Fairness in Perpetual Participatory Budgeting

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

Participatory Budgeting (PB) is a process of collective decision-making in which citizens of a municipality have a direct say in the way public funds are spent. This has recently inspired a vast amount of mathematical and computational research into the way that public funds are to be allocated given the preferences of the inhabitants. An important desideratum is that this allocation should be fair to everybody participating. In the currently dominant mathematical models of PB – where PB is modelled as a one-shot process – fair allocations cannot always be guaranteed to exist. This thesis is an investigation into the extent to which fairer allocations can be guaranteed by taking into account previous rounds of the PB process, thereby building on the recent model of Perpetual Participatory Budgeting by Lackner, Maly, and Rey (2021).

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Chapter 1

Introduction

Participatory Budgeting (PB) is an exciting new method of public decision making (A. Shah, 2007). The core idea of PB is that residents have a direct say in how public funds are spent (Aziz and N. Shah, 2021). For example, a municipality may divide some amount of money among several districts. The residents of these districts can think of several different proposals for some projects they would like to see funded. After these project proposals are made and after there has been some preliminary selection of the viable projects, the people of the district go on to vote for their favourite projects. After the voting, the 'best' set of projects is selected and these are then actually realised.

In practice PB is mainly applied to letting residents decide on the funding of *public projects*. Typical examples of such public projects include building a playground in a particular neighbourhood or planting more trees in specific streets. However, in theory PB could be applied to *any* public spending. PB could, for example, also be applied to more general funding issues, such as whether to allocate more money to improving a country's infrastructure or to improving a country's healthcare system.

1.1 History of PB

PB started out as a radical democratic project in 1989 in Porto Allegre, Brazil (Cabannes, 2004). This was an initiative led by the Workers Party. Olívio Dutra, the mayor at that time and one of the founding members of the Workers Party, initiated the process of PB as a reaction to the non-transparent and non-democratic ways of decision making of the previous twenty years (Abers, 1998).

Since its start in Porto Allegre, PB has spread rapidly to many municipalities across the world (A. Shah, 2007). In total, more than 1500 municipalities are implementing PB as a way of deciding how to spend public funds (Aziz and N. Shah, 2021). It has been implemented by municipalities all across the world. In the US and Canada, for example, PB has been implemented in more than 29 cities, with more than 300 million euro of public funds being allocated to public projects. And in Paris, more than 100 million euro is spent via PB each year (Legendre, Madénian, and Scully, 2018). And new regions are still joining. For example, Toronto, the state of New South Wales (Australia) (Aziz and N. Shah, 2021) and also Amsterdam[1] recently started with implementing PB.

1.2 Advantages of PB

PB has been considered to have many advantages compared to classical ways of deciding on how to spend public funds (A. Shah, 2007). We'll name three.

- First of all, PB is a way to empower groups previously excluded from power. *Every* resident can participate in PB. In particular, then, also previously socially excluded groups of people can participate. Moreover, PB can be a way to dedicate even *more* power to these groups. After all, with PB the municipality does not have to spread the total available budget evenly across all districts (A. Shah, 2007). Instead, it can dedicate – relative to the number of residents of that district – more available budget to low-income districts than to high-income districts. In this way previously excluded groups, such as the poor, can execute more influence over the decision making than the rich, in the sense that they have a say in how to allocate a relatively larger budget.

- Secondly, PB makes for a more transparent decision-making process. Suppose that a PB-round has taken place and that a set of projects has been selected. It is then clear why the selected projects have been chosen: residents have voted on the projects and these projects received (in some sense to be made precise later) the 'most' votes. This is different from some other ways of decision making. For example, when local administrators decide on what public projects to fund it can be unclear why the administrators chose to fund those specific projects, instead of others. This non-transparency opens the door for corruption and clientelism. For example, this characterized the pre-PB situation in Porto Allegre (Abers, 1998).

- Thirdly, PB acts as a so-called 'citizenship school' (A. Shah, 2007). Due to PB, citizens are actively involved in the decision-making process. They have to, amongst other things, deliberate with other citizens and the municipality on how to spend scarce resources, on the feasibility of projects and on practical considerations such as the maintenance of the projects. In this way, citizens learn about local politics: what kind of responsibilities the municipality has (and hence what they can expect from the municipality) and what kind of rights they have as citizens. PB can thus function as a citizenship school; a way of enhancing citizens' knowledge about local politics.

[1] See https://buurtbudget.amsterdam.nl.

1.3 The Mathematical Study of PB

PB seems straightforward, but, in fact, it is not so straightforward to specify how *exactly* it must be executed (Bhatnagar et al., 2003). For example, what is the best way of letting people vote on projects, and how do you best pick a certain amount of projects based on these votes? Since PB is still a new topic, it is important that these questions are settled sooner rather than later: "Whatever the best approach to participatory budgeting is, now is the time to identify it, before various heuristics become hopelessly ingrained" (Benadè et al., 2021).

The need to answer these questions has recently initiated a large body of research in the field of *Computational Social Choice*. Computational Social Choice is an interdisciplinary research area at the intersection of social choice theory and computer science (Brandt et al., 2016). Mainly, this entails the studying of social choice questions from a mathematical and algorithmic perspective.

More precisely, this research strand within Computational Social Choice that studies PB from a mathematical and algorithmic perspective is concerned with two main questions. Firstly: how to elicit the preferences of the residents? And secondly: given a certain way of eliciting the preferences of the residents, how to select a set of projects based on these preferences?

There are multiple ways of asking for the agents' preferences. In this thesis we will work with so-called 'Approval Voting', where the agents vote for the projects that they approve of (Aziz and N. Shah, 2021). That is, for each of the projects, an agent can report to either like or dislike the project. Later in this chapter we will consider this in more detail. This thesis focuses primarily on giving an answer to the second question; given the approval votes of the agents, what projects should we select?

To illustrate the gist of this question, consider Figure 1.1 below, which depicts a simple example of PB with approval votes.

From left to right the voters are: Jessy (approving of building a new park and a swimming pool), Maureen (approving of building a new playground and a football field), Paul (approving of building a park bench), John (approving of building a football field and a playground with gymnastics equipment) and George (approving of building a football field).

Note now three things about Figure 1.3. As can be seen from the figure, some voters might 'agree' with each other on what to fund. For example, Maureen, Paul and John agree on building a football field. Further, note that their preferences are not ordered: in Approval Voting we do not know whether Jessy prefers building a park to building a swimming pool, or vice versa. And finally note that not everybody needs to necessarily approve of the same amount of projects. While Jessy, Maureen and John approve of two projects, both Paul and George approve of only one project.

Now suppose that we can only afford to fund one project. There are several different approaches that could be used in determining which set of projects to fund (Aziz and

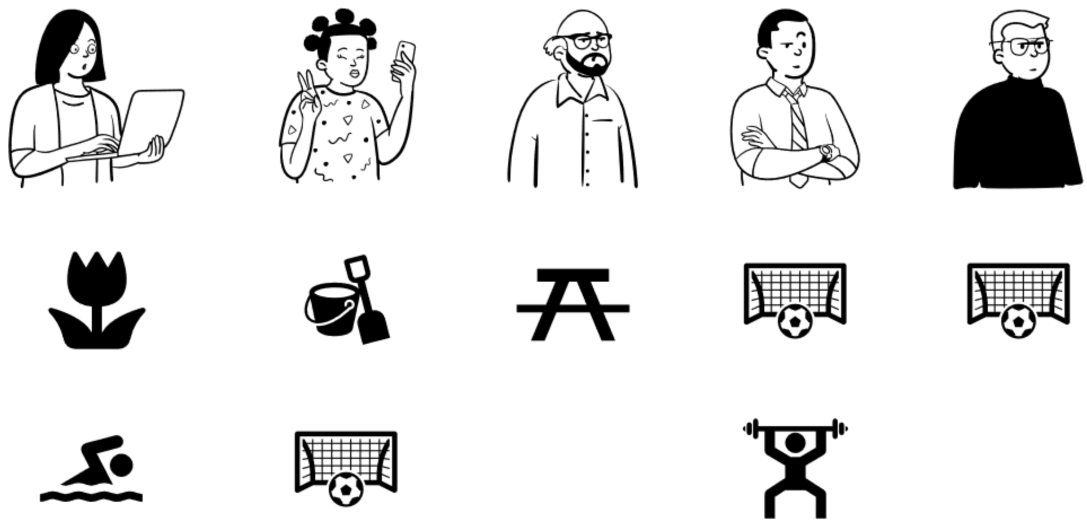

Figure 1.1: Five agents and their approval votes

N. Shah, 2021). For example, we could want to *maximise* the 'welfare' – roughly corresponding to the extent to which the voters are happy with a certain allocation – of the voters. Though this is a natural objective, we should at least also want our allocation to satisfy some notion of fairness. Otherwise, if welfare maximisation would be the only desideratum, then the outcome might satisfy some voters much more than others, thereby thwarting the democratic objective of PB. We will come back to this consideration later in more detail.

