Paradefinite Zermelo-Fraenkel Set Theory: A Theory of Inconsistent and Incomplete Sets

MSc Thesis *(Afstudeerscriptie)*

written by

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Contents

Abstract

A paradefinite logic is a logic that is both paraconsistent and paracomplete. In this thesis, we present a set theory in a four-valued paradefinite logic that can be viewed as the result of enriching the standard von Neumann universe for ZFC with various non-classical sets.

Our approach differs from most previous attempts at paraconsistent or paracomplete set theory in that we do not chase increasingly general comprehension principles. Rather, we prioritise an intuitive treatment of non-classical sets so as to make our set theory accessible to the classical mathematician who is used to working in classical ZFC. Moreover, as we work in a paradefinite logic, we provide a unified account of paraconsistent and paracomplete set theory.

We provide a natural model of our set theory starting from classical ZFC. We also show that within our theory, we can construct a class that acts as a model of classical ZFC. This allows us to translate back and forth between our theory and classical ZFC. Finally, we will generalize the construction of Boolean-valued models for classical set theory to obtain algebra-valued models of our theory.

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Introduction

The *principle of explosion* states that from a proposition φ together with its negation $\sim\varphi$, all other propositions follow. A logic is said to be *paraconsistent* if it does not validate this principle [24]. Dually, a *paracomplete* logic rejects the *law of excluded middle*, which states that for any given proposition φ, either φ is true or $\sim\varphi$ is true [18]. Finally, a logic is called *paradefinite* it is both paraconsistent and paracomplete [2].

In [5, 6], Belnap motivates a four-valued paradefinite logic by envisioning a computer having access to a database that contains possibly inconsistent and incomplete information being asked by a user whether a given proposition is true or false. He argues that the computer should start by organizing the information available to it by marking any given atomic proposition with the sign 'told True' if it has information saying that the proposition is true, and marking the proposition with the sign 'told False' if it has information saying that the proposition is false. It should then assign the proposition one of the truth values *only true* (1), *only false* (0), *both true and false* ($\mathfrak{b}$) and *neither true nor false* ($\mathfrak{n}$).[1] The labels 'told True' and 'told False' are then assigned to compound statements involving negation, conjunction and disjunction in a natural manner and propositions get their truth value accordingly. When the user then asks the computer about a particular statement, the computer responds by giving the truth value of the statement. So if the database contains conflicting information regarding the statement, the computer will reply something along the lines "I have both been told that this statement is true and that this statement false."

This idea has since been expanded upon by keeping the same basic setup, but adding new connectives besides negation, conjunction and disjunction. In [3], a natural implication connective is added, and in [23], the logic $BS4$ is given by keeping the aforementioned implication and adding a so-called *classicality operator*. This operator expresses that a proposition has the truth value 1 or 0. A predicate version is also given.

The aim of this thesis is to develop an axiomatic set theory in the predicate version of $BS4$ which allows us to represent both inconsistent and incomplete information by allowing statement of the form $a \in b$ to take any one of the truth values 1, 0, $\mathfrak{b}$ or $\mathfrak{n}$. So $a \in b$ can be only true, only false, both true nor false, or it can be neither true nor false.

[1] Belnap refers to them as told values and denotes them by **T**, **F**, **Both** and **Neither**.

A bit of terminology: A set a will be called *classical* if the proposition $x \in a$ has the truth value 1 or 0 for every x. Similarly, a will be called *consistent* if for all x, the truth value of $x \in a$ is 1, 0 or $\mathfrak{n}$, and a will be called *complete* if for all x, the truth value of $x \in a$ is 1, 0 or $\mathfrak{b}$.[2]

The thesis is divided into three parts. Part I serves as an introduction to the logic $BS4$. Chapter 1 contains an informal introduction to the logic, and Chapter 2 covers the syntax and semantics of $BS4$ and introduces a few useful defined connectives. In Chapter 3, we introduce algebraic semantics for $BS4$, based on so called "twist-structures", from [10] and [30], originally developed for Nelson's constructive logic with strong negation from [20].

Part II contains the main results of the thesis. In Chapter 4, we give two axiomatic set theories in the the predicate version of the logic $BS4$ called $PZFC$ and $BZFC$. The theory $PZFC$ is arrived at by giving natural versions of the ZFC axioms in $BS4$, and the theory $BZFC$ is then obtained by adding an axiom called the *anti-classicality axiom*[3], abbreviated as $AClA$, postulating the existence of various non-classical sets. We can think of $PZFC$ as ZFC without the implicit assumption that all sets are classical. The theory $BZFC$ can, in turn, be thought of as ZFC with said assumption replaced with the anti-classicality axiom.

In Chapter 5, we construct a natural model, which we will call W, of $BZFC$ within classical ZFC. This implies that $BZFC$ is not trivial[4], assuming that ZFC is consistent. We also show that the classical universe V can be embedded into W. So W can be seen as the result of extending V by adding various non-classical sets. In Chapter 6, we reverse the situation, and show that the class of hereditarily classical sets, abbreviated as HCl, is definible in $PZFC$ and that it satisfies the classical ZFC axioms. Here, a hereditarily classical set is a classical set whose members are classical sets, and so forth. This implies that ZFC consistent if $PZFC$ is non-trivial. We then go on to show that a sentence is a theorem of $BZFC$ if and only if ZFC proves that it holds in W. Similarly, a sentence is a theorem of ZFC if and only if $BZFC$ proves that it holds in HCl. So from the point of view of ZFC, $BZFC$ is the theory of W, and from the point of view of $BZFC$, ZFC is the theory of hereditarily classical sets.

Part III contains the Chapters 7 and 8. In Chapter 7, we give an application of $BZFC$. We show that by taking advantage of non-classical sets, we can give semantics for $BS4$ that are in a sense more natural than is possible in a classical set theory. In Chapter 8, we briefly review the construction of Boolean-valued models for set theory and go on to generalize said construction to get algebra-valued models for the theories $PZFC$ and $BZFC$. Our models will be similar to the ones found in [19] and [8] for paraconsistent set theory. However, by giving slightly different interpretation of the atomic formulas, we get models of the full theories rather than just the negation-free fragments as was the case in [19] and [8].

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[2] All of these are expressible in $BS4$.

[3] The name is inspired by Aczel's anti-foundation axiom from [1].

[4] Recall that a theory is said to be trivial if every sentence is derivable from it.

Part I

Logic

Chapter 1

An Informal Introduction to the Logic

In this chapter, we shall get acquainted with the four-valued logic $BS4$ from [23]. The logic $BS4$ with its four truth values and multiple implication connectives can at times seem counter-intuitive to the classically inclined mathematician/ logician. In order to make our introduction to $BS4$ as seamless as possible, we shall follow along a fictional character, Alice, as she gradually discovers $BS4$ when trying to organize all the information available to her while planning a big celebration.

It should be noted that none of the material in this chapter is original. However, notation and terminology has been used that is not standard in the literature.

1.1 Simple partial logic

Our story begins in early December. Our protagonist, Alice, who is an acclaimed logician and is known for throwing grand parties, decides to throw an extravagant New Year's Eve celebration. She plans to have a dinner in the evening, followed by a party that will go on long into the night. She sends out invitations to her friends and colleagues. She realizes that someone might want to have New Year's Eve dinner at home with their families but still attend the party. Conversely, someone might want to attend the dinner but not the party. Therefore, she asks in the invitations that people reply letting her know whether they will be attending, and to specify which they plan to attend.

As replies begin to arrive, Alice decides to organize the information contained in them by making five lists: I, D^+, D^-, P^+ and P^-. In I, she writes the names of everyone whom she has invited. She writes a name n in D^+/P^+ if she has received a reply stating that n will attend the dinner/party, and she writes n in D^-/P^- if she has received a reply stating that n will not attend the dinner/party.

In order to represent the information obtained from her lists, she introduces the predicate symbols D and P. She will regard $Q(n)$ to be true (T) if and only if n appears on the list Q^+ and regard $Q(n)$ to be false (F) if and only if n appears on the list Q^-, where Q is either D or P. So, for example, saying that $D(Bob)$ is T means that Bob has replied saying that he will be attending the dinner, while saying that $D(Bob)$ is F means that Bob has replied that he will not be attending the dinner.

Now, it is possible that a name is neither in D^+ nor D^- since someone might not yet have replied, or forgotten to specify whether they will attend the dinner. The same observation holds for P^+ and P^-. For the moment we shall assume, and so does Alice, that no name is in both Q^+ and Q^-.

She quickly realizes that at some point she will want to represent information more complex than just "is $D(n)$ T?" or "is $D(n)$ F?" For example, she might want to know whether someone has replied that they will attend the dinner but not the party. She therefore considers formulas of the form $\sim\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\exists x \varphi(x)$ and $\forall x \varphi(x)$. These formulas are read as "not φ", "φ and ψ", "φ or ψ", "for all x, $\varphi(x)$" and "there exists x such that $\varphi(x)$", respectively. She settles on the following interpretations:[1]

$$\sim\varphi \text{ is } \begin{cases} \text{T} & \text{iff } \varphi \text{ is F} \\ \text{F} & \text{iff } \varphi \text{ is T.} \end{cases}$$

$$\varphi \wedge \psi \text{ is } \begin{cases} \text{T} & \text{iff } \varphi \text{ is T and } \psi \text{ is T} \\ \text{F} & \text{iff } \varphi \text{ is F or } \psi \text{ is F.} \end{cases}$$

$$\varphi \vee \psi \text{ is } \begin{cases} \text{T} & \text{iff } \varphi \text{ is T or } \psi \text{ is T} \\ \text{F} & \text{iff } \varphi \text{ is F and } \psi \text{ is F.} \end{cases}$$

$$\exists x \varphi(x) \text{ is } \begin{cases} \text{T} & \text{iff } \varphi(x) \text{ is T for some } x \in I \\ \text{F} & \text{iff } \varphi(x) \text{ is F for all } x \in I. \end{cases}$$

$$\forall x \varphi(x) \text{ is } \begin{cases} \text{T} & \text{iff } \varphi(x) \text{ is T for all } x \in I \\ \text{F} & \text{iff } \varphi(x) \text{ is F for some } x \in I. \end{cases}$$

As an example the statement $\exists x[D(x) \wedge \sim P(x)]$ gets evaluated as follows:

$$\exists x[D(x) \wedge \sim P(x)] \text{ is } \begin{cases} \text{T} & \text{iff } D(x) \text{ is T and } P(x) \text{ is F for some } x \in I \\ \text{F} & \text{iff } D(x) \text{ is F or } P(x) \text{ is T for all } x \in I. \end{cases}$$

So $\exists x[D(x) \wedge \sim P(x)]$ is T if someone has said that they will come to the dinner but not the party, and it is F if everyone has either stated that they will not come to the dinner or that they will come to the party.

[1] I am taking this way of introducing connectives from [7].

Remark. It is important to note that "φ is T" should not be read as saying that "φ is necessarily true" or "Alice knows φ". Take, for example, the statement "Bob will attend the party or Bob will not attend the party". Clearly, this is necessarily true, and Alice knows this. However, $P(Bob) \vee {\sim}P(Bob)$ is T if and only if Bob has replied and specified whether he will come to the party.

At this point she sees that every sentence φ she can write down so far can be T, it can be F, or it can be neither T nor F. To keep track of the three possibilities, she defines the truth value of φ, which she denotes as $[\![\varphi]\!]$, as follows:

$$[\![\varphi]\!] := \begin{cases} 1 & \text{if, } \varphi \text{ is T} \\ \mathfrak{n} & \text{if, } \varphi \text{ is neither T nor F} \\ 0 & \text{if, } \varphi \text{ is F.} \end{cases}$$

She now has on her hands a three-valued logic with truth values 1, $\mathfrak{n}$ and 0, and with 1 as its only designated value. The value of the connectives are given by the following truth tables:

	$\sim$
1	0
$\mathfrak{n}$	$\mathfrak{n}$
0	1

$\wedge$	1	$\mathfrak{n}$	0
1	1	$\mathfrak{n}$	0
$\mathfrak{n}$	$\mathfrak{n}$	$\mathfrak{n}$	0
0	0	0	0

$\vee$	1	$\mathfrak{n}$	0
1	1	1	1
$\mathfrak{n}$	1	$\mathfrak{n}$	$\mathfrak{n}$
0	1	$\mathfrak{n}$	0

By ordering the truth values by $0 \leq \mathfrak{n} \leq 1$, she gets a complete lattice where the "meet" and "join" are given by the tables above. Moreover, $[\![\exists x\varphi(x)]\!] = \bigvee_{x\in I}[\![\varphi(x)]\!]$ and $[\![\forall x\varphi(x)]\!] = \bigwedge_{x\in I}[\![\varphi(x)]\!]$.

The logic she has now described is called *Simple Partial Logic* in [7]. It is also a predicate version of *Kleene's Strong Three-Valued Logic* $K3$ from [16].

1.2 Adding an implication

When trying to decide how to formalize the statement "if $D(Bob)$, then $P(Bob)$" or rather "$D(Bob)$ implies $P(Bob)$", she notices something strange. First she imagines that it is New Year's Eve and the party has already started. Then the statements "Bob attended the dinner" and "Bob attended the party" have both turned out to be either true or false. Moreover, the statement "if Bob attended the dinner, then Bob attended the party" will have the same truth value as "Bob did not come to the dinner or Bob came to the party." Going back to the present day, she arrives at "Bob will not attend the dinner or Bob will attend the party." She therefore defines a connective $\supset$ by $\varphi \supset \psi := {\sim}\varphi \vee \psi$. It has the following truth table:

$\supset$	1	$\mathfrak{n}$	0
1	1	$\mathfrak{n}$	0
$\mathfrak{n}$	1	$\mathfrak{n}$	$\mathfrak{n}$
0	1	1	1

Even though Alice sees that $\supset$ has an important role to play, she decides against formalizing the implication this way. Her reason being that $\supset$ does not allow her to carry out much deductive reasoning. To see why, suppose for a moment that Alice wants to know if she will, at all, see Bob on New Year's Eve. So she wants to evaluate $D(Bob) \vee P(Bob)$. By consulting the truth table for $\vee$, she sees that if $D(Bob)$ is T, then so is $D(Bob) \vee P(Bob)$. She would like to express this by saying that "if $D(Bob)$, then $D(Bob) \vee P(Bob)$" is T. However, if Bob has not yet replied to the invitation, then $D(Bob) \supset D(Bob) \vee P(Bob)$ is not T.

In order to remedy this, she decides to introduce a new connective $\to$ which is designed to more closely represent the reasoning she, herself, can carry out. So $\varphi \to \psi$ should correspond to something like "ψ, under the assumption that φ." She settles on the following interpretation for $\to$:

$$\varphi \to \psi \text{ is } \begin{cases} \text{T} & \text{iff } (\varphi \text{ is T}) \text{ implies } (\psi \text{ is T}) \\ \text{F} & \text{iff } \varphi \text{ is T and } \psi \text{ is F.} \end{cases}$$

This gives the following truth table:

$\to$	1	$\mathfrak{n}$	0
1	1	$\mathfrak{n}$	0
$\mathfrak{n}$	1	1	1
0	1	1	1

The propositional fragment of this logic is called $K3^{\to}$ in [13].

Remark. Before moving on we should emphasize the following point: Formulas such as $P(Bob) \to P(Carol)$ should **not** be read as "on New Years Eve it will be the case that Bob is at the party implies that Carol is at the party." To see why, simply note that if Bob has not replied, then $P(Bob) \to P(Carol)$ is T even though it is still possible that Bob actually comes to the party and Carol stays at home.

1.3 Dealing with contradictions

The following day disaster strikes! Alice is working in her system when she discovers that both $D(Bob)$ and $\sim D(Bob)$ are T. She realizes that from this contradiction she can derive every sentence. This means that she cannot trust anything she has derived so far.

Rather than giving up completely, she decides to call Bob and see what is going on. Bob informs her that when he first saw the invitation, he decided to attend both the dinner and the party. So he sent a reply stating as much. Later, his parents invited him to have dinner with them on New Year's Eve, so he wrote a new reply stating that he would attend the party but not the dinner. What happened is that Alice wrote Bob's name in D^+ and P^+ when she received the first reply. When she received the second reply, she also wrote his name in D^- without removing it from D^+. She was therefore able to derive that both $D(Bob)$ and $\sim D(Bob)$ were T.

With this information at hand, Alice decides to remove Bob's name from D^+, thereby eliminating the contradiction. She does, however, worry that this was not the only contradiction in her lists. So she can no longer assume that no name appears both on Q^+ and Q^-, where as usual Q is either D or P.

To account for this possibility she does not have to change much. She leaves the T/F-conditions for $D(n)$, $P(n)$, $\sim\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \to \psi$, $\exists x\varphi(x)$ and $\forall x\varphi(x)$ unchanged. By doing so, she now gets four possibilities for each φ: φ can be only T, only F, neither T nor F, or φ can be both T and F. She denotes the four possibilities by $[\![\varphi]\!] = 1$, $[\![\varphi]\!] = 0$, $[\![\varphi]\!] = \mathfrak{n}$ and $[\![\varphi]\!] = \mathfrak{b}$, respectively. She therefore has on her hands a four-valued logic with 1, $\mathfrak{b}$, $\mathfrak{n}$, and 0 as truth values and 1 and $\mathfrak{b}$ as designated values. The connectives now have the following truth tables:

	$\sim$
1	0
$\mathfrak{b}$	$\mathfrak{b}$
$\mathfrak{n}$	$\mathfrak{n}$
0	1

$\wedge$	1	$\mathfrak{b}$	$\mathfrak{n}$	0
1	1	$\mathfrak{b}$	$\mathfrak{n}$	0
$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{b}$	0	0
$\mathfrak{n}$	$\mathfrak{n}$	0	$\mathfrak{n}$	0
0	0	0	0	0

$\vee$	1	$\mathfrak{b}$	$\mathfrak{n}$	0
1	1	1	1	1
$\mathfrak{b}$	1	$\mathfrak{b}$	1	$\mathfrak{b}$
$\mathfrak{n}$	1	1	$\mathfrak{n}$	$\mathfrak{n}$
0	1	$\mathfrak{b}$	$\mathfrak{n}$	0

$\to$	1	$\mathfrak{b}$	$\mathfrak{n}$	0
1	1	$\mathfrak{b}$	$\mathfrak{n}$	0
$\mathfrak{b}$	1	$\mathfrak{b}$	$\mathfrak{n}$	0
$\mathfrak{n}$	1	1	1	1
0	1	1	1	1

She can now order the truth values by $0 \leq \mathfrak{n} \leq 1$ and $0 \leq \mathfrak{b} \leq 1$ and once again get a complete lattice with join $\vee$ and meet $\wedge$ and $[\![\exists x\varphi(x)]\!] = \bigvee_{x\in I}[\![\varphi(x)]\!]$ and $[\![\forall x\varphi(x)]\!] = \bigwedge_{x\in I}[\![\varphi(x)]\!]$. This lattice is called $L4$ in [5] and the propositional fragment of the logic is called $4CL$ in [2], $B_4^{\to}$ in [21] and $FDE^{\to}$ in [13].

Alice can now handle receiving contradictory replies without trivializing her system. For example, if Carol replied stating that she will attend the party and replied stating that she will not attend the party, then the proposition $P(Carol)$ is both T and F, i.e., $[\![P(Carol)]\!] = \mathfrak{b}$. This does not imply that every statement is T. It simply means that Carol has provided contradictory replies.

Finally, in light of what happened with Bob, she decides to add a connective to be able to express that a proposition has the truth value 1 or 0. She therefore defines the unary connective $\circ$ by

$$\circ\varphi \text{ is } \begin{cases} \text{T} & \text{iff } [\![\varphi]\!] = 1 \text{ or } 0 \\ \text{F} & \text{iff } [\![\varphi]\!] = \mathfrak{b} \text{ or } \mathfrak{n}. \end{cases}$$

It has the following truth table:

	$\circ$
1	1
$\mathfrak{b}$	0
$\mathfrak{n}$	0
0	1

As an example, $\circ P(Carol)$ is T iff Carol has either replied that she will attend the party or she has replied that she will not attend the party, but not both. On the other hand, $P(Carol)$ is F iff Carol has not provided a reply concerning the party or she has provided contradictory replies.

Alice has now arrived at the predicate version of $BS4$ from [23] but without equality.

Chapter 2

The Logic $BS4$

In this chapter, we make precise the logic $BS4$ outlined in Chapter 1. We will also introduce a few concepts and connectives that will be of use in later chapters.

Our presentation will differ slightly from [23], and notation and terminology has been used that is not standard in the literature.

2.1 Syntax

Let us start by fixing the logical symbols we shall be working with. They are the following:

1. A countable infinite set of variables;
2. the logical connectives $\sim$, $\wedge$, $\vee$ and $\rightarrow$;
3. the propositional constant $\perp$;
4. the quantifiers $\exists$ and $\forall$;
5. the equality symbol $=$;
6. the brackets (,), [and].

From here the syntax is defined exactly as usual per classical predicate logic.

Remark. For practical reasons, we opted to include the propositional constant $\perp$ in our basic syntax, rather than the connective $\circ$ from Chapter 1. However, it will become apparent that this results in an definitionally equivalent logic.

2.2 Semantics

While the syntax is the same as for classical predicate logic, the semantics is very slightly different since we need to take into account the separation of truth from falsity.

Definition 2.2.1. A T/F-*structure* or *model* $\mathcal{M}$ in a language L consists of

1. a non-empty set M, called the *domain* of $\mathcal{M}$;
2. an element $c^{\mathcal{M}} \in M$ for every constant symbol c in L;
3. a function $f^{\mathcal{M}} \colon M^n \to M$ for every n-ary function symbols f from L;
4. a pair of n-ary relations $R^+_{\mathcal{M}} \subseteq M^n$ and $R^-_{\mathcal{M}} \subseteq M^n$ for every n-ary relation symbol R in L;
5. a pair of binary relations $=^+_{\mathcal{M}} \subseteq M \times M$ and $=^-_{\mathcal{M}} \subseteq M \times M$ such that for all $m, n \in M$,
   (a) $m =^+_{\mathcal{M}} n$ iff $m = n$, i.e., $=^+_{\mathcal{M}}$ is the real equality on M, and
   (b) $m =^-_{\mathcal{M}} n$ iff $n =^-_{\mathcal{M}} m$.[1]

We let L_M denote the language obtained by adding a new constant symbol c_m to L for each $m \in M$. We will regard $\mathcal{M}$ as a T/F-structure in L_M with $(c_m)^{\mathcal{M}} = m$ and usually just write m instead of c_m.

Definition 2.2.2. Let $\mathcal{M}$ be a T/F-model and φ be a sentence in L_M. We recursively define what it means for φ to be *true* (T) or *false* (F) in $\mathcal{M}$ as follows:

$$\bot \text{ is } \begin{cases} \text{T} & \text{never} \\ \text{F} & \text{always.} \end{cases}$$

$$t = s \text{ is } \begin{cases} \text{T} & \text{iff } t^{\mathcal{M}} =^+_{\mathcal{M}} s^{\mathcal{M}} \\ \text{F} & \text{iff } t^{\mathcal{M}} =^-_{\mathcal{M}} s^{\mathcal{M}}. \end{cases}$$

$$R(t_1, ..., t_n) \text{ is } \begin{cases} \text{T} & \text{iff } (t_1^{\mathcal{M}}, ..., t_n^{\mathcal{M}}) \in R^+_{\mathcal{M}} \\ \text{F} & \text{iff } (t_1^{\mathcal{M}}, ..., t_n^{\mathcal{M}}) \in R^-_{\mathcal{M}}. \end{cases}$$

$$\sim\varphi \text{ is } \begin{cases} \text{T} & \text{iff } \varphi \text{ is F} \\ \text{F} & \text{iff } \varphi \text{ is T.} \end{cases}$$

$$\varphi \wedge \psi \text{ is } \begin{cases} \text{T} & \text{iff } \varphi \text{ is T and } \psi \text{ is T} \\ \text{F} & \text{iff } \varphi \text{ is F or } \psi \text{ is F.} \end{cases}$$

$$\varphi \vee \psi \text{ is } \begin{cases} \text{T} & \text{iff } \varphi \text{ is T or } \psi \text{ is T} \\ \text{F} & \text{iff } \varphi \text{ is F and } \psi \text{ is F.} \end{cases}$$

$$\varphi \to \psi \text{ is } \begin{cases} \text{T} & \text{iff } \varphi \text{ is T implies } \psi \text{ is T} \\ \text{F} & \text{iff } \varphi \text{ is T and } \psi \text{ is F.} \end{cases}$$

$$\exists x \varphi(x) \text{ is } \begin{cases} \text{T} & \text{iff } \varphi(m) \text{ is T for some } m \in M \\ \text{F} & \text{iff } \varphi(m) \text{ is F for all } m \in M. \end{cases}$$

[1] The condition (b) is not included in the original formulation from [23].

$$\forall x\varphi(x) \text{ is } \begin{cases} \text{T} & \text{iff } \varphi(m) \text{ is T for all } m \in M \\ \text{F} & \text{iff } \varphi(m) \text{ is F for some } m \in M. \end{cases}$$

We write $\mathcal{M} \vDash_4 \varphi$, and say that $\mathcal{M}$ is a T/F-model of φ, if φ is true in $\mathcal{M}$.

Definition 2.2.3. Let $\mathcal{M}$ be a T/F-model, and Σ and Δ be theories. We write $\mathcal{M} \vDash_4 \Sigma$, and call $\mathcal{M}$ a T/F-model of Σ, if $\mathcal{M} \vDash_4 \varphi$ for all $\varphi \in \Sigma$. We write $\Sigma \vDash_4 \Delta$ if every T/F-model of Σ is a T/F-model of Δ. We say that Σ is *trivial* if $\Sigma \vDash_4 \bot$.

Definition 2.2.4. Let $\mathcal{M}$ be a T/F-model and φ be an L_M-sentence. We define the *truth value* $[\![\varphi]\!]^{\mathcal{M}}$ of φ in $\mathcal{M}$ by

$$[\![\varphi]\!]^{\mathcal{M}} := \begin{cases} 1 & \text{if } \varphi \text{ is T} \\ \mathfrak{b} & \text{if } \varphi \text{ is both T and F} \\ \mathfrak{n} & \text{if } \varphi \text{ is neither T nor F} \\ 0 & \text{if } \varphi \text{ is F.} \end{cases}$$

Now, $[\![\bot]\!]^{\mathcal{M}} = 0$ and the truth value of $\sim\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$ and $\varphi \to \psi$ are obtained by the following truth tables:

	$\sim$
1	0
$\mathfrak{b}$	$\mathfrak{b}$
$\mathfrak{n}$	$\mathfrak{n}$
0	1

$\wedge$	1	$\mathfrak{b}$	$\mathfrak{n}$	0
1	1	$\mathfrak{b}$	$\mathfrak{n}$	0
$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{b}$	0	0
$\mathfrak{n}$	$\mathfrak{n}$	0	$\mathfrak{n}$	0
0	0	0	0	0

$\vee$	1	$\mathfrak{b}$	$\mathfrak{n}$	0
1	1	1	1	1
$\mathfrak{b}$	1	$\mathfrak{b}$	1	$\mathfrak{b}$
$\mathfrak{n}$	1	1	$\mathfrak{n}$	$\mathfrak{n}$
0	1	$\mathfrak{b}$	$\mathfrak{n}$	0

$\to$	1	$\mathfrak{b}$	$\mathfrak{n}$	0
1	1	$\mathfrak{b}$	$\mathfrak{n}$	0
$\mathfrak{b}$	1	$\mathfrak{b}$	$\mathfrak{n}$	0
$\mathfrak{n}$	1	1	1	1
0	1	1	1	1

By ordering the truth values by $0 \leq \mathfrak{n} \leq 1$ and $0 \leq \mathfrak{b} \leq 1$, we get the complete lattice $L4$ from [5]. It has the join $\vee$ and meet $\wedge$ and $[\![\exists x\varphi(x)]\!] = \bigvee_{x \in M}[\![\varphi(x)]\!]$ and $[\![\forall x\varphi(x)]\!] = \bigwedge_{x \in M}[\![\varphi(x)]\!]$.

Definition 2.2.5. Let $\mathcal{M}$ be a T/F-model and φ be a sentence. We say that φ is *classical in* $\mathcal{M}$ if $[\![\varphi]\!]^{\mathcal{M}} \in \{1, 0\}$. Similarly, we say that a sentence is *consistent* in $\mathcal{M}$ if $[\![\varphi]\!]^{\mathcal{M}} \neq \mathfrak{b}$ and *complete* if $[\![\varphi]\!]^{\mathcal{M}} \neq \mathfrak{n}$. A sentence is said to be *classical/consistent/complete* if it is classical/consistent/complete in all T/F-models.

2.3 Defined connectives

At this point we have become fairly well acquainted with the logic $BS4$. Now we will examine a few additional connectives, defined in terms of the primitive ones, that that will prove useful in our later treatment of set theory.

