BI-INTERMEDIATE LOGICS OF TREES AND CO-TREES

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ABSTRACT. A bi-Heyting algebra validates the Gödel-Dummett axiom $(p \to q) \vee (q \to p)$ iff the poset of its prime filters is a disjoint union of co-trees (i.e., order duals of trees). Bi-Heyting algebras of this form are called *bi-Gödel algebras* and form a variety that algebraizes the extension bi-LC of bi-intuitionistic logic axiomatized by the Gödel-Dummett axiom. In this paper we initiate the study of the lattice $\Lambda(\text{bi-LC})$ of extensions of bi-LC. We develop the method of Jankov formulas for bi-Gödel algebras and use them to prove that $\Lambda(\text{bi-LC})$ has the size of the continuum. We also show that bi-LC is not locally tabular and give a criterion of locall tabularity in $\Lambda(\text{bi-LC})$.

1. INTRODUCTION

Bi-intuitionistic logic bi-IPC is the conservative extension of intuitionistic logic IPC obtained by adding a new binary connective $\leftarrow$ to the language, called the *co-implication* (or exclusion, or subtraction), which behaves dually to $\to$. In this way, bi-IPC achieves a symmetry, which IPC lacks, between the connectives $\wedge, \top, \to$ and $\vee, \bot, \leftarrow$, respectively.

The Kripke semantics of bi-IPC [52] provides a transparent interpretation of co-implication: given a Kripke model $\mathfrak{M}$, a point x in $\mathfrak{M}$, and formulas ϕ, ψ, we define

$$\mathfrak{M}, x \models \phi \leftarrow \psi \iff \exists y \leqslant x \ (\mathfrak{M}, y \models \phi \text{ and } \mathfrak{M}, y \not\models \psi).$$

Equipped with this new connective, bi-IPC achieves significantly greater expressivity than IPC. For instance, if the points of a Kripke frame are interpreted as states in time, the language of bi-IPC is expressive enough to talk about the past, something that is not possible in IPC. With this example in mind, Wolter [58] extended Gödel's embedding of IPC into S4 to an embedding of bi-IPC into tense-S4. In particular, he proved a version of the Blok-Esakia Theorem [11, 26] stating that the lattice $\Lambda(\text{bi-IPC})$ *bi-intermediate logics* (i.e., consistent axiomatic* extensions of bi-IPC) is isomorphic to that of consistent normal tense logics containing Grz.t, see also [16, 56].

The greater symmetry of bi-IPC with respect to IPC is reflected in the fact that bi-IPC is algebraized in the sense of [13] by the variety bi-HA of *bi-Heyting algebras* [51], i.e., Heyting algebras whose order duals are also Heyting algebras. As a consequence, the lattice $\Lambda(\text{bi-IPC})$ is dually isomorphic to that of nontrivial varieties of bi-Heyting algebras. The latter, in turn, is amenable to the methods of universal algebra and duality theory because the category of bi-Heyting algebras is dually isomorphic to that of *bi-Esakia spaces* [25], see also [7].

The theory of bi-Heyting algebras was developed in a series of papers by Rauszer and others motivated by the connection with bi-intuitionistic logic (see, e.g., [2, 40, 50, 51, 52, 54]). However, bi-Heyting algebras arise naturally in other fields of research as well such as topos theory [45, 46, 53]. Furthermore, the lattice of open sets of an Alexandrov space is always a bi-Heyting algebra, and so is the lattice of subgraphs of an arbitrary graph (see, e.g., [57]). Similarly, every quantum system can be associated with a complete bi-Heyting algebra [20].

The lattice $\Lambda(\text{IPC})$ of intermediate logics (i.e., consistent extensions of IPC) has been thoroughly investigated (see, e.g., [17]). On the other hand, the lattice $\Lambda(\text{bi-IPC})$ of bi-intermediate logics lacks such an in-depth analysis, but for some recent developments see, e.g., [1, 10, 30, 31, 55]. In this paper we contribute to filling this gap by studying a simpler, yet nontrivial, sublattice of $\Lambda(\text{bi-IPC})$: the lattice of consistent extensions of the *bi-intuitionistic linear calculus* (or the bi-Gödel-Dummett's logic),

$$\text{bi-LC} := \text{bi-IPC} + (p \to q) \vee (q \to p).$$

*From now on we will use *extension* are a synonym of *axiomatic extension*.

Notably, the properties of $\Lambda(\mathsf{bi\text{-}IPC})$ and its extensions diverge significantly from those of its intermediate counterpart, i.e., the *intuitionistic linear calculus* (or the Gödel-Dummett's logic) $\mathsf{LC} := \mathsf{IPC} + (p \to q) \vee (q \to p)$ [21, 29].

The choice of bi-LC as a case study was motivated by some of its properties that make it an interesting logic on its own. In particular, bi-LC is complete in the sense of Kripke semantics with respect to the class of *co-trees* (i.e., order duals of trees). Moreover, we prove that the bi-intuitionistic logic of linearly ordered Kripke frames is a proper extension of bi-LC (Theorem 4.25). This contrasts with the case of intermediate logics, where LC is both the logic of the class of linearly ordered Kripke frames and of co-trees. Because of this, the language of bi-IPC seems more appropriate to study tree-like structures than that of IPC. Furthermore, because of the symmetric nature of bi-intuitionistic logic, our results on extensions of bi-LC can be extended in a straightforward manner to the extensions of the bi-intermediate logic of trees by replacing in what follows every formula φ by its dual $\neg\varphi^\partial$, where φ^∂ is the formula obtained from φ by replacing each occurrence of $\wedge, \top, \to$ by $\vee, \bot, \leftarrow$ respectively, and every algebra of Kripke frame by its order dual.

Also the logic bi-LC admits a form of a classical reductio ad absurdum (Theorem 4.1). Recall that a deductive system $\vdash$ is said to have a *classical inconsistency lemma* if, for every nonnegative integer n, there exists a finite set of formulas $\Psi_n(p_1, \dots, p_n)$, which satisfies the equivalence

$$(1) \qquad \Gamma \cup \Psi_n(\varphi_1, \dots, \varphi_n) \text{ is inconsistent in } \vdash \iff \Gamma \vdash \{\varphi_1, \dots, \varphi_n\},$$

for all sets of formulas $\Gamma \cup \{\alpha_1, \dots, \alpha_n\}$ [49] (see also [15, 44, 43]). As expected, the only intermediate logic having a classical inconsistency lemma is CPC (with $\Phi_n := \{\neg(p_1 \wedge \dots \wedge p_n)\}$). This contrasts with the case of bi-intermediate logics where every member of $\Lambda(\mathsf{bi\text{-}LC})$ has a classical inconsistency lemma witnessed by

$$\Phi_n := \{\sim\neg\sim(p_1 \wedge \dots \wedge p_n)\},$$

where $\neg p$ and $\sim p$ are shorthands for $p \to \bot$ and $\top \leftarrow p$ (see, e.g., [42, Chpt. 4]). Accordingly, logics in $\Lambda(\mathsf{bi\text{-}LC})$ exhibit a certain balance between the classical and intuitionistic behavior of negation connectives.

The main contributions of the paper can be summarized as follows. In order to classify extensions of bi-LC, we develop theories of Jankov, subframe and canonical formulas for them. We then employ Jankov formulas to obtain a characterization of splittings in $\Lambda(\mathsf{bi\text{-}LC})$ and to show that this lattice has the cardinality of the continuum (Theorems 4.11 and 4.17), cf. [8]. This contrasts with the case of $\Lambda(\mathsf{LC})$ which is well known to be a chain of order type $(\omega+1)^\partial$ [17]. Moreover, we show that canonical formulas provide a uniform axiomatization for all the extensions of bi-LC (Theorem 4.7). Lastly, subframe formulas can be used to describe the fine structure of co-trees, by governing the embeddability of finite co-trees in arbitrary co-trees (Lemma 4.23). Using this property of subframe formulas we prove the most challenging result of this paper, the characterization of locally tabular extensions of bi-LC. More precisely, we show that an extension L of bi-LC is locally tabular iff at least one of the finite co-trees in Figure 4 does not embed into any model of L (Theorem 5.1). As a consequence, we obtain that bi-LC is not locally tabular, which contrasts with the well-known fact that LC is locally tabular [33].

2. Preliminaries

In this section, we review the basic concepts and results that we will need throughout this paper. For a more in-depth study of bi-IPC and bi-Heyting algebras, see, e.g., [50, 51, 52, 42, 57], while for universal algebra, see, e.g., [3, 14]. Henceforth, given $n \in \omega$, the notation $i \leqslant n$ will always mean that $i \in \{0, \dots, n\}$.

2.1. **Bi-intuitionistic propositional logic.**

Given a formula φ, we write $\neg\varphi$ and $\sim\varphi$ as a shorthand for $\varphi \to \bot$ and $\top \leftarrow \varphi$. *Bi-intuitionistic logic* bi-IPC is least set of formulas in the language $\wedge, \vee, \to, \leftarrow, \top, \bot$ (built up from a denumerable set *Prop* of variables) that contains IPC and the eight axioms below and is closed under uniform substitutions, modus ponens, and the *double negation rule* (DN for short) "from ϕ infer $\neg\sim\phi$".

1. $p \to (q \vee (p \leftarrow q))$,
2. $(p \leftarrow q) \to \sim(p \to q)$,
3. $((p \leftarrow q) \leftarrow r) \to (p \leftarrow q \vee r)$,
4. $\neg(p \leftarrow q) \to (p \to q)$,
5. $(p \to (q \leftarrow q)) \to \neg p$,
6. $\neg p \to (p \to (q \leftarrow q)$,
7. $((p \to p) \leftarrow q) \to \sim q$,
8. $\sim q \to ((p \to p) \leftarrow q)$,

It turns out that bi-IPC is a conservative extension of IPC. Furthermore, we may identify *classical propositional calculus* CPC with the proper extension of bi-IPC obtained by adding the *law of excluded middle* $p \vee \neg p$. Notably, in CPC the connective $p \leftarrow q$ is term-definable as $p \wedge \neg q$. Consequently, the DN rule becomes superfluous, as it translates to "from ϕ infer ϕ".

A set of formulas L closed under the three inference rules (modus ponens, uniform substitution, and DN) is called a *super-bi-intuitionistic logic* if it contains bi-IPC. Given a formula ϕ and a super-bi-intuitionistic logic L, we say that ϕ is a *theorem* of L, denoted by $L \vdash \phi$, if $\phi \in L$. Otherwise, write $L \nvdash \phi$. We call L *consistent* if $L \nvdash \bot$, and *inconsistent* otherwise. Given another super-bi-intuitionistic logic L', we say that L' is an *extension* of L if $L \subseteq L'$. Consistent extensions of bi-IPC are called *bi-intermediate logics*, and it can be shown that L is a bi-intermediate logic iff bi-IPC $\subseteq L \subseteq$ CPC. Finally, given a set of formulas Σ, we denote the least (with respect to inclusion) bi-intuitionistic logic containing $L \cup \Sigma$ by $L + \Sigma$. If Σ is a singleton $\{\phi\}$, we simply write $L + \phi$. Given another formula ψ, we say that ϕ and ψ are *L-equivalent* if $L \vdash \phi \leftrightarrow \psi$.

2.2. **Varieties of algebras.** We denote by $\mathbb{H}, \mathbb{S}, \mathbb{P}, \mathbb{I}$, and $\mathbb{P}_U$ the class operators of closure under homomorphic images, subalgebras, isomorphic copies, direct products, and ultraproducts. A variety V is a class of (similar) algebras closed under homomorphic images, subalgebras, and (direct) products. By Birkhoff's Theorem, varieties coincide with classes of algebras that can be axiomatized by sets of equations (see, e.g., [14, Thm. II.11.9]). The smallest variety $\mathbb{V}(\mathsf{K})$ containing a class K of algebras is called the *variety generated by* K and coincides with $\mathbb{HSP}(\mathsf{K})$. If $\mathsf{K} = \{\mathfrak{A}\}$, we simply write $\mathbb{V}(\mathfrak{A})$.

Given an algebra $\mathfrak{A}$, we denote by $Con(\mathfrak{A})$ its congruence lattice. An algebra $\mathfrak{A}$ is said to be *subdirectly irreducible*, or SI for short, (resp. *simple*) if $Con(\mathfrak{A})$ has a second least element (resp. has exactly two elements: the identity relation Id_A and the total relation A^2). Consequently, every simple algebra is subdirectly irreducible. Given a class K of algebras, we denote by K_F, K_{SI}, and K_{FSI} the classes of finite members of K, SI members of K, and finite SI members of K, respectively. In view of the Subdirect Decomposition Theorem, if K is a variety, then $\mathsf{K} = \mathbb{V}(\mathsf{K}_{SI})$ (see, e.g., [14, Thm. II.8.6]).

Definition 2.1. A variety V is said to

(i) be *semi-simple* if its SI members are simple;
(ii) be *locally finite* if its finitely generated members are finite;
(iii) have the *finite model property* (FMP for short) if it is generated by its finite members;
(iv) be *congruence distributive* if every member of V has a distributive lattice of congruences;
(v) have *equationally definable principal congruences* (EDPC for short) if there exists a conjunction $\Phi(x, y, z, v)$ of finitely many equations such that for every $\mathfrak{A} \in \mathsf{V}$ and all $a, b, c, d \in A$,
$$c\theta_{a,b}d \iff \mathfrak{A} \models \Phi(a, b, c, d),$$
where $\theta_{a,b}$ is the least congruence of $\mathfrak{A}$ that identifies a and b;
(vi) be a *discriminator variety* if there exists a *discriminator term* $t(x, y, z)$ for V, i.e., a ternary term such that for every $\mathfrak{A} \in \mathsf{V}_{SI}$ and all $a, b, c \in A$, we have
$$t^{\mathfrak{A}}(a, b, c) = \begin{cases} c & \text{if } a = b, \\ a & \text{if } a \neq b. \end{cases}$$

The next result collects some of the relations between these properties.

Proposition 2.2. *If V is a variety and K a class of similar algebras, then the following conditions hold:*

(i) *If V is locally finite, then its subvarieties have the FMP;*
(ii) *V has the FMP iff $\mathsf{V} = \mathbb{V}(\mathsf{V}_{FSI})$;*
(iii) *If V has EDPC, then V is congruence distributive and $\mathbb{HS}(\mathsf{K}) = \mathbb{SH}(\mathsf{K})$ for all $\mathsf{K} \subseteq \mathsf{V}$;*

(iv) Jónsson's Lemma: *if* $\mathbb{V}(\mathsf{K})$ *is congruence distributive, then* $\mathbb{V}(\mathsf{K})_{SI} \subseteq \mathbb{HSP}_{\mathrm{U}}(\mathsf{K})$;
(v) *If* V *is discriminator, then it is semi-simple and it has EDPC.*

Proof. Condition (i) holds because every variety is generated by its finitely generated members (see, e.g., [3, Thm. 4.4]), while condition (ii) is exactly the definition of the FMP. The first part of condition (iii) was established in [40] and the second in [19]. For condition (iv), see, e.g., [14, Thm. VI.6.8]. Lastly, for the first part of condition (v) see, e.g., [14, Lem. IV.9.2(b)] and for the second [12, Exa. 6 p. 200]. □

The following result provides a useful description of locally finite varieties of finite type (see, e.g., [4]).

Theorem 2.3. *If* V *is a variety of a finite type, then the following conditions are equivalent:*
(i) V *is locally finite;*
(ii) $\forall n \in \omega, \exists m(n) \in \omega, \forall \mathfrak{A} \in \mathsf{V}\ (\mathfrak{A} \text{ is } n\text{-generated} \implies |A| \leqslant m(n))$;
(iii) $\forall n \in \omega, \exists m(n) \in \omega, \forall \mathfrak{A} \in \mathsf{V}_{SI}\ (\mathfrak{A} \text{ is } n\text{-generated} \implies |A| \leqslant m(n))$.

2.3. **Bi-Heyting algebras.** Given a subset U of a poset $\mathfrak{F}$, let $max(U)$ be the set the *maximal* elements of U viewed as a subposet of $\mathfrak{F}$, and if U has a *maximum* (i.e., a greatest element), we denote it by $Max(U)$. Similarly, we define $min(U)$ and $Min(U)$. We denote the *upset generated* by U by

$$\uparrow U := \{x \in \mathfrak{F} : \exists u \in U\ (u \leqslant x)\},$$

and if $U = \uparrow U$, then U is called an *upset*. If $U = \{u\}$, we simply write $\uparrow u$ and call it a *principal upset*. We define the *downsets* of $\mathfrak{F}$ in a similar way. A set that is both an upset and a downset is an *updownset*. We denote the set of upsets of $\mathfrak{F}$ by $Up(\mathfrak{F})$, of downsets by $Do(\mathfrak{F})$, and of updownsets by $UpDo(\mathfrak{F})$. Given two distinct points $x, y \in \mathfrak{F}$, x is an *immediate predecessor* of y, denoted by $x \prec y$, if $x \leqslant y$ and no point of $\mathfrak{F}$ lies between them (i.e., if $z \in \mathfrak{F}$ is such that $x \leqslant z \leqslant y$, then either $x = z$ or $y = z$). If this is the case, we call y an *immediate successor* of x.

Definition 2.4. A *bi-Heyting algebra* is a Heyting algebra $\mathfrak{A}$ whose order-dual is also a Heyting algebra. Equivalently, $\mathfrak{A}$ is both a Heyting and a co-Heyting algebra, i.e., $\mathfrak{A}$ is a bounded distributive lattice such that for every $a, b \in A$, there are elements $a \to b, a \leftarrow b \in A$ satisfying

$$\big(c \leqslant a \to b \iff a \wedge c \leqslant b\big) \text{ and } \big(a \leftarrow b \leqslant c \iff a \leqslant b \vee c\big),$$

for all $c \in A$. In this case, we use the abbreviations $\neg a := a \to 0$ and $\sim a := 1 \leftarrow a$.

It is well known that the class $\mathsf{bi\text{-}HA}$ of Heyting algebras is a variety. The following properties of bi-Heyting algebras will be useful throughout.

Proposition 2.5. *If* $\mathfrak{A} \in \mathsf{bi\text{-}HA}$ *and* $a, b, c \in \mathfrak{A}$, *then:*

1. $a \to b = \bigvee\{d \in A : a \wedge d \leqslant b\}$,
2. $a \to b = 1 \iff a \leqslant b$,
3. $\neg a = 1 \iff a = 0$,
4. $a \wedge \neg a = 0$,
5. $a \leftarrow b = \bigwedge\{d \in A : a \leqslant d \vee b\}$,
6. $a \leftarrow b = 0 \iff a \leqslant b$,
7. $\sim a = 0 \iff a = 1$,
8. $a \vee \sim a = 1$.

Example 2.6. Here we present some standard examples of bi-Heyting algebras.
(i) Every finite Heyting algebra $\mathfrak{A}$ can be viewed as a bi-Heyting algebra, since $a \leftarrow b = \bigwedge\{d \in A : a \leqslant d \vee b\}$ is then a meet of finitely many elements, and thus the operation $\leftarrow$ is well-defined on $\mathfrak{A}$;
(ii) Every Boolean algebra $\mathfrak{A}$ can be viewed as a bi-Heyting algebra, where the co-implication is given by $a \leftarrow b = a \wedge \neg b$;
(iii) Given a poset $\mathfrak{F} = (W, \leqslant)$, then

$$(Up(\mathfrak{F}), \cup, \cap, \to, \leftarrow, \varnothing, W)$$

is a bi-Heyting algebra, where the implications are defined by

$$U \to V := W \smallsetminus \downarrow(U \smallsetminus V) \text{ and } U \leftarrow V := \uparrow(U \smallsetminus V).$$

Recall that a *valuation* on a bi-Heyting algebra $\mathfrak{A}$ is a map $v\colon Prop \to A$, where $Prop$ is the denumerable set of propositional variables of our language, and that any such valuation can be extended uniquely to a homomorphism from the term algebra to $\mathfrak{A}$. We say that a formula ϕ is *valid* on $\mathfrak{A}$, denoted by $\mathfrak{A} \models \phi$, if $v(\phi) = 1$ for all valuations v on $\mathfrak{A}$. On the other hand, if $v(\phi) \neq 1$ for some valuation v on $\mathfrak{A}$, we say that $\mathfrak{A}$ *refutes* ϕ via v, and write $\mathfrak{A} \not\models \phi$. If K is a class of bi-Heyting algebras such that $\mathfrak{A} \models \phi$ for all $\mathfrak{A} \in \mathsf{K}$, we write $\mathsf{K} \models \phi$. Otherwise, write $\mathsf{K} \not\models \phi$.

