Hereditary Structural Completeness over K4: Rybakov's Theorem Revisited

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

A deductive system is said to be structurally complete if its admissible rules are derivable, and moreover is hereditarily structurally complete if all its finitary extensions are structurally complete. Citkin (1997) established a characterisation of hereditarily structurally complete intermediate logics and Rybakov (1995) gave a characterisation for transitive modal logics. Both their proofs are difficult in their own way, however recently Bezhanishvili and Moraschini (2019) gave a self-contained proof of Citkin's result based on Esakia duality. The aim of this project is to do the same for Rybakov's result using a duality for modal algebras. In doing so we will identify and correct for an error in Rybakov's characterisation.

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Chapter 1

Introduction

In deductive systems, a rule is said to be *admissible* if the tautologies of the system are closed under its applications and *derivable* if the rule itself holds in the system [24]. Whilst every derivable rule is admissible, whether the converse holds varies between deductive systems. As one might expect, this converse holds in the classical propositional calculus (**CPC**), but it fails for many non-classical systems including the intuitionistic propositional calculus (**IPC**) [5]. This gap has motivated an in depth study in the criteria for admissibility. In 1975 Friedman [16] posed the problem of determining if it was decideable that a given rule is admissible for **IPC** or not. Rybakov undertook an extensive study on the criteria of admissibility (for example [28, Chapter 3]), including solving Friedman's problem [27]. Building on the the work of Ghilardi on unification [17], the problem of finding bases for admissible rules was solved for **IPC** by Iemhoff [19] and independently by Rozière [26]. Jeřábek [20] obtained similar results for modal and Łukasiewicz logics.

A classical problem in the area is to determine which deductive systems share with **CPC** the property of all admissible rules being derivable, that is are *structurally complete*. Prucnal [23] showed that all finitary extensions of the implicative fragment of **IPC** are structurally complete and a similar result that all finitary extensions of Gödel-Dummet logic are structurally complete was obtained by Dzik and Wroński [13]. One outcome from these investigations was a suggestion that even if a full characterisation of the structurally complete modal and intuitionistic logics was out of reach, it might be possible to precisely characterise the *hereditarily structurally complete* (HSC) systems, those which are not only structurally complete themselves but whose finitary extensions are too. This proved a fruitful question, Citkin [12] produced a characterisation for intermediate logics, and Rybakov [29, 28] did so for transitive modal logics. Both these characterisations take a similar form. In Citkin's case, an intermediate logic is hereditarily structurally complete if and only if the variety of Heyting algebras associated with it omits five finite algebras [12]. In Rybakov's case, a transitive modal logic is hereditarily structurally complete if and only if it is not included in the logic of a list of 20 frames (see section 4.1 or [28, pg 274] for the list of frames).

However, both these milestone results are difficult in their own way. A detailed version of Citkin's proof has only been published in Russian [11] and the proof Rybakov gives is difficult, working with a construction of so-called characterising models and free algebras [29]. Recently, Bezhanishvili and Moraschini [5] gave a new proof of Citkin's theorem. Their approach utilises two different theories. First is the theory of algebraic logic. Using techniques from this field, it is possible to identify so-called *algebraizable* logics, those which have an associated class of algebras in which various algebraic properties reflect logical properties of interest [6].

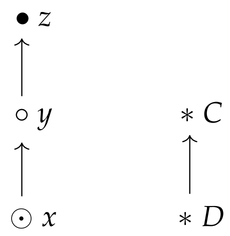
Intermediate logics are just such a logic, they are algebraizable with the variety of Heyting algebras as their associated class of algebras [5]. Second is the theory of Esakia duality. Esakia duality, like Stone duality, formalises a link between a class of algebras and class of order-topological spaces, in this case Heyting algebras and Esakia spaces. Together, these two theories enable the question of which intermediate logics are hereditarily structurally complete to be investigated through both algebraic and topological methods.

Notably a similar framework exists for modal logics; modal logic is algebraizable with the variety of modal algebras their associated class of algebras [15]. Then, modal algebras are themselves are linked by Jónnson-Tarski duality to the class of modal spaces. This provides the motivation of this project, to do for Rybakov's result what Bezhanishvili and Moraschini did for Citkin's and investigate HSC modal logics through this duality. A benefit of this approach is that, in contrast to Rybakov's original proof, we avoid having to work with free algebras and characterising models, instead relying on results from universal algebra in combination with the duality to complete the proof.

This is not the sole benefit to this approach. Utilising the results from universal algebra illuminates a mistake in Rybakov's characterisation. The list of frames given by Rybakov is too restrictive, including the frame F_3' but there are HSC modal logics included in the logic of that frame. Our aim then is more than to simply provide a new proof of Rybakov's characterisation, but to correct this error establishing an adjusted characterisation using our algebraic and topological methods. Our adjustment illustrates that the area of HSC transitive modal logics is more complex than originally thought, with a new group of logics determined to be HSC.

Our work is organised as follows. In chapter 2 we introduce the first theory central to our main task - Jónsson-Tarski duality. We also undergo some extensive study of transitive spaces. In chapter 3 we introduce the other important theory for our project - algebraic logic. The theory of algebraic logic describes a precise relationship between logic and algebra and we'll explain how this lets us recast our central question into characterising the primitive varieties of K4-algebras. We will further reduce this problem by establishing a necessary and sufficient condition for any variety to be primitive and discuss how in the modal case the logic-algebra relationship can be further extended to incorporate topology. Once the theoretical basis is in place, in chapter 4 we introduce Rybakov's characterisation and explain where the mistake lies. We then give our adjusted characterisation. The proof of our new characterisation is quite technical, so before proceeding with the proof itself we give an overview of strategy (refer to section 4.2.1 for this overview). We then give the first direction of the characterisation, proving that primitive varieties of K4-algebras must omit the algebras in the new characterisation (lemma 4.2). The other harder direction is split across chapters 5 and 6. In chapter 5 we work through a series of results describing the structure of algebras in our interested varieties, culminating in a precise description of their non-trivial, finitely generated subdirectly irreducible members (theorem 5.11). Finally, we complete the proof of the main theorem in chapter 6 (theorem 6.3 and corollary 6.4).

A brief note on notation. Throughout our work we will be working with transitive relational structures for which a pictorial representation is especially helpful. We will adopt the same notation as Rybakov in [28]. As all our diagrams refer to transitive relations, much like Hasse diagrams we will not draw transitive arrows. We will use • to denote a reflexive point, ∘ for an irreflexive point and ⊙ for a point that may be reflexive or irreflexive. We also use $*$ to denote an arbitrary finite collection of points all of whom relate to each other (when the collection is just a single point this can be reflexive or irreflexive, otherwise all these points are obviously reflexive). For example in the following:



A $\qquad$ B

A represents the set $\{x, y, z\}$ under either the relation:

$$R := \{(x, x), (x, y), (x, z), (y, z), (z, z)\} \text{ or } R' = \{(x, y), (x, z), (y, z), (z, z)\}.$$

B represents the family of relational structures where for $n, m \in \omega$ we have:

$$C = \{c_i : 1 \leq i \leq n\} \text{ and } D = \{d_j : 1 \leq j \leq m\}.$$

We consider the set $C \cup D$ under the relation:

$$R := C^2 \cup D^2 \cup \{(c_i, d_j) : 1 \leq i \leq n, 1 \leq j \leq m\}.$$

Chapter 2

Jónsson-Tarski Duality

Just as the study of Boolean and Heyting algebras is aided by their (order-)topological representations known as *Stone duality* and *Esakia duality*, we can study modal algebras through the *Jónsson-Tarski duality*. In this chapter we properly introduce modal algebras and their topological dual, modal spaces. We'll then give the duality between them and expand on this a little, before embarking on some extensive study of transitive modal spaces.

2.1 The Duality

We begin by introducing our two structures and the duality between them.

2.1.1 Algebra

The algebraic structures we are interested in are modal algebras. Here we briefly recall the definition of modal algebras, assuming a familiarity with Boolean algebras and standard algebraic notions such as subalgebras, quotient algebras, direct product and so on. We'll also recall some basic properties of transitive modal algebras, known as K4-algebras. For a more detailed study the reader may consult [10, Section 7.5].

Definition 2.1. A *modal algebra* is a structure $(A, \wedge, \vee, \neg, \bot, \top, \Box)$ where $\Box$ is a unary function on A such that:

(i) $(A, \wedge, \vee, \neg, \bot, \top)$ is a Boolean algebra;

(ii) $\forall a, b \in A\ \Box(a \wedge b) = \Box a \wedge \Box b$;

(iii) $\Box\top = \top$.

Equivalently, $\Box$ is a unary operation such that $\Box(a \to b) = \Box a \to \Box b$ and $\Box\top = \top$. We define an operator $\Diamond$ dual to $\Box$ as $\Diamond := \neg\Box\neg$.

A modal homomorphism between two modal algebras A and B is a Boolean homomorphism $f : A \to B$ satisfying $\forall a \in A\ f(\Box a) = \Box f(a)$. We let MA denote the category of modal algebras with modal homomorphisms.

A modal algebra is called a *K4-algebra* iff $\forall a \in A, \Box a \leq \Box\Box a$ and an *S4-algebra* iff it is a K4-algebra and moreover $\forall a \in A\ \Box a \leq a$. We let K4-A and S4-A denote the full subcategory of MA consisting of K4-algebras and S4-algebras respectively.

Our work is entirely focused on K4-algebras, and there are a number of basic properties and concepts associated with them.

There is a useful extension of the $\Box$ and $\Diamond$ operators. Given $A \in$ *K4-A* and $a \in A$ we define:

$$\Box^+ a := a \wedge \Box a,\ \Diamond^+ a := a \vee \Diamond a.$$

A *filter* F of a K4-algebra A is a non-empty set $F \subseteq A$ such that:

(i) If $a \in F$ and $a \leq b$ then $b \in F$;

(ii) If $a, b \in F$ then $a \wedge b \in F$.

A filter F is called a *modal* (or *open*) filter iff $\forall a \in A$ if $a \in F$ then $\Box a \in F$.

Let $a \in A$. The smallest modal filter containing a is the set:

$$\uparrow\Box^+ a = \{b \in A : \Box^+ a \leq b\}.$$

A modal filter F is *principal* iff $\exists a \in A : F = \uparrow\Box^+ a$.

A *congruence* of a K4-algebra A is an equivalence relation θ on A such that $\forall a, b, c, d \in A$:

(i) If $(a, b) \in \theta$ and $(c, d) \in \theta$ then $(a \wedge c, b \wedge d) \in \theta$;

(ii) If $(a, b) \in \theta$ and $(c, d) \in \theta$ then $(a \vee c, b \vee d) \in \theta$;

(iii) If $(a, b) \in \theta$ then $(\neg a, \neg b) \in \theta$;

(iv) If $(a, b) \in \theta$ then $(\Box a, \Box b) \in \theta$.

We say a congruence $\sim$ of A is *completely $\wedge$-irreducible* in the congruence lattice of A iff for any collection of congruences $\{\theta_i\}_{i \in I}$ of A if $\theta = \bigcap_{i \in I} \theta_i$ then $\exists i \in I$ such that $\theta = \theta_i$.

We say a congruence θ is *$\wedge$-irreducible* in the congruence lattice of A iff for any congruences θ_1 and θ_2 of A if $\theta_1 \wedge \theta_2 = \theta$ then either $\theta_1 = \theta$ or $\theta_2 = \theta$.

We say that A is *subdirectly irreducible* or SI (*finitely subdirectly irreducible* or FSI) iff the identity relation is completely $\wedge$-irreducible ($\wedge$-irreducible) in the congruence lattice of A.

Lemma 2.2. The lattice of modal filters of a K4-algebra is isomorphic to the lattice of its congruences.

Proof. The isomorphism is given by $F \mapsto \theta_F$ with

$$\theta_F := \{(a, b) \in A^2 : (a \to b) \wedge (b \to a) \in F\}.$$

And in reverse, $\theta \mapsto F_\theta := \top/\theta$. □

Letting $A \in$ *K4-A*, $B \subseteq A$ and $c \in A \setminus \{\top\}$, we say that c is an *opremum* of B iff $\forall a \in B \setminus \{\top\}\ \Box^+ a \leq c$, i.e. $c \in \uparrow\Box^+ a$. This need not be unique.

Theorem 2.3. (Rautenberg's Criterion)
Let $A \in$ *K4-A*. A is SI iff A has an opremum. Moreover A is FSI iff every finite subset of A has an opremum.

Proof. See [25]. □

On top of the familiar algebraic constructions, we also make occasional use of another known as a relativisation. (See [3] for more details).

Letting $a \in A$ we define the set $A_a := \{x \in A : x \leq a\}$ and given $x, y \in A_a$ we let:

$$x \vee_a y := x \vee y, \neg_a x := a \wedge \neg x \text{ and } \Box_a x := a \wedge \Box(a \to x).$$

Then, $(A_a, \vee_a, \neg_a, a, \bot, \Box_a)$ is a modal algebra called the *relativisation* of A by a.

2.1.2 Topology

We now introduce the topological structure central to our investigation. We assume a familiarity with rudimentary topological notions such as open, closed and clopen sets, continuous maps, basis and so on. For a more detailed study the reader may consult [10, Chapter 8].

Our topological structure is an expansion of Stone spaces.

Definition 2.4. A *Stone space* is a topological space $\mathcal{X} = (X, \tau)$ such that:

(i) $\mathcal{X}$ is compact, i.e. every open cover of X has a finite sub-cover;

(ii) $\mathcal{X}$ is Hausdorff, i.e. $\forall x, y \in X$ such that $x \neq y$ $\exists U, V \in \tau$ such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$;

(iii) $\mathcal{X}$ has a basis of clopens.

We will use X^* to denote the set of clopen subsets of X.

We now list some basic well-known properties of Stone spaces.

Lemma 2.5. Let $\mathcal{X}$ be a Stone space. The following hold:

1. $\forall x, y \in X$ such that $x \neq y$ there exists $U \in X^*$ such that $x \in U$ and $y \notin U$. This property is known as Stone separation.
2. $\forall x \in X$, $\{x\}$ is closed.
3. The topology of a finite Stone space is necessarily discrete, and any finite set equipped with the discrete topology is a Stone space.

Proof. $\mathcal{X}$ having a basis of clopens implies 1, $\mathcal{X}$ being Hausdorff implies 2, and 2 implies 3. □

Definition 2.6. A *frame* (or *Kripke frame*) is a pair (X, R) where X is a set and $R \subseteq X^2$ a relation on X. For $x \in X$ we define:

$$R[x] := \{y \in X : xRy\} \text{ and } R^{-1}[x] := \{y \in X : yRx\}.$$

We extend this for $U \subseteq X$ by:

$$R[U] := \bigcup_{x \in U} R[x] \text{ and } R^{-1}[U] := \bigcup_{x \in U} R^{-1}[x].$$

A *modal space* (or *descriptive Kripke frame*) is a triple $\mathcal{X} = (X, \tau, R)$ where (X, R) is a frame, (X, τ) is a Stone space and $R \subseteq X^2$ is such that:

(i) $R[x]$ is closed for all $x \in X$;

(ii) $R^{-1}[U]$ is clopen for all clopen $U \subseteq X$.

Equivalently $R[x]$ is closed for all $x \in X$ and $\Box U := \{x \in X : R[x] \subseteq U\}$ is clopen for all clopen $U \subseteq X$.

A *p-morphism* or *bounded morphism* between two frames is a map $f : X \to Y$ such that $f[R_X[x]] = R_Y[f(x)]$ for every $x \in X$.

We let MS denote the category of modal spaces with continuous p-morphisms.

A modal space is called a *transitive space* iff its relation is transitive and a *quasi-ordered space* iff its relation is reflexive and transitive. We let TS and QS denote the full subcategory of MS consisting of transitive spaces and quasi-ordered spaces respectively.

Once again, we will focus exclusively on transitive spaces. Later we will substantially develop the theory of transitive spaces, but for now we recall some basic properties and concepts.

Given $\mathcal{X} \in TS$, we say elements $x, y \in X$ are *comparable* iff either xRy, yRx or $x = y$. Otherwise, we say x and y are *incomparable*, denoted $x||y$.

Given $\mathcal{X} \in TS$, we say an element $x \in X$ is isolated iff $\{x\}$ is open. Recalling that in Stone spaces all finite sets are closed, we immediately have that x is isolated iff $\{x\}$ is clopen.

We can define a similarly useful extension of the relation in a transitive space. Letting $Y \subseteq X$ we define:

$$R^+[Y] = Y \cup R[Y].$$

Note that for $x \in X$ by lemma 2.5 $\{x\}$ is closed and by the definition of a transitive space $R[x]$ is closed, so $R^+[X]$ is closed.

We say $x \in \mathcal{X}$ is a *root* iff $X = R^+[x]$ and $\mathcal{X}$ is *rooted* iff it has a root.

We say $Y \subseteq X$ is an *upset* iff for all $y \in Y$, $R[y] \subseteq Y$, i.e. it is closed under R. The smallest upset containing Y is $R^+[Y]$.

A *modal subspace* (*M-subspace*) of $\mathcal{X}$ is a closed upset of $\mathcal{X}$ equipped with the subspace topology and the restricted relation, and is itself a modal space.

Notably, given $x \in X$ the set $R^+[x]$ is closed and clearly an upset, and thus forms an M-subspace of $\mathcal{X}$ when equipped with the subspace topology.

We say an equivalence relation E on $\mathcal{X}$ is a *modal equivalence* iff $\forall x, y, \in X$:

(i) If xEy & xRz then $\exists w \in X$ such that yRw & zEw;

(ii) If $x \not{E} y$ then $\exists U$ clopen such that $x \in U, y \notin U$ and U is a union of equivalence classes of E.

We then denote by $\mathcal{X}/E$ the modal space $(X/E, \tau_E, R_E)$ where τ_E is the quotient topology and R_E is defined by:

$$[x]R_E[y] \text{ iff } \exists xEx', yEy' : x'Ry'.$$

The map $x \mapsto [x]$ is a continuous p-morphism and for any continuous p-morphism $f : \mathcal{X} \to \mathcal{Y}$ the relation $ker(f) := \{(x, x') \in X^2 : f(x) = f(x')\}$ is a modal equivalence.

Letting $\mathcal{X}_1, ..., \mathcal{X}_n$ be modal spaces, we denote by $\coprod_{i=1}^{n} \mathcal{X}_i$ the modal space obtained by taking the disjoint union of the X_i endowed with the disjoint topology and under the disjoint relation.

A *subframe* of $\mathcal{X}$ is a clopen set equipped with the subspace topology and restricted relation. It is itself a modal space [3].

2.1.3 Duality

With the two structures introduced we can give the central bridging result between them.

Theorem 2.7 (Jónsson-Tarski Duality)**.**
The category MA is dually equivalent to the category MS. Moreover this duality restricts to a dual equivalence between the categories K4-A & TS and S4-A & QS respectively.

Proof. We give just a sketch of the proof. The functors $(-)_* : MA \leftrightarrow MS : (-)^*$ that establish this equivalence are defined as follows.

Given $A \in MA$, we denote its set of ultrafilters filters by A_* and define the map $\varphi : A \to \mathcal{P}(A_*)$ by $\varphi(a) := \{F \in A_* : a \in F\}$. Then, (A_*, τ, R) is a modal space, where τ is the topology with clopen basis $\varphi[A]$ and FRF' iff $\forall a \in A$ if $\Box a \in F$ then $a \in F'$. We call R the dual of $\Box$. Note we use A_* to denote the modal space and the underlying set of ultrafilters interchangeably. For a modal homomorphism $f : A \to B$ we define $f_* : B_* \to A_*$ by $f_*(F) := f^{-1}(F)$.

Given $\mathcal{X} \in MS$, $\mathcal{X}^* = (X^*, \Box)$ is a modal algebra where X^* is the Boolean algebra of clopens of $\mathcal{X}$ and $\Box U := \{x \in X : R[x] \subseteq U\}$. Note that $\Diamond U = R^{-1}[U]$. For a continuous p-morphism $f : \mathcal{X} \to \mathcal{Y}$ we define $f^* : \mathcal{Y}^* \to \mathcal{X}^*$ by $f^*(U) := f^{-1}(U)$. □

2.2 Specifying the duality

Let us spell out some specific consequences of our duality of categories.

Given $A \in$ *K4-A* the isomorphism $A \cong (A_*)^*$ is given by φ and given $\mathcal{X} \in TS$ the isomorphism $\mathcal{X} \cong (\mathcal{X}^*)_*$ is given by $\psi : \mathcal{X} \to (\mathcal{X}^*)_*$ where $\psi(x) := \{U \in \mathcal{X}^* : x \in U\}$.

Lemma 2.8. The following hold:

(i) Given $A \in$ *K4-A*, if $G \subseteq A$ is a modal filter then $G_+ = \bigcap_{a \in G} \varphi(a)$ is a closed upset of A_*.

(ii) Given $\mathcal{X} \in TS$, if $B \subseteq X$ is a closed upset then $B^+ = \bigcap_{x \in B} \psi(x)$ is a modal filter of $\mathcal{X}^*$.

Moreover $\varphi(G) = (G_+)^+$ and $\psi(B) = (B^+)_+$.

Proof. (i); As each $\varphi(a)$ is clopen, G_+ is clearly closed. Letting $F \in G_+$ and FRF', then if $a \in G$ as G is a modal filter $\Box a \in G$. As $F \in G_+$ $G \subseteq F$, so $\Box a \in F$ and then FRF' implies $a \in F'$ and $F' \in \varphi(a)$. Therefore, $F' \in G_+$ and G_+ is an upset.

(ii); Let $U, V \in B^+$. Then $\forall x \in B$ we have $U, V \in \psi(x)$ so $x \in U \cap V$ and $U \cap V \in B^+$, and if $U \in B^+$ and $U \subseteq V$, then $\forall x \in B$ we have $U \in \psi(x)$ so $x \in U \subseteq V$ and $V \in B^+$. Therefore, B^+ is a filter. If $U \in B^+$ then letting $x \in B$, we have $U \in \psi(x)$ so $x \in U$. As U is an upset $R[x] \subseteq U$ and so $x \in \Box U$ and $\Box U \in B^+$. Therefore, B^+ is a modal filter.

The moreover follows from the definitions and φ and ψ being isomorphisms in their categories. □

Lemma 2.9. The following hold:

(i) A K4-algebra A is SI iff $Int(\{F \in A_* : F \text{ is a root}\}) \neq \varnothing$.

(ii) A K4-algebra A is FSI iff A_* is rooted.

Proof. (i); This is established in a more general setting in [31]. It is worth noting the partial result that for $F \in A_*$, F is a root iff $\forall a \neq \top, \uparrow\Box^+ a \not\subseteq F$.

(ii); Suppose A_* is rooted, i.e. $A_* = R^+[F]$ for $F \in A_*$. We first claim that $\forall a \in A : a \neq \top\ \Box^+ a \notin F$. Let $a \neq \top$, then $\neg a \neq \bot$ so $\exists F' \in A_* : \neg a \in F'$, i.e. $a \notin F'$. Now $F' \in \{F\} \cup R[F]$ so either $a \notin F$ and so $\Box^+ a \notin F$ or $a \notin F'$ with FRF' and then $\Box a \notin a$ and so again $\Box^+ a \notin F$.

Now, for any finite subset $B \subseteq A$, letting $b_1, ...b_n \in B \setminus \{\top\}$ we have $\Box^+ b_i \notin F$. Consider:

$$c := \bigvee_{1 \leq n} \Box^+ b_i.$$

As F is prime $c \notin F$ and so $c \neq \top$. Moreover $\forall 1 \leq i \leq n\ \Box^+ b_i \leq c$, so c is an opremum for B. Therefore, every finite subset $B \subseteq A$ has an opremum and by Rautenberg's criterion A is FSI.

Suppose A is FSI. Again by Rautenberg's criterion every finite subset of A has an opremum. We claim $\forall a, b \in A$ if $(\Box^+ a) \vee (\Box^+ b) = \top$ then either $a = \top$ or $b = \top$. We proceed by contraposistion, let $a, b \neq \top$. Then, $\exists c \neq \top$ which is an opremum for $\{a, b\}$, i.e. such that $\Box^+ a \leq c$ and $\Box^+ b \leq c$. Then, $(\Box^+ a) \vee (\Box^+ b) \leq c < \top$ and in particular $(\Box^+ a) \vee (\Box^+ b) \neq \top$. This naturally extends to finite collections of elements. Now we can consider:

$$B := \downarrow\{(\Box^+ a_1) \vee ... \vee (\Box^+ a_n) \in A : n \in \omega, a_i \neq \top\}.$$

This is an ideal and moreover $\top \notin B$ as otherwise $\top = (\Box^+ a_1) \vee ... \vee (\Box^+ a_n)$ so from above $\exists 1 \leq i \leq n : a_i = \top$ which is a contradiction.

Therefore, by the prime filter theorem for Boolean algebras $\exists F \in A_* : \{\top\} \subseteq F$ and $F \cap B = \varnothing$. In particular $\forall a \neq \top\ \Box^+ a \notin F$, so $\uparrow\Box^+ a \not\subseteq F$ and F is a root for A_*. □

Lemma 2.10. The following hold:

(i) There is a dual lattice isomorphism σ between the lattice of congruences of $A \in \mathit{K4\text{-}A}$ and lattice of M-subspaces of A_* such that for any congrunence θ of A, $\sigma(\theta) \cong (A/\theta)_*$ and for any M-subspace Y of A_*, $Y^* \cong A/\sigma^{-1}(Y)$.

(ii) There is a dual lattice isomorphism ρ between subalgebras of $A \in \mathit{K4\text{-}A}$ and modal equivalences on A_* such that for any sub-algebra B of A, $B_* \cong A_*/\rho(B)$ and for any modal equivalence E on A_*, $\rho^{-1}(E) \cong (A_*/E)^*$.

(iii) There is a dual lattice isomorphism between relativisations of $A \in \mathit{K4\text{-}A}$ and subframes of A_* given by φ and such that for any $a \in A$, $\varphi(a) \cong (A_a)_*)$ and for any clopen $Y \subseteq A_*$, $Y^* \cong A_{\varphi^{-1}(Y)}$

(iv) The disjoint union of finitely many transitive spaces $\mathcal{X}_1, ..., \mathcal{X}_n$ is isomorphic to the dual of the direct product of the K4-algebras $\mathcal{X}_1^*, ..., \mathcal{X}_n^*$.

Proof. (i); The isomorphism is given by:

$$\sigma(\theta) := \{F \in A_* : F_\theta \subseteq F\} \text{ and } \sigma^{-1}(Y) := \theta_{\varphi^{-1}[Y]}.$$

Checking this is a dual lattice isomorphism is straightforward. Then $\sigma(\theta) \cong (A/\theta)_*$ is witnessed by $G \mapsto \{[a] \in A/\theta : \exists a' \in G : (a, a') \in \theta\}$ and $Y^* \cong A/\sigma^{-1}(Y)$ is witnessed by $Y \cap \varphi(a) \mapsto [a]$. Checking these are isomorphisms in their categories is straightforward.

(ii); The isomorphism is given by:

$$F\rho(B)F' \text{ iff } F \cap B = F' \cap B.$$

$$\rho^{-1}(E) = \{a \in A : \varphi(a) \text{ can be written as a union of equivalence classes of } E\}.$$

Checking this is a dual lattice isomorphism is mostly straightforward with the exception of checking condition (i) for $\rho(B)$ being a modal equivalence, which we'll present. We let $F, F', G \in A_* : F \cap B = F' \cap B$ and FRG. We must show $\exists G' \in A_* : F'RG'$ and $G \cap B = G' \cap B$. Note that $F \cap B = F' \cap B$ and FRG means $\forall b \in B\ \Box b \in F'$ implies $b \in G$. As B is closed under $\wedge$, so too is $G \cap B$ and so $\uparrow\{a \wedge b \in A : \Box a \in F', b \in G \cap B\}$ is a filter. As G is a prime filter of A, $G \cap B$ is a prime filter of B and so $B \setminus G$ is a prime ideal of B and $\downarrow B \setminus G$ is an ideal of A. We claim that the filter and ideal are disjoint, then by the prime filter theorem for Boolean algebras $\exists G' \in A_* : \{a \in A : \Box a \in F'\} \subseteq G'$, $G \cap B \subseteq G'$ and $B \setminus G \cap G' = \varnothing$, i.e. $F'RG'$ and $G \cap B = G' \cap B$.

For the claim, suppose $\exists a \in A : \Box a \in F', b \in G$ and $d \notin G$ such that $a \wedge b \leq r \leq d$. Then $a \wedge b \leq a \wedge d$ so $a = a \wedge (\neg b \vee b) = (a \wedge \neg b) \vee (a \wedge b) \leq (a \wedge \neg b) \vee (a \wedge d) = a \wedge (\neg b \vee d) = a \wedge (b \to d) \leq b \to d$. In other words $a \leq b \to d$, therefore $\Box a \leq \Box(b \to d)$ and $\Box(b \to d) \in F'$. Finally $b \to d \in B$ and so from our note $b \to d \in G$, but $b \in G$ so this implies $d \in G$ which is a contradiction.

Then, $B_* \cong A_*/\rho(B)$ is witnessed by $G \mapsto \{F \in A_* : F \cap B = G\}$ and $\rho^{-1}(E) \cong (A_*/E)^*$ by $a \mapsto \{[F] \in A_*/E : [F] \subseteq \varphi(a)\}$. Again, checking these are isomorphisms in their categories is mostly straightforward, aside from checking that the first is continuous. This requires establishing the non-trivial claim that if $a \notin B$ then $\varphi(a)$ is not closed under $\rho(B)$, i.e. $\exists F, F' \in A_* : a \in F, a \notin F'$ and $F \cap B = F' \cap B$. We do this in two constructions. Firstly, $\downarrow(\downarrow a \cap B)$ is an ideal of A disjoint from $\uparrow a$, so by the prime filter theorem $\exists F \in A_* : a \in F$ and $a \notin \uparrow(F \cap B)$. Secondly, we consider $\uparrow(F \cap B)$ as a filter of A and $\downarrow\{a \vee b \in A : b \notin F, b \in B\}$ as an ideal of A. Similar reasoning to the claim above establishes these are disjoint, then using the prime filter theorem again gives the desired F'.

(iii); See [3, Prop 4.5].

(iv); The isomorphism is $\chi : \coprod_{i=1}^{n} X_i \to (\prod_{i=1}^{n} X_i^*)_*$ by:

$$\chi(x, j) := \{(U_i) \in \prod_{i=1}^{n} X_i^* : x \in U_i\}.$$

Again, checking the various claims is straightforward. $\square$

2.3 Advanced Transitive Spaces

The theory of transitive spaces is a rich area in its own right, and drawing on these results will prove immensely helpful as we work towards our main result.

Our first result is an extension of Stone Separation. For those familiar with Esakia spaces it is an analog to the Priestly Separation axiom.

Lemma 2.11 (Modal Separation)**.**
Let $\mathcal{X} \in TS$ and $x, y \in X : y \notin R^{\omega}[x]$. Then $\exists U : U$ is a clopen upset and $x \in U$, $y \notin U$.

Proof. We have $y \in X \setminus R^{\omega}[x]$ which is open as $R^{+}[X]$ is closed. As $\mathcal{X}$ is a Stone space, it has a basis of clopens and so there is a collection of clopens $\{U_i\}_{i \in I}$ such that:

$$X \setminus R^{+}[x] = \bigcup_{i \in I} U_i.$$

In particular there is a clopen U' such that $y \in U' \subseteq X \setminus R^{\omega}[x]$. Now, $R^{-1}[U']$ is clopen, and so $R^{-\omega}[U']$ is a clopen downset containing y. Therefore, taking $U := X \setminus R^{-\omega}[U']$ we have $x \in U, y \notin U$ where U is a clopen upset as required. □

A very useful concept in transitive frames is the cluster. Let (X, R) be a transitive frame. A *cluster* C of X is a set of mutually comparable elements, or a single irreflexive element. If C has exactly one element we say it is *improper*, otherwise it is *proper* and if C is the singleton containing a single irreflexive element we call it *degenerate*.

Let (X, R) be a transitive frame and C and D be clusters of X. We say that C sees D iff $\exists x \in C, \exists y \in D : xRy$ or $C = D$. One can view this 'seeing relation' as the reflexive and anti-symmetric closure of R, and it is easy to see that the clusters of X under R' form a poset.

2.3.1 Reductions

As we noted earlier, the topology of finite Stone spaces and by extension finite modal spaces is in a sense trivialised. This helps give a characterisation for the existence of a surjective continuous p-morphisms between finite modal spaces. This is a generalisation of the similar characterisation in the case of Esakia spaces found in [4, lemma 3.1.6, 3.1.7].

Lemma 2.12. Let $\mathcal{X} \in TS$ be finite.

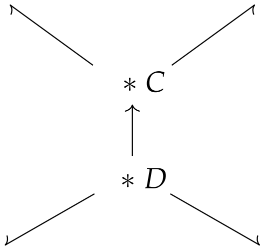
(i) Let $C = \{c_i \in X : 1 \leq i \leq n\}$ and $D = \{d_i \in X : 1 \leq i \leq m\}$ be distinct clusters of X with $m \leq n$ such that:

  (a) D sees C.

  (b) C is non-degenerate.

  (c) $\forall x \in X \setminus (C \cup D)$ $x \in R[C]$ iff $x \in R[D]$.

Pictorially:



We define the binary relation E on X:

$$E := \{(c_i, d_i), (d_i, c_i) \in X^2 : 1 \leq i \leq m\} \cup \{(u, u) \in X^2 : u \in X\}.$$

That is, E is an equivalence relation that pairs each element of D to a unique element of C whilst all other elements relate only to themselves.

Then E is a modal equivalence and we call the canonical map $f : \mathcal{X} \to \mathcal{X}/E$ an *α-reduction*.

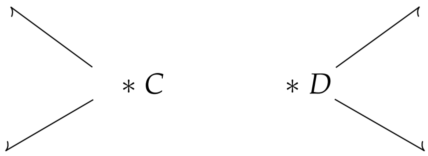
(ii) Let $C = \{c_i \in X : 1 \leq i \leq n\}$ and $D = \{d_i \in X : 1 \leq i \leq n\}$ be distinct clusters of X of the same size such that:

(a) C and D do not see each other.

(b) C is degenerate iff D is degenerate.

(c) $\forall x \in X \setminus (C \cup D)$ $x \in R[C]$ iff $x \in R[D]$.

Pictorially:



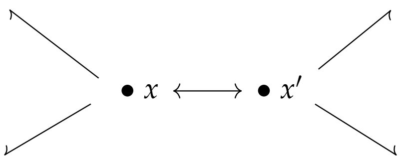
We define the binary relation E on X:

$$E := \{(c_i, d_i), (d_i, c_i) \in X^2 : 1 \leq i \leq n\} \cup \{(u, u) \in X^2 : u \in X\}.$$

That is, E is an equivalence relation that pairs off the elements of C and D whilst all other elements relate only to themselves.

Then E is a modal equivalence and we call the canonical map $f : \mathcal{X} \to \mathcal{X}/E$ a *β-reduction*.

(iii) Let $x, x' \in X$ be distinct elements in the same cluster, i.e. $x \neq x'$, xRx' and $x'Rx$. Pictorially:



We define the binary relation E on X:

$$E := \{(x, x'), (x', x) \in X^2\} \cup \{(u, u) \in X^2 : u \in X\}.$$

That is, E is the smallest equivalence relation on X such that xEx'.

Then E is a modal equivalence and we call the canonical map $f : \mathcal{X} \to \mathcal{X}/E$ a *γ-reduction*.

Proof. In each case E trivially fulfils condition (ii) for being a modal equivalence as $\mathcal{X}$ is finite and so have the discrete topology. Condition (i) follows straightforwardly from the conditions in each case. □

Lemma 2.13. Let $\mathcal{X}$ and $\mathcal{Y}$ be finite transitive spaces. Suppose there exists a surjective continuous p-morphism $f : \mathcal{X} \to \mathcal{Y}$ which identifies exactly two points. Then there is a α, β or γ-reduction $f_E : \mathcal{X} \to \mathcal{X}/E$ such that $\mathcal{X}/E \cong \mathcal{Y}$.

Proof. Let $u, v \in X : u \neq v$ and $f(u) = f(v)$. Note that $\forall x, y \in X$ if $x \notin \{u, v\}$ and $f(x) = f(y)$ then $x = y$. Either u and v are in the same cluster or they are not. If they are in the same cluster, we can by lemma 2.12 consider the γ-reduction $f_E : \mathcal{X} \to \mathcal{X}/E$.

Now suppose that u and v are in different clusters, we let C be the cluster containing u and D be the cluster containing v. As f is a p-morphism we have $f[R[u]] = R[f(u)] = R[f(v)] = f[R[v]]$. Now, let $x \in X$ such that $x \notin \{u, v\}$ and suppose $x \in R[C]$. In particular $x \in R[u]$ so $f(x) \in f[R[u]] = f[R[v]]$ and $\exists y \in X : f(x) = f(y)$ and $y \in R[v]$. Since $x \notin \{u, v\}$ we have $x = y$ and therefore $x \in R[v]$ and by extension $x \in R[D]$. This holds symmetrically for $x \in R[D]$ so we have $x \in R[C]$ iff $x \in R[D]$. In particular, this implies that $C = \{u\}$ and $D = \{v\}$ and $\forall x \in X \setminus (C \cup D)$ $x \in R[C]$ iff $x \in R[D]$.

Moreover, if $u \in R[C]$ then in particular $u \in R[u]$ and $f(u) \in f[R[u] = f[R[v]]$ so $\exists y \in X : f(u) = f(y)$ and $y \in R[v]$. If $y \notin \{u, v\}$ then from above $y = u$ and we have a contradiction, so $y \in \{u, v\}$. Therefore, either $u \in R[v]$ and $u \in R[D]$ or $v \in R[v]$ and $v \in R[D]$.

As C and D are distinct clusters at least one does not see the other. We may assume w.l.o.g that C does not see D. Then, either D sees C or D does not see C. If D sees C then $u \in R[D]$ which from above implies either $u \in R[C]$ or $v \in R[C]$ but the latter is impossible as C does not see D. Therefore $u \in R[C]$ and C is non-degenerate. Therefore, from lemma 2.12 C and D fulfill the conditions to define a α-reduction $f_E : \mathcal{X} \to \mathcal{X}/E$.

Finally suppose D does not see C, so C and D do not see each other. Now if C is degenerate then in particular $u \notin R[C]$ and as D does not see C $v \notin R[C]$ which from the above constraint implies $v \notin R[D]$ and D is degenerate. Symmetrically, if D is degenerate then C is degenerate. So C is degenerate iff D is degenerate. Therefore, from lemma 2.12 C and D fulfil the conditions to define a β-reduction $f_E : \mathcal{X} \to \mathcal{X}/E$.

In each of the possible cases we have a defined modal equivalence E and some reduction map $f : \mathcal{X} \to \mathcal{X}/E$. To finish the base case we must find in each case an isomorphism $g : \mathcal{X}/E \to \mathcal{Y}$ such that $f_E \circ g = f$. We define g the same way in all cases, by $g([x]) = f(x)$. This is well defined as letting $x, x' \in X : xEx'$, either $x \notin \{u, v\}$ and so $x = x'$ and $f(x) = f(x')$ or $x \in \{u, v\}$ and so $x' \in \{u, v\}$ and $f(x) = f(x')$. Checking g is an isomorphism is straightforward.

□

Lemma 2.14. Let $\mathcal{X}$ and $\mathcal{Y}$ be finite modal spaces. There exists a surjective continuous p-morphism from $\mathcal{X}$ to $\mathcal{Y}$ iff there is a finite sequence of α, β and γ-reductions

$\langle f_i : Z_i \to Z_{i+1}\rangle_{i=1}^n$ such that $\mathcal{X} = Z_1$ and $Z_{n+1} \cong \mathcal{Y}$, i.e. $\mathcal{X}$ can be transformed into $\mathcal{Y}$ be a finite sequence of α, β and γ-reductions.

Proof. The if direction is almost immediate, simply take the composition of the sequence of reductions and the final isomorphism as the desired map.

For the only if direction: letting $f : \mathcal{X} \twoheadrightarrow \mathcal{Y}$, as X and Y are finite, f identifies some finite number $k \in \omega$ of points in X. We proceed by induction on the number of points f identifies. We may assume that $k \geq 2$ as otherwise f is an isomorphism and we are done immediatly. The base case of $k = 2$ is just lemma 2.13.

Inductive Step: Let $n \in \omega$ and suppose $\forall 2 \leq k < n$ that if g is a surjective continuous p-morphism between finite modal spaces identifying k points we can find a sequence of reductions as described. We let $f : \mathcal{X} \to \mathcal{Y}$ identify n points. We claim that we can find a reduction $f_E : \mathcal{X} \to \mathcal{X}/E$ such that $E \subseteq ker(f)$. Given such a reduction, we again want to define a map $g : \mathcal{X}/E \to \mathcal{Y}$ such that $f_E \circ g = f$, which we do by taking $g([x]) := f(x)$. This is well defined as $E \subseteq ker(f)$, and it is easy to check that g is a surjective continuous p-morphism that identifies less points than f. Applying the induction hypothesis to g and adjoining our f_E reduction then gives the desired sequence of reductions.

It remains to prove the claim, and we proceed through a series of cases. Firstly, either $\exists u, v \in X$ such that u and v are in the same cluster and $f(u) = f(v)$ or not. If we can find two such points, then by lemma 2.12 we can consider the γ-reduction $f_E : \mathcal{X} \to \mathcal{X}/E$ and as $f(u) = f(v)$ we have $E \subseteq ker(f)$ as required.

So now suppose that $\forall u, v \in X$ if $f(u) = f(v)$ then u and v are in distinct clusters. As Y is finite we can consider a maximal cluster B in Y such that $\exists y \in B : |f^{-1}(y)| \geq 2$. Our second distinction is whether B is a degenerate cluster or not. If it is degenerate, then letting $x, x' \in f^{-1}[B] : xRx'$ we have $f(x') \in f[R[x]] = R[f(x)]$ so $f(x)Rf(x')$ but $f(x), f(x') \in B$ contradicting B being degenerate. Therefore, $R \subseteq f^{-1}[B]^2 = \varnothing$, i.e. the pre-image of B is an anti-chain of irreflexive points. As $|f^{-1}[B]| \geq 2$ we can choose $u, v \in f^{-1}[B] : u \neq v$ and consider the clusters $C = \{u\}$ and $D = \{v\}$. These do not see each other and both are degenerate. Finally, letting $x \in X \setminus (C \cup D)$ and supposing $x \in R[C]$ then $f(x) \in f[R[u]] = R[f(u)]] = R[B] = R[f(v)]] = f[R[v]]$ so $\exists x' \in X : vRx'$ and $f(x) = f(x')$. As uRx we have that $x \notin f^{-1}[B]$ but $x \in R[B]$ so by the maximality of B $x = x'$ and in fact $x \in R[v] = R[D]$. Therefore, from lemma 2.12 we can consider the β-reduction $f_E : \mathcal{X} \to \mathcal{X}/E$ defined from C and D. Note that as $f(u) = f(v)$ we have $E \subseteq ker(f)$ as required.

Now suppose that B is not degenerate. As X is finite we can consider the maximal clusters C_i of X such that $f[C_i] \cap B = \varnothing$. Let $x_i \in C_i$ be such that $f(x_i) \in B$.

We claim that f restricted to a given maximal cluster C_i is a bijection. Consider $y \in B$. As f is surjective $\exists x' \in X : f(x') = y$. Now as $y \in B$ and B is not degenerate $f(x_i)Ry$ and so $f(x') \in R[f(x)] = f[R[x]]$. Therefore, $\exists x'' \in X : f(x'') = y$ and x_iRx'' and the maximaility of C_i implies that $x'' \in C_i$. So, every point in B is mapped to by some point in C_i. Moreover, as $x_i, x'' \in C_i$ is such that x_iRx'' C_i is not a degenerate cluster.