We define the bi-implication connective $\leftrightarrow$ by letting

$$\varphi \leftrightarrow \psi := \varphi \to \psi \wedge \psi \to \varphi.$$

Its T/F-conditions are given by

$$\varphi \leftrightarrow \psi \text{ is } \begin{cases} \text{T} & \text{iff } (\varphi \text{ is T}) \text{ if and only if } (\psi \text{ is T}) \\ \text{F} & \text{iff } (\varphi \text{ is T and } \psi \text{ is F}) \text{ or } (\varphi \text{ is F and } \psi \text{ is T}). \end{cases}$$

and it has the following truth table:

$\leftrightarrow$	1	$\mathfrak{b}$	$\mathfrak{n}$	0
1	1	$\mathfrak{b}$	$\mathfrak{n}$	0
$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{n}$	0
$\mathfrak{n}$	$\mathfrak{n}$	$\mathfrak{n}$	1	1
0	0	0	1	1

We read $\leftrightarrow$ as "if and only if". The main appeal of $\leftrightarrow$ is that if φ and ψ are sentences and $\Gamma \vDash_4 \varphi \leftrightarrow \psi$, then φ and ψ are true in precisely the same models of Γ. Moreover, if χ is a sentence and χ' is obtained from χ by replacing an occurrence of φ in χ, that is not in the scope of a $\sim$-negation symbol, by ψ, then χ and χ' are true in precisely the same models of Γ.

In order to motivate our next pair of connectives, we first point out what the connectives $\to$ and $\leftrightarrow$ do not tell us: Consider a model $\mathcal{M}$ and sentences φ and ψ with $[\![\varphi]\!]^{\mathcal{M}} = 1$ and $[\![\psi]\!]^{\mathcal{M}} = \mathfrak{b}$. Then $\mathcal{M} \vDash_4 \varphi \to \psi$, but $\mathcal{M} \nvDash_4 {\sim}\psi \to {\sim}\varphi$. So we do not have contraposition for $\to$, i.e.,

$$\varphi \to \psi \nvDash_4 {\sim}\psi \to {\sim}\varphi.$$

Moreover, $\mathcal{M} \vDash_4 \varphi \leftrightarrow \psi$, but $\mathcal{M} \nvDash_4 {\sim}\psi \leftrightarrow {\sim}\varphi$. Therefore,

$$\varphi \leftrightarrow \psi \nvDash_4 {\sim}\psi \leftrightarrow {\sim}\varphi.$$

This second point is particularly important. It tells us that even if $\mathcal{M} \vDash_4 \varphi \leftrightarrow \psi$ and $\mathcal{M} \vDash_4 \chi$, we cannot conclude $\mathcal{M} \vDash_4 \chi'$, where χ' is obtained from χ by replacing φ with ψ in the scope of a $\sim$-negation symbol. The connective $\leftrightarrow$ is therefore not a good notion of equivalence.

With this in mind we introduce the connectives $\Rightarrow$ and $\Leftrightarrow$ by letting

$$\varphi \Rightarrow \psi := \varphi \to \psi \wedge {\sim}\psi \to {\sim}\varphi \text{ and } \varphi \Leftrightarrow \psi := \varphi \Rightarrow \psi \wedge \psi \Rightarrow \varphi.$$ [2]

Their T/F-conditions, for a given T/F-model $\mathcal{M}$, are then given by

$$\varphi \Rightarrow \psi \text{ is } \begin{cases} \text{T} & \text{iff } [\![\varphi]\!]^{\mathcal{M}} \leq [\![\psi]\!]^{\mathcal{M}} \\ \text{F} & \text{iff } \varphi \text{ is T and } \psi \text{ is F.} \end{cases}$$

[2] To the best of my knowledge, the connectives $\Leftrightarrow$ and $\Rightarrow$ first appeared in [25] and chapter XII of [26], respectively.

$$\varphi \Leftrightarrow \psi \text{ is } \begin{cases} \text{T} & \text{iff } [\![\varphi]\!]^{\mathcal{M}} = [\![\psi]\!]^{\mathcal{M}} \\ \text{F} & \text{iff } (\varphi \text{ is T and } \psi \text{ is F) or } (\varphi \text{ is F and } \psi \text{ is T}). \end{cases}$$

They have the following truth tables:

$\Rightarrow$	1	$\mathfrak{b}$	$\mathfrak{n}$	0
1	1	0	$\mathfrak{n}$	0
$\mathfrak{b}$	1	$\mathfrak{b}$	$\mathfrak{n}$	0
$\mathfrak{n}$	1	$\mathfrak{n}$	1	$\mathfrak{n}$
0	1	1	1	1

$\Leftrightarrow$	1	$\mathfrak{b}$	$\mathfrak{n}$	0
1	1	0	$\mathfrak{n}$	0
$\mathfrak{b}$	0	$\mathfrak{b}$	$\mathfrak{n}$	0
$\mathfrak{n}$	$\mathfrak{n}$	$\mathfrak{n}$	1	$\mathfrak{n}$
0	0	0	$\mathfrak{n}$	1

Now, if $\varphi \Leftrightarrow \psi$ is true in $\mathcal{M}$, then φ and ψ have the same truth value in $\mathcal{M}$. We can therefore substitute instances of φ and ψ for each other in a sentence without changing the truth value of that sentence. On the other hand, $\varphi \Leftrightarrow \psi$ is false in $\mathcal{M}$ iff one of φ and ψ is true and the other is false. We will $\Leftrightarrow$ as "is equivalent to".

We define the *classical negation* $\neg$ by letting $\neg\varphi := \varphi \to \bot$. It has the following truth table:

	$\neg$
1	0
$\mathfrak{b}$	0
$\mathfrak{n}$	1
0	1

The classical negation allows us to express the absence of truth, in the sense that $\neg\varphi$ is true in a model $\mathcal{M}$ precisely when φ is not true in $\mathcal{M}$, i.e., $\mathcal{M} \vDash_4 \neg\varphi$ iff $\mathcal{M} \nvDash_4 \varphi$. Similarly, $\neg\varphi$ is false in $\mathcal{M}$ iff φ is true in $\mathcal{M}$. This gives the following T/F-conditions:

$$\neg\varphi \text{ is } \begin{cases} \text{T} & \text{iff } \varphi \text{ is not T} \\ \text{F} & \text{iff } \varphi \text{ is T.} \end{cases}$$

It follows that $\neg\varphi$ is a classical sentence.

Now that we have the classical negation, we can introduce unary connectives ! and ? by letting

$$!\varphi := {\sim}\neg\varphi \text{ and } ?\varphi := \neg{\sim}\varphi.$$ [3]

Their T/F-conditions are

$$!\varphi \text{ is } \begin{cases} \text{T} & \text{iff } \varphi \text{ is T} \\ \text{F} & \text{iff } \varphi \text{ is not T.} \end{cases}$$

$$?\varphi \text{ is } \begin{cases} \text{T} & \text{iff } \varphi \text{ is not F} \\ \text{F} & \text{iff } \varphi \text{ is F.} \end{cases}$$

[3] I am taking the connectives ! and ? from linear logic. See, e.g., [29].

So ! expresses the presence of truth, while ? expresses the absence of falsity. They have the following truth tables:

	!
1	1
$\mathfrak{b}$	1
$\mathfrak{n}$	0
0	0

	?
1	1
$\mathfrak{b}$	0
$\mathfrak{n}$	1
0	0

A key feature of ! and ? is that $!\varphi$ and $?\varphi$ are classical sentences. Moreover, in a given T/F-model,

$$\varphi \text{ is T if and only if } !\varphi \text{ is T, and}$$
$$\varphi \text{ is F if and only if } ?\varphi \text{ is F.}$$

We are therefore able to completely describe the T/F-conditions of a sentence in terms of a pair of classical sentences. That is, φ has the T-condition of $!\varphi$ and F-condition of $?\varphi$. The following observation will also prove useful:

$$\varphi \textit{ is classical in } \mathcal{M} \text{ iff } [\![!\varphi]\!]^{\mathcal{M}} = [\![?\varphi]\!]^{\mathcal{M}};$$
$$\varphi \textit{ is consistent in } \mathcal{M} \text{ iff } [\![!\varphi]\!]^{\mathcal{M}} \leq [\![?\varphi]\!]^{\mathcal{M}};$$
$$\varphi \textit{ is complete in } \mathcal{M} \text{ iff } [\![?\varphi]\!]^{\mathcal{M}} \leq [\![!\varphi]\!]^{\mathcal{M}}.$$

Accordingly, we introduce the connectives $\circ$, $\circ_{con}$ and $\circ_{com}$ by letting

$$\circ\varphi := !\varphi \Leftrightarrow ?\varphi, \;\; \circ_{con}\varphi := !\varphi \Rightarrow ?\varphi \text{ and } \;\circ_{com}\,\varphi := ?\varphi \Rightarrow !\varphi.$$

They have the following truth tables:

	$\circ$
1	1
$\mathfrak{b}$	0
$\mathfrak{n}$	0
0	1

	$\circ_{con}$
1	1
$\mathfrak{b}$	0
$\mathfrak{n}$	1
0	1

	$\circ_{com}$
1	1
$\mathfrak{b}$	1
$\mathfrak{n}$	0
0	1

Remark. Before moving on, we should address the following point: Some might find it distasteful to include the constant $\perp$ when working in a paraconsistent logic because it allows us to define the classical negation. However, if L is a language with finitely many relation symbols, then we could just as well have defined $\perp$ by

$$\perp = \forall x \forall y (x = y \wedge x \neq y) \wedge \bigwedge_{P \in L} \forall x_1, ..., x_n (P(x_1, ..., x_n) \wedge {\sim} P(x_1, ..., x_n)).$$

Since we will focus on set theory, we do not have any reservations about $\perp$.

2.4 Proofs in $BS4$

Considering that the aim of this thesis is to develop an axiomatic set theory is $BS4$, it stands to reason that we dedicate a little space discussing proofs $BS4$. Here, we are going to provide a sound and complete Hilbert-style proof system for $BS4$. The system is a slight modification on the one originally given in [23].

First, we notice that the T-conditions for the connectives $\wedge$, $\vee$ and $\to$ are just the ones we are used to from the semantics for classical logic. For example, $\varphi \wedge \psi$ is true in a particular T/F-model if and only if both φ and ψ are true in said T/F-model. The same observation goes for the quantifiers, $\perp$ and the equality symbol. This tells us that we should expect the axioms and inference rules that determine the behaviour of these symbols in classical logic to stay the same in $BS4$. We therefore introduce our first batch of axioms and inference rules.

- The first batch of axioms:

  1. $\varphi \to (\psi \to \varphi)$
  2. $(\varphi \to (\psi \to \chi)) \to$
     $((\varphi \to \psi) \to (\varphi \to \chi))$
  3. $\varphi \vee (\varphi \to \psi)$
  4. $(\varphi \wedge \psi) \to \varphi$
  5. $(\varphi \wedge \psi) \to \psi$
  6. $\varphi \to (\psi \to (\varphi \wedge \psi))$
  7. $\varphi \to (\varphi \vee \psi)$
  8. $\psi \to (\varphi \vee \psi)$
  9. $(\varphi \to \chi) \to ((\psi \to \chi) \to$
     $((\varphi \vee \psi) \to \chi))$
  10. $\perp \to \varphi$
  11. $\forall x \varphi(x) \to \varphi(t)$
  12. $\varphi(t) \to \exists x \varphi(x)$
  13. $x = x$
  14. $x = y \to [\varphi(x) \to \varphi(y)]$

- The inference rules:

  – From φ and $\varphi \to \psi$, infer ψ (modus ponens).
  – Infer $\varphi \to \forall x \psi$ from $\varphi \to \psi$, provided x does not occur free in φ.
  – Infer $\exists x \varphi \to \psi$ from $\varphi \to \psi$, provided x does not occur free in ψ.

We still need axioms that determine the behavior of the $\sim$-negation. These are obtained by looking at the F-conditions for formulas, and noting that $\sim\varphi$ should be true iff φ is false.

- Additional axioms for $BS4$:

  15. $\sim\sim\varphi \leftrightarrow \varphi$
  16. $\sim(\varphi \wedge \psi) \leftrightarrow (\sim\varphi \vee \sim\psi)$
  17. $\sim(\varphi \vee \psi) \leftrightarrow (\sim\varphi \wedge \sim\psi)$
  18. $\sim(\varphi \to \psi) \leftrightarrow (\varphi \wedge \sim\psi)$
  19. $\varphi \to \sim\perp$
  20. $\sim\forall x \varphi \leftrightarrow \exists x \sim\varphi$
  21. $\sim\exists x \varphi \leftrightarrow \forall x \sim\varphi$
  22. $\sim(x = y) \to \sim(y = x)$.[4]

[4]Axiom 22. is not a part of the original formulation.

If $\Sigma \cup \{\varphi\}$ is a set of formulas, then we write $\Sigma \vdash_{BS4} \varphi$ to indicate that φ is derivable from Σ in this system.

Proposition 2.4.1. *If $\Sigma \cup \{\varphi, \psi\}$ is a set of formulas, then*

$$\Sigma, \varphi \vdash_{BS4} \psi \textit{ if and only if } \Sigma \vdash_{BS4} \varphi \to \psi.$$

Proof. This is established by the usual proof using axioms 1. and 2. together with modus ponens. $\square$

The completeness of this system is a consequence of Corollary 5.15 from [27].

Theorem 2.4.2. *If $\Sigma \cup \{\varphi\}$ is a set of sentences, then*

$$\Sigma \vdash_{BS4} \varphi \textit{ if and only if } \Sigma \vDash_4 \varphi.$$

Proposition 2.4.3. *The following formulas are theorems of $BS4$:*

i. $\sim\sim\varphi \Leftrightarrow \varphi$

ii. $\sim(\varphi \wedge \psi) \Leftrightarrow \sim\varphi \vee \sim\psi$

iii. $\sim(\varphi \vee \psi) \Leftrightarrow \sim\varphi \wedge \sim\psi$

iv. $\sim\exists x\varphi \Leftrightarrow \forall x\sim\varphi$

v. $\sim\forall x\varphi \Leftrightarrow \exists x\sim\varphi$

vi. $x = y \to [\varphi(x) \Leftrightarrow \varphi(y)]$

vii. $(\varphi \to \psi) \Leftrightarrow (\neg\varphi \vee \psi)$

viii. $\sim(\varphi \Rightarrow \psi) \leftrightarrow (\varphi \wedge \sim\psi)$

ix. $\sim(\varphi \Leftrightarrow \psi) \leftrightarrow [(\varphi\wedge\sim\psi)\vee(\sim\varphi\wedge\psi)]$

x. $\varphi \leftrightarrow !\varphi$

xi. $\sim\varphi \leftrightarrow \sim?\varphi$

xii. $\neg\varphi \Leftrightarrow \sim!\varphi$

xiii. $\varphi(x) \Leftrightarrow \exists y[\varphi(y) \wedge !(x = y)]$.

Proof. We will only prove xiii. Just as in classical logic, we have

$$\vdash_{BS4} \varphi(x) \leftrightarrow \exists y[\varphi(y) \wedge x = y].$$

Using x, we get

$$\vdash_{BS4} \varphi(x) \leftrightarrow \exists y[\varphi(y) \wedge !(x = y)].$$

On the other hand

$$\begin{aligned}\vdash_{BS4} \sim\exists y[\varphi(y) \wedge !(x = y)] &\Leftrightarrow \forall y[\sim\varphi(y) \vee \sim!(x = y)]\\ &\Leftrightarrow \forall y[\sim\varphi(y) \vee \neg(x = y)]\\ &\Leftrightarrow \forall y[x = y \to \sim\varphi(y)]\end{aligned}$$

and

$$\vdash_{BS4} \sim\varphi(x) \leftrightarrow \forall y[x = y \to \sim\varphi(y)].$$

Thus

$$\vdash_{BS4} \varphi(x) \Leftrightarrow \exists y[\varphi(y) \wedge !(x = y)].$$

$\square$

Chapter 3

Algebraic Semantics

In this chapter, we will follow Fidel [10], Vakarelov [30] and Odinstov [22] and introduce a class of algebras called twist-structures. We then go on to define twist-valued models for $BS4$.

It should be noted that what we call a twist-structure is a special case of a twist-structure from [22], and that our twist-valued models are a straightforward generalization of similar models from [9].

3.1 Twist-structures

Let us suppose we have a T/F-structure $\mathcal{M}$ and a sentence φ. The truth value $[\![\varphi]\!]$ of φ in $\mathcal{M}$ represents two things. First, it represents whether φ is true in $\mathcal{M}$, and second, it represents whether φ is false in $\mathcal{M}$. We can therefore view $[\![\varphi]\!]$ as a pair of bits $([\![\varphi]\!]^+, [\![\varphi]\!]^-) \in \{0,1\}^2$, where $[\![\varphi]\!]^+ = 1$ if and only if φ is true in $\mathcal{M}$, and $[\![\varphi]\!]^- = 1$ if and only if φ is false in $\mathcal{M}$. We can now represent the four truth values 1, $\mathfrak{b}$, $\mathfrak{n}$ and 0 as follows:

$$1 = (1,0),\ \mathfrak{b} = (1,1),\ \mathfrak{n} = (0,0) \text{ and } 0 = (0,1).$$

Moreover, if we view $\{1,0\}$ as the two element Boolean algebra, we can calculate the truth values of $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \to \psi$ and $\sim\varphi$ as follows:

$$\begin{aligned}
[\![\varphi \wedge \psi]\!] &= ([\![\varphi]\!]^+ \wedge [\![\psi]\!]^+, [\![\varphi]\!]^- \vee [\![\psi]\!]^-) \\
[\![\varphi \vee \psi]\!] &= ([\![\varphi]\!]^+ \vee [\![\psi]\!]^+, [\![\varphi]\!]^- \wedge [\![\psi]\!]^-) \\
[\![\varphi \to \psi]\!] &= ([\![\varphi]\!]^+ \to [\![\psi]\!]^+, [\![\varphi]\!]^+ \wedge [\![\psi]\!]^-) \\
[\![\sim\varphi]\!] &= ([\![\varphi]\!]^-, [\![\varphi]\!]^+).
\end{aligned}$$

This leads us to the following definition.

Definition 3.1.1. Let $\mathcal{B} = (B, \wedge, \vee, \to, 1, 0)$ be a Boolean algebra. The *full twist-structure* $\mathcal{B}^{\bowtie}$ *over* $\mathcal{B}$ is the algebra $(B \times B, \wedge, \vee, \to, \sim, 1, 0)$, where $1 :=$

$(1, 0)$, $0 := (0, 1)$ and for all $(a, b), (c, d) \in B \times B$,

$$(a, b) \wedge (c, d) := (a \wedge c, b \vee d)$$
$$(a, b) \vee (c, d) := (a \vee c, b \wedge d)$$
$$(a, b) \to (c, d) := (a \to c, a \wedge d)$$
$$\sim(a, b) := (b, a).$$

A *twist-structure* over $\mathcal{B}$ is any subalgebra $\mathcal{A} = (A, \wedge, \vee, \to, \sim, 1, 0)$ of $\mathcal{B}^{\bowtie}$ such that $\pi_1[A] = B$, where $\pi_1 : B \times B \to B$ is the projection onto the first coordinate.

Remark. Notice that $\pi_1[A] = \pi_2[A]$ since $\mathcal{A}$ is closed under $\sim$. The condition that $\pi_1[A] = B$ ensures that $\mathcal{B}$ is a subalgebra of $\mathcal{A}$: We let $\mathcal{B}^*$ be the twist-structure over $\mathcal{B}$ given by

$$B^* := \{(x, \neg x) : x \in B\}.$$

Then it is easy to check that $\mathcal{B}^* \cong \mathcal{B}$ and that $\mathcal{B}^*$ is a subalgebra of $\mathcal{A}$. We will therefore identify $\mathcal{B}^*$ with $\mathcal{B}$ and view any Boolean algebra as a twist structure satisfying $\neg a = \sim a$, where $\neg a$ is defined as $a \to 0$.

It is also worth noting that any subalgebra of a twist-structure is again a twist structure. However, it need not be a twist-structure over the same Boolean algebra.

Example 3.1.2. There are four twist structures over the two element Boolean algebra $\{1, 0\}$. Namely, $\{1, 0\}$ itself, $\{1, \mathfrak{b}, 0\}$, $\{1, \mathfrak{n}, 0\}$ and the full twist structure $\{1, 0\}^{\bowtie} = \{1, \mathfrak{b}, \mathfrak{n}, 0\}$.

Definition 3.1.3. Let $\mathcal{A}$ be a twist-structure over a Boolean algebra $\mathcal{B}$ and let $a \in A$. We let a^+ and a^- be the elements of B such that

$$a = (a^+, a^-),$$

i.e., $a^+ := \pi_1(a)$ and $b^- := \pi_2(a)$. Moreover, we let

$$X^+ := \pi_1[X] \text{ and } X^- := \pi_2[X]$$

for $X \subseteq A$.

Example 3.1.4. Twist-structures can often help us better understand defined connectives. For example, if $\mathcal{A}$ is a twist-structure and $a \in A$, then

$$\neg a = (\neg a^+, a^+),\ !a = (a^+, \neg a^+) \text{ and } ?a = (\neg a^-, a^-).$$

We can view any twist-structure $\mathcal{A}$ as a lattice with the ordering $a \leq b$ iff $a \wedge b = a$. This gives

$$a \leq b \text{ iff } a^+ \leq b^+ \text{ and } b^- \leq a^-$$

for all $a, b \in A$. Moreover, 1 and 0 are its top and bottom elements, respectively. The following proposition is immediate.

Proposition 3.1.5. *If $\mathcal{B}$ is a complete Boolean algebra, then $\mathcal{B}^{\bowtie}$ is a complete lattice with*

$$\bigvee X = (\bigvee X^+, \bigwedge X^-) \text{ and}$$
$$\bigwedge X = (\bigvee X^+, \bigwedge X^-)$$

for all $X \subseteq B \times B$.

Definition 3.1.6. We say that a twist-structure $\mathcal{A}$ over a Boolen algebra $\mathcal{B}$ is *complete* if $\mathcal{B}$ is a complete Boolean algebra and $\mathcal{A}$ is a complete sublattice of $\mathcal{B}^{\bowtie}$.

Definition 3.1.7. Let $\mathcal{A}$ be a twist-structure over a Boolean algebra $\mathcal{B}$. We define the relations $\preceq$ and $\approx$ on A by letting

$$a \preceq b \text{ iff } a^+ \leq b^+, \text{ and}$$
$$a \approx b \text{ iff } a^+ = b^+$$

for all $a, b \in A$.

Proposition 3.1.8. *If $\mathcal{A}$ is a twist-structure and $a, b \in A$, then*

$$(a \to b) \approx 1 \textit{ iff } a \preceq b, \textit{ and}$$
$$(a \Rightarrow b) \approx 1 \textit{ iff } a \leq b.$$

3.2 Twist-valued models

We can now generalize the notion of a T/F-model for $BS4$, where the truth value of sentences are elements of the twist-structure $\{1, \mathfrak{b}, \mathfrak{n}, 0\}$, to models where the truth value of sentences are elements any fixed complete twist-structure. This also generalizes the notion of a Boolean-valued model since any complete Boolean algebra is also a complete twist-structure.

Definition 3.2.1. An *twist-valued model* $\mathcal{M}$ in a language L consists of

1. a non-empty set M, called the *domain* of $\mathcal{M}$;
2. a complete twist-structure $\mathcal{A}$;
3. an element $c^{\mathcal{M}} \in M$ for every constant symbol c in L;
4. a function $f^{\mathcal{M}}\colon M^n \to M$ for every n-ary function symbols f from L;
5. an n-ary function $R^{\mathcal{M}} : M^n \to \mathcal{A}$ for every n-ary relation symbol R in L;
6. a function $eq^{\mathcal{M}} : M \times M \to \mathcal{A}$ such that for all $a, b, c, a_1, ..., a_n, b_1, ..., b_n \in M$, and for every n-ary function symbol f and n-ary relation symbol R,
   (a) $eq^{\mathcal{M}}(a, a) \approx 1$;

(b) $eq^{\mathcal{M}}(a,b) = eq^{\mathcal{M}}(b,a)$;

(c) $eq^{\mathcal{M}}(a,b) \preceq eq^{\mathcal{M}}(a,c) \Leftrightarrow eq^{\mathcal{M}}(b,c)$;

(d) $eq^{\mathcal{M}}(a_1,b_1) \wedge ... \wedge eq^{\mathcal{M}}(a_n,b_n) \preceq eq^{\mathcal{M}}(f^{\mathcal{M}}(a_1,...,a_n), f^{\mathcal{M}}(b_1,...,b_n))$;

(e) $eq^{\mathcal{M}}(a_1,b_1) \wedge ... \wedge eq^{\mathcal{M}}(a_n,b_n) \preceq R^{\mathcal{M}}(a_1,...,a_n) \Leftrightarrow R^{\mathcal{M}}(b_1,...,b_n)$.

Definition 3.2.2. Let $\mathcal{M}$ be a twist-valued model and φ be a sentence in L_M. We recursively define the *truth value* $[\![\varphi]\!]^{\mathcal{M}}$ of φ in $\mathcal{M}$ by letting

1. $[\![\bot]\!]^{\mathcal{M}} := 0$,
2. $[\![a = b]\!]^{\mathcal{M}} := eq^{\mathcal{M}}(a,b)$ for all $a, b \in M$;
3. $[\![R(a_1,...,a_n)]\!]^{\mathcal{M}} := R^{\mathcal{M}}(a_1,...,a_n)$ for all $a_1,...,a_n \in M$,
4. $[\![{\sim}\varphi]\!]^{\mathcal{M}} := {\sim}[\![\varphi]\!]^{\mathcal{M}}$;
5. $[\![\varphi * \psi]\!]^{\mathcal{M}} := [\![\varphi]\!]^{\mathcal{M}} * [\![\psi]\!]^{\mathcal{M}}$ for $* \in \{\vee, \wedge, \rightarrow\}$;
6. $[\![\exists x\varphi]\!]^{\mathcal{M}} := \bigvee_{x\in M}[\![\varphi]\!]^{\mathcal{M}}$ and $[\![\forall x\varphi]\!]^{\mathcal{M}} := \bigwedge_{x\in M}[\![\varphi]\!]^{\mathcal{M}}$.

We write $\mathcal{M} \vDash_{Tw} \varphi$ and say that φ is *true in* $\mathcal{M}$ if $[\![\varphi]\!]^{\mathcal{M}} \approx 1$, i.e., $([\![\varphi]\!]^{\mathcal{M}})^+ = 1$.

Theorem 3.2.3. *If Σ be a theory in a language L and φ is an L-sentence, then*

$$\Sigma \vdash_{BS4} \varphi \textit{ iff } \Sigma \vDash_{Tw} \varphi.$$

Proof. As soundness is routine to verify, we will only show that $\Sigma \vDash_{Tw} \varphi$ implies $\Sigma \vdash_{BS4} \varphi$. By a standard argument it suffices to show that if Σ is non-trivial, then it has a twist-valued model. Now, every non-trivial theory has a T/F-model by the completeness theorem for $BS4$. Since every T/F-model is also a twist-valued model, the result follows. $\square$

Part II

Paradefinite Zermelo–Fraenkel Set Theory

Chapter 4

The Axioms

We are now ready to begin begin our investigation of set theory in the logic $BS4$. We should note that the aim of our set theory is not to solve any of the paradoxes of naive set theory, or to allow the formation of paradoxical sets such as the Russell set or the universal set.[1] Rather, we aim to provide a set theory that is able to represent both inconsistent and incomplete information in an intuitive manner.

In this chapter, we will lay down the axioms of our set theory, introduce its basic concepts and definitions, and derive a few of its consequences. Throughout this chapter we will work in the logic $BS4$. We work in the language of set theory which has the binary symbol $\in$ as its only non-logical symbol. Our domain of discourse will contain only sets, meaning that the variables will range over sets only. Just as in classical set theory, we are going to use informal arguments, formulated in English, which can be translated into $BS4$.

4.1 Extensionality

Let us begin by introducing our axiom of extensionality. Our axiom is inspired by similar axioms from [11] and [14].

Axiom 1 (Extensionality)**.**

$$\forall u \forall v[u = v \Leftrightarrow \forall x(x \in u \Leftrightarrow x \in v)].$$

Our motivation for this axiom is as follows: If u is a set, then it is natural to think of u as the extension of the predicate $x \in u$. Moreover, if v is also a set, then it is natural to interpret $u = v$ as saying that the predicates $x \in u$ and $x \in v$ have the same extensions. When we say that the predicates $x \in u$ and $x \in v$ have the same extensions, we mean that they are equivalent for every x. Since we use $\Leftrightarrow$ to express equivalence, we get our axiom.

[1] I highly recommend [15] for an overview of set theories that go in this direction.

Definition 4.1.1. Let a and b be sets. We say that a is a *subset* of b and write $a \subseteq b$ if $\forall x(x \in a \Rightarrow x \in b)$.

We easily obtain the following.

Proposition 4.1.2. *For all u and v,*

$$u = v \Leftrightarrow u \subseteq v \wedge v \subseteq u.$$

Definition 4.1.3. Let u and v be sets. We write $u \notin v$, $u \neq v$ and $u \nsubseteq v$ as abbreviations for $\sim(u \in v)$, $\sim(u = v)$ and $\sim(u \subseteq v)$, respectively.

Proposition 4.1.4. *For all u and v,*

i. $u \nsubseteq v \leftrightarrow \exists x(x \in u \wedge x \notin v)$

ii. $u \neq v \leftrightarrow \exists x(x \in u \wedge x \notin v) \vee \exists x(x \notin u \wedge x \in v)$.

Proof. We have

$$\begin{aligned} u \nsubseteq v &\Leftrightarrow \sim\forall x(x \in u \Rightarrow x \in v) \\ &\Leftrightarrow \exists x \sim(x \in u \Rightarrow x \in v) \\ &\leftrightarrow \exists x(x \in u \wedge x \notin v) \end{aligned}$$

and

$$\begin{aligned} u \neq v &\Leftrightarrow u \nsubseteq v \vee v \nsubseteq u \\ &\leftrightarrow \exists x(x \in u \wedge x \notin v) \vee \exists x(x \notin u \wedge x \in v). \end{aligned}$$

$\square$

Thus two sets are unequal if and only if one set contains an element that the other does not.