Given that we do not have any information about what happened in previous rounds, we should fund the football field, as this maximises the welfare and since there is not a fairer alternative available. Note however that this is *not* a perfectly fair outcome: while Maureen, John and George got a project that they approved of, Jessy and Paul got none of their approved projects.

So suppose now that in the next year, given that we funded the football field, the votes change as specified in Figure 1.2.

That is, since their football field has now been funded, Maureen, John and George now approve of something else: a playground with gymnastics equipment. And Paul changed his mind: instead of approving of a park bench, he now approves of a swimming pool. Given these votes, and given that we can still only afford one project, the question is what project to fund next.

Note that if we would only be interested in maximising welfare, we would select the playground with gymnastics equipment. However, as mentioned above, we also want our outcome to be 'fair' in some sense. Observe, then, that if we would select the swimming pool, Jessy and Paul would get what they want. Hence, after 2 rounds of PB, every voter got something of their liking, and we might therefore argue that this allocation is 'fair'

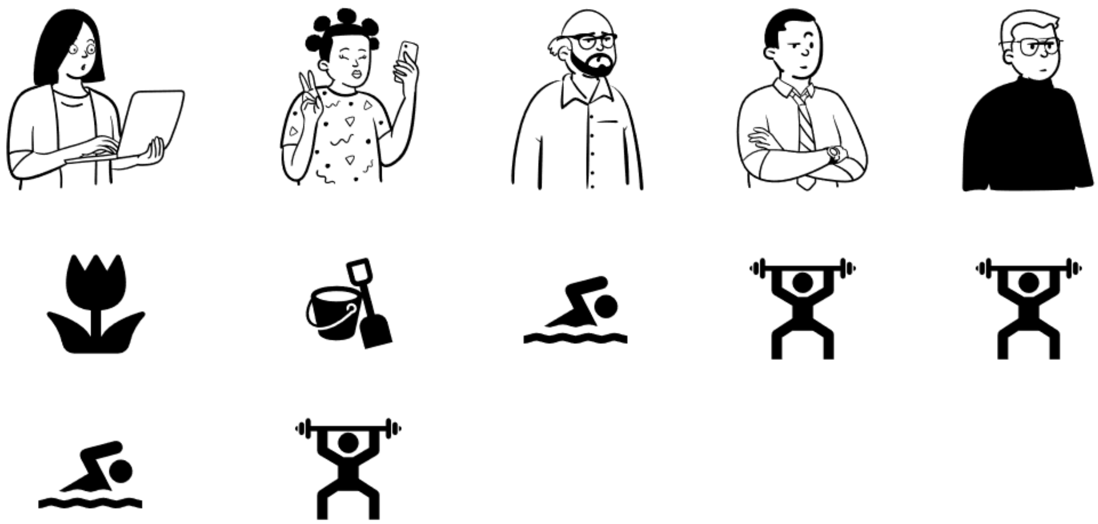

Figure 1.2: Five agents and their approval votes in a subsequent round

to everyone participating.

This example illustrates the main topic of the thesis. In PB we want our selection of projects to be ‘fair’ in some sense. However, as the simple example above illustrates, it is not always possible in *single* rounds to be perfectly fair to every voter. In the first round, for example, both Jessy and Paul were dissatisfied, since they got none of their approved projects funded. And we were not able to improve on this result: whatever project we would choose, we would always leave some voters dissatisfied.

One possible solution to this problem is to try to guarantee fairness over time. We cannot require that an allocation is always fair to each voter in every individual round, but possibly we can be fair to every voter on the long run. That is, by dedicating more budget to the voters that were previously left dissatisfied. This solution is illustrated by the example given above: in the second round we dedicated more budget to Jessy and Paul, who were dissatisfied in the previous round, thereby overruling the welfare maximisation consideration.

This thesis is an investigation to the extent to which such a fair solution over time can always be guaranteed to exist. The research question of the current thesis is therefore: *can fairness over time always be guaranteed in PB?* We will show that this depends on the way that ‘fairness’ and the related notion of ‘welfare’ are defined: for some definitions of fairness and welfare we *can* always guarantee a fair outcome, while for others this cannot be guaranteed.

1.4 Perpetual Participatory Budgeting

The initial research into the possibilities of realising fairness over time in PB has only recently been conducted, namely by Lackner, Maly, and Rey (2021). Their research contributes in two ways to answering the fairness-question.

Firstly, they defined the formal (perpetual) framework needed to study whether fairness in the long term can be realised. This framework is called ‘Perpetual Participatory Budgeting’ (PPB) and will be adopted in this thesis. Secondly, they investigated for several different notions of welfare and fairness whether fairness is possible in PPB.

Lackner, Maly, and Rey (2021) define three different notions of fairness, and three different notions of welfare. A ‘fair’ outcome is an outcome with a certain distribution of welfare among the voters. If fairness is taken to be an outcome that generates an *equal* amount of welfare for each group of agents, then for none of the studied notions of welfare such an outcome can always be guaranteed to exist. When fairness is taken to be an outcome that minimises the inequality of welfare (as measured by the Gini-coefficient), then such an outcome can – by definition – always be guaranteed to exist. However, it is computationally hard to find this outcome. Finally, when fairness is taken to be an outcome that, in the limit, generates an equal amount of welfare for each group of agents, then such an outcome can for one notion of welfare be guaranteed to exist, provided that there are at most two groups (or: ‘types’) of agents.

1.5 Thesis Contributions

In this thesis we will elaborate on the work already done by Lackner, Maly, and Rey (2021) and extend their results. We show that when defining welfare as ‘satisfaction’ and fairness as convergence to equal welfare, even when there are only seven agents, we cannot guarantee a fair outcome (Proposition 3.2.1). However, when there are at most four agents and at most three types, a fair outcome can be guaranteed to exist (and we will show how to compute this outcome) (Theorem 3.1.1). Further, we show that computing a fair outcome (taken to be an outcome generating exactly the same amount of ‘share’ for each group of agents) is computationally hard (Theorem 4.0.1). Finally, when we define welfare as ‘relative satisfaction’ and fairness as convergence to equal welfare, we show that we can guarantee, given some assumptions, the existence of a fair outcome for three groups with an arbitrary amount of agents (Theorem 5.3.1).

In the rest of the chapter we will first give an informal description of the formal model of PB that we will be using to study our main question (we will present the formal model itself in the next chapter), discuss research done in different PB models alongside this description and provide an overview of the chapters to come.

1.6 Research Done in Different PB Models

The current research is a strand within the field of Computational Social Choice. For an overview of this field we recommend the Handbook on Computational Social Choice by Brandt et al. (2016).

For a historical and philosophical perspective on PB, we recommend the book on PB which was published by the World Bank (A. Shah, 2007). In particular the first chapter is relevant for this thesis.

The mathematical work on PB is: recent, vast and diverse. For example, there are many different mathematical models that describe PB. In a recent overview article, Aziz and N. Shah (2021) describe a general mathematical model that captures all the current particular PB models as special cases. That is, the general model consists of certain parameters and all the specific models are particular instantiations of these parameters. For an overview of the current mathematical approaches to PB, we therefore recommend this overview article by Aziz and N. Shah (2021).

We briefly describe the four main parameters – or design choices – and state the design choices that we make.

Decision space. The first design choice is the space of possible outcomes: it can either be discrete – meaning that projects can be either fully funded or not funded at all – or continuous or 'divisible', meaning that projects can be funded to some fractional degree.[2] In this thesis, we will assume that the decision space is discrete, which is the default choice in PB research (see e.g. the work by Legendre, Madénian, and Scully (2018), Conitzer, Freeman, and N. Shah (2017) and Delort, Spanjaard, and Weng (2011)).

Work on fairness in *divisible* models has for example been done by Garg, Kulkarni, and Murhekar (2021), which showed that a fair outcome – formalised as an outcome that is 'in the core' – cannot always be guaranteed to exist, though can be approximated. Fain, Goel, and Munagala (2016) have studied proportional representation in the divisible model.

Preference modelling and ballot design. As mentioned above, there are multiple ways in which voters can vote. Since different voting mechanisms elicit different types of information about the voters' preferences, connected to the choice for a particular voting mechanism is the choice for the type of information that one wants to elicit.

In this thesis, we will work with Approval Voting, where voters vote on projects that they approve of. We therefore model agents' preferences as *dichotomous preferences*, which is a preference relation that divides the set of projects into two subsets: those

[2] For example, consider again our earlier Figure 1.1. Suppose that after funding the football field, we still have some budget available to fund part of the gymnastics park. We might not be able to fund all of the gymnastics equipment, but we can fund a few pieces of equipment. A mathematical PB model that allows for this possibility is called continuous/divisible.

which are liked and those which are disliked.