Then, letting $x \in C_i$ x_iRx which as f is a p-morphism implies $f(x_i)Rf(x)$ and similarly xRx_i we have $f(x)Rf(x_i)$, that is $f(x) \in B$. So every point in C_1 maps into B and moreover, as $\forall u, v \in X$ if $f(u) = f(v)$ then u and v belong to different clusters, each point in C_1 must map to a distinct point in B. So f restricted to the maximal cluster C_i is indeed a bijection onto B.

Our final distinction is whether there is one or multiple maximal clusters. Suppose there is more than one such cluster, let C_1 and C_2 be two of them. As we just established f restricted to either C_1 and C_2 is a bijection onto B, so in particular $|C_1| = |B| = |C_2|$. They are also both non-degenerate and from the maximality condition on the C_i they do not see each other. Now, let $x \in X \setminus (C_1 \cup C_2)$ and suppose $x \in R[C_1]$. Now $f(x) \in f[R[C_1]] = R[f[C_1]] = R[B]$, and $x \in R[C_1] \setminus C_1$. So by the maximality of C_1 $f(x) \notin B$. Therefore we have $f(x) \in R[B] \setminus B$ which by the maximality of B implies that $\forall x' \in X$ if $f(x') = f(x)$ then $x = x'$. Now, $f(x) \in f[R[C_1]] = R[f[C_1]] = R[f[C_2]] = f[R[C_2]]$ so $\exists x' \in X$ such that C_2 sees x' and $f(x') = f(x)$. Then, as we just noted $f(x) = f(x')$ implies $x = x'$ and in fact $x \in R[C_2]$. Symmetrically we get that if $x \in R[C_2]$ then $x \in R[C_1]$.

Therefore, from lemma 2.12 we can consider the β-reduction $f_E : \mathcal{X} \to \mathcal{X}/E$ defined from C and D. Note that from f restricted to both C_1 and C_2 being a bijection onto B, we have $E \subseteq ker(f)$ as required.

Finally then, we assume that there is just one such maximal cluster C. This means that $\forall x \in X$ if $f(x) \in B$ then $x \in R^{-1}[C]$. Again, f resricted to C is a bijection onto B and C is not degenerate. Now, as $\exists y \in B : f^{-1}[y]| \geq 2$ $f^{-1}[B] \not\subseteq C$ and so from the maximality of C $f^{-1}[B] \cap R^{-1}[C] \setminus C \neq \varnothing$ and we can again consider a maximal cluster D in $f^{-1}[B] \cap R^{-1}[C] \setminus C$. Let $x_d \in D : f(x_d) \in B$. Obviously D sees C. If $|D| > 1$ then D is not degenerate and letting $x \in D$ $x_d R x$ and $x R x_d$ and as f is a p-morphism this implies $f(x_d) R f(x)$ and $f(x) R f(x_d)$ so $f(x) \in B$. Again, as $\forall u, v \in X$ if $f(u) = f(v)$ then u and v belong in different clusters each point in D must map to a distinct point in B. Therefore $|D| \leq |B| = |C|$. If $|D| = 1$ then again each point in D maps to a distinct point in B and $|D| \leq |C|$. So in all cases each point in D maps to a distinct point in B and $|D| \leq |B|$. Finally, as D sees C $R[C] \subseteq R[D]$. Moreover, letting $x \in X : x \in R[D]$ then $f(x) \in f[R[D]] = R[f[D]] = R[B]$. If $f(x) \in B$ then $x \in R^{-1}[C]$ and by the maximality of D $x \in C \cup D$. If $f(x) \in R[B] \setminus B$ then by the maximality of B $\forall x' \in X$ if $f(x') = f(x)$ then $x = x'$ and as $f(x) \in f[R[D]] = R[f[D]] = R[f[C]] = f[R[C]]$ $\exists x' \in X$ such that C sees x' and $f(x') = f(x)$, so $x' = x$ and $x \in R[C]$. In particular $\forall x \in X \setminus (C \cup D) x \in R[C]$ iff $x \in R[D]$.

Therefore, from lemma 2.12 we can consider the α-reduction $f_E : \mathcal{X} \to \mathcal{X}/E$ defined from C and D. Note that from f restricted to C being bijection onto B and f restricted to D being injective, we have $E \subseteq ker(f)$ as required. This completes the proof of the claim. □

2.3.2 Modal Equivalences

Next we turn to a group of results that define a useful concept and describe common modal equivalences related to them.

Somewhat naturally, when studying transitive spaces the focus is often on the behaviour of clusters rather than points. However, at times we can effectively ignore them thanks to the following.

Lemma 2.15. Let $\mathcal{X} \in TS$. Define a binary relation E on X by xEy iff x and y are mutually comparable or $x = y$, i.e. E identifies elements in the same cluster. Then E is a modal equivalence.

Proof. That E is an equivalence relation is clear, so we must check conditions (i) and (ii) for being a modal equivalence.

For (i); let uEv and uRw. Then, either $u = v$ and so vRw or uRv and vRu so vRw. In both cases, vRw and then wEw so we may take w as witness.

For (ii); suppose $u\not{E}v$, then $u \neq v$ and either $u\not{R}v$ or $v\not{R}u$. If $u\not{R}v$ then $v \notin R^\omega[u]$ so by modal separation $\exists U : U$ is a clopen upset, $u \in U$ and $v \notin U$. Then, if $w \in U$ and wEt, either $w = t$ and $t \in U$ or wRt and so $t \in U$. So U is closed under E, i.e. it is a union of E-classes, and we may take U as witness. If $v\not{R}u$ the case is symmetric. □

Another frequently useful equivalence is the following:

Lemma 2.16. Let $\mathcal{X} \in TS$. Let $U \subseteq X$ be a clopen upset such that $\forall x \in U\ R[x] \neq \varnothing$, and define a binary relation E on X by xEy iff $x = y$ or $x, y \in U$, i.e. E is the smallest equivalence relation identifying points in U. Then E is a modal equivalence.

Proof. That E is an equivalence relation is clear, so we must check conditions (i) and (ii) for being a modal equivalence.

For (i); let uEv and uRw. If $u = v$ then vRw and wEw so we may take w as witness. If $u \neq v$, then $u, v \in U$ and as U is an upset $w \in U$. Then $R[v] \neq \varnothing$ so letting $t \in R[v]$ again as U is an upset $t \in U$ and wEt so we may take t as witness.

For (ii); let $u\not{E}v$, then $u \neq v$ and at least one of $u, v \notin U$. If both $u, v \notin U$, as X is a Stone space $\exists V : V$ is clopen, $u \notin V$ and $v \in V$. Then $X \setminus (U \cup V)$ is clopen with $u \in X \setminus (U \cup V)$ and $v \notin X \setminus (U \cup V)$. Now, if $w \in X \setminus U \cup V$ and wEt as $w \notin U$ $w = t$ so $t \in X \setminus U \cup V$. Therefore $X \setminus (U \cup V)$ is closed under E and separates u and v as required.

If exactly one of u or v are in U, either $u \in U$ and $v \notin U$ or $u \in X \setminus U$ and $v \notin X \setminus U$. U is an E-class so certainly a union of them, and moreover $X \setminus U$ is a union of E-classes, so we either U or $X \setminus (U \cup V)$ separates u and v as required. □

In reality, this lemma is a particular case of a broad group of equivalences. Let $\mathcal{X} \in TS$ and $\{U_i\}_{i=1}^n$ be a finite collection of pairwise disjoint clopen subsets of X. We say this collection is an *M-partition* of X iff $\forall 1 \leq i, j \leq n$ if $u, v \in U_i$ and uRw then $\exists t \in U_j : vRt$ if $w \in U_j$ and vRw otherwise.

Lemma 2.17. Let $\mathcal{X} \in TS$ and $\{U_i\}_{i=1}^n$ be an M-partition of X. Define a binary relation E on X by xEy iff $u = v$ or $\exists 1 \leq i \leq n : u, v \in U_i$, i.e. E is the smallest equivalence relation that identifies points within each U_i. Then E is a modal equivalence.

Proof. That E is an equivalence relation follows easily from the U_i being pairwise disjoint.

For (i); let uEv and uRw. If $u = v$ then as usual we can simply take w as witness. If $u \neq v$ then $u, v \in U_i$ for some $1 \leq i \leq n$. If $w \in U_j$ for some $1 \leq j \leq n$ then by the definition of an M-partition $\exists t \in U_j : vRt$. Then wEt so we may take w as witness. If $w \notin U_j\ \forall\, 1 \leq j \leq n$ then by the definition of an M-partition vRw and we may take w as witness.

For (ii); let $u\not{E}v$. If $u \in U_i$ for some $1 \leq i \leq n$ then $v \notin U_i$ where U_i is clopen and an E-class so separates u and v as required. If $v \in U_i : 1 \leq i \leq n$ then $u \in X \setminus U_i$, $v \notin X \setminus U_i$ and this too is clopen and a union of E-classes so separates u and v as required. If $\forall 1 \leq i \leq n\ u, v \notin U_i$ then $u \neq v$ and so by Stone separation $\exists V : V$ is clopen, $u \in V$ and $v \notin V$. We define:

$$U =: V \cup \bigcup_{i=1}^{n} U_i$$

Then we have $u \in U$, $v \notin U$ and U is clopen and a union of E-classes so separates u and v as required. □

Another useful concept is that of depth, and it too comes with an associated modal equivalence.

Let (X, R) be a transitive frame. We define the *depth* of x as the maximal number of clusters in maximal chains of clusters rooted at x, including the cluster containing x. If there there is no such maximal (or an infinite chain of clusters rooted at x) we say that it is ω-deep. We use $d(x) \in \omega \cup \{\omega\}$ to denote the depth of x. The depth of X is $d(X) := max\{d(x) \in \omega \cup \{\omega\} : x \in X\}$ if this exists and $d(X) := \omega$ otherwise.

We define $Sl_n(X) := \{x \in X : d(x) = n\}$ and $Sl_\omega(X)$ similarly. We also define $S_n(X) := \bigcup_{m \leq n} SL_n(X)$.

Remarks. There are some basic properties of depth worth bearing in mind.

1. If $d(x) = n \in \omega$ and xRy then $d(y) \leq n$ and if $d(y) = n$ then yRx.
2. If $d(x) = n \in \omega$ then $\forall\, m < n\ \exists y \in X : xRy$ and $d(y) = m$.

Lemma 2.18. Let $\mathcal{X} \in TS$ and suppose that:

(a) $\forall\, x, y \in Sl_\omega(X)\ \{n \in \omega : R[x] \cap Sl_n(X) \neq \varnothing\} = \{n \in \omega : R[y] \cap Sl_n(X) \neq \varnothing\}$;

(b) $\forall\, n \in \omega$ either $\forall x \in Sl_n(X)\ xRx$ or $\forall x \in Sl_n(X)\ x\not{R}x$;

(c) $\forall\, n \in \omega\ Sl_n(X)$ is clopen.

We define the binary relation E on X by xEy iff $d(x) = d(y)$. Then E is a modal equivalence.

Proof. We would like to simply say that because by (c) the $Sl_n(X)$ form a pairwise disjoint collection of clopens and moreover by (b) form an M-partition the result follows immediatly from lemma 2.17. However, the collection of sets we are taking is possibly infinite so technically may not form a genuine M-partition and we have to adjust slightly. Clearly E is an equivalence relation.

For (i); letting uEv and uRw if $d(u) = n = d(v)$ for $n \in \omega$ then as $uRw\ d(w) \leq n$. If $d(w) = n$ then wEv and wRu so uRu. Then by (b) vRv and we may take v as witness. If $d(w) < n = d(v)$ then $\exists t \in Sl_{d(w)}(X) : vRt$ so wEt and we may take t as witness. If $d(u) = d(v) = \omega$ then either $d(w) = n \in \omega$ or $d(w) = \omega$. If the former then by (a) $\exists t \in Sl_{d(w)}(X) : vRt$, then wEt and we may take t as witness.

Suppose the latter, we need to show that $\exists t \in Sl_\omega(X) : vRt$, as in that case wEt' and we may take t as witness. Suppose for contradiction that $R[v] \cap Sl_\omega(X) = \varnothing$, then $R[v] \subseteq \bigcup_{n \in \omega} Sl_n(X)$ and is closed. By (c) each $Sl_n(X)$ is clopen, so this is an open cover for $R[v]$, and so by compactness there is some finite subcover of $R[v]$. Thus, $\exists n \in \omega : R[v] \subseteq Sl_n(X)$ but then $d(v) = n + 1$ which is a contradiction.

For (ii); suppose $u\not{E}v$. Then by the definition of E either $d(u) \in \omega$ and $d(v) \neq d(u)$ or $d(u) = \omega$ and $d(v) \in \omega$. If the former, then $u \in Sl_{d(u)}(X)$ and $v \notin Sl_{d(u)}(X)$. This is clopen by (c) and an E-class so separates u and v as required. If the latter then $u \in X \setminus Sl_{d(v)}(X)$ and $v \notin X \setminus Sl_{d(v)}(X)$. This is also clopen by (c) and a union of E-classes so separates u and v as required. □

2.3.3 Finitely Generated Spaces

The consideration of finitely generated members in a class of algebras is a frequent technique in the study of that class. As a result understanding the dual spaces to finitely generated algebras is quite helpful. Fortunately, these spaces have been extensively studied (for example see [10, Section 8.6]). For the sake of completion, we will present these results and their proofs in full detail.

Let $\mathcal{X} \in MS$. We say that $\mathcal{X}$ is *finitely generated* iff $\mathcal{X}^*$ is finitely generated as a modal algebra, i.e. $\exists U_1, ..., U_n \in \mathcal{X}^*$ such that every clopen subset of X is expressible in terms of $U_1, ..., U_n$ using $\cap$, $\cup$, $\setminus$ and $\Box$. We say that $\mathcal{X}$ is n-generated for some natural number n to mean that $\mathcal{X}$ is finitely generated by a collection of n elements.

The key result for understanding finitely generated spaces is the colouring theorem. Letting $A \in MA$ and $g_1, ...g_n \in A$, for each $x \in A_*$ we define $col(x) := \langle j_i \rangle_{i=1}^n$ where:

$$j_i = \begin{cases} 0 & \text{if } g_i \notin x \\ 1 & \text{if } g_i \in x \end{cases}$$

Theorem 2.19 (Colouring Theorem)**.**
Let $A \in MA$ and $g_1, ..., g_n \in A$. The following are equivalent:

(i) A is generated by $g_1, ..., g_n$;

(ii) For every proper surjective continuous p-morphism $f : A_* \to \mathcal{X}$ there exist points $u, v \in A_* : f(u) = f(v)$ and $col(u) \neq col(v)$;

(iii) For every proper modal equivalence E of A_* there exists an E-class containing points of different colours.

Proof. This result and its proof are an adaptation of the Esakia space equivalent in [4, Theorem 3.1.5]. The relationship between modal equivalences and surjective continuous p-morphisms gives the equivalence of (ii) and (iii) immediately. As such, we will just cover (i) iff (iii).

Suppose A is generated by $g_1, ..., g_n$ and E is a proper modal equialence of A_*. From lemma 2.10 $\rho^{-1}(E)$ is a proper subalgebra of A and as A is generated by $g_1, ..., g_n$ $\exists i \leq n : g_i \notin \rho^{-1}(E)$. From the definition of $\rho^{-1}(E)$, this means $\varphi(g_i)$ is not a union of E-classes and therefore not closed under E. That is $\exists u, v \in A_* : uEv$, $g_i \in u$ and $g_i \notin v$, therefore u and v are in the same E-class and $col(u) \neq col(v)$.

Conversely, suppose A is not generated by $g_1, ...g_n$. Let B be the subalgebra of A generated by $g_n, ...g_n$, i.e. $B = \langle g_1, ..., g_n \rangle$. Then B is a proper subalgebra of A and by lemma 2.10 $\rho(B)$ is a proper modal equivalence of A_*. Letting $[u]$ be a $\rho(B)$-class, $\forall v \in [u]$ $u\rho(B)v$, i.e. $u \cap B = v \cap B$. So $\forall 1 \leq i \leq n$, $g_i \in u$ iff $g_i \in v$ and $col(u) = col(v)$. Therefore, we have found a proper modal equivalence which has only monochrome equivalence classes. $\Box$

The colouring theorem helps establish some useful insights into the structure of finitely generated transitive spaces.

Lemma 2.20. Let $\mathcal{X} \in TS$ be n-generated. Let C be a cluster in X, then $|C| \leq 2^n$.

Proof. We will prove that each element in a given cluster C must have a unique colour, then as there are 2^n colours $|C| \leq 2^n$.

Suppose for contradiction that $u, v \in X : u \neq v$ and u and v are in the same cluster, i.e. uRv,vRu and $col(u) = col(v)$. We consider the relation:

$$E := \{(u, v), (v, u)\} \cup \{(x, x) \in X^2 : x \in X\}.$$

That is the smallest equivalence relation identifying u and v. We claim this is a modal equivalence.

For (i); letting $x, y \in X : xEy$ and xRz either $x = y$ and the case is trivial or we are considering uEv and uRw or vEu and vRw. Then either $vRuRw$ and wEw or $uRvRw$ and wEw.

For (ii); let $x, y \in X : x\not{E}y$. Either $x = u, y \neq v$, $x = v, y \neq u$, $x \neq v, y = u$, $x \neq u, y = v$ or $x, y \notin \{u, v\}$. If $x = u$ and $y \neq v$, we apply Stone separation to v and y to find a clopen U_v^y such that $v \in U_v^y$ and $y \notin U_v^y$ and also to x and y to find a similar clopen U_x^y. Then $U_x^y \cup U_v^y$ is clopen and closed under E so seperates x and y as required. The other cases where x or $y \in \{u, v\}$ are similar.

Suppose $x, y \notin \{u, v\}$, then we apply Stone separation to x with u, v and y in turn to find clopens U, V and W such that $x \in U, V$ and W whilst $u \notin U$, $v \notin V$ and $y \notin W$. Then $U \cap V \cap W$ is clopen and a union of E classes so separates x and y as required.

Thus, E is a proper modal equivalence, but as $col(u) = col(v)$ all its equivalence classes are monochrome, contradicting the colouring theorem. □

The next result concerns finitely generated spaces of finite width. Letting $\mathcal{X} \in TS$ and $x \in X$ we define the *width* of x as the maximal number of points in a maximal anti-chain in $R^+[x]$. If there is no maximal anti-chain, (or an anti-chain with infinitely many points) we say x has width ω. Then, the width of X is the maximal width of its elements should that exists, and ω otherwise. Moreover, letting $A \in MA$ we define the width of A as the width of A_*.

Lemma 2.21. Let $\mathcal{X} \in MS$ be such that it contains no infinite anti-chains. Then every infinite non-descending sequence of distinct points in X contains an infinite ascending subsequence. More precisely, let $\langle x_n \rangle_{n\in\omega}$ be an infinite sequence such that:

(i) $\forall i, j \in \omega\ i \neq j$ implies $x_i \neq x_j$;

(ii) $\forall i, j \in \omega$ if $i < j$ then $x_j \not{R} x_i$.

Then, there exists a sub-subsequence $\langle x_{i_n} \rangle_{n\in\omega}$ such that $\forall n, m \in \omega$ if $n < m$ then $x_{i_n} R x_{i_m}$.

Proof. This is a specification of [10, lemma 10.33]. Let $\langle x_n \rangle$ be such a sequence. First, observe that $\exists i \in \omega : X_i = \{x_j : j > i \,\&\, x_i R x_j\}$ is infinite, as otherwise by defining $i_0 = 0$ and $i_{k+1} = max(\{i_k\} \cup \{i : x_i \in X_{i_k}\})$, we find $x_{i_0}, x_{i_1}, ...$ that form an infinite anti-chain.

Now, let x_{i_0} be the first $i \in \omega : X_i$ is infinite. Then supposing x_{i_n} has been defined where X_{i_n} is infinite, let $x_{i_{n+1}}$ be the first point in the infinite non-descending chain X_{i_n} with infinite $X_{i_{n+1}}$. Then $\langle x_{i_n} \rangle_{n\in\omega}$ is an infinite ascending sequence. □

Theorem 2.22. Let $\mathcal{X} \in TS$ be finitely generated and of finite width. Then X contains no infinite ascending chains.

Proof. This is a specification of [10, Theorem 10.34]. From the duality, letting $A \in MA$ be such that $A_* \cong \mathcal{X}$, we work on A_* as opposed to $\mathcal{X}$. Let g_i be the generate A.

We will call a point $x \in A_*$ *deep* iff there is an infinite ascending chain of distinct points in A_* starting at x_0. Our goal is to prove A_* has no deep points. Suppose for contradiction that A_* has a deep point. Then for $x \in A*$ we define:

$$U_x := \{u \in R[x] : u \text{ is not deep}\}.$$

We call a deep point *static* iff $\forall y \in R[x]$ deep $U_x = U_y$. We claim that A_* contains a deep static point. Consider some deep point as x_0. If x_0 is static we are done. If not, then $\exists x_1 \in A_* : x_0 R x_1$, x_1 is deep and $U_{x_0} \neq U_{x_1}$. As $x_0 R x_1$, $U_{x_1} \subseteq U_{x_0}$, so $U_{x_1} \subset U_{x_0}$ and $x_1 \not R x_1$. Then, either x_1 is static or not. Continuing in this way, if A_* contains no static points we find $x_0 R x_1 R x_2 ...$ such that $U_{x_0} \supset U_{x_1} \supset U_{x_2} ....$ Then we consider $y_i \in U_{x_i} \setminus U_{x_{i+1}}$. Each of these points is not deep, if $i > j$ then $y_j \in U_{x_j}$ and $y_j \notin U_{x_{j+1}}$. So $x_{j+1} \not R y_i$, and $j + 1 \leq i$ so $y_i \in U_{x_{j+1}}$, $x_{j+1} R y_i$ and $y_i \not R y_j$.

Thus, we have a non-descending sequence $\langle y_i \rangle_{i \in \omega}$ in $R^+[x]$. As A_* is of finite width $R^\omega[x]$ contains no infinite anti-chains, thus by 2.21 our non-descending sequence has an infinite ascending sub-sequence contradicting that all the y_i are not deep. So A_* contains a deep static point x.

Letting $x R x_1 R x_2 ...$ be an infinite ascending chain starting at x, we note that $\forall n \in \omega$ x_n is deep and if $x_n R y$ such that y is deep, then $x R y$ so $U_{x_n} = U_x = U_y$, so x_n is static. We also define for $y \in X$ the set

$$V_y := \{col(z) \in 2^n : yRz \text{ and } z \text{ is deep}\}.$$

We say a deep point is stationary iff $\forall y R z : z$ is deep $V_y = V_z$. As if yRz $V_z \subseteq V_y$ and each V_y is finite, every infinite ascending chain contains a stationary point, so in particular $\exists n \in \omega : x_n$ is stationary. So we have found a static and stationary point.

We now argue by induction that $\forall U \subseteq A_*$ clopen, that $\forall y, z \in R[x_n] : y$ and z are deep and $col(y) = col(z)$ that $y \in U$ iff $z \in U$.

Base Case: $U = \varphi(g_i)$. Then, letting $y, z \in R[x_n]$ as above, $col(y) = col(z)$ $y \in U$ iff $g_i \in y$ iff $g_i \in z$ iff $z \in U$.

Induction step: $\cap$ and $\setminus$ are trivial, so let $U = \Box V = \{u \in X : R[u] \subseteq V\}$, where V is clopen and has the property. Letting $y, z \in R[x_n]$ be deep and $col(y) = col(z)$, if $y \in U$ then $y \in \Box V$ so $R[y] \subseteq V$. Letting $w \in R[z]$ either w is not deep and $w \in U_z = U_{x_n} = U_y$ so yRw and $w \in V$ or w is deep, so $col(w) \in V_z = V_{x_n} = V_y$, so $\exists v \in R[y] : v$ is deep and $col(w) = col(v)$. Both $w, v \in R[x_n]$, so by induction $w \in V$ iff $v \in V$. $v \in R[y] \subseteq V$ so $w \in V$. Either way $w \in V$ so $R[z] \subseteq V$ and $z \in U$. If $z \in U$ the case is symmetric.

Finally, x_n sees infinitely many deep points, so $\exists y, z \in R[x_n] : y \neq z$, y and z are deep and $col(y) = col(z)$. But then $\forall U \subseteq X$ clopen $y \in U$ iff $z \in U$ contradicting Stone separation. □

Another useful property of finitely generated transitive spaces is how well regulated their points of finite depth are. These results are adaptions of similar results established for Esakia spaces in [4, Chapter 3].

Given $\mathcal{X} \in TS$ and two clusters C and D of X, we recall that C sees D iff $\exists x \in C, \exists y \in D : xRy$ or $C = D$ and that the clusters of X under the seeing relation form a poset.

Lemma 2.23. Let $\mathcal{X} \in TS$, then X contains maximal clusters.

Proof. We aim to apply Zorn's lemma, so let $\langle C_\alpha \rangle_{\alpha \in I}$ be an R'-chain of clusters in X indexed by an arbitrary set I. We may assume w.l.o.g that they are distinct. Moreover, note that letting $x' \in C$ and $y' \in D : CR'D$ where C and D are distinct clusters then $\exists x \in C, \exists y \in D : xRy$ and so $x'RxRyRy'$ and $x'Ry'$. So, letting $x_\alpha \in C_\alpha$ we obtain an R-chain $\langle x_\alpha \rangle_{\alpha \in I}$ of X. Then, if $x \in X$ is an upper bound for $\langle x_\alpha \rangle$ then letting $\alpha \in I$ $x_\alpha R x$ so $C_\alpha R'[x]$, i.e. the cluster containing x is an upper bound for the cluster chain. So, it is sufficient to show that every strict R-chain in X has an upper bound.

From our duality we may consider the $A \in K4\text{-}A : A_* \cong \mathcal{X}$, then it is sufficient to check that every R-chain in A_* has an upper bound. Let $\langle F_\alpha \rangle_{\alpha \in I}$ be a strict chain of prime filters in A_* and consider:

$$F' := \bigcup_{\alpha \in \omega} \{a \in A : \Box a \in F_\alpha\}.$$

Letting $a, b \in F'$ $\Box a \in F_\alpha$ and $\Box b \in F_\beta$ for $\alpha, \beta \in I$. Taking $\gamma = max\{\alpha, \beta\} + 1$, $\alpha, \beta < \gamma$ so $F_\alpha R F_\gamma$ and $F_\beta R F_\gamma$. Then $\Box a \leq \Box\Box a$ so $\Box\Box a \in F_\alpha$ and $\Box a \in F_\gamma$, similarly $\Box b \in F_\gamma$. So $\Box(a \wedge b) = \Box a \wedge \Box b \in F_\gamma$ and so $a \wedge b \in F'$. If $a \in F'$ and $a \leq b$ then $\Box a \in F_\alpha : \alpha \in I$ and $\Box a \leq \Box b$ so $\Box b \in F_\alpha$ and $b \in F'$. So F' is a filter of A. Moreover, if $\Box\bot \in F_\alpha$ for some $\alpha \in I$ then $\bot \in F_{\alpha+1}$ which is a contradiction. So $\bot \notin F'$.

Thus, by the prime filter theorem for Boolean algebras $\exists F \in A_* : F' \subseteq F$. Then letting $\Box a \in F_\alpha$ $a \in F'$ so $a \in F$, so $F_\alpha R F$ and F is an upper bound for the chain as required. □

Corollary 2.24. Let $\mathcal{X} \in TS$ and $Y \subseteq X$ be clopen. Then Y contains R' maximal clusters.

Proof. This is an immediate consequence of applying lemma 2.23 to the sub-frame Y. This can also be checked directly via the correspondence of sub-frames and relativisations. The proof proceeds as above except we use lemma 2.10 to find $a \in A$ such that $Y = \varphi(a)$ and then take a chain of primes filters within $\varphi(a)$. Finally, we must much check the filter F' defined in the previous proof also contains a and so is still within $\varphi(a)$. □

Letting $\mathcal{X} \in TS$, we say a point $x \in X$ is *maximal* iff $\forall y \in X$ if xRy then either yRx or $y = x$, i.e. x belongs to an R' maximal cluster. We define $max(X)$ as the set of maximal points of X.

Lemma 2.25. Let $\mathcal{X} \in TS$ be finitely generated and consist only of improper clusters. Then letting $max(X)$ is finite and clopen.

Proof. From the duality, letting $A \in K4\text{-}A : A_* \cong \mathcal{X}$ we will work on A_* as opposed to $\mathcal{X}$. Let $g_1, ..., g_n$ generate A. We first check finiteness. Letting $x, y \in max(A_*) \cap \{u \in A_* : uRu\}$, we consider the relation:

$$E := \{(x, y), (y, x)\} \cup \{(u, u) \in A_*^2 : u \in A_*\}.$$

That is, the smallest equivalence relation E on $A_* : xEy$. We claim E is a modal equivalence.

For (i); if uEv and $uRw : u, v \notin \{x, y\}$ then $u = v$ so vRw and we may take w as witness. For xEy and xRz, as $x \in max(A_*)$ zRx and x and z are in the same clusters. A_* consists of only improper clusters so $x = z$ Then yRy and yEx so we may take y as witness. The case when yEx and yRz is symmetric.

For (ii); if $u \not E v$ then if $u \in \{x, y\}$ then $v \notin \{x, y\}$. We apply Stone separation on x and v and y and v to find clopens U_x^v and U_y^v respectively, then $u \in U_x^v \cap U_y^v$ and $v \notin U_x^v \cup U_y^v$ which is clopen and a union of E classes so seperates u and v as required. If $u \notin \{x, y\}$ and $v \in \{x, y\}$ we use the compliment of $U_x^u \cup U_y^u$, and if $u, v \notin \{x, y\}$ then apply Stone separation to u and v and then $U_u^v \cup U_x^v \cup U_y^v$ is clopen and a union of E-classes so seperates u and v as required.

So E is a proper modal equivlaence, and so by the colouring theorem has a class containing points of different colours. The only non-singular E-class is $\{x, y\}$ so $col(x) \neq col(y)$. So any reflexive maximal points in A_* have different colours, and there are only 2^n different colours and so $max(A_*) \cap \{u \in A_* : uRu\}$ is finite. We can similarly consider $max(A_*) \cap \{u \in A_* : u\not{R}u\}$, running the same proof as above, except that when considering xEy or yEx for condition (i) as $x, y \in max(A_*) \cap \{u \in A_* : u\not{R}u\}$ we have $R[x] = \varnothing = R[y]$ so the case is trivial. Then $max(A_*) = (max(A_*) \cap \{u \in A_* : uRu\}) \cup (max(A_*) \cap \{u \in A_* : u\not{R}u\})$ so $max(A_*)$ is finte.

Next we check clopenness. Consider the element $g \in A$ defined by:

$$g := \bigwedge_{i=1}^{n} ((g_i \to \Box g_i) \wedge (\neg g_i \to \Box \neg g_i)).$$

We will prove that $\varphi(g) = max(A_*)$. If $x \in max(A_*)$ then for each $1 \leq i \leq n$ either $g_i \in x$ or $\neg g_i \in x$. If $g_i \in x$ then $x \in \varphi(\neg g_i \to \Box \neg g_i)$ and as $x \in max(A_*)$ and A_* consists only of improper clusters $R[x] = \{x\}$ or $\varnothing$ so $R[x] \subseteq \varphi(g_i)$, giving $x \in \Box\varphi(g_i) = \varphi(\Box g_i)$. Thus, $x \in \varphi(g \to \Box g_i)$ and $x \in \varphi(g)$. If $\neg g_i \in x$ then symmetrically $x \in \varphi(g)$, and so $max(A_*) \subseteq \varphi(g)$.

Now, let $x \in A_* : x \in \varphi(g)$. We define the sets J and J' and the element $\eta \in A$ by:

$$J := \{g_i \wedge \Box g_i \in A : g_i \in x\} \text{ and } J' := \{\neg g_i \wedge \Box \neg g_i \in A : \neg g_i \in x\}.$$

$$\eta := \bigwedge J \wedge \bigwedge J'.$$

Consider $\varphi(\eta)$. This is clopen, it is also an upset; letting $u \in \varphi(\eta)$ and uRv then letting $g_i \wedge \Box g_i \in J$ $u \in \varphi(\eta)$ implies $\Box g_i \in u$ so $g_i \in v$ and letting $w \in A_* : vRw$ uRw so again $g_i \in w$ so $\Box g_i \in v$. Similarly for $\neg g_i \wedge \Box \neg g_i \in J'$. So $\eta \in v$ and $v \in \varphi(\eta)$.

It is also monochrome; letting $u \in \varphi(\eta)$ for each $1 \leq i \leq n$ either $g_i \in x$ and so $g_i \wedge \Box g_i \in J$ and $g_i \wedge \Box g_i \in u$ so $g_i \in u$ or $\neg g_i \in x$ and $\neg g_i \in u$, so $col(u) = col(x)$.

Finally, $x \in \varphi(\eta)$, as letting $g_i \wedge \Box g_i \in J$, then $g_i \in x$ and as $g \in x$ $g_i \to \Box g_i \in x$ so $\Box g_i \in x$ and $g_i \wedge \Box g_i \in x$. Similarly if $\neg g_i \wedge \neg \Box g_i \in J'$ then $\neg g_i \wedge \Box \neg g_i \in x$, so $\eta \in x$ and $x \in \varphi(\eta)$.

So $\varphi(\eta)$ is a clopen upset, by lemma 2.16 we can consider the modal equivalence E identifying points within it. Then the only possibly non-singleton E-class of this equivalence is $\varphi(\eta)$ itself, which is monochrome. So all the E-classes are monochrome and by the colouring theorem E cannot be proper, i.e. all E-classes are singletons. Then, as $x \in \varphi(\eta)$, $\varphi(\eta) = \{x\}$. Finally, as $\varphi(\eta)$ is an upset, $\forall y \in A_*$ if xRy then $y \in \varphi(\eta)$ so $y = x$, i.e. $x \in max(A_*)$ and we have $\varphi(g) \subseteq max(A_*)$. □

Corollary 2.26. Let $\mathcal{X} \in TS$ be finitely generated. Then $max(X)$ is finite and clopen.

Proof. We consider the modal equivalence on $\mathcal{X}$ induced by lemma 2.15 identifying points in the same cluster. Then $\mathcal{X}/E \in TS$ and is also finitely generated, by lemma 2.10 $(\mathcal{X}/E)^*$ is isomorphic to a subalgebra of $\mathcal{X}^*$ and so is finitely generated as a K4-algebra. It also consists of only improper clusters, so applying lemma 2.25, $max(X/E)$ is finite and clopen. Then $max(X)$ is the inverse image of $max(X/E)$ and so is clopen and contains finitely many clusters. Then, by 2.20 these clusters are themselves finite, so $max(X)$ is also finite. □

We can extend this result beyond the maximal points in a finitely generated transitive space to all its points of finite depth.

Theorem 2.27. Let $\mathcal{X} \in TS$ be finitely generated. Then, $\forall n \in \omega\ Sl_n(X)$ is finite and $S_n(X)$ is clopen. Moreover, $\forall n \in \omega\ Sl_n(A_*)$ is clopen.

Proof. We proceed by induction on n, again letting $A \in K4\text{-}A$ be such that $A_* \cong \mathcal{X}$ we work on A_* instead. We let $g_1, ..., g_k$ generate A. As $Sl_1(A_*) = max(A_*)$ the base case is just corollary 2.26, so let $n \in \omega : \forall m \leq n\ Sl_m(A_*)$ is finite and $S_m(A_*)$ is clopen. Consider the clopen subset $A_* \setminus S_n(A_*)$ and from lemma 2.10 let $A_n \in K4\text{-}A$ be the corresponding relativisation of A. We claim that A_{n_*} is finitely generated.

If so, then by corollary 2.26 $max(A_{n_*})$ is finite and clopen. Letting $x \in Sl_{n+1}(A_*)$, $x \notin S_n(A_*)$ and letting $xRy : y \in A_{n_*}$ then as $y \in A_{n_*}$ $d(y) \geq n+1$ and as xRy $d(y) \leq n+1$ so $d(y) = n+1$ and yRx. So either yRx or $y = x$, i.e. $x \in max(A_{n_*})$. If $x \notin Sl_{n+1}(A_*)$ then either $x \in S_n(A_*)$ and so $x \notin A_{n_*}$ or $d(x) > n+1$. As by theorem 2.22 A_* has no infinitely ascending chains we have $\forall k < d(x)\ R[x] \cap Sl_k(A_*) \neq \varnothing$ so $\exists y \in Sl_{n+1}(A_*) : xRy$ and $y\not{R}x$. Then $y \in A_{n_*}$ and so $x \notin max(A_{n_*})$. Together, this means $Sl_{n+1}(A_*) = max(A_{n_*})$, and so is finite and clopen. Then $S_{n+1}(A_*) = S_n(A_*) \cup Sl_{n+1}(A_*)$ and so is also clopen completing the induction.

It remains to prove the claim. As $S_n(A_*)$ is clopen $\exists a \in A : \varphi(a) = S_n(A_*)$, moreover as $S_n(A_*)$ is finite and clopen, it has a finite number of subsets all of which are clopen as well. In particular, its upsets are clopen, so letting $\{U_j\}_{j=1}^m$ be those upsets, we let $a_j \in A : \varphi(a_j) = U_j$.

We consider two collections of elements; first we define the elements of A:

$$g'_i := a \vee g_i \text{ and } g'_{k+j} := a \vee \Box(a \to a_j)$$

. Second, we let $g''_1, ..., g''_{k+m}$ be the corresponding elements of A_n, i.e. $g''_i \in A_n : \varphi(g''_i) = \varphi(g'_i) \cap A_{n_*}$. These new elements define their own colouring of A_{n_*}, letting $x \in A_{n_*}$ we'll use $col(x)$ to denote its colour by the g_i and $col_n(x)$ the colour by g''_i. We claim that the g''_i generate A_{n_*}.
Note, letting $x, y \in A_{n_*}$ if $col_n(x) = col_n(y)$ then, if $g_i \in x$, $g'_i \in x$ so $g'_i \in y$ and then $y \in A_{n_*}$ means $y \notin S_n(A_*)$ so $a \notin y$. So $g_i \in y$. Similarly, if $g_i \in y$ then $g_i \in x$, so $col(x) = col(y)$.

Now, suppose A_{n_*} is not generated by the g''_i, then by the colour theorem there is a proper modal equivalence E of A_{n_*} such that all E-classes are monochrome.

We define a relation Q on A_* as follows:

$$Q := E \cup \{(u,u) \in A_*^2 : u \in S_n(A_*)\}.$$

That is, the smallest equivalence relation on A_* containing E. We claim this is a modal equivalence.

For (i); letting xQy and xRz, if x or $y \in S_n(A_*)$ then $x = y$ so yRz and we may take z as witness. If $x, y \in A_{n_*}$ then xEy. Now, if $z \in A_{n_*}$ as E is a modal equivalence on A_{n_*} $\exists v \in A_{n_*} : yRv$ and zEv. Then zQv and we may take v as witness. If $z \notin A_{n_*}$ either yRz so or $y\not{R}z$. If, $y\not{R}z$ then moreover $z \neq y$ (as $y \in A_{n_*}$) and so by modal separation there is a clopen upset $U : y \in U, z \notin U$. $U \cap S_n(A_*)$ is an upset contained in $S_n(A_*)$ so equals U_j for some $1 \leq j \leq m$. Letting $u \in S_n(A_*) : yRu$ then $u \in U$ so $u \in U_j$, therefore $y \in \Box\varphi(a \to a_j) = \varphi(\Box(a \to a_j))$, i.e. $g'_{n+j} \in y$. Therefore, $g''_{n+j} \in y$.

By contrast, xRz, $z \in S_n(A_*)$ and $z \notin U$ so $z \notin U_j$, i.e. $x \notin \varphi(\Box(a \to a_j)$ Moreover, $x \notin S_n(A_*)$ means $a \notin x$ so $g'_{n+j} \notin x$ and so $g''_{n+j} \notin x$. So $col_n(x) \neq col_n(y)$, contradicting that xEy. So, we must have yRz and then again we may take z as witness.

For (ii); letting xQy, if $x \in S_n(A_*)$ then $x \neq y$, we apply Stone separation to find a clopen $U \subseteq A_*$ separating x and y, then $U \cap S_n(A_*)$ is clopen, a union of Q-classes and separates x and y as required. If $y \in S_n(A_*)$ the case is symmetric. If $x, y \notin S_n(A_*)$, then $x\not{E}y$ so there is a clopen subset $U \subseteq A_{n_*}$ separating x and y which is a union of E-classes. Then $U \cap A_{n_*}$ is a clopen subset of A_* and a union of Q classes so separates x and y as required.

So Q is a modal equivalence, and is proper as E was proper. Then $\forall x, y \in A_* :$ xQy, if x or $y \in S_n(A_*)$ then $x = y$ and $col(x) = col(y)$ and if $x, y \in A_{n_*}$ then xEy so $col_n(x) = col_n(y)$ and $col(x) = col(y)$ as we noted earlier. So Q is a proper modal equivalence of A_* which is monochrome by the g_i colouring. Thus, by the colouring theorem A_* is not generated by the g_i which is a contradiction. □

This completes our study of transitive spaces, along with our presentation of Jónsson-Tarski duality. Having introduced the both K4-algebras and transitive spaces and their basic properties, . We also highlighted a number of dual properties that we will frequently use in the main investigation (lemmas 2.9 & 2.10). To aid our main investigation we greatly expanded our understanding of transitive spaces. We've provided a number of useful tools for our proof work, including a separation axiom (lemma 2.11), an alternative way to think about surjective maps between finite transitive spaces in the form of reductions (lemma 2.14) and a group of important modal equivalences we can use to simply spaces we are working with (lemmas 2.15, 2.16, 2.17 & 2.18). We also have a much better understanding of the behaviour of finite transitive spaces, establishing that they are conversely well founded (lemma 2.20 & theorem 2.22) and how their elements of finite depth behave (theorem 2.27).

Chapter 3

Algebraic Logic

The core theory that enables investigations like ours is the tight relationship one can establish between logic and algebra.

3.1 Algebraic Modal Logic

3.1.1 Universal Algebra

The central idea of algebraic logic is to use the tools of universal algebra to investigate logic. In universal algebra, we abstract away from particular algebraic structures such as rings or modal algebras and consider an algebra as a set accompanied by a collection of constant terms and operators. We will give a brief introduction to the concepts most relevant for our primary investigation. For a more detailed study, the reader may consult [1, 9, 18].

Definition 3.1. A *language*, or *signature*, is a collection $\mathcal{L}$ of function symbols each with an associated arity. We call function symbols with arity 0 constants.

An $\mathcal{L}$-*algebra* is a set A accompanied by an element of the set for each constant in $\mathcal{L}$ and a function from A to A for each function symbol in $\mathcal{L}$ with the same arity. We frequently use A to refer to an algebra and its underlying set interchangeably.

An $\mathcal{L}$-*morphism* is a map between two $\mathcal{L}$-algebras that respects the terms and operators of $\mathcal{L}$.

Given a set of variables P, we define the *term algebra* for $\mathcal{L}$ as follows.

We define the set of terms over P as the least set $Fm_{\mathcal{L}}(P)$ such that:

(i) $P \subseteq Fm_{\mathcal{L}}(P)$;

(ii) If c is a constant in $\mathcal{L}$ then $c \in Fm_{\mathcal{L}}(P)$;

(iii) If $\varphi_1, ..., \varphi_n \in Fm_{\mathcal{L}}(P)$ and f is a function symbol in $\mathcal{L}$ with artiy n, then $f(\varphi_1, ..., \varphi_n) \in Fm_{\mathcal{L}}(P)$.