4.2 Classes and separation

If u is a set and $\varphi(x)$ is a formula such that $\forall x[x \in u \Leftrightarrow \varphi(x)]$, then we denote u by the expression $\{x : \varphi(x)\}$. Now, the axiom of extensionality tells us that if $\{x : \varphi(x)\}$ denotes a set, then it is unique. However, the expression $\{x : \varphi(x)\}$ need not denote any set at all.

To see why, consider the class $R := \{x : \neg(x \in x)\}$, i.e., the Russell class w.r.t. the classical negation. If R denotes a set, then either $R \in R$ or $\neg(R \in R)$. If $R \in R$, then $\neg(R \in R)$ and therefore $\perp$. One the other hand, if $\neg(R \in R)$, then $R \in R$, so $\perp$. In either case, we get $\perp$.

With the above in mind we introduce the informal notion of a class. Given a formula $\varphi(x)$, we denote the *class* or *collection* of sets x such that $\varphi(x)$ by the expression $\{x : \varphi(x)\}$. We give the following definition to make this notion formal.

Definition 4.2.1. Let u be a set, and let $\varphi(x)$ and $\psi(x)$ be formulas with x as a free variable. We introduce the following abbreviations:

$$\begin{aligned}
u = \{x : \varphi(x)\} \; &:\Leftrightarrow \; \forall x[x \in u \Leftrightarrow \varphi(x)] \\
\{x : \varphi(x)\} = u \; &:\Leftrightarrow \; \forall x[\varphi(x) \Leftrightarrow x \in u] \\
\{x : \varphi(x)\} = \{x : \psi(x)\} \; &:\Leftrightarrow \; \forall x[\varphi(x) \Leftrightarrow \psi(x)] \\
u \in \{x : \varphi(x)\} \; &:\Leftrightarrow \; \varphi(u) \\
\{x : \varphi(x)\} \in u \; &:\Leftrightarrow \; \exists y[y \in u \land !\forall x(\varphi(x) \Leftrightarrow x \in y)] \\
\{x : \varphi(x)\} \in \{x : \psi(x)\} \; &:\Leftrightarrow \; \exists y[\psi(y) \land !\forall x(\varphi(x) \Leftrightarrow x \in y)].
\end{aligned}$$

Remark. Recall from Proposition 2.4.3 that

$$\vdash_{BS4} \psi(x) \Leftrightarrow \exists y[\psi(y) \land !(x = y)].$$

This explains the appearance of the !-connective in the definition above.

Definition 4.2.2. We define the classes

$$V := \{x : !(x = x)\} \text{ and } \emptyset := \{x : \neg(x = x)\},$$

called the *universe* and the *empty set*, respectively.

We easily get the following proposition.

Proposition 4.2.3. *For all x,*

$$x \in V \Leftrightarrow \top \text{ and } x \in \emptyset \Leftrightarrow \bot.$$

Notice that given a class A, $A \in V \Leftrightarrow \exists x!(x = A)$. We therefore give the following definition.

Definition 4.2.4. We say that a class A is a *set* if $\exists x!(x = A)$. A class is said to be a *proper class* if it is not a set.

Let us now give our version of the axiom schema of separation.

Axiom 2 (Separation)**.**

$$\forall u \exists v \forall x[x \in v \Leftrightarrow x \in u \land \varphi(x)],$$

where v is not free in $\varphi(x)$.

It follows that given a set u and a formula $\varphi(x)$, the class $\{x \in u : \varphi(x)\}$ is a set. Here, $\{x \in u : \varphi(x)\}$ is shorthand for $\{x : x \in u \land \varphi(x)\}$.

Remark. Strictly speaking, our axiom schema states that the class $\{x \in u : \varphi(x)\}$ is *equal* to a set, which is slightly different than saying that $\{x \in u : \varphi(x)\}$

is a set according to Definition 4.2.4. If we wanted a axiom schema that directly states that $\{x \in u : \varphi(x)\}$ is a set, we could have taken

$$\forall u \exists v ! [v = \{x : x \in u \wedge \varphi(x)\}]$$

as our axiom schema. Writing this out explicitly gives

$$\forall u \exists v ! \forall x [x \in v \Leftrightarrow x \in u \wedge \varphi(x)].$$

This is true if and only if our separation axiom is true.

Proposition 4.2.5. *The class $\emptyset$ is a set and V is a proper class.*

Proof. By virtue of the logic alone, we know that some set u exists. Moreover, $\perp \Rightarrow x \in u$ for all x, and therefore $\emptyset \subseteq u$. By applying the axiom of separation, we see that $\emptyset$ is indeed a set.

To see why V is a proper class, we note that if V is a set, then $R = \{x : \neg(x \in x)\}$ is also a set. As we have already seen, this leads to $\perp$. $\square$

Definition 4.2.6. We define the operations of *union*, *intersection* and *compliment* on classes the classes A and B by letting

$$\begin{aligned} A \cup B &:= \{x : x \in A \vee x \in B\}; \\ A \cap B &:= \{x : x \in A \wedge x \in B\}; \\ A \setminus B &:= \{x : x \in A \wedge x \notin B\}, \end{aligned}$$

respectively.

4.3 Classical sets

Recall from Section 2.3 that we can express that a formula is classical using the connective $\circ$. That is to say, $\circ\varphi$ is true iff φ is either true or false but not both. Moreover, the formula $\circ\varphi$ is itself classical, so $\circ\varphi$ is false iff φ is both true and false or φ is neither true nor false. We repeat the truth table for $\circ$ here for easy reference.

	$\circ$
1	1
$\mathfrak{b}$	0
$\mathfrak{n}$	0
0	1

Definition 4.3.1. We say that a set u is *classical* and write $Cl(u)$ if $\forall x[\circ(x \in u)]$.

The nice thing about classical sets is that we are already familiar with them from classical set theory. As usual, we can represent a classical set u by drawing a circle and declaring that the elements of u are the things appearing inside the circle, and anything outside the circle is not an element of u. (See Figure 4.1.)

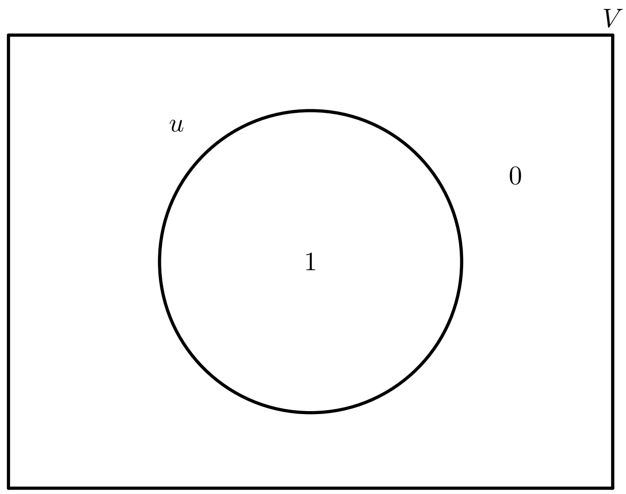

Figure 4.1: The truth value of $x \in u$, where u is a classical set. Here, the circle represents the classical set u. The number 1 inside the circle means that for any element x inside the circle, the statement $x \in u$ gets the truth value 1. The 0 outside the circle means that for every element x outside the circle, the statement $x \in u$ gets the truth value 0.

An immediate example of a classical set is $\emptyset$, and V is a classical class.

The notion of a classical set allows us to use much of what we know from classical set theory. As an example, we easily get the following proposition, where $\sim$ is replaced by $\neg$, $\Rightarrow$ is replaced by $\rightarrow$, and $\Leftrightarrow$ is replaced by $\leftrightarrow$.

Proposition 4.3.2. *For all classical sets u and v,*

i. $\forall x[x \notin u \Leftrightarrow \neg(x \in u)]$;

ii. $u \subseteq v \Leftrightarrow \forall x(x \in u \rightarrow x \in v)$;

iii. $u = v \Leftrightarrow \forall x(x \in u \leftrightarrow x \in v)$;

iv. $u \neq v \Leftrightarrow \neg(u = v)$.

Proof. The proof is left to the reader. □

We introduce the following axiom in order to simplify our development of set theory.

Axiom 3 (Classical supersets)**.**

$$\forall u \exists v[Cl(v) \wedge u \subseteq v].$$

The axiom states that each set has a classical superset. The main appeal is that it will allow us to describe non-classical sets in terms of classical ones.

Recall from Section 2.3 that given a formula φ, we defined the pair of classical formulas $!\varphi$ and $?\varphi$ with the property that φ is true iff $!\varphi$ is true, and φ is false iff $?\varphi$ is false. As a reminder, their truth tables are the following:

	!
1	1
$\mathfrak{b}$	1
$\mathfrak{n}$	0
0	0

	?
1	1
$\mathfrak{b}$	0
$\mathfrak{n}$	1
0	0

Definition 4.3.3. Given a set u, we define the classes

$$u^! := \{x : !(x \in u)\} \text{ and } u^? := \{x : ?(x \in u)\}.$$

Notice that both $u^!$ and $u^?$ are classical, and for all x,

$$x \in u \leftrightarrow x \in u^! \text{ and } x \notin u \leftrightarrow x \notin u^?.$$

So $u^!$ and $u^?$ are classical classes that together completely describe u. (See Figure 4.2.)

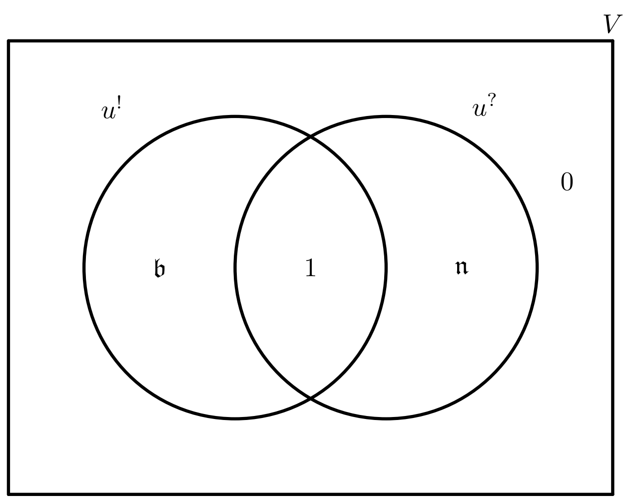

Figure 4.2: The truth value of $x \in u$. The left circle represents $u^!$, while the right circle represents $u^?$. Notice that $x \in u$ is true iff x is in the interior of the left circle, while $x \in u$ is false iff x is in the exterior of the right circle.

Proposition 4.3.4. *For all u and v,* *i.* $Cl(u) \Leftrightarrow u^! = u^?$*;*

ii. $u \subseteq v$ *iff* $u^! \subseteq v^!$ *and* $u^? \subseteq v^?$*;*

iii. $u \nsubseteq v$ *iff* $u^! \nsubseteq v^?$*;*

iv. $u = v$ *iff* $u^! = v^!$ *and* $u^? = v^?$*;*

v. $u \neq v$ *iff* $u^! \nsubseteq v^?$ *or* $v^! \nsubseteq u^?$*.*

Proof. i. Follows by definition of $Cl(u)$.

ii. We have

$$\begin{aligned} u \subseteq v \text{ iff } & \forall x(x \in u \to x \in v) \wedge \forall x(x \notin v \to x \notin u) \\ \text{iff } & \forall x(x \in u^! \to x \in v^!) \wedge \forall x(x \notin v^? \to x \notin u^?) \\ \text{iff } & \forall x(x \in u^! \to x \in v^!) \wedge \forall x(\neg(x \in v^?) \to \neg(x \in u^?)) \\ \text{iff } & \forall x(x \in u^! \to x \in v^!) \wedge \forall x((x \in u^? \to x \in v^?) \\ \text{iff } & u^! \subseteq v^! \wedge u^? \subseteq v^?. \end{aligned}$$

iii. We have

$$\begin{aligned} u \nsubseteq v \text{ iff } & \exists x(x \in u \wedge x \notin v) \\ \text{iff } & \exists x(x \in u^! \wedge x \notin v^?) \\ \text{iff } & u^! \nsubseteq v^?. \end{aligned}$$

iv. and v. easily follow. □

Definition 4.3.5. Fix a set u. We define the *realm* of u by

$$rlm(u) := u^! \cup u^?.$$

The reason we care about $rlm(u)$ is that it is the least classical class containing u in the sense of the following proposition.

Lemma 4.3.6. *For all u, the class $rlm(u)$ is classical, and if X is a classical class such that $u \subseteq X$, then $rlm(u) \subseteq X$.*

Proof. That $rlm(u)$ is classical follows from $u^!$ and $u^?$ being classical. Now, let X be a classical class with $u \subseteq X$. We have $X^! = X^?$, since X is classical, and $u^! \subseteq X^!$ and $u^? \subseteq X^?$, since $u \subseteq X$. This gives $u^! \subseteq X$ and $u^? \subseteq X$. Hence $rlm(u) \subseteq X$. □

Theorem 4.3.7. *If u is a set, then the classes $u^!$, $u^?$ and $rlm(u)$ are sets.*

Proof. By the axiom of classical supersets, there is a classical set v such that $u \subseteq v$. We have $rlm(u) \subseteq v$ by Lemma 4.3.6. The axiom of separation now tells us that $rlm(u)$ is a set. Clearly, $u^!, u^? \subseteq rlm(u)$, so both $u^!$ and $u^?$ are sets. □

We can now see that every set u can be described in terms of the classical sets $u^!$ and $u^?$. We therefore introduce the following notation.

Definition 4.3.8. If u, v and w are sets such that $w^! = u$ and $w^? = v$, then we denote w by the expression $\langle u, v\rangle$.

So given classical sets u and v, $\langle u, v\rangle$ is the unique set with $\langle u, v\rangle^! = u$ and $\langle u, v\rangle^? = v$ if such a set exists. (See Figure 4.3.)

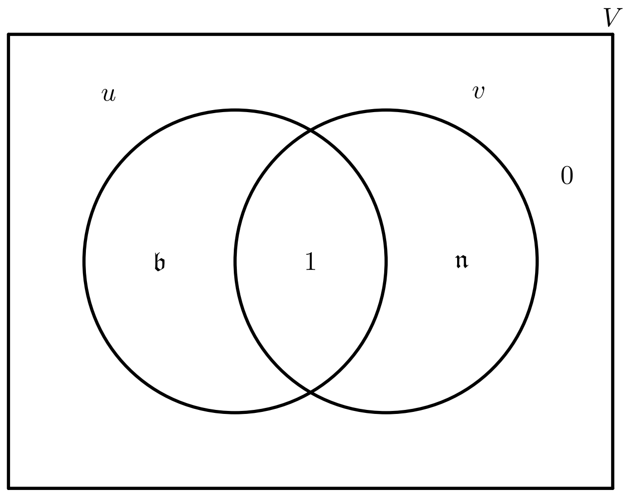

Figure 4.3: The truth value of $x \in \langle u, v\rangle$.

There is another way to describe sets in terms of classical sets that is perhaps slightly more intuitive than the one that we have given. However, it has the drawback of requiring more classical sets to achieve the same goal. Suppose that we are given a set u and a classical set X such that $u \subseteq X$. We can then form the subsets

$$u_X^+ := \{x \in X : !(x \in u)\} \text{ and } u_X^- := \{x \in X : !(x \notin u)\}$$

of X. Now, both u_X^+ and u_X^- are classical sets, and for all x,

$$x \in u \leftrightarrow x \in u_X^+ \text{ and } x \notin u \leftrightarrow x \in u_X^- \cup (V \setminus X).$$

So the classical sets X, u_X^+ and u_X^- together completely describe u. (See Figure 4.4)

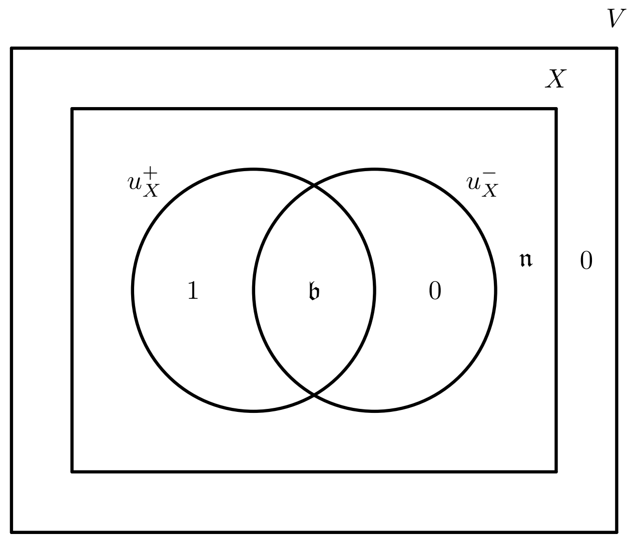

Figure 4.4: The truth value of $x \in u$.

4.4 Inconsistent and incomplete sets

For this section, we recall from Section 2.3 that we defined the connectives $\circ_{con}$ and $\circ_{com}$ that express that a given formula is consistent and complete, respectively. As a reminder, their truth tables are the following:

	$\circ_{con}$
1	1
$\mathfrak{b}$	0
$\mathfrak{n}$	1
0	1

	$\circ_{com}$
1	1
$\mathfrak{b}$	1
$\mathfrak{n}$	0
0	1

Definition 4.4.1. We say that a set u is *consistent* and write $Con(A)$ if $\forall x[\circ_{con}(x \in u)]$. We say that u is *complete* and write $Com(u)$ if $\forall x[\circ_{com}(x \in u)]$. A set is said to be *inconsistent* if it is not consistent and *incomplete* if it is not complete.

We easily get the following proposition.

Proposition 4.4.2. *For all u,*

$$Con(u) \Leftrightarrow u^! \subseteq u^? \text{ and } Com(u) \Leftrightarrow u^? \subseteq u^!.$$

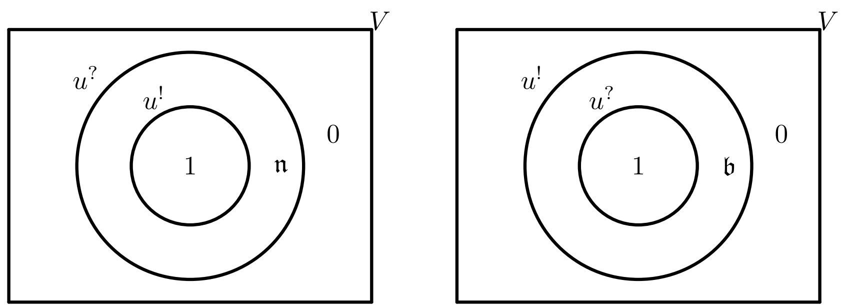

Figure 4.5: The truth value of $x \in u$. In the picture on the left, u is assumed to be consistent, and u is assumed to be complete in the picture on the right.

Clearly, for each w, there are classical sets u and v such that $w = \langle u, v \rangle$, namely $u = w^!$ and $u = w^?$. A more interesting question is, given classical sets u and v, when is there is a set w such that $w = \langle u, v \rangle$? The following theorem essentially tells us that as soon as we know that there exists a single inconsistent set and a single incomplete set, then we can conclude that $\langle u, v \rangle$ exists for all classical u and v.

Theorem 4.4.3. *Suppose that there exists both an inconsistent set and an incomplete set. Then for all classical sets u and v, such that $u \cup v$ is a set, there is a set w such that*

$$w = \langle u, v \rangle,$$

i.e., for all x,

$$x \in w \leftrightarrow x \in u \textit{ and } x \notin w \leftrightarrow x \notin v.$$

Proof. Since there exist both an inconstant set and an incomplete set, there are sets a, b, c and d such that $a \in b \wedge a \notin b$ and $\neg(c \in d \vee c \notin d)$. We can therefore enrich our language with the propositional constants $\bot_{\mathfrak{b}}$ and $\bot_{\mathfrak{n}}$ with the property

$$\bot_{\mathfrak{b}} \wedge {\sim}\bot_{\mathfrak{b}} \text{ and } \neg(\bot_{\mathfrak{n}} \vee {\sim}\bot_{\mathfrak{n}}).$$

We let

$$w := \{x \in u \cup v : x \in u \cap v \vee (x \in u \setminus v \wedge \bot_{\mathfrak{b}}) \vee (x \in v \setminus u \wedge \bot_{\mathfrak{n}})\}.$$

This gives

$$\begin{aligned} x \in w &\text{ iff } x \in u \cap v \vee (x \in u \setminus v \wedge \bot_{\mathfrak{b}}) \vee (x \in v \setminus u \wedge \bot_{\mathfrak{n}}) \\ &\text{ iff } x \in u \cap v \vee x \in u \setminus v \\ &\text{ iff } x \in u \end{aligned}$$

and

$$x \notin w \text{ iff } x \notin u \cap v \wedge (x \notin u \setminus v \vee \bot_{\mathfrak{b}}) \wedge (x \notin v \setminus u \vee \bot_{\mathfrak{n}})$$

iff $x \notin u \cap v \wedge x \notin v \setminus u$
iff $x \notin v$.

□

Remark. In the above theorem, we needed to add the caveat that $u \cup v$ is a set. This is because we have not yet introduced an axiom of union which guarantees this. Of course, we will later add such an axiom, so this bit can be safely ignored.

Corollary 4.4.4. *Let u, v and X be classical sets with $u, v \subseteq X$. If there there exists an inconsistent set and an incomplete set, then there exists a set $w \subseteq X$ such that for all x,*

$$x \in w \leftrightarrow x \in u \text{ and } x \notin w \leftrightarrow x \in v \cup (V \setminus X).$$

Proof. We let $w := \langle u, X \setminus v \rangle$ and apply Theorem 4.4.3. □

4.5 Replacement

Definition 4.5.1. By an *operation* we mean a classical formula $\varphi(x, y)$ with x and y free such that

$$\forall x \exists y[\varphi(x, y) \wedge \forall z(\varphi(x, z) \to !(y = z))].$$

The intuition is that we think of an operation as a process that takes in an input and produces an output. So if we have an operation given by the formula $\varphi(x, y)$, we think of $\varphi(u, v)$ as saying that the operation outputs v on the input u.

Remark. The reason we require an operation to be given by a classical formula is that given any input, the operation should, in no uncertain terms, produce a well-defined output. To see why we used the !-connective in our definition, consider the formula $\varphi(x, y) :\Leftrightarrow !(x = x) \wedge !(y = a)$, where a is some inconsistent set. Now, $\varphi(x, y)$ is a classical formula that is true iff $y = a$. So we can think of $\varphi(x, y)$ as representing the operation that always outputs a. However, since a is inconsistent, we have that $a \neq a$. This means that the formula

$$\forall x \exists y[\varphi(x, y) \wedge \forall z(\varphi(x, z) \to y = z)]$$

is both true and false. So if we had not included the !-connective in our definition, the formula $\varphi(x, y)$ would both be and not be an operation. This does not seem right since $\varphi(x, y)$ always produces a well defined output, and the fact that said output happens to be an inconsistent set is irrelevant.

If $\varphi(x, y)$ is an operation, then we can introduce a new function symbol F_φ via the defining axiom

$$\forall x \varphi(x, F_\varphi(x)).$$

Since $\varphi(x, y)$ is a classical formula, we can easily show that

$$\psi(F_\varphi(x)) \Leftrightarrow \exists y[\psi(y) \wedge \varphi(x, y)]$$

for all x and any formula $\psi(y)$. This means that any formula containing the symbol F_φ can be rewritten as an equivalent formula without an occurrence of F_φ.

Definition 4.5.2. Let A be a class and F an operation defined by the classical formula $\varphi(x, y)$. We define the *image* $\{F(x) : x \in A\}$ of A under F by

$$\{F(x) : x \in A\} := \{y : \exists x \in A\varphi(x, y)\}.$$

Now, $y \in \{F(x) : x \in A\}$ can be read as saying that there is an $x \in A$ such that F maps x to y.

Remark. Someone might claim that $\{y : \exists x \in A[y = F(x)]\}$ is a more natural definition for $\{F(x) : x \in A\}$. To see why this definition does not work, consider the identity operation *id* which maps each element to itself. Clearly, we want $\{id(x) : x \in A\}$ to be the same thing as $\{x : x \in A\}$, i.e., A itself. Now, if A is the class $\{x : !(x = a)\}$, where a is some inconsistent set, then $a \neq a$. So $a \notin \{y : \exists x \in A[y = id(x)]\}$, but $\neg(a \notin A)$.

Axiom 4 (Replacement)**.**

$$\forall u \exists v[v = \{F(x) : x \in u\}],$$

where F is an operation, and v is not a free variable in the formula defining F.

4.6 Union

Let u be a set and φ be a formula. We introduce the abbreviations $\exists x \in u\varphi$ and $\forall x \in u\varphi$ for $\exists x(x \in u \wedge \varphi)$ and $\forall x(x \in u \to \varphi)$, respectively.

Definition 4.6.1. Given a class A we define the *union* of A by

$$\bigcup A := \{x : \exists y \in A(x \in y)\}.$$

Moreover, if $\exists x(x \in A)$, then we define the *intersection* of A by

$$\bigcap A := \{x : \forall y \in A(x \in y)\}.$$

Axiom 5 (Union)**.**

$$\forall u \exists v \forall x[x \in v \Leftrightarrow \exists y \in u(x \in y)].$$

4.7 Pairing

Definition 4.7.1. Given sets u and v, we define the *unordered pair* $\{u, v\}$ by

$$\{u, v\} := \{x : !(x = u \vee x = v)\}.$$

Now, $\{u, v\}$ is the classical set having u and v as elements. The reader might be curious why we did not use the class $\{x : x = u \vee x = v\}$. There are three reasons for this: First, when specifying $\{u, v\}$ we would simply like to point to u and point to v and say that these are the elements of $\{u, v\}$. This is different than pointing u and v and specifying the elements that are *equal* to one of these, which would give the class $\{x : x = u \vee x = v\}$. Second, $\{x : !(x = u \vee x = v)\}$ tends to be much easier to work with than $\{x : x = u \vee x = v\}$. The third reason, which also plays into the second reason, is that if there exist an incomplete set, then $\{x : x = u \vee x = v\}$ is always a proper class.

Proposition 4.7.2. *If there exists an incomplete set, then for all u and v, $\{x : x = u \vee x = v\}$ is a proper class.*

Proof. We prove the special case where $u = v$ and the general case easily follows. Assume that $\{x : x = u\}$ is a set. Then $\{x :?(x = u)\}$ is also a set.

Since there exists an incomplete set, we have that for every classical set x, $\langle \emptyset, x\rangle$ exists. That is, if x is classical set, then there is a set w such that $w^! = \emptyset$ and $w^? = x$. We have

$$\begin{aligned} u \neq \langle \emptyset, x\rangle \text{ iff } & \exists z[z \in u \wedge z \notin \langle \emptyset, x\rangle] \vee \exists z[z \notin u \wedge z \in \langle \emptyset, x\rangle] \\ \text{iff } & \exists z[z \in u^! \wedge z \notin \langle \emptyset, x\rangle^?] \vee \exists z[z \notin u^? \wedge z \in \langle \emptyset, x\rangle^!] \\ \text{iff } & \exists z[z \in u^! \wedge z \notin x] \vee \exists z[z \notin u^? \wedge z \in \emptyset] \\ \text{iff } & \exists z[z \in u^! \wedge z \notin x] \\ \text{iff } & \neg\forall z[z \in u^! \Rightarrow z \in x] \\ \text{iff } & \neg(u^! \subseteq x). \end{aligned}$$

So for every classical x,

$$u^! \subseteq x \to \neg(u \neq \langle \emptyset, x, \rangle),$$

i.e.,

$$u^! \subseteq x \to ?(u = \langle \emptyset, x\rangle).$$

Both $\{\langle \emptyset, x\rangle : Cl(x) \wedge u^! \subseteq x\}$ and $\{x :?(x = u)\}$ are classical, so

$$\{\langle \emptyset, x\rangle : Cl(x) \wedge u^! \subseteq x\} \subseteq \{x :?(x = u)\}.$$

This tells us that $\{\langle \emptyset, x\rangle : Cl(x) \wedge u^! \subseteq x\}$ is a set. Using replacement, we get that $\{x : Cl(x) \wedge u^! \subseteq x\}$ is a set. This last set is just $\{x \cup u^! : Cl(x)\}$ and using replacement one more time, we get that $\{x : Cl(x)\}$ is a set. But this implies that $\{x : Cl(x) \wedge \neg(x \in x)\}$ is a set. We leave it to the reader to show that this implies $\perp$. □

We now introduce our pairing axiom.

Axiom 6 (Pairing)**.**

$$\forall u \forall v \exists w \forall x [x \in w \Leftrightarrow !(x = u \vee x = v)].$$

It follows that $\{u, v\}$ is a set for all u and v. Moreover, since $u \cup v = \bigcup\{u, v\}$, $u \cup v$ is also a set.

4.8 Ordered pairs and relations

We now turn to the problem of defining the ordered pair (u, v). We would like our notion of ordered pairs to satisfy

$$(u, v) = (z, w) \Leftrightarrow u = z \wedge v = w.$$

This means that we cannot use the standard Kuratowski definition which defines (u, v) as the pair $\{\{u\}, \{u, v\}\}$ since $\{\{u\}, \{u, v\}\}$ is a classical set. So $\{\{u\}, \{u, v\}\} = \{\{z\}, \{z, w\}\}$ is a classical formula, whereas $u = z \wedge v = w$ can be non-classical. We will therefore opt for a different definition. Said definition comes from [28], and was originally formulated in classical set theory.