Using the well-known Lindenbaum-Tarski construction (see, e.g., [17, 28]) we obtain the following equivalence: $\mathsf{bi\text{-}IPC} \vdash \phi$ iff $\mathsf{bi\text{-}HA} \models \phi$. This phenomenon, known as the algebraic completeness of bi-IPC, can be extended to all other super-bi-intuitionistic logics. Let L be such a logic, and denote the *variety of* L by $\mathsf{V}_L := \{\mathfrak{A} \in \mathsf{bi\text{-}HA} \colon \mathfrak{A} \models L\}$. On the other hand, given a subvariety $\mathsf{V} \subseteq \mathsf{bi\text{-}HA}$, we denote its *logic* by $L_{\mathsf{V}} := Log(\mathsf{V}) = \{\phi\colon \mathsf{V} \models \phi\}$. Again using the standard Lindenbaum-Tarski construction, it can be shown that L is *sound* and *complete* with respect to V_L, i.e., for all formulas ϕ, we have $L \vdash \phi$ iff $\mathsf{V}_L \models \phi$. It follows that this correspondence between extensions of bi-IPC and subvarieties of bi-Heyting algebras is one-to-one, and therefore the following theorem can now be easily proved.

Theorem 2.7. *Let L be a super-bi-intuitionistic logic. Then the lattice of extensions of L is dually isomorphic to the lattice of subvarieties of V_L. Equivalently, if V is a variety of bi-Heyting algebras, then the lattice of subvarieties of V is dually isomorphic to the lattice of extensions of L_{V}.*

2.4. **Bi-Esakia spaces.** Given an ordered topological space $\mathcal{X}$, we denote its set of open sets by $Op(\mathcal{X})$, closed sets by $Cl(\mathcal{X})$, clopen sets by $Cp(\mathcal{X})$, clopen upsets by $CpUp(\mathcal{X})$, and closed updownsets by $ClUpDo(\mathcal{X})$.

Definition 2.8. Let $\mathfrak{F} = (X, \leqslant)$ and $\mathfrak{G} = (W, \leqslant)$ be posets. A map $f\colon X \to W$ is called a *bi-p-morphism*, denoted $f\colon \mathfrak{F} \to \mathfrak{G}$, if it satisfies the following conditions:

- *Order preserving:* $\forall x, y \in X \big(x \leqslant y \implies f(x) \leqslant f(y)\big)$;
- *Forth:* $\forall x \in X, \forall u \in W \big(f(x) \leqslant u \implies \exists y \in {\uparrow}x\, (f(y) = u)\big)$;
- *Back:* $\forall x \in X, \forall v \in W \big(v \leqslant f(x) \implies \exists z \in {\downarrow}x\, (f(z) = v)\big)$.

If f is moreover surjective, then $\mathfrak{G}$ is a *bi-p-morphic image* of $\mathfrak{F}$, denoted by $f\colon \mathfrak{F} \twoheadrightarrow \mathfrak{G}$.

Proposition 2.9. *If $f\colon \mathfrak{F} \to \mathfrak{G}$ is a bi-p-morphism, then:*

(i) *$f[{\uparrow}w] = {\uparrow}f(w)$ and $f[{\downarrow}w] = {\downarrow}f(w)$, for all $w \in \mathfrak{F}$;*
(ii) *$f[max(\mathfrak{F})] \subseteq max(\mathfrak{G})$ and $f[min(\mathfrak{F})] \subseteq min(\mathfrak{G})$. If f is surjective, then these are equalities;*
(iii) *If both $Max(\mathfrak{F})$ and $Max(\mathfrak{G})$ exist, then $f\big(Max(\mathfrak{F})\big) = Max(\mathfrak{G})$ and f is necessarily surjective.*

Proof. Condition (i) follows immediately from the definition of f, while the other two are direct consequences of (i). □

Definition 2.10. A triple $\mathcal{X} = (X, \tau, \leqslant)$ is a *bi-Esakia space* if it is both an Esakia and a co-Esakia space, i.e., (X, τ) is a topological space equipped with a partial order $\leqslant$ such that:

- (X,τ) is compact;
- $\forall U \in Cp(\mathcal{X})\, \big({\uparrow}U, {\downarrow}U \in Cp(\mathcal{X})\big)$;
- *Priestley separation axiom* (PSA for short):

$$\forall x, y \in X \big(x \nleqslant y \implies \exists U \in CpUp(\mathcal{X})\, (x \in U \text{ and } y \notin U)\big).$$

A map $f\colon \mathcal{X} \to \mathcal{Y}$ is a *bi-Esakia morphism* if it is a continuous bi-p-morphism between bi-Esakia spaces. If f is moreover bijective, then $\mathcal{X}$ and $\mathcal{Y}$ are said to be *isomorphic*, denoted by $\mathcal{X} \cong \mathcal{Y}$. Finally, we define the following operations on $CpUp(\mathcal{X})$:

$$U \to V := X \smallsetminus {\downarrow}(U \smallsetminus V) = \big\{x \in X\colon \forall y \in {\uparrow}x\, \big(y \in U \implies y \in V\big)\big\}$$
$$U \leftarrow V := {\uparrow}(U \smallsetminus V) = \big\{x \in X\colon \exists y \in U \smallsetminus V\, \big(y \leqslant x\big)\big\}$$
$$\neg U := U \to \varnothing = X \smallsetminus {\downarrow}U$$
$$\sim U := X \leftarrow U = {\uparrow}(X \smallsetminus U).$$

Example 2.11. Every finite poset can be viewed as a bi-Esakia space, when equipped with the discrete topology. In fact, since (bi-)Esakia spaces are Hausdorff, this is the only way to view a finite poset as a bi-Esakia space. Furthermore, since maps between spaces equipped with the discrete topology are always continuous, it follows that every bi-p-morphism between finite posets is a bi-Esakia morphism.

The celebrated Esakia duality restricts to a duality between the category of bi-Heyting algebras and bi-Heyting morphisms, and that of bi-Esakia spaces and bi-Esakia morphisms [25] (for a proof, see [42]). Here we just recall the main constructions establishing this duality. Given a bi-Heyting algebra $\mathfrak{A}$, we denote its *bi-Esakia dual* by $\mathfrak{A}_* := (A_*, \tau, \subseteq)$, where A_* is the set of prime filters of $\mathfrak{A}$ and τ is the topology generated by the subbasis

$$\{\varphi(a) \colon a \in A\} \cup \{A_* \smallsetminus \varphi(a) \colon a \in A\},$$

where $\varphi(a) := \{F \in A_* \colon a \in F\}$. Moreover, it can be shown that $CpUp(\mathfrak{A}_*) = \{\varphi(a) \colon a \in A\}$. Furthermore, if $f \colon \mathfrak{A} \to \mathfrak{B}$ is a bi-Heyting morphism, then its dual is the restricted inverse image map $f_* := f^{-1} \colon \mathfrak{B}_* \to \mathfrak{A}_*$.

Conversely, given a bi-Esakia space $\mathcal{X}$ we denote its *bi-Heyting* (or *algebraic*) *dual* by $\mathcal{X}^* := (CpUp(\mathcal{X}), \cup, \cap, \to, \leftarrow, \emptyset, X)$, and if $f \colon \mathcal{X} \to \mathcal{Y}$ is a bi-Esakia morphism, then its dual is the restricted inverse image map $f^* := f^{-1} \colon \mathcal{Y}^* \to \mathcal{X}^*$. We note that $\mathfrak{A}$ and $(\mathfrak{A}_*)^*$ are isomorphic as bi-Heyting algebras, while $\mathcal{X}$ and $(\mathcal{X}^*)_*$ are isomorphic as bi-Esakia spaces.

The following result collects some useful properties of bi-Esakia spaces.

Proposition 2.12. *The following conditions hold for a bi-Esakia space $\mathcal{X}$:*
- (i) *$\mathcal{X}$ is Hausdorff;*
- (ii) *If $Z \in Cl(\mathcal{X})$, then ${\downarrow}Z, {\uparrow}Z \in Cl(\mathcal{X})$. Consequently, ${\downarrow}x, {\uparrow}x \in Cl(\mathcal{X})$ for all $x \in X$;*
- (iii) *If $x \in X$, then there are $y \in min(\mathcal{X})$ and $z \in max(\mathcal{X})$ satisfying $y \leqslant x \leqslant z$;*
- (iv) *$\neg{\sim}U = \{x \in X \colon {\downarrow}{\uparrow}x \subseteq U\}$, for all $U \in CpUp(\mathcal{X})$.*

Proof. The results stated in the first three conditions are either well-known results for Esakia spaces, or their order-dual versions (for co-Esakia spaces). For a proof of the former, see [27]. We now prove (iv). By spelling out the definition of $\neg{\sim}U$, we have

$$\neg{\sim}U = X \smallsetminus {\downarrow}{\uparrow}(X \smallsetminus U) = \big\{x \in X \colon \forall y \in X \ \big(\exists z \in X \smallsetminus U \ (z \leqslant y) \implies x \nleqslant y\big)\big\}.$$

Suppose that $x \in \neg{\sim}U$ and let $u \in {\downarrow}{\uparrow}x$, so there exists $v \in {\uparrow}x$ such that $u \leqslant v$. By the equality above, $x \leqslant v$ entails that for all $z \in X \smallsetminus U$, we have $z \nleqslant v$. Hence u must be in U and we conclude ${\downarrow}{\uparrow}x \subseteq U$. Thus, $\neg{\sim}U \subseteq \{x \in X \colon {\downarrow}{\uparrow}x \subseteq U\}$.

To prove the right to left inclusion, suppose ${\downarrow}{\uparrow}x \subseteq U$, for some $x \in X$. Let $y \in X$ be such that there exists a $z \in (X \smallsetminus U) \cap {\downarrow}y$. If $x \leqslant y$, then we would have $z \in {\downarrow}{\uparrow}x \subseteq U$, a contradiction. Thus $x \nleqslant y$, and we conclude $x \in \neg{\sim}U$, as desired. □

Next we define the three standard methods of generating bi-Esakia spaces. Let $\mathcal{X} = (X, \tau, R)$, $\mathcal{Y} = (Y, \pi, S), \mathcal{X}_1 = (X_1, \tau_1, R_1), \dots, \mathcal{X}_n = (X_n, \tau_n, R_n)$ be bi-Esakia spaces. We say that:
- (i) $\mathcal{Y}$ is a *bi-generated subframe* of $\mathcal{X}$ if $Y \in ClUpDo(\mathcal{X})$, π is the subspace topology, and $S = Y^2 \cap R$;
- (ii) $\mathcal{Y}$ is a *bi-Esakia morphic image* of $\mathcal{X}$, denoted by $\mathcal{X} \twoheadrightarrow \mathcal{Y}$, if there exists a surjective bi-Esakia morphism from $\mathcal{X}$ to $\mathcal{Y}$;
- (iii) $\mathcal{X} = \biguplus_{i=1}^n \mathcal{X}_i$ is the *disjoint union* of the collection $\{\mathcal{X}_1, \dots, \mathcal{X}_n\}$ if (X, R) is the disjoint union $\biguplus_{i=1}^n (X_i, R_i)$ of the various posets and (X, τ) is the topological sum of the (X_i, τ_i).

As is the case with the analogous notions for Esakia spaces, the above definitions can be translated (using the bi-Esakia duality) into terms of bi-Heyting algebras (for a proof, see [42]).

Proposition 2.13. *Let $\{\mathfrak{A}, \mathfrak{B}\} \cup \{\mathfrak{A}_1, \dots, \mathfrak{A}_n\}$ and $\{\mathcal{X}_1, \dots, \mathcal{X}_n\}$ be finite sets of bi-Heyting algebras and bi-Esakia spaces, respectively. The following conditions hold:*
- (i) *$\mathfrak{B}$ is a homomorphic image of $\mathfrak{A}$ iff $\mathfrak{B}_*$ is (isomorphic to) a bi-generated subframe of $\mathfrak{A}_*$;*
- (ii) *$\mathfrak{B}$ is (isomorphic to) a subalgebra of $\mathfrak{A}$ iff $\mathfrak{B}_*$ is a bi-Esakia morphic image of $\mathfrak{A}_*$;*
- (iii) *$(\prod_{i=1}^n \mathfrak{A}_i)_* \cong \biguplus_{i=1}^n \mathfrak{A}_{i*}$ and $(\biguplus_{i=1}^n \mathcal{X}_i)^* \cong \prod_{i=1}^n \mathcal{X}_i^*$.*

Let $\mathcal{X}$ be a bi-Esakia space. A map $V\colon Prop \to CpUp(\mathcal{X})$ is called a *valuation* on $\mathcal{X}$, and the pair $\mathfrak{M} := (\mathcal{X}, V)$ a *bi-Esakia model* (on $\mathcal{X}$). Moreover, we define the *validity* of a formula ϕ in $\mathcal{X}$ by $\mathcal{X} \models \phi$ iff $\mathcal{X}^* \models \phi$. In other words, $V(\phi) = X$, for all valuations V on $\mathcal{X}$. Otherwise, write $\mathcal{X} \not\models \phi$. Since the validity of a formula is preserved under taking homomorphic images, subalgebras, and direct products of bi-Heyting algebras, it follows from the previous proposition that the validity of a formula is preserved under taking bi-generated subframes, bi-Esakia morphic images, and finite disjoint unions of bi-Esakia spaces.

Finally, we review the Coloring Theorem, a characterization of the finitely generated bi-Heyting algebras. To this end, we need to define the notions of bi-bisimulations equivalences and colorings on bi-Esakia spaces.

Definition 2.14. Let $\mathcal{X} = (X, \tau, \leqslant)$ be a bi-Esakia space and E an equivalence relation on X. We say that E is a *bi-bisimulation equivalence* on $\mathcal{X}$ if it satisfies the following conditions:

- *Forth:* $\forall w, w', v' \in X\ (wEw' \text{ and } w' \leqslant v' \implies \exists v \in [\![v']\!]_E\ (w \leqslant v))$;
- *Back:* $\forall w, w', u' \in X\ (wEw' \text{ and } u' \leqslant w' \implies \exists u \in [\![u']\!]_E\ (u \leqslant w))$;
- *Refined:* Any two non-E-equivalent elements of X are *separated* by an *E-saturated* clopen upset, that is, for every $w, v \in X$, if $\neg(wEv)$ then there exists $U \in CpUp(\mathcal{X})$ such that $E[U] = \{x \in X \colon \exists u \in U\ (uEx)\} = U$, and exactly one of w and v is contained in U.

We call a bi-bisimulation equivalence E on $\mathcal{X}$ *trivial* if $E = X^2$, and *proper* otherwise.

Let $\mathfrak{M} = (\mathcal{X}, V)$ be a bi-Esakia model and $p_1, \dots, p_n \in Prop$ a finite number of fixed distinct propositional variables. With every point $w \in \mathfrak{M}$, we associate the sequence $col(w) := i_1 \dots i_n$ defined by

$$i_k := \begin{cases} 1 & \text{if } w \models p_k, \\ 0 & \text{if } w \not\models p_k, \end{cases}$$

for $k \in \{1, \dots, n\}$. We call $col(w)$ the *color* of w (relative to $p_1, \dots, p_n$), and call a valuation $V\colon \{p_1, \dots, p_n\} \to CpUp(\mathcal{X})$ a *coloring* of $\mathfrak{M}$.

Now, note that thinking of a bi-Heyting algebra $\mathfrak{A}$ endowed with some fixed elements $a_1, \dots, a_n$ is the same as equipping $\mathfrak{A}$ with a valuation $v\colon \{p_1, \dots, p_n\} \to A$ defined by $v(p_i) := a_i$, for $i \leqslant n$. The dual of the tuple $(\mathfrak{A}, v)$ is the bi-Esakia model $\mathfrak{M} = (\mathfrak{A}_*, V)$, where V satisfies $V(p_i) = \varphi(a_i) = \{x \in A_* \colon g_i \in x\}$, and this valuation clearly defines a coloring of $\mathfrak{M}$.

The following theorem can be proven using the same argument as for the analogous result for Heyting algebras (see, e.g., [9, Thm. 3.1.5]). Notice the use of the notation $\mathfrak{A} = \langle a_1, \dots, a_n \rangle$ for "$\mathfrak{A}$ is generated (as a bi-Heyting algebra) by $\{a_1, \dots, a_n\}$".

Theorem 2.15 (Coloring Theorem)**.** *Let $\mathfrak{A}$ be a bi-Heyting algebra, $a_1, \dots, a_n \in A$ a finite number of fixed elements, and $(\mathcal{X}, V)$ the corresponding bi-Esakia model. Then $\mathfrak{A} = \langle a_1, \dots, a_n \rangle$ iff every proper bi-bisimulation equivalence E on $\mathcal{X}$ identifies points of different colors.*

3. The bi-intuitionistic linear calculus

The *bi-intuitionistic linear calculus* (or *bi-Gödel-Dummett's logic*) is the bi-intermediate logic

$$\mathsf{bi\text{-}LC} := \mathsf{bi\text{-}IPC} + (p \to q) \vee (q \to p).$$

This terminology hints that bi-LC is a bi-superintuionistic analogue of the linear calculus LC axiomatized by the same axiom over IPC. In view of Theorem 2.7, bi-LC is algebraized by the variety

$$\mathsf{bi\text{-}GA} := L_{\mathsf{bi\text{-}LC}} = \{\mathfrak{A} \in \mathsf{bi\text{-}HA} \colon \mathfrak{A} \models (p \to q) \vee (q \to p)\},$$

whose elements will be called *bi-Gödel algebras*. Furthermore, there exists a dual isomorphism between the lattice $\Lambda(\mathsf{bi\text{-}LC})$ of extensions of bi-LC and that of subvarieties of bi-GA.

In this section, we will restrict bi-Esakia duality to a duality between the bi-Gödel algebras and bi-Esakia co-forests. This allows us to obtain a transparent description of the SI members of bi-GA and, therefore, to conclude that bi-GA is a discriminator variety.

To this end, recall that a *co-tree* is a poset with a greatest element (called the *co-root*) and whose principal upsets are chains, and that a disjoint union of co-trees is called a *co-forest*. Moreover, a *bi-Esakia co-forest* (respectively, *bi-Esakia co-tree*) is a bi-Esakia space whose underlying poset is a co-forest (respectively, co-tree).

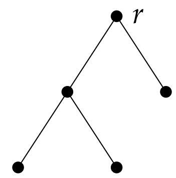

FIGURE 1. A co-tree.

Theorem 3.1. *If $\mathfrak{A} \in \mathsf{bi\text{-}HA}$, then $\mathfrak{A}$ is a bi-Gödel algebra iff $\mathfrak{A}_*$ is a bi-Esakia co-forest.*

Proof. Observe that a bi-Heyting algebra $\mathfrak{A}$ validates the axiom $(p \to q) \vee (q \to p)$ iff so does its Heyting algebra reduct $\mathfrak{A}^-$. Since the latter condition is equivalent to the demand that $\mathfrak{A}^-_*$ is a co-forest [32, Thm. 2.4] and $\mathfrak{A}_* = \mathfrak{A}^-_*$, we are done. □

Before we characterize the SI bi-Gödel algebras, let us recall the standard characterization of simple and SI bi-Heyting algebras by means of their bi-Esakia duals, as well as prove that the existence of a greatest prime filter of $\mathfrak{A} \in \mathsf{bi\text{-}HA}$ is a sufficient condition for $\mathfrak{A}$ to be simple.

Theorem 3.2. *If $\mathfrak{A} \in \mathsf{bi\text{-}HA}$, then the following conditions are equivalent:*
- (i) *$\mathfrak{A}$ is SI;*
- (ii) *$\big(Con(\mathfrak{A}) \smallsetminus \{Id_A\}, \subseteq\big)$ has a least element;*
- (iii) *$\big(ClUpDo(\mathfrak{A}_*) \smallsetminus \{A_*\}, \subseteq\big)$ has a greatest element.*

Proof. Using Proposition 2.13, it can be shown that the lattice of congruences on $\mathfrak{A}$ is dually isomorphic to that of closed updownsets of $\mathfrak{A}_*$, yielding the result. □

Corollary 3.3. *If $\mathfrak{A} \in \mathsf{bi\text{-}HA}$, then the following conditions are equivalent:*
- (i) *$\mathfrak{A}$ is simple;*
- (ii) *$Con(\mathfrak{A}) = \{Id_A, A^2\}$ and $Id_A \neq A^2$;*
- (iii) *$ClUpDo(\mathfrak{A}_*) = \{\emptyset, A_*\}$ and $\emptyset \neq A_*$.*

Proof. This result follows immediately from the definition of a simple algebra and the aforementioned dual isomorphism between the lattice of congruences on $\mathfrak{A}$ and that of closed updownsets of $\mathfrak{A}_*$. □

Proposition 3.4. *If $\mathfrak{A} \in \mathsf{bi\text{-}HA}$ and $\mathfrak{A}_*$ has a greatest element, then $\mathfrak{A}$ is simple.*

Proof. Firstly, let us note that if $\mathfrak{A}_*$ has a greatest element x, then $A_* \neq \emptyset$ and every nonempty upset of $\mathfrak{A}_*$ contains x. Since we also have $\downarrow x = A_*$, it now follows $UpDo(\mathfrak{A}_*) = \{\emptyset, A_*\}$, hence also $ClUpDo(\mathfrak{A}_*) = \{\emptyset, A_*\}$. Therefore, by Corollary 3.3 we have that $\mathfrak{A}$ is simple. □

Remark 3.5. The converse of Proposition 3.4 fails in general because $Up(\mathfrak{F})$ is a simple bi-Heyting algebra for every nonempty finite connected poset $\mathfrak{F}$.