Approval Voting is the default way of eliciting preferences (Aziz and N. Shah, 2021), and there are different ways of doing it. In the least restrictive version of Approval Voting, voters can approve of any set of projects (see e.g. the work by Aziz, Brill, et al. (2017) and Aziz, Lee, and Talmon (2018)). They might even approve of all of the available projects. As we will later see, we will sometimes put some restrictions on the set of projects that voters can approve of. For example, we might require that voters can only approve of sets of projects whose total cost does not exceed the budget that is available. So-called 'Knapsack Voting' is a special case of this and has been studied in the context of PB for example by Goel et al. (2020).

Work on PB where voters submit ranked ballots instead of approval ballots has for example been done by Airiau et al. (2019). They studied several voting rules in this context that maximise the social welfare, and showed that they also satisfy several notions of fairness. In this work, preferences are modelled correspondingly as *ordinal preferences*, which is a ranking over the set of available projects, stating which projects are preferred to which other projects, though not specifying *to what extent* a project is preferred to another project.

We work with approval ballots instead of ranked ballots for two reasons. First, ranking projects gives *relative* but not absolute information: the fact that a voter ranked a project last does not imply that the voter thinks this project is not a good project, only that it is less preferred to the other projects. For example, a voter might think all of the proposed projects are good projects. Second, asking agents for a full ranking of all of the available projects can present agents with too many choices. It has been shown by Iyengar and Lepper (2000) that a greater choice can decrease the agents' subsequent satisfaction of their choice.

Vote aggregation. The third design choice is on how to aggregate the votes into a single set of chosen projects. As mentioned above, there are different goals that can steer this choice.

One common goal in selecting a suitable set of projects is to maximise the welfare of the voters. The work by Hershkowitz et al. (2021) and Fluschnik et al. (2019) are examples.

In this thesis, however, our goal is not to maximise the welfare of the agents, but to generate a fair outcome. The work most similar to this thesis is the work done by Lackner, Maly, and Rey (2021).

Sequentiality. The last and most distinguishing design choice is whether PB is described as a one-shot process, or as a sequential process (i.e., as a process in which agents vote during multiple rounds). Almost all work on PB models PB as a one-shot process.

Only recently, Lackner (2020) introduced perpetual (sequential) voting in the context of classical voting. It was then introduced to the context of PB by Lackner, Maly, and

Rey (2021). The perpetual model is called ‘Perpetual Participatory Budgeting’ (PPB). We will adopt this formal model.

Decision space	Ballot design	Vote aggregation	Sequentiality
Divisible	**Approval**	Welfare maximisation	**Sequential**
Discrete	Ranked	**Fairness**	Non-sequential

Table 1.1: The design choices of the current thesis

Finally, in this thesis our main focus is on the welfare of *groups* of agents, instead of on the welfare of individual agents. In particular, we are interested in the welfare of districts – being a particular group of agents. Currently, PB is usually done per district, as opposed to city-wide. Recent work by Hershkowitz et al. (2021) has demonstrated that a city-wide PB is a Pareto-improvement on the district-level PB. This means that every district would get at least as much welfare as in district-level PB and at least one district gets strictly more welfare.

This work shares its point of departure with this thesis: PB should be done city-wide, and subsequently the way of selecting projects should take the preferences of the districts into account. The exact interpretation thereof, however, is different. While Hershkowitz et al. (2021) focus on maximising the welfare of the districts, the current thesis puts the focus on generating fairness for the separate districts.

1.7 Overview

In **Chapter** 2, we define (P)PB and formulate a fairness theory for PPB following Lackner, Maly, and Rey (2021). In particular, we define three different notions of fairness and three different ways of measuring welfare. Furthermore, we will highlight several recent results about the extent to which fair outcomes can be achieved in PPB.

In **Chapter** 3, we extend these results by examining the extent to which solutions can be guaranteed that converge to *equal-satisfaction*, which is one of the ways of defining fairness and welfare. We show that this depends on the amount of agents that are involved. If there are more than seven agents, a converging solution is not guaranteed to exist. However, if there are less than four agents – divided into at most three types – a converging solution can be guaranteed to exist.

Chapter 4 is an analysis of the computational complexity of determining whether a fair outcome exists – where a fair outcome is defined as an outcome in which all types get an equal *share* of the available budget. We show that we cannot always efficiently compute an outcome that satisfies the property of equal-share.

In **Chapter** 5, we investigate the possibilities of achieving fair outcomes when these outcomes are taken to be outcomes that converge to equal-relative satisfaction (which is also a way of defining fairness and welfare). We show that when there are three groups of agents, we can guarantee that – given certain assumptions – such an outcome exists. Not only do we show *that* such an outcome must exist, we also show how to compute it.

In the conclusion, we summarise all of the results (in particular, Table 6.1 summarises all the results discussed in the thesis), discuss the implications of these results and present some directions for future research.

Chapter 2

Preliminaries

In this chapter we will build the foundation that is necessary to understand the following chapters and results. We will first define the formal model of Participatory Budgeting (PB), followed by the formal model of Perpetual Participatory Budgeting (PPB). This formal model – introduced by Lackner, Maly, and Rey (2021) – adds a "perpetual" dimension to the classical PB model. Perpetual voting had already been considered in the context of classical voting by Lackner (2020). Since the notion of *fairness* is defined in terms of the notion of *welfare*, we will – in the second part of the chapter – define several notions of welfare that we will be using. Finally, we will give an overview of some important results that apply to the framework of PPB.

In Chapter 4 we will prove a complexity result. This requires basic knowledge about complexity theory, such as knowledge about the complexity classes **P** and **NP**, but since this is not a core result, we will not elaborate on this here. For an introduction to complexity theory we recommend the handbook by Arora and Barak (2009).

2.1 Participatory Budgeting

Let $\mathcal{N}$ be a finite set of n voters – also called agents – such that $\mathcal{N} = \{1, \ldots, n\}$. And let $\mathcal{P} = \{p_1, \ldots, p_m\}$ with $m \in \mathbb{N}$ be a finite set of *projects* on which the agents can vote. Each agent $i \in \mathcal{N}$ approves of a set of projects, which is expressed by the *approval function*:

Definition 2.1.1 (Approval function)**.** *The approval function is a function* $A : \mathcal{N} \to 2^{\mathcal{P}} \setminus \{\emptyset\}$ *giving for every* $i \in \mathcal{N}$ *the set of projects* $A(i) \subseteq \mathcal{P}$ *the agent approves of.*

Given these notions, the *budgeting problem for PB* can be defined as a tuple consisting of the set of available projects, the available budget and the approval function. This is called a *budgeting problem*, because it intuitively – almost completely – describes the problem that a policy-maker faces when she has to decide on how to allocate the available money among the available projects.

However, another crucial part of the policy maker's problem is the *cost* of the different projects, while the cost function is not part of the formal definition. The (technical) reason for this will become apparent later.

Definition 2.1.2 (Budgeting problem for PB)**.** *A budgeting problem I for PB, also called a budgeting instance, is a tuple $I = \langle \mathcal{P}, b, A \rangle$, where:*

- $\mathcal{P} = \{p_1, \ldots, p_m\}$ *is the finite set of available projects,*
- $b \in \mathbb{N}_{>0}$ *is the available budget,*
- $A : \mathcal{N} \to 2^{\mathcal{P}} \backslash \{\emptyset\}$ *is the approval function. For any $i \in \mathcal{N}$, $A(i)$ is called a (approval)* ballot*.*

All projects have an associated cost. This is expressed by the *cost function*, which is a function that maps every available project to a natural number:

Definition 2.1.3 (Cost function)**.** *Given the set $\mathcal{P} = \{p_1, \ldots, p_m\}$ of available projects, a* cost function *is a function $c^{PB} : \mathcal{P} \to \mathbb{N}$ that maps every available project to its associated cost.*

The outcome of a budgeting problem I is a *budget allocation*. A budget allocation intuitively corresponds to a set of projects that could be selected after the voting procedure. A budget allocation can have different desirable properties. Two minimal ones that we will usually require are that a budget allocation (1) is *feasible* – which intuitively expresses that the budget allocation respects the available budget – and (2) is *exhaustive* – which intuitively expresses that the budget allocation uses all the available budget, though not more than that.

Definition 2.1.4 (Budget allocation, feasibility and exhaustiveness)**.** *Given a budgeting problem $I = \langle \mathcal{P}, b, A \rangle$ and a cost function c^{PB}, a budget allocation is a set $\pi \subseteq \mathcal{P} \backslash \{\emptyset\}$. A budget allocation π is called:*

- *Feasible iff $\sum_{p \in \pi \subseteq \mathcal{P}} c(p) \leq b$,*
- *Infeasible iff π is not feasible, and*
- *Exhaustive iff π is feasible and there does not exist a project $p \in \mathcal{P}$ such that $p \notin \pi$ and $c(\pi \cup \{p\}) \leq b$.*

As mentioned in the introduction, our main interest is in the welfare of *types* of agents, instead of in the welfare of *individual* agents. A type is defined as a subset of the set of agents, and, accordingly, the set of types is a set of subsets of the set of agents.

