Simultaneous Substitution Algebras

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

In this thesis we introduce simultaneous substitution algebras as an abstraction of simultaneous substitution operations on terms and on functions. The class of simultaneous substitution algebras is defined by a set of equations, and we prove that the equational theory generated by this set is decidable and complete with the class of term simultaneous substitution algebras and of polynomial simultaneous substitution algebras. We also prove that each simultaneous substitution algebra can be represented as a quotient of a function simultaneous substitution algebra, and each locally finite-dimensional one can be represented as a polynomial simultaneous substitution algebra. Relevant results in singular substitution algebras can be derived from the results in this thesis.

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Contents

Chapter 1

Introduction

Substitution is the operation which replaces the (free) occurrences of variables in an expression by occurrences of other expressions in many formal systems, like propositional logic, first-order logic, and lambda calculus. When we replace the occurrences of a single variable by the occurrences of another expression, we call this operation singular substitution; when we replace the occurrences of some variables (say $x_1, \ldots, x_n$) by occurrences of expressions (say $e_1, \ldots, e_n$) respectively at the same time, we call it simultaneous substitution.

In the study of the algebraization of formal systems, substitution operations can be defined in algebras, for instance in cylindric algebras, algebraization of first-order logic (Henkin, Monk, and Tarski [HMT71]), and in lambda abstraction algebras, algebraization of lambda calculus (Pigozzi and Salibra [PS95]). Substitution can also be treated as basic operations in algebras; in [Pin73], Pinter defines a class of Boolean algebras with substitution operations, and shows that this class of algebras is definitionally equivalent to the class of cylindric algebras.

In [Fel82], a class of algebras where substitution operations are the only primitive operations, called substitution algebras, is introduced by Feldman. It is an abstraction of singular substitution on functions and on terms. Feldman proves that the first-order axioms of substitution algebras and a non-first-order condition of local finiteness[1] characterize the class of polynomial substitution algebras, a specific class of substitution algebras of functions. Furthermore, Feldman provides several equivalent conditions for a substitution algebra to be representable as a function substitution algebra in [Fel15].

However, the discussion is based on singular substitution in [Fel82]. In many formal systems we are familiar with, simultaneous substitution can be defined with singular substitution: since the expressions in these formal systems are finite and there are infinitely many variables, we can always use new variables not occurring in a given expression to simulate simultaneous substitution with singular ones. In algebras, "local finite-dimensionality" is the name for a similar phenomenon that only finitely many variables "matter to" each element, and the method to simulate simultaneous substitution by singular substitution doesn't always work without local finite-dimensionality.

In our work, we follow the path taken by Feldman and introduce simultaneous substitution algebras, aiming to characterize the simultaneous substitution operation on terms and on

[1] We call it local finite-dimensionality in our discussion.

operations over a set. The axiom schemas will be given in Chapter 2; we will also present several classes of simultaneous substitution algebras we are interested in, namely the class of term simultaneous substitution algebras (TSSA), of function simultaneous substitution algebras (FSSA), and of polynomial simultaneous substitution algebras (PSSA).

In Chapter 3 we will discuss the relation between simultaneous substitution algebras and singular substitution algebras. It is natural to view simultaneous substitution as a complicated version of singular substitution, and indeed we can show that every simultaneous substitution algebra can be reduced to a singular substitution algebra. We will also show that each locally finite-dimensional singular substitution algebra can be expanded to a simultaneous substitution algebra.

In Chapter 4, we will prove a key property of simultaneous substitution: each term of the type of simultaneous substitution algebras has a normal form. With the normal form theorem for simultaneous substitution, we can arrive at the first important result in our study: the decidability of the equational theory generated by our axioms of simultaneous substitution algebras, and the completeness of it with respect to the class of TSSAs and of PSSAs.

The representation problem of simultaneous substitution algebras will be considered in Chapter 5. We will prove that every simultaneous substitution algebra is isomorphic to a quotient of a TSSA in a broader sense, and to a quotient of a FSSA. Moreover, we will pay special attention to locally finite-dimensional simultaneous substitution algebras and demonstrate their representability.

We will also derive relevant results (completeness and decidability of equational theory, and representability as in [Fel82]) in locally finite-dimensional singular substitution algebras from our main results in Chapter 4 and 5.

Chapter 2

Simultaneous substitution algebras

In this chapter we introduce simultaneous substitution algebras. We provide the axiom schemas, several examples, and some basic definitions and lemmas we will use in the following chapters.

At the beginning we introduce some notations we will use throughout our discussion. We write the set of all functions from a set B to a set A as A^B. Let $a_b \in A$ for each $b \in B$, then we also write the function $f : B \to A$ such that $f(b) = a_b$ for each $b \in B$ as $\langle a_b \rangle_{b \in B}$.

Finite sequences and *permutations* are defined as functions in our discussion. We define the empty sequence (the sequence of length 0) as the empty function and denote it by $()$. For finite sequences of length n, $n > 0$, we define them as functions with domain $n = \{0, \dots, n-1\}$; for n elements $a_0, a_1, \dots, a_{n-1}$, we use $(a_0, \dots, a_{n-1})$ to denote the finite sequence $f : n \to \{a_0, \dots, a_{n-1}\}$ such that $f(i) = a_i$ for all $i \in n$. For each set A, we use $A^{\#}$ to denote the set of all finite sequences without repetitions of elements of A, i.e.,

$$A^{\#} = \{()\} \cup \bigcup_{n \in \mathbb{N}^+} \{(a_1, \dots, a_n) \mid a_1, \dots, a_n \in D, a_i \neq a_j \text{ for all } i, j \text{ with } 1 \leq i < j \leq n\};$$

we also use $\vec{a}$ to denote sequences in $A^{\#}$.

For positive integers i, j, n with $1 \leq i < j \leq n$, we use $[i, j]_n$ to denote the permutation p of $\{1, \dots, n\}$ such that $p(i) = j$, $p(j) = i$, and $p(x) = x$ for all $x \in \{1, \dots, n\} \setminus \{i, j\}$; we also call $[i, j]_n$ a *transposition*.

Let A be an arbitrary nonempty set; for each positive integer n, an *n-ary operation* on A is a function from A^n to A. We generalize this definition and allow the arity to be any set: for each set X, a *X-ary operation* on A is a function from A^X to X. When X is a set of variables, a X-ary operation can be viewed as an assignment of elements in A to variables; hence we also call a X-ary operation an *assignment to X*.

2.1 Axiom schemas

Definition 2.1. Let D be a set (we also call elements in D dimensions in the following) and A be a nonempty set. Let c_d be an element in A for each $d \in D$, and $S^{\vec{d}}$ be a $(n+1)$-ary operation on A for each $\vec{d} \in D^{\#}$ of length n. Then $\mathbf{A} = \langle A, \langle c_d \rangle_{d \in D}, \langle S^{\vec{d}} \rangle_{\vec{d} \in D^{\#}} \rangle$ is a *D-dimensional simultaneous* *substitution algebra* (D-SSA) if for all $n \geq 1$, all pairwise distinct dimensions $d, d_1, \ldots, d_n \in D$, and all elements $a, a_1, \ldots, a_n, b_1, \ldots, b_n \in A$,

(ss1) $S^{()}(a) = a$;

(ss2) $S^{(d_1,\ldots,d_n)}(a_1, \ldots, a_n, c_{d_1}) = a_1$;

(ss3) $S^{(d_1,\ldots,d_n)}(c_{d_1}, a_2 \ldots, a_n, a) = S^{(d_2,\ldots,d_n)}(a_2, \ldots, a_n, a)$;

(ss4) $S^{(d_1,\ldots,d_n)}(a_1, \ldots, a_n, c_d) = c_d$;

(ss5) $S^{(d_1,\ldots,d_n)}(b_1, \ldots, b_n, S^{(d_1,\ldots,d_n)}(a_1, \ldots, a_n, a)) = S^{(d_1,\ldots,d_n)}(S^{(d_1,\ldots,d_n)}(b_1, \ldots, b_n, a_1), \ldots,$ $S^{(d_1,\ldots,d_n)}(b_1, \ldots, b_n, a_n), a)$;

(ss6) $1 \leq i < j \leq n, p = [i, j]_n \Rightarrow S^{(d_1,\ldots,d_n)}(a_1, \ldots, a_n, a) = S^{(d_{p(1)},\ldots,d_{p(n)})}(a_{p(1)}, \ldots, a_{p(n)}, a)$.

Some useful lemmas can be derived from the axiom schemas (ss1)-(ss6):

Lemma 2.1. Let $\mathbf{A} = \langle A, \langle c_d \rangle_{d \in D}, \langle S^{\vec{d}} \rangle_{\vec{d} \in D^\#} \rangle$ be a D-SSA. For all $(d_1, \ldots, d_n), (d'_1, \ldots, d'_m) \in D^\#$ and $a, a_1, \ldots, a_n, b_1, \ldots, b_m \in A$,

(a) $S^{(d_1,\ldots,d_n)}(a_1, \ldots, a_n, c_{d_i}) = a_i$, $1 \leq i \leq n$;

(b) p a permutation of $\{1, \ldots, n\} \Rightarrow S^{(d_1,\ldots,d_n)}(a_1, \ldots, a_n, a) = S^{(d_{p(1)},\ldots,d_{p(n)})}(a_{p(1)}, \ldots, a_{p(n)}, a)$;

(c) $\{u_1, \ldots, u_k, v_1, \ldots, v_{n-k}\} = \{1, \ldots, n\}$, $a_{v_i} = c_{d_{v_i}}$ for all i with $1 \leq i \leq n-k \Rightarrow$ $S^{(d_1,\ldots,d_n)}(a_1 \ldots, a_n, a) = S^{(d_{u_1},\ldots,d_{u_k})}(a_{u_1}, \ldots, a_{u_k}, a)$;

(d) $\{d_1, \ldots, d_n\} \setminus \{d'_1, \ldots, d'_m\} = \{d_{v_1}, \ldots, d_{v_k}\}$ with pairwise distinct $v_1, \ldots, v_k \in \{1, \ldots, n\} \Rightarrow$ $S^{(d_1,\ldots,d_n)}(a_1, \ldots, a_n, S^{(d'_1,\ldots,d'_m)}(b_1, \ldots, b_m, a)) =$ $S^{(d'_1,\ldots,d'_m,d_{v_1},\ldots,d_{v_k})}\big(S^{(d_1,\ldots,d_n)}(a_1, \ldots, a_n, b_1), \ldots, S^{(d_1,\ldots,d_n)}(a_1, \ldots, a_n, b_m), a_{v_1}, \ldots, a_{v_k}, a\big)$.

Proof. (a) If $i = 1$, just take (ss2). Otherwise $1 < i \leq n$; let $p = [1, i]_n$, we have

$$\begin{aligned} S^{(d_1,\ldots,d_n)}(a_1, \ldots, a_n, c_{d_i}) &\overset{\text{(ss6)}}{=} S^{(d_{p(1)},\ldots,d_{p(n)})}(a_{p(1)}, \ldots, a_{p(n)}, c_{d_i}) \\ &= S^{(d_i,d_{p(2)}\ldots,d_{p(n)})}(a_i, a_{p(2)} \ldots, a_{p(n)}, c_{d_i}) \\ &\overset{\text{(ss2)}}{=} a_i. \end{aligned}$$

(b) Since each permutation p of $\{1, \ldots, n\}$ is a composition of transpositions, we can use (ss6) several times to obtain this lemma.

(c) Assume that $\{u_1, \ldots, u_k, v_1, \ldots, v_{n-k}\} = \{1, \ldots, n\}$ and $a_{v_i} = c_{d_{v_i}}$ for all i with $1 \leq i \leq n-k$, then we have $S^{(d_1,\ldots,d_n)}(a_1, \ldots, a_n, a) \overset{\text{(b)}}{=} S^{(d_{v_1},\ldots,d_{v_{n-k}},d_{u_1},\ldots,\ldots,d_{u_k})}(a_{v_1}, \ldots, a_{v_{n-k}}, a_{u_1}, \ldots, a_{u_k}, a)$. Using (ss3) $n-k$ times, we get $S^{(d_1,\ldots,d_n)}(a_1, \ldots, a_n, a) = S^{(d_{u_1},\ldots,d_{u_k})}(a_{u_1}, \ldots, a_{u_k}, a)$.

(d) Assume that $\{d_1, \ldots, d_n\} \setminus \{d'_1, \ldots, d'_m\} = \{d_{v_1}, \ldots, d_{v_k}\}$ with pairwise distinct $v_1, \ldots, v_k \in \{1, \ldots, n\}$. Then $\{d'_1, \ldots, d'_m\} \setminus \{d_1, \ldots, d_n\}$ contains $l = m + k - n$ different dimensions, and we call them $d_{n+1}, \ldots, d_{n+l}$. Let $\vec{d} = (d_1, \ldots, d_{n+l})$ and $\vec{d'} = (d'_1, \ldots, d'_m, d_{v_1}, \ldots, d_{v_k})$; it is easy to see that both $\vec{d}$ and $\vec{d'}$ has no repetitions and $\{d_1, \ldots, d_{n+l}\} = \{d'_1, \ldots, d'_m, d_{v_1}, \ldots, d_{v_k}\}$, hence we can define a permutation p of $\{1, \ldots, m + k\}$ such that

$$d_{p(i)} = \begin{cases} d'_i, & 1 \leq i \leq m, \\ d_{v_{i-m}}, & m+1 \leq i \leq m+k; \end{cases}$$

so we have $(d_{p(1)}, \ldots, d_{p(n+l)}) = (d'_1, \ldots, d'_m, d_{v_1}, \ldots, d_{v_k}) = \vec{d'}$. Let a_{n+i} be $c_{d_{n+i}}$ for all i with $1 \leq i \leq l$, then

$$\begin{aligned}
&S^{(d_1,\ldots,d_n)}(a_1, \ldots, a_n, S^{(d'_1,\ldots,d'_m)}(b_1, \ldots, b_m, a)) \\
&\overset{\text{(c)}}{=} S^{(d_1,\ldots,d_{n+l})}(a_1, \ldots, a_{n+l}, S^{(d'_1,\ldots,d'_m)}(b_1, \ldots, b_m, a)) \\
&\overset{\text{(c)}}{=} S^{(d_1,\ldots,d_{n+l})}(a_1, \ldots, a_{n+l}, S^{(d'_1,\ldots,d'_m,d_{v_1},\ldots,d_{v_k})}(b_1, \ldots, b_m, c_{d_{v_1}}, \ldots, c_{d_{v_k}}, a)) \\
&\overset{\text{(b)}}{=} S^{(d_{p(1)},\ldots,d_{p(n+l)})}(a_{p(1)}, \ldots, a_{p(n+l)}, S^{(d'_1,\ldots,d'_m,d_{v_1},\ldots,d_{v_k})}(b_1, \ldots, b_m, c_{d_{v_1}}, \ldots, c_{d_{v_k}}, a)) \\
&= S^{\vec{d'}}(a_{p(1)}, \ldots, a_{p(n+l)}, S^{\vec{d'}}(b_1, \ldots, b_m, c_{d_{v_1}}, \ldots, c_{d_{v_k}}, a)) \\
&\overset{\text{(ss5)}}{=} S^{\vec{d'}}(S^{\vec{d'}}(a_{p(1)}, \ldots, a_{p(n+l)}, b_1), \ldots, S^{\vec{d'}}(a_{p(1)}, \ldots, a_{p(n+l)}, b_m), \\
&\qquad S^{\vec{d'}}(a_{p(1)}, \ldots, a_{p(n+l)}, c_{d_{v_1}}), \ldots, S^{\vec{d'}}(a_{p(1)}, \ldots, a_{p(n+l)}, c_{d_{v_k}}), a).
\end{aligned}$$

For each b_i, $1 \leq i \leq m$, we have

$$\begin{aligned}
S^{\vec{d'}}(a_{p(1)}, \ldots, a_{p(n+l)}, b_i) &= S^{(d_{p(1)},\ldots,d_{p(n+l)})}(a_{p(1)}, \ldots, a_{p(n+l)}, b_i) \overset{\text{(b)}}{=} S^{\vec{d}}(a_1, \ldots, a_{n+l}, b_i) \\
&= S^{(d_1,\ldots,d_{n+l})}(a_1, \ldots, a_n, c_{d_{n+1}}, \ldots, c_{d_{n+l}}, b_i) \overset{\text{(c)}}{=} S^{(d_1,\ldots,d_n)}(a_1, \ldots, a_n, b_i).
\end{aligned}$$

Also, for each $c_{d_{v_i}}$, $1 \leq i \leq k$, we have $S^{\vec{d'}}(a_{p(1)}, \ldots, a_{p(n+l)}, c_{d_{v_i}}) \overset{\text{(b)}}{=} S^{\vec{d}}(a_1, \ldots, a_{n+l}, c_{d_{v_i}}) \overset{\text{(a)}}{=} a_{v_i}$. Hence we get $S^{(d_1,\ldots,d_n)}(a_1, \ldots, a_n, S^{(d'_1,\ldots,d'_m)}(b_1, \ldots, b_m, a)) = S^{\vec{d'}}(S^{(d_1,\ldots,d_n)}(a_1, \ldots, a_n, b_1), \ldots, S^{(d_1,\ldots,d_n)}(a_1, \ldots, a_n, b_m), a_{v_1}, \ldots, a_{v_k}, a)$. □

2.2 Examples

Here are some examples of simultaneous substitution algebras.

2.2.1 Term simultaneous substitution algebras

First let us consider the simultaneous substitution algebras of terms. Let $\mathcal{S}$ be an arbitrary similarity type and X be an arbitrary set of variables[1] such that $\mathcal{S} \cap X = \emptyset$ and $\mathcal{S}_0 \cup X \neq \emptyset$ ($\mathcal{S}_0$ is the set of all constant symbols in $\mathcal{S}$). Let $T_{\mathcal{S}}(X)$ be the set of terms of type $\mathcal{S}$ over variables X; more precisely, $T_{\mathcal{S}}(X)$ is the smallest set such that

[1]Variables play the same role as dimensions in the last section.

(i) for each $x \in X$, $(x) \in T_{\mathcal{S}}(X)$;

(ii) if $t_1, \dots, t_n \in T_{\mathcal{S}}(X)$ and $Q \in \mathcal{S}$ is an n-ary operation symbol, then $(Q, (t_1, \dots, t_n)) \in T_{\mathcal{S}}(X)$.

We usually omit the parentheses and commas, and represent (x) by x and $(Q, (t_1, \dots, t_n))$ by $Qt_1 \dots t_n$.

Recall that $X^{\#}$ is the set of finite sequences without repetitions of elements of X. For $() \in X^{\#}$, let $S^{(),\mathbf{T}}$ be the identical operation on $T_{\mathcal{S}}(X)$. For each sequence $\vec{x} = (x_1, \dots, x_n) \in X^{\#}$, we define $S^{\vec{x},\mathbf{T}}$ as the $(n+1)$-ary operation on $T_{\mathcal{S}}(X)$ such that for all terms $t_1, \dots, t_{n+1} \in T_{\mathcal{S}}(X)$, $S^{\vec{x},\mathbf{T}}(t_1, \dots, t_{n+1})$ is the term obtained by replacing all occurrences of x_i in t_{n+1} by t_i for all i with $1 \leq i \leq n$ simultaneously; formally, for each $\vec{x} = (x_1, \dots, x_n) \in X^{\#}$, $S^{\vec{x},\mathbf{T}}$ is defined by recursion:

(i) $S^{\vec{x},\mathbf{T}}(t_1, \dots, t_n, x) = \begin{cases} t_i, & x = x_i \text{ with } 1 \leq i \leq n, \\ x, & x \neq x_i \text{ for all } i \text{ with } 1 \leq i \leq n; \end{cases}$

(ii) $S^{\vec{x},\mathbf{T}}(t_1, \dots, t_n, Qt'_1 \dots t'_m) = QS^{\vec{x},\mathbf{T}}(t_1, \dots, t_n, t'_1) \dots S^{\vec{x},\mathbf{T}}(t_1, \dots, t_n, t'_m)$.

