text{ finite}\} \in [A \to A]$,

(iii) $G(sup\{f_{a,X} \mid X \subseteq N \;\wedge\; X \text{ finite}\}) =_1 a$.

PROOF. (i) Observe that $\{ax \mid x \in X \;\wedge\; x \sqsubseteq a'\} \sqsubseteq aa'$, since $*$ is monotone. Hence $sup\{ax \mid x \in X \;\wedge\; x \sqsubseteq a'\} \in A$, by 7.4.2(iii). Thus $f_{a,X} : A \rightarrow A$. Now let $A' \subseteq A$ be directed. Then $f_{a,X}(supA') =$

$= sup\{ax \mid x \in X \;\wedge\; x \sqsubseteq supA')\}$

$= sup\{ax \mid x \in X \;\wedge\; x \sqsubseteq a' \;\wedge\; a' \in A'\}$, since x is compact

$= sup\{sup\{ax \mid x \in X \; x \sqsubseteq a')\} \mid a' \in A'\} = sup\{f_{a,X}(a') \mid a' \in A'\}$.

For (ii) observe that $\{f_{a,X} \mid X \subseteq N \;\wedge\; X \text{ finite}\}$ is directed.
(iii) We have to prove that $sup\{f_{a,X} \mid X \subseteq N \;\wedge\; X \text{ finite}\}(a') = aa'$, for all $a' \in N$. So let $a' \in N$. Then

$$aa' = f_{a,\{a'\}}(a') \sqsubseteq sup\{f_{a,X} \mid X \subseteq N \;\wedge\; X \text{ finite}\}(a').$$

The converse follows from the fact that $f_{a,X}(a') \sqsubseteq aa'$, for all $X \subseteq N$. □

These canonical approximations will now reveal the finite amount of information needed by a type-2 functional to determine its value for a given argument.

Proposition 7.4.5 Let M be a CF-pca$^+$ and $a \in IT_2$, $a' \in IT_1$. Then there is a finite $X \subseteq N$ such that

$$\forall a'' \in IT_1(\forall x \in X(\, a'x = a''x\,) \longrightarrow aa' = aa''\,).$$

PROOF. Let $a \in IT_2$ and recall that $IT_2 = ET_2$, since M is ft-extensional. Let $a' \in IT_1$. Then $aa' =$

$= a(G(sup\{f_{a',X} \mid X \subseteq N \;\wedge\; X \text{ finite}\}))$, by 7.4.4(iii)

$= a(sup\{G(f_{a',X}) \mid X \subseteq N \;\wedge\; X \text{ finite}\})$, since G is continuous

$= sup\{a(G(f_{a',X})) \mid X \subseteq N \;\wedge\; X \text{ finite}\}$, since $*$ is continuous.

Hence, since aa' is compact, $aa' = a(G(f_{a',X}))$ for some finite $X \subseteq N$. Now let $a'' \in IT_1$ be such that $a'x = a''x$, for all $x \in X$. Then

$$f_{a',X}(a''') = sup\{a'x \mid x \in X \ \wedge \ x \sqsubseteq a'''\} =$$

$$sup\{a''x \mid x \in X \ \wedge \ x \sqsubseteq a'''\} = f_{a'',X}(a''').$$

Thus $f_{a',X} = f_{a'',X}$ and therefore $G(f_{a',X}) = G(f_{a'',X})$. So

$$aa' = a(G(f_{a'',X})) \sqsubseteq a(G(sup\{f_{a'',X} \mid X \in N \ \wedge \ X \text{ finite}\})) = aa''.$$

Thus $aa' = aa''$, since numerals are consistent. □

The last step is the coding of the global description of a type-2 functional into a type-1 object. As a preliminary we shall first show that every $f \in \omega^\omega$ is numerically representable in every CF-pca$^+$.

Lemma 7.4.6 Let M be a CF-pca$^+$ and $f \in \omega^\omega$. Then there is an $a \in A$ such that

$$\forall n \in \omega(\, a\overline{n} = \overline{f(n)}\,).$$

PROOF. Let $f \in \omega^\omega$. For $a \in A$, finite $X \subseteq \omega$, define $f_X(a)$ by

$$f_X(a) = sup\{\overline{f(x)} \mid x \in X \ \wedge \ \overline{x} \sqsubseteq a\}.$$

We shall first prove that $f_X : A \to A$. This is done by induction on the cardinality of X. Let $a \in A$. If $|X| \leq 1$ or $|X| = l + 2$ but $|\{\overline{f(x)} \mid x \in X \ \wedge \ \overline{x} \sqsubseteq a\}| \leq 1$, this is obvious. If $|X| = l + 2$ and $|\{\overline{f(x)} \mid x \in X \ \wedge \ \overline{x} \sqsubseteq a\}| \geq 2$, pick $x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $\overline{x_1} \sqsubseteq a$, $\overline{x_2} \sqsubseteq a$. By the induction hypothesis $f_{X\setminus\{x_2\}}(a) \in A$. Then

$$f_{X\setminus\{x_2\}}(a) = D\ f_{X\setminus\{x_2\}}(a)\ \overline{f(x_2)}\ \overline{x_1}\ \overline{x_2} \sqsubseteq D\ f_{X\setminus\{x_2\}}(a)\ \overline{f(x_2)}\ \overline{x_1}\ a,$$

$$\overline{f(x_2)} = D\ f_{X\setminus\{x_2\}}(a)\ \overline{f(x_2)}\ \overline{x_1}\ \overline{x_1} \sqsubseteq D\ f_{X\setminus\{x_2\}}(a)\ \overline{f(x_2)}\ \overline{x_1}\ a,$$

since M is a monotone pca$^+$. Therefore $sup\{f_{X\setminus\{x_2\}}(a), \overline{f(x_2)}\} \in A$, by 7.4.2(iii). But

$$sup\{f_{X\setminus\{x_2\}}(a), \overline{f(x_2)}\} = sup\{\overline{f(x)} \mid x \in X \ \wedge \ \overline{x} \sqsubseteq a\}.$$

Hence $f_X(a) \in A$.
Then also $f_X \in [A \to A]$, since f_X is monotone and numerals are

compact.
Now, since $\{f_X \mid X \subseteq \omega \ \wedge \ X \text{ finite}\}$ is directed, one has

$$sup\{f_X \mid X \subseteq \omega \ \wedge \ X \text{ finite}\} \in [A \to A]$$

and therefore

$$G(sup\{f_X \mid X \subseteq \omega \ \wedge \ X \text{ finite}\}) \in A.$$

Finally, observe that

$$f_X(\overline{n}) = \begin{cases} \overline{f(n)} & \text{if } n \in X \\ \bot & \text{otherwise,} \end{cases}$$

since numerals are consistent. Thus $G(sup\{f_X \mid X \subseteq \omega \wedge X \text{ finite}\})\overline{n} =$

$$sup\{f_X(\overline{n}) \mid X \subseteq \omega \ \wedge \ X \text{ finite}\}) = \overline{f(n)},$$

for all $n \in \omega$. □

Theorem 7.4.7 Let M be a CF-pca$^+$. Then $CF_2 = IT_2$.

PROOF. By the definition one has, $CF_2 \subseteq IT_2$. For the converse, let $a \in IT_2$. Then $aa' \in N$, for all $a' \in CF_1$. We have to prove that a has an associate. Since $Ass(a') = a'$, for all $a' \in CF_1$, this comes down to the construction of a $b \in IT_1$ such that

(1) $\exists \overline{m}(\, b(\overline{a'}(\overline{m})) > \overline{0}\,)$,

(2) $\forall \overline{m}(\, b(\overline{a'}(\overline{m})) > \overline{0} \longrightarrow b(\overline{a'}(\overline{m})) = (aa') + 1\,)$,

for all $a' \in CF_1$. To this end define $f \in \omega^\omega$ by

$$f(k) = \begin{cases} m+1 & \text{if } \Phi(k,m,a) \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\Phi(k,m,a) \longleftrightarrow \exists a' \in IT_1 \exists l \in \omega(\, aa' = \overline{m} \ \wedge \ \overline{a'}(\overline{l}) = \overline{k} \ \wedge$$
$$\forall a'' \in IT_1(\, \overline{a''}(\overline{l}) = \overline{k} \longrightarrow aa' = aa''\,)\,).$$

Then

$$(1)\ \ \forall a' \in IT_1 \exists k, l \in \omega(\,\overline{a'}(\bar{l}) = \bar{k}\ \wedge\ f(k) > 0\,)\ :$$

Let $a' \in IT_1$. By proposition 7.4.5 there is a finite $X \subseteq N$ such that for all $a'' \in IT_1$

$$\forall x \in X(\,a'x = a''x\,) \longrightarrow aa' = aa''.$$

We can thus pick $l, k \in \omega$ such that

$$\bar{k} =< a'\bar{0}, ..., a'\bar{l} >$$

and $l \geq x$, for all $\bar{x} \in X$. Every $a'' \in IT_1$ satisfying $\overline{a''}(\bar{l}) = \bar{k}$, then, has the same values on X as a' has. Therefore $aa' = aa''$. Hence $f(k) > 0$.

$$(2)\ \forall a' \in IT_1 \forall k, l \in \omega(\,\overline{a'}(\bar{l}) = \bar{k}\ \wedge\ f(k) > 0 \longrightarrow \overline{f(k)} = aa' + 1\,)\ :$$

This follows immediately from the definition.