The next theorem lists equivalent conditions for a bi-Gödel algebra to be SI. In particular, it provides us with a transparent characterization of these algebras: they are exactly the bi-Heyting algebras whose duals are bi-Esakia co-trees. Recall that in a bounded distributive lattice $\mathfrak{L}$, the element 0 is said to be *$\wedge$-irreducible* if for all $a, b \in L$, $a \wedge b = 0$ implies $a = 0$ or $b = 0$.

Theorem 3.6. *If $\mathfrak{A} \in \mathsf{bi\text{-}GA}$, then the following conditions are equivalent:*
- (i) *$\mathfrak{A}$ is SI;*
- (ii) *$\mathfrak{A}_*$ is a bi-Esakia co-tree;*
- (iii) *$\mathfrak{A}_*$ has a greatest element;*
- (iv) *$\mathfrak{A}$ is simple;*
- (v) *$\mathfrak{A}$ is nontrivial and $\neg a = 0$, for all $a \in A \smallsetminus \{0\}$;*
- (vi) *$\mathfrak{A}$ is nontrivial and 0 is $\wedge$-irreducible.*

Proof. $\boxed{\text{(i)} \Longrightarrow \text{(ii)}}$ Let $\mathfrak{A}$ be an SI bi-Gödel algebra. Since $\mathfrak{A}_*$ is a bi-Esakia co-forest by Theorem 3.1, we can write $A_* = \biguplus\{\downarrow x \colon x \in max(\mathfrak{A}_*)\}$ as a disjoint union of co-trees. As $\mathfrak{A}$ is SI, Theorem 3.2 entails that $ClUpDo(\mathfrak{A}_*) \smallsetminus \{A_*\}$ has a greatest element U. Since U is a proper downset, there must be a maximal point w of $\mathfrak{A}_*$ not in U. Note that, since w is maximal and principal upsets are chains, it follows that $\downarrow w$ is an upset of $\mathfrak{A}_*$. By definition, it is also a downset. Moreover, we know by Proposition 2.12 that $\downarrow w$ is closed, hence it is a closed updownset not contained in U. By the definition of U, it follows that $\downarrow w = A_*$, so $\mathfrak{A}_*$ is indeed a bi-Esakia co-tree.

$\boxed{\text{(ii)} \Longrightarrow \text{(iii)} \Longrightarrow \text{(iv)} \Longrightarrow \text{(i)}}$ The first implication follows directly from the definition of a co-tree, the second is an immediate consequence of Proposition 3.4, while the third follows from the fact that simple algebras are SI.

$\boxed{\text{(iii)} \Longrightarrow \text{(v)}}$ Suppose that $\mathfrak{A}_*$ has a greatest element, i.e., that $\mathfrak{A}$ has a greatest prime filter r. It is an immediate consequence of the Prime Filter Theorem and the definition of r that $a \in A \smallsetminus \{0\}$ iff a is contained in some prime filter iff $a \in r$. If $a \in A \smallsetminus \{0\}$, then $a \wedge \neg a = 0 \notin r$ entails $\neg a \notin r$, i.e., $\neg a = 0$ by the previous remark.

$\boxed{\text{(v)} \Longrightarrow \text{(vi)}}$ Suppose that $\mathfrak{A}$ satisfies condition (v), and $a \wedge c = 0$, for some $a, c \in A$. If $a \neq 0$, then $\neg a = 0$, so $c \in \{b \in A \colon b \wedge a \leqslant 0\}$ now entails $c = 0 = \neg a = \bigvee\{b \in A \colon b \wedge a \leqslant 0\}$. Therefore, 0 is $\wedge$-irreducible.

$\boxed{\text{(vi)} \Longrightarrow \text{(iii)}}$ Suppose that $\mathfrak{A}$ is a nontrivial bi-Gödel algebra, whose 0 is $\wedge$-irreducible. It is easy to see that $A \smallsetminus \{0\}$ is not only a prime filter, but is in fact the greatest such filter. □

Corollary 3.7. bi-GA *is a discriminator variety. Consequently, it is semi-simple and has EDPC.*

Proof. We will prove that

$$t(x, y, z) := \big((x + y) \wedge z\big) \vee \big(\neg(x + y) \wedge x\big)$$

is a discriminator term for bi-GA, where $x + y$ is defined by

$$x + y := \neg\big((x \leftarrow y) \vee (y \leftarrow x)\big).$$

Let $\mathfrak{A} \in \text{bi-GA}_{SI}$ and $a, b \in A$. If $a = b$, then $a \leftarrow b = 0 = b \leftarrow a$, hence $a + b = \neg 0 = 1$. On the other hand, if $a \neq b$, we can suppose without loss of generality that $a \nleqslant b$. By Proposition 2.5, this is equivalent to $a \leftarrow b \neq 0$, and therefore $0 < \big((a \leftarrow b) \vee (b \leftarrow a)\big)$. As $\mathfrak{A}$ is SI, it follows from Theorem 3.6 that $a + b = \neg\big((a \leftarrow b) \vee (b \leftarrow a)\big) = 0$. This discussion yields

$$a + b = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{if } a \neq b, \end{cases}$$

and it is now an easy check to see that the term t is a discriminator term for $\mathfrak{A}$. Hence, bi-GA is a discriminator variety. The second part of the statement follows from Proposition 2.2(v). □

Remark 3.8. In [57] it is shown that a variety V of bi-Heyting algebras is discriminator iff there exists some $n \in \omega$ such that

$$\mathsf{V} \models (\neg\sim)^{n+1} x \approx (\neg\sim)^n x.$$

It is easy to see that in the case of bi-GA this holds for $n = 1$.

Corollary 3.9. *If $\mathfrak{A}$ is a subalgebra of an SI bi-Gödel algebra, or is an ultraproduct of a family of SI bi-Gödel algebras, then $\mathfrak{A}$ is also SI.*

Proof. It is well known that in every discriminator variety V, V_{SI} forms a universal class (see, e.g., [14, Thm. IX.9.4.(c)]). Since universal sentences are preserved under taking subalgebras and ultraproducts, the desired result now follows from (ii) of the previous corollary. To see this, note that by Proposition 3.6, $\mathfrak{A}$ is SI iff $\mathfrak{A}$ is nontrivial and 0 is $\wedge$-irreducible iff $\mathfrak{A}$ satisfies the following universal sentence: $0 \neq 1$ and $\forall x, y \; \big(x \wedge y = 0 \implies (x = 0 \text{ or } y = 0)\big)$. □

4. Stable, Jankov and subframe formulas for bi-Gödel algebras

In this section, we develop the theories of stable canonical and subframe formulas for bi-Gödel algebras. For an overview of these formulas and their use in superintuitionistic and modal logics we refer to [8] and [17], respectively. We use the former class of formulas to provide a uniform axiomatization of all extensions of bi-LC. By focusing on a particular subclass, that of the Jankov formulas, we fully characterize the splitting logics of the lattice $\Lambda(\text{bi-LC})$ of extensions of bi-LC, and prove that $|\Lambda(\text{bi-LC})| = 2^{\aleph_0}$. As for the subframe formulas, we utilize them to establish a straightforward axiomatization of some notable extensions of bi-LC, such as the logic of co-trees of depth (respectively, width) less than $n \in \omega$. But our main use for these formulas will come in the following section.

4.1. Stable canonical formulas and Jankov formulas.

Let ϕ be a formula and $\mathfrak{M} = (\mathcal{X}, V)$ a model on a bi-Esakia co-tree $\mathcal{X}$. Suppose $\mathfrak{M}, w \models \sim\neg\phi$, and note the following equivalences:

$$w \models \sim\neg\phi \iff w \models \top \leftarrow \neg\phi \iff \exists v \in {\downarrow} w\ (v \models \top \text{ and } v \not\models \neg\phi) \iff$$

$$\iff \exists v \in {\downarrow} w\ (v \not\models \neg\phi) \iff \exists u \in {\uparrow}{\downarrow} w\ (u \models \phi).$$

Hence, we have that $\sim\neg\phi$ holds in some point of $\mathfrak{M}$ iff ϕ holds in some point of $\mathfrak{M}$. Thus, in the setting of bi-Esakia co-trees, $\sim\neg$ can be viewed as an analogue to the notion of the universal diamond from modal logic. Similarly, suppose that $\mathfrak{M}, w \models \neg\sim\phi$, i.e.,

$$w \models \neg\sim\phi \iff \forall v \in {\uparrow} w\ (v \not\models \sim\phi) \iff \forall v \in {\uparrow} w\ (v \not\models \top \leftarrow \phi) \iff$$

$$\iff \forall u \in {\downarrow}{\uparrow} w\ (u \not\models \top \text{ or } u \models \phi) \iff \forall u \in {\downarrow}{\uparrow} w\ (u \models \phi).$$

Since $\mathcal{X}$ is a co-tree, it follows that for each point $w \in X$ we have $X = {\downarrow} r = {\downarrow}{\uparrow} w$, where r is the co-root of $\mathcal{X}$. Thus, the equivalences above imply that $\neg\sim\phi$ holds in some point of $\mathfrak{M}$ iff ϕ holds everywhere in $\mathfrak{M}$. Therefore, in the setting of bi-Esakia co-trees, $\neg\sim$ can be viewed as an analogue to the notion of the universal box from modal logic.

This discussion not only provides some intuition for what follows, by highlighting a similarity with the construction of the Jankov-Fine formulas for modal algebras, but it helps us show that extensions of bi-LC admit a metalogical classical inconsistency lemma as in condition (1). These types of lemmas were formally introduced and studied in [49], see also [15, 44, 43]. Given an extension L of bi-LC, let us define a consequence relation $\vdash_L$ in the following manner: given a set of formulas $\Sigma \cup \{\varphi\}$, we write $\Sigma \vdash_L \varphi$ iff for every model $\mathfrak{M} = \langle \mathcal{X}, V \rangle$ such that $\mathcal{X}$ is a co-tree and $\mathcal{X} \models L$, if $\mathfrak{M} \models \Sigma$ then $\mathfrak{M} \models \varphi$.

Theorem 4.1. *Let L be an extension of* bi-LC*. If $\Sigma \cup \{\varphi\}$ is a set of formulas, then*

$$\Sigma \cup \{\sim\neg\sim\varphi\} \vdash_L \bot \iff \Sigma \vdash_L \varphi.$$

Consequently, L has a classical inconsistency lemma.

Proof. Suppose first that $\Sigma \cup \{\sim\neg\sim\varphi\} \vdash_L \bot$ and consider a model $\mathfrak{M} = \langle \mathcal{X}, V \rangle$ such that $\mathcal{X}$ is a co-tree validating L and $\mathfrak{M} \models \Sigma$. Since $\mathfrak{M}$ is nonempty, $\mathfrak{M} \not\models \bot$. As $\Sigma \cup \{\sim\neg\sim\varphi\} \vdash_L \bot$ and $\mathfrak{M} \models \Sigma$, this implies $\mathfrak{M} \not\models \sim\neg\sim\varphi$, i.e., that there exists $x \in X$ such that $x \not\models \sim\neg\sim\varphi$. It now follows that $x \models \neg\sim\varphi$, and by our previous discussion on the connective $\neg\sim$, we conclude $\mathfrak{M} \models \varphi$.

For the converse, let us suppose $\Sigma \vdash_L \varphi$. We can show that $\Sigma \cup \{\sim\neg\sim\varphi\} \vdash_L \bot$ holds by proving that $\mathfrak{M} \not\models \Sigma \cup \{\sim\neg\sim\varphi\}$, for every model $\mathfrak{M} = \langle \mathcal{X}, V \rangle$ such that $\mathcal{X}$ is a co-tree and $\mathcal{X} \models L$. Accordingly, let $\mathfrak{M}$ be such a model and suppose that $\mathfrak{M} \models \Sigma \cup \{\sim\neg\sim\varphi\}$. By our assumption, $\mathfrak{M} \models \Sigma$ implies $\mathfrak{M} \models \varphi$, and by our discussion on the connective $\sim\neg$, $\mathfrak{M} \models \sim\neg\sim\varphi$ entails the existence of a point $x \in X$ such that $x \models \sim\varphi$. But now we have $y \not\models \varphi$, for some $y \in {\downarrow} x$, contradicting $\mathfrak{M} \models \varphi$. Therefore, $\mathfrak{M} \not\models \Sigma \cup \{\sim\neg\sim\varphi\}$, as desired. □

In view of Theorem 4.1, extensions of bi-LC exhibit an appealing balance between the classical and intuitionistic behavior of negation connectives.

Let $\mathfrak{A} \in \mathsf{bi\text{-}GA}_{FSI}$, $D \subseteq A^2$, and introduce a propositional variable $p_a \in Prop$, for each $a \in A$. Let Γ be the formula describing the Heyting algebra structure of $\mathfrak{A}$ fully, while the behavior of the operator $\leftarrow$ is only given for elements of D, i.e.,

$$\begin{aligned}\Gamma := &\bigwedge\{p_{a\vee b} \leftrightarrow (p_a \vee p_b) \colon a, b \in A\} \wedge \\ &\bigwedge\{p_{a\wedge b} \leftrightarrow (p_a \wedge p_b) \colon a, b \in A\} \wedge \\ &\bigwedge\{p_{a\to b} \leftrightarrow (p_a \to p_b) \colon a, b \in A\} \wedge \\ &\bigwedge\{p_{a\leftarrow b} \leftrightarrow (p_a \leftarrow p_b) \colon (a, b) \in D\} \wedge \\ &\{p_0 \leftrightarrow \bot\} \wedge \{p_1 \leftrightarrow \top\}.\end{aligned}$$

We define the *stable canonical formula* associated with $\mathfrak{A}$ and D by

$$\gamma(\mathfrak{A}, D) := \neg \sim\Gamma \to \neg \bigwedge \{p_a \leftarrow p_b \colon a, b \in A \text{ and } a \nleqslant b\}.$$

Moreover, given $\mathfrak{B} \in \mathsf{bi\text{-}GA}$ and a Heyting homomorphism $h\colon \mathfrak{A} \to \mathfrak{B}$, we call D a *$\leftarrow$-closed domain* of $\mathfrak{A}$ if for all $(a, b) \in D$, we have $h(a \leftarrow b) = h(a) \leftarrow h(b)$. In this case, we say that h satisfies the *$\leftarrow$-stable domain condition* ($\mathsf{SDC}_{\leftarrow}$ for short) for D.

Before we discuss the fundamental properties of the stable canonical formulas, we recall four elementary facts about bi-Heyting algebras, that will be used without further reference in what follows.

Lemma 4.2. *If $\mathfrak{A} \in \mathsf{bi\text{-}HA}$ and $a, b \in A$, then:*

(i) $\neg a = 1 \iff a = 0$,
(ii) $\neg \sim a = 1 \iff a = 1$,
(iii) $a \to b = 1 \iff a \leqslant b$,
(iv) $a \leftarrow b \neq 0 \iff a \nleqslant b$.

Henceforth, we will make extensive use of the fact that every finite Gödel algebra can be regarded as a finite bi-Gödel algebra.

Lemma 4.3. *(Stable Jankov Lemma) Let $\mathfrak{A}, \mathfrak{B} \in \mathsf{bi\text{-}GA}$ and $D \subseteq A^2$. If $\mathfrak{A} \in \mathsf{bi\text{-}GA}_{FSI}$, then $\mathfrak{B} \not\models \gamma(\mathfrak{A}, D)$ iff there exists $\mathfrak{C} \in \mathbb{H}(\mathfrak{B})_{SI}$ and a Heyting algebra embedding $h\colon \mathfrak{A} \hookrightarrow \mathfrak{C}$ satisfying the $\mathsf{SDC}_{\leftarrow}$ for D.*

Proof. We start by proving the left to right implication. Suppose that $\mathfrak{B} \not\models \gamma(\mathfrak{A}, D)$. By the bi-Esakia duality, this is equivalent to $(\mathfrak{B}_*, V), w \not\models \gamma(\mathfrak{A}, D)$, for some valuation V on $\mathfrak{B}_*$ and some $w \in B_*$. Since $\mathfrak{B}_*$ is a bi-Esakia co-forest, we know that w lies below a single maximal point r of $\mathfrak{B}_*$, and that ${\downarrow} r \in ClUpDo(\mathfrak{B}_*)$, i.e., ${\downarrow} r$ is a bi-generated subframe of $\mathfrak{B}_*$. Again by the bi-Esakia duality, it follows that $\mathfrak{C} := ({\downarrow} r)^*$ is a homomorphic image of $\mathfrak{B}$, and since ${\downarrow} r$ is a co-tree, $\mathfrak{C}$ is SI. Furthermore, $({\downarrow} r, V{\restriction}_{{\downarrow} r}), w \not\models \gamma(\mathfrak{A}, D)$ implies $v\big(\gamma(\mathfrak{A}, D)\big) \neq 1$, where v is the valuation on $\mathfrak{C}$ corresponding to $V{\restriction}_{{\downarrow} r}$.

Now, define a map $h\colon A \to C$ by setting $h(a) := v(p_a)$, for $a \in A$. Since $\mathfrak{C}$ is SI, it follows from Theorem 3.6 that for all $c \in C \smallsetminus \{0\}$, we have $\neg c = 0$. Consequently, if we had $v(\sim\Gamma) \neq 0$, then $\neg v(\sim\Gamma) = 0$ would follow, yielding $v\big(\gamma(\mathfrak{A}, D)\big) = 1$, a contradiction. Thus, we must have $v(\sim\Gamma) = 0$, and now the equivalences

$$v(\sim\Gamma) = 0 \iff \neg v(\sim\Gamma) = \neg \sim v(\Gamma) = 1 \iff v(\Gamma) = 1$$

imply $v(\Gamma) = 1$. By the definitions of h and the formula Γ, it is easy to see that $v(\Gamma) = 1$ iff h is a Heyting homomorphism which satisfies the $\mathsf{SDC}_{\leftarrow}$ for D, hence the only thing that remains to show is that h is injective. To this end, let $a \neq b \in A$ and suppose, without loss of generality, that $a \nleqslant b$. Notice that, since $v\big(\gamma(\mathfrak{A}, D)\big) \neq 1$, we must have

$$v\big(\neg \bigwedge \{p_x \leftarrow p_y \colon x, y \in A \text{ and } x \nleqslant y\}\big) \neq 1,$$

which is equivalent to

$$\bigwedge \{v(p_x) \leftarrow v(p_y) \colon x, y \in A \text{ and } x \nleqslant y\}\big) \neq 0.$$

By the definition of h, it now follows

$$\bigwedge \{h(x) \leftarrow h(y) \colon x, y \in A \text{ and } x \nleq y\}) \neq 0,$$

thus $h(a) \leftarrow h(b) \neq 0$, i.e., $h(a) \nleq h(b)$. We conclude that h is indeed a Heyting embedding satisfying the SDC$_{\leftarrow}$ for D, as desired.

For the converse, let $\mathfrak{C} \in \mathbb{H}(\mathfrak{B})_{SI}$, $h \colon \mathfrak{A} \hookrightarrow \mathfrak{C}$ be a Heyting algebra embedding that satisfies the SDC$_{\leftarrow}$ for D, and $v \colon Prop \to C$ a valuation satisfying $v(p_a) = h(a)$, for each $a \in A$. We shall prove that $\mathfrak{C}$ refutes $\gamma(\mathfrak{A}, D)$ via v, hence showing that $\mathfrak{B} \not\models \gamma(\mathfrak{A}, D)$ holds (recall that the validity of a formula is preserved under taking homomorphic images). By the definitions of h and Γ, it is clear that $v(\Gamma) = 1$, which is equivalent to $\neg{\sim}v(\Gamma) = v(\neg{\sim}\Gamma) = 1$. Now, if $a \nleq b$, for $a, b \in A$, then we have $a \to b \neq 1$. Since h is a Heyting algebra embedding, it follows that $h(a \to b) = h(a) \to h(b) \neq 1$, i.e., $h(a) \nleq h(b)$, which in turn is equivalent to $h(a) \leftarrow h(b) = v(p_a) \leftarrow v(p_b) \neq 0$. Thus, we have $0 \notin \{v(p_a) \leftarrow v(p_b) \colon a, b \in A \text{ and } a \nleq b\}$. As $\mathfrak{C}$ is SI by assumption, $0_{\mathfrak{C}}$ is $\wedge$-irreducible (see Theorem 3.6), hence

$$v\big(\bigwedge\{p_a \leftarrow p_b \colon a, b \in A \text{ and } a \nleq b\}\big) \neq 0.$$

Consequently, $\neg v\big(\bigwedge\{p_a \leftarrow p_b \colon a, b \in A \text{ and } a \nleq b\}\big) \neq 1$, and we conclude that $v\big(\gamma(\mathfrak{A}, D)\big) \neq 1$, as desired. □

The following lemma is essential for what follows. Notice that it makes crucial use of the fact that the HA-reduct of bi-GA is locally finite, much like the analogous version of this result for Heyting algebras relies on the local finiteness of the lattice reduct of HA.