Definition 4.8.1. Let u and v be sets. We define the *ordered pair* (u, v) by

$$(u, v) := \{\{\{x\}\} : x \in u\} \cup \{\{\{x\}, \emptyset\} : x \in v\}.$$

We put the proof that this definition satisfies our requirement in Appendix A.

Definition 4.8.2. We recursively define the *n-tuple*, $(u_1, ..., u_n)$, by letting $(u_1) := u_1$ and $(u_1, ..., u_n) := (u_1, (u_2, ..., u_n))$ for $n \geq 2$

Definition 4.8.3. Let $A, A_1, ..., A_n$ be sets. We define the *n-ary product*, $A_1 \times ... \times A_n$ by

$$A_1 \times ... \times A_n := \{(x_1, ..., x_n) : x_1 \in A_1 \wedge ... \wedge x_n \in A_n\}$$

and let

$$A^n := \underbrace{A \times ... \times A}_{n \text{ times}}.$$

Proposition 4.8.4. *For all u and v, the product $u \times v$ is a set.*

Proof. Using replacement, we see that for each y, the class $\{(x, y) : x \in u\}$ is a set. Now,

$$\{(x, y) : x \in u \wedge y \in v\} = \bigcup_{y \in v} \{(x, y) : x \in u\}.$$

The axiom of union now tells us that $u \times v$ is a set. □

Definition 4.8.5. We say that a set R is a *binary relation* if

$$R \subseteq V \times V.$$

The *domain* of R is given by

$$dom(R) := \{x : \exists y[(x, y) \in R]\}$$

and the *range* of R is given by

$$ran(R) := \{y : \exists x[(x, y) \in R]\}.$$

The *inverse* of R is

$$R^{-1} := \{(y, x) : (x, y) \in R\}.$$

Definition 4.8.6. Fix a set X. We say that a relation $E \subseteq X \times X$ is an *equivalence relation* on X if the following holds for all $x, y, z \in X$:

1. $(x, x) \in E$,
2. $(x, y) \in E \Leftrightarrow (y, x) \in E$, and
3. $(x, y) \in E \rightarrow [(x, z) \in E \Leftrightarrow (y, z) \in E]$.

The *equivalence class* of $x \in rlm(X)$ w.r.t E is given by

$$[x]_E := \{y : xEy\}.$$

If X is a set, then we define the *quotient of X by E* by letting

$$X/E := \{[x]_E : x \in X\}.$$

Proposition 4.8.7. *Let E be an equivalence relation on the class X. Then for all $x, y \in rlm(X)$,*

$$(x, y) \in E \Leftrightarrow [x]_E = [y]_E.$$

Proof. It follows from 3. that $(x, y) \in E \rightarrow [x]_E = [y]_E$. If $[x]_E = [y]_E$, then $(y, y) \in E \rightarrow (x, y) \in E$. So 1. gives $(x, y) \in E$. We have $(x, y) \in E \leftrightarrow [x]_E = [y]_E$.

Now, assume that $(x, y) \notin E$, i.e., $y \notin [x]_E$. By 1., we have $y \in [y]_E$ and therefore $[x]_E \neq [y]_E$. Finally, assume that $[x]_E \neq [y]_E$. There is then a z such that $(x, z) \in E \wedge (y, z) \notin E$ or $(x, z) \notin E \wedge (y, z) \in E$. If the former holds, then $(z, x) \in E$ and $(z, y) \notin E$ by 2. Using 3., we get $(x, y) \notin E$. Similarly, if $(x, z) \notin E \wedge (y, z) \in E$, then $(x, y) \notin E$. Hence $(x, y) \in E \Leftrightarrow [x]_E = [y]_E$. $\square$

4.9 Functions

Let us now turn to finding a suitable notion of a function. There are many possible definitions we could give; each having their own advantages and disadvantages. The definition we give here should therefore not be seen as the one true definition of a function. Rather, it is simply the definition simply that I have found the most useful.

We start by giving a preliminary definition.

Definition 4.9.1.

(a) By a *classical function* we mean a classical relation f such that

$$\forall x, y, z[(x, y) \in f) \wedge (x, z) \in f \to !(x = y)].$$

(b) If A and B are classical sets and f is a classical function, then we say that f *goes from* A *to* B, and write $f : A \to B$, if

$$dom(f) = A \text{ and } ran(f) \subseteq B.$$

(c) For $x \in A$, we let $f(x)$ denote the the unique element such that $(x, f(x)) \in f$.

(d) The *restriction* of f to a set $X \subseteq A$ is

$$f{\restriction}X := \{(x, f(x)) : x \in X\}.$$

In short, classical functions are functions that behave as we would expect from classical set theory. The intuition is that a classical function is a process that takes an input from its domain and produces an output. Most of the functions we will encounter in this thesis will be classical.

Now, suppose we have a classical function f with the domain A, and suppose that X is a non-classical subset of A. We can then think of the restriction $g := f{\restriction}X$ as a non-classical process with the domain X. We think of X as the set of inputs for g, and we think of $(x, y) \in g$ as saying that g maps x to y. Now, if $x \in X \wedge x \notin X$, then x both is and is not an input for g and, accordingly, g both produces and does not produce an output for x.

Definition 4.9.2.

(a) A set f is said to be a *function* if $rlm(f)$ is a classical function.

(b) If A and B are sets and f is a function, then we say that f *goes from* A *to* B, and write $f : A \to B$, if

$$!(dom(f) = A) \text{ and } ran(f) \subseteq B.$$

(c) For $x \in rlm(A)$, we let $f(x)$ denote the the unique element such that $(x, f(x)) \in rlm(f)$.

(d) The *restriction* of f to a set $X \subseteq A$ is

$$f{\restriction}X := \{(x, f(x)) : x \in X\}.$$

Let us devote a little space to unpack the definition of the formula $f : A \to B$. First, we note that if f, A and B are classical sets, then $f = rlm(A)$ and $dom(f) = A \Leftrightarrow !(dom(f) = A)$. So our definition of a function agrees with our definition of a classical function.

Next, we notice that if f and A are sets, then

$$f : A \to V \Leftrightarrow \exists g[!(f = g{\restriction}A) \wedge g : rlm(A) \to V].$$

By Definition 4.5.2, we also have

$$f \in \{g{\restriction}A : g : rlm(a) \to V\} \Leftrightarrow \exists g[!(f = g{\restriction}A) \wedge g : rlm(A) \to V].$$

Taking these two together, we get

$$f : A \to V \Leftrightarrow f \in \{g{\restriction}A : g : rlm(a) \to V\}.$$

Now, $\{g{\restriction}A : g : rlm(a) \to V\}$ is the class obtained by restricting the classical functions from $rlm(A)$ to A. So the formula $f : A \to V$ is saying that f is the result of restricting some function from $rlm(A)$ to A. That is to say, to get a function from A, we take a classical function from $rlm(A)$ and restrict it to A. Similarly, the formula $f : A \to B$ simply states that f is a function from A and the range of f is a subset of B.

Definition 4.9.3. Let A and B be sets. We say that a set f is an *injection* from A to B if

$$f : A \to B \text{ and } f^{-1} \text{ is a function}$$

and that f is a *surjection* from A to B if

$$f : A \to B \text{ and } ran(F) = B.$$

We say that f is an a *bijection* between A and B if

$$f : A \to B \text{ and } f^{-1} : B \to A.$$

4.10 Power set

Recall that we say that u is a subset of v, and write $u \subseteq v$, if $\forall x(x \in u \Rightarrow x \in u)$.

Definition 4.10.1. Given a set u, we let

$$\mathcal{P}(u) := \{x : x \subseteq u\}.$$

Sadly, the following proposition will tell us that we cannot expect $\mathcal{P}(u)$ to be a set.

Proposition 4.10.2. *If there exists an incomplete set, then $\mathcal{P}(u)$ is a proper class for all u.*

Proof. Assume that $\mathcal{P}(u)$ is a set. Then $\{x :?(x \subseteq u)\}$ is also a set.

Since there exists an incomplete set, we have that for every classical set x, $\langle \emptyset, x\rangle$ exists. Now,

$$\begin{aligned} u \not\subseteq \langle \emptyset, x\rangle &\text{ iff } \exists z[z \in u \wedge z \notin \langle \emptyset, x\rangle] \\ &\text{ iff } \exists z[z \in u^! \wedge z \notin \langle \emptyset, x\rangle^?] \\ &\text{ iff } \exists z[z \in u^! \wedge z \notin x] \\ &\text{ iff } \neg\forall z[z \in u^! \Rightarrow z \in x] \\ &\text{ iff } \neg(u^! \subseteq x). \end{aligned}$$

It follows that for every classical x, $u^! \subseteq x$ implies $\neg(u \not\subseteq \langle \emptyset, x, \rangle)$. In other words, $u^! \subseteq x$ implies $?(u \subseteq \langle \emptyset, x, \rangle)$ for all classical x. Thus

$$\{\langle \emptyset, x\rangle : Cl(x) \wedge u^! \subseteq x\} \subseteq \{x :?(x \subseteq u)\}.$$

But, as we saw in the proof of Proposition 4.7, $\{\langle \emptyset, x\rangle : Cl(x) \wedge u^! \subseteq x\}$ is a proper class. □

This means that we cannot add an axiom stating that $\mathcal{P}(u)$ is a set for all u. What goes wrong is that if $\mathcal{P}(u)$ is a set, then $\mathcal{P}^?(u) := \{x : ?(x \subseteq u)\}$ would also be a set. But, $\mathcal{P}^?(u)$ is to big to be a set assuming that there exist an incomplete set. We will therefore have to settle for a weaker axiom.

Definition 4.10.3. Given a set u, we let

$$\mathcal{P}_{Cl}(u) := \{x : Cl(x) \wedge x \subseteq u\}.$$

Axiom 7 (Classical power set)**.**

$$\forall u \exists v \forall x[x \in v \Leftrightarrow Cl(x) \wedge x \subseteq u].$$

Our motivation for this axiom is as follows: If u is a classical set, then surely we expect the class of all classical subsets of u to be a set. Moreover, if u is any set, then $rlm(u)$ is a classical set. So we expect $\mathcal{P}_{Cl}(rlm(u))$ to be a set. Since $u \subseteq rlm(u)$, we get that $\mathcal{P}_{Cl}(u) \subseteq \mathcal{P}_{Cl}(rlm(u))$. We therefore expect $\mathcal{P}_{Cl}(u)$ to be a set for all u.

Definition 4.10.4. Given a set u, we let

$$\mathcal{P}^!(u) := \{x : !(x \subseteq u)\}.$$

Now, $\mathcal{P}^!(u)$ is the classical class such that for all x, $x \in \mathcal{P}^!(u) \leftrightarrow x \subseteq u$.

Proposition 4.10.5. *For all u, the class $\mathcal{P}^!(u)$ is a set.*

Proof. Fix a set u, and consider the class $A := \{(x^!, x^?) : !(x \subseteq u)\}$. Notice that $A \subseteq \mathcal{P}_{Cl}(rlm(u)) \times \mathcal{P}_{Cl}(rlm(u))$, so A is a set. We can define a bijection $f : \mathcal{P}^!(u) \to A$ by letting

$$f(x) := (x^!, x^?).$$

It follows that $\mathcal{P}^!(u)$ is a set by replacement. $\square$

The set $\mathcal{P}^!(u)$ will provide us with a suitable alternative to $\mathcal{P}(u)$ for most applications.

4.11 Infinity and ordinals

In this section, we introduce the ordinal numbers. We will not embark on an investigation of well ordered sets. Rather, we will simply recruit the classical von Neumann ordinals and show that we can still give definitions by recursion and carry out proofs by induction.

Axiom 8 (Infinity)**.**

$$\exists u[\emptyset \in u \wedge \forall x \in u(x \cup \{x\} \in u)].$$

Let us call a set u *inductive* if $\emptyset \in u \wedge \forall x \in u[x \cup \{x\} \in u]$. We can now form the first von Neumann ordinal ω by letting

$$\omega := \{x : Cl(x) \wedge x \text{ is inductive}\}.$$

As usual, we can encode the natural as elements of ω by letting $0 := \emptyset$, $1 := \{0\}$, $2 := \{0, 1\}$, $3 := \{0, 1, 2\}$ and so on.

Before we give the next definition, recall that in classical set theory, an ordinal can be defined as transitive set of transitive sets such that every nonempty subset has a $\in$-least element.

Definition 4.11.1. We say that a set u is *transitive* if $\forall x \in u(x \subseteq u)$. An *ordinal* is a classical transitive set u of classical transitive sets such that

$$\forall X \in \mathcal{P}_{Cl}(u)[X \neq \emptyset \to \exists x \in X \neg\exists y(y \in x \wedge x \in u)].$$

We denote the class of ordinals by Ord.

In short, an ordinal is a classical transitive set of classical transitive sets such that every nonempty classical subset has a $\in$-least element. There is nothing strange going on here as we are only dealing with classical sets.

We easily obtain the following proposition.

Proposition 4.11.2.

i. *The class of ordinals is classical.*

ii. $0 = \emptyset$ *is an ordinal.*

iii. If α is a ordinal, then so is $\alpha + 1 := \alpha \cup \{\alpha\}$.

iv. If X is a classical set of ordinals, then $\bigcup X$ is an ordinal.

v. ω is an ordinal.

Theorem 4.11.3 (Induction)**.** *For every formula $\varphi(x)$, we have*

$$[\forall\alpha \in Ord(\forall\beta \in \alpha\varphi(\beta) \to \varphi(\alpha))] \to \forall\alpha \in Ord\varphi(\alpha).$$

Proof. Suppose that $\forall\alpha \in Ord(\forall\beta \in \alpha\varphi(\beta) \to \varphi(\alpha))$, and assume that there is an ordinal α such that $\neg\varphi(\alpha)$. Let $X := \{\beta \in \alpha : \neg\varphi(\beta)\}$. Note that X is a classical subset of α. If $X = \emptyset$, then $\forall\beta \in \alpha\varphi(\beta)$, and therefore $\varphi(\alpha)$ by assumption. This gives $\perp$. On the other hand, if $X \neq \emptyset$, then there is $\beta \in X$ such that $\neg\varphi(\beta)$ and $\forall\gamma \in \beta\varphi(\gamma)$. Once again, this gives $\perp$. We can therefore conclude that $\forall\alpha\varphi(\alpha)$. □

The proof of the following theorem is entirely standard, and is therefore omitted.

Theorem 4.11.4 (Recursion)**.** *For every class function $G : V \to V$ there is a unique $F : Ord \to V$ such that for every ordinal α,*

$$F(\alpha) = G(F\restriction\alpha).$$

Definition 4.11.5. We say that $\alpha \in Ord$ is a *successor ordinal* if α is of the form $\beta + 1$ for some ordinal β. We say α a *limit ordinal* if $\alpha \neq 0$ and α is not a successor ordinal.

4.12 Foundation

We now introduce our axiom schema of foundation.

Axiom 9 (Foundation)**.**

$$\forall x[\forall y \in rlm(x)\varphi(y) \to \varphi(x)] \to \forall x\varphi(x),$$

where y is not a free variable of $\varphi(x)$.

The purpose of foundation is to allow us to view the universe as being formed in stages; one stage for each ordinal. The intuition is that if u is a set such that each $x \in rlm(u)$ has been formed at stage α, then u is formed at stage $\alpha + 1$.

Definition 4.12.1. We let

$$\begin{aligned} V_0 :=& \emptyset \\ V_{\alpha+1} :=& \{x : rlm(x) \subseteq V_\alpha\} \\ V_\lambda :=& \bigcup_{\alpha<\lambda} V_\alpha, \text{ if } \lambda \text{ is a limit ordinal.} \end{aligned}$$

Definition 4.12.2. For $u \in \bigcup_\alpha V_\alpha$, we let $rnk(u)$ be the least ordinal α such that $u \in V_{\alpha+1}$.

Theorem 4.12.3. $V = \bigcup_\alpha V_\alpha$

Proof. Since $\bigcup_\alpha V_\alpha \subseteq V$ and both $\bigcup_\alpha V_\alpha$ and V are are classical, we need only show that for every x, there is some ordinal α such that $x \in V_\alpha$.

Suppose that x is a set such that $\forall y \in rlm(x) \exists \alpha (y \in V_\alpha)$. We let $\beta := \bigcup\{rnk(y) : y \in rlm(x)\}$. Since both $rlm(x)$ and V_β are classical, we get that $rlm(x) \subseteq V_\beta$, and therefore $x \in V_{\beta+1}$. Foundation now tells us that $\forall x \exists \alpha (x \in V_\alpha)$. □

Proposition 4.12.4. *For all α, $V_{\alpha+1} = \mathcal{P}^!(V_\alpha)$.*

Proof. Notice that both $V_{\alpha+1}$ and $\mathcal{P}^!(V_\alpha)$ are classical, so we only need to show that for all x, $x \in V_{\alpha+1} \leftrightarrow x \in \mathcal{P}^!(V_\alpha)$. If $x \in V_{\alpha+1}$, then $rlm(x) \subseteq V_\alpha$. So $x \subseteq V_\alpha$, and therefore $x \in \mathcal{P}^!(V_\alpha)$. Conversely, if $x \in \mathcal{P}^!(V_\alpha)$, then $x \subseteq V_\alpha$. Since $rlm(x)$ is the least classical superset of x, and V_α is classical, we get $rlm(x) \subseteq V_\alpha$, i.e., $x \in V_{\alpha+1}$. □

4.13 Choice

Definition 4.13.1. We call a set u *inhabited* if $\exists x(x \in u)$.

Definition 4.13.2. Let u be a set of inhabited sets. A *choice function* for u is a function from u such that

$$\forall x \in u(f(x) \in x).$$

Axiom 10 (Choice)**.**

$$\forall u[\forall x \in u \exists y(y \in x) \to (\exists f : u \to V)\forall x \in u(f(x) \in x)].$$

So our axiom of choice simply states that every set of inhabited sets has a choice function.

Proposition 4.13.3. *The axiom of choice holds if and only if every classical set of inhabited sets has a choice function.*

Proof. Assume that every classical set of inhabited sets has a choice function and let u be an arbitrary set of inhabited sets. Then $u^!$ is a classical set of inhabited sets, so there exists a choice function g for $u^!$. We define $f : u \to V$ by letting

$$f(x) := \begin{cases} g(x) & \text{if } x \in u^!, \\ \emptyset & \text{if } x \in rlm(u) \setminus u^!. \end{cases}$$

Now f is a choice function for u. □

4.14 The anti-classicality axiom

In Section 4.4, we saw that from the assumption that there exists a single non-classical set, we can generate a whole host of new non-classical sets. Since we are interested in both inconsistent and incomplete sets, we introduce the *anti-classicality axiom* which states that there exists an inconsistent set and an incomplete set.

Axiom 11 (*AClA*).

$$\exists u{\sim}Con(u) \wedge \exists u{\sim}Com(u).$$

Using Theorem 4.4.3, we see that for all classical sets u and v, there exists a set w such that

$$w = \langle u, v\rangle.$$

In particular, this tells us that there are subsets $\mathfrak{b}$ and $\mathfrak{n}$ of 1 such that

$$\emptyset \in \mathfrak{b} \wedge \emptyset \notin \mathfrak{b} \text{ and } \neg(\emptyset \in \mathfrak{n} \vee \emptyset \notin \mathfrak{n}).$$

Recall that we defined 1 as $\{\emptyset\}$.

Definition 4.14.1. We define the *set of truth values* by $\Omega := \mathcal{P}^!(1)$. Given a sentence φ, we put $\{\emptyset : \varphi\} := \{x : !(x = \emptyset) \wedge \varphi\}$, and call $\{\emptyset : \varphi\}$ the *truth value* of φ.

The anti-classicality axiom now tells us that

$$\Omega = \{1, \mathfrak{b}, \mathfrak{n}, 0\}.$$

Notice that $\varphi \Leftrightarrow \emptyset \in \{\emptyset : \varphi\}$. So φ is true if and only if $\emptyset \in \{\emptyset : \varphi\}$, and is false if and only if $\emptyset \notin \{\emptyset : \varphi\}$. Looking at this from the meta-theoretic perspective for a moment, we see that $[\![\varphi]\!] = 1$ iff $\{\emptyset : \varphi\} = 1$, $[\![\varphi]\!] = \mathfrak{b}$ iff $\{\emptyset : \varphi\} = \mathfrak{b}$, $[\![\varphi]\!] = \mathfrak{n}$ iff $\{\emptyset : \varphi\} = \mathfrak{n}$, and $[\![\varphi]\!] = 0$ iff $\{\emptyset : \varphi\} = 0$, justifying the name 'truth value of φ' for $\{\emptyset : \varphi\}$.

Example 4.14.2. Recall from chapter 1 that Alice had made five lists concerning the celebration: I, D^+, D^-, P^+ and P^-. By using the anti-classicality axiom, she can now represent the same information by viewing I as a classical set, and defining the sets $D, P \subseteq I$ by

$$n \in D \leftrightarrow n \in D^+ \text{ and } n \notin D \leftrightarrow n \in D^-,$$

and

$$n \in P \leftrightarrow n \in P^+ \text{ and } n \notin P \leftrightarrow n \in P^-$$

for all $n \in I$. So, for example, if Bob provided contradictory replies whether he will attend the dinner, then Bob $\in D \wedge$ Bob $\notin D$. So the truth value of Bob $\in D$ is $\mathfrak{b}$. Similarly, if Bob has not replied specifying whether he will attend the dinner, then the sentence Bob $\in D$ gets the truth value $\mathfrak{n}$. This can be expressed internally using Definition 4.14.1.

4.15 The theories $PZFC$ and $BZFC$

The axioms we have given in this chapter are the following:

1. Extensionality: $\forall u \forall v[u = v \Leftrightarrow \forall x(x \in u \Leftrightarrow x \in v)]$.
2. Separation: $\forall u \exists v \forall x[x \in v \Leftrightarrow x \in u \wedge \varphi(x)]$,
   where v is not a free variable in $\varphi(x)$.
3. Classical supersets: $\forall u \exists v[Cl(v) \wedge u \subseteq v]$.
4. Replacement: $\forall u \exists v[v = \{F(x) : x \in u\}]$,
   where F is an operation, and v is not a free variable in the formula defining F.
5. Union: $\forall u \exists v \forall x[x \in v \Leftrightarrow \exists y \in u(x \in y)]$.
6. Pairing: $\forall u \forall v \exists w \forall x[x \in w \Leftrightarrow !(x = u \vee x = v)]$.
7. Classical power set: $\forall u \exists v \forall x[x \in v \Leftrightarrow Cl(x) \wedge x \subseteq u]$.
8. Infinity: $\exists u[\emptyset \in u \wedge \forall x \in u(x \cup \{x\} \in u)]$.
9. Foundation: $\forall x[\forall y \in rlm(x)\varphi(y) \rightarrow \varphi(x)] \rightarrow \forall x \varphi(x)$,
   where y is not a free variable in $\varphi(x)$.
10. Choice: $\forall u[\forall x \in u \exists y(y \in x) \rightarrow \exists(f : u \rightarrow V)\forall x \in u(f(x) \in x)]$.
11. $AClA$: $\exists u{\sim}Con(u) \wedge \exists u{\sim}Com(u)$.

Definition 4.15.1. By *paradefinite Zermelo–Fraenkel set theory PZF* we mean the theory given by the axioms 1.–9. and $PZFC$ is PZF together with the axiom of choice.

Notice that $PZFC$ does not prove the existence of any non-classical sets and that $PZFC$ together with $\forall x Cl(x)$ is just classical ZFC with two symbols for the same negation. We now have the following theorem.

Theorem 4.15.2. *If ZFC is consistent, then $PZFC$ is non-trivial. (Recall that a theory is said to be non-trivial if $\perp$ is not derivable from it.)*

Definition 4.15.3. We let $BZF := PZF + AClA$, and $BZFC$ is the theory BZF together with the axiom of choice, i.e., axioms 1.–11.

Remark. If the reader is only interested in inconsistent sets, then they can consider the theory $PZFC + \exists u{\sim}Con(u) + \forall u Com(u)$. Similarly, if the reader is only interested in incomplete sets, then they can consider the theory $PZFC + \forall u Con(u) + \exists u{\sim}Com(u)$. In either case, all the major results of this thesis will have straightforward analogues for these theories.

Chapter 5

A model of $BZFC$

In this chapter, we show that $BZFC$ is not trivial, assuming that ZFC is consistent. We do this by constructing a natural T/F-model W of $BZFC$. Throughout this chapter we will work in classical ZFC.

5.1 T/F-models of set theory

We start by slightly broadening our definition of a T/F-model to allow for models with domains that are proper classes.

Definition 5.1.1. A T/F-*model* $\mathcal{M}$ of set theory consists of

1. a non-empty class M, called the *domain* of $\mathcal{M}$;
2. a pair of binary relations $\in^+_{\mathcal{M}} \subseteq M \times M$ and $\in^-_{\mathcal{M}} \subseteq M \times M$;
3. a pair of binary relations $=^+_{\mathcal{M}} \subseteq M \times M$ and $=^-_{\mathcal{M}} \subseteq M \times M$ such that for all $m, n \in M$,
   (a) $m =^+_{\mathcal{M}} n$ iff $m = n$, and
   (b) $m =^-_{\mathcal{M}} n$ iff $n =^-_{\mathcal{M}} m$.

Let $\mathcal{M}$ be a model of set theory and φ be a sentence with parameters from M. We keep the T/F-conditions for φ from Definition 2.2.2. We again write $\mathcal{M} \vDash_4 \varphi$ if φ is true in $\mathcal{M}$, and if Σ is a set of sentences, then we write $\mathcal{M} \vDash_4 \Sigma$ if every sentence from Σ is true in $\mathcal{M}$.

Remark. We should point out that we cannot formally define the class $\{\varphi : \mathcal{M} \vDash_4 \varphi\}$ within ZFC since that would contradict Tarski's undefinability theorem. So $\mathcal{M} \vDash_4 \varphi$ has to be defined in the metatheory. However, given a particular sentence φ, the statement $\mathcal{M} \vDash_4 \varphi$ is definable within ZFC by recursion on the complexity of φ.

The following definition is inspired by [17].

Definition 5.1.2. Let $\mathcal{M}$ be a T/F-model of set theory and $\varphi(x)$ a formula in L_M. We define the *interpretation* of the class $A := \{x : \varphi(x)\}$ in $\mathcal{M}$ by $A^{\mathcal{M}} := (A^+_{\mathcal{M}}, A^-_{\mathcal{M}})$, where

$$A^+_{\mathcal{M}} := \{x \in M : \mathcal{M} \vDash_4 \varphi(x)\} \text{ and } A^-_{\mathcal{M}} := \{x \in M : \mathcal{M} \vDash_4 \sim\varphi(x)\}.$$

If $A^+_{\mathcal{M}} = M \setminus A^-_{\mathcal{M}}$, then we identify $A^{\mathcal{M}}$ with $A^+_{\mathcal{M}}$. Moreover, if $a \in M$ such that

$$\mathcal{M} \vDash_4 \forall x(x \in a \Leftrightarrow x \in A),$$

then we identify $A^{\mathcal{M}}$ with a.

5.2 A T/F-model of $BZFC$

We now define our T/F-model W of $BZFC$. The basic idea behind W is that every element $a \in W$ is of the form (a_1, a_2), where a_1 and a_2 represent $a^!$ and $a^?$, respectively.

Definition 5.2.1. We recursively define the class W by

$$\begin{aligned} W_0 &:= \emptyset \\ W_{\alpha+1} &:= \mathcal{P}(W_\alpha) \times \mathcal{P}(W_\alpha) \\ W_\lambda &:= \bigcup_{\alpha<\lambda} W_\alpha, \text{ if } \lambda \text{ is a limit ordinal} \\ W &:= \bigcup_{\alpha} W_\alpha. \end{aligned}$$

We define the relations $\in^+_W$, $\in^-_W$, $=^+_W$ and $=^-_W$ on W by letting

$$\begin{aligned} &a \in^+_W b \text{ iff } a \in b_1, \\ &a \in^-_W b \text{ iff } a \notin b_2, \\ &a =^+_W b \text{ iff } a = b, \text{ and} \\ &a =^-_W b \text{ iff } \exists x \in W[(x \in^+_W a \wedge x \in^-_W b) \text{ or } (x \in^-_W a \wedge x \in^+_W b)] \end{aligned}$$

for $a = (a_1, a_2)$ and $b = (b_1, b_2)$.

Theorem 5.2.2. *The axioms of $BZFC$ are true in W.*

Proof. We will only show that the axioms of extensionality, seperation, classical power set and choice hold in W. The other axioms are left as an exercise.

Extensionality: Let $a, b \in W$. We have

$$\begin{aligned} a = b &\text{ iff } a_1 = b_1 \text{ and } a_2 = b_2 \\ &\text{ iff } \forall x \in W(x \in a_1 \text{ iff } x \in b_1) \text{ and } \forall x \in W(x \in a_2 \text{ iff } x \in b_2) \\ &\text{ iff } \forall x \in W(x \in a_1 \text{ iff } x \in b_1) \text{ and } \forall x \in W(x \notin a_2 \text{ iff } x \notin b_2) \end{aligned}$$

iff $\forall x \in W(x \in^+_W a \text{ iff } x \in^+_W b)$ and $\forall x \in W(x \in^-_W a \text{ iff } x \in^-_W b)$
iff $W \vDash_4 \forall x(x \in a \Leftrightarrow x \in b)$.

So
$$W \vDash_4 a = b \leftrightarrow \forall x(x \in a \Leftrightarrow x \in b).$$
Moreover,
$$W \vDash_4 a \neq b \leftrightarrow {\sim}\forall x(x \in a \Leftrightarrow x \in b)$$
follows immediately from the definition of $=^-_W$.