Now let $b \in A$ be any numerical representation of f. Then b is an associate for a, by (1) and (2). From the preceding lemma it therefore follows that $Ass(a) \neq \emptyset$. Hence $a \in CF_2$. □

Corollary 7.4.8 Let M be a pca$^+$ having standard integers. If M is an expansion of

(i) D_A or P_ω and N is a collection of finite sets, or

(ii) T^ω and N is a collection of finite elements, or

(iii) an extensional, (p)-reflexive, complete lattice and N is a collection of compact elements,

then $CF_2 = IT_2$. □

We conjecture that the argument used above can be extended to all pure finite types, i.e. $CF(M) =< IT_n >_{n \in \omega}$, for every CF-pca$^+$ M.

Theorem 7.4.7 does not cover H_ω, since it is not a reflexive cpo. It is also unlikely that there is any pca$^+$-expansion of H_ω in which every type-2 functional is countable, since application in H_ω essentially allows operators the values of which depend on an infinite amount of information about the arguments involved. We shall illustrate this phenomenon by an example.

Proposition 7.4.9 There is a pca$^+$-expansion M of H_ω having standard integers such that

(i) $\overline{n} = \{n\}$, for all $n \in \omega$,

(ii) $CF_2 \neq IT_2$.

PROOF. We assume some coding of finite sequences such that 0 is the code of the empty sequence and let M be the expansion of H_ω such that

- $S_N = \{< 0, m, n+1 > \mid m, n \in \omega \ \wedge \ e_m = \overline{n}\}$,
- $P_N = \{< 0, m, n > \mid m, n \in \omega \ \wedge \ e_m = \overline{n+1}\}$,
- $D = \{< 0, k, < 0, l, < 0, m, < 0, n, o >>>> \mid$

$$k, l, m, n, o \in \omega \ \wedge \ \exists p, q \in \omega((e_m = \overline{p} \ \wedge \ e_n = \overline{q}) \ \wedge$$

$$((p = q \ \wedge \ o \in e_l) \ \vee \ (p \neq q \ \wedge \ o \in e_k)))\}.$$

We leave the verification that M is a pca$^+$ to the reader. We shall now show that there is an $F \in IT_2$ such that for all $f \in IT_1$ one has

$$(1) \quad Ff = \overline{0} \longleftrightarrow \forall n \in \omega(f\overline{n} = \overline{0}),$$

$$(2) \quad Ff \neq \overline{0} \longleftrightarrow \exists n \in \omega(f\overline{n} \neq \overline{0}).$$

Such an F does not belong to CF_2, since
- $f := \{< 0, n, 0 > \mid n \in \omega\}$ is the constant zero-function and therefore $Ff = \overline{0}$,
- for every $n \in \omega$,

$$f_n := \{< 0, l, 0 > \mid \exists m \leq n(e_l = \overline{m})\} \cup \{< 0, l, 1 > \mid \exists m > n(e_l = \overline{m})\}$$

is a function such that

$$f_n\overline{m} = \begin{cases} \overline{0} & \text{if } m \leq n \\ \overline{1} & \text{otherwise.} \end{cases}$$

Hence for every finite $X \subseteq N$ there is a $f_n \in IT_1$ such that

$$\forall x \in X(fx = f_n x) \ \wedge \ Ff \neq Ff_n.$$

Thus F has no associate.

For the construction of F we shall write $< n > *\overline{f}(p)$ for the code of the sequence $n, f(0), ..., f(p-1)$ and let $(.,.)$ be some coding of pairs of natural numbers such that $(0,0) = 0$. Then F is defined by $F = F_1 \cup F_2$ where

$$F_1 := \{<< n > *\overline{f}(p), m, 0 > \mid f \in \omega^\omega \ \wedge \ n, m, p \in \omega \ \wedge$$

$$\exists e_k \subseteq \overline{n}(< \overline{f}(p), k, 0 > \in e_m)\},$$

$$F_2 := \{< \overline{f}(p), m, (n, l+1) > \mid f \in \omega^\omega \ \wedge \ l, m, n, p \in \omega \ \wedge$$

$$\exists e_k \subseteq \overline{n} \exists p' \leq p(< \overline{f}(p'), k, l+1 > \in e_m) \ \wedge$$

$$\forall r < n \exists e_k \subseteq \overline{r} \exists p' \leq p(< \overline{f}(p'), k, 0 > \in e_m)\}.$$

Now let $g \in IT_1$. Then $0 \in Fg \longleftrightarrow$

$$\longleftrightarrow \forall f \exists p \exists e_m \subseteq g(< \overline{f}(p), m, 0 > \in F)$$

$$\longleftrightarrow \forall f \exists p \exists e_m \subseteq g(< \overline{f}(p), m, 0 > \in F_1)$$

$$\longleftrightarrow \forall n \in \omega \forall f \exists p \exists e_m \subseteq g(<< n > *\overline{f}(p), m, 0 > \in F_1)$$

$$\longleftrightarrow \forall n \in \omega \forall f \exists p \exists e_k \subseteq \overline{n}(< \overline{f}(p), k, 0 > \in g)$$

$$\longleftrightarrow \forall n \in \omega(g\overline{n} = \overline{0}).$$

Moreover, $(n, l+1) \in Fg \longleftrightarrow$

$$\longleftrightarrow \forall f \exists p \exists e_m \subseteq g(< \overline{f}(p), m, (n, l+1) > \in F)$$

$$\longleftrightarrow \forall f \exists p \exists e_m \subseteq g(< \overline{f}(p), m, (n, l+1) > \in F_2)$$

$$\longleftrightarrow \forall f \exists p \exists e_m \subseteq g(\exists e_k \subseteq \overline{n} \exists p' \leq p(< \overline{f}(p'), k, l+1 > \in e_m)$$

$$\wedge \ \forall r < n \exists e_k \subseteq \overline{r} \exists p' \leq p(\overline{f}(p'), k, 0 > \in e_m))$$

$$\longleftrightarrow \forall f \exists p \exists e_k \subseteq \overline{n}(< \overline{f}(p), k, l+1 > \in g)$$

$$\wedge \forall r < n \forall f \exists p \exists e_k \subseteq \overline{r}(< \overline{f}(p), k, 0 > \in g\ )\ (*)$$

$$\longleftrightarrow g\overline{n} = \overline{l+1}\ \wedge\ \forall r < n(\ g\overline{r} = \overline{0}\ ).$$

(*): For $\leftarrow$ pick $q, q_0, ..., q_{n-1}$ such that $< \overline{f}(q), k, l+1 > \in g$, for some $e_k \subseteq \overline{n}$, and $< \overline{f}(q_i), k_i, 0 > \in g$, for some $e_{k_i} \subseteq \overline{i}$. Now let $p = max\{q, q_0, ..., q_{n-1}\}$ and

$$e_m = \{< \overline{f}(q), k, l+1 >, < \overline{f}(q_0), k_0, 0 >, ..., < \overline{f}(q_{n-1}), k_{n-1}, 0 >\}.$$

Thus F satisfies (1) and (2) above. □

The *countably based* functionals $CbF = < CbF_n >_{n \in \omega}$ were first introduced by Hartley [Ha]. These functionals can be roughly characterized by the fact that values are determined by a countable amount of information about the argument. Moreover, they are globally described by a continuum of information coded in an associate which is now a type-2 object. Since every pca$^+$-expansion of H_ω is ft-extensional, we conjecture that in pca$^+$-expansions of H_ω having standard integers the pure finite type functionals coincide with the countably based functionals.