Lemma 4.4. *(Stable Filtration Lemma) Let ϕ be a formula and $\mathfrak{B} \in$ bi-GA. If $\mathfrak{B} \not\models \phi$, then there exists a finite Heyting subalgebra $\mathfrak{A}$ of $\mathfrak{B}$ such that $\mathfrak{A} \in$ bi-GA and $\mathfrak{A} \not\models \phi$. If $\mathfrak{B}$ is moreover SI, then so is $\mathfrak{A}$.*

Proof. Suppose that $\mathfrak{B} \not\models \phi$. Then $\phi^{\mathfrak{B}}(\overline{a}) \neq 1$ for some tuple $\overline{a} \in B$. Let

$$\Sigma := \{\psi(\overline{a}) \colon \psi \text{ is a subformula of } \phi\}$$

and $\mathfrak{A}$ be the Heyting subalgebra of $\mathfrak{B}$ generated by Σ. Since Σ is finite and bi-GA has a locally finite HA-reduct, it follows that $\mathfrak{A}$ is a finite Heyting algebra, hence also a finite bi-Heyting algebra (see Example 2.6), although not necessarily a bi-Heyting subalgebra of $\mathfrak{B}$. Moreover, since bi-GA is axiomatized by $\leftarrow$-free formulas and $\mathfrak{A}$ is a Heyting subalgebra of $\mathfrak{B}$, then clearly $\mathfrak{B} \in$ bi-GA implies $\mathfrak{A} \in$ bi-GA.

As $\mathfrak{A}$ is a bi-Heyting algebra, it has a well-defined $\leftarrow^{\mathfrak{A}}$ operation. And although said operation need not coincide with $\leftarrow^{\mathfrak{B}}$, it crucially does so when $a {\leftarrow^{\mathfrak{B}}} b \in \Sigma$. To see this, just note that $A \subseteq B$ implies

$$a{\leftarrow^{\mathfrak{B}}}b = \bigwedge\{c \in B \colon a \leqslant c \vee b\} \leqslant \bigwedge\{c \in A \colon a \leqslant c \vee b\} = a{\leftarrow^{\mathfrak{A}}}b,$$

for all $a, b \in A$. Moreover, if $a{\leftarrow^{\mathfrak{B}}}b \in A$, then clearly $a{\leftarrow^{\mathfrak{B}}}b \in \{c \in A \colon a \leqslant c \vee b\}$, hence $a{\leftarrow^{\mathfrak{A}}}b \leqslant a{\leftarrow^{\mathfrak{B}}}b$. It now follows that if $a{\leftarrow^{\mathfrak{B}}}b \in A$ (in particular, if $a{\leftarrow^{\mathfrak{B}}}b \in \Sigma$), then $a{\leftarrow^{\mathfrak{B}}}b = a{\leftarrow^{\mathfrak{A}}}b$. Therefore, using a simple argument by induction on the complexity of ϕ, we can conclude that $\phi^{\mathfrak{B}}(\overline{a}) = \phi^{\mathfrak{A}}(\overline{a})$, hence $\phi^{\mathfrak{A}}(\overline{a}) \neq 1$ and $\mathfrak{A} \not\models \phi$ as desired.

Suppose now that $\mathfrak{B}$ is SI, hence $0_{\mathfrak{B}}$ is $\wedge$-irreducible by Theorem 3.6. Since $\mathfrak{A}$ is a Heyting subalgebra of $\mathfrak{B}$, it is nontrivial and $0_{\mathfrak{A}}$ must also be $\wedge$-irreducible. Again using Theorem 3.6, we conclude that $\mathfrak{A}$ is SI. □

Corollary 4.5. *The variety* bi-GA *has the FMP.*

Equipped with the two previous lemmas, we can start our proof of the first main result of this subsection, which establishes a uniform axiomatization of all extensions of bi-LC by means of stable canonical formulas. This is in analogy with the intuitionistic case, see e.g., [7, 17]. However, a similar axiomatization technique for arbitrary bi-intermediate logics cannot be obtained as we discuss below.

Fix a formula $\phi \notin$ bi-LC and set $n := |Sub(\phi)|$. Since the HA-reduct of bi-GA is locally finite, there exists a bound $c(\phi) \in \omega$ on the size of n-generated Heyting algebras belonging to this reduct. Accordingly, let $\mathfrak{A}_1, \dots, \mathfrak{A}_{m(n)}$ be the list of (up to isomorphism) all n-generated SI bi-Gödel algebras such that $|A_i| \leqslant c(\phi)$ and $\mathfrak{A}_i \not\models \phi$. Now, for each of these bi-Gödel algebra $\mathfrak{A}_i$ refuting ϕ via a valuation $\mathfrak{v}$, we let $\Theta := \mathfrak{v}[Sub(\phi)]$ and

$$D_i^{\leftarrow} := \{(a, b) \in \Theta^2 \colon a \leftarrow b \in \Theta\}.$$

Consider a new list $(\mathfrak{A}_1, D_1^{\leftarrow}), \dots, (\mathfrak{A}_{k(n)}, D_{k(n)}^{\leftarrow})$ (notice that $k(n)$ need not be smaller than $m(n)$, since each $\mathfrak{A}_i$ may refute ϕ through distinct valuations), whose elements we call the *refutation patterns* for ϕ. Keeping this discussion in mind, we have the following theorem:

Theorem 4.6. *If $\mathfrak{B}$ is an SI bi-Gödel algebra, then:*

(i) *$\mathfrak{B} \not\models \phi$ iff there exists $i \leqslant k(n)$ and a Heyting algebra embedding $h \colon \mathfrak{A}_i \hookrightarrow \mathfrak{B}$ satisfying the SDC$_\leftarrow$ for $D_i^{\leftarrow}$;*

(ii) $\mathfrak{B} \models \phi \iff \mathfrak{B} \models \bigwedge_{i=1}^{k(n)} \gamma(\mathfrak{A}_i, D_i^{\leftarrow})$.

Proof. (i): Firstly, note that right to left implication follows immediately from $(\mathfrak{A}_i, D_i^{\leftarrow}) \not\models \phi$ and the definition of $D_i^{\leftarrow}$, since if a Heyting algebra embedding $h \colon \mathfrak{A}_i \hookrightarrow \mathfrak{B}$ satisfies the SDC$_\leftarrow$ for $D_i^{\leftarrow}$, then we clearly have $\mathfrak{B} \not\models \phi$. To prove the converse, suppose that $\mathfrak{B} \not\models \phi$. As $\mathfrak{B} \in \mathsf{bi\text{-}GA}_{SI}$, it follows from Lemma 4.4 that there is a finite Heyting subalgebra $\mathfrak{A}$ of $\mathfrak{B}$ such that $\mathfrak{A} \in \mathsf{bi\text{-}GA}_{SI}$ and $\mathfrak{A}$ refutes ϕ via some valuation $\mathfrak{v}$. Thus, there exists a Heyting embedding $h \colon \mathfrak{A} \hookrightarrow \mathfrak{B}$, and by looking at the proof of Lemma 4.4, we not only see that $\mathfrak{A}$ is n-generated for $n = |Sub(\phi)|$ (as a Heyting algebra), but also that $a \leftarrow b \in \mathfrak{v}[Sub(\phi)]$ implies $h(a \leftarrow b) = h(a) \leftarrow h(b)$, for all $a, b \in A$. It is now easy to see that h satisfies the SDC$_\leftarrow$ for

$$D^{\leftarrow} := \{(a, b) \in \mathfrak{v}[Sub(\phi)]^2 \colon a \leftarrow b \in \mathfrak{v}[Sub(\phi)]\}.$$

Therefore, the pair $(\mathfrak{A}, D^{\leftarrow})$ must be one of the $(\mathfrak{A}_i, D_i^{\leftarrow})$ listed above, hence we showed that the right side of the desired equivalence holds, as desired.

(ii): This follows immediately from (i) and the Stable Jankov Lemma 4.3. □

As a consequence, stable canonical formulas can be used to axiomatized extensions of $\mathsf{bi\text{-}LC}$.

Theorem 4.7. *Every extension of* $\mathsf{bi\text{-}LC}$ *is axiomatizable by stable canonical formulas. Moreover, if L is finitely axiomatized, then L is axiomatizable by finitely many stable canonical formulas.*

Proof. Suppose that $L = \mathsf{bi\text{-}LC} + \{\phi_i \colon i \in I\}$, so we can assume without loss of generality that $\mathsf{bi\text{-}LC} \nvdash \phi_i$, for all $i \in I$. By the previous theorem, we know that for each $i \in I$ there is a list of refutation patterns $(\mathfrak{A}_{i,1}, D_{i,1}^{\leftarrow}), \dots, (\mathfrak{A}_{i,k(i)}, D_{i,k(i)}^{\leftarrow})$ such that

$$\mathsf{bi\text{-}LC} + \phi_i = \mathsf{bi\text{-}LC} + \bigwedge_{j=1}^{k(i)} \gamma(\mathfrak{A}_{i,j}, D_{i,j}^{\leftarrow}).$$

Thus, we have

$$L = \mathsf{bi\text{-}LC} + \{\phi_i \colon i \in I\} = \mathsf{bi\text{-}LC} + \{\bigwedge_{j=1}^{k(i)} \gamma(\mathfrak{A}_{i,j}, D_{i,j}^{\leftarrow}) \colon i \in I\}.$$

The last part of the statement clearly follows from the previous equality. □

Corollary 4.8. *Let $L' \subseteq L$ be extensions of* $\mathsf{bi\text{-}LC}$. *Then L is axiomatizable over L' by stable canonical formulas. Moreover, if L is finitely axiomatized over L', then L is axiomatizable over L' by finitely many stable canonical formulas.*

Proof. This is an immediate consequence of the proof of the previous theorem. □

We will now focus on a particular class of stable canonical formulas: the Jankov formulas [34, 35, 36]. Recall that $\mathsf{bi\text{-}GA}_{FSI}$ is the class of finite SI Gödel algebras. For each $\mathfrak{A} \in \mathsf{bi\text{-}GA}_{FSI}$, we call $\mathcal{J}(\mathfrak{A}) := \gamma(\mathfrak{A}, A^2)$ the *Jankov formula* of $\mathfrak{A}$. We compile the defining properties of these formulas in the following lemma, and subsequently use them to characterize the splitting logics of the lattice $\Lambda(\mathsf{bi\text{-}LC})$ of extensions of $\mathsf{bi\text{-}LC}$, as well as finding its cardinality.

Lemma 4.9. *(Jankov Lemma) If* $\mathfrak{B} \in \mathsf{bi\text{-}GA}$ *and* $\mathfrak{A} \in \mathsf{bi\text{-}GA}_{FSI}$*, then the following conditions are equivalent:*

(i) $\mathfrak{B} \not\models \mathcal{J}(\mathfrak{A})$;
(ii) *there exists* $\mathfrak{C} \in \mathbb{H}(\mathfrak{B})_{SI}$ *and a bi-Heyting algebra embedding* $h \colon \mathfrak{A} \hookrightarrow \mathfrak{C}$;
(iii) $\mathfrak{A} \in \mathbb{SH}(\mathfrak{B})$;
(iv) $\mathfrak{A} \in \mathbb{HS}(\mathfrak{B})$.

Proof. Firstly, let us note that the equivalence (i) $\iff$ (ii) is just a particular instance of Lemma 4.3, and that (ii) clearly implies (iii). The equivalence (iii) $\iff$ (iv) follows from Proposition 2.2(iii) and the fact that $\mathsf{bi\text{-}GA}$ has EDPC (see Corollary 3.7). Finally, (iv) $\implies$ (i) follows from the easily checked fact that $\mathfrak{A} \not\models \mathcal{J}(\mathfrak{A})$, and by noting that the operators $\mathbb{H}$ and $\mathbb{S}$ preserve the validity of formulas. □

Corollary 4.10. *If* $\mathfrak{B} \in \mathsf{bi\text{-}GA}_{SI}$*, then* $\mathbb{V}(\mathfrak{B})_{FSI} = \mathbb{IS}(\mathfrak{B})_F$.

Proof. We start by noting that $\mathbb{IS}(\mathfrak{B})_F \subseteq \mathbb{V}(\mathfrak{B})_{FSI}$ follows directly from Corollary 3.9. To prove the other inclusion, we use the Jankov Lemma and the fact that the product of algebras preserves the validity of formulas to deduce that if $\mathfrak{A} \in \mathbb{V}(\mathfrak{B})_{FSI}$, then $\mathfrak{B} \not\models \mathcal{J}(\mathfrak{A})$, i.e., $\mathfrak{A} \in \mathbb{SH}(\mathfrak{B})$. As $\mathsf{bi\text{-}GA}$ is a semi-simple variety (see Corolary 3.7) and simple algebras have no nontrivial homomorphic images, $\mathfrak{B} \in \mathsf{bi\text{-}GA}_{SI}$ now implies that $\mathfrak{A} \in \mathbb{IS}(\mathfrak{B})_F$, as desired. □

Given a lattice $\mathfrak{L}$ and elements $a, b \in \mathfrak{L}$, we call (a, b) a *splitting pair* for $\mathfrak{L}$ if $a \nleq b$ and $\mathfrak{L} = {\uparrow} a \uplus {\downarrow} b$. In particular, if $\mathfrak{L} = \Lambda(\mathsf{bi\text{-}LC})$ then a is said to be a *splitting logic*.

Theorem 4.11. *(Splitting Theorem) If* $L \in \Lambda(\mathsf{bi\text{-}LC})$*, then:*

(i) *L is a splitting logic iff L is axiomatized by a single Jankov formula,*
(ii) *L is a join of splitting logics iff L is axiomatized by Jankov formulas.*

Proof. We start by noting that (ii) is a straightforward consequence of (i), hence we only prove the latter equivalence. Suppose that (L, L') is a splitting pair for $\Lambda(\mathsf{bi\text{-}LC})$, for some $L' \in \Lambda(\mathsf{bi\text{-}LC})$. As $\mathsf{bi\text{-}GA}$ is a congruence distributive variety with the FMP (Corollary 4.5), it follows from a result by McKenzie [48] that $L' = Log(\mathfrak{A})$, for some $\mathfrak{A} \in \mathsf{bi\text{-}GA}_{FSI}$. Using the definition of a splitting pair together with the fact $\mathfrak{A} \not\models \mathcal{J}(\mathfrak{A})$, it is easy to see that the equivalence $\mathfrak{B} \models \mathcal{J}(\mathfrak{A})$ iff $\mathfrak{B} \models L$ holds for all $\mathfrak{B} \in \mathsf{bi\text{-}GA}$. Thus, $L = \mathsf{bi\text{-}LC} + \mathcal{J}(\mathfrak{A})$.

Conversely, assume $L = \mathsf{bi\text{-}LC} + \mathcal{J}(\mathfrak{A})$ for some $\mathfrak{A} \in \mathsf{bi\text{-}GA}_{FSI}$. Set $L' := Log(\mathfrak{A})$ and notice that $\mathfrak{A} \not\models \mathcal{J}(\mathfrak{A})$ implies $L \nsubseteq L'$. Now, take $E \in \Lambda(\mathsf{bi\text{-}LC})$ and suppose $L \nsubseteq E$, i.e., $\mathcal{J}(\mathfrak{A}) \notin E$. By a simple application of the Jankov Lemma 4.9, this implies $\mathfrak{A} \in \mathsf{V}_E = \{\mathfrak{B} \in \mathsf{bi\text{-}HA} \colon \mathfrak{B} \models E\}$. Equivalently, $E \subseteq Log(\mathfrak{A}) = L'$. We just proved that for $E \in \Lambda(\mathsf{bi\text{-}LC})$, $E \notin {\uparrow} L$ entails $E \in {\downarrow} L'$, i.e., that $\Lambda(\mathsf{bi\text{-}LC}) = {\uparrow} L \uplus {\downarrow} L'$. Therefore, (L, L') is a splitting pair for $\Lambda(\mathsf{bi\text{-}LC})$. □

Translating the Splitting Theorem into algebraic terms yields a characterization of the *splitting algebras* of the variety $\mathsf{bi\text{-}GA}$, that is, elements $\mathfrak{A} \in \mathsf{bi\text{-}GA}_{SI}$ for which there exists a largest subvariety $\mathsf{V} \subseteq \mathsf{bi\text{-}GA}$ omitting $\mathfrak{A}$. In other words, $(\mathbb{V}(\mathfrak{A}), \mathsf{V})$ is a splitting pair for the lattice of nontrivial subvarieties of bi-Gödel algebras $\Lambda(\mathsf{bi\text{-}GA})$.

Theorem 4.12. *The splitting algebras of* $\mathsf{bi\text{-}GA}$ *are exactly the finite SI bi-Gödel algebras.*

Remark 4.13. The equality between splitting algebras and finite SI algebras holds more in general for every variety of finite type with EDPC and the FMP, as shown in [12, Cor. 3.2] and [48].

It is well known that the analogue of the previous theorem holds for the variety of Heyting algebras (see, e.g., [17]): the splitting algebras of HA are exactly its finite SI elements. However, this is far from the case for bi-Heyting algebras; a result by Wolter [58] shows that the only splitting algebras in $\mathsf{bi\text{-}HA}$ are the two-element and three-element chains. This is the main reason why the theories of stable canonical formulas cannot be developed for $\mathsf{bi\text{-}IPC}$.

4.2. **The cardinality of $\Lambda(\text{bi-LC})$.** The goal of this section is to prove that the cardinality of the lattice $\Lambda(\text{bi-LC})$ is $2^{\aleph_0}$. Accordingly, let us define a partial order $\leqslant$ on the class bi-GA_{FSI} by $\mathfrak{A} \leqslant \mathfrak{B}$ iff $\mathfrak{A} \in \mathbb{HS}(\mathfrak{B})$. Note that, by Corollary 4.10, we have

$$\mathfrak{A} \leqslant \mathfrak{B} \iff \mathfrak{A} \in \mathbb{IS}(\mathfrak{B}).$$

We will show $|\Lambda(\text{bi-LC})| = 2^{\aleph_0}$ by proving that there exists a countably infinite $\leqslant$-antichain $\Omega \subseteq \text{bi-GA}_{FSI}$ (that is, the elements of Ω are pairwise $\leqslant$-incomparable). That the existence of Ω suffices to establish the desired equality follows easily from the next proposition, as we shall see in a moment.

Proposition 4.14. *Let $\Omega \subseteq \text{bi-GA}_{FSI}$ be a countably infinite $\leqslant$-antichain. If $\Omega_1, \Omega_2 \in \mathcal{P}(\Omega)$ are different, then*

$$\text{bi-LC} + \mathcal{J}(\Omega_1) \neq \text{bi-LC} + \mathcal{J}(\Omega_2),$$

where $\mathcal{J}(\Omega_i) := \{\mathcal{J}(\mathfrak{A}) : \mathfrak{A} \in \Omega_i\}$.

Proof. Without loss of generality, suppose that there exists $\mathfrak{B} \in \Omega_1 \smallsetminus \Omega_2$. Since $\mathfrak{B} \not\models \mathcal{J}(\mathfrak{B})$, it is clear that $\mathfrak{B} \not\models \text{bi-LC} + \mathcal{J}(\Omega_1)$. On the other hand, if $\mathfrak{B} \not\models \text{bi-LC} + \mathcal{J}(\Omega_2)$ then there is $\mathfrak{A} \in \Omega_2$ such that $\mathfrak{B} \not\models \mathcal{J}(\mathfrak{A})$. By the Jankov Lemma, it follows $\mathfrak{A} \leqslant \mathfrak{B}$. But this is a contradiction, since $\mathfrak{A}$ and $\mathfrak{B}$ are in Ω, an $\leqslant$-antichain. Therefore, $\mathfrak{B} \models \text{bi-LC} + \mathcal{J}(\Omega_2)$, and we conclude

$$\text{bi-LC} + \mathcal{J}(\Omega_1) \neq \text{bi-LC} + \mathcal{J}(\Omega_2). \qquad \square$$

Suppose that we have Ω satisfying the conditions of the previous proposition. As our language is countable, we know that there are at most continuum many extensions of bi-LC, that is, $|\Lambda(\text{bi-LC})| \leqslant 2^{\aleph_0} = |\mathcal{P}(\Omega)|$. But we just proved that distinct elements of $\mathcal{P}(\Omega)$ give rise to distinct extensions of bi-LC, hence it follows $|\mathcal{P}(\Omega)| \leqslant |\Lambda(\text{bi-LC})| = 2^{\aleph_0}$.

We end this discussion by noting that we can use the bi-Esakia duality to translate the partial order $\leqslant$ into one on the class of finite co-trees: $\mathfrak{F} \leqslant \mathfrak{G}$ iff $\mathfrak{F}$ is a bi-p-morphic image of $\mathfrak{G}$. It is now clear that to find our desired $\leqslant$-antichain of finite SI bi-Gödel algebras, it suffices to find a countably infinite $\leqslant$-antichain of finite co-trees. In order to do this, we rely on the following observation.