Separation: Let $a \in W$ and $\varphi(x)$ be a formula with x as its only free variable and parameters from W. We let $b_1 := \{x \in a_1 : W \vDash_4 \varphi(x)\}$ and $b_2 := \{x \in a_2 : W \nvDash_4 {\sim}\varphi(x)\}$. Now, for all $x \in W$,
$$\begin{aligned} W \vDash_4 x \in b &\text{ iff } x \in a_1 \text{ and } W \vDash_4 \varphi(x) \\ &\text{ iff } W \vDash_4 x \in a \wedge \varphi(x) \end{aligned}$$
and
$$\begin{aligned} W \vDash_4 x \notin b &\text{ iff } x \notin b_2 \\ &\text{ iff } x \notin a_2 \text{ or } W \vDash_4 {\sim}\varphi(x) \\ &\text{ iff } W \vDash_4 x \notin a \vee {\sim}\varphi(x), \end{aligned}$$
Classical power set: It suffices to show that
$$W \vDash_4 \text{‘every classical set has a classical power set.’}$$
Let $a \in W$ be such that $W \vDash_4 Cl(a)$. Notice that $a_1 = a_2$, and for all $x \in W$,
$$W \vDash_4 Cl(x) \wedge x \subseteq a \text{ iff } x_1 = x_2 \wedge x_1 \subseteq a_1.$$
We therefore let $b_1 := \{(x, x) : x \subseteq a_1\}$ and $b_2 := b_1$, and get
$$W \vDash_4 \forall x[x \in b \leftrightarrow Cl(x) \wedge x \subseteq a].$$
Since $W \vDash_4 Cl(b)$ and the formula $x \in b \leftrightarrow Cl(x) \wedge x \subseteq a$ is classical in W, we get
$$W \vDash_4 \forall x[x \in b \Leftrightarrow Cl(x) \wedge x \subseteq a].$$
Choice: By Proposition 4.13.3, we only need to show that
$$W \vDash_4 \text{‘every classical set of inhabited sets has a choice function.’}$$
Let $a \in W$ be such that $W \vDash Cl(a) \wedge \forall x \in a \exists y(y \in x)$. Then $a_1 = a_2$ and $x_1 \neq \emptyset$ for all $x \in a_1$. By the axiom of choice for V, there is a function $g : a_1 \to W$ such that $g(x) \in x_1$ for all $x \in a_1$. We let
$$f_1 := \{(x, g(x))^W : x \in a_1\}$$
and $f_2 := f_1$. Then $W \vDash_4 f : a \to V$ and $W \vDash_4 \forall x(f(x) \in x)$. □

Theorem 5.2.3. *If ZFC is consistent, then $BZFC$ is not trivial.*

Proof. Suppose we have a proof of $\bot$ from the $BZFC$ axioms. We can then argue in ZFC that $W \vDash_4 \bot$ and conclude $ZFC \vdash_{CL} \bot$. □

5.3 Embedding V into W

We now show that W has a submodel that is, in a certain sense, isomorphic to the classical universe of sets V. So W can be thought of as the result of adding non-classical sets to V.

Definition 5.3.1. We define the map $x \mapsto \check{x}$ by letting

$$\check{x} := (\{\check{y} : y \in x\}, \{\check{y} : y \in x\})$$

for all $x \in V$. Moreover, we let

$$\check{V} := \{\check{x} : x \in V\}$$

and let $\in^+_{\check{V}}$, $\in^+_{\check{V}}$, $=^+_{\check{V}}$ and $=^-_{\check{V}}$ be the restrictions of $\in^+_W$, $\in^-_W$, $=^+_W$ and $=^-_W$ to $\check{V}$, respectively.

Theorem 5.3.2. *The map $x \mapsto \check{x}$ is a bijection from V to $\check{V}$, and for each formula $\varphi(x_1, ..., x_n)$ and all $a_1, ..., a_n \in V$,*

$$\varphi(a_1, ..., a_n) \textit{ iff } \check{V} \vDash_4 \varphi(\check{a}_1, ..., \check{a}_n).$$

Proof. First we show that the map is injective: Fix $x \in V$ and assume that for all y with $rank(y) < rank(x)$, $\forall z(\check{y} = \check{z}$ implies $y = z)$. Fix z such that $\check{x} = \check{z}$. For all $y \in x$, we have $\check{y} \in \check{x}_1 = \check{z}_1 = \{\check{w} : w \in z\}$. So $\check{y} = \check{w}$ for some $w \in z$. Our induction hypothesis therefore gives that $y \in z$. Conversely, if $y \in z$, then there an $w \in x$ such that $\check{y} = \check{w}$, and therefore $y \in x$. We have that $y = z$ and by induction we get that the map is injective. It follows that $x \mapsto \check{x}$ is a bijection from V to $\check{V}$.

Next we show that for all $x, y \in V$,

$$x \in y \text{ iff } \check{x} \in^+_W \check{y}, \text{ and}$$
$$x \notin y \text{ iff } \check{x} \in^-_W \check{y}.$$

If $x \in y$, then $\check{x} \in y_1$ and therefore $\check{x} \in^+_W \check{y}$. Conversely, if $\check{x} \in^+_W \check{y}$, then $\check{x} = \check{z}$ for some $z \in y$, and therefore $x \in y$. Now suppose that $\check{x} \in^-_W \check{y}$. Then $\check{x} \notin \check{y}_2$, so $x \notin y$. Lastly, if $x \notin y$, then $\check{x} \notin \{\check{x} : x \in y\} = \check{y}_2$ and therefore $\check{x} \in^-_W \check{y}$.

Since $x \mapsto \check{x}$ is injective, we get that for all $x, y \in V$,

$$x = y \text{ iff } \check{x} =^+_W \check{y}.$$

For all $x, y \in V$, we have

$$\begin{aligned} x \neq y &\text{ iff } \exists z[(z \in x \wedge z \notin y) \text{ or } (z \notin x \wedge z \in y)] \\ &\text{ iff } \exists \check{z} \in \check{V}[(\check{z} \in^+_{\check{V}} \check{x} \wedge \check{z} \in^-_{\check{V}} \check{y}) \text{ or } (\check{z} \in^-_{\check{V}} \check{x} \wedge \check{z} \in^+_{\check{V}} \check{y})] \\ &\text{ iff } \check{x} =^- \check{y}. \end{aligned}$$

The result now follows by an induction on the complexity of φ. □

Chapter 6

Connection to Classical Set Theory

In this chapter, we will see that in $PZFC$, we can define the class HCl of hereditarily classical sets, and show that it interprets the classical ZFC axioms. We will then go on to show that the class HCl defined in $PZFC$, together with the T/F-model W defined in ZFC (see Section 5.2), act as a bridge between the theories $BZFC$ and ZFC, allowing us to translate from one theory into the other.

6.1 Models of set theory within $PZFC$

We work in $PZFC$ throughout this section.

Definition 6.1.1. A *model* of set theory is a pair (M, R) where M is a classical class and R is a relation on M. Given a sentence φ with parameters from M, we define $(M, R) \vDash \varphi$ by letting

$$
\begin{aligned}
&(M, R) \vDash a = b :\Leftrightarrow a = b \\
&(M, R) \vDash a \in b :\Leftrightarrow aRb \\
&(M, R) \vDash {\sim}\varphi :\Leftrightarrow (M, R) \nvDash \varphi \\
&(M, R) \vDash \varphi \wedge \psi :\Leftrightarrow (M, R) \vDash \varphi \text{ and } (M, R) \vDash \psi \\
&(M, R) \vDash \varphi \vee \psi :\Leftrightarrow (M, R) \vDash \varphi \text{ or } (M, R) \vDash \psi \\
&(M, R) \vDash \varphi \rightarrow \psi :\Leftrightarrow (M, R) \vDash \varphi \text{ implies } (M, R) \vDash \psi \\
&(M, R) \vDash \exists x \varphi(x) :\Leftrightarrow (\exists x \in M)(M, R) \vDash \varphi(x) \\
&(M, R) \vDash \forall x \varphi(x) :\Leftrightarrow (\forall x \in M)(M, R) \vDash \varphi(x) \\
&(M, R) \vDash \bot :\Leftrightarrow \bot.
\end{aligned}
$$

If Γ is a set of sentences with parameters from M and $(M, R) \vDash \varphi$ for all $\varphi \in \Gamma$, then we write $(M, R) \vDash \Gamma$. Whenever R is clear from the context, we refer to (M, R) by M.

If $A = \{x : \varphi(x)\}$ is a class where φ is formula with parameters from M, then we define the *interpretation* $A^{(M,R)}$ of A in (M, R) by

$$A^{(M,R)} := \{x \in M : (M, R) \vDash \varphi(x)\}.$$

If there is an $a \in M$ such that $(M, R) \vDash \forall x[x \in a \Leftrightarrow \varphi(x)]$, then we identify $A^{(M,R)}$ with a.

An $\in$-*model* is a model of set theory of the form $(M, \in \cap (M \times M))$, and a *transitive model* is an $\in$-model such that M is transitive. Finally, we say that M is an *inner model* of a set of sentences Γ if M is a transitive model of Γ and $Ord \subseteq M$.

Remark. The reason we require M to be classical, for a model (M, R) of set theory, is that we want our models to behave like universes of sets. In particular, we want $V^{(M,R)}$ to be classical. Since $V^{(M,R)} = M$, we must require M to be classical. This does not mean that R is a classical relation, however, as can be seen by considering the model $(V, \in)$.

It will also prove useful to keep our notion of T/F-models of set theory from Chapter 5, with the added requirement that all the classes involved are classical. That is to say, a T/F-*model* $\mathcal{M}$ *of set theory* is a non-empty classical class M together with four classical relations $\in^+_{\mathcal{M}}$, $\in^-_{\mathcal{M}}$, $=^+_{\mathcal{M}}$ and $=^-_{\mathcal{M}}$ such that for all $a, b \in M$,

$$a =^+_{\mathcal{M}} b \text{ iff } a = b \text{ and}$$
$$a =^-_{\mathcal{M}} b \text{ iff } b =^-_{\mathcal{M}} a.$$

Notice that this implies that $a =^+_{\mathcal{M}} b \Leftrightarrow !(a = b)$.

6.2 Absoluteness

We will continue to work in in $PZFC$ throughout this section.

Definition 6.2.1. Let M be a transitive model and $\varphi(x_1, ..., x_n)$ be a formula with the free variables $x_1, ..., x_n$ and no constants. We say that φ is *absolute over* M if

$$\varphi(a_1, ..., a_n) \Leftrightarrow M \vDash \varphi(a_1, ..., a_n),$$

for all $a_1, ..., a_n \in M$. We say that φ is *absolute upwards over* M if

$$M \vDash \varphi(a_1, ..., a_n) \Rightarrow \varphi(a_1, ..., a_n),$$

for all $a_1, ..., a_n \in M$. Finally, φ is *absolute downwards over* M if

$$\varphi(a_1, ..., a_n) \Rightarrow M \vDash \varphi(a_1, ..., a_n),$$

for all $a_1, ..., a_n \in M$. The formula φ is *absolute (upwards/downwards)* if it is absolute (upwards/downwards) over all transitive models. Classes and functions are said to be *absolute (upwards/downwards)* over M if they are given by formulas that are absolute (upwards/downwards) over M.

Proposition 6.2.2. *A formula φ is absolute (upwards/downwards) over a transitive model M if and only if the formulas $!\varphi$ and $?\varphi$ are.*

Proof. We have

$$\begin{aligned}
\varphi \text{ is absolute over } M &\text{ iff } \varphi \Leftrightarrow M \vDash \varphi \\
&\text{ iff } !\varphi \leftrightarrow !\,(M \vDash \varphi) \text{ and } ?\varphi \leftrightarrow ?\,(M \vDash \varphi) \\
&\text{ iff } !\varphi \Leftrightarrow !\,(M \vDash \varphi) \text{ and } ?\varphi \Leftrightarrow ?\,(M \vDash \varphi) \\
&\text{ iff } !\varphi \Leftrightarrow M \vDash !\varphi \text{ and } ?\varphi \Leftrightarrow M \vDash ?\varphi \\
&\text{ iff } !\varphi \text{ and } ?\varphi \text{ are absolute over } M.
\end{aligned}$$

The proofs of the upwards and downwards parts are similar. □

Definition 6.2.3. A formula is said to be a Δ_0-*formula* if it is formed by the following rules:

1. The formulas $x \in y$, $x = y$ and $\perp$ are Δ_0-formulas.
2. If φ and ψ are Δ_0-formulas, then so are $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \to \psi$, and $\sim\varphi$. (It follows that $!\varphi$ and $?\varphi$ are Δ_0.)
3. If φ is a Δ_0-formula, then so are $\exists x \in rlm(y)\varphi$ and $\forall x \in rlm(y)\varphi$. (Recall that $x \in rlm(y)$ is an abbreviation of $!(x \in y) \vee ?(x \in y)$.)

A formula is said to be a Σ_1-*formula* if it is formed by the rules:

1. All Δ_0-formulas are Σ_1-formulas.
2. If φ and ψ are Σ_1-formulas, then so are $\varphi \wedge \psi$, $\varphi \vee \psi$, $!\varphi$ and $?\varphi$.
3. If φ is a Σ_1-formula, then so is $\exists x\varphi$.
4. If φ is a Σ_1-formula, then so are $\exists x \in rlm(y)\varphi$ and $\forall x \in rlm(y)\varphi$.

Notice that for any φ,

$$\begin{aligned}
\exists x \in y\varphi &\Leftrightarrow \exists x \in rlm(y)(x \in y \wedge \varphi), \text{ and} \\
\forall x \in y\varphi &\Leftrightarrow \forall x \in rlm(y)(x \in y \to \varphi).
\end{aligned}$$

So if φ is a Δ_0-formula (Σ_1-formula), then the formulas $\exists x \in y\varphi$ and $\forall x \in y\varphi$ are both equivalent to Δ_0-formulas (Σ_1-formulas). Similarly, the formulas $\exists x \in y^!\varphi$, $\forall x \in y^!\varphi$, $\exists x \in y^?\varphi$, $\forall x \in y^?\varphi$, $\forall x(x \in y \supset \varphi)$ and $\forall x(x \in y \Rightarrow \varphi)$ are all equivalent to Δ_0-formulas (Σ_1-formulas).

Proposition 6.2.4. *All Δ_0-formulas are absolute and all Σ_1-formulas are absolute upwards.*

Proof. We will only show that if φ is absolute over a transitive model M, then so is $\exists x \in rlm(y)\varphi$.

We have for $b \in M$,

$$\begin{aligned} M \vDash \exists x \in rlm(b)\varphi(x) &\Leftrightarrow \exists x \in M\big[M \vDash [!(x \in b) \vee ?(x \in b)] \wedge M \vDash \varphi(x)\big] \\ &\Leftrightarrow \exists x \in M\big[[!(x \in b) \vee ?(x \in b)] \wedge \varphi(x)\big] \\ &\Leftrightarrow \exists x \in M[x \in rlm(b) \wedge \varphi(x)]. \end{aligned}$$

Now, M is classical and transitive, so $rlm(b) \subseteq M$. Hence

$$\exists x \in M[x \in rlm(x) \wedge \varphi(x)] \Leftrightarrow \exists x[x \in rlm(b) \wedge \varphi(x)].$$

□

Proposition 6.2.5. *Every transitive model satisfies extensionality.*

Proof. The formula $\forall x(x \in u \Leftrightarrow x \in v)$ is equivalent to a Δ_0-formula and is therefore absolute. So for a transitive model M and $a, b \in M$, we get

$$\begin{aligned} a = b &\Leftrightarrow \forall x(x \in a \Leftrightarrow x \in b) \\ &\Leftrightarrow M \vDash \forall x(x \in a \Leftrightarrow x \in b). \end{aligned}$$

□

6.3 Hereditarily classical sets

We will work in $PZFC$ throughout this section.

Definition 6.3.1. We let

$$\begin{aligned} HCl_0 :=& \emptyset \\ HCl_{\alpha+1} :=& \mathcal{P}_{Cl}(HCl_\alpha) \\ HCl_\lambda :=& \bigcup_{\alpha<\lambda} HCl_\alpha, \text{ if } \lambda \text{ is a limit ordinal} \\ HCl :=& \bigcup_\alpha HCl_\alpha. \end{aligned}$$

We say that a set x is *hereditarily classical* if $x \in HCl$ and that a proper class X is hereditarily classical if X is classical and $X \subseteq HCl$.

Theorem 6.3.2. *The class HCl is an inner model of ZFC.*

Proof. Cearly, HCl is a transitive class and $Ord \subseteq HCl$, so HCl is an inner model. We will only show that the axioms of exensionality, seperation and choice hold in HCl.

Extensionality: Follows from HCl being a transitive model.

Separation: Let $a \in HCl$ and $\varphi(x)$ be a formula with parameters from HCl. An easy induction on the complexity of $\varphi(x)$ shows that it is a classical formula for all $x \in HCl$. It follows that $b := \{x \in HCl : x \in a \wedge HCl \vDash \varphi(x)\}$ is a hereditarily classical set.

Choice: Let $a \in HCl$ such that $HCl \vDash \forall x \in a(x \neq \emptyset)$. Then $\forall x \in a(x \neq \emptyset)$ and by the axiom of choice for V, there there is a function f from a such that $f(x) \in x$ for all $x \in a$. Now, both the domain and range of f are hereditarily classical sets so it easy to check that $f \in HCl$. Since the formulas "x is a function" and $!(dom(x) = y)$ are both (equivalent to) Δ_0-formulas, we get $HCl \vDash \forall x \in u(f(x) \in x)$.

□

Theorem 6.3.3. *If $PFZC$ is not trivial, then classical ZFC is consistent.*

Proof. If there is a proof of $\perp$ from the ZFC axioms, then we argue in $PZFC$ that $HCl \vDash \perp$. Thus $PZFC \vdash_{BS4} \perp$. □

6.4 Connecting ZFC and $BZFC$

In this section, we will provide translations between ZFC and $BZFC$. In particular, we will show that given any sentence φ in the language of set theory,

$$ZFC \vdash \varphi \text{ if and only if } BZFC \vdash HCl \vDash \varphi$$

and

$$BZFC \vdash \varphi \text{ if and only if } ZFC \vdash W \vDash_4 \varphi.$$

Recall from Chapter 5 that when working in ZFC, we can define a T/F-model W of $BZFC$ and a map $x \mapsto \check{x}$ that embeds V into W. It follows that $\check{V} = \{\check{x} : x \in V\}$ is a model of ZFC that is isomorphic to V. On the other hand, we can internalize the construction of HCl to W and get the model HCl^W of ZFC.

Theorem 6.4.1 (In ZFC). $HCl^W = \check{V}$

Proof. We start out by showing that $Ord^W = \{\check{\alpha} : \alpha \in Ord\}$. Let $a = (a_1, a_2) \in W$, and assume that for all $x \in a_1$, $x \in Ord^W$ iff $x \in \{\check{\alpha} : \alpha \in Ord\}$. We have

$$\begin{aligned} W \vDash_4 a \in Ord \text{ iff } & W \vDash_4 Cl(a) \wedge \forall x \in a[x \in Ord \wedge !(x \subseteq a)] \\ \text{iff } & a_1 = a_2 \wedge \forall x \in a_1(x \in \{\check{\alpha} : \alpha \in Ord\} \wedge x_1 \subseteq a_1) \\ \text{iff } & a \in \{\check{\alpha} : \alpha \in Ord\}. \end{aligned}$$

Next we show by induction that $HCl^W_{\check{\alpha}} = \check{V}_\alpha$ for all α. Fix an ordinal α and assume that $HCl^W_{\check{\beta}} = \check{V}_\beta$ for all $\beta \in \alpha$. For all $a = (a_1, a_2) \in W$, we have

$$a \in HCl^W_{\check{\alpha}} \text{ iff } W \vDash_4 \exists \beta \in \check{\alpha}\,[Cl(a) \wedge a \subseteq HCl_\beta]$$

$$\begin{aligned}
&\text{iff } \exists\beta\in\alpha\Big(W\vDash_4\Big[Cl(a)\wedge a\subseteq HCl_{\check\beta}\Big]\Big)\\
&\text{iff } \exists\beta\in\alpha\Big(a_1=a_2 \text{ and } W\vDash_4\forall x\Big[x\in a\Rightarrow x\in HCl_{\check\beta}\Big]\Big)\\
&\text{iff } \exists\beta\in\alpha\Big(a_1=a_2 \text{ and } W\vDash_4\forall x\Big[x\in a\rightarrow x\in HCl_{\check\beta}\Big]\Big)\\
&\text{iff } \exists\beta\in\alpha(a_1=a_2 \text{ and } a_1\subseteq HCl_{\check\beta}^W)\\
&\text{iff } \exists\beta\in\alpha(a_1=a_2 \text{ and } a_1\subseteq \check V_\beta)\\
&\text{iff } a\in\check V_\beta.
\end{aligned}$$

Finally,

$$\begin{aligned}
a\in HCl^W &\text{ iff } W\vDash_4 a\in HCl\\
&\text{ iff } W\vDash_4 \exists\alpha(a\in HCl_\alpha)\\
&\text{ iff } \exists\alpha\big(W\vDash_4 a\in HCl_{\check\alpha}\big)\\
&\text{ iff } \exists\alpha\big(a\in\check V_\alpha\big)\\
&\text{ iff } a\in\check V
\end{aligned}$$

for all a. □

Theorem 6.4.2. *Let φ be a sentence in the language of set theory. We have*

$$ZFC\vdash\varphi \textit{ if and only if } BZFC\vdash HCl\vDash\varphi.$$

Proof. Suppose that $ZFC\vdash\varphi$. Since $BZFC\vdash HCl\vDash ZFC$, we get $BZFC\vdash HCl\vDash\varphi$.

Now, suppose that $BZFC\vdash HCl\vDash\varphi$. We have $ZFC\vdash\check V=HCl^W$ and $ZFC\vdash V\cong\check V$, so $ZFC\vdash\big(\varphi\leftrightarrow HCl^W\vDash\varphi\big)$. Hence $ZFC\vdash HCl^W\vDash\varphi$. □

We have seen that when working in $BZFC$, we obtain a model of ZFC by restricting our attention to the class of hereditarily classical sets HCl. We can then construct W in HCl to obtain the T/F-model W^{HCl} of $BZFC$.

Lemma 6.4.3 (In $BZFC$)**.** *We have*

$$\begin{aligned}
W_0^{HCl}&=\emptyset\\
W_{\alpha+1}^{HCl}&=\mathcal{P}_{Cl}(W_\alpha^{HCl})\times\mathcal{P}_{Cl}(W_\alpha^{HCl})\\
W_\lambda^{HCl}&=\bigcup_{\alpha<\lambda}W_\alpha^{HCl},\ \textit{if } \lambda \textit{ is a limit ordinal}\\
W^{HCl}&=\bigcup_\alpha W_\alpha^{HCl}.
\end{aligned}$$

Proof. We only show that $W_{\alpha+1}^{HCl}=\mathcal{P}_{Cl}(W_\alpha^{HCl})\times\mathcal{P}_{Cl}(W_\alpha^{HCl})$ and $W^{HCl}=\bigcup_\alpha W_\alpha^{HCl}$.

For all $a\in HCl$, we have

$$a\in W_{\alpha+1}^{HCl}\Leftrightarrow HCl\vDash a\in W_{\alpha+1}$$

$$\begin{aligned}
&\Leftrightarrow HCl \vDash a \in \mathcal{P}(W_\alpha) \times \mathcal{P}(W_\alpha) \\
&\Leftrightarrow HCl \vDash \exists a_1, a_2 [a_1 \subseteq W_\alpha \wedge a_2 \subseteq W_\alpha \wedge !(a = (a_1, a_2))] \\
&\Leftrightarrow \exists a_1, a_2 \in HCl \big[HCl \vDash [a_1 \subseteq W_\alpha \wedge a_2 \subseteq W_\alpha \wedge !(a = (a_1, a_2))] \big].
\end{aligned}$$

The formulas $x \subseteq y$ and $!(x = (y, z))$ are absolute, so

$$a \in W_{\alpha+1}^{HCl} \Leftrightarrow \exists a_1, a_2 \in HCl [a_1 \subseteq W_\alpha^{HCl} \wedge a_2 \subseteq W_\alpha^{HCl} \wedge !(a = (a_1, a_2))].$$

Since W_α^{HCl} is hereditarily classical, we get

$$\begin{aligned}
a \in W_{\alpha+1}^{HCl} &\Leftrightarrow \exists a_1, a_2 [Cl(a_1) \wedge Cl(a_2) \wedge a_1, a_2 \subseteq W_\alpha^{HCl} \wedge !(a = (a_1, a_2))] \\
&\Leftrightarrow a \in \mathcal{P}_{Cl}(W_\alpha^{HCl}) \times \mathcal{P}_{Cl}(W_\alpha^{HCl}).
\end{aligned}$$

Hence

$$W_{\alpha+1}^{HCl} = \mathcal{P}_{Cl}(W_\alpha^{HCl}) \times \mathcal{P}_{Cl}(W_\alpha^{HCl}).$$

To see that $W^{HCl} = \bigcup_\alpha W_\alpha^{HCl}$, we first note that the class Ord is absolute, and $Ord \subseteq HCl$. Therefore,

$$\begin{aligned}
a \in W^{HCl} &\Leftrightarrow HCl \vDash (\exists \alpha \in Ord \wedge a \in W_\alpha) \\
&\Leftrightarrow \exists \alpha \in Ord\, (HCl \vDash a \in W_y) \\
&\Leftrightarrow a \in \bigcup_\alpha W_\alpha^{HCl}
\end{aligned}$$

for all $a \in HCl$. $\square$

Theorem 6.4.4 (In $BZFC$)**.** *There is a bijection $\mu : V \to W^{HCl}$ such that*

$$\varphi(u_1, ..., u_n) \text{ iff } W^{HCl} \vDash_4 \varphi(\mu(u_1), ..., \mu(u_n))$$

for all $u_1, ..., u_n \in V$ and every formula $\varphi(x_1, ..., x_n)$.

Proof. We recursively define the function μ by letting

$$\mu(u) := (\mu[u^!], \mu[u^?])$$

for all $u \in V$.

Let $\alpha \in Ord$ and assume that μ restricted to V_β is a bijection to W_β^{HCl} for all $\beta \in \alpha$. First we show that $\mu[V_\alpha] \subseteq W_\alpha^{HCl}$: We let $u \in V_\alpha$. There is a $\beta \in \alpha$ such that $rlm(u) \subseteq V_\beta$, and therefore $u^! \subseteq V_\beta$ and $u^? \subseteq V_\beta$. Now, $\mu[u^!] \subseteq \mu[V_\beta] = W_\beta^{HCl}$, and similarly $\mu[u^?] \subseteq W_\beta^{HCl}$. Since both $u^!$ and $u^?$ are classical, we get

$$\begin{aligned}
\mu[u] &= (\mu[u^!], \mu[u^?]) \\
&\in \mathcal{P}_{Cl}(W_\beta^{HCl}) \times \mathcal{P}_{Cl}(W_\beta^{HCl}) \\
&= W_\beta^{HCl}.
\end{aligned}$$

Next we show that $\mu\restriction V_\alpha$ is an injection: Let $u, v \in V_\alpha$ be such that $\mu(u) = \mu(v)$. Then $\mu[u^!] = \mu[v^!]$ and $\mu[u^?] = \mu[v^?]$. Now, $u^!$, $u^?$, $v^!$ and $v^?$ are elements of V_α, so there is a $\beta \in \alpha$ such that they are all subsets of V_β. Since μ is injective on V_β, we get $u^! = v^!$ and $u^? = v^?$. Hence $u = v$.

Now we show that $W_\alpha^{HCl} \subseteq \mu[V_\alpha]$: Let $(a_1, a_2) \in W_\alpha^{HCl}$. There is a $\beta \in \alpha$ such that $a_1, a_2 \subseteq W_\beta^{HCl}$. By the induction hypothesis, we have that $\mu \restriction V_\beta$ is a bijection from V_β to W_β^{HCl}. So $\mu^{-1}[a_1] \subseteq V_\beta$ and $\mu^{-1}[a_2] \subseteq V_\beta$, and therefore $\langle \mu^{-1}[a_1], \mu^{-1}[a_2] \rangle \in V_\alpha$. We conclude that $\mu\restriction V_\alpha$ is a bijection between V_α and W_β^{HCl}. By induction, we get that μ is a bijection between V and W^{HCl}.

For the second part, we have for all $u, v \in V$.

$$\begin{aligned} W^{HCl} \vDash_4 \mu(u) \in \mu(v) &\text{ iff } \mu(u) \in \mu[v^!] \\ &\text{ iff } u \in v^! \\ &\text{ iff } u \in v \end{aligned}$$

and

$$\begin{aligned} W^{HCl} \vDash_4 \mu(u) \notin \mu(v) &\text{ iff } \mu(u) \notin \mu[v^?] \\ &\text{ iff } u \notin v^? \\ &\text{ iff } u \notin v, \end{aligned}$$

An easy induction on the complexity of $\varphi(x_1, ..., x_n)$ now gives that $\varphi(u_1, ..., u_n)$ if and only if $W^{HCl} \vDash_4 \varphi(\mu(u_1), ..., \mu(u_n))$ for all $u_1, ..., u_n \in V$ and every formula $\varphi(x_1, ..., x_n)$. $\square$

Theorem 6.4.5. *Let φ be a sentence in the language of set theory. We have*

$$BZFC \vdash \varphi \textit{ if and only if } ZFC \vdash W \vDash_4 \varphi.$$

Proof. Suppose we have a proof of φ from the $BZFC$ axioms. We saw in Chapter 5 that $ZFC \vdash W \vDash_4 BZFC$. We can therefore argue in ZFC that $W \vDash_4 \varphi$ and conclude $ZFC \vdash W \vDash_4 \varphi$.