Then we call the algebra

$$\mathbf{T}^{\mathrm{ss}}_{\mathcal{S}}(X) = \langle T_{\mathcal{S}}(X), \langle x \rangle_{x \in X}, \langle S^{\vec{x},\mathbf{T}} \rangle_{\vec{x} \in X^{\#}} \rangle$$

the *X-dimensional term simultaneous substitution algebra* (X-TSSA) of type $\mathcal{S}$. The "ss" in superscript represents "simultaneous substitution"; we write $T_{\mathcal{S}}(X)$ as T, $\mathbf{T}^{\mathrm{ss}}_{\mathcal{S}}(X)$ as $\mathbf{T}^{\mathrm{ss}}$ or $\mathbf{T}$, and $S^{\vec{x},\mathbf{T}}$ as $S^{\vec{x}}$ when there is no confusion. It can be verified that $\mathbf{T}$ satisfies the axiom schemas of X-SSAs. Take (ss3) as an example; we prove that for all pairwise distinct $x_1, \dots, x_n \in X$, and all $t_2, \dots, t_n, t \in T$, $S^{(x_1,\dots,x_n)}(x_1, t_2, \dots, t_n, t) = S^{(x_2,\dots,x_n)}(t_2, \dots, t_n, t)$ by induction on the structure of t:

(1) $t = x$, $x \in X$: if $x = x_1$, then $S^{(x_1,\dots,x_n)}(x_1, t_2, \dots, t_n, x) = x_1 = S^{(x_2,\dots,x_n)}(t_2, \dots, t_n, x)$;
else if $x = x_i$, $2 \leq i \leq n$, then $S^{(x_1,\dots,x_n)}(x_1, t_2, \dots, t_n, x) = t_i = S^{(x_2,\dots,x_n)}(t_2, \dots, t_n, x)$;
else, $x \neq x_i$ for all i with $1 \leq i \leq n$, then $S^{(x_1,\dots,x_n)}(x_1, t_2, \dots, t_n, x) = x = S^{(x_2,\dots,x_n)}(t_2, \dots, t_n, x)$;

(2) $t = Qt'_1 \dots t'_m$ for some m-ary operation symbol Q and some $t'_1, \dots, t'_m \in T$:

$$\begin{aligned} S^{(x_1,\dots,x_n)}(x_1, t_2, \dots, t_n, t) &= S^{(x_1,\dots,x_n)}(x_1, t_2, \dots, t_n, Qt'_1 \dots t'_m) \\ &= QS^{(x_1,\dots,x_n)}(x_1, t_2, \dots, t_n, t'_1) \dots S^{(x_1,\dots,x_n)}(x_1, t_2, \dots, t_n, t'_m) \\ &\overset{\text{IH}}{=} QS^{(x_2,\dots,x_n)}(t_2, \dots, t_n, t'_1) \dots S^{(x_2,\dots,x_n)}(t_2, \dots, t_n, t'_m) \\ &= S^{(x_2,\dots,x_n)}(t_2, \dots, t_n, Qt'_1 \dots t'_m) \\ &= S^{(x_2,\dots,x_n)}(t_2, \dots, t_n, t). \end{aligned}$$

For convenience in later discussions, we also introduce term algebras here. For each n-ary operation symbol $Q \in \mathcal{S}$, let $Q^{\mathbf{T}_{\mathcal{S}}(X)}$ be the n-ary operation on $T_{\mathcal{S}}(X)$ such that for all $t_1, \ldots, t_n \in T_{\mathcal{S}}(X)$,

$$Q^{\mathbf{T}_{\mathcal{S}}(X)}(t_1, \ldots, t_n) = Qt_1 \ldots t_n.$$

Then $\mathbf{T}_{\mathcal{S}}(X) = \langle T_{\mathcal{S}}(X), \langle Q^{\mathbf{T}_{\mathcal{S}}(X)} \rangle_{Q \in \mathcal{S}} \rangle$ is the *term algebra of type $\mathcal{S}$ over X*. Notice that $\mathbf{T}_{\mathcal{S}}(X)$ is an algebra of type $\mathcal{S}$.

2.2.2 Function simultaneous substitution algebras

The next example is a class of simultaneous substitution algebras of functions. Let D be an arbitrary set of dimensions and A be an arbitrary nonempty set, then the functions we consider are D-ary operations on A. Let $F_D(A) = A^{A^D}$. For each $d \in D$, let $e_d \in F_D(A)$ be the d-th projection function, i.e., $e_d(\alpha) = \alpha(d)$ for every $\alpha : D \to A$. For each $\alpha : D \to A$, $\vec{d} = (d_1, \ldots, d_n) \in D^\#$ and $\vec{a} = (a_1, \ldots, a_n) \in A^n$, let $\alpha\langle \vec{d}, \vec{a} \rangle : D \to A$ be the assignment such that

$$\alpha\langle \vec{d}, \vec{a} \rangle(d) = \begin{cases} a_i, & d = d_i \text{ with } 1 \leq i \leq n, \\ \alpha(d), & d \neq d_i \text{ for all } i, 1 \leq i \leq n. \end{cases}$$

Then for each $(d_1, \ldots, d_n) \in D^\#$, let $S^{(d_1,\ldots,d_n),\mathbf{F}}$ be the $(n+1)$-ary operation on $F_D(A)$ such that for all $f_1, \ldots, f_n, f \in F_D(A)$ and all $\alpha : D \to A$,

$$S^{(d_1,\ldots,d_n),\mathbf{F}}(f_1, \ldots, f_n, f)(\alpha) = f(\alpha\langle (d_1, \ldots, d_n), (f_1(\alpha), \ldots, f_n(\alpha)) \rangle).$$

Besides, let $S^{(),\mathbf{F}}$ be the identical function on $F_D(A)$. Then we call

$$\mathbf{F}_D^{\mathrm{ss}}(A) = \langle F_D(A), \langle e_d \rangle_{d \in D}, \langle S^{\vec{d},\mathbf{F}} \rangle_{\vec{d} \in D^\#} \rangle$$

the *full D-dimensional function simultaneous substitution algebra* with base A. It can be checked that $\mathbf{F}_D^{\mathrm{ss}}(A)$ is a D-SSA. Subalgebras of $\mathbf{F}_D^{\mathrm{ss}}(A)$ are called *D-dimensional function simultaneous substitution algebras* (D-FSSA) with base A.

2.2.3 Polynomial simultaneous substitution algebras

Then we introduce a class of function simultaneous substitution algebras that are closely connected with term simultaneous substitution algebras. Let $\mathcal{S}$ be an arbitrary similarity type such that $\mathcal{S} \cap D = \emptyset$ and $\mathcal{S}_0 \cup D \neq \emptyset$, and let $\mathbf{A}$ be an arbitrary algebra of type $\mathcal{S}$. Then each term $t \in T_{\mathcal{S}}(D)$ can be interpreted as a *term operation* (also called polynomials in [Fel82]), which is a D-ary operation $t^{\mathbf{A}}$ over A (hence $t^{\mathbf{A}} \in F_D(A)$). We define term operations recursively: for each assignment to dimensions $\alpha : D \to A$,

$$d^{\mathbf{A}}(\alpha) = \alpha(d) \text{ for each } d \in D,$$
$$(Qt_1 \ldots t_n)^{\mathbf{A}}(\alpha) = Q^{\mathbf{A}}(t_1^{\mathbf{A}}(\alpha), \ldots, t_n^{\mathbf{A}}(\alpha)) \text{ for each } Qt_1 \ldots t_n \in T.$$

Lemma 2.2. $t \mapsto t^{\mathbf{A}}$ is a homomorphism from $\mathbf{T}^{\mathrm{ss}}_{\mathcal{S}}(D)$ to $\mathbf{F}^{\mathrm{ss}}_{D}(A)$.

Proof. First we show that for all $\vec{d} \in D^{\#}$ of length n and all $t_1, \ldots, t_{n+1} \in T_{\mathcal{S}}(D)$,

$$(S^{\vec{d},\mathbf{T}}(t_1, \ldots, t_{n+1}))^{\mathbf{A}} = S^{\vec{d},\mathbf{F}}(t_1^{\mathbf{A}}, \ldots, t_{n+1}^{\mathbf{A}}).$$

If $\vec{d} = ()$, then $(S^{(),\mathbf{T}}(t_1))^{\mathbf{A}} = t_1^{\mathbf{A}} = S^{(),\mathbf{F}}(t_1^{\mathbf{A}})$. Else, assume that $\vec{d} = (d_1, \ldots, d_n)$. Take arbitrary $\alpha : D \to A$, we prove $(S^{\vec{d},\mathbf{T}}(t_1, \ldots, t_{n+1}))^{\mathbf{A}}(\alpha) = S^{\vec{d},\mathbf{F}}(t_1^{\mathbf{A}}, \ldots, t_{n+1}^{\mathbf{A}})(\alpha)$ by induction on the structure of t_{n+1}. For convenience of expression, let $\alpha' = \alpha\langle \vec{d}, (t_1^{\mathbf{A}}(\alpha), \ldots, t_n^{\mathbf{A}}(\alpha))\rangle$, then we have $S^{\vec{d},\mathbf{F}}(t_1^{\mathbf{A}}, \ldots, t_n^{\mathbf{A}}, t^{\mathbf{A}})(\alpha) = t^{\mathbf{A}}(\alpha')$ for all $t \in T$.

(1) $t_{n+1} = d_i$, $1 \le i \le n$:

$$\begin{aligned}(S^{\vec{d},\mathbf{T}}(t_1, \ldots, t_n, d_i))^{\mathbf{A}}(\alpha) &= t_i^{\mathbf{A}}(\alpha) = \alpha\langle (d_1, \ldots, d_n), (t_1^{\mathbf{A}}(\alpha), \ldots, t_n^{\mathbf{A}}(\alpha))\rangle(d_i) \\ &= \alpha'(d_i) = d_i^{\mathbf{A}}(\alpha') = S^{\vec{d},\mathbf{F}}(t_1^{\mathbf{A}}, \ldots, t_n^{\mathbf{A}}, d_i^{\mathbf{A}})(\alpha) = S^{\vec{d},\mathbf{F}}(t_1^{\mathbf{A}}, \ldots, t_{n+1}^{\mathbf{A}})(\alpha);\end{aligned}$$

(2) $t_{n+1} = d$, $d \notin \{d_1, \ldots, d_n\}$:

$$\begin{aligned}(S^{\vec{d},\mathbf{T}}(t_1, \ldots, t_n, d))^{\mathbf{A}}(\alpha) &= d^{\mathbf{A}}(\alpha) = \alpha(d) = \alpha\langle (d_1, \ldots, d_n), (t_1^{\mathbf{A}}(\alpha), \ldots, t_n^{\mathbf{A}}(\alpha))\rangle(d) \\ &= \alpha'(d) = d^{\mathbf{A}}(\alpha') = S^{\vec{d},\mathbf{F}}(t_1^{\mathbf{A}}, \ldots, t_n^{\mathbf{A}}, d^{\mathbf{A}})(\alpha) = S^{\vec{d},\mathbf{F}}(t_1^{\mathbf{A}}, \ldots, t_{n+1}^{\mathbf{A}})(\alpha);\end{aligned}$$

(3) $t_{n+1} = Qt'_1 \ldots t'_m$:

$$\begin{aligned}&(S^{\vec{d},\mathbf{T}}(t_1, \ldots, t_n, Qt'_1 \ldots t'_m))^{\mathbf{A}}(\alpha) \\ &= (QS^{\vec{d},\mathbf{T}}(t_1, \ldots, t_n, t'_1) \ldots S^{\vec{d},\mathbf{T}}(t_1, \ldots, t_n, t'_m))^{\mathbf{A}}(\alpha) \\ &= Q^{\mathbf{A}}((S^{\vec{d},\mathbf{T}}(t_1, \ldots, t_n, t'_1))^{\mathbf{A}}(\alpha), \ldots, (S^{\vec{d},\mathbf{T}}(t_1, \ldots, t_n, t'_m))^{\mathbf{A}}(\alpha)) \\ &\overset{\text{IH}}{=} Q^{\mathbf{A}}(S^{\vec{d},\mathbf{F}}(t_1^{\mathbf{A}}, \ldots, t_n^{\mathbf{A}}, t_1'^{\mathbf{A}})(\alpha), \ldots, S^{\vec{d},\mathbf{F}}(t_1^{\mathbf{A}}, \ldots, t_n^{\mathbf{A}}, t_m'^{\mathbf{A}})(\alpha)) \\ &= Q^{\mathbf{A}}(t_1'^{\mathbf{A}}(\alpha'), \ldots, t_m'^{\mathbf{A}}(\alpha')) \\ &= (Qt'_1 \ldots t'_m)^{\mathbf{A}}(\alpha') \\ &= S^{\vec{d},\mathbf{F}}(t_1^{\mathbf{A}}, \ldots, t_n^{\mathbf{A}}, (Qt'_1 \ldots t'_m)^{\mathbf{A}})(\alpha) \\ &= S^{\vec{d},\mathbf{F}}(t_1^{\mathbf{A}}, \ldots, t_{n+1}^{\mathbf{A}})(\alpha).\end{aligned}$$

Thus $(S^{\vec{d},\mathbf{T}}(t_1, \ldots, t_{n+1}))^{\mathbf{A}}(\alpha) = S^{\vec{d},\mathbf{F}}(t_1^{\mathbf{A}}, \ldots, t_{n+1}^{\mathbf{A}})(\alpha)$ for all $\alpha : D \to A$, which means that $(S^{\vec{d},\mathbf{T}}(t_1, \ldots, t_{n+1}))^{\mathbf{A}} = S^{\vec{d},\mathbf{F}}(t_1^{\mathbf{A}}, \ldots, t_{n+1}^{\mathbf{A}})$.

Besides, for each $d \in D$, $d^{\mathbf{A}}$ is the d-th projection e_d in $F_D(A)$. Therefore, $t \mapsto t^{\mathbf{A}}$ is a homomorphism from $\mathbf{T}^{\mathrm{ss}}_{\mathcal{S}}(D)$ to $\mathbf{F}^{\mathrm{ss}}_{D}(A)$. □

Let $\mathrm{Clo}_D(\mathbf{A})$ be the least set of D-ary operations on A that contains the D-ary projection operations and is closed under composition by the basic operations of $\mathbf{A}$; it can be shown that

$\text{Clo}_D(\mathbf{A}) = \{t^{\mathbf{A}} \mid t \in T\}$. By the lemma above, $\text{Clo}_D(\mathbf{A})$ is a subuniverse of $\mathbf{F}_D(A)$. Let $\mathbf{Clo}^{\text{ss}}_D(\mathbf{A})$ be the subalgebra of $\mathbf{F}^{\text{ss}}_D(A)$ taking $\text{Clo}_D(\mathbf{A})$ as its universe, and we call it the *D-dimensional polynomial simultaneous substitution algebra* (D-PSSA) induced by $\mathbf{A}$. The next theorem describes the connection between D-PSSAs and D-TSSAs.

Theorem 2.3. (a) Let $\phi : T_{\mathcal{S}}(D) \to \text{Clo}_D(\mathbf{A})$ be such that $\phi(t) = t^{\mathbf{A}}$ for all $t \in T_{\mathcal{S}}(D)$, then $\mathbf{T}^{\text{ss}}_{\mathcal{S}}(D)/\ker(\phi)$ is isomorphic to $\mathbf{Clo}^{\text{ss}}_D(\mathbf{A})$.

(b) $\mathbf{T}^{\text{ss}}_{\mathcal{S}}(D)$ is isomorphic to $\mathbf{Clo}^{\text{ss}}_D(\mathbf{T}_{\mathcal{S}}(D))$.

Proof. (a) By Lemma 2.2, ϕ is a homomorphism from $\mathbf{T}^{\text{ss}}_{\mathcal{S}}(D)$ onto $\mathbf{Clo}^{\text{ss}}_D(\mathbf{A})$, hence we have $\mathbf{T}^{\text{ss}}_{\mathcal{S}}(D)/\ker(\phi) \cong \mathbf{Clo}^{\text{ss}}_D(\mathbf{A})$ by the Homomorphism Theorem.

(b) Notice that the term algebra $\mathbf{T}_{\mathcal{S}}(D)$ is of type $\mathcal{S}$. Let $\iota : D \to T$ be such that $\iota = \langle d \rangle_{d \in D}$, then it is easy to check that $t^{\mathbf{T}_{\mathcal{S}}(D)}(\iota) = t$ for all $t \in T$. Hence $t \mapsto t^{\mathbf{T}_{\mathcal{S}}(D)}$ is injective, then we have $\mathbf{T}^{\text{ss}}_{\mathcal{S}}(D) \cong \mathbf{Clo}^{\text{ss}}_D(\mathbf{T}_{\mathcal{S}}(D))$ by (a). $\square$

2.2.4 Generalization of term simultaneous substitution algebras

Normally, the arities of operation symbols are natural numbers and the terms we have discussed so far are all finitary. However, a broader definition of terms is in order, in view of the fact that we are dealing with algebras of possibly infinitary character. For this purpose, we allow the arity of an operation symbol to be any set and consider a sort of "generalized terms" in the sequel. Let $\mathcal{I}$ be a set of sets (we call sets in $\mathcal{I}$ arities), $\mathcal{F}$ be a set of operation symbols, and $\pi : \mathcal{F} \to \mathcal{I}$ be the function associating each $Q \in \mathcal{F}$ with its arity $\pi(Q)$; we call $\mathcal{F}$ a *generalized type*. Let X be a set of variables such that $X \cap \mathcal{F} = \emptyset$ and $X \cup \{Q \in \mathcal{F} \mid \pi(Q) = \emptyset\} \neq \emptyset$.

Let $T_{\mathcal{F}}(X)$ be the least set such that

(i) for each $x \in X$, $(x) \in T_{\mathcal{F}}(X)$;

(ii) if $Q \in \mathcal{F}$ and $f : \pi(Q) \to T_{\mathcal{F}}(X)$, then $(Q, f) \in T_{\mathcal{F}}(X)$.

We call elements in $T_{\mathcal{F}}(X)$ (generalized) terms, and usually represent (x) by x and (Q, f) by Qf. Let $S^{(),\mathbf{T}}$ be the identical operation on $T_{\mathcal{F}}(X)$. For each $\vec{x} = (x_1, \ldots, x_n)$ and $t_1, \ldots, t_{n+1} \in T_{\mathcal{F}}(X)$, we define $S^{\vec{x},\mathbf{T}}(t_1, \ldots, t_{n+1})$ by induction on the structure of t_{n+1}:

(i) $S^{\vec{x},\mathbf{T}}(t_1, \ldots, t_n, x) = \begin{cases} t_i, & x = x_i \text{ with } 1 \ \leq i \leq n, \\ x, & x \notin \{x_1, \ldots, x_n\}; \end{cases}$

(ii) $S^{\vec{x},\mathbf{T}}(t_1, \ldots, t_n, Qf) = Qf'$ where $f' = \langle S^{\vec{x},\mathbf{T}}(t_1, \ldots, t_n, f(a)) \rangle_{a \in \pi(Q)}$.

Then we consider $\mathbf{T}^{\text{ss}}_{\mathcal{F}}(X) = \langle T_{\mathcal{F}}(X), \langle x \rangle_{x \in X}, \langle S^{\vec{x},\mathbf{T}} \rangle_{\vec{x} \in X^{\#}} \rangle$, and call it the *full X-dimensional term simultaneous substitution algebra of generalized type $\mathcal{F}$*, and subalgebras of $\mathbf{T}^{\text{ss}}_{\mathcal{F}}(X)$ *X-dimensional term simultaneous substitution algebras* (X-TSSA) *of generalized type $\mathcal{F}$*. It can be checked that X-TSSAs of generalized type $\mathcal{F}$ are X-SSAs. Notice that our definitions here coincide with the definitions in 2.2.1 when $\mathcal{F}$ is a type of algebras, i.e., the set of arities $\mathcal{I}$ is a subset of ω; hence using the same notations for terms and term simultaneous substitution algebras here is not troublesome. To make a distinction, we call the X-TSSAs in 2.2.1 (in other words, the X-TSSAs of types in which the arities of operation symbols are all natural numbers) X-TSSAs *of type of algebras*, or simply, X-TSSAs, and call X-TSSAs defined in this subsection (in other words, X-TSSAs of arbitrary generalized type) X-TSSAs *of generalized type*, or X-TSSAs *in a broader sense*.

Finally, we generalize Theorem 2.3(b) to show that every $\mathbf{T}^{\mathrm{ss}}_{\mathcal{F}}(X)$ is isomorphic to a X-FSSA with base $T_{\mathcal{F}}(X)$. Let e_x be the x-th projection function in $F_X(T_{\mathcal{F}}(X))$ for each $x \in X$. For each assignment $\alpha : X \to T_{\mathcal{F}}(X)$ and $t \in T_{\mathcal{F}}(X)$, we define the (generalized) term operation $t^{\mathbf{T}}(\alpha)$ by induction on the structure of t:

$$x^{\mathbf{T}}(\alpha) = \alpha(x),$$
$$Qf^{\mathbf{T}}(\alpha) = Qf', \text{ where } f' = \langle f(a)^{\mathbf{T}}(\alpha)\rangle_{a\in\pi(Q)}.$$

Then $t^{\mathbf{T}}$ is a X-ary operation on $T_{\mathcal{F}}(X)$ for each $t \in T_{\mathcal{F}}(X)$.

Lemma 2.4. $t \mapsto t^{\mathbf{T}}$ is an injective homomorphism from $\mathbf{T}^{\mathrm{ss}}_{\mathcal{F}}(X)$ to $\mathbf{F}^{\mathrm{ss}}_{X}(T_{\mathcal{F}}(X))$.