Lemma 4.15. *Let $f \colon \mathfrak{F} \to \mathfrak{G}$ be a bi-p-morphism between co-trees. If $x \prec y \in \mathfrak{F}$ and $f(x) \neq f(y)$, then $f(x) \prec f(y)$.*

Proof. Suppose $\neg(f(x) \prec f(y))$. Since $x \prec y$ entails $f(x) \leqslant f(y)$, there must exist $u \in \mathfrak{G}$ such that $f(x) < u < f(y)$. By the forth condition of the definition of p-morphisms, we now have $f(z) = u$, for some $z \in {\uparrow}x \backslash \{x\}$. As $\mathfrak{F}$ is a co-tree, its principal upsets are chains, so $x \prec y$ implies $y \leqslant z$, hence $f(y) \leqslant f(z) = u$. But this yields a contradiction, since $u < f(y)$ and partial orders are anti-symmetric. $\square$

Let $\mathcal{T} := \{T_n : n \in \omega\}$ be the sequence of finite co-trees depicted in Figure 2. The next result proves that this is an $\leqslant$-antichain of finite co-trees.[†] Notice that the proof makes extensive use without reference of Proposition 2.9, and of a direct consequence of Lemma 4.15 (if $x \prec y$, then $f(x) = f(y)$ or $f(x) \prec f(y)$).

Proposition 4.16 (Hodkinson)**.** *The set $\mathcal{T} := \{T_n : n \in \omega\}$ is a countably infinite $\leqslant$-antichain of finite co-trees.*

Proof. Firstly, we prove $T_0 \nleqslant T_n$, for all $n \in \omega \smallsetminus \{0\}$. Suppose otherwise, i.e., that there exists a surjective bi-p-morphism $f \colon T_n \to T_0$. Denote $T_0 = \{a', b', c', d'\}$. We know that the co-root u_1 of T_n must be mapped to the co-root a' of T_0, hence $f(v_1) \in \{a', b'\}$ by Lemma 4.15. But this already yields a contradiction, since both cases imply that at least one of the nontrivial predecessors of v_1 (which are both minimal points) is mapped to a non-minimal point. Note as well that $|T_m| < |T_n|$ entails $T_n \nleqslant T_m$, for $m < n \in \omega$.

[†]This $\leqslant$-antichain was constructed (and proved to be one) by Ian Hodkinson (personal communication). We use a different method to show that this is in fact an $\leqslant$-antichain, closer to the ones we used in Section 3.

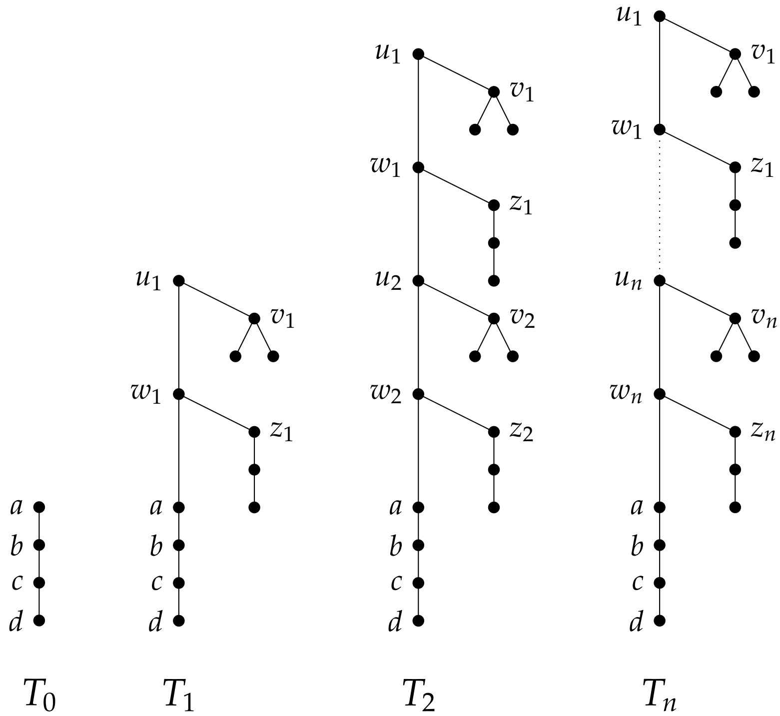

FIGURE 2. The co-trees T_0, T_1, T_2, and T_n.

It remains to show $T_m \not\leqslant T_n$, for $0 < m < n \in \omega$. Denote the elements of T_m by u_1', v_1', a', b', etc. Moreover, let $U_m := \{u_j' \in T_m \colon j \leqslant m\}$ and $U_n := \{u_i \in T_n \colon i \leqslant n\}$. In a similar way, define the sets V_m and V_n, W_m and W_n, Z_m and Z_n. Suppose now that there exists a surjective bi-p-morphism $f \colon T_n \to T_m$. It is easy to see that

$$f[V_n] \cap (U_m \cup W_m \cup Z_m \cup \{a'\}) = \emptyset,$$

since for all $i \leqslant n$ and $j \leqslant m$, the inequality $|{\downarrow} v_i| = 3 < Min\{|{\downarrow} u_j'|, |{\downarrow} w_j|\}$ already implies $f(v_i) \notin \{u_j', w_j'\}$. Moreover, by the same reasoning used above (to show $f(v_1) \notin \{a', b'\}$) we get $f(v_i) \notin \{a', z_j'\}$. On the other hand,

$$f[Z_n] \cap (U_m \cup W_m \cup V_m \cup \{a'\}) = \emptyset$$

follows easily by noting the inequality $|{\downarrow} z_i| = 3 < Min\{|{\downarrow} u_j'|, |{\downarrow} w_j|, |{\downarrow} a'|\}$, and that $f(z_i) \neq v_j'$, since a chain (such as ${\downarrow} z_i$) cannot be mapped onto a set with incomparable elements (such as ${\downarrow} v_j'$).

Suppose now that $f(u_i) = w_j'$. It follows from Lemma 4.15 that $f(v_i) \in \{w_j', z_j', x'\}$ (where $x' = u_{j+1}'$ if $j < m$, and $x' = a'$ otherwise). But this contradicts what was said above about $f[V_n]$, and thus we have $f[U_n] \cap W_m = \emptyset$. In a similar way, now using the equality above involving $f[Z_n]$, it can be shown that $f[W_n] \cap U_m = \emptyset$.

We now prove by complete induction that for all $i \leqslant m$,

$$f(u_i) = u_i',\ f(v_i) = v_i',\ f(w_i) = w_i', \text{ and } f(z_i) = z_i'.$$

Take $i \leqslant m$ and suppose that the induction hypothesis holds for all $j < i$. Since $f(w_{i-1}) = w_{i-1}'$ and $f[U_n] \cap W_m = \emptyset$, we must have $f(u_i) \in \{u_i', z_{i-1}'\}$ by Lemma 4.15. But $f(u_i) = z_{i-1}'$ cannot happen, as this would entail $f[{\downarrow} u_i] = {\downarrow} z_{i-1}'$. Looking at the poset structure of both T_n and T_m, and by using our induction hypothesis, $f[{\downarrow} u_i] = {\downarrow} z_{i-1}'$ yields, e.g., that $a' \notin f[T_n]$, which contradicts the surjectivity of f. Thus, $f(u_i) = u_i'$. In a similar way, it can be proven that $f(w_i) = w_i'$. We end this short proof by induction by noting that the equalities $f(v_i) = v_i'$ and $f(z_i) = z_i'$

follow from what was said above about the sets $f[V_n]$ and $f[Z_n]$, respectively, and by using Lemma 4.15.

Finally, since we now know $f(w_m) = w'_m$, $f[U_n] \cap W_m = \emptyset$, and $f(u_{m+1}) \neq z'_m$ (see the previous paragraph), it follows from Lemma 4.15 that $f(u_{m+1}) = a'$. But by the same argument used in the first paragraph, this yields yet another contradiction. Therefore, f cannot exist, and we showed $T_m \not\leq T_n$, as desired. □

By our previous discussion, the following theorem follows immediately.

Theorem 4.17. *The cardinality of the lattice* $\Lambda(\text{bi-LC})$ *is* $2^{\aleph_0}$.

4.3. **Subframe formulas.** Let $\mathfrak{A} \in \text{bi-GA}_{FSI}$ and introduce, for each $a \in A$, a propositional variable $p_a \in Prop$. Let Γ be the formula describing the algebraic structure of the $(\vee, \leftarrow)$-reduct of $\mathfrak{A}$, i.e.,

$$\Gamma := \bigwedge\{p_{a\vee b} \leftrightarrow (p_a \vee p_b) \colon a, b \in A\} \wedge \bigwedge\{p_{a\leftarrow b} \leftrightarrow (p_a \leftarrow p_b) \colon a, b \in A\}.$$

We define the *subframe formula* of $\mathfrak{A}$ by

$$\beta(\mathfrak{A}) := \neg\sim\Gamma \to \neg \bigwedge\{p_a \leftarrow p_b \colon a, b \in A \text{ and } a \not\leq b\}.$$

In order to state the analogue of the Stable Jankov Lemma for subframe formulas, we need to introduce the notion of a $(\vee, \leftarrow)$-*homomorphism* between co-Heyting algebras, i.e., a map $f \colon \mathfrak{A} \to \mathfrak{B}$ that preserves both $\vee$ and $\leftarrow$. Notice that any such map must always preserves 0, since the equation $x \leftarrow x \approx 0$ is valid on all co-Heyting algebras, and therefore

$$f(0) = f(a \leftarrow a) = f(a) \leftarrow f(a) = 0.$$

If f is moreover injective, then it is called a $(\vee, \leftarrow)$-*embedding*, denoted by $f \colon \mathfrak{A} \hookrightarrow \mathfrak{B}$.

As before, we use the facts stated in Lemma 4.2 without further reference in the following proof.

Lemma 4.18. *(Subframe Jankov Lemma) If* $\mathfrak{B} \in \text{bi-GA}$ *and* $\mathfrak{A} \in \text{bi-GA}_{FSI}$*, then* $\mathfrak{B} \not\models \beta(\mathfrak{A})$ *iff there exists a* $(\vee, \leftarrow)$*-embedding* $h \colon \mathfrak{A} \hookrightarrow \mathfrak{C}$*, for some* $\mathfrak{C} \in \mathbb{H}(\mathfrak{B})_{SI}$.

Proof. To show that the left to right implication holds, we can apply the argument used to prove the same direction of the Stable Jankov Lemma 4.3, but since now the formula Γ only describes the algebraic structure of the $(\vee, \leftarrow)$-reduct of $\mathfrak{A}$, the injective map $h \colon A \to C$ is simply a $(\vee, \leftarrow)$-embedding, as desired.

Conversely, suppose there exists a $(\vee, \leftarrow)$-embedding $h \colon \mathfrak{A} \hookrightarrow \mathfrak{C}$, for some $\mathfrak{C} \in \mathbb{H}(\mathfrak{B})_{SI}$. Let $v \colon Prop \to C$ be a valuation on $\mathfrak{C}$ satisfying $v(p_a) = h(a)$, for all $a \in A$. We show that $\mathfrak{C}$ refutes $\beta(\mathfrak{A})$ via v, which is a sufficient condition for $\mathfrak{B} \not\models \beta(\mathfrak{A})$. Firstly, note that since h is a $(\vee, \leftarrow)$-homomorphism, we have for that all $a, b \in A$,

$$v(p_{a\vee b}) = h(a \vee b) = h(a) \vee h(b) = v(p_a) \vee v(p_b),$$

hence $v(p_{a\vee b} \leftrightarrow p_a \vee p_b) = 1$. Similarly, we have

$$v(p_{a\leftarrow b}) = h(a \leftarrow b) = h(a) \leftarrow h(b) = v(p_a) \leftarrow v(p_b),$$

and thus $v(p_{a\leftarrow b} \leftrightarrow p_a \leftarrow p_b) = 1$. By the equalities above, we see that $v(\Gamma) = 1$, and therefore $v(\neg\sim\Gamma) = \neg\sim v(\Gamma) = \neg\sim 1 = 1$. Now, if $a, b \in A$ are such that $a \not\leq b$, i.e., $a \leftarrow b \neq 0$, then it follows $0 \neq h(a \leftarrow b) = v(p_{a\leftarrow b})$, since h is an injective map that preserves 0. This proves that $0 \notin \{v(p_a \leftarrow p_b) \colon a, b \in A \text{ and } a \not\leq b\}$. As $\mathfrak{C}$ is SI by assumption, $0_{\mathfrak{C}}$ is $\wedge$-irreducible (see Theorem 3.6), and thus we get $\bigwedge\{v(p_a \leftarrow p_b) \colon a, b \in A \text{ and } a \not\leq b\} \neq 0$. Equivalently,

$$\neg \bigwedge\{v(p_a \leftarrow p_b) \colon a, b \in A \text{ and } a \not\leq b\} \neq 1,$$

and we conclude

$$v(\beta(\mathfrak{A})) = 1 \to \neg \bigwedge\{v(p_a \leftarrow p_b) \colon a, b \in A \text{ and } a \not\leq b\} \neq 1.$$ □

Next we introduce partial co-Esakia morphisms, which are necessary to translate the Subframe Jankov Lemma into terms of bi-Esakia spaces.

Definition 4.19. Let $\mathcal{X}$ and $\mathcal{Y}$ be co-Esakia spaces. A partial map $f\colon \mathcal{X} \to \mathcal{Y}$ is a *partial co-Esakia morphism* if it satisfies the following conditions:

(i) $\forall x, z \in dom(f)\ \big(x \leqslant z \implies f(x) \leqslant f(z)\big)$;
(ii) $\forall x \in dom(f), \forall y \in Y\ \big(y \leqslant f(x) \implies \exists z \in {\downarrow}x\ (f(z) = y)\big)$;
(iii) $\forall x \in X\ \big(x \in dom(f) \iff \exists y \in Y\ (f[{\downarrow}x] = {\downarrow}y)\big)$;
(iv) $\forall x \in X\ \big(f[{\downarrow}x] \in Cl(\mathcal{Y})\big)$;
(v) $\forall U \in CpUp(\mathcal{Y})\ \big({\uparrow}f^{-1}U \in CpUp(\mathcal{X})\big)$.

Proposition 4.20. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be co-Heyting algebras, while $\mathcal{X}$ and $\mathcal{Y}$ are co-Esakia spaces.*

(i) *If $f\colon \mathcal{X} \to \mathcal{Y}$ is a partial co-Esakia morphism, then setting $f^*(U) := {\uparrow}f^{-1}U$, for $U \in CpUp(\mathcal{Y})$, yields a $(\vee, \leftarrow)$-homomorphism $f^*\colon \mathcal{Y}^* \to \mathcal{X}^*$ between co-Heyting algebras. Moreover, if f is surjective then f^* is a $(\vee, \leftarrow)$-embedding;*
(ii) *If $h\colon \mathfrak{A} \to \mathfrak{B}$ is a $(\vee, \leftarrow)$-homomorphism between co-Heyting algebras, then setting $dom(h_*) := \{x \in B_*\colon h^{-1}x \in A_*\}$ and $h_*(x) := h^{-1}x$, for $x \in dom(h_*)$, yields a partial co-Esakia morphism $h_*\colon \mathfrak{B}_* \to \mathfrak{A}_*$ between co-Esakia spaces. Moreover, if h is a $(\vee, \leftarrow)$-embedding then h_* is surjective.*

Proof. Both (i) and (ii) can be proven simply by order-dualizing the proofs of the analogous results for partial Esakia morphisms and $(\wedge, \to)$-homomorphisms between Heyting algebras (see, e.g., [6]). □

Before we present the Dual Subframe Lemma and some of its equivalent conditions, we need one more definition and a subsequent lemma.

Definition 4.21. A map $f\colon \mathfrak{F} \hookrightarrow \mathfrak{G}$ between posets is an *order-embedding* if it is *order-invariant* $\big($i.e., $w \leqslant v \iff f(w) \leqslant f(v)\ \big)$. In this case, we say that $\mathfrak{F}$ *order-embeds into* $\mathfrak{G}$.

Lemma 4.22. *If $\mathfrak{F}$ is a finite co-tree and $\mathcal{X}$ a co-Esakia space, then $\mathfrak{F}$ order-embeds into $\mathcal{X}$ iff there exists a surjective partial co-Esakia morphism $f\colon \mathcal{X} \twoheadrightarrow \mathfrak{F}$.*

Proof. This is exactly the order-dual version of [5, Thm. 3.6] (and thus we omit the proof). □

We are finally ready to translate the Subframe Jankov Lemma into the language of bi-Esakia co-forests.

Lemma 4.23. *(Dual Subframe Jankov Lemma) If $\mathfrak{B} \in$ bi-GA and $\mathfrak{A} \in$ bi-GA$_{FSI}$, then the following conditions are equivalent:*

1. $\mathfrak{B} \not\models \beta(\mathfrak{A})$;
2. *there exists a $(\vee, \leftarrow)$-embedding $h\colon \mathfrak{A} \hookrightarrow \mathfrak{C}$, for some $\mathfrak{C} \in \mathbb{H}(\mathfrak{B})_{SI}$;*
3. *there exists a surjective partial co-Esakia morphism $f\colon \mathfrak{C}_* \twoheadrightarrow \mathfrak{A}_*$, for some $\mathfrak{C} \in \mathbb{H}(\mathfrak{B})_{SI}$;*
4. *$\mathfrak{A}_*$ order-embeds into $\mathfrak{C}_*$, for some $\mathfrak{C} \in \mathbb{H}(\mathfrak{B})_{SI}$;*
5. *$\mathfrak{A}_*$ order-embeds into $\mathfrak{B}_*$;*
6. *there exists a surjective partial co-Esakia morphism $f\colon \mathfrak{B}_* \to \mathfrak{A}_*$;*
7. *there exists a $(\vee, \leftarrow)$-embedding $h\colon \mathfrak{A} \to \mathfrak{B}$.*

Proof. The equivalence (1) $\iff$ (2) is just the Subframe Jankov Lemma 4.18, while (2) $\iff$ (3) follows immediately from the duality between $(\vee, \leftarrow)$-homomorphisms of co-Heyting algebras and partial co-Esakia morphisms stated in Proposition 4.20. Notice that this result also yields (6) $\iff$ (7). As an immediate consequence of Lemma 4.22, we have that both (3) $\iff$ (4) and (5) $\iff$ (6) hold true.

Finally, to see that (4) $\implies$ (5), let $\mathfrak{C} \in \mathbb{H}(\mathfrak{B})_{SI}$ and note that if $\mathfrak{A}_*$ order-embeds into a bi-generated subframe of $\mathfrak{B}_*$, such as $\mathfrak{C}_*$, then clearly $\mathfrak{A}_*$ order-embeds into $\mathfrak{B}_*$. Conversely, suppose that $\mathfrak{A}_*$ order-embeds into $\mathfrak{B}_*$. Since $\mathfrak{A}$ is nontrivial, $\mathfrak{A}_*$ is nonempty, hence so is $\mathfrak{B}_*$. Therefore, we can write $B_* = \biguplus_{i \in I} T_i$ as a nonempty disjoint union of co-trees. By the definition of an order-embedding, it is clear that the co-tree $\mathfrak{A}_*$ is mapped to a single co-tree T_i. Since we can view T_i as a bi-generated subframe of $\mathfrak{B}_*$, by equipping T_i with the subspace topology, we conclude by duality that (4) holds. Thus, we proved (5) $\implies$ (4). □

Before we present some applications of subframe formulas, we need a few definitions. Let $\mathfrak{F}$ be a co-tree and $n \in \omega$. If $\mathfrak{F}$ has a chain with n elements, and all chains of $\mathfrak{F}$ have at most n elements, we say that $\mathfrak{F}$ has *depth* n, and write $dp(\mathfrak{F}) = n$. Otherwise, we say that $\mathfrak{F}$ has *infinite depth*. Furthermore, if $\mathfrak{F}$ has an antichain (i.e., a subposet whose elements are pairwise incomparable) with n elements, and all antichains in $\mathfrak{F}$ have at most n elements, we say that $\mathfrak{F}$ has *width* n, and write $wd(\mathfrak{F}) = n$. Otherwise, we say that $\mathfrak{F}$ has *infinite width*.

We prove that if $n \in \omega \smallsetminus \{0\}$, then the (bi-intuitionistic) logic of co-trees of depth (respectively, width) less than n can be axiomatized by a single subframe formula.

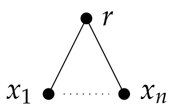

FIGURE 3. The n-co-fork $\mathfrak{F}_n$.