Now suppose that $ZFC \vdash W \vDash_4 \varphi$. We have seen that $BZFC \vdash HCl \vDash ZFC$, which gives $BZFC \vdash W^{HCl} \vDash \varphi$. Now, $BZFC \vdash \left(\varphi \leftrightarrow W^{HCl} \vDash \varphi\right)$, which allows us to conclude $BZFC \vdash \varphi$. $\square$

The main takeaway from this chapter is that we can view ZFC as the theory of hereditarily finite sets, whereas $BZFC$ describes a larger universe that properly contains HCl. So we can think of classical mathematics as taking place in HCl, which is described by ZFC. If we then encounter a phenomenon we think is better described using incomplete or inconsistent sets, we can switch to $BZFC$, and take full advantage of the anti-classicality axiom. Finally, if one is determined to keep a classical metatheory, then the whole process can be formalized in ZFC as statements about W.

Part III

Topics in Paradefinite Set Theory

Chapter 7

Model Theory Within $BZFC$

My original motivation for devising the set theory $BZFC$, which not only tolerates non-classical sets, but also has an axiom guaranteeing their existence, was to be able to provide sound and complete semantics for $BS4$ closer to the Tarskian semantics for classical logic. In particular, I wanted models such that $\mathfrak{A} \vDash \varphi \wedge {\sim}\varphi$ would really mean that both $\mathfrak{A} \vDash \varphi$ and $\mathfrak{A} \nvDash \varphi$.

In this chapter, we will work in $BZFC$ and assume that we have suitable encodings for the notions of languages, terms, formulas, theories and proofs, with the added caveat that they all are encoded as hereditarily classical sets.

Definition 7.0.1. A *model* $\mathfrak{A}$ in a language L consists of

1. a non-empty classical set A, called the *domain* of $\mathfrak{A}$;
2. an element $c^{\mathfrak{A}} \in A$ for every constant symbol c in L;
3. a function $f^{\mathfrak{A}} \colon A^n \to A$ for every n-ary function symbol f from L;
4. an n-ary relation $R^{\mathfrak{A}} \subseteq A^n$ for every n-ary relation symbol R in L.

Again, we let L_A denote the language obtained by adding a new constant symbol c_a to L for each $a \in A$. We will regard $\mathfrak{A}$ as a model in L_A, with $(c_a)^{\mathfrak{A}} = a$ and write a instead of c_a. The interpretation of a closed term is given in the usual way.

Definition 7.0.2. Let $\mathfrak{A}$ be a L-model and φ be a sentence in L_A. We recursively define $\mathfrak{A} \vDash \varphi$ as follows:

$$\mathfrak{A} \vDash \bot :\Leftrightarrow \bot$$
$$\mathfrak{A} \vDash a = b :\Leftrightarrow a = b$$
$$\mathfrak{A} \vDash R(a_1, ..., a_n) :\Leftrightarrow (a_1, ..., a_n) \in R^{\mathfrak{A}}$$

$$
\begin{aligned}
&\mathfrak{A} \models {\sim}\varphi :\Leftrightarrow \mathfrak{A} \not\models \varphi \\
&\mathfrak{A} \models \varphi \wedge \psi :\Leftrightarrow \mathfrak{A} \models \varphi \text{ and } \mathfrak{A} \models \psi \\
&\mathfrak{A} \models \varphi \vee \psi :\Leftrightarrow \mathfrak{A} \models \varphi \text{ or } \mathfrak{A} \models \psi \\
&\mathfrak{A} \models \varphi \rightarrow \psi :\Leftrightarrow \mathfrak{A} \models \varphi \text{ implies } \mathfrak{A} \models \psi \\
&\mathfrak{A} \models \exists x \varphi(x) :\Leftrightarrow (\exists x \in A) \mathfrak{A} \models \varphi(x) \\
&\mathfrak{A} \models \forall x \varphi(x) :\Leftrightarrow (\forall x \in A) \mathfrak{A} \models \varphi(x).
\end{aligned}
$$

If Σ is a theory and $\mathfrak{A} \models \varphi$ for all $\varphi \in \Sigma$, then we write $\mathfrak{A} \models \Sigma$. We write $\Sigma \models \varphi$ if $\mathfrak{A} \models \varphi$ for every model $\mathfrak{A}$ of Σ.

Before we show that $BS4$ is complete with respect to the above semantics, let us compare our new models with the T/F-models from Chapter 2. When working in $BZFC$ we keep the definition of a T/F-model just as we did in Definition 2.2.1, with the added requirement that all the sets involved are hereditarily classical. Let us for the moment consider the simpler situation of $BS4$ without equality and in a language L which only contains a single binary relation symbol R.

A T/F-model $\mathcal{M}$ in L consists of a hereditarily classical set M together with two classical subsets $R^+_{\mathcal{M}}$ and $R^-_{\mathcal{M}}$ of $M \times M$. Since we are working in $BZFC$, we can represent the information present in $R^+_{\mathcal{M}}$ and $R^-_{\mathcal{M}}$ by a single (possibly non-classical) set $R^{\mathfrak{A}} \subseteq M \times M$ given by

$$
\begin{aligned}
&(m,n) \in R^{\mathfrak{A}} \text{ iff } (m,n) \in R^+_{\mathcal{M}}, \text{ and} \\
&(m,n) \notin R^{\mathfrak{A}} \text{ iff } (m,n) \in R^-_{\mathcal{M}}.
\end{aligned}
$$

We can then define a model $\mathfrak{A}$ with the domain M and $R^{\mathfrak{A}}$ as the interpretation of R. We see that for all $m, n \in M$,

$$
\begin{aligned}
\mathfrak{A} \models R(m,n) &\text{ iff } (m,n) \in R^{\mathfrak{A}} \\
&\text{ iff } (m,n) \in R^+_{\mathcal{M}} \\
&\text{ iff } \mathcal{M} \models_4 R(m,n)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathfrak{A} \models {\sim}R(m,n) &\text{ iff } \mathfrak{A} \not\models R(m,n) \\
&\text{ iff } (m,n) \notin R^{\mathfrak{A}} \\
&\text{ iff } (m,n) \in R^-_{\mathcal{M}} \\
&\text{ iff } \mathcal{M} \models_4 {\sim}R(m,n).
\end{aligned}
$$

Moreover, if φ is a sentence, then a simple induction on the complexity of φ gives

$$\mathfrak{A} \models \varphi \text{ if and only if } \mathcal{M} \models_4 \varphi.$$

Proposition 7.0.3 (In $BZFC$)**.** *If $\mathcal{M}$ is T/F-model in a language L, then there is a model $\mathfrak{A}$ in the same language such that for every L-sentence φ,*

$$\mathfrak{A} \models \varphi \textit{ if and only if } \mathcal{M} \models_4 \varphi.$$

Proof. For this proof, the reader is advised to review Definition 4.8.6 and Proposition 4.8.7 for the notion of an equivalence relation.

By Theorem 4.4.3, we can define an equivalence relation E on M by letting

$$(m, n) \in E \text{ iff } \mathcal{M} \vDash_4 m = n, \text{ and}$$
$$(m, n) \notin E \text{ iff } \mathcal{M} \vDash_4 m \neq n.$$

We define the model $\mathfrak{A}$ as follows:

1. $A := M/E$, i.e., $A = \{[m]_E : m \in M\}$
2. If c is a constant symbol, we let $c^{\mathfrak{A}} := [c^{\mathcal{M}}]$
3. If f is an n-ary function symbol, then we put

$$f^{\mathfrak{A}}([a_1], ..., [a_n]) := [f^{\mathcal{M}}(a_1, ..., a_n)]$$

4. If R is an n-ary relation symbol, we let $R^{\mathfrak{A}}$ be the n-ary relation on A such that

$$([a_1], ..., [a_n]) \in R^{\mathfrak{A}} \text{ iff } (a_1, ..., a_n) \in R^{+}_{\mathcal{M}}, \text{ and}$$
$$([a_1], ..., [a_n]) \notin R^{\mathfrak{A}} \text{ iff } (a_1, ..., a_n) \in R^{-}_{\mathcal{M}}.$$

The result is now proved by an induction on the complexity of φ. $\square$

Theorem 7.0.4 (In $BZFC$)**.** *If Σ is a theory and φ is a sentence, then*

$$\Sigma \vdash_{BS4} \varphi \textit{ if and only if } \Sigma \vDash \varphi.$$

Proof. We will only show that $\Sigma \vDash \varphi$ implies $\Sigma \vdash_{BS4} \varphi$, as soundness is easy to verify. By a standard argument it suffices to show that every non-trivial theory has a model. We therefore assume that $\Sigma \nvdash_{BS4} \perp$ and show that Σ has a model.

Since HCl is a model of classical ZFC, we can carry out the semantics from Chapter 2 inside of HCl and show that there is a T/F-model of Σ. We can now apply Proposition 7.0.3 to get a model of Σ. $\square$

Chapter 8

Algebra-Valued Models for Paradefinite Set Theory

We will work in classical ZFC throughout this chapter unless otherwise specified.

Boolean-valued models of set theory were introduced by Scott, Solovay and Vopěnka in order to provide an intuitive framework for Cohen's method of forcing. The main idea is that given a complete Boolean algebra $\mathcal{B}$, one can construct a Boolean-valued model $V^{(\mathcal{B})}$ of $ZF(C)$ which behaves much like V except that propositions take their truth value in $\mathcal{B}$, rather than $\{1, 0\}$.

The construction has since been adopted to provide models set theories in various different logics. For example, in [12], Grayson shows that $\mathcal{B}$ can be replaced by any complete Heyting algebra $\mathcal{H}$ to get a Heyting-valued model $V^{(\mathcal{H})}$ of IZF. In [19], Löwe and Tarafder introduce a class of algebras called *reasonable implication algebras*, and construct models that validate the axioms of the negation-free fragment of Zermelo-Fraenkel set theory in a paraconsistent logic.

In this chapter, we will generalize the notion of Boolean-valued models for set theory by allowing any complete twist-structure $\mathcal{A}$ to take the place of the complete Boolean algebra in the construction of the model. In doing so, we will get a twist-valued model $V^{(\mathcal{A})}$ that validates all of the axioms of $PZFC$. Moreover, if the twist structure happens to be full, then we get a model that validates the axioms of $BZFC$. We will also see that if $\mathcal{A}$ is a twist structure over a complete Boolean algebra $\mathcal{B}$, then $V^{(\mathcal{B})}$ represents the class of hereditarily classical sets in $V^{(\mathcal{A})}$ in a natural way.

The presentation of our construction will very closely follow the Boolean-valued account as presented in [4]. The rest of this chapter depends heavily on the material from Chapter 3, so the reader is advised to review the ideas from that chapter before proceeding.

8.1 Class-sized twist-valued models

Before we get into the meat of this chapter, we first need to slightly broaden our definition of twist-valued model from Chapter 3 to allow for domains that are proper classes. Here we will only consider the special case of models in the language of set theory.

Definition 8.1.1. A *twist-valued model* $\mathcal{M}$ in the language of set theory consists of

1. a non-empty class M, called the *domain* of $\mathcal{M}$,
2. a complete twist-structure $\mathcal{A}$,
3. a function $[\![\cdot \in \cdot]\!]^{\mathcal{M}} : M \times M \to A$, and
4. a function $[\![\cdot = \cdot]\!]^{\mathcal{M}} : M \times M \to A$ such that for all $a, b, c, d \in M$,
   (a) $[\![a = a]\!]^{\mathcal{M}} \approx 1$,
   (b) $[\![a = b]\!]^{\mathcal{M}} = [\![b = a]\!]^{\mathcal{M}}$,
   (c) $[\![a = b]\!]^{\mathcal{M}} \preceq [\![a = c]\!]^{\mathcal{M}} \Leftrightarrow [\![b = c]\!]^{\mathcal{M}}$, and
   (d) $[\![a = b]\!]^{\mathcal{M}} \wedge [\![c = d]\!]^{\mathcal{M}} \preceq [\![a \in b]\!]^{\mathcal{M}} \Leftrightarrow [\![c \in d]\!]^{\mathcal{M}}$.

If $\mathcal{M}$ is a twist-valued model in the language of set theory, we let $L_M := \{\in\} \cup M$ and regard $\mathcal{M}$ as a twist-valued model in L_M, with each element of M being its own interpretation.

For $a, b \in M$, we refer to $[\![a \in b]\!]^{\mathcal{M}}$ and $[\![a = b]\!]^{\mathcal{M}}$ as the *truth values* of the sentences $a \in b$ and $a = b$, respectively. Given a particular sentence φ of L_M, we can define the *truth value* $[\![\varphi]\!]^{\mathcal{M}}$ of φ in $\mathcal{M}$ by letting

1. $[\![\bot]\!]^{\mathcal{M}} := 0$,
2. $[\![\sim\varphi]\!]^{\mathcal{M}} := \sim[\![\varphi]\!]^{\mathcal{M}}$,
3. $[\![\varphi * \psi]\!]^{\mathcal{M}} := [\![\varphi]\!]^{\mathcal{M}} * [\![\psi]\!]^{\mathcal{M}}$ for $* \in \{\vee, \wedge, \to\}$, and
4. $[\![\exists x\varphi(x)]\!]^{\mathcal{M}} := \bigvee_{x \in M} [\![\varphi(x)]\!]^{\mathcal{M}}$ and $[\![\forall x\varphi(x)]\!]^{\mathcal{M}} := \bigwedge_{x \in M} [\![\varphi(x)]\!]^{\mathcal{M}}$.

Notice that $\bigvee_{x \in M} [\![\varphi(x)]\!]^{\mathcal{M}}$ and $\bigwedge_{x \in M} [\![\varphi(x)]\!]^{\mathcal{M}}$ are well-defined. This is because

$$\bigvee_{x \in M} [\![\varphi(x)]\!]^{\mathcal{M}} = \bigvee\{[\![\varphi(x)]\!] : x \in M\} \text{ and } \bigwedge_{x \in M} [\![\varphi(x)]\!]^{\mathcal{M}} = \bigwedge\{[\![\varphi(x)]\!] : x \in M\}.$$

The definable class $\{[\![\varphi(x)]\!] : x \in M\}$ is a subset of A, and is therefore a set.

Remark. We should point out that we cannot formally construct a map that takes every L_M sentence φ to its truth value $[\![\varphi]\!]^{\mathcal{M}}$, since that would contradict Tarski's undefinability theorem. So the map $\varphi \mapsto [\![\varphi]\!]^{\mathcal{M}}$ has to be defined in the metatheory. Nevertheless, given a particular sentence φ, the truth value $[\![\varphi]\!]^{\mathcal{M}}$ can still be calculated within ZFC as described above.

Definition 8.1.2. We write $\mathcal{M} \vDash_{Tw} \varphi$ and say that φ is *true in* $\mathcal{M}$ if $[\![\varphi]\!]^{\mathcal{M}} \approx 1$. We will often write $[\![\varphi]\!]$ rather than $[\![\varphi]\!]^{\mathcal{M}}$ if $\mathcal{M}$ is clear from the context. If Σ is a set of sentences, then we write $\mathcal{M} \vDash_{Tw} \Sigma$ to indicate that $\mathcal{M} \vDash_{Tw} \varphi$ for each $\varphi \in \Sigma$.

8.2 Boolean-valued models of set theory

In this section, we give a brief review of Boolean-valued models of set theory. This review is by no means intended as a comprehensive introduction to the topic. Rather, it is simply a summary of the standard motivation for the construction of Boolean-valued models, which we can then use as a guide when we construct our twist-valued models of set theory. For a more comprehensive account of Boolean-valued models of set theory, see [4].

Recall that in classical set theory, a *characteristic function* χ_a of a set a is a function such that $a \subseteq dom(\chi_a)$ and for all $x \in dom(\chi_a)$,

$$\chi_a(x) = \begin{cases} 1 & \text{if } \ x \in a \\ 0 & \text{else.} \end{cases}$$

Now, the function χ_a completely describes a, so each set can be represented by a function taking values in the Boolean algebra $2 = \{1, 0\}$, or a *two-valued function* for short. Similarly, the elements of a can themselves be represented by two-valued functions. So a can be represented by a two-valued function whose domain consists of two-valued functions.

By carrying out this process out to its extreme, we see that each set can be represented by an element of the *universe of two-valued sets* $V^{(2)}$, where $V^{(2)}$ is defined recursively as follows:

$$\begin{aligned} V_0^{(2)} :=& \emptyset \\ V_{\alpha+1}^{(2)} :=& \{u : fun(u) \wedge dom(u) \subseteq V_\alpha^{(2)} \wedge ran(u) \subseteq A\} \\ V_\lambda^{(2)} :=& \bigcup_{\alpha\in\lambda} V_\alpha^{(2)}, \text{ if } \lambda \text{ is a limit ordinal} \\ V^{(2)} :=& \bigcup_{\alpha} V_\alpha^{(2)}. \end{aligned}$$

If we now take any complete Boolean algebra $\mathcal{B}$, and let it play the role of 2 in the construction above, we get the *universe of $\mathcal{B}$-valued sets* $V^{(\mathcal{B})}$, where $V^{(\mathcal{B})}$ is given by

$$\begin{aligned} V_0^{(\mathcal{B})} :=& \emptyset \\ V_{\alpha+1}^{(\mathcal{B})} :=& \{u : fun(u) \wedge dom(u) \subseteq V_\alpha^{(\mathcal{B})} \wedge ran(u) \subseteq A\} \\ V_\lambda^{(\mathcal{B})} :=& \bigcup_{\alpha\in\lambda} V_\alpha^{(\mathcal{B})}, \text{ if } \lambda \text{ is a limit ordinal} \end{aligned}$$

$$V^{(\mathcal{B})} := \bigcup_{\alpha} V_{\alpha}^{(\mathcal{B})}.$$

In order to turn $V^{(\mathcal{B})}$ into a Boolean-valued model, we still have to specify how it interprets sentences of the form $u \in v$ and $u = v$, where $u, v \in V^{(\mathcal{B})}$. To simplify our notation, we will denote the truth value of a sentence φ in $V^{(\mathcal{B})}$ by $[\![\varphi]\!]^{\mathcal{B}}$. Moreover, if $\mathcal{B}$ is understood from context, we will simply write $[\![\varphi]\!]$.

Before defining $[\![u \in v]\!]^{\mathcal{B}}$ and $[\![u = v]\!]^{\mathcal{B}}$, we first consider some natural requirements that the definition should satisfy.

First, we note that the formula $u \in v \leftrightarrow \exists y \in v(u = v)$ is provable in classical logic, which tells us that we should require

$$[\![u \in v]\!]^{\mathcal{B}} = [\![\exists y \in v(u = y)]\!]^{\mathcal{B}}.$$

Moreover, we would like $V^{(\mathcal{B})}$ to satisfy the axiom of extensionality, which in classical logic is $\forall u, v[u = v \leftrightarrow \forall x(x \in u \leftrightarrow x \in v)]$. We must therefore have

$$[\![u = v]\!]^{\mathcal{B}} = [\![\forall x(x \in u \leftrightarrow x \in v)]\!]^{\mathcal{B}}.$$

Finally, suppose we have a formula $\varphi(x)$. We would like to be able to assign truth values to the sentences $\exists x \in u\varphi(x)$ and $\forall x \in u\varphi(x)$ by only allowing x to range over elements from $dom(u)$. We therefore require

$$[\![\exists x \in u\varphi(x)]\!]^{\mathcal{B}} = \bigvee_{x \in dom(u)} \left[u(x) \wedge [\![\varphi(x)]\!]^{\mathcal{B}}\right] \text{ and}$$
$$[\![\forall x \in u\varphi(x)]\!]^{\mathcal{B}} = \bigwedge_{x \in dom(u)} \left[u(x) \to [\![\varphi(x)]\!]^{\mathcal{B}}\right].$$

These considerations require us to take

$$[\![u \in v]\!]^{\mathcal{B}} := \bigvee_{y \in dom(v)} \left[v(y) \wedge [\![u = y]\!]^{\mathcal{B}}\right] \text{ and}$$

$$[\![u = v]\!]^{\mathcal{B}} := \bigwedge_{x \in \mathrm{dom}(u)} \left[u(x) \to [\![x \in v]\!]^{\mathcal{B}}\right] \wedge \bigwedge_{y \in \mathrm{dom}(v)} \left[v(y) \to [\![y \in u]\!]^{\mathcal{B}}\right].$$

This does indeed define $[\![u \in v]\!]^{\mathcal{B}}$ and $[\![u = v]\!]^{\mathcal{B}}$ via recursion on the well founded relation

$$(x, y) < (u, v) \text{ iff } (x \in dom(u) \text{ and } y = v) \text{ or } (y \in dom(v) \text{ and } x = u).$$

It can now be shown that $V^{(\mathcal{B})}$ is a Boolean-valued model [4, Theorem 1.17], and that all of the ZFC axioms are true in $V^{(\mathcal{B})}$ [4, Theorem 1.33].

8.3 Twist-valued models of set theory

In this section, we will generalize the construction from the previous section to allow for arbitrary complete twist structures. For the rest of this chapter we will fix a complete Boolean algebra $\mathcal{B}$ and a complete twist-structure $\mathcal{A}$ over $\mathcal{B}$.

Definition 8.3.1. The *universe of $\mathcal{A}$-valued sets* $V^{(\mathcal{A})}$ is defined as follows:

$$
\begin{aligned}
V_0^{(\mathcal{A})} :=& \emptyset \\
V_{\alpha+1}^{(\mathcal{A})} :=& \{u : fun(u) \wedge dom(u) \subseteq V_\alpha^{(\mathcal{A})} \wedge ran(u) \subseteq A\} \\
V_\lambda^{(\mathcal{A})} :=& \bigcup_{\alpha\in\lambda} V_\alpha^{(\mathcal{A})}, \text{ if } \lambda \text{ is a limit ordinal} \\
V^{(\mathcal{A})} :=& \bigcup_{\alpha} V_\alpha^{(\mathcal{A})}.
\end{aligned}
$$

We still need to specify how $V^{(\mathcal{A})}$ interprets sentences of the form $u \in v$ and $u = v$. This time around we are constrained by the requirements that

$$
\begin{aligned}
&[\![u \in v]\!]^{\mathcal{A}} = [\![\exists y \in v[!(u = y)]]\!]^{\mathcal{A}} \text{ and} \\
&[\![u = v]\!]^{\mathcal{A}} = [\![\forall x(x \in u \Leftrightarrow x \in v)]\!]^{\mathcal{A}}.
\end{aligned}
$$

The first requirement comes from the fact that $\vdash_{BS4} u \in v \Leftrightarrow \exists y \in v[!(u = y)]$, and the second requirement comes from our desire for $V^{(\mathcal{A})}$ to satisfy our extensionality axiom. We will also require that

$$
\begin{aligned}
[\![\exists x \in u\varphi(x)]\!]^{\mathcal{A}} &= \bigvee_{x\in dom(u)} \left[u(x) \wedge [\![\varphi(x)]\!]^{\mathcal{A}}\right] \text{ and} \\
[\![\forall x[x \in u \Rightarrow \varphi(x)]]\!]^{\mathcal{A}} &= \bigwedge_{x\in dom(u)} \left[u(x) \Rightarrow [\![\varphi(x)]\!]^{\mathcal{A}}\right].
\end{aligned}
$$

We now arrive at the following definition.

Definition 8.3.2. We recursively define $[\![u = v]\!]^{\mathcal{A}}$ and $[\![u \in v]\!]^{\mathcal{A}}$ for $u, v \in V^{(\mathcal{A})}$ by letting

$$
[\![u \in v]\!]^{\mathcal{A}} := \bigvee_{y\in \mathrm{dom}(v)} \left[v(y) \wedge ![\![y = u]\!]^{\mathcal{A}}\right] \text{ and}
$$

$$
[\![u = v]\!]^{\mathcal{A}} := \bigwedge_{x\in \mathrm{dom}(u)} \left[u(x) \Rightarrow [\![x \in v]\!]^{\mathcal{A}}\right] \wedge \bigwedge_{y\in \mathrm{dom}(v)} \left[v(y) \Rightarrow [\![y \in u]\!]^{\mathcal{A}}\right].
$$

Remark. Notice that if $\mathcal{A}$ happens to be a Boolean algebra, then $a \wedge !b = a \wedge b$ and $a \Rightarrow b = a \to b$ for all $a, b \in A$. Our definition of $V^{(\mathcal{A})}$ is therefore a generalization of the Boolean-valued case.

Theorem 8.3.3. *$V^{(\mathcal{A})}$ is a twist-valued model.*

Proof. We will show that for all $u, v, w \in V^{(\mathcal{A})}$,

i. $[\![u = u]\!] \approx 1$;

ii. $u(x) \leq [\![x \in u]\!]$ for all $x \in dom(u)$;

iii. $[\![u = v]\!] = [\![v = u]\!]$;

iv. $[\![u=v]\!] \wedge [\![u=w]\!] \preceq [\![v=w]\!]$;

v. $[\![u=v]\!] \wedge [\![u \in w]\!] \preceq [\![v \in w]\!]$;

vi. $[\![u=v]\!] \wedge [\![w \in u]\!] \preceq [\![w \in v]\!]$;

vii. $[\![u=v]\!] \wedge [\![u \notin w]\!] \preceq [\![v \notin w]\!]$;

viii. $[\![u=v]\!] \wedge [\![w \notin u]\!] \preceq [\![w \notin v]\!]$;

ix. $[\![u=v]\!] \wedge [\![u \neq w]\!] \preceq [\![v \neq w]\!]$.

i. Let $u \in V^{(\mathcal{A})}$, and assume that $[\![x=x]\!] \approx 1$ for all $x \in dom(u)$. Then

$$\begin{aligned} [\![x \in u]\!] &= \bigvee_{y \in dom(u)} \big[u(y) \wedge ![\![x=y]\!]\big] \\ &\geq u(x) \wedge ![\![x=x]\!] \\ &= u(x). \end{aligned}$$

So

$$u(x) \leq [\![x \in u]\!],$$

and therefore

$$\begin{aligned} [\![u=u]\!] &= \bigwedge_{x \in \mathrm{dom}(u)} \big[u(x) \Rightarrow [\![x \in u]\!]\big] \\ &\approx 1. \end{aligned}$$

ii. We have

$$\begin{aligned} 1 &\approx [\![u=u]\!] \\ &= \bigwedge_{x \in \mathrm{dom}(u)} \big[u(x) \Rightarrow [\![x \in u]\!]\big]. \end{aligned}$$

So $(u(x) \Rightarrow [\![x \in u]\!]) \approx 1$ for all $x \in dom(u)$.

iii. This holds by symmetry.

iv. Fix $u \in V^{(\mathcal{A})}$. We take as our induction hypothesis that for all $x \in \mathrm{dom}(u)$ and all $v, w \in V^{(\mathcal{A})}$,

$$[\![x=v]\!] \wedge [\![x=w]\!] \preceq [\![v=w]\!].$$

Let $v, w \in V^{(\mathcal{A})}$ and $x \in \mathrm{dom}(u)$, $y \in \mathrm{dom}(v)$ and $z \in \mathrm{dom}(w)$. We have

$$[\![u=v]\!] \wedge v(y) \leq \bigwedge_{y \in \mathrm{dom}(v)} \big[v(y) \Rightarrow [\![y \in u]\!]\big] \wedge v(y)$$

$$\preceq [\![y \in u]\!],$$

and similarly

$$[\![u = w]\!] \wedge u(x) \preceq [\![x \in w]\!].$$

This gives

$$\begin{aligned}
[\![u = w]\!] \wedge [\![u = v]\!] \wedge v(y) &\preceq [\![u = w]\!] \wedge [\![y \in u]\!] \\
&\approx \bigvee_{x \in dom(u)} \big[[\![u = w]\!] \wedge u(x) \wedge [\![y = x]\!] \big] \\
&\preceq \bigvee_{x \in dom(u)} \big[[\![x \in w]\!] \wedge [\![y = x]\!] \big].
\end{aligned}$$

By the induction hypothesis, we have

$$\begin{aligned}
[\![x \in w]\!] \wedge [\![y = x]\!] &\approx \bigvee_{z \in dom(w)} \big[w(z) \wedge [\![x = z]\!] \wedge [\![y = x]\!] \big] \\
&\preceq \bigvee_{z \in dom(w)} \big[w(z) \wedge [\![y = z]\!] \big] \\
&\approx [\![y \in w]\!].
\end{aligned}$$

We have

$$[\![u = v]\!] \wedge [\![u = w]\!] \wedge v(y) \preceq [\![y \in w]\!],$$

and therefore

$$[\![u = v]\!] \wedge [\![u = w]\!] \preceq v(y) \to [\![y \in w]\!]. \tag{8.1}$$

A similar argument gives

$$[\![u = v]\!] \wedge [\![u = w]\!] \preceq w(z) \to [\![z \in v]\!]. \tag{8.2}$$

We are now halfway there! We still need to show that

$$[\![u = v]\!] \wedge [\![u = w]\!] \preceq [\![y \notin w]\!] \to \sim v(y) \tag{8.3}$$

and

$$[\![u = v]\!] \wedge [\![u = w]\!] \preceq [\![z \notin v]\!] \to \sim w(z). \tag{8.4}$$

We only prove the former as the latter will follow by a similar argument.

