Single-Peaked Electorates in Liquid Democracy

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

Liquid democracy is a voting system that allows citizens to vote directly, or to delegate their votes to a trusted individual. If delegations occur, the preferences of the electorate (set of voters choosing to vote directly) can exhibit different properties than the preferences of the entire society. One well-studied property of particular interest is *single-peakedness* since it guarantees problem-free aggregation of preferences into a collective choice. We investigate conditions under which delegations generate single-peaked electorates out of non-single-peaked societies. We find that the willingness of voters to delegate is critical for the existence of single-peaked electorates.

Chapter 1

Introduction

Democratic decision-making has a long history but its story has not been told to the end. Two forms of democratic voting systems prevailed, representative and direct democracy, but recently a contender has entered the stage: *liquid democracy* (Brill, 2021). In liquid democracy the ability to cast a vote directly, like in a direct democracy, is extended by *delegations* allowing for political representation, like in a representative democracy. Voters can choose to delegate their vote to somebody they trust, or vote directly. The voters deciding *not* to delegate are often referred to as *gurus*, and the set of gurus is called the *electorate*. Generally seen as a hybrid between representative and direct democracy, liquid democracy sets out to bring together the best of both worlds (Blum and Zuber, 2015).

One main advantage of representative democracy is that voters can choose a representative, who is arguably an expert, or a trusted individual. At the same time, representative democracy is confronted with a democratic deficit. Voters can only choose representatives every couple of years, which then might not serve the public's best interest. A state of (political) lethargy can be the consequence. On the other hand, direct democracy improves democratic participation. Everyone is treated equally in every vote. However, it is caught up in a dilemma: Either voters need immense effort to understand underlying issues, or voters are, potentially, uneducated with respect to these issues. Liquid democracy, arguably, alleviates these shortcomings and improves democratic participation, giving everyone the chance to vote, while it furthermore allows representation through delegations in case voters do not feel confident enough to submit an own ballot.

Arguably, liquid democracy brings benefits along but, like any other system, it is nonetheless confronted with a classical problem in social choice theory: the

(fair) aggregation of individual preferences into a collective choice. Citizens rank alternatives to their liking, and collectively decide on a best alternative, or possibly best ranking of alternatives. Finding an agreement is not only hard in real life, it poses mathematical problems and even leads to paradoxes. One fundamental paradox concerns the (ir)rationality of preferences. Consider three friends, Alice, Bob, and Charlie, who have to decide on a vacation destination. Alice, an environmentalist who dislikes aviation, suggests the Ijsselmeer, Bob loves the urban life and wants to go to Berlin, and Charlie prefers beaches as they have them in Cuba. They rank their friends' suggestions with the following result.

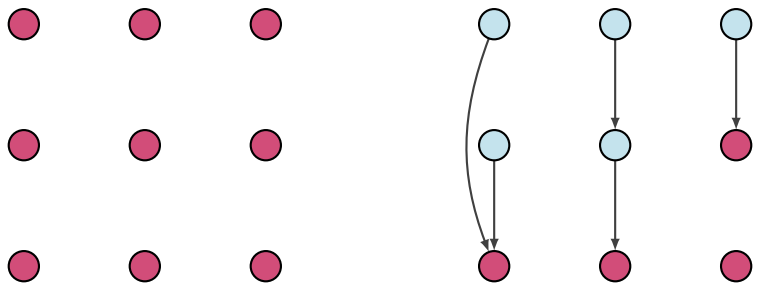
Alice	Bob	Charlie
$Ijsselmeer$	$Berlin$	$Cuba$
$Berlin$	$Cuba$	$Ijsselmeer$
$Cuba$	$Ijsselmeer$	$Berlin$

While each of the three friends have acyclic preferences (arguably a minimal requirement for rationality), the majority preferences are not acyclic. A majority of voters (Alice and Charlie) prefer the $Ijsselmeer$ to $Berlin$, a majority of voters prefers $Berlin$ over $Cuba$ (Alice and Bob), and yet a majority prefers $Cuba$ over the $Ijsselmeer$ (Bob and Charlie). The fact that individually rational preferences do not necessarily result in rational societal preferences is called *Condorcet's Paradox* (Brandt et al., 2016). A further classical problem has been advanced in a seminal work by Arrow (1950) who showed that, under reasonable assumptions for fairness, any fair voting rule is a dictatorship. In a dictatorship a single voter decides over the collective outcome. Gibbard (1973) and Satterthwaite (1975) seem to bury the hopes for good voting rules by showing that the only voting rule that cannot be manipulated must be a dictatorship. A manipulation occurs when voters do not submit their actual preferences, but can affect the collective outcome to their advantage by submitting a manipulated ballot. Naturally, the question arises how individual preferences can be aggregated into a collective choice, while avoiding Condorcet's Paradox, and the threat of dictatorship and manipulation. We call this the *aggregation problem*.

A mathematically elegant solution to the aggregation problem are restricted preferences: If certain preferences are not admitted, and thus restricted, then the aggregation problem does not arise. The restriction discussed in Black (1948) and shown to avoid the aggregation problem is *single-peakedness*. A society is called single-peaked, if its citizens can agree on an order of the alternatives such that each voter prefers alternatives less, if they are further away in this order from their most preferred preference (Brandt et al., 2016). This ordering

comes about naturally in some scenarios, for example in discussions about the legal drinking age. If one's preferred legal drinking age is 18, one will probably consider 15 as a legal drinking age more inappropriate than 16, and 21 more inappropriate than 19. However, single-peaked preferences are not generally realistic.[1] This poses a problem, as voters' preferences cannot be simply made single-peaked if they are not. Declaring non-single-peaked preferences invalid would be extremely undemocratic, and hardly an improvement to a dictatorship, the very thing a democracy sets out to avoid.

Liquid democracy might come to rescue and offer a way out of this dilemma. Starting with a non-single-peaked society, delegations can bring about single-peaked electorates. Consider again the example of our three vacationers Alice, Bob, and Charlie. Together they form a non-single-peaked society. However, note that any two of them together are single-peaked. If any of them were to delegate their vote, a single-peaked electorate can be formed. Compare this process more generally to direct democracy. Given the same society, a single-peaked electorate can therefore be generated through delegations from a non-single-peaked society.



(a) Direct Democracy (b) Liquid Democracy

As we have argued above, if the electorate, which is a subset of all citizens, is single-peaked, the results from Black (1948) apply, and the aggregation problem is solved. Instead of undemocratically taking the voting rights away from non-single-peaked citizens, liquid democracy could avoid the aggregation problem by bringing about single-peakedness with democratic means, namely, delegations. The goal of the thesis is to identify conditions under which liquid democracy guarantees single-peaked electorates—and when it does not.

[1] In fact, the real life data set we will analyze in Chapter 5 of this thesis does not contain a single single-peaked profile.

1.1 Related Literature

Behrens (2017) traces back the origins of liquid democracy to Dodgsons (1884). With technological advances like the internet, a revival of liquid democracy came along (Behrens, 2017). Discussions on political and philosophical foundations as well as implications can be found in Blum and Zuber (2015) and Valsangiacomo (2021). Furthermore, liquid democracy has become a trend in the field of computational social choice. One standard model of liquid democracy and some standard problems it is exposed to is provided in Brill (2019). Various papers have analyzed and criticized liquid democracy, e.g. for its performance in discovering an underlying ground truth (Kahng et al., 2021; Caragiannis and Micha, 2019), a lack of individual rationality (Brill and Talmon, 2018; Christoff and Grossi, 2017), or potential tendencies to aggregate power in the hands of few individuals (Zhang and Grossi, 2021). Liquid democracy has been tested in some scenarios, for example in the German Pirate party (Litvinenko, 2012), and in Google Votes (Hardt and Lopes, 2015). However, scalability remains a problem since, similarly to direct democracy, it is hard to implement it on a large scale. For a more optimistic view on scalability see Brill (2021). Recently, liquid democracy has been discovered in the world of blockchains, and cryptocurrencies (Zhang et al., 2018; Fan et al., 2019).

Brandt et al. (2016) provide an excellent introduction to the field of computational social choice. The chapter by Zwicker ('Introduction to the Theory of Voting', Chapter 2) describes the three problems of Condorcet's paradox, Arrow's Theorem, as well as the result by Gibbard and Saitherwaite, and lays out how single-peaked domain restrictions solve these problems. Elkind et al. (2017) discuss the relevance of domain restrictions, like single-peakedness, beyond the aggregation problem by laying out their benefits from an algorithmic perspective. Essentially, single-peakedness can reduce the complexity of certain computational problems.

The theoretical foundations of this work are largely inspired by Escoffier et al. (2019), and Escoffier et al. (2020), who provide us with a model and analysis of stable electorates. This thesis is, to the best of our knowledge, the first discussion of single-peaked (stable) electorates in liquid democracy. The idea that single-peakedness has to be in some sense 'brought about' also appears in the literature on deliberative democracy (List and Dryzek, 2003; List et al., 2013; List, 2018). List (2018) discusses whether deliberation processes can bring about a meta-agreement between citizens. While not reaching substantive consensus about a topic, citizens can agree on what they disagree on (for example they 'agree that their disagreement concerns a trade-off be-

tween the economy and the environment', List, 2018). One interpretation of single-peaked electorates in liquid democracy could therefore go along the lines of meta-agreements. However, it requires a conceptual analysis what meta-agreement in liquid democracy means which we do not provide in this thesis.

1.2 Our Contribution

The main concern of this thesis will be to investigate the conditions under which single-peaked electorates can be generated in liquid democracy. To the best of our knowledge, this is the first work on single-peaked electorates in liquid democracy. We will, however, not analyze the whole class of single-peaked electorates, but a subclass, the *stable*, single-peaked electorates. A game-theoretical notion of stability, has been proposed by Escoffier et al. (2019), and Escoffier et al. (2020). An electorate is stable, if every voter is happy with their final delegate after all voters have delegated. The relevance of stability becomes clear if we make ourselves aware of the fact, that liquid democracy allows for transitive delegations. If Alice delegates to Bob, and Bob delegates to Charlie, Charlie is the guru of Alice. However, Alice might not accept Charlie as her guru, and prefers to change her delegation. This would be a case of unstable delegations. The concept of acceptance of a guru will be cashed out in mathematical symbolism through two notions. First, we introduce a distance measure between voters. While there are many distance measures on preference orders, the Kendall Tau distance is arguably the most representative for differences in opinions represented by strict linear orders. For two preference orders, it counts the number of adjacent swaps needed to turn one order into the other.[2] Besides preferences, each citizen is associated with a delegation threshold. Intuitively, this threshold can be seen as an indicator of how opinionated citizens are. If this threshold is larger than the distance to another voter, the latter will not be accepted as a guru, if it is smaller, she will be accepted. This setting including a (symmetric) distance measure and delegation thresholds is called *distance based* in Escoffier et al. (2020), and it always admits stable electorates. We will analyze what restrictions we have to make on the delegation thresholds, in order to guarantee that the stable electorates are furthermore *single-peaked.*

[2]Compare this to the Cayley distance which counts the number of not necessarily adjacent swaps. The two orders $a \succ b \succ c$, and $c \succ' b \succ' a$, have Kendall Tau distance three, and Cayley distance one. As they are inverses of each other, they represent diametrically different opinions, and we take the Kendall Tau distance to be more appropriate for our setting.

1.3 Overview

In **Chapter 2** we introduce the main concepts and tools of this thesis. We lay the preliminary groundwork by introducing the standard components of preferences and profiles, the main property of single-peakedness, and some graph theoretical basics. Each of these are treated independently, and partially brought together when the Kendall tau distance is defined, and some basic results are proven.

Chapter 3 forms the theoretical core of this thesis, and proves the main result. We discuss the model of liquid democracy defined in Escoffier et al. (2020), and connect it to single-peakedness. We prove a characterization theorem for the existence of single-peaked, stable electorates in liquid democracy. Essentially, we prove a lower bound on the delegation thresholds: Single-peaked, stable electorates exist if and only if voters are (in a to be defined way) *open-minded*. The proof is based on three assumptions: Every possible preference is submitted once (complete preference domain), every voter has the same delegation threshold (homogeneity of thresholds), and everyone is allowed to delegate to anyone (complete social network). Analytical results are hard to obtain for relaxed assumptions. In **Chapter 4** we will computationally analyze the impact of dropping the assumption of the complete preference domain via a Monte Carlo simulation. We will see that the bound on the delegation threshold established in the existence characterization extends (mostly) to the universal preference domain. Finally, we consider real life data, and additionally drop the assumption of the homogeneity of thresholds in **Chapter 5**. With a further Monte Carlo simulation, we show that the bound on the delegation threshold alone is not fine-grained enough to indicate the existence of single-peaked equilbria. It remains, however, a crucial factor. A generalization to any social network remains future work, which together with a summary, and discussion of the results is provided in the conclusive **Chapter 6**. Furthermore, we relate the results to the motivating aggregation problem.

Chapter 2

Preliminaries

This chapter formally introduces some of the main concepts. The thesis spans from combinatorial arguments on preference orders, to graph theoretical notions. Most of these notions are introduced independently here, and brought together in Chapter 3.

2.1 Preference Orders and Profiles

We denote by $A = \{a_1, ..., a_m\}$ the set of *alternatives* with $m = |A|$. A *strict linear order* $\succ$ is a binary relation over A, satisfying the following three properties for all $a, b, c \in A$: Irreflexivity (not $a \succ a$), transitivity (if $a \succ b$ and $b \succ c$, then $a \succ c$), and connectedness (if $a \neq b$ then $a \succ b$ or $b \succ a$). For a strict linear order $\succ$ we sometimes refer to the maximal element of $\succ$ as its *peak*. Generally, we denote by $\overline{\succ}$ the *reverse* of $\succ$, that is for all distinct alternatives $a, b \in A$, we have $a \,\overline{\succ}\, b$ if and only if $b \succ a$. Intuitively, one can think of a strict linear order as a ranking of alternatives. We denote by $\mathcal{L}(A)$ the set of all strict linear orders over A.

If $\succ$ is a strict linear order on A, we say that $\succ'$ is a *suborder* of $\succ$ if it is a restriction of $\succ$ to a subset $A' \subset A$. If $\succ'$ is a suborder of $\succ$, and only contains the highest ranked alternatives of $\succ$, we say that $\succ'$ is a *prefix* of $\succ$. Analogously, we define the *suffix* if $\succ'$ contains the lowest ranked alternatives of $\succ$. We define the operation $+$ to be the concatenation of two strict linear orders $\succ'$ and $\succ''$ on disjoint sets A' and A'', such that $\succ' + \succ'' = \succ$ where $\succ$ is a strict linear order on $A = A' \cup A''$.

Example 1. $\succ$ is a strict linear order over $A = (a_1, a_2, a_3)$.

$$\begin{aligned} \succ &= (a_1, a_2, a_3) & \succ' &= (a_1, a_3) \\ \succ'' &= (a_1, a_2) & \succ''' &= (a_3) \end{aligned}$$

The order $\succ'$ is a suborder but neither prefix nor suffix of $\succ$. While $\succ''$ is a prefix, $\succ'''$ is a suffix of $\succ$. Note furthermore that $\succ'' + \succ''' = \succ$. $\triangle$

Given a set of alternatives, each of a finite set of agents $N = \{1, ..., n\}$ expresses a preference $\succ_i \in \mathcal{L}(A)$. This gives rise to a vector of preferences orders which we call a *preference profile* $\boldsymbol{P} = (\succ_1, ..., \succ_n) \in \mathcal{L}(A)^n$. The *domain* of a preference profile $\mathcal{D}(\boldsymbol{P})$ is the set $\mathcal{D}(\boldsymbol{P}) \subseteq \mathcal{L}(A)$ containing the orders corresponding to each preference relation $\succ_i$ in the profile $\boldsymbol{P}$. In the setting of profiles we will make use of the preference orders indexed by the agents, e.g. $\succ_i$, while we omit the index when reasoning about domains.

2.2 Single-Peakedness

We begin by introducing the basic definitions of single-peakedness, and talk through some observations. Let $\max(\succ)$ denote the peak of the order $\succ \in \mathcal{L}(A)$, and $\min(\succ)$ the minimal element. For a domain $\mathcal{D} \subseteq \mathcal{L}(A)$, we let $\min(\mathcal{D})$ denote the set of all bottom alternatives, i.e. $\min(\mathcal{D}) = \bigcup_{\succ \in \mathcal{D}} \min(\succ)$. For a preference profile $\boldsymbol{P}$ the set containing the least preferred alternatives of all voters, is just $\min(\mathcal{D}(\boldsymbol{P}))$, which we denote by $\min(\boldsymbol{P})$ for short.

Definition 1. Given a designated order $\succ \in \mathcal{L}(A)$, an order $\succ' \in \mathcal{L}(A)$ is single-peaked with respect to $\succ$ if we have for every triple of distinct alternatives $\max(\succ), a, b \in A$:

$$(\max(\succ') \succ a \succ b \text{ or } b \succ a \succ \max(\succ')) \text{ implies } a \succ' b\,.$$

We say a set $D \subseteq \mathcal{L}(A)$ is $\succ$-single-peaked if all its members are single-peaked with respect to $\succ$. Denote by $\mathcal{SP}_\succ$ the maximal $\succ$-single-peaked domain, i.e. $\mathcal{SP}_\succ = \{\succ' \in \mathcal{L}(A) \mid \succ' \text{ is } \succ\text{-single-peaked}\}$.

The following observation follows from closer inspection of Definition 1:

Observation 1. *An order $\succ' \in \mathcal{L}(A)$ is $\succ$-single-peaked if and only if $\succ'$ is single-peaked with respect to the reverse of $\succ$.*

The definition of single-peakedness can be naturally expanded to preference profiles as follows:

Definition 2. Given some order $\succ \in \mathcal{L}(A)$, a preference profile $\boldsymbol{P}$ is single-peaked with respect to $\succ$ for voter i if we have for every triple of alternatives $\max(\succ_i), a, b \in A$:

$$(\max(\succ_i) \succ a \succ b \text{ or } b \succ a \succ \max(\succ_i)) \text{ implies } a \succ_i b\,.$$

A preference profile is single-peaked with respect to $\succ$ if it is single-peaked with respect to $\succ$ for all voters. We then write that $\boldsymbol{P}$ is $\succ$-single-peaked. Note that the basic definition of single-peakedness is insensitive to the number of voters as it only considers which orders are submitted.

Observation 2. *A preference profile $\boldsymbol{P}$ is $\succ$-single-peaked if and only if the domain of the profile $\mathcal{D}(\boldsymbol{P})$ is $\succ$-single-peaked.*

Instead of reasoning about different preference *profiles* with the same domain, we can therefore reason about the set-based notion of *domains* of preference profiles. By Proposition 2 this allows us to draw conclusions about all profiles with the same domain.

The following theorem shows that given an order $\succ$, any $\succ$-single-peaked order can be divided into two suborders.[1]

Theorem 1. *Let $\succ = (a_1, ..., a_m)$, and let $\succ'$ be an order with $a_i = \max(\succ')$ and $1 \leq i \leq m$. Then $\succ'$ is $\succ$-single-peaked if and only if*

$$a_i \succ' a_{i-1} \succ' ... \succ' a_1 \text{ and } a_{i+1} \succ' a_{i+2} \succ' ... \succ' a_m \quad .$$

Proof. Let $\succ = (a_1, ..., a_m)$, and $a_i = \max(\succ')$.
($\Rightarrow$) For contradiction, assume $\succ'$ is $\succ$-single-peaked, and not $a_i \succ' a_{i-1} \succ' ... \succ' a_1$ or not $a_{i+1} \succ' a_{i+2} ... \succ' a_m$. In the first case, there are a_j and a_k such that $j < k < i$, and $a_j \succ' a_k$. Since $\succ = (a_1, ..., a_m)$, we know $a_i \succ a_k \succ a_j$. By the definition of single-peakedness it follows that $a_k \succ' a_j$, a contradiction. The second case works analogously.
($\Leftarrow$) Again, there are two cases to consider. In the first case we have to show that for any triple $a_i, a_j, a_k \in A$ if $a_i \succ a_j \succ a_k$ then $a_j \succ' a_k$. Since $\succ = (a_1, ..., a_m)$, we know that $i > j > k$. Since $a_i \succ' a_{i-1} \succ' ... \succ' a_1$, it follows immediately that $a_j \succ' a_k$. For the second case consider any triple $a_j \succ a_k \succ a_i$. Since $j < k < i$, and $a_{i+1} \succ' a_{i+2} ... \succ' a_m$, we derive by analogous reasoning that $a_j \succ' a_k$ which concludes the proof. □

[1] The equivalence is well-known. In fact, the definition of single-peakedness varies from author to author. Here we essentially prove the equivalence of two definitions.

Given the peak of an order, Theorem 1 makes checking for $\succ$-single-peaked-ness relatively easy. Once the indexation $\{1, ..., m\}$ of the alternatives in $\succ$ is fixed, simply check whether alternatives with lower indexes than the peak are ranked in descending order in $\succ'$, and alternatives with a larger index than the peak are ranked in ascending order in $\succ'$.

Example 2. Let $\succ = (a_1, ..., a_6)$. We want to check whether the following orders are $\succ$-single-peaked:

$$\succ' = (a_4, a_3, a_2, a_1, a_5, a_6)$$
$$\succ'' = (a_4, a_5, a_3, a_6, a_2, a_1)$$
$$\succ''' = (a_4, a_6, a_3, a_2, a_5, a_1)$$

The first two orders are indeed $\succ$-single-peaked, since they both contain the suborders (a_4, a_3, a_2, a_1) and (a_5, a_6). The last order however contains $a_6 \succ''' a_5$ which violates Theorem 1, and is thus not $\succ$-single-peaked. △

With Theorem 1 at hand, it is easy to prove that any single-peaked profile can have at most two different minimal alternatives.

Observation 3. *If $\boldsymbol{P}$ is single-peaked, then $|\min(\boldsymbol{P})| \leq 2$.*

Proof. Assume for contradiction that $\boldsymbol{P}$ is single-peaked with respect to some order $\succ = (a_1, ..., a_m)$, and $|\min(\boldsymbol{P})| > 2$. From the latter, we deduce that there exists an order $\succ'$ with minimal element $a_i \neq a_1 \neq a_m$. But then $\succ'$ is not $\succ$-single-peaked by Theorem 1. □

Theorem 1 furthermore motivates the partition of the single-peaked domain according to the peaks of the orders. It provides us with structural knowledge which we will exploit later in this thesis. Let $\mathcal{SP}^{a_i}_{\succ}$ denote the set of $\succ$-single-peaked orders with peak a_i.

Observation 4. *The $\succ$-single-peaked domain can be partitioned into disjunct subdomains induced by the peaks a_i of the orders $\succ' \in \mathcal{SP}_{\succ}$ as follows:*

$$\mathcal{SP}_{\succ} = \mathcal{SP}^{a_1}_{\succ} \cup ... \cup \mathcal{SP}^{a_m}_{\succ}$$

We will show later in this chapter that each subset is non-empty (Corollary 4.1). Furthermore, note that $\mathcal{SP}^{a_1}_{\succ}$ and $\mathcal{SP}^{a_m}_{\succ}$ each contain only one element: the orders $\succ$, and $\overline{\succ}$ respectively. To give a taste of what is yet to come, Observation 4 will come in handy when measuring the distance between an order $\succ'$ and a $\succ$-single-peaked domain. As this task turns out to be rather difficult, we will instead measure the distances between $\succ'$ and each of the subdomains $\mathcal{SP}^{a_1}_{\succ}$,..., $\mathcal{SP}^{a_m}_{\succ}$. The minimal distance of $\succ'$ to any single-peaked subdomain is then the distance to the whole single-peaked $\mathcal{SP}_{\succ}$.

2.3 Elements of Graph Theory

A *directed graph* is a pair $G = (V, E)$ consisting of a set of vertices V and edges $E \subseteq V \times V$, a subset of the Cartesian product of V. A directed graph is *symmetric* if for all edges $u, v \in V$ we have $(u, v) \in E$ if and only if $(v, u) \in E$. In a symmetric graph the direction of an edge loses its significance, and we will refer to them as *undirected graphs*. In a directed graph $G = (V, E)$, a subset of vertices $S \subseteq V$ is *independent* if there is no edge between any two vertices in S. We say that S is maximal independent if no vertex can be added to S without losing the property of independence. A subset $S \subseteq V$ is *absorbing* if for every vertex $u \notin S$, there exists $v \in S$ such that $(u, v) \in E$ (then we say that v absorbs u). A *kernel* of G is a subset of vertices that is both independent and absorbing. A *clique* $C \subseteq V$ is a subset of vertices, such that all vertices in C are adjacent. Intuitively, a clique is the opposite of an independent set. The complement $\overline{G}$ of a graph G, is a graph on the same vertices, and contains an edge between two vertices whenever G does not contain an edge, and vice versa.

Observation 5. *In an undirected graph G, a set S is independent in G if and only if S is a clique in the complement graph $\overline{G}$.*

Observation 5 follows directly from the definitions of independent sets and cliques. Next, we show that a kernel is equivalent to a maximal independent set if the graph is undirected.

Observation 6. *If $G = (V, E)$ is an undirected graph, a set $K \subseteq$ of vertices is a kernel if and only if it is a maximal independent set.*

Proof. ($\Rightarrow$): Since K is a kernel, any vertex $v \notin K$ is absorbed by some vertex u in K. Thus, $(u, v) \in E$, and v cannot be added to K without losing independence.

($\Leftarrow$): Assume for contradiction that there is a vertex v that is not absorbed by K. Since G is undirected, there is no edge between v and any element of K, and v can be added to K without losing independence, resulting in a contradiction. Thus, K is absorbing, and therefore a kernel. □

Notably, we only used the assumption that the graph was undirected in the right-to-left direction. In directed graphs not every maximal independent set is a kernel, but every kernel is a maximal independent set.

An *isomorphism* between two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ is a bijection $f : V_1 \to V_2$ between the vertices such that $(u, v) \in E_1$ if and only if $(f(u), f(v)) \in E_2$. Isomorphisms will play an important role in Chapter 3 since

they preserve kernels. This is straightforward to see since by definition edges are preserved in isomorphism.

Observation 7. *Kernels are preserved under isomorphisms.*

2.4 Kendall Tau Distance

For any two distinct alternatives $a, b \in A$, two strict linear orders $\succ, \succ' \in \mathcal{L}(A)$ agree on the rankings of a and b if $a \succ b \Leftrightarrow a \succ' b$, and they disagree on the rankings if $a \succ b \Leftrightarrow b \succ' a$. We define $\tau_{a,b}(\succ, \succ') = 0$ if $\succ$ and $\succ'$ agree on the order of a and b, and $\tau_{a,b}(\succ, \succ') = 1$ if they disagree. The Kendall tau distance is defined by the following formula

$$\tau(\succ, \succ') = \sum_{a,b \in A} \tau_{a,b}(\succ, \succ')$$

In words, the Kendall tau distance between two orders equals the number of pairs they disagree on. If two orders disagree on all pairs the Kendall tau distance is maximal (denoted by τ_m), and one order is the *reverse* of the other. Since there are $\binom{m}{2}$ pairs of distinct alternatives we have that:

Observation 8. *The maximal Kendall tau distance τ_m between any two orders of length m is $\binom{m}{2} = \frac{m(m-1)}{2}$.*

A more intuitive way to think of the Kendall tau distance is as the minimum number of adjacent swaps needed to reach an order $\succ$ from $\succ'$. Each swap of adjacent alternatives changes whether two orders agree, or disagree on the ranking of these alternatives. A swap therefore either increases or decreases the Kendall tau distance between two orders by 1. The minimal number of swaps is therefore equivalent to the Kendall tau distance. We will often write that $\succ$ and $\succ'$ are k swaps away from each other, instead of $\tau(\succ, \succ') = k$.

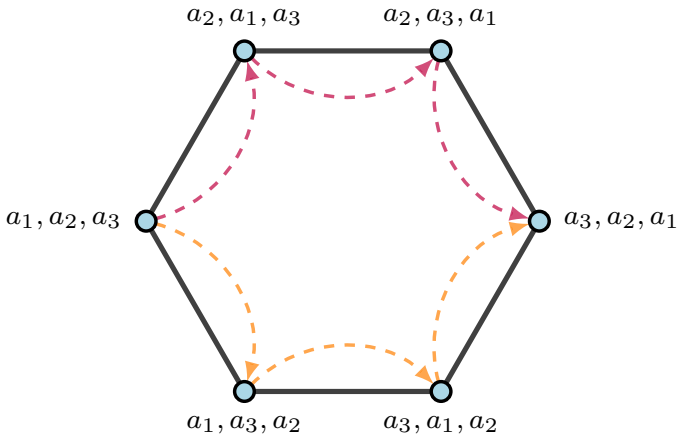
Example 3. Consider the orders $\succ = (a_1, a_2, a_3)$, and $\succ' = (a_3, a_2, a_1)$. As they disagree on the ranking of all pairs, we get $\tau(\succ, \succ') = \binom{3}{2} = 3$. In the sequence $((a_1, a_2, a_3), (a_2, a_1, a_3), (a_2, a_3, a_1), (a_3, a_2, a_1))$ three swaps are performed. First a_1 and a_2, then a_1 and a_3, and finally a_2 and a_3 are swapped, resulting in the same answer. △

2.4.1 τ-Graph

For the pursuit of the following chapters it will prove helpful to visualize the complete domain $\mathcal{L}(A)$ in terms of the Kendall tau distance. We introduce the

τ-graph $G_\tau = (\mathcal{L}(A), E_\tau)$, an undirected graph consisting of all strict linear orders on a set of alternatives A as vertices, and edges between $\succ$ and $\succ'$ if $\tau(\succ, \succ') = 1$, or equivalently two orders are one swap away from each other. The d-τ-graph is a generalization of the τ-graph, where edges between $\succ$ and $\succ'$ exist if $\tau(\succ, \succ') \leq d$. A *path* from $\succ^1$ to $\succ^k$ is a sequence $P = (\succ^1, ..., \succ^k)$ with $k \geq 2$ such that for all $j \in \{1, ..., k-1\}$ we have that $\succ^j$ and $\succ^{j+1}$ are neighbors, i.e. $\tau(\succ^j, \succ^{j+1}) = 1$.[2] The *length* $\ell(P)$ equals the number of elements in P which are distinct from the starting order $\succ^1$. The Kendall tau distance between two orders $\succ, \succ'$ therefore corresponds to the length of a shortest path from $\succ$ to $\succ'$ in the τ-graph.