Then, the term algebra is the unique algebra with underlying set $Fm_{\mathcal{L}}(P)$ accompanied by a basic n-ary operation f' defined, for every $\varphi_1, ..., \varphi_n \in Fm_{\mathcal{L}}(P)$, as

$$f'(\varphi_1, ..., \varphi_n) := f(\varphi_1, ..., \varphi_n).$$

We frequently shorten this to simply Fm when $\mathcal{L}$ and P are understood.

Definition 3.2. Am $\mathcal{L}$-*equation* is an expression of the form $\epsilon \approx \delta$ where $\epsilon, \delta \in Fm_{\mathcal{L}}(P)$.

We say that a $\mathcal{L}$-algebra A satisfies an equation $\epsilon \approx \delta$ iff $h(\epsilon) = h(\delta)$ for all $\mathcal{L}$-morphisms $h : Fm \to A$. We denote this $A \models \epsilon \approx \delta$.

Given a class of $\mathcal{L}$-algebras $\mathcal{A}$, we define the equational consequence relation relative to $\mathcal{A}$, $\models_{\mathcal{A}}$, as follows. Let $\Theta \cup \{\epsilon \approx \delta\}$ be a set of equations. Then $\Theta \models_{\mathcal{A}} \epsilon \approx \delta$ iff $\forall A \in \mathcal{A}$ and all $\mathcal{L}$-morphisms $h : Fm \to A$ if $\forall \varphi \approx \psi \in \Theta\ h(\varphi) = h(\psi)$ then $h(\epsilon) = h(\delta)$.

A *variety* is a class of algebras $\mathcal{A}$ that is equationally definable, that is there is a set of equations Θ such that for any algebra A we have that $A \in \mathcal{A}$ iff $A \models \epsilon \approx \delta$ for all $\epsilon \approx \delta \in \Theta$.

Given a class of algebras $\mathcal{A}$ we denote by $\mathbb{V}(\mathcal{A})$ the least variety containing $\mathcal{A}$.

An alternative way to look at varieties is through class operations. We denote by $\mathbb{I}, \mathbb{H}, \mathbb{S}, \mathbb{P}$ and $\mathbb{P}_U$ the class operators of closure under isomorphism, homomorphic images, subalgebras, direct products and ultraproducts respectively. We assume direct products and ultra products of empty families of algebras are trivial algebras.

Theorem 3.3 (Birkhoff's Theorem)**.**
A class of algebras $\mathcal{A}$ is a variety iff it is closed under $\mathbb{H}, \mathbb{S}$ and $\mathbb{P}$.

Proof. See [10, Theorem 7.79]. □

Theorem 3.4 (Tarski's Theorem)**.**
Given a class of algebras $\mathcal{A}$, $\mathbb{V}(\mathcal{A}) = \mathbb{HSP}(\mathcal{A})$.

Proof. See [10, Theorem 7.8]. □

We can generalise this set up slightly further.

Definition 3.5. A *quasi-equation* is an expression of the form

$$\Phi = \bigwedge_{i \leq n} \varphi_i \approx \psi_i \to \epsilon \approx \delta.$$

Note that an equation $\epsilon \approx \delta$ can be effectively identified with the quasi-equation $\varnothing \to \epsilon \approx \delta$.

We say that an $\mathcal{L}$-algebra satisfies a quasi-equation $\Phi = \bigwedge_{i \leq n} \varphi_i \approx \psi_i \to \epsilon \approx \delta$, denoted $A \models \Phi$, iff for all $\mathcal{L}$-morphisms $h : Fm \to A$ if for all $1 \leq i \leq n\ h(\varphi_i) = h(\psi_i)$ then $h(\epsilon) = h(\delta)$.

Given a class of algebras $\mathcal{A}$ we say that Φ is valid in $\mathcal{A}$ iff $\{\varphi_i \approx \psi_i : i \leq n\} \models_{\mathcal{A}} \epsilon \approx \delta$.

A *quasi-variety* is a class of algebras $\mathcal{A}$ that is quasi-equationally definable, that is there is a set of quasi-equations Θ such that for any algebra A we have that $A \in \mathcal{A}$ iff $A \models \Phi$ for all $\Phi \in \Theta$.

Given a class of algebras $\mathcal{A}$, we denote by $\mathbb{Q}(\mathcal{A})$ the quasi-variety containing $\mathcal{A}$

Theorem 3.6 (Maltsev's Theorem)**.**
A class of algebras is a quasi-variety iff it is closed under $\mathbb{I}$,$\mathbb{S}$,$\mathbb{P}$ and $\mathbb{P}_U$.

Proof. See [9, Theorem V2.25]. □

Theorem 3.7. Give a class of algebras $\mathcal{A}$, $\mathbb{Q}(\mathcal{A}) = \mathbb{ISPP}_U(\mathcal{A})$.

Proof. See [9, Theorem V2.25]. □

An important property of varieties for our investigation is that of being primitive. This is because, as we will formalise shortly, being primitive is an 'algebraic counterpart' to hereditary structural completeness.

A class $M \subseteq \mathcal{A}$ is a *sub-variety* or *subquasi-variety* of $\mathcal{A}$ iff M is a variety or quasi-variety. A variety $\mathcal{A}$ is said to be *primitive* iff every subquasi-variety M of $\mathcal{A}$ is a variety.

3.1.2 Logic

In our context we start with a very abstract notion of a logic, that of consequence relations and deductive systems. The advantage of this very abstract framing is that we can give a very precise correspondence between finitary deductive systems and quasi-varieties of algebras. Again, here we give a only brief overview of this process. For a more detailed study, the reader may consult [6, 15].

Definition 3.8. Given a set A a *consequence relation* on A is a relation $\vdash \subseteq \mathcal{P}(A) \times A$ such that for every $X \cup Y \cup \{x\} \subseteq A$:

(i) If $x \in X$ then $X \vdash x$;

(ii) If $\forall y \in Y\ X \vdash y$ and $Y \vdash x$ then $X \vdash x$.

Then, given a signature $\mathcal{L}$ and set of variables P, a *deductive system* is a consequence relation of Fm such that for any $\mathcal{L}$-morphism $\sigma : Fm \to Fm$, $\forall \Gamma \cup \{\varphi\} \subseteq Fm$ if $\Gamma \vdash \varphi$ then $\sigma[\Gamma] \vdash \sigma(\varphi)$. We call $\mathcal{L}$-morphisms like σ substitutions and consequence relations with the above property are called substitution invariant.

A deductive system $\vdash$ is *finitary* iff $\forall \Gamma \cup \{\varphi\} \subseteq Fm$, if $\Gamma \vdash \varphi$ then $\exists \Delta \subseteq \Gamma$ which is finite and such that $\Delta \vdash \varphi$.

Given a set of formulas in at most two variables $\Delta(x, y)$ and a set of equations $\Theta \cup \{\epsilon \approx \delta\}$ we define:

$$\Delta(\epsilon \approx \delta) := \{\varphi(\epsilon, \delta) : \varphi(x, y) \in \Delta(x, y)\}$$

$$\Delta[\Theta] := \bigcup_{\epsilon\approx\delta\in\Theta} \Delta(\epsilon \approx \delta)$$

Similarly, given a set of equations in at most one variable $\tau(x)$ and $\Gamma \cup \{\varphi\} \subseteq Fm$, we define:

$$\tau(\varphi) := \{\epsilon(\varphi) \approx \delta(\varphi) : \epsilon(x) \approx \delta(x) \in \tau(x)\}$$

$$\tau[\Gamma] := \bigcup_{\varphi\in\Gamma} \tau(\varphi)$$

Definition 3.9. A finitary deductive system $\vdash$ is said to be *algebraizable* iff there exists a quasi-variety $\mathcal{A}$, a set of equations $\tau(x)$ and set of formulas $\Delta(x, y)$ such that for all sets of equations $\Theta \cup \{\epsilon \approx \delta\}$ and sets of formulas $\Gamma \cup \{\varphi\}$:

Alg1 $\Gamma \vdash \varphi$ iff $\tau[\Gamma] \models_{\mathcal{A}} \tau(\varphi)$;

Alg2 $\Delta[\Theta] \vdash \Delta(\epsilon, \delta)$ iff $\Theta \models_{\mathcal{A}} \epsilon \approx \delta$;

Alg3 $\varphi \dashv\vdash \Delta[\tau(\varphi)]$;

Alg4 $\epsilon \approx \delta \mathrel{=\!\mid\!\models_{\mathcal{A}}} \tau[\Delta(\epsilon, \delta)]$.

Equivalently, when $\Gamma \vdash \varphi$ iff $\tau[\Gamma] \models_{\mathcal{A}} \tau(\varphi)$ and $x \approx y \mathrel{=\!\mid\!\models_{\mathcal{A}}} \tau[\Delta(x, y)]$ [6, Corollary 2.9].

We call $\mathcal{A}$ an *equivalent algebraic semantics* (EAS) for $\vdash$. Every algebraizable finitary deductive system has a unique equivalent algebraic semantics [6, Theorem 2.15].

For example, **IPC** is a finitary deductive system for $\mathcal{L} = \{\wedge, \vee, \rightarrow, \top, \bot\}$ and has the variety of Heyting algebras as its EAS under $\tau(x) = \{x \approx \top\}$ and $\Delta(x,y) = \{x \rightarrow y, y \rightarrow x\}$[5].

Of course, we are specifically interested in modal logic. Briefly recalling the basic set up, a normal modal logic (NML) is a set of formulas λ in signature $(\wedge, \vee\neg, \Box, \top)$ such that:

(i) λ contains all the classical tautologies.

(ii) $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q) \in \lambda$ for all propositional variables p and q.

(iii) λ is closed under Modus Ponens, necessitation ($\varphi \in \lambda$ implies $\Box\varphi \in \lambda$) and substitution.

The least NML is called K, and given $\varphi \in Fm$ we use $K + \varphi$ to denote the least NML containing φ, e.g. $K4 = K + \Box p \rightarrow \Box\Box p$ and $S4 = K4 + \Box p \rightarrow p$.

In the basic set up NMLs are identified with sets of formulas, but we wish to view them as finitary deductive systems. There are at least two ways to define a finitary deductive system given a NML λ, which are as follows.

Definition 3.10. Let λ be a normal modal logic.

We define the finitary deductive system λ_g by $\forall \Gamma \cup \{\varphi\} \subseteq Fm$ $\Gamma \vdash_{\lambda_g} \varphi$ iff φ is derivable from Γ using the theorems of λ and the inference rules Modus Ponens and necessitation.

We define the finitary deductive system λ_l by $\forall \Gamma \cup \{\varphi\} \subseteq Fm$ $\Gamma \vdash_{\lambda_l} \varphi$ iff φ is derivable from Γ using the theorems of λ and the inference rule Modus Ponens.

Our focus is on λ_g precisely because it has an EAS witnessed by a variety of modal algebras whilst this is not the case for λ_l [6, Corollary 5.6].

Theorem 3.11 (Algebraic Modal Logic)**.**
Let λ be a normal modal logic. Then $\vdash_{\lambda_g}$ is algebraizable with EAS:

$$\mathcal{A}_\lambda = \{A \in MA : \forall \varphi \in \lambda, \models_A \varphi \approx \top\};$$

$$\tau(x) = \{x \approx \top\} \text{ and } \Delta(x,y) = \{x \rightarrow y, y \rightarrow x\}.$$

Moreover, $\mathcal{A}_{K_g} = $ MA, $\mathcal{A}_{K4_g} = $ K4-A and $\mathcal{A}_{S4_g} = $ S4-A.

Proof. See [15, Examples 2.17, Proposistion 3.15]. □

Going forward we'll suppress a lot of this notation and talk of a normal modal logic λ to refer to the finitary deductive system λ_g. We will also say that a normal modal logic λ has EAS $\mathcal{A}$ to refer to λ_g having EAS $\mathcal{A}_\lambda$, $\tau(x)$ and $\Delta(x,y)$.

3.1.3 Hereditary Structural Completeness & Primitive Varieties

The correspondence between logic and algebra becomes useful because properties of logical systems frequently have natural and well understood algebraic mirrors. This is the framework for our project.

Let $\vdash$ be a deductive system. A deductive system $\vdash'$ in the same language is said to be an *extension* of $\vdash$ iff for every $\Gamma \cup \{\varphi\} \subseteq Fm$ if $\Gamma \vdash \varphi$ then $\Gamma \vdash' \varphi$.

A *rule* is an expression of the form $\Gamma \rhd \varphi$ where $\Gamma \cup \{\varphi\}$ is a finite subset of Fm.

Definition 3.12. A rule $\Gamma \rhd \varphi$ is said to be *admissible* in $\vdash$ iff for all substitutions σ if $\forall\gamma \in \Gamma \vdash \sigma(\gamma)$ then $\vdash \sigma(\varphi)$.

A rule $\Gamma \rhd \varphi$ is said to be *derivable* in $\vdash$ iff $\Gamma \vdash \varphi$.

Accordingly we say that $\vdash$ is *structurally complete* (SC) iff every rule that is admissible in $\vdash$ is also derivable in $\vdash$ and $\vdash$ is *hereditarily structurally complete* (HSC) iff every finitary extension of $\vdash$ is structurally complete.

Remarks. Hereditary structural completeness is sometimes defined as the property that every *axiomatic* extension of $\vdash$ is structurally complete. This is equivalent to our given definition by theorem 3.2 in [24].

The critical comparison for our purposes is the following two theorems.

Theorem 3.13. Let $\vdash$ be a algebraizable finitary deductive system with variety $\mathcal{A}$ as its EAS.

The lattice of axiomatic extensions of $\vdash$ is dually isomorphic to that of sub-varieties of K.

Proof. Included in [5, Section 2]. □

Theorem 3.14. Let $\vdash$ be a algebraizable finitary deductive system with variety $\mathcal{A}$ as its EAS.

$\vdash$ is hereditarily structurally complete iff $\mathcal{A}$ is primitive.

Proof. Included in [5, Section 2]. □

As axiomatic extensions of K4 have EAS witnissed by K4-A, the task of characterising hereditarily structurally complete axiomatic extensions of K4 is equivalent to that of characterising primitive sub-varieties of K4-A.

To this end, we will employ some standard results from universal algebra. These will allow us to give a sufficient and necessary condition for a variety to be primitive which centre around the algebraic property of being weakly projective. Let $\mathcal{A}$ be a variety. An algebra $A \in \mathcal{A}$ is *weakly projective in* $\mathcal{A}$ iff for every $B \in \mathcal{A}$, if $A \in \mathbb{H}(B)$ then $A \in \mathbb{IS}(B)$.

Our necessary condition for a variety to be primitive is straightforward.

Lemma 3.15. Let $\mathcal{A}$ be a primitive variety with finite signature. The finite non-trivial FSI members of $\mathcal{A}$ are weakly projective in $\mathcal{A}$.

Proof. See [5, lemma 2.1]. □

Establishing our sufficient condition requires a little more work. We start with sufficiency condition for varieties with the additional property of being locally finite. A variety is said to be *locally finite* when its finitely generated members are finite.

Theorem 3.16. A locally finite variety $\mathcal{A}$ is primitive iff its finite, non-trivial FSI members are weakly projective in $\mathcal{A}$.

Proof. See [5, Theorem 2.2]. □

As explored by Bezhanisvhili and Moraschini [5], in the case of intermediate logics and Esakia spaces theorem 3.16 is sufficient for the broader characterisation because primitive varieties of Heyting algebras are locally finite. In our main investigation we will be working with varieties that are not necessarily locally finite. Therefore, we need to establish a more general version of theorem 3.16 and for this we need a small amount of additional theory. First, there is another mirror of properties between logic and algebra.

Given an algebra A and elements $c, d \in A$ we denote the smallest congruence of A identifying c and d as $Cong_A(c,d)$. We say that a variety $\mathcal{A}$ has *equationally definable principal congruences* (EDPC) iff there is a finite set of equations $\Phi(x,y,z,w)$ such that $\forall A \in \mathcal{A}$ and $\forall a,b,c,d \in A$ $(a,b) \in Cong_A(c,d)$ iff $A \models \Phi(a,b,c,d)$.

Lemma 3.17. Let $\mathcal{A}$ be a variety with EDPC witnessed by $\Phi(x,y,z,w)$. Then letting $\Theta \cup \{\varphi \approx \psi, \epsilon \approx \delta\}$ be a set of equations; $\Theta, \varphi \approx \psi \models_{\mathcal{A}} \epsilon \approx \delta$ iff $\Theta \models_{\mathcal{A}} \Phi(\varphi, \psi, \epsilon, \delta)$.

Proof. See [7, Def 3.11, Theorem 5.4]. □

A finitary deductive system $\vdash$ has a *deduction detatchment theorem* (DDT) iff there exists a finite set of formulas I(x,y) such that for every set of formulas $\Gamma \cup \{\varphi, \psi\}$, $\Gamma, \varphi \vdash \psi$ iff $\Gamma \vdash I(\varphi, \psi)$.

Theorem 3.18. Let $\vdash$ be a algebraizable finitary deductive system with variety $\mathcal{A}$ as its EAS. $\vdash$ has a DDT iff $\mathcal{A}$ has EDPC.

Proof. See [7, Theorem 5.5]. □

Whilst the varieties of K4-algebras we will work with can fail to be locally finite, they all have EDPC. In fact this is a property of any variety of K4-algebras.

Lemma 3.19. Every variety of K4-algebras has EDPC.

Proof. Discussed in detail in [8]. In particular, theorem 5.4 and the examples discussed on page 597 imply our lemma. □

We also make use of another way to think about hereditary structural completeness. We say that a variety $\mathcal{A}$ has the *finite model property* (FMP) iff for any equation $\epsilon \approx \delta$ such that $\not\models_{\mathcal{A}} \epsilon \approx \delta$ there exists a finite algebra $A \in \mathcal{A}$ such that $A \not\models \epsilon \approx \delta$.

We denote free countably generated algebra of a variety as $F_{\mathcal{A}}(\omega)$.

Lemma 3.20. Let $\vdash$ be a algebraizable finitary deductive system with variety $\mathcal{A}$ as its EAS.

(i) $\vdash$ is SC iff $\mathcal{A} = \mathbb{Q}(F_{\mathcal{A}}(\omega))$;

(ii) $\vdash$ is HSC iff for all subvareities M of $\mathcal{A}$, $M = \mathbb{Q}(F_M(\omega))$.

Proof. For (i) see [24, Theorem 6.4]. From our earlier remark, $\vdash$ is HSC iff all its axiomatic extensions are SC. Then, (ii) follows by (i) and theorem 3.14. □

Lemma 3.21. Let $\mathcal{A}$ be a variety with the FMP and EDPC, and let $\mathcal{A}_{FinSI}$ be the class of finite, SI members of $\mathcal{A}$. Then $\mathcal{A} = \mathbb{Q}(\mathcal{A}_{FinSI})$

Proof. As $\mathcal{A}_{FinSI} \subseteq \mathcal{A}$, the inverse inclusion is immediate. For $\subseteq$; as $\mathbb{Q}(\mathcal{A}_{FinSI})$ is a quasi-variety it is quasi-equationally definable, so it is sufficient to show that if a quasi-equation fails in $\mathcal{A}$ it also fails in $\mathcal{A}_{FinSI}$. Repeated application of lemma 3.17 makes it is sufficient to consider a single premise quasi-equation.

Let $\varphi \approx \psi \to \alpha \approx \delta$ be a quasi equation failing in $\mathcal{A}$, i.e. $\exists A \in \mathcal{A}$ and $h : Fm \to A : h(\varphi) = h(\psi)$ and $h(\alpha) \neq h(\beta)$. Then, $(h(\alpha), h(\beta)) \notin Cong_A(h(\varphi), h(\psi))$ and so by EDPC $\not\models_A \Phi(h(\varphi), h(\psi), h(\alpha), h(\beta))$, and in particular $\not\models_{\mathcal{A}} \Phi(\varphi, \psi, \alpha, \beta)$. Then, as

$\mathcal{A}$ has FMP, and as a variety is generated by its SI elements, we can find $B \in \mathcal{A}_{FinSI}$ and $h' : Fm \to B$ such that $\not\models_B \Phi(h'(\varphi), h'(\psi), h'(\alpha), h'(\beta))$, and so by EDPC again $(h'(\alpha), h'(\beta)) \notin Cong_B(h'(\varphi), h'(\psi))$.

Now, every congruence of B is an intersection of completely $\cap$-irreducible congruences, therefore there is a completely $\cap$-irreducible congruence θ of B such that $Cong_B(h'(\varphi), h'(\psi)) \subseteq \theta$ and $(h'(\alpha), h'(\beta)) \notin \theta$. Then $B/\theta \in \mathcal{A}_{FinSI}$, and this alongside the quotient map composed with h' witness that $\varphi \approx \psi \not\models_{\mathcal{A}_{FinSI}} \alpha \approx \beta$. □

We can now establish our alternative to theorem 3.16 and sufficiency condition for a variety being primitive.

Theorem 3.22. Let $\mathcal{A}$ be a variety with EDPC and such that all its sub-varieties have FMP. If the finite, non-trivial FSI members of $\mathcal{A}$ are weakly projective in $\mathcal{A}$ then $\mathcal{A}$ is primitive.

Proof. By lemma 3.20 and theorem 3.14 it is sufficient to check that for all sub-varieties M of $\mathcal{A}$ that $M = \mathbb{Q}(F_M(\omega))$. So let M be a sub-variety of $\mathcal{A}$, by assumption M has EDPC and FMP. As $F_M(\omega) \in M$, $\mathbb{Q}(F_M(\omega)) \subseteq M$ immediately.
For the other inclusion; by lemma 3.21, $M = \mathbb{Q}(M_{FinSI})$. Then, letting $A \in M_{FinSI}$, as A is finite it is countably generated and in particular $A \in \mathbb{H}(F_M(\omega)) \subseteq M \subseteq \mathcal{A}$. A is finite and SI (and in particular FSI) and therefore by assumption weakly projective in $\mathcal{A}$. Therefore, $A \in \mathbb{IS}(F_M(\omega))$, which gives $M = \mathbb{Q}(M_{FinSI}) \subseteq \mathbb{Q}(F_M(\omega))$ as required. □

Putting our two conditions together in the context of K4-algebras reduces the problem of characterising their primitive varieties to the following.

Lemma 3.23. Let $\mathcal{A}$ be a variety of K4-algebras.

(i) If $\mathcal{A}$ is primitive then the finite, non-trivial, FSI members of $\mathcal{A}$ are weakly projective in $\mathcal{A}$.

(ii) Suppose all sub-varieties of $\mathcal{A}$ have the FMP. If the finite, non-trivial FSI members of $\mathcal{A}$ are weakly projective in $\mathcal{A}$ then $\mathcal{A}$ is primitive.

Proof. (i) is exactly lemma 3.15 whilst (ii) follows from theorem 3.22 and lemma 3.19. □

3.2 Order-Topological Semantics for Modal Logic

The theory established so far is essentially sufficient for our project. As K4 is algebrized by the variety K4-A, to characterise the hereditary structurally complete transitive modal logics it is sufficient to characterise the primitive varieties of K4-algebras. In a moment we will begin to undertake that task with our modal duality doing a lot of the heavy lifting.

However, in the modal case we can say a little more about the relationship between logic, algebra and topology. This is because we can give a direct order-toplogical semantics for NMLs, one which lines up with the picture already described and provides some additional context to neatly tie up the main theory. This semantics is a generalisation of the familiar Kripke semantics for modal logic (hence the alternative naming for modal spaces as descriptive Kripke frames), which we briefly recall here. For more detail see [10, Section 8] and [3].

Definition 3.24. Let (X, R) be a frame, $x \in X$ an element of the frame and $V : P \to \mathcal{P}(X)$ a valuation on the frame. Given a modal formula $\varphi \in Fm$ we define the truth of φ at x under V, denoted $x, V \models \varphi$ inductively as follows:

$$x, V \models p \text{ iff } x \in V(P).$$
$$x, V \models \psi \wedge \lambda \text{ iff } x, V \models \psi \text{ and } x, V \models \lambda.$$
$$x, V \models \psi \vee \lambda \text{ iff } x, V \models \psi \text{ or } x, V \models \lambda.$$
$$x, V \models \neg\psi \text{ iff } x, V \not\models \psi.$$
$$x, V \models \Box\psi \text{ iff } \forall y \in X : xRyy, V \models \psi.$$
$$x, V \models \Diamond\psi \text{ iff } \exists y \in X : xRy \text{ and } y, V \models \psi.$$

Then, given a modal space $\mathcal{X}$ and recalling that X^* denotes the set of clopen subsets of X, we define a consequence relation for $\mathcal{X}$, $\models_{\mathcal{X}}$, by $\forall \Gamma \cup \{\varphi\} \subseteq Fm, \Gamma \models_{\mathcal{X}} \varphi$ iff for all valuations $V : P \to X^*$ if $\forall x \in X\, \forall \gamma \in \Gamma, x, V \models \gamma$ then $\forall x \in Xx, V \models \varphi$.

Letting λ be a NML and $\mathcal{X} \in MS$, we say that $\mathcal{X}$ is a *λ-space* iff $\forall \Gamma \cup \{\varphi\} \subseteq Fm$ if $\Gamma \vdash_{\lambda} \varphi$ then $\Gamma \models_{\mathcal{X}} \varphi$.

Then, the natural relationship one would hope for holds.

Theorem 3.25. Let λ be an NML with EAS $\mathcal{A}$ and $\mathcal{X} \in MS$. Then $\mathcal{X}^* \in \mathcal{A}$ iff $\mathcal{X}$ is a λ-space.

Proof. The basic idea is that a valuation on a modal space $\mathcal{X}$ induces a modal homomorphism from Fm to $\mathcal{X}^*$ and vice versa.

Let $\mathcal{X} \in MS$ and let $V : P \to X^*$. We define $h_V : Fm \to \mathcal{X}^*$ by:

$$h_V(\varphi) := \{x \in X : x, V \models \varphi\}.$$

We claim this is a modal homomorphism. The $\wedge$, $\vee$ and $\neg$ cases are trivial, for $\Box\varphi$; $x \in h_V(\Box\varphi)$ iff $x, V \models \Box\varphi$ iff $\forall y \in R[x]\ y, V \models \varphi$ iff $\forall y \in R[x]\ y \in h_V(\varphi)$ iff $R[x] \subseteq h_V(\varphi)$ iff $x \in \Box h_V(\varphi)$.

Conversely, given $h : Fm \restriction P \to \mathcal{X}^*$, we define $V_h : Fm \to X^*$ by $V_h := h \restriction P$. This is clearly a valuation on $\mathcal{X}$, moreover by induction we can check that $\forall \varphi \in Fm$ $h(\varphi) = \{x \in X : x, V \models \varphi\}$. The base case is simply the definition of V_h and $x, V \models p$ for $p \in P$, the inductive step on $\wedge$, $\vee$ and $\neg$ is trivial. For $\varphi = \Box\psi$; $x \in h(\Box\psi)$ iff $x \in \Box h(\psi)$ iff $\forall y \in R[x]\ y \in h(\psi)$ iff $\forall y \in R[x]\ y, V \models \psi$ iff $x, V \models \Box\psi$.

Now, let $\Gamma \cup \{\varphi\} \subseteq Fm$. Suppose there is a modal homomorphism $h : Fm \to \mathcal{X}$ such that $\forall \gamma \in \Gamma\ h(\gamma) = X$ but $h(\varphi) \neq X$. Then $V_h : P \to X^*$ is a valuation on $\mathcal{X}$ such that $\forall x \in X\, \forall \gamma \in \Gamma\ x, V_h \models \gamma$ but $\exists x \in X : x, V_h \not\models \varphi$. Conversely, if $V : P \to X^*$ is a valuation on $\mathcal{X}$ such that $\forall x \in X\ \forall \gamma \in \Gamma\ x, V \models \gamma$ but $\exists x \in X : x, V \not\models \varphi$ then $h_V : Fm \to \mathcal{X}$ is a modal homomorphism such that $\forall \gamma \in \Gamma\ h_V(\gamma) = X$ but $h_V(\varphi) \neq X$. That is, $\forall \Gamma \cup \{\varphi\} \subseteq Fm, \Gamma \approx \top \models_{\mathcal{X}^*} \varphi \approx \top$ iff $\Gamma \models_{\mathcal{X}} \varphi$.

So finally, let $\mathcal{X} \in MS$. Suppose $\mathcal{X}^* \in \mathcal{A}$, then $\forall \Gamma \cup \{\varphi\} \subseteq Fm : \Gamma \vdash_{\lambda} \varphi$ we have $\Gamma \approx \top \models_{\mathcal{X}^*} \varphi \approx \top$ and therefore $\forall \Gamma \cup \{\varphi\} \subseteq Fm : \Gamma \vdash_{\lambda} \varphi$ we have $\Gamma \models_{\mathcal{X}} \varphi$, i.e. $\mathcal{X}$ is a λ-space.

Suppose $\mathcal{X}$ is a λ-space. From theorem 3.11 $\mathcal{X}^* \in \mathcal{A}$ iff $\forall \varphi \in Fm : \vdash_{\lambda} \varphi \models_{\mathcal{X}^*} \varphi$. Letting $\varphi \in Fm : \vdash_{\lambda} \varphi$ then as $\mathcal{X}$ is a λ-space we have $\models_{\mathcal{X}} \varphi$ and therefore $\models_{\mathcal{X}^*} \varphi$ as required. $\Box$

This alongside the duality allows us to convert the entire preceding section from a discussion about the relationship between normal modal logics and their algebraic modes to one about the relationship between normal modal logics λ and their λ-spaces. In particular, we can re-frame the question of characterising HSC normal modal logics once more.

Corollary 3.26. Every normal modal logic λ is sounds and complete with respect to its class of λ-spaces.

Proof. Follows from theorem 3.11 and 3.25. □

We say that a NML λ has the *finite model property* (FMP) iff for any $\varphi \in Fm$ if there is a λ-space $\mathcal{X}$ with $\not\models_{\mathcal{X}} \varphi$ then there is a finite λ-space $\mathcal{Y}$ such that $\not\models_{\mathcal{Y}} \varphi$.

We say that a modal space $\mathcal{X}$ is *weakly projective* for a NML λ iff for every λ-space $\mathcal{Y}$, if $\mathcal{X}$ is a closed upset of $\mathcal{Y}$ then there is a surjective continuous p-morphism $f : \mathcal{Y} \to \mathcal{X}$.

Corollary 3.27. Let λ be a logic extending K4.

(i) If λ is HSC then the finite, non-trivial rooted λ-spaces are weakly projective for λ.

(ii) Suppose all the axiomatic extensions of λ have FMP. Then, if the finite, non-trivial, rooted λ-spaces are weakly projective for λ, then λ is HSC.

Proof. This is simply a translation of lemma 3.23 using theorem 3.25 & 3.18 and lemmas 2.9 & 2.10. □

With this we have all the requisite background theory for our main task. Through the notion of algebraizable logics and its application to modal logic (definition 3.9, theorem 3.11) we have re-characterised the problem of determining which transitive modal logics are hereditarily structurally complete to the problem of determining which varieties of K4-algebras are primitive (theorem 3.14). We have also gone some way into reducing that task providing both a necessary (3.15) and sufficient (3.22) condition for being primitive. This sets the stage for our main investigation, where we will look to solve this algebraic problem using Jónsson-Tarski duality as an aid. However, we have also given a direct order-topological semantics for transitive modal logics (3.24) which fits in neatly with the rest of the theory (3.25). This provides a different framing of the problem, one which avoids any reference to algebra (3.27).

Chapter 4

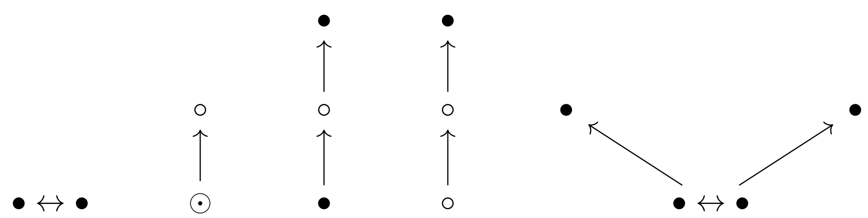
Understanding the Problem

With the relevant background theory covered, we now turn to our central question - when is a variety of K4-algebras primitive?

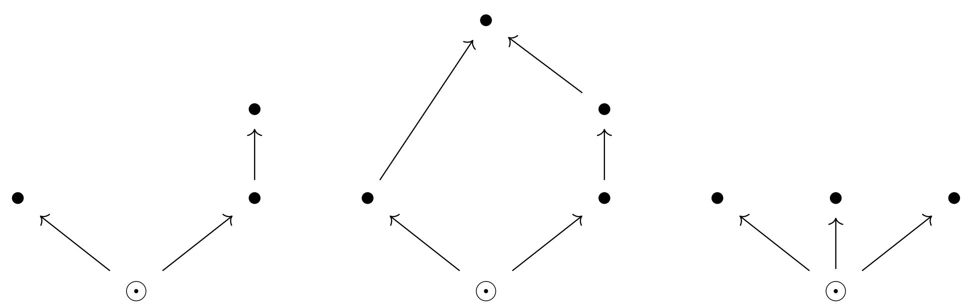
4.1 Rybakov's Characterisation of HSC logics over K4

Our starting point is the characterisation given by Rybakov for HSC logics over K4 [28, Theorem 4.5].

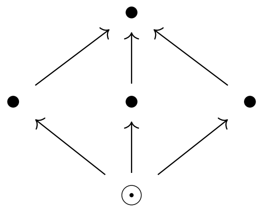
Rybakov's Theorem In order for a modal logic λ over K4 to be HSC, it is necessary and sufficient that λ not be included in any of the logics $\lambda(F_i) : 1 \leq i \leq 13$ and $\lambda(F_3')$.



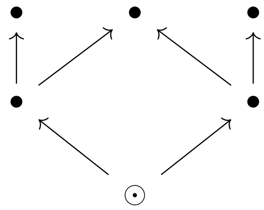
F_1 F_2 F_3 F_3' F_4



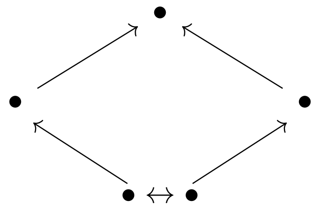
F_5 F_6 F_7



F_8

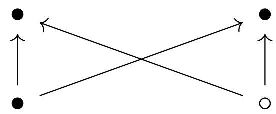


F_9

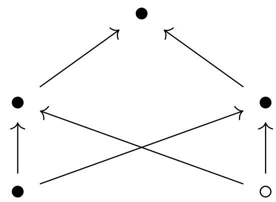


F_{10}

F_{11}



F_{12}



F_{13}

Remarks. In the case of $i \in \{2, 5, 6, 7, 8, 9\}$ the root of the frame can be either reflexive or irreflexive and so F_i represents two frames. When we say that λ is not included in the logic $\lambda(F_i)$, this is shorthand for saying that λ is not included in the logic of the reflexive version of the frame F_i and λ is not included in the logic of the irreflexive version of the frame.

Rybakov discusses hereditary structural completeness as a property of a logic λ, that is a set of formulas. To make sense of derivability, which is sensitive to the postulated inference rules for the logic, Rybakov always assumes those rules to be modus ponens and necessitation [28, pg 477]. As explained in the previous chapter, this lines up with our focus on λ_g.

Rybakov defines HSC for a logic λ as every logic λ' *extending* λ being structurally complete. Translating to the deductive system terminology, this means Rybakov is following the axiomatic extension version of HSC as explained in definition 3.12.

Each of the transitive frames above is naturally a transitive space under the discrete topology.

Bearing this in mind and our work in the previous chapter, we can put Rybakov's characterisation into terms more amenable to our investigation:

Claim Let $\mathcal{A}$ be a variety of $K4$-algebras. Then $\mathcal{A}$ is primitive iff $\mathcal{A}$ omits $F_i^* : 1 \leq i \leq 13$ and $(F_3')^*$.

However, it is not our aim to prove this. By re-framing the problem in algebraic terms we illuminate a mistake in Rybakov's characterisation regarding F_3'.

Theorem 4.1. The variety generated by $(F_3')^*$ is primitive.

Proof. Let $A = (F_3')^*$, so $A_* \cong F_3'$. Let $\mathcal{A}$ be the variety generated by A, i.e. $\mathcal{A} = \mathbb{HSP}(A)$. First, we recall that a variety generated by a finite collection of finite algebras is locally finite [1, Theorem 3.49], so $\mathcal{A}$ is locally finite. Thus, by theorem 3.16, to show $\mathcal{A}$ is primitive it is sufficient to show that each finite non-trivial FSI member of $\mathcal{A}$ is weakly projective in $\mathcal{A}$. We start with a structural claim:

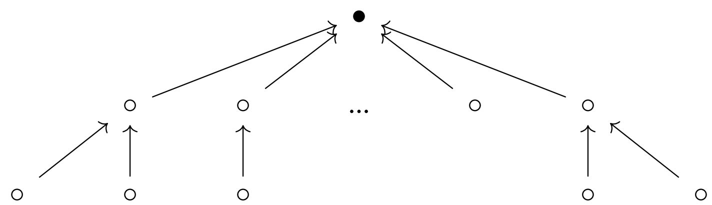
Claim: Letting $\mathcal{A}_\omega$ denote the class of finite members of $\mathcal{A}$, then $\forall B \in \mathcal{A}_\omega$ B is finite with $d(B_*) \leq 3$ and $\forall x \in B_*$:

(a) If $d(x) = 1$ then $R[x] = \{x\}$;

(b) If $d(x) = 2$ then $R[x] = \{y\}$ for some $y \in B$ such that $d(y) = 1$;

(c) If $d(x) = 3$ then $R[x] = \{y, z\}$ for some $y, z \in B$ such that $d(y) = 2, d(z) = 1$ and $R[y] = \{z\}$.

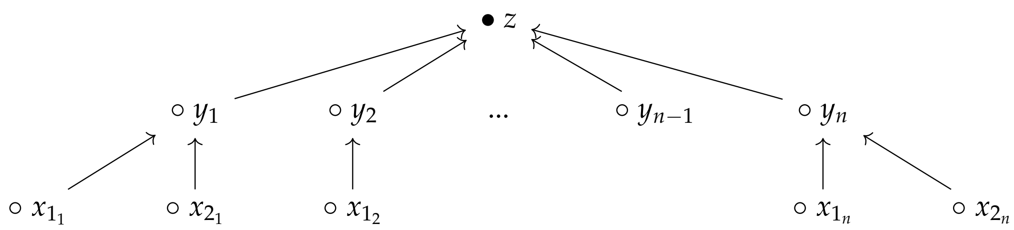
That is B_* is a disjoint union of a finite collection of spaces, each of which has as underlying frame a tree of depth 3 where the element of depth 1 is reflexive and all other elements are irreflexive. For example:



Now, $\mathcal{A}_\omega = \mathbb{HSP}_\omega(A)$ where $\mathbb{P}_\omega$ denotes the operation of taking finite products. From lemma 2.10 we know that $\mathbb{H}$ is dual to $\mathbb{M}$, the operation of taking M-subspaces, $\mathbb{S}$ to $\mathbb{Q}$, the operation of taking quotients of modal equivalences and $\mathbb{P}_\omega$ to $\mathbb{U}_\omega$, the operation of taking finite disjoint unions. So, letting S be the set of finite transitive spaces of depth at most 3 satisfying conditions (a), (b) and (c), to establish our claim it is sufficient to check $S = \mathbb{MQU}_\omega(A_*)$. We will use a, b and c to denote the element of depth $3, 2$ and 1 in A_* respectively.

For $\subseteq$: Let $\mathcal{X} \in S$. If X has no elements of depth 1 then $X = \varnothing$ and $\mathcal{X} \in \mathbb{MQU}_\omega(A_*)$. Suppose X has elements of depth 1, indeed for now assume that X has exactly one element of depth 1 which we denote by z. Now, if X has no elements of depth 2 then $X = \{z\}$, and by (a) zRz, i.e., X is a single reflexive point. This is an M-subspace of A_*, so $\mathcal{X} \in \mathbb{MQU}_\omega(A_*)$.

Suppose X has elements of depth 2. As X is finite we may list them $\{y_i\}_{i=1}^n$. Note that $\forall y_i \in X$ by (b) $R[y_i] = \{z\}$. Then, for each $1 \leq i \leq n$ we list any elements of depth 3 in X that see y_i as $\{x_{j_i}\}_{j=1}^{n_i}$. If there are no such elements, we will take $n_i = 1$ and add a placeholder x_{1_i}. using our example above, we would have the following labels for the elements of X and add a placeholder element $x_{1_{n-1}}$:



Then we define the following:

$$J := \bigcup_{i=1}^{n} \{(j_i) \in \omega : 1 \leq j_i \leq n_i\};\ \mathcal{Y} := \coprod_{(i,j_i) \in n \times J} A_*.$$

That is $\mathcal{Y}$ is the disjoint union of $n \times |J|$ copies of A_*. Note that by definition $\mathcal{Y} \in \mathbb{U}_\omega(A_*)$ and by construction $\mathcal{Y} \in S$.

We then define E on $\mathcal{Y}$ as follows:

(i) $(c,(i,j_i))E(c,(i',j'_{i'}))\ \forall (i,j_i),(i',(j'_{i'}) \in n \times J$;

(ii) $(b,(i,j_i))E(b,(i',j'_{i'}))$ iff $i = i'$;

(iii) $(a,(i,j_i))E(a,(i',j'_{i'}))$ iff $i = i'$ and $j = j'$.

That is we identify all elements of depth 1 together and for each $1 \leq i \leq n$ we identify all the elements of depth 2 with index $i \in n$ together. Each element of depth 3 is in its own singleton equivalence class. This is clearly an equivalence relation, we moreover claim it is a modal equivalence. As we are working with finite spaces condition (ii) is trivial. For condition (i); Let $u, v, w \in Y$ such that uEv and uRw. Either $d(u) = 1$, $d(u) = 2$ or $d(u) = 3$. If $d(u) = 1$ then $d(v) = 1$ and wEv. Moreover vRv so we may take v itself as witness. If $d(u) = 2$ then $d(v) = 2$, u and v have the same index $1 \leq i \leq n$ and vRv. Now $d(u) = 2$ with uRw so because $\mathcal{Y} \in S$ we have $R[u] = \{w\}$ with $d(w) = 1$. Similarly, $R[v] = \{t\}$ for some $t \in Y$ such that $d(t) = 1$, then vRt and wEt so we may take t as witness. Finally, if $d(u) = 3$ then $u = v$ so vRw and we may take w as witness.

Therefore $\mathcal{Y}/E \in \mathbb{QU}_\omega(A_*)$. We then consider the closed upset of $\mathcal{Y}/E$:

$$Z := \bigcup_{R^{-1}[y_i] \neq \varnothing} R[a,(i,j_i)].$$

That is we cut out the singleton equivalence classes $[a,(i,j_i)]$ where $n_i = 1$ was a placeholder. Letting $\mathcal{Z}$ be the M-subspace with underlying set Z, $\mathcal{Z} \in \mathbb{MQU}_\omega(A_*)$. Moreover, the construction demonstrates that the map $z \mapsto [c,(1,1_1)]$, $y_i \mapsto [b,(i,j_1)]$ and $x_{j_i} \mapsto [a(i,j_i)]$ is an isomorphism from $\mathcal{X}$ to $\mathcal{Z}$, so $\mathcal{X} \in \mathbb{MQU}_\omega(A_*)$.

In the case that $\mathcal{X}$ has more than one element of depth 1, letting $\{z_i\}_{i=1}^n$ be those elements, for each z_i, $R^{-1}[z_i]$ is a closed upset with exactly one element of depth 1, and so the M-subspace with it as underlying set is in S and moreover is also in $\mathbb{MQU}_\omega(A_*)$ by the argument above. Then $\mathcal{X} \cong \coprod_{i=1}^{n} R^{-1}[z_i]$, so $\mathcal{X} \in \mathbb{MQU}_\omega(A_*)$.

For $\supseteq$: Inspecting A_* it is clear that A_*. To conclude we must check S is closed under our three operations. Let $\mathcal{X} \in S$.