Definition 2.1.5 (Type)**.** *Given a budgeting instance $I = \langle \mathcal{P}, b, A \rangle$, a type t is a finite subset $t \subseteq \mathcal{N}$. We denote by $\mathcal{T} = \{t_1, \ldots, t_n\} \subseteq 2^{\mathcal{N}}$ a set of $n \in \mathbb{N}$ types so that $\bigcap_{t_i \in \mathcal{T}} t_i = \emptyset$.*

Similar to the approval function, the *type function* is a function that gives for every agent its associated type.

Definition 2.1.6 (Type function)**.** *Given a set of agents $\mathcal{N}$, a budgeting instance $I = \langle \mathcal{P}, b, A \rangle$, and a set $\mathcal{T} \subseteq 2^{\mathcal{N}}$ of types, the type function is a function $T : \mathcal{N} \to \mathcal{T}$ that maps every agent $i \in \mathcal{N}$ to its associated type.*

Example 2.1.7. Consider the following scenario with a set of agents $\mathcal{N} = \{1, 2, 3, 4\}$, a type function $T(1) = T(2) = t_1$ and $T(3) = T(4) = t_2$, and a budgeting problem $I = \langle \mathcal{P}, b, A \rangle$, with:

- $\mathcal{P} = \{p_1, p_2, p_3, p_4\}$,
- $b = 10$,
- $A(1) = \{p_1\}, A(2) = \{p_1, p_2\}$ and $A(3) = \{p_3, p_4\}$ and $A(4) = \{p_4\}$.

Suppose that we are given the following cost function: $c(p_1) = 5, c(p_2) = 5$, $c(p_3) = 3$ and $c(p_4) = 5$, and consider a budget allocation $\pi = \{p_1, p_3\}$. Note now that $\pi \cap A(1) = A(1)$. That is, agents 1's full ballot is funded, while significantly less projects are funded of other agents' approval ballots. For example, it holds that $\pi \cap A(4) = \emptyset$. Intuitively, then, the budget allocation π doesn't seem to be 'fair' for individual agents participating. Also on the level of *types* of agents, the budget allocation intuitively doesn't seem fair. For both agents $1, 2 \in t_1$ at least one project of their choice is funded, while this is only the case for *one* of the agents of t_2. We will later make this intuition about the fairness of a budget allocation more precise.

Finally, note that the ballots of all the agents are feasible, since $c(\{p_1\}) = c(\{p_4\}) = 5 \leq b = 10$, $c(\{p_3, p_4\}) = 8 \leq b = 10$ and $c(\{p_1, p_2\}) = 10 \leq b = 10$.

However, not all ballots are exhaustive. To illustrate, agent 2's ballot *is* exhaustive, as $c(\{p_1, p_2\} \cup \{p_3\}) > b$ and $c(\{p_1, p_2\} \cup \{p_4\}) > b$. However, the ballot of agent 4 is not exhaustive. For example, for agent 4 we have that $c(\{p_4\} \cup \{p_1\}) = 10 \leq b = 10$. $\triangle$

To conclude, one note about notation. In the following we will usually write a vector $\boldsymbol{x} = (x_1, \ldots, x_n)$ in boldface. We write – with minor abuse of notation – $x \in \boldsymbol{x}$ to express that x is an element of the vector $\boldsymbol{x}$.

2.2 Perpetual Participatory Budgeting

The model of Perpetual Participatory Budgeting (PPB) is an extension of the PB model.

In PPB, we consider a sequence $\boldsymbol{I} = (I_1, \ldots, I_k)$ of k budgeting problems with $k \in \mathbb{N} \cup \{\infty\}$, where I_j denotes the jth entry of $\boldsymbol{I}$.

Definition 2.2.1 (k-PPB Instance)**.** *A k-PPB Instance $\boldsymbol{I}$ is a sequence $\boldsymbol{I} = (I_1, \ldots, I_k)$ of k budgeting instances with $k \in \mathbb{N} \cup \{\infty\}$, where $I_j = \langle \mathcal{P}_j, b_j, A_j \rangle$ denotes the jth entry of $\boldsymbol{I}$.*

Formally, a k-PPB instance is any sequence of k budgeting instances. Intuitively, this sequence is meant to represent a *temporal* sequence of budgeting instances. It is meant to represent the fact that PB is a recurring process, in which agents vote on projects possibly for several times in a row. Given such a sequence of budgeting problems, we denote by a 'round' the place that a specific budgeting problem takes in that sequence:

Definition 2.2.2 (Rounds)**.** *Given a k-PPB Instance $\boldsymbol{I} = (I_1, \ldots, I_k)$, and given any budgeting instance $I_j \in \boldsymbol{I}$, j is called a* round *of $\boldsymbol{I}$. If clear from the context, we will omit the $\boldsymbol{I}$ of which j is a round.*

We assume that the set $\mathcal{N}$ of agents stays the same in each round. And we denote by $\mathcal{A}(I_j)$ the set of all feasible budget allocations for I_j.

As mentioned before, the PPB model is an extension of the PB model, in a way that the PPB model should allow us to capture the recurring nature of PB instances. For example, we require the notion of the set of projects available in a certain round, the budget available in that round, and the approval function of that round. This allows for the possibility that the set of available projects, the budget, and the approval function differ per round. Nevertheless, it is sometimes useful to be able to denote *all* the projects that occur during a k-PPB instance.

Definition 2.2.3 (The set of all projects)**.** *Given a k-PPB instance $\boldsymbol{I} = (I_1, \ldots, I_k)$ with $k \in \mathbb{N} \cup \{\infty\}$, and given a set of available projects $\mathcal{P}_j$ for each round $j \in \{1, \ldots, k\}$, we denote by $\mathfrak{P} = \{p_1, \ldots, p_z\}$ with $z \in \mathbb{N}$ the set of projects such that $\mathfrak{P} = \bigcup_{j \in \{1,\ldots,k\}} \mathcal{P}_j$.*

Example 2.2.4. Consider the 2-PPB instance $\boldsymbol{I} = (I_1, I_2)$. Then $I_1 = \langle \mathcal{P}_1, b_1, A_1 \rangle$ is the budgeting instance of the first round and $I_2 = \langle \mathcal{P}_2, b_2, A_2 \rangle$ is the budgeting instance of the second round.

By assumption, the set $\mathcal{N}$ of agents is the same in both rounds. The set of all projects $\mathfrak{P}$ equals the union of all the available projects in each round, so $\mathfrak{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. △

Furthermore, the cost function for PPB is an extension of the cost function for PB. It now maps every possible project to its associated cost, as opposed to mapping merely every *available* project to its associated cost.

Definition 2.2.5 (Cost function for PPB)**.** *Given the set $\mathfrak{P} = \{p_1, \ldots, p_z\}$ with $z \in \mathbb{N}$ of possible projects, a* cost function for PPB *is a function $c : \mathfrak{P} \to \mathbb{N}$ that maps every possible project to its associated cost.*

Given this definition, we can explain the aforementioned reason for not including the cost function in the definition of a budgeting problem. The reason is that by excluding the cost function from the definition of a budgeting problem, we want to stress that, while the available projects, budget and approval function might differ per round, the cost of the projects is the same in every round.

Given the notion of a k-PPB instance, we are able to define the notion of a *sequence* of budget allocations, which we refer to as a *solution*.

As mentioned in the introduction, our main interest lies with analysing the possible welfare that can be guaranteed to agents if, in the choice for a budget allocation, the budget allocations of *previous* rounds are taken into account. Hence, the definition of a solution is a fundamental one, for it allows us exactly to reason about these sequences of budget allocations.

Definition 2.2.6 (Solution)**.** *Given a k-PPB instance $\boldsymbol{I} = (I_1, \dots, I_k)$ with $k \in \mathbb{N} \cup \{\infty\}$, a cost function c, and given a budget allocation $\pi_j \subseteq \mathcal{P}_j$ for any round $j \in \{1, \dots, k\}$, the vector $\boldsymbol{\pi} = (\pi_1, \dots, \pi_k)$ is called a* solution *for $\boldsymbol{I}$. A solution $\boldsymbol{\pi}$ is called:*

- *Feasible iff all budget allocations $\pi_j \in \boldsymbol{\pi}$ are feasible*
- *Infeasible iff $\boldsymbol{\pi}$ is not feasible.*
- *Exhaustive iff all budget allocations $\pi_j \in \boldsymbol{\pi}$ are exhaustive.*

The notions of feasibility, infeasibility and exhaustiveness also apply to ballots. Feasibility and exhaustiveness express two properties that ballots can have. It will later turn out to be useful to require ballots to have these properties. More precisely: some desirable results about solutions only hold if we require that the ballots satisfy feasibility and exhaustiveness.