Example 4. The τ-graph for $A = \{a_1, a_2, a_3\}$.



Recall the sequence of swaps performed to reverse (a_1, a_2, a_3) in Example 3:

$$P = ((a_1, a_2, a_3), (a_2, a_1, a_3), (a_2, a_3, a_1), (a_3, a_2, a_1))$$

This sequence corresponds to the purple path in the τ-graph. △

As can be seen in the τ-graph in Example 4, there are two shortest paths (purple and orange) resulting in the reversion of $\succ^1 = (a_1, a_2, a_3)$. Note that the sequence P follows a specific logic. First, the top alternative in $\succ$ is moved to the bottom: The order $\succ^3 = (a_2, a_3, a_1)$ agrees with $\overline{\succ}$ on all pairs containing

[2]To avoid confusion, a remark about the notation is to be made. Although both paths and profiles are sequences of orders, we refrain from using the bold symbol $\boldsymbol{P}$ for paths since it is already used for profiles. A profile is associated with a set of agents N which is not necessarily the case for paths.

a_1. Next, the same is done for a_2, until the reversion is completed. The reversion path following the same logic looks as follows for four alternatives:

$$P' = ((\overset{\frown}{a_1, a_2}, a_3, a_4), (a_2, \overset{\frown}{a_1, a_3}, a_4), (a_2, a_3, \overset{\frown}{a_1, a_4}), (\overset{\frown}{a_2, a_3}, a_4, a_1),$$
$$(a_3, \underset{\smile}{a_2, a_4}, a_1), (\underset{\smile}{a_3, a_4}, a_2, a_1), (a_4, a_3, a_1, a_1))$$

As $\ell(P') = 6 = \binom{4}{2}$, P' is a shortest reversion path. The paths induced by this method fulfill a property which we are aiming to investigate throughout this thesis: single-peakedness.

2.4.2 Reversion-Algorithm

We generalize the reversion approach above by defining an algorithm which always outputs a shortest reversion path. As an auxiliary operation, we define Push-Down$(\succ, a)$ to output the order which is like $\succ$ except it switches the alternative a with its lower adjacent neighbor. If there is no lower adjacent neighbor, the operation is undefined. In the definition of P' above, the Push-Down-operation corresponds to the arcs.

Algorithm 1

Input: Strict linear order $\succ = (a_1, ..., a_m) \in \mathcal{L}(A)$
Output: Shortest reversion path P

1: $\succ' \leftarrow \succ$
2: $P \leftarrow (\succ)$
3: **for** $i \in \{1, ..., m-1\}$ **do**
4: **while** $\tau_{a_i, a_m}(\succ', \overline{\succ}) = 0$ **do**
5: $\succ' \leftarrow$ Push-Down$(\succ', a_i)$
6: $P \leftarrow P + (\succ')$
7: **end while**
8: **end for**
9: **return** $P = (\succ, ..., \overline{\succ})$

Algorithm 1 iteratively builds a reversion path based on an input order $\succ = (a_1, ..., a_m)$. Ordered by the ranks, an alternative a_i is picked, swapped with its lower adjacent neighbor, and the resulting order $\succ'$ is added to the path. Since the goal is to reach the order $\overline{\succ}$, this is repeated until the order $\succ'$ and $\overline{\succ}$ agree on the ranking of a_i and a_m. In practice, the while-loop (line 4) is exited once a_i and a_m have been swapped. The while-loop is then repeated with alternative

a_{i+1}. Intuitively, after each completed while-loop, the last element $\succ'$ of the path P can be divided into two concatenated suborders:

$$\begin{aligned}\succ' &= (a_i, ..., a_m, a_{i-1}, ..., a_1) \\ &= (a_i, ..., a_m) + (a_{i-1}, ..., a_1)\end{aligned}$$

The first part contains elements that have not been reversed, they are ranked according to the input order $\succ$. The second suborder is reversed, and its elements are ranked such that they agree with $\overline{\succ}$. As a_m cannot swap the position with itself, the algorithm terminates, once alternatives a_{m-1} and a_m have been swapped.

Correctness

We need to show that the algorithm indeed produces a shortest reversion path. Formally, we show that for an order $\succ$ of length m, Algorithm 1 produces a path P such that (a) the final element of P is $\overline{\succ}$, and (b) $\ell(P) = \binom{m}{2}$. Firstly, note that it takes $m-1$ swaps to turn a maximal element of an order of length m into the minimal element. Formally,

Observation 9. *If $\succ = (a_1, a_2, ..., a_m)$, and $\succ' = (a_2, ..., a_m, a_1)$, then*

$$\tau(\succ, \succ') = m - 1.$$

Next, we prove the following Lemma by induction on the ranking i of the alternatives. Part *a.* formally represents the intuition that the order $\succ'$ can be divided into two suborders, the first agreeing with $\succ$, and the second agreeing with $\overline{\succ}$. Part *b.* establishes the amount of swaps performed—equivalently the length of P—after i while-loops.

Lemma 2. *Given $\succ = (a_1, ..., a_m)$ as the input for Algorithm 1, let $\succ'$ be the order such that a_i has been swapped with a_m, and $\succ'$ has been added to the path P,*

a. the orders $\succ'$ and $\overline{\succ}$ agree on the ranking of all pairs (a_j, a_k), for $1 \leq j < k \leq i$,

b. $\ell(P) = \sum_{j=1}^{i} (m - j)$

Proof. Let $\succ = (a_1, ..., a_m)$.
Base Case: $i = 1$. Immediate from Observation 9.
Induction Step: Assume Lemma 2 holds for alternative a_{i-1}. We need to show that if also holds for a_i. The last element of P before entering the while-loop for a_i is therefore

$$\succ' = (a_i, ...a_m, a_{i-1}, ..., a_1)$$

as otherwise part *a.* of the induction hypothesis would be violated. Let P^+ be the path after exiting the while loop for a_i, and $\succ^+$ its last element. Proving *a.* is trivial, as the while-loop is exited once a_i and a_m are swapped. Thus,

$$\succ^+ = (a_{i+1}, ...a_m, a_i, a_{i-1}, ..., a_1)$$

as this amounts to turning the maximal element of subsequence $(a_i, ..., a_m)$ of length $m-(i-1)$ into the minimal element, the while-loop performs $m-i$ swaps. We therefore get

$$\begin{aligned}\ell(P^+) &= \ell(P) + (m-1) \\ &= \sum_{j=1}^{i-1}(m-j) + (m-i) \\ &= \sum_{j=1}^{i}(m-j)\end{aligned}$$

where the second step follows from the induction hypothesis. □

With Lemma 2 in hand, we continue to prove the correctness of Algorithm 1, in words we need to prove that Algorithm 1 produces a shortest reversion path.

Theorem 3. *Algorithm 1 is correct and terminates.*

Proof. Consider the last iteration of the while-loop $i = m-1$. From Lemma 2.*a* we deduce that the orders $\succ'$ and $\overline{\succ}$ agree on all pairs, and therefore $\succ' = \overline{\succ}$. Furthermore, $\ell(P) = \sum_{j=1}^{m-1}(m-j) = \binom{m}{2}$. This concludes the correctness of Algorithm 1. From above considerations together with the fact that the order $\succ$ is of finite length, termination of Algorithm 1 follows as well. □

Single-Peakedness

The purpose of the—admittedly—lengthy discussion of Algorithm 1 becomes clear when we consider its relation to the property of single-peakedness. Before

we finally bring the three threads of kernels, single-peakedness, and the Kendall tau distance together in the next chapter, the following paragraph is devoted to show that all elements of the path induced by Algorithm 1 are single-peaked with respect to its input order. For short, we say that P is single-peaked with respect to its input.[3]

Lemma 4. *Algorithm 1 produces an output P which is single-peaked with respect to its input $\succ$.*

Proof. Recall that we can test $\succ$-single-peakedness of an order $\succ'$ by checking whether given a peak $\max(\succ')$, and $\succ$-rank-indexes $i \in \{1, ..., m\}$, alternatives with a lower index than $\max(\succ')$ are ordered descendingly, while higher indexed alternatives are ordered ascendingly. This condition is only violated if at one point in the algorithm an alternative a_i is swapped with an alternative a_j such that $j < i$. However, since alternatives are only swapped with lower adjacent neighbors, i.e. higher indexed alternatives, until the adjacent neighbor is a_m, this is never the case. □

Algorithm 1 gives us some valuable insights about the single-peaked domain. We say that a set $S \subseteq \mathcal{L}(A)$ of strict-linear orders is *minimally-rich* if every alternative $a \in A$ is the maximal element of at least one order $\succ \in S$. Furthermore, S is of *maximal width* if S contains two reversed orders.

Corollary 4.1. *The single-peaked domain is minimally-rich and of maximal width.*[4]

Proof. Note that the set containing the elements of path P output by Algorithm 1 is minimally-rich and of maximal width. As P is single-peaked with respect to $\succ$ by Lemma 4, the $\succ$-single-peaked domain is minimally rich and of maximal width. □

2.4.3 An Important Lemma

We conclude this chapter by proving a more technical lemma which we will utilise throughout the next chapter. The lemma states that the distances of an order $\succ'$ to an order $\succ$ and its reverse $\overline{\succ}$ sum up to the maximal distance τ_m. We first prove an auxiliary lemma.

[3]Recall that we defined both paths and a profiles to be vector of orders. A path, however, is not associated with a set of agents N.

[4]These are two of four properties that form a characterization of the single-peaked domain (Puppe, 2018).

Lemma 5. *Given any order $\succ$ and its reverse $\overline{\succ}$, it holds for every third order $\succ'$ that for any two distinct alternatives $a, b \in A$,*

$$\tau_{a,b}(\succ, \succ') = 1 \text{ iff } \tau_{a,b}(\overline{\succ}, \succ') = 0$$

Proof. This follows immediately from the observation that $a \succ b$ whenever $b \overline{\succ} a$, and vice versa. Therefore, whenever $\succ$ and $\succ'$ agree on the order of a and b, $\overline{\succ}$ and $\succ'$ disagree on the order of a and b. □

Lemma 6. *Let $\succ$ and $\overline{\succ}$ be two orders, one the reverse of the other. Then for any order $\succ'$ we have*

$$\tau(\succ, \succ') + \tau(\overline{\succ}, \succ') = \tau_m$$

Proof. We start by expanding the distances according to the definition of the Kendall tau distance.

$$\begin{aligned}\tau(\succ, \succ') + \tau(\overline{\succ}, \succ') &= \sum_{a,b \in A} \tau_{a,b}(\succ, \succ') + \sum_{a,b \in A} \tau_{a,b}(\overline{\succ}, \succ') \\ &= \sum_{a,b \in A} \underbrace{[\tau_{a,b}(\succ, \succ') + \tau_{a,b}(\overline{\succ}, \succ')]}_{=1}\end{aligned}$$

From Lemma 5 we can straightforwardly derive that $\tau_{a,b}(\succ, \succ') + \tau_{a,b}(\overline{\succ}, \succ') = 1$ for all distinct $a, b \in A$. Since there are $\binom{m}{2}$ many pairs we get

$$\tau(\succ, \succ') + \tau(\overline{\succ}, \succ') = \binom{m}{2} = \tau_m.$$ □

On an intuitive level, Lemma 6 proves a form of symmetry on the distances of orders with respect to a designated pair of reversed orders. If an order $\succ'$ is close to $\succ$, it is far away from $\overline{\succ}$, and vice versa.

Chapter 3

Single-Peaked Electorates

The main endeavour of this thesis is to study stable delegations in liquid democracy that give rise to single-peaked electorates. In this chapter we introduce one model of liquid democracy formally, and explore in game-theoretic terms what it means for a delegation to be stable. Escoffier et al. (2019) show that the game-theoretical concept of stable delegations corresponds to the graph theoretical concept of *kernels*. We will then see that stable delegations exist if voters choose to delegate based on a symmetric distance measure (Escoffier et al., 2020). The second part of the chapter is devoted to the analysis of stable, single-peaked electorates under a specific symmetric distance measure, the Kendall tau distance. We show that the search for single-peaked, stable electorates can be reduced to single-peaked kernels in the τ-graph, an undirected graph consisting of linear orders and edges depending on their respective distances. The main result of this chapter is a characterization theorem for the existence of stable, single-peaked electorates. We conclude this chapter by counting the stable, single-peaked electorates, and discussing their structure.

3.1 Stable Electorates in Liquid Democracy

In liquid democracy the voters in a preference profile $\boldsymbol{P}$ are connected in a social network, restricting the admissible delegations of individual voters. The network is represented by an undirected graph $G = (N, E)$ in which vertices are agents, and there is an edge $\{i, j\} \in E$ between two agents if they can delegate votes to each other. In large networks not all voters might be aware of one another, so delegations might be limited to, for example, colleagues, friends and family. Each voter i is either allowed to vote herself, or to delegate to one of her neighbors $\mathtt{Nb}(i) = \{j \in N \mid \{i, j\} \in E\}$. A *delegation function* is a

function $d : N \to N$ such that $d(i) = i$ if voter i decides to vote herself, and $d(i) = j \in \texttt{Nb}(i)$ if she delegates her vote to her neighbor j.[1]

Given a delegation function d, the electorate $\texttt{Elct}(d)$ under d is the set of agents that vote themselves, i.e. $\texttt{Elct}(d) = \{i \in N \mid d(i) = i\}$. The delegation function d is transitive, in words, if i delegates to j, and j delegates to k, then i delegates her vote to whomever k delegates her vote to. If one follows this sequence of delegations until there is an agent l with $d(l) = l$, then we call l the *guru* $\texttt{gu}(i, d)$ of i. However, transitive delegations do not come without problems as the following example illustrates.

Example 5. Our three friends Alice, Bob and Charlie agreed to disagree on the vacation plans, and are now trying to arrange their dinner. Alice, blindly in love with Bob, is willing to go anywhere Bob wants to go, so she delegates her vote to Bob. Charlie has been praising a new restaurant, so Bob trusts him with the choice, and delegates his vote to Charlie. However, Charlie picks a Steak house for dinner which is a rather troublesome decision for Alice, who is vegetarian. She is not happy with this choice whatsoever, and would like to change her delegation. △

The underlying delegation function in Example 5 is not *stable* since there is an agent, namely Alice, who would like to change her vote. The example motivates the study of conditions that give rise to stable delegation, which we will introduce formally later in this section. To tackle this challenge, Escoffier et al. (2019) introduce preferences over possible gurus. This allows them to check whether each voter accepts their guru, or favors a change of her delegation. While we will make use of their framework, in our setting the preferences over gurus are not given but rather induced from the voters' preferences over alternatives.

Let *dist* be some symmetric distance measure over preference relations. We denote by $dist(\succ_i, \succ_j)$ the distance between the preference relations $\succ_i$ and $\succ_j$ of voters i and j under that distance measure. Each voter has an *acceptability threshold* $\rho_i \in \mathbb{R}^+$: she accepts as possible gurus only the voters that are at distance at most ρ_i from her. For a voter i we call the set containing the acceptable gurus $\texttt{Acc}(i)$.

$$\forall j \in N \setminus \{i\}, \quad j \in \texttt{Acc}(i) \Leftrightarrow dist(i, j) \leq \rho_i$$

Given some preference profile $\boldsymbol{P}$, let us denote by $\boldsymbol{\rho} = (\rho_1, ..., \rho_n)$ the *threshold profile* assigning to each voter i an acceptability threshold ρ_i. If all voters have

[1] This model can be straightforwardly extended to permit abstentions by redefining the delegation function $d : N \to N \cup \{0\}$, where $d(i) = 0$ if agent i chooses to abstain.

the same acceptability threshold, we say $\boldsymbol{\rho}$ is *homogeneous*, and we write ρ instead of ρ_i for specific voters. If voters have different delegation thresholds we say $\boldsymbol{\rho}$ is *heterogeneous*. We call the triple $\langle G, \boldsymbol{P}, \boldsymbol{\rho}\rangle$ a *delegation structure*, where G is the social network, $\boldsymbol{P}$ is a preference profile, and $\boldsymbol{\rho}$ specifies the acceptability threshold of each voter.

Definition 3. Given a delegation structure $\langle G, \boldsymbol{P}, \boldsymbol{\rho}\rangle$, a symmetric distance measure *dist* induces a ranking $\rhd_i$ over acceptable gurus for voter i as follows

$$\begin{array}{lll} (i) & & dist(i,i) = \rho_i \\ (ii) & \forall j,k \in N, & dist(i,j) \leq dist(i,k) \Leftrightarrow j \rhd_i k \\ (iii) & \forall j,k \in N, & dist(i,j) > dist(i,k) \Leftrightarrow k \rhd_i j \end{array}$$

Proposition 1. *The ranking $\rhd_i$ is a total preorder over N.*[2]

Proof. Firstly, note that by the definition of $dist(i,i)$ as ρ_i, the position of agent i with respect to her own order $\rhd_i$ is defined by its delegation threshold ρ_i.

To show transitivity, assume $h \rhd_i j$, and $j \rhd_i k$, for $i,j,k,h \in N$. From (ii) of Definition 3, we get $dist(i,h) \leq dist(i,j)$, and $dist(i,j) \leq dist(i,k)$. Since $\leq$ is transitive, we get $dist(i,h) \leq dist(i,k)$, and consequently $h \rhd_i k$.

For total connectedness, note that for any $j,k \in N$, either $dist(i,j) \leq dist(i,k)$, or $dist(i,j) > dist(i,k)$. From both cases we derive by definitions (ii) and (iii) that $j \rhd_i k$ or $j \lhd_i k$. □

With these tools in hand we can define the notion of a *Nash-stable* delegation function.

Definition 4. Given a delegation structure $\langle G, \boldsymbol{P}, \boldsymbol{\rho}\rangle$, a delegation function d is Nash-stable for voter i if

$$\mathtt{gu}(i,d) \rhd_i g \quad \forall g \in (\mathtt{Elct}(d) \cup \{i\}) \setminus \{\mathtt{gu}(i,d)\}.$$

In words, d is Nash-stable for i, whenever i prefers her guru over herself and any member of the electorate under d. If a delegation function d is Nash-stable for all voters, d is Nash-stable, and we call it an *equilibrium*.[3] Note that while the definition of Nash-stability does not explicitly mention the preference profile $\boldsymbol{P}$,

[2]Note that this order is not strict. It can easily be made strict by introducing a tiebreaking rule.

[3]The underlying intuition is the same as for *pure Nash equilibria* in a normal-form game: No player, voter in our case, can do better by unilaterally deviating from her assigned action, delegation in our case.

the induced rankings $\rhd_i$ for voters $i \in N$ relies on the preference orders of each agent i. Furthermore, since the rankings $\rhd$ are total preorders (Proposition 1), the definition of Nash-stability through $\rhd$ is sensible, as $\rhd$ compares any two voters (total connectedness), and is transitive.

3.1.1 Acceptability Digraphs and Kernels

Given the rankings $\rhd_i$ that have been induced for all voters, an equivalent definition for the set of acceptable gurus is $\texttt{Acc}(i) = \{j \in N \mid j \rhd_i i\}$, i.e. the set of voters that i prefers over voting directly. It is a necessary condition for Nash-stability that the guru of any voter i is also an acceptable guru for i. Consequently, all voters ranked lower than i in $\rhd_i$ have no impact on equilibria. We proceed by introducing *delegation-acceptability digraphs*, in which an edge from i to j exists if and only if i accepts j as a guru.

Definition 5. The *delegation acceptability digraph* is the directed graph $G_\rho = (N, E_\rho)$, with $E_\rho = \{(i, j) \mid j \in \text{Acc}(i)\}$

We briefly recapitulate some graph theoretical terminology, and discuss the connection between acceptability digraphs and kernels. Given a digraph $G = (V, A)$, a subset of vertices $K \subseteq V$ is *independent* if there is no arc between two vertices of K. It is *absorbing* if for every vertex $u \notin K$, there exists $v \in K$ such that $(u, v) \in A$ (then we say that v absorbs u). A *kernel* of G is a subset of vertices that is both independent and absorbing.

Escoffier et al. (2019) prove that an equilibrium (*stable* electorate) in a delegation structure is equivalent to a kernel in the acceptability digraph. Stable delegations (which are functions) can therefore be reduced to a graph theoretical concept.[4]

Theorem 7 (Escoffier et al. 2019)**.** *Given a delegation structure $\langle G, \boldsymbol{P}, \rho\rangle$ and a subset $K \subseteq N$ there exists an equilibrium d such that $\texttt{Elct}(d) = K$ if and only if K is a kernel of the acceptability digraph induced from $\langle G, \boldsymbol{P}, \rho\rangle$.*

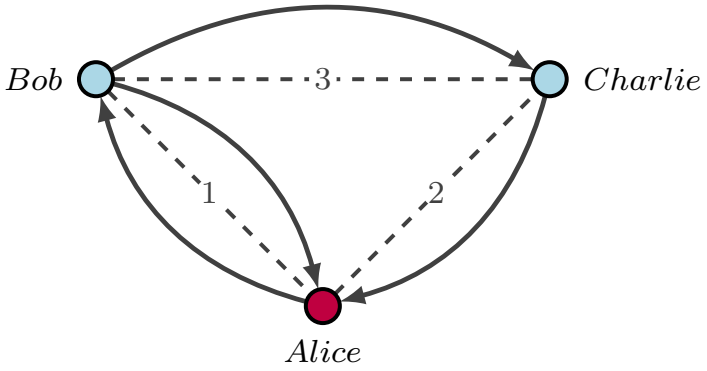
Theorem 7 allows us to treat equilibria in a delegation structure and kernels in the associated acceptability digraph equivalently. When it is clear from context, we will use these notions interchangeably. With the newly gained concepts, let us return to the dinner plans of Alice, Bob and Charlie.

[4]The connection between kernels and game theory is not novel. Von Neumann and Morgenstern proposed kernels as a solution to cooperative games (Boros and Gurvich (2006), Berge (1985)).

Example 5 (continued)**.** Assume that both Bob and Charlie appreciate Alice's vegetarianism, and are happy to let her choose a restaurant. Formally, the scenario can be represented by a delegation structure $\langle G, \boldsymbol{P}, \boldsymbol{\rho} \rangle$ with G the complete graph over $N = \{Alice, Bob, Charlie\}$, a set of restaurants, representing the alternatives $A = \{Veggie, Steak, Pizza\}$, and the following preferences and delegation thresholds.

$$\begin{array}{lll} \succ_A = (V, P, S) & \succ_B = (P, V, S) & \succ_C = (S, V, P) \\ \rho_A = 1 & \rho_B = 3 & \rho_C = 2 \end{array}$$

Based on the Kendall tau distances, the sets of acceptable gurus for each agent are $\texttt{Acc}(\text{Alice}) = \{Bob\}$, $\texttt{Acc}(Bob) = \{Alice, Charlie\}$, and $\texttt{Acc}(Charlie) = \{Alice\}$. The acceptability digraph looks as follows, where the dashed lines depict the Kendall tau distances.



The electorate under the first (unstable) delegation d from Example 5 was $\texttt{Elct}(d) = \{Charlie\}$. Since Alice is not absorbed, $\texttt{Elct}(d)$ is not a kernel, and by Theorem 7 d is not an equilibrium. However, consider $K = \{Alice\}$. K absorbs both Bob and $Charlie$, is independent since it is a singleton, and thus a kernel. It is indeed the only kernel. Therefore, any delegation function d' such that $\texttt{Elct}(d') = K$ is an equilibrium. △

As Escoffier et al. (2020) point out, this leads to the following interesting observation: Given a kernel K of the acceptability digraph G_ρ, one can easily construct an equilibrium d such that $\texttt{Elct}(d) = K$. Every member of K votes directly, and every voter not in K delegates to her most preferred voter in K. Consequently, for any equilibrium d, there is an equilibrium d' with $\texttt{gu}(i, d) = \texttt{gu}(i, d')$ for every voter i, where each voter delegates *directly* to her guru in d'.

We define $[d]$ to be the equivalence class containing all delegation functions d' such that $\texttt{Elct}(d) = \texttt{Elct}(d')$. In the following we are not interested in particular delegation functions, but reason about their equivalence classes. This is justified by the observation above.

Before continuing to explore the connection between acceptability digraphs and the Kendall tau distance, we present a general result concerning the existence of kernels in acceptability digraphs. In Definition 3 we defined how a symmetric distance measure induces a ranking over acceptable gurus. Based on this ranking, together with the delegation thresholds for each voter, Definition 5 defined the delegation acceptability digraph. Here, a kernel corresponds to a stable delegation, which we call an equilibrium. Theorem 8 states that if the social network is complete, and the distance used to induce the ranking over acceptable gurus is symmetric, an equilibrium always exist. This is particularly interesting since the Kendall tau distance is symmetric.

Theorem 8 (Escoffier et al., 2020)**.** *For a delegation structure $\langle G, \boldsymbol{P}, \boldsymbol{\rho} \rangle$, if the acceptable gurus are induced by a symmetric distance measure, and G is the complete graph, then an equilibrium always exists.*

Corollary 8.1. *For a delegation structure $\langle G, \boldsymbol{P}, \boldsymbol{\rho} \rangle$, if the acceptable gurus are induced by the Kendall tau distance, and G is the complete graph, then an equilibrium always exists.*

In this section we investigated how kernels in acceptability digraphs correspond to stable delegations, also called equilibria, in liquid democracy. In Section 3.1.2 we show that under some assumptions the acceptability digraph is isomorphic to the τ-graph introduced in Section 2.4. We will prove that an isomorphism exist which preserves kernels as well as single-peakedness. Due to this bridge between equilibria and the Kendall tau distance, the search for single-peaked equilibria in liquid democracy can therefore be reduced to single-peaked kernels in the τ-graph.

3.1.2 Acceptability Digraphs and τ-Graph: Isomorphism

In the previous chapter we introduced the Kendall tau distance, and associated to it the τ-graph. Recall the definition of the Kendall tau distance between two linear orders over the same set of alternatives, which equals the number of pairs of alternatives the two linear orders disagree on. In the τ-graph vertices are preference orders, and there is an edge between two vertices if their Kendall tau distance is 1. We extended the τ-graph to the d-τ-graph, where an edge between

two vertices u, v exists if $\tau(u, v) \leq d$. We show that under three assumptions, the acceptability digraph induced from a delegation structure $\langle G, \boldsymbol{P}, \boldsymbol{\rho} \rangle$ is isomorphic to the d-τ-graph. This enables us to shift our focus to the d-τ-graph, as kernels are preserved under isomorphisms.

For a delegation structure $\langle G, \boldsymbol{P}, \boldsymbol{\rho} \rangle$ we assume the following three assumptions for the remainder of the chapter unless stated otherwise.

1. Every preference order is submitted exactly once, entailing $\mathcal{D}(\boldsymbol{P}) = \mathcal{L}(A)$,[5]
2. G is the complete graph, i.e. every voter is allowed to delegate to every other voter,
3. $\boldsymbol{\rho}$ is homogeneous, i.e. all voters have the same acceptability threshold.

Let $\langle G, \boldsymbol{P}, \boldsymbol{\rho} \rangle$ be some delegation structure satisfying the assumptions, and $G_\rho = (N, E_\rho)$ the associated acceptability digraph. We will prove that G_ρ is isomorphic to the d-τ-graph $G_\tau = (\mathcal{L}(A), E_\tau)$, where $d = \rho$. In words the acceptability digraph with homogeneous delegation threshold ρ is isomorphic to the ρ-τ-graph.

Before proving the isomorphism, we first show that the delegation acceptability digraph is symmetric, i.e., for all $i, j \in N$, $(i, j) \in E_\rho \Leftrightarrow (j, i) \in E_\rho$. Recall that the edges in the acceptability digraph are based on the sets of acceptable gurus. Symmetry of the acceptability digraph therefore follows, if for any two voters $i, j \in N$, voter i accepts j as a guru, if and only if j accepts i as a guru.