Proof. First we show that $t \mapsto t^{\mathbf{T}}$ is a homomorphism. By definition, $x^{\mathbf{T}} = e_x$ for each $x \in X$. For $() \in D^{\#}$ and each $t \in T_{\mathcal{F}}(X)$, we have $(S^{(),\mathbf{T}}(t))^{\mathbf{T}} = t^{\mathbf{T}} = S^{(),\mathbf{F}}(t^{\mathbf{T}})$. Then we prove that for all $\vec{x} = (x_1, \dots, x_n) \in X^{\#}$ and $t_1, \dots, t_{n+1} \in T_{\mathcal{F}}(X)$, $(S^{\vec{x},\mathbf{T}}(t_1, \dots, t_{n+1}))^{\mathbf{T}} = S^{\vec{x},\mathbf{F}}(t_1^{\mathbf{T}}, \dots, t_{n+1}^{\mathbf{T}})$ by induction on the structure of t_{n+1}:

(1) $t_{n+1} = x_i$, $1 \le i \le n$:

$$(S^{\vec{x},\mathbf{T}}(t_1, \dots, t_n, x_i))^{\mathbf{T}} = t_i^{\mathbf{T}} = S^{\vec{x},\mathbf{F}}(t_1^{\mathbf{T}}, \dots, t_n^{\mathbf{T}}, e_{x_i}) = S^{\vec{x},\mathbf{F}}(t_1^{\mathbf{T}}, \dots, t_n^{\mathbf{T}}, x_i^{\mathbf{T}});$$

(2) $t_{n+1} = x$, $x \notin \{x_1, \dots, x_n\}$:

$$(S^{\vec{x},\mathbf{T}}(t_1, \dots, t_n, x))^{\mathbf{T}} = x^{\mathbf{T}} = e_x = S^{\vec{x},\mathbf{F}}(t_1^{\mathbf{T}}, \dots, t_n^{\mathbf{T}}, e_x) = S^{\vec{x},\mathbf{F}}(t_1^{\mathbf{T}}, \dots, t_n^{\mathbf{T}}, x^{\mathbf{T}});$$

(3) $t_{n+1} = Qf$: let $f' = \langle S^{\vec{x},\mathbf{T}}(t_1, \dots, t_n, f(a))\rangle_{a\in\pi(Q)}$, then $S^{\vec{x},\mathbf{T}}(t_1, \dots, t_n, Qf) = Qf'$. Take an arbitrary assignment $\alpha : X \to T_{\mathcal{F}}(X)$,

$$(S^{\vec{x},\mathbf{T}}(t_1, \dots, t_n, Qf))^{\mathbf{T}}(\alpha) = Qf'^{\mathbf{T}}(\alpha) = Qf''$$

where $f''(a) = f'(a)^{\mathbf{T}}(\alpha) = (S^{\vec{x},\mathbf{T}}(t_1, \dots, t_n, f(a)))^{\mathbf{T}}(\alpha) \overset{\mathrm{IH}}{=} S^{\vec{x},\mathbf{F}}(t_1^{\mathbf{T}}, \dots, t_n^{\mathbf{T}}, f(a)^{\mathbf{T}})(\alpha)$ for all $a \in \pi(Q)$; meanwhile, we have

$$(S^{\vec{x},\mathbf{F}}(t_1^{\mathbf{T}}, \dots, t_n^{\mathbf{T}}, Qf^{\mathbf{T}}))(\alpha) = Qf^{\mathbf{T}}(\alpha\langle\vec{x}, (t_1^{\mathbf{T}}(\alpha), \dots, t_n^{\mathbf{T}}(\alpha))\rangle) = Qf'''$$

where $f'''(a) = f(a)^{\mathbf{T}}(\alpha\langle\vec{x}, (t_1^{\mathbf{T}}(\alpha), \dots, t_n^{\mathbf{T}}(\alpha))\rangle) = (S^{\vec{x},\mathbf{F}}(t_1^{\mathbf{T}}, \dots, t_n^{\mathbf{T}}, f(a)^{\mathbf{T}}))(\alpha)$ for all $a \in \pi(Q)$. Therefore $(S^{\vec{x},\mathbf{T}}(t_1, \dots, t_n, Qf))^{\mathbf{T}}(\alpha) = (S^{\vec{x},\mathbf{F}}(t_1^{\mathbf{T}}, \dots, t_n^{\mathbf{T}}, Qf^{\mathbf{T}}))(\alpha)$ for all α, i.e., $(S^{\vec{x},\mathbf{T}}(t_1, \dots, t_n, Qf))^{\mathbf{T}} = S^{\vec{x},\mathbf{F}}(t_1^{\mathbf{T}}, \dots, t_n^{\mathbf{T}}, Qf^{\mathbf{T}})$.

Then we show that $t \mapsto t^{\mathbf{T}}$ is injective. Let $\iota : X \to T_{\mathcal{F}}(X)$ be such that $\iota = \langle x\rangle_{x\in X}$, then it's easy to check that $t^{\mathbf{T}}(\iota) = t$ for all $t \in T_{\mathcal{F}}(X)$. So $t_a^{\mathbf{T}} = t_b^{\mathbf{T}}$ implies $t_a = t_a^{\mathbf{T}}(\iota) = t_b^{\mathbf{T}}(\iota) = t_b$ for all $t_a, t_b \in T_{\mathcal{F}}(X)$. Hence $t \mapsto t^{\mathbf{T}}$ is injective. □

Since $t \mapsto t^{\mathbf{T}}$ is an injective homomorphism from $\mathbf{T}^{\text{ss}}_{\mathcal{F}}(X)$ to $\mathbf{F}_X(T_{\mathcal{F}}(X))$, we have the following theorem:

Theorem 2.5. $\mathbf{T}^{\text{ss}}_{\mathcal{F}}(X)$ is isomorphic to the subalgebra of $\mathbf{F}_X(T_{\mathcal{F}}(X))$ with $\{t^{\mathbf{T}} \mid t \in T_{\mathcal{F}}(X)\}$ as its universe.

2.3 Dimension sets and local finite-dimensionality

In a term simultaneous substitution algebra, a variable x may not matter to a term t, that is to say, x doesn't occur in t. The following concept helps us to generalize this phenomenon to all non-trivial simultaneous substitution algebras:

Definition 2.2. Let $\mathbf{A} = \langle A, \langle c_d \rangle_{d \in D}, \langle S^{\vec{d}} \rangle_{\vec{d} \in D^\#} \rangle$ be a non-trivial D-SSA. For each $a \in A$, the *dimension set of a in $\mathbf{A}$* is

$$\Delta^{\mathbf{A}} a = \{d \in D \mid \exists a' \in D \; S^{(d)}(a', a) \neq a\}.$$

The superscript $^{\mathbf{A}}$ will be omitted where it is clear which algebra is being discussed.

By the definition, $d \notin \Delta a$ iff $S^{(d)}(a', a) = a$ for all $a' \in A$. The following lemmas are useful in the proof of the representability of simultaneous substitution algebras in Chapter 5.

Lemma 2.6. Let D be a nonempty set and $\mathbf{A} = \langle A, \langle c_d \rangle_{d \in D}, \langle S^{\vec{d}} \rangle_{\vec{d} \in D^\#} \rangle$ be a non-trivial D-SSA. For all $(d_1, \dots, d_n) \in D^\#$ and $a, a_1, \dots, a_n \in A$,

(a) $d_1 \notin \Delta a \Rightarrow S^{(d_1,\dots,d_n)}(a_1, \dots, a_n, a) = S^{(d_2,\dots,d_n)}(a_2, \dots, a_n, a)$;

(b) $\{u_1, \dots, u_k, v_1, \dots, v_{n-k}\} = \{1, \dots, n\}, d_{v_1}, \dots, d_{v_{n-k}} \notin \Delta a \Rightarrow$
$S^{(d_1,\dots,d_n)}(a_1, \dots, a_n, a) = S^{(d_{u_1},\dots,d_{u_k})}(a_{u_1}, \dots, a_{u_k}, a)$.

Proof. (a) If $n = 1$, then $S^{(d_1)}(a_1, a) \overset{d_1 \notin \Delta a}{=} a \overset{\text{(ss1)}}{=} S^{()}(a)$. Else, we have $n \geq 2$. Since $d_1 \notin \Delta a$, we have $a = S^{(d_1)}(c_{d_2}, a)$. Then

$$\begin{aligned}
S^{(d_1,\dots,d_n)}(a_1, \dots, a_n, a) &= S^{(d_1,\dots,d_n)}(a_1, \dots, a_n, S^{(d_1)}(c_{d_2}, a)) \\
&\overset{2.1\text{(d)}}{=} S^{(d_1,\dots,d_n)}(S^{(d_1,\dots,d_n)}(a_1, \dots, a_n, c_{d_2}), a_2, \dots, a_n, a) \\
&\overset{2.1\text{(a)}}{=} S^{(d_1,\dots,d_n)}(a_2, a_2, \dots, a_n, a) \\
&\overset{2.1\text{(a)}}{=} S^{(d_1,\dots,d_n)}(S^{(d_2,\dots,d_n)}(a_2, \dots, a_n, c_{d_2}), a_2, \dots, a_n, a) \\
&\overset{2.1\text{(d)}}{=} S^{(d_2,\dots,d_n)}(a_2, \dots, a_n, S^{(d_1)}(c_{d_2}, a)) \\
&= S^{(d_2,\dots,d_n)}(a_2, \dots, a_n, a).
\end{aligned}$$

(b) By Lemma 2.1(b),

$$S^{(d_1,\dots,d_n)}(a_1, \dots, a_n, a) = S^{(d_{v_1},\dots,d_{v_{n-k}},d_{u_1},\dots,d_{u_k})}(a_{v_1}, \dots, a_{v_{n-k}}, a_{u_1}, \dots, a_{u_k}, a).$$

Apply (a) $n - k$ times, then we get $S^{(d_1,\dots,d_n)}(a_1, \dots, a_n, a) = S^{(d_{u_1},\dots,d_{u_k})}(a_{u_1}, \dots, a_{u_k}, a)$. □

Lemma 2.7. Let D be a nonempty set and $\mathbf{A} = \langle A, \langle c_d \rangle_{d\in D}, \langle S^{\vec{d}} \rangle_{\vec{d}\in D^\#} \rangle$ be a non-trivial D-SSA.

(a) For each $d \in D$, $\Delta c_d = \{d\}$.

(b) For each $(d_1, \dots, d_n) \in D^\#$ and $a_1, \dots, a_n, a \in A$, $\Delta S^{(d_1,\dots,d_n)}(a_1, \dots, a_n, a) \subseteq (\Delta a \setminus \{d_1, \dots, d_n\}) \cup \bigcup_{1\leq i\leq n} \Delta a_i$.

Proof. (a) Since $\mathbf{A}$ is non-trivial, we can take $a \in A$ such that $a \neq c_d$, so $S^{(d)}(a, c_d) \overset{\text{(ss2)}}{=} a \neq c_d$, hence $d \in \Delta c_d$. For each $d' \in D \setminus \{d\}$, $S^{(d')}(a, c_d) \overset{\text{(ss4)}}{=} c_d$ for all $a \in A$, so $d' \notin \Delta c_d$. Therefore $\Delta c_d = \{d\}$.

(b) Let $D_0 = (\Delta a \setminus \{d_1, \dots, d_n\}) \cup \bigcup_{1\leq i\leq n} \Delta a_i$. To see that $\Delta S^{(d_1,\dots,d_n)}(a_1, \dots, a_n, a) \subseteq D_0$, suppose $d \notin D_0$. Then for each i with $1 \leq i \leq n$, $d \notin \Delta a_i$, i.e., $S^{(d)}(a', a_i) = a_i$ for all $a' \in A$. There are two cases.

Case 1: $d \in \{d_1, \dots, d_n\}$. For all $a' \in A$,

$$\begin{aligned} S^{(d)}(a', S^{(d_1,\dots,d_n)}(a_1, \dots, a_n, a)) &\overset{2.1\text{(d)}}{=} S^{(d_1,\dots,d_n)}(S^{(d)}(a', a_1), \dots, S^{(d)}(a', a_n), a) \\ &= S^{(d_1,\dots,d_n)}(a_1, \dots, a_n, a). \end{aligned}$$

Case 2: $d \notin \{d_1, \dots, d_n\}$, so $d \notin \Delta a$. For all $a' \in A$,

$$\begin{aligned} S^{(d)}(a', S^{(d_1,\dots,d_n)}(a_1, \dots, a_n, a)) &\overset{2.1\text{(d)}}{=} S^{(d_1,\dots,d_n,d)}(S^{(d)}(a', a_1), \dots, S^{(d)}(a', a_n), a', a) \\ &= S^{(d_1,\dots,d_n,d)}(a_1, \dots, a_n, a', a) \\ &\overset{2.6\text{(b)}}{=} S^{(d_1,\dots,d_n)}(a_1, \dots, a_n, a). \end{aligned}$$

So $d \notin \Delta S^{(d_1,\dots,d_n)}(a_1, \dots, a_n, a)$ for all $d \notin D_0$. By contraposition, $\Delta S^{(d_1,\dots,d_n)}(a_1, \dots, a_n, a) \subseteq D_0$. □

Definition 2.3. Let D be an infinite set and $\mathbf{A} = \langle A, \langle c_d \rangle_{d\in D}, \langle S^{\vec{d}} \rangle_{\vec{d}\in D^\#} \rangle$ be a non-trivial D-SSA. $\mathbf{A}$ is *locally finite-dimensional* if Δa is finite for all $a \in A$.

It is easy to see that for each infinite D and each similarity type $\mathcal{S}$ of algebras such that $\mathcal{S} \cap D = \emptyset$ and $\mathcal{S}_0 \cup D \neq \emptyset$, $\mathbf{T}^{\text{ss}}_{\mathcal{S}}(D)$ is locally finite-dimensional; it can also be shown that a quotient algebra of a locally finite-dimensional D-SSA is still locally finite-dimensional, hence for each algebra $\mathbf{B}$ of type $\mathcal{S}$, $\mathbf{Clo}^{\text{ss}}_D(\mathbf{B})$ is locally finite-dimensional.

Chapter 3

Simultaneous substitution algebras and (singular) substitution algebras

In this chapter we discuss the relation between simultaneous substitution algebras and singular substitution algebras. We also say substitution algebras instead of singular substitution algebras as in [Fel82] and [Fel15]. Intuitively, singular substitution is a simple version of simultaneous substitution; a question is whether the complex version can be built up from the simple one, and a partial answer will be given in our discussion. Our axiom schemas of substitution algebras are based on the axiom schemas given by Feldman in [Fel82], but differ in the choice of (s6).

Definition 3.1. Let A, D be two nonempty sets; for each $x \in D$, let $c_x \in A$ be a distinguished element, and S^x be a binary operation of A; then $\mathbf{A} = \langle A, \langle c_x, S^x \rangle_{x \in D} \rangle$ is a D-*dimensional substitution algebra* (D-SA) if for all $x, y \in D$ and $a, b, d \in A$,

(s1) $S^x(a, c_x) = a$;

(s2) $S^x(c_x, a) = a$;

(s3) $x \neq y \Rightarrow S^x(a, c_y) = c_y$;

(s4) $S^x(d, S^x(b, a)) = S^x(S^x(d, b), a)$;

(s5) $x \neq y,\ S^x(c_y, d) = d \Rightarrow S^y(d, S^x(b, a)) = S^x(S^y(d, b), S^y(d, a))$;

(s6) $S^y(b, S^x(c_y, a)) = S^x(b, S^y(c_x, a))$. [1]

In the following we will also write $S^x_b a$ instead of $S^x(b, a)$, and $S^x_y a$ instead of $S^x(c_y, a)$. Again we can think of algebras of terms as an example. Let $\mathcal{S}$ be an arbitrary type of algebras, then $T_\mathcal{S}(D)$ is the set of all terms of type $\mathcal{S}$ over variables D. Remember that for each $x \in D$, $S^{(x)}$ is the binary operation over $T_\mathcal{S}(D)$ such that for all terms t and t', $S^{(x)}(t', t)$ is the term obtained by replacing the occurrences of x by t' in t. It can be verified that $\mathbf{T}^{\mathrm{s}}_\mathcal{S}(D) = \langle T_\mathcal{S}(D), \langle x, S^{(x)} \rangle_{x \in D} \rangle$

[1] (s5) is equivalent to $x \neq y \Rightarrow S^y(S^x(c_y, d), S^x(b, a)) = S^x(S^y(S^x(c_y, d), b), S^y(S^x(c_y, d), a))$ under (s3)(s4), thus the class of D-SAs can be defined by a set of equations.

is a D-SA, and we call this algebra the *D-dimensional term substitution algebra* (D-TSA) of type $\mathcal{S}$.

Similarly to the previous chapter, we can also define *D-dimensional function substitution algebras* (D-FSA), *D-dimensional polynomial substitution algebras* (D-PSA), and D-TSAs *in a broader sense*; we can verify they satisfy (s1)-(s6). Given an arbitrary nonempty set A and an arbitrary algebra $\mathbf{B}$ of type $\mathcal{S}$, we denote the full D-FSA with base A by $\mathbf{F}^{\mathrm{s}}_D(A)$ and the D-PSA induced by $\mathbf{B}$ by $\mathbf{Clo}^{\mathrm{s}}_D(\mathbf{B})$.

Notice that $\mathbf{T}^{\mathrm{s}}_{\mathcal{S}}(D)$, $\mathbf{F}^{\mathrm{s}}_D(A)$, $\mathbf{Clo}^{\mathrm{s}}_D(\mathbf{B})$ are reducts of $\mathbf{T}^{\mathrm{ss}}_{\mathcal{S}}(D)$, $\mathbf{F}^{\mathrm{ss}}_D(A)$, $\mathbf{Clo}^{\mathrm{ss}}_D(\mathbf{B})$ respectively. In fact, it can be shown that each D-SSA can be reduced to a D-SA:

Proposition 3.1. For each D-SSA $\mathbf{A}^{\mathrm{ss}} = \langle A, \langle c_d\rangle_{d\in D}, \langle S^{\vec{d}}\rangle_{\vec{d}\in D\#}\rangle$, the structure $\mathbf{A}^{\mathrm{s}} = \langle A, \langle c_d, S^{(d)}\rangle_{d\in D}\rangle$ is a D-SA.

Proof. Let $\mathbf{A}^{\mathrm{ss}} = \langle A, \langle c_d\rangle_{d\in D}, \langle S^{\vec{d}}\rangle_{\vec{d}\in D\#}\rangle$ be an arbitrary D-SSA. We can show that $\mathbf{A}^{\mathrm{s}} = \langle A, \langle c_d, S^{(d)}\rangle_{d\in D}\rangle$ satisfies the axiom schemas (s1)-(s6). We check (s6) as an example. For all $x, y \in D$ and $a, b \in A$, we have

$$S^{(y)}(b, S^{(x)}(c_y, a)) \overset{2.1(\mathrm{d})}{=} S^{(x,y)}(S^{(y)}(b, c_y), b, a) \overset{(\mathrm{ss2})}{=} S^{(x,y)}(b, b, a);$$

similarly, $S^{(x)}(b, S^{(y)}(c_x, a)) = S^{(y,x)}(b, b, a)$. Hence

$$S^{(y)}(b, S^{(x)}(c_y, a)) = S^{(x,y)}(b, b, a) \overset{(\mathrm{ss6})}{=} S^{(y,x)}(b, b, a) = S^{(x)}(b, S^{(y)}(c_x, a)).$$

Therefore, $\mathbf{A}^{\mathrm{s}}$ is a D-SA. □

The next question is whether each D-SA can be expanded to a D-SSA. To start our discussion, we need definitions of dimension sets and local finite-dimensionality, coming from [Fel82].

Definition 3.2. Let $\mathbf{A} = \langle A, \langle c_x, S^x\rangle_{x\in D}\rangle$ be a D-SA. For $a \in A$, the *dimension set of a in $\mathbf{A}$* is

$$\Delta^{\mathbf{A}} a = \{x \in D \mid \exists b \in A \ S^x_b a \neq a\}.$$

The superscript will be omitted where it is clear which algebra is being discussed.

Notice that by this definition, a dimension $x \notin \Delta a$ iff $S^x_b a = a$ for all $b \in A$. This condition can be weakened when there are at least two dimensions in D (in other words, x is not the only dimension). The following lemma comes from Theorem 2.1 in [Fel82]:

Lemma 3.2. Let D be a set with $|D| \geq 2$, $\mathbf{A}$ be a D-SA, $a \in A$, and $x \in D$. Then $x \notin \Delta a$ (i.e., $S^x_b a = a$ for all $b \in A$) if and only if there exists $y \in D \setminus \{x\}$ such that $S^x_y a = a$.

With this lemma we can change the antecedent in (s5):

$$\text{(s5')} \qquad x \neq y, x \notin \Delta a \Rightarrow S^y(a, S^x(d, b)) = S^x(S^y(a, d), S^y(a, b)).$$

Definition 3.3. Let D be an infinite set and $\mathbf{A} = \langle A, \langle c_x, S^x \rangle_{x\in D}\rangle$ be a D-SA.
(a) $\mathbf{A}$ is *locally finite-dimensional* if for all $a \in A$, Δa is finite.
(b) $\mathbf{A}$ is *dimension-complemented* if for all finite $A_0 \subseteq A$, $D \setminus \bigcup\{\Delta a \mid a \in A_0\}$ is infinite.

It is easy to see that each locally finite-dimensional D-SA is also dimension-complemented. Then we will show that a D-SA can expanded to a D-SSA under the conditions of local finite-dimensionality (or dimension-complementedness). We prove some lemmas first. Part of the following lemma is substantially the same as Theorem 2.2 in [Fel15].

Lemma 3.3. Let D be a nonempty set and $\mathbf{A} = \langle A, \langle c_d, S^d \rangle_{d\in D}\rangle$ be a D-SA.

(a) For all $x, y \in D$ with $x \neq y$, $y \notin \Delta c_x$.

(b) For all $a, b, d \in A$ and $x, y \in D$ with $x \neq y$, $x \notin \Delta d$, and $y \notin \Delta b$,

$$S^y(d, S^x(b, a)) = S^x(b, S^y(d, a)) \text{ (or } S^y_d S^x_b a = S^x_b S^y_d a).$$

(c) For all $a, b \in A$ and $x, y \in D$ with $x \notin \Delta a \cup \Delta b$, $x \notin \Delta S^y_b a$.