Let $\mathcal{Y}$ be an M-subspace of $\mathcal{X}$. Then $Y \subseteq X$ is a closed upset. As $\mathcal{X}$ is finite and $d(X) \leq 3$, $\mathcal{Y}$ is also finite and as it is an upset $d(Y) \leq 3$ and moreover $\forall x \in Y$ $R_Y[x] = R_X[x]$ and so $\mathcal{Y}$ immediately satisfies conditions (a) (b) and (c). So $\mathcal{Y} \in S$.

Let E be a modal equivalence on $\mathcal{X}$. First, we quickly note that X/E is finite. Now, consider $x \in X$. $R_E[x] = \{[u] \in X/E : \exists x E x', u E u' : x' R u'\}$ and moreover as E is a modal equivalence if $[u] \in R_E[x]$ then $\exists u'' E u' E u : x R u''$.

Now, if $d(x) = 1$, $R[x] = \{x\}$ so $u'' = x$, $[x] = [u]$ and $R_E[x] = \{[x]\}$. Note also that $d([x])]1$.

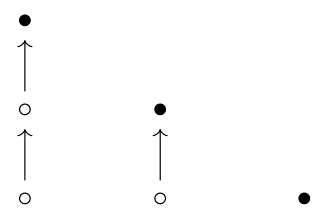
If $d(x) = 2$, $R[x] = \{y\} : d(y) = 1$. Then $u'' = y$ and $[u] = [y]$. So $R_E[x] = \{[y]\}$. Now, either xEy or $x\not{E}y$. If xEy then $R_E[x] = \{[x]\}$ and $d([x]) = 1$. If $x\not{E}y$ then $[y] \neq [x]$ and $R_E[x] = \{[y]\}$. We just noted that $d(y) = 1$ implies $d([y]) = 1$ and so $d([x]) = 2$.

If $d(x) = 3$ then $R[x] = \{y, z\}$, $d(y) = 2, d(z) = 1$ and $R[y] = \{z\}$. So $u'' = y$ or $u'' = z$ and $R_E[x] \subseteq \{[y], [z]\}$. If yEz then $R_E[x] = \{[y]\}$ and as in the previous case we have either $R_E[x] = \{[x]\}$ and $d([x]) = 1$ or $R_E[x] = \{[y]\}$, $d([y]) = 1$ and $d([x]) = 2$. If $y\not{E}z$, $R_E[x] = \{[y], [z]\}$. Moreover, as yRz from the previous case we have $R_E[y] = \{[z]\}$, $d([z]) = 1$ and $d([y]) = 2$. If xEy then as E is a modal equivalence $\exists w \in X : yRw$ and yEw. Then xRw so $w = y$ or z. As $y\not{E}z$ $w \neq z$, so $w = y$ and yRy, but $R[y] = \{z\}$ so we have a contradiction. So $x\not{E}y$. If xEz then again $\exists w \in X : wEy$ and zRw, then xRw so $w = y$ or z. $y\not{E}z$ so $w \neq z$ but as $d(z) = 1$ $R[z] = \{z\}$ so we have a contradiction. So $x\not{E}z$. So $[x]$, $[y]$ and $[z]$ are all distinct and $d([x]) = 3$.

This exhausts the possibilities for $d(x)$, so $\forall [x] \in X/E$ $d([x]) \leq 3$, $d(X/E) \leq 3$ and in every case conditions (a), (b) and (c) held, so $\mathcal{X}/E \in S$.

Let $\{\mathcal{X}\}_{i=1}^{n} \subseteq S$. Let $(x, j) \in \coprod_{i=1}^{n} X_i$. Then $\mathcal{X}_j \in S$ and $x \in X_j$. If $d(x, j) = 1$ then $d(x) = 1$, $R_j[x] = \{x\}$ and so $R[x, j] = \{(x, j)\}$. If $d(x, j) = 2$ then $d(x) = 2$, $R_j[x] = \{y\} : d(y) = 1$ and so $R[x, j] = \{(y, j)\} : d(y, j) = 1$ and if $d(x, j) = 3$ then $R_j[x] = \{y, z\} : d(y) = 2, d(z) = 1$ and $R[y] = \{z\}$ and so $R[x, j] = \{(y, j), (x, j)\} : d(y, j) = 2, d(z, j) = 1$ and $R[y, j] = \{z, j\}$. So $\coprod_{i=1}^{n} \mathcal{X}_j \in S$. This completes the proof of the claim.

Now, let $B \in \mathcal{A}$ be finite, non-trivial and FSI. $B \in \mathcal{A}_\omega$, therefore lemma 2.9 and the claim together imply that B_* is one of the following frames:



(i) (ii) (iii)

Letting $C \in \mathcal{A} : B \in \mathbb{H}(C)$, as B is finite there exists a subalgebra D of C which is finitely generated and such that $B \in \mathbb{H}(D)$. If $B \in \mathbb{IS}(D)$ then $B \in \mathbb{IS}(C)$, so to check that B is weakly projective in $\mathcal{A}$ it is enough to establish $B \in \mathbb{IS}(D)$. As D is finitely generated and $\mathcal{A}$ is locally finite, D is finite, so $D \in \mathcal{A}_\omega$. Now we have three cases:

(i); As $B \in \mathbb{H}(D)$, by lemma 2.10, B_* is an M-subspace of D_* and so D_* has an element of depth 3. Now, $d(D_*) \leq 3$ so D_* satisfies condition (a) in lemma 2.18 trivially and is finite so satisfied condition (c) trivially as well. Letting $x \in D_*$, if $x \in Sl_1(D_*)$ then $R[x] = \{x\}$ and xRx, if $x \in Sl_2(D_*)$ then $R[x] = \{y\}$ so $x\not{R}x$ and if $x \in Sl_3(D_*)$ then $R[x] = \{y, z\}$ so again $x\not{R}x$. So D_* satisfied condition (b) of lemma 2.18. So, letting E be the modal equivalence identifying points at the same depth, as

D_* has an element of depth 3 we have that $D_*/E \cong B_*$ and so $D_* \twoheadrightarrow B_*$.

(ii); D_* has an element of depth 2. As above we may take the modal equivalence E identifying points at the same depth on D_*. Then, either D_* has no elements of depth 3, $D_*/E \cong B_*$ and so $D_* \twoheadrightarrow B_*$, or it does have an element of depth 3. Then we obtain B_* from D_*/E by applying an α-reduction and so again $D_* \twoheadrightarrow B_*$.

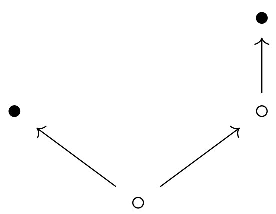
(iii); D_* has an element of depth 1. Once more we take the modal equivalence E identifying points at the same depth on D_*. This time, D_* either has an element of depth 3, has no elements of depth 3 and an element of depth 2 or only elements of depth 1. In the first two cases we obtain B_* from D_*/E by applying α-reductions, and in the third $D_*/E \cong B_*$, so in all cases $D_* \twoheadrightarrow B_*$.

In all cases $D_* \twoheadrightarrow B_*$ and so $B \in \mathbb{IS}(D)$ and we are done. □

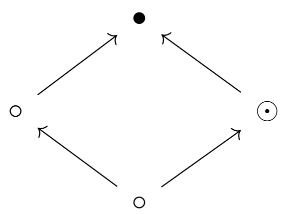
4.2 A New Characterisation of HSC logics over K4

With theorem 4.1 we know there are primitive varieties that include $(F_3')^*$ and we need to adjust the characterisation. This is not as simple as just dropping F_3' from the characterisation. Whilst that frame, along with a family of frames like it, should be in the characterisation its presence in Rybakov's characterisation was preventing a large collection of genuinely problematic algebras from appearing. Moreover, as we will make precise in the next section, whilst in isolation frames of this family are not problematic together they can present a problem.

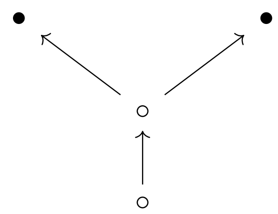
In addition to those already introduced, the following frames and spaces will play a special role in our considerations:



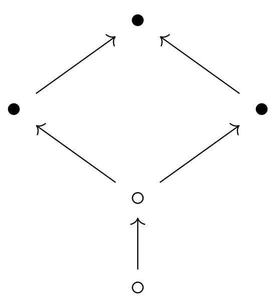
F_{14}



F_{15}

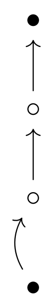


F_{16}

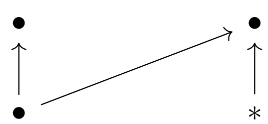


F_{17}

G_n



G_ω



H

Remarks. The frame G_n where $n \in \omega$ refers to a reflexive point preceded by a chain of n irreflexive points and $|G_n| = n + 1$. G_ω is the transitive space $(\mathbb{N} \cup \{\omega\}, \tau, R)$ where:

$$R[x] = \begin{cases} \mathbb{N} \cup \{\omega\} & \text{if } x = \omega \\ \{m \in \mathbb{N} : m < x\} & \text{if } x \in \mathbb{N} \\ \{0\} & \text{if } x = 0 \end{cases}$$

Also, τ is the one-point compactification of $\mathbb{N}$, i.e. $U \subseteq \mathbb{N} \cup \{\omega\}$ is clopen iff U is any finite subset of $\mathbb{N}$ exlcuding ω or $U = U' \cup \{\omega\}$ where U' is a cofinite subsets of $\mathbb{N}$.

Our characterisation for primitive varieties of K4-algebras and the main theorem of this project becomes:

Theorem (Primitive Varieties of K4-algebras).
Let $\mathcal{A}$ be a variety of $K4$-algebras. Then $\mathcal{A}$ is primitive iff $\mathcal{A}$ omits $F_i^* : 1 \leq i \leq 17$ and $\exists n > 0 : \mathcal{A}$ omits G_n^*.

Once established, in line with our discussions in chapter 3 this gives our characterisation of HSC transitive modal logics.

Corollary (Hereditarily Strucutrally Complete Logics over K4).
Let λ be a normal modal logic with equivalent algebraic semantics $\mathcal{A}$. The following are equivalent:

(i) λ is HSC.

(ii) $\mathcal{A}$ is primitive.

(iii) For all $1 \leq i \leq 17$ F_i is not a λ-space and $\exists n > 0$ such that G_n is not a λ-space.

4.2.1 The Proof Strategy

The proof of our characterisation for primitive varieties of K4-algebras is quite technical and fairly dense. In an effort not to miss the forest for the trees, we should take a moment now to comment on our overall proof strategy and plan for the rest of our investigation.

In the previous chapter we used the theory of algebraic logic alongside results from universal algebra to establish a necessary and sufficient condition for a variety of K4-algebras to be primitive (lemma 3.23). This forms the backbone of our proof strategy. First we establish that primitive varieties of K4-algebras must omit the given algebras as otherwise they violate the necessary condition. Second we establish that any variety of K4-algebras omitting the given algebras satisfies the sufficient condition and is therefore primitive. In each case, we employ Jónnson-Tarski to tackle these algebraic problems using topological methods. The first task is straightforward and covered in the next section (lemma 4.2). To show that a primitive variety $\mathcal{A}$ omits one of the barred algebras A we argue that we can, through the operations of disjoint union and quotient, construct form A a finite, non-trivial FSI algebra which is not weakly projective in $\mathcal{A}$.

The second task is much more involved. Our sufficiency condition means that given a variety $\mathcal{A}$ omitting the given algebras we need to establish two things, that all its sub-varieties have the FMP and that all the finite, non-trivial FSI members of the variety are weakly projective in $\mathcal{A}$. In both cases, the bulk of the work lies in establishing a detailed description of the structure of the finitely generated, non-trivial SI members of the varieties (theorem 5.11). This is the focus of chapter 5. The idea is to establish a group of results that in each case demonstrate a particular frame substructure never appears in our interested spaces. These results then drive the proof of the description.

A helpful comparison for this part of our investigation is the work done by Bezhanishvili and Moraschini in [5, Section 6] where they attempt to do the same for intermediate logic and Esakia spaces. Each of the frame substructure proofs follow a similar pattern. In each case, we assume the substructure does appear in a space whose dual is in one of our varieties. Then by taking M-subspaces and quotients we eventually recover one of our barred spaces, this implies via lemma 2.10 that the dual of the barred space is in the variety which is a contradiction. Whilst Bezhanishvili and Moraschini only had to consider finite spaces, we cannot make that restriction. Accounting for this changes the timbre of the proofs a little, forcing us to repeatedly find clopen subsets to work with in place of individual points, but the ideas are very similar.

We can also compare the wider strategy to establish the desired description. The three central results (lemmas 5.6, 5.7 & 5.8) that drive the proof of the main theorem are the same three structural results that Bezhanishvili and Moraschini establish (lemmas 6.4, 6.8 & 6.10 respectively). However, because our spaces are built from frames rather than posets we have to worry about clusters and irreflexive points. Whilst clusters can be effectively ignored throughout via lemma 2.15, irreflexive points are more problematic, even making the work done within each proof markedly more difficult. As such, we start with some results to better understand how irreflexive points behave in our spaces (lemmas 5.3 & 5.4). Once the three central results are established, we need a few more results related to irreflexive points (lemmas 5.9 & 5.10) before we are then in position to establish our description of the structure of dual spaces to finitely generated, non-trivial SI algebras in our varieties.

Once that detailed description of the finitely generated, non-trivial SI members is in place we will be in position to complete the proof of our characterisation in chapter 6. Recall that our aim was to establish given any variety omitting the given algebras all its sub-varieties have the FMP and all its finite, non-trivial FSI members are weakly projective. For the FMP result, because any sub-variety must omit all algebras that its larger variety does, it is sufficient to check that any variety $\mathcal{A}$ omitting the given algebras has the FMP (theorem 6.1). We do this through a variation on K. Fine's drop point technique [14, Theorem 4], we take an algebra $A \in \mathcal{A}$, which we can assume is finitely generated, non-trivial and SI, that invalidates a given formula. Our assumptions mean the dual space A_* has the structure described by theorem 5.11 and based on this we demonstrate how to construct a finite M-subspace of A_* such that its dual algebra also invalidates the given formula.

Finally we give the weakly projective result. Our description of the finitely generated, non-trivial SI members of a variety $\mathcal{A}$ omitting the given algebras has as a corollary a description of the *finite*, non-trivial and FSI members of $\mathcal{A}$ (corollary 5.12). Given such an algebra $A \in \mathcal{A}$ we can reduce the problem of it being weakly projective in $\mathcal{A}$ to demonstrating that if A_* is a closed upset of B_* where $B \in \mathcal{A}$ is finitely generated then there is a surjective continuous p-morphism $f : B_* \twoheadrightarrow A_*$. With our description we can do this recursively, collapsing B_* into the elements of A_* of depth 0, then depth 1 and so on.

It is interesting to compare our work to Rybakov's own proof strategy in his original characterisation [29]. Much of our work is quite similar, in particular there is a clear mirror between our structural work in chapter 5 and similar results from Rybakov in [29, Section 3]. We both put some controls on irreflexive points (lemmas 5.1, 5.3 & 5.4 for us vs lemmas 3.1 and 3.2 for Rybakov [29]), a width criteria (lemma 5.6 vs lemma 3.3) and how clusters relate to each other (lemmas 5.7 & 5.8 vs lemma 3.4). This is used to give the desired description (theorem 5.11 vs lemma 3.6 & corollary 3.8). As one would expect, the description itself is quite similar with the differences arising in line with our adjusted characterisation; we allow for some additional behaviour amongst irreflexive points beyond solely being the root (as Rybakov requires). Rybakov also establishes that the logics λ omitting his frames have the FMP using K. Fine's drop point technique (theorem 6.1 vs lemma 3.9).

However, when it comes to how these results are used to complete the proof of the characterisation there is a sharp differences in the approaches. As discussed, we use results in universal algebra to complete the sufficient direction of the characterisation via our weakly projective result (lemma 6.2). By contrast, Rybakov reduces the problem to showing that every FSI modal algebra in $\mathcal{A}$ is a subalgebra of some free algebra of finite rank in $\mathcal{A}$, where $\mathcal{A}$ is the equivalent algebraic semantics of a logic omitting his frames (theorem 2.2). To prove this, he relies on the construction of a sequence of n-characterising models $Ch_n(\lambda)$ for a logic λ over K4 (lemma 4.3). This difference also appears in the necessary direction, where again our focus is on the notion of weak projectivity (lemma 4.2) whereas Rybakov employs the n-characterising models, reducing the problem to showing that there is no p-morphism from $Ch_{\lambda(F_i)}(k)$ for all $k \in \omega$ onto some rooted generated subframe E of the F_i, where F_i is one of his given frames.

4.3 The First Direction

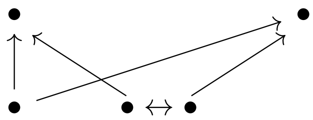
One direction of our main theorem can be established relatively easily via lemma 3.15.

Lemma 4.2. Primitive varieties of K4-algebras omit $F_i^* : 1 \leq i \leq 17$ and G_n^* for some $n > 0$.

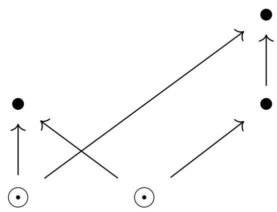
Proof. Supposing $\mathcal{A}$ is primitive, by lemma 3.15 all its finite, non-trivial FSI memebrs are weakly projective in $\mathcal{A}$. Therefore, to show $\mathcal{A}$ omits some algebra A it is sufficient to show that if $A \in \mathcal{A}$ then there is a finite, non-trivial FSI member of $\mathcal{A}$ that is not weakly projective. This is the plan for each of the F_i and F_i', following the proof strategy of lemma 5.1 in [5].

For $i \in \{1, 4, 5, 6, 7, 8, 9, 10, 14, 15;$ consider the following frames:

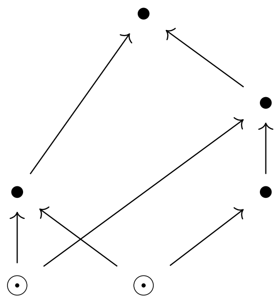
X_1



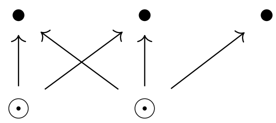
X_4



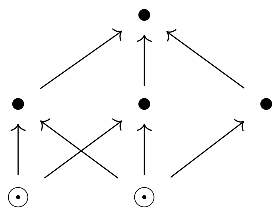
X_5



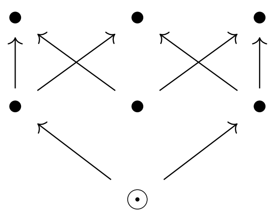
X_6



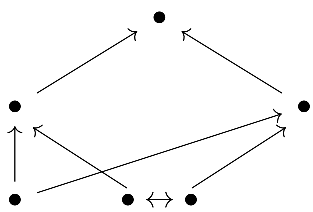
X_7



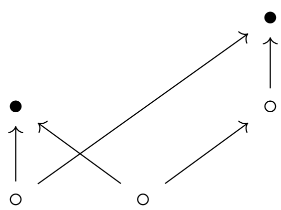
X_8



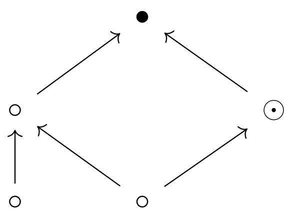
X_9



X_{10}



X_{14}

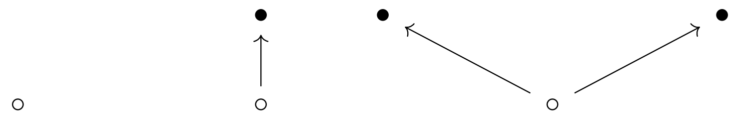


X_{15}

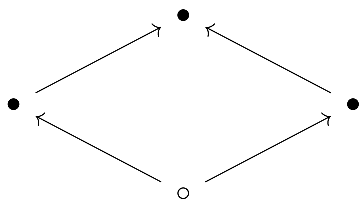
Where relevant, we insist the $\odot$ points within a frame match, i.e. they are either both reflexive or both irreflexive. First, observe that each F_i is an M-subspace of X_i so by lemma 2.10 $F_i^* \in \mathbb{H}(X_i^*)$. Second, by inspection we can see that there is no way to reduce each X_i to F_i by α, β or γ reductions, so by lemma 2.14 there is no surjective continuous p-morphism from X_i to F_i and in turn by lemma 2.10, $F_i^* \notin \mathbb{IS}(X_i^*)$. By contrast, we can reduce the disjoint union $F_i \coprod F_i$ to X_i and so once more by lemma 2.10, $X_i^* \in \mathbb{ISP}(F_i^*) \subseteq \mathcal{A}$. Thus, $X_i^* \in \mathcal{A}$, $F_i^* \in \mathbb{H}(X_i^*)$ but $F_i^* \notin \mathbb{IS}X_i^*)$, so F_i^* is not weakly projective in $\mathcal{A}$ but is finite, non-trivial and FSI (from lemma 2.9 and F_i

rooted).

For $i \in \{2, 3, 11, 12, 13, 16, 17\}$; consider the following frames:



$X_2 = X_{11}$ X_3 $X_{12} = X_{16}$



$X_{13} = X_{17}$

Then, X_i is an M-subspace of F_i so $X_i^* \in \mathbb{H}(F_i^*) \subseteq \mathcal{A}$. By inspection we cannot reduce F_i to X_i and so $X_i^* \notin \mathbb{IS}(F_i^*)$. So X_i^* is finite, non-trivial and FSI but not weakly projective in $\mathcal{A}$.

Now we need to show that $\mathcal{A}$ omits G_n^* for some $n > 0$. Consider G_1 which is a M-subspace of G_ω so $G_1 \in \mathbb{H}(G_\omega)$. Now, let $f : G_\omega \to G_1$ be a p-morphism. Then, $f[R[\omega]] = f[\mathbb{N} \cup \{\omega\}]$. Letting y be the maximal reflexive element of G_1 and x its irreflexive root, either $f(\omega) = y$ or $f(\omega) = x$. If the latter, then $x \in f[\mathbb{N} \cup \{\omega\}]$ but $R[f(\omega)] = R[x] = \{y\}$ and so $R[f(\omega)] \neq f[R[\omega]]$ contradicting f being a p-morphism. So $f(\omega) = y$, and then $f[\mathbb{N} \cup \{\omega\}] = f[R[\omega]] = R[f(\omega)] = R[y] = \{y\}$, so f is not surjective. Thus, there is no surjective continuous p-morphism from G_ω to G_1 and once again $G_1^* \notin \mathbb{IS}(G_\omega^*)$. Now, G_1^* finite, non-trivial and FSI so must be weakly projective in $\mathcal{A}$, and so as $G_1^* \in \mathbb{H}(G_\omega^*)$ and $G_1^* \notin \mathbb{I}S(G_\omega^*)$ we have that $\mathcal{A}$ must omit G_ω^*.

Now, each G_n is a closed upset of G_ω, so by lemma 2.10 $G_n^* \in \mathbb{H}(G_\omega^*)$, in fact it is easy to check that $f_n : G_\omega^* \to G_n^*$ by $U \mapsto U \cap G_n$ is the resulting surjective homomorphism.

Consider a Fm-equation $\epsilon \approx \delta$ such that $\not\models_{G_\omega^*} \epsilon \approx \delta$. Then $\exists h : Fm \to G_\omega^* : h(\epsilon) \neq h(\delta)$. We assume w.l.o.g that $h(\epsilon) \not\subseteq h(\delta)$ i.e. $\exists x \in \mathbb{N} \cup \{\omega\} : x \in h(\epsilon)$ and $x \notin h(\delta)$. If $x = n \in \mathbb{N}$ then we can consider G_n^*, and $x \in h(\epsilon) \cap G_n$ but $x \notin h(\delta) \cap G_n$, i.e. $f_n \circ h : Fm \to G_n^* : f_n(h(\epsilon)) \neq f_n(h(\delta))$ and $\not\models_{G_n^*} \epsilon \approx \delta$. If $x = \omega$, then $\omega \in h(\epsilon)$ and so from the topology on G_ω we get $h(\epsilon) \cap \mathbb{N}$ is cofinite in $\mathbb{N}$. Also, $\omega \notin h(\delta)$, so again from the topology on G_ω we get $h(\delta)$ is a finite subset of $\mathbb{N}$. So, we can find $n \in \omega : n \in h(\epsilon)$ and $n \notin h(\delta)$ and proceeding as before we obtain $\not\models_{G_n^*} \epsilon \approx \delta$. That is, if $\not\models_{G_\omega^*} \epsilon \approx \delta$ then $\exists n \in \omega : \not\models_{G_n^*} \epsilon \approx \delta$.

Finally, recall that as a variety $\mathcal{A}$ is equationally definable. Let Θ be a set of defining equations for $\mathcal{A}$, then as it omits G_ω^* $\exists \epsilon \approx \delta \in \Theta : \not\models_{G_\omega^*} \epsilon \approx \delta$, which from above implies $\exists n \in \omega : \not\models_{G_n^*} \epsilon \approx \delta$. Then $G_n^* \notin \mathcal{A}$ as required. □

This completes the necessary direction of our main result. We have now identified the mistake in Rybakov's original characterisations (theorem 4.1) and provided a new corrected characterisation. Our main task is to establish the new characterisations, with the first and more straightforward direction already provided (lemma 4.2).

Chapter 5

Structural Results

In this chapter we are going to start describing the dual frame structure to the algebras in the varieties we are interested in. This culminates in a detailed description of the dual spaces to finitely generated, non-trivial SI members of those varieties.

5.1 Handling Irreflexive Points

We start with three lemmas related to the behaviour of irreflexive points. These are not only important structural results in their own right, but will also make the remainder of our work far easier by allowing us to control for where irreflexive points in our spaces appear.

Lemma 5.1. Let $\mathcal{A}$ be a variety omitting $F_i^* : 1 \leq i \leq 17$ and G_n^* for some $n > 0$. Let $A \in \mathcal{A}$.
Then, either the maximal points of A_* are reflexive or A_* is an anti-chain of irreflexive points.

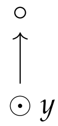
Proof. Suppose that A_* is not an anti-chain of irreflexive points, i.e. $R \neq \varnothing$ and that A_* has maximal irreflexive points, i.e. $\exists x \in A_* : R[x] = \varnothing$.

First, we suppose $\exists y \in A_* : \forall z \in R^+[y], R[z] \neq \varnothing$. Then $y \in R^+[y]$ so $R[y] \neq \varnothing$ and $y \neq x$. Let $\mathcal{X}$ be the M-subspace of A_* with underlying set $R^+[x] \cup R^+[y] = \{x\} \cup R^+[y]$, by lemma 2.10 $\mathcal{X} \in \mathcal{A}$. Now, $\varnothing$ is a clopen in $\mathcal{X}$ and so $\Box\varnothing = \{z \in X : R[z] = \varnothing\} = \{x\}$ by the assumption on y. So $\{x\}$ is clopen as and moreover so is its complement $R^+[y]$. Then $R^+[y]$ is an upset with $\forall z \in R^+[y]\ R[z] \neq \varnothing$, so a small adaption of lemma 2.16 let us consider the modal equivalence E identifying all points in $R^+[y]$. Then, again by lemma 2.10, $(\mathcal{X}/E)^* \in \mathcal{A}$, so me may w.l.o.g assume E is the identity on X, i.e. $R^+[y] = \{y\}$ and X is the following frame is isomorphic to F_{11}. So $F_{11}^* \in \mathcal{A}$ which is a contradiction.

$$\bullet\, x \quad \circ\, y$$

So, now suppose $\forall y \in A_*\ \exists z \in R^+[y] : R[z] = \varnothing$. Next, we suppose $\exists y \in A_* : R[y] \neq \varnothing$ and $\forall z \in R^+[y] \setminus \{y\}\ R[z] = \varnothing$. Then, we consider the M-subspace $\mathcal{X}$ with underlying set $R^+[y]$. Then, once more $\varnothing$ is clopen in $\mathcal{X}$, so $\Box\varnothing = R^+[y] \setminus \{y\}$ is clopen and an upset. We claim that the relation E identifying all points in $R^+[y] \setminus \{y\}$ is a modal equivalence. Condition (ii) holds from it being clopen, for (i), if uEv then either $u = y = v$ so if uRw, vRw as well or $u, v \in R^+[y] \setminus \{y\}$ and so $R[u] = \varnothing = R[v]$ and the condition holds trivially. Then, assuming w.l.o.g that E is the identity on X

then X is the following frame isomorphic to F_2 which is a contradiction.



So, now we may suppose that firstly $\forall y \in A_* \; \exists z \in R^+[y] : R[z] = \varnothing$ and secondly either $R[y] = \varnothing$ or $\exists z \in R^+[y] \setminus \{y\} : R[z] \neq \varnothing$. We define E on A_* by:

$$E := \{(u,v) \in A_*^2 : R[u] = \varnothing = R[v]\} \cup \{(u,v) \in A_*^2 : R[u], R[v] \neq \varnothing\}.$$

We claim this is a modal equialvance. For (i) in the definition of a modal equivalence; let uEv and uRw, then $R[u] \neq \varnothing$ and so $R[v] \neq \varnothing$. Then, by our conditions $\exists t_1 \in R^+[v] : R[t_1] = \varnothing$ and $\exists t_2 \in R^+[v] \setminus \{v\} : R[t_2] \neq \varnothing$. As $R[v] \neq \varnothing$, $t_1 \neq v$, so vRt_1. Then, either $R[w] = \varnothing$ so vRt_1 and wEt_1 as required, or $R[w] \neq \varnothing$ so vRt_2 and wEt_2 as required. For (ii); once more $\varnothing$ is clopen in A_* and so $\Box\varnothing = \{z \in A_* : R[z] = \varnothing\}$ is clopen, meaning it or its complement will separate any $u \not{E} v$ as required. Now, $R \neq \varnothing$ means that $\exists y \in A_* : R[y] \neq \varnothing$, which by assumption sees a point sees a point $z : R[z] = \varnothing$. Therefore, assuming w.l.o.g that E is the identity on A_*, A_* is the following frame isomorphic to F_2 which is again a contradiction.

□

As hinted at in the proof itself, there is a useful consequence of this first lemma.

Corollary 5.2. Let $\mathcal{A}$ be a variety omitting $F_i^* : 1 \leq i \leq 17$ and G_n^* for some $n > 0$. Let $A \in \mathcal{A}$.

If $R \neq \varnothing$, then the maximal points of A_* are reflexive and $\forall x \in A_* \; R[x] \neq \varnothing$.

In almost all of our work to follow we will be working with spaces where $R \neq \varnothing$, and we will routinely use this corollary without direct reference to find a point in $R[x]$.

Lemma 5.3. Let $\mathcal{A}$ be a variety omitting $F_i^* : 1 \leq i \leq 17$ and G_n^* for some $n > 0$. Let $A \in \mathcal{A}$ and $x \in A_* : x \not{R} x$. Then:

(i) $R^{-1}[x]$ is well founded and any ascending chain in $R^{-1}[x]$ is finite.

(ii) $R^{-1}[x]$ is conversely well founded and any descending chain in $R^{-1}[x]$ is finite.

(iii) $\forall y \in R^{-1}[x]$, $y \not{R} y$.

Proof. Suppose not, so we have $A \in \mathcal{A}$ such that $\exists x \in A$ for which either (i), (ii) or (iii) fails. Firstly, by lemma 2.15 we may consider $(A_*/E)^* \in \mathcal{A}$ where E is the modal equivalence identifying all elements in the same cluster. Note that if (i) fails in A_* then it also does so in A_*/E, similarly for (ii) and (iii). So w.l.o.g we may assume E is the identity on A_*, i.e. A_* consists of only improper clusters.

Now, $R[x]$ is closed and non-empty, with $x \notin R[x]$. Letting $z \in R[x]$ again as $x \not{R} x$, $x \notin R^+[z]$. So, by applying modal separation, we find a clopen upset U_z^x containing

z and omitting x. Then, $R[x] \subseteq \bigcup_{z \in R[x]} U^x_z$, and by compactness $\exists \{z_i\}_{i=1}^m \subseteq R[x]$ such that $R[x] \subseteq \bigcup_{i=1}^{m} U^x_{z_i}$. Letting U' be that union, we have U' a clopen upset and $R[x] \subseteq U'$. Then $R^{-1}[U']$ is a clopen downset, making $X \setminus R^{-1}[U']$ a clopen upset and finally $U = U' \cup X \setminus R^{-1}[U']$ a clopen upset. Note that $x \notin U'$ and considering $u \in R[x] \subseteq U'$ we see $x \in R^{-1}[U']$, so $x \notin U$, and as $u \in U$, $U \neq \varnothing$.

Next, we consider:

$$V := (A_* \setminus U) \setminus R^{-1}[A_* \setminus U].$$

This too is clopen, moreover $x \notin U$, and as $R[x] \subseteq U' \subseteq U$, $x \notin R^{-1}[A_* \setminus U]$, so $x \in V$. Letting $z_1, z_2 \in V$, then $z_2 \in A_* \setminus U$ and $z_1 \notin R^{-1}[A_* \setminus U]$ so $z_1 \not R z_2$, so V consists of an anti-chain of irreflexive points. Furthermore, letting $z_1 \in V$ and $z_1 R z_2$, $z_2 \notin V$, i.e. either $z_2 \in U$ or $z_2 \in R^{-1}[A_* \setminus U]$. However, as $z_1 R z_2$ and $z_1 \notin R^{-1}[A_* \setminus U]$ we have $z_2 \notin R^{-1}[A_* \setminus U]$ and so $z_2 \in U$. That is, $\forall z \in V$, $R[z] \subseteq U$.

We also consider:

$$W := U \cup A_* \setminus (V \cup R^{-1}[V]).$$

We claim that W and V form an M-partition of A_*. From their definitions we immediately see they are both clopen and disjoint. Then, W is the union of two upsets and so is itself an upset and trivially satisfies the M-partition condition. Then, letting $u, v \in V$ and uRw, then $w \in W$ and letting $t \in R[v]$, $t \in W$ and so we may take it as witness for the M-partition condition.

So, by lemma 2.17 we may consider the modal equivalence E identifying points within W and V. By lemma 2.10, $(\mathcal{A}_*/E)^* \in \mathcal{X}$ and so we may w.l.o.g assume that E is the identity on A_*, i.e. W is a singleton and $V = \{x\}$. We let $\top$ denote the unique element in W. Moreover, consider $u \in A_* : u \notin \{\top, x\}$. Then $u \notin W$ and $u \notin V$. As $u \notin W$ we get $u \notin U$ and $u \in V \cup R^{-1}[V]$, then $u \notin V$ in fact $u \in R^{-1}[V]$, i.e. uRx. So $A_* = \{\top, x\} \cup R^{-1}[x]$.

We now make a case distinction, either $\exists z \in A_* : 1 < d(z) < \omega$ and zRz or not. We start with the former, and let $z \in A_*$ be of minimal depth such that zRz and consider the M-subspace $\mathcal{X}$ with underlying set $X = R^+[z]$. Note that as $Sl_2(A_*) = \{x\}$, $d(z) > 2$. Now, condition (a) of lemma 2.18 holds trivially. For (b); $Sl_{d(z)}(X) = \{z\}$, $Sl_1(X) = \{\top\}$ and $Sl_2(X) = \{x\}$. Then, letting $1 < k < d(z)$, by the minimality of $d(z)$, $\forall u \in Sl_k(X)\ u \not R u$, giving condition (b).

For (c); we already have $Sl_1(X) = \{\top\}$ and $Sl_2(X) = \{x\}$ are clopen. Assuming that $Sl_{k-1}(X)$ is clopen for $1 < k < d(z)$, then again by the minimality of $d(z)$ $\forall u \in Sl_k(X) \cup Sl_{k-1}(X)\ u \not R u$. Therefore, we can make the identifications:

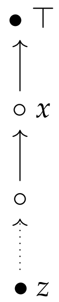
$$R^{-1}[Sl_{k-1}(X)] = \{u \in X : d(u) \geq k\};$$

$$R^{-1}[R^{-1}[Sl_{k-1}(X)]] = \{u \in X : d(u) > k\};$$

$$Sl_k(X) = \{u \in X : d(u) \geq k\} \setminus \{u \in X : d(u) > k\}.$$

As $Sl_{k-1}(X)$ is clopen, the first two sets are clopen and in turn $Sl_k(X)$ s clopen. So, by induction $\forall k < d(z)\ Sl_k(X)$ is clopen, and finally $Sl_{d(z)}(X) = X \setminus S_{d(z)-1}(X)$ and so is clopen, giving condition (c).

So, applying lemma 2.18 we may assume w.l.o.g that $Sl_k(X)$ is a singelton $\forall k \leq d(z)$, i.e. X is the following frame.



Then, we can reduce X to F_3 through a series of α- reduction, so $F_3^* \in \mathcal{A}$ which is a contradiction.

So now suppose that $\forall z \in A_* : 1 < d(z) < \omega$ that $z \not R z$. Letting $n \in \omega$ be such that $\mathcal{A}$ omits G_n^*, we claim that $Sl_k(A_*) = \varnothing \; \forall k \geq n+1$. Suppose that $Sl_{n+1} \neq \varnothing$, because $\forall z \in A_* : 1 < d(z) < \omega \; z \not R z$, we can easily repeat the argument from the previous case to establish that $S_{n+1}(A_*)$ is clopen, and so a M-subspace of A_*, and that conditions (a), (b) and (c) of lemma 2.18 hold. So, applying the lemma we may w.l.o.g assume each $Sl_k(S_{n+1}(A_*)) : k \leq n+1$ is a singleton, and then $S_{n+1}(A_*) \cong G_n \notin \mathcal{A}$ which is a contradiction. Then, letting $k \geq n+1$, $Sl_{n+1}(A_*) = \varnothing$ implies $Sl_k(A_*) = \varnothing$.

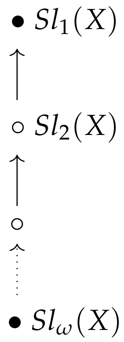
Now, if (i) fails for A, $R^{-1}[x]$ contains an infinite descending chain $\langle x_k \rangle_{k \in \omega}$, but as $Sl_k(A_*) = \varnothing$ for all $k \geq n+1$ and $x_k \notin Sl_{k'}(A_*)$ for all $k' < k$ we get that $d(x_k) = \omega$ for all $k \geq n+1$. In particular $Sl_\omega(A_*) \neq \varnothing$. If (ii) fails for A, then $R^{-1}[x]$ contains an infinite ascending chain, and so all points in that chain have depth ω and $Sl_\omega(A_*) \neq \varnothing$. If (iii) fails, then $\exists y \in R^{-1}[x] : yRy$, which by assumption implies $d(y) = \omega$, so once more $Sl_\omega(A_*) \neq \varnothing$. So, in all cases $Sl_\omega(A_*) \neq \varnothing$.

Now, letting $k \geq n+1$, $Sl_k(A_*) = \varnothing$ implies that $Sl_\omega(A_*) \cap A_* \setminus R^{-1}[Sl_k(A_*)] = Sl_\omega(A_*) \neq \varnothing$. Thus, we may consider the least $m \in \omega$ such that $Sl_\omega(A_*) \cap A_* \setminus R^{-1}[Sl_{m+1}(A_*)] \neq \varnothing$. Note that as $A_* = \{\top, x\} \cup R^{-1}[x]$ we have $m \geq 1$. Moreover, the minimality of m means that $\forall k < m \; Sl_\omega(A_*) \cap A_* \setminus R^{-1}[Sl_k(A_*)] = \varnothing$, i.e. $Sl_\omega(A_*) \subseteq R^{-1}[Sl_k(A_*)]$.

Now take a $z \in Sl_\omega(A_*) \cap A_* \setminus R^{-1}[Sl_{m+1}(A_*)]$ and consider the M-subspace $\mathcal{X}$ with underlying set $X = R^+[z] = (R^+[z] \cap Sl_\omega(A_*)) \cup S_m(A_*)$. Letting $u, v \in X : d(u) = d(v) = \omega$, as $u, v \in R^+[z]$, $u, v \notin R^{-1}[Sl_{m+1}(X)]$ and so $\forall m+1 \leq k \leq n+1$ $u, v \notin R^{-1}[Sl_k(X)]$, i.e. $\forall m+1 \leq k \leq n+1 \; R[u] \cap Sl_k(X) = \varnothing = R[v] \cap Sl_k(X)$. Additionally, as $u, v \in Sl_\omega(X)$ as noted above $\forall k \leq m \; u, v \in R^{-1}[Sl_k(A_*)]$, i.e. $\forall k \leq m \; R[u] \cap Sl_k(X) \neq \varnothing \neq R[v] \cap Sl_k(X)$. So condition (a) of lemma 2.18 holds. Conditions (b) and (c) hold in familiar fashion to our previous cases, so once more by lemma 2.18 we may consider the resulting modal equivalence E and the quotient space $\mathcal{X}/E$. The elements of $\mathcal{X}/E$ are $Sl_k(X) : k \leq m$ and $Sl_\omega(X)$.

As $z \in Sl_\omega(A_*)$ and $z \notin R^{-1}[Sl_{m+1}(A_*)]$, we must have an infinite ascending chain starting from z, which is contained in X. Letting z' be in that chain, $z' \in X$ and it too has an infinite ascending chain starting from it in X so $z' \in Sl_\omega(X)$. So then $[z] = Sl_\omega(X) = [z']$ and as zRz' $Sl_\omega(X) R_E Sl_\omega(X)$. Each $Sl_k(X) : 2 < k < m$

consisted only of irreflexive points, $Sl_k(X) \not R_E Sl_k(X)$. Finally, as $\forall k \leq m\ Sl_\omega(A_*) \subseteq R^{-1}[Sl_k(A_*)]$ we have $Sl_\omega(X) R_E Sl_k(X)$. Putting this all together, we obtain X/E is the following frame.



Again, this can be reduced to F_3 via α-reductions, giving a contradiction. □

Lemma 5.4. Let $\mathcal{A}$ be a variety omitting $F_i^* : 1 \leq i \leq 17$ and G_n^* for some $n > 0$. Let $A \in \mathcal{A}$ and $x \in \mathcal{A}_* : x \not R x$. Then, if $y \in A_* : yRx$ then $\forall z \in A_* : yRz$, z and x are comparable.

Proof. Suppose not, so we have $A \in \mathcal{A}$ such that $\exists x, y, z \in A : x \not R x\ yRx$, yRz and $x||z$. By lemma 5.3, $R^{-1}[x]$ is conversely well founded, noting $y \in \{u \in R^{-1}[x] : \exists v \in A_* : uRv\ \&\ x||v\}$, we can assume y is maximal with this property. By lemma 2.15 we may w.l.o.g assume that A_* consists of only improper clusters. Taking the M-subspace $R^+[y]$ we may w.l.o.g assume y is the root of A_*, then the maximality of y means that $\forall u \in A_* \setminus \{y\}$ if uRx then $\forall v \in R[u]$ x and v are comparable.

Now, $R[x]$ is closed and non-empty, with $x \notin R[x]$. Letting $z' \in R[x]$, again as $x \not R x$, $x \notin R^+[z']$, so applying modal separation we find a clopen upset $U_{z'}^x$ containing z' and omitting x. Moreover, as $x||z\ z \notin R^+[z']$, applying modal separation gives a clopen upset $U_{z'}^z$ containing z' and omitting z. Then, $R[x] \subseteq \bigcup_{z' \in R[x]} U_{z'}^x \cap U_{z'}^z$, and by compactness $\exists \{z_i'\}_{i=1}^n \subseteq R[x]$ such that $R[x] \subseteq \bigcup_{i=1}^n U_{z'}^x \cap U_{z'}^z$. Letting U be that union, we have a clopen upset U such that $R[x] \subseteq U$, $x \notin U$ and $z \notin U$. By lemma 2.16 we may assume w.l.o.g that this is a singleton, i.e. letting $\top$ be its sole element we have $\top$ is isolated and maximal in A_*, $\top \neq z$ and $R[x] = \{\top\}$. We make our first case distinction, either $\exists u \in A_* : x||u$ and $uR\top$ or not.