Lemma 9. *If ρ is homogeneous, then for all $i, j \in N$:*

$$j \in \mathtt{Acc}(i) \Leftrightarrow i \in \mathtt{Acc}(j)$$

Proof. Recall the definition of the set of acceptable gurus:

$$\forall j \in N \setminus \{i\}, \quad j \in \mathtt{Acc}(i) \Leftrightarrow dist(i, j) \leq \rho_i$$

It suffices to show that $dist(i, j) \leq \rho_i \Leftrightarrow dist(j, i) \leq \rho_j$. Since ρ is homogeneous, $\rho_i = \rho_j$. Since the Kendall tau distance is symmetric, $\tau(i, j) = \tau(j, i)$. □

[5]The assumption can be generalized to 'at least once'. If there were voters with the same preference, we would need to argue about equivalence classes of voters with the same preferences. This complicates the following proofs, but we are confident they go through. In essence, a kernel in the τ-graph does not correspond to a kernel in the acceptability digraph, but one would need to pick out one single voter from the equivalence class of voters.

Corollary 9.1. *The delegation acceptability digraph is symmetric if ρ is homogeneous.*

With Corrollary 9.1 at hand we proceed to prove the isomorphism. In words, we show that the function mapping each voter to her preference order is an isomorphism between the delegation acceptability digraph and the d-τ-graph, where d is equal to the (homogeneous) delegation threshold ρ. Crucially, the proof makes use of all three assumptions.

Theorem 10. *Given a delegation structure $\langle G, \boldsymbol{P}, \boldsymbol{\rho}\rangle$, the induced delegation acceptability graph $G_\rho = (N, E_\rho)$ is isomorphic to the d-τ-graph $G_\tau = (\mathcal{L}(A), E_\tau)$ with $d = \rho$.*

Proof. We show that

$$\begin{aligned} f : N &\to \mathcal{L}(A) \\ i &\mapsto \succ_i \end{aligned}$$

is a bijective function that satisfies the adjacency condition $(i, j) \in E_\rho$ if and only if $\Leftrightarrow (\succ_i, \succ_j) \in E_\tau$. Bijectivity follows from the assumption that $\mathcal{D}(\boldsymbol{P}) = \mathcal{L}(A)$. As the social network is complete, delegations are not restricted, and therefore only based on the thresholds and distances of voters. Since the threshold is homogeneous, the adjacency condition follows from these equivalences.

$$\begin{aligned} (i, j) \in E_\rho &\Leftrightarrow j \in Acc(i) \\ &\Leftrightarrow dist(i, j) \leq \rho_i \\ &\Leftrightarrow \tau(\succ_i, \succ_j) \leq d \\ &\Leftrightarrow (\succ_i, \succ_j) \in E_\tau \end{aligned}$$

Note that the third equivalence follows from Corollary 9.1 together with the assumption that $d = \rho$. □

We showed in the preliminaries that kernels are preserved under isomorphisms (Observation 7). Generally, isomorphisms do not guarantee the preservation of *single-peaked* kernels. The isomorphism f mapping an agent i to her preference $\succ_i$, however, does preserve single-peaked kernels. Recall that we showed in the preliminaries (Observation 2) that the profile-based definition of single-peakedness corresponds to the set-based definition: A profile is single-peaked if and only if the domain of the profile is single-peaked. Since the isomorphism f maps each voter to her respective preference order, any single-peaked kernel in the acceptability digraph is a single-peaked kernel in the corresponding τ-graph.

Proposition 2. *Single-peaked kernels are preserved under the isomorphism f mapping an agent i to her preference $\succ_i$.*

3.2 Characterization Theorem for Existence of Single-Peaked Equilibria

Having concluded a substantial part of the preliminary work we can finally analyze under which assumptions liquid democracy gives rise to single-peaked (stable) electorates. In particular, we analyze the class of stable, single-peaked electorates where acceptability digraphs are induced by the Kendall tau distance. Recall that we also call stable electorates *equilibria*.

Before we begin the proof of the main theorem, we provide a recap of the previous sections. Starting with a delegation structure $\langle G, \boldsymbol{P}, \boldsymbol{\rho} \rangle$, a ranking over acceptable gurus for each voter is induced through the Kendall tau distance measure, which in turn gives rise to a set of acceptable gurus for each voter. Based on this set, the delegation acceptability digraph is constructed. Escoffier et al. (2019) prove that any kernel in this graph constitutes a stable delegation (equilibrium). We are not merely interested in the existence of equilibria induced by the Kendall tau distance, but the existence of *single-peaked* equilibria. In this chapter we prove a characterization for the existence of single-peaked equilibria, based on the delegation acceptability threshold ρ, and under the three assumptions:

1. Every preference order is submitted exactly once, entailing $\mathcal{D}(\boldsymbol{P}) = \mathcal{L}(A)$,
2. G is the complete graph, i.e. every voter is allowed to delegate to every other voter,
3. $\boldsymbol{\rho}$ is homogeneous, i.e. all voters have the same acceptability threshold.

The main result claims that there is a crucial delegation threshold that is decisive for the existence of single-peaked equilibria. In a delegation structure $\langle G, \boldsymbol{P}, \boldsymbol{\rho} \rangle$ we say that a voter $i \in N$ is *opinionated* if the delegation threshold ρ_i is strictly smaller than $\lfloor \frac{\tau_m}{2} \rfloor$, half of the maximal Kendall tau distance. If $\rho_i \geq \lfloor \frac{\tau_m}{2} \rfloor$, we say that voter i is *open-minded*. On a more intuitive level, the opinionated voter only delegates her vote to voters that have a similar opinion, where similarity is based on the Kendall tau distance. On the other hand, an open-minded voter is willing to delegate to voters with drastically different views. If every voter is open-minded (opinionated) we sometimes speak of

an open-minded (opinionated) society. In words, the main result states that single-peaked equilibria exist if and only if all voters are open-minded.[6]

Theorem 22. *Single-peaked equilibria exist if and only if voters are open-minded.*

This is bad news for liquid democracy as it is highly unlikely that a society in real life is open-minded. Furthermore, we will see that the only single-peaked equilibria that exist are made up of (probably) at most two voters with drastically different preferences. The successful search for single-peaked equilibria is thus not only unlikely if it succeeds liquid democracy has strong polarizing effects.
The proof of the main theorem proceeds in two parts. First, we show that for an open-minded society, single-peaked equilibria always exist (right-to-left direction). Based on insights on the Kendall tau distance we gained in Chapter 2 we can show that any two voters with reversed preferences constitute a single-peaked equilbrium. To prove that the delegation threshold $\lfloor \frac{\tau_m}{2} \rfloor$ is furthermore necessary for the existence of single-peaked equilibria (left-to-right), we will need to utilize some of the technical machinery introduced in the previous chapters. Essentially, we show that for any $\succ$-single-peaked domain, there is always a voter whose distance to the single-peaked domain is $\lfloor \frac{\tau_m}{2} \rfloor$. If that voter has a delegation threshold smaller than $\lfloor \frac{\tau_m}{2} \rfloor$, she will never accept anyone from the $\succ$-single-peaked domain as a guru. Therefore, she will not be absorbed by any $\succ$-single-peaked voter, and no $\succ$-single-peaked, stable electorate can exist.

3.2.1 Open-Minded Societies

Recall Lemma 6 which stated that the distance between any order, and two reversed orders sums up to the maximal distance τ_m. With this lemma at hand Theorem 11 is relatively straightforward. We show that for a single-peaked domain $\mathcal{SP}_{\succ}$, any two voters with reversed preference orders $\succ$ and $\overline{\succ}$ form a kernel. For the special case where $\rho = \tau_m$, independence fails for said kernel. However, any singleton is a kernel, as any voter is willing to delegate to any other voter.

Lemma 11. *If voters are open-minded, then single-peaked equilibria exist.*

[6] During the proofs we will often talk about kernels while we generally use the term equilibria. Recall that in this thesis kernels are the graph theoretical equivalent of the game theoretical notion of equilibria.

Proof. There are two cases.
Case 1: $\rho = \tau_m$. As any voter is willing to delegate to any other voter, any singleton is absorbing. Trivially, a singleton is independent, and therefore a kernel.
Case 2: $\lfloor \frac{\tau_m}{2} \rfloor \leq \rho < \tau_m$. Let $K = \{i, j\}$ for two voters $i, j \in N$ with reversed preference relations $\succ$ and $\overline{\succ}$. In order to prove that K is indeed a kernel, we show that K is independent and absorbing. From Lemma 6 we can derive that any order that is more than $\lfloor \frac{\tau_m}{2} \rfloor$ swaps away from $\succ$ is less than $\lfloor \frac{\tau_m}{2} \rfloor$ swaps away from $\overline{\succ}$. Since $\rho \geq \lfloor \frac{\tau_m}{2} \rfloor$, any order is therefore reachable from the kernel within $\lfloor \frac{\tau_m}{2} \rfloor$ swaps, and K is absorbing. Furthermore, $\tau(\succ, \overline{\succ}) = \tau_m > \lfloor \frac{\tau_m}{2} \rfloor = \rho$, and K is thus independent. Adding any other voter to K would result in loss of independence of K, thus K can only contain two voters with reversed preference relations. Since every kernel is an equilibrium (Theorem 7), K is a single-peaked equilibrium. □

We have therefore found a sufficient condition for the existence of single-peaked equilibria: an open-minded society. Note that we can relax the assumption of homogeneity of the delegation thresholds. Instead of homogeneity, we can simply require that all voters have a threshold larger than $\lfloor \frac{\tau_m}{2} \rfloor$. The proof works analogously to Case 2 from the proof of Theorem 11, and extends the result to heterogeneous thresholds.

Corollary 11.1. *If* $\lfloor \frac{\tau_m}{2} \rfloor \leq \rho_i < \tau_m$ *for all* $i \in N$*, then single-peaked equilibria exist.*

3.2.2 Single-Peaked Kernels in the τ-Graph

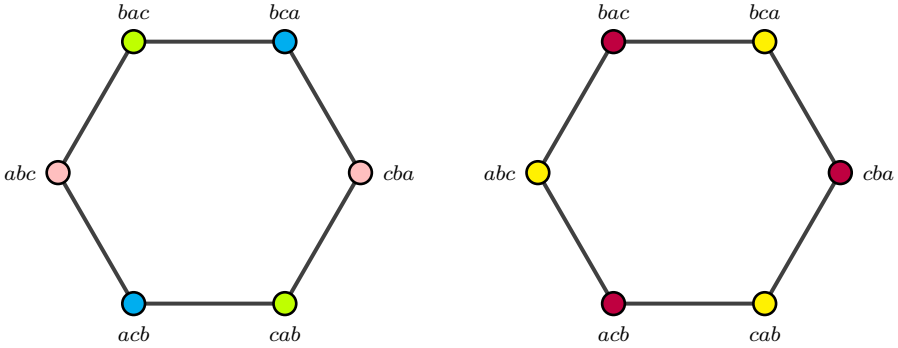
We proceed to show that a (homogeneous) delegation threshold of at least $\lfloor \frac{\tau_m}{2} \rfloor$ is necessary for the existence of single-peaked, stable electorates.

Lemma 21. *If voters are opinionated, no single-peaked equilibria exist.*

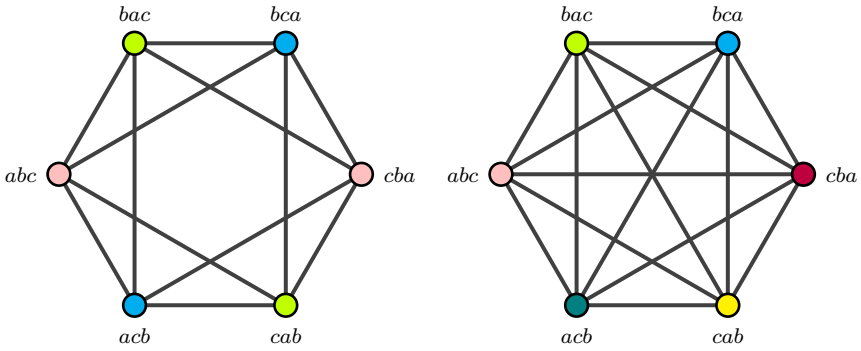
The proof of Lemma 21 is more involved, and we will need to utilize some of the technical machinery introduced in the previous chapters. For one, Algorithm 1 plays an important role. Recall that Algorithm 1 produces a reversion path of an input order. It was only defined on the τ-graph, while our attention shifted to the acceptability digraph. However, we found an isomorphism between the acceptability digraph and the τ-graph which preserves single-peaked kernels (Corollary 2). The proof strategy of Lemma 21 consists of finding an order in the τ-graph of maximal distance to the $\succ$-single-peaked domain. We then show that the maximal distance is $\lfloor \frac{\tau_m}{2} \rfloor$. Thus, in opinionated societies voters with

that order are not willing to delegate to voters associated to $\succ$. To support the reader's intuition for (single-peaked) kernels in the τ-graph, we start with an example.[7]

Example 6. There are five kernels in the τ-graph for three alternatives a, b, c. Recall that there is an edge between two orders in the τ-graph if their Kendall tau distance is equal to 1, and an edge in the d-τ-graph if the Kendall tau distance equals d. Vertices with the same color form a kernel. However, only the three kernels in the left graph are single-peaked. Note that the single-peaked kernels consist of two reversed orders.



In the 2-τ-graph, there are three kernels, all of which are single-peaked (left). In the 3-τ-graph, each singleton is a single-peaked kernel (right).

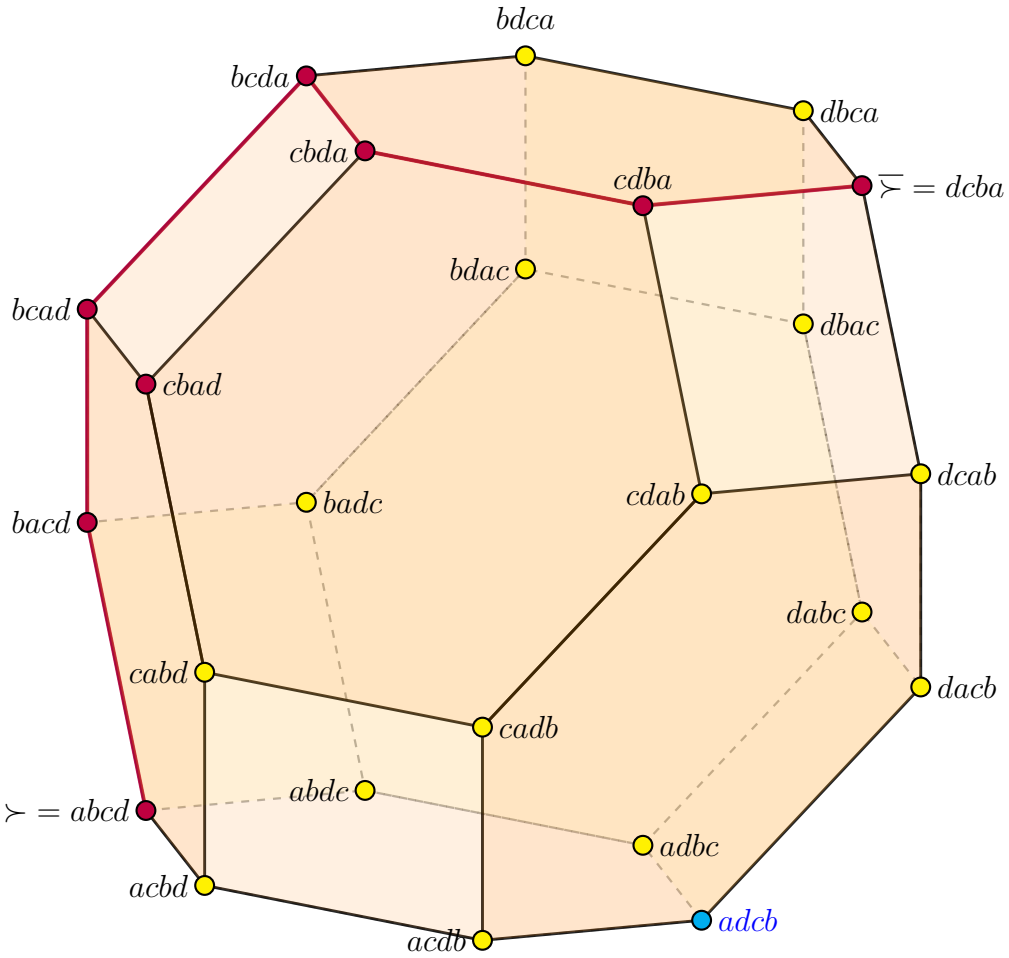


△

[7]Recall that we showed that kernels in the d-τ-graph are equivalent to kernels in the acceptability digraph which are equivalent to equilibria in a delegation structure. Kernels and equilibria are thus equivalent notions. When talking about delegation structures we will use the notion of 'equilibria' while we make use of 'kernels' in more graph theoretical settings.

The τ-graph for three alternatives is a special case. Recall from Theorem 11 that kernels exist if the delegation threshold is larger than $\lfloor \frac{\tau_m}{2} \rfloor$. For three alternatives, this lower bound equals 1, so kernels exist starting at the lowest threshold that allows for any delegation to happen (if $\rho = 0$, trivially, there are no delegations). The case for four alternatives is more elucidating.

Example 7. The orders (vertices) that are single-peaked with respect to the order (a, b, c, d) are colored in red. It covers only a relatively small set of vertices (8), not evenly distributed throughout the graph, but rather spanning over one side of the graph. The reversion path (edges) of Algorithm 1 with input (a, b, c, d) is additionally added in purple.



Recall that the d-τ-graph simply adds edges according to the Kendall tau distance d between two orders. As the graph becomes unreadable already for d=2,

we ask the interested reader to manually check for the shortest paths for d-τ-graphs. It can then be verified that the set $K = \{(a, b, c, d), (d, c, b, a)\}$ forms a single-peaked kernel in the 3-τ-graph. Recall our aim to prove that for any threshold strictly smaller than $\frac{\tau_m}{2}$ (here $\frac{\tau_m}{2} = 3$), no single-peaked kernels exists. It suffices to find an order which is at least 3 swaps away from the $\succ$-single-peaked domain. While there are multiple such orders, we can construct the order (a, d, c, b) (colored in blue) with the help of Algorithm 1. We leave it to the reader to verify that indeed the distance of (a, d, c, b) to any $\succ$-single-peaked order is at least three which can be checked by counting the edges in the τ-graph. Crucially, the order (a, d, c, b) is the reverse of the order (b, c, d, a) which lies on the *reversion path* of Algorithm 1 such that it is *equidistant* to (a, b, c, d) and (d, c, b, a).

Recall that the d-τ-graph is isomorphic to the delegation acceptability graph of a delegation structure $\langle G, \boldsymbol{P}, \boldsymbol{\rho}\rangle$ with $\rho = d$. If $\rho < 3$, in words voters are opionated, the voter associated to the constructed order (a, d, c, b) is then not willing to delegate to any voter single-peaked with respect to (a, b, c, d) since the closest order is at least 3 swaps away. △

In fact, the procedure sketched in Example 7 generalizes to all strict linear orders of finite length. For any order $\succ$ we pick the order that is of distance $\lceil \frac{\tau_m}{2} \rceil$ to $\succ$ on the reversion path of Algorithm 1, reverse it, and obtain an order with distance of $\lfloor \frac{\tau_m}{2} \rfloor$ to the $\succ$-single-peaked domain. Note that this order is equidistant to $\succ$ and $\overline{\succ}$ if the path is of even length. Since the distance to the $\succ$-single-peaked domain is at least $\lfloor \frac{\tau_m}{2} \rfloor$, this means for opinionated societies—delegation thresholds are smaller than $\lfloor \frac{\tau_m}{2} \rfloor$—that for every single-peaked domain, there is a voter *not* willing to delegate to any member of that domain. Consequently, no single-peaked kernels exist. In the following section this construction is analyzed in more detail, and generalized.

3.2.3 The π-Algorithm

The construction of the orders described in the previous section can be formalized by the π-Algorithm which essentially proceeds in two steps. First, it partially inverts the input order $\succ$ following the inversion routine of Algorithm 1, and then outputs the inverse of the partially inverted order. The output $\pi(\succ) = \succ^{\pi}$ is not single-peaked with respect to its input $\succ$, and the distance between input and output is $\lfloor \frac{\tau_m}{2} \rfloor$. The π-Algorithm will prove helpful as we will see in Section 3.2.4 that the distance not only from $\pi(\succ)$ to $\succ$ is $\lfloor \frac{\tau_m}{2} \rfloor$ but also from $\pi(\succ)$ to the whole $\succ$-single-peaked domain.

Recall that Algorithm 1 induces an inversion path P for an input order $\succ$. The path P is of length τ_m which can be even or odd. If P is of even length, there is an order which is equidistant from $\succ$ and $\overline{\succ}$. If it is odd, there is an order which is $\lceil \frac{\tau_m}{2} \rceil$ swaps from the input $\succ$, and $\lfloor \frac{\tau_m}{2} \rfloor$ swaps from $\overline{\succ}$. Let $\succ^\star \in P$ be the order such that $\tau(\succ, \succ^\star) = \lceil \frac{\tau_m}{2} \rceil$. Intuitively, this order is in the middle of the path P.[8]

Algorithm 2 π-Algorithm

Input: Strict linear order $\succ = (a_1, ..., a_m) \in \mathcal{L}(A)$
Output: $\pi(\succ) = \succ^\pi$

1: $P \quad \leftarrow$ Algorithm 1 with input $\succ$
2: $\succ^\star \quad \leftarrow (\lceil \frac{\tau_m}{2} \rceil + 1)$-th element of P
3: **return** $\succ^\pi = \overline{\succ^\star}$

The π-Algorithm makes use of Algorithm 1 as a subroutine, and picks out the order $\succ^\star$ from the path P. Recall that all elements of P are $\succ$-single-peaked, and therefore that $\succ^\star$ is $\succ$-single-peaked. However, the inverse $\overline{\succ^\star}$ fulfills two requirements:

1. $\overline{\succ^\star}$ is not $\succ$-single-peaked, and
2. its distance to $\succ$ is $\lfloor \frac{\tau_m}{2} \rfloor$.

The output of the π-Algorithm is then $\pi(\succ) = \overline{\succ^\star}$.

Analysis and Correctness

To understand the structure of the order $\pi(\succ)$, it is useful to take a closer look at the order $\succ^\star$. Recall the routine of Algorithm 1 with output P. The maximal element of the order $\succ = (a_1, ..., a_m)$ is repeatedly swapped with its lower adjacent neighbor until it is swapped with a_m. This procedure is repeated for all a_i with $i \in \{1, ..., m-1\}$, until finally a_{m-1} and a_m are swapped. We saw that after an alternative a_i swapped positions with a_m, the resulting order $\succ'$ can be split into two concatenated suborders, where the first agrees with $\succ$ on the ranking of the alternatives, and the second with $\overline{\succ}$.

$$\succ' = (a_{i+1}, ..., a_m) + (a_i, ..., a_1)$$

[8] More precisely, an exact middle only exists if the path is of even length. If it is uneven, the order is picked that is one swap farther away from $\succ$ than from $\overline{\succ}$.

If we now consider the order $\succ^\star$ from Algorithm 2, we notice that it is not guaranteed that $\succ^\star$ can be split into two suborders like above. As the following example shows, there are two cases.

Example 8. Let $\succ_1 = (a_1, ..., a_4)$, and $\succ_2 = (a_1, ..., a_5)$ be two orders of length four and five respectively. The inversion paths induced by Algorithm 1 is as follows, where the alternative that has been swapped with its lower adjacent neighbor is highlighted in red.

$$P_1 = (\,(a_1, a_2, a_3, a_4), (a_2, \textcolor{red}{a_1}, a_3, a_4), (a_2, a_3, \textcolor{red}{a_1}, a_4), \overbrace{(a_2, a_3, a_4, \textcolor{red}{a_1})}^{\succ_1^\star}, ...,$$
$$(a_4, \textcolor{red}{a_3}, a_2, a_1)\,)$$

$$P_2 = ((\,a_1, a_2, a_3, a_4, a_5), (a_2, \textcolor{red}{a_1}, a_3, a_4, a_5), (a_2, a_3, \textcolor{red}{a_1}, a_4, a_5), (a_2, a_3, a_4, \textcolor{red}{a_1}, a_5),$$
$$(a_2, a_3, a_4, a_5, \textcolor{red}{a_1}), \underbrace{(a_3, \textcolor{red}{a_2}, a_4, a_5, a_1)}_{\succ_2^\star}, ..., (a_5, \textcolor{red}{a_4}, a_3, a_2, a_1)\,)$$

The order $\succ_1^\star$ can indeed be split into two concatenated suborders as described above. However, in $\succ_2^\star$ the alternative a_2 has been swapped with its lower adjacent neighbor, but not yet with a_5. The alternatives in the prefix of $\succ_2^\star$ are not in ascending order with respect to their index. △

Let $a^\star$ be the alternative which has been swapped with its lower adjacent neighbors at least once, but not yet with a_m if it exists. While we do not know its exact position, we can divide the order $\succ^\star$ into two concatenated suborders as follows if $a^\star$ exists.

$$\begin{aligned}\succ^\star &= (a_{i+2}, ..., a^\star, ..., a_m, a_i, ..., a_1) \\ &= \underbrace{(a_{i+2}, ..., a^\star, ..., a_m)}_{\succ_\alpha} + \underbrace{(a_{i-1}, ..., a_1)}_{\succ_\beta}\end{aligned}$$

The sequence $\succ_\beta$ contains the alternatives that completed the inversion routine, and agrees with $\overline{\succ}$ on the ranking of the alternatives. On the other hand, $\succ_\alpha$ does not agree with $\succ$ on the rankings. Note that $a^\star = a_{i+1}$, and since $i < i{+}1$ the alternatives are not ranked ascendingly with respect to their indexes. However, if we delete the culprit $a^\star$, the sequence $\succ_\alpha \setminus (a^\star)$ is a suborder of $\succ$ once again. The analysis of $\succ^\star$ can be described as follows. A number of alternatives, say i many, have swapped their position with a_m and constitute the suborder $\succ_\beta$. Possibly, there is an alternative $a^\star = a_{i+1}$ for which a number of adjacent swaps have been performed, however $a^\star$ has not swapped positions

with a_m. The prefix of $\succ^\star$ is therefore an order $\succ_\alpha$, such that $\succ_\alpha \setminus a^\star$ is a suborder of $\succ$.

This gives us valuable insights into the structure of $\pi(\succ)$, as it is simply the inversion of $\succ^\star$.

$$\begin{aligned} \succ^\star &= \overbrace{(a_{i+2}, ..., a^\star, ..., a_m)}^{\succ_\alpha} + \overbrace{(a_i, ..., a_1)}^{\succ_\beta} \\ \pi(\succ) &= \underbrace{(a_1, ..., a_i)}_{\overline{\succ_\beta}} + \underbrace{(a_m, ..., a^\star, ..., a_{i+2})}_{\overline{\succ_\alpha}} \end{aligned} \tag{3.1}$$

The orders $\pi(\succ)$ and $\succ$ therefore share a prefix, while the suffix of $\pi(\succ)$ agrees with $\overline{\succ}$ on the rankings of alternatives $\{a_{i+2}, ..., a_m\} \setminus \{a^\star\}$. If $a^\star$ does not exist the structure is simple, as $\pi(\succ)$ and $\overline{\succ}$ share the suffix $\{a_{i+1}, ..., a_m\}$.

Correctness

For the correctness of π we need to show that (i) the distance of $\pi(\succ)$ to $\succ$ is $\lfloor \frac{\tau_m}{2} \rfloor$, and (ii) $\pi(\succ)$ is not $\succ$-single-peaked. Correctness thus follows from the following two lemmas.

Lemma 12. *The distance between $\succ$ and $\succ^\pi$ is $\lfloor \frac{\tau_m}{2} \rfloor$.*

Proof. Recall Lemma 6: The distance between an order and two reversed orders sums up to τ_m. For our case, let the two reversed orders be $\succ^\pi$ and $\succ^\star$, and the third order $\succ$. Since we know by construction of $\succ^\star$ that $\tau(\succ, \succ^\star) = \lceil \frac{\tau_m}{2} \rceil$, we get that $\tau(\succ, \succ^\pi) = \lfloor \frac{\tau_m}{2} \rfloor$. □

Lemma 13. *The order $\pi(\succ)$ is not $\succ$-single-peaked.*

Proof. Since any single-peaked domain contains at most two reversed orders (Observation 3), $\pi(\succ)$ cannot be $\succ$-single-peaked. □

Correctness of the π-Algorithm follows from Lemmas 11 and 12, while termination is trivial, since the subroutine Algorithm 1 terminates.

Theorem 14. *Algorithm 2 is correct and terminates.*

In summary, the π-Algorithm generalizes the construction of an order $\pi(\succ)$ which is not $\succ$-single-peaked, and $\lfloor \frac{\tau_m}{2} \rfloor$ swaps away from its input $\succ$. In the following section we show that not only is this the distance to $\succ$ but to the whole $\succ$-single-peaked domain $\mathcal{SP}_\succ$.

3.2.4 Opinionated Societies

Let us take a step back, and look at the big picture again. Our aim is to find a delegation acceptability threshold ρ, such that any values smaller than ρ guarantee that single-peaked, stable delegations do not exist. We will show that this is the case if voters are opinionated, i.e. $\rho < \frac{\tau_m}{2}$.