Definition 2.2.7 (Feasible, infeasible and exhaustive ballots)**.** *Given a k-PPB instance $\boldsymbol{I} = (I_1, \dots, I_k)$ with $k \in \mathbb{N} \cup \{\infty\}$, any $I_j = \langle \mathcal{P}_j, b_j, A_j \rangle \in \boldsymbol{I}$, a cost function c and a solution $\boldsymbol{\pi} = (\pi_1, \dots, \pi_k)$ for $\boldsymbol{I}$, then for any agent $i \in \mathcal{N}$ and any round $j \in \{1, \dots, k\}$ a ballot $A_j(i)$ is called:*

- *Feasible iff $\sum_{p \in A_j(i)} c(p) \leq b_j$,*
- *Infeasible iff π is not feasible, and*
- *Exhaustive iff $A_j(i)$ is feasible, and there does not exist a project $p \in \mathcal{P}_j$ such that $p \notin A_j(i)$ and $c(A_j(i) \cup \{p\}) \leq b_j$. Exhaustive ballots are also called* knapsack ballots *(Goel et al., 2020).*

Finally, note that while the available budget might differ per round, there will sometimes be a certain 'bound' such that the available budget will not be higher than that bound:

Definition 2.2.8 (Budget bound)**.** *Given a set $\mathcal{N}$ of agents, a finite set $\mathfrak{P}$ of possible projects, a k-PPB instance $\boldsymbol{I} = (I_1, \dots, I_k)$ with $k \in \mathbb{N} \cup \{\infty\}$, a* budget bound $B^* \in \mathbb{N}$ *denotes a constant such that $b_j \leq B^*$ for all $j \in \{1, \dots, k\}$. We say that B^* is a bound for $\boldsymbol{I}$.*

2.3 A Fairness Theory for PPB

In this section we will describe the work that has already been done on analysing the extent to which fair solutions can be guaranteed in PPB. In the first part of the section we will provide definitions of welfare and fairness. In the second part of the section we will give several results about the guaranteed existence of fair solutions in this framework.

2.3.1 Definitions

In the first part of this section, we will consider several notions of fairness. These notions are based on certain conceptualisations of welfare. We will therefore also present different measures of welfare. These definitions are based on the work done by Lackner, Maly, and Rey (2021).

First, we define the notion of a *welfare measure*. A welfare measure intuitively expresses how much welfare a certain solution generates for a type. Formally, it is a function that maps a k-PPB instance $\boldsymbol{I}$, a solution $\boldsymbol{\pi}$ for $\boldsymbol{I}$, a type t and a round j to a real number, expressing the amount of welfare generated by an instance, with a solution, for a type and in a specific round.

Definition 2.3.1 (Welfare measure)**.** *A* welfare measure *F is a function that takes as input a k-PPB instance $\boldsymbol{I}$, a solution $\boldsymbol{\pi}$ for $\boldsymbol{I}$, a type $t \in \mathcal{T}$ and a round $j \in \{1, \ldots, k\}$, and outputs a welfare score $F(\boldsymbol{I}, \boldsymbol{\pi}, t, j) \in \mathbb{R}$ for type t of the solution $\boldsymbol{\pi}$ for the first j rounds of $\boldsymbol{I}$.*

As mentioned above, there are different ways of measuring welfare, and we will consider three of these later on in this section. Furthermore, aside from having a certain way of measuring types' welfare, we require a way of determining whether this distribution of welfare is 'fair'. There are several useful 'fairness criteria'. First, consider the following fairness criterion of Equal-F.

Definition 2.3.2 (Equal-F)**.** *Given a k-PPB instance $\boldsymbol{I} = (I_1, \ldots, I_k)$ with $k \in \mathbb{N} \cup \{\infty\}$, a solution $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_k)$ for $\boldsymbol{I}$, a set of types $\mathcal{T}$, a round $j \in \{1, \ldots, k\}$, we say $\boldsymbol{\pi}$ satisfies* equal-*F at round j if for every two types $t, t' \in \mathcal{T}$, we have:*

$$F(\boldsymbol{I}, \boldsymbol{\pi}, t, j) = F(\boldsymbol{I}, \boldsymbol{\pi}, t', j).$$

Moreover, a solution $\boldsymbol{\pi}$ satisfies equal-F if it satisfies equal-F at round j for all rounds $j \in \{1, \ldots, k\}$.

Example 2.3.3. Consider the 4-PPB instance $\boldsymbol{I}$ with four agents $1, 2, 3$ and 4 and two types t_1 and t_2, such that agents 1 and 2 are of type t_1 and agents 3 and 4 are of type t_2. Given some solution $\boldsymbol{\pi}$, consider the welfare measure F that generates a welfare score of 2 for each type in the first round, a score of 4 in the second, a score of 6 in the third and a score of 8 in the final (fourth) round. So, for example, $F((I_1, I_2), (\pi_1, \pi_2), t_1, 2) =$

$F((I_1, I_2), (\pi_1, \pi_2), t_2, 2) = 4$. Clearly, the solution $\boldsymbol{\pi}$ satisfies equal-F at round j for all $j \in \{1, 2, 3, 4\}$. △

We can show, however, that the fairness criterion of equal-F is an extremely demanding criterion. We show in Example 2.3.10 that there are k-PPB instances with only a limited number of agents and projects that do not allow solutions that satisfy equal-F.

This forms the main motivation for introducing another fairness-criterion: convergence to equal-F. While we might often not be able to find solutions that are perfectly fair, in the sense of satisfying equal-F, we might be able to find solutions that *converge* to equal-F on the long run. We will be mainly using the notion of convergence to equal-F as our fairness criterion, because it is not too stringent, while still being an appealing fairness criterion.

The definition intuitively states that a solution converges to equal-F if all the different types have an equal amount of welfare when the amount of rounds that have taken place tends to infinity.

Definition 2.3.4 (Convergence to equal-F)**.** *Given a set $\mathcal{N}$ of agents, an ∞-PPB instance $\boldsymbol{I}$, a solution $\boldsymbol{\pi} = (\pi_1, \pi_2 \ldots)$ for $\boldsymbol{I}$, a set of types $\mathcal{T} \subseteq 2^{\mathcal{N}}$, a round $j \in \{1, 2, \ldots\}$ with $I_j = \langle \mathcal{P}_j, b_j, A_j \rangle$, and a welfare measure F, we say that the solution $\boldsymbol{\pi}$ for $\boldsymbol{I}$ converges to equal-F iff for any two types $t_i, t_{i'} \in \mathcal{T}$:*

$$\frac{F(\boldsymbol{I}, \boldsymbol{\pi}, t_i, k)}{F(\boldsymbol{I}, \boldsymbol{\pi}, t_{i'}, k)} \xrightarrow[k \to +\infty]{} 1.$$

Next to different fairness criteria, of which equal-F and convergence to equal-F are examples, there are also many different possible welfare measures. Now consider the following three specific welfare measures that we will be mainly using: satisfaction, relative satisfaction and share.

There are different ways of defining satisfaction, three of which can be found in the recent article by Talmon and Faliszewski (2019). Given a budgeting problem $I = \langle \mathcal{P}, b, A \rangle$, a budget allocation $\pi \subseteq \mathcal{P}$, and some agent $i \in \mathcal{N}$, the satisfaction of agent i can for example be defined as the *amount* of projects that are funded and which i approves of. Satisfaction could also be defined as being 0 if none of i's approved projects are funded, and 1 otherwise (i.e., if at least one of agent i's approved projects is funded).

In this thesis, we will define the satisfaction of agent i as the total cost of all the projects that are funded *and* approved by i.

The intuitive reason behind defining satisfaction in this way, rather than in one of the two ways above, is that there seems to be a positive correlation between the satisfaction of an agent and the cost of the funded projects which are approved by the agent (Talmon and Faliszewski, 2019). For example, if an agent approves of both building a new park bench, and of building an entire swimming pool, than funding the swimming pool is intuitively more likely to generate more satisfaction for the agent than building the bench.

The formal definition starts with defining the *marginal satisfaction of an agent for a round*. This definition closely resembles the intuitive reasoning above. This basic notion is then extended to a welfare measure in a natural way.