7.5 Extensionality, Weak Extensionality and FT-Extensionality

Having reached a point where we talk about three notions of extensionality, the question about the interdependencies of these notions arises. There is the well-known fact that extensionality implies weak extensionality. This however - and showing this is the principal aim of this section - is the only dependency. That is,

(1) extensionality does not imply ft-extensionality;

(2) weak extensionality does not imply extensionality;

(3) weak extensionality does not imply ft-extensionality;

(4) ft-extensionality does not imply extensionality;

(5) ft-extensionality does not imply weak extensionality.

(2) is known from the literature and in the previous section we have already encountered examples for (4) and (5): D_A and H_ω are both ft-extensional but neither is the first extensional nor is the second weakly extensional. In the remainder of this section we shall prove (1) and (3) by constructing an extensional ca that is not ft-extensional.

The construction described below owes much to Plotkin's and Scott's account and differs from their P_ω-models essentially only in that the resulting model is not closed under arbitrary unions. Although it may be interesting to investigate more closely the global and local structure of this model, we shall restrict ourselves entirely to proving that it is an extensional ca in which the extensional finite type structure and the intensional one do not coincide. Other investigations are far beyond the scope of this section.

The construction is inspired by the following diagnosis. In D_ω with the interpretation of the various constants defined as in example 7.1.3 there are two 0-functions of type 1 having different global behaviour:

- $f_1 = \{(\emptyset, 0)\}$,
- $f_2 = \{(\{n\}, 0) \mid n \in \omega\}$.

The global behaviour of these functions differs precisely at $\emptyset$ where $f_1\emptyset = \{0\}$ but $f_2\emptyset = \emptyset$. The reason for the absence of an operator in D_ω that takes this difference in the global behaviour of f_1 and f_2 into account while mapping the functions of type 1 into N is threefold: firstly, $f_3 := f_1 \cup f_2$ is also a function of type 1; secondly, application is monotone; thirdly, the numerals are consistent. The third fact is inevitable. Moreover, ever since Scott's approach in 1969 it seems hard to construct a ca that is not monotone. The basic idea for the construction, then, is to exclude operators such as f_3 from the universe.

We shall work in P_ω rather than D_A. The reason for taking the coded graphmodel lies in one of our aims, namely in constructing an *extensional* ca. The coded version, then, is less opposed to forcing extensionality into the model, since every natural number functions under the coding as an elementary instruction. Sticking to the well-founded D_A instead would mean an additional complication, since one also has to force elements of A to act. However, making this choice also means that the construction is less transparent, since the interesting properties of the model seem to be caused by the miracles of coding.

Because of reasons of heredity, we shall not work in the whole of ω but shall restrict the carrier set to a proper subset. Prior to the moment of being used, we shall not comment on the several clauses of the definition of the universe given below. The patient reader will discover, while following the proofs, that the definition resembles precisely those conditions which need to be met in order to reach the aim.

The construction can be carried out for arbitrary bijective codings (n, m) and e_n. However, for reasons of simplicity, we shall fix the following standard codings where $(0,0) = 0$, $(1,0) = 1$ and $e_0 = \emptyset$, $e_1 = \{0\}$:

Definition 7.5.1 For $n, m \in \omega$,

(i) let $(n, m) = \frac{1}{2}(n+m)(n+m+1) + m$,

(ii) define the finite set e_n as follows:

$$e_n = \{k_0, ..., k_{m-1}\} \text{ with } k_0 < ... < k_{m-1} \longleftrightarrow n = \textstyle\sum_{i<m} 2^i. \square$$

Definition 7.5.2 ($M(\omega)$)
(i) For all $X, Y \subseteq \omega$, define

$$X * Y := \{m \mid \exists e_n \subseteq Y(\, (n, m) \in X\,)\}.$$

(ii) Define $M(\omega) \subseteq \omega$ recursively by

(1) $M_0 = \{0, 1\}$;

(2) put

$$\Gamma_n(p) \longleftrightarrow$$

$$\forall (s,t), (s',t) \in e_p(\, \exists (l,t') \in M_n(\, e_l = e_s \cup e_{s'}\,) \longrightarrow s = s'\,),$$

$$\Delta_n(p) \longleftrightarrow \forall (s,t) \in M_n \exists (l,t') \in M_n(\, e_l = e_p * e_s\,),$$

$$M_{n+1} = M_n \cup \{(p,q) \mid e_p \subseteq M_n \ \wedge\ q \in M_n \ \wedge\ \Gamma_n(p) \ \wedge\ \Delta_n(p)\};$$

(3) $M(\omega) = \cup_{n\in\omega} M_n$. □

It is easily seen that for all n

$$(p, q) \in M_{n+1} \longrightarrow e_p \subseteq M_n \;\wedge\; q \in M_n.$$

We shall first work towards a characterization of $M(\omega)$ that is more suitable for our purpose.

Lemma 7.5.3 For all $n \in \omega$, $(p, q) \in M_n$

(i) $\neg(\, 0 \in e_p \;\wedge\; 1 \in e_p\,)$;

(ii) $\forall e_{p'} \subseteq e_p(\,(p', 0) \in M_n\,)$;

(iii) $\forall m(\, e_p \subseteq M_m \longrightarrow (p, 0) \in M_{m+1}\,)$.

PROOF. We prove (i)-(iii) by induction on n. First recall that

$$0 = (0,0),\; 1 = (1,0),\; e_0 = \emptyset,\; e_1 = \{0\}.$$

For $n = 0$, this is trivial.
Let $(p, q) \in M_{n+1} \backslash M_n$. Then $e_p \subseteq M_n$ and $\Gamma_n(p)$, $\Delta_n(p)$ hold.

(i) Assume $(0,0), (1,0) \in e_p$. Then $e_0 \cup e_1 = e_1$ and $(1,0) = 1 \in M_0 \subseteq M_n$. But $0 \neq 1$, contradicting $\Gamma_n(p)$.

(ii) Let $e_{p'} \subseteq e_p$. To prove $(p', 0) \in M_{n+1}$, we shall verify $\Gamma_n(p')$ and $\Delta_n(p')$. $\Gamma_n(p')$ follows from $\Gamma_n(p)$. To prove $\Delta_n(p')$, let $(s, t) \in M_n$ and put $e_{l'} = e_{p'} * e_s$. By $\Delta_n(p)$, let $(l, t) \in M_n$ be such that $e_l = e_p * e_s$. Then $e_{l'} \subseteq e_l$. Hence $(l', 0) \in M_n$, by the induction hypothesis.

(iii) Suppose $e_p \subseteq M_m$. If $n \leq m$, then $(p, 0) \in M_{n+1} \subseteq M_{m+1}$. Thus assume $m < n$. If $m = 0$, then $e_p \in \{\emptyset, \{0\}, \{1\}\}$, by (i). Clearly, $\Gamma_0(p)$ holds. For $\Delta_0(p)$ observe that for all (s, t), $e_p * e_s \in \{\emptyset, \{0\}\}$. But $0, 1 \in M_0$. Hence also $\Delta_0(p)$ holds. Therefore $(p, 0) \in M_1$. Finally, assume $0 < m < n$. Then $\Gamma_m(p)$ follows from $\Gamma_n(p)$, since $M_m \subseteq M_n$. To prove $\Delta_m(p)$, let $(s, t) \in M_m$. Then also $(s, t) \in M_n$. It thus follows from $\Delta_n(p)$ that $e_p * e_s = e_l$, for some $(l, t) \in M_n$. Observe that $e_l \subseteq M_{m-1}$, since $e_p \subseteq M_m$. Thus $(l, 0) \in M_m$, by the induction hypothesis. Hence also $\Delta_m(p)$ holds and therefore $(p, 0) \in M_{m+1}$. □

It follows that membership of $M(\omega)$ is characterizable in terms of the following two conditions, which we shall often use instead of the original definition.

Proposition 7.5.4 For all $(p,q) \in \omega$, $(p,q) \in M(\omega)$ iff $e_p \subseteq M(\omega)$, $q \in M(\omega)$ and

(I) $\forall (s,t),(s',t) \in e_p(\, \exists (l,t') \in M(\omega)(\, e_l = e_s \cup e_{s'}\,) \longrightarrow s = s'\,)$,

(II) $\forall (s,t) \in M(\omega) \exists (l,t') \in M(\omega)(\, e_l = e_p * e_s\,)$.

PROOF. $\rightarrow$: Assume $(p,q) \in M(\omega)$.

If $(p,q) \in M_0$, then $e_p \in \{\emptyset, \{0\}\}$. Hence (I) holds trivially. For (II) observe that $e_p * e_s \in \{\emptyset, \{0\}\}$, for all $(s,t) \in M(\omega)$. But $0,1 \in M_0 \subseteq M(\omega)$. Hence also (II) holds.