Theorem 21. *If voters are opinionated, no single-peaked equilibria exist.*

We have seen that these types of delegations correspond to single-peaked kernels in the acceptability digraph, which in turn is isomorphic to the τ-graph under assumptions of a complete social network, a complete domain, and homogeneous delegation thresholds. We are therefore searching for single-peaked kernels in the τ-graph. The proof of Lemma 21 is based on two steps. First, we show that the order $\succ^\pi$ constructed by Algorithm 2 is $\lfloor \frac{\tau_m}{2} \rfloor$ swaps away from the $\succ$-single-peaked domain $\mathcal{SP}_\succ$. In a second step we conclude, that no single-peaked kernel can absorb voters with order $\succ^\pi$ if the delegation acceptability threshold ρ is strictly smaller than $\lfloor \frac{\tau_m}{2} \rfloor$. We will prove the former in Lemma 20, and then conclude with the main theorem of this section.

Lemma 20 claims that every $\succ$-single-peaked order is at least $\lfloor \frac{\tau_m}{2} \rfloor$ swaps away from $\pi(\succ)$.

Lemma 20. *If* $\succ' \in \mathcal{SP}_\succ$, *then* $\tau(\succ', \succ^\pi) \geq \lfloor \frac{\tau_m}{2} \rfloor$

We first provide an outline of the proof, and argue that the proof can be split into five auxiliary lemmas. Unless stated otherwise, let $\succ = (a_1, ...a_m)$ for the remainder of this section. Recall that we can partition any single-peaked domain $\mathcal{SP}_\succ$ into disjunct subdomains, according to the peaks a_p (Observation 4). In fact, each of these subdomains is non-empty, as we showed in Corollary 4.1. Furthermore, we have shown in the analysis of Algorithm 2 (Equation 3.1), that $\pi(\succ)$ is divisible into two subsequences $\overline{\succ_\alpha}$, and $\overline{\succ_\beta}$. These insights are the cornerstones of the proof. Let a_i be the last alternative that has been completely inversed, the alternative that has possibly undergone a partial inversion is therefore $a^\star = a_{i+1}$, and the minimal element of $\pi(\succ)$ is thus a_{i+2}.

$$\mathcal{SP}_\succ = \mathcal{SP}_\succ^{a_1} \cup ... \cup \mathcal{SP}_\succ^{a_m}$$

$$\pi(\succ) = \underbrace{(a_1, ..., a_i}_{\overline{\succ_\beta}}, \underbrace{a_m, ..., a^\star, ..., a_{i+2})}_{\overline{\succ_\alpha}}$$

We show that the distance between $\pi(\succ)$ and any subdomain of $\mathcal{SP}_\succ$ with peaks a_2 to a_m is greater than the distance between $\pi(\succ)$ and $\mathcal{SP}_\succ^{a_1} = \{\succ\}$.

Since we know the latter distance to be $\lfloor \frac{\tau_m}{2} \rfloor$ the desired result follows. In order to prove that the distance from $\pi(\succ)$ to each subdomain is larger than $\lfloor \frac{\tau_m}{2} \rfloor$, we group the subdomains based on which subsequence $\overline{\succ_\alpha}$ or $\overline{\succ_\beta}$ the maximal element is part of. The division of $\pi(\succ)$ into subsequences enables us to reconstruct the distance to any subdomain in terms of (partial) inversions of these subsequences. Formally, the proof is divided into five claims, each depending on which alternative is the peak, and in which subsequence of $\succ^\pi$ the peak appears.

$$\mathcal{SP}_\succ = \underbrace{\mathcal{SP}_\succ^{a_1}}_{\text{Lemma 15}} \cup \underbrace{\mathcal{SP}_\succ^{a_2} \cup ... \cup \mathcal{SP}_\succ^{a_i}}_{\text{Lemma 16}} \cup \underbrace{\mathcal{SP}_\succ^{a^\star}}_{\text{Lemma 17}} \cup \underbrace{\mathcal{SP}_\succ^{a_{i+2}} ... \cup \mathcal{SP}_\succ^{a_{m-1}}}_{\text{Lemma 19}} \cup \underbrace{\mathcal{SP}_\succ^{a_m}}_{\text{Lemma 18}}$$

We remind the reader of three propositions from the Preliminaries (Section 2). Firstly, the inversion of an order of length m requires $\binom{m}{2}$ swaps. Secondly, a path through the τ-graph is a shortest path from $\succ$ to $\succ'$ if only pairs of alternatives are swapped on which the two orders disagree. And finally, a shortest path through the τ-graph corresponds to the minimal number of swaps needed to make two orders agree on all pairs of alternatives, which in turn equals the Kendall Tau distance.

In the first auxiliary lemma, the Kendall tau distance between $\pi(\succ)$ and $\mathcal{SP}_\succ^{a_1}$ is calculated. In words, the distance equals the inversion of an order of length $m-i-1$, added to the number of swaps needed to reach agreement between $a^\star$ and all other alternatives, which we denote by $R \in \mathbb{N}$.

Lemma 15. $\tau\left(\succ^\pi, \mathcal{SP}_\succ^{a_1}\right) = \binom{m-i-1}{2} + R, \quad$ *with* $R \in \mathbb{N}$

Proof. Note that the only order with peak a_1 that is is single-peaked with respect to $\succ$ is $\succ$ itself.

$$\succ = \overbrace{(a_1, ..., a_i}^{\overline{\succ_\beta}}, a^\star, \overbrace{a_{i+2}, ..., a_m)}^{\succ_\alpha \setminus a^\star}$$
$$\pi(\succ) = \underbrace{(a_1, ..., a_i}_{\overline{\succ_\beta}}, \underbrace{a_m, ..., a^\star, ..., a_{i+2})}_{\overline{\succ_\alpha}}$$

We describe the number of swaps needed to turn $\pi(\succ)$ into $\succ$. The two orders share the prefix $\overline{\succ_\beta}$, therefore no swaps are required to reach agreement on the prefix. Next, consider the alternative $a^\star$. As $a^\star = a_{i+1}$, it needs to be turned into the lower adjacent neighbor of a_i. Let $R \in \mathbb{N}$ be the minimal number of swaps needed for this. The intermediary order $\pi(\succ)'$ is then

$$\pi(\succ)' = \underbrace{(a_1, ..., a_i}_{\overline{\succ_\beta}}, a^\star, \underbrace{a_m, ..., ..., a_{i+2})}_{\overline{\succ_\alpha} \setminus a^\star}$$

The suffix $(a_m, ..., a_{i+2})$ of $\pi(\succ)'$ is the inverse of the suffix of $\succ$. The length of the suffix is $m-i-1$, and thus $\binom{m-i-1}{2}$ swaps are needed to reach agreement between $\pi(\succ)'$ and $\succ$. It is clear that the paths from $\pi(\succ)$ to $\pi(\succ)'$ and from $\pi(\succ)'$ to $\succ$ are shortest paths. Generally, the concatenation of two shortest paths is not necessarily a shortest path. However, note that in the concatenation of the two paths, $a^\star$ is only swapped with alternatives the two orders $\succ$ and $\pi(\succ)$ disagree on. The same holds for the inversion steps. Thus, the number of swaps calculated is minimal, and we conclude

$$\tau\left(\succ^\pi, \succ\right) = \binom{m-i-1}{2} + R. \qquad \square$$

The second auxiliary lemma argues that the distance of $\succ^\pi$ to any member $\succ'$ of subdomains of $\mathcal{SP}_\succ$ with maximal elements $a_p \in \{a_2, ..., a_i\}$ is strictly smaller than the distance of $\succ^\pi$ to $\succ$. In the proof of Lemma 15, we calculated the distance between $\pi(\succ)$ and $\mathcal{SP}_\succ^{a_1} = \{\succ\}$. For Lemma 16 the distance to a subdomain of $\mathcal{SP}_\succ$ is calculated which is not a singleton. The proof therefore requires a slightly different approach. First, we calculate the distance between $\pi(\succ)$ and a designated order $\succ'$, and in a second step prove that any other order with the same peak as $\succ'$ requires more swaps. The minimal distance between $\pi(\succ)$ and $\mathcal{SP}_\succ^{a_p}$ is therefore $\tau(\succ^\pi, \succ')$ which we prove to be strictly larger than $\tau(\succ^\pi, \succ)$.

Lemma 16. $\tau\left(\succ^\pi, \mathcal{SP}_\succ^{a_p}\right) > \tau\left(\succ^\pi, \succ\right), \quad$ *for* $a_p \in \{a_2, ..., a_i\}$

Proof. From Theorem 1 we know that any order $\succ'$ with peak a_p is $\succ$-single-peaked if and only if it can be divided into two suborders as follows.

$$a_p \succ' a_{p-1} \succ' ... \succ' a_1 \text{ and } a_{p+1} \succ' a_{p+2} \succ' ... \succ' a_m \tag{3.2}$$

Consider the order $\succ'$, which is a concatenation of the two suborders described in 3.2, and compare it to $\pi(\succ)$.

$$\begin{aligned} \succ' &= (a_p, ..., a_1, a_{p+1}, ..., a_m) \\ \pi(\succ) &= \underbrace{(a_1, ...a_p, ..., a_i}_{\overline{\succ_\beta}}, \underbrace{a_m, ..., a^\star, ..., a_{i+2})}_{\overline{\succ_\alpha}} \end{aligned}$$

Recall that $a_p \in \{a_2, ..., a_i\}$. In order to reach agreement on the prefix between $\pi(\succ)$ and $\succ'$, the first p alternatives of $\pi(\succ)$ need to be reversed in $\binom{p}{2}$ swaps. In order to reach agreement on the suffix, exactly the same steps as in Lemma 15 need to be performed. First, move $a^\star$ into the correct position in R swaps,

and then inverse the remaining suborder in $\binom{m-i-1}{2}$ swaps. Since the set of alternatives swapped in the inversion of $(a_1, ..., a_p)$ is disjunct to the set of alternatives swapped to reach agreement on the suffix $\overline{\succ_\alpha}$, the concatenation of the two paths is a shortest path. We therefore get,

$$\tau(\succ^\pi, \succ') = \underbrace{\binom{m-i-1}{2} + R}_{\tau(\succ^\pi, \succ)} + \binom{p}{2} > \tau(\succ^\pi, \succ)$$

We proceed to show that for any other $\succ$-single-peaked order with peak a_p, the distance to $\pi(\succ)$ is larger. In order to maintain the condition (3.2) for $\succ$-single-peakedness, the two subsequences can be 'zipped together'. There are multiple options to do this, the easiest two are to move a_1 into the order $a_p \succ' a_{p+1} \succ' ... \succ' a_m$, or equivalently, to move a_{p+1} into the order $a_p \succ' a_{p-1} \succ^{a_p} ... \succ' a_1$.

$$\begin{aligned} \succ' &= (a_p, a_{p-1}, ..., a_1, a_{p+1}, ..., a_m) \\ \succ'' &= (a_p, a_{p-1}, ..., a_{p+1}, a_1, ..., a_m) \end{aligned}$$

Compared to $\succ'$, $\pi(\succ)$ additionally disagrees with $\succ''$ on the pair (a_1, a_{p+1}), while the rest of disagreements stays the same. The crucial point is that the more one 'zips' the two suborders into each other, the more additional disagreements arise. Thus, $\succ'$ is the closest order to $\pi(\succ)$ from the subdomain $\mathcal{SP}^{a_p}_{\succ}$. □

Next, for $\succ$-single-peaked orders with maximal element $a^\star$, we proof that the distance to $\succ^\pi$ is strictly greater than $\tau(\succ^\pi, \succ)$.

Lemma 17. $\tau\left(\succ^\pi, \mathcal{SP}^{a^\star}_{\succ}\right) > \tau(\succ^\pi, \succ)$

Proof. Compare the orders $\succ'$ with peak $a^\star$ and $\pi(\succ)$.

$$\begin{aligned} \succ' &= (a^\star, a_i, ..., a_1, a_{i+2}, ..., a_m) \\ \pi(\succ) &= \underbrace{(a_1, ..., a_i}_{\overline{\succ_\beta}}, \underbrace{a_m, ..., a^\star, ..., a_{i+2})}_{\overline{\succ_\alpha}} \end{aligned}$$

Again, we consider the number of swaps required to turn $\pi(\succ)$ into $\succ'$. In order to make $a^\star$ the peak, it needs to be moved to the top position in $R' > R > 0$ swaps. Then $\overline{\succ_\beta}$ needs to be reversed in $\binom{i}{2}$ swaps, to guarantee $a^\star \succ' a_i \succ' ... \succ' a_1$. Furthermore, $\overline{\succ_\alpha}$ needs to be reversed in $\binom{m-i-1}{2}$ swaps, to guarantee $a_{i+2} \succ' a_{i+3} \succ' ... \succ' a_m$. An analogous argument to the proof for Lemma 16

shows that for any other $\succ$-single-peaked order with maximal element $a^\star$ more swaps are required. Thus,

$$\tau\left(\succ, \mathcal{SP}_{\succ}^{a^\star}\right) = \underbrace{\binom{m-i-1}{2} + R'}_{>\tau(\succ^\pi,\succ)} + \binom{i}{2} > \tau(\succ^\pi, \succ) \qquad \square$$

In Lemma 18 we will prove that the distance between $\succ^\pi$ and $\overline{\succ}$ is the partial inversion of $\succ^\pi$ plus some positive rest R''. The partial inversion is performed for the alternatives a_1 to a_i, as they need to be inversed and moved to the bottom of the order in order to reach agreement with $\overline{\succ}$.

Lemma 18. *$\tau\left(\succ^\pi, \overline{\succ}\right) = \sum_{h=1}^{i}(m-h) + R''$, with $R'' \in \mathbb{N}$*

Proof. As $\overline{\succ}$ is the only $\succ$-single-peaked order with maximal element a_m, compare $\pi(\succ)$ with $\overline{\succ}$.

$$\begin{aligned}\overline{\succ} &= \overbrace{(a_m, ..., a_{i+2})}^{\succ_\alpha \setminus a^\star}, a^\star, \overbrace{a_i, ..., a_1)}^{\succ_\beta} \\ \pi(\succ) &= \underbrace{(a_1, ..., a_i)}_{\overline{\succ_\beta}}, \underbrace{a_m, ..., a^\star, ..., a_{i+2})}_{\overline{\succ_\alpha}}\end{aligned}$$

First, make $a^\star$ the minimal element to receive the intermediary order

$$\pi(\succ)' = (a_1, ..., a_i, a_m, ..., a^\star)$$

in R'' steps. The orders $\pi(\succ)'$ and $\overline{\succ}$ agree on the suffix $(a_m, ..., a^\star)$. In order to reach agreement on all pairs, we move, one after the other, the top elements to the bottom of the order. First, make a_1 the minimal element of the order in $m-1$ steps, then move a_2 to the second last spot in $m-2$ swaps, until a_i has been moved to i-th last place in $m-i$ swaps. Note that this is essentially the reversion of the π-Algorithm. This takes in total $\sum_{j=1}^{i}(m-j) + R''$ swaps. We leave it to the readers to convince themselves that only pairs are swapped that the orders disagree on. $\square$

Finally, we prove that it requires some additional swaps for $\pi(\succ)$ to reach any $\succ$-single-peaked orders with maximal elements $a_p \in \{a_{i+2}, ..., a_{m-1}\}$.

Lemma 19. *$\tau\left(\succ^\pi, \mathcal{SP}_{\succ}^{a_p}\right) > \tau\left(\succ^\pi, \overline{\succ}\right), \quad$ for $a_p \in \{a_{i+2}, ..., a_{m-1}\}$*

Proof. Analogously to the proof of Lemma 18, first move $a^\star$ to the bottom to receive the order $(a_1, ..., a_j, a_m, ..., a^\star)$ in R'' swaps. Again, just like in Lemma 18, move the first i alternatives to the bottom in $\sum_{j=1}^{i}(m-j)$ swaps, to arrive at the order

$$\overline{\succ} = (a_m, ..., a_{p+1}, a_p, ..., a^\star, a_i, ...a_1).$$

Since we require a_p to be the peak, the sequence $(a_m, ..., a_{p+1})$ additionally needs to be reversed and moved behind a_p. Let R''' be the number of swaps necessary for this operation which, notice, is always strictly larger than 0. Therefore for $a_p \in \{a_{i+2}, ..., a_{m-1}\}$,

$$\tau\left(\succ, \mathcal{SP}_{\succ}^{a_p}\right) = \sum_{h=1}^{j}(m-h)+R''+R''' > \tau(\succ^\pi, \overline{\succ}) \qquad \square$$

We have seen in Lemmas 16 and 17 that the distance between any $\succ$-single-peaked orders with peaks $a_p \in \{a_1, ..., a_i, a^\star\}$ and $\pi(\succ)$ is strictly smaller than the distance between $\pi(\succ)$ and $\succ$. Furthermore, Lemma 19 proved that for $\succ$-single-peaked orders with peaks $a_p \in \{a_{i+2}, ..., a_{m-1}\}$, the distance to $\pi(\succ)$ is strictly smaller than the distance to $\overline{\succ}$. We furthermore know by the construction of $\pi(\succ)$,

$$\tau(\succ^\pi, \succ) = \left\lfloor \frac{\tau_m}{2} \right\rfloor \leq \left\lceil \frac{\tau_m}{2} \right\rceil = \tau(\succ^\pi, \overline{\succ}).$$

Therefore, the closest order to $\pi(\succ)$ from the $\succ$-single-peaked domain is $\succ$ itself, and we conclude

Lemma 20. *If $\succ' \in \mathcal{SP}_\succ$, then $\tau(\succ', \succ^\pi) \geq \lfloor \frac{\tau_m}{2} \rfloor$.*

We are now finally ready to prove the main theorem of this chapter. Intuitively, if $\rho < \lfloor \frac{\tau_m}{2} \rfloor$, no agent is willing to delegate to anyone who is farther than $\lfloor \frac{\tau_m}{2} \rfloor$ swaps away from her. The proof is based on the observation that for any single-peaked domain $\mathcal{SP}_\succ$ there is an order $\pi(\succ)$ which is more than $\rho < \lfloor \frac{\tau_m}{2} \rfloor$ swaps away from $\mathcal{SP}_\succ$ (Lemma 20). Therefore, the agent associated with that order will not delegate to anyone with a $\succ$-single-peaked order.

Lemma 21. *If voters are opinionated, no single-peaked equilibria exist.*

Proof. Take a delegation structure $\langle G, \boldsymbol{P}, \boldsymbol{\rho} \rangle$, and consider $\mathcal{SP}_\succ$ for some order $\succ \in \mathcal{L}(A)$. Construct $\pi(\succ)$ from $\succ$ according to Algorithm 2. Recall our assumption that every preference order is submitted, so let i be the agent submitting the preference $\pi(\succ)$. Furthermore, since ρ is homogeneous, and we assume $\rho < \lfloor \frac{\tau_m}{2} \rfloor$, we have that $\rho_i < \lfloor \frac{\tau_m}{2} \rfloor$. By contraposition of Lemma 20, it

follows that the distance between $\pi(\succ)$ and any $\succ$-single-peaked order is greater than $\lfloor \frac{\tau_m}{2} \rfloor$, and therefore greater than the delegation acceptability threshold for agent i. Thus, any kernel K, containing only members with $\succ$-single-peaked preference orders, will not absorb agent i, since i is not willing to delegate to any member of K. In turn, any $K \cup \{j\}$ such that i is willing to delegate to j, i.e. $\tau(\succ_i, \succ_j) \leq \rho_i$, is indeed absorbing, but not single-peaked by Lemma 20. □

From the necessary condition for the existence of single-peaked equilibria (Lemma 21) and the sufficient condition (Theorem 11), we derive the main result of this chapter.

Theorem 22. *Single-peaked equilibria exist if and only if voters are open-minded.*

To summarize this section, we have investigated the existence of single-peaked equilibria under the three assumptions that (1) every order is submitted, (2) every voter can delegate to every other voter, and (3) all voters have the same acceptability threshold. We found that the society needs to be open-minded for single-peaked kernels to exist.

3.3 Number and Structure of Single-Peaked Equilibria

After the exploration of the characterization theorem for the existence of single-peaked kernels, we conclude this chapter with an analysis of the *number* of single-peaked kernels, and their *structure*, in particular their cardinality. At first sight, the number of single-peaked equilibria does not seem to be of interest. In the light of the following result proved by Escoffier et al. (2020) the picture changes. In this thesis we are mainly concerned with the *existence* of single-peaked equilibria, and do not inspect whether equilibria are actually reached in some delegation dynamics. A brief remark about the latter shows that extending our interest from the mere existence to the number of single-peaked equilibria is valuable. Escoffier et al. (2020) analyze whether a "delegation process (necessarily) converges", and come to the conclusion that this is the case whenever a delegation structure is distance-based. Since some equilibrium is reached, we might be able to estimate how likely it is that this equilibrium is furthermore single-peaked by counting both single-peaked equilibria and non-single-peaked equilibria.

Theorem 23 (Escoffier et al., 2020)**.** *If delegation structures are distance-based, then a best response dynamic (BRD) always converges (in three rounds).*

We will omit diving into the details of convergence of (best) response dynamics in detail, and merely briefly sketch the underlying intuition. In a best response dynamic voters iteratively vote for their most preferred guru. A round is finished, after each voter submitted their vote. All delegations are made public, and everyone is given the opportunity to retract their delegation and change their guru. Theorem 23 establishes that after three rounds a best response dynamic converges to an equilibrium. This equilibrium is not necessarily single-peaked, but given the number of single-peaked equilibria and total equilibria, we gain insights into the likelihood that a best response dynamic converges to a *single-peaked* equilibrium.[9] Here, we only provide a lower bound for the number of *single-peaked* equilibria. However, in Chapter 4 we will put the number of single-peaked and total equilibria into relation with each other. The proof of the lower bound also provides insights into the structure of single-peaked equilibria. It is based on the construction of a particular single-peaked equilibrium, containing two voters with reversed preferences. We will see that a single-peaked equilibrium always contains voters with relatively opposed opinions (large distance between their preferences).

3.3.1 Counting Single-Peaked Electorates

Before we consider the upper and lower bounds, it is worthwhile to consider the question of how many single-peaked electorates, not necessarily equilibria, can possibly exist. Again, we assume that every preference is submitted once, and thus this question is equivalent to counting the number of single-peaked profiles. Given a set A containing m many alternatives, Escoffier et al. (2008) show that for every order $\succ \in \mathcal{L}(A)$ there are 2^{m-1} orders that are $\succ$-single-peaked. In our terminology this means that the $\succ$-single-peaked domain is of size 2^{m-1}, or formally $|\mathcal{SP}_{\succ}| = 2^{m-1}$. Note that any subset of $\mathcal{SP}_{\succ}$ forms a $\succ$-single-peaked electorate. As the number of subsets of a set of size k equals 2^k, there are $2^{2^{m-1}} = 2^{2(m-1)}$ possible $\succ$-single-peaked electorates. We will see that requiring single-peaked electorates to be equilibria reduces this number significantly.

[9] Unfortunately, we do not have knowledge whether certain equilibria are more likely to be reached than others. While it might be the case that single-peaked equilibria are never reached (although they exist), it is also possible that every best response dynamic converges to a single-peaked equilibrium if one exists. It is certain however, that a single-peaked equilibrium is reached if *all* equilibria are single-peaked.

The main result from the previous section (Theorem 22) showed that whether an electorate is a single-peaked equilirium depends on the delegation threshold ρ, which we, once again, assume to be homogeneous for the remainder of this section. We analyze how the number of possible $\succ$-single-peaked electorates is reduced if we furthermore require the electorates to be equilibria for two cases: First we consider $\rho = \tau_m$, the maximal Kendall tau distance, and secondly, we count the number of single-peaked equilibria if $\rho = \lfloor \frac{\tau_m}{2} \rfloor$. We then generalize this to the number of equilibria single-peaked with respect to any order.[10] Let us start with a general result that holds for all delegation thresholds.

Theorem 24. *The number of single-peaked equilibria is bounded from below by $\frac{m!}{2}$, the number of pairs of reversed orders.*

Proof. We have seen in the proof of Theorem 11 that any two reversed orders form a single-peaked kernel if $\lfloor \frac{\tau_m}{2} \rfloor \leq \rho < \tau_m$. Note that there are $\frac{m!}{2}$ many pairs of reversed orders. The lower bound for the number of single-peaked kernels is therefore $\frac{m!}{2}$. □

Furthermore, recall that each singleton is a single-peaked kernel if the delegation threshold is maximal, i.e. $\rho = \tau_m$. Since singletons are the only kernels for maximal ρ (otherwise independence would be violated), the number of single-peaked equilibria equals the number of singletons, in other words the number of orders.

Proposition 3. *If $\rho = \tau_m$, then the number of single-peaked equilibria is $m!$.*

As every singleton forms a single-peaked equilibrium, and there are 2^{m-1} orders single-peaked with respect to some designated order, there are 2^{m-1} single-peaked equilibria for each single-peaked domain. If we compare this result with the number of possible $\succ$-single-peaked electorates, not necessarily equilibria, we notice this number reduces from $2^{2(m-1)}$ to 2^{m-1} if we require the electorates to be equilibria.

In Theorem 3 we considered the *maximal* delegation threshold $\rho = \tau_m$. It is furthermore interesting to consider the *minimal* delegation threshold $\rho = \lfloor \frac{\tau_m}{2} \rfloor$. We show that the number of single-peaked kernels is exactly $\frac{m!}{2}$ if τ_m is odd, and exactly $3 \cdot \frac{m!}{2}$ if τ_m even. Crucially, if τ_m is even we have $\lfloor \frac{\tau_m}{2} \rfloor = \frac{\tau_m}{2}$, and if τ_m is odd we have $\lfloor \frac{\tau_m}{2} \rfloor \neq \frac{\tau_m}{2}$. The parity of τ_m thus plays a surprising

[10] It would furthermore be interesting to analyze how the number of possible equilibria reduces if we additionally require them to be single-peaked. Recall that an equilibrium corresponds to a kernel in the appropriate delegation acceptability graph. However, we are not aware of results counting kernels in graphs.

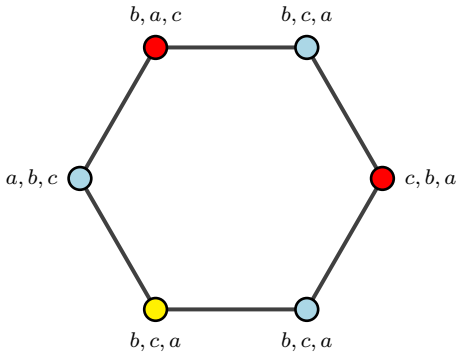


Figure 3.1: The τ-graph for $\rho = 1$ and $A = \{a, b, c\}$. Since $\frac{\tau_m}{2} = 1.5$, the only $\succ$-single-peaked kernels are unordered pairs of reversed orders (Proposition 4). Let $\succ = (a, b, c)$, and therefore $\succ^+ = (b, a, c)$. The set $S = \{\succ^+, \overline{\succ}\}$ (the red vertices) does not absorb the order $\pi(\succ) = (b, c, a)$ (yellow vertex).

role for the number of single-peaked equilibria. Figure 3.1 provides a visual representation for the following proposition, in particular for three alternatives and odd maximal distance $\tau_m = 3$.

Proposition 4. *If $\rho = \lfloor \frac{\tau_m}{2} \rfloor$ and τ_m is odd, then the number of single-peaked equilibria is exactly $\frac{m!}{2}$.*

Proof. Recall that any kernel is an equilibrium and vice versa. In order to prove that $\frac{m!}{2}$ is also an upper bound, we show that any set $S \subseteq \mathcal{L}(A)$ that does not consist of two reversed orders cannot be a single-peaked kernel. First, note that adding any third order to a set of two reversed orders results in the loss of independence of the set, and therefore such an extension is not a kernel. Let $\succ \in \mathcal{L}(A)$ be an arbitrary order. We show that any $\succ$-single-peaked kernel must contain either $\succ$ or $\overline{\succ}$. Let $\mathcal{SP}^-_\succ = \mathcal{SP}_\succ \setminus \{\succ, \overline{\succ}\}$. Recall that the distance from $\pi(\succ)$ to the $\mathcal{SP}^-_\succ$ is strictly larger than $\lfloor \frac{\tau_m}{2} \rfloor$ (Lemma 20). Thus, any set $S \subseteq \mathcal{SP}_\succ$ such that $S \cap \{\succ, \overline{\succ}\} = \emptyset$ does not absorb the order $\pi(\succ)$. Any $\succ$-single-peaked kernel must therefore contain $\succ$ or $\overline{\succ}$. We claim that for any $\succ' \in \mathcal{SP}^-_\succ$, the set $\{\succ', \overline{\succ}\}$ is not a $\succ$-single-peaked kernel. In particular, the set $\{\succ', \overline{\succ}\}$ does not absorb the order $\pi(\succ)$. As $\frac{\tau_m}{2} \notin \mathbb{N}$, the distance from $\pi(\succ)$ to $\succ$ is $\lfloor \frac{\tau_m}{2} \rfloor$, and from $\pi(\succ)$ to $\overline{\succ}$ the distance is $\lceil \frac{\tau_m}{2} \rceil$. Thus, since $\rho = \lfloor \frac{\tau_m}{2} \rfloor$, the order $\pi(\succ)$ is not absorbed by $\overline{\succ}$. Furthermore, recall that the distance from $\pi(\succ)$ to $\mathcal{SP}^-_\succ$ is strictly larger than $\lfloor \frac{\tau_m}{2} \rfloor$ (Lemma 20). Therefore, $\succ'$ does not absorb $\pi(\succ)$ neither, and $\{\succ', \overline{\succ}\}$ is not a $\succ$-single-peaked kernel. The proof for the set $\{\succ', \succ\}$ works analogously. The only single-peaked equilibria for each

single-peaked domain therefore consist of the two reversed orders. As there are $\frac{m!}{2}$ many, there are exactly $\frac{m!}{2}$ many single-peaked equilibria if τ_m is odd. □

Next, we consider the case where τ_m is even. As noted above, we have $\lfloor \frac{\tau_m}{2} \rfloor = \frac{\tau_m}{2}$ for even τ_m. One of the intuitive consequences is that the order constructed through the π-Algorithm is equidistant to its input $\succ$ and $\overline{\succ}$. We prove that two reversed orders are not the single-peaked equilibria. In particular, we will show that the $\succ$-single-peaked neighbor an order $\succ \in \mathcal{L}(A)$ forms a single-peaked kernel together with the order $\overline{\succ}$. Unlike in the proof of Proposition 4, $\pi(\succ)$ is absorbed by $\overline{\succ}$, since $\pi(\succ)$ is equidistant to both $\succ$ and $\overline{\succ}$. Analogously, the $\succ$-single-peaked neighbor of $\overline{\succ}$ forms a single-peaked kernel together with $\succ$. Together with $\{\succ, \overline{\succ}\}$, there are thus three single-peaked kernels for every single-peaked domain.