Proposition 4.24. *Let n be a positive integer, $\mathfrak{C}_n$ be the n-chain, and $\mathfrak{F}_n$ be the n-co-fork (see Figure 3). If $\mathcal{X}$ is bi-Esakia co-tree, then:*

(i) $\mathcal{X} \models \beta(\mathfrak{C}_n^*) \iff dp(\mathcal{X}) < n$. *Equivalently,* bi-LC $+\beta(\mathfrak{C}_n^*)$ *is the logic of co-trees of depth less that n;*

(ii) $\mathcal{X} \models \beta(\mathfrak{F}_n^*) \iff wd(\mathcal{X}) < n$. *Equivalently,* bi-LC $+\beta(\mathfrak{F}_n^*)$ *is the logic of co-trees of width less that n.*

Proof. The desired equivalences follows immediately from Lemma 4.23, noting that by the definition of an order-embedding, we clearly have $\mathfrak{C}_n$ does not order-embed into $\mathcal{X}$ iff $dp(\mathcal{X}) < n$, and that $\mathfrak{F}_n$ does not order-embed into $\mathcal{X}$ iff $wd(\mathcal{X}) < n$. The last part of both statements are now an immediate consequence of the algebraic completeness of bi-LC and the bi-Esakia duality. □

The above result can also be used to characterize another notable class of algebras. A bounded distributive lattice is said to be *linear* if its partial order $\leqslant$ is linear. Note that any such lattice can be viewed as a bi-Heyting algebra, as it has well-defined operations:

$$a \to b = \begin{cases} 1 & \text{if } a \leqslant b, \\ b & \text{if } b < a, \end{cases} \qquad \text{and} \qquad a \leftarrow b = \begin{cases} 0 & \text{if } a \leqslant b, \\ a & \text{if } b < a. \end{cases}$$

Thus, the terms linear distributive lattice, *linear Heyting algebra*, and *linear bi-Heyting algebra* are all equivalent. Using the bi-Esakia duality, it is easy to see that $\mathfrak{A}$ is a nontrivial linear Heyting algebra iff (the underlying poset of its bi-Esakia dual) $\mathfrak{A}_*$ is a nonempty chain iff $\mathfrak{A}_*$ is a co-tree of width 1. In other words, $\mathfrak{A}$ is an SI bi-Gödel algebra satisfying $\beta(\mathfrak{F}_2^*)$.

Let us denote by bi-LA the variety generated by linear bi-Heyting algebras. From the above discussion, we obtain the following.

Theorem 4.25. *The bi-intermediate logic of totally ordered Kripke frames is algebraized by* bi-LA *and coincides with* bi-LC $+\beta(\mathfrak{F}_2^*)$.

5. LOCALLY TABULAR EXTENSIONS OF bi-LC

A bi-intermediate logic L is said to be *locally tabular* when V_L is locally finite. Equivalently, L is locally tabular when for every positive integer n there are finitely many formulas $\varphi_1, \ldots, \varphi_m$ in variables $x_1, \ldots, x_n$ such that for every other formula ψ in variables $x_1, \ldots, x_n$ there exists $i \leqslant m$ such that L contains $\varphi_i \leftrightarrow \psi$.

In this section we present a characterization of locally tabular extensions of bi-LC. To this end, for each positive integer n, we define $\mathfrak{C}_n := (C_n, \leqslant_n)$ as the finite co-tree depicted in Figure 4, and call it the *n-comb*. Our aim is to prove the following result:

Theorem 5.1. *An extension of* bi-LC *if locally tabular iff it contains $\beta(\mathfrak{C}_n^*)$ for some $n \in \omega$.*

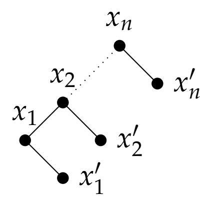

FIGURE 4. The n-comb $\mathfrak{C}_n$.

The first step in the proof of Theorem 5.1 consists in establishing the next result.

Theorem 5.2. *Let $\mathfrak{F}$ be a co-tree and n a positive integer. If $\mathfrak{C}_n$ order-embeds into $\mathfrak{F}$, then $\mathfrak{C}_n$ is a bi-p-morphic image of $\mathfrak{F}$.*

To this end, recall that given a poset $(X, \leqslant)$ and a subset $U \subseteq X$, we denote the minimum (respectively, the maximum) of U, if it exists, by $Min(U)$ (respectively, $Max(U)$). Let us fix a positive integer n and a co-tree $\mathfrak{F} = (X, \leqslant)$ such that $\mathfrak{C}_n = (C_n, \leqslant_n)$ order-embeds into $\mathfrak{F}$. In order to prove Theorem 5.2, it suffices to show that $\mathfrak{C}_n$ is a bi-p-morphic image of $\mathfrak{F}$.

First, as $\mathfrak{C}_n$ order-embeds into $\mathfrak{F}$, we can view $\mathfrak{C}_n$ as a subposet of $\mathfrak{F}$, i.e., $C_n \subseteq X$ and $\leqslant_n = C_n^2 \cap \leqslant$. Furthermore, given $a, b \in X$, we write $a \prec_{\mathfrak{C}_n} b$ to express that b is the (the use of the word "the" is justified, since in a co-tree, points have at most one immediate successor) immediate successor of a in the subposet $\mathfrak{C}_n$ of $\mathfrak{F}$. Notice that b need not be an immediate successor of a in $\mathfrak{F}$. For each $x \in X$, define the following conditions:

- $(\mathbf{A_1^x})$ $C_n \cap {\uparrow}x = \emptyset$;
- $(\mathbf{A_2^x})$ $\exists a_x^- := Min(C_n \cap {\uparrow}x)$;
- $(\mathbf{B_1^x})$ $C_n \cap {\downarrow}x = \emptyset$;
- $(\mathbf{B_2^x})$ $\exists b_x^+ := Max(C_n \cap {\downarrow}x)$;
- $(\mathbf{B_3^x})$ $max(C_n \cap {\downarrow}x) = \{c_x, d_x\}$, $c_x \neq d_x$ and there exists $p_x \in C_n$ such that $c_x, d_x \prec_{\mathfrak{C}_n} p_x$;
- $(\mathbf{D^x})$ $\forall z \in {\downarrow}x \; (\neg \mathbf{B_1^z})$;
- $(\mathbf{E^x})$ $\mathbf{A_2^x}$ & $\forall z < x(\mathbf{A_2^z}$ & $\mathbf{B_1^z} \implies a_z^- < a_x^-)$

We rely on the following observation.

Lemma 5.3. *Let $\mathfrak{F}$ be as above and $x, y \in X$. If $x \leqslant y$, then the following conditions hold:*

1. $x \in C_n \implies \neg \mathbf{B_1^y}$;
2. $\mathbf{A_1^x} \implies \mathbf{A_1^y}$;
3. $(\mathbf{A_2^x}$ *and* $a_x^- \leqslant y) \implies \neg \mathbf{B_1^y}$;
4. $(\mathbf{A_2^x}$ *and* $y \leqslant a_x^-) \implies (\mathbf{A_2^y}$ *and* $a_y^- = a_x^-)$;
5. $\mathbf{B_2^x} \implies \neg \mathbf{B_1^y}$;
6. $(\mathbf{A_2^x}$ & $\mathbf{B_2^x}$ *and* $x \in C_n) \implies b_x^+ = x = a_x^-$;
7. $(\mathbf{A_2^x}$ & $\mathbf{B_2^x}$ *and* $x \notin C_n) \implies b_x^+ \prec_{\mathfrak{C}_n} a_x^-$;
8. $\mathbf{B_3^x} \implies (\mathbf{A_2^x}$ *and* $p_x = a_x^-)$;
9. $\mathbf{B_3^x} \implies \neg \mathbf{B_1^y}$;
10. $\neg \mathbf{D^x} \implies \neg \mathbf{D^y}$;

Proof. Immediate from the definitions, noting that $\mathfrak{F}$ is a co-tree, so both ${\uparrow}x$ and ${\uparrow}y$ are chains, and using the inclusions ${\uparrow}y \subseteq {\uparrow}x$ and ${\downarrow}x \subseteq {\downarrow}y$. ☐

As we mentioned, our aim is to construct a surjective bi-p-morphism $f_n^{\mathfrak{F}} \colon \mathfrak{F} \to \mathfrak{C}_n$. To this end, let $f_n^{\mathfrak{F}} \colon X \to C_n$ be the map defined by

$$f_n^{\mathfrak{F}}(x) := \begin{cases} x & \text{if } x \in C_n, \\ x_n & \text{if } x_n < x, \\ x'_n & \text{if } \mathbf{A_1^x} \ \& \ \mathbf{B_1^x}, \\ g_n(a_x^-) & \text{if } \mathbf{A_2^x} \ \& \ \mathbf{B_1^x}, \\ b_x^+ & \text{if } (\mathbf{A_2^x} \ \& \ \mathbf{B_2^x} \ \& \ \mathbf{D^x}) \text{ or } (\mathbf{A_2^x} \ \& \ \mathbf{B_2^x} \ \& \neg\mathbf{D^x} \ \& \ \mathbf{E^x}), \\ a_x^- & \text{if } (\mathbf{A_2^x} \ \& \ \mathbf{B_2^x} \ \& \neg\mathbf{D^x} \ \& \neg\mathbf{E^x}) \text{ or } \mathbf{B_3^x}, \end{cases}$$

where $g_n(x_i) := x'_i$ and $g_n(x'_i) := x'_i$, for all $i \leqslant n$.

Lemma 5.4. *Let $\mathfrak{F}$ be as above. The map $f_n^{\mathfrak{F}} \colon X \to C_n$ is well defined.*

Proof. Let $x \in X$ and $f := f_n^{\mathfrak{F}}$. Firstly, let us note that the conditions $\mathbf{A_1^x}$ and $\mathbf{A_2^x}$, and the conditions $\mathbf{B_1^x}$, $\mathbf{B_2^x}$, and $\mathbf{B_3^x}$ are, respectively, pairwise exclusive. If $x \in C_n$, we have $\mathbf{A_2^x}$ and $\mathbf{B_2^x}$, where $b_x^+ = x = a_x^-$ by condition (6) of the previous lemma. In this case, by the previous comment we know that $\mathbf{A_1^x}$, $\mathbf{B_1^x}$, and $\mathbf{B_3^x}$ all fail. Furthermore, we have $x_n \not< x$, since x_n is the greatest element of $\mathfrak{C}_n$. Therefore, from the definition of f it follows that necessarily $f(x) = x$.

Suppose now that $\mathbf{A_1^x}$ holds, hence clearly $x \notin C_n$. If $\mathbf{B_1^x}$ also holds, then we have $x \notin C_n$, $x_n \not< x$, and that both $\mathbf{B_2^x}$ and $\mathbf{B_3^x}$ fail. Thus, we must have $f(x) = x'_n$. On the other hand, if $\mathbf{B_1^x}$ fails, i.e., $C_n \cap {\downarrow}x \neq \emptyset$, then the condition $\mathbf{A_1^x}$ and the co-tree structure of $\mathfrak{F}$ entail $b_x^+ = Max(C_n \cap {\downarrow}x) = x_n$, hence $\mathbf{B_2^x}$ holds and we are in the case $x_n < x$. Notice that the previous comment yields the implication ($\mathbf{A_1^x} \,\&\, \neg\mathbf{B_1^x} \implies \mathbf{A_1^x} \,\&\, \mathbf{B_2^x}$), so the case $\mathbf{A_1^x} \,\&\, \mathbf{B_3^x}$ cannot happen, since $\mathbf{B_1^x}$, $\mathbf{B_2^x}$, and $\mathbf{B_3^x}$ are mutually exclusive. It is easy to see that $x_n < x$ implies both $\mathbf{A_1^x}$ and $\mathbf{B_2^x}$, hence by above we have the equivalence $x_n < x$ iff $\mathbf{A_1^x} \,\&\, \mathbf{B_2^x}$. It is now clear that if $x_n < x$, then the only possibility for $f(x)$ is x_n, and that this discussion covers all possible cases where $\mathbf{A_1^x}$ holds.

It now remains to consider the case where $x \notin C_n$ and $\neg\mathbf{A_1^x}$. Accordingly, suppose $x \notin C_n$. If $\mathbf{A_1^x}$ does not hold, then since $\mathfrak{F}$ is a co-tree and C_n is finite, we must have $\mathbf{A_2^x}$. To see this, notice that $C_n \cap {\uparrow}x$ is then a finite nonempty chain, and thus has a minimum. Moreover, since $\leqslant$ is transitive and anti-symmetric, it follows from $C_n \cap {\uparrow}x \neq \emptyset$ that $x_n \not< x$. By this previous discussion and since the conditions $\mathbf{B_1^x}$, $\mathbf{B_2^x}$, and $\mathbf{B_3^x}$ are mutually exclusive, if $\neg\mathbf{A_1^x}$ and $\mathbf{B_1^x}$ hold, then we have $\mathbf{A_2^x} \,\&\, \mathbf{B_1^x}$ and necessarily $f(x) = g_n(a_x^-)$. Suppose now that $\mathbf{B_2^x}$ holds, so we have $\mathbf{A_2^x} \,\&\, \mathbf{B_2^x}$. By what was said until now, and as we clearly have either $\mathbf{D^x}$ or $\neg\mathbf{D^x} \,\&\, \mathbf{E^x}$, or $\neg\mathbf{D^x} \,\&\, \neg\mathbf{E^x}$(and these possibilities are obviously mutually exclusive), then we must have either $f(x) = b_x^+$, or $f(x) = a_x^-$, respectively. Thus, if $\mathbf{A_2^x} \,\&\, \mathbf{B_2^x}$ holds, then $f(x)$ is well defined.

Finally, we prove that $\neg\mathbf{B_1^x} \,\&\, \neg\mathbf{B_2^x}$ implies $\mathbf{B_3^x}$. Suppose $\neg\mathbf{B_1^x} \,\&\, \neg\mathbf{B_2^x}$, and that $x_i \notin {\downarrow}x$, for all $i \leqslant n$. It then follows that $x'_i, x'_j \in {\downarrow}x$, for some $i < j \leqslant n$. As $x_i \notin {\downarrow}x$, by assumption, then ${\uparrow}x'_i$ being a chain in $\mathfrak{F}$ and $x, x_i \in {\uparrow}x'_i$ now imply $x'_i < x < x_i$, and thus we have $x'_j < x < x_i$, contradicting the poset structure of $\mathfrak{C}_n$. Thus, not only there exists $x_i \in {\downarrow}x$, for some $i < n$ (notice that this inequality is indeed strict, since $x_n \in {\downarrow}x$ implies $x_n = Max(C_n \cap {\downarrow}x)$, and we assumed $\neg\mathbf{B_2^x}$), but since C_n is finite, there exists $m := Max(\{i \colon i < n \text{ and } x_i \in {\downarrow}x\})$. By the definition of m, we know $x_{m+1} \notin {\downarrow}x$, and since ${\uparrow}x_m$ is a chain in $\mathfrak{F}$ containing both x_{m+1} and x, it now follows $x_m < x < x_{m+1}$. Recall that we assumed $\neg\mathbf{B_2^x}$, i.e., that $C_n \cap {\downarrow}x$ does not have a greatest element. In particular, this implies that $C_n \cap {\downarrow}x$ has an element not lying below x_m. By looking at the structure of $\mathfrak{C}_n$ and by the definition of m, we see that we must have $x'_j \in {\downarrow}x$, for some $j \in \{m+1, \dots, n\}$. Note that the only j satisfying the previous condition is $j = m+1$, since $x'_j < x < x_{m+1}$ implies $j \leqslant m+1$, again by the poset structure of $\mathfrak{C}_n$. We can now conclude that $max(C_n \cap {\downarrow}x) = \{x_m, x'_{m+1}\}$. Since x_{m+1} is the unique immediate successor in $\mathfrak{C}_n$ of both x_m and x'_{m+1}, $\mathbf{B_3^x}$ holds, as desired.

Then it only remains to consider the case were $\mathbf{B_3^x}$ holds and $x \notin C_n$ (and $\neg\mathbf{A_1^x}$). Since $\mathbf{B_3^x}$ implies $\neg\mathbf{B_1^x}$, we have $x_n \nleq x$. As $\mathbf{B_3^x}$ is incompatible with $\mathbf{B_1^x}$ and $\mathbf{B_2^x}$, this yields $f(x) = a_x^-$. As we covered all possible cases for $x \in X$, we conclude that f is well defined. □

Lemma 5.5. *Let $\mathfrak{F}$ be as above. The map $f_n^{\mathfrak{F}} \colon \mathfrak{F} \to \mathfrak{C}_n$ is a surjective bi-p-morphism.*

Proof. Let $f := f_n^{\mathfrak{F}}$. We start by proving that f is order preserving. Consider $x, y \in X$ such that $x \leqslant y$. Notice that the case $x = y$ is trivial. Furthermore, if $x_n < y$ then $f(y) = x_n$, so $f(x) \leqslant_n x_n = f(y)$, since x_n is the maximum of $\mathfrak{C}_n$. We end this preliminary discussion by noting that if $y \in C_n$, then $f(x) \leqslant_n f(y)$. To see this, just note that: the case $x \in C_n$ is immediate; $x_n < x$ cannot happen; if $\mathbf{A}_2^{\mathbf{x}} \,\&\, \mathbf{B}_1^{\mathbf{x}}$ then $y \in C_n \cap {\uparrow} x$ entails

$$f(x) = g_n(a_x^-) \leqslant_n a_x^- = Min(C_n \cap {\uparrow} x) \leqslant_n y = f(y);$$

the previous argument also covers the cases where $f(x) = a_x^-$; and if $f(x) = b_x^+$, then we have

$$f(x) = b_x^+ = Max(C_n \cap {\downarrow} x) \leqslant x \leqslant y = f(y),$$

hence $f(x) \leqslant_n f(y)$. By this previous discussion, we can assume, without loss of generality, that $x < y$, $x_n \nless y$, and $y \notin C_n$. We proceed by cases, noting that $x_n < x$ cannot happen by our assumptions.