If $(p,q) \in M_1 \backslash M_0$, then $e_p \in \{\emptyset, \{0\}, \{1\}\}$, by 7.5.3(i). Therefore (I) holds. (II) follows again from the fact that $e_p * e_s \in \{\emptyset, \{0\}\}$, for all $(s,t) \in M(\omega)$.

Now suppose $(p,q) \in M_{n+2} \backslash M_{n+1}$. Then $e_p \subseteq M_{n+1} \subseteq M(\omega)$, $q \in M_{n+1} \subseteq M(\omega)$ and $\Gamma_{n+1}(p)$, $\Delta_{n+1}(p)$ hold.

To prove (I) let $(s,t),(s',t) \in e_p$ be such that $e_s \cup e_{s'} = e_l$ for some $(l,t') \in M(\omega)$. Observe that $e_l \subseteq M_n$, since $e_p \subseteq M_{n+1}$. Hence $(l,0) \in M_{n+1}$, by 7.5.3(iii). It then follows from $\Gamma_{n+1}(p)$ that $s = s'$.

Finally, for (II) let $(s,t) \in M(\omega)$. Put

$$e_{s''} = \bigcup \{ e_{s'} \subseteq e_s \mid \exists m\ (s',m) \in e_p \}.$$

Then $e_{s''} \subseteq M_n$, since $e_p \subseteq M_{n+1}$. Hence $(s'',0) \in M_{n+1}$, by 7.5.3.(iii). It therefore follows from $\Delta_{n+1}(p)$ that $e_p * e_{s''} = e_l$, for some $(l,t') \in M_{n+1}$. But $e_p * e_{s''} = e_p * e_s$. Thus $e_p * e_s = e_l$, for some $(l,t') \in M(\omega)$.

$\leftarrow$: Assume $e_p \subseteq M(\omega)$, $q \in M(\omega)$ and (I),(II) hold. Choose $n+1$ such that $e_p \subseteq M_{n+1}$ and $q \in M_{n+1}$. We shall prove $(p,q) \in M_{n+2}$ by verifying $\Gamma_{n+1}(p)$ and $\Delta_{n+1}(p)$. $\Gamma_{n+1}(p)$ follows from (I). For $\Delta_{n+1}(p)$ let $(s,t) \in M_{n+1}$. Then $e_p * e_s = e_l$ for some $(l,t') \in M(\omega)$, by (II). But $e_l \subseteq M_n$, since $e_p \subseteq M_{n+1}$. Hence $(l,0) \in M_{n+1}$, by 7.5.3(iii). □

We shall now define our universe. We shall not simply take the whole of $\mathcal{P}(M(\omega))$ but restrict ourselves to those sets which essentially share the properties (I) and (II) of the preceding proposition.

Definition 7.5.5 (U). Define $U \subseteq \mathcal{P}(M(\omega))$ by

$$X \in U \longleftrightarrow \forall e_n \subseteq X \exists m \in M(\omega)(\,(n,m) \in M(\omega)\,).\square$$

Lemma 7.5.6 U is closed under subsets, i.e.

$$\forall X \in U \forall X' \subseteq X(\, X' \in U\,).$$

Moreover, for all $(n,m) \in M(\omega)$, $e_n \in U$.

PROOF. The first claim follows immediately from definition 7.5.5. For the second apply lemma 7.5.3(ii). □

Proposition 7.5.7 For all $X \subseteq M(\omega)$, $X \in U$ iff

(I) $\forall (s,t),(s',t) \in X(\, e_s \cup e_{s'} \in U \longrightarrow s = s'\,)$,

(II) $\forall Y \in U(\, X * Y \in U\,)$.

PROOF. $\rightarrow$: Assume $X \in U$. To prove (I) let $(s,t),(s',t) \in X$ be such that $e_s \cup e_{s'} \in U$. Put $e_l = e_s \cup e_{s'}$. Then $(l,t) \in M(\omega)$, for some $t \in M(\omega)$. Put, moreover, $e_n = \{(s,t),(s',t)\}$. Then $(n,t') \in M(\omega)$, for some $t' \in M(\omega)$, since $e_n \subseteq X \in U$. Hence $s = s'$, by 7.5.4(I).

For (II) let $Y \in U$ and $e_n \subseteq X*Y$. We have to prove that $(n,m) \in M(\omega)$, for some $m \in M(\omega)$. To this end observe that one can choose $e_p \subseteq X$, $e_s \subseteq Y$ such that $e_n = e_p * Y$ and $e_s = \bigcup\{e_{s'} \mid \exists m\ (s',m) \in e_p\}$. Then $(p,t),(s,t') \in M(\omega)$, for some $t,t' \in M(\omega)$, since U is closed under subsets. Hence $e_p * e_s = e_l$, for some $(l,t'') \in M(\omega)$, by 7.5.4(II). But $e_n = e_l$. Thus $(n,t'') \in M(\omega)$.

$\leftarrow$: Assume $X \subseteq M(\omega)$ and (I),(II) hold. Let $e_n \subseteq X$. We have to prove that $(n,m) \in M(\omega)$, for some $m \in M(\omega)$. We shall show that $(n,0) \in M(\omega)$ by applying proposition 7.5.4. Clearly, $e_n \subseteq M(\omega)$ and $0 \in M(\omega)$. 7.5.4(I) follows from (I) and 7.5.6. For 7.5.4(II) let $(s,t) \in M(\omega)$. Then $e_s \in U$, by 7.5.6. Hence $X * e_s \in U$, by (II). Now put $e_l = e_n * e_s$. Then $e_l \subseteq X * e_s$. Therefore $e_l \in U$, since U is closed under subsets. Thus for some $t' \in M(\omega)$ $(l,t') \in M(\omega)$, by definition 7.5.5. □

Next, we shall show that $(U, *)$ is a nontrivial, total, extensional applicative structure.

Proposition 7.5.8 For all $n \in M(\omega)$, $\{n\} \in U$. Moreover, for all $X,Y \in U$

(i) $X * Y \in U$,

(ii) $\forall Z \in U(X * Z = Y * Z) \longrightarrow X = Y.$

PROOF. One proves by induction on m

$$\forall n \in M_m(\{n\} \in U),$$

by applying proposition 7.5.7. Clearly, 7.5.7(I) holds for all singletons. If $m = 0$, then $n = 0$ or $n = 1$. Hence $\{n\} * Y \in \{\emptyset, \{0\}\}$. But since $0, 1 \in M(\omega)$, it follows from lemma 7.5.6 that $\emptyset, \{0\} \in U$. The induction step is left to the reader.

(i) follows from proposition 7.5.7(II). To prove (ii) let $X, Y \in U$ be such that $X * Z = Y * Z$, for all $Z \in U$. We shall only prove $X \subseteq Y$. The converse follows by symmetry. Let $n \in X$. Say, $n = (l, m)$. Then $e_l \in U$. Hence $m \in X * e_l = Y * e_l$. Thus $(p, m) \in Y$, for some $e_p \subseteq e_l$. Therefore also $m \in Y * e_p = X * e_p$. So $(q, m) \in X$, for some $e_q \subseteq e_p \subseteq e_l$. Then $e_q \cup e_l = e_l$ and $(q, m), (l, m) \in X$. Therfore $q = l$, by 7.5.7(I). Hence also $p = l$, i.e. $(l, m) \in Y$. □

It remains to show that the universe is not too restricted, i.e. we have to prove, firstly, that $(U, *)$ is combinatory complete and secondly, that the additional constants can be defined in a way such that the finite type structures do not coincide. To these ends we shall first characterize the representable functions.

Observe firstly, that U is a cpo with bottom $\emptyset$ and $supD = \cup D$, for directed D. U is *not* a complete lattice, since both $\{0\}, \{1\} \in U$, by the preceding proposition, but $\{0, 1\} \notin U$, by lemma 7.5.3(i). We thus have already achieved one aim, namely the exclusion of the coded version of the operator f_3 mentioned in the beginning of this section. Secondly, observe that $*$ is continuous with respect to the Scott topology on U. It follows that every representable function is continuous. The converse, however, is not true.

Lemma 7.5.9 $\forall n, m \in \omega(e_n \in U \wedge m \in M(\omega) \longrightarrow (n, m) \in M(\omega)).$

PROOF. Let $e_n \in U$ and $m \in M(\omega)$. Then $(n, t) \in M(\omega)$, for some $t \in M(\omega)$. Thus (n, t) satisfies (I) and (II) of proposition 7.5.4. Hence also $(n, m) \in M(\omega)$. □.