To facilitate the proof we first show an auxiliary lemma concerning the distance of $\pi(\succ)$ to the $\succ$-single-peaked domain. Note that the orders $\succ$ and $\overline{\succ}$ have only one $\succ$-single-peaked neighbor in the τ-graph, and the respective distance of the neighbors to $\pi(\succ)$ is $\frac{\tau_m}{2}+1$. Lemma 25 states that the distance of any other $\succ$-single-peaked order to $\pi(\succ)$ is at least $\tau(\succ^\pi, \succ)+2$. For the following sections the superscript $(\cdot)^+$ denotes the neighbor of an order in the τ-graph which we will specify in the given context.

Lemma 25. *Let* $\succ = (a_1, a_2, ..., a_m)$, $\succ^+ = (a_2, a_1, ..., a_m)$ *and furthermore* $\mathcal{SP}^-_\succ = \mathcal{SP}_\succ \setminus \{\succ, \succ^+\}$. *Then for any* $\succ' \in \mathcal{SP}^-_\succ$,

$$\tau(\succ', \succ^\pi) \geq \tau(\succ, \succ^\pi)+2 \ .$$

Proof. Recall the structuring of the order $\pi(\succ)$ into two suborders, depending on the fully reversed alternatives $a_1, ..., a_i$, and the partially reversed alternative $a^\star$. Claim 2 and 3 from Lemma 20 stated that $\tau(\succ', \succ^\pi) \geq \tau(\succ, \succ^\pi)+\binom{p}{2}$, where a_p designates the maximal element of $\succ'$. p can therefore take values between 2 and the rank of the alternative $a^\star$ in $\succ$. If a_2 is the maximal element, the distance between $\succ^+$ and $\pi(\succ)$ is $\tau(\succ, \succ^\pi)+\binom{2}{2} = \tau(\succ, \succ^\pi)+1$. For any other peaks, the distance is therefore at least $\tau(\succ, \succ^\pi)+\binom{3}{2} = \tau(\succ, \succ^\pi)+2$. □

We continue to prove Proposition 5 by showing that for every single-peaked domain there are three single-peaked kernels, and thus the total number of single-peaked equilibria is $3 \cdot \frac{m!}{2}$. Figure 3.2 provides a visual representation for four alternatives and even maximal distance $\tau_m = 6$.

Proposition 5. *If* $\rho = \lfloor \frac{\tau_m}{2} \rfloor$ *and* τ_m *is even, then the number of single-peaked equilibria is exactly* $3 \cdot \frac{m!}{2}$.

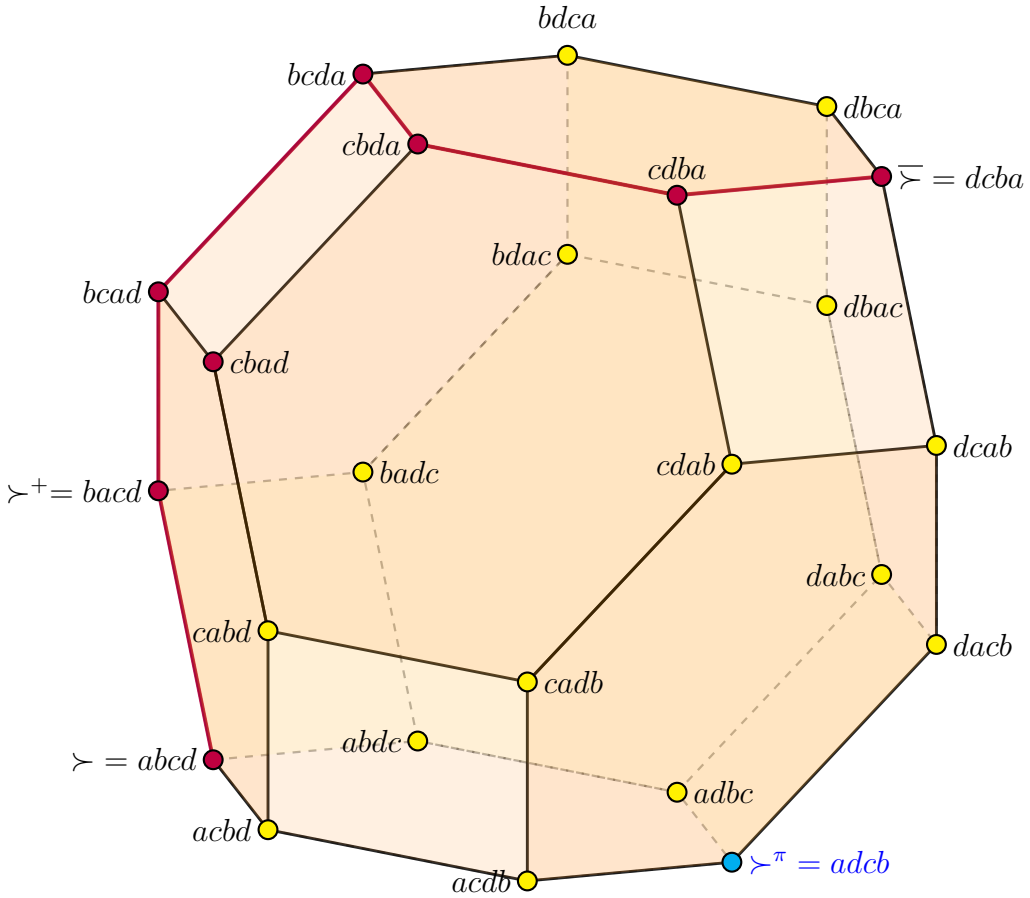


Figure 3.2: The τ-graph for $A = \{a, b, c, d\}$. Let $\succ = (a, b, c, d)$, then $\pi(\succ) = (a, d, c, b)$ (blue vertex), and $\succ^+ = (b, a, c, d)$. If $\rho = 3$, the set $K' = \{\succ^+, \overline{\succ}\}$ absorbs all orders, including $\pi(\succ)$.

Proof. The proof makes use of the isomorphism f mapping an agent i to her preference $\succ_i$, and we will argue about distances and sets of orders, instead of the associated voters. The proof proceeds in two parts. We first show that there are three single-peaked kernels for a single-peaked domain, and then in a second step that there are no other single-peaked kernels.

Let $\succ = (a_1, ..., a_m)$, $\succ^+ = (a_2, a_1, ..., a_m)$, $\overline{\succ}^+ = (a_{m-1}, a_m, ..., a_1)$, and $K = \{\succ, \overline{\succ}\}$. The orders with the + superscript are $\succ$-single-peaked neighbors of $\succ$ and $\overline{\succ}$ respectively. We prove that $K' = \{\succ^+, \overline{\succ}\}$ and $K'' = \{\succ, \overline{\succ}^+\}$ are $\succ$-single-peaked kernels. As the proofs work analogously, we only consider $K' = \{\succ^+, \overline{\succ}\}$. By assumption, $\rho = \frac{\tau_m}{2}$. For independence of K' we thus need to prove that $\tau(\succ^+, \overline{\succ}) > \frac{\tau_m}{2}$. Since $\succ$ and $\succ^+$ are neighbors we have $\tau(\succ^+, \overline{\succ}) = \tau_m - 1$. Recall that $\tau_m = \frac{m(m-1)}{2}$. We leave it to the reader to verify that

$$\tau(\succ^+, \overline{\succ}) = \frac{m(m-1)}{2} - 1 > \frac{m(m-1)}{4} = \frac{\tau_m}{2}$$

from which we conclude the *independence* of K'.

Recall that $K = \{\succ, \overline{\succ}\}$ is a $\succ$-single-peaked kernel. In order to prove that K' is *absorbing*, assume for contradiction that there is an order $\succ' \in \mathcal{L}(A)$ that is absorbed by K but *not* by K'. Consequently, $\tau(\succ, \succ') \leq \frac{\tau_m}{2}$, $\tau(\succ^+, \succ') > \frac{\tau_m}{2}$, and $\tau(\overline{\succ}, \succ') > \frac{\tau_m}{2}$. There are two cases to consider:

Case 1: $\tau(\succ, \succ') = \frac{\tau_m}{2}$. Then also $\tau(\overline{\succ}, \succ') = \frac{\tau_m}{2}$, and $\succ'$ is absorbed by $\overline{\succ}$.

Case 2: $\tau(\succ, \succ') < \frac{\tau_m}{2}$. This is the case if and only if $\tau(\succ, \succ') \leq \frac{\tau_m}{2} - 1$. Since $\succ$ and $\succ^+$ are neighbors, $\succ^+$ absorbs $\succ'$.

As both cases result in a contradiction, no order that is absorbed by K but not by K' exists, and consequently K' is absorbing. Thus, K' is a single-peaked kernel, since it is absorbing and independent.

So far we found an improved lower bound for $\rho = \frac{\tau_m}{2}$ with even τ_m. We continue to prove that there are no other $\succ$-single-peaked kernels than K, K', and K''. As we have seen in the proof of Proposition 4 any $S \subseteq \mathcal{SP}_\succ$ that does not contain $\succ$ or $\overline{\succ}$ does not absorb $\pi(\succ)$. Any potential kernel must therefore contain either of these orders.

Let $\mathcal{SP}^-_\succ = \mathcal{SP}_\succ \setminus \{\succ, \overline{\succ}, \succ^+\}$. Consider the order $\pi(\succ)^+$ which is the neighbor of $\pi(\succ)$ on a shortest path to $\succ$. From Lemma 20 we know that the distance between $\pi(\succ)$ and $\mathcal{SP}^-_\succ$ is at least $\tau(\succ, \succ^\pi)+2$. The distance between $\pi(\succ)^+$ and $\mathcal{SP}^-_\succ \setminus \{\succ^+\}$ is therefore at least $\tau(\succ, \succ^\pi)+1$. Since $\rho = \frac{\tau_m}{2}$, the order $\pi(\succ)^+$ is therefore never absorbed by any order in $\mathcal{SP}^-_\succ \setminus \{\succ^+\}$. Consequently, the only three kernels are K, K', and K''. For every single-peaked domain there are thus three unique single-peaked kernels. Since there are $\frac{m!}{2}$ single-peaked domains, the number of single-peaked peaked kernels is $3 \cdot \frac{m!}{2}$ if $\rho = \frac{\tau_m}{2}$ and τ_m

is even. Note that we can simply multiply by $\frac{m!}{2}$: No double-counting of the kernels takes place since the kernels K, K', and K'' are single-peaked *only* with respect to the order $\succ$. □

In Propositions 4 and 5 we considered the special case where $\rho = \lfloor \frac{\tau_m}{2} \rfloor$ in more detail. We have seen that for $\rho = \lfloor \frac{\tau_m}{2} \rfloor$, the number of possible $\succ$-single-peaked electorates reduces from $2^{2(m-1)}$ to, depending on the parity of τ_m, one or three if we additionally require them to be equilibria. This shows how strong the definition of an equilibrium is.

3.3.2 Structure of Single-Peaked Equilibria

We believe that for delegation thresholds larger than $\lfloor \frac{\tau_m}{2} \rfloor$, the single-peaked equilibria can be constructed along the technique used in the proof of Proposition 5. The consequences are that single-peaked equilibria always consist of two voters with relatively opposed opinions. Here we can see the conflict between the property of absorbance and independence. If ρ increases, the distance between the orders in a potential kernel can increase, as they absorb orders that are relatively far away. At the same time, the distance between the two voters must always exceed ρ, as otherwise the voters are not independent.

We conclude this chapter with a remark about the cardinality of a single-peaked equilibrium. We conjecture that the cardinality of a single-peaked equilibrium is bounded from above by two, and it is one if and only if $\rho = \tau_m$. The latter is easy to see, as any singleton is a single-peaked equilibrium if $\rho = \tau_m$. For lower thresholds, no voter can absorb the voter with the reversed order. Thus, at least two voters are required to achieve an equilibrium. However, we conjecture that the addition of any third voter results in the loss of independence, and thus an electorate containing three voters cannot be an equilibrium.

Conjecture 1. *The cardinality of a single-peaked equilibrium is at most two.*

If the conjecture turns out to be true, single-peaked equilibria in liquid democracy are a reduction of the multitude of opinions to merely two opinions. We are still concerned with *equilibria*, and therefore each voter is accepting this reduction (otherwise we would not have an equilibrium). However, from a more abstract democratic theoretical view, a single-peaked equilibrium does not seem to be a desirable outcome. If we treat liquid democracy as a model aiming to increase participation, the reduction of an electorate to two voters seems to undermine the entire endeavour of liquid democracy. The aggregation of voting power into the hands of only a few individuals is a phenomenon that has

been confirmed by Kling et al. (2015), and theoretically discussed by Gölz et al. (2018). Our analysis of single-peaked equilibria confirms this aggregation tendency of liquid democracy.

We will see in the following chapter, that for smaller thresholds than $\lfloor \frac{\tau_m}{2} \rfloor$ single-peaked kernels can exist if we relax the assumptions. In particular, there are single-peaked kernels of size larger than two if not all orders are submitted. We can already point at a necessary condition: the order $\pi(\succ)$ cannot be submitted, as otherwise it cannot be absorbed by any $\succ$-single-peaked voter/order.

Chapter 4

Counting Single-Peaked Electorates: Simulations

In the previous chapter we investigated the existence of single-peaked equilibria in delegation structures $\langle G, \boldsymbol{P}, \boldsymbol{\rho} \rangle$ under the three assumptions

1. Every preference order is submitted exactly once, entailing $\mathcal{D}(\boldsymbol{P}) = \mathcal{L}(A)$,
2. G is the complete graph, i.e. every voter is allowed to delegate to every other voter,
3. $\boldsymbol{\rho}$ is homogeneous, i.e. all voters have the same acceptability threshold.

A complete generalization of the results in Chapter 3 would require an analysis of all graphs, every possible profile, and every possible delegation threshold. We will see, partly in this chapter, partly in the next, that generalizing the result analytically is a complex if not impossible endeavour. The consequences of relaxing the assumptions will thus not be investigated analytically but computationally. Essentially, each constituent of a delegation structure $\langle G, \boldsymbol{P}, \boldsymbol{\rho} \rangle$ can be treated as a random variable. This chapter is dedicated to the relaxation of the complete domain assumption, and we will therefore treat the profile $\boldsymbol{P}$ as a random variable. We generate random profiles and analyze these for single-peaked equilibria. In Chapter 5 we shift our attention to real life data, which we take from the PrefLib data set (Mattei and Walsh, 2013). Under the assumption that the underlying social network G is complete, we randomize over the (heterogeneous) delegation thresholds of each voter.

The goal of this chapter is to take a step from the complete preference domain $\mathcal{D}(\boldsymbol{P}) = \mathcal{L}(A)$ towards the universal preference domain $\mathcal{D}(\boldsymbol{P}) \subseteq \mathcal{L}(A)$ in which every possible domain is admitted. For a small number of alternatives,

the number of possible preferences is relatively small, so it is likely that every preference is held by one voter. However, the number of possible orders grows factorially with the number of alternatives. For three alternatives there are $3! = 6$, for four $4! = 24$, and for five alternatives already $5! = 120$ possible preferences. While it may be realistic that for three alternatives each of the possible six orders is submitted, this becomes less likely as the number of alternatives grows. This is in fact supported by the PrefLib data sets. In the vast majority of profiles containing three or four alternatives, each order is submitted at least once, while for five or more alternatives many possible orders are not submitted.

Ultimately, our interest lies not in the domains but the profiles. However, recall that if $\boldsymbol{\rho}$ is homogeneous, then all voters with the same preference order can be put into the same equivalence class. Since—for now—only the complete domain assumption is relaxed, results about the set of all possible domains $\mathcal{D}(\boldsymbol{P}) \subseteq \mathcal{L}(A)$ extend to the set of all possible profiles $\boldsymbol{P} \subset \mathcal{L}(A)^n$.

In the previous chapter we have proven a characterization for the existence of single-peaked equilibria if the domain is complete, in particular, if every order is submitted exactly once. We can make use of this result and achieve the generalization to the universal domain by iteratively deleting voters from the complete profile $\boldsymbol{P}$ with $\mathcal{D}(\boldsymbol{P}) = \mathcal{L}(A)$. Note that we assume that every order is submitted exactly once, thus voter deletion is equivalent to the deletion of orders from the complete domain. Since any domain is a subdomain of the complete domain, this approach covers all possible domains. Our approach can easily be generalized to allow for multiple submission of the same order by different voters by deleting all voters with the same order. For reasons of simplicity this chapter is following the assumptions that orders are submitted exactly once.[1]

As hinted at above, the impact of deleting voters from the complete profile, and thus orders from the domain, on the existence and number of single-peaked equilibria is hard to generalize analytically. In Section 4.1 we prove that the deletion of a voter can have two effects: Either equilibria that included the deleted order cease to be kernels, or the deletion results in a new equilibrium. Unfortunately, this observation bears little informational value, which motivates us to perform computational simulations—the main endeavour of this chapter.

[1] Consider the case where one order is submitted multiple times. Under the assumptions that the social network is complete, and thresholds are homogeneous, the deletion of a single voter does not impact the existence of single-peaked equilibria. There are still voters with the exact same delegation behaviour: They can delegate to everyone, have the same preference, and same delegation threshold as the deleted voter. The deletion of single voters becomes relevant only once thresholds are heterogeneous, or the social network is not complete.

4.1 Theoretical Observations

Given a delegation structure $\langle G, \boldsymbol{P}, \boldsymbol{\rho}\rangle$ a *deletion sequence* $s = (s_1, ..., s_n)$ is a sequence of voters with agent $s_1 \in N$, and agents $s_i \in N \setminus \{s_1, ..., s_{i-1}\}$ that are successively deleted from the profile $\boldsymbol{P}$. A deletion sequence therefore gives rise to profiles $\boldsymbol{P}_1$ to $\boldsymbol{P}_n$, and consequently to delegation structures $\langle G_1, \boldsymbol{P}_1, \boldsymbol{\rho}_1\rangle$ to $\langle G_n, \boldsymbol{P}_n, \boldsymbol{\rho}_n\rangle$ each containing one voter less than its predecessor. The number of deletion sequences is therefore $n\times(n-1)\times...\times 1 = n!$. By the assumptions, each voter expresses one of $m!$ many strict linear orders over A. This gives rise to $(m!)!$ total distinct deletion sequences.[2]

Let $\langle G, \boldsymbol{P}, \boldsymbol{\rho}\rangle$ be a delegation structure where G is the complete graph, $\boldsymbol{\rho}$ is homogeneous, $\mathcal{D}(\boldsymbol{P}) \subseteq \mathcal{L}(A)$, and $\boldsymbol{s} = (s_1, ..., s_n)$ is a deletion sequence. We first show that the number of single-peaked equilibria lost by deleting a voter, cannot exceed the number of single-peaked equilibria of which the deleted voter was a member.

Proposition 6. *Let s_i be a deleted voter, and let k be the number of single-peaked equilibria of which s_i is a member in the delegation structure $\langle G_{i-1}, \boldsymbol{P}_{i-1}, \boldsymbol{\rho}_{i-1}\rangle$. Then the number of deleted equilibria in $\langle G_i, \boldsymbol{P}_i, \boldsymbol{\rho}_i\rangle$ is bounded from above by k.*

Proof. Let K be a single-peaked equilibrium in $\langle G_{i-1}, \boldsymbol{P}_{i-1}, \boldsymbol{\rho}_{i-1}\rangle$. If $s_i \notin K$, trivially K remains a single-peaked equilibrium in $\langle G_i, \boldsymbol{P}_i, \boldsymbol{\rho}_i\rangle$. Next, assume $s_i \in K$. Since s_i is no longer a voter in the delegation structure $\langle G_i, \boldsymbol{P}_i, \boldsymbol{\rho}_i\rangle$, trivially K cannot be an equilibrium. More interestingly, we check whether $K' = K \setminus \{s_i\}$ is a single-peaked equilibrium. Let S be the set of voters that delegated to s_i in $\langle G_{i-1}, \boldsymbol{P}_{i-1}, \boldsymbol{\rho}_{i-1}\rangle$. If all members of S are willing to delegate to someone in K', then K' remains a single-peaked equilibrium. However, if there is a voter in S that is not willing to delegate to someone in K', that voter is not absorbed, and K' ceases to be a equilibrium in $\langle G_i, \boldsymbol{P}_i, \boldsymbol{\rho}_i\rangle$. Thus, the number of equilibria lost by deleting s_i is bounded from above by k, the number of equilibria s_i was part of in $\langle G_{i-1}, \boldsymbol{P}_{i-1}, \boldsymbol{\rho}_{i-1}\rangle$. □

Whether the deleted equilibria actually equals the number of equilibria the deleted order was part of, depends on the willingness of voters to delegate to the reduced equilibria. This is a big 'if' that is based on the number of voters left, whether the distance between the voters is large, and generally on the specific

[2]Note that the number of possible domains is $2^{m!} << (m!)!$. This can be explained by the fact that multiple deletion sequences can express the same profile. For example, first deleting voter 1, and then voter 2, results in the same profile as first deleting voter 2 and then voter 1.

structure of the profile. Without this knowledge it is hard to get more specific results. An analogous argument can be made about the following proposition in which we analyze whether new single-peaked equilibria come about by deleting a voter.

Proposition 7. *Let s_i be a deleted voter. A set S of voters is a single-peaked equilibrium in $\langle G_i, \boldsymbol{P}_i, \boldsymbol{\rho}_i \rangle$, and not in $\langle G_{i-1}, \boldsymbol{P}_{i-1}, \boldsymbol{\rho}_{i-1} \rangle$ if and only if S is a single-peaked, independent set and the only voter not absorbed by S in $\langle G_{i-1}, \boldsymbol{P}_{i-1}, \boldsymbol{\rho}_{i-1} \rangle$ is s_i.*

Proof. ($\Leftarrow$): Trivial. ($\Rightarrow$): If S is a single-peaked equilibrium in $\langle G_i, \boldsymbol{P}_i, \boldsymbol{\rho}_i \rangle$, and not in $\langle G_{i-1}, \boldsymbol{P}_{i-1}, \boldsymbol{\rho}_{i-1} \rangle$, then either S was not absorbing in $\langle G_{i-1}, \boldsymbol{P}_{i-1}, \boldsymbol{\rho}_{i-1} \rangle$, not independent, or not single-peaked. If it did not absorb other voters than s_i, it is still not an equilibrium since only s_i is deleted. Since $s_i \notin S$, deleting s_i does not impact whether S is independent and single-peaked. Thus, s_i must be the only voter not absorbed by S in $\langle G_{i-1}, \boldsymbol{P}_{i-1}, \boldsymbol{\rho}_{i-1} \rangle$. □

Unfortunately, Propositions 6 and 7 are too general to bear informational value. Whether or not equilibria are lost or added depends too much on the structure of the profiles. Without the structure at hand, there is little to say about the existence and number of single-peaked equilibria. Once we have the structure, on the other hand, the number of single-peaked equilibria can be computed. We therefore proceed by computations whose design is in line with the theoretical observations we have obtained—the iterated deletion of voters.

4.2 Monte Carlo Simulation

An analytical solution to the analysis of single-peaked equilibria for the universal domain requires analyzing $(m!)!$ deletion sequences. This ceases to be tractable already for $m > 4$, which pulls us towards probabilistic modelling, in particular we will run a *Monte Carlo Simulation*. The underlying idea of Monte Carlo simulations is that "anything we want to know about a random variable θ can be learned by sampling many times from $f(\theta)$, the density of θ" (Jackman, 2009). We will treat the deletion sequences as random variables, and therefore sample from their—to be defined—distribution. Before we devote our attention to describing the random variable and the Monte Carlo simulation in more detail, some technical preliminaries are required.

Assume we are given a complete profile $\boldsymbol{P}$ consisting of voters $i \in N$ with strict, linear preferences over a set of alternatives A with $|A| = m$. The goal is to

understand the impact of voter deletion on the number and structure of single-peaked equilibria, for fixed, homogeneous delegation thresholds $\rho \in \{1, ..., \tau_m\}$. Unless stated otherwise, we assume the voters to be part of a complete social network G. For readability, we will refer to the social network as G, and to the delegation thresholds as $\boldsymbol{\rho}$ even if voters are deleted from the network.

Recall that a deletion sequence $\boldsymbol{s} \in \Sigma$ gives rise to n-many profiles $\boldsymbol{P}_1$ to $\boldsymbol{P}_n$. Each profile $\boldsymbol{P}_k$ together with a (homogeneous) delegation threshold ρ gives rise to a delegation structure $\langle G, \boldsymbol{P}_k, \boldsymbol{\rho} \rangle$, where the social network G is completely specified since we assume it to be complete. Therefore, there is a function with a deletion sequence $\boldsymbol{s}$ and delegation threshold ρ as inputs, and a sequence of delegations structures as output. Given a delegation structure $\langle G, \boldsymbol{P}_k, \rho \rangle$, one can check for every subset of voters K whether K is a single-peaked equilibrium. Therefore, there is a function g mapping a delegation structure to a set of single-peaked equilibria $\mathcal{K} \subseteq \mathcal{P}(N)$, the power set of N. Consequently, there must be a compositional function h, taking as input a deletion sequence $\boldsymbol{s} = (s_1, ..., s_n)$, and delegation threshold ρ, mapping these to a sequence of sets of single-peaked equilibria:

$$\begin{aligned} h : \Sigma \times \{1, ..., \tau_m\} &\to (\mathcal{P}(\mathcal{P}(N)), ..., \mathcal{P}(\mathcal{P}(N))) \\ ((s_1, ..., s_n), \rho) &\mapsto (\mathcal{K}_1, ..., \mathcal{K}_n) \end{aligned} \tag{4.1}$$

Intuitively, given a delegation threshold, the function h maps n profiles to their single-peaked equilibria, where the profiles result from deleting voters from the complete profile $\boldsymbol{P}$ according to some deletion sequence $\boldsymbol{s}$=$(s_1, ..., s_n)$. As we have argued above, we cannot consider all possible deletion sequences. The underlying idea of the Monte Carlo simulations is to randomly draw from the set Σ and then compute for each $\rho \in \{1, ..., \tau_m\}$ a sequence of sets of single-peaked equilibria, which we then analyze with respect to the average number of single-peaked equilibria, the smallest single-peaked equilibrium, and the largest single-peaked equilibrium.