Definition 2.3.5 (Satisfaction)**.** *Given a set $\mathcal{N}$ of agents, a finite set $\mathfrak{P}$ of possible projects, a k-PPB instance $\boldsymbol{I} = (I_1, \ldots, I_k)$ with $k \in \mathbb{N} \cup \{\infty\}$, a solution $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_k)$ for $\boldsymbol{I}$, a cost function $c : \mathfrak{P} \to \mathbb{N}$, a set of types $\mathcal{T} \subseteq 2^{\mathcal{N}}$, a round $j \in \{1, \ldots, k\}$ with $I_j = \langle \mathcal{P}_j, b_j, A_j \rangle$, we define the marginal satisfaction of agent $i \in \mathcal{N}$ as:*

$$sat_j^m(\boldsymbol{I}, \pi_j, i) = c(\pi_j \cap A_j(i)).$$

Moreover, the marginal satisfaction *of a type $t \in \mathcal{T}$ for round $j \in \{1, \ldots, k\}$ is defined as:*

$$sat_j^m(\boldsymbol{I}, \pi_j, t) = \frac{1}{|t|} \sum_{i \in t} sat_j^m(\boldsymbol{I}, \pi_j, i).$$

And, finally, the satisfaction of a type *$t \in \mathcal{T}$ for some round $j \in \{1, \ldots, k\}$ is defined as:*

$$sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t) = \sum_{1 \le j^* \le j} sat_{j^*}^m(\boldsymbol{I}, \pi_{j^*}, t).$$

Another welfare measure is the notion of *relative satisfaction*. It intuitively expresses an agents' welfare – given a specific round and a budget allocation for that round – to be equal to the amount of money spent on her preferences, relative to the amount of money that *could* have been spend on her preferences. Then this notion is extended in a similar way as above to a welfare measure.

Definition 2.3.6 (Relative satisfaction)**.** *Given a set $\mathcal{N}$ of agents, a finite set $\mathfrak{P}$ of possible projects, a k-PPB instance $\boldsymbol{I} = (I_1, \ldots, I_k)$ with $k \in \mathbb{N} \cup \{\infty\}$, a solution $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_k)$ for $\boldsymbol{I}$, a cost function $c : \mathfrak{P} \to \mathbb{N}$, a set of types $\mathcal{T} \subseteq 2^{\mathcal{N}}$, a round $j \in \{1, \ldots, k\}$ with $I_j = \langle \mathcal{P}_j, b_j, A_j \rangle$, we define the marginal relative satisfaction of agent $i \in \mathcal{N}$ as:*

$$rsat_j^m(\boldsymbol{I}, \pi_j, i) = \frac{c(\pi_j \cap A_j(i))}{max\{c(A) | A \subseteq A_j(i) \text{ and } c(A) \le b_j\}}.$$

Moreover, the marginal relative satisfaction *of a type $t \in \mathcal{T}$ for round $j \in \{1, \ldots, k\}$ is defined as:*

$$rsat_j^m(\boldsymbol{I}, \pi_j, t) = \frac{1}{|t|} \sum_{i \in t} rsat_j^m(\boldsymbol{I}, \pi_j, i).$$

And, finally, the satisfaction of a type *$t \in \mathcal{T}$ for some round $j \in \{1, \ldots, k\}$ is defined as:*

$$rsat_j(\boldsymbol{I}, \boldsymbol{\pi}, t) = \sum_{1 \le j^* \le} rsat_{j^*}^m(\boldsymbol{I}, \pi_{j^*}, t).$$

Intuitively, both the notion of satisfaction and of relative satisfaction are intended as ways to approximate agents' underlying satisfaction functions (i.e., functions that take as input an allocation and produce some level of satisfaction). Our notion of 'satisfaction', for example, is an approximation that is based on the assumption that agents' underlying satisfaction functions depend on the cost of the projects that they approve of. It states that the satisfaction generated for an agent due to an approved project is equal to the cost of this project. This correlation is probably not perfect, and therefore our notion of satisfaction is an *approximation*.

Nevertheless, two drawbacks of this approach are (1) that it is not clear what these underlying satisfaction functions are (and that we therefore cannot know how good our approximations are), and (2) that it might be the case that no such underlying satisfactions functions exist at all.

Another approach is to refrain from approximating agents' underlying satisfaction functions and instead consider the amount of effort at satisfying the agents (not bothering with the *extent* to which projects generate satisfaction). This conception of welfare is formalised in the notion of 'share'. The share of an agent i due to some allocation π intuitively corresponds to the amount of effort that has been put – by selecting π – into satisfying i. An agent's share due to an approved project increases the fewer other agents also approve of this project. For example, if an agent's approved project is funded while no other agent approves of this project, then this agent's welfare should be high, since it has received a large 'share' of the total budget.

Definition 2.3.7 (Share)**.** *Given a set $\mathcal{N}$ of agents, a finite set $\mathfrak{P}$ of possible projects, a k-PPB instance $\boldsymbol{I} = (I_1, \dots, I_k)$ with $k \in \mathbb{N} \cup \{\infty\}$, a solution $\boldsymbol{\pi} = (\pi_1, \dots, \pi_k)$ for $\boldsymbol{I}$, a cost function $c : \mathfrak{P} \to \mathbb{N}$, a set of types $\mathcal{T} \subseteq 2^{\mathcal{N}}$, a round $j \in \{1, \dots, k\}$ with $I_j = \langle \mathcal{P}_j, b_j, A_j \rangle$, we define the marginal share of agent $i \in \mathcal{N}$ as:*

$$share^m_j(\boldsymbol{I}, \pi_j, i) = \sum_{p \in \pi_j \cap A_j(i)} \frac{c(p)}{|\{i' \in \mathcal{N} | p \in A_j(i')\}|}$$

Moreover, the marginal share *of a type $t \in \mathcal{T}$ for round $j \in \{1, \dots, k\}$ is defined as:*

$$share^m_j(\boldsymbol{I}, \pi_j, t) = \frac{1}{|t|} \sum_{i \in t} share^m_j(\boldsymbol{I}, \pi_j, i).$$

And, finally, the share of a type *$t \in \mathcal{T}$ for some round $j \in \{1, \dots, k\}$ is defined as:*

$$share_j(\boldsymbol{I}, \boldsymbol{\pi}, t) = \sum_{1 \leq j^* \leq j} share^m_{j^*}(\boldsymbol{I}, \pi_{j^*}, t).$$

Now that we have defined the three welfare measures satisfaction, relative satisfaction and share, we can make precise in what sense the allocation π of Example 2.1.7 is not fair.

Example 2.3.8 (Computing the (relative) satisfaction of Example 2.1.7)**.** So consider again Example 2.1.7. We have four agents, called $1, 2, 3, 4$, so that agents 1 and 2 are of type t_1 and agents 3 and 4 are of type t_2. And we have four available projects: p_1, p_2 and p_4 with a cost of 5, and p_3 with a cost of 3. And suppose that we have a budget allocation $\pi = \{p_1, p_3\}$.

Suppose w.l.o.g. that the budgeting problem I is the jth entry of a k-PPB instance $\boldsymbol{I}$ with $j = 1$. We thus focus only on computing the welfare – and determining the corresponding fairness – in the *first* round.

We can now compute the satisfaction and relative satisfaction scores for the agents and the types (we will calculate the share scores in Example 2.3.9). First, we compute the satisfaction scores.

First we compute the marginal satisfaction scores for the different agents.

- ***Agent 1.*** By definition of satisfaction, we have that $sat_1^m(\boldsymbol{I}, \pi, 1) = c(\pi \cap A_1(1)) = c(\{p_1, p_3\} \cap \{p_1\}) = c(\{p_1\}) = 5$.
- ***Agent 2.*** By definition of satisfaction, we have that $sat_1^m(\boldsymbol{I}, \pi, 2) = c(\pi \cap A_1(2)) = c(\{p_1, p_3\} \cap \{p_1, p_2\}) = c(\{p_1\}) = 5$.
- ***Agent 3.*** By definition of satisfaction, we have that $sat_1^m(\boldsymbol{I}, \pi, 3) = c(\pi \cap A_1(3)) = c(\{p_1, p_3\} \cap \{p_3, p_4\}) = c(\{p_3\}) = 3$.
- ***Agent 4.*** By definition of satisfaction, we have that $sat_1^m(\boldsymbol{I}, \pi, 4) = c(\pi \cap A_1(4)) = c(\{p_1, p_3\} \cap \{p_4\}) = c(\emptyset) = 0$.

This allows us to compute the marginal satisfaction of a type $t \in \mathcal{T}$ for round $1 \in \{1, \ldots, k\}$ as follows:

- ***Type 1.*** By definition of satisfaction, we have

$$sat_1^m(\boldsymbol{I}, \pi, t_1) = \frac{1}{|t_1|} \sum_{i \in t_1} sat_1^m(\boldsymbol{I}, \pi, i) = \frac{1}{2} \cdot 10 = 5.$$

  Now, let $\boldsymbol{\pi}$ be any solution such that $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_k)$. Since $j = 1 \in \{1, \ldots, k\}$ it follows that: $sat_1(\boldsymbol{I}, \boldsymbol{\pi}, t_1) = sat_1^m(\boldsymbol{I}, \pi, t_1) = 5$.