Proposition 7.5.10 There is an $f \in [U \rightarrow U]$ such that f is not representable in U.

PROOF. Define $f : U \to U$ by

$$f(X) = \begin{cases} \emptyset & \text{if } X = \emptyset \\ \{0\} & \text{otherwise.} \end{cases}$$

Clearly, $f \in [U \to U]$. Suppose f is representable, i.e. there is an $X_f \in U$ such that $X_f * \emptyset = \emptyset$ and $X_f * Y = \{0\}$, for all $Y \in U \backslash \{\emptyset\}$. From the existence of such an X_f it follows that for all $e_n, e_{n'} \in U \backslash \{\emptyset\}$

$$e_n \cup e_{n'} \in U \longrightarrow e_n \cap e_{n'} \neq \emptyset :$$

Observe that $(0,0) \notin X_f$, since $X_f * \emptyset = \emptyset$. Now let $e_n, e_{n'} \in U \backslash \{\emptyset\}$. Then $X_f * e_n = \{0\} = X_f * e_{n'}$. Hence there are $(s,0), (s',0) \in X_f$ such that $e_s \subseteq e_n$ and $e_{s'} \subseteq e_{n'}$. Assume $e_n \cup e_{n'} \in U$. Then $e_s \cup e_{s'} \subseteq e_n \cup e_{n'} \in U$. Thus $e_s \cup e_{s'} \in U$, since U is closed under subsets. Therefore $s = s'$, i.e. $e_s \subseteq e_n \cap e_{n'}$. But $e_s \neq \emptyset$.

We shall now derive a contradiction by constructing two nonempty sets $e_n, e_{n'} \in U$ with $e_n \cup e_{n'} \in U$ and $e_n \cap e_{n'} = \emptyset$. Put $e_n = \{1\}$. Then $e_n \in U$, by proposition 7.5.8. Hence also $e_{n'} := \{(n,1)\} \in U$, by 7.5.9 and 7.5.8. Clearly, $e_n \cap e_{n'} = \emptyset$. To prove that $e_n \cup e_{n'} \in U$ we shall apply proposition 7.5.7. Recall that $1 = (1,0)$. Hence (I) holds. For (II) let $Y \in U$. Assume $(e_n \cup e_{n'}) * Y = \{0,1\}$. Then $e_1, e_n \subseteq Y$, i.e. $\{0,1\} \subseteq Y$. Contradiction with 7.5.3(i). Thus $(e_n \cup e_{n'}) * Y \in \{\emptyset, \{0\}, \{1\}\} \subseteq U$. □

As shown above not every continuous function preserves $\cap$ for consistent sets, i.e. continuous functions do not satisfy in general

$$Y \cup Y' \in U \longrightarrow f(Y) \cap f(Y') = f(Y \cap Y').$$

$*$, however, is distributive over $\cap$ in both arguments for consistent sets, that is

Lemma 7.5.11 For all $n \in \omega$ and all $X_0 \cup Y_0, \ldots, X_{n+1} \cup Y_{n+1} \in U$

$$(X_0 * \ldots * X_{n+1}) \cap (Y_0 * \ldots * Y_{n+1}) = (X_0 \cap Y_0) * \ldots * (X_{n+1} \cap Y_{n+1}).$$

PROOF. By induction on n.

For $n = 0$, observe that $(X_0 \cap Y_0) * (X_1 \cap Y_1) \subseteq (X_0 * X_1) \cap (Y_0 * Y_1)$, since $*$ is monotone. For the converse let $t \in (X_0 * X_1) \cap (Y_0 * Y_1)$. Then there are $e_s \subseteq X_1$, $e_{s'} \subseteq Y_1$ such that $(s,t) \in X_0$ and $(s',t) \in Y_0$. But $e_s \cup e_{s'} \in U$, since $e_s \cup e_{s'} \subseteq X_1 \cup Y_1 \in U$ and U is closed under subsets. Moreover, $(s,t),(s',t) \in X_0 \cup Y_0 \in U$. Therefore $s = s'$, by 7.5.7(I). Hence $e_s \subseteq X_1 \cap Y_1$ and $(s,t) \in X_0 \cap Y_0$. Thus $t \in (X_0 \cap Y_0) * (X_1 \cap Y_1)$.

Now let $X_0 \cup Y_0, ..., X_{n+1} \cup Y_{n+1}, X_{n+2} \cup Y_{n+2} \in U$. Then

$$(X_0 \cap Y_0) * ... * (X_{n+1} \cap Y_{n+1}) * (X_{n+2} \cap Y_{n+2}) =$$

$$= ((X_0 * ... * X_{n+1}) \cap (Y_0 * ... * Y_{n+1})) * (X_{n+2} \cap Y_{n+2}).$$

In order to apply the induction hypothesis a second time we have to show that $(X_0 * ... * X_{n+1}) \cup (Y_0 * ... * Y_{n+1}) \in U$. To this end observe that

$$(X_0 * ... * X_{n+1}) \cup (Y_0 * ... * Y_{n+1}) \subseteq (X_0 \cup Y_0) * ... * (X_{n+1} \cup Y_{n+1}),$$

since application is monotone. Thus $(X_0 * ... * X_{n+1}) \cup (Y_0 * ... * Y_{n+1}) \in U$ and therefore $((X_0 * ... * X_{n+1}) \cap (Y_0 * ... * Y_{n+1})) * (X_{n+2} \cap Y_{n+2}) =$

$$(X_0 * ... * X_{n+2}) \cap (Y_0 * ... * Y_{n+2}).$$

□

It follows that the representable functions can be characterized in the following way:

Definition 7.5.12 Define

(i) for $n \in \omega$,

$$Rep_{n+1}(U) = \{f \in [U^{n+1} \to U] \mid \forall X_0, ..., X_n, Y_0, ..., Y_n \in U$$

$$(X_0 \cup Y_0, ..., X_n \cup Y_n \in U \longrightarrow$$

$$f(X_0, ..., X_n) \cap f(Y_0, ..., Y_n) = f(X_0 \cap Y_0, ..., X_n \cap Y_n))\},$$

(ii) for $f \in Rep_1(U)$,

$$G(f) = \{(s,t) \mid e_s \in U \ \wedge \ t \in f(e_s) \ \wedge \ \forall e_{s'} \subset e_s (t \notin f(e_{s'}))\}.$$ □

Lemma 7.5.13 For all $f \in Rep_1(U)$

(i) $\forall X \in U(\, G(f) * X = f(X)\,)$,

(ii) $G(f) \in U$.

PROOF. Let $f \in Rep_1(U)$.

(i) Let $X \in U$ and $t \in G(f) * X$. Then $t \in f(e_s)$, for some $e_s \subseteq X$. Hence $t \in f(X)$, since f is monotone. Therefore $G(f) * X \subseteq f(X)$.

For the converse let $t \in f(X)$. Then $t \in \bigcup\{f(e_n) \mid e_n \subseteq X\}$, since f is continuous. Fix $e_n \subseteq X$ such that $t \in f(e_n)$ and put $s = min\{s' \mid e_{s'} \subseteq e_n \;\wedge\; t \in f(e_{s'})\}$. Then $e_s \subseteq X$, $t \in f(e_s)$ and $t \notin f(e_{s'})$, for all $e_{s'} \subset e_s$. So there is an $e_s \in X$ such that $(s,t) \in G(f)$. Therefore $t \in G(f) * X$. Hence also $f(X) \subseteq G(f) * X$.

(ii) $G(f) \subseteq M(\omega)$, by lemma 7.5.9. To prove that $G(f) \in U$ we shall invoke proposition 7.5.7. For (I) let $(s,t),(s',t) \in G(f)$ be such that $e_s \cup e_{s'} \in U$. Then $t \in f(e_s)$, $t \in f(e_{s'})$. Now put $e_l = e_s \cap e_{s'}$. Then $t \in f(e_s) \cap f(e_{s'}) = f(e_l)$, since $f \in Rep_1(U)$. Hence $\neg(\, e_l \subset e_s\,)$ and $\neg(\, e_l \subset e_{s'}\,)$. But $e_l \subseteq e_s$ and $e_l \subseteq e_{s'}$. Therefore $e_s = e_l = e_{s'}$, i.e. $s = s'$. (II) follows from (i). □

We leave it to the reader to check that $Rep_1(U)$ is in fact a cpo. Defining as usual for $X, Y \in U$, $F(X)(Y) = X * Y$ one then has $F \in [U \rightarrow Rep_1(U)]$, since application is continuous in both its arguments and distributive over $\cap$ for consistent sets. Then $F \circ G = id_{Rep_1(U)}$ and moreover, $G \circ F = id_U$, since $(U, *)$ is extensional. Hence $U \cong Rep_1(U)$. We shall now extend this result to continuous, distributive functions with arbitrary arity.