- <u>**Case 1:** $x \in C_n$, so $f(x) = x$;</u>
  By assumption, we have $x_n \nless y$, $y \notin C_n$, and $\neg \mathbf{B}_1^{\mathbf{y}}$ (see condition (1) of Lemma 5.3). Consequently, by the definition of f it suffices to consider the following two cases:
  - <u>**Case 1.1:** $(\mathbf{A}_2^{\mathbf{y}} \,\&\, \mathbf{B}_2^{\mathbf{y}} \,\&\, \mathbf{D}^{\mathbf{y}})$ or $(\mathbf{A}_2^{\mathbf{y}} \,\&\, \mathbf{B}_2^{\mathbf{y}} \,\&\, \neg \mathbf{D}^{\mathbf{y}} \,\&\, \mathbf{E}^{\mathbf{y}})$, so $f(y) = b_y^+$;</u>
    Since $x < y$ and $x \in C_n$, by assumption, it follows $x \in C_n \cap {\downarrow} y$. It is now clear that
    $$f(x) = x \leqslant_n Max(C_n \cap {\downarrow} y) = b_y^+ = f(y).$$
  - <u>**Case 1.2:** $(\mathbf{A}_2^{\mathbf{y}} \,\&\, \mathbf{B}_2^{\mathbf{y}} \,\&\, \neg \mathbf{D}^{\mathbf{y}} \,\&\, \neg \mathbf{E}^{\mathbf{y}})$ or $\mathbf{B}_3^{\mathbf{y}}$, so $f(y) = a_y^-$;</u>
    Since $x < y$ by assumption, and $y \leqslant Min(C_n \cap {\uparrow} y) = a_y^- = f(y)$, it follows
    $$f(x) = x < y \leqslant a_y^- = f(y),$$
    hence $f(x) \leqslant_n f(y)$.
- <u>**Case 2:** $\mathbf{A}_1^{\mathbf{x}} \,\&\, \mathbf{B}_1^{\mathbf{x}}$, so $f(x) = x_n'$;</u>
  By assumption, we have $x_n \nless y$, $y \notin C_n$, and $\mathbf{A}_1^{\mathbf{y}}$ (see condition (2) Lemma 5.3). Thus, by the definition of f, we need only consider the case where $\mathbf{A}_1^{\mathbf{y}} \,\&\, \mathbf{B}_1^{\mathbf{y}}$ holds, which follows immediately from $f(x) = x_n' = f(y)$.
- <u>**Case 3:** $\mathbf{A}_2^{\mathbf{x}} \,\&\, \mathbf{B}_1^{\mathbf{x}}$, so $f(x) = g(a_x^-)$;</u>
  Notice that $y, a_x^- \in {\uparrow} x$, by our assumption $x < y$ and by the definition of $a_x^- = Min(C_n \cap {\uparrow} x)$. Furthermore, as $\mathfrak{F}$ is a co-tree, ${\uparrow} x$ is a chain, and since we assumed $y \notin C_n$, we have $a_x^- < y$ or $y < a_x^-$. Hence $\neg \mathbf{B}_1^{\mathbf{y}}$or $(\mathbf{A}_2^{\mathbf{y}}$ and $a_y^- = a_x^-)$, respectively, by conditions (3) and (4) of Lemma 5.3. Thus, the case $\mathbf{A}_1^{\mathbf{y}} \,\&\, \mathbf{B}_1^{\mathbf{y}}$ cannot happen. Since we also supposed $x_n \nless y$ and $y \notin C_n$, it follows from the definition of f that it suffices to consider the following cases:
  - <u>**Case 3.1:** $\mathbf{A}_2^{\mathbf{y}} \,\&\, \mathbf{B}_1^{\mathbf{y}}$, so $f(y) = g(a_y^-)$;</u>
    If $\mathbf{A}_2^{\mathbf{y}} \,\&\, \mathbf{B}_1^{\mathbf{y}}$ holds, then by condition (3) of Lemma 5.3 we must have $a_x^- \nleqslant y$. Since $x \leqslant y, a_x^-$ and ${\uparrow} x$ is an upset, this implies $y < a_x^-$, so condition (4) of said lemma yields $a_y^- = a_x^-$. Thus, we have $f(x) = g_n(a_x^-) = g(a_y^-) = f(y)$.
  - <u>**Case 3.2:** $(\mathbf{A}_2^{\mathbf{y}} \,\&\, \mathbf{B}_2^{\mathbf{y}} \,\&\, \mathbf{D}^{\mathbf{y}})$ or $(\mathbf{A}_2^{\mathbf{y}} \,\&\, \mathbf{B}_2^{\mathbf{y}} \,\&\, \neg \mathbf{D}^{\mathbf{y}} \,\&\, \mathbf{E}^{\mathbf{y}})$, so $f(y) = b_y^+$;</u>
    Notice that $x \leqslant y$ and $\mathbf{B}_1^{\mathbf{x}}$ imply $\neg \mathbf{D}^{\mathbf{y}}$, by the definition of $\mathbf{D}^{\mathbf{y}}$. Since we assumed both $x < y$ and $\mathbf{A}_2^{\mathbf{x}} \,\&\, \mathbf{B}_1^{\mathbf{x}}$, then $\mathbf{A}_2^{\mathbf{y}} \,\&\, \mathbf{B}_2^{\mathbf{y}} \,\&\, \neg \mathbf{D}^{\mathbf{y}} \,\&\, \mathbf{E}^{\mathbf{y}}$ must hold. By the definition of $\mathbf{E}^{\mathbf{y}}$, $x < y$ and $\mathbf{A}_2^{\mathbf{x}} \,\&\, \mathbf{B}_1^{\mathbf{x}}$ imply $a_x^- < a_y^-$. By our comment in the beginning of Case 3, the strict inequality $a_x^- < a_y^-$ entails $a_x^- < y$. Hence we have $a_x^- \in C_n \cap {\downarrow} y$, so $a_x^- \leqslant Max(C_n \cap {\downarrow} y) = b_y^+$. It is now clear that $f(x) = g_n(a_x^-) \leqslant_n a_x^- \leqslant_n b_y^+ = f(y)$.
  - <u>**Case 3.3:** $(\mathbf{A}_2^{\mathbf{y}} \,\&\, \mathbf{B}_2^{\mathbf{y}} \,\&\, \neg \mathbf{D}^{\mathbf{y}} \,\&\, \neg \mathbf{E}^{\mathbf{y}})$ or $\mathbf{B}_3^{\mathbf{y}}$, so $f(y) = a_y^-$;</u>
    From $x \leqslant y$ it follows that $a_x^- \leqslant_n a_y^-$. Consequently, $f(x) = g_n(a_x^-) \leqslant_n a_x^- \leqslant_n a_y^- = f(y)$.

- **Case 4:** $(\mathbf{A}_2^{\mathbf{x}}\ \&\ \mathbf{B}_2^{\mathbf{x}}\ \&\ \mathbf{D}^{\mathbf{x}})$ or $(\mathbf{A}_2^{\mathbf{x}}\ \&\ \mathbf{B}_2^{\mathbf{x}}\ \&\neg\mathbf{D}^{\mathbf{x}}\ \&\ \mathbf{E}^{\mathbf{x}})$, so $f(x) = b_x^+$;
  By condition (5) of Lemma 5.3, we have $\neg\mathbf{B}_1^{\mathbf{y}}$. Moreover, since we assumed $x_n \not< y$ and $y \notin C_n$, it follows from the definition of f that the only possibilities for $f(y)$ are either $f(y) = b_y^+$ or $f(y) = a_y^-$. Since from $x \leqslant y$ it follows $b_x^+ \leqslant b_y^+ \leqslant a_y^+$ (where the inequalities hold, provided that the relevant elements exist), we obtain
  $$\big(f(x) = b_x^+ \leqslant_n b_y^+ = f(y)\big) \text{ or } \big(f(x) = b_x^+ \leqslant_n a_y^- = f(y)\big).$$
- **Case 5:** $(\mathbf{A}_2^{\mathbf{x}}\ \&\ \mathbf{B}_2^{\mathbf{x}}\ \&\ \neg\mathbf{D}^{\mathbf{x}}\ \&\neg\mathbf{E}^{\mathbf{x}})$ or $\mathbf{B}_3^{\mathbf{x}}$, so $f(x) = a_x^-$.
  We start by assuming that $\mathbf{A}_2^{\mathbf{x}}\ \&\ \mathbf{B}_2^{\mathbf{x}}\ \&\neg\mathbf{D}^{\mathbf{x}}\ \&\neg\mathbf{E}^{\mathbf{x}}$ holds. By conditions (5) and (10) of Lemma 5.3, $\neg\mathbf{B}_1^{\mathbf{y}}$ and $\neg\mathbf{D}^{\mathbf{y}}$ must also hold. This, together with our assumption $x_n < y \notin C_n$, is enough to infer that either $f(y) = a_y^-$ or $f(y) = b_y^+$.
  The case $f(y) = a_y^-$ is clear, since $x < y$ then entails $f(x) = a_x^- \leqslant_n a_y^- = f(y)$. Then suppose that $f(y) = b_y^+$. As ${\uparrow}x$ is a chain and $x \leqslant a_x^-, y$, either $a_x^- \leqslant y$ or $y < a_x^-$. If $a_x^- \leqslant y$, then clearly $f(x) = a_x^- \leqslant_n Max(C_n \cap {\downarrow}y) = b_y^+ = f(y)$. We now prove that $y < a_x^-$ cannot happen under our assumptions. For suppose this inequality is true. Notice that our hypothesis $\neg\mathbf{E}^{\mathbf{x}}$ yields a point $z < x$ for which $\mathbf{A}_2^{\mathbf{z}}\ \&\ \mathbf{B}_1^{\mathbf{z}}$ holds, but $a_z^- = a_x^-$. Since $x < y < a_x^-$, we also have the equality $a_x^- = a_y^-$. Consequently, $\neg\mathbf{E}^{\mathbf{y}}$ is also true. But under our previous assumptions (in particular, since $\neg\mathbf{B}_1^{\mathbf{y}}$ and $\neg\mathbf{D}^{\mathbf{y}}$), the case $f(y) = b_y^+$ can only happen if $\mathbf{A}_2^{\mathbf{y}}\ \&\ \mathbf{B}_2^{\mathbf{y}}\ \&\ \neg\mathbf{D}^{\mathbf{y}}\ \&\ \mathbf{E}^{\mathbf{y}}$, a contradiction.
  Finally, we consider the case where $\mathbf{B}_3^{\mathbf{x}}$ is true. By using condition (9) of Lemma 5.3 together with our assumptions $x_n \not< y$ and $y \notin C_n$, we see that the only possibilities for the value of $f(y)$ are either a_y^- or b_y^+. The case $f(y) = a_y^-$ follows immediately from the fact that $f(x) = a_x^- \leqslant_n a_y^-$. Suppose now that $f(y) = b_y^+ = Max(C_n \cap {\downarrow}y)$. Note that the inclusion $C_n \cap {\downarrow}x \subseteq C_n \cap {\downarrow}y$ entails that the two maximal points c_x and d_x of $C_n \cap {\downarrow}x$ (these points exist, since we assumed $\mathbf{B}_3^{\mathbf{x}}$), satisfy $c_x, d_x < b_y^+ = Max(C_n \cap {\downarrow}y)$. We now use condition (8) of said lemma to infer that a_x^- is the immediate successor in $\mathfrak{C}_n$ of these maximal points, and we conclude that
  $$f(x) = a_x^- \leqslant_n b_y^+ = f(y),$$
  by the definition of an immediate successor and using the fact that $\mathfrak{C}_n$ is co-tree, so its principal upsets are chains.

We conclude that f is indeed an order preserving map, as desired.

The next step of this proof is to show that f satisfies the forth condition, i.e., for all $x \in X$ and all $u \in C_n$, we have
$$f(x) \leqslant_n u \implies \exists y \in {\uparrow}x\ \big(f(y) = u\big).$$
To this end, let $x \in X$, $u \in C_n$, and suppose that $f(x) \leqslant_n u$. Notice that the case $f(x) = u$ is trivial, as we can take $y := x$. Moreover, if $x \in C_n$, then $y := u$ satisfies the desired conditions. Lastly, if $x \leqslant u$, then we of course can take $y := u$, and if $x_n < x$, then we have $f(x) = x_n \leqslant_n u \leqslant_n x_n$ since x_n is the maximum of $\mathfrak{C}_n$, hence $u = x_n$ and we set $y := x$. Because of this, we may assume without loss of generality that $f(x) <_n u$, $x \notin C_n \cup {\uparrow}x_n$ and $x \nleqslant u$.

If $f(x) = x_n'$, then we must have $u = x_n$, since x_n is the unique element of $\mathfrak{C}_n$ strictly greater than x_n' and $x_n' = f(x) <_n u$. As $\mathfrak{F}$ is a co-tree, it has a co-root r. By the definition of a co-root $x, x_n \leqslant r$. Moreover, by the definition of f, we have $f(r) = x_n$. Thus, we can take $y := r$.

Now, note that the cases $f(x) = a_x^-$ and $f(x) = g_n(a_x^-)$ cannot happen. The former is clear, since then we would have
$$x \leqslant Min(C_n \cap {\uparrow}x) = a_x^- = f(x) < u,$$
but we assumed $x \nleqslant u$. As for the latter, we know, by the definition of the map g_n, that either $g_n(a_x^-) = a_x^-$ or $g_n(a_x^-) \prec_{\mathfrak{C}_n} a_x^-$. Both cases yield $x \leqslant a_x^- \leqslant u$, again contradicting $x \nleqslant u$.

By what was said until now and by the definition of f, the only case that remains to consider is when $(\mathbf{A}_2^{\mathbf{x}}\ \&\ \mathbf{B}_2^{\mathbf{x}}\ \&\ \mathbf{D}^{\mathbf{x}})$ or $(\mathbf{A}_2^{\mathbf{x}}\ \&\ \mathbf{B}_2^{\mathbf{x}}\ \&\neg\mathbf{D}^{\mathbf{x}}\ \&\ \mathbf{E}^{\mathbf{x}})$ hold, and therefore $f(x) = b_x^+$. If this is the case, then note that our assumption $f(x) = b_x^+ = Max(C_n \cap {\downarrow}x) <_n u$ implies

$$u \in C_n \cap {\uparrow} b_x^+ = \big(C_n \cap {\uparrow} a_x^-\big) \cup \{b_x^+\},$$

where the equality above follows from $b_x^+ \prec_{\mathfrak{C}_n} a_x^-$ (see condition (7) of Lemma 5.3) and the fact that in a co-tree, such as $\mathfrak{C}_n$, points have at most one immediate successor. Since u cannot be an element of ${\uparrow} a_x^-$, as we assumed $x \nleqslant u$ and we have the inclusion ${\uparrow} a_x^- \subseteq {\uparrow} x$, then $u \in \big(C_n \cap {\uparrow} a_x^-\big) \cup \{b_x^+\}$ implies $u = b_x^+$. Thus, by taking $y := x$ we are done. Therefore, f satisfies the forth condition, as desired.

Finally, we prove that f satisfies the back condition, i.e., for all $x \in X$ and all $u \in C_n$, we have

$$u \leqslant_n f(x) \implies \exists z \in {\downarrow} x\big(f(z) = u\big).$$

Let $x \in X$, $u \in C_n$, and suppose that $u \leqslant_n f(x)$. As before, we can immediately take care of some cases, allowing us to make some useful assumptions. The case $u = f(x)$ is trivial, and if $u \leqslant x$, we can always take $z := u$. Notice that this already takes care of the case $x \in C_n$. Moreover, if $x_n < x$, then we have $u \leqslant x_n < x$, since x_n is the maximum of $\mathfrak{C}_n$, and we again have $u \leqslant x$. So, we can suppose without loss of generality that $u < f(x)$, $u \nleqslant x$ and $x \notin C_n \cup {\uparrow} x_n$.

Note that neither $f(x) = x_n'$ or $f(x) = g_n(a_x^-)$ can happen, since x_n' and $g_n(a_x^-)$ are minimal points in $\mathfrak{C}_n$, and we assumed $u <_n f(x)$. If we have $f(x) = b_x^+$, then we can take $z := u$, since $u <_n f(x) = b_x^+ = Min(C_n \cap {\downarrow} x)$ and $f(u) = u$.

By our previous discussions and by the definition of f, the only remaining case is when $(\mathbf{A_2^x} \mathbin{\&} \mathbf{B_2^x} \mathbin{\&} \neg \mathbf{D^x} \mathbin{\&} \neg \mathbf{E^x})$ or $\mathbf{B_3^x}$ hold, and thus $f(x) = a_x^-$.

If $\mathbf{B_3^x}$ holds, then x lies above the two immediate predecessors, c_x and d_x, of a_x^- in $\mathfrak{C}_n$, (see condition (8) of Lemma 5.3). Notice that $u <_n f(x) = a_x^-$ yields either $u \leqslant c_x \leqslant x$ or $u \leqslant d_x \leqslant x$, by the poset structure of $\mathfrak{C}_n$ and by the definition of an immediate predecessor. Since both possibilities clearly contradict $u \nleqslant x$, we conclude that $\mathbf{B_3^x}$ cannot be true. Consequently, in this case we must have $\mathbf{A_2^x} \mathbin{\&} \mathbf{B_2^x} \mathbin{\&} \neg \mathbf{D^x} \mathbin{\&} \neg \mathbf{E^x}$. By condition (7) of Lemma 5.3, we know that $b_x^+ \prec_{\mathfrak{C}_n} a_x^-$. Furthermore, since we assumed that $u \nleqslant x$ and we have $b_x^+ \leqslant x$, by definition of b_x^+, then by looking at the poset structure of $\mathfrak{C}_n$, we see that the only way that our assumption $u <_n f(x) = a_x^-$ can hold is if we have $u = g_n(a_x^-) \neq a_x^-$ and $b_x^+ \neq g_n(a_x^-)$. With this previous comment in mind, we suppose $u = g_n(a_x^-) \neq a_x^-$. Since $\neg \mathbf{E^x}$ holds, there exists a $y < x$ such that $\mathbf{A_2^y} \mathbin{\&} \mathbf{B_1^y}$ and $a_y^- = a_x^-$. It now follows that taking $z := y$ satisfies the desired conditions, since $y \in {\downarrow} x$ and $f(y) = g_n(a_y^-) = g_n(a_x^-) = u$. Therefore, f satisfies the back condition.

We conclude that f is a bi-p-morphism, and since $C_n \subseteq X$ and $f[C_n] = C_n$, f is surjective, as desired. ☐

Theorem 5.2 is an immediate consequence of Lemmas 5.4 and 5.5. As a consequence of it, we obtain the following.

Corollary 5.6. *A finite bi-Esakia co-tree $\mathcal{X}$ refutes $\mathcal{J}(\mathfrak{C}_n^*)$ iff it refutes $\beta(\mathfrak{C}_n^*)$.*

Proof. To prove the left to right implication, suppose that a finite bi-Esakia co-tree $\mathcal{X}$ refutes $\beta(\mathfrak{C}_n^*)$. By the Dual Subframe Jankov Lemma 4.23, this is equivalent to $\mathfrak{C}_n$ order-embeds into $\mathcal{X}$. Since a bi-p-morphism between posets equipped with the discrete topology (in particular, between finite bi-Esakia spaces) is always continuous, it follows from Theorem 5.2 that $\mathfrak{C}_n$ is a bi-Esakia morphic image of $\mathcal{X}$. Thus, $\mathcal{X}$ must also refute $\mathcal{J}(\mathfrak{C}_n^*)$, by the Jankov Lemma 4.9.

As for the other implication, just note that $\mathcal{X}$ is a co-tree, hence it has no nontrivial bi-generated subframes. By the Jankov Lemma 4.9, $\mathcal{X}$ refuting $\mathcal{J}(\mathfrak{C}_n^*)$ is then equivalent to the existence of a surjective bi-Esakia morphism from $\mathcal{X}$ onto $\mathfrak{C}_n$. Since bi-Esakia morphisms are obviously partial co-Esakia morphisms, it now follows from the Dual Subframe Jankov Lemma 4.23 that $\mathcal{X}$ refutes $\beta(\mathfrak{C}_n^*)$, as desired. ☐

The second step in the proof of Theorem 5.1 consists in establishing the following:

Lemma 5.7. *$\mathfrak{C}_n^*$ is a 1-generated bi-Gödel algebra, for every $n \in \omega$.*

Proof. Notice that the bi-Heyting algebras dual to the various $\mathfrak{C}_n$ are bi-Gödel algebras, because each $\mathfrak{C}_n$ is a finite co-tree. Therefore, it only remains to prove that $\mathfrak{C}_n^*$ is 1-generated for every positive integer n.

First, the algebraic dual of the 1-comb is the three element chain, which is generated as a bi-Heyting algebra by its only element distinct from 0 and 1. Let then $n \geqslant 2$ and recall that by the Coloring Theorem 2.15, to show that $\mathfrak{C}_n^*$ is 1-generated, i.e., that there exists $U \in Up(\mathfrak{C}_n)$ such that $\mathfrak{C}_n^* = \langle U \rangle$, it suffices to show that every proper bi-bisimulation equivalence on $\mathfrak{C}_n$ (see Definition 2.14) identifies points of different colors, where the coloring of $\mathfrak{C}_n = (C_n, \leqslant_n)$ is defined by

$$col(w) = \begin{cases} 1 & \text{if } w \in U, \\ 0 & \text{if } w \notin U, \end{cases}$$

for all $w \in C_n$. To this end, let

$$U := \{x_1\} \cup {\uparrow}\{x'_i \in C_n : i \leqslant n \text{ is even}\},$$

and E be a proper bi-bisimulation equivalence on $\mathfrak{C}_n$.

First we show that for all $x, y, z \in C_n$, $x \leqslant_n y \leqslant z$ and xEz imply xEy (note that the following argument also works for an arbitrary bi-bisimulation equivalence on an arbitrary bi-Esakia space). Suppose that $\langle x, y \rangle \notin E$. Then the refined condition of E yields the existence of an E-saturated (clopen) upset V separating x and y. If $x \leqslant_n y$, then y must be the point contained in V, since V is an upset. But then $y \leqslant_n z$ entails $z \in V$. As xEz and V is E-saturated, we now have that $x \in V$, contradicting the definition of V. In particular, this discussion shows that if $x_i E x_j$ for some $i < j \leqslant n$, then $x_i E x_{i+1}$.

Next we prove by complete induction on $i < n$ that $x_i E x_{i+1}$ implies that E identifies points of different colors. Suppose $x_i E x_{i+1}$ and that our induction hypothesis holds true for all $j < i$. By the definition of E we have $x'_{i+1} \leqslant x_{i+1}$ and $x_i E x_{i+1}$ entail wEx'_{i+1}, for some $w \in {\downarrow}x_i$. If $w = x'_i$, then $x'_i E x'_{i+1}$ and $col(x'_i) \neq col(x'_{i+1})$, as desired. If $w = x'_j$ for some $j < i$, then applying the forth condition to $x'_j \leqslant x_j$ and $x'_j = wEx'_{i+1}$ yields zEx_j, for some $z \in {\uparrow}x'_{i+1}$.

If $z = x'_{i+1}$, then by the back condition and noting that x'_{i+1} is minimal, it follows that ${\downarrow}x_j \subseteq [\![x'_{i+1}]\!]_E$, where $[\![x'_{i+1}]\!]_E$ is the E-equivalence class of x'_{i+1}. Hence $x'_1 E x'_{i+1} E x_1$ and we are done, since $col(x'_1) = 0 \neq 1 = col(x_1)$. Note that the previous argument can also be used to prove the case where the $w \in {\downarrow}x_i$ satisfying wEx'_{i+1} is equal to x_l, for $l \leqslant i$. On the other hand, if $z \neq x'_{i+1}$, then $z = x_t$ for some $t \geqslant i+1$. As $x_j E z = x_t$, the discussion above entails $x_j E x_{j+1}$, which falls under our induction hypothesis since $j < i < t$. Thus, E identifies points of different colors, as desired.