- ***Type 2.*** By definition of satisfaction, we have

$$sat_1^m(\boldsymbol{I}, \pi, t_2) = \frac{1}{|t_2|} \sum_{i \in t_2} sat_1^m(\boldsymbol{I}, \pi, i) = \frac{1}{2} \cdot 3 = 1.5.$$

  Now, let $\boldsymbol{\pi}$ be any solution such that $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_k)$. Since $j = 1 \in \{1, \ldots, k\}$ it follows that: $sat_1(\boldsymbol{I}, \boldsymbol{\pi}, t_2) = sat_1^m(\boldsymbol{I}, \pi, t_2) = 1.5$.

Therefore, $sat_1(\boldsymbol{I}, \boldsymbol{\pi}, t_1) \neq sat_1(\boldsymbol{I}, \boldsymbol{\pi}, t_2)$. Hence, by definition, the solution $\boldsymbol{\pi}$ does not satisfy equal-satisfaction at round 1. In this sense, then, the allocation π_1 is not *fair*.

Next, we compute the relative satisfaction scores at round 1. First we compute the marginal relative satisfaction scores for the different agents.

- ***Agent 1.*** By definition of relative satisfaction, we have that

$$rsat_1^m(\boldsymbol{I}, \pi, 1) = \frac{c(\pi \cap A_1(1))}{max\{c(A) | A \subseteq A_1(1) \text{ and } c(A) \leq b_1\}} = \frac{5}{5} = 1.$$

- ***Agent 2.*** By definition of relative satisfaction, we have that

$$rsat_1^m(\boldsymbol{I}, \pi, 2) = \frac{c(\pi \cap A_1(2))}{max\{c(A) | A \subseteq A_1(2) \text{ and } c(A) \leq b_1\}} = \frac{5}{10} = \frac{1}{2}.$$

- ***Agent 3.*** By definition of relative satisfaction, we have that

$$rsat_1^m(\boldsymbol{I}, \pi, 3) = \frac{c(\pi \cap A_1(3))}{max\{c(A) | A \subseteq A_1(3) \text{ and } c(A) \leq b_1\}} = \frac{3}{8}.$$

- ***Agent 4.*** By definition of relative satisfaction, we have that

$$rsat_1^m(\boldsymbol{I}, \pi, 4) = \frac{c(\pi \cap A_1(4))}{max\{c(A) | A \subseteq A_1(4) \text{ and } c(A) \leq b_1\}} = \frac{0}{5} = 0.$$

In a similar way to the case of satisfaction, this allows us to compute the relative satisfaction of a type $t \in \mathcal{T}$ for round $j \in \{1, \ldots, k\}$. We then get the following:

- ***Type 1.*** We get that $rsat_1(\boldsymbol{I}, \boldsymbol{\pi}, t_1) = \frac{1+\frac{1}{2}}{2} = 0.75$.

- ***Type 2.*** We get that $rsat_1(\boldsymbol{I}, \boldsymbol{\pi}, t_2) = \frac{\frac{3}{8}+0}{2} = \frac{3}{16}$.

Therefore, $rsat_1(\boldsymbol{I}, \boldsymbol{\pi}, t_1) \neq rsat_1(\boldsymbol{I}, \boldsymbol{\pi}, t_2)$. Hence, by definition, the solution $\boldsymbol{\pi}$ also does not satisfy equal-relative satisfaction at round 1. In this sense too, then, the allocation π is not *fair*.

Note, however, that it is still possible to reach fairness in the long run. For example, suppose that for every round $i \in \{1, \ldots, k\}$ we have that $I_1 = I_i$. That is, we have the k-PPB instance $\boldsymbol{I} = (I_1, \ldots, I_k)$ with $I_1 = I_i$ for each $I_i \in \boldsymbol{I}$. Then define $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_k)$ as follows. For each $\pi_i \in \boldsymbol{\pi}$, set:

$$\pi_i = \begin{cases} \{p_1\}, & \text{if } i \text{ is odd} \\ \{p_4\}, & \text{if } i \text{ is even.} \end{cases}$$

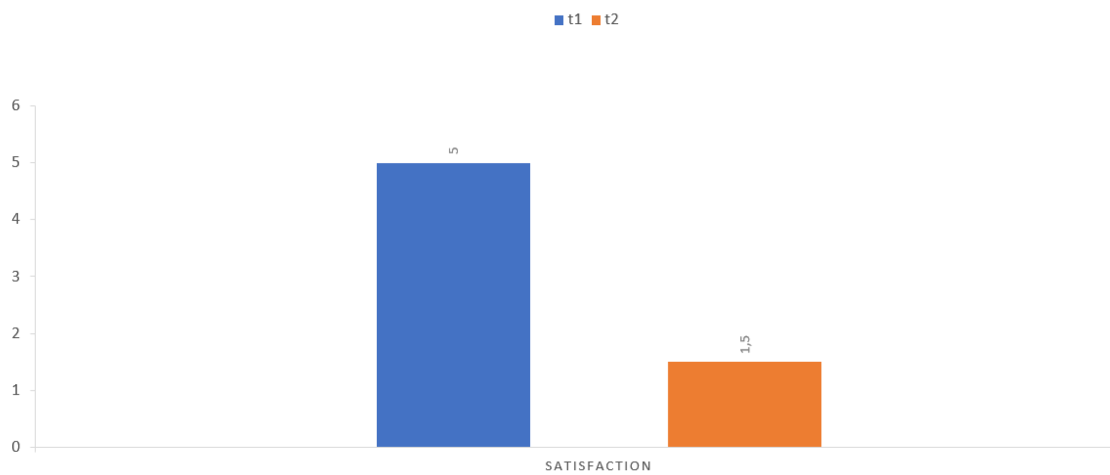

Figure 2.1: The satisfaction scores of types t_1 and t_2 of Example 2.1.7

Then, $sat_i^m(\boldsymbol{I}, \boldsymbol{\pi}, t_1) = 5$ (and $sat_i^m(\boldsymbol{I}, \boldsymbol{\pi}, t_2) = 0$) for each odd round i and $sat_{i^*}^m(\boldsymbol{I}, \boldsymbol{\pi}, t_2) = 5$ (and $sat_{i^*}^m(\boldsymbol{I}, \boldsymbol{\pi}, t_1) = 0$) for each even round i^*.

Hence, we have

$$\frac{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1)}{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2)} \xrightarrow[j \to +\infty]{} 1.$$

Thus, by definition, the solution $\boldsymbol{\pi}$ for $\boldsymbol{I}$ converges to equal-satisfaction. In this sense, then, though we couldn't acquire perfect fairness in the first round, there exists some solution that *converges* to a fair situation in the long run. $\triangle$

Figure 2.1 illustrates the concepts above. In the figure, the satisfaction scores of types t_1, t_2 in round 1 are visualised, though since by definition all welfare measures output a welfare score $F(\boldsymbol{I}, \boldsymbol{\pi}, t, j) \in \mathbb{R}$, the figure could equally well be an illustration of any other welfare measure. Conceptualising the welfare of types in such a visual way will be useful later on.

We now also briefly illustrate the definition of share by computing some of the share-scores of Example 2.1.7.

Example 2.3.9 (Computing the share of Example 2.1.7)**.** We compute only the marginal share of agents 1 and 2 for round $1 \in \{1, \ldots, k\}$, given the budget allocation $\pi = \pi_1 = \{p_1, p_3\}$. Suppose we're given the same k-PPB instance $\boldsymbol{I}$ as in Example 2.3.8.

- ***Agent 1.*** By definition of share, we have:

$$share_1^m(\boldsymbol{I}, \{p_1, p_3\}, 1) = \sum_{p \in \pi_1 \cap A_1(1)} \frac{c(p)}{|\{i' \in \mathcal{N} | p \in A_1(i')\}|} = \frac{c(p_1)}{|\{1, 2\}|} = \frac{5}{2} = 2.5.$$

- ***Agent 2.*** By definition of share, we have:

$$\mathit{share}_1^m(\boldsymbol{I}, \{p_1, p_3\}, 2) = \sum_{p \in \pi_1 \cap A_1(2)} \frac{c(p)}{|\{i' \in \mathcal{N} | p \in A_1(i')\}|} = \frac{c(p_1)}{|\{1,2\}|} = \frac{5}{2} = 2.5.$$

$\triangle$

As Example 2.3.8 shows, there are k-PPB instances that do not satisfy Equal-F in some *specific* round, but that allow for a solution that converges to Equal-F in the long run. As mentioned before, in general Equal-F is an extremely demanding criterion. There are examples of k-PPB instances that allow *no* solution that satisfies equal-F at any round.

The following k-PPB instance is one such example. It is based on an example given by Lackner, Maly, and Rey (2021).

Example 2.3.10 (Equal-F is an extremely demanding criterion)**.** Let $\mathcal{N} = \{1, 2, 3, 4\}$ and let $T(1) = T(2) = t_1$ and $T(3) = T(4) = t_2$. Let $\mathfrak{P} = \{p_1, p_2, p_3, p_4\}$, with $c(p) = 1$ for all $p \in \mathfrak{P}$. We define the k-PPB instance $\boldsymbol{I}$ as follows.