Proposition 7.5.14 For all $n \in \omega$ and all $f : U^{n+1} \rightarrow U$

$$f \text{ is representable in } U \text{ iff } f \in Rep_{n+1}(U).$$

PROOF: $\rightarrow$ follows from the fact that $*$ is continuous and distributive over $\cap$ for consistent sets.

$\leftarrow$: By induction on n. The basis case is given by lemma 7.5.13. Let $f \in Rep_{n+2}(U)$. For $X_0, \ldots, X_n \in U$ define $f_{X_0,\ldots,X_n} : U \rightarrow U$ by

$$f_{X_0,\ldots,X_n}(X) = f(X_0, \ldots, X_n, X).$$

Then $f_{X_0,\ldots,X_n} \in Rep_1(U)$. By the induction hypothesis we can pick for all $X_0, \ldots, X_n \in U$ an $Y_{f,X_0,\ldots,X_n} \in U$ representing $f_{X_0,\ldots,X_n}$. Then

$$(1) \quad Y_{f,X_0,\ldots,X_n} \cup Y_{f,Z_0,\ldots,Z_n} \in U,$$

for all $X_0 \cup Z_0, ..., X_n \cup Z_n \in U$: We shall invoke proposition 7.5.7.

For (I) let $(s,t),(s',t) \in Y_{f,X_0,...,X_n} \cup Y_{f,Z_0,...,Z_n}$ be such that $e_s \cup e_{s'} \in U$. If $(s,t),(s',t) \in Y_{f,X_0,...,X_n}$ or $(s,t),(s',t) \in Y_{f,Z_0,...,Z_n}$ then $s = s'$. Thus assume $(s,t) \in Y_{f,X_0,...,X_n}$ and $(s',t) \in Y_{f,Z_0,...,Z_n}$. Then

$$t \in Y_{f,X_0,...,X_n} * e_s = f(X_0, ..., X_n, e_s)$$

and

$$t \in Y_{f,Z_0,...,Z_n} * e_{s'} = f(Z_0, ..., Z_n, e_{s'}).$$

Hence

$$t \in f(X_0, ..., X_n, e_s) \cap f(Z_0, ..., Z_n, e_{s'}) =$$

$$= f(X_0 \cap Z_0, ..., X_n \cap Z_n, e_s \cap e_{s'}) \subseteq f(X_0, ..., X_n, e_s \cap e_{s'}) =$$

$$= Y_{f,X_0,...,X_n} * (e_s \cap e_{s'}),$$

since f is distributive and monotone. Thus $(r,t) \in Y_{f,X_0,...,X_n}$, for some $e_r \subseteq (e_s \cap e_{s'})$. Since $(s,t) \in Y_{f,X_0,...,X_n}$ and $e_r \cup e_s = e_s \in U$, one has $r = s$. So $e_s \subseteq (e_s \cap e_{s'}) \subseteq e_{s'}$. Therefore $s \leq s'$. Similarly, one shows that $s' \leq s$. Hence $s = s'$.

For (II) observe that $(Y_{f,X_0,...,X_n} \cup Y_{f,Z_0,...,Z_n}) * X =$

$$= (Y_{f,X_0,...,X_n} * X) \cup (Y_{f,Z_0,...,Z_n} * X), \text{ by the definition of } *$$

$$= f(X_0, ..., X_n, X) \cup f(Z_0, ..., Z_n, X) \subseteq f(X_0 \cup Z_0, ..., X_n \cup Z_n, X),$$

since f is montone. Thus $(Y_{f,X_0,...,X_n} \cup Y_{f,Z_0,...,Z_n}) * X \in U$, since U is closed under subsets.

From (1) it now follows that for all $X_0 \cup Z_0, ..., X_n \cup Z_n \in U$,

$$(2) \quad Y_{f,X_0,...,X_n} \cap Y_{f,Z_0,...,Z_n} = Y_{f,X_0 \cap Z_0,...,X_n \cap Z_n} \quad :$$

Let $X \in U$. Then $(Y_{f,X_0,...,X_n} \cap Y_{f,Z_0,...,Z_n}) * X =$

$$= (Y_{f,X_0,...,X_n} * X) \cap (Y_{f,Z_0,...,Z_n} * X), \text{ by (1) and 7.5.11}$$

$$= f(X_0, ..., X_n, X) \cap f(Z_0, ..., Z_n, X)$$

$$= f(X_0 \cap Z_0, ..., X_n \cap Z_n, X) = Y_{f,X_0 \cap Z_0,...,X_n \cap Z_n} * X.$$

Thus $Y_{f,X_0,...,X_n} \cap Y_{f,Z_0,...,Z_n} = Y_{f,X_0 \cap Z_0,...,X_n \cap Z_n}$, by extensionality.
Now define $h : U^{n+1} \to U$ by

$$h(X_0, ..., X_n) = Y_{f,X_0,...,X_n}.$$

We shall show that $h \in Rep_{n+1}(U)$. The distributivity of h for consistent sets follows from (2). In order to prove that $h \in [U^{n+1} \to U]$ it is sufficient to prove that h is continuous in its arguments separately. Let $0 \leq i \leq n$ and $D \subseteq U$ be directed. Then for all $X \in D$,

$$h(X_0, ..., X_{i-1}, X, X_{i+1}, ..., X_n) =$$

$$= h(X_0, ..., X_{i-1}, X \cap (\cup D), X_{i+1}, ..., X_n)$$

$$= h(X_0, ..., X_{i-1}, X, X_{i+1}, ..., X_n) \cap h(X_0, ..., X_{i-1}, \cup D, X_{i+1}, ..., X_n)$$

$$\subseteq h(X_0, ..., X_{i-1}, \cup D, X_{i+1}, ..., X_n).$$

Hence $\cup\{h(X_0, ..., X_{i-1}, X, X_{i+1}, ..., X_n) \mid X \in D\} \in U$, since U is closed under subsets. Now let $Z \in U$. Then

$$(\cup\{h(X_0, ..., X_{i-1}, X, X_{i+1}, ..., X_n) \mid X \in D\}) * Z =$$

$$= \cup\{h(X_0, ..., X_{i-1}, X, X_{i+1}, ..., X_n) * Z \mid X \in D\}$$

$$= \cup\{f(X_0, ..., X_{i-1}, X, X_{i+1}, ..., X_n, Z) \mid X \in D\}$$

$$= f(X_0, ..., X_{i-1}, \cup D, X_{i+1}, ..., X_n, Z), \text{ since } f \text{ is continuous}$$

$$= h(X_0, ..., X_{i-1}, \cup D, X_{i+1}, ..., X_n) * Z.$$

Thus

$$\cup\{h(X_0, ..., X_{i-1}, X, X_{i+1}, ..., X_n) \mid X \in D\} =$$

$$= h(X_0, ..., X_{i-1}, \cup D, X_{i+1}, ..., X_n),$$

by extensionality.
This proves that $h \in Rep_{n+1}(U)$. By the induction hypothesis there is an $X_h \in U$ representing h. Then

$$X_h * X_0 * ... * X_n * X_{n+1} = h(X_0, ..., X_n) * X_{n+1} = f(X_0, ..., X_n, X_{n+1}),$$

for all $X_0, ..., X_n, X_{n+1}$. Thus f is representable. □

Theorem 7.5.15 There are $K, S \in U$ such that $(U, *, K, S)$ is an extensional ca.

PROOF. $(U, *)$ is an extensional applicative structure by proposition 7.5.8. Define $f_K : U^2 \to U$ and $f_S : U^3 \to U$ by

$$f_K(X, Y) = X \ , \ f_S(X, Y, Z) = X * Z * (Y * Z),$$

for all $X, Y, Z \in U$. Clearly, $f_K \in Rep_2(U)$. f_S is continuous, since $*$ is so. For the distributivity let $X \cup X', Y \cup Y', Z \cup Z' \in U$. Then

$$(Y * Z) \cup (Y' * Z') \subseteq (Y \cup Y') * (Z \cup Z'),$$

since $*$ is monotone. Therefore also $(Y * Z) \cup (Y' * Z') \in U$. Thus

$$f_S(X, Y, Z) \cap f_S(X', Y', Z') = ((X * Z * (Y * Z)) \cap ((X' * Z' * (Y' * Z')) =$$

$$(X \cap X') * (Z \cap Z') * ((Y * Z) \cap (Y' * Z')) =$$

$$= (X \cap X') * (Z \cap Z') * ((Y \cap Y') * (Z' \cap Z')) = f_S(X \cap X', Y \cap Y', Z \cap Z'),$$

by 7.5.11. Hence also $f_S \in Rep_3(U)$. Now let $K, S \in U$ be the representations of f_K and f_S, respectively. □

We shall now define an appropriate pca$^+$-expansion. The numerals will be singletons which have pairwise no upper bound. They differ from the numerals in the standard pca$^+$-expansion of P_ω where $N = \{\{n\} \mid n \in \omega\}$.