Formally, a *random variable* X is a mapping from a sample space Ω to measurable space E. The delegation thresholds ρ are fixed, and we only treat the deletion sequences as random events. An outcome $\omega \in \Omega$ is the result of drawing a deletion sequence $\boldsymbol{s} \in \Sigma$. We are finally ready to define the random variables. Let X_ρ be the random variable, mapping a sample space Ω to a sequence of natural numbers $(x_1, ..., x_n)$, where each x_i equals the number of single-peaked equilibria according to the function $h(s_i, \rho)_i$ (Equation 4.1). If

$(\mathcal{K}_1, ..., \mathcal{K}_n)$ is the output of $h(\boldsymbol{s}, \rho)$, then $(|\mathcal{K}_1|, ..., |\mathcal{K}_n|)$ is the output of $X_\rho(\boldsymbol{s})$.

$$\begin{aligned} X_\rho : \quad & \Omega \to \mathbb{N}^n \\ & (s_1, ..., s_n) \mapsto |h(s_i, \rho)_i| \quad \text{for } 1 \leq i \leq n \end{aligned}$$

We are not only interested in the number, but also in the size of single-peaked equilibria. For a set $\mathcal{K}$ of single-peaked equilibria, let $\text{card}(\mathcal{K}) = \{|K| \mid K \in \mathcal{K}\}$ be the set containing the size of each single-peaked equilibrium. If $(\mathcal{K}_1, ..., \mathcal{K}_n)$ is the output of $h(\boldsymbol{s}, \rho)$, then $(\text{card}(\mathcal{K}_1), ..., \text{card}(\mathcal{K}_n))$ is the output of $Y_\rho(\boldsymbol{s})$.

$$\begin{aligned} Y_\rho : \quad & \Omega \to \mathbb{N}^n \\ & (s_1, ..., s_n) \mapsto \text{card}(h(s_i, \rho)_i) \quad \text{for } 1 \leq i \leq n \end{aligned}$$

In order to show that the random variables are well-defined we need to show that Ω is part of a *probability triple* $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}$ is a set of events where an event $E \subseteq \Omega$ is a set of outcomes in the sample space Ω, and $\mathbb{P}$ is a probability distribution over Ω. As is commonly done on a discrete sample space, let $\mathcal{F} = \mathcal{P}(\Omega)$ be the power set of Ω. Any subset of Ω is therefore an admissible event. Next, consider the probability distribution $\mathbb{P}$. Each $\omega \in \Omega$ is the outcome of drawing a deletion sequence $\boldsymbol{s} \in \Sigma$. Recall that there are $(m!)!$ many deletion sequences, each equally likely. Thus, $\mathbb{P}$ is a uniform distribution with $\mathbb{P}(\omega) = \frac{1}{(m!)!}$ for all $\omega \in \Omega$.[3] Since all random variables defined above share the same sample space Ω, we can assume that they furthermore share the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

With the formal definitions of the random variables at hand, we proceed to describe the Monte Carlo simulation in more detail. Generally, one knows the probability distribution of a random variable θ but cannot draw conclusions about functions of θ. In our case, we know the distribution of the deletion sequences, but we cannot from this alone infer anything about, for example, the number of single-peaked equilibria. A Monte Carlo simulation offers a way out: By sampling many times from the distribution, we can make statistically valid statements about the number of single-peaked equilibria. The mathematical foundation of this approach, like so often in statistics, is the law of large numbers. For a formal description, and the relation of the law of large numbers to the Monte Carlo simulation, we refer the reader to Chapter 3 in Jackman (2009), in particular pages 138 ff.

[3] To show that $(\Omega, \mathcal{F}, \mathbb{P})$ is indeed a probability space is straightforward. As Ω is a discrete set, the power set guarantees that the event space is measurable. Trivially, the uniform probability distribution adds up to 1.

Our Monte Carlo Simulation essentially follows Jackman (2009), Chapter 3, with a slight variation. Recall that the random variables we defined output a sequence of numbers. To estimate the expected value $\mathbb{E}$, we therefore cannot take the mean, but take the mean of each member of the sequence. The 'mean' of a random variable is therefore a sequence of means. We draw a random deletion sequence $\boldsymbol{s} \in \Sigma$ and then calculate the random variables based on $\boldsymbol{s}$.

Algorithm 3 Monte Carlo Estimate for the Mean

Input: Set of all deletion sequences Σ

1: **for** $t = 1$ to T **do**
2: draw $\boldsymbol{s}^{(t)} \in \Sigma$
3: calculate $\theta(\boldsymbol{s}^{(t)})$ where $\theta \in \{X_\rho, Y_\rho, Z_\rho\}$ and $\rho \in \{1, ..., \tau_m\}$
4: **end for**
5: Estimate $E(\theta)$ with $\frac{1}{T}\sum_{t=1}^{T} \theta^{(t)}$

Furthermore, if we are interested not in the mean but the probability that a variable takes on values larger than a constant c we can use Algorithm 4. In particular, we will be interested in the probability that single-peaked equilibria exist, i.e. the probability that the number of single-peaked equilibria is larger than zero.

Algorithm 4 Monte Carlo Estimate of Cumulative Probability

Input: Set of all deletion sequences Σ

1: **for** $t = 1$ to T **do**
2: draw $\boldsymbol{s}^{(t)} \in \Sigma$
3: $z^{(t)} \leftarrow \mathcal{I}\left(c < \theta(\boldsymbol{s}^{(t)})\right)$ where $\theta \in \{X_\rho, Y_\rho, Z_\rho\}$ and $\rho \in \{1, ..., \tau_m\}$, and $\mathcal{I}(\cdot)$ is a binary indicator function
4: **end for**
5: Estimate $\mathbb{P}\left(c < \theta(\boldsymbol{s}^{(t)})\right)$ with $\frac{1}{T}\sum_{t=1}^{T} z^{(t)}$

For the random variable Y_ρ that counts the size of single-peaked equilibria, we will use a different approach. We will simply count the number each size occurs.

4.3 Experimental Planning and Setup

We expounded the general approach of the Monte Carlo simulation in the previous section. In the following section, we explain the Python implementation

of the simulation. Furthermore, three variables need to be fixed in order to run the simulation: (i) the number of alternatives, (ii) the delegation thresholds, and (iii) the number of runs. The main criterion for the parameter choice has been the running time of the resulting simulation.

4.3.1 Python Implementation

In every delegation structure $\langle G, \boldsymbol{P}, \boldsymbol{\rho} \rangle$ we assume the social network G to be complete. Delegations thus only depend on the Kendall Tau distances between voters, and their delegation thresholds. We follow the structure of profiles provided by PrefLib (Mattei and Walsh (2013), Mattei and Walsh (2017)). For every deletion sequence, we create the profiles resulting from that sequence. For every profile we then create delegation acceptability digraphs based on the Kendall Tau distance, and the delegation thresholds. We make use of the Python package NetworkX (Hagberg et al., 2008) to analyze the graphs for kernels. Recall that homogeneous thresholds result in symmetric acceptability digraphs, which we can simply reduce to undirected graphs. In the preliminaries we proved that kernels in an undirected graph correspond to maximal indepedent sets (Observation 6), which in turn correspond to maximal cliques in the complement of that graph (Observation 5). The built-in function `find_cliques`($\overline{G}$) in NetworkX outputs all maximal cliques of a graph $\overline{G}$, and thus all kernels of G. In a final step we check which of these kernels are single-peaked, and store the result in a Pandas Dataframe (Wes McKinney (2010)). Effectively, the dataframe stores the output of the function h (Equation 4.1). The values of all random variables can be derived from the dataframe, as they are based on the function h.

We just argued that the search for kernels in a graph G can be reduced to maximal cliques in its complement $\overline{G}$. It is a well known result that the maximal clique problem is NP-hard, and brute force requires exponential running times. However, the Bron–Kerbosch algorithm improves on brute force, and can solve the problem in $\mathcal{O}(3^{n/3})$, where n is number of vertices (Bron and Kerbosch, 1973). In NetworkX the function `find_cliques`(`G`) returns all maximal cliques of a graph G based on the Bron-Kerbosch algorithm. Together with the function `is_single_peaked`(`S`) from PrefLib, which checks each maximal clique S of G for single-peakedness, this builds the pipeline for our analysis of the graph.

m \ ρ	1	2	3	5	7	9
3	$\sim 1.2s$	$\sim 1.2s$	-	-	-	-
4	$\sim 1.3s$	$\sim 1.5s$	$\sim 1.5s$	$\sim 2.0s$	-	-
5	$>30min^*$	$>30min^*$	$\sim 182s$	$\sim 2.2s$	$\sim 2.2s$	$\sim 1.8s$

Table 4.1: Average Running times for checking the *complete* profiles for single-peaked kernels. Experiments were run on a MacBook Pro with 2 GHz Quad-Core Intel Core i5, 16Gb Ram, running MacOS Big Sur 11.6.
$(\cdot)^*$: Did not terminate after 30 min.

4.3.2 Parameter Selection

As the running time grows exponentially with the number of vertices, the choice of parameters is based on the running time. A number of test runs yield the running times depicted in Table 4.1. As the number of alternatives, and thus vertices in the acceptability graph grows, the number of kernels increases drastically. For example, if $\rho = 3$, there are 88 kernels for $m = 4$, and already 6899432 for $m = 5$. All of these then additionally have to be checked for single-peakedness. Based on Table 4.1 we choose the following parameters. For $m = 3$, and $m = 4$ we perform 30.000 runs with $\rho \in \{0, 1, ..., \tau_m\}$. In other words, we perform a Monte Carlo simulation with a sample size of $T = 30.000$. As the running times increase drastically for $m = 5$, we perform $T = 1000$ runs for $\rho \in \{0, 3, 4, 5, 7, 9\}$. Additionally, we do not analyze the profiles for each deletion of a deletion sequence. A complete deletion sequence is of length $5! = 120$, however we restrict the analysis to profiles after every tenth deletion. Thus, we only analyze 12 profiles for each deletion sequence.

4.3.3 Hypotheses

Recall from Section 4.1 the impact one deletion has on the number of single-peaked equilibria. All equilibria the deleted voter was part of are lost, and potentially new kernels, that did not absorb the deleted voter, come into existence. On average, we expect the deletion of voters to result in a decrease of single-peaked kernels. It is furthermore interesting whether the random variable X_ρ is *strictly* decreasing.

Hypothesis 4.1. The number of single-peaked equilibria is inversely proportional to the number of deletions.

One of the main points of interests in this thesis is to explore the delegation threshold $\lfloor \frac{\tau_m}{2} \rfloor$. In the previous chapter it was proven, that no single-peaked equilibria exist if voters are opinionated. We expect a slightly weakened result to hold if the domain of a profile is not complete. Generally speaking, for a $\succ$-single-peaked equilibrium in a delegation structure $\langle G, \boldsymbol{P}, \boldsymbol{\rho} \rangle$ to exist, all voters with a delegation threshold smaller than ρ must be deleted. If d is the number of these voters, at least d many deletions need to be performed. The probability that $\succ$-single-peaked kernels exist is thus depended on the likelihood that all d voters are deleted. As the number of deletions increases, the likelihood that all d voters are deleted increases.

Hypothesis 4.2. If voters are opinionated, the likelihood for the existence of single-peaked equilibria increases as the number of deletion increases.

If voters are open-minded, we expect the result from Chapter 3 to generalize to the universal domain.

Hypothesis 4.3. If voters are open-minded, single-peaked equilibria almost always exist.

In the previous chapter we conjectured that the size of single-peaked equilibria in complete domains is bounded from above by two (Conjecture 1). This conjecture should not be impacted by voter deletion. If the upper bounds holds for the complete domain, the bound generalizes to any domain if voters are open-minded.

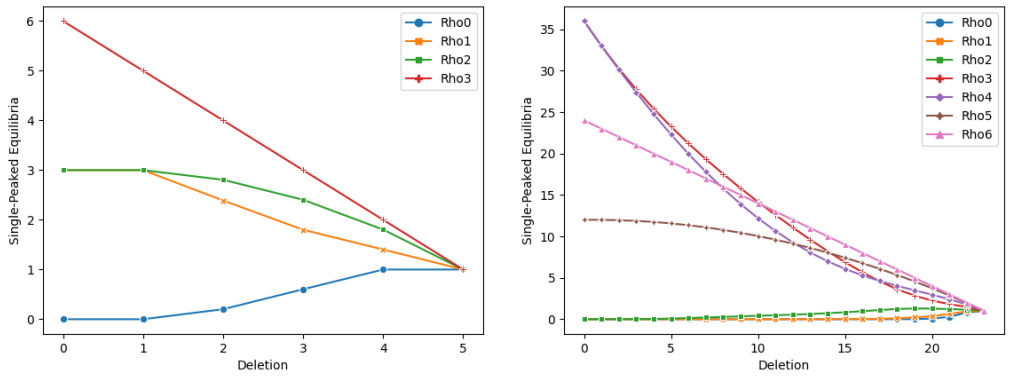
Hypothesis 4.4. If voters are open-minded, the size of a single-peaked equilibrium is bounded from above by 2.

On the other hand, if voters are opinionated, we expect larger equilibria. In the extreme case, where no one is willing to delegate, i.e. ρ=0, it is possible, though unlikely, that only a single-peaked domain remains as voters. As the size of a single-peaked domain is 2^{n-1}, the maximal size of a single-peaked equilibrium is 2^{n-1}. If the threshold is 1, at most every second voter of a single-peaked domain can be in an equilibrium, reducing its size to $\frac{2^{n-1}}{2}$. If the threshold is 2, only every third voter can be part of an equilibrium, and so on.

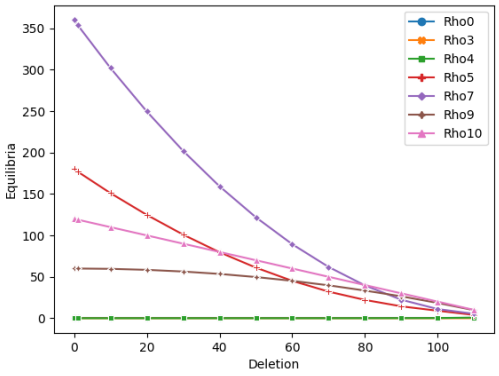
Hypothesis 4.5. If voters are opinionated, the size of a single-peaked equilibrium is bounded from above by $\frac{2^{n-1}}{\rho+1}$.

4.4 Results

We discuss the hypothesis individually and conclude this chapter with a more general discussion of the results (Section 4.5).



(a) $m = 3$ (b) $m = 4$

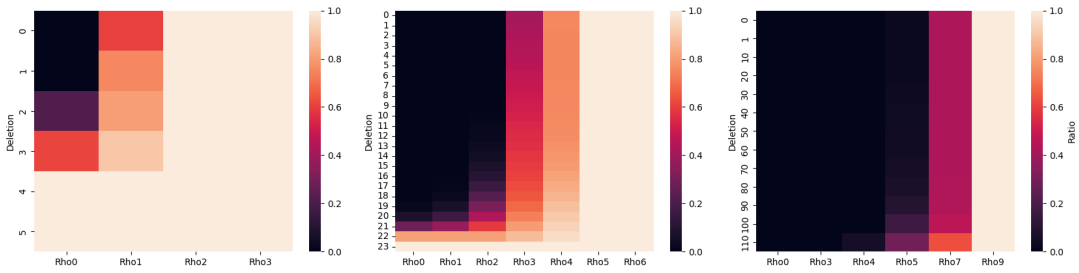


(c) $m = 5$

Figure 4.1: Number of deleted voters plotted against the mean of the number of single-peaked equilibria. Each colour represents a different value for the delegation threshold ρ.

Hypothesis 4.1

As expected, the number of single-peaked equilibria decreases as voters are deleted. If $\rho = \tau_m$ every singleton is a kernel, thus one deletion results in the linear decrease of one kernel. As can be seen in Fig 4.1, the decay is slower than linear if $\rho = \tau_m - 1$. To understand this phenomenon, it helps to take a look at the structure of the equilibria. If $\rho = \tau_m - 1$, the only voter not absorbed by a voter with order $\succ$, is the voter associated with the reversed order $\overline{\succ}$. Thus, a single-peaked equilibrium consists of two voters with reversed orders.

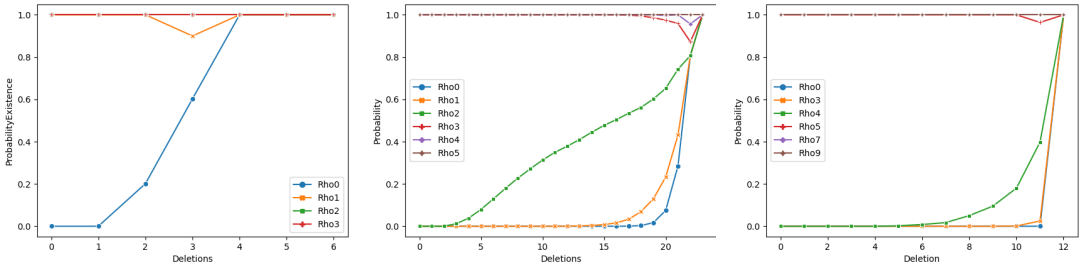


(a) $m = 3$ (b) $m = 4$ (c) $m = 5$

Figure 4.2: Each cell represents a combination of the delegation threshold ρ and number of deletions. The colour depicts the ratio between single-peaked and total equilibria. Since equilibria always exist (Theorem 8), this value is always defined, and ranges from zero to one.

A deletion however does not necessarily result in the loss of an equilibrium. As we have established in Proposition 6, an equilibrium K can be replaced by a subset $K' \subset K$. If $\succ$ is the deleted order, the singleton containing the voter with order $\overline{\succ}$ is an equilibrium, since $\succ$ is the only order not absorbed by $\overline{\succ}$. Any single-peaked equilibrium containing two reversed orders is replaced by a singleton subset. This is confirmed by Figure 4.4 depicting the size of the single-peaked equilibria. For $\rho = \tau_m - 1$, this number is equal for size one and two. Only once the likelihood that two reversed orders are deleted, the decrease of single-peaked equilibria becomes visible (Fig 4.1). If voters are opinionated, the number of single-peaked equilibria becomes larger than zero, only after some deletions have been performed. For $m = 4$, only after 71% of voters have been deleted, there is more than one single-peaked equilibrium. For $m = 5$ the mean never exceeds one.

If voters are open-minded the mean of the number of single-peaked equilibria decreases faster than linearly. This is unsurprising, as the same can be observed for the number of equilibria—including non-single-peaked ones. Curiously though, the latter decreases faster than the former. Figure 4.2 depicts the ratio of *single-peaked* equilibria to total equilibria. For large delegation thresholds, this ratio equals one: Every equilibrium is a single-peaked equilibrium. As can be seen especially for $m = 4$, the ratio increases as voters are deleted. The ratio is noticeably bigger than 0 after many deletions, even though voters are opinionated. The absolute number of single-peaked equilibria is small, however the probability that an equilibrium is single-peaked is not negligible. We will discuss the implication of this result in Section 4.5. The reasons for the



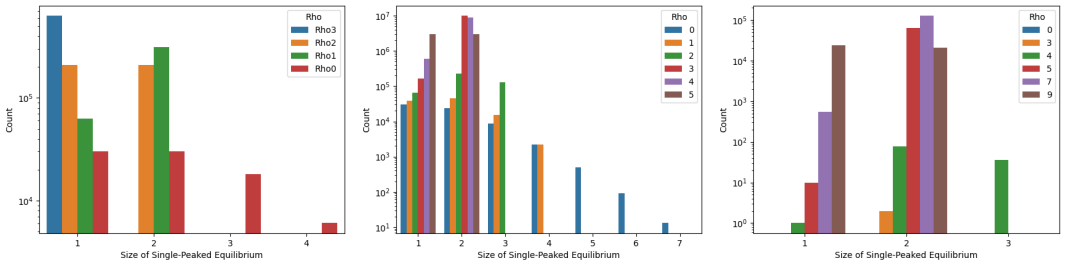
(a) $m = 3$ (b) $m = 4$ (c) $m = 5$

Figure 4.3: Number of deleted voters plotted against the probability that single-peaked equilibrium exists, i.e. that the number of single-peaked equilibria is larger than 1. Each colour represents a different value for the delegation threshold ρ.

increasing ratio shine a light on the role of single-peakedness in Proposition 7 which concerned how the deletion of a voter s_i results in a new single-peaked equilibrium S. Essentially, Proposition 7 established that s_i is not absorbed by S, and the deletion then makes S a single-peaked equilibrium. The following scenario explains the phenomenon of the increasing ratio. Assume $S \cup \{s_i\}$ is kernel, but not single-peaked kernel due to s_i. Then if s_i is the only voter not absorbed by S, S is single-peaked kernel if s_i deleted. Therefore, there is a new kernel. Disregarding the single-peakedness, this scenario cannot be the case since $S \cup \{s_i\}$ already is a kernel. Thus, no 'new kernel' is created but only *replaced* with the subset $S \subset S \cup \{s_i\}$. This furthermore explains, why the number (not the mean!) of *total* equilibria is strictly decreasing, while the number of *single-peaked* equilibria is not.

Hypotheses 4.2 & 4.3

For delegation structures with open-minded voters, there almost always exist single-peaked equilibria, as can be seen in Figure 4.3. Only in small profiles does the average probability drops below one. The importance of the threshold $\rho = \lfloor \frac{\tau_m}{2} \rfloor$ can be seen in Figure 4.3. For opinionated voters, the probability that single-peaked equilibria exists is very low. The only exception is for $\rho = 2$ where $m = 4$. We observe continuous increase while this increase fails to materialize for the analogous threshold of $\rho = 4$ for $m = 5$. One explanation can be the difference in the number of runs performed (30,000 for $m = 4$, and 1,000 for $m = 5$). However, the number of voters in the domains that are of maximal



(a) $m = 3$ (b) $m = 4$ (c) $m = 5$

Figure 4.4: Size of single-peaked equilibria. Each colour represents a different value of the delegation thresholds ρ.

distance to a single-peaked domain seems to be a more significant difference. If voters are opinionated, for a $\succ$-single-peaked equilibrium to exist, all voters with a distance larger than $\frac{\tau_m}{2}$ to the $\succ$-single-peaked domain need to be deleted. The number of these voters seems to grow faster than linearly with respect to the number of alternatives. Thus, the likelihood that these voters are all deleted decreases. If this conjecture is true, the threshold $\lfloor \frac{\tau_m}{2} \rfloor$ becomes only more rigid, as the number of alternatives grows.

Hypotheses 4.4 & 4.5

Figure 4.4 confirms Hypotheses 4.4, and 4.5. In open-minded societies, we do not observe any single-peaked equilibria of size larger than two. For opinionated voters the picture looks different. For $m = 3$, we observe that kernels of size three and four occur, although less frequently than of size one or two. This holds analogously for $m = 4$, where kernels of size up to seven exist if $\rho = 0$, up to size four if $\rho = 1$, and up to size three if $\rho = 2$. We take this to be evidence for the upper bound of $\frac{2^{n-1}}{\rho+1}$ proposed in Hypothesis 4.5.

4.5 Discussion

Before discussing the results, we devote a paragraph to analyzing the robustness of the data. While the sample size of $30,000$, and $1,000$, for three and four alternatives, and for five alternatives respectively, is not particularly large, regularities in the data speak for its representativeness. For each deletion, the number of single-peaked equilibria seem to be normally distributed around the

mean, with relatively small standard deviation (see the Appendix (Figure A.1) for a selection and brief discussion of graphs). We furthermore observe similar, if not the same, results for three, four and five alternatives.[4] Arguably, making similar observation counts as evidence for reliability. While each observation is not conclusive evidence for the robustness of the data, we take the multitude of evidence as sufficient for the robustness.

The results confirm that the distinction between opinionated voters and open-minded voters is not only sensible, but marks a crucial threshold for the existence of single-peaked equilibria. Although the results from Chapter 3 do not generalize in its entirety to the universal domain, a slight weakening holds. We have seen that single-peaked equilibria exist even though voters are opionated, but only after a substantial amount of deletions have been performed. Recall the main result from Chapter 3 under the assumptions of a complete social network, homogeneous thresholds, and complete domains:

Theorem 22. *Single-peaked equilibria exist if and only if voters are open-minded.*

Generalizing to the universal domain, we take the Monte Carlo simulation to be evidence for the following weakening of Theorem 22.

- If voters are open-minded, single-peaked equilibria are "very likely" to exist.
- If voters are opinionated, single-peaked equilibria only exist if the domain is "small".

However, the probability that single-peaked equilibria exist is not the only relevant result. We have seen that the ratio of single-peaked equilibria to all equilibria increases, as voters are deleted. Thus, the share of single-peaked equilibria increases if the profile decreases in size. This is particularly interesting in light of the convergence result (Theorem 23) by Escoffier et al. (2020). Theorem 23 stated that a best response dynamic always converges to an equilibrium. Crucially, this is the case for both opinionated and open-minded voters. Thus, even though the total number of single-peaked equilibria might be low, the probability that an actual vote under the best response dynamic converges to a single-peaked equilibrium is not negligible.

[4] With the slight exception of the probability for the existence of equilibria for $m = 4$, and $\rho = 2$. However, as argued in the previous section, we believe to have found an explanation for this phenomenon.

We conclude this chapter with a remark about the cardinality of single-peaked equilibria. In open-minded societies, the maximal cardinality is two. Furthermore, most single-peaked equilibria consist of voters with two reversed orders. Essentially, this means that single-peaked equilibria consist of people with opposite opinions. Delegations therefore lead to an absolute polarization of opinions, paired with an aggregation of voting power in only two hands. Chapter 6 concludes with a more thorough discussion of this phenomenon, and implications for the 'democraticity' of liquid democracy.

Chapter 5

Counting Single-Peaked Electorates: Real Life Data

In the previous chapter we generalized the existence characterization from the complete domain to the universal domain with the help of a Monte Carlo simulation. We randomly created synthetic profiles through voter deletion, and concluded that while the delegation threshold $\lfloor \frac{\tau_m}{2} \rfloor$ is no longer a logical necessity, it remains a crucial factor. This chapter takes the step from theoretical results to analyzing real life data collected in the PrefLib library (Mattei and Walsh, 2013). Again, we assume the social network to be complete, thus the missing constituent to form a delegation structure $\langle G, \boldsymbol{P}, \boldsymbol{\rho} \rangle$ is the delegation thresholds profile $\boldsymbol{\rho}$. In order to make our analysis as realistic as possible, we dismiss the assumption of homogeneity, and for the first time in this thesis dedicate our attention to *heterogeneous* thresholds. Similarly to the findings in Chapter 3, an analytical analysis is not feasible. And similarly to our proceeding in Chapter 3, a Monte Carlo simulation offers a good computational alternative. By drawing individual delegation thresholds from a normal distribution, we randomize over $\boldsymbol{\rho}$. The resulting delegation structures $\langle G, \boldsymbol{P}, \boldsymbol{\rho} \rangle$ are then analyzed for single-peaked equilibria.

We begin this chapter with a description of the Monte Carlo simulation, and formally introduce the random variables. We proceed with a brief discussion of the PrefLib data, and our choice of profiles. The parameter choice for the simulation pose some problems (Section 5.2.3), which we will solve with a probabilistic argument, about the likelihood that single-peaked equilibria exist, leading us to the simulation and analysis thereof (Section 5.4).

5.1 Monte Carlo Simulation

Once again, we assume that voters are part of a complete social network G, and can therefore delegate to every voter. Given a profile $\boldsymbol{P}$, there is therefore a function from the delegation thresholds $\boldsymbol{\rho}$ to a delegation structure $\langle G, \boldsymbol{P}, \boldsymbol{\rho} \rangle$. Recall from Chapter 3 the function assigning a delegation structure to a set $\mathcal{K} \subseteq \mathcal{P}(N)$ containing all single-peaked equilibria in $\langle G, \boldsymbol{P}, \boldsymbol{\rho} \rangle$. Given a profile $\boldsymbol{P}$, we define the function g to map a delegation threshold $\boldsymbol{\rho} = (\rho_1, ..., \rho_n) \in \mathbb{R}^n$ with $\rho_i \in \mathbb{R}$ for each voter $i \in N$ to the set of single-peaked equilibria $\mathcal{K} \subseteq \mathcal{P}(N)$ in the delegation structure $\langle G, \boldsymbol{P}, \boldsymbol{\rho} \rangle$.

$$\begin{aligned} g : \mathbb{R}^n &\to \mathcal{P}(\mathcal{P}(N)) \\ \boldsymbol{\rho} &\mapsto \mathcal{K} \end{aligned} \tag{5.1}$$

Based on the function g, we can define random variables analogously to Chapter 4. The sample space Ω contains as outcomes every possible heterogeneous delegation threshold $\boldsymbol{\rho} \in \mathbb{R}^n$. Defining the sample space only on the delegation thresholds suffices since the social network G and the profile $\boldsymbol{P}$ are given. We define the random variable X as a mapping from Ω to the number of single-peaked equilibria $\mathcal{K}$ in $\langle G, \boldsymbol{P}, \boldsymbol{\rho} \rangle$ based on the function g (Equation 5.1).

$$\begin{aligned} X : \mathbb{R}^n &\to \mathbb{R} \\ \boldsymbol{\rho} &\mapsto |g(\boldsymbol{\rho})| \end{aligned} \tag{5.2}$$

Furthermore, we let X^{tot} be the random variable counting *all* equilibria, including the ones which are not single-peaked.

In order to get a deeper understanding of the foundation of the Monte-Carlo simulation, and to show that the random variables are well-defined, we show that Ω is part of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For all $i \in N$, we define Ω_i to contain all outcomes of drawing an individual delegation threshold $\rho_i \in \mathbb{R}$. Since Ω is a vector of individual delegation thresholds, let $\Omega = \Omega_1 \times ... \times \Omega_n$. Furthermore, we assume that all delegation thresholds ρ_i are identically, independently *normally* distributed. In particular, we assume $\rho_i \sim \mathcal{N}(\mu, \sigma^2)$ for all $i \in N$, where μ denotes the mean and σ the standard deviation. Since the normal distribution is a well-defined probability distribution, for each voter $i \in N$, so is the probability space $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$. Since all $\mathbb{P}_i$ are independent, and $\mathbb{P}_i(\Omega_i) = 1$, we have $\mathbb{P}(\Omega) = 1$, and consequently $(\Omega, \mathcal{F}, \mathbb{P})$ forms a probability space. We refer the reader to Remark 1 for a more formal definition of the probability space, including a breve exploration of the event space $\mathcal{F}$.

A single run of the Monte Carlo simulation consists of drawing delegation thresholds $\rho_i \in \mathbb{R}$ independently from the normal distribution $\mathcal{N}(\mu, \sigma^2)$ for all

voters $i \in N$. This gives rise to a delegation threshold $\boldsymbol{\rho} = (\rho_1, ..., \rho_n) \in \mathbb{R}^n$, on which the random variables were defined. In the following paragraphs, first the Python implementation, and then the choice of the parameters μ (mean), and σ (standard deviation), as well as the choice of profiles from PrefLib, and the number of runs is explained.