(d) For all $i, n \in \mathbb{N}$ with $1 \leq i \leq n$, and for all pairwise distinct dimensions $d_1, \ldots, d_n, d^*_1, \ldots, d^*_n, d^{**}_i \in D$ and all $a, a_1, \ldots, a_n \in A$ such that $d^*_1, \ldots, d^*_n, d^{**}_i \notin \Delta a \cup \bigcup_{1\leq j\leq n} \Delta a_j$,

$$S^{d^*_n}_{a_n} \ldots S^{d^*_1}_{a_1} S^{d_n}_{d^*_n} \ldots S^{d_1}_{d^*_1} a = S^{d^*_n}_{a_n} \ldots S^{d^{**}_i}_{a_i} \ldots S^{d^*_1}_{a_1} S^{d_n}_{d^*_n} \ldots S^{d_i}_{d^{**}_i} \ldots S^{d_1}_{d^*_1} a$$

(the expression on the right is obtained by replacing $S^{d^*_i}_{a_i}$, $S^{d_i}_{d^*_i}$ with $S^{d^{**}_i}_{a_i}$, $S^{d_i}_{d^{**}_i}$ respectively).

(e) For all pairwise distinct dimensions $d_1, \ldots, d_n, d^*_1 \ldots, d^*_n, d^{**}_1, \ldots, d^{**}_n \in D$ and $a, a_1, \ldots, a_n \in A$ such that $d^*_1, \ldots, d^*_n, d^{**}_1, \ldots, d^{**}_n \notin \Delta a \cup \bigcup_{1\leq i\leq n} \Delta a_i$,

$$S^{d^*_n}_{a_n} \ldots S^{d^*_1}_{a_1} S^{d_n}_{d^*_n} \ldots S^{d_1}_{d^*_1} a = S^{d^{**}_n}_{a_n} \ldots S^{d^{**}_1}_{a_1} S^{d_n}_{d^{**}_n} \ldots S^{d_1}_{d^{**}_1} a.$$

Proof. (a) For all $x, y \in D$ with $x \neq y$, we have $S^y_a c_x = c_x$ for all $a \in A$ by (s3), hence $y \notin \Delta c_x$.
(b) For all $a, b, d \in A$ and $x, y \in D$ with $x \neq y$, $x \notin \Delta d$ and $y \notin \Delta b$,

$$S^y(d, S^x(b, a)) \overset{\text{(s5')},x\neq y,x\notin\Delta d}{=} S^x(S^y(d, b), S^y(d, a)) \overset{y\notin\Delta b}{=} S^x(b, S^y(d, a)).$$

(c) Take arbitrary $a, b \in A$ and $x, y \in D$ with $x \notin \Delta a \cup \Delta b$. Then for all $d \in A$,

$$\begin{aligned} S^x_d S^y_b a &\overset{x\notin\Delta a}{=} S^x_d S^y_b S^x_y a \overset{\text{(s6)}}{=} S^x_d S^x_b S^y_x a \overset{\text{(s4)}}{=} S^x(S^x_d b, S^y_x a) \\ &\overset{x\notin\Delta b}{=} S^x_b S^y_x a \overset{\text{(s6)}}{=} S^y_b S^x_y a \overset{x\notin\Delta a}{=} S^y_b a. \end{aligned}$$

Thus $x \notin \Delta S^y_b a$.

(d) First concerning $S^{d_n^*}_{a_n} \cdots S^{d_1^*}_{a_1} S^{d_n}_{d_n^*} \cdots S^{d_1}_{d_1^*} a$. Since $d_1, \ldots, d_n, d_1^*, \ldots, d_n^*$ are pairwise distinct, we have $d_1, \ldots, d_n \notin \Delta c_{d_j^*}$ for all j with $1 \leq j \leq n$ by (a), hence we can exchange $S^{d_i}_{d_i^*}$ with $S^{d_{i+1}}_{d_{i+1}^*}, \ldots, S^{d_n}_{d_n^*}$ in turn by (b); because $d_1^*, \ldots, d_n^* \notin \bigcup_{1 \leq j \leq n} \Delta a_j$, we can also exchange $S^{d_i^*}_{a_i}$ with $S^{d_{i-1}^*}_{a_{i-1}}, \ldots, S^{d_1^*}_{a_1}$ in turn; for convenience of formulation, we write $S^{d_n^*}_{a_n} \ldots S^{d_{i+1}^*}_{a_{i+1}} S^{d_{i-1}^*}_{a_{i-1}} \ldots S^{d_1^*}_{a_1}$ and $S^{d_n}_{d_n^*} \ldots S^{d_{i+1}}_{d_{i+1}^*} S^{d_{i-1}}_{d_{i-1}^*} \ldots S^{d_1}_{d_1^*}$ as $\boldsymbol{S}_1$, $\boldsymbol{S}_2$ respectively, then we have

$$S^{d_n^*}_{a_n} \ldots S^{d_1^*}_{a_1} S^{d_n}_{d_n^*} \ldots S^{d_1}_{d_1^*} a = \boldsymbol{S}_1 S^{d_i^*}_{a_i} S^{d_i}_{d_i^*} \boldsymbol{S}_2 a.$$

Since $d_i^* \notin \Delta a$ and $d_i^* \notin \{d_j^* \mid 1 \leq j \leq n, j \neq i\}$, we have $d_i^* \notin \Delta \boldsymbol{S}_2 a$ by applying (c) $n-1$ times. Then we have

$$\boldsymbol{S}_1 S^{d_i^*}_{a_i} S^{d_i}_{d_i^*} \boldsymbol{S}_2 a \overset{\text{(s6)}}{=} \boldsymbol{S}_1 S^{d_i}_{a_i} S^{d_i^*}_{d_i} \boldsymbol{S}_2 a \overset{d_i^* \notin \Delta \boldsymbol{S}_2 a}{=} \boldsymbol{S}_1 S^{d_i}_{a_i} \boldsymbol{S}_2 a.$$

Hence $S^{d_n^*}_{a_n} \ldots S^{d_1^*}_{a_1} S^{d_n}_{d_n^*} \ldots S^{d_1}_{d_1^*} a = \boldsymbol{S}_1 S^{d_i}_{a_i} \boldsymbol{S}_2 a$.

Similarly, we can show that $S^{d_n^*}_{a_n} \ldots S^{d_i^{**}}_{a_i} \ldots S^{d_1^*}_{a_1} S^{d_n}_{d_n^*} \ldots S^{d_i}_{d_i^{**}} \ldots S^{d_1}_{d_1^*} a = \boldsymbol{S}_1 S^{d_i}_{a_i} \boldsymbol{S}_2 a$. Thus the equation we want holds.

(e) Apply (d) n times. ☐

Lemma 3.4. Let D be an infinite set and $\mathbf{A}$ be a locally finite-dimensional D-SA. For all $d_1, \ldots, d_n, d_1^*, \ldots, d_n^*, d_1^{**}, \ldots, d_n^{**} \in D$ and $a, a_1, \ldots, a_n \in A$ such that $d_1, \ldots, d_n, d_1^*, \ldots, d_n^*$ are pairwise distinct, $d_1, \ldots, d_n, d_1^{**}, \ldots, d_n^{**}$ are pairwise distinct, and $d_1^*, \ldots, d_n^*, d_1^{**}, \ldots, d_n^{**} \notin \Delta a \cup \bigcup_{1 \leq i \leq n} \Delta a_i$, we have

$$S^{d_n^*}_{a_n} \ldots S^{d_1^*}_{a_1} S^{d_n}_{d_n^*} \ldots S^{d_1}_{d_1^*} a = S^{d_n^{**}}_{a_n} \ldots S^{d_1^{**}}_{a_1} S^{d_n}_{d_n^{**}} \ldots S^{d_1}_{d_1^{**}} a.$$

Proof. Notice that $\{d_1^*, \ldots, d_n^*\}$ and $\{d_1^{**}, \ldots, d_n^{**}\}$ can overlap. Take n different dimensions $d_1^{***}, \ldots, d_n^{***}$ which are not in $\{d_i, d_i^*, d_i^{**} \mid 1 \leq i \leq n\} \cup \Delta a \cup \bigcup_{1 \leq i \leq n} \Delta a_i$; this can be done because D is infinite and $\mathbf{A}$ is locally finite-dimensional. By Lemma 3.3(e), we have

$$S^{d_n^*}_{a_n} \ldots S^{d_1^*}_{a_1} S^{d_n}_{d_n^*} \ldots S^{d_1}_{d_1^*} a = S^{d_n^{***}}_{a_n} \cdots S^{d_1^{***}}_{a_1} S^{d_n}_{d_n^{***}} \cdots S^{d_1}_{d_1^{***}} a = S^{d_n^{**}}_{a_n} \ldots S^{d_1^{**}}_{a_1} S^{d_n}_{d_n^{**}} \ldots S^{d_1}_{d_1^{**}} a.$$

☐

Theorem 3.5. Let D be an infinite set and $\mathbf{A} = \langle A, \langle c_d, S^d \rangle_{d \in D} \rangle$ be a locally finite-dimensional D-SA, then $\mathbf{A}$ can be expanded to a D-SSA.

Proof. We show that simultaneous substitution can be defined in $\mathbf{A}$. For each $(d_1, \ldots, d_n) \in D^\#$, $n > 0$, we define $S^{(d_1, \ldots, d_n)}$ as the $(n+1)$-ary operation such that given arbitrary $a_1, \ldots, a_n, a \in A$, $S^{(d_1, \ldots, d_n)}(a_1, \ldots, a_n, a)$ is $S^{d_n^*}_{a_n} \ldots S^{d_1^*}_{a_1} S^{d_n}_{d_n^*} \ldots S^{d_1}_{d_1^*} a$ where $d_1^*, \ldots, d_n^*$ are n different dimensions outside $\{d_1, \ldots, d_n\} \cup \Delta a \cup \bigcup_{1 \leq i \leq n} \Delta a_i$ (there exist such dimensions because $\mathbf{A}$ is locally finite-dimensional); Lemma 3.4 ensures that our choice of $d_1^*, \ldots, d_n^*$ doesn't affect the final result. Besides, for the empty sequence $() \in D^\#$, we define $S^{()}$ as the identical operation.

It can be shown that the structure $\langle A, \langle c_d \rangle_{d\in D}, \langle S^{\vec{d}} \rangle_{\vec{d}\in D^{\#}} \rangle$ we have defined satisfies (ss1)-(ss6). Take (ss2) as an example:

$$
\begin{aligned}
S^{(d_1,\dots,d_n)}(a_1,\dots,a_n,c_{d_1}) &= S^{d_n^*}_{a_n}\dots S^{d_1^*}_{a_1} S^{d_n}_{d_n^*}\dots S^{d_1}_{d_1^*} c_{d_1} && \\
&= S^{d_n^*}_{a_n}\dots S^{d_1^*}_{a_1} S^{d_n}_{d_n^*}\dots S^{d_2}_{d_2^*} c_{d_1^*} && \text{(s1)}\\
&= S^{d_n^*}_{a_n}\dots S^{d_1^*}_{a_1} c_{d_1^*} && (d_2,\dots,d_n \notin \Delta c_{d_1^*})\\
&= S^{d_n^*}_{a_n}\dots S^{d_2^*}_{a_2} a_1 && \text{(s1)}\\
&= a_1. && (d_2^*,\dots,d_n^* \notin \Delta a_1)
\end{aligned}
$$

Hence each locally finite-dimensional D-SA can be expanded to a D-SSA. □

Remark 1. Notice that the proofs of Lemma 3.4 and Theorem 3.5 still hold if we replace local finite-dimensionality by dimension-complementedness, which means we can relax the condition to dimension-complementedness.

Remark 2. We provide two examples to show why we need the conditions that D is infinite and the substitution algebra is local finite-dimensionality in Theorem 3.5.

Example 1: let $D = \{x, y\}$ with $x \neq y$ (hence D is finite) and $\mathcal{S} = \{f\}$ where f is a binary operation symbol, then $\mathbf{T}^{\mathrm{s}}_{\mathcal{S}}(D)$ is the D-TSA of type $\mathcal{S}$. Consider the subalgebra of $\mathbf{T}_{\mathcal{S}}(D)$ generated by $\{fxy\}$; the term fyx is not in the universe of this subalgebra, while it can be obtained by substituting y, x for x, y simultaneously in fxy.

Example 2: Let ω be the set of variables and $\mathcal{F} = \{Q\}$ with $\pi(Q) = \omega$. Then $\mathbf{T}^{\mathrm{s}}_{\mathcal{F}}(\omega)$ is the ω-TSA of generalized type $\mathcal{F}$. Let $f_0, f_1 : \omega \to \omega$ be such that $f_0(n) = n$ for all $n \in \omega$, and

$$
f_1(n) = \begin{cases} 1 & n = 0, \\ 0 & n = 1, \\ n & n \geq 2. \end{cases}
$$

Then Qf_0 and Qf_1 are in $T_{\mathcal{F}}(\omega)$, and $\Delta Qf_0 = \Delta Qf_1 = \omega$. Consider the subalgebra of $\mathbf{T}^{\mathrm{s}}_{\mathcal{F}}(\omega)$ generated by $\{Qf_0\}$; it can be shown that Qf_1 is not in the universe of this algebra, while Qf_1 can be obtained by substituting 1,0 for 0,1 simultaneously in Qf_0.

Notice that in both of these examples, the substitution algebra can be superexpanded to a simultaneous subsitution algebra. The question whether every substitution algebra can be superexpanded to a simultaneous subsitution algebra remains open.

Chapter 4

Decidability and completeness

The goal of this chapter is to show our axiom schemas (ss1)-(ss6) actually characterize the class of term simultaneous substitution algebras and of polynomial simultaneous substitution algebras; what's more, the equational theory generated by our set of axioms is decidable. These results come from the normal form theorem for simultaneous substitution, which we will introduce in the first section.

4.1 Normal form theorem for simultaneous substitution

For each $d \in D$, let $\boldsymbol{d}$ be a corresponding constant symbol; for each $\vec{d} \in D^{\#}$ of length n, let $\boldsymbol{S}^{\vec{d}}$ be a corresponding $(n+1)$-ary operation symbol. Let $\mathcal{S}_D^{\text{ss}} = \{\boldsymbol{d} \mid d \in D\} \cup \{\boldsymbol{S}^{\vec{d}} \mid \vec{d} \in D^{\#}\}$, then $\mathcal{S}_D^{\text{ss}}$ is the similarity type of D-SSAs; we omit the superscript when there is no confusion. Let X be an arbitrary countable set which is disjoint with $\mathcal{S}_D$, then $T_{\mathcal{S}_D}(X)$, the set of terms of type $\mathcal{S}_D$ over X, is the least set such that

(i) $X \subseteq T_{\mathcal{S}_D}(X)$;

(ii) $\{\boldsymbol{d} \mid d \in D\} \subseteq T_{\mathcal{S}_D}(X)$;

(iii) If $\vec{d} \in D^{\#}$ is of length n and $t_1, \ldots, t_{n+1} \in T_{\mathcal{S}_D}(X)$, then $\boldsymbol{S}^{\vec{d}} t_1 \ldots t_{n+1} \in T_{\mathcal{S}_D}(X)$.

Among all the terms in $T_{\mathcal{S}_D}(X)$, we say a term t is in *normal form* when $t \in NF_{\mathcal{S}_D}(X)$, where $NF_{\mathcal{S}_D}(X) \subseteq T_{\mathcal{S}_D}(X)$ is the least set such that

(i) $X \subseteq NF_{\mathcal{S}_D}(X)$;

(ii) $\{\boldsymbol{d} \mid d \in D\} \subseteq NF_{\mathcal{S}_D}(X)$;

(iii) If $(d_1, \ldots, d_n) \in D^{\#}$, $x \in X$, $t_1, \ldots, t_n \in NF_{\mathcal{S}_D}(X)$ and $t_i \neq \boldsymbol{d_i}$ for all i with $1 \leq i \leq n$, then $\boldsymbol{S}^{(d_1,\ldots,d_n)} t_1 \ldots t_n x \in NF_{\mathcal{S}_D}(X)$.

Lemma 4.1. Let D, D' be sets of dimensions and X, X' be sets of variables such that $D' \subseteq D$, $X' \subseteq X$, and $\mathcal{S}_D \cap X = \emptyset$, then $NF_{\mathcal{S}_{D'}}(X') = NF_{\mathcal{S}_D}(X) \cap T_{\mathcal{S}_{D'}}(X')$.

Proof. First we prove that for each $t \in NF_{\mathcal{S}_{D'}}(X')$, $t \in NF_{\mathcal{S}_D}(X)$ by induction on the structure of t:

(1) $t = x$, $x \in X'$ or $t = \boldsymbol{d}$, $d \in D'$: $t \in NF_{\mathcal{S}_D}(X)$ because $X' \subseteq X$ and $D' \subseteq D$.

(2) $t = \boldsymbol{S}^{(d_1,\ldots,d_n)} t_1 \ldots t_n x$ with $(d_1, \ldots, d_n) \in D'^{\#}$, $t_1, \ldots, t_n \in NF_{\mathcal{S}_{D'}}(X')$, and $t_i \neq \boldsymbol{d_i}$ for all i with $1 \leq i \leq n$: since $D' \subseteq D$, $(d_1, \ldots, d_n) \in D^{\#}$; by IH, $t_i \in NF_{\mathcal{S}_D}(X)$ for all i; hence $t \in NF_{\mathcal{S}_D}(X)$.

As we also have $NF_{\mathcal{S}_{D'}}(X') \subseteq T_{\mathcal{S}_{D'}}(X')$, $NF_{\mathcal{S}_{D'}}(X') \subseteq NF_{\mathcal{S}_D}(X) \cap T_{\mathcal{S}_{D'}}(X')$.

To show $NF_{\mathcal{S}_{D'}}(X') \supseteq NF_{\mathcal{S}_D}(X) \cap T_{\mathcal{S}_{D'}}(X')$, we show that for each $t \in T_{\mathcal{S}_{D'}}(X')$, if $t \in NF_{\mathcal{S}_D}(X)$ then $t \in NF_{\mathcal{S}_{D'}}(X')$ by induction on the structure of t:

(1) $t = x$, $x \in X'$ or $t = \boldsymbol{d}$, $d \in D'$: then $t \in NF_{\mathcal{S}_{D'}}(X')$.

(2) $t = \boldsymbol{S}^{\vec{d}} t_1 \ldots t_{n+1}$ with $\vec{d} \in D^{\#}$ of length n and $t_1, \ldots, t_{n+1} \in T_{\mathcal{S}_{D'}}(X')$: assume that $t \in NF_{\mathcal{S}_D}(X)$, then $t_i \in NF_{\mathcal{S}_D}(X)$ and $t_i \neq \boldsymbol{d_i}$ for all i, $1 \leq i \leq n$; by IH, $t_i \in NF_{\mathcal{S}_{D'}}(X')$ for all i, so we have $t \in NF_{\mathcal{S}_{D'}}(X')$.

Therefore, $NF_{\mathcal{S}_{D'}}(X') = NF_{\mathcal{S}_D}(X) \cap T_{\mathcal{S}_{D'}}(X')$. □

With the observation in this lemma, we can say a term is in normal form without explicitly stating the type and the set of variables we are talking about.

Let D-SSA be the set of equations corresponding to (ss1)-(ss6) in Chapter 2; it is the set of axioms of D-SSAs. We will prove that every term in $T_{\mathcal{S}_D}(X)$ is equivalent to a term in normal form under D-SSA. We write $T_{\mathcal{S}_D}(X)$, $NF_{\mathcal{S}_D}(X)$, D-SSA as T, NF, SSA respectively when there is no confusion.

Theorem 4.2. For each term $t \in T_{\mathcal{S}_D}(X)$, there exists a term $t' \in NF_{\mathcal{S}_D}(X)$ such that D-SSA $\vdash t \approx t'$.

Proof. First we introduce two measurements of the number of substitution operators in a term, called w_1 and w_2. Definitions are given recursively:

(i) for all $x \in X$ and $d \in D$, $w_1(x) = w_2(x) = w_1(\boldsymbol{d}) = w_2(\boldsymbol{d}) = 0$;

(ii) for all $\vec{d} \in D^{\#}$ of length n and $t_1, \ldots, t_{n+1} \in T$,

$$w_1(\boldsymbol{S}^{\vec{d}} t_1 \ldots t_{n+1}) = w_1(t_{n+1}) + 1,$$
$$w_2(\boldsymbol{S}^{\vec{d}} t_1 \ldots t_{n+1}) = \begin{cases} w_2(t_{n+1}) + 1, & n = 0, \\ \max\{w_2(t_1), \ldots, w_2(t_n)\} + w_2(t_{n+1}) + 1, & n \geq 1. \end{cases}$$

Notice that for each term t, we have

$$w_1(t) = 0 \Leftrightarrow w_2(t) = 0 \Leftrightarrow t = x \text{ for some } x \in X \text{ or } t = \boldsymbol{d} \text{ for some } d \in D.$$

First we claim that

(∗) each term t with $w_1(t) \geq 1$ is equivalent to a term t' under SSA such that $w_1(t') = 1$ and $w_2(t') \leq w_2(t)$.

We prove (∗) by induction on $w_1(t)$:

(1) $w_1(t) = 1$: take $t' = t$.

(2) Assume (∗) holds for all terms t with $w_1(t) = n \geq 1$. Let t be an arbitrary term with $w_1(t) = n + 1$. Since $w_1(t) \geq 2$, t is of the form $\boldsymbol{S}^{\vec{d}} t_1 \dots t_{m+1}$ where t_{m+1} is of the form $\boldsymbol{S}^{\vec{d'}} t'_1 \dots t'_{l+1}$ for some $\vec{d} \in D^{\#}$ of length m and $\vec{d'} \in D^{\#}$ of length l.