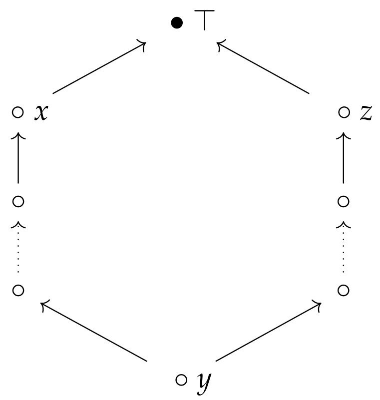
Case 1: If there is such a u, then yRu and $x||u$ so we may assume that $u = z$. Now, $A_* \setminus R^{-1}[\top]$ is a clopen upset, and as $\top$ is maximal in A_*, $\{\top\}$ is also an upset. So $A_* \setminus R^{-1}[\top] \cup \{\top\}$ is a clopen upset, then by lemma 2.16 and $x, y, z \in R^{-1}[\top]$ we may w.l.o.g assume the set is a singleton, i.e. $A_* \setminus R^{-1}[\top] \cup \{\top\} = \{\top\}$ and $A_* = R^{-1}[\top]$. Now, consider the set:

$$V := (A_* \setminus \{\top\}) \setminus R^{-1}[A_* \setminus \{\top\}].$$

It is easy to check that $V = \{u \in A_* \setminus \{\top\} : R[u] = \{\top\}\}$ which implies $x \in V$ and that V consists of an anti-chain of irreflexive points. Additionally V is clopen as $\top$ is isolated. We make a second case distinction, either $V = \{x\}$ or not.

Case 1a: If $V \neq \{x\}$, then letting $u \in V : u \neq x$ we have $uR\top$, $x||u$ and yRu, so we can again assume $u = z$. Firstly, we consider $A_* \setminus (V \cup R^{-1}[V])$ which is an upset, is clopen because V is clopen, and contains $\top$. Then applying lemma 2.16 once more we may w.l.o.g assume the set is a singleton, i.e. $A_* \setminus (V \cup R^{-1}[V]) = \{\top\}$ and $A_* = \{\top\} \cup V \cup R^{-1}[V]$.

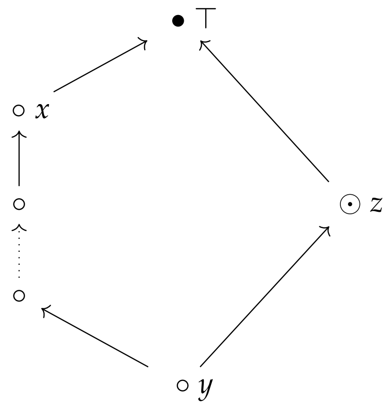
Applying modal separation to x and z to obtain U_x^z, we let $A = V \cap U_x^z$ and $B = V \cap (A_* \setminus U_x^z)$. These are clearly clopen and pairwise disjoint as they partition V, and moreover as $\forall u \in A \cup B = V$ $R[u] = \{\top\}$ they form an M-partition. So, by lemma 2.17 we may assume w.l.o.g they are singletons, i.e. $\{x\}$ and $\{z\}$ are clopen, $V = \{x, z\}$ and $A_* = \{\top, x, z\} \cup R^{-1}[x, z]$. Notably, by lemma 5.3 the only reflexive point in A_* is $\top$ and $d(y) < \omega$. The maximality assumption on y then gives that the underlying frame of A_* is the following. Then, by repeatedly applying α-reductions we can reduce A_* to F_{15}, thus $F_{15}^* \in \mathcal{A}$ which is a contradiction.



Case 1b: Suppose $V = \{x\}$, then x is isolated and $\forall u \in A_* \setminus \{\top, x\}$, $R[u] \setminus \{\top\} \neq \varnothing$. Now, consider the set

$$U := A_* \setminus (R^{-1}[x] \cup \{\top, x\}).$$

Note that $z \in U$ and by definition $A_* = \{\top, x\} \cup U \cup R^{-1}[x]$. We claim that U is an M-partition, it is clopen as $\{\top\}$ and $\{x\}$ are both clopen. Then, let $u, v \in U$ and uRw, note $u, v \in A_* \setminus \{\top, x\}$. Either $w = \top$ or not, if $w = \top$ then $vR\top$ so vRw. If $w \neq \top$, then as $u \not R x$ $w \neq x$ and $w \not R x$, so $w \in U$. Now, letting $t \in R[v] : t \neq \top$, we again have $v \not R x$ so $t \neq x$ and $t \not R x$, so $t \in U$, and we may take it as witness. So, by lemma 2.17 we may assume w.lo.g that U is a singelton, i.e. $U = \{z\}$ and $A_* = \{\top, x, z\} \cup R^{-1}[x]$. Again, by lemma 5.3, every point in $R^{-1}[x]$ is irreflexive, and $d(y) < \omega$. Finally, the maximality assumption on y then gives that the underlying frame of A_* is the following:

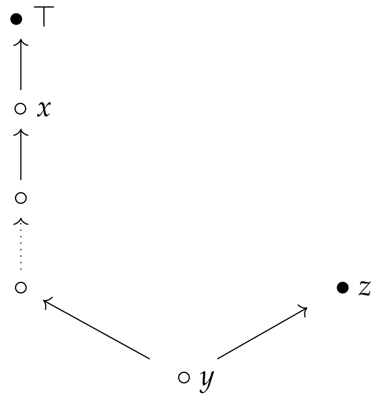


Then, by repeatedly applying α-reductions, we can reduce A_* to F_{15}, thus $F_{15}^* \in \mathcal{A}$ which is a contradiction.

Case 2: $\forall u \in A_*$ if $x||u$ then $u \not R \top$. In particular $z \not R \top$. We consdier the set

$$U := A_* \setminus R^{-1}[\top].$$

Note that $z \in U$ and by definition $A_* = R^{-1}[\top] \cup U$. Then, by our assumption if $uR\top$ then it is comparable to x and as $R[x] = \{\top\}$ this further implies $u \in \{\top, x\} \cup R^{-1}[x]$. Additionally, $\{\top, x\} \cup R^{-1}[x] \subseteq R^{-1}[\top]$, so we have $R^{-1}[\top] = \{\top, x\} \cup R^{-1}[x]$ and $A_* = \{\top, x\} \cup U \cup R^{-1}[x]$. Now U is is clopen as $\top$ is isolated and is an upset. So, by lemma 2.16 we may w.l.o.g assume it is a singelton, i.e. $U = \{z\}$ and $A_* = \{\top, x, z\} \cup R^{-1}[x]$. Then, by corollary 5.2 and lemma 5.3 we get that zRz, every point in $R^{-1}[x]$ is irreflexive and $d(y) < \omega$. Finally, the maximality assumption on y then gives that the underlying frame of A_* is the following:



Then, by repeatedly applying α-reductions, we can reduce A_* to F_{14}, thus $F_{14}^* \in \mathcal{A}$ which is a contradiction. □

Corollary 5.5. Let $\mathcal{A}$ be a variety omitting $F_i^* : 1 \leq i \leq 17$ and G_n^* for some $n > 0$. Let $A \in \mathcal{A}$ and $x \in \mathcal{A}_* : x \not R x$. Then, $R^{-1}[x] \cup \{x\}$ is a tree of irreflexive point of finite depth.

Proof. Assume for the moment that $R[x] = \{\top\}$ for some $\top \in A_*$. Then, from lemma 5.3 $R^{-1}[x] \cup \{x\}$ has finite depth, and from lemma 5.4 we have that for any $n > 2$ and $u \in Sl_n(A_*) \cap R^{-1}[x]$ that $u \not R u$ and $\forall 2 < k < n \; \exists! v \in Sl_k(A_*) : uRv$, i.e. $R^{-1}[x] \cup \{x\}$ is a tree of irreflexive points of finite depth.

Now, dropping the assumption that $[x] = \{\top\}$, just as in the proof of lemma 5.3 we find a clopen upset $U : R[x] \subseteq U$. We take the induced modal equivalence from lemma 2.16. Considering A_*/E, we have $(A_*/E)^* \in \mathcal{A}$, $R_E[[x]] = U$ and $Sl_2(A_*/E) = \{[x]\}$. From our previous consideration we have the required structure for A_*/E. Then as E only identified points in $R[x]$, $R^{-1}[x]$ has the same structure in A_* as $R_E^{-1}[[x]]$ does in A_*/E. □

5.2 Three Central Results

With some control over irreflexive points we can now prove the three central structural results that drive the main theorem of this chapter. The first is of particular importance and relates to another important concept when working with transitive frames - width.

Let (X, R) be a transitive frame. We define the *width* of an element $x \in X$ as the maximal number of points in a maximal anti-chain of points in $R^+[x]$. If there is no maximal anti-chain, or an anti-chain with infinitely many points we say that x is ω-wide and use $w(x) \in \omega \cup \{\omega\}$ to denote the width of x. The width of X is $w(X) = max\{w(x) \in \omega \cup \{\omega\} : x \in X\}$ if this exists, and $w(X) = \omega$ otherwise.

Given a $K4$-algebra A, we say A has width equal to A_*.

Lemma 5.6. Let $\mathcal{A}$ be a variety omitting $F_i^* : 1 \leq i \leq 17$ and G_n^* for some $n > 0$. Then $\forall A \in \mathcal{A}, w(A) \leq 2$.

Proof. Suppose not, that is suppose $\exists A \in \mathcal{A}$ with width > 2. We make the following claim:
Claim: $\exists B \in \mathcal{A} : B_*$ is rooted and has three incomparable, isolated points x_1, x_2, x_3 such that either:

(i) x_1, x_2 & x_3 are maximal and $B_* = R^{-1}[x_1, x_2, x_3]$;

(ii) B_* has a maximum $\top$ that is isolated and $B_* \setminus \{\top\} = R^{-1}[x_1, x_2, x_3]$;

(iii) B_* has an isolated point $\top : x_1 R \top$, $x_2 R \top$, $\top$ and x_3 are maximal, $B_* \setminus \{\top\} = R^{-1}[x_1, x_2, x_3]$ and $R^{-1}[\top] \cap R^{-1}[x_3] \subseteq R^{-1}[x_1, x_2]$.

As A has width > 2 we can find $\bot, x_1, x_2, x_3 \in A_*$ such that $\bot R x_1, \bot R x_2, \bot R x_3$ and x_1, x_2, x_3 are all incomparable. As ever, by considering the M-subspace $R^+[\bot]$ of A_* we may assume w.l.o.g that $\bot$ is the root of A_*, and by lemma 2.15 may assume A_* consists only of improper clusters.

Let $u \in A_* : u \not R u$, then either $\bot = u$ or $\bot R u$. We have that $\bot R x_1$, $\bot R x_2$ and $\bot R x_3$, therefore by lemma 5.4 u is comparable with x_1, x_2 and x_3. In particular $u \notin \{x_1, x_2, x_3\}$ and therefore $x_1 R x_1$, $x_2 R x_2$ and $x_3 R x_3$. Moreover, as $x_1 R x_1$ and $u \not R u$ by lemma 5.3 $x_1 \not R u$ so in fact $u R x_1$. Similarly, $u R x_2$ and $u R x_3$, that is $u \in R^{-1}[x_1] \cap R^{-1}[x_2] \cap R^{-1}[x_3]$. As u was arbitrary, the only irreflexive points in A_* belong to $R^{-1}[x_1] \cap R^{-1}[x_2] \cap R^{-1}[x_3]$.

As x_1, x_2 and x_3 are incomparable, we may by modal separation find clopen upsets U_i for $i \in \{1, 2, 3\}$ such that $x_i \in U_j$ iff $i = j$. We then define the following sets:

$$U := (U_1 \cap U_2) \cup (U_1 \cap U_3) \cup (U_2 \cap U_3).$$

$$V_i := \begin{cases} U_i \setminus R^{-1}[U] & \text{if } x_i \notin R^{-1}[U] \\ U_i \cap R^{-1}[U] \setminus U & \text{if } x_i \in R^{-1}[U] \end{cases}$$

Note that the V_i are clopen and $x_i \in V_j$ iff $i = j$. As we are choosing 2 options for 3 sets at least two will match, so we may w.l.o.g assume either:

(a) $V_1 = U_1 \setminus R^{-1}[U]$ and $V_2 = U_2 \setminus R^{-1}[U]$ or

(b) $V_1 = U_1 \cap R^{-1}[U] \setminus U$ and $V_2 = U_2 \cap R^{-1}[U] \setminus U$.

First assume (a) holds. If $u \in V_1$ then $u \in U_1$ and $u \notin U$, so by the definition of U $u \notin U_2$ and $u \notin U_3$. This in turn implies $u \notin V_2$, so $V_1 \cap V_2 = \varnothing = V_1 \cap U_3$. Similarly, $V_2 \cap U_3 = \varnothing$. Therefore V_1, V_2 & U_3 are all disjoint clopen upsets of A_*. We further define:

$$W := U_3 \cup A_* \setminus R^{-1}[V_1 \cup V_2 \cup U_3].$$

W is a clopen upset still disjoint from V_1 and V_2. Therefore, V_1, V_2 and W form a collection of pairwise disjoint clopen sets, and as each is an upset they moreover form an M-partition. By lemma 2.17 we can consider the modal equivalence E that identifies points within those sets and consider $(A_*/E)^* \in \mathcal{A}$.

We claim that A_*/E satisfies case (i). Note that $x_1 \in V_1$, $x_2 \in V_2$ and $x_3 \in W$ so each set really is an element of A_*/E, whilst $\bot$ is not in any of them so $[\bot] = \{\bot\} \in A_*/E$. As V_1, V_2 and W were clopen in A_* they are isolated in A_*/E, and as they were upsets in A_* they are maximal in A_*/E. Moreover as each is maximal they are all incomparable. Letting $u \in A_*$, then either $u \in R^{-1}[V_1 \cup V_2 \cup U_3]$ and then $[u] \in R_E^{-1}[V_1, V_2, W]$ or $u \notin R^{-1}[V_1 \cup V_2 \cup U_3]$ and then $u \in W$ and $[u] = W$. So $A_*/E = R_E^{-1}[V_1, V_2, W]$ and A_*/E satisfies case (i).

Now assume (b) holds, we consider:

$$V^+ := A_* \setminus R^{-1}[V_1 \cup V_2 \cup V_3].$$

V^+ is clopen. Moreover, supposing $u \in U \setminus V^+$ then as $u \notin V^+$ we have $u \in R^{-1}[V_i]$ for some $1 \leq i \leq 3$ and so $\exists v \in V_i : uRv$. Then, $u \in U$ and U is an upset so $v \in U$ and $U \cap V_i \neq \varnothing$ which is a contradiction. Therefore, $U \subseteq V^+$. We already have that V_1, V_2 and V_3 are pairwise disjoint and as every element in U is reflexive $V_1 \cup V_2 \cup V_3 \subseteq R^{-1}[V_1 \cup V_2 \cup V_3]$ so $\forall 1 \leq i \leq 3$ $V^+ \cap V_i = \varnothing$. Thus, $\{V_1, V_2, V_3, V^+\}$ is a collection of pairwise disjoint clopen sets. We claim moreover that they form an M-partition. As V^+ is an upset the M-partition conditions holds for it immediately.

For V_i: Let $u, v \in V_i$ and uRw. Now, $u \in U_i$ which is an upset so $w \in U_i$. If $w \in V_i$ then as $v \in V_i \subseteq U_i$ we have $vRv \in V_i$ and so we may take v itself as witness. If $w \notin V_i$ then $V_i \neq U_i \setminus R^{-1}[U]$ (as the latter is an upset and $u \in V_i$) and instead $V_i = U_i \cap R^{-1}[U] \setminus U$. We have $w \in U_i$ and $w \notin V_i$. If $w \notin V^+$ then $w \in R^{-1}[V_j]$ for $j \in \{1, 2, 3\}$, so $\exists t \in V_j : wRt$. Then $uRwRt$ so $t \in U_i$, and moreover $j = i$ as otherwise we have $t \in U$, uRt and $u \notin R^{-1}[U]$ which is a contradiction. But then, $t \in R^{-1}[U]$ and so $w \in R^{-1}[U]$ and $w \notin U$, i.e. $w \in V_i$ which is a contradiction. So, $w \in V^+$. Then as $v \in V_i = U_i \cap R^{-1}[U] \setminus U$ we have $\exists t \in U \subseteq V^+ : vRt$ which we may take as witness.

So, again we consider by lemma 2.17 the resulting space A_*/E and the elements $V_1, V_2, V_3, V^+ \in A_*/E$. They are all isolated as they were clopen in A_*. The V_i are also incomparable; let $u \in V_i$ and $v \in V_j : uRv$, then $u \in U_i$ implies $v \in U_i$ so $v \in U \subseteq V^+$ and $v \notin V_j$ which is a contradiction. Now, suppose $u \notin V^+$, then $u \in R^{-1}[V_1 \cup V_2 \cup V_3]$ and so $uRv : v \in V_i$ for some $1 \leq i \leq 3$. Therefore, $[u] \in R_E^{-1}[V_1, V_2, V_3]$, that is $A_*/E = \{V^+\} \cup R_E^{-1}[V_1, V_2, V_3]$.

If V^+ is the maximum of A_*/E then case (ii) holds, so suppose it is not the maximum. By (b), $x_1 \in V_1 = U_1 \cap R^{-1}[U] \setminus U$ and so $\exists u \in U \subseteq V^+ : x_1 Ru$, so $V_1 R_E V^+$. Similarly, $V_2 R_E V^+$. Now, if $V_3 = U_3 \cap R^{-1}[U] \setminus U$ then again $V_3 R_E V^+$, but then as $A_*/E = \{V^+\} \cup R_E^{-1}[V_1, V_2, V_3]$ this would make V^+ the maximum of A_*/E, so instead $V_3 = U_3 \setminus R^{-1}[U]$ and so an upset of A_* and maximal in A_*/E. Similarly, V^+ is an upset of A_* so is maximal in A_*/E. Finally, if $R_E^{-1}[V^+] \cap R_E^{-1}[V_3] \subseteq R_E^{-1}[V_1, V_2]$ then case (iii) holds, so again suppose it does not. We define:

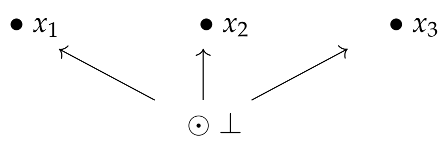
$$W_1 := R_E^{-1}[V^+] \cap R_E^{-1}[V_3] \setminus R_E^{-1}[V_1, V_2];\ W_2 := \{V^+\} \cup A_*/E \setminus R^{-1}[V^+].$$

W_1 and W_2 are disjoint clopen subsets, by our assumption $W_1 \neq \varnothing$ and moreover we claim they form an M-partition. W_2 is an upset so the condition holds trivially. Letting $[u], [v] \in W_1$ and $[u] R_E [w]$, if $[w] \in W_1$ then we have vRv so $[v] R_E [v] \in W_1$

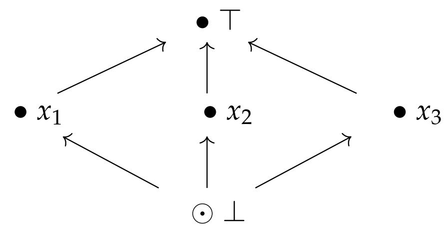
and we may take $[v]$ itself as witness. If $[w] \notin W_1$, as $[u]R_E[v]$ and $[u] \in W_1$ we have $[w] \notin R_E^{-1}[V_1, V_2]$, so then either $[w]\not{R}_E V^+$ and $[w] \in W_2$ or $[w]\not{R}_E V_3$ so then $[w] = V^+ \in W_2$. So $[w] \in W_2$ and $[v]R_E V^+ \in W_2$ so we may take it as witness.

Once more, in line with lemma 2.17 we can consider the modal equivalence E' on A_*/E identifying points within W_1 and W_2. We now aim to show that $Y = (A_*/E)/E'$ satisfies case (ii) with W_1, $[V_1] = \{V_1\}$, $[V_2] = \{V_2\}$ and W_2 respectively. As each was clopen in A_*/E they are isolated in Y. That $[V_1]$ and $[V_2]$ are incomparable is immediate from V_1 and V_2 being incomparable in A_*/E. Then $W_1 \cap R_E^{-1}[V_1] = \varnothing$ so $W_1 \not{R}_Y [V_1]$, if $[V_1]R_Y W_1$ then $V_1 R_E [u]$ for some $[u] \in W_1$, then $[u]R_E V_3$ so $V_1 R_E V_3$ which is a contradiction. So $[V_1]$ and W_1 are incomparable, and similarly $[V_2]$ and W_1 are incomparable. Now, letting $[S] \in Y \setminus \{W_2\}$, then $S \in A_*/E \setminus \{V^+\}$ so $SR_E V_1, V_2$ or V_3. If the first two then $[S]R_Y[V_i]R_Y W_2$, if $SR_E V_3$, we note that $V_3, V^+ \in W_2$ so $[V_3] = [V^+] = W_2$ and $[S]R_Y W_2$. Thus W_2 is the maximum of Y. Finally, if $[S] \in Y \setminus \{W_2\}$ then $S \notin W_2$ and so $SR_Y W_2$ and $S \in R_E^{-1}[V_1, V_2, V_3]$, if $SR_E V_1$ or $SR_E V_2$ then $[S]R_Y[V_1]$ or $[S]R_Y[V_2]$, if $SR_E V_3$ and $S\not{R}_E V_1$ and $S\not{R}_E V_2$ then $S \in W_1$ so $[S]R_Y W_1$, so $Y \setminus \{W_2\} = R_Y^{-1}[[V_1], [V_2], W_1]$ as required.

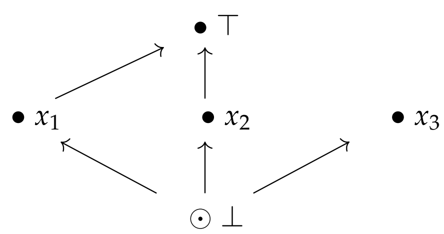
This completes the proof of the claim, so we have $B \in \mathcal{A}$ such that B_* contains one of the following substructures where each x_i and $\top$ is isolated:



(i)



(ii)



(iii)

We may still by lemma 2.15 assume that B_* consists of only improper clusters. Moreover, as x_1, x_2 and x_3 are isolated, the set $R^{-1}[x_1] \cap R^{-1}[x_2] \cap R^{-1}[x_3]$ is clopen. Then, from corollary 2.24 we can take a maximal cluster in the set, which as B_* consists of only improper clusters is in fact a maximal point p in the set. Of course, p sees x_1, x_2 and x_3 and by considering the M-subspace of B_* rooted at p, we can assume w.l.o.g $\bot = p$ and $R^{-1}[x_1] \cap R^{-1}[x_2] \cap R^{-1}[x_3] = \{\bot\}$. Note, just as we established earlier that the only irreflexive points of A_* could be in $R^{-1}[x_1] \cap R^{-1}[x_2] \cap R^{-1}[x_3]$, we can check this also holds for B_*, i.e. the only irreflexive point in B_* is possibly $\bot$. Now, we define the following clopen sets:

$$W_1 := R^{-1}[x_1] \setminus R^{-1}[x_2, x_3]; \qquad W_2 := R^{-1}[x_2] \setminus R^{-1}[x_1, x_3];$$
$$W_3 := R^{-1}[x_3] \setminus R^{-1}[x_1, x_2]; \qquad W_4 := R^{-1}[x_1] \cap R^{-1}[x_2] \setminus R^{-1}[x_3];$$
$$W_5 := R^{-1}[x_1] \cap R^{-1}[x_3] \setminus R^{-1}[x_2]; \qquad W_6 := R^{-1}[x_2] \cap R^{-1}[x_3] \setminus R^{-1}[x_1].$$

These are clearly pairwise disjoint, and from claim A we have:

$$B_* = \bigcup_{i=1}^{6} W_6 \cup \{\bot, \top\}.$$

We claim moreover that they are an M-partition. So, let $u, v \in W_i$ and uRw with:

i=1; We have $u \notin R^{-1}[x_2, x_3]$ and so $w \notin R^{-1}[x_2, x_3]$. Then, either wRx_1 and so $w \in W_1$, and $vRx_1 \in W_1$ so we may take x_1 as witness, or $w = \top$ and $vR\top$.

i=2; As the $i = 1$ case.

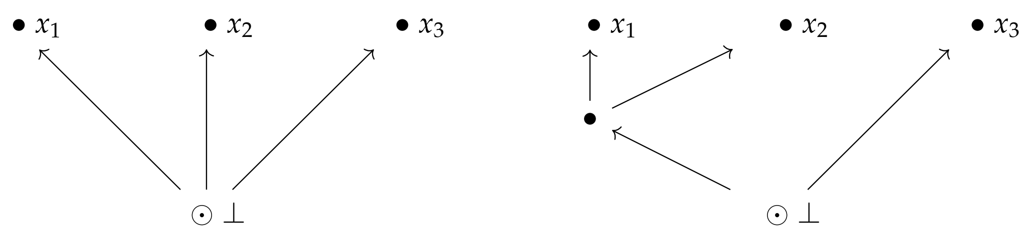
i=3; As the $i = 1$ case, except we note that if $w = \top$ as $u \in W_3$ we must be in a case for B_* where $x_3R\top$ so $vR\top$ as needed.

i=4; We have $u \notin R^{-1}[x_3]$ so $w \notin R^{-1}[x_3]$. Thus, $w \in W_1$, W_2, W_4 or $w = \top$ and we have $vRx_1 \in W_1$, $vRx_2 \in W_2$, $vRv \in W_4$ and $vR\top$ for witnesses.

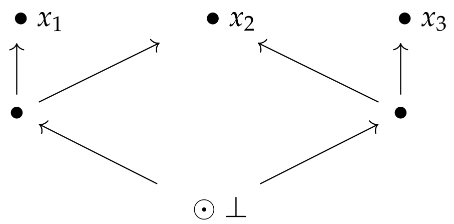
i=5; As the $i = 4$ case.

i=6; As the $i = 6$ case.

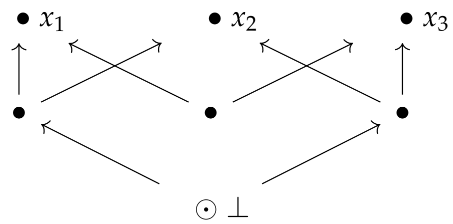
Then, as ever by lemma 2.17 we may consider the modal equivalence identifying the points within these sets and assume w.l.o.g it is the identity on B_*. Now, whilst $x_1 \in W_1$, $x_2 \in W_2$ and $x_3 \in W_3$ and so this amounts to assuming W_1, W_2 and W_3 are singletons, each of W_4, W_5 and W_6 may or may not be empty, i.e. may or may not exist as elements of B_*. Combining with the three possible substructures of B_* listed earlier, and eliminating some via isomorphism, this leaves us with the following possible underlying frames for B_*:



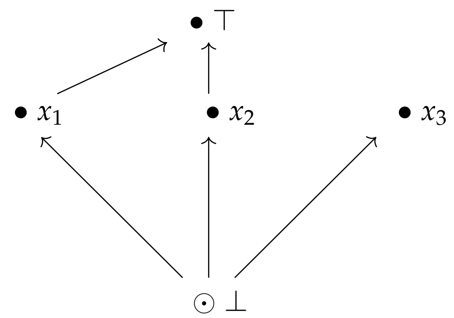
(a) (b)


x_1
x_2
x_3
$\bot$

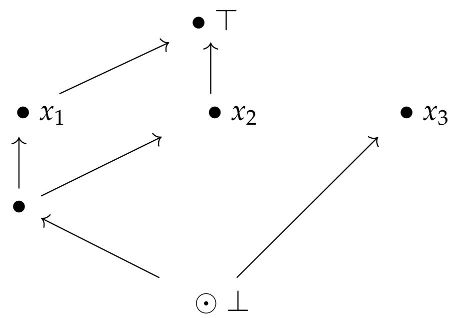
(c)


x_1
x_2
x_3
$\bot$

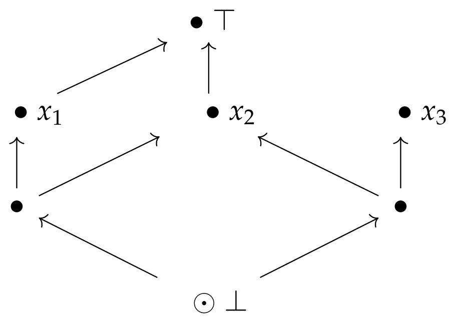
(d)


$\top$
x_1
x_2
x_3
$\bot$

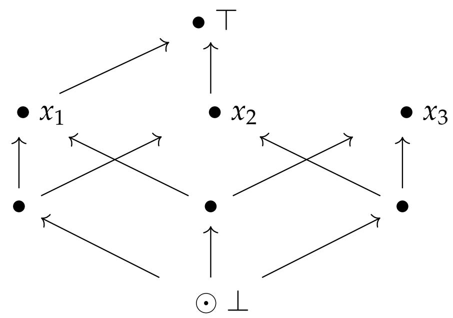
(e)


$\top$
x_1
x_2
x_3
$\bot$

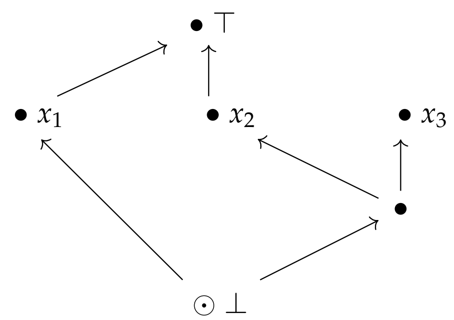
(f)


$\top$
x_1
x_2
x_3
$\bot$

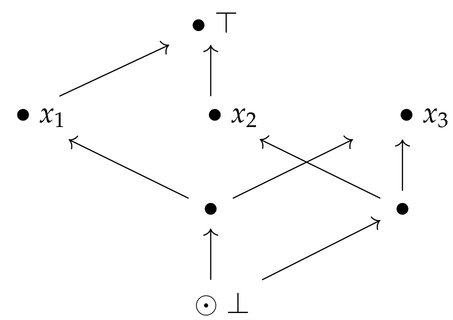
(g)


$\top$
x_1
x_2
x_3
$\bot$

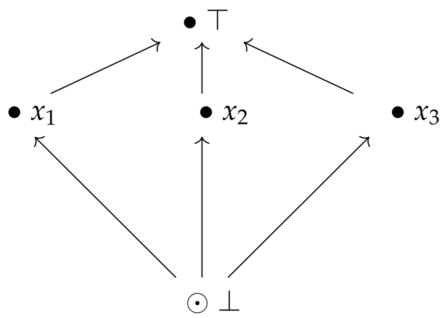
(h)


$\top$
x_1
x_2
x_3
$\bot$

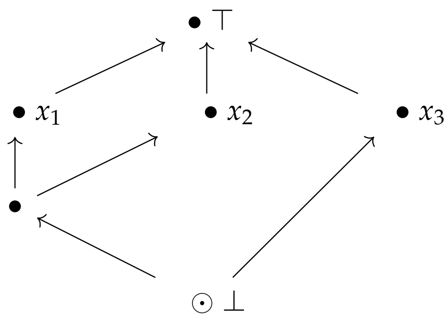
(i)


$\top$
x_1
x_2
x_3
$\bot$

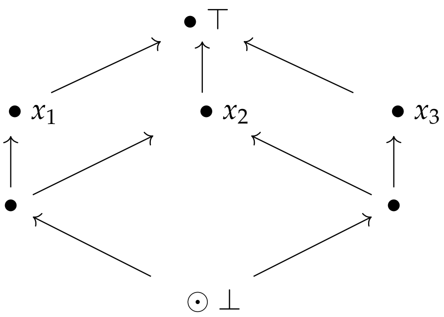
(j)



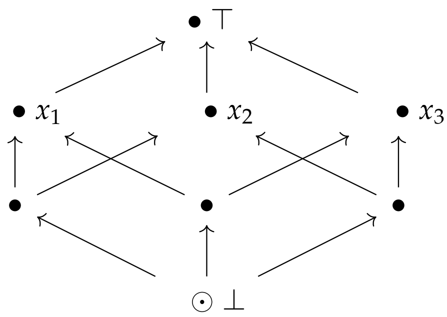
(k)



(l)



(m)



(n)

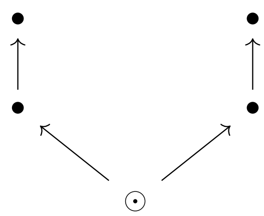
Finally, using α, β and γ-reductions, we can reduce each of these as follows:

$$(a) \mapsto F_7, (b) \mapsto F_5, (c) \mapsto F_9, (d) \mapsto F_8, (e) \mapsto F_5, (f) \mapsto F_5, (g) \mapsto F_6, (h) \mapsto F_5,$$

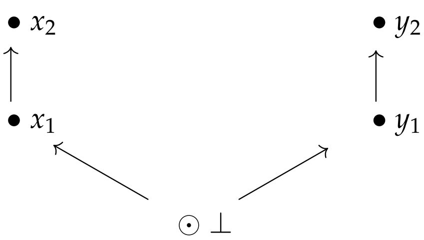
$$(i) \mapsto F_6, (j) \mapsto F_6, (k) \mapsto F_8, (l) \mapsto F_6, (m) \mapsto F_6, (n) \mapsto F_8$$

. Thus, in all cases we have $F_i^* \in \mathcal{A}$ for some $1 \leq i \leq 17$ which is a contradiction. □

Lemma 5.7. Let $\mathcal{A}$ be a variety omitting $F_i^* : 1 \leq i \leq 17$ and G_n^* for some $n > 0$. Then $\forall A \in \mathcal{A}$, A_* does not contain the following substructure:



Proof. Suppose not, let $A \in \mathcal{A}$ have the substructure described above which we label as follows:



We make the following claim:
Claim: $\exists B \in \mathcal{A} : B_*$ is rooted and contains the substructure witnessed by four elements x_1, x_2, y_1, y_2 such that either:

(i) x_2 & y_2 are maximal with $B_* = R^{-1}[x_2, y_2]$

(ii) B_* has a maximum element $\top$ that is isolated, x_2 and y_2 are isolated and $B_* = \{\top\} \cup R^{-1}[x_2, y_2]$

(iii) B_* has an isolated point $\top : x_2 R \top$ and $y_1 R \top$, $\top$ & y_2 are maximal and $B_* = \{\top\} \cup R^{-1}[x_2, y_2]$

As ever, by lemma 2.15 and taking an M-subspace we may assume that A_* consists of only improper clusters and is rooted by $\bot$. Also note, that letting $u \in A_* : u \not R u$ then either $\bot = u$ or $\bot R u$. Now, $\bot R x_1$ and $\bot R y_1$ and $x_1 || y_1$ so by lemma 5.4, u is comparable with x_1 and y_1. In particular $u \notin \{x_1, y_1\}$ and so $x_1 R x_1$ and $y_1 R y_1$. Then, by lemma 5.3 $x_1 \not R u$ and $y_1 \not R u$, so in fact $u R x_1$ and $u R y_1$. That is, the only irreflexive points in A_* belong to $R^{-1}[x_1] \cap R^{-1}[y_1]$.

Now, by modal separation, we can find clopen upsets U_1 and V_1 such that $x_1 \in U_1$ and $y_1, y_2 \notin U_1$ and $y_1 \in V_1$ and $x_1, x_2 \notin V_1$. Either $U_1 \cap V_1 = \varnothing$ or not.

Suppose $U_1 \cap V_1 = \varnothing$; We can also find by modal separation clopen upsets U_2, V_2 such that $x_2 \in U_2$ and $x_1, y_2 \notin U_2$ and $y_2 \in V_2$ and $x_2, y_1 \notin V_2$. Now $U_1 \cap U_2$ is a clopen upset and $x_2 \in U_1 \cap U_2$. Similarly, $y_2 \in V_1 \cap V_2$. Moreover, as $U_1 \cap V_1 = \varnothing$ $(U_1 \cap U_2) \cap (V_1 \cap V_2) = \varnothing$. Therefore, $U_1 \cap U_2$ and $V_1 \cap V_2$ are clopen and disjoint and as they are also upsets they easily form an M-partition.

So, by lemma 2.17 we may consider the modal equivalence E identifying points within those sets. As $U_1 \cap U_2$ was clopen in A_* it is isolated in A_*/E, and as it was an upset it is maximal in A_*/E. Similarly, $V_1 \cap V_2$ is an isolated maximal point of A_*/E. Then we have $\bot, x_1, y_1 \notin U_1 \cap U_2$ or $V_1 \cap V_2$, so $[\bot] = \{\bot\}, [x_1] = \{x_1\}, [x_2] = \{x_2\} \in A_*/E$ and together with $U_1 \cap U_2$ and $V_1 \cap V_2$ witness the substructure (that $[x_1] R_E U_1 \cap U_2$ and so on is clear, then we note that if $x_1 R z \in V_1 \cap V_2$ then $z \in U_1$ so $U_1 \cap V_1 \neq \varnothing$, so $[x_1] \not R_E V_1 \cap V_2$ and similarly $[y_1] \not R_E U_1 \cap U_2$).

Finally, letting $u \in A_*$, by lemma 5.6 A_* has width ≤ 2 and $x_2 || y_2$ so u must be comparable with either x_2 or y_2. If comparable with x_2 then either $u \in U_1 \cap U_2$ and $[u] = U_1 \cap U_2$ or not, and then $u \neq x_2$ and $x_2 \not R u$ so $u R x_2$ and $[u] R_E U_1 \cap U_2$ Similarly, if u is comparable with y_2 then either $[u] = V_1 \cap V_2$ or $[u] R_E V_1 \cap V_2$. So, $A_*/E = R_E^{-1}[U_1 \cap U_2, V_1 \cap V_2]$. Thus, we have A_*/E satisfying case (i).

So now suppose $U_1 \cap V_1 \neq \varnothing$. We consider $R^{-1}[U_1]$ and $R^{-1}[V_1]$. If $x_1 \notin R^{-1}[V_1]$, then $x \in U_1 \setminus R^{-1}[v_1]$ which is a clopen upset, $y_1 \notin U_1 \setminus R^{-1}[V_1]$ and $U_1 \setminus R^{-1}[V_1] \cap V_1 = \varnothing$. So, replacing U_1 with $U_1 \setminus R^{-1}[V_1]$ we can proceed as in the previous case. Similarly, if $y_1 \notin R^{-1}[U_1]$. So, suppose $x_1 \in R^{-1}[V_1]$ and $y_1 \in R^{-1}[U_1]$. We have either $x_2 \in R^{-1}[V_1]$ or not and either $y_2 \in R^{-1}[U_1]$ or not. We make a case distinction, either both inclusions hold, exactly one holds or neither holds.

Suppose neither holds, then $x_2 \in U_1 \setminus R^{-1}[V_1]$, $x_1, y_1 \notin U_1 \setminus R^{-1}[V_1]$ and this set is a clopen upset, and similarly, $y_2 \in V_1 \setminus R^{-1}[U_1]$, $x_1, y_1 \notin V_1 \setminus R^{-1}[U_1]$ and this set is a clopen upset. Moreover, $(U_1 \setminus R^{-1}[V_1]) \cap (V_1 \setminus R^{-1}[U_1]) = \varnothing$. So we can proceed as in the previous case with $U_1 \setminus R^{-1}[V_1]$ and $V_1 \setminus R^{-1}[U_1]$ replacing $U_1 \cap U_2$

and $V_1 \cap V_2$ respectively.

Suppose exactly one holds, say $x_2 \in R^{-1}[V_1]$ and $y_2 \notin R^{-1}[U_1]$. Then, by modal separation we can find a clopen upset $V_2 : x_2 \in V_2$ and $x_1 \notin V_2$. We then define the following sets:

$$W_1 := (U_1 \cap V_1) \cup U_1 \setminus R^{-1}[V_1];\ W_2 := U_1 \cap U_2 \cap R^{-1}[V_1] \setminus V_1;$$
$$W_3 := V_1 \setminus R^{-1}[U_1].$$

We note that $x_2 \in W_2$ and $y_2 \in W_3$. Then, the W_i are pairwise disjoint and clopen. We claim moreover that they form an M-partition. For W_1 and W_3, these are both upsets so the condition holds trivially. So let $u, v \in W_2$ and uRw. Either $w \in V_1$ or $w \notin V_1$. If $w \in V_1$ as $u \in U_1$ and uRw then $w \in U_1 \cap V_1 \subseteq W_1$. Then $v \in W_2$ so $\exists t \in R[v] : t \in V_1$ and again $t \in U_1 \cap V_1 \subseteq W_1$ so may be taken as witness. Similarly, if $w \notin R^{-1}[V_1]$ we also have $w \in U_1 \setminus R^{-1}[V_1] \subseteq W_1$ and so we can take this t as witness again. If $w \notin V_1$ and $w \in R^{-1}[V_1]$, as uRw and $u \in U_1 \cap U_2$ which is an upset $w \in U_1 \cap U_2$ so $w \in W_2$. Then $vRv \in W_2$ so we may take v as witness.

Then, by lemma 2.17 we may consider the modal equivalence E identifying points within W_1, W_2 and W_3 and consider A_*/E. As each W_i was clopen in A_* they are isolated points in A_*/E and W_1 and W_3 being upsets in A_* make them maximal in A_*/E. Then, we have $\perp, x_1, y_1 \notin W_i$ for $1 \leq i \leq 3$ so $[\perp] = \{\perp\}$, $[x_1] = \{x_1\}$ and $[y_1] = \{y_1\} \in A_*/E$ and together with W_2 and W_3 witness the substructure.

Finally, as $x_2 \in R^{-1}[V_1]$ $\exists z \in V_1 : x_2Rz$ and $x_2 \in U_1$ which is an upset so $z \in U_1 \cap V_1 \subseteq W_1$. Then $x_2 \in W_2$ means $W_2R_EW_1$. Similarly, $y_1 \in V_1$ and $y_1 \in R^{-1}[U_1]$ so $\exists z \in U_1 : y_1Rz$ and V_1 is an upset so $z \in U_1 \cap V_1 \subseteq W_1$. So $[y_1]R_EW_1$. Then, letting $u \in A_*$ we have u comparable with x_2 or y_2. If uRx_2 or $x_2 = u$ then $[u]R_EW_2$ and if uRy_2 or $y_2 = u$ then $[u]R_EW_3$. If x_2Ru then either $u \in W_2$ and $[u]R_EW_2$ or $u \notin W_2$, but then $x_2 \in U_1 \cap U_2$ which is an upset implies $u \in U_1 \cap U_2$ so in fact $u \notin R^{-1}[V_1] \setminus V_1$, i.e. either $u \in V_1$ and $u \in U_1 \cap V_1 \subseteq W_1$ or $u \notin R^{-1}[V_1]$ and $u \in U_1 \setminus R^{-1}[V_1] \subseteq W_1$. So $u \in W_1$ and $[u] = W_1$. If y_2Ru then $u \in W_3$ and $[u]R_EW_3$. Taken together, $A_*/E = \{W_1\} \cup R_E^{-1}[W_2, W_3]$ and so A_*/E satisfies case (iii).

Now suppose both $x_2 \in R^{-1}[V_1]$ and $y_2 \in R^{-1}[U_1]$, this time we use modal separation to find $U_2 : x_2 \in U_2, x_1 \notin U_2$ and $V_2 : y_2 \in V_2$ and $y_1 \notin V_2$ and define the following sets:

$$W_1 := (U_1 \cap V_1) \cup U_1 \setminus R^{-1}[V_1] \cup V_1 \setminus R^{-1}[U_1];\ W_2 := U_1 \cap U_2 \cap R^{-1}[V_1] \setminus V_1;$$
$$W_3 := V_1 \cap V_2 \cap R^{-1}[U_1] \setminus U_1.$$

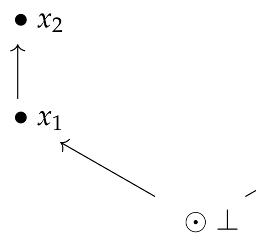
We note that $x_2 \in W_2$ and $y_2 \in W_3$. Then, again the W_i are pairwise disjoint and clopen. They moreover form an M-partition, W_1 is an upset so the condition holds trivially and W_2 and W_3 follow as the W_2 argument for the previous case.