Remark 1. *More formally, the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is defined as a finite product of probability spaces $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$. By defining a probability space $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$ for each individual delegation threshold ρ_i, we can define the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as the product of each individual probability space as follows.*

$$\begin{aligned} \Omega &= \Omega_1 \times ... \times \Omega_n = \mathbb{R}^n \\ \mathcal{F} &= \mathcal{B}(\mathbb{R}^n) \\ \mathbb{P}(\omega_1, ..., \omega_n) &= \prod_{i=1}^{n} \mathbb{P}_i(\omega_i) \end{aligned}$$

This definition can be explained straightforwardly for the sample space Ω, as well as the probability distribution $\mathbb{P}$. However, the definition for the event space $\mathcal{F}$ requires some measure theoretical, and topological work, which we will omit diving into. Essentially, the complexity arises due to uncountability of the sample space. Unlike for the countable case (Chapter 4), admitting every possible subset of an uncountable set as events, leads to paradox (Tao (2011), Chapter 1) We refer the interested reader to Tao (2011) for an introduction to measure theory, and reasons why the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$ represents a suitable event space.

5.2 Experimental Planning and Setup

We discuss the Python implementation of the Monte Carlo simulation as well as the data set selection. We choose the parameters mean and standard deviation mainly based on the running time while a probabilistic argument will be of further aid to justify our choice. Finally, we propose a number of hypotheses.

5.2.1 Python Implementation

Again, we assume the social network G to be complete for every delegation structure $\langle G, \boldsymbol{P}, \boldsymbol{\rho} \rangle$. Since the synthetic profiles constructed in Chapter 4 are of

the same structure as the PrefLib profiles, we can follow a similar setup to Chapter 4. The first difference is that profiles are not generated but given. Secondly, we need to generate heterogeneous delegation thresholds drawn from the normal distribution instead of homogeneous thresholds. For each voter the function `random.normal`(μ, σ) from the NumPy package (Harris et al., 2020) generates random delegation thresholds from the normal distribution. Since thresholds are heterogeneous, the induced acceptability digraph is no longer symmetric. The function `list_all_cliques`(G) in NetworkX, which is defined only on symmetric graphs, is thus no longer sufficient for kernels. However, two vertices are independent in a directed graph if and only if they are independent in the underlying undirected graph. In order to list all kernels, we therefore first list all maximally independent sets in the undirected graph of the acceptability (di)graph. We then use a naive algorithm to check whether a maximally independent set is absorbing in the directed graph. In a final step, each kernel is inspected for single-peakedness, and the single-peaked kernels are stored in a Pandas dataframe. Effectively, the dataframe stores the output of the function g (Equation 5.1). The values of all random variables can be derived from the dataframe, as they are based on the function g.

5.2.2 Data Set Construction

The PrefLib data set contains 315 profiles with strict linear orders, none of which are single-peaked, making all of them in principal eligible for the investigation of single-peaked equilibria. The main criterion, as in the previous chapter, is the running time of the simulation. As we have seen, the running time increases exponentially with the number of voters. Some test runs yield that maximally ~ 450 voters are feasible leaving us with 191 profiles. From these profiles, there are 65 profiles in which the only possible single-peaked equilibria are singletons, as no two voters in these profiles are single-peaked together. This is mainly due to the fact, that these profiles contain few voters with large ballots. We furthermore exclude all profiles contain less than 30 voters, as delegation dynamics are limited with few voters. This leaves us with ~ 80 profiles with at most 450 voters with ballots of size three or four, 15 profiles with ballot size nine to 15 and 30 voters, and two profiles with ~ 150 voters and ballots of size seven and nine.

We consider the last two profiles the most interesting, as they provide a relatively large number of voters, with an *incomplete* preference domain. Furthermore, we pick two profiles with ballot sizes three and four, as their underlying domain is *complete*. With these profiles we analyze the impact of heteroge-

neous thresholds on complete domains, while the former two provide us with data on incomplete domains. The profiles chosen are 'ED-00004-00000005' (448 voters, ballot size 3), 'ED-00004-00000096' (371 voters, ballot size 3), 'ED-00004-00000140' (352 voters, ballot size 4), 'ED-00004-00000160' (350 voters, ballot size 4), 'ED-00009-00000001' (146 voters, ballot size 9), and 'ED-00009-00000002' (153 voters, ballot size 7).

5.2.3 Parameter Choice

Since the running times for the profiles 'ED-00009-00000001' and 'ED-00009-00000002' are relatively low, our parameter choice is liberal. We perform 500 runs for means $\mu \in \{15, ..., 31\}$, and $\mu \in \{8, ..., 20\}$ respectively, with $\sigma \in \{0.25, 0.5, 0.75, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0\}$. The values for the mean μ are mainly motivated by the distances $\lfloor \frac{\tau_m}{2} \rfloor$ and τ_m. The maximal distance τ_m is 36, and $\lfloor \frac{\tau_m}{2} \rfloor$ is 18 for 'ED-00009-00000001', and $\tau_m = 21$ with $\lfloor \frac{\tau_m}{2} \rfloor = 10$ for 'ED-00009-00000002'. Through this choice the impact of both the mean and the standard deviation on the single-peaked equilibria can be studied.

For the profiles with ballot size three and four however, the number of voters is with >350 relatively large, resulting in long running times. For $\mu < \lfloor \frac{\tau_m}{2} \rfloor$ multiple runs are not feasible. However, a probabilistic argument shows that the likelihood that single-peaked equilibria exist is low if $\mu < \lfloor \frac{\tau_m}{2} \rfloor$. Based on the profile $\boldsymbol{P}$, the mean μ and standard deviation σ of the normal distribution, we calculate a lower bound for the probability that *no* single-peaked equilibrium exists. A large lower bound on the probability that *no* single-peaked equilibrium exists is equivalent to a small upper bound on the probability that single-peaked equilibria exist since they are complementary events. In cases where this bound is sufficiently small, we can therefore refrain from running the simulations: It is highly unlikely that we will find single-peaked equilibria. Fortunately for us, we will see that this is exactly the case whenever running times are very large. We present the argument in the following subsection, and then continue to apply it to the parameter choice.

Probabilistic Argument

In the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we defined in Section 5.1 for the Monte Carlo simulation, let $K \subseteq \mathcal{F}$ be the event that *no* single-peaked equilibrium exists, and $E \subseteq \mathcal{F}$ the event that for every order $\succ$ there is a voter i with order $\succ^\pi$ and threshold $\rho_i < \lfloor \frac{\tau_m}{2} \rfloor$. We show that $E \subseteq K$, and thus $\mathbb{P}(E) \leq \mathbb{P}(K)$. In words, the probability of the event E constitutes a lower bound for the probability that single-peaked equilibria do not exist. We proved in Chapter 3, if $\rho_i < \lfloor \frac{\tau_m}{2} \rfloor$

for all $i \in N$, then no single-peaked kernels exist (Theorem 21). Recall the foundation of the proof: The existence of an order $\succ^\pi$ which is 'too far' from the $\succ$-single-peaked domain $\mathcal{SP}_\succ$ for it to be absorbed by any subset $S \subseteq \mathcal{SP}_\succ$. More precisely, if a voter with preference $\succ^\pi$ has a threshold smaller than $\lfloor \frac{\tau_m}{2} \rfloor$, S does not absorb $\succ^\pi$, and is therefore not an equilibrium. In terms of the events E and K, this implication translates into $E \subseteq K$, and consequently $\mathbb{P}(E) \leq \mathbb{P}(K)$.

We can calculate the probability of event E, and thus a lower bound on $\mathbb{P}(K)$, the probability that single-peaked equilibria exist. First, partition the set of voters N according to their associated orders. Let $N(\succ) = \{i \in N | \succ_i = \succ\}$ contain all voters $i \in N$ with preference $\succ$.

Proposition 8. *If delegation thresholds are identically and independently distributed, and every order is submitted at least once, then the probability that no single-peaked equilibria exist* $\mathbb{P}(K)$ *is bounded from below by*

$$\mathbb{P}(K) \geq \prod_{\succ \in \mathcal{L}(A)} 1 - p^{|N(\succ)|}$$

where p *denotes the probability that a voter is open-minded.*

Proof. Let $E_\succ$ be the event that for all $i \in N(\succ)$, $\rho_i \geq \lfloor \frac{\tau_m}{2} \rfloor$, in words every voter with preference $\succ$ has a delegation threshold larger than $\lfloor \frac{\tau_m}{2} \rfloor$. As above, let E denote the event that for every order $\succ$ there is a voter i with order $\succ^\pi$ and threshold $\rho_i < \lfloor \frac{\tau_m}{2} \rfloor$. Thus, E denotes the event that for all orders $\succ$, $E_\succ$ is not the case.

$$E = \bigcap_{\succ \in \mathcal{L}(A)} \overline{E_\succ}$$

Assume that each threshold ρ is drawn independently from the same distribution (in our case $\rho \sim \mathcal{N}(\mu, \sigma^2)$). The probability of E is therefore the product of the probabilities $\overline{E_\succ}$ of all $\succ \in \mathcal{L}(A)$.

$$\begin{aligned} \mathbb{P}(E) &= \prod_{\succ \in \mathcal{L}(A)} \mathbb{P}(\overline{E_\succ}) \\ &= \prod_{\succ \in \mathcal{L}(A)} 1 - \mathbb{P}(E_\succ) \end{aligned}$$

For all voters, the probability that their delegation threshold is larger than $\lfloor \frac{\tau_m}{2} \rfloor$ is identical. Denote this probability by p. The event that all voters with preference $\succ$ have a delegation threshold larger than $\lfloor \frac{\tau_m}{2} \rfloor$ is therefore $p^{|N(\succ)|}$.

$$\mathbb{P}(E) = \prod_{\succ \in \mathcal{L}(A)} 1 - p^{|N(\succ)|}$$

Since $E \subseteq K$ we get

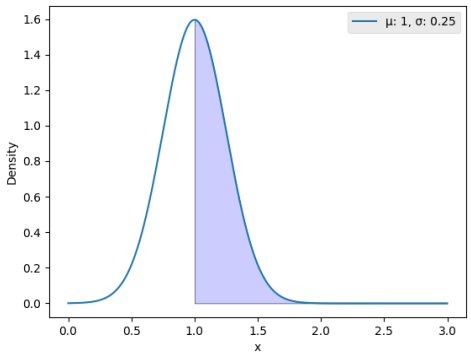
$$\mathbb{P}(K) \geq \prod_{\succ \in \mathcal{L}(A)} 1 - p^{|N(\succ)|} \qquad \square$$

Furthermore, note that given any distribution $\mathbb{P}$ of the delegation thresholds ρ with probability density f, the probability p that a voter $i \in N$ is open-minded, can be calculated in standard practice through the following integral.

$$\begin{aligned} p &= \mathbb{P}\left(\rho_i \geq \frac{\tau_m}{2}\right) \\ &= \int_{\lfloor \frac{\tau_m}{2} \rfloor}^{\infty} f_\rho(x)dx \end{aligned} \tag{5.3}$$

We apply Proposition 8 to support the readers intuition of its consequences.

Example 9. The profile ED-0004-0000005 contains 448 voters, where the number of voters per order are as follows: 191, 119, 41, 38, 32, 27. Note that $\frac{\tau_m}{2} = 1.5$ since the ballots are of size three. For $\rho \sim \mathcal{N}(\mu, \sigma^2)$ with $\mu = \lfloor \frac{\tau_m}{2} \rfloor = 1$ we get that $p = 0.5$. In the graph below, the shaded area depicts the probability p.



$$\begin{aligned} \mathbb{P}(E) &= \prod_{\succ \in \mathcal{L}(A)} 1 - 0.5^{|N(\succ)|} \\ &= (1 - 0.5^{191}) \cdot \ldots \cdot (1 - 0.5^{27}) \\ &\approx 0.999999992 \end{aligned}$$

Since $\mathbb{P}(E) \leq \mathbb{P}(K)$, the probability that no single-peaked equilibrium exists is therefore large if the mean is $\mu = 1$. $\triangle$

Let us briefly discuss the limitations of Proposition 8. In Proposition 8 it is easy to verify, that both an increase in the probability p that a voter is open-minded, and an increase in the number of voters, result in a larger likelihood

that a single-peaked equilibrium exists. Unfortunately, Proposition 8 only bears informational value if p does not exceed some threshold. In case it does, the calculated lower bound is not a good estimation for the probability of the *existence* of single-peaked equilibria as the following example shows. Consider again Example 9, but this time with $\rho \sim \mathcal{N}(1.5, 0.25^2)$ which yields $p = 0.9772$. This results in $\mathbb{P}(K) \geq 0.008$, in words the probability that *no* single-peaked equilibria exist is with 0.008 very low.[1] However, this does not mean that the probability that a equilibrium actually exists is high. We can deduce that the probability that a single-peaked kernel exists is *at most* $1 - 0.008 = 0.992$. This is not informative since the actual probability can still be any number between 0 and 0.992. In particular, in a test simulation with 50 runs with $\rho \sim \mathcal{N}(1.5, 0.25^2)$, there were single-peaked kernels in only four runs.

Fortunately, we can run simulations for the cases in which the lower bound calculated through Proposition 8 is not informative, while we can exclude simulations with long running times based on Proposition 8.

Running Times

Recall again the purpose of Proposition 8: Excluding certain probability distributions from the simulations by showing that the probability that single-peaked equilibria exist is small. Table 5.1 depicts the lower bound probabilities for the non-existence of single-peaked equilibria $\mathbb{P}(K)$ and running times for a selection of means μ and standard deviations σ. The probability that no equilibrium exists $\mathbb{P}(K)$ has been calculated analogously to Example 9. The probability p that a voter is open-minded, is calculated according to Equation 5.3, where the density is given by the normal distribution with mean μ and standard deviation σ stipulated in Table 5.1. The first two profiles ('ED-...005', 'ED-...0096') contain three alternatives, thus $\lfloor \frac{\tau_m}{2} \rfloor = 1$. For the means and standard deviations depicted in Table 5.1, the running time is relatively large. We believe that the large running times are due to a large number of independent sets (for one test run there were 260,493,280 many independent sets which had to

[1] It is hard to exactly define a threshold on p but we can stipulate a minimal likelihood c for the event K, and require $\mathbb{P}(K) \geq c$. We then solve analytically for p, leading us to an upper bound on p, which in turn gives us insights about the choice of the parameters μ and σ of the normal distribution. For example, if we want to be 99% sure that no single-peaked equilibria exist (i.e. $\mathbb{P}(E) > 0.99$) in the profile from Example 9, it can be calculated that $p < 0.837$. Thus, any distribution that results in $p < 0.837$ will lead to no single-peaked equilibria with a certainty of at least 99%. However, while this analytic approach gives us interesting insights, the aim of the argument is to facilitate the parameter selection based mainly on the running times of the Monte Carlo simulation.

Profile	μ	σ	p	$\mathbb{P}(K)$	**Running Time**
ED-...005	1	0.25	0.5	≈ 0.9999	>30min
	1	0.5	0.5	≈ 0.9999	>30min
	1.5	0.5	0.8413	0.9844	10min
ED-...096	1	0.25	0.5	≈ 0.9999	>30min
	1	0.5	0.5	≈ 0.9999	>30min
	1.5	0.5	0.8413	0.9971	10min
ED-...140	2	0.25	3.167e-05	≈ 1	>30min
	2	0.5	0.0228	≈ 0.9999	>30min
	2.5	0.25	0.0228	≈ 0.9999	>30min
	2.5	0.5	0.1587	0.9992	>30min
	3	0.25	0.5	0.8712	14min
ED-...160	2	0.25	3.167e-05	≈ 0.9999	>30min
	2	0.5	0.0228	0.9767	>30min
	2.5	0.25	0.0228	0.9767	>30min
	2.5	0.5	0.1587	0.8135	>30min
	3	0.25	0.5	0.2539	11min

Table 5.1: Running times, probability p that a voter is open-minded, and $\mathbb{P}(K)$ for the complete profiles depending on the mean and standard deviation. Experiments were run on a MacBook Pro with 2 GHz Quad-Core Intel Core i5, 16Gb Ram, running MacOS Big Sur 11.6.

be checked for absorbance and single-peakedness for $\mu = 1$, $\sigma = 0.5$). Fortunately, the probabilities $\mathbb{P}(K)$ are large as well, and we can discard simulations for these values since the probability that single-peaked equilibria exist is very small. The same argument holds for the profile 'ED-...140', where $\lfloor \frac{\tau_m}{2} \rfloor = 3$. For the profile 'ED-...160' the red values in Table 5.1 pose a problem, since the running time is large while the probability $\mathbb{P}(K)$ is not sufficiently large to discard this mean-standard deviation combination. We therefore exclude all means and standard deviations with large running times based on the the large probability that single-peaked equilibria do not exist. The parameters chosen are as follows:

- ED-...05: 100 runs for $\mu \in \{1.75, 2.0, 2.5\}$ and $\sigma \in \{0.25, 0.5\}$, and $\mu = 1.5$ with $\sigma = 0.25$
- ED-...96: 100 runs for $\mu \in \{1.75, 2.0, 2.5\}$ and $\sigma \in \{0.25, 0.5\}$
- ED-...140: 100 runs for $\mu \in \{3.0, 3.5, 4.0, 5.0\}$ and $\sigma \in \{0.25, 0.5\}$

- ED-...160: 100 runs for $\mu \in \{3.0, 3.5, 4.0, 5.0\}$ and $\sigma \in \{0.25, 0.5\}$, and $\mu \in \{3.5, 4.0, 5.0\}$ with $\sigma = 1.0$

5.2.4 Hypotheses

We expect that the delegation threshold of $\lfloor \frac{\tau_m}{2} \rfloor$ materializes as an important value once again. Since in this chapter the delegation thresholds are subject to randomness, we first formulate a probabilistic variant of previous results. Like before, in a delegation structure $\langle G, \boldsymbol{P}, \boldsymbol{\rho} \rangle$ denote by p the probability that a voter i is open-minded, i.e. $\rho_i \geq \frac{\tau_m}{2}$, according to $\rho_i \sim \mathcal{N}(\mu, \sigma^2)$.

Hypothesis 5.1. As p increases, the number of single-peaked equilibria, as well as the probability that single-peaked equilibria exist, increases.

The following two hypothesis concern the role of the mean and the standard deviation.

Hypothesis 5.2. If $\mu \leq \lfloor \frac{\tau_m}{2} \rfloor$, the likelihood that single-peaked equilibria exist is small.

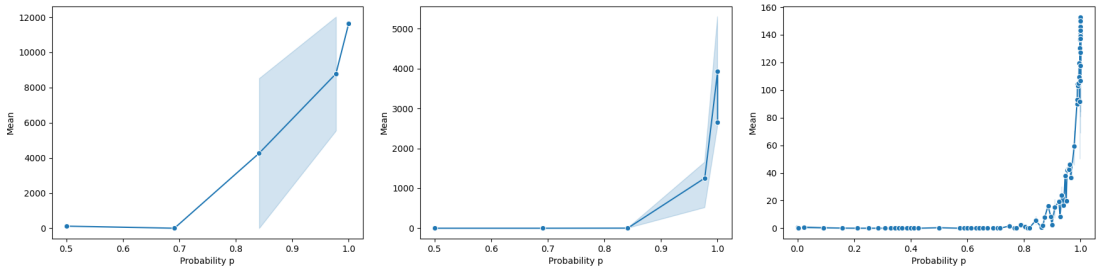
For the case where $\mu > \lfloor \frac{\tau_m}{2} \rfloor$, we expect the standard deviation to play an important role. As an elucidating example, suppose $\mu = \lfloor \frac{\tau_m}{2} \rfloor + 1$. If the standard deviation is very small, most of the probability mass of $\mathcal{N}(\mu, \sigma^2)$ lies above $\lfloor \frac{\tau_m}{2} \rfloor$, and we expect that single-peaked equilibria exist. However, if σ is large, a substantial part of the probability mass is located below $\lfloor \frac{\tau_m}{2} \rfloor$, resulting in a lower likelihood for the existence of single-peaked equilibria. It will be interesting to see if we can quantify an upper bound for σ based on μ.

Hypothesis 5.3. If $\mu > \lfloor \frac{\tau_m}{2} \rfloor$, the likelihood that single-peaked equilibria exist depends on σ.

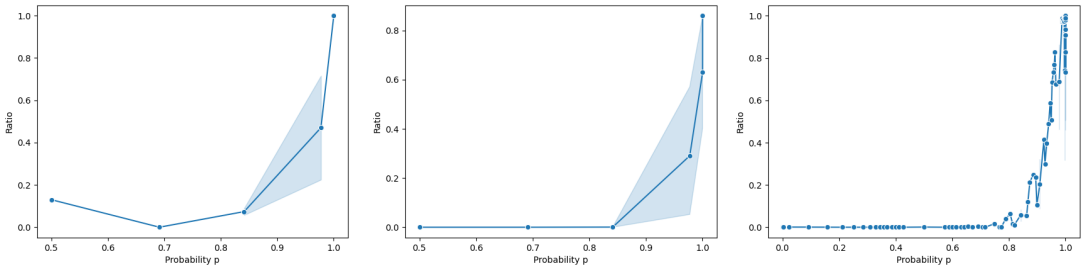
- If σ is small, the likelihood that single-peaked equilibria exist is large.
- If σ is large, likelihood that single-peaked equilibria exist is small.

Finally, we return to the question of the cardinality of single-peaked equilibria. We expect analogous results to Chapter 4. In line with the conjecture that for delegation structures with thresholds larger than $\lfloor \frac{\tau_m}{2} \rfloor$, the cardinality of single-peaked equilibria is bounded from above by two (Conjecture 1), our final hypothesis is as follows.

Hypothesis 5.4. If $\mu \geq \lfloor \frac{\tau_m}{2} \rfloor$, the cardinality of single-peaked equilibria is bounded from above by two.



(a) Number of single-peaked equilibria



(b) Existence probability for single-peaked equilibria

Figure 5.1: Mean (top row) and existence probability (bottom row) for profiles 'ED-00004-...005' (left), 'ED-00004-...160' (middle), and 'ED-00009-...002' (right).

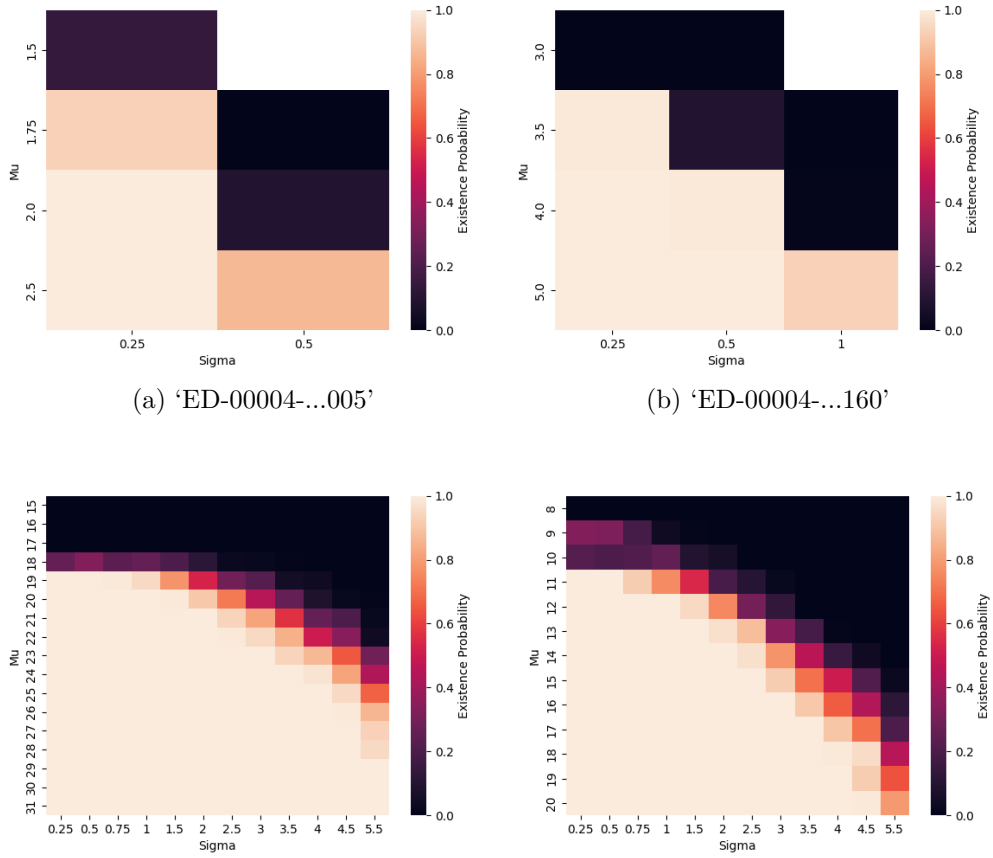
5.3 Results

Hypothesis 5.1

The hypothesis that an increase in the probability that an individual voter has a delegation thresholds larger than $\lfloor \frac{\tau_m}{2} \rfloor$ is positively correlated to (i) the number of and (ii) existence probability for single-peaked equilibria can only be partially confirmed. As can be seen in Figure 5.1, such a trend exists. However, only two of the graphs are monotonically increasing (top left and bottom middle of Figure 5.1). Especially for the profile 'ED-00009-...002' containing seven alternatives, many 'jumps' can be observed. The probability p seems to be an important but not decisive factor.

Hypothesis 5.2 & 5.3

Hypothesis 5.2 stated that the likelihood for single-peaked equilibria is low if the mean is smaller than $\lfloor \frac{\tau_m}{2} \rfloor$. For the profiles containing three and four al-



(a) 'ED-00004-...005'

(b) 'ED-00004-...160'

(c) 'ED-00009-...001'

(d) 'ED-00009-...002'

Figure 5.2: Each cell represents a combination of the mean μ and standard deviation σ. The colour of the cells depicts the probability that a single-peaked equilibrium exists. The two white cells in the top row have not been calculated, and do not contain values.

ternatives, we did not run the Monte Carlo simulation for $\mu < \lfloor \frac{\tau_m}{2} \rfloor$. However, as we can see in Figure 5.2.b the probability is zero for $\mu = \lfloor \frac{\tau_m}{2} \rfloor = 3$. Together with the probabilistic argument in Section 5.2.3 the evidence confirms the hypothesis for three and four alternatives. Note that $\lfloor \frac{\tau_m}{2} \rfloor = 18$ for profile 'ED-00009-...001', and $\lfloor \frac{\tau_m}{2} \rfloor = 10$ for 'ED-00009-...002'. In Figure 5.2 it can be seen that $\mu = 18$, and $\mu = 10$ respectively, mark crucial thresholds for the existence of single-peaked equilibria. For values smaller than these, the probability is very low. We consider Hypothesis 5.2 therefore to be confirmed.

Hypothesis 5.3 can also be confirmed by Figure 5.2. Consider especially subfigures (c) and (d). If $\mu > \lfloor \frac{\tau_m}{2} \rfloor$, the standard deviation plays a crucial role. For values of μ close to the threshold $\lfloor \frac{\tau_m}{2} \rfloor$, a large standard deviation σ impacts the probability negatively. This impact decreases as μ increases. For large μ, it is almost always certain that single-peaked equilibria exist. This is unsurprising, since we have seen that single-peaked equilibria exist if all voters have a delegation threshold larger than $\lfloor \frac{\tau_m}{2} \rfloor$.

Hypothesis 5.4

We cannot confirm Hypothesis 5.4 as the cardinality of single-peaked kernels is not bounded from above by two as we can see in Figure 5.3. Rather unsurprisingly, the profiles containing three or four alternatives do not admit singletons as single-peaked equilibria. As these profiles contain all possible strict linear orders (complete domain), this would require delegation thresholds close to the maximal threshold τ_m for many voters. As the likelihood is small for this to happen, no single-peaked equilibria of size one exist. The opposite is the cases for the profiles containing seven and nine alternatives. The vast majority of single-peaked equilibria are singletons (note the logarithmic scale in Figure 5.3). Since these profiles only contain a small number of preferences with respect to the total number of possible preferences (incomplete domain), and the maximal distance between any two preferences is with 27 and 15 respectively significantly smaller than the maximal possible Kendall tau distance (36 and 21 respectively), the property of independence between more than one or two voters is more easily violated. For these profiles, the maximal size of single-peaked equilibria is indeed two, as claimed in the hypothesis.

However, this maximal size does not hold up for the profiles with three and four alternatives (left and middle of Figure 5.3). As we draw the delegation thresholds randomly, the possibility that voters are associated with thresholds smaller than $\lfloor \frac{\tau_m}{2} \rfloor$ remains. We checked the delegation thresholds of the voters of all single-peaked equilibria larger than two, and found that at most one voter has a delegation thresholds larger than $\lfloor \frac{\tau_m}{2} \rfloor$. While we cannot confirm Hypothesis 5.3, Conjecture 1 is not affected.