It is easy to show that if $m = 0$ or $l = 0$, i.e., $\vec{d} = ()$ or $\vec{d'} = ()$, then t is equivalent to some t' with $w_1(t') = 1$ and $w_2(t') \leq w_2(t)$ by IH. Then we consider the case that $m \neq 0$ and $l \neq 0$. Let $\vec{d} = (d_1, \dots, d_m)$ and $\vec{d'} = (d'_1, \dots, d'_l)$, then there exist $t_1, \dots, t_m, t'_1, \dots, t'_l, t'_{l+1} \in T$ such that

$$t = \boldsymbol{S}^{(d_1,\dots,d_m)} t_1 \dots t_m \boldsymbol{S}^{(d'_1,\dots,d'_l)} t'_1 \dots t'_l t'_{l+1}.$$

Take integers $u_1, \dots, u_k$ such that $1 \leq u_1 < \dots < u_k \leq m$ and $\{d_{u_1}, \dots, d_{u_k}\} = \{d_1, \dots, d_m\} \setminus \{d'_1, \dots, d'_l\}$. Take

$$t' = \boldsymbol{S}^{(d'_1,\dots,d'_l,d_{u_1},\dots,d_{u_k})} t''_1 \dots t''_l t_{u_1} \dots t_{u_k} t'_{l+1}$$

where $t''_i = \boldsymbol{S}^{(d_1,\dots,d_m)} t_1 \dots t_m t'_i$ for all i with $1 \leq i \leq l$; by Lemma 2.1(d), we have SSA $\vdash t \approx t'$. Because $w_1(t) = n + 1$, $w_1(t) = w_1(\boldsymbol{S}^{(d'_1,\dots,d'_l)} t'_1 \dots t'_{l+1}) + 1 = w_1(t'_{l+1}) + 2$, and $w_1(t') = w_1(t'_{l+1}) + 1$, we have $w_1(t') = n$.

Then we show that $w_2(t) = w_2(t')$. Let $a = \max\{w_2(t_1), \dots, w_2(t_m)\}$ and $b = \max\{w_2(t'_1), \dots, w_2(t'_l)\}$. Then

$$\begin{aligned} w_2(t) &= \max\{w_2(t_1), \dots, w_2(t_m)\} + w_2(\boldsymbol{S}^{(d'_1,\dots,d'_l)} t'_1 \dots t'_l t'_{l+1}) + 1 \\ &= a + (\max\{w_2(t'_1), \dots, w_2(t'_l)\} + w_2(t'_{l+1}) + 1) + 1 \\ &= a + b + w_2(t'_{l+1}) + 2. \end{aligned}$$

For each i with $1 \leq i \leq l$, $w_2(t''_i) = w_2(\boldsymbol{S}^{(d_1,\dots,d_m)} t_1 \dots t_m t'_i) = \max\{w_2(t_1), \dots, w_2(t_m)\} + w_2(t'_i) + 1 = a + w_2(t'_i) + 1$, hence

$$\begin{aligned} w_2(t') &= \max\{w_2(t''_1), \dots, w_2(t''_l), w_2(t_{u_1}), \dots, w_2(t_{u_l})\} + w_2(t'_{l+1}) + 1 \\ &= \max\{a + w_2(t'_1) + 1, \dots, a + w_2(t'_l) + 1, w_2(t_{u_1}), \dots, w_2(t_{u_k})\} + w_2(t'_{l+1}) + 1 \\ &= \max\{a + w_2(t'_1) + 1, \dots, a + w_2(t'_l) + 1\} + w_2(t'_{l+1}) + 1 \quad (\text{since } w_2(t_{u_i}) \leq a) \\ &= a + \max\{w_2(t'_1), \dots, w_2(t'_l)\} + 1 + w_2(t'_{l+1}) + 1 \\ &= a + b + w_2(t'_{l+1}) + 2. \end{aligned}$$

Thus $w_2(t') = w_2(t)$. By IH, there exists a term t'' such that SSA $\vdash t' \approx t''$, $w_1(t'') = 1$ and $w_2(t'') \leq w_2(t')$. Thus we have SSA $\vdash t \approx t''$ and $w_2(t'') \leq w_2(t)$.

Then we prove that each term t is equivalent to a term t' in normal form under SSA by induction on $w_2(t)$:

(3) $w_2(t) = 0$: then $t = x$ for some $x \in X$ or $t = \boldsymbol{d}$ for some $d \in D$, and in both cases we have $\mathsf{SSA} \vdash t \approx t$ where t itself is in normal form.

(4) Assume that each term t with $0 \leq w_2(t) \leq n$ is equivalent to a term t' in normal form under SSA. Let t be an arbitrary term with $w_2(t) = n + 1$; since $w_2(t) \geq 1$, $w_1(t) \geq 1$ as well, hence there exists t' such that $\mathsf{SSA} \vdash t \approx t'$, $w_1(t') = 1$ and $w_2(t') \leq w_2(t) = n + 1$ by $(*)$. Because $w_1(t') = 1$, t' starts with $\boldsymbol{S}^{\vec{d}}$ for some $\vec{d} \in D^{\#}$. If $\vec{d} = ()$, then $t' = \boldsymbol{S}^{()}t_1$ for some $t_1 \in T$ with $w_1(t_1) = 0$, hence $w_2(t_1) = 0$, hence we have $\mathsf{SSA} \vdash \boldsymbol{S}^{()}t_1 \approx t_1$ by (ss1) and t_1 is in normal form; hence t is equivalent to a normal form.

Else, $\vec{d}$ is of length m, $m \geq 1$. Let $\vec{d} = (d_1, \ldots, d_m)$, then we have $t' = \boldsymbol{S}^{\vec{d}}t_1 \ldots t_m t_{m+1}$ for some $t_1, \ldots, t_{m+1} \in T$ with $w_1(t_{m+1}) = 0$. Since $w_2(t') = \max\{w_2(t_1), \ldots, w_2(t_m)\} + w_2(t_{m+1}) + 1 = n + 1$, we have $\max\{w_2(t_1), \ldots, w_2(t_m)\} = n$, hence $w_2(t_i) \leq n$ for all i with $1 \leq i \leq m$. We consider three cases with regard to t_{m+1}:

Case 1: $t_{m+1} = x$, $x \in X$. By IH, for each i with $1 \leq i \leq m$, there exists t'_i in normal form such that $\mathsf{SSA} \vdash t_i \approx t'_i$, hence $t' = \boldsymbol{S}^{(d_1,\ldots,d_m)}t_1 \ldots t_m x$ is equivalent to $\boldsymbol{S}^{(d_1,\ldots,d_m)}t'_1 \ldots t'_m x$ under SSA. Take integers $u_1, \ldots, u_l$ such that $1 \leq u_1 < \cdots < u_l \leq m$ and $\{u_1, \ldots, u_l\} = \{i \mid 1 \leq i \leq m, t'_i \neq \boldsymbol{d_i}\}$. By Lemma 2.1(c), $\mathsf{SSA} \vdash \boldsymbol{S}^{(d_1,\ldots,d_m)}t'_1 \ldots t'_m x \approx \boldsymbol{S}^{(d_{u_1},\ldots,d_{u_l})}t'_{u_1} \ldots t'_{u_l} x$. Since each t'_{u_i} is in normal form and $t'_{u_i} \neq \boldsymbol{d_{u_i}}$, $\boldsymbol{S}^{(d_{u_1},\ldots,d_{u_l})}t'_{u_1} \ldots t'_{u_l} x$ is in normal form as well. Because t is equivalent to $t' = \boldsymbol{S}^{(d_1,\ldots,d_m)}t'_1 \ldots t'_m x$ under SSA, t is also equivalent to $\boldsymbol{S}^{(d_{u_1},\ldots,d_{u_l})}t'_{u_1} \ldots t'_{u_l} x$.

Case 2: $t_{m+1} = \boldsymbol{d}_i$, $1 \leq i \leq m$. By Lemma 2.1(a), $\mathsf{SSA} \vdash \boldsymbol{S}^{(d_1,\ldots,d_m)}t_1 \ldots t_m \boldsymbol{d_i} \approx t_i$; by IH, there exists t'_i in normal form such that t_i is equivalent to t'_i; hence $\mathsf{SSA} \vdash t \approx t'_i$.

Case 3: $t_{m+1} = \boldsymbol{d}$ for some $d \in D$ such that $d \neq d_i$ for all i, $1 \leq i \leq m$. Then $\mathsf{SSA} \vdash \boldsymbol{S}^{(d_1,\ldots,d_m)}t_1 \ldots t_m \boldsymbol{d} \approx \boldsymbol{d}$ by (ss4).

Therefore, for each term t, there exists a term t' in normal form such that $\mathsf{SSA} \vdash t \approx t'$. □

This proof not only shows the existence of an equivalent normal form, but also implies an algorithm to compute such a normal form.

Proposition 4.3. There is an algorithm such that for each term $t \in T_{\mathcal{S}_D}(X)$, it outputs a term t' in normal form such that $\mathsf{SSA} \vdash t \approx t'$.

Proof. We sketch the basic idea here. For a term t of the form $\boldsymbol{S}^{(d_1,\ldots,d_n)}t_1 \ldots t_{n+1}$, we can use the method in Lemma 2.1(d) for at most $w_1(t) - 1$ times to lower w_1 and obtain an equivalent term t' of the form $\boldsymbol{S}^{(d'_1,\ldots,d'_m)}t'_1 \ldots t'_m x$ or $\boldsymbol{S}^{(d'_1,\ldots,d'_m)}t'_1 \ldots t'_m d$; notice that $w_2(t'_1), \ldots, w_2(t'_m) < w_2(t)$. Repeat this procedure on $t'_1, \ldots, t'_m$ and other new terms obtained in the loop, and simplify the terms with (ss1), (ss4), Lemma 2.1(a) and Lemma 2.1(c) during the process, until a normal form has been reached. □

4.2 Decidability and completeness

To show the decidability and completeness of the equational theory generated by SSA, there is still some work to be done. To see whether an equation is valid under SSA, i.e., whether the two terms in the equation are equivalent, our idea is to use the normal form theorem and check whether their normal forms are equivalent. Notice that this cannot be done by simply checking whether two terms in normal form are identical, as a term can be equivalent to more than one term in normal form under SSA. For example, take two different dimensions $d_1, d_2 \in D$ and three variables $x, y, z \in X$ (x, y, z can be the same), then $\boldsymbol{S}^{(d_1,d_2)}xyz$ and $\boldsymbol{S}^{(d_2,d_1)}yxz$ are two different terms in normal form; at the same time, we have $\mathsf{SSA} \vdash \boldsymbol{S}^{(d_1,d_2)}xyz \approx \boldsymbol{S}^{(d_2,d_1)}yxz$. The problem is that elements in simultaneous substitution algebras stay the same after a rearrangement according to a permutation, while our definition of normal form distinguishes such different arrangements. To solve it, we can define an equivalence relation on NF to represent the invariance under permutations. Let $\sim_P \subseteq NF^2$ be the least relation such that

(i) $x \sim_P x$ for all $x \in X$;

(ii) $\boldsymbol{d} \sim_P \boldsymbol{d}$ for all $d \in D$;

(iii) If $n \geq 1$, p is a permutation of $\{1, \ldots, n\}$, $t_i \sim_P t'_i$ for all i with $1 \leq i \leq n$, and $\boldsymbol{S}^{(d_1,\ldots,d_n)}t_1 \ldots t_n x$, $\boldsymbol{S}^{(d_{p(1)},\ldots,d_{p(n)})}t'_{p(1)} \ldots t'_{p(n)}x \in NF$, then $\boldsymbol{S}^{(d_1,\ldots,d_n)}t_1 \ldots t_n x \sim_P \boldsymbol{S}^{(d_{p(1)},\ldots,d_{p(n)})}t'_{p(1)} \ldots t'_{p(n)}x$.

Below are some basic properties of $\sim_P$:

Proposition 4.4. (a) $\sim_P$ is an equivalence relation.

(b) For all $t_a, t_b \in NF$, if $t_a \sim_P t_b$ then $\mathsf{SSA} \vdash t_a \approx t_b$.

(c) $\{(t_a, t_b) \in NF^2 \mid t_a \sim_P t_b\}$ is decidable.

Proof. (a) $\sim_P$ is reflective: for each $t \in NF$, we show $t \sim_p t$ by induction on the structure of t:

(1) $t = x$, $x \in X$ or $t = \boldsymbol{d}$, $d \in D$: then $t \sim_P t$ by (i)(ii);

(2) $t = \boldsymbol{S}^{(d_1,\ldots,d_n)}t_1 \ldots t_n x$: since $t \in NF$, each t_i is in normal form, so $t_i \sim_P t_i$ for each i, $1 \leq i \leq n$ by IH, hence $\boldsymbol{S}^{(d_1,\ldots,d_n)}t_1 \ldots t_n x \sim_P \boldsymbol{S}^{(d_1,\ldots,d_n)}t_1 \ldots t_n x$ by (iii).

$\sim_P$ is symmetric: we prove that for all $t_a, t_b \in NF$, if $t_a \sim_P t_b$ then $t_b \sim_P t_a$ by induction on the structure of t_a:

(3) $t_a = x$, $x \in X$ or $t_a = \boldsymbol{d}$, $d \in D$: by definition of $\sim_P$, $t_b = t_a$, so $t_a \sim_P t_b$;

(4) $t_a = \boldsymbol{S}^{(d_1,\ldots,d_n)}t_1 \ldots t_n x$: then $t_b = \boldsymbol{S}^{(d_{p(1)},\ldots,d_{p(n)})}t'_{p(1)} \ldots t'_{p(n)}x$ for some permutation p of $\{1, \ldots, n\}$ and some $t'_1, \ldots, t'_n$ such that $t_i \sim_P t'_i$ for all i, $1 \leq i \leq n$; by IH, $t'_i \sim_P t_i$ for all i, $1 \leq i \leq n$; since p^{-1} is also a permutation, we have $t_b = \boldsymbol{S}^{(d_{p(1)},\ldots,d_{p(n)})}t'_{p(1)} \ldots t'_{p(n)}x \sim_P \boldsymbol{S}^{(d_1,\ldots,d_n)}t_1 \ldots t_n x = t_a$ by (iii).

$\sim_P$ is transitive: we prove that for all $t_a, t_b, t_c \in NF$, if $t_a \sim_P t_b$ and $t_b \sim_P t_c$, then $t_a \sim_P t_c$, by induction on the structure of t_a:

(5) $t_a = x$, $x \in X$ or $t_a = \boldsymbol{d}$, $d \in D$: then $t_a = t_b = t_c$, hence $t_a \sim_P t_c$;

(6) $t_a = \boldsymbol{S}^{(d_1,\dots,d_n)} t_1 \dots t_n x$: then $t_b = \boldsymbol{S}^{(d_{p(1)},\dots,d_{p(n)})} t'_{p(1)} \dots t'_{p(n)} x$ for some permutation p of $\{1,\dots,n\}$ and some $t'_1,\dots,t'_n$ such that $t_i \sim_P t'_i$ for all i, $1 \le i \le n$, and $t_c = \boldsymbol{S}^{(d_{q(p(1))},\dots,d_{q(p(n))})} t''_{q(p(1))} \dots t''_{q(p(n))} x$ for some permutation q of $\{1,\dots,n\}$ and some $t''_{p(1)}, \dots, t''_{p(n)}$ such that $t'_{p(i)} \sim_P t''_{p(i)}$ for all i, $1 \le i \le n$; since $q \circ p$ is also a permutation of $\{1,\dots,n\}$ and $t_i \sim_P t''_i$ for all i by IH, then $t_a \sim_P t_c$.

Hence $\sim_P$ is an equivalence relation.

(b) Induction on the structure of t_a:

(1) $t_a = x$, $x \in X$ or $t_a = \boldsymbol{d}$, $d \in D$: then $t_b = t_a$, hence $\mathsf{SSA} \vdash t_a \approx t_b$.

(2) $t_a = \boldsymbol{S}^{(d_1,\dots,d_n)} t_1 \dots t_n x$: then $t_b = \boldsymbol{S}^{(d_{p(1)},\dots,d_{p(n)})} t'_{p(1)} \dots t'_{p(n)} x$ for some permutation p of $\{1,\dots,n\}$ and some $t'_1,\dots,t'_n$ such that $t_i \sim_P t'_i$ for all i, $1 \le i \le n$; by IH, $\mathsf{SSA} \vdash t_i \approx t'_i$ for all i, so $\mathsf{SSA} \vdash \boldsymbol{S}^{(d_1,\dots,d_n)} t_1 \dots t_n x \approx \boldsymbol{S}^{(d_{p(1)},\dots,d_{p(n)})} t'_{p(1)} \dots t'_{p(n)} x$ by the congruence rule and Lemma 2.1(b).

(c) The algorithm is recursive:

(1) $t_a = x$, $x \in X$ or $t_a = \boldsymbol{d}$, $d \in D$: if $t_b = t_a$, the algorithm outputs 1, otherwise it outputs 0;

(2) $t_a = \boldsymbol{S}^{(d_1,\dots,d_n)} t_1 \dots t_n x$: first the algorithm checks the first symbol of t_b, and outputs 0 if it is not a $\boldsymbol{S}^{(d'_1,\dots,d'_m)}$ with $m = n$ and $\{d_1,\dots,d_n\} = \{d'_1,\dots,d'_m\}$; otherwise, t_b is of the form $\boldsymbol{S}^{(d'_1,\dots,d'_n)} t'_1 \dots t'_n x$, then for each i with $1 \le i \le n$, it finds the j such that $d'_j = d_i$, runs the same algorithm with input (t_i, t'_j), and outputs 0 if the result is 0. When all results are 1, the algorithm outputs 1.

It is easy to see that for each input (t_a, t_b), if the algorithm outputs 1 then $t_a \sim_P t_b$, and if the algorithm outputs 0 then $t_a \nsim_P t_b$ since $\sim_P$ is the least relation satisfying (i)-(iii). Hence $\{(t_a, t_b) \in NF^2 \mid t_a \sim_P t_b\}$ is decidable. □

We want to show that $NF/\sim_P$ expresses the inequivalence under SSA, i.e., if $t_a \nsim_P t_b$ then $\mathsf{SSA} \nvdash t_a \approx t_b$; we tackle this problem semantically, by providing a special D-SSA that invalidates such equations. Let $\mathcal{S}_0 = \{f^n_m \mid n, m \in \mathbb{N}\}$ where each f^n_m is a n-ary operation symbol that is not in D; this type contains countable many n-ary operation symbols for each $n \in \mathbb{N}$. The lemma below shows that $\mathbf{T}^{\mathrm{ss}}_{\mathcal{S}_0}(D)$, the D-TSSA of type $\mathcal{S}_0$, is the algebra we want:

Lemma 4.5. (a) For each finite set of dimensions $D' \subseteq D$, each finite set of variables $X' \subseteq X$, and each pair of terms $t_a, t_b \in NF_{\mathcal{S}_{D'}}(X')$ with $t_a \nsim_P t_b$, $\mathbf{T}^{\mathrm{ss}}_{\mathcal{S}_0}(D) \nvDash t_a \approx t_b$.

(b) For all terms $t_a, t_b \in NF_{\mathcal{S}_D}(X)$,

$$\mathsf{SSA} \vdash t_a \approx t_b \;\Leftrightarrow\; \mathbf{T}^{\mathrm{ss}}_{\mathcal{S}_0}(D) \vDash t_a \approx t_b \;\Leftrightarrow\; t_a \sim_P t_b.$$

Proof. (a) We abbreviate $\mathbf{T}^{\text{ss}}_{\mathcal{S}_0}(D)$ to $\mathbf{T}$ in this proof. Let D' be an arbitrary finite set of dimensions and X' be an arbitrary finite set of variables. Let $d_1, \ldots, d_n$ be an enumeration of D' without repetition, $x_1, \ldots, x_m$ be an enumeration of X' without repetition, and $\alpha : X \to T_{\mathcal{S}_0}(D)$ be an assignment that maps each x_i to $f_i^n d_1 \ldots d_n$, $1 \le i \le m$. Recall w_2 in the proof of Theorem 4.2; we show that $t_a^{\mathbf{T}}(\alpha) \neq t_b^{\mathbf{T}}(\alpha)$ for all $t_a, t_b \in NF_{\mathcal{S}_{D'}}(X')$ with $t_a \not\approx_P t_b$ by induction on $\max\{w_2(t_a), w_2(t_b)\}$.

(1) $\max\{w_2(t_a), w_2(t_b)\} = 0$: since $t_a \not\approx_P t_b$, $t_a \neq t_b$. There are four cases.

Case 1: $t_a = \boldsymbol{d_i}$, $t_a = \boldsymbol{d_j}$ with $1 \le i, j \le n$, and $i \neq j$. Then $t_a^{\mathbf{T}}(\alpha) = d_i \neq d_j = t_b^{\mathbf{T}}(\alpha)$.

Case 2: $t_a = \boldsymbol{d_i}$, $t_a = x_j$ with $1 \le i \le n$ and $1 \le j \le m$. Then $t_a^{\mathbf{T}}(\alpha) = d_i \neq f_j^n d_1 \ldots d_n = t_b^{\mathbf{T}}(\alpha)$.