Then, by lemma 2.17 we may consider the modal equivalence identifying points in those sets and consider A_*/E. As each W_i was clopen in A_* they are isolated points in A_*/E. Then, we have $\perp, x_1, y_1 \notin W_i$ for all $1 \leq i \leq 3$ so $[\perp] = \{\perp\}$, $[x_1] = \{x_1\}$, $[y_1] = \{y_1\} \in A_*/E$ and together with W_2 and W_3 witness the substructure.

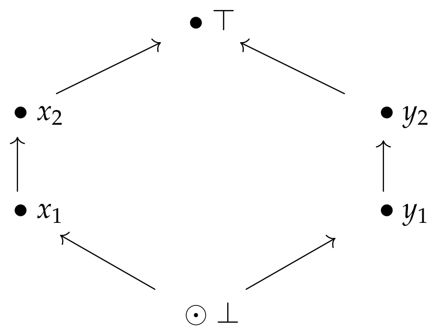
Finally, as $x_2 \in R^{-1}[V_1]$ $\exists z \in R[x] : z \in V_1$, and $x_2 \in U_1$ which is an upset so $z \in U_1 \cap V_1 \subseteq W_1$. Then $x_2 \in W_2$ means $W_2R_EW_1$. Similarly, $y_2 \in R^{-1}[U_1]$ implies $W_3R_EW_1$. Then, letting $u \in A_*$ we have u comparable with x_2 or y_2. If uRx_2 or $u = x_2$ then $[u]R_EW_2R_EW_1$ and if uRy_2 or $u = y_2$ then $[u]R_EW_3R_EW_1$. If x_2Ru then either

$u \in W_2$ and $[u]R_E W_2 R_E W_1$ or $u \notin W_2$, but then $x_2 \in U_1 \cap U_2$ which is an upset implies $u \in U_1 \cap U_2$ so in fact $u \notin R^{-1}[V_1] \setminus V_1$, i.e. either $u \in V_1$ and $u \in U_1 \cap V_1 \subseteq W_1$ or $u \notin R^{-1}[V_1]$ and $u \in U_1 \setminus R^{-1}[V_1] \subseteq W_1$. So $u \in W_1$ and $[u]R_E W_1$. Symmetrically, if $y_2 R u$ either $[u]R_E W_3 R_E W_1$ or $[u]R_E W_1$. Taken together, W_1 is a maximum element of A_*/E and $A_*/E = \{W_1\} \cup R_E^{-1}[W_2, W_3]$ and so A_*/E satisfies case (ii).

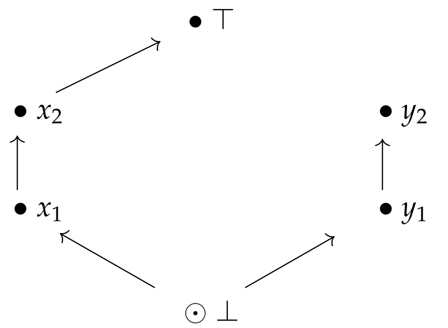
This completes the proof of the claim, so we have $B \in \mathcal{A}$ such that B_* contains one of the following substructures, where x_2, y_2 and $\top$ are isolated:



(i)



(ii)



(iii)

We may still by lemma 2.15 assume that B_* consists of only improper clusters. Moreover, by modal separation we can find a clopen upset $U : x_1 \in U$ and $y_2 \notin U$ and clopen upset $V : y_1 \in V$ and $x_2 \notin V$. Then letting:

$$U' := U \cap (R^{-1}[x_2] \setminus \{x_2\}) \setminus \{\top\},\ V' := V \cap (R^{-1}[y_2] \setminus \{y_2\}) \setminus \{\top\}.$$

These are both clopen as x_2 and y_2 are isolated and have $x_1 \in U'$, $y_1, y_2 \notin U'$ and $y_1 \in V'$, $x_1, x_2 \notin V'$. Moreover, if $u \in U' \cap V'$ then $u \in U$ and uRy_2 so $y_2 \in U$ which is a contradiction, so $U' \cap V' = \varnothing$. So U' and V' a pairwise disjoint clopens, and indeed they form an M-partition. Letting $u, v \in U'$ and uRw, we have from claim A that wRx_2, wRy_2 or $w = \top$. As uRw $w \in U$ so $w\not R y_2$. If wRx_2 then either $w \neq x_2$ and so $w \in U'$, then $vRv \in U'$ so we may take v itself as witness, or $w = x_2$ and then vRw. If $w = \top$ then vRw. The case for V' is similar, except we note when $w = \top$ that as uRw we are not in case (iii) and have $vR\top$ as needed. So, applying lemma 2.17 we may w.l.o.g assume that U' and V' are singletons, i.e. x_1 and y_1 are isolated.

Moreover, $R^{-1}[x_1] \cap R^{-1}[y_1]$ is clopen and so by corollary 2.24 we can consider a maximal cluster in it, which as B_* consists of only improper cluster is in fact a maximal point in the set. This point sees x_1 and y_1, so by considering the M-subspace of B_* rooted at this point we can assume w.l.o.g that the point is $\bot$ and $R^{-1}[x_1] \cap R^{-1}[y_1] = \{\bot\}$. Note, just as we established earlier that the only irreflexive points of A_* could be in $R^{-1}[x_1] \cap R^{-1}[y_1]$, we can check this also holds for B_*, i.e. the only irreflexive point in B_* is possibly $\bot$.

We now show each case leads to a contradiction. Case (i); We consider the following clopen sets:

$$W_1 := R^{-1}[x_2] \setminus R^{-1}[y_2, x_1]; \qquad W_2 := R^{-1}[y_2] \setminus R^{-1}[x_2, y_1];$$
$$W_3 := R^{-1}[x_1] \setminus R^{-1}[y_2]; \qquad W_4 := R^{-1}[y_1] \setminus R^{-1}[x_2];$$
$$W_5 := R^{-1}[x_1] \cap R^{-1}[y_2] \setminus R^{-1}[y_1]; \qquad W_6 := R^{-1}[y_1] \cap R^{-1}[x_2] \setminus R^{-1}[x_1].$$

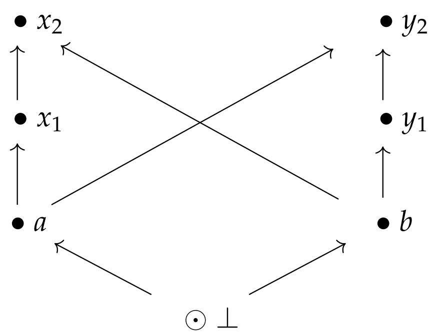
By inspection these sets are pairwise disjoint. Moreover, letting $u \in B_* \setminus \{\bot\}$ as $x_1 || y_1$, $\bot R u, \bot R x_1$, $\bot R y_1$ and by lemma 5.6 B_* has width ≤ 2 we get that u is comparable with either x_1 or y_1. As $u \neq \bot$ either $u \not R x_1$ or $u \not R y_1$. If $u \not R x_1$ then either $x_1 R u$, $u \not R y_2$, $u R x_2$ and $u \in W_1 \cup W_3$; $y_1 R u$, $u \not R x_2$, $u R y_2$ and $u \in W_2 \cup W_4$ or $u R y_1$ and $u \in W_4 \cup W_6$. If $u \not R x_1$ then either $y_1 R u$, $u \not R x_2$, $u R y_2$ and $u \in W_2 \cup W_4$; $x_1 R u$, $u \not R y_2$, $u R x_2$ and $u \in W_1 \cup W_3$ or $u R x_1$ and $u \in W_3 \cup W_6$. So:

$$B_* = \bigcup_{i=1}^{6} W_i \cup \{\bot\}.$$

We claim that the W_i form an M-partition. So, let $u, v \in W_i$ with uRw with:

- i=1; We have $u \notin R^{-1}[y_2, x_1]$ so $w \notin R^{-1}[y_2, x_1]$ and $w \neq \bot$. Then $w \in W_j$ for some $1 \leq j \leq 6$, and as $w \notin R^{-1}[y_2, x_1]$ we must have $j = 1$, then $vRx_2 \in W_1$.
- i=2; As the $i = 1$ case.
- i=3; We have $u \notin R^{-1}[y_2]$ so $w \notin R^{-1}[y_2]$ and $w \neq \bot$. Then $w \in W_j$ for some $1 \leq j \leq 6$, and as $w \notin R^{-1}[y_2]$ we must have $j = 1$ or $j = 3$. If $j = 1$ then $vRx_1Rx_2 \in W_1$ and if $j = 3$ then $vRx_1 \in W_3$.
- i=4; As the $i = 3$ case.
- i=5; We have $u \notin R^{-1}[y_1]$, so $w \notin R^{-1}[y_1]$ and $w \neq \bot$. Then $w \in W_j$ for some $1 \leq j \leq 6$, and as $w \notin R^{-1}[y_1]$ we must have $j = 1$ or $j = 3$ or $j = 5$. If $j = 1$ then $vRx_1Rx_2 \in W_1$, if $j = 3$ then $vRx_1 \in W_3$ and if $j = 5$ then $vRv \in W_5$.
- i=6; As the $i = 5$ case.

Then, as ever by lemma 2.17 we may consider the modal equivalence identifying the points within these sets and assume w.l.o.g it is the identity on B_*. Now, whilst $x_2 \in W_1, y_2 \in W_2, x_1 \in W_3$ and $y_1 \in W_4$ and so this amounts to assuming W_1, W_2, W_3 and W_4 are singeletons, both W_5 and W_6 may or may not be empty, i.e. may or may not exist as elements of B_*. In other words, B_* has the following underlying frame where the elements labelled a and b may or may not be present:

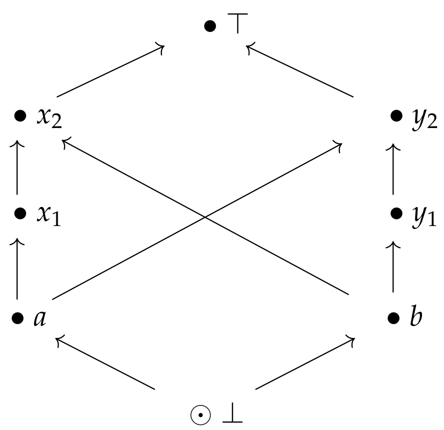


If either a or b are present, the M-subspace rooted at a or b respectively is isomorphic to F_5 so $F_5^* \in \mathcal{A}$ which is a contradiction. If neither a or b are present we can reduce B_* to F_6, so $F_6^* \in \mathcal{A}$ which is also a contradiction.

Case (ii); We consider the same clopen sets as in case (i) and proceed as before, however this time we have:

$$B_* = \bigcup_{i=1}^{6} W_i \cup \{\bot, \top\}.$$

Additionally, when checking the M-partition criteria we have in each case the possibility that $w = \top$, but $vR\top$ in all cases as needed. This yields that B_* has the following underlying frame where the elements labelled a and b may or may not be present:



Again, if either a or b is present the M-subspace rooted at a or b respectively is isomorphic to F_6 so $F_6^* \in \mathcal{A}$ which is a contradiction. if neither are present, we can reduce B_* to F_6 also implying that $F_6^* \in \mathcal{A}$ and a contradiction.

Case (iii); We consider the following clopen sets:

$$W_1 := R^{-1}[\top, y_2] \setminus R^{-1}[x_2, y_1]; \qquad W_2 := R^{-1}[x_2] \setminus R^{-1}[y_2, x_1];$$
$$W_3 := R^{-1}[y_1] \setminus R^{-1}[x_2]; \qquad W_4 := R^{-1}[x_1] \setminus R^{-1}[y_1];$$
$$W_5 := R^{-1}[x_2] \cap R^{-1}[y_1] \setminus R^{-1}[x_1].$$

By inspection these are pairwise disjoint. Moreover, letting $u \in B_* \setminus \{\bot, \top\}$, as $x_1 || y_1$, $\bot R u$, $\bot R x_1$, $\bot R y_1$ and by lemma 5.6 B_* has width ≤ 2 we get that u is comparable with either x_1 or y_1. As $u \neq \bot$, either $u \not R x_1$ or $u \not R y_1$. If $u \not R x_1$, then either $x_1 R u$, $u \not R y_2$, $u R x_2$ and $u \in W_2$; $y_1 R u$, $u \not R x_2$, $u R y_2$ and $u \in W_1 \cup W_3$ or $u R y_1$ and $u \in W_3 \cup W_5$. If $u \not R y_1$, then either $y_1 R u$, $u \not R x_2$, $u R y_2$ and $u \in W_1 \cup W_3$; $x_1 R u$, $u \not R y_2$, $u R x_2$ and $u \in W_2 \cup W_4$ or $u R x_1$ and $u \in W_4$. So:

$$B_* = \bigcup_{i=1}^{5} W_i \cup \{\bot, \top\}.$$

We claim that they form an M-partition. So, let $u, v \in W_i$ and uRw with:

i=1; We have $u \notin R^{-1}[x_2, y_1]$ so $w \notin R^{-1}[x_2, y_1]$ and $w \neq \bot$. If $w = \top$ then $w \in W_1$ and $vRv \in W_1$. If $w \neq \top$, then $w \in W_j : 1 \leq j \leq 5$ and as $w \notin R^{-1}[x_2, y_1]$ we must have $j = 1$ and again $vRv \in W_1$.

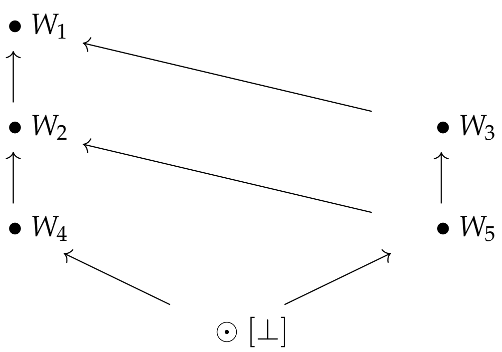
i=2; We have $u \notin R^{-1}[y_2, x_1]$ so $w \notin R^{-1}[y_2, x_1]$ and $w \notin \{\bot\}$. If $w = \top$ then $vRx_2R\top$ so vRw. If $w \neq \top$, then $w \in W_j : 1 \leq j \leq 5$ and as $w \notin R^{-1}[y_2, x_1]$ we must have $j = 1$ or $j = 2$. If $j = 1$ then again $vRx_2R\top \in W_1$ and if $j = 2$ then $vRx_2 \in W_2$.

i=3; We have $u \notin R^{-1}[x_2]$ so $w \notin R^{-1}[x_2]$ and $w \neq \top$. If $w = \top$ then $vRy_1Ry_2 \in W_1$. If $w \neq \top$ then $w \in W_j : 1 \leq j \leq 6$ and as $w \notin R^{-1}[x_2]$ we must have $j = 1$ or $j = 3$. If $j = 1$ then $vRy_1Ry_2 \in W_1$ and if $j = 3$ then $vRy_1 \in W_3$.

i=4; We have $u \notin R^{-1}[y_1]$ so $w \notin R^{-1}[y_1]$ and $w \neq \bot$. If $w = \top$ then $vRx_1Rx_2R\top$ so vRw. If $w \neq \top$, then $w \in W_j : 1 \leq j \leq 5$ and as $w \notin R^{-1}[y_1]$ we must have $j = 1$ or $j = 2$ or $j = 4$. If $j = 1$ then $vRx_1Rx_2Rx\top \in W_1$, if $j = 2$ then $vRx_1Rx_2 \in W_2$ and if $j = 4$ then $vRx_1 \in W_4$.

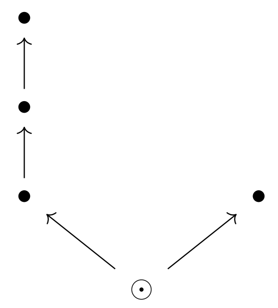
i=5; We have $u \notin R^{-1}[x_1]$ so $w \notin R^{-1}[x_1]$ and $w \neq \bot$. If $w = \top$ then $vRx_2R\top$ so vRw. If $w \neq \top$ then $w \in W_j : 1 \leq j \leq 5$ and as $w \notin R^{-1}[x_1]$ we must have $j = 1$, $j = 2$, $j = 3$ or $j = 5$. If $j = 1$ then $vRx_2R\top \in W_1$, if $j = 2$ then $vRx_2 \in W_2$, if $j = 3$ then $vRy_1 \in W_3$ and if $j = 5$ then $vRv \in W_5$.

Then, as ever by lemma 2.17 we may consider the modal equivalence identifying the points within these sets and consider B_*/E. Now, whilst $\top \in W_1$, $x_2 \in W_2$, $y_1 \in W_3$ and $x_1 \in W_4$, W_5 may or may not be empty, i.e. may or may not exist as an element of B_*/E. In other words, B_*/E has the following underlying frame where the element W_5 may or may not be present:

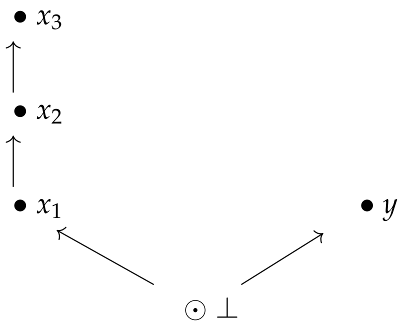


If W_5 is not present then $B_*/E \cong F_6$ so $F_6^* \in \mathcal{A}$ and we have a contradiction. If W_5 is present we can reduce B_*/E to F_6, again giving $F_6^* \in \mathcal{A}$ and a contradiction. □

Lemma 5.8. Let $\mathcal{A}$ be a variety omitting $F_i^* : 1 \leq i \leq 17$ and G_n^* for some $n > 0$. Then $\forall A \in \mathcal{A}$, A_* does not contain the following substructure:



Proof. Suppose not, let $A \in \mathcal{A}$ have the substructure described above which we label as follows:



We make the following claim:
Claim: $\exists B \in \mathcal{A} : B_*$ is rooted and contains the substructure witnessed by four elements elements x_1, x_2, x_3, y such that either

(i) x_3 and y are maximal and isolated with $B_* = R^{-1}[x_3, y]$.

(ii) B_* has a maximum element $\top$ that is isolated, $x_3 R \top$ and is isolated, $yR\top$ and is isolated and $B_* = \{\top\} \cup R^{-1}[x_3, y]$.

(iii) B_* has a maximal element $\top$ that is isolated, x_3 is maximal and isolated, $x_2 R \top$, $yR\top$ and is isolated and $B_* = \{\top\} \cup R^{-1}[x_3, y]$.

(iv) B_* has a maximal element $\top$ that is isolated, x_3 is maximal and isolated, $x_1 R \top$, $yR\top$ and is isolated, $x_2 \not R \top$ and $B_* = \{\top\} \cup R^{-1}[x_3, y]$.

As ever, by lemma 2.15 and taking an M-subspace we may assume that A_* consists of only improper clusters and is rooted by $\bot$. Once again, we also note if $u \in A_* : u \not R u$ then either $\bot = u$ or $\bot R u$. Now $\bot R x_1$ and $\bot R y$ and $x_1 || y$, so by lemma 5.4, u is comparable with x_1 and y_1. In particular, $u \notin \{x_1, y\}$ and so $x_1 R x_1$ and yRy. Then by lemma 5.3, $x_1 \not R u$ and $y \not R u$ so in fact uRx_1 and uRy. That is, the only irreflexive points in A_* belong to $R^{-1}[x_1] \cap R^{-1}[y]$.

By modal separation we can find clopen upsets U and V such that $x_1 \in U$ and $y \notin U$ and $y \in V$ and $x_1, x_2, x_3 \notin U$. We make our first case distinction, either $U \cap V = \varnothing$ or not.

If $U \cap V = \varnothing$; then moreover $R^{-1}[V] \cap U = \varnothing$ and $R^{-1}[U] \cap V = \varnothing$. By modal separation we also find a clopen upset $U' : x_3 \in U'$ and $x_2 \notin U'$ and let $W = U' \cap U$. We note that $W \cap V = \varnothing$, $x_3 \in W$, $y \notin W$, $y \in V$ and $x_3 \notin W$. Now W and V are disjoint clopen upsets, so easily form an M-partition. By lemma 2.17 we consider the modal equivalence E identifying points within W and V and $A_*/E \in \mathcal{A}$. As they were clopen in A_* they are isolated points in A_*/E, and as they were upsets in A_* they are maximal in A_*/E. Then we have $\bot, x_1, x_2 \notin W$, or V so $[\bot] = \{\bot\}, [x_1] = \{x_1\}, [x_2] = \{x_2\} \in A_*/E$. As $R^{-1}[V] \cap U = \varnothing$ and $x_1, x_2 \in U$ we have $x_1, x_2 \notin R^{-1}[V] = R_E^{-1}[V]$, and similarly, $W \cap V = \varnothing$ implies $R^{-1}[W] \cap V = \varnothing$ and $y \in V$ so $y \notin R_E^{-1}[W]$. Therefore, $[x_1], [x_2], W$ and V witness the substructure.

Finally, letting $u \in A_*$, by lemma 5.6 A_* has width ≤ 2 and $x_3||y$ so u must be comparable with either x_3 or y, that is x_3Ru, uRx_3, yRu or uRy. If x_3Ru then $u \in W$ and $[u] = W$, if uRx_3 then $[u]R_EW$. Similarly, if yRu then $[u] = V$ and if uRy then $[u]R_EV$. So $A_*/E = R_E^{-1}[W, V]$ and we have A_*/E satisfying case (i).

Now we suppose $U \cap V \neq \varnothing$; if $x_1 \notin R^{-1}[V]$ then $U \setminus R^{-1}[V]$ is a clopen upset such that $x_1 \in U \setminus R^{-1}[V]$, $y \notin U \setminus R^{-1}[V]$, $y \in V$, $x_1, x_2, x_3 \notin V$ and $U \setminus R^{-1}[V] \cap V = \varnothing$. So we can work as in the previous case with $U \setminus R^{-1}[V]$ in place of U. Similarly, if $y \notin R^{-1}[U]$ we can work as in the previous case with $V \setminus R^{-1}[U]$ in place of V. So suppose $x_1 \in R^{-1}[V]$ and $y \in R^{-1}[U]$. We make our second case distinction, either $x_3 \in R^{-1}[V]$ or not.

Suppose not; we again use modal separation to find a clopen upset $U' : x_3 \in U'$ and $x_2 \notin U'$. Then, we define the following clopen sets:

$$W_1 := U \cap V \cup V \setminus R^{-1}[U];\ W_2 := U' \cap U \setminus R^{-1}[V];$$
$$W_3 := V \cap R^{-1}[U] \setminus U.$$

We note that $x_3 \in W_2, y \in W_3$, and by inspection the W_i are pairwise disjoint. We claim moreover that they form an M-partition. For W_1 and W_2, these are both upsets so the condition holds trivially. So let $u, v \in W_3$ and uRw. We have $u \in V$ so $w \in V$. If $w \in U$ then $w \in W_1$, if $w \notin U$ and $w \in R^{-1}[U]$ then $w \in W_3$ and if $w \notin U$ and $w \notin R^{-1}[U]$ then $w \in W_1$, so $w \in W_1$ or $w \in W_3$. If $w \in W_1$, as $v \in W_3$ we can find $t \in U : vRt$ then $t \in W_1$ so may be taken as witness. If $w \in W_3$ then $vRv \in W_3$.

So, by lemma 2.17 we may consider the modal equivalence E identifying points within these sets and consider $A_*/E \in \mathcal{A}$. As each was clopen in A_* they are isolated points in A_*/E and as W_1 and W_2 were upsets they are maximal in A_*. We have $\bot, x_1, x_2 \notin W_i : 1 \leq i \leq 3$ so $[\bot] = \{\bot\}$,$[x_1] = \{x_1\}$,$[x_2] = \{x_2\} \in A_*/E$. If $W_2R_EW_3$ then $\exists u \in W_2$ and $v \in W_3 : uRv$, but then $u \in U$ so $v \in U$ but $v \in W_3$ gives $v \notin U$ so we have a contradiction. Therefore $W_2 \not\!R_E W_3$. If $W_3R_EW_2$ then $\exists u \in W_3$ and $v \in W_2 : uRv$, but then $u \in V$ and so $v \in V$, but $v \in W_2$ gives $v \notin R^{-1}[V] \supseteq V$ so we have a contradiction. Therefore $W_3 \not\!R_E W_2$. If $[x_2]R_EW_3$ then x_2Ru for some $u \in W_3$ but $x_2 \in U$ so $u \in U$ and $u \in W_3$ implies $u \notin U$, a contradiction. Therefore, $[x_2] \not\!R_E W_3$ and similarly $[x_1] \not\!R_E W_3$. If $W_3R_E[x_2]$ then $\exists u \in W_3 : uRx_2$, but then $u \in V$ and $x_2 \notin V$ so we have a contradiction. Therefore, $W_3 \not\!R_E [x_2]$ and similarly, $W_3 \not\!R_E [x_1]$. Putting this all together, $[x_1], [x_2], W_2$ and W_3 witness the substructure. We moreover have that $W_3R_EW_1$, as $y \in W_3$ and $y \in R^{-1}[U]$ so $\exists u \in U : yRu$ and then $u \in U \cap V \subseteq W_1$.

Letting $u \in A_*$, by lemma 5.6 A_* has width ≤ 2 and $x_3||y$ so u must be comparable with x_3 or y, that is x_3Ru, uRx_3, yRu or uRy. If x_3Ru, then $u \in W_2$ and $[u] = W_2$, if uRx_3 then $[u]R_EW_2$. If yRu, then $u \in V$ and as noted earlier this implies either $u \in W_1$ and so $[u] = W_1$ or $u \in W_3$ and $[u] = W_3$, if uRy then $[u]R_EW_3$. So $A_*/E = \{W_1\} \cup R^{-1}[W_2, W_3]$.

Finally, either $x_2 \in R^{-1}[V]$ or $x_2 \notin R^{-1}[V]$. If the former, then $[x_2]R_EW_1$ and we are in case (iii), and if the latter $[x_2] \not\!R_E W_1$ and we are in case (iv).

Suppose $x_3 \in R^{-1}[V]$; once more we use modal separation to find a clopen upset U' such that $x_3 \in U'$ and $x_2 \notin U'$. This time, we define the following clopen sets:

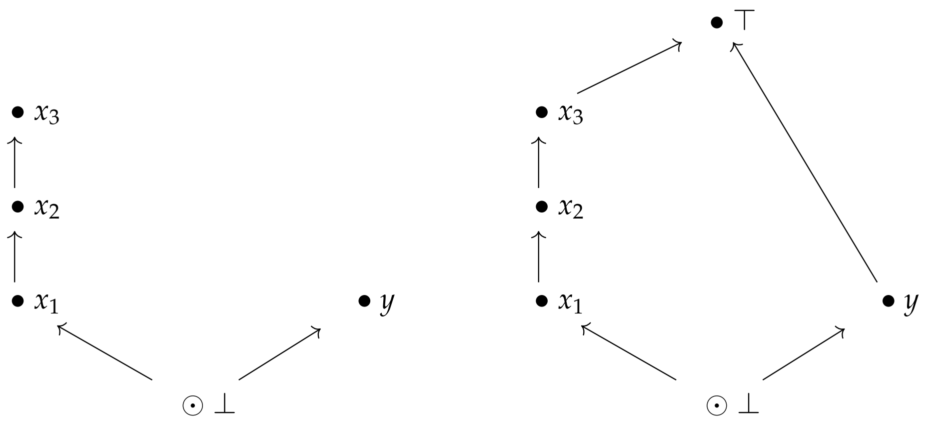
$$W_1 := U \cap V \cup U \setminus R^{-1}[V] \cup V \setminus R^{-1}[U];\ W_2 := U \cap U' \cap R^{-1}[V] \setminus V;$$
$$W_3 := V \cap R^{-1}[U] \setminus U.$$

We note that $x_3 \in W_2$, $y \in W_3$, and by inspection the W_i are pairwise disjoint We claim moreover that they form an M-partition. For W_1 this is an upset so the condition holds trivally. Letting $u, v \in W_2$ and uRw, then $w \in U \cap U'$ and this is an upset. Then either $w \in V$ and so $w \in W_1$, $w \notin R^{-1}[V]$ and $w \in W_1$ or $W \notin V$ and $W \in R^{-1}[V]$ and so $w \in W_2$. So $w \in W_1$ or W_2, if the former then as $v \in W_2$ $\exists t \in V : vRt$ and then $t \in W_1$ so we may take it as witness, and if $w \in W_2$ then $vRv \in W_2$. Letting $u, v \in W_3$ and uRw we have $w \in V$ as it is an upset. Then either $w \in U$ and so $w \in W_1$, $w \notin R^{-1}[U]$ and so $w \in W_1$ or $w \notin U$ and $w \in R^{-1}[U]$ and so $w \in W_3$. So $w \in W_1$ or W_2, if the former then as $v \in W_2$ $\exists t \in U : vRt$ and then $t \in W_1$ so we may take it as witness, and if $w \in W_2$ then $vRv \in W_2$.

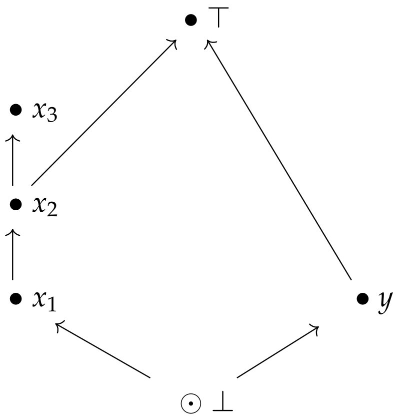
So, by lemma 2.17 we may consider the modal equivalence E identifying points within these sets and consider $A_*/E \in \mathcal{A}$. As each was clopen in A_* they are isolated points in A_*/E, and as W_1 was an upset it is maximal in A_*/E. We have $\bot, x_1, x_2 \notin W_i : 1 \leq i \leq 3$ so $[\bot] = \{\bot\}$,$[x_1] = \{x_1\}$,$[x_2] = \{x_2\} \in A_*/E$. If $W_2 R_E W_3$ then we have $u \in W_2$ and $v \in W_3$ such that uRv, but then $u \in U$ implies $v \in U$ and $v \in W_3$ implies $v \notin U$ which is a contradiction. So $W_2 \not{R} W_3$, and similarly $W_3 \not{R}_E W_2$. Then, just as in the previous case $[x_2] \not{R}_E W_3$ and $[x_1] \not{R}_E W_3$. So $[x_1], [x_2], W_2$ and W_3 witness the substructure. We moreover have that $W_E R_E W_1$ and $W_2 R_E W_1$.

Finally, letting $u \in A_*$ we have just as in previous cases that $x_3 R u$, $u R x_3$, $y R u$ or $u R y$. If $x_3 R u$ then as $x_3 \in W_2$ as argued earlier this implies either $u \in W_1$ and $[u] = W_1$ or $u \in W_2$ and $[u] = W_2$. If $u R x_3$ then $[u] R_E W_2$. Similarly, if $y R u$ then $[u] = W_1$ or $[u] = W_3$ and if $u R y$ then $[u] R_E W_3$. So $A_*/E = \{W_1\} \cup R^{-1}[W_2, W_3]$ and satisfies case (ii).

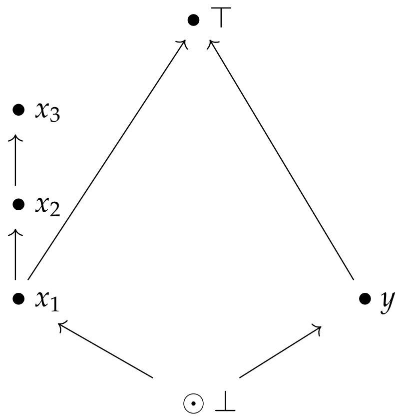
This completes the proof of the claim, so we have $B \in \mathcal{A}$ such that B_* contains one of the following substructures with x_3, y and $\top$ isolated:



(i) (ii)



(*iii*)



(*iv*)

We may still be lemma 2.15 assume that B_* consists of only improper clusters. We further claim that we may w.l.o.g assume x_2 is isolated.

For cases (i) and (ii) we use modal separation to find a clopen upset U such that $x_2 \in U$ and $x_1, y \notin U$. We then consider:

$$W := U \cap R^{-1}[x_3] \setminus \{x_3\}.$$

Then, $x_1, x_3, \top, y \notin W$ and $x_2 \in W$ and W is clopen. Letting $u, v \in W$ and uRw, $w \in U$ and from our case distinction we know either wRx_3, wRy or $w = \top$. As $y \notin U$ we have $w\not{R}y$. If wRx_3 then either $w = x_3$ and so vRw or $w \neq x_3$ and $w \in W$ and $vRv \in W$. If $w = \top$, then we are in case (ii) and vRw. In other words $\{W\}$ forms an M-partition, and we by lemma 2.17 may consider the modal equivalence identifying points within W and $B_*/E \in \mathcal{A}$. Noting that for $u \in B_*$ if yRu then $u \in \{y, \top\}$ and so $[y]\not{R}_E W$, it is easy to see that if case (i) held for B_* then it does for B_*/E with $[x_1], W, [x_3]$ and $[y]$, and similarly for case (ii). Therefore, we may assume w.l.o.g that E is the identity on B_*, i.e. $W = \{x_2\}$ and it is isolated.

In case (iii) we consider:

$$W_1 := R^{-1}[x_3] \setminus R^{-1}[\top]; \; W_2 := U \cap R^{-1}[\top] \setminus \{\top\}.$$

W_1 and W_2 are clopen and disjoint and we claim moreover an M-partition. For W_1, as x_3 is maximal this is an upset so the condition holds trivially. Letting $u, v \in W_2$ and uRw, then $w \in U$ and so $w\not{R}y$. Now, if $w = \top$ then vRw, and if $w \neq \top$ by case (iii) wRx_3. So then, either $wR\top$, $w \in W_2$ and $vRv \in W_2$ or $w\not{R}\top$ and so $w \in W_1$ and $vRx_3 \in W_1$. So, by lemma 2.17 we may consider the modal equivalence E identifying points within those two sets and $B_*/E \in \mathcal{A}$. Moting that for $u \in B_*$ if yRu then $u \in \{y, \top\}$ and so $[y]\not{R}_E W_1$ and $[y]\not{R}_E W_2$, it is easy to see that B_* still satisfies case (iii) with $[x_1], W_2, W_1, [y]$. So we may assume w.l.o.g that E is the identity on B_*, i.e. $W_2 = \{x_2\}$ and it is isolated.

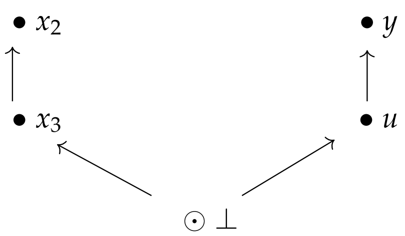
Now for case (iv) we use modal separation to find a clopen upset U such that $x_2 \in U$ and $x_1, y, \top \notin U$ and consider:

$$W := U \cap R^{-1}[x_3] \setminus \{x_3\}.$$

This is clopen. Letting $u, v \in W$ and uRw then $w \in U$ and so $w \neq \top$ and $w \not R y$. So by case (iv) we have wRx_3, so either $w = x_3$ and vRw or $w \neq x_3$, $w \in W$ and $vRv \in W$. By lemma 2.17 we may consider the modal equivalence identifying points in W and $B_*/E \in \mathcal{A}$. Noting that for $u \in B_*$ if yRu then $u \in \{y, \top\}$ and so $[y] \not R_E W$ it is easy to see that B_* still satisfied case (iv) with $[x_1], W, [x_3]$ and $[y]$. So we may assume w.l.o.g that E is the identity on B_*, i.e. $W = \{x_2\}$ and it is isolated.

A similar process lets us assume w.l.o.g that x_1 is isolated in B_*. Then, $R^{-1}[x_1] \cap R^{-1}[y]$ is clopen and so by corollary 2.24 we can consider a maximal cluster in it, which as B_* consists of only improper clusters is in fact a maximal point p in the set. This point sees x_1 and y, so by considering the M-subspace of B_* rooted at p we can assume w.l.o.g that the point is $\bot$ and $R^{-1}[x_1] \cap R^{-1}[y] = \{\bot\}$. Finally, just as we established earlier that the only irreflexive points in A_* could be in $R^{-1}[x_1] \cap R^{-1}[y]$, we can check this also holds for B_*.

We now show each case leads to a contradiction. However, we first note the following. Suppose $\exists u \in B_*$ such that $R[u] \cap \{x_1, x_2, x_3, y\} = \{x_3, y\}$. Then, as uRx_3 $u \neq y$ and as $u \not R x_1$ $\bot \neq u$. As uRy, $x_2 \not R u$ and $x_1 \not R u$, in other words A_* contains the following substrcture:



This contradictions lemma 5.7.

Case (i); We consider the following clopen sets:

$W_1 := R^{-1}[x_3] \setminus R^{-1}[x_2, y]$; $W_2 := R^{-1}[x_2] \setminus R^{-1}[x_1, y]$;
$W_3 := R^{-1}[x_1] \setminus R^{-1}[y]$; $W_4 := R^{-1}[y] \setminus R^{-1}[x_2]$;
$W_5 := R^{-1}[y] \cap R^{-1}[x_2] \setminus R^{-1}[x_1]$.

By inspection these sets are pairwise disjoint. Moreover, letting $u \in B_* \setminus \{\bot\}$, we have from case (i) and $R^{-1}[x_1] \cap R^{-1}[y] = \{\bot\}$ that either uRx_3 or uRy and $u \not R x_1$. If the latter then then $u \in W_4 \cup W_5$. If the former, either uRy or $u \not R y$, if uRy then as noted above $R[u] \cap \{x_1, x_2, x_3, y\} \neq \{x_3, y\}$ so either uRx_1 and $u \in W_4$ or $u \not R x_1$ so uRx_2 and $u \in W_5$. If $u \not R y$ then $u \in W_1 \cup W_2 \cup W_3$. So:

$$B_* = \bigcup_{i=1}^{5} W_i \cup \{\bot\}$$

We claim that the sets form an M-partition, so let $u, v \in W_i$ and uRw with:

i=1; We have $u \notin R^{-1}[x_2, y]$ so $w \notin R^{-1}[x_2, y]$ and $w \neq \bot$. Then $w \in W_j : 1 \leq j \leq 5$ and as $w \notin R^{-1}[x_2, y]$ we must have $j = 1$ and $vRv \in W_1$.

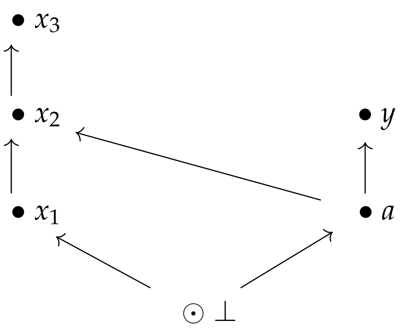
i=2; We have $u \notin R^{-1}[x_1, y]$ so $w \notin R^{-1}[x_1, y]$ and $w \neq \bot$. Then $w \in W_j : 1 \leq j \leq 5$ and ass $w \notin R^{-1}[x_1, y]$ we must have $j = 1$ or $j = 2$. If $j = 1$ then $vRx_2Rx_3 \in W_1$ and if $j = 2$ then $vRv \in W_2$.

i=3; We have $u \notin R^{-1}[y]$ so $w \notin R^{-1}[y]$ and $w \neq \bot$. Then $w \in W_j : 1 \leq j \leq 5$ and as $w \notin R^{-1}[y]$ we must have $j = 1$, $j = 2$ or $j = 3$. If $j = 1$ then $vRx_1Rx_3 \in W_1$, if $j = 2$ then $vRx_1Rx_2 \in W_2$ and if $j = 3$ then $vRv \in W_3$.

i=4; We have $u \notin R^{-1}[x_2]$ so $w \notin R^{-1}[x_2]$ and $w \neq \bot$. Then $w \in W_j : 1 \leq j \leq 5$ and as $w \notin R^{-1}[x_2]$ we must have $j = 1$ or $j = 4$. However, if $j = 1$ then $uRwRx_3$ but we noted earlier that $u\not{R}x_3$. So $j = 4$ and $vRv \in W_4$.

i=5; We have $u \notin R^{-1}[x_1]$ so $w \notin R^{-1}[x_1]$ and $w \neq \bot$. Then $w \in W_j : 1 \leq j \leq 5$ and as $w \notin R^{-1}[x_1]$ we must have $j = 1$, $j = 2$, $j = 4$ or $j = 5$. If $j = 1$ then $vRx_2Rx_3 \in W_1$m if $j = 2$ then $vRx_2 \in W_2$, if $j = 4$ then $vRy \in W_4$ and if $j = 5$ then $vRv \in W_5$.

Then, as ever by lemma 2.17 we may consider the modal equivalence identifying the points within these sets and assume w.l.o.g it is the identity on B_*. Now, whilst $x_3 \in W_1$, $x_2 \in W_2$, $x_1 \in W_3$ and $y \in W_4$ and so this amounts to assuming these sets are singletons, W_5 may or may not be empty, i.e. may or may not exists as an element of B_*. In other words, B_* has the following underlying frame where the element labelled a may or may not be present:

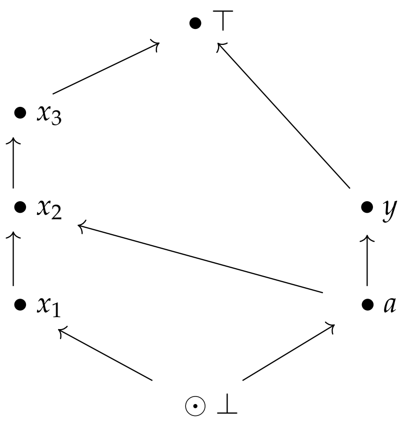


If a is present then the M-subspace of B_* rooted at a is isomorphic to F_5 and $F_5^* \in \mathcal{A}$ which is a contradiction. If it is not present then we can reduce B_* to F_5, again giving $F_5^* \in \mathcal{A}$ and a contradiction.

Case (ii); We consider the same clopen sets as in case (i) and proceed as before, except now we have:

$$B_* = \bigcup_{i=1}^{5} W_i \cup \{\bot, \top\}.$$

Additionally, when checking the M-partition for each W_i we have the possibility of $w = \top$, but then vRw. Applying lemma 2.17 gives that B_* has the following underlying frame where the element labelled a may or may not be present:



If a is present then the M-subspace of B_* rooted at a is isomorphic to F_5 and $F_6^* \in \mathcal{A}$ which is a contradiction. If it is not present then we can reduce B_* to F_5, again giving $F_6^* \in \mathcal{A}$ and a contradiction.

Case (iii); We proceed almost as case (i). We define our clopen sets as before except we take

$$W_1 := \{\top\} \cup R^{-1}[x_3] \setminus R^{-1}[x_2, y].$$

Then, we check that:

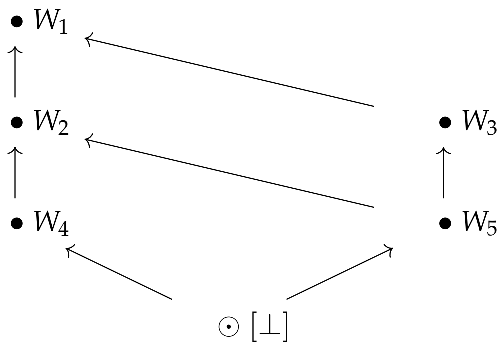
$$B_* = \bigcup_{i=1}^{5} W_i \cup \{\bot\}.$$

This proceeds as case (i) except we begin by noting either $w = \top$ and then $w \in W_1$ or $w \neq \top$ and then either wRx_3 or wRy.