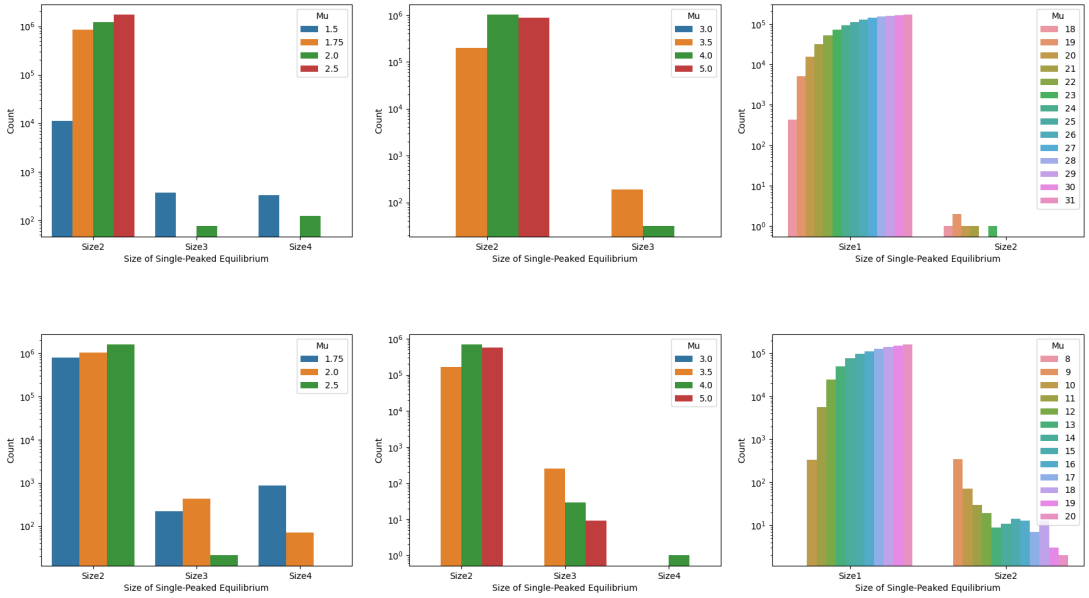


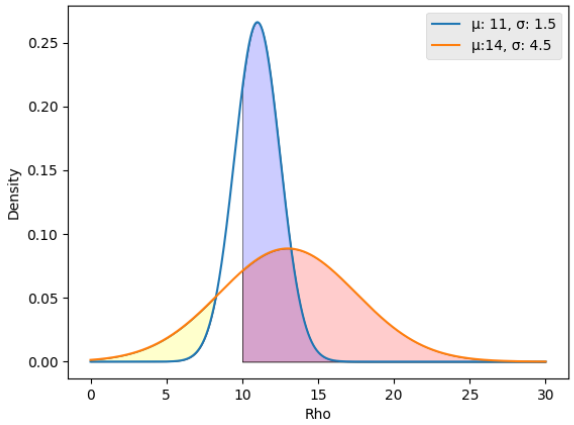
Figure 5.3: Size of single-peaked equilibria for profiles 'ED-00004-...005', 'ED-00004-...140', 'ED-00009-...001' (top row), and 'ED-00004-...096', 'ED-00004-...160', 'ED-00009-...002' (top row). The profiles underneath each other are comparable (either same size, or similar structure).

5.4 Discussion

We have investigated the effect of heterogeneous thresholds on profiles with a complete domain, and on profiles with relatively small domains. For all profiles we can establish $\lfloor \frac{\tau_m}{2} \rfloor$ as an important mark for heterogeneous delegation thresholds. The Monte Carlo simulation was based on the normal distribution, and we found that the likelihood that single-peaked equilibria exist is strongly dependent on the mean μ. Interestingly, neither the mean alone nor the probability p that an individual threshold is larger than $\lfloor \frac{\tau_m}{2} \rfloor$, are fine-grained enough to capture the existence of single-peaked equilibria. If the probability p were fine-grained enough, we could—arguably—extend the results to any distribution. But we have seen that the standard deviation σ plays a decisive role which cannot be captured by p alone. With the following example we propose that the likelihood that voters with particularly small delegation thresholds exist is a critical notion.

Example 10. For the profile 'ED-00009-...002', the probability p equals 0.75 for

both $\rho \sim \mathcal{N}(11, 1.5^2)$ and $\rho \sim \mathcal{N}(13, 4.5^2)$ (colored in blue and red respectively in the graph below). However, the likelihood that a single-peaked equilibrium exists, is with 0.54 significantly larger for the former distribution than for the latter, for which no single-equilibria have been observed.



Our explanation for this difference is the area colored in yellow in the graph. The likelihood that a delegation threshold is significantly smaller than $\lfloor \frac{\tau_m}{2} \rfloor = 10$, is larger for the distribution with a larger standard deviation. These voters are more likely to not delegate to anyone, and consequently prevent the existence of single-peaked equilibria. △

In conclusion, the delegation threshold $\lfloor \frac{\tau_m}{2} \rfloor$ is important but on its own not decisive. The binary distinction between opinionated and open-minded voters is not sufficiently fine-grained for heterogeneous thresholds as Example 10 showed. A further distinction within the category of opinionated voters seems to be necessary, in order to provide a full account for the existence of single-peaked equilibria in delegation structures with heterogenous thresholds. One may want to introduce the notion of *stubborn* voters which have a very small delegation threshold. A more precise definition, and experiments concerning the necessity of this notion remain future work.[2]

[2]The hypothesis that stubborn voters are critical could be investigated through a Monte Carlo simulation on multimodal distributions. Such a distribution, for example, could have one peak within the—to be defined—range of stubborn voters, and another within the range of open-minded voters.

Chapter 6

Conclusions

In this thesis we investigated under which conditions liquid democracy guarantees the existence of single-peaked electorates. Liquid democracy provides voters with the opportunity to delegate votes. Consequently, not every voter submits their ballot—this would be direct democracy—but only a subset of voters, those which choose not delegate their vote (the electorate). *Single-peaked* electorates are particularly interesting since they guarantee problem-free aggregation of individual preferences into a collective choice. In particular, the aggregation problem (Condorcet's paradox, and the threat of dictatorship and manipulation) is avoided if the electorate is single-peaked. We identified conditions under which liquid democracy generates single-peaked electorates out of a non-single-peaked society, thus guaranteeing problem-free aggregation of preferences.

To the framework of Escoffier et al. (2020) which analyzes stable electorates—called equilibria—we added the condition of single-peakedness, and investigated delegations induced by the Kendall tau distance. Besides a preference, each voter is associated with a delegation threshold. Each agent is willing to delegate to anyone closer, and not willing to delegate to anyone with a distance larger than this threshold. Throughout the thesis one delegation threshold manifested itself as a crucial condition for the existence of single-peaked equilibria: half of the maximal Kendall tau distance. We called voters with a smaller threshold *opinionated*, and with a larger threshold *open-minded*.

In Chapter 3 we formally introduced the concept of delegation structures $\langle G, \boldsymbol{P}, \boldsymbol{\rho} \rangle$, and proved under the assumptions of completeness of the social network G, completeness of the preference domain $\mathcal{D}(\boldsymbol{P})$, and homogeneity of the delegation thresholds $\boldsymbol{\rho}$, that voters ought to be open-minded for single-peaked equilibria to exist. In other words, if voters are opinionated liquid democ-

racy never generates single-peaked equilibria under these assumptions. Unfortunately, an open-minded society is a strong requirement, at least depending on the setting and topic. In the political sphere we can readily label this requirement as impossible. In more private settings, choosing a restaurant for example, this requirement is possibly more realistic. However, not only the open-mindedness of voters, but also the three assumptions on the delegation structure are strong. In Chapter 4 we relaxed the complete domain assumption, and established that the distinction between open-minded and opinionated voters upholds even in small profiles. Computationally validated for profiles of ballot size up to five, we argue that the results extend to larger ballot sizes. Leaving the synthetic scenery of Chapters 3 and 4, we performed a Monte Carlo simulation on real life data. To make the simulation as realistic as possible, we dropped the assumption of the homogeneity of the delegation thresholds, and assumed that delegation thresholds are normally distributed. In this setting the distinction between open-minded and opionated voters once again draws the line between the existence and non-existence of single-peaked equilibria. We have seen, however, that this threshold is not decisive on its own, and the general structure of the distribution plays an important role (standard deviation, possible multiple peaks).

All in all, can liquid democracy guarantee problem-free aggregation by bringing about single-peaked (stable) electorates? While the following conclusion does not hold with absolute certainty, this work essentially provides a 'characterization' for the existence of single-peaked, stable electorates:

Conclusion. *For a delegation structure $\langle G, \boldsymbol{P}, \boldsymbol{\rho} \rangle$, if the social network G is complete, liquid democracy guarantees single-peaked, stable electorates if and only if the society is (relatively) open-minded.*

An open-minded society is generally a rather strong requirement, or at least topic sensitive. The importance of topic sensitivity is exemplified in the example of strangers discussing politic, where open-minded voters seem unlikely, compared to friends agreeing on a restaurant, assuming the friend group is more willing to compromise.

Besides the—arguably unrealistic—requirement of an open-minded society, our work undermines a further point of what liquid democracy set out to do. If liquid democracy is motivated by increasing democratic participation, the aggregation of all voting power into the hands of only a few voters has the contrary effect. In all chapters we have seen that an electorate size of larger than two is either unlikely, or impossible. This underpins doubts raised against liquid democracy by Gölz et al. (2018), and empirically confirmed by Kling

et al. (2015). This work furthermore undermines hopes that liquid democracy increases the willingness to compromise. Not only does a single-peaked equilibrium only contain at most two voters (in most cases), these two voters have drastically different if not completely opposite preferences. Instead of deliberating effects, liquid democracy thus seems to have polarizing effects.

There is a final remark to be made, which we did not touch on before. The distinction between open-minded and opinionated voters only makes sense in a society where at least some voters have different opinions. If the maximal distance between any two voters is at most $\lfloor \frac{\tau_m}{2} \rfloor$, the distinction looses its meaning, since an open-minded voter would be willing to delegate to any other voter. We purposely selected the real life data in Chapter 5 to exclude this case, however were not concerned with this in Chapter 4. A further point of investigation would thus include the distinction between homogeneous and heterogeneous *opinions* within the society.

Future Work

The most natural continuation of this work considers a generalization on the social network of a delegation structure $\langle G, \boldsymbol{P}, \boldsymbol{\rho} \rangle$. In Chapter 3 we made assumptions on all three constituents of a delegation structure, dropped the assumption on the profiles $\boldsymbol{P}$ in Chapter 4, and finally dropped the assumption on delegation thresholds $\boldsymbol{\rho}$ in Chapter 5. Since we assumed the social network G to be complete in all chapters, it is natural to analyze the impact of different graph structures of the social network. If G_ρ is the delegation acceptability graph constructed under the complete social network, and G is some restriction of the social network, the final delegation acceptability digraph to be investigated for single-peaked equilibria is $G \cap G_\rho$. There are multiple ways for an analysis. Firstly, one can look for analytical results for trees, stars, and various other common graph structures. If this turns out to be an unfruitful investigation, a Monte Carlo simulation once again is of help. In line with the proceeding in Chapter 4, one could delete edges between voters, instead of vertices. The deletion of an edge thus restricts the range of admissible delegations for individual voters.

In this work we made an assumption that poses some conceptual problems: Each voter is associated with a strict linear order. If everyone knows their preferences, and the preferences are complete, the question arises why anyone would delegate. Delegations are more likely to happen if voters are uncertain about their own preferences. One way to model this scenario would be via *incomplete* preferences. A voter might know that they prefer alternative a over

b, but does not have an opinion on alternatives c.

Furthermore, the Kendall tau distance is by far not the only distance measure. The *Cayley distance* (Diaconis, 1988), for example, is given by the minimum number of (not necessarily adjacent) transpositions of any pair of alternatives. The *Hamming* (Diaconis, 1988) and *Duddy–Piggins* distances (Duddy and Piggins, 2011) are further examples. More creative distance measures can be introduced. The notion of a distance measure could be furthermore extended by sociological factors like trust. Two voters with drastically different preferences might be willing to delegate to each other since they trust each other. This could be mathematically cashed out by introducing a factor by which, for example, the Kendall tau distance between two voters i, j is multiplied.

$$dist(i, j) = \text{trust}(i, j) \cdot \tau(\succ_i, \succ_j)$$

Finally, it is worthwhile to note that single-peakedness is not the only domain restriction that avoids the aggregation problem. While we conclude this work with rather negative results for liquid democracy concerning single-peaked electorates, different properties are possibly generated through the same delegation mechanisms.

Appendix A

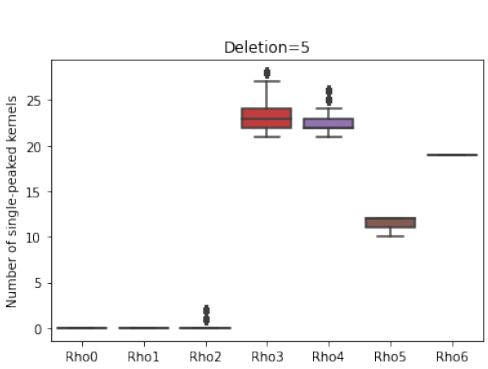
Graphs

A.1 Chapter 4: Robustness of Data

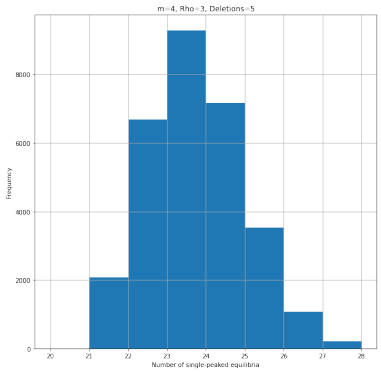
Figure A.1 depicts the boxplot (left) and selected distribution of number of single-peaked kernels (right) for $m = 4$. Note that in subfigures (b), and (c) the minimal and maximal values (whiskers) are spread farther apart as well as more 'outliers' can be noted than in subfigure (a).[1] This can be explained through through the fact that there are significantly more distinct deletion sequences if we delete 12 or 15 voters compared to 5 voters. For 12 deletions there are $\binom{24}{12} = 2,704,156$, for 15 there are $\binom{24}{12} = 1,307,504$, and for 5 there are merely $\binom{24}{5} = 42,504$ distinct deletion sequences. Since distinct deletion sequences lead to distinct profiles, it is not surprising that the larger variety in profiles leads to a larger variety in the maximum and minimum number of single-peaked kernels. Interestingly, the interquartile range (the 'box'), i.e. the range between first and third quartile, does not increase significantly. We take these boxplots to be evidence for the robustness of the data.

In subfigures (b), (d), and (f) we additionally plotted the number of single-peaked kernels against their occurrences. In particular, the subfigure (b) is a closer analysis for parameter 'Rho3' (red box) in subfigure (a). In all subfigures we can see that the single-peaked kernels are roughly normally distributed. We suggest that this is further evidence for the representativeness of the data.

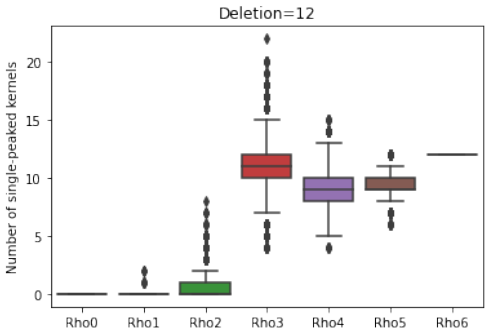
[1] For more details about the function that generates the 'outliers' we refer the reader to the Python package 'Seaborn' with which the boxplot was created. In particular see https://seaborn.pydata.org/generated/seaborn.boxplot.html#seaborn.boxplot



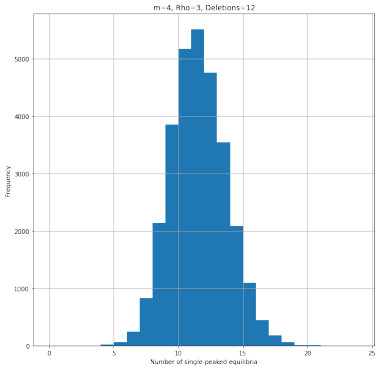
(a) 5 deletions



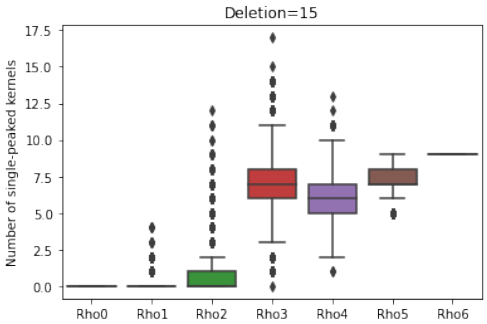
(b) 5 deletions, Rho=3



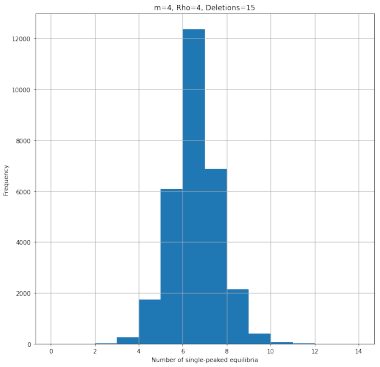
(c) 12 deletions



(d) 5 deletions, Rho=3



(e) 15 deletions

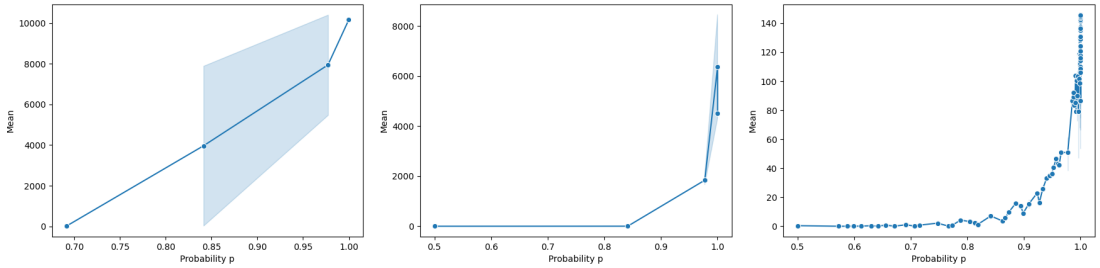


(f) 5 deletions, Rho=4

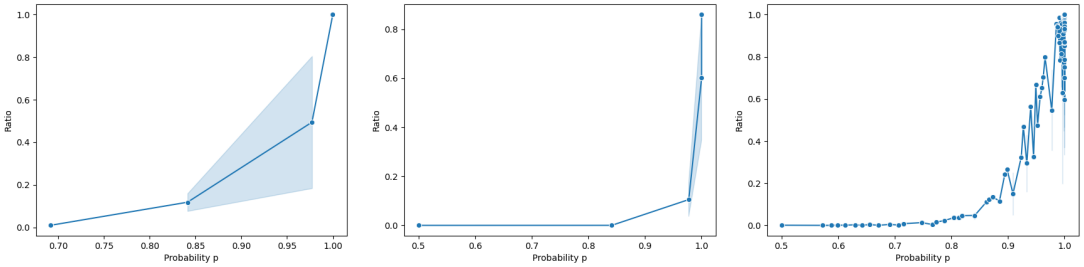
Figure A.1: Mean and standard deviation (left) and selected distribution of number of single-peaked kernels (right), for $m = 4$.

A.2 Chapter 5: Results

This section presents graphs we omitted in the main text for the sake of readability. Figure A.2 presents the mean and existence probability of single-peaked equilibria depending on the probability p that a voter is open-minded for profiles 'ED-00004-...096', 'ED-00004-...140', and 'ED-00009-...001'. Figure A.3 depicts the existence probability of single-peaked equilibria depending on the mean and standard deviation of a normal distribution from which the delegation thresholds of voters were drawn for profiles 'ED-00004-...096' and 'ED-00004-...140'.

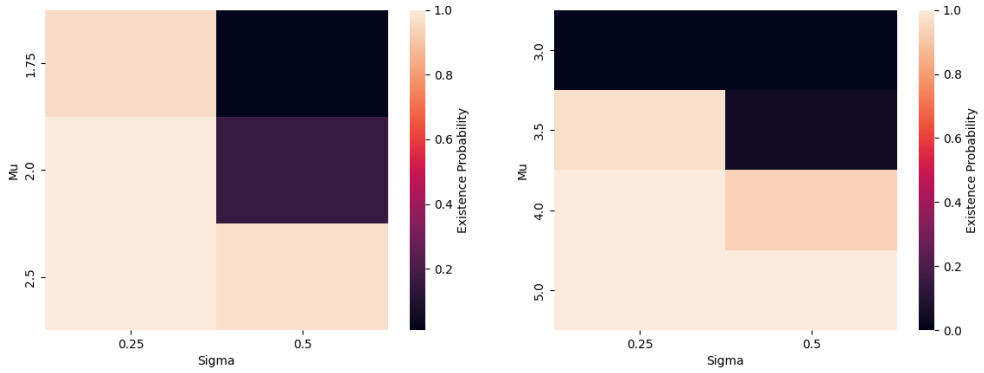


(a) Number of single-peaked equilibria



(b) Existence Probability for single-peaked equilibria

Figure A.2: Mean (top row) and existence probability (bottom row) for profiles 'ED-00004-...096' (left), 'ED-00004-...140' (middle), and 'ED-00009-...001' (right).



(a) 'ED-00004-...0096'

(b) 'ED-00004-...140'

Figure A.3: Each cell represents a combination of the mean μ and standard deviation σ. The colour of the cells depicts the probability that a single-peaked equilibrium exists. The white cells in the top row have not been calculated, and do not contain values.

Bibliography

Arrow, K. J. (1950). A difficulty in the concept of social welfare. *Journal of Political Economy*, 58(4):328–346.

Ballester, M. and Haeringer, G. (2011). A characterization of the single-peaked domain. *Social Choice and Welfare*, 36:305–322.

Behrens, J. (2017). The origins of liquid democracy. *The Liquid Democracy Journal*, 5.

Berge, C. (1985). *Graphs and Hypergraphs*. Elsevier Science Ltd.

Black, D. (1948). On the rationale of group decision-making. *Journal of Political Economy*, 56(1):23–34.

Blum, C. and Zuber, C. (2015). Liquid democracy: Potentials, problems, and perspectives. *Journal of Political Philosophy*, 24.

Boros, E. and Gurvich, V. (2006). Perfect graphs, kernels, and cores of cooperative games. *Discrete Mathematics*, 306(19):2336–2354.

Brandt, F., Conitzer, V., Endriss, U., Lang, J., and Procaccia, A. D. (2016). *Handbook of Computational Social Choice*. Cambridge University Press.

Brill, M. (2019). *Interactive Democracy: New Challenges for Social Choice Theory*, pages 59–66. Springer International Publishing, Cham.

Brill, M. (2021). From computational social choice to digital democracy. In Zhou, Z.-H., editor, *Proceedings of the Thirtieth International Joint Conference on Artificial Intelligence, IJCAI-21*, pages 4937–4939. International Joint Conferences on Artificial Intelligence Organization. Early Career.

Brill, M. and Talmon, N. (2018). Pairwise liquid democracy. In *Proceedings of the 27th International Joint Conference on Artificial Intelligence*, IJCAI'18, page 137–143. AAAI Press.

Bron, C. and Kerbosch, J. (1973). Algorithm 457: Finding all cliques of an undirected graph. *Commun. ACM*, 16(9):575–577.

Bruner, M.-L. and Lackner, M. (2017). On the likelihood of single-peaked preferences. *Social Choice and Welfare*, 48.

Caragiannis, I. and Micha, E. (2019). A contribution to the critique of liquid democracy. In *Proceedings of the 28th International Joint Conference on Artificial Intelligence*, IJCAI'19, page 116–122. AAAI Press.

Christoff, Z. and Grossi, D. (2017). Binary voting with delegable proxy: An analysis of liquid democracy. In *Proceedings of the 16th Conference on Theoretical Aspects of Rationality and Knowledge (TARK)*, page 134–150.

Diaconis, P. (1988). *Group representations in probability and statistics*. Institute of Mathematical Statistics Lecture Notes—Monograph Series, 11. Institute of Mathematical Statistics, Hayward, CA.

Dodgsons, C. L. (1884). *The Principles of Parliamentary Representation*. Harrison and Sons.

Duddy, C. and Piggins, A. (2011). A measure of distance between judgment sets. social choice and welfare. *Social Choice and Welfare*, 39:855–867.

Elkind, E., Lackner, M., and Peters, D. (2017). Structured preferences. In *Trends in Computational Social Choice. AI Access Foundation*.

Endriss, U. (2017). *Trends in Computational Social Choice*. AI Access.

Escoffier, B., Gilbert, H., and Pass-Lanneau, A. (2019). *The Convergence of Iterative Delegations in Liquid Democracy in a Social Network*, pages 284–297.

Escoffier, B., Gilbert, H., and Pass-Lanneau, A. (2020). Iterative delegations in liquid democracy with restricted preferences. *Proceedings of the AAAI Conference on Artificial Intelligence*, 34:1926–1933.

Escoffier, B., Lang, J., and Öztürk, M. (2008). Single-peaked consistency and its complexity. In *Proceedings of the 18th European Conference on Artificial Intelligence*, pages 366–370.

Fan, X., Li, P., Zeng, Y., and Zhou, X. (2019). Implement liquid democracy on ethereum: A fast algorithm for realtime self-tally voting system. *CoRR*, abs/1911.08774.

Gibbard, A. (1973). Manipulation of voting schemes: A general result. *Econometrica*, 41(4):587–601.

Gölz, P., Kahng, A., Mackenzie, S., and Procaccia, A. D. (2018). The fluid mechanics of liquid democracy. *Lecture Notes in Computer Science*, page 188–202.

Hagberg, A. A., Schult, D. A., and Swart, P. J. (2008). Exploring network structure, dynamics, and function using networkx. In Varoquaux, G., Vaught, T., and Millman, J., editors, *Proceedings of the 7th Python in Science Conference*, pages 11 – 15, Pasadena, CA USA.

Hardt, S. and Lopes, L. C. R. (2015). Google votes: A liquid democracy experiment on a corporate social network. In *Technical Disclosure Commons*.

Harris, C. R., Millman, K. J., van der Walt, S. J., Gommers, R., Virtanen, P., Cournapeau, D., Wieser, E., Taylor, J., Berg, S., Smith, N. J., Kern, R., Picus, M., Hoyer, S., van Kerkwijk, M. H., Brett, M., Haldane, A., del Río, J. F., Wiebe, M., Peterson, P., Gérard-Marchant, P., Sheppard, K., Reddy, T., Weckesser, W., Abbasi, H., Gohlke, C., and Oliphant, T. E. (2020). Array programming with NumPy. *Nature*, 585(7825):357–362.

Jackman, S. D. (2009). *Bayesian Analysis for the Social Sciences*. Wiley Series in Probability and Statistics. Wiley.

Kahng, A., Mackenzie, S., and Procaccia, A. (2021). Liquid democracy: An algorithmic perspective. *J. Artif. Int. Res.*, 70:1223–1252.

Kling, C. C., Kunegis, J., Hartmann, H., Strohmaier, M., and Staab, S. (2015). Voting behaviour and power in online democracy: A study of liquidfeedback in germany's pirate party.

List, C. (2018). Democratic deliberation and social choice: A review. In Bächtiger, A., Mansbridge, J., Dryzek, J., and Warren, M., editors, *Oxford Handbook of Deliberative Democracy*. Oxford: Oxford University Press.

List, C. and Dryzek, J. (2003). Social choice theory and deliberative democracy: A reconciliation. *British Journal of Political Science*, 33(1):1–28.

List, C., Luskin, R. C., Fishkin, J. S., and McLean, I. (2013). Deliberation, single-peakedness, and the possibility of meaningful democracy: Evidence from deliberative polls. *Journal of Politics*, 75(1):80–95.

Litvinenko, A. (2012). Social media and perspectives of liquid democracy: The example of political communication in the pirate party in germany. In *Proceedings of the European Conference on e-Government, ECEG*, Proceedings of the European Conference on e-Government, ECEG, pages 403–407.

Mattei, N. and Walsh, T. (2013). Preflib: A library of preference data HTTP://PREFLIB.ORG. In *Proceedings of the 3rd International Conference on Algorithmic Decision Theory (ADT 2013)*, Lecture Notes in Artificial Intelligence. Springer.

Mattei, N. and Walsh, T. (2017). Apreflib. org retrospective: Lessons learned and new directions. *Trends in Computational Social Choice. AI Access Foundation*, pages 289–309.

Puppe, C. (2018). The single-peaked domain revisited: A simple global characterization. *Journal of Economic Theory*, 176.

Satterthwaite, M. A. (1975). Strategy-proofness and arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions. *Journal of Economic Theory*, 10(2):187–217.

Tao, T. (2011). *An Introduction to Measure Theory Vol. 126*. Amer Mathematical Society.

Valsangiacomo, C. (2021). Political representation in liquid democracy. *Frontiers in Political Science*, 3.

Wes McKinney (2010). Data Structures for Statistical Computing in Python. In Stéfan van der Walt and Jarrod Millman, editors, *Proceedings of the 9th Python in Science Conference*, pages 56 – 61.

Zhang, B., Oliynykov, R., and Balogun, H. (2018). A treasury system for cryptocurrencies: Enabling better collaborative intelligence. *IACR Cryptol. ePrint Arch.*, 2018:435.

Zhang, Y. and Grossi, D. (2021). Power in liquid democracy. In *AAAI*.