Case 3: $t_a = x_i$, $t_b = \boldsymbol{d_j}$ with $1 \le i \le m$ and $1 \le j \le n$. Similar as Case 2.

Case 4: $t_a = x_i$, $t_b = x_j$ with $1 \le i, j \le m$, and $i \neq j$. Then $t_a^{\mathbf{T}}(\alpha) = f_i^n d_1 \ldots d_n \neq f_j^n d_1 \ldots d_n = t_b^{\mathbf{T}}(\alpha)$.

(2) $\max\{w_2(t_a), w_2(t_b)\} = h + 1$, $h \ge 0$: assume without loss of generality that $w_2(t_a) = h + 1$, hence $w_2(t_b) \le h+1$. Since $w_2(t_a) \ge 1$, there exist pairwise distinct $u_1, \ldots, u_k \in \{1, \ldots, n\}$, $i \in \{1, \ldots, m\}$, and $t_1, \ldots, t_k \in NF_{D'}(X')$ such that

$$t_a = \boldsymbol{S}^{(d_{u_1}, \ldots, d_{u_k})} t_1 \ldots t_k x_i.$$

Then we have $t_a^{\mathbf{T}}(\alpha) = S^{(d_{u_1}, \ldots, d_{u_k})}(t_1^{\mathbf{T}}(\alpha), \ldots, t_k^{\mathbf{T}}(\alpha), x_i^{\mathbf{T}}(\alpha)) = S^{(d_{u_1}, \ldots, d_{u_k})}(t_1^{\mathbf{T}}(\alpha), \ldots, t_k^{\mathbf{T}}(\alpha), f_i^n d_1 \ldots d_n)$. Notice that $t_a^{\mathbf{T}}(\alpha)$ is a term starting with f_i^n.

If $w_2(t_b) = 0$, consider three cases with respect to t_b.

Case 1: $t_b = \boldsymbol{d_j}$, $1 \le j \le n$. Then $t_b^{\mathbf{T}}(\alpha) = d_j \neq t_a^{\mathbf{T}}(\alpha)$ since $t_a^{\mathbf{T}}(\alpha)$ starts with f_i^n.

Case 2: $t_b = x_i$. Then $t_b^{\mathbf{T}}(\alpha) = f_i^n d_1 \ldots d_n$. Since t_a is in normal form, t_1 is also in normal form and $t_1 \neq \boldsymbol{d_{u_1}}$; besides, $w_2(t_1) \le h$ because $w_2(t_a) = h+1$, hence $\max\{w_2(t_1), w_2(\boldsymbol{d_{u_1}})\} \le h$; by IH, we have $t_1^{\mathbf{T}}(\alpha) \neq \boldsymbol{d}_{\boldsymbol{u_1}}^{\mathbf{T}}(\alpha) = d_{u_1}$. Since the u_1-th argument of f_i^n in $t_a^{\mathbf{T}}(\alpha)$ is not d_{u_1}, $t_a^{\mathbf{T}}(\alpha) \neq f_i^n d_1 \ldots d_n$.

Case 3: $t_b = x_j$, $1 \le j \le m$ and $i \neq j$. Then $t_a^{\mathbf{T}}(\alpha) \neq t_b^{\mathbf{T}}(\alpha)$ because $t_a^{\mathbf{T}}(\alpha)$ starts with f_i^n and $t_b^{\mathbf{T}}(\alpha)$ starts with f_j^n.

Else, $w_2(t_b) > 0$, then there are pairwise distinct $v_1, \ldots, v_l \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$ and $t_{k+1}, \ldots, t_{k+l} \in NF_{D'}(X')$ such that

$$t_b = \boldsymbol{S}^{(d_{v_1}, \ldots, d_{v_l})} t_{k+1} \ldots t_{k+l} x_j.$$

Then $t_b^{\mathbf{T}}(\alpha) = S^{(d_{v_1}, \ldots, d_{v_l})}(t_{k+1}^{\mathbf{T}}(\alpha), \ldots, t_{k+l}^{\mathbf{T}}(\alpha), x_j^{\mathbf{T}}(\alpha)) = S^{(d_{v_1}, \ldots, d_{v_l})}(t_{k+1}^{\mathbf{T}}(\alpha), \ldots, t_{k+l}^{\mathbf{T}}(\alpha), f_j^n d_1 \ldots d_n)$, which is a term starting with f_j^n. If $i \neq j$, then $t_a^{\mathbf{T}}(\alpha)$ starts with f_i^n and $t_b^{\mathbf{T}}(\alpha)$ starts with f_j^n, hence $t_a^{\mathbf{T}}(\alpha) \neq t_b^{\mathbf{T}}(\alpha)$. Else, $i = j$, there are two cases.

Case 1: $\{u_1, \ldots, u_k\} = \{v_1, \ldots, v_l\}$. In this case, we have either $\{u_1, \ldots, u_k\} \setminus \{v_1, \ldots, v_l\} \neq \emptyset$ or $\{v_1, \ldots, v_l\} \setminus \{u_1, \ldots, u_k\} \neq \emptyset$. If $\{u_1, \ldots, u_k\} \setminus \{v_1, \ldots, v_l\} \neq \emptyset$, take $p \in \{1, \ldots, k\}$ such that $u_p \notin \{v_1, \dots, v_l\}$; since t_a is in normal form, t_p is also in normal form and $t_p \neq \boldsymbol{d_{u_p}}$; as we also have $\max(w_2(t_p), w_2(\boldsymbol{d_{u_p}})\} = w_2(t_p) < w_2(t_a) = h+1$, so $t_p^{\mathbf{T}}(\alpha) \neq d_{u_p}$ by IH. While both $t_a^{\mathbf{T}}(\alpha)$ and $t_b^{\mathbf{T}}(\alpha)$ are obtained by a simultaneous substitution on $f_i^n d_1 \dots d_n$, we need to replace the d_{u_p} to get $t_a^{\mathbf{T}}(\alpha)$ and keep d_{u_p} unchanged to get $t_b^{\mathbf{T}}(\alpha)$. Thus $t_a^{\mathbf{T}}(\alpha) \neq t_b^{\mathbf{T}}(\alpha)$. Otherwise, we have $\{v_1, \dots, v_l\} \setminus \{u_1, \dots, u_k\} \neq \emptyset$; similarly we can show $t_a^{\mathbf{T}}(\alpha) \neq t_b^{\mathbf{T}}(\alpha)$.

Case 2: $\{u_1, \dots, u_k\} = \{v_1, \dots, v_l\}$. Because $t_a \nsim_P t_b$, there exist $p, q \in \{1, \dots, k\}$ and $r \in \{1, \dots, n\}$ such that $u_p = v_q = r$ and $t_p \nsim_P t_{k+q}$. Notice that the r-th arguments of f_i^n in $t_a^{\mathbf{T}}(\alpha), t_b^{\mathbf{T}}(\alpha)$ are $t_p^{\mathbf{T}}(\alpha), t_{k+q}^{\mathbf{T}}(\alpha)$ respectively. By definition of w_2, we have $\max\{w_2(t_p), w_2(t_{k+q})\} < \max\{w_2(t_a), w_2(t_b)\} = h+1$. By IH, $t_p^{\mathbf{T}}(\alpha) \neq t_{k+q}^{\mathbf{T}}(\alpha)$, hence $t_a^{\mathbf{T}}(\alpha) \neq t_b^{\mathbf{T}}(\alpha)$.

Therefore for all $t_a, t_b \in NF_{\mathcal{S}_{D'}}(X')$ with $t_a \nsim_P t_b$, we have $t_a^{\mathbf{T}}(\alpha) \neq t_b^{\mathbf{T}}(\alpha)$, hence $\mathbf{T} \nvDash t_a \approx t_b$.

(b) $\mathsf{SSA} \vdash t_a \approx t_b \;\Rightarrow\; \mathbf{T} \vDash t_a \approx t_b$: this holds because $\mathbf{T}$ is a D-SSA.

$\mathbf{T} \vDash t_a \approx t_b \;\Rightarrow\; t_a \sim_P t_b$: assume that $\mathbf{T} \vDash t_a \approx t_b$. Let D' be the set of all dimensions occurring in t_a, t_b and X' be the set of all variables occurring in t_a, t_b. By Lemma 4.1, $t_a, t_b \in NF_{\mathcal{S}_{D'}}(X')$. Because the length of a term is finite, D' and X' are also finite. Hence we have $t_a \sim_P t_b$ by (a).

$t_a \sim_P t_b \;\Rightarrow\; \mathsf{SSA} \vdash t_a \approx t_b$: see Proposition 4.4(b). □

With all the preliminary work, now we are ready to prove the final results in this chapter:

Theorem 4.6 (Decidability of D-SSA)**.** $\{(t_a, t_b) \in T_{\mathcal{S}_D^{\text{ss}}}(D)^2 \mid D\text{-}\mathsf{SSA} \vdash t_a \approx t_b\}$ is decidable.

Proof. We describe an algorithm as follows: given arbitrary $(t_a, t_b) \in T^2$, first compute two terms $t'_a, t'_b \in NF$ such that $\mathsf{SSA} \vdash t_a \approx t'_a$ and $\mathsf{SSA} \vdash t_b \approx t'_b$ as in Proposition 4.3. Then it uses the algorithm in Proposition 4.4(c) and outputs the result (i.e., whether $t'_a \sim_P t'_b$). By Lemma 4.5(b), $\mathsf{SSA} \vdash t'_a \approx t'_b$ iff $t'_a \sim_P t'_b$, hence this is an algorithm deciding $\{(t_a, t_b) \in T^2 \mid \mathsf{SSA} \vdash t_a \approx t_b\}$. □

Theorem 4.7 (Completeness of D-SSA with D-TSSAs and with D-PSSAs)**.** Let $K_{D\text{-TSSA}}$ be the class of all D-TSSAs and $K_{D\text{-PSSA}}$ be the class of all D-PSSAs.

(a) D-SSA is complete with $K_{D\text{-TSSA}}$.

(b) D-SSA is complete with $K_{D\text{-PSSA}}$.

Proof. (a) Let t_a, t_b be a pair of terms in T. If D-$\mathsf{SSA} \vdash t_a \approx t_b$, then $K_{D\text{-TSSA}} \vDash t_a \approx t_b$ since every D-TSSA is a D-SSA. If D-$\mathsf{SSA} \nvdash t_a \approx t_b$, then D-$\mathsf{SSA} \nvdash t'_a \approx t'_b$ where t'_a is a normal form of t_a and t'_b is a normal form of t_b, hence $\mathbf{T}^{\text{ss}}_{\mathcal{S}_0}(D) \nvDash t'_a \approx t'_b$ by Lemma 4.5(b), hence $K_{D\text{-TSSA}} \nvDash t'_a \approx t'_b$; as $K_{D\text{-TSSA}} \vDash t_a \approx t'_a$ and $K_{D\text{-TSSA}} \vDash t_b \approx t'_b$, we have $K_{D\text{-TSSA}} \nvDash t_a \approx t_b$.

(b) By Lemma 2.3(b), we have $\mathbf{T}^{\text{ss}}_{\mathcal{S}_0}(D) \cong \mathbf{Clo}^{\text{ss}}_D(\mathbf{T}_{\mathcal{S}_0}(D))$ where $\mathbf{Clo}^{\text{ss}}_D(\mathbf{T}_{\mathcal{S}_0}(D)) \in K_{D\text{-PSSA}}$, then we can use the same argument as in (a). □

Finally, we show that some results in (singular) substitution algebras can be derived in light of these results in simultaneous substitution algebras. Let $\mathcal{S}^{\text{s}}_D = \{\boldsymbol{d} \mid d \in D\} \cup \{\boldsymbol{S}^{(d)} \mid d \in D\}$ be the type of D-SAs and D-SA be the set of equations corresponding to (s1)-(s6) in Chapter 3.

Theorem 4.8. Let D be an infinite set and $K_{\mathrm{lf}D\text{-SA}}$ be the class of all locally finite-dimensional D-SAs, $K_{D\text{-TSA}}$ be the class of all D-TSAs, and $K_{D\text{-PSA}}$ be the class of all D-PSAs.

(a) $\mathrm{Th}_X(K_{\mathrm{lf}D\text{-SA}}) = \mathrm{Th}_X(K_{D\text{-TSA}}) = \mathrm{Th}_X(K_{D\text{-PSA}})$.

(b) $\mathrm{Th}_X(K_{\mathrm{lf}D\text{-SA}})$ is decidable.

Proof. (a) First we show that $\mathrm{Th}_X(K_{\mathrm{lf}D\text{-SA}}) = \mathrm{Th}_X(K_{D\text{-TSA}})$. Since each D-TSA is a locally finite-dimensional D-SA, we have $\mathrm{Th}_X(K_{\mathrm{lf}D\text{-SA}}) \subseteq \mathrm{Th}_X(K_{D\text{-TSA}})$.

For each $t_a, t_b \in T_{\mathcal{S}_D^{\mathrm{s}}}(X)$ such that $K_{\mathrm{lf}D\text{-SA}} \not\models t_a \approx t_b$, there exists a locally finite-dimensional D-SA $\mathbf{A}^{\mathrm{s}}$ such that $\mathbf{A}^{\mathrm{s}} \not\models t_a \approx t_b$. By Theorem 3.5, $\mathbf{A}^{\mathrm{s}}$ can be expanded to a D-SSA $\mathbf{A}^{\mathrm{ss}}$, hence $\mathbf{A}^{\mathrm{ss}} \not\models t_a \approx t_b$. Thus $\mathsf{SSA} \nvdash t_a \approx t_b$, so $\mathbf{T}^{\mathrm{ss}}_{\mathcal{S}_0}(D) \not\models t_a \approx t_b$ by Lemma 4.5(b). Since $\mathbf{T}^{\mathrm{s}}_{\mathcal{S}_0}(D)$ is the reduct of $\mathbf{T}^{\mathrm{ss}}_{\mathcal{S}_0}(D)$, we have $\mathbf{T}^{\mathrm{s}}_{\mathcal{S}_0}(D) \not\models t_a \approx t_b$, so $K_{D\text{-TSA}} \not\models t_a \approx t_b$.

As $\mathbf{T}^{\mathrm{ss}}_{\mathcal{S}_0}(D) \cong \mathbf{Clo}^{\mathrm{ss}}_D(\mathbf{T}_{\mathcal{S}_0}(D))$, we can show $\mathrm{Th}_X(K_{\mathrm{lf}D\text{-SA}}) = \mathrm{Th}_X(K_{D\text{-PSA}})$ with a similar argument.

(b) By Theorem 4.6 and 4.7, $\mathrm{Th}_X(K_{D\text{-TSSA}})$ is the equational theory generated by D-SSA, hence it is decidable. Since D-TSAs are just reducts of D-TSSAs, $\mathrm{Th}_X(K_{D\text{-TSA}})$ is also decidable. Therefore $\mathrm{Th}_X(K_{\mathrm{lf}D\text{-SA}})$ is decidable by (a). $\square$

As a result, we can check whether an equation $t_a \approx t_b$ is valid for all locally finite-dimensional D-SAs by finding normal forms t'_a, t'_b of t_a and t_b, and checking whether $t'_a \sim_P t'_b$.

Chapter 5

Representation of simultaneous substitution algebras

In this chapter we will show that each simultaneous substitution algebra is isomorphic to a quotient of a term simultaneous substitution algebra of generalized type and a quotient of a function simultaneous substitution algebra. We will also show that under the condition of local finite-dimensionality, a simultaneous substitution algebra is isomorphic to a polynomial simultaneous substitution algebra and a quotient of a term simultaneous substitution algebra; with this result we can provide another proof of the representation theorem of locally finite-dimensional substitution algebras in [Fel82].

5.1 Representation of simultaneous substitution algebras

The representability of trivial D-SSAs is easy to see, so we only consider the non-trivial cases in the following. Let $\mathbf{A} = \langle A, \langle c_d \rangle_{d \in D}, \langle S^{\vec{d},\mathbf{A}} \rangle_{\vec{d} \in D^\#} \rangle$ be an arbitrary non-trivial D-SSA. We show that $\mathbf{A}$ is isomorphic to a quotient of a term simultaneous substitution algebra of generalized type, then a quotient of a function simultaneous substitution algebra.

For each $a \in A$, let $\boldsymbol{Q}_a$ be a corresponding symbol of arity Δa; we require that $\boldsymbol{Q}_a \neq \boldsymbol{Q}_{a'}$ for all $a, a' \in A$ with $a \neq a'$, and $\boldsymbol{Q}_a \neq d$ for all $a \in A$ and $d \in D$. Let $\mathcal{F}_A = \{\boldsymbol{Q}_a \mid a \in A\}$, then $T_{\mathcal{F}_A}(D)$ is the set of terms of type $\mathcal{F}_A$ over D. Consider the following sequence of sets of terms defined by recursion:

$$\begin{aligned} T_A^0 &= D, \\ T_A^{n+1} &= T_A^n \cup \{\boldsymbol{Q}_a f \mid a \in A, f : \Delta a \to T_A^n, f(d) \neq d \text{ for finitely many } d \in \Delta a\}. \end{aligned}$$

Let $T_A = \bigcup_{n<\omega} T_A^n$. It is easy to see that $T_A^0 \subseteq T_A^1 \subseteq \cdots \subseteq T_A^n \subseteq \cdots$ and $T_A \subseteq T_{\mathcal{F}_A}(D)$. For each $t \in T_A$, let $\operatorname{depth}(t)$ be the least natural number n such that $t \in T_A^n$. In the following lemma we show that T_A is a subuniverse of $\mathbf{T}^{\mathrm{ss}}_{\mathcal{F}_A}(D) = \langle T_{\mathcal{F}_A}(D), \langle d \rangle_{d \in D}, \langle S^{\vec{d},\mathbf{T}} \rangle_{\vec{d} \in D^\#} \rangle$.

Lemma 5.1. (a) For all $\vec{d} = (d_1, \ldots, d_n) \in D^\#$, $t_1, \ldots, t_n \in T_A^m$ and $t \in T_A$, $S^{\vec{d},\mathbf{T}}(t_1, \ldots, t_n, t) \in T_A^{m+\operatorname{depth}(t)}$.

(b) T_A is a subuniverse of $\mathbf{T}^{\mathrm{ss}}_{\mathcal{F}_A}(D)$.

Proof. (a) Assume that $\vec{d} = (d_1, \ldots, d_n) \in D^\#$ and $t_1, \ldots, t_n \in T_A^m$. We show $S^{\vec{d}}(t_1, \ldots, t_n, t) \in T_A^{m+\mathrm{depth}(t)}$ for all $t \in T_A$ by induction on depth(t):

(1) depth$(t) = 0$: then $t = d$ for some $d \in D$. If $d = d_i$ for some i with $1 \le i \le n$, then $S^{\vec{d}}(t_1, \ldots, t_n, t) = t_i \in T_A^m$; else, $S^{\vec{d}}(t_1, \ldots, t_n, t) = d \in T_A^0 \subseteq T_A^m$.

(2) Assume that our claim holds for all $t \in T_A$ with depth$(t) \le k$. Take arbitrary $t \in T_A$ with depth$(t) = k+1$, then $t = \boldsymbol{Q}_a f$ where $f : \Delta a \to T_A^k$ and $f(d) \neq d$ for finitely many $d \in \Delta a$. Then $S^{\vec{d}}(t_1, \ldots, t_n, t) = S^{\vec{d}}(t_1, \ldots, t_n, \boldsymbol{Q}_a f) = \boldsymbol{Q}_a f'$, where $f' = \langle S^{\vec{d}}(t_1, \ldots, t_n, f(d))\rangle_{d \in \Delta a}$. For each $d \in \Delta a$, we have $f(d) \in T_A^k$, so depth$(f(d)) \le k$, hence $f'(d) = S^{\vec{d}}(t_1, \ldots, t_n, f(d)) \in T_A^{m+\mathrm{depth}(f(d))} \subseteq T_A^{m+k}$ by our assumption. Thus f' is a function from Δa to T_A^{m+k}.

Because $f(d) \neq d$ for finitely many $d \in \Delta a$, $\{d_1, \ldots, d_n\} \cup \{d \in \Delta a \mid f(d) \neq d\}$ is finite. For each d in Δa such that $d \notin \{d_1, \ldots, d_n\} \cup \{d \in \Delta a \mid f(d) \neq d\}$,

$$f'(d) = S^{\vec{d}}(t_1, \ldots, t_n, f(d)) \overset{f(d)=d}{=} S^{\vec{d}}(t_1, \ldots, t_n, d) \overset{d \notin \{d_1,\ldots,d_n\}}{=} d.$$

By contraposition, we have $\{d \in \Delta a \mid f'(d) \neq d\} \subseteq \{d_1, \ldots, d_n\} \cup \{d \in \Delta a \mid f(d) \neq d\}$, hence $f'(d) \neq d$ for finitely many $d \in \Delta a$.

Therefore $S^{\vec{d}}(t_1, \ldots, t_n, t) = \boldsymbol{Q}_a f' \in T_A^{m+k+1}$.

(b) We need to show that T_A is closed under the basic operations of $\mathbf{T}^{\mathrm{ss}}_{\mathcal{F}_A}(D)$. For each $d \in D$, we have $d \in T_A^0 \subseteq T_A$. Then we show that for all $\vec{d} \in D^\#$ of length n and all $t_1, \ldots, t_{n+1} \in T_A$, $S^{\vec{d}}(t_1, \ldots, t_{n+1}) \in T_A$. If $n = 0$, we have $\vec{d} = ()$, hence $S^{\vec{d}}(t_1) = t_1 \in T_A$. If $n > 0$, let $m = \max\{\mathrm{depth}(t_1), \ldots, \mathrm{depth}(t_n)\}$, then $t_1, \ldots, t_n \in T_A^m$, hence $S^{\vec{d}}(t_1, \ldots, t_{n+1}) \in T_A^{m+\mathrm{depth}(t_{n+1})} \subseteq T_A$ by (a).