When checking for the M-partition, when $i \neq 4$, this is as case (i) except we may have $w = \top$ and $w \in W_1$. For $i = 1$ we have $vRv \in W_1$, and for $i \neq 1$ then vRw.

For $i = 4$; as $u \notin R^{-1}[x_2]$ we have $w \notin R^{-1}[x_2]$ and $w \neq \bot$. So $w \in W_j : 1 \leq j \leq 5$ and as $w \notin R^{-1}[x_2]$ we must have $j = 1$ or $j = 4$. If $j = 1$ then as we noted earlier uRy and $u\not{R}x_2$ implies $u\not{R}x_3$ so $w\not{R}x_3$, therefore, $w = \top$, and $vRyRw$. If $j = 4$ then $vRv \in W_4$.

As ever, by lemma 2.17 we consider the modal equivalence identifying points within these sets and B_*/E. Now, whilst $\top \in W_1$, $x_2 \in W_2$, $x_1 \in W_3$ and $y \in W_4$, W_5 may or may not be empty, i.e. may or may not exist as an element of B_*/E. In other words, B_*/E has the following underlying frame where the element W_5 may or may not be present:



If W_5 is not present then $B_*/E \cong F_6$ so $F_6^* \in \mathcal{A}$ and we have a contradiction. If W_5 is present we can reduce B_*/E to F_6, again giving $F_6^* \in \mathcal{A}$ and a contradiction.

Case (iv); For this case we proceed slightly differently. We consider the M-subspace $\mathcal{X}$ of B_* rooted at x_1, note that by case (iv) we have $X = \{\top\} \cup R^{-1}[x_3]$. We define the following clopen sets:

$$W_1 := R^{-1}[x_3] \setminus (R^{-1}[x_2] \cup \{\top\}); \qquad W_2 := R^{-1}[x_2] \setminus R^{-1}[\top];$$
$$W_3 := \{\top\}; \qquad W_4 := R^{-1}[x_2] \cap R^{-1}[\top].$$

By inspection these sets are pairwise disjoint. Moreover letting $u \in X$ either $u = \top$ and $u \in W_3$ or $u \neq \top$, so uRx_3 and $u \in W_1 \cup W_2 \cup W_4$. So:

$$X = \bigcup_{i=1}^{4} W_i$$

.

We claim that the sets form an M-partition, so let $u, v \in W_i$ and uRw with:

i=1; We have $u \notin R^{-}1x_2$ so $w \notin R^{-1}[x_2]$ and as uRx_3 by case (iv) we have $w \neq \top$ and wRx_3. So $w \in W_1$ and $vRv \in W_1$.

i=2; We have $u \notin R^{-1}[\top]$ so $w \notin R^{-1}[\top]$ and in particular $w \neq \top$. So, either $w \in W_1$ and then $vRx_2Rx_3 \in W_1$ or $w \in W_2$ and $vRv \in W_2$.

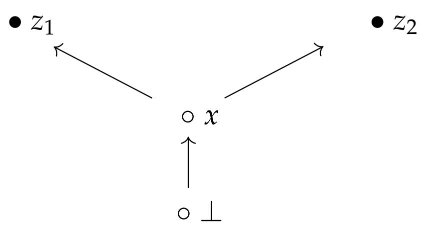
i=3; $u = \top = v$ so $w = \top$ and vRw.

i=4; $w \in W_j$ for $1 \leq j \leq 4$, if $j = 1$ then $vRx_2Rx_3 \in W_1$, if $j = 2$ then $vRx_2 \in W_2$, if $j = 3$ then $vR\top \in W_3$ and if $j = 4$ then $vRv \in W_4$.

So, by lemma 2.17 we may consider the modal equivalence identifying the points within these sets, and assume w.l.o.g it is the identity on $\mathcal{X}$, i.e. $X \cong F_5$, so $F_5^* \in \mathcal{A}$ which is a contradiction. □

These three lemmas 5.6, 5.7 and 5.8 are sufficient to establish the core of our main theorem for this chapter. Before we do so, there are two additional structures we'll want to control for.

Lemma 5.9. Let $\mathcal{A}$ be a variety omitting $F_i^* : 1 \leq i \leq 17$ and G_n^* for some $n > 0$. Then $\forall A \in \mathcal{A}$, A_* does not contain the following substructure, where $R^{-1}[z_i] \cap R[x] = \{u \in A_* : z_iRu \;\&\; uRz_i\}$, i.e. each z_i is an immediate successor to x:



Proof. Suppose not, let $A \in \mathcal{A}$ have the substructure described and labelled as above. Now, by lemma 5.3 we may consider a maximal cluster in $R^{-1}[x]$ which is an irreflexive point, and may assume w.l.o.g that this point is $\perp$, and also that $\perp$ is the root of A_* by taking the M-subspace with $\perp$ as the root. Then $R^{-1}[x] = \{\perp\}$. Moreover, by lemma 5.4 we have $A_* = \{\perp\} \cup R[\perp] = \{\perp\} \cup \{x\} \cup R[x]$. We can also by lemma 2.15 assume A_* consists of only improper clusters, note this implies that

$R^{-1}[z_i] \cap R[x] = \{z_i\}$. Additionally, by lemma 5.6 A_* has width ≤ 2, so letting $u \in A_* \setminus \{\bot, x\}$, $u \in R[x]$ and as $z_1||z_2$, xRz_1 and xRz_2 u must be comparable with either z_1 or z_2. Then as $R^{-1}[z_i] \cap R[x] = \{z_i\}$ we in fact have $u \in R[z_1] \cup R[z_2]$. In summary, $A_* = \{\bot, x\} \cup R[z_1] \cup R[z_2]$, and then from lemma 5.3 we have $\forall u \in R[z_1] \cup R[z_2]$ that uRu, i.e. $\bot$ and x are the only irreflexive points in A_*.

Now, by modal separation we can find clopen upsets U_1 and U_2 such that $z_i \in U_j$ iff $i = j$. As $z_i \in U_i$ we have $R[z_i] \subseteq U_i$ which further implies $A_* = \{\bot, x\} \cup U_1 \cup U_j$. Now, if $U_1 \cap U_2 = \varnothing$, they are pairwise disjoint clopen upsets so easily form an M-partition. As usual, we may assume w.l.o.g that they are singletons, i.e. A_* is exactly the labelled frame and so $A_* \cong F_{16}$ so $F_{16}^* \in \mathcal{A}$ which is a contradiction.

So now suppose $U_1 \cap U_2 \neq \varnothing$. We consider $R^{-1}[U_1]$ and $R^{-1}[U_2]$. If $z_1 \notin R^{-1}[U_2]$, then $z_2 \in U_1 \setminus R^{-1}[U_2]$ which is a clopen upset, $z_2 \notin U_1 \setminus R^{-1}[U_2]$ and $U_1 \setminus R^{-1}[U_2] \cap U_2 = \varnothing$. So, replacing U_1 with $U_1 \setminus R^{-1}[U_2]$ we can proceed as in the previous case. Similarly, if $z_2 \notin R^{-1}[U_1]$ we can replace U_2 with $U_2 \setminus R^{-1}[U_2]$. So suppose $z_1 \in R^{-1}[U_2]$ and $z_2 \in R^{-1}[U_1]$. We define the following clopen sets:

$$W_1 := U_1 \cap U_2 \cap U_1 \setminus R^{-1}[U_2] \cap U_2 \setminus R^{-1}[U_1];\ W_2 := U_1 \cap R^{-1}[U_2] \setminus U_2;$$
$$W_3 := U_2 \cap R^{-1}[U_1] \setminus U_1.$$

By inspection these sets are pairwise disjoint. Moreover, letting $u \in A_* \setminus \{\bot, x\}$ then $u \in R[z_1] \cup R[z_2]$. If $u \in R[z_1]$ then $u \in U_1$, so then if $u \in U_2$ or $u \notin R^{-1}[U_2]$ then $u \in W_1$ and if $u \notin U_2$ and $u \in R^{-1}[U_2]$ then $u \in W_2$. Similarly, if $u \in U_2$ then either $u \in W_1$ or $u \in W_3$. So:

$$A_* = \{\bot, \top\} \cup \bigcup_{i=1}^{3} W_i.$$

We claim moreover that the sets form an M-partition, so let $u, v \in W_i$ and uRw with:

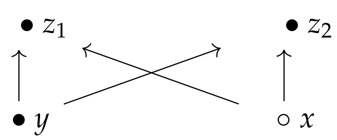
i=1; W_1 is an upset so $w \in W_1$ and $vRv \in W_1$.

i=2; $u \notin U_2$ so $w \notin U_2$, and $w \neq \bot$ and $w \neq x$. So $w \in W_1 \cup W_2$, if $w \in W_2$ then $vRv \in W_2$ and if $w \in W_1$, we have $v \in R^{-1}[U_2]$ so $\exists t \in U_2 : vRt$ then $t \in U_1 \cap U_2 \subseteq W_1$.

i=3; As the $i = 2$ case.

So, by lemma 2.17 we may consider the modal equivalence identifying points within these sets and assume w.l.o.g that it is the identity on A_*. Now, $z_1 \in W_2$, $z_2 \in W_3$ and $U_1 \cap U_2 \neq \varnothing$ so $W_1 \neq \varnothing$, so this amounts to assuming that these sets are singletons, i.e. $A_* \cong F_{17}$ and $F_{17}^* \in \mathcal{A}$ which is a contradiction. □

Lemma 5.10. Let $\mathcal{A}$ be a variety omitting $F_i^* : 1 \leq i \leq 17$ and G_n^* for some $n > 0$. Then $\forall A \in \mathcal{A}$, A_* does not contain the following substructure, where $R^{-1}[z_i] \cap R[x] = \{u \in A_* : z_iRu \ \&\ uRz_i\}$, i.e. each z_i is an immediate successor to x:



Proof. Suppose not, let $A \in \mathcal{A}$ have the substructure described and labelled as above. We may by lemma 2.15 assume w.l.o.g that A_* consists of only improper clusters which in particular implies $R^{-1}[z_i] \cap R[x] = \{z_i\}$, and by considering the M-subspace of A_* $R^+[x] \cup R^+[y]$ we may assume $A_* = R^+[x] \cup R^+[y]$. Additionally, by lemma 5.6 A_* has width ≤ 2, so letting $u \in R[x]$, as $z_1 || z_2$, xRz_1 and xRz_2 u must be comparable with either z_1 or z_2. Then as $R^{-1}[z_i] \cap R[x] = \{z_i\}$ we in fact have $u \in R[z_1] \cup R[z_2]$. So $R[x] = R[z_1] \cup R[z_2]$. Finally, as yRy if yRu then by lemma 5.3 uRu and if xRu then $u \in R[z_1] \cup R[z_2]$ so again by lemma 5.3 uRu. That is, the only irreflexive point in A_* is x.

Now, by modal separation we can find clopen upsets U_1 and U_2 such that $z_i \in U_j$ iff $i = j$. Note that $x, y \notin U_1 \cup U_2$. Once more, either $U_1 \cap U_2 = \varnothing$ or not. Suppose $U_1 \cap U_2 = \varnothing$; They are pariwise disjoint clopen upsets, so easily form an M-partition. As usual, we may assume w.l.o.g that they are singletons, that is $U_1 = \{z_1\}$, $U_2 = \{z_2\}$, and z_1 and z_2 are isolated and maximal in A_*. Then $\{z_1\} \cup A_* \setminus R^{-1}[z_1, z_2]$ is a clopen upset, so again assuming w.l.o.g it is a singleton we have $A_* = R^{-1}[z_1, z_2]$. Finally, we consider the clopen set $V = R^{-1}[z_1] \cap R^{-1}[z_2]$. Now, $x \in V \setminus R^{-1}[V]$ which is clopen, and if $u \in V \setminus R^{-1}[V]$ then $u \in V$ but $u \notin R^{-1}[V]$ means $u \not R u$, which as we noted earlier means $u = x$. So $V \setminus R^{-1}[V] = \{x\}$ and x is isolated in A_*. We then define the following clopen sets:

$$W_1 := V \setminus \{x\};\ W_2 := R^{-1}[z_1] \setminus V;$$
$$W_3 := R^{-1}[z_2] \setminus V.$$

By inspection these are pairwise disjoint. We claim moreover that they form an M-partition. Letting $u, v \in W_i$ and uRw with:

i=1; we have that $w \in R^{-1}[z_1, z_2]$, and as uRw $w \neq x$. So either $w \in V$ and $w \in W_1$, or $w \notin V$ in which case either $w \in R^{-1}[z_1]$ and $w \in W_2$ or $w \in R^{-1}[z_2]$ and $w \in W_3$. If $w \in W_1$ then as $v \neq x$ $vRv \in W_1$, if $w \in W_2$ then $vRz_1 \in W_2$ and if $w \in W_3$ then $vRz_2 \in W_3$.

i=2; $u \in R^{-1}[z_1]$ and $u \notin V$ means $u \notin R^{-1}[z_2]$, so $w \notin R^{-1}[z_2]$. As $A_* = R^{-1}[z_1, z_2]$, we get $w \in R^{-1}[z_1]$ and $w \notin V$ so $w \in W_2$. Then $vRv \in W_2$.

i=3; As the $i = 2$ case.

So we may by lemma 2.17 consider the modal equivalence identifying points within these sets and assume w.l.o.g that it is the identity on A_*. Then $y \in W_1$, $z_1 \in W_2$ and $z_2 \in W_3$ means that $A_* \cong F_{12}$, so $F_{12}^* \in \mathcal{A}$ which is a contradiction.

Now suppose $U_1 \cap U_2 \neq \varnothing$; We consider $R^{-1}[U_1]$ and $R^{-1}[U_2]$. If $z_1 \notin R^{-1}[U_2]$ then $z_1 \in U_1 \setminus R^{-1}[U_2]$ which is a clopen upset, $z_2 \notin U_1 \setminus R^{-1}[U_2]$ and $U_1 \setminus R^{-1}[U_2] \cap U_2 = \varnothing$. So, replacing U_1 with $U_1 \setminus R^{-1}[U_2]$ we can proceed as in the previous case. Similarly, if $z_2 \notin R^{-1}[U_1]$, we can replace U_2 with $U_2 \setminus R^{-1}[U_1]$. So suppose $z_1 \in R^{-1}[U_2]$ and $z_2 \in R^{-1}[U_1]$. We define the following clopen sets:

$$W_1 := U_1 \cap U_2 \cap A_* \setminus R^{-1}[U_2] \cap A_* \setminus R^{-1}[U_1];\ W_2 := U_1 \cap R^{-1}[U_2] \setminus U_2;$$
$$W_3 := U_2 \cap R^{-1}[U_1] \setminus U_1.$$

By inspection these sets are pairwise disjoint. We claim moreover that the sets form an M-partition, so let $u, v \in W_i$ and uRw with:

i=1; W_1 is an upset so $w \in W_1$, $x \notin W_1$ so $v \neq x$ and $vRv \in W_1$.

i=2: $u \in U_1$ which is an upset so $w \in U_1$. If $w \in U_2$ then $w \in W_1$, if $w \notin R^{-1}[U_2]$ then $w \in W_1$ and if $w \notin U_2$ and $w \in R^{-1}[U_2]$ then $w \in W_2$, so $w \in W_1 \cup W_2$. If $w \in W_1$, then $v \in R^{-1}[U_2]$ so $\exists t \in U_2 : vRt$, then $t \in U_1$ so $t \in W_1$ and may be taken as witness. If $w \in W_2$, then $x \notin W_2$ so $v \neq x$ and $vRv \in W_2$.

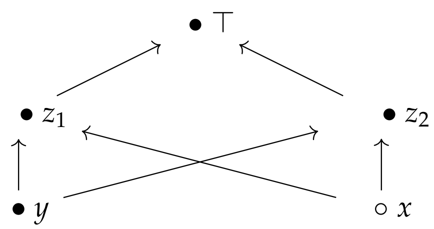
i=3; As the $i = 2$ case.

So, by lemma 2.17 we may consider the modal equivalence identifying points within these sets and assume w.l.o.g that it is the identity on A_*. Now, as $U_1 \cap U_2 \neq \varnothing$, $W_1 \neq \varnothing$, and $z_1 \in W_2$ and $z_2 \in W_3$, so this amounts to assuming that these sets are singletons.

Moreover, let $u \in A_*$. If $u \in R^{-1}[U_1]$, then $uRw \in U_1$ and as we argued in the $i = 1$ case, $w \in U_1$ implies $w \in W_1 \cup W_2$, so $u \in R^{-1}[W_1, W_2]$. Then, letting $v \in W_2$ we have $v \in R^{-1}[U_2]$ and $v \in U_1$ so $\exists t \in W_1 : vRt$ and $R^{-1}[W_2] \subseteq R^{-1}[W_1]$. So $u \in R^{-1}[W_1]$. Similarly, if $u \in R^{-1}[U_2]$ then $u \in R^{-1}[W_1]$. Then, if $u \notin R^{-1}[U_1]$ and $u \notin R^{-1}[U_2]$ we have $u \in W_1$ and $u \in R^{-1}[W_1]$. On our assumption that $W_1 = \{\top\}$, this means $\forall u \in A_*$ we have $u \in R^{-1}[\top]$, i.e. $A_* = R{-}1[\top]$.

Then, let $u \in A_* : z_1Ru$, as $z_1 \in U_1$ we have $u \in U_1$ so $u \in W_1 \cup W_2$ i.e. $u = z_1$ or $u = \top$ and $R[z_1] = \{\top, z_1\}$. Similarly, $R[z_2] = \{\top, z_1\}$. Finally, letting $u \in A_*$ we have either yRu or xRu and $z_1||z_2$, yRz_1, yRz_2, xRz_1, xRz_2 and by lemma 5.6 A_* has width ≤ 2. So u is comparable with z_1 or z_2.

Putting this all together, A_* contains the following substructure, where z_1, z_2 and $\top$ are all isolated and $A_* = \{\top\} \cup R^{-1}[z_1, z_2]$:



Now, we can proceed exactly as we did for the $U_1 \cap U_2 = \varnothing$ case except when checking the given clopen sets form an M-partition we always have the possibility that $w = \top$, but then vRw as needed. This lets us assume w.l.o.g that $A_* \cong F_{13}$, so $F_{13}^* \in \mathcal{A}$ which is a contradiction. □

5.3 The Main Theorem

We are now finally in position to prove our main structural result.

Given two disjoint transitive frames (X, R_X) and (Y, R_Y) we define their *sequential composition*, denoted $X \oplus Y$ as the frame $X \cup Y$ under the relation $R = R_X \cup R_Y \cup \{(y, x) \in (X \cup Y)^2 : x \in X, y \in Y\}$. That is, we paste Y below X and insists that every element of Y sees every element of X. This naturally extends to collections of frames with $\bigoplus_{i \in I} X_i$ for a linear order I.

Theorem 5.11. Let $\mathcal{A}$ be a variety omitting $F_i^* : 1 \leq i \leq 17$ and G_n^* for some $n > 0$. Let $A \in V$ be finitely generated, non-trivial and SI. Then the frame underlying A_* is a sequential composition of frames $\bigoplus_{\alpha \leq \beta} Q_\alpha$ for some $\beta \in Ord$ and such that:

$$Q_\alpha \text{ is } \begin{cases} \text{a single cluster} & \text{if } \alpha = \beta \text{ or } \alpha \text{ is a limit ordinal} \\ \text{a single cluster, a two cluster anti-chain or } H & \text{if } \alpha = 0 \\ \text{a single cluster or a two cluster anti-chain} & \text{otherwise} \end{cases}$$

Note that when we say Q_α is a two cluster anti-chain we mean that Q_α consists of two arbitrary disjoint clusters that do not see each other.

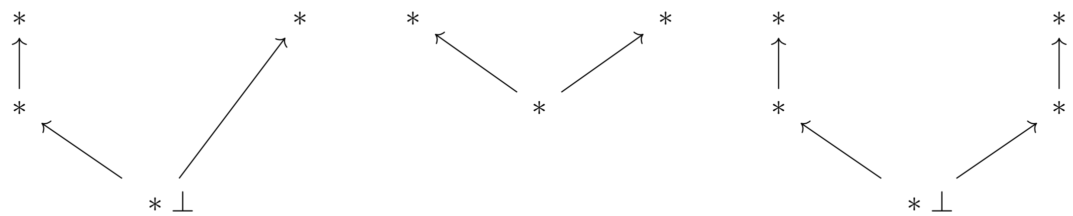
Moreover, any maximal clusters are single reflexive points, if Q_α is a two cluster anti-chain then clusters in $Q_{\alpha+1}$ are improper.

If A_* contains an irreflexive point we will say it is *i*-type, otherwise it is *r*-type, and if A_* is *i*-type then $\beta = \lambda + n$ for some limit ordinal λ, $n \neq 0$ and $\exists 0 < m \leq n : \forall \alpha < \lambda + m\ Q_\alpha$ contains no irreflexive points, $\forall k \geq m\ Q_{\lambda+k}$ is a single irreflexive point and if $m < n$ then $Q_{\lambda+m-1}$ is a single cluster.

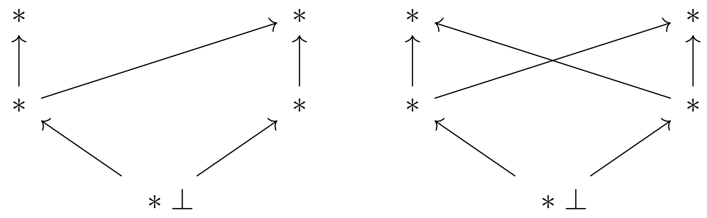
Proof. By lemma 5.6 A has finite width, and A is finitely generated, therefore by theorem 2.22 A_* contains no infinite ascending chains. In particular, A_* is conversely well founded amongst clusters, that is letting $X \subseteq A_*$ be non-empty, X contains maximal clusters. As A is non-trivial and SI, A_* is non-empty and the interior of its set of topo-roots is non-empty. In particular, it is rooted. We start by proving the main structure through ordinal recursion.

For Q_0 and Q_1: A_* conversely well founded and non-empty gives $Sl_0(A_*) \neq \varnothing$. Moreover, A_* is rooted and by lemma 5.6 has width ≤ 2, so $Sl_1(A_*)$ contains at most two clusters. If there is just a single cluster we take Q_0 as that cluster. Now, either $A_* = Q_0$ and we are done, or $A_* \neq Q_0$. Then, as A_* is conversely well founded we may consider a the maximal clusters in $A_* \setminus Q_0$. These must be of depth 1, and again A_* rooted and of width ≤ 2 means there are at most two. Letting Q_1 be those clusters we have $Q_0 \otimes Q_1$ and as we took maximal clusters in $A_* \setminus Q_0$ we also have $A_* \setminus (Q_0 \cup Q_1) \subseteq R^{-1}[Q_1]$ as required.

Now suppose there are two clusters of depth 1; we proceed as above except now we have a number of possible substructures of A_*:



(i) *(ii)* *(iii)*



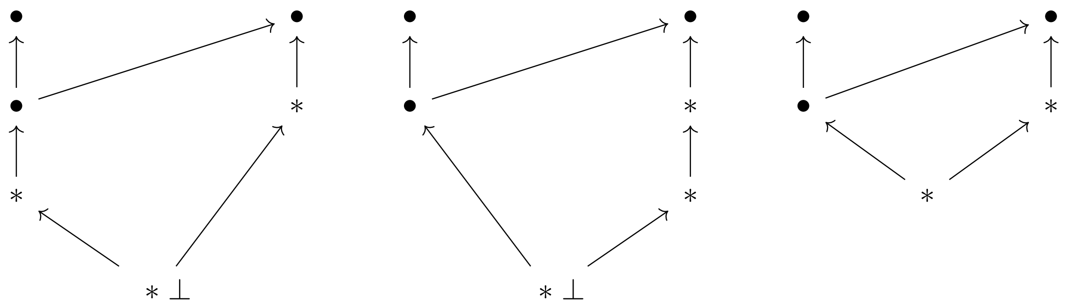
(iv) (v)

In case (ii), the cluster of depth 2 may or may not be the root, in the other cases the root of A_* cannot be of depth 2 by construction, and we include it in the substructure as the labelled cluster $\bot$.

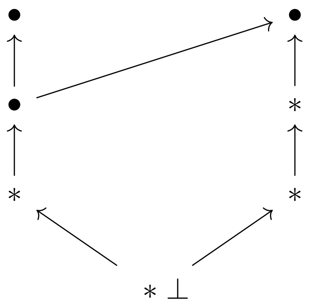
In cases (ii) and (v), we let Q_1 be the single cluster/two cluster anti-chain at depth 2 and Q_0 be the clusters of depth 1 and have $Q_0 \oplus Q_1$. Again, as we took maximal clusters in $A_* \setminus Q_0$ we also have $A_* \setminus (Q_0 \cup Q_1) \subseteq R^{-1}[Q_1]$ as required. We can rule out case (iii) as it contradicts lemma 5.7.

For case (i), let X denote top right cluster, that is the cluster at depth 0 not seen by the cluster of depth 1. Now, $\bot \in R^{-1}[Sl_2(A_*)]$ so it is non-empty and we may consider a maximal cluster in this set, which will be of depth 3 and we call Y. Now, either Y sees X or not. If it does, then we take M-subspace of A_* rooted at Y. Note that by lemma 5.3 Y is the only cluster in the subspace that can be a single irreflexive point and all four clusters in the subspace are finite, so the subspace itself finite and each cluster can be reduce to a single point via γ-reductions. The resulting frame is F_5, so $F_5^* \in \mathcal{A}$ which is a contradiction. If it does not, $Y \neq \bot$, and then $\{\bot, Y\} \cup S_2(A_*)$ gives a substructure of A_* contradicting lemma 5.8. So we can also rule out case (i).

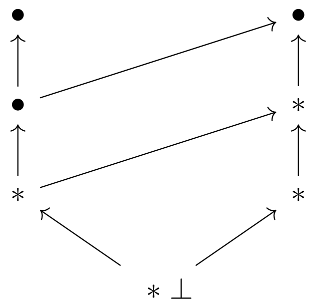
This just leaves case (iv), which we handle a little differently. Namely, the two clusters of depth 1 cannot be irreflexive points by lemma 5.1 and also must be improper as otherwise F_1 is an M-subspace of A_* and $F_1^* \in \mathcal{A}$ which is a contradiction. Similarly, the cluster of depth 2 which sees both clusters of depth 1 cannot be an irreflexive point by lemma 5.9, and must be proper otherwise we obtain F_4 as an M-subspace of A_* which is also a contradiction. Thus, the four clusters at depth 1 and 2 form a frame of type H which we take for Q_0. We then repeat our process, $A_* \neq Q_0$ so we consider maximal clusters in $A_* \setminus Q_0$ which will be clusters of depth 3 and there can be at most two of them. Thus, we have the following as possible substructures of A_*:



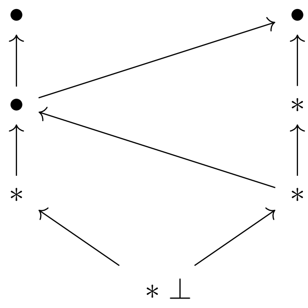
(a) (b) (c)



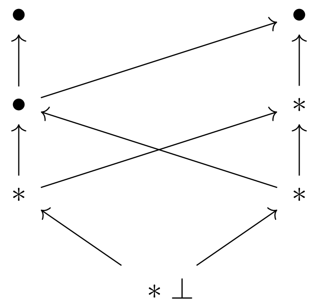
(d)



(e)

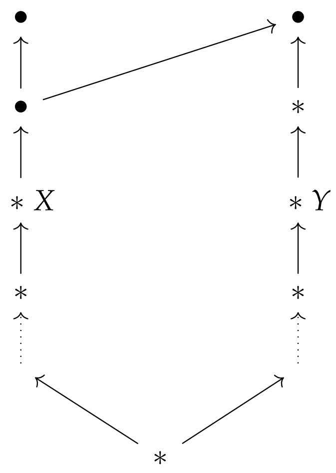


(f)

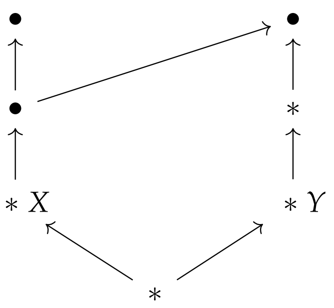


(g)

Note that in case (c), the cluster of depth 3 may or may not be the root. In cases (c) and (g) we may take Q_1 to be the single cluster or two cluster anti-chain at depth 3 and have $Q_0 \oplus Q_1$ as required, and once more we have $A_* \setminus (Q_0 \cup Q_1) \subseteq R^{-1}[Q_1]$ as we took maximal clusters. Cases (a), (b), (d) and (e) contradict lemma 5.8 so we can rule them out. For case (f), letting X and Y denote the two clusters of depth 3, we have $R^{-1}[X] \cap R^{-1}[Y] \neq \varnothing$, and so we may consider a maximal cluster in that set. Then, taking the M-subspace of A_* rooted at this cluster, it has the following as its underlying frame:



Now, using modal separation and taking differences we can recover the cluster X which implies it is a clopen subset of A_*. This holds similarly for Y. Then $R^{-1}[X] \setminus R^{-1}[Y]$ and $R^{-1}[Y] \setminus R^{-1}[X]$ are easily seen to be an M-partition. Taking the resulting quotient space from lemma 2.17, we find $\mathcal{X}$ such that $\mathcal{X}^* \in \mathcal{A}$ where the frame underlying $\mathcal{A}$ is:



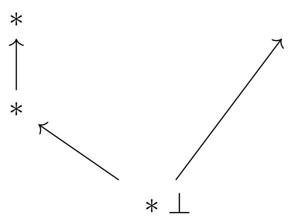
Finally, we can reduce this to F_6, so $F_6^* \in \mathcal{A}$ which is a contradiction, ruling out case (f). This completes the construction of $Q_0 \oplus Q_1$.

Now, let $\alpha \geq 1$ and suppose we have completed the construction for all $\gamma \leq \alpha$, that is we have $\forall \gamma \leq \alpha$ a substructure Q_γ such that:

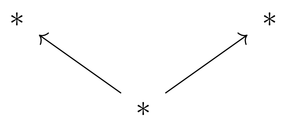
$$\bigcup_{\gamma \leq \alpha} Q_\gamma = \bigoplus_{\gamma \leq \alpha} Q_\gamma \text{ and } A_* \setminus \bigcup_{\gamma \leq \alpha} Q_\gamma \subseteq R^{-1}[Q_\alpha].$$

If $A_* = \bigoplus_{\gamma \in \alpha+1} Q_\gamma$ then we are done, and if not we may can consider the maximal clusters in $A_* \setminus \bigcup_{\gamma \leq \alpha} Q_\gamma$. As A_* has width ≤ 2 there are at most two such clusters and we take $Q_{\alpha+1}$ as these clusters. Now, from the construction and $\alpha \neq 0$ we have Q_α is either a single cluster or two cluster anti-chian. If Q_α was a single cluster then as $A_* \setminus \bigcup_{\gamma \leq \alpha} Q_\gamma \subseteq R^{-1}[Q_\alpha]$ all clusters in $Q_{\alpha+1}$ see it. As we took maximal clusters we have our $\bigoplus_{\gamma \leq \alpha+1} Q_\gamma$ with $A_* \setminus \bigcup_{\gamma \leq \alpha+1} Q_\gamma \subseteq R^{-1}[Q_{\alpha+1}]$ as required.

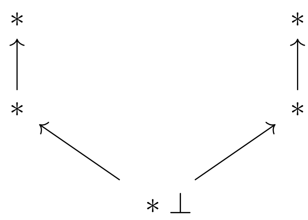
If Q_α is a two cluster anti-chain, we have the following possible subframes of A_*:



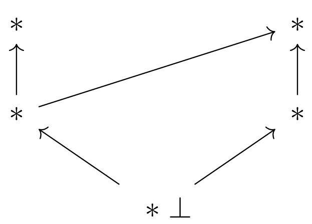
(i)



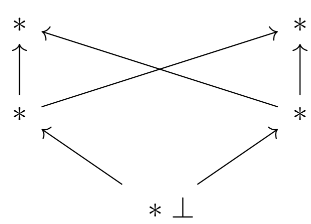
(ii)



(iii)



(iv)

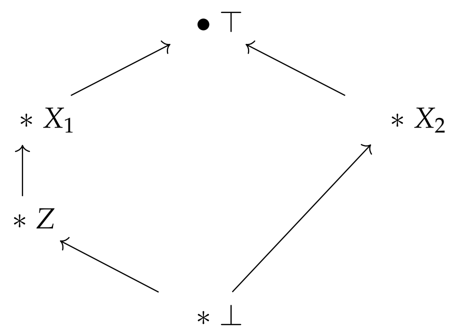


(v)

In case (ii), the single cluster in $Q_{\alpha+1}$ may or may not be the root, in the other cases the root of A_* cannot be in $Q_{\alpha+1}$ by construction, and we include it in the substructure as the labelled cluster $\bot$.

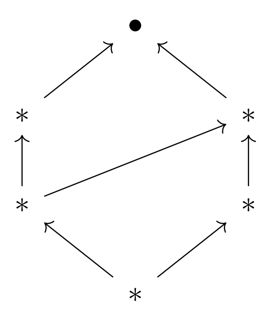
In cases (ii) and (v), we have $\bigoplus_{\gamma \leq \alpha+1} Q_\gamma$ and as we took maximal clusters we also have $A \setminus \bigcup_{\gamma \leq \alpha+1} Q_\gamma \subseteq R^{-1}[Q_{\alpha+1}]$ as required. Case (iii) contradicts lemma 5.7 so we can rule it out.

For case (i); First, we note that as Q_α is a two cluster anti-chain it cannot be a limit ordinal. Then, by using modal separation on clusters in $Q_{\alpha-1}$ with the clusters of Q_α we have that $\bigoplus_{\gamma \leq \alpha-1} Q_\gamma$ is a clopen subset of A_*, it is also by the construction an upset. Applying lemma 2.16 we find a modal equivalence E which identifies all points in it and consider A_*/E, which contains the following substructure with the X_i the two clusters in Q_α and Z the single cluster in $Q_{\alpha+1}$:



Note that, by the construction $A_*/E = \{\top\} \cup X_1 \cup R^{-1}[Z, X_2]$. Now, $\bot \in R^{-1}[Z]$ so it is non-empty and we may consider a maximal cluster in this set, which we call Y. Now, either Y sees X_2 or not. If it does, then we take the M-subspace of A_*/E rooted at Y, which has as underlying set $\{\top\} \cup X_1 \cup X_2 \cup Z \cup Y$. Note that by lemma 5.5 the clusters X_1, X_2 and Z cannot be single irreflexive points and all the clusters are finite, so the subspace is finite and each cluster can be reduced to a single point via γ-reductions. The resulting frame is F_6, so $F_6^* \in \mathcal{A}$ which is a contradiction. If it does not, $Y \neq \bot$, and then taking an element from $\bot, Y, Z, X_1$ and X_2 respectively we find a substructure of A_*/E contradicting lemma 5.8. So we can also rule out case (i).

For case (iv), we have that $R^{-1}[Q_{\alpha+1}] \neq \varnothing$, so we take a maximal cluster in it and consider the M-subspace of A_* rooted there. Then, once again we can collapse $\bigoplus_{\gamma \leq \alpha-1} Q_\gamma$ into a single point which gives $\mathcal{X}^* \in \mathcal{A}$ where $\mathcal{X}$ has the following underlying frame:



As A was finitely generated, by lemma 2.20 all its clusters are finite, so the clusters in $\mathcal{X}$ are finite making the whole frame finite. Then we reduce $\mathcal{X}$ to F_6, giving that

$F_6^* \in \mathcal{A}$ which is a contradiction.

Now, let $\alpha \geq 1$ be a limit ordinal and suppose we have completed the construction for all $\gamma < \alpha$, that is we have $\forall \gamma < \alpha$ a substructure Q_γ such that:

$$\bigcup_{\gamma<\alpha} Q_\gamma = \bigoplus_{\gamma<\alpha} Q_\gamma \text{ and } A_* \setminus \bigcup_{\eta\leq\gamma} Q_\eta \subseteq R^{-1}[Q_\gamma].$$

Now, A_* is rooted so $A_* \neq \bigoplus_{\gamma<\alpha} Q_\gamma$, and we may consider the maximal clusters in $A_* \setminus \bigcup_{\gamma<\alpha} Q_\gamma$. As A_* has width ≤ 2 there are at most two such clusters, and we take Q_α as these clusters. Now, letting $x \in Q_\alpha$ we have $\forall \gamma < \alpha$ that $\gamma + 1 < \alpha$ and so $x \notin Q_{\gamma+1}$, $x \in A_* \setminus \bigcup_{\eta\leq\gamma} Q_\eta$ and $x \in R^{-1}[Q_{\gamma+1}]$. Then, letting $y \in Q_\gamma$ and $z \in Q_{\gamma+1} : xRz$ by construction zRy and so xRy. Therefore, we have $\bigoplus_{\gamma<\alpha} Q_\gamma \oplus Q_\alpha$, and as we took maximal clusters we also have $A_* \setminus \bigcup_{\gamma\leq\alpha} Q_\gamma \subseteq R^{-1}[Q_\alpha]$ as required.

To finish the limit case we need to argue that in fact Q_α can only be a single cluster and not a two-cluster anti-chain. Suppose for contradiction it is a two cluster anti-chain, we label these clusters X_1 and X_2. Then root of A_* is in $A_* \setminus \bigcup_{\gamma\leq\alpha} Q_\gamma$, so it is non-empty and we can find maximal clusters in it. Applying modal separation to X_1 with X_2 and these clusters respectively, we find a clopen upset U_1 such that $X_1 \subseteq U_1$ and $U_1 \cap (X_2 \cup A_* \setminus \bigcup_{\gamma\leq\alpha} Q_\gamma) = \varnothing$. Then, as X_1 sees every point in $\bigoplus_{\gamma<\alpha} Q_\gamma$ we have $\bigoplus_{\gamma<\alpha} Q_\gamma \cup X_1 \subseteq U_1$. So, in fact $U_1 = \bigoplus_{\gamma<\alpha} Q_\gamma \cup X_1$. Similarly, we can find a clopen upset U_2 such that $U_2 = \bigoplus_{\gamma<\alpha} Q_\gamma \cup X_2$. Then, taking their intersection implies $\bigoplus_{\gamma<\alpha} Q_\gamma$ is a clopen subset of A_*. However, letting $\gamma < \alpha$, we similarly apply modal separation on clusters in Q_γ against clusters in $Q_{\gamma+1}$ to check that $\bigoplus_{\eta\leq\gamma} Q_\eta$ is clopen. Then, the collection $\{\bigoplus_{\eta\leq\gamma} Q_\eta\}_{\gamma<\alpha}$ form an infinite open cover of $\bigoplus_{\gamma<\alpha} Q_\gamma$ with no finite sub-cover, contradicting that A_* is compact.

This completes the construction of $A_* = \bigoplus_{\alpha\leq\beta} Q_\alpha$, which must terminate as A_* is rooted. We now check our additional claims. If A_* has a maximal cluster that is not a single reflexive point then it is either a single reflexive point contradicting lemma 5.1 or a proper cluster, and taking the M-subspace of A_* that is just this cluster we obtain F_1 so $F_1^* \in \mathcal{A}$ which is a contradiction. Letting Q_α be a two cluster anti-chain and consider any cluster X in $Q_{\alpha+1}$. This cluster sees both clusters in Q_α and we consider the M-subspace of A_* rooted at X. If $\alpha = 0$ then this space can be reduced to F_4 giving $F_4^* \in \mathcal{A}$ which is a contradiction. If $\alpha \neq 0$, we also have that α is not a limit ordinal, and so we can consider the clopen upset $\bigoplus_{\gamma\leq\alpha-1} Q_\gamma$. Collapsing this via lemma 2.16, we can reduce the resulting space to F_{10} and so $F_{10}^* \in \mathcal{A}$ which is again a contradiction.

Finally, let A_* be i-type and let δ be the least ordinal such that Q_δ contains an irreflexive point. Now, by cor 5.5 there can only be finitely many ordinals $\gamma : \delta \leq \gamma \leq \beta$, so then $\beta = \lambda + n$ for some $n \in \omega$, and $\lambda \leq \delta \leq \beta$. If $n = 0$ then $\lambda = \delta = \beta$, so then Q_λ is a single cluster, which in fact is a single irreflexive point we label x.

Moreover $R[x] = \bigoplus_{\gamma<\lambda} Q_\gamma$ is closed, but then as we argued earlier for each $\gamma < \lambda$ $\bigoplus_{\eta\leq\gamma} Q_\eta$ is clopen, and so together form an open cover of $R[x]$ with no finite subcover, which is a contradiction with A_* being compact. So $n \neq 0$ and $\exists 0 < m \leq n$: $\delta = \lambda + m$. Then, by the definition of δ we have $\forall \alpha < \lambda + m$ that Q_α contains no irreflexive points and again by corollary 5.5 $\forall k \geq m$ $Q_{\lambda+k}$ is a single irreflexive point. Then, if $m < n$ then $Q_{\lambda+m-1}$ is defined and it cannot be a two cluster anti-chain by lemma 5.9 so is a single cluster. □

We will also consider *finite*, non-trivial and FSI members of our varieties, for which we can instantiate the previous theorem.

Corollary 5.12. Let $\mathcal{A}$ be a variety omitting $F_i^* : 1 \leq i \leq 17$ and G_n^* for some $n > 0$. Let $A \in \mathcal{A}$ be finite, non-trivial and FSI. Then the frame underlying A_* is a sequential composistion of frames $Q_0 \oplus ... \oplus Q_n$ such that:

$$Q_k \text{ is } \begin{cases} \text{a single cluster} & \text{if } k = n \\ \text{a single cluster, a two cluster anti-chain or } H & \text{if } k = 0 \\ \text{a single cluster or a two cluster anti-chain} & \text{otherwise} \end{cases}$$

Moreover, any maximal clusters are single reflexive points, if Q_k is a two cluster anti-chain then clusters in Q_{k+1} are improper.

If A_* contains an irreflexive point we will say it is *i*-type, otherwise it is *r*-type, and if A_* is *i*-type then $n \neq 0$ and $\exists 0 < m \leq n$: $\forall k < m$ Q_k contains no irreflexive points, $\forall k \geq m$ Q_k is a single irreflexive point and if $m < n$ then Q_{m-1} is a single cluster.

Proof. A is finite so obviously finitely generated. Then we apply theorem 5.11 noting that we must have $\beta = n \in \omega$ as A_* is finite. □

With theorem 5.11 in place we now have our detailed description of the dual spaces to finitely generated, non-trivial SI members in the varieties we are interested in. This represents the bulk of the work needed to establish the sufficient direction of our main result, which we will conclude in the next chapter. Along the way, we have provided a detailed description of how irreflexive points behave in these spaces (corollary 5.5 and lemmas 5.9 & 5.10) which we will continue to make use of.

Chapter 6

Primitive Varieties of K4-algebras

With a detailed description of the finitely generated SI members of the varieties in question, we can complete the characterisation of primitive $K4$-algebras. We effectively do this in two stages, the first is to establish the FMP for our varieties.

Theorem 6.1. Let $\mathcal{A}$ be a variety omitting $F_i^* : 1 \leq i \leq 17$ and G_n^* for some $n > 0$. Then $\mathcal{A}$ has the FMP.

Proof. We follow the same proof strategy as Rybakov [29, Lemma 3.9], which itself is a variation on the drop-point technique of K. Fine [14, Theorem 4].

Let $\varphi \in Fm$ and $A \in \mathcal{A}$ be such that $\not\models_A \varphi$. As $\mathcal{A}$ is generated by its SI members [10, ex 7.24], we may assume it is SI, and w.l.o.g that it is generated by $h(p_i) \in A$ where the $\{p_i\}_{i \leq n}$ are the propositional variables occurring in φ. Therefore, A is finitely generated and SI. A is also non-trivial as the trivial algebra validates all formulas. Thus we may apply theorem 5.11 giving $A_* = \bigoplus_{\alpha \leq \beta} Q_\alpha$.