By the induction hypothesis we have

$$\begin{aligned}
[\![x = y]\!] \wedge [\![x = z]\!] \wedge [\![y \notin w]\!] &\preceq [\![y = z]\!] \wedge [\![y \notin w]\!] \\
&= [\![y = z]\!] \wedge \bigwedge_{z \in dom(w)} \big[[\![z = y]\!] \to \sim w(z) \big] \\
&\preceq \sim w(z),
\end{aligned}$$

and therefore

$$[\![x = y]\!] \wedge [\![y \notin w]\!] \preceq [\![x = z]\!] \to \sim w(z).$$

Taking the infimum over z now gives

$$[\![x = y]\!] \wedge [\![y \notin w]\!] \preceq [\![x \notin w]\!].$$

Now,

$$\begin{aligned}[\![x = y]\!] \wedge [\![y \notin w]\!] \wedge [\![u = w]\!] &\preceq [\![x \notin w]\!] \wedge \bigwedge_{x \in dom(u)} \left[[\![x \notin w]\!] \to {\sim}u(x)\right] \\ &\preceq {\sim}u(x),\end{aligned}$$

and therefore

$$[\![u = w]\!] \wedge [\![y \notin w]\!] \preceq [\![x = y]\!] \to {\sim}u(x).$$

Taking the infimum over x gives

$$[\![u = w]\!] \wedge [\![y \notin w]\!] \preceq [\![y \notin u]\!].$$

We have

$$\begin{aligned}[\![u = v]\!] \wedge [\![u = w]\!] \wedge [\![y \notin w]\!] &\preceq [\![u = v]\!] \wedge [\![y \notin u]\!] \\ &\preceq [[\![y \notin u]\!] \to {\sim}v(y)] \wedge [\![y \notin u]\!] \\ &\preceq {\sim}v(y).\end{aligned}$$

This establishes (8.3), and a similar argument gives (8.4). Taking (8.1)–(8.4) together gives

$$[\![u = v]\!] \wedge [\![u = w]\!] \preceq [\![v = w]\!].$$

v. Let $z \in dom(w)$. By iv., we have

$$\begin{aligned}[\![u = v]\!] \wedge ![\![u = z]\!] \wedge w(z) &\approx [\![u = v]\!] \wedge [\![u = z]\!] \wedge w(z) \\ &\preceq [\![v = z]\!] \wedge w(z) \\ &\approx ![\![v = z]\!] \wedge w(z).\end{aligned}$$

By taking the supremum over z, we get

$$[\![u = v]\!] \wedge [\![u \in w]\!] \preceq [\![v \in w]\!].$$

vi. Let $x \in dom(u)$. By the definition of $[\![u = v]\!]$, we have

$$[\![u = v]\!] \wedge u(x) \preceq [\![x \in v]\!].$$

This taken together with v. gives

$$\begin{aligned}[\![u = v]\!] \wedge u(x) \wedge ![\![w = x]\!] &\preceq [\![x \in v]\!] \wedge ![\![w = x]\!] \\ &\approx [\![x \in v]\!] \wedge [\![w = x]\!] \\ &\preceq [\![w \in v]\!].\end{aligned}$$

By taking the supremum over x, we get

$$[\![u = v]\!] \wedge [\![w \in u]\!] \preceq [\![w \in v]\!]$$

vii. For all $z \in dom(w)$, we have

$$\begin{aligned}
[\![u = v]\!] \wedge [\![u \notin w]\!] \wedge [\![z = v]\!] &\preceq [\![z = u]\!] \wedge [\![u \notin w]\!] \\
&\leq [\![z = u]\!] \wedge ([\![z = u]\!] \to \sim w(z)) \\
&\preceq \sim w(z)
\end{aligned}$$

and therefore

$$[\![u = v]\!] \wedge [\![u \notin w]\!] \preceq [\![z = u]\!] \to \sim w(z).$$

Thus

$$[\![u = v]\!] \wedge [\![u \notin w]\!] \preceq [\![v \notin w]\!].$$

viii. Fix $y \in dom(v)$. By vii, we have

$$[\![w \notin u]\!] \wedge [\![w = y]\!] \preceq [\![y \notin u]\!],$$

and therefore

$$\begin{aligned}
[\![u = v]\!] \wedge [\![w \notin u]\!] \wedge [\![w = y]\!] &\preceq [\![u = v]\!] \wedge [\![y \notin u]\!] \\
&\preceq \bigwedge_{y' \in dom(v)} \big[[\![y' \notin u]\!] \to \sim v(y')\big] \wedge [\![y \notin u]\!] \\
&\preceq \sim v(y).
\end{aligned}$$

This gives

$$[\![u = v]\!] \wedge [\![w \notin u]\!] \preceq [\![w = y]\!] \to \sim v(y).$$

By taking the infimum over y, we get

$$[\![u = v]\!] \wedge [\![w \notin u]\!] \preceq [\![w \notin v]\!].$$

ix. Fix $x \in dom(u)$. We have

$$\begin{aligned}
[\![u = v]\!] \wedge u(x) \wedge [\![x \notin w]\!] &\preceq [\![x \in v]\!] \wedge [\![x \notin w]\!] \\
&\approx \bigvee_{y \in dom(v)} \big[[\![x = y]\!] \wedge v(y)\big] \wedge [\![x \notin w]\!] \\
&\approx \bigvee_{y \in dom(v)} \big[v(y) \wedge [\![x = y]\!] \wedge [\![x \notin w]\!]\big] \\
&\preceq \bigvee_{y \in dom(v)} \big[v(y) \wedge [\![y \notin w]\!]\big],
\end{aligned}$$

and therefore

$$[\![u = v]\!] \wedge \bigvee_{x \in dom(u)} \big[u(x) \wedge [\![x \notin w]\!]\big] \preceq \bigvee_{y \in dom(v)} \big[v(y) \wedge [\![y \notin w]\!]\big].$$

Now notice that

$$[\![u \neq w]\!] \approx \bigvee_{y \in dom(u)} \big[u(y) \wedge [\![y \notin w]\!]\big] \vee \bigvee_{z \in dom(w)} \big[w(z) \wedge [\![z \notin u]\!]\big]$$

and

$$[\![v \neq w]\!] \approx \bigvee_{y \in dom(v)} \big[v(y) \wedge [\![y \notin w]\!]\big] \vee \bigvee_{z \in dom(w)} \big[w(z) \wedge [\![z \notin v]\!]\big].$$

Taking this together with the above gives

$$[\![u = v]\!] \wedge [\![u \neq w]\!] \preceq [\![v \neq w]\!].$$

□

Proposition 8.3.4. *For every formula $\varphi(x)$ and $u \in V^{(\mathcal{A})}$, the following holds:*

i. $[\![\exists x \in u \varphi(x)]\!] = \bigvee_{x \in dom(u)} \big[u(x) \wedge [\![\varphi(x)]\!]\big]$

ii. $[\![\forall x \in u \varphi(x)]\!] = \bigwedge_{x \in dom(u)} \big[u(x) \to [\![\varphi(x)]\!]\big]$

iii. $[\![\forall x\big(\varphi(x) \to x \notin u\big)]\!] = \bigwedge_{x \in dom(u)} \big[[\![\varphi(x)]\!] \to {\sim} u(x)\big]$

iv. $[\![\forall x\big(x \in u \Rightarrow \varphi(x)\big)]\!] = \bigwedge_{x \in dom(u)} \big[u(x) \Rightarrow [\![\varphi(x)]\!]\big].$

Proof. i. We have

$$\begin{aligned}
[\![\exists x \in u \varphi(x)]\!] &= \bigvee_{v \in V^{(\mathcal{A})}} \big[[\![v \in u]\!] \wedge [\![\varphi(v)]\!]\big] \\
&= \bigvee_{v \in V^{(\mathcal{A})}} \Big[\bigvee_{x \in dom(u)} \big(u(x) \wedge ![\![v = x]\!]\big) \wedge [\![\varphi(v)]\!]\Big] \\
&= \bigvee_{v \in V^{(\mathcal{A})}} \bigvee_{x \in dom(u)} \big[\big(u(x) \wedge ![\![v = x]\!]\big) \wedge [\![\varphi(v)]\!]\big] \\
&= \bigvee_{v \in V^{(\mathcal{A})}} \bigvee_{x \in dom(u)} \big[u(x) \wedge \big(![\![v = x]\!] \wedge [\![\varphi(v)]\!]\big)\big] \\
&= \bigvee_{x \in dom(u)} \big[u(x) \wedge \bigvee_{v \in V^{(\mathcal{A})}} \big(![\![v = x]\!] \wedge [\![\varphi(v)]\!]\big)\big] \\
&= \bigvee_{x \in dom(u)} \big[u(x) \wedge [\![\varphi(x)]\!]\big].
\end{aligned}$$

ii. We have

$$
\begin{aligned}
\bigwedge_{x\in dom(u)} \big[u(x) \to [\![\varphi(x)]\!]\big] &= \bigwedge_{x\in dom(u)} \Big[u(x) \to \bigwedge_{v\in V^{(\mathcal{A})}} \big[[\![x=v]\!] \to [\![\varphi(v)]\!]\big]\Big] \\
&= \bigwedge_{x\in dom(u)} \bigwedge_{v\in V^{(\mathcal{A})}} \big[u(x) \wedge [\![x=v]\!] \to [\![\varphi(v)]\!]\big] \\
&= \bigwedge_{v\in V^{(\mathcal{A})}} \bigwedge_{x\in dom(u)} \big[u(x) \wedge [\![x=v]\!] \to [\![\varphi(v)]\!]\big] \\
&= \bigwedge_{v\in V^{(\mathcal{A})}} \Big[\bigvee_{x\in dom(u)} \big[u(x) \wedge [\![x=v]\!]\big] \to [\![\varphi(v)]\!]\Big] \\
&= \bigwedge_{v\in V^{(\mathcal{A})}} \big[[\![v\in u]\!] \to [\![\varphi(v)]\!]\big] \\
&= [\![\forall x \in u \varphi(x)]\!]
\end{aligned}
$$

iii. We have

$$
\begin{aligned}
\bigwedge_{x\in dom(u)} \big[[\![\varphi(x)]\!] \to {\sim} u(x)\big] &= \bigwedge_{x\in dom(u)} \Big[\bigvee_{v\in V^{(\mathcal{A})}} \big[[\![\varphi(v)]\!] \wedge [\![x=v]\!]\big] \to {\sim} u(x)\Big] \\
&= \bigwedge_{x\in dom(u)} \bigwedge_{v\in V^{(\mathcal{A})}} \Big[\big[[\![\varphi(v)]\!] \wedge [\![x=v]\!]\big] \to {\sim} u(x)\Big] \\
&= \bigwedge_{x\in dom(u)} \bigwedge_{v\in V^{(\mathcal{A})}} \Big[[\![\varphi(v)]\!] \to \big[[\![x=v]\!] \to {\sim} u(x)\big]\Big] \\
&= \bigwedge_{v\in V^{(\mathcal{A})}} \bigwedge_{x\in dom(u)} \Big[[\![\varphi(v)]\!] \to \big[[\![x=v]\!] \to {\sim} u(x)\big]\Big] \\
&= \bigwedge_{v\in V^{(\mathcal{A})}} \Big[[\![\varphi(v)]\!] \to \bigwedge_{x\in dom(u)} \big[[\![x=v]\!] \to {\sim} u(x)\big]\Big] \\
&= \bigwedge_{v\in V^{(\mathcal{A})}} \big[[\![\varphi(v)]\!] \to [\![v \notin u]\!]\big] \\
&= [\![\forall x(\varphi(x) \to x \notin u)]\!].
\end{aligned}
$$

ix. follows from ii. and iii. □

8.4 Models of subalgebras

In this section, we will see that if $\mathcal{A}'$ is a complete subalgebra of $\mathcal{A}$, then $V^{(\mathcal{A}')}$ can be regarded as a submodel of $V^{(\mathcal{A})}$.

Recall that the formulas $x \in y^!$, $x \in y^?$ and $x \in rlm(y)$ are abbreviations for $!(x \in y)$, $?(x \in y)$ and $!(x \in y) \vee ?(x \in y)$, respectively. We note that if

$f : A \to A$ is a function and $u \in V^{(\mathcal{A})}$, then $f \circ u$ is an element of $V^{(\mathcal{A})}$ with the same domain as u. We will write $!u$ instead of $! \circ u$ and so forth.

Lemma 8.4.1. *For all $u, v \in V^{(\mathcal{A})}$,*

$$\begin{aligned}
[\![u \in v^{!}]\!] =& [\![u \in\, !v]\!], \\
[\![u \in v^{?}]\!] =& [\![u \in\, ?v]\!], \text{ and} \\
[\![u \in rlm(v)]\!] =& [\![u \in (!v \vee ?v)]\!].
\end{aligned}$$

Proof. We have

$$\begin{aligned}
[\![u \in v^{!}]\!] &= [\![!(u \in v)]\!] \\
&= ![\![u \in v]\!] \\
&= !\Big(\bigvee_{y \in dom(v)} \big[v(y) \wedge ![\![u = y]\!]\big] \Big) \\
&= \bigvee_{y \in dom(v)} \big[!v(y) \wedge ![\![u = y]\!]\big] \\
&= [\![u \in\, !v]\!].
\end{aligned}$$

So $[\![u \in v^{!}]\!] = [\![u \in\, !v]\!]$, and similarly $[\![u \in v^{?}]\!] = [\![u \in\, ?v]\!]$. The third point easily follows. □

In the following theorem, we are using Definition 6.2.3 for Δ_0 and Σ_1-formulas.

Theorem 8.4.2. *Let $\mathcal{A}'$ be a complete subalgebra of $\mathcal{A}$ and $u_1, ..., u_n \in A'$. If $\varphi(x_1, ..., x_n)$ is a Δ_0-formula, then*

$$[\![\varphi(u_1, ..., u_n)]\!]^{\mathcal{A}'} = [\![\varphi(u_1, ..., u_n)]\!]^{\mathcal{A}},$$

and if $\varphi(x_1, ..., x_n)$ is a Σ_1-formula, then

$$[\![\varphi(u_1, ..., u_n)]\!]^{\mathcal{A}'} \leq [\![\varphi(u_1, ..., u_n)]\!]^{\mathcal{A}}.$$

Proof. First we show that for all $u, v \in \mathcal{A}'$,

$$[\![u \in v]\!]^{\mathcal{A}'} = [\![u \in v]\!]^{\mathcal{A}} \text{ and } [\![u = v]\!]^{\mathcal{A}'} = [\![u = v]\!]^{\mathcal{A}}.$$

We fix $v \in V^{(\mathcal{A}')}$ and take as our induction hypothesis that for all $y \in dom(v)$ and all $u \in V^{(\mathcal{A}')}$,

$$\begin{aligned}
&[\![y \in u]\!]^{\mathcal{A}'} = [\![y \in u]\!]^{\mathcal{A}}, \\
&[\![u \in y]\!]^{\mathcal{A}'} = [\![u \in y]\!]^{\mathcal{A}}, \text{ and} \\
&[\![u = y]\!]^{\mathcal{A}'} = [\![u = y]\!]^{\mathcal{A}}.
\end{aligned}$$

Now, for all $u \in V^{(\mathcal{A}')}$,

$$\begin{aligned} [\![u \in v]\!]^{\mathcal{A}'} &= \bigvee_{y \in dom(v)} v(y) \wedge ![\![u = y]\!]^{\mathcal{A}'} \\ &= \bigvee_{y \in dom(v)} v(y) \wedge ![\![u = y]\!]^{\mathcal{A}} \\ &= [\![u \in v]\!]^{\mathcal{A}} \end{aligned}$$

and

$$\begin{aligned} [\![u = v]\!]^{\mathcal{A}'} &= \bigwedge_{x \in \mathrm{dom}(u)} \big[u(x) \Rightarrow [\![x \in v]\!]^{\mathcal{A}'}\big] \wedge \bigwedge_{y \in \mathrm{dom}(v)} \big[v(y) \Rightarrow [\![y \in u]\!]^{\mathcal{A}'}\big] \\ &= \bigwedge_{x \in \mathrm{dom}(u)} \big[u(x) \Rightarrow [\![x \in v]\!]^{\mathcal{A}}\big] \wedge \bigwedge_{y \in \mathrm{dom}(v)} \big[v(y) \Rightarrow [\![y \in u]\!]^{\mathcal{A}}\big] \\ &= [\![u = v]\!]^{\mathcal{A}}. \end{aligned}$$

The proof now proceeds by an induction on the complexity of $\varphi(x_1, ..., x_n)$. We already have the base case. For the induction step we will only show that if $\psi(x)$ is a formula with parameters from $V^{\mathcal{A}'}$ such that $[\![\psi(x)]\!]^{\mathcal{A}'} \leq [\![\psi(x)]\!]^{\mathcal{A}}$ for all $x \in V^{(\mathcal{A}')}$, then

$$\begin{aligned} [\![\exists x \psi(x)]\!]^{\mathcal{A}'} &\leq [\![\exists x \psi(x)]\!]^{\mathcal{A}}, \\ [\![\exists x \in u \psi(x)]\!]^{\mathcal{A}'} &\leq [\![\exists x \in u \psi(x)]\!]^{\mathcal{A}}, \text{ and} \\ [\![\exists x \in rlm(u) \psi(x)]\!]^{\mathcal{A}'} &\leq [\![\exists x \in rlm(u) \psi(x)]\!]^{\mathcal{A}} \end{aligned}$$

for all $u \in V^{(\mathcal{A}')}$. We have

$$\begin{aligned} [\![\exists x \psi(x)]\!]^{\mathcal{A}'} &= \bigvee_{x \in V^{(\mathcal{A}')}} [\![\psi(x)]\!]^{\mathcal{A}'} \\ &\leq \bigvee_{x \in V^{(\mathcal{A}')}} [\![\psi(x)]\!]^{\mathcal{A}} \\ &\leq \bigvee_{x \in V^{(\mathcal{A})}} [\![\psi(x)]\!]^{\mathcal{A}} \\ &= [\![\exists x \psi(x)]\!]^{\mathcal{A}} \end{aligned}$$

and

$$\begin{aligned} [\![\exists x \in u \psi(x)]\!]^{\mathcal{A}'} &= \bigvee_{x \in dom(u)} \big[u(x) \wedge [\![\psi(x)]\!]^{\mathcal{A}'}\big] \\ &\leq \bigvee_{x \in dom(u)} \big[u(x) \wedge [\![\psi(x)]\!]^{\mathcal{A}}\big] \\ &= [\![\exists x \in u \psi(x)]\!]^{\mathcal{A}}. \end{aligned}$$

Finally, we note that $(!u \vee ?u) \in V^{(\mathcal{A}')}$, and therefore

$$\begin{aligned}
[\![\exists x \in rlm(u)\psi(x)]\!]^{\mathcal{A}'} &= [\![\exists x \in (!u \vee ?u)\psi(x)]\!]^{\mathcal{A}'} \\
&\leq [\![\exists x \in (!u \vee ?u)\psi(x)]\!]^{\mathcal{A}} \\
&\leq [\![\exists x \in rlm(u)\psi(x)]\!]^{\mathcal{A}}.
\end{aligned}$$

□

8.5 The BZF axioms in $V^{(\mathcal{A})}$

In this section, we show that the PZF axioms are true in $V^{(\mathcal{A})}$ and that $V^{(\mathcal{A})}$ is a model of BZF iff $\mathcal{A}$ is the full twist structure over $\mathcal{B}$. We will postpone the proof that $V^{(\mathcal{A})}$ satisfies the axiom of choice to a later section.

Theorem 8.5.1. *The axioms of PZF are true in* $V^{(\mathcal{A})}$.

Proof. Extensionality: Let $u, v \in V^{(\mathcal{A})}$. We have

$$\begin{aligned}
[\![\forall x(x \in u \Leftrightarrow x \in v)]\!] &= [\![\forall x(x \in u \Rightarrow x \in v)]\!] \wedge [\![\forall y(y \in v \Rightarrow y \in u)]\!] \\
&= \bigwedge_{x \in dom(u)} \big[u(x) \Rightarrow [\![x \in v]\!]\big] \wedge \bigwedge_{y \in dom(v)} \big[v(y) \Rightarrow [\![y \in u]\!]\big] \\
&= [\![u = v]\!].
\end{aligned}$$

Hence

$$[\![u = v \Leftrightarrow \forall x(x \in u \Leftrightarrow x \in v)]\!] \approx 1.$$

Seperation: Fix a $u \in V^{(\mathcal{A})}$ and let $\varphi(x)$ be a formula with parameters from $V^{(\mathcal{A})}$. We define $v \in V^{(\mathcal{A})}$ by letting $dom(v) := dom(u)$ and

$$v(x) := u(x) \wedge [\![\varphi(x)]\!]$$

for $x \in dom(u)$. For all $w \in V^{(\mathcal{A})}$,

$$\begin{aligned}
[\![z \in u]\!] \wedge [\![\varphi(z)]\!] &= \bigvee_{x \in dom(u)} \big[u(x) \wedge ![\![z = x]\!]\big] \wedge [\![\varphi(z)]\!] \\
&= \bigvee_{x \in dom(u)} \big[u(x) \wedge [\![\varphi(z)]\!] \wedge ![\![z = x]\!]\big] \\
&= \bigvee_{x \in dom(u)} \big[u(x) \wedge [\![\varphi(x)]\!] \wedge ![\![z = x]\!]\big] \\
&= [\![z \in v]\!].
\end{aligned}$$

Classical supersets: Fix $u \in V^{(\mathcal{A})}$. Notice that the sentence $\forall y \forall x[x \in y \Rightarrow !(x \in y) \vee ?(x \in y)]$ is provable in $BS4$, so

$$V^{(\mathcal{A})} \vDash_{Tw} \forall x[x \in u \Rightarrow !(x \in u) \vee ?(x \in u)].$$

We let $v := !u \vee ?u$ and get $[\![z \in v]\!] = [\![z \in rlm(u)]\!]$ for all $z \in V^{(\mathcal{A})}$. It follows that

$$V^{(\mathcal{A})} \vDash_{Tw} Cl(v) \wedge u \subseteq v.$$

Replacement: We will show that $V^{(\mathcal{A})}$ validates the schema

$$\forall u[\forall x \in u \exists y \varphi(x, y) \to \exists v \forall x \in u \exists y \in v \varphi(x, y)],$$

and leave it as an exercise to show that this implies the axiom schema of replacement.

Fix $u \in V^{(\mathcal{A})}$ and let $\varphi(x, y)$ be a formula with parameters from $V^{(\mathcal{A})}$. For each $x \in dom(u)$, the definable class $D_x := \{[\![\varphi(x, y)]\!] : y \in V^{(\mathcal{A})}\}$ is a subset of A and is therefore a set. For all $a \in D_x$, there is a $y \in V^{(\mathcal{A})}$ such that $a = [\![\varphi(x, y)]\!]$. There is therefore an ordinal α_x such that for all $a \in D_x$, there is a $y \in V^{(\mathcal{A})}_{\alpha_x}$ with $a = [\![\varphi(x, y)]\!]$. In particular, this means that

$$\bigvee_{y \in V^{(\mathcal{A})}} [\![\varphi(x, y)]\!] = \bigvee_{y \in V^{(\mathcal{A})}_{\alpha_x}} [\![\varphi(x, y)]\!].$$

Taking $\alpha := \bigcup\{\alpha_x : x \in dom(u)\}$ we get that for all $x \in dom(u)$,

$$\bigvee_{y \in V^{(\mathcal{A})}} [\![\varphi(x, y)]\!] = \bigvee_{y \in V^{(\mathcal{A})}_{\alpha}} [\![\varphi(x, y)]\!].$$

We define $v \in V^{(\mathcal{A})}$ by letting $dom(v) := V^{(\mathcal{A})}_\alpha$ and $v(y) := 1$ for all $y \in V^{(\mathcal{A})}_\alpha$. Now,

$$\begin{aligned}
[\![\forall x \in u \exists y \varphi(x, y)]\!] &= \bigwedge_{x \in dom(u)} \Big[u(x) \to \bigvee_{y \in V^{(\mathcal{A})}} [\![\varphi(x, y)]\!]\Big] \\
&= \bigwedge_{x \in dom(u)} \Big[u(x) \to \bigvee_{y \in V^{(\mathcal{A})}_\alpha} [\![\varphi(x, y)]\!]\Big] \\
&= \bigwedge_{x \in dom(u)} \Big[u(x) \to \bigvee_{y \in dom(v)} v(y) \wedge [\![\varphi(x, y)]\!]\Big] \\
&= [\![\forall x \in u \exists y \in v \varphi(x, y)]\!] \\
&\leq [\![\exists v \forall x \in u \exists y \in v \varphi(x, y)]\!].
\end{aligned}$$

Union: Fix $u \in V^{(\mathcal{A})}$. We define $v \in V^{(\mathcal{A})}$ by letting $dom(v) := \bigcup_{y \in dom(u)} dom(y)$ and $v(x) := 1$ for all $x \in dom(v)$. Then

$$\begin{aligned}
[\![\forall x[\exists y(x \in y \wedge y \in u) \to x \in v]]\!] &= [\![\forall y \in u \forall x \in y(x \in v)]\!] \\
&= \bigwedge_{y \in dom(u)} \bigwedge_{x \in dom(y)} \Big[u(y) \wedge y(x) \to [\![x \in v]\!]\Big] \\
&\geq \bigwedge_{y \in dom(u)} \bigwedge_{x \in dom(y)} \Big[u(y) \wedge y(x) \to v(x)]\!]\Big]
\end{aligned}$$

$$= 1.$$

We have
$$V^{(\mathcal{A})} \vDash_{Tw} \forall u \exists v \forall x [\exists y (x \in y \wedge y \in u) \rightarrow x \in v].$$

Using the axioms of classical supersets, replacement and seperation, we get

$$V^{(\mathcal{A})} \vDash_{Tw} \text{“}\bigcup \{rlm(x) : x \in rlm(u)\} \text{ is a set for all } u\text{.”}$$

Finally, separation gives

$$V^{(\mathcal{A})} \vDash_{Tw} \forall u \exists v \forall x [x \in v \Leftrightarrow \exists y \in u (x \in y)].$$

Pairing: For $u, v \in V^{(\mathcal{A})}$, we let $w := \{u, v\} \times \{1\}$. Clearly,

$$[\![u \in w \wedge v \in w]\!] = 1.$$

Classical power set: We will show that

$$V^{(\mathcal{A})} \vDash_{Tw} \forall u \exists v \forall x [x \subseteq u \rightarrow x \in v],$$

and leave it as an exercise to show that this implies the axiom of classical power set.

Fix $u \in V^{(\mathcal{A})}$. We define $v \in V^{(\mathcal{A})}$ by letting
$dom(v) := \{x \in V^{(\mathcal{A})} : dom(x) = dom(u)\}$ and $v(x) := 1$ for $x \in dom(v)$.
Let $x \in V^{(\mathcal{A})}$. Our goal is to show that

$$[\![x \subseteq u]\!] \preceq [\![x \in v]\!].$$

We let $x' \in V^{(\mathcal{A})}$ be given by $dom(x') := dom(u)$ and $x'(y) := [\![y \in x]\!]$ for all $y \in dom(u)$. We easily see that

$$[\![x' \in v]\!] = 1 \text{ and } [\![x' \subseteq x]\!] \approx 1.$$

We also have

$$\begin{aligned}
[\![y \in u \wedge y \in x]\!] &= \bigvee_{z \in dom(u)} \big[[\![y = z]\!] \& u(z)\big] \wedge [\![y \in x]\!] \\
&= \bigvee_{z \in dom(u)} \big[![\![y = z]\!] \wedge u(z)\big] \wedge [\![y \in x]\!] \\
&\leq \bigvee_{z \in dom(u)} \big[![\![y = z]\!]\big] \wedge [\![y \in x]\!] \\
&= \bigvee_{z \in dom(u)} \big[![\![y = z]\!] \wedge [\![z \in x]\!]\big] \\
&= [\![y \in x']\!]
\end{aligned}$$

and therefore
$$[\![u \cap x \subseteq x']\!] \approx 1.$$

Finally,

$$\begin{aligned} [\![x \subseteq u]\!] &\preceq [\![x = x']\!] \wedge [\![x' \in v]\!] \\ &\preceq [\![x \in v]\!]. \end{aligned}$$

Foundation: Let $\varphi(x)$ be a formula with parameters from $V^{(\mathcal{A})}$ and put

$$a := [\![\forall x[\forall y \in rlm(x)\varphi(y) \to \varphi(x)]]\!].$$

We want to show that

$$a \preceq [\![\varphi(x)]\!]$$

for all $x \in V^{(\mathcal{A})}$. Fix x and assume that $a \preceq [\![\varphi(y)]\!]$ for all $y \in dom(x)$. Then

$$\begin{aligned} a &\preceq \bigwedge_{y \in dom(x)} [\![\varphi(y)]\!] \\ &\leq \bigwedge_{y \in dom(x)} \big[(!x(y) \vee ?x(y)) \to [\![\varphi(y)]\!]\big] \\ &= [\![\forall y \in rlm(x)\varphi(y)]\!]. \end{aligned}$$

We also have

$$a \leq [\![\forall y \in rlm(x)\varphi(y)]\!] \to [\![\varphi(x)]\!]$$

by the definition of a. Hence

$$a \preceq [\![\varphi(x)]\!].$$

Infinity: Note that the sentence $\exists u[\emptyset \in u \wedge \forall x \in u \exists y \in u(x \in y)]$ is a Σ_1-formula. Since $\mathcal{B}$ is a complete subalgebra of $\mathcal{A}$ and $V^{(\mathcal{B})} \vDash_{Tw} \exists u[\emptyset \in u \wedge \forall x \in u \exists y \in u(x \in y)]$, we get

$$V^{(\mathcal{A})} \vDash_{Tw} \exists u[\emptyset \in u \wedge \forall x \in u \exists y \in u(x \in y)].$$

□

Now that we have seen that the PZF axioms hold in $V^{(\mathcal{A})}$, the question becomes: what about the BZF axioms? In other words, when does the $AClA$ hold in $V^{(\mathcal{A})}$? Clearly, we cannot expect it to hold for every twist structure since it fails in $V^{(\mathcal{B})}$. On the other hand, it should be just as clear that if $\mathcal{A}$ is the full twist structure over $\mathcal{B}$, then the $AClA$ is true in $V^{(\mathcal{A})}$. The following theorem tells us that the $AClA$ holds in $V^{(\mathcal{A})}$ just in the case that $\mathcal{A}$ is the full twist structure over $\mathcal{B}$.