Thus T_A is closed under the basic operations of $\mathbf{T}^{\mathrm{ss}}_{\mathcal{F}_A}(D)$, so T_A is a subuniverse of $\mathbf{T}^{\mathrm{ss}}_{\mathcal{F}_A}(D)$. □

Let $\mathbf{T}_A$ be the subalgebra of $\mathbf{T}^{\mathrm{ss}}_{\mathcal{F}_A}(D)$ taking T_A as its universe. We want to show that $\mathbf{A}$ is isomorphic to a quotient of $\mathbf{T}_A$ by giving a homomorphism from $\mathbf{T}_A$ to $\mathbf{A}$. Let $\phi : T_A \to A$ be such that for all $d, \mathbf{Q}_a f \in T_A$,

$$\phi(d) = c_d,$$
$$\phi(\boldsymbol{Q}_a f) = S^{(d_1,\ldots,d_n),\mathbf{A}}(\phi(f(d_1)), \ldots, \phi(f(d_n)), a),$$

where $\{d_1, \ldots, d_n\} = \{d \in \Delta a \mid f(d) \neq d\}$; notice that for each permutation p of $\{1, \ldots, n\}$,

$$S^{(d_1,\ldots,d_n),\mathbf{A}}(\phi(f(d_1)), \ldots, \phi(f(d_n)), a) = S^{(d_{p(1)},\ldots,d_{p(n)}),\mathbf{A}}(\phi(f(d_{p(1)})), \ldots, \phi(f(d_{p(n)})), a)$$

by Lemma 2.1(b), hence ϕ is well-defined. The following lemma shows that ϕ is the homomorphism we want:

Lemma 5.2. (a) For all $a \in A$, $(d_1, \dots, d_n) \in D^\#$ and $t_1, \dots, t_n, \boldsymbol{Q}_a f \in T_A$ such that $\{d \in \Delta a \mid f(d) \neq d\} \subseteq \{d_1, \dots, d_n\} \subseteq \Delta a$, $S^{(d_1,\dots,d_n),\mathbf{A}}(\phi(f(d_1)), \dots, \phi(f(d_n)), a) = \phi(\boldsymbol{Q}_a f)$.

(b) For all $\vec{d} \in D^\#$ of length n and $t_1, \dots, t_{n+1} \in T_A$, $\phi(S^{\vec{d},\mathbf{T}}(t_1, \dots, t_{n+1})) = S^{\vec{d},\mathbf{A}}(\phi(t_1), \dots, \phi(t_{n+1}))$.

(c) ϕ is a homomorphism from $\mathbf{T}_A$ onto $\mathbf{A}$.

Proof. (a) Take integers $v_1, \dots, v_m$ such that $1 \le v_1 < \dots < v_m \le n$ and $\{d_{v_1}, \dots, d_{v_m}\} = \{d \in \Delta a \mid f(d) \neq d\}$; let $u_1, \dots, u_{n-m}$ be such that $\{d_{u_1}, \dots, d_{u_{n-m}}\} = \{d_1, \dots, d_n\} \setminus \{d \in \Delta a \mid f(d) \neq d\}$, then $f(d_{u_i}) = d_{u_i}$ for all i, $1 \le i \le n-m$. Then

$$\begin{aligned}
&S^{(d_1,\dots,d_n)}(\phi(f(d_1)), \dots, \phi(f(d_n)), a)\\
\overset{2.1(b)}{=}\ & S^{(d_{v_1},\dots,d_{v_m},d_{u_1},\dots,d_{u_{n-m}})}(\phi(f(d_{v_1})), \dots, \phi(f(d_{v_m})), \phi(f(d_{u_1})), \dots, \phi(f(d_{u_{n-m}})), a)\\
=\ & S^{(d_{v_1},\dots,d_{v_m},d_{u_1},\dots,d_{u_{n-m}})}(\phi(f(d_{v_1})), \dots, \phi(f(d_{v_m})), \phi(d_{u_1}), \dots, \phi(d_{u_{n-m}}), a)\\
=\ & S^{(d_{v_1},\dots,d_{v_m},d_{u_1},\dots,d_{u_{n-m}})}(\phi(f(d_{v_1})), \dots, \phi(f(d_{v_m})), c_{d_{u_1}}, \dots, c_{d_{u_{n-m}}}, a)\\
\overset{2.1(c)}{=}\ & S^{(d_{v_1},\dots,d_{v_m})}(\phi(f(d_{v_1})), \dots, \phi(f(d_{v_m})), a)\\
=\ & \phi(\boldsymbol{Q}_a f).
\end{aligned}$$

(b) If $n = 0$, then $\vec{d} = ()$, hence $\phi(S^{(),\mathbf{T}}(t_1)) = \phi(t_1) = S^{(),\mathbf{A}}(\phi(t_1))$. Else we have $n \ge 0$, then assume that $\vec{d} = (d_1, \dots, d_n)$. We show $\phi(S^{\vec{d},\mathbf{T}}(t_1, \dots, t_n, t_{n+1})) = S^{\vec{d},\mathbf{A}}(\phi(t_1), \dots, \phi(t_n), \phi(t_{n+1}))$ by induction on depth(t_{n+1}):

(1) depth$(t_{n+1}) = 0$: then $t_{n+1} = d$ for some $d \in D$. If $d = d_i$ for some i, $1 \le i \le n$, then

$$\begin{aligned}
&\phi(S^{\vec{d},\mathbf{T}}(t_1, \dots, t_{n+1})) = \phi(S^{(d_1,\dots,d_n),\mathbf{T}}(t_1, \dots, t_n, d_i)) = \phi(t_i)\\
&= S^{(d_1,\dots,d_n),\mathbf{A}}(\phi(t_1), \dots, \phi(t_n), c_{d_i}) = S^{(d_1,\dots,d_n),\mathbf{A}}(\phi(t_1), \dots, \phi(t_n), \phi(d_i)).
\end{aligned}$$

Else, $d \neq d_i$ for all i with $1 \le i \le n$, then

$$\begin{aligned}
&\phi(S^{\vec{d},\mathbf{T}}(t_1, \dots, t_{n+1})) = \phi(S^{(d_1,\dots,d_n),\mathbf{T}}(t_1, \dots, t_n, d)) = \phi(d) = c_d\\
&= S^{(d_1,\dots,d_n),\mathbf{A}}(\phi(t_1), \dots, \phi(t_n), c_d) = S^{(d_1,\dots,d_n),\mathbf{A}}(\phi(t_1), \dots, \phi(t_n), \phi(d)).
\end{aligned}$$

(2) Assume that our claim holds for all $t_{n+1} \in T_A$ with depth$(t_{n+1}) \le m$. Take arbitrary $t_{n+1} \in T_A$ with depth$(t_{n+1}) = m+1$, hence $t_{n+1} = \boldsymbol{Q}_a f$ where $f : \Delta a \to T_A^m$ and $f(d) \neq d$ for finitely many d. Let $f' = \langle S^{\vec{d},\mathbf{T}}(t_1, \dots, t_n, f(d)) \rangle_{d \in \Delta a}$, then $S^{\vec{d},\mathbf{T}}(t_1, \dots, t_{n+1}) = \boldsymbol{Q}_a f'$. By Lemma 5.1, we have $\boldsymbol{Q}_a f' \in T_A$, so $f'(d) \neq d$ for finitely many d. Hence we can take pairwise distinct $d'_1, \dots, d'_l \in \Delta a$ such that $\{d'_1, \dots, d'_l\} = \{d \in \Delta a \mid f(d) \neq d\} \cup \{d \in \Delta a \mid f'(d) \neq d\} \cup (\{d_1, \dots, d_n\} \cap \Delta a)$.

Take integers $v_1, \dots, v_k$ such that $1 \le v_1 < \dots < v_k \le n$ and $\{d_{v_1}, \dots, d_{v_k}\} = \{d_1, \dots, d_n\} \setminus \{d'_1, \dots, d'_l\}$; since $\{d_1, \dots, d_n\} \cap \Delta a \subseteq \{d'_1, \dots, d'_l\}$, we have $d_{v_1}, \dots, d_{v_k} \notin \Delta a$. Let $\vec{d'} = (d'_1, \dots, d'_l)$. Then

$$\phi(S^{\vec{d},\mathbf{T}}(t_1, \dots, t_{n+1})) = \phi(\boldsymbol{Q}_a f')$$

$$
\begin{aligned}
&\overset{\text{(a)}}{=} S^{\vec{d}',\mathbf{A}}(\phi(f'(d'_1)),\ldots,\phi(f'(d'_l)),a)\\
&= S^{\vec{d}',\mathbf{A}}(\phi(S^{\vec{d},\mathbf{T}}(t_1,\ldots,t_n,f(d'_1))),\ldots,\phi(S^{\vec{d},\mathbf{T}}(t_1,\ldots,t_n,f(d'_l))),a)\\
&\overset{\text{IH}}{=} S^{\vec{d}',\mathbf{A}}(S^{\vec{d},\mathbf{A}}(\phi(t_1),\ldots,\phi(t_n),\phi(f(d'_1))),\ldots,S^{\vec{d},\mathbf{A}}(\phi(t_1),\ldots,\phi(t_n),\phi(f(d'_l))),a)\\
&\overset{\text{2.6(b)}}{=} S^{(d'_1,\ldots,d'_l,d_{v_1},\ldots,d_{v_k}),\mathbf{A}}(S^{\vec{d},\mathbf{A}}(\phi(t_1),\ldots,\phi(t_n),\phi(f(d'_1))),\ldots,S^{\vec{d},\mathbf{A}}(\phi(t_1),\ldots,\phi(t_n),\\
&\qquad \phi(f(d'_l))),S^{\vec{d},\mathbf{A}}(\phi(t_1),\ldots,\phi(t_n),c_{d_{v_1}}),\ldots,S^{\vec{d},\mathbf{A}}(\phi(t_1),\ldots,\phi(t_n),c_{d_{v_k}}),a)\\
&\overset{\text{2.1(d)}}{=} S^{\vec{d},\mathbf{A}}\ (\phi(t_1),\ldots,\phi(t_n),S^{(d'_1,\ldots,d'_l,d_{v_1},\ldots,d_{v_k}),\mathbf{A}}(\phi(f(d'_1)),\ldots,\phi(f(d'_l)),c_{d_{v_1}},\ldots,c_{d_{v_k}},a))\\
&\overset{\text{2.1(c)}}{=} S^{\vec{d},\mathbf{A}}(\phi(t_1),\ldots,\phi(t_n),S^{\vec{d}',\mathbf{A}}(\phi(f(d'_1)),\ldots,\phi(f(d'_l)),a))\\
&\overset{\text{(a)}}{=} S^{\vec{d},\mathbf{A}}(\phi(t_1),\ldots,\phi(t_n),\phi(\boldsymbol{Q}_a f))\\
&= S^{\vec{d},\mathbf{A}}(\phi(t_1),\ldots,\phi(t_{n+1})).
\end{aligned}
$$

(c) First we show that ϕ is a homomorphism. For each $d \in D$, we have $\phi(d) = c_d$. For each $\vec{d} \in D^{\#}$ of length n, we have $\phi(S^{\vec{d},\mathbf{T}}(t_1,\ldots,t_{n+1})) = S^{\vec{d},\mathbf{A}}(\phi(t_1),\ldots,\phi(t_{n+1}))$ for all $t_1,\ldots,t_{n+1} \in T_A$ by (b). Therefore, ϕ is a homomorphism.

Then we show that ϕ is surjective. For each $a \in A$, let $f_a : \Delta a \to T^0_A$ be such that $f_a(d) = d$ for all $d \in \Delta a$, and let $\tau_a = \boldsymbol{Q}_a f_a$, then $\tau_a \in T^1_A$; by definition of ϕ, $\phi(\tau_a) = a$. Thus ϕ is surjective. □

Theorem 5.3 (Representation of D-SSAs)**.** Let $\mathbf{A}$ be a D-SSA.

(a) $\mathbf{A}$ is isomorphic to a quotient of a D-TSSA of generalized type.

(b) $\mathbf{A}$ is isomorphic to a quotient of a D-FSSA.

Proof. (a) If $\mathbf{A}$ is trivial, then $\mathbf{A}$ is isomorphic to a trivial D-TSSA. Else, $\mathbf{A}$ is non-trivial; by Lemma 5.2(c), there is a homomorphism ϕ from $\mathbf{T}_A$ onto $\mathbf{A}$, hence we have $\mathbf{T}_A/\ker(\phi) \cong \mathbf{A}$ by the Homomorphism Theorem.

(b) If $\mathbf{A}$ is trivial, then $\mathbf{A}$ is isomorphic to a trivial D-FSSA. Else, $\mathbf{A}$ is non-trivial. By Theorem 2.5, $\mathbf{T}_A$ is isomorphic to the D-FSSA with base $T_{\mathcal{F}_A}(D)$ of which the universe is $\{t^{\mathbf{T}^{\text{ss}}_{\mathcal{F}_A}(D)} \mid t \in T_{\mathcal{F}_A}(D)\}$, so $\mathbf{A}$ is isomorphic to a quotient of this D-FSSA by (a). □

5.2 Representation of locally finite-dimensional simultaneous substitution algebras

We have already shown that each simultaneous substitution algebra can be represented as a quotient of a function simultaneous substitution algebra. Moreover, with the condition of local finite-dimensionality, we can improve the result a bit: each locally finite-dimensional simultaneous substitution algebra can be represented as a polynomial simultaneous substitution algebra.

First we will show that each non-trivial locally finite-dimensional simultaneous substitution algebra can be represented as a quotient of simultaneous substitution algebra of finitary terms.

We assume that D is infinite in this section. The proof is essentially the same as the proof in the last section. As we want to represent elements in simultaneous substitution algebras by terms in the narrow sense, it will help if we have a well-ordering of the set of dimensions beforehand. Let $\kappa = |D|$, and $\{d_\lambda \mid \lambda < \kappa\}$ be an enumeration of D without repetition.

Let $\mathbf{A} = \langle A, \langle c_d \rangle_{d \in D}, \langle S^{\vec{d},\mathbf{A}} \rangle_{\vec{d} \in D^\#} \rangle$ be an arbitrary non-trivial locally finite-dimensional D-SSA. For each $a \in A$, let $\boldsymbol{Q}_a$ be a corresponding operation symbol of arity $|\Delta a|$. Let $\mathcal{S}_A = \{\boldsymbol{Q}_a \mid a \in A\}$, then $T_{\mathcal{S}_A}(D)$ is the set of terms of type $\mathcal{S}_A$ over D. For each element a, let $n = |\Delta a|$ and

$$\tau_a = \boldsymbol{Q}_a d_{\lambda_1} \dots d_{\lambda_n}, \text{ where } \lambda_1 < \dots < \lambda_n < \kappa \text{ and } \Delta a = \{d_{\lambda_1}, \dots, d_{\lambda_n}\};$$

our idea is to represent a by the term τ_a (more precisely, an equivalence class containing τ_a).

We define a mapping $f : T_{\mathcal{S}_\mathbf{A}}(D) \to A$ by induction on the structure of terms:

(i) For each $d \in D$, $f(d) = c_d$;

(ii) For each $a \in A$ with $|\Delta a| = n$ and $t_1, \dots, t_n \in T_{\mathcal{S}_\mathbf{A}}(D)$,

$$f(\boldsymbol{Q}_a t_1 \dots t_n) = S^{(d_{\lambda_1}, \dots, d_{\lambda_n}),\mathbf{A}}(f(t_1), \dots, f(t_n), a)$$

where $\lambda_1 < \dots < \lambda_n < \kappa$ and $\Delta a = \{d_{\lambda_1}, \dots, d_{\lambda_n}\}$.

Notice that $f(\tau_a) = a$ for each $a \in A$ by this definition. Then we show that f is a homomorphism from $\mathbf{T}^{\text{ss}}_{\mathcal{S}_\mathbf{A}}(D)$ onto $\mathbf{A}$:

Lemma 5.4. f is a homomorphism from $\mathbf{T}^{\text{ss}}_{\mathcal{S}_A}(D)$ onto $\mathbf{A}$.

Proof. First we show that f is a homomorphism from $\mathbf{T}^{\text{ss}}_{\mathcal{S}_A}(D)$ to $\mathbf{A}$. For each $d \in D$, $f(d) = c_d$ by definition of f. Then we need to show that for all $\vec{d} \in D^\#$ of length n and for all terms $t_1, \dots, t_{n+1} \in T_{\mathcal{S}_A}(D)$,

$$f(S^{\vec{d},\mathbf{T}}(t_1, \dots, t_{n+1})) = S^{\vec{d},\mathbf{A}}(f(t_1), \dots, f(t_{n+1})).$$

If $n = 0$, we have $\vec{d} = ()$, so $f(S^{\vec{d},\mathbf{T}}(t_1)) = f(t_1) = S^{\vec{d},\mathbf{A}}(f(t_1))$. Then assume that $n > 0$ and $\vec{d} = (d_{\theta_1}, \dots, d_{\theta_n})$; we show $f(S^{\vec{d},\mathbf{T}}(t_1, \dots, t_{n+1})) = S^{\vec{d},\mathbf{A}}(f(t_1), \dots, f(t_{n+1}))$ by induction on the structure of t_{n+1}.

(1) $t_{n+1} = d_{\theta_i}$, $1 \le i \le n$: then $f(S^{\vec{d},\mathbf{T}}(t_1, \dots, t_{n+1})) = f(S^{\vec{d},\mathbf{T}}(t_1, \dots, t_n, d_{\theta_i})) = f(t_i) = S^{\vec{d},\mathbf{A}}(f(t_1), \dots, f(t_n), c_{d_{\theta_i}}) = S^{\vec{d},\mathbf{A}}(f(t_1), \dots, f(t_n), f(d_{\theta_i}))$.

(2) $t_{n+1} = d_\theta$, $\theta \in \kappa \setminus \{\theta_1, \dots, \theta_n\}$: then $f(S^{\vec{d},\mathbf{T}}(t_1, \dots, t_{n+1})) = f(S^{\vec{d},\mathbf{T}}(t_1, \dots, t_n, d_\theta)) = f(d_\theta) = c_{d_\theta} = S^{\vec{d},\mathbf{A}}(f(t_1), \dots, f(t_n), c_{d_\theta}) = S^{\vec{d},\mathbf{A}}(f(t_1), \dots, f(t_n), f(d_\theta))$.

(3) $t_{n+1} = \boldsymbol{Q}_a$, $|\Delta a| = 0$: then $d_{\theta_1}, \dots, d_{\theta_n} \notin \Delta a$, so we have

$$f(S^{\vec{d},\mathbf{T}}(t_1, \dots, t_{n+1})) = f(S^{\vec{d},\mathbf{T}}(t_1, \dots, t_n, \boldsymbol{Q}_a)) = f(\boldsymbol{Q}_a) = a$$
$$\overset{2.6\text{(b)}}{=} S^{\vec{d},\mathbf{A}}(f(t_1), \dots, f(t_n), a) = S^{\vec{d},\mathbf{A}}(f(t_1), \dots, f(t_n), f(\boldsymbol{Q}_a)).$$

(4) $t_{n+1} = \boldsymbol{Q}_a t'_1 \dots t'_m$ with $m > 0$: then $|\Delta a| = m$. Take ordinal numbers $\lambda_1, \dots, \lambda_m$ such that $\lambda_1 < \cdots < \lambda_m < \kappa$ and $\Delta a = \{d_{\lambda_1}, \dots, d_{\lambda_m}\}$. Then take integers $v_1, \dots, v_k$ such that $1 \leq v_1 < \cdots < v_k \leq n$ and $\{d_{\theta_{v_1}}, \dots, d_{\theta_{v_k}}\} = \{d_{\theta_1}, \dots, d_{\theta_n}\} \setminus \{d_{\lambda_1}, \dots, d_{\lambda_m}\}$. Let $\vec{d'} = (d_{\lambda_1}, \dots, d_{\lambda_m})$ and $\vec{d''} = (d_{\lambda_1}, \dots, d_{\lambda_m}, d_{\theta_{v_1}}, \dots, d_{\theta_{v_k}})$, then

$$\begin{aligned}
&S^{\vec{d},\mathbf{A}}(f(t_1), \dots, f(t_{n+1})) \\
&= S^{\vec{d},\mathbf{A}}(f(t_1), \dots, f(t_n), f(\boldsymbol{Q}_a t'_1 \dots t'_m)) \\
&= S^{\vec{d},\mathbf{A}}(f(t_1), \dots, f(t_n), S^{\vec{d'},\mathbf{A}}(f(t'_1), \dots, f(t'_m), a)) \\
&\overset{2.1(\text{d})}{=} S^{\vec{d''},\mathbf{A}}(S^{\vec{d},\mathbf{A}}(f(t_1), \dots, f(t_n), f(t'_1)), \dots, S^{\vec{d},\mathbf{A}}(f(t_1), \dots, f(t_n), f(t'_m)), \\
&\qquad f(t_{v_1}), \dots, f(t_{v_k}), a) \\
&\overset{\text{IH}}{=} S^{\vec{d''},\mathbf{A}}(f(S^{\vec{d},\mathbf{T}}(t_1, \dots, t_n, t'_1)), \dots, f(S^{\vec{d},\mathbf{T}}(t_1, \dots, t_n, t'_m)), f(t_{v_1}), \dots, f(t_{v_k}), a) \\
&\overset{2.6(\text{b})}{=} S^{\vec{d'},\mathbf{A}}(f(S^{\vec{d},\mathbf{T}}(t_1, \dots, t_n, t'_1)), \dots, f(S^{\vec{d},\mathbf{T}}(t_1, \dots, t_n, t'_m)), a) \\
&= f(\boldsymbol{Q}_a S^{\vec{d},\mathbf{T}}(t_1, \dots, t_n, t'_1) \dots S^{\vec{d},\mathbf{T}}(t_1, \dots, t_n, t'_m)) \\
&= f(S^{\vec{d},\mathbf{T}}(t_1, \dots, t_n, \boldsymbol{Q}_a t'_1 \dots t'_m)).
\end{aligned}$$

Therefore f is a homomorphism. Besides, it is easy to see that $f(\tau_a) = a$ for each $a \in A$, hence f is surjective. □

This lemma immediately implies that $\mathbf{A}$ is isomorphic to a quotient of $\mathbf{T}^{\text{ss}}_{\mathcal{S}_A}(D)$. What's more, we want to show that $\mathbf{A}$ can also be represented as a polynomial simultaneous substitution algebra.