We are finally ready to prove that if E is a proper bi-bisimulation equivalence on $\mathfrak{C}_n$, then E identifies points of different colors, and therefore the Coloring Theorem ensures $\mathfrak{C}_n^* = \langle U \rangle$, as desired. Let $i < j \leqslant n$. If $x_i E x_j$, then $x_i E x_{i+1}$ by above, and the result now follows by our previous discussion. If $x'_i E x'_j$ or $x_i E x'_j$, using the same argument as in the last part of the proof by induction above yields E-equivalent points of different colors, as desired. The only cases that remain are either $x'_i E x_i$ or $x'_i E x_j$, which are clear, since the minimality of x'_i implies either ${\downarrow}x_i \subseteq [\![x'_i]\!]_E$ or ${\downarrow}x_j \subseteq [\![x'_i]\!]_E$, respectively, hence $x'_1 E x_i E x_1$. Since we have $col(x'_1) \neq col(x_1)$, the result follows. □

As a consequence, we obtain the left to right implication in Theorem 5.1:

Corollary 5.8. *If an extension of* bi-LC *if locally tabular, then it contains $\beta(\mathfrak{C}_n^*)$ for some $n \in \omega$.*

Proof. Let L be a locally tabular extension of bi-LC and suppose, with a view towards contradiction, that it omits the subframe formulas of all the (algebraic duals of the) finite combs. Notice that V_L must have the FMP by Proposition 2.2, so given a positive integer n there exists a finite SI algebra $\mathfrak{A}$ which validates L but refutes $\beta(\mathfrak{C}_n^*)$. Since $\mathfrak{A} \not\models \beta(\mathfrak{C}_n^*)$ iff $\mathfrak{A} \not\models \mathcal{J}(\mathfrak{C}_n^*)$ by Corollary 5.6, it now follows from the Jankov Lemma 4.9 that $\mathfrak{C}_n^*$ must also validate L.

As n was arbitrary in the discussion above, we proved that the algebraic duals of the finite combs, which are all 1-generated (see Lemma 5.7), are in V_L. In particular, this entails that V_L

has arbitrarily large finite 1-generated members. By Theorem 2.3, V_L cannot be locally finite and, therefore, L cannot be locally tabular, which contradicts the assumptions. ☐

The third and last step in the proof of Theorem 5.1 consists in finding a natural bound for the size of m-generated SI bi-Gödel algebras whose bi-Esakia duals do not admit the n-comb as a subposet. We need a few auxiliary lemmas.

Definition 5.9. Given a poset $\mathfrak{F}$ and a chain $H \subseteq \mathfrak{F}$ with a least element m_0 and a greatest element m_1, we say that H is an *isolated chain* (in $\mathfrak{F}$) if

$$\downarrow m_1 \smallsetminus H = \downarrow m_0 \smallsetminus \{m_0\} \text{ and } \uparrow m_0 \smallsetminus H = \uparrow m_1 \smallsetminus \{m_1\}.$$

Example 5.10. Consider the poset $\mathfrak{F}$ depicted in Figure 5. The set $H := \{f, e, d\}$ forms an isolated chain in $\mathfrak{F}$, since $\downarrow d \smallsetminus H = \{g, h\} = \downarrow f \smallsetminus \{f\}$ and $\uparrow f \smallsetminus H = \{b, c, a\} = \uparrow d \smallsetminus \{d\}$. On the other hand, the chain $G := \{a.b, d\}$ is not isolated in $\mathfrak{F}$, since, for example, $c \in \downarrow a \smallsetminus G$ but $c \notin \downarrow d \smallsetminus \{d\}$.

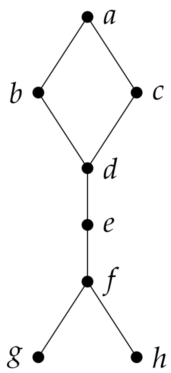

FIGURE 5. The poset $\mathfrak{F}$

Lemma 5.11. *If $\mathcal{X} = (X, \tau, \leqslant)$ is a bi-Esakia space and H is an isolated chain in $\mathcal{X}$, then the least equivalence relation identifying the points in H, $E := H^2 \cup Id_X$, is a bi-bisimulation equivalence on $\mathcal{X}$.*

Proof. That E is an equivalence relation that satisfies the back and forth conditions follows immediately from the definition of E and that of an isolated chain. It remains to show that E is refined, i.e., that every two non-E-equivalent points are separated by an E-saturated clopen upset of $\mathcal{X}$ (notice that, by the definition of E, a clopen upset U is E-saturated iff $H \subseteq U$ or $H \cap U = \emptyset$). To this end, let $w, v \in X$ and suppose $\neg(wEv)$. There are only two possible cases: either $w, v \notin H$, or, without loss of generality, $w \in H$ and $v \notin H$.

We first suppose that $w, v \notin H$. Since $\neg(wEv)$, we have $w \neq v$, and we can suppose without loss of generality that $w \nleqslant v$. By the PSA, there exists $U \in CpUp(\mathcal{X})$ satisfying $w \in U$ and $v \notin U$. If $w < m_1 := Max(H)$, then H being an isolated chain in $\mathcal{X}$ and $w \notin H$ imply $w < m_0 := Min(H)$, hence we have $H \subseteq U$, since U is an upset containing w. Thus, U is an E-saturated clopen upset that separates w from v. On the other hand, if $w \nleqslant m_1$, then by the PSA there exists $V \in CpUp(\mathcal{X})$ such that $w \in V$ and $m_1 \notin V$. Since V is an upset not containing m_1, it follows $H \cap V = \emptyset$, and it is easy to see that $U \cap V$ is an E-saturated clopen upset that separates w from v, as desired.

Suppose now that $w \in H$ and $v \notin H$. If $m_1 \nleqslant v$, then we also have $m_0 \nleqslant v$, by the definition of an isolated chain. The PSA now yields $U \in CpUp(\mathcal{X})$ satisfying $m_0 \in U$ and $v \notin U$. Since U is an upset containing m_0, we have $H \subseteq U$ and clearly U satisfies our desired conditions. On the other hand, if $m_1 \leqslant v$, we must have $m_1 < v$ because $v \notin H$. Consequently, in this case, we have $v \nleqslant m_1$. Therefore, we can apply the PSA obtaining some $V \in CpUp(\mathcal{X})$ such that $v \in V$ and $m_1 \notin V$. Since V is an upset, it follows $H \cap V = \emptyset$, and we conclude that V is an E-saturated clopen upset that separates v from w, as desired. ☐

Recall that an *order-isomorphism* is an order-invariant bijection between posets (in other words, a surjective order-embedding), and that given two points w and v in a poset $\mathfrak{F} = (W, \leqslant)$, we denote $[w, v] := \{x \in W : w \leqslant x \leqslant v\}$. Notice that if $\mathfrak{F}$ is a co-tree, then $[w, v]$ is a chain.

Lemma 5.12. *Let $\mathcal{X} = (X, \tau, \leqslant)$ be a bi-Esakia co-tree and $w, v \in X$ two distinct points with a common immediate successor. If both $\downarrow w$ and $\downarrow v$ are finite, and there exists an order-isomorphism $f : \downarrow w \to \downarrow v$, then*

$$E := \{(x, y) \in X^2 : (x \in \downarrow w \text{ and } f(x) = y) \text{ or } (x \in \downarrow v \text{ and } f(y) = x)\} \cup Id_X$$

is a bi-bisimulation equivalence on $\mathcal{X}$.

Proof. We start by noting that, by its definition, E is clearly an equivalence relation. Furthermore, that E satisfies the back condition is immediate from the definition of E and that of an order-isomorphism. Since we assumed that w and v share an immediate successor, and since in a co-tree points have at most one immediate successor, it follows that E satisfies the forth condition. To see this, let us denote the unique immediate successor of both w and v by u, and note that, for $x \in \downarrow w$ (or $x \in \downarrow v$), we have a description $\uparrow x = [x, w] \cup \uparrow u$ (respectively, $\uparrow x = [x, v] \cup \uparrow u$), since the principal upsets of $\mathcal{X}$ are chains. By this previous comment, the definition of E, and that of an order-isomorphism, it is now clear that E satisfies the forth condition.

We now show that E is refined, thus ensuring that E is a bi-bisimulation equivalence on $\mathcal{X}$. Let $x, y \in X$ and suppose that $\neg(xEy)$. So $x \neq y$, and we can suppose without loss of generality that $x \nleqslant y$. We proceed by cases:

- **Case 1:** $\{x, y\} \cap (\downarrow w \cup \downarrow v) = \varnothing$;

  In this case, we have $x \nleqslant w$ and $x \nleqslant v$. Since we also assumed $x \nleqslant y$, by the PSA there are $U_y, U_w, U_v \in CpUp(\mathcal{X})$ all containing x, and such that $y \notin U_y$, $w \notin U_w$, and $v \notin U_v$. As U_w is an upset not containing w, we have $\downarrow w \cap U_w = \varnothing$. Similarly, it follows $\downarrow v \cap U_v = \varnothing$. Thus, $U := U_y \cap U_w \cap U_v$ is an E-saturated (since $U \cap (\downarrow w \cup \downarrow v) = \varnothing$) clopen upset separating x from y, as desired.
- **Case 2:** $x \notin (\downarrow w \cup \downarrow v)$ and $y \in (\downarrow w \cup \downarrow v)$;

  By assumption, we have $x \nleqslant w$ and $x \nleqslant v$, so by the PSA there are $U_w, U_v \in CpUp(\mathcal{X})$, both containing x, satisfying $w \notin U_w$ and $v \notin U_v$. As U_w is an upset not containing w, we have $\downarrow w \cap U_w = \varnothing$. Similarly, it follows $\downarrow v \cap U_v = \varnothing$. Thus, $U := U_w \cap U_v$ is an E-saturated (since $U \cap (\downarrow w \cup \downarrow v) = \varnothing$) clopen upset separating x from y, since we assumed $y \in (\downarrow w \cup \downarrow v)$. We note that this previous argument can also be used when $y \notin (\downarrow w \cup \downarrow v)$ and $x \in (\downarrow w \cup \downarrow v)$, by replacing x with y, and vice-versa.
- **Case 3:** $x, y \in (\downarrow w \cup \downarrow v)$.

  Without loss of generality, we suppose that $x \in \downarrow w$ and $y \in \downarrow w$ (if $y \in \downarrow v$ or $x \in \downarrow v$, we can replace y or x in the following argument by $f^{-1}(y)$ or $f^{-1}(x)$, respectively, where f^{-1} is the inverse of the order-isomorphism f). As $\downarrow w$ is finite by hypothesis, we can enumerate $\downarrow w \smallsetminus \downarrow y := \{x_1, \ldots, x_n\}$. Notice that for all $i \leqslant n$, we have $x_i \nleqslant y$ by the definition of x_i, and $x_i \nleqslant f(y)$, since $f(y) \in \downarrow v$ and $x_i \notin \downarrow v$ (recall that w and v are distinct points in a co-tree with a common immediate successor, hence we have $\downarrow w \cap \downarrow v = \varnothing$). Using the same argument as in the previous cases, $x_i \nleqslant y$ and $x_i \nleqslant f(y)$ imply, by the PSA, that there exists $U_i \in CpUp(\mathcal{X})$ satisfying $x_i \in U_i$ and $y, f(y) \notin U_i$. As U_i is an upset, it follows that $U_i \cap (\downarrow y \cup \downarrow f(y)) = \varnothing$. Furthermore, by the definition of an order-isomorphism, $x_i \nleqslant y$ entails $f(x_i) \nleqslant f(y)$, and since we have $f(x_i) \in \downarrow v$ and $y \in \downarrow w$, it follows $f(x_i) \nleqslant y$. Again, the PSA yields some $V_i \in CpUp(\mathcal{X})$ satisfying $f(x_i) \in V_i$ and $V_i \cap (\downarrow y \cup \downarrow f(y)) = \varnothing$. Let $U := \bigcup_{i=1}^{n} U_i \cup \bigcup_{i=1}^{n} V_i$, and note that this is a clopen upset satisfying $\{x_1, \ldots, x_n, f(x_1), \ldots, f(x_n)\} \subseteq U$ and $U \cap (\downarrow y \cup \downarrow f(y)) = \varnothing$. As we assumed $x \in \downarrow w$ and $x \nleqslant y$, it now follows $x \in \{x_1, \ldots, x_n\} = \downarrow w \smallsetminus \downarrow y$, and thus $x \in U$. By the way we defined U and E, we conclude that U is an E-saturated clopen upset separating x from y, as desired. □

We now have all the necessary tools to obtain the desired bound.

Proposition 5.13. *If n and m are positive integers, there is a natural bound $k(n,m)$ (only dependent on n and m) for the size of m-generated SI bi-Gödel algebras whose bi-Esakia duals do not admit the n-comb as a subposet.*

Proof. Let $\mathcal{X} = (X, \tau, \leqslant)$ be a bi-Esakia co-tree which does not admit the n-comb as a subposet and suppose that $\mathcal{X}^*$ is m-generated, so there are $U_1, \dots, U_m \in CpUp(\mathcal{X})$ such that $\mathcal{X}^* = \langle U_1, \dots, U_m \rangle$. By the Coloring Theorem 2.15, every proper bi-bisimulation equivalence on $\mathcal{X}$ identifies points of different colors, where the coloring of $\mathcal{X}$ is defined by $V(p_i) = U_i$, for $i \leqslant m$.

First we prove that if $w \in min(\mathcal{X})$ then $|{\uparrow}w| \leqslant (m+1) \cdot n$. Take $w \in min(\mathcal{X})$ and notice that since the U_i are upsets, we can re-enumerate them so they satisfy ${\uparrow}w \cap U_1 \subseteq \dots \subseteq {\uparrow}w \cap U_m$. Set $H_i := {\uparrow}w \cap \left(\bigcap_{j=i}^{m} U_j\right)$ for each $i \leqslant m$, $H_{m+1} := {\uparrow}w \smallsetminus U_m$, and notice ${\uparrow}w = \bigcup_{i=1}^{m+1} H_i$. We now show that $|H_i| \leqslant n$, for all $i \leqslant m+1$. For suppose this is not the case, i.e., that $|H_i| > n$ for some $i \leqslant m+1$. As H_i is contained in the chain ${\uparrow}w$, H_i is also a chain, thus there are points $a_1 < \dots < a_n < a_{n+1} \in H_i$. Let $j \leqslant n$ and suppose that $[a_j, a_{j+1}]$ is an isolated chain in $\mathcal{X}$. By the definitions of our coloring of $\mathcal{X}$ and of H_i, $[a_j, a_{j+1}] \subseteq H_i$ implies that all the points in this isolated chain have the same color. But now Lemma 5.11 yields a proper (since $a_j \neq a_{j+1}$) bi-bisimulation equivalence on $\mathcal{X}$ which does not identify points of different colors, contradicting the Coloring Theorem. Thus, the chain $[a_j, a_{j+1}]$ cannot be isolated. Since $\mathcal{X}$ is a co-tree, it is clear that ${\uparrow}a_j \smallsetminus [a_j, a_{j+1}] = {\uparrow}a_{j+1} \smallsetminus \{a_{j+1}\}$ and ${\downarrow}a_j \smallsetminus \{a_j\} \subseteq {\downarrow}a_{j+1} \smallsetminus [a_j, a_{j+1}]$, and therefore we must have ${\downarrow}a_{j+1} \smallsetminus [a_j, a_{j+1}] \nsubseteq {\downarrow}a_j \smallsetminus \{a_j\}$. Equivalently, there exists $x_j \in [a_j, a_{j+1}] \smallsetminus \{a_j\}$ such that ${\downarrow}x_j \smallsetminus ([a_j, a_{j+1}] \cup {\downarrow}a_j) \neq \emptyset$. Fix a $x_j' \in {\downarrow}x_j \smallsetminus ([a_j, a_{j+1}] \cup {\downarrow}a_j)$. As $j \leqslant n$ was arbitrary, we have found a subposet of $\mathcal{X}$, $\left(\{x_j \colon j \leqslant n\} \cup \{x_j' \colon j \leqslant n\}, \leqslant\right)$, which is clearly a copy of the n-comb $\mathfrak{C}_n$, contradicting our hypothesis. Therefore, there can be no chain $a_1 < \dots < a_n < a_{n+1} \in H_i$, i.e., $|H_i| \leqslant n$. Consequently, we conclude that ${\uparrow}w = \bigcup_{i=1}^{m+1} H_i$ consists of at most $m+1$ pieces, each of size at most n, that is, $|{\uparrow}w| \leqslant (m+1) \cdot n$ as desired.

Since every point in a bi-Esakia space lies above a minimal one (see Proposition 2.12), it now follows from the definition of the depth of a co-tree (see the end of Section 4) that $dp(\mathcal{X}) \leqslant (m+1) \cdot n$. Notice that $\mathcal{X}$ being a co-tree of finite depth entails that every point distinct from its co-root r has a unique immediate successor. Let $\{w_i\}_{i \in I} \subseteq min(\mathcal{X})$, and suppose they all share their unique immediate successor, v. Note that there are only 2^m distinct colors, and that $i \neq j \in I$ implies $col(w_i) \neq col(w_j)$, otherwise Lemma 5.12 would contradict the Coloring Theorem. Thus, we have $|I| \leqslant 2^m$ and $|{\downarrow}v| \leqslant 2^m + 1$.

Now, let $u \in X$ be such that all of its strict predecessors are either minimal, or are immediate successors of minimal points. Set $\{v_i\}_{i \in I} := \{y \in X \colon y \prec u\}$, and notice that for all $i \in I$, we have $|{\downarrow}v_i| \leqslant 2^m + 1$ by above. Moreover, since there are only 2^m distinct colors, there exists a natural bound $b(m)$ for the number of possible distinct colored configurations (by which we mean poset structure together with a coloring) of the posets ${\downarrow}v_i$. As the v_i all share their unique immediate successor, we cannot have that for $i \neq j \in I$, ${\downarrow}v_i$ and ${\downarrow}v_j$ have both the same poset structure (i.e., there exists an order-isomorphism from ${\downarrow}v_i$ to ${\downarrow}v_j$) and coloring, otherwise Lemma 5.12 would contradict the Coloring Theorem. Hence $|I| \leqslant b(m)$, and we now have $|{\downarrow}u| \leqslant (2^m + 1) \cdot b(m) + 1$.

Since we have a natural bound for the depth of $\mathcal{X}$, we can now iterate the above argument a finite number of times (namely, at most $(m+1) \cdot n$ times) to find a bound $k_0(n,m) \in \omega$ for the size of X, i.e., $|X| = |{\downarrow}r| \leqslant k_0(n,m)$. By the nature of the argument that led to this bound, $k_0(n,m)$ depends only on n and m, and not on $\mathcal{X}$.

As there are only finitely many co-trees of size less than or equal to $k_0(n,m)$, it follows that there are only finitely many bi-Esakia co-trees which do not admit $\mathfrak{C}_n$ as a subposet and whose algebraic dual is m-generated. Therefore, we can now find a natural bound $k(n,m)$ (only dependent on $k_0(n,m)$) for the size of the bi-Heyting duals of these bi-Esakia co-trees, as desired. ∎

We are finally ready to complete the proof of Theorem 5.1.

Proof. The implication from left to right is Corollary 5.8. To prove the other direction, let L be an extension of bi-LC containing $\beta(\mathfrak{C}_n^*)$, for some $n \in \omega$. In particular, we have that for all $m \in \omega$, if $\mathfrak{A}$ is an SI m-generated algebra which validates L, then $\mathfrak{A} \models \beta(\mathfrak{C}_n^*)$. By the Dual Subframe Jankov Lemma 4.23, this is equivalent to $\mathfrak{C}_n$ does not order-embed into $\mathfrak{A}_*$. In other words, $\mathfrak{A}_*$ does not admit $\mathfrak{C}_n$ as a subposet. It now follows from Proposition 5.13 that $|A| \leqslant k(n, m)$, and we can use Theorem 2.3 to conclude that V_L is locally finite, i.e., that L is locally tabular, as desired. □

We close this paper by comparing some properties of the logic bi-LC (algebraized by bi-GA) with those of the thoroughly investigated linear calculus LC which is algebraized by the variety GA of Gödel algebras, i.e., the class of Heyting algebras satisfying the Gödel-Dummett axiom. In the table below, SRC is a short hand for *strongly rooted chain*, i.e., a chain with an isolated minimum. The fact that Λ(bi-LC) is not a chain is an immediate consequence of the proof of Theorem 4.17.

LC = IPC + $(p \to q) \vee (q \to p)$	bi-LC = bi-IPC + $(p \to q) \vee (q \to p)$
$\mathfrak{A} \in \mathsf{GA}_{SI} \iff \mathfrak{A}_*$ is a SRC	$\mathfrak{A} \in \text{bi-}\mathsf{GA}_{SI} \iff \mathfrak{A}_*$ is a co-tree
LC has the FMP	bi-LC has the FMP
LC is locally tabular	bi-LC is not locally finite
Λ(LC) is a chain of order-type $(\omega + 1)^{\partial}$	Λ(bi-LC) is of size $2^{\aleph_0}$ and is not a chain

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