Definition 7.5.16 For $n \in \omega$, define $\overline{n} = \{(\psi(n), 0)\}$ where

$$e_{\psi(n)} := \{(2^{\phi(m)}, \phi(m)) \mid m \leq n\}$$

and $\phi : \omega \to \omega$ is defined by

(i) $\phi(0) = 0$,

(ii) $\phi(n + 1) = (2^{\phi(n)}, 0)$. □

Lemma 7.5.17 ϕ is injective and $range\phi \subseteq M(\omega)$. Moreover, for all $n, m \in \omega$

(i) $e_{\psi(n)} \in U$,

(ii) $\overline{n} \in U$,

(iii) $\overline{n} \cup \overline{m} \in U \longrightarrow n = m$.

PROOF. First recall that 2^m is the code for the singleton $\{m\}$. Clearly, ϕ is injective, since $(.,.)$ is injective. One proves $range\phi \subseteq M(\omega)$ by induction on n. $\phi(0) \in M(\omega)$ by definition 7.5.2. The induction step follows from lemma 7.5.9.

For (i) we shall invoke 7.5.7. Clearly, (I) holds. To prove (II) observe that

$$e_{\psi(n)} * X = \{\phi(m) \mid m \leq n \ \wedge \ \phi(m) \in X\} \subseteq X,$$

for all $X \in U$.

(ii) follows from (i) and lemma 7.5.9. For (iii) let $\overline{n} \cup \overline{m} \in U$, i.e. $\{(\psi(n), 0), (\psi(m), 0)\} \in U$. Observe that either $e_{\psi(n)} \subseteq e_{\psi(m)}$ or $e_{\psi(m)} \subseteq e_{\psi(n)}$. Hence $e_{\psi(n)} \cup e_{\psi(m)} \in U$. Thus $\psi(n) = \psi(m)$, by 7.5.7(I). Therefore $n = m$. □

Proposition 7.5.18 There is a pca$^+$-expansion of $(U, *)$ such that

$$N = \{\overline{n} \mid n \in \omega\}.$$

PROOF. For $X \in U$ define $f_{S_N}(X), f_{P_N}(X)$ by

$$f_{S_N}(X) = \begin{cases} \overline{n+1} & \text{if } \overline{n} \subseteq X \\ \emptyset & \text{otherwise} \end{cases}$$

and

$$f_{P_N}(X) = \begin{cases} \overline{n} & \text{if } \overline{n+1} \subseteq X \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $f_{S_N} \in Rep_1(U)$: $f_{S_N}(X)$ is well-defined by 7.5.17(iii). Moreover, f_{S_N} is continuous, since the numerals are singletons. Thus $f_{S_N} \in [U \to U]$. For the distributivity let $X \cup X' \in U$. If $f_{S_N}(X) = \emptyset$ or $f_{S_N}(X') = \emptyset$ then

$$f_{S_N}(X) \cap f_{S_N}(X') = \emptyset = f_{S_N}(X \cap X').$$

Thus assume $f_{S_N}(X) = \overline{n+1}$ and $f_{S_N}(X') = \overline{m+1}$. Then $\overline{n} \subseteq X$ and $\overline{m} \subseteq X'$ and therefore $\overline{n} \cup \overline{m} \subseteq X \cup X' \in U$. Hence $n = m$ and $\overline{n} \subseteq X \cap X'$. So

$$f_{S_N}(X) \cap f_{S_N}(X') = \overline{n+1} = f_{S_N}(X \cap X').$$

Similarly $f_{P_N} \in Rep_1(U)$.
For $X, Y, Z_1, Z_2 \in U$ define

$$f_D(X,Y,Z_1,Z_2) = \begin{cases} Y & \text{if } \exists n \in \omega(\overline{n} \subseteq Z_1 \cap Z_2) \\ X & \text{if } \exists n,m \in \omega(n \neq m \ \wedge \ \overline{n} \subseteq Z_1 \ \wedge \ \overline{m} \subseteq Z_2) \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $f_D \in Rep_4(U)$: Observe that $f_D(X,Y,Z_1,Z_2)$ is well-defined by 7.5.17. Thus $f_D : U^4 \to U$. To prove that f_D is continuous it is sufficient to prove that f_D is continuous in its arguments separately. Clearly, f_D is continuous in X and Y. For Z_1 observe that f_D is monotone. Hence

$$\bigcup\{f_D(X,Y,Z,Z_2) \mid Z \in U'\} \subseteq f_D(X,Y,\bigcup U',Z_2),$$

for directed $U' \subseteq U$. The converse follows from the fact that the numerals are singletons. By symmetry, f_D is also continuous in Z_2. So $f_D \in [U^4 \to U]$. Now let $X \cup X', Y \cup Y', Z_1 \cup W_1, Z_2 \cup W_2 \in U$. Again one has

$$f_D(X \cap X', Y \cap Y', Z_1 \cap W_1, Z_2 \cap W_2) = \emptyset,$$

if $f_D(X,Y,Z_1,Z_2) = \emptyset$ or $f_D(X',Y',W_1,W_2) = \emptyset$. Thus assume $f_D(X,Y,Z_1,Z_2) \neq \emptyset \neq f_D(X',Y',W_1,W_2)$. We we shall distinguish four cases.
(1) $f_D(X,Y,Z_1,Z_2) = Y$ and $f_D(X',Y',W_1,W_2) = Y'$. Then there are $n, m \in \omega$ such that $\overline{n} \subseteq Z_1 \cap Z_2$ and $\overline{m} \subseteq W_1 \cap W_2$. Hence

$$\overline{n} \cup \overline{m} \subseteq (Z_1 \cap Z_2) \cup (W_1 \cap W_2) \subseteq Z_1 \cup W_1.$$

So $n = m$ and therefore

$$\overline{n} \subseteq Z_1 \cap Z_2 \cap W_1 \cap W_2 = (Z_1 \cap W_1) \cap (Z_2 \cap W_2).$$

Thus $f_D(X \cap X', Y \cap Y', Z_1 \cap W_1, Z_2 \cap W_2) = Y \cap Y'$.
(2) $f_D(X,Y,Z_1,Z_2) = Y$ and $f_D(X',Y',W_1,W_2) = X'$. Then there are $n, m, l \in \omega$ such that $\overline{n} \subseteq Z_1 \cap Z_2$, $\overline{m} \subseteq W_1$, $\overline{l} \subseteq W_2$ and $m \neq l$. Then

$$\overline{n} \cup \overline{m} \subseteq (Z_1 \cap Z_2) \cup W_1 \subseteq Z_1 \cup W_1.$$

Hence $n = m$. Moreover,

$$\overline{n} \cup \overline{l} \subseteq (Z_1 \cap Z_2) \cup W_2 \subseteq Z_2 \cup W_2.$$

Hence $n = l$. So $m = l$. Contradiction.
(3) $f_D(X, Y, Z_1, Z_2) = X$ and $f_D(X', Y', W_1, W_2) = Y'$. As (2).
(4) $f_D(X, Y, Z_1, Z_2) = X$ and $f_D(X', Y', W_1, W_2) = X'$. Then there are $l, m, n, p \in \omega$ such that $\overline{l} \subseteq Z_1$, $\overline{m} \subseteq Z_2$, $\overline{n} \subseteq W_1$, $\overline{p} \subseteq W_2$ and $l \neq m$, $n \neq p$. Hence

$$\overline{l} \cup \overline{n} \subseteq Z_1 \cup W_1.$$

Hence $l = n$ and therefore $\overline{l} \subseteq Z_1 \cap W_1$. Similarly, $\overline{m} \subseteq Z_2 \cap W_2$. Thus $f_D(X \cap X', Y \cap Y', Z_1 \cap W_1, Z_2 \cap W_2) = X \cap X'$.
This proves that $f_D \in Rep_4(U)$. Now let $S_N, P_N, D \in U$ be the representations of f_{S_N}, f_{P_N} and f_D, respectively. □

Finally, we shall show that the finite type structures do not coincide in any pca$^+$-expansion of $(U, *)$ having $\{\overline{n} \mid n \in \omega\}$ as its set of numerals. The reason for this disagreement is twofold: firstly, the type-2 functional

$$H(f) = \begin{cases} \overline{0} & \text{if } f * \emptyset \neq \emptyset \\ \overline{1} & \text{if } f * \emptyset = \emptyset \end{cases}$$

is representable in such a pca$^+$-expansion and secondly, there are equivalent type-1 functions f_1, f_2 with $f_1 * \emptyset = \emptyset$ and $f_2 * \emptyset \neq \emptyset$.