From $\mathbf{A}$, we define an algebra $\mathfrak{S}(\mathbf{A})$ of type $\mathcal{S}_\mathbf{A}$: we take A as the universe; for each $a \in A$, with $\Delta^\mathbf{A} a = \{d_{\lambda_1}, \dots, d_{\lambda_n}\}$, where $\lambda_1 < \cdots < \lambda_n < \kappa$, let $\boldsymbol{Q}_a^{\mathfrak{S}(\mathbf{A})} = g_a$ where g_a is the n-ary operation that sends $a_1, \dots, a_n \in A$ to $S^{(d_{\lambda_1}, \dots, d_{\lambda_n}),\mathbf{A}}(a_1, \dots, a_n, a)$.

Consider the D-PSSA induced by $\mathfrak{S}(\mathbf{A})$: recall that $\text{Clo}_D(\mathfrak{S}(\mathbf{A})) = \{t^{\mathfrak{S}(\mathbf{A})} \mid t \in T_{\mathcal{S}_\mathbf{A}}(D)\}$ is a set of D-ary operations on A, the universe of $\mathfrak{S}(\mathbf{A})$, and e_d is the d-th projection operation for each $d \in D$; besides we have defined simultaneous substitution operations $S^{\vec{d},\mathbf{F}}$ on D-ary operations on A. Using the same notation for their restriction to $\text{Clo}_D(\mathfrak{S}(\mathbf{A}))$, we define $\mathbf{Clo}^{\text{ss}}_D(\mathfrak{S}(\mathbf{A}))$ as $\langle \text{Clo}_D(\mathfrak{S}(\mathbf{A})), \langle e_d \rangle_{d \in D}, \langle S^{\vec{d},\mathbf{F}} \rangle_{\vec{d} \in D^\#} \rangle$. Let $\phi : T_{\mathcal{S}_\mathbf{A}}(D) \to \text{Clo}_D(\mathfrak{S}(\mathbf{A}))$ be such that $\phi(t) = t^{\mathfrak{S}(\mathbf{A})}$ for each $t \in T_{\mathcal{S}_\mathbf{A}}(D)$.

Lemma 5.5. (a) For each $t \in T_{\mathcal{S}_\mathbf{A}}(D)$, $(\tau_{f(t)})^{\mathfrak{S}(\mathbf{A})} = t^{\mathfrak{S}(\mathbf{A})}$.

(b) Let $\iota : D \to A$ be such that $\iota = \langle c_d \rangle_{d \in D}$, then $t^{\mathfrak{S}(\mathbf{A})}(\iota) = f(t)$ for each $t \in T_{\mathcal{S}_\mathbf{A}}(D)$.

(c) $\ker(f) = \ker(\phi)$.

Proof. (a) Notice that for each $a \in A$ and $\alpha : D \to A$,

$$\begin{aligned}
(\tau_a)^{\mathfrak{S}(\mathbf{A})}(\alpha) = (\boldsymbol{Q}_a d_{\lambda_1} \dots d_{\lambda_n})^{\mathfrak{S}(\mathbf{A})}(\alpha) &= \boldsymbol{Q}_a^{\mathfrak{S}(\mathbf{A})}(d_{\lambda_1}^{\mathfrak{S}(\mathbf{A})}(\alpha), \dots, d_{\lambda_n}^{\mathfrak{S}(\mathbf{A})}(\alpha)) \\
&= S^{(d_{\lambda_1}, \dots, d_{\lambda_n})}(\alpha(d_{\lambda_1}), \dots, \alpha(d_{\lambda_n}), a)
\end{aligned}$$

where $\lambda_1 < \cdots < \lambda_n$ and $\Delta^{\mathbf{A}} a = \{d_{\lambda_1}, \ldots, d_{\lambda_n}\}$. Take arbitrary $\alpha : D \to A$, we prove that for all $t \in T_{\mathcal{S}_{\mathbf{A}}}(D)$, $(\tau_{f(t)})^{\mathfrak{S}(\mathbf{A})}(\alpha) = t^{\mathfrak{S}(\mathbf{A})}(\alpha)$ by induction on the structure of t:

(1) $t = d$, $d \in D$: then $(\tau_{f(d)})^{\mathfrak{S}(\mathbf{A})}(\alpha) = (\tau_{c_d})^{\mathfrak{S}(\mathbf{A})}(\alpha) \overset{2.7(a)}{=} S^{(d),\mathbf{A}}(\alpha(d), c_d) \overset{(ss2)}{=} \alpha(d) = d^{\mathfrak{S}(\mathbf{A})}(\alpha)$;

(2) $t = \boldsymbol{Q}_a t_1 \ldots t_n$, $a \in A$: since $\mathbf{A}$ is locally finite-dimensional, $\Delta^{\mathbf{A}} a \cup \bigcup_{1 \le i \le n} \Delta^{\mathbf{A}} f(t_i)$ is finite. Take ordinal numbers $\lambda_1, \ldots, \lambda_m$ such that $\lambda_1 < \cdots < \lambda_m < \kappa$ and $\{d_{\lambda_1}, \ldots, d_{\lambda_m}\} = \Delta^{\mathbf{A}} a \cup \bigcup_{1 \le i \le n} \Delta^{\mathbf{A}} f(t_i)$. First we prove the following claim:

$(*)$ for each i, $1 \le i \le n$, $S^{(d_{\lambda_1},\ldots,d_{\lambda_m}),\mathbf{A}}(\alpha(d_{\lambda_1}), \ldots, \alpha(d_{\lambda_m}), f(t_i)) = t_i^{\mathfrak{S}(\mathbf{A})}(\alpha)$.

Since $\Delta^{\mathbf{A}} f(t_i) \subseteq \{d_{\lambda_1}, \ldots, d_{\lambda_m}\} = \Delta^{\mathbf{A}} a \cup \bigcup_{1 \le i \le n} \Delta^{\mathbf{A}} f(t_i)$, we can take integers $v_1, \ldots, v_l$ such that $1 \le v_1 < \cdots < v_l \le m$ and $\Delta^{\mathbf{A}} f(t_i) = \{d_{\lambda_{v_1}}, \ldots, d_{\lambda_{v_l}}\}$, hence we have

$$\begin{aligned} S^{(d_{\lambda_1},\ldots,d_{\lambda_m}),\mathbf{A}}(\alpha(d_{\lambda_1}), \ldots, \alpha(d_{\lambda_m}), f(t_i)) &\overset{2.6(b)}{=} S^{(d_{\lambda_{v_1}},\ldots,d_{\lambda_{v_l}}),\mathbf{A}}(\alpha(d_{\lambda_{v_1}}), \ldots, \alpha(d_{\lambda_{v_l}}), f(t_i)) \\ &= (\tau_{f(t_i)})^{\mathfrak{S}(\mathbf{A})}(\alpha) \overset{\text{IH}}{=} t_i^{\mathfrak{S}(\mathbf{A})}(\alpha). \end{aligned}$$

Let p be a permutation of $\{1, \ldots, m\}$ such that $p(1) < \cdots < p(n)$ and $\Delta^{\mathbf{A}} a = \{d_{\lambda_{p(1)}}, \ldots, d_{\lambda_{p(n)}}\}$. By Lemma 2.7(b), $\Delta^{\mathbf{A}} f(t) = \Delta^{\mathbf{A}} S^{(d_{\lambda_{p(1)}},\ldots,d_{\lambda_{p(n)}})}(f(t_1), \ldots, f(t_n), a) \subseteq (\Delta^{\mathbf{A}} a \setminus \{d_{\lambda_{p(1)}}, \ldots, d_{\lambda_{p(n)}}\}) \cup \bigcup_{1 \le i \le n} \Delta^{\mathbf{A}} f(t_i) = \bigcup_{1 \le i \le n} \Delta^{\mathbf{A}} f(t_i)$. Let q be a permutation of $\{1, \ldots, m\}$ such that $q(1) < \cdots < q(k)$ and $\Delta^{\mathbf{A}} f(t) = \{d_{\lambda_{q(1)}}, \ldots, d_{\lambda_{q(k)}}\}$. Then we have

$$\begin{aligned} (\tau_{f(t)})^{\mathfrak{S}(\mathbf{A})}(\alpha) &= S^{(d_{\lambda_{q(1)}},\ldots,d_{\lambda_{q(k)}})}(\alpha(d_{\lambda_{q(1)}}), \ldots, \alpha(d_{\lambda_{q(k)}}), f(t)) \\ &\overset{2.6(b)}{=} S^{(d_{\lambda_{q(1)}},\ldots,d_{\lambda_{q(m)}})}(\alpha(d_{\lambda_{q(1)}}), \ldots, \alpha(d_{\lambda_{q(m)}}), f(t)) \\ &\overset{2.1(b)}{=} S^{(d_{\lambda_1},\ldots,d_{\lambda_m})}(\alpha(d_{\lambda_1}), \ldots, \alpha(d_{\lambda_m}), f(t)) \\ &= S^{(d_{\lambda_1},\ldots,d_{\lambda_m})}(\alpha(d_{\lambda_1}), \ldots, \alpha(d_{\lambda_m}), S^{(d_{\lambda_{p(1)}},\ldots,d_{\lambda_{p(n)}})}(f(t_1), \ldots, f(t_n), a)) \\ &\overset{2.1(d)}{=} S^{(d_{\lambda_{p(1)}},\ldots,d_{\lambda_{p(m)}})}(S^{(d_{\lambda_1},\ldots,d_{\lambda_m})}(\alpha(d_{\lambda_1}), \ldots, \alpha(d_{\lambda_m}), f(t_1)), \ldots, \\ &\qquad S^{(d_{\lambda_1},\ldots,d_{\lambda_m})}(\alpha(d_{\lambda_1}), \ldots, \alpha(d_{\lambda_m}), f(t_n)), \alpha(d_{\lambda_{p(n+1)}}), \ldots, \alpha(d_{\lambda_{p(n+m)}}), a) \\ &\overset{(*)}{=} S^{(d_{\lambda_{p(1)}},\ldots,d_{\lambda_{p(m)}})}(t_1^{\mathfrak{S}(\mathbf{A})}(\alpha), \ldots, t_n^{\mathfrak{S}(\mathbf{A})}(\alpha), \alpha(d_{\lambda_{p(n+1)}}), \ldots, \alpha(d_{\lambda_{p(n+m)}}), a) \\ &\overset{2.6(b)}{=} S^{(d_{\lambda_{p(1)}},\ldots,d_{\lambda_{p(n)}})}(t_1^{\mathfrak{S}(\mathbf{A})}(\alpha), \ldots, t_n^{\mathfrak{S}(\mathbf{A})}(\alpha), a) \\ &= g_a(t_1^{\mathfrak{S}(\mathbf{A})}(\alpha), \ldots, t_n^{\mathfrak{S}(\mathbf{A})}(\alpha)) \\ &= (\boldsymbol{Q}_a t_1 \ldots t_n)^{\mathfrak{S}(\mathbf{A})}(\alpha). \end{aligned}$$

Therefore, given arbitrary $\alpha : D \to A$, $(\tau_{f(t)})^{\mathfrak{S}(\mathbf{A})}(\alpha) = t^{\mathfrak{S}(\mathbf{A})}(\alpha)$ for all t. Hence $(\tau_{f(t)})^{\mathfrak{S}(\mathbf{A})} = t^{\mathfrak{S}(\mathbf{A})}$ for all t.

(b) We show that $t^{\mathfrak{S}(\mathbf{A})}(\iota) = f(t)$ by induction on t:

(1) $t = d$, $d \in D$: then $t^{\mathfrak{S}(\mathbf{A})}(\iota) = d^{\mathfrak{S}(\mathbf{A})}(\iota) = \iota(d) = c_d = f(d) = f(t)$.

(2) $t = \boldsymbol{Q}_a t_1 \dots t_n$, $a \in A$ with $\lambda_1 < \dots < \lambda_n < \kappa$ and $\{d_{\lambda_1}, \dots, d_{\lambda_n}\} = \Delta a$: then

$$t^{\mathfrak{S}(\mathbf{A})}(\iota) = (\boldsymbol{Q}_a t_1 \dots t_n)^{\mathfrak{S}(\mathbf{A})}(\iota) = g_a(t_1^{\mathfrak{S}(\mathbf{A})}(\iota), \dots, t_n^{\mathfrak{S}(\mathbf{A})}(\iota)) \overset{\text{IH}}{=} g_a(f(t_1), \dots, f(t_n))$$
$$= S^{(d_{\lambda_1}, \dots, d_{\lambda_n})}(f(t_1), \dots, f(t_n), a) = f(\boldsymbol{Q}_a t_1 \dots t_n) = f(t).$$

(c) Assume that $f(t) = f(t')$, then $\phi(t) = t^{\mathfrak{S}(\mathbf{A})} \overset{\text{(a)}}{=} (\tau_{f(t)})^{\mathfrak{S}(\mathbf{A})} = (\tau_{f(t')})^{\mathfrak{S}(\mathbf{A})} \overset{\text{(a)}}{=} t'^{\mathfrak{S}(\mathbf{A})} = \phi(t')$.

Assume that $\phi(t) = \phi(t')$, then $f(t) \overset{\text{(b)}}{=} t^{\mathfrak{S}(\mathbf{A})}(\iota) = \phi(t)(\iota) = \phi(t')(\iota) = t'^{\mathfrak{S}(\mathbf{A})}(\iota) \overset{\text{(b)}}{=} f(t')$.

Hence $f(t) = f(t')$ iff $\phi(t) = \phi(t')$. Thus we have $\ker(f) = \{(t, t') \in T^2 \mid f(t) = f(t')\} = \{(t, t') \in T^2 \mid \phi(t) = \phi(t')\} = \ker(\phi)$. □

Combining these results, we can get the theorem:

Theorem 5.6. The following claims are equivalent:

(i) $\mathbf{A}$ is a locally finite-dimensional D-SSA;

(ii) $\mathbf{A}$ is isomorphic to $\mathbf{Clo}^{\text{ss}}_D(\mathbf{B})$ for some $\mathbf{B}$;

(iii) $\mathbf{A}$ is isomorphic to a quotient of $\mathbf{T}^{\text{ss}}_{\mathcal{S}}(D)$ for some type of algebras $\mathcal{S}$.

Proof. (i) $\Rightarrow$ (ii): assume that $\mathbf{A}$ is a locally finite-dimensional D-SSA. If $\mathbf{A}$ is trivial, then it is easy to see that $\mathbf{Clo}^{\text{ss}}_D(\mathbf{A})$ is also trivial and $\mathbf{A} \cong \mathbf{Clo}^{\text{ss}}_D(\mathbf{A})$. Else, $\mathbf{A}$ is not trivial, then by Lemma 5.4 and the Homomorphism Theorem, $\mathbf{A} \cong \mathbf{T}^{\text{ss}}_{\mathcal{S}_{\mathbf{A}}}(D)/\ker(f)$. We also have $\mathbf{T}^{\text{ss}}_{\mathcal{S}_{\mathbf{A}}}(D)/\ker(\phi) \cong \mathbf{Clo}^{\text{ss}}_D(\mathfrak{S}(\mathbf{A}))$ by Theorem 2.3. Since $\ker(f) = \ker(\phi)$ by Lemma 5.5(c), we have $\mathbf{A} \cong \mathbf{Clo}^{\text{ss}}_D(\mathfrak{S}(\mathbf{A}))$.

(ii) $\Rightarrow$ (iii): assume that $\mathbf{A}$ is isomorphic to $\mathbf{Clo}^{\text{ss}}_D(\mathbf{B})$ for some $\mathbf{B}$. Let $\mathcal{S}$ be the type of $\mathbf{B}$ and $\phi : T_{\mathcal{S}}(D) \to \text{Clo}_D(\mathbf{B})$ such that $\phi(t) = t^{\mathbf{B}}$ for all t. Then by Theorem 2.3, $\mathbf{T}^{\text{ss}}_{\mathcal{S}}(D)/\ker(\phi) \cong \mathbf{Clo}^{\text{ss}}_D(\mathbf{B})$. Hence $\mathbf{A} \cong \mathbf{T}^{\text{ss}}_{\mathcal{S}}(D)/\ker(\phi)$.

(iii) $\Rightarrow$ (i): assume that $\mathbf{A}$ is isomorphic to a quotient algebra of $\mathbf{T}^{\text{ss}}_{\mathcal{S}}(D)$ for some type of algebras $\mathcal{S}$. Since $\mathbf{T}^{\text{ss}}_{\mathcal{S}}(D)$ is locally finite-dimensional and a quotient algebra of a locally finite-dimensional D-SSA is also locally finite-dimensional, $\mathbf{A}$ is locally finite-dimensional. □

Then we can give another proof of the representation theorem (Theorem 3.1) in [Fel82].

Corollary 5.7. The following claims are equivalent:

(i) $\mathbf{A}$ is a locally finite-dimensional D-SA;

(ii) $\mathbf{A}$ is isomorphic to $\mathbf{Clo}^{\text{s}}_D(\mathbf{B})$ for some $\mathbf{B}$;

(iii) $\mathbf{A}$ is isomorphic to a quotient algebra of $\mathbf{T}^{\text{s}}_{\mathcal{S}}(D)$ for some type of algebras $\mathcal{S}$.

Proof. (i) $\Rightarrow$ (ii): by Theorem 3.5, $\mathbf{A}$ can be expanded to a D-SSA $\mathbf{A}^{\text{ss}}$. By Theorem 5.6, $\mathbf{A}^{\text{ss}} \cong \mathbf{Clo}^{\text{ss}}_D(\mathbf{B})$ for some $\mathbf{B}$, then we can take their reducts and get $\mathbf{A} \cong \mathbf{Clo}^{\text{s}}_D(\mathbf{B})$.

(ii) $\Rightarrow$ (iii): by Theorem 2.3, we have $\mathbf{Clo}^{\text{ss}}_D(\mathbf{B})$ is isomorphic to a quotient algebra of $\mathbf{T}^{\text{ss}}_{\mathcal{S}}(D)$, hence $\mathbf{Clo}^{\text{s}}_D(\mathbf{B})$ is isomorphic to a quotient algebra of $\mathbf{T}^{\text{s}}_{\mathcal{S}}(D)$.

(iii) $\Rightarrow$ (i): it is easy to see that every D-TSA of type of algebras is locally finite-dimensional, and every quotient algebra of a locally finite-dimensional D-SA is locally finite-dimensional, thus the implication holds. $\square$

Chapter 6

Conclusion and discussion

While the previous works in the study of substitution algebras treated singular substitution as the footstone, we attempted to put simultaneous substitution in the central place and explored what this perspective can bring us in this thesis. Given a set of dimensions D, we defined the class of D-dimensional simultaneous substitution algebras by a set of equations, to characterize simultaneous substitution operation on terms over variables from D and on D-ary operations on a nonempty set. Comparing with singular substitution algebras, simultaneous substitution algebras equip with more basic operations, which seem cumbersome at first glance. However, rich with these simultaneous substitution operations, simultaneous substitution algebras seems to be simpler in nature: without any additional condition like local finite-dimensionality, the decidability and completeness of equational theory, and representability have been shown in Chapter 4 and Chapter 5.

As we proved that every locally finite-dimensional substitution algebra can be expanded to a simultaneous substitution algebra in Chapter 3, decidability, completeness, and representability of locally finite-dimensional singular substitution algebras were easily derived. It is noticeable that local finite-dimensionality remains a key condition in our study, and we still don't know whether each singular substitution algebra can be superexpanded to a simultaneous substitution algebra.

We also point out a possible direction of future work here. In Introduction, we mentioned substitution operation in cylindric algebras and lambda abstraction algebras; there is substantial difference between substitution we have discussed and substitution in these two kinds of algebras, since both cylindric algebras and lambda abstraction algebras are algebraizations of formal systems which have free and bound variables, whereas variables (or dimensions) we have discussed in this thesis are all free in this sense. It remains to be investigated that how to characterize the substitution operations in formal systems containing bound variables uniformly in algebras, and what the two perspectives (putting singular/simultaneous substitution at the central place) might bring us in the new study.

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