For now let us assume A_* is r-type. Recall from the proof of theorem 3.25 that h induces a valuation V on A_* such that $V \not\models_{A_*} \varphi$. We extend this valuation in the natural way to define a clopen subset $V(\psi)$ for any $\psi \in Fm$. Also recall from lemma 2.20 that as A_* is n-generated each cluster of A_* has at most 2^n elements.

For each sub-formula ψ of φ consider:

$$\alpha_\psi := min\{\alpha \in Ord : \alpha \leq \beta \text{ and } V(\neg\psi) \cap Q_\alpha \neq \varnothing\}.$$

$$B := \{\alpha_\psi \in Ord : \psi \text{ is a subformula of } \varphi\}.$$

Note that as there are finitely many sub-formulas of φ, the set B is finite. Moreover, we claim that each α_ψ is a successor ordinal. Suppose not, then $V(\neg\psi)$ is by definition clopen, and we have $\forall \alpha < \alpha_\psi \; V(\neg\psi) \cap Q_\alpha = \varnothing$. Then, α_ψ is a limit ordinal so Q_{α_ψ} is a single cluster giving that $\bigoplus_{\alpha_\psi \leq \alpha \leq \beta} Q_\alpha = R^{-1}[V(\neg\psi)]$ which is clopen. Thus, its complement which is $\bigoplus_{\alpha < \alpha_\psi} Q_\alpha$ is also clopen. Now, by modal separation we can easily show that for any $\alpha \leq \beta$ $\bigoplus_{\gamma \leq \alpha} Q_\gamma$ is clopen, and so the collection $\{\bigoplus_{\gamma \leq \alpha} Q_\gamma\}_{\alpha < \alpha_\psi}$ together cover $\bigoplus_{\alpha < \alpha_\psi} Q_\alpha$. This then is an open cover of $\bigoplus_{\alpha < \alpha_\psi} Q_\alpha$ with no finite subcover, contradicting compactness.

Now, we define $M \subseteq A_*$ as follows: For each subformula ψ of φ, either Q_{α_ψ} is of type H, a two cluster anti-chain or a single cluster. If it is either of type H or a two cluster anti-chain, we choose one of the possible two single element clusters in $Q_{\alpha_\psi + 1}$ and label its element x_ψ. If Q_{α_ψ} is a single cluster, we simply choose one of the elements in the cluster for x_ψ. We then define:

$$M := \bigcup_{\alpha_\psi \in B} Q_{\alpha_\psi} \cup \{x_\psi \in A_* : \alpha_\psi \in B\} \cup Q_0.$$

Note that as B is finite and each Q_α has at most four finite clusters M itself is finite. We claim moreover that M is clopen. As each α_ψ is not a limit ordinal, each $x \in M$ belongs to Q_α where α is not a limit ordinal. Then, as noted earlier both $\bigoplus_{\gamma \leq \alpha - 1} Q_\gamma$ and $\bigoplus_{\gamma \leq \alpha} Q_\gamma$ are clopen, and so Q_α which is their intersection is clopen. Then, as Q_α is finite by Stone separating x from each other member of Q_α and taking intersections we get $\{x\}$ clopen. Thus, M is a finite collection of isolated points and is clopen. This means M is a finite sub-frame of A_*. Our aim now is two-fold. We want to find a surjective continuous p-morphism $f : A_* \twoheadrightarrow M$ and a valuation $W : P \to \mathcal{P}(M)$ such that $W \not\models_M \varphi$. Then, by the duality $M^* \in \mathcal{A}$ and $\not\models_{M^*} \varphi$, which gives $\mathcal{A}$ has FMP as required.

For the map f; For each $y \in A_* \setminus M$ we consider:

$$\alpha_{\psi_y} := min\{\alpha_\psi \in B : y \in Q_\alpha \text{ and } \alpha_\psi \leq \alpha\}.$$

Then, let $f : A_* \twoheadrightarrow M$ be the map defined by

$$f(y) := \begin{cases} y & \text{if } y \in M \\ x_{\psi_y} & \text{if } y \notin M \end{cases}$$

This is clearly surjective, therefore we only need to check it is a continuous p-morphism.

For continuity; as M is finite its topology is discrete, and so it is sufficient to check that $\forall y \in M\, f^{-1}(y)$ is clopen. If $y \in M$ then y is isolated in A_*, and if $y \neq x_\psi$ for any $\psi \in \varphi$ then $f^{-1}(y) = \{y\} \subseteq M$ so this is clopen. So now consider $x_\psi : \psi \in \varphi$. Let $\lambda \in \varphi$ be the subformula of $\varphi : \alpha_\lambda = min\{\alpha_\eta \in B : \alpha_\psi < \alpha_\eta\}$, i.e. Q_{α_λ} is the greatest layer of A_* intersecting M after Q_{α_ψ} Then, by the definition of f we get:

$$f^{-1}(x_\psi) = \bigcup_{\alpha_\psi < \alpha < \alpha_\lambda} Q_\alpha \cup \{x_\psi\}.$$

As $\alpha_\lambda \in B$, α_λ is not a limit ordinal, and so:

$$\bigcup_{\alpha_\psi < \alpha < \alpha_\lambda} Q_\alpha = \bigcup_{\alpha \leq \alpha_\lambda - 1} Q_\alpha \setminus \bigcup_{\alpha \leq \alpha_\psi} Q_\alpha.$$

This is therefore clopen in A_*. Then, x_ψ is isolated in A_* and $f^{-1}(x_\psi)$ is clopen.

For $R[f(y)] = f[R[y]]$; let $y \in A_*$. We make two observations. First, suppose that $y \not R f(y)$, then from structure of A_* we have $y \in Q_\alpha$ and $f(y) \in Q_\gamma$ with $\alpha \leq \gamma$. Then, from the defintion of f we also have $\gamma \leq \alpha$, so in fact $\gamma = \alpha$ and y and $f(y)$ are in different clusters. This means Q_α is a two cluster anti-chain. Moreover, $y \neq f(y)$ so $y \notin M$ and therefore $f(y) = x_\psi$ for some $\psi \in \varphi$. Then, as $y \notin M$ $M \cap Q_\alpha \neq Q_\alpha$, and so $x_\psi \notin Q_{\alpha_\psi}$, that is $\alpha = \alpha_\psi + 1$ and Q_{α_ψ} is also a two cluster anti-chain. So, both clusters in Q_α are improper. In summary, either $yRf(y)$ or $y, f(y) \in Q_\alpha$ where Q_α is an anti-chain of two points.

Second, let $u, v \in A_*$ such that uRv. If $u \in M$, then either $vRf(v)$ so $f(u) = uRvRf(v)$ or $v, f(v) \in Q_\alpha$ where Q_α is an anti-chain of two points. Then $u \neq v$ and

uRv means $u \in Q_\gamma$ such that $\gamma > \alpha$ and so $uRf(v)$ and $f(u) = uRf(v)$. If $u \notin M$ and $v \in M$ then by definition $\alpha_{\psi_u} \geq \gamma$ where $v = f(v) \in Q_\gamma$. If $\alpha_{\psi_u} > \gamma$ then $f(u)Rf(v)$, if $\alpha_{\psi_u} = \gamma$ we have either Q_γ is of type H or a two cluster anti-chain, $f(u) = x_{\psi_u} \in Q_{\gamma+1}$ and $f(u)Rf(v)$ or Q_γ is a single cluster and so $f(u) = x_{\psi_u}Rf(v)$. If $u \notin M$ and $v \notin M$ then as uRv $\alpha_{\psi_u} \geq \alpha_{\psi_v}$, if $\alpha_{\psi_u} > \alpha_{\psi_v}$ then $f(u)Rf(v)$ and if $\alpha_{\psi_u} = \alpha_{\psi_v}$ then $f(u)x_{\psi_u} = x_{\psi_v} = f(v)$ and $f(u)Rf(v)$. In all cases, $f(u)Rf(v)$, i.e. f is R-preserving.

Then, letting $z \in M$ such that $f(y)Rz$, we have $y \in Q_\alpha$ and $z \in Q_\gamma$ and $z = f(z)$. If $yRf(y)$ then $yRf(y)Rz$ and $z = f(z)$ so $f(z) \in f[R(y)]$. If $y, f(y) \in Q_\alpha$ and Q_α is an anti-chain of two points, then as $f(y)Rz$ either $f(y) = z$ and then yRy so $f(z) \in f[R(y)]$ or $\gamma < \alpha$ and so yRz and again $f(z) \in f[R(y)]$. So $R[f(y)] \subseteq f[R[y]]$.

Finally, letting $z \in M$ such that $z = f(u)$ where yRu, then $f(y)Rf(u) = z$ and $z \in R[f(y)]$. So $f[R[y]] \subseteq R[f(y)]$.

For the valuation; for each $p_i \in \varphi$ we define $W(p_i) = V(p_i) \cap M$. We claim that $\forall x \in M\ \forall \psi \in \varphi\ x, V \models \psi$ iff $x, W \models \psi$. Then, in particular as $V \not\models_{A_*} \varphi$ we can consider $\alpha_\varphi \in B$ and have $V(\neg\varphi) \cap Q_{\alpha_\varphi} \neq \varnothing$. So, letting $x \in V(\neg\varphi) \cap Q_{\alpha_\varphi}$, $x \in M$ and $x, V \not\models \varphi$, so $x, W \not\models \varphi$ and $W \not\models_M \varphi$ as required. We proceed by induction:

For $\psi = p_i \in \varphi$; letting $x \in M$ $x, V \models p_i$ iff $x \in V(p_i)$ iff $x \in W(p_i)$ iff $x, W \models p_i$.

For $\psi = \lambda \wedge \eta$; letting $x \in M$ $x, V \models \psi$ iff $x, V \models \lambda$ and $x, V \models \eta$ iff $x, W \models \lambda$ and $x, W \models \eta$ iff $x, W \models \psi$

For $\psi = \neg\lambda$; letting $x \in M$ $x, V \models \psi$ iff $x, V \not\models \lambda$ iff $x, W \not\models \lambda$ iff $x, W \models \psi$.

For $\psi = \Box\lambda$; letting $x \in M$ if $x, V \models \psi$ then letting $y \in M : xRy$ $y, V \models \lambda$ so $y, W \models \lambda$ and then $x, W \models \lambda$. If $x, V \not\models \psi$ then $\exists y \in A_* : y, V \not\models \lambda$, so letting $y \in Q_\alpha$ we have $V(\neg\lambda) \cap Q_\alpha \neq \varnothing$. Therefore $\alpha_\lambda \leq \alpha$. If $\alpha_\lambda < \alpha$ then we have $z \in Q_{\alpha_\lambda} \subseteq M$ such that $z, V \not\models \lambda$ and yRz, so then $z, W \not\models \lambda$ and xRz, therefore $x, W \not\models \psi$. If $\alpha_\lambda = \alpha$ then $y \in M$ and $y, W \not\models \lambda$ so $x, W \not\models \psi$.

We still need to check the case when A_* is i-type. Recall that this means A_* has at its base some finite number of irreflexive points in a chain. As such, when constructing M we also include each layer of A_* that is a single irreflexive point. The proof then follows as in the r-type case. Continuity is maintained as the irreflexive points still only appear in successor ordinal layers and so they are isolated in A_* which covers the additional continuity requirements. Then, $R[f(y)] = f[R[y]]$ when y is an irreflexive point is immediate from them forming a chain. Finally, constructing the valuation proceeds exactly as above. □

The second stage and final major result of our investigations relates to weak projectivity.

Lemma 6.2. Let $\mathcal{A}$ be a variety omitting $F_i^* : 1 \leq i \leq 17$ and G_n^* for $n > 0$. Then every finite, non-trivial FSI member of $\mathcal{A}$ is weakly projective in $\mathcal{A}$.

Proof. Let $A \in \mathcal{A}$ be finite, non-trivial and FSI. Then we may apply corollary 5.12 giving $A_* = \bigoplus_{m=0}^{n} Q_m$. Let k be the least i such that $Sl_i(A_*)$ is a single irreflexive point or the root of A_*.

Let also $C \in \mathcal{A}$ such that $A \in \mathbb{H}(C)$. Since A is finite, $\exists B \leq C : B$ is finitely generated and $A \in \mathbb{H}(B)$. Moreover, if $A \in \mathbb{IS}(B)$ then $A \in \mathbb{IS}(C)$, so it is sufficient to check the former. By the duality (lemma 2.10), this amounts to assuming A_* is a closed upset of B_* and we must show there is a surjective continuous p-morphism $f : B_* \twoheadrightarrow A_*$. The plan is to do this recursively by collapsing points in

$R^{-1}[Sl_i(A_*)] \setminus R^{-1}[Sl_{i-1}(A_*)]$ into A_*. More formally, we are going to define a series of modal equivlaences where we can identify the underlying set of the resulting quotient space with $A_* \cup R^{-1}[Sl_1(A_*)]$, $A_* \cup R^{-1}[Sl_2(A_*)]$,..., $A_* \cup R^{-1}[Sl_k(A_*)]$ and finally A_* respectively.

For E_0; By the structure of A_*, $Sl_1(A_*)$ is either a single reflexive point or an antichain of two reflexive points. We label these a_0 and b_0 respectively. Now, consider:

$$B_* \setminus R^{-1}[A_*] \cup \{a_0\}.$$

This is an upset and moreover we claim it is clopen. B_* is finitely generated so by theorem 2.27 $\forall i \in \omega$ $Sl_i(B_*)$ is finite and clopen. Then, considering $x \in A_*$ $x \in Sl_i(B_*)$ for some $i \leq n+1$, and by first stone separating from each other element in $Sl_i(B_*)$ and then taking intersections we get that $\{x\}$ is clopen, i.e. A_* is a finite collection of isolated points and so it and any subset of A_* is clopen. This in turn implies $B_* \setminus R^{-1}[A_*] \cup \{a_0\}$ is clopen.

Now, by lemma 2.16 we take the modal equivalence identifying points in the set as E_0. We define $B_0 := B_*/E_0$ and use R_0 to denote its relation. Of course $B_* \twoheadrightarrow B_0$ and B_0 is finitely generated. Moreover, consider the set:

$$A_0 := \{[x] \in B_0 : [x] \cap A_* \neq \varnothing\}.$$

Notably, letting $[x] \in A_0$, and $x' \in [x] \cap A_*$, either $x' = a_0$, $[x] = [a_0]$ and $[a_0] \cap A_* = \{a_0\}$ or $x' \neq a_0$, $[x] \neq [a_0]$ so $[x] = \{x'\}$ and $[x] \cap A_* = \{x'\}$. That is, if $x \in A_*$ then $[x] \in A_0$ and $[x] \cap A_* = \{x\}$.

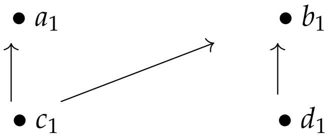
Letting $[y] \in B_0$, either $y \in R^{-1}[A_*]$ or not. If $y \in R^{-1}[A_*]$ then $[y] \in R_0[A_0]$, and if $y \notin R^{-1}[A_*]$ then $[y] = [a_0] \in R_0^{-1}[A_0]$. So $B_0 = R_0^{-1}[A_0] = A_0 \cup R_0^{-1}[A_0]$. The pre-image of A_0 in B_* is $A_* \cup B_* \setminus R^{-1}[A_*]$ which is clopen and so A_0 is clopen in B_0.

Then, letting $x, y \in A_*$ if xRy then $[x]R_0[y]$. If $[x]R_0[y]$ so $\exists x'E_0x$ and $y'E_0y$: $x'Ry'$. If $x = a_0$ then $[x] = [a_0]$ so $[y] = [a_0]$, $y = a_0$ and xRy. If $y = a_0$ then as $[b_0] = \{b_0\}$ and $R[b_0] = \{b_0\}$ we have $[b_0]\not{R}_0[a_0]$ so $[x] \neq [b_0]$ and $x \neq b_0$ and so xRy. If $x \neq a_0$ and $y \neq a_0$ then $[x] = \{x\}$, $[y] = \{y\}$ and xRy. Therefore, $\forall x, y \in A_*$ xRy iff $[x]R_0[y]$. Now, we consider A_0 as a transitive space under the restricted relation and discrete topology (as it is finite). Considering the quotient map $x \mapsto [x]$ restricted to A_*, as both A_* and A_0 are finite it is trivially continuous and open, and as xRy iff $[x]R_0[y]$ it moreover a p-morphism. It is clearly surjective, and letting $x, y \in A_* : [x] = [y]$ we have $\{x\} = [x] \cap A_* = [y] \cap A_* = \{y\}$ so $x = y$. So the map is an isomorphism and $A_0 \cong A_*$.

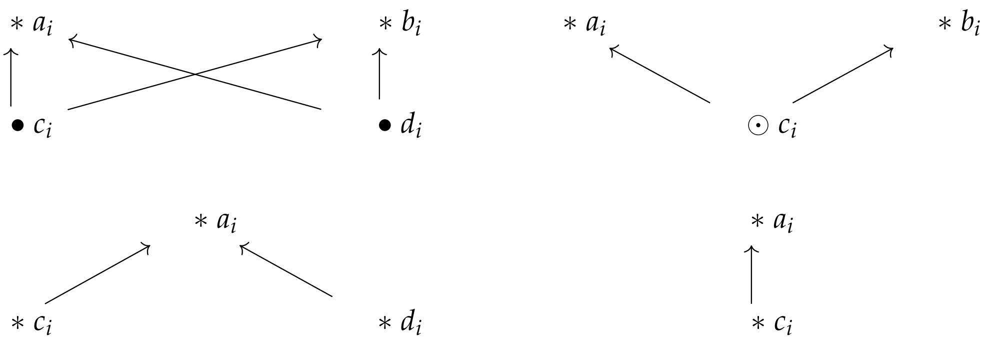
In summary, we have constructed B_0 and $A_0 \subseteq B_0$ such that $B_* \twoheadrightarrow B_0$, B_0 is finitely generated, $B_0 = A_0 \cup R_0^{-1}[A_0] = A_0 \cup R_0^{-1}[Sl_1(A_0)]$, A_0 is clopen in B_0 and $A_0 \cong A_*$.

For $E_i : 1 \leq i < k$; Suppose we have constructed B_{i-1} & $A_{i-1} \subseteq B_{i-1}$ such that $B_{i-2} \twoheadrightarrow B_{i-1}$, B_{i-1} is finitely generated, $B_{i-1} = A_{i-1} \cup R_{i-1}^{-1}[Sl_i(A_{i-1})]$, A_{i-1} is clopen in B_{i-1} & $A_{i-1} \cong A_{i-2}$. Note that $A_{i-1} \cong A_{i-2} \cong ....A_0 \cong A_*$ so has the same structure as A_*, so $Sl_i(A_{i-1})$ is either a single cluster or two cluster anti-chain. We choose a point in each cluster as a_i and b_i respectively. Moreover, $Sl_{i+1}(A_{i-1})$ is also either a single cluster or two cluster anti-chain. If $i = 1$ and Q_0 is of type H then we set up our labels as follows, otherwise we simply choose elements in the clusters as

c_i and d_i.



Notably, in both cases $c_i R_{i-1} a_i$ and $c_i R_{i-1} b_i$, and as $i < k$ $a_i R_{i-1} a_i$ and $a_i R_{i-1} b_i$. The case for $i = 2$ and Q_0 is of type H is slightly different and we will cover it separately, for now assume either $i \neq 2$ or Q_0 is not of type H, i.e. applying corollary 5.12, we have that $Sl_i(A_{i-1})$ and $Sl_{i+1}(A_{i-1})$ is one of the following:



We will detail the first case, all others are recoverable by deleting references to b_i and d_i as required.

Letting $x \in B_{i-1} \setminus (A_{i-1} \cup R_{i-1}^{-1}[Sl_{i+1}(A_{i-1})])$, as $B_{i-1} = A_{i-1} \cup R_{i-1}^{-1}[Sl_i(A_{i-1})]$ we have $xR_{i-1}a_i$ or $xR_{i-1}b_i$. We define $m(x) = R_{i-1}[x] \cap \{a_i, b_i\}$. We then define three sets:

$U_1 := \{x \in B_{i-1} \setminus (A_{i-1} \cup R_{i-1}^{-1}[Sl_{i+1}(A_{i-1})]) : m(x) = \{a_i\}\} \cup \{a_i\}$;
$U_2 := \{x \in B_{i-1} \setminus (A_{i-1} \cup R_{i-1}^{-1}[Sl_{i+1}(A_{i-1})]) : m(X) = \{b_i\}\} \cup \{b_i\}$;
$U_3 = \{x \in B_{i-1} \setminus (A_{i-1} \cup R_{i-1}^{-1}[Sl_{i+1}(A_{i-1})]) : m(x) = \{a_i, b_i\}\} \cup \{c_i\}$.

By inspection these are pairwise disjoint. We claim they form an M-partition. Now, as A_{i-1} is clopen and finite we can stone separate each $x \in A_{i-1}$ from the other elements and take intersections to find $\{x\}$ is clopen and A_{i-1} is a finite collection of isolate points. In particular, all subsets of A_{i-1} are clopen. Then, letting $x \in B_{i-1} \setminus (A_{i-1} \cup R_{i-1}^{-1}[Sl_{i+1}(A_{i-1})])$, $m(x) = \{a_i\}$ iff $xR_{i-1}a_i$ and $x \not R_{i-1} b_i$ iff $x \in R_{i-1}^{-1}[a_i] \setminus R_{i-1}^{-1}[b_i]$. Therefore we can express U_1 as:

$$U_1 = (B_{i-1} \setminus (A_{i-1} \cup R_{i-1}^{-1}[Sl_{i+1}(A_{i-1})]) \cap (R_{i-1}^{-1}[a_i] \setminus R_{i-1}^{-1}[b_i])) \cup \{a_i\}.$$

Therefore, U_1 is clopen. Similarly, we can express U_2 and U_3 as:

$$U_2 = (B_{i-1} \setminus (A_{i-1} \cup R_{i-1}^{-1}[Sl_{i+1}(A_{i-1})]) \cap (R_{i-1}^{-1}[b_i] \setminus R_{i-1}^{-1}[a_i])) \cup \{b_i\}.$$

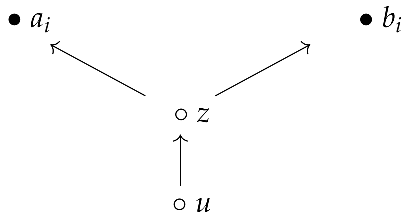
$$U_3 = (B_{i-1} \setminus (A_{i-1} \cup R_{i-1}^{-1}[Sl_{i+1}(A_{i-1})]) \cap (R_{i-1}^{-1}[a_i] \cap R_{i-1}^{-1}[b_i])) \cup \{c_i\}.$$

Therefore, both are clopen as well.

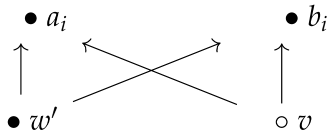
Then, letting $u, v \in U_1$ and $uR_{i-1}w$, as $uR_{i-1}w$ we have $u \notin R_{i-1}^{-1}[Sl_{i+1}(A_{i-1})]$. If $w \in A_{i-1}$ then again $u \not R_{i-1} w$ and $u \not R_{i-1} b_i$ implies $w \not R_{i-1} b_i$ so w is in the same cluster as a_i or $w \in S_{i-1}(A_{i-1})$, either way $a_i R_{i-1} w$, and $vR_{i-1}a_i$ so vRw. If $w \notin A_{i-1}$ then $m(w)$ is defined and as $w \not R_{i-1} b_i$, $m(w) = \{a_i\}$, so $w \in U_1$ and $vR_{i-1}a_i \in U_1$. The case for $u, v \in U_2$ is symmetric.

So now let $u, v \in U_3$ and $uR_{i-1}w$. Now, either $u = c_i$ or $u \notin R^{-1}[Sl_{i+1}(A_{i-1})]$. In the latter, $w \notin R^{-1}[Sl_{i+1}(A_{i-1})]$. Again, if $w \in A_{i-1}$ then $w \in S_i(A_{i-1})$ so either $a_iR_{i-1}w$ and then $vR_{i-1}a_i$ or $b_iR_{i-1}w$ and then $vR_{i-1}b_i$. If $w \notin A_{i-1}$ then $m(w)$ is defined and $w \in U_1 \cup U_2 \cup U_3$. If $w \in U_1$ then $vR_{i-1}a_i \in U_1$, and if $w \in U_2$ then $vR_{i-1}b_i \in U_2$.

So suppose $w \in U_3$, we want to find $t \in U_3$ such that $vR_{i-1}t$. Now, suppose that there is no $w' \in R_{i-1}[u]$ such that $w'R_{i-1}a_i$, $w'R_{i-1}b_i$ and $w'R_{i-1}w'$. Then, firstly, in particular $u\not R_{i-1}u$ and considering $w \in R_{i-1}[u] \cap R_{i-1}^{-1}[a_i] \cap R_{i-1}^{-1}[b_i]$, as B_* is finitely generated by lemma 2.22 we can consider an R_{i-1}-maximal cluster in this set which again is a single irreflexive point which we denote as z. Then a_i, b_i, z and u witness the following substructure in B_* contradicting lemma 5.9:



So, in fact $\exists w' \in R_{i-1}[u]$ such that $w'R_{i-1}a_i$, $w'R_{i-1}b_i$ and $w'R_{i-1}w'$. Then, if there was no $t \in U_3$ such that $vR_{i-1}t$ in particular have $v\not R_{i-1}v$, $R_{i-1}^{-1}[a_i] \cap R_{i-1}[v] = \{a_i\}$ and $R_{i-1}^{-1} \cap R_{i-1}[v] = \{b_i\}$ and so v, w', a_i and b_i witness the following substructure in B_* contradicting lemma 5.10:



So, in fact $\exists t \in U_3 : vRt$ as required.

Finally, if $u = c_i$ then $w \in A_{i-1} \cap R_{i-1}[c]$. Noting that as $c_iR_{i-1}a_i$, $c_iR_{i-1}b_i$ and $a_i||b_i$ from corollary 5.12 we have the cluster containing c_i is improper, i.e. is exctly $\{c_i\}$. So either $wR_{i-1}u$ and $w = c_i \in U_3$ and we proceed as we did in this case before, or $w\not R_{i-1}u$ so $w \in S_i(A_{i-1})$ and we again proceed as we did in this case before.

So, applying lemma 2.17 we take E_i as the modal equivalence identifying points within U_1, U_2 and U_3 and define $B_i := B_{i-1}/E_i$ and

$$A_i := \{[x] \in B_i : [x] \cap A_{i-1} \neq \varnothing\}.$$

Once more, we have $B_{i-1} \twoheadrightarrow B_i$ and B_i is finitely generated. Letting $[x] \in A_i$ and $x' \in [x] \cap A_{i-1}$, either $x' = a_i$, $[x] = U_1$ and $[x] \cap A_{i-1} = \{a_i\}$, $x' = b_i$, $[x] = U_2$ and $[x] \cap A_{i-1} = \{b_i\}$, $x' = c_i$, $[x] = U_3$ and $[x] \cap A_{i-1} = \{c_i\}$ or $x' \notin \{a_i, b_i, c_i\}$ so $[x] = \{x'\}$ and $[x] \cap A_{i-1} = \{x'\}$. That is, if $x \in A_{i-1}$ then $[x] \in A_i$ and $[x] \cap A_* = \{x\}$.

Letting $[y] \in B_i$, either $y \in R_{i-1}^{-1}[Sl_{i+1}(A_{i-1})]$ or not. If it is, then $[y] \in R_i^{-1}[Sl_{i+1}(A_i)]$ and if it isn't then either $y \in A_{i-1}$ and $[y] \in A_i$ or $y \notin A_{i-1}$ so $y \in U_1 \cup U_2 \cup U_3$ and then $[y] \in \{U_1, U_2, U_3\}$ and $[y] \in A_i$. So $B_i = A_i \cup R_i^{-1}[Sl_{i+1}(A_i)]$. The pull back of A_i to B_{i-1} is $A_{i-1} \cup U_1 \cup U_2 \cup U_3$ which is clopen in B_{i-1}, so A_i is clopen in B_i.

Then, letting $x, y \in A_{i-1}$, if $xR_{i-1}y$ then $[x]R_i[y]$. If $[x]R_i[y]$ then $\exists x'E_ix$ and $y'E_iy : x'R_{i-1}y'$. Then either $x, y \notin \{a_i, b_i, c_i\}$ and so $[x] = \{x\}$ $[y] = \{y\}$ and $xR_{i-1}y$ or $x, y \in \{a_i, b_i, c_i\}$ in which case we have one of the following cases:

$x = a_i = y$ then $a_i R_{i-1} a_i$ so $x R_{i-1} y$.

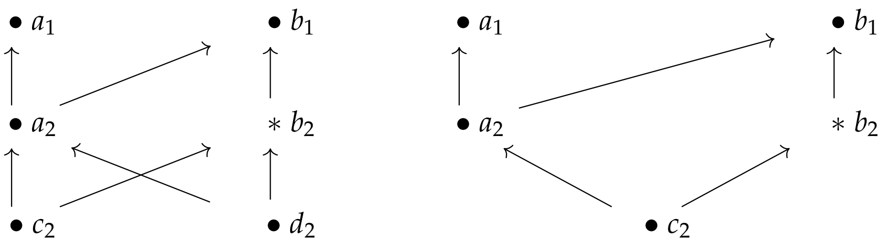
$x = a_i$, $y = b_i$ then $x' \in U_1$ and $y' \in U_2$ so $x' R_{i-1} y' R_{i-1} b_i$ but $x' \not R_{i-1} b_i$ so this is impossible. In a similar manner, the cases for $x = a_i$, $y = c_i$ or $x = b_i$, $y = a_i$ or $x = b_i$, $y = c_i$ are impossible.

$x = b_i = y_i$ then $b_i R_{i-1} b_i$ so $x R_{i-1} y$.

$x = c_i$ and $y = a_i$ or $y = b_i$, then $c_i R_{i-1} y$ so $x R_{i-1} y$.

Once again then, $\forall x, y \in A_{i-1}$ $x R_{i-1} y$ iff $[x] R_i [y]$. Now we consider A_i as a transitive space under the restricted relation and discrete topology. Considering the quotient map $x \mapsto [x]$ restricted to A_{i-1}, as both A_{i-1} and A_i are finite it is trivially continuous and open, and as $x R_{i-1} y$ iff $[x] R_i [y]$ it is moreover a p-morphism. It is clearly surjective, and letting $x, y \in A_{i-1} : [x] = [y]$ we have $\{x\} = [x] \cap A_{i-1} = [y] \cap A_{i-1} = \{y\}$ so $x = y$. So the map is an isomorphism and $A_{i-1} \cong A_i$.

In the specific case that $i = 2$ and Q_0 is of type H then Q_0 and $Sl_3(A_2)$ form one of the following labelled sub-frames:



This case proceeds as before, except for a specific part of checking the M-partition requirement. Namely, where $u, v \in U_2$ and $u R_1 w$ and $w \in A_i$. Then as $u \not R_1 a_2$ we have $w \not R_1 a_2$ so either w is in the same cluster as b_2, $w = b_1$ or $w = a_1$. If w is in the same cluster as b_2 then $b_2 R_1 w$ and $v R_1 w$. If $w = b_1$ then $v R_1 b_2 R_1 w$. If $w = a_1$, then $u R_1 w$ so $R_1[u] \cap Q_0 = \{a_1, b_1, b_2\}$. Considering $R_1^{\omega}[u] \subseteq B_1$ as a closed up-set, we have $R_1^{\omega}[u]^* \in \mathbb{H}(B_1^*)$, B_1^* is finitely generated, and so $R_1^{\omega}[u]$ is also finitely generated and so $R_1^{\omega}[u]$ has the structure $\bigoplus_{\alpha \leq \beta} P_\alpha$ as described by theorem 5.11. Moreover, $a_1, b_1, b_2 \in R_1^{\omega}[u]$ and have the same depth, so we have $b_2 \in Sl_2(R_1^{\omega}[u])$ and $a_1 \in Sl_1(R_1^{\omega}[u])$ with $b_2 \not R_1 a_1$. This forces P_0 to also be of type H, thus $\exists t \in R_1^{\omega}[u]$ such that $t \in Sl_2(R_1^{\omega}[u])$ and $t R_1 a_1$, $t R_1 b_1$. But then, $t \in Sl_2(B_1)$ and $t_1 R a_1$ and $t R_1 b_1$, i.e. $t = a_2$ and $u R_1 a_2$ which is a contradiction.

Repeatedly applying our process, we obtain B_{k-1} and A_{k-1} such that $B_* \twoheadrightarrow B_{k-1}$, $B_{k-1} = A_{k-1} \cup R_{k-1}^{-1}[Sl_k(A_{k-1})$, A_{k-1} is clopen in B_{k-1} and $A_{k-1} \cong A_{k-2} \cong ... \cong A_*$. Then $Sl_k(A_{k-1})$ is either the first irreflexive point in A_{k-1} or not an irreflexive point and so the cluster containing all of the roots of A_{k-1}.

Suppose it the root cluster of A_{k-1}, we again choose an element from it and label it a_k. Then, we define E_k on A_{k-1} by:

$$E_k := \{(x, a_k), (a_k, y), (x, y) \in B_{k-1}^2 : x, y \in B_{k-1} \setminus A_{k-1}\} \cup \{(u, u) \in B_{k-1}^2 : u \in B_{k-1}\}.$$

That is, E is the smallest equivalence relation identifying all points outside A_{k-1} with a_k. We claim this is a modal equivalence. Letting $u E_k v$ and $u R_{k-1} w$, either $u, v \in A_{k-1} \setminus \{a_k\}$ and so $u = v$ and $v R_{k-1} w$ or $u, v \in (B_{k-1} \setminus A_{k-1}) \cup \{a_k\}$. Then, as $B_{k-1} = A_{k-1} \cup R_{k-1}^{-1}[Sl_k(A_{k-1})]$, $u, v \in R^{-1}[Sl_k(A_{k-1})]$ and so both u and v see a_k and by extension all of A_{k-1}. So either $w \in A_{k-1}$ and $v R w$ or $w \notin A_{k-1}$ so $v R a_k$ and $w E_k a_k$.

Then, letting $u \not{E}_k v$ either $u \in (B_{k-1} \setminus A_{k-1}) \cup \{a_k\}$ or $u \in A_{k-1}$. In the former, $v \notin (B_{k-1} \setminus A_{k-1}) \cup \{a_k\}$ and this set is a clopen E_k-class, therefore it separates u and v as required. In the latter $u \in A_{k-1} \setminus \{a_k\}$ and $v \neq u$ then $u \in A_{k-1}$ implies it is isolated so $\{u\}$ separates u and v as required.

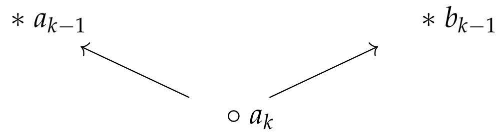
So, we finally let $B_k := B_{k-1}/E_k$ and once more:

$$A_k := \{[x] \in B_k : [x] \cap A_{k-1} \neq \varnothing\}.$$

Now, letting $[y] \in B_k$, either $y \in A_{k-1}$ and $[y] \in A_k$ or $y \notin A_{k-1}$ and $[y] = [a_k] \in A_k$ So $B_k = A_k$ making it finite and its topology discrete.

Letting $x, y \in A_{k-1}$ if $xR_{k-1}y$ then $[x]R_k[y]$. If $[x]R_k[y]$ either $x, y \neq a_k$, $x = a_k$ or $y = a_k$. If $x, y \neq a_k$ then so $[x] = \{x\}$, $[y] = \{y\}$ and $xR_{k-1}y$. If $y = a_k$ then $\exists x' E_k x$ and $y' E_k y$ such that $x' R_{k-1} y'$ then $y = a_k$ implies $y' \in B_{k-1} \setminus A_{k-1} \cup \{a_k\}$ which again means $y' R_{k-1} a_k$ so $x' R_{k-1} a_k$ and $[x] = [x'] = [a_k]$ so $x = a_k$, so this reduces to the $x = a_k$ case, and then $xR_{k-1}y$ as its a root for A_{k-1}. So, once more $\forall x, y \in A_{k-1}$ we have $xR_{k-1}y$ iff xR_ky and the quotient map restricted to A_{k-1} is an isomorphism and $A_{k-1} \cong A_k$. Finally then, $B_* \twoheadrightarrow B_{k-1} \twoheadrightarrow B_k \cong A_k \cong A_{k-1} \cong A_*$. So $B_* \twoheadrightarrow A_*$ as required.

Now suppose A_{k-1} is i-type and $Sl_k(A_{k-1})$ is the first layer of A_{k-1} that is a single irreflexive point which we label a_k. Now, either Q_{n-1} is a two cluster anti-chain or not. If it is, then firstly by lemma 5.4 the points in it are reflexive so $k > n-1$ and in fact $k = n$, i.e. the base of A_{k-1} is the following:

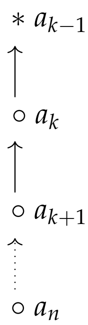


Notably:

$$R_{k-1}^{-1}[a_{k-1}] \cap R_{k-1}[a_k] = \{u \in B_{k-1} : a_{k-1}R_{k-1}u \;\&\; uR_{k-1}a_{k-1}\}.$$

So, by lemma 5.9 $R_{k-1}^{-1}[Sl_k(A_{k-1})] = R_{k-1}^{-1}[a_k] = \varnothing$. So $B_{k-1} = A_{k-1}$ and $B_* \twoheadrightarrow B_{k-1} \cong A_{k-1} \cong A*$. So $B_* \twoheadrightarrow A_*$ as required.

Finally, suppose Q_{n-1} is a single cluster, then recalling corollary 5.12 the base A_{k-1} is the following:



Moreover, as $B_{k-1} \in \mathcal{A}$ and $Sl_k(A_{k-1})$ is a single irreflexive point, from corollary 5.5 we have that $R_{k-1}^{-1}[Sl_k(A_{k-1})]$ is a tree of irreflexive points of depth $l \in \omega$ such that $l + k \geq n$. Then, $B_{k-1} = A_{k-1} \cup R_{k-1}^{-1}[Sl_k(A_{k-1})]$ and A_{k-1} is finite, so B_{k-1} is finite and of depth $k + l$.

Consider the collection of sets $\{Sl_r(B_{k-1})\}_{k\leq r\leq k+l}$. As B_{k-1} is finite each of these is trivially clopen. Given any $u \in Sl_r(B_{k-1})$, u is irreflexive, sees a point of depth $r' : r' < r$ and only such points, so the collection forms an M-partition. Taking E as the modal equivalence induced by lemma 2.17 and considering $B' := B_{k-1}/E$ we have $B' \cong \bigoplus_{m=0}^{k-1} Q_m \oplus \{a_k\} \oplus \{a_{k+1}\} \oplus ... \oplus \{a_{k+l}\}$. Then, applying $l+k-n$ α-reductions to B' we obtain A_{k-1} and so $B_{k-1} \twoheadrightarrow B' \twoheadrightarrow A_{k-1}$. Finally, $B_* \twoheadrightarrow B_{k-1} \twoheadrightarrow A_{k-1} \cong A_*$, so $B_* \twoheadrightarrow A_*$ as required. □

With the hard work done the final proof of our characterisation of primitive K4-algebras is straightforward.

Theorem 6.3 (Primitive Varieties of K4-algebras)**.**
Let $\mathcal{A}$ be a variety of K4-algebras. Then $\mathcal{A}$ is primitive iff $\mathcal{A}$ omits $F_i^* : 1 \leq i \leq 17$ and $\exists n > 0 : \mathcal{A}$ omits G_n^*.

Proof. The only if direction is exactly lemma 4.2. For the if direction, suppose $\mathcal{A}$ omits $F_i^* : 1 \leq i \leq 17$ and $G_n^* : n > 0$. By lemma 3.19 $\mathcal{A}$ has EDPC. Letting M be a sub-varietiy of $\mathcal{A}$, M also omits $F_i^* : 1 \leq i \leq 17$ and G_n^*, so by theorem 6.1 M has FMP. By lemma 6.2 each non-trivial, finite FSI member of $\mathcal{A}$ is weakly projective in $\mathcal{A}$. So from theorem 3.22 we conclude that $\mathcal{A}$ is primitive. □

Finally, we obtain as a corollary the characterisation of HSC transitive modal logics.

Corollary 6.4 (Hereditarily Structurally Complete Logics over K4)**.**
Let λ be a normal modal logic with equivalent algebraic semantics $\mathcal{A}$. The following are equivlaent:

(i) λ is HSC;

(ii) $\mathcal{A}$ is primitive;

(iii) For all $1 \leq i \leq 17$ F_i is not a λ-space and $\exists n > 0$ such that G_n is not a λ-space.

Proof. Combination of theorems 6.3, 3.14 and 3.25. □

With theorem 6.3 and corollary 6.4 we have completed our main task, establishing our new characterisation of primitive varieties of K4-algebras and by extension of hereditarily structurally complete transitive modal logics. With the structural results of the preceding chapter in place, we were able to firstly establish that our varieties have the FMP (theorem 6.1) and secondly that that their finite, non-trivial FSI members were weakly projective (theorem 6.2) in the variety. This completed the necessary steps to employ our sufficient condition and complete the proof of the characterisation.

Chapter 7

Conclusions

By utilising the relationships between logic, algebra and topology we have both corrected Rybakov's characterisation of the hereditarily structurally complete transitive modal logics and given a new, detailed proof strategy of the new characterisation. Whilst our strategy added a substantive theoretical load to the proof in the form of Jónnson-Tarski duality and algebraic logic, that additional theory helped illuminate a group of HSC transitive modal logics missed by Rybakov's characterisation and clarified exactly how component parts to the main proof progressed.

To close let us consider a few areas of further study. The central theory that enabled our investigation was the joining together of transitive modal logic being algebraizable and a duality theory for its associated class of algebras. A natural expansion to our investigation is to look for other logics which share this set up. When introducing the equivalent algebraic semantics and the Jónsson-Tarski duality we worked with modal logic generally before specialising to the transitive case, so the picture is readily present here. However, there are significant problems with attempting our proof strategy in the general modal case. In order to utilise the sufficiency condition we gave the variety of algebras we work with needed to have EDPC, but there are varieties of modal algebras that lack the EDPC [8, Theorem 5.4, pg 597]. This means a more general version of theorem 3.22 would be required which drops the EDPC requirement. A potential candidate is that for any variety $\mathcal{A}$ if the *finitely generated*, non-trivial SI members of $\mathcal{A}$ are weakly projective in $\mathcal{A}$ then the variety is primitive [24, p. 4.7]. One would then have to attempt a proof of theorem 6.2 without transitivity and working with a finitely generated space in place of a finite one, which would in contrast to our work (theorem 2.27) necessitate understanding the behaviour of finitely generated modal spaces beyond their elements of finite depth.

A more modest generalisation one could attempt would be to consider weakly transitive modal logic (wK4), which is algebrized by the variety of weakly transitive modal algebras (wK4-algebras), modal algebras A such that forall $a \in A$ $a \wedge \Box a \leq \Box\Box a$. wK4-algebras do have EDPC [8, Pg597], so one could study HSC wK4 logics through a similar proof strategy. There would be some significant subtleties to work out, beyond determining the potential characterisation the assumption of transitivity is woven throughout the development of our proof. One would want to make sense of reductions in the weakly transitive setting, and develop new techniques for defining modal equivalences on weakly transitive spaces to make the eventual proof manageable. Finally, as mentioned any algebraizable logic whose equivalent algebraic semantics has a duality theory is potential ground for an investigation in our style. As examples, intuitionistic modal logics [32, 30] has a corresponding class of algebras [2] which moreover have a duality available [22], as do multi-modal algebras [21].

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