Proposition 7.5.19 Let M be a pca$^+$-expansion of $(U, *)$ such that $N = \{\overline{n} \mid n \in \omega\}$. Then $IT(M) \neq ET(M)$.

PROOF. For $X \in U$, define

$$f_H(X) = \begin{cases} \overline{0} & \text{if } \exists n \in \omega(\overline{n} \subseteq X * \emptyset) \\ \overline{1} & \text{if } \exists n \in \omega(\overline{n} \subseteq X * \overline{0} \ \wedge \ \neg(\overline{n} \subseteq X * \emptyset)) \\ \emptyset & \text{otherwise.} \end{cases}$$

Let us first prove that $f_H(X)$ is well-defined. Suppose there are $n, m \in \omega$ with $\overline{n} \subseteq X * \emptyset$, $\overline{m} \subseteq X * \overline{0}$ and $\neg(\overline{m} \subseteq X * \emptyset)$. Then also

$$\overline{n} \subseteq X * \emptyset \subseteq X * \overline{0}.$$

Hence $\overline{n} \cup \overline{m} \subseteq X * \overline{0}$. Therefore $n = m$ and thus $\overline{m} \subseteq X * \emptyset$. Contradiction.
So $f_H : U \to U$. In order to prove that $f_H \in Rep_1(U)$ we shall first show that f_H is monotone. So let $X \subseteq X' \in U$. If $f_H(X) = \emptyset$ or $f_H(X) = \overline{0}$ then clearly $f_H(X) \subseteq f_H(X')$. Assume $f_H(X) = \overline{1}$. Then $\overline{n} \subseteq X * \overline{0}$ and $\neg(\overline{n} \subseteq X * \emptyset)$, for some $n \in \omega$. Thus $(2^{(\psi(0),0)}, (\psi(n), 0)) \in X \subseteq X'$. Therefore $(0, (\psi(n), 0)) \notin X'$, by 7.5.7(I). Hence $\overline{n} \subseteq X' * \overline{0}$ and $\neg(\overline{n} \subseteq X' * \emptyset)$, i.e. $f_H(X') = \overline{1}$.
From the monotonicity it now follows that

$$\cup\{f_H(X) \mid X \in U'\} \subseteq f_H(\cup U'),$$

for directed $U' \subseteq U$. The converse is proved by combining the continuity of $*$ and the fact that the numerals are singletons. So $f_H \in [U \to U]$. For the distributivity let $X \cup X' \in U$. If $f_H(X) = \emptyset$ or $f_H(X') = \emptyset$ then clearly

$$f_H(X) \cap f_H(X') = \emptyset = f_H(X \cap X').$$

The remaining four cases are:
(1) $f_H(X) = \overline{0}$ and $f_H(X') = \overline{0}$. Then there are $n, m \in \omega$ such that $\overline{n} \subseteq X * \emptyset$ and $\overline{m} \subseteq X' * \emptyset$. Thus

$$\overline{n} \cup \overline{m} \subseteq (X * \emptyset) \cup (X' * \emptyset) \subseteq (X \cup X') * \emptyset.$$

Hence $n = m$ and therefore $\overline{n} \subseteq (X * \emptyset) \cap (X' * \emptyset) = (X \cap X') * \emptyset$. So $f_H(X \cap X') = \overline{0}$.
(2) $f_H(X) = \overline{0}$ and $f_H(X') = \overline{1}$. Then there are $n, m \in \omega$ such that $\overline{n} \subseteq X * \emptyset$, $\overline{m} \subseteq X' * \overline{0}$ and $\neg(\overline{m} \subseteq X' * \emptyset)$. Thus

$$\overline{n} \subseteq X * \emptyset \subseteq X * \overline{0} \subseteq (X \cup X') * \overline{0}$$

and

$$\overline{m} \subseteq X' * \overline{0} \subseteq (X \cup X') * \overline{0}.$$

Hence $n = m$ and therefore

$$\overline{m} \subseteq (X * \emptyset) \cap (X' * \overline{0}) = (X \cap X') * (\emptyset \cap \overline{0}) = (X \cap X') * \emptyset \subseteq X' * \emptyset.$$

Contradiction.
(3) $f_H(X) = \overline{1}$ and $f_H(X') = \overline{0}$. As (2).
(4) $f_H(X) = \overline{1}$ and $f_H(X') = \overline{1}$. Then there are $n, m \in \omega$ such that $\overline{n} \subseteq X * \overline{0}$, $\overline{m} \subseteq X' * \overline{0}$, $\neg(\overline{n} \subseteq X * \emptyset)$ and $\neg(\overline{m} \subseteq X' * \emptyset)$. Thus

$$\overline{n} \cup \overline{m} \subseteq (X * \overline{0}) \cup (X' * \overline{0}) \subseteq (X \cup X') * \overline{0}.$$

Hence $n = m$ and therefore $\overline{n} \subseteq (X * \overline{0}) \cap (X' * \overline{0}) = (X \cap X') * \overline{0}$. Moreover, $\neg(\overline{n} \subseteq (X \cap X') * \emptyset)$, since $X \cap X' \subseteq X$. So $f_H(X \cap X') = \overline{1}$. This proves that $f_H \in Rep_1(U)$. Now let $H \in U$ be the representation of f_H. Clearly $H \in IT_2$. Define $f_1, f_2 : U \to U$ by $f_1 = \lambda X.\overline{0}$ and

$$f_2(X) = \begin{cases} \overline{0} & \text{if } \exists n \in \omega(\overline{n} \subseteq X) \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $f_1, f_2 \in Rep_1(U)$. Let $X_1, X_2 \in U$ be the representations of f_1 and f_2, respectively, and observe that $X_1 =_1 X_2$ and

$$H * X_1 = \overline{0} \neq \overline{1} = H * X_2. \;\square$$

Corollary 7.5.20 There is an extensional pca$^+$ M such that

$$IT(M) \neq ET(M). \;\square$$

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Index

Samenvatting

Dit proefschrift bevat een vijftal artikelen over verschillende onderwerpen binnen het raamwerk van de zogenaamde combinatorische algebra's, dwz. modellen van de combinatorische logica. De artikelen worden voorafgegaan door een algemene inleiding en een korte opsomming van uit de literatuur bekende definities en feiten die voor het vervolg van belang zijn.

In hoofdstuk 3 laten we zien hoe uitgaande van een van de eenvoudigste model-constructies, het graph-model, een extensioneel model kan worden geconstrueerd. Deze constructie berust op de techniek van de extensionele collaps.

Hoofdstuk 4 modificeert deze techniek voor de verkrijging van niet-totale extensionele modellen. De beschrijving van deze standaardmethode wordt voorafgegaan door een belichting van enkele eigenschappen van niet-totale extensionele modellen en de invoering van p-reflexive volledige partiële ordeningen, die een raamwerk vormen voor niet-totale topologische modellen, dwz. modellen waarin iedere continue functie representeerbaar is.

Hoofdstuk 5 behandelt cardinaliteitsaspecten van topologische modellen; in het bijzonder wordt aangetoond, dat niet-totale topologische combinatorische algebra's overaftelbaar zijn.

In hoofdstuk 6 laten wij zien, dat iedere partiële applicative structuur kan worden ingebed in een extensioneel topologisch model. Hierbij maken wij gebruik van de constructies uit hoofdstuk 3 en 4.

Het laatste en meest omvangrijke hoofdstuk gaat over eindige typenstructuren binnen combinatorische algebra's. Het centrale thema is hierbij finite-type-extensionaliteit, dwz. extensionaliteit op eindige types. Er worden bekende modellen op deze eigenschap heen getoetst. Aangetoond wordt, dat de meeste voorbeelden uit de literatuur ft-extensioneel zijn ongeacht hun graad van globale extensionaliteit. Om te laten zien, dat er geen verband bestaat tussen locale en globale extensionaliteit wordt een extensioneel model geconstrueerd, dat niet ft-extensioneel is.