Theorem 8.5.2. *$V^{(\mathcal{A})} \vDash_{Tw} AClA$ iff $\mathcal{A} = \mathcal{B}^{\bowtie}$.*

Proof. We only show that $V^{(\mathcal{A})} \vDash_{Tw} AClA$ implies $\mathcal{A} = \mathcal{B}^{\bowtie}$.

Assume that $V^{(\mathcal{A})} \vDash_{Tw} AClA$, and let

$$\begin{aligned} \varphi &:= \exists x, y[Com(y) \wedge x \in y \wedge x \notin y] \text{ and} \\ \psi &:= \forall x, y[Con(y) \to (x \in y \vee x \notin y)]. \end{aligned}$$

It is then straightforward to show $BZF \vdash_{BS4} \varphi \wedge \sim\varphi$ and $BZF \vdash_{BS4} \neg(\psi \vee \sim\psi)$. It follows that $[\![\varphi]\!]^{\mathcal{A}} = \mathfrak{b}$ and $[\![\varphi]\!]^{\mathcal{A}} = \mathfrak{n}$, and therefore $\mathfrak{b}, \mathfrak{n} \in A$.

Now, for all $(x, y) \in B \times B$,

$$\begin{aligned}(x, y) &= (x, 1) \vee (0, y) \\ &= [(x, \neg x) \wedge \mathfrak{b}] \vee [(\neg y, y) \wedge \mathfrak{n}] \\ &\in A.\end{aligned}$$

Hence $\mathcal{A} = \mathcal{B}^{\bowtie}$. □

8.6 Standard elements and the ordinals in $V^{(\mathcal{A})}$

Definition 8.6.1. For each $x \in V$, we let

$$\hat{x} := \{(\hat{y}, 1)\}.$$

The elements of the form $\hat{x}$ are called the *standard elements of* $V^{(\mathcal{A})}$.

It can be shown (see [4, Theorem 1.23]) that the map $x \mapsto \hat{x}$ is an injection from V into $V^{(2)}$, and for each formula $\varphi(x_1, ..., x_n)$ and all $a_1, ..., a_n$,

$$\varphi(a_1, ..., a_n) \text{ iff } V^{(2)} \vDash_{Tw} \varphi(\hat{a}_1, ..., \hat{a}_n).$$

It follows that $x \mapsto \hat{x}$ is an injection from V to $V^{(\mathcal{A})}$ and that for all $x, y \in V$, both $[\![\hat{x} \in \hat{y}]\!]^{\mathcal{A}}$ and $[\![\hat{x} = \hat{y}]\!]^{\mathcal{A}}$ are elements of the Boolean algebra 2.

Proposition 8.6.2. *For all $x \in V$ and $u \in V^{(\mathcal{A})}$,*

$$[\![u \in \hat{x}]\!] = \bigvee_{y \in x} ![\![u = \hat{y}]\!].$$

Proof. We have

$$\begin{aligned}[\![u \in \hat{x}]\!] &= \bigvee_{y \in x} \hat{x}(\hat{y}) \wedge ![\![u = \hat{y}]\!] \\ &= \bigvee_{y \in x} ![\![u = \hat{y}]\!].\end{aligned}$$

□

The following proposition tells us that the class $\{\hat{\alpha} : \alpha \in Ord\} \subseteq V^{(2)}$ represents the class of ordinals in $V^{(\mathcal{A})}$ in a natural way.

Proposition 8.6.3. *For all $u \in V^{(\mathcal{A})}$,*

$$[\![u \in Ord]\!] = \bigvee_{\alpha \in Ord} ![\![u = \hat{\alpha}]\!].$$

Proof. We have

$$
\begin{aligned}
\bigvee_{\alpha \in Ord} ![\![u = \hat{\alpha}]\!] &= \bigvee_{\alpha \in Ord} [\![\hat{\alpha} \in Ord]\!] \wedge ![\![u = \hat{\alpha}]\!] \\
&\leq \bigvee_{\alpha \in Ord} [\![u \in Ord]\!] \\
&= [\![u \in Ord]\!].
\end{aligned}
$$

For each $x \in dom(u)$, we let

$$D_x := \{\beta \in Ord : ![\![\hat{\beta} = x]\!] \neq 0\}.$$

Let us for a moment fix some $x \in dom(u)$. We define a function $f : D_x \to A$ by letting

$$f(\beta) := [\![\hat{\beta} = x]\!].$$

I claim that f is an injection: Suppose that $\beta_1, \beta_2 \in D_x$ such that $f(\beta_1) = f(\beta_2)$. Then

$$
\begin{aligned}
0 &\neq ![\![\hat{\beta}_1 = x]\!] \\
&= ![\![\hat{\beta}_1 = x]\!] \wedge ![\![\hat{\beta}_2 = x]\!] \\
&\leq ![\![\hat{\beta}_1 = \hat{\beta}_2]\!] \\
&= [\![\hat{\beta}_1 = \hat{\beta}_2]\!].
\end{aligned}
$$

Since $[\![\hat{\beta}_1 = \hat{\beta}_2]\!] \in \{0, 1\}$, we can conclude that $\beta_1 = \beta_2$. We have shown that $f : D_x \to A$ is an injection, so D_x is a set for all $x \in dom(u)$.

We let β be an ordinal not in $\bigcup_{x \in dom(u)} D_x$, and see that $[\![\hat{\beta} = x]\!] = 0$ for all $x \in dom(u)$. Hence

$$[\![\hat{\beta} \in u]\!] = \bigvee_{x \in dom(u)} \left[u(x) \wedge ![\![\hat{\beta} = x]\!]\right] = 0.$$

It is easy to check that

$$PZF \vdash_{BS4} \forall x, y\, (x \in Ord \wedge y \in Ord \Rightarrow !(x \in y) \vee !(x = y) \vee !(y \in x)).$$

Thus

$$
\begin{aligned}
[\![u \in Ord]\!] &\leq ![\![u \in \hat{\beta}]\!] \vee ![\![u = \hat{\beta}]\!] \vee ![\![\hat{\beta} \in u]\!] \\
&= ![\![u = \hat{\beta}]\!] \vee ![\![u = \hat{\beta}]\!] \\
&= \bigvee_{\alpha \in \beta} ![\![u \in \hat{\alpha}]\!] \vee ![\![u = \hat{\beta}]\!] \\
&\leq \bigvee_{\alpha \in Ord} ![\![u = \hat{\alpha}]\!].
\end{aligned}
$$

□

Proposition 8.6.4. *If $\varphi(x)$ is formula, then*

$$[\![\exists\alpha \in Ord\ \varphi(\alpha)]\!] = \bigvee_{\alpha\in Ord} [\![\varphi(\hat{\alpha})]\!], \text{ and}$$
$$[\![\forall\alpha \in Ord\ \varphi(\alpha)]\!] = \bigwedge_{\alpha\in Ord} [\![\varphi(\hat{\alpha})]\!].$$

Proof. We have

$$\begin{aligned}[\![\exists\alpha \in Ord\ \varphi(\alpha)]\!] &= \bigvee_{u\in V^{(\mathcal{A})}} \left[[\![u \in Ord]\!] \wedge [\![\varphi(u)]\!]\right] \\ &= \bigvee_{u\in V^{(\mathcal{A})}} \bigvee_{\alpha\in Ord} \left[![\![u = \hat{\alpha}]\!] \wedge [\![\varphi(u)]\!]\right] \\ &= \bigvee_{\alpha\in Ord} [\![\exists u(\varphi(u) \wedge !(u = \hat{\alpha}))]\!] \\ &= \bigvee_{\alpha\in Ord} [\![\varphi(\hat{\alpha})]\!].\end{aligned}$$

The proof of the second equality is similar. □

8.7 Hereditarily classical sets in $V^{(\mathcal{A})}$

In Section 8.4, we saw that if $\mathcal{A}'$ is a complete subalgebra of $\mathcal{A}$, then $V^{(\mathcal{A}')}$ can be regarded as a submodel of $V^{(\mathcal{A})}$. In particular, this tells us that we can regard $V^{(\mathcal{B})}$ as a submodel of $V^{(\mathcal{A})}$. The following theorem tells us that $V^{(\mathcal{B})}$ represents HCl in $V^{(\mathcal{A})}$ in natural way.

Theorem 8.7.1. *For all $u \in V^{(\mathcal{A})}$,*

$$[\![u \in HCl]\!]^{\mathcal{A}} = \bigvee_{v\in V^{(\mathcal{B})}} ![\![u = v]\!]^{\mathcal{A}}.$$

Proof. We let $\Phi(\alpha)$ be the property

$$[\![u \in HCl_{\hat{\alpha}}]\!]^{\mathcal{A}} = \bigvee_{v\in V_{\hat{\alpha}}^{(\mathcal{B})}} ![\![u = v]\!]^{\mathcal{A}}$$

and show by induction that $\Phi(\alpha)$ for all $\alpha \in Ord$.

For $\Phi(0)$, we have

$$[\![u \in HCl_{\hat{0}}]\!]^{\mathcal{A}} = 0 = \bigvee_{v\in V_{\hat{0}}^{(\mathcal{B})}} ![\![u = v]\!]^{\mathcal{A}}.$$

Let λ be a limit ordinal such that such that $\Phi(\alpha)$ for all $\alpha \in \lambda$. Since the statement “x is a limit ordinal” is Δ_0, we get

$$[\![u \in HCl_{\hat{\lambda}}]\!] = [\![\exists\alpha \in \hat{\lambda}(u \in HCl_\alpha)]\!]$$

$$
\begin{aligned}
&= \bigvee_{\alpha\in\lambda} [\![u \in HCl_{\hat{\alpha}}]\!] \\
&= \bigvee_{\alpha\in\lambda} \bigvee_{v\in V_\alpha^{(\mathcal{B})}} ![\![u = v]\!] \\
&= \bigvee_{v\in V_\alpha^{(\mathcal{B})}} ![\![u = v]\!].
\end{aligned}
$$

We now let $\alpha \in Ord$ such that $\Phi(\alpha)$ and wish to show that $\Phi(\alpha+1)$. We define $w \in V_{\alpha+1}^{(\mathcal{B})}$ by letting $dom(w) := V_\alpha^{(\mathcal{B})}$ and for all $v \in V_\alpha^{(\mathcal{B})}$,

$$w(v) := ![\![v \in u]\!]^{\mathcal{A}}.$$

We note that the formula $x \in Ord \wedge y \in Ord \wedge y = x+1$ is Δ_0 and $PZF \vdash_{BS4} \forall x(Cl(x) \Rightarrow x = x^!)$. This gives

$$[\![u \in HCl_{\widehat{\alpha+1}}]\!] = [\![u \in HCl_{\hat{\alpha}+1}]\!]$$

and

$$[\![Cl(u)]\!] \leq [\![u = u^!]\!].$$

Now,

$$
\begin{aligned}
[\![u \in HCl_{\hat{\alpha}+1}]\!] &= [\![Cl(u) \wedge u \subseteq HCl_{\hat{\alpha}}]\!] \\
&\leq [\![u = u^!]\!] \wedge [\![u \subseteq HCl_{\hat{\alpha}}]\!] \\
&= [\![u = u^!]\!] \wedge \bigwedge_{x\in dom(u)} \left[u(x) \Rightarrow [\![x \in HCl_{\hat{\alpha}}]\!]\right] \\
&= [\![u = u^!]\!] \wedge \bigwedge_{x\in dom(u)} \left[u(x) \Rightarrow \bigvee_{v\in V_\alpha^{(\mathcal{B})}} ![\![x = v]\!]\right] \\
&= [\![u^! \subseteq u]\!] \wedge [\![u \subseteq u^!]\!] \wedge \bigwedge_{x\in dom(u)} \left[u(x) \Rightarrow \bigvee_{v\in V_\alpha^{(\mathcal{B})}} ![\![x = v]\!]\right] \\
&= [\![u^! \subseteq u]\!] \wedge \bigwedge_{x\in dom(u)} \left[u(x) \Rightarrow \bigvee_{v\in V_\alpha^{(\mathcal{B})}} \left[![\![x \in u]\!] \wedge ![\![x = v]\!]\right]\right] \\
&= [\![u^! \subseteq u]\!] \wedge \bigwedge_{x\in dom(u)} \left[u(x) \Rightarrow \bigvee_{v\in V_\alpha^{(\mathcal{B})}} \left[![\![v \in u]\!] \wedge ![\![x = v]\!]\right]\right] \\
&= [\![u^! \subseteq u]\!] \wedge [\![u \subseteq w]\!] \\
&= \bigwedge_{v\in V^{(\mathcal{A})}} \left[![\![v \in u]\!] \Rightarrow [\![v \in u]\!]\right] \wedge [\![u \subseteq w]\!] \\
&\leq \bigwedge_{v\in V_{\hat{\alpha}}^{(\mathcal{B})}} \left[![\![v \in u]\!] \Rightarrow [\![v \in u]\!]\right] \wedge [\![u \subseteq w]\!] \\
&= [\![u = w]\!].
\end{aligned}
$$

Since $PZF \vdash_{BS4} Cl(HCl_\beta)$ for all $\beta \in Ord$, we also get

$$\begin{aligned}\llbracket u \in HCl_{\hat{\alpha}+1} \rrbracket &= !\llbracket u \in HCl_{\hat{\alpha}+1} \rrbracket \\ &\leq !\llbracket u = w \rrbracket.\end{aligned}$$

Thus

$$\llbracket u \in HCl_{\hat{\alpha}+1} \rrbracket \leq \bigvee_{v \in V^{(\mathcal{B})}_{\alpha+1}} !\llbracket u = v \rrbracket.$$

For the reverse inequality, we note that for all $v \in V^{(\mathcal{B})}_{\alpha+1}$,

$$\begin{aligned}\llbracket v \in HCl_{\hat{\alpha}+1} \rrbracket &= \llbracket Cl(v) \rrbracket \wedge \llbracket v \subseteq HCl_{\hat{\alpha}} \rrbracket \\ &= \llbracket v \subseteq HCl_{\hat{\alpha}} \rrbracket \\ &= \bigwedge_{y \in dom(v)} \Big[v(y) \Rightarrow \bigvee_{v' \in V^{(\mathcal{B})}_{\alpha}} !\llbracket y = v' \rrbracket \Big] \\ &= 1\end{aligned}$$

and

$$\begin{aligned}!\llbracket u = v \rrbracket &= !\llbracket u = v \rrbracket \wedge \llbracket w \in HCl_{\hat{\alpha}+1} \rrbracket \\ &\leq \llbracket u \in HCl_{\hat{\alpha}+1} \rrbracket.\end{aligned}$$

We conclude that

$$\bigvee_{v \in V^{(\mathcal{B})}_{\alpha+1}} !\llbracket u = v \rrbracket \leq \llbracket u \in HCl_{\hat{\alpha}+1} \rrbracket.$$

□

Proposition 8.7.2. *If $\varphi(x)$ is formula, then*

$$\begin{aligned}\llbracket \exists u \in HCl\ \varphi(u) \rrbracket^{\mathcal{A}} &= \bigvee_{u \in V^{(\mathcal{B})}} \llbracket \varphi(u) \rrbracket^{\mathcal{A}}, \text{ and} \\ \llbracket \forall u \in HCl\ \varphi(u) \rrbracket^{\mathcal{A}} &= \bigwedge_{u \in V^{(\mathcal{B})}} \llbracket \varphi(u) \rrbracket^{\mathcal{A}}.\end{aligned}$$

Proof. Similar to the proof of Proposition 8.6.4. □

8.8 The axiom of choice holds in $V^{(\mathcal{A})}$

The work done in the previous section will help to give a relatively simple proof that the axiom of choice holds in $V^{(\mathcal{A})}$. But first, we need one more result of PZF.

Lemma 8.8.1 (In PZF)**.** *If every hereditarily classical set of non-empty sets has a choice function, then the axiom of choice is true.*

Proof. We will show that for each classical set a, there is a hereditarily classical set b and a bijection from a to b. The result will then follow by Proposition 4.13.3.

Just as in the proof of Theorem 6.4.4, we can define a function $\mu : V \to HCl$ by letting

$$\mu(x) := (\mu[x^!], \mu[x^?])$$

for all $x \in V$. By a simple induction on the rank of sets, we see that that μ is an injection. It follows that if a is classical set, then $\mu \restriction a$ is a bijection from a to $\mu[a] \in HCl$. $\square$

Theorem 8.8.2. *The axiom of choice is true in $V^{(\mathcal{A})}$.*

Proof. We let $\varphi(x)$ be the formula

$$\forall y \in x \exists z (z \in y) \to \exists (f : x \to V) \forall y \in x (f(y) \in y).$$

By the previous lemma, it suffices to show

$$V^{(\mathcal{A})} \vDash_{Tw} \forall u \in HCl \ \varphi(u).$$

Notice that the formula $\forall y \in x \exists z (z \in y)$ is Δ_0, and the formula $\exists (f : x \to V) \forall y \in x (f(y) \in y)$ is Σ_1. So $[\![\varphi(u)]\!]^{\mathcal{B}} \leq [\![\varphi(u)]\!]^{\mathcal{A}}$ for all $u \in V^{(\mathcal{B})}$. Now,

$$\begin{aligned} [\![\forall u \in HCl \ \varphi(u)]\!]^{\mathcal{A}} &= \bigwedge_{u \in V^{(\mathcal{B})}} [\![\varphi(u)]\!]^{\mathcal{A}} \\ &\geq \bigwedge_{u \in V^{(\mathcal{B})}} [\![\varphi(u)]\!]^{\mathcal{B}} \\ &= [\![\forall u \varphi(u)]\!]^{\mathcal{B}} \\ &= 1. \end{aligned}$$

$\square$

Conclusion

In this thesis, we developed a set theory that is able to represent both inconsistent and incomplete information in an intuitive way. We reformulated the familiar ZFC axioms in the four-valued paradefinite logic $BS4$ to obtain the theory $PZFC$. We then added the powerful anti-classically axiom to get the theory $BZFC$, which allows us represent both inconsistent and incomplete information by using inconsistent and incomplete sets.

We provided a T/F-model W of $BZFC$ starting from the classical ZFC axioms, and showed that a sentence is a theorem of $BZFC$ if and only if it holds in W. On the other hand, starting from the $PZFC$ axioms, we provided the class HCl of hereditarily classical sets, which interprets the classical ZFC axioms, and showed that sentence is a theorem of ZFC if and only if it holds in HCl. These two results allow us to translate back and forth between the theories ZFC and $BZFC$.

As an application of $BZFC$, we used non-classical sets to give natural semantics for $BS4$.

Finally, we generalized the construction of Boolean-valued models for ZFC to obtain algebra-valued models of the theories $PZFC$ and $BZFC$.

Bibliography

[1] Aczel, P. *Non-well-founded sets*. Center for the Study of Language and Information, Stanford, CA, 1988.

[2] Arieli, O., and Avron, A. Four-valued paradefinite logics. *Studia Logica 105*, 6 (2017), 1087–1122.

[3] Avron, A. Natural 3-valued logics—characterization and proof theory. *J. Symbolic Logic 56*, 1 (1991), 276–294.

[4] Bell, J. L. *Set theory*, third ed., vol. 47 of *Oxford Logic Guides*. The Clarendon Press, Oxford University Press, Oxford, 2005. Boolean-valued models and independence proofs.

[5] Belnap, N. D. How a computer should think. In *Contemporary Aspects of Philosophy*, G. Ryle, Ed. Oriel Press, Stocksfield, 1976, pp. 30–56.

[6] Belnap, N. D. A useful four-valued logic. In *Modern Uses of Multiple-Valued Logic*, J. M. Dunn and G. Epstein, Eds. Springer Netherlands, Dordrecht, 1977, pp. 5–37.

[7] Blamey, S. Partial logic. In *Handbook of Philosophical Logic*, D. M. Gabbay and F. Guenthner, Eds., vol. 5. Springer Netherlands, Dordrecht, 2002, pp. 261–353.

[8] Carnielli, W., and Coniglio, M. Twist-valued models for three-valued paraconsistent set theory. *Logic and Logical Philosophy 0*, 0 (2020), 1–40.

[9] Coniglio, M. E., Figallo-Orellano, A., and Golzio, A. C. First-order swap structures semantics for some logics of formal inconsistency. *Journal of Logic and Computation 30*, 6 (06 2020), 1257–1290.

[10] Fidel, M. M. An algebraic study of a propositional system of Nelson. In *Mathematical logic (Proc. First Brazilian Conf., State Univ. Campinas, Campinas, 1977)* (1978), vol. 39 of *Lecture Notes in Pure and Appl. Math.*, Dekker, New York, pp. 99–117.

[11] Gilmore, P. C. The consistency of partial set theory without extensionality. In *Axiomatic set theory (Proc. Sympos. Pure Math., Vol. XIII, Part II, Univ. California, Los Angeles, Calif., 1967)* (1974), pp. 147–153.

[12] Grayson, R. J. Heyting-valued models for intuitionistic set theory. In *Applications of sheaves (Proc. Res. Sympos. Appl. Sheaf Theory to Logic, Algebra and Anal., Univ. Durham, Durham, 1977)* (1979), vol. 753 of *Lecture Notes in Math.*, Springer, Berlin, pp. 402–414.

[13] Hazen, A. P., and Pelletier, F. J. K3, Ł3, LP, RM3, A3, FDE, M: How to make many-valued logics work for you. In *New Essays on Belnap-Dunn Logic*, H. Omori and H. Wansing, Eds. Springer International Publishing, Cham, 2019, pp. 155–190.

[14] Hinnion, R. Naive set theory with extensionality in partial logic and in paradoxical logic. *Notre Dame Journal of Formal Logic 35* (01 1994).

[15] Hinnion, R. About the coexistence of "classical sets" with "non-classical" ones: A survey. *Logic and Logical Philosophy 0*, 11-12 (2003), 79–90.

[16] Kleene, S. C. On notation for ordinal numbers. *Journal of Symbolic Logic 3*, 4 (1938), 150–155.

[17] Libert, T. Models for a paraconsistent set theory. *Journal of Applied Logic 3*, 1 (2005), 15–41. A Paraconsistent Decagon.

[18] Loparić, A., and da Costa, N. C. A. Paraconsistency, paracompleteness, and valuations. *Logique et Analyse 27*, 106 (1984), 119–131.

[19] Löwe, B., and Tarafder, S. Generalized algebra-valued models of set theory. *The Review of Symbolic Logic 8*, 1 (2015), 192–205.

[20] Nelson, D. Constructible falsity. *The Journal of Symbolic Logic 14*, 1 (1949), 16–26.

[21] Odintsov, S. *Constructive Negations and Paraconsistency*, vol. 26 of *Trends in Logic*. Springer Netherlands, Dordrecht, 2008.

[22] Odintsov, S. P. Algebraic Semantics for Paraconsistent Nelson's Logic. *Journal of Logic and Computation 13*, 4 (08 2003), 453–468.

[23] Omori, H., and Waragai, T. Some observations on the systems LFI1 and LFI1. In *2011 22nd International Workshop on Database and Expert Systems Applications* (2011), pp. 320–324.

[24] Priest, G. Paraconsistent logic. In *Handbook of Philosophical Logic*, D. M. Gabbay and F. Guenthner, Eds., vol. 6. Springer Netherlands, Dordrecht, 2002, pp. 287–393.

[25] Rasiowa, H. $\mathcal{N}$-lattices and constructive logic with strong negation. *Fund. Math. 46* (1958), 61–80.

[26] Rasiowa, H. *An Algebraic Approach to Non-classical Logics*. Studies in logic and the foundations of mathematics. North-Holland Publishing Company, 1974.

[27] Sano, K., and Omori, H. An expansion of first-order Belnap–Dunn logic. *Logic Journal of the IGPL 22*, 3 (11 2013), 458–481.

[28] Schneider, H. H. Some remarks concerning a definition of ordered pairs. *The American Mathematical Monthly 84*, 8 (1977), 636–638.

[29] Shulman, M. Linear logic for constructive mathematics, 2018. arXiv:1805.07518 [math.LO].

[30] Vakarelov, D. Notes on $\mathcal{N}$-lattices and constructive logic with strong negation. *Studia Logica 36*, 1-2 (1977), 109–125.

Appendix A

Ordered Pairs

We work in PZF throughot this appendix. Our aim is to show that for all u, v, z and w,

$$(u, v) = (z, w) \Leftrightarrow u = z \wedge v = w.$$

Definition A.0.1. We say that an operation $\varphi(x, y)$ is an injective operation if

$$\forall x, x', y[\varphi(x, y) \wedge \varphi(x', y) \rightarrow !(x = x')].$$

Our strategy will be to show tha show that if $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are injective operations such that $\mathfrak{F}_1[V] \cap \mathfrak{F}_2[V] = \emptyset$, then defining (u, v) as $\mathfrak{F}_1[u] \cup \mathfrak{F}_2[v]$ satisfies our requirement.

Lemma A.0.2. *If $\mathfrak{F}$ is an injective operation, then for all u and v*

$$u = v \Leftrightarrow \mathfrak{F}[u] = \mathfrak{F}[v].$$

Proof. First note that $\mathfrak{F}$ being injective implies that for all x,

$$x \in u \Leftrightarrow \mathfrak{F}(x) \in \mathfrak{F}[u]$$

and

$$x \in v \Leftrightarrow \mathfrak{F}(x) \in \mathfrak{F}[v].$$

That $\mathfrak{F}$ is injective also implies

$$\mathfrak{F}[u] = \mathfrak{F}[v] \Leftrightarrow \forall x(\mathfrak{F}(x) \in \mathfrak{F}[u] \Leftrightarrow \mathfrak{F}(x) \in \mathfrak{F}[v]).$$

We now have

$$\begin{aligned} u = v &\Leftrightarrow \forall x(x \in u \Leftrightarrow x \in v) \\ &\Leftrightarrow \forall x(\mathfrak{F}(x) \in \mathfrak{F}[u] \Leftrightarrow \mathfrak{F}(x) \in \mathfrak{F}[v]) \\ &\Leftrightarrow \mathfrak{F}[u] = \mathfrak{F}[v]. \end{aligned}$$

□

Lemma A.0.3. *If* $\mathfrak{F}_1$ *and* $\mathfrak{F}_2$ *are definable injective operations such that* $\mathfrak{F}_1[V] \cap \mathfrak{F}_2[V] = \emptyset$*, then for all* u*,* v*,* z *and* w*,*

$$\mathfrak{F}_1[u] \cup \mathfrak{F}_2[v] = \mathfrak{F}_1[z] \cup \mathfrak{F}_2[w] \Leftrightarrow u = z \wedge v = w.$$

Proof. If $u = z$ and $v = w$, then, clearly, $\mathfrak{F}_1[u] \cup \mathfrak{F}_2[v] = \mathfrak{F}_1[z] \cup \mathfrak{F}_2[w]$.

If $\mathfrak{F}_1[u] \cup \mathfrak{F}_2[v] = \mathfrak{F}_1[z] \cup \mathfrak{F}_2[w]$, then

$$\begin{aligned} \mathfrak{F}_1[u] &= \mathfrak{F}_1[V] \cap (\mathfrak{F}_1[u] \cup \mathfrak{F}_2[v]) \\ &= \mathfrak{F}_1[V] \cap (\mathfrak{F}_1[z] \cup \mathfrak{F}_2[w]) \\ &= \mathfrak{F}_1[z]. \end{aligned}$$

So $u = z$, and similarly $v = w$.

Now, suppose that $\mathfrak{F}_1[u] \cup \mathfrak{F}_2[v] \neq \mathfrak{F}_1[z] \cup \mathfrak{F}_2[w]$. We can then also assume, w.l.o.g., that

$$\exists y(y \in \mathfrak{F}_1[u] \wedge y \notin \mathfrak{F}_1[z] \wedge y \notin \mathfrak{F}_2[w]).$$

It follows that

$$\exists x(x \in u \wedge x \notin z \wedge x \notin w),$$

and therefore $u \neq z$.

Finally, suppose that $u \neq z$ or $v \neq w$. We will assume, w.l.o.g., that

$$\exists x(x \in u \wedge x \notin z).$$

Then

$$\exists x(\mathfrak{F}_1(x) \in \mathfrak{F}_1[u] \wedge \mathfrak{F}_1(x) \notin \mathfrak{F}_1[z]).$$

Using that $\mathfrak{F}_1[u] \cap \mathfrak{F}_2[w] = \emptyset$, we get

$$\exists x(x \in \mathfrak{F}_1[u] \cup \mathfrak{F}_2[v] \wedge x \notin \mathfrak{F}_1[z] \cup \mathfrak{F}_2[w]).$$

□

Proposition A.0.4. *For all* u, v, z *and* w*,*

$$(u, v) = (z, w) \Leftrightarrow u = z \wedge v = w.$$

Proof. We let $\mathfrak{F}_1(x) := \{\{x\}\}$ and $\mathfrak{F}_2(x) := \{\{x\}, \emptyset\}$ and apply Lemma A.0.3.

□