Let $b_i = 1$ for each $i \in \{1, \ldots, k\}$. In the first round, $\mathcal{P} = \{p_1, p_2, p_3, p_4\}$ and $A_1(1) = \{p_1\}, A_2(2) = \{p_2\}, A_3(3) = \{p_3\}, A_4(4) = \{p_4\}$. In all other rounds $j \neq 1$ we have $A_j(n) = \{p_1\}$ for each $n \in \mathcal{N}$.
Suppose w.l.o.g. that $\pi_1 = \{p_1\}$. Then consider any arbitrary solution $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_k)$.

First we compute the satisfaction scores for the first round. For agent 1, we have $\mathit{sat}_1^m(\boldsymbol{I}, \pi_1, 1) = c(\pi_1 \cap A_1(1)) = c(\{p_1\} \cap \{p_1\}) = 1$. For all other agents $i \in \mathcal{N}$ with $i \neq 1$ we have that $\mathit{sat}_1^m(\boldsymbol{I}, \pi_1, i) = c(\pi_1 \cap A_1(i)) = c(\{p_1\} \cap \{p_i\}) = 0$. Hence the marginal satisfaction scores of the types for round 1 are:

$$\mathit{sat}_1^m(\boldsymbol{I}, \pi_1, t_1) = \frac{1}{|t_1|} \sum_{i \in t_1} \mathit{sat}_1^m(\boldsymbol{I}, \{p_1\}, i) = 0.5,$$

and

$$\mathit{sat}_1^m(\boldsymbol{I}, \pi_1, t_2) = \frac{1}{|t_2|} \sum_{i \in t_2} \mathit{sat}_1^m(\boldsymbol{I}, \{p_1\}, i) = 0.$$

Now we compute the satisfaction scores for the next rounds. Since $A_j(i) = A_j(i')$ for any round $j \in \{2, \ldots, k\}$ and for any agents $i, i' \in \mathcal{N}$, it holds that $\mathit{sat}_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) = \mathit{sat}_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2) + 0.5$. Hence $\mathit{sat}_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) \neq \mathit{sat}_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2)$ for any round $j \in \{2, \ldots, k\}$. Therefore, since $\boldsymbol{\pi}$ was arbitrary, it holds that there exists no solution for $\boldsymbol{I}$ that satisfies equal-satisfaction.

The same result holds for relative satisfaction. To see this, note that for any round $j \in \{1, \ldots, k\}$ and for any type $t \in \mathcal{T}$, we have that – by definition of the approval function A_j and the available budget b_j – $max\{c(A) | A \subseteq A_j(i) \ and \ c(A) \leq b_j\} = 1$. Thus, for any agent $i \in t$:

$$\frac{c(\pi_j \cap A_j(i))}{max\{c(A)|A \subseteq A_j(i) \text{ and } c(A) \leq b_j\}} = \frac{c(\pi_j \cap A_j(i))}{1} = c(\pi_j \cap A_j(i)).$$

Therefore it follows that for any budget allocation $\pi_j \subseteq \mathcal{P}_j$, for any round $j \in \{1, \ldots, k\}$ and for any type $t \in \{t_1, t_2\}$, we have $sat^m_j(\boldsymbol{I}, \pi_j, t) = rsat^m_j(\boldsymbol{I}, \pi_j, t)$. Hence it follows that there exists no solution for $\boldsymbol{I}$ that satisfies equal-relative satisfaction. $\triangle$

As will be apparent later on, in proofs it will be useful to be able to refer to the type that, in some round, has the lowest, middle or highest welfare.

The definition of a middle type, though similar to those of the lowest and highest type, is somewhat different, since it's defined only if there are exactly three types.[1]

Definition 2.3.11 (Lowest, middle and highest welfare for types)**.** *Given a set $\mathcal{N}$ of agents, a finite set $\mathfrak{P}$ of possible projects, a k-PPB instance $\boldsymbol{I} = (I_1, \ldots, I_k)$ with $k \in \mathbb{N} \cup \{\infty\}$, a solution $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_k)$ for $\boldsymbol{I}$, a cost function $c : \mathfrak{P} \to \mathbb{N}$, a set of types $\mathcal{T} \subseteq 2^{\mathcal{N}}$ with $t \in \mathcal{T}$, a round $j \in \{1, \ldots, k\}$ with $I_j = \langle \mathcal{P}_j, b_j, A_j \rangle$, and a welfare measure $F : (\boldsymbol{I}, \boldsymbol{\pi}, t, j) \to \mathbb{R}$, we define:*

1. *$t^{j,F}_-$ to be the type $t \in \mathcal{T}$ such that $F(\boldsymbol{I}, \boldsymbol{\pi}, t, j) \leq F(\boldsymbol{I}, \boldsymbol{\pi}, t_i, j)$ for any type $t_i \in \mathcal{T}$. We will omit the superscript F when F is clear from the context.*

2. *$t^{j,F}_+$ to be the type $t \in \mathcal{T}$ such that $F(\boldsymbol{I}, \boldsymbol{\pi}, t, j) \geq F(\boldsymbol{I}, \boldsymbol{\pi}, t_i, j)$ for any type $t_i \in \mathcal{T}$. We will omit the superscript F when F is clear from the context.*

3. *If $|\mathcal{T}| = 3$, then we define $t^{j,F}_0$ to be the type $t \in \mathcal{T}$ such that:*

$$F(\boldsymbol{I}, \boldsymbol{\pi}, t^{j,F}_-, j) \leq F(\boldsymbol{I}, \boldsymbol{\pi}, t^{j,F}_0, j) \leq F(\boldsymbol{I}, \boldsymbol{\pi}, t^{j,F}_+, j).$$

We will omit the superscript F when F is clear from the context.

As mentioned above, one important desideratum for an optimal solution is that it converges to equal-F. For this it is desired, if not necessary, that the difference between any two types' welfare doesn't get arbitrarily large. Therefore, it is useful to be able to directly refer to the difference in welfare, in some round, between the type with the highest welfare and the type with the lowest welfare.

Definition 2.3.12 (The big difference)**.** *Given a set $\mathcal{N}$ of agents, a finite set $\mathfrak{P}$ of possible projects, a k-PPB instance $\boldsymbol{I} = (I_1, \ldots, I_k)$ with $k \in \mathbb{N} \cup \{\infty\}$, a solution $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_k)$ for $\boldsymbol{I}$, a cost function $c : \mathfrak{P} \to \mathbb{N}$, a set of types $\mathcal{T} \subseteq 2^{\mathcal{N}}$ with $t \in \mathcal{T}$, a round $j \in \{1, \ldots, k\}$ with $I_j = \langle \mathcal{P}_j, b_j, A_j \rangle$, a welfare measure $F : (\boldsymbol{I}, \boldsymbol{\pi}, t, j) \to \mathbb{R}$, a type $t^{j,F}_-$ and a type $t^{j,F}_+$ we define the big, or total, difference DIF^F_j as $DIF^F_j = F(\boldsymbol{I}, \boldsymbol{\pi}, t^{j,F}_+, j) - F(\boldsymbol{I}, \boldsymbol{\pi}, t^{j,F}_-, j)$. We will omit the superscript F when F is clear from the context.*

[1]We could relax this assumption. However, this would complicate the definition. And since for our proofs this definition suffices, we stick to this definition.

- *Given a budget allocation $\pi \subseteq \mathcal{P}_j$ and $\pi \in \boldsymbol{\pi}$, we define the marginal increase of DIF^F_j due to π as $DIF^{\pi,F}_j = DIF^F_j - DIF^F_{j-1}$.*

Similarly, for sake of brevity, it will turn out to be useful to directly refer to the amount of welfare that a specific type has gained during some sequence of rounds.

Definition 2.3.13 (Increase of a type)**.** *Given a set $\mathcal{N}$ of agents, a finite set $\mathfrak{P}$ of possible projects, a k-PPB instance $\boldsymbol{I} = (I_1, \ldots, I_k)$ with $k \in \mathbb{N} \cup \{\infty\}$, a solution $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_k)$ for $\boldsymbol{I}$, a cost function $c : \mathfrak{P} \to \mathbb{N}$, a set of types $\mathcal{T} \subseteq 2^{\mathcal{N}}$, a type $t_i \in \mathcal{T}$, two rounds $j, l \in \{1, \ldots, k\}$ with $I_j = \langle \mathcal{P}_j, b_j, A_j \rangle$ and $I_l = \langle \mathcal{P}_l, b_l, A_l \rangle$, and a welfare measure $F : (\boldsymbol{I}, \boldsymbol{\pi}, t_i, j) \to \mathbb{R}$ we define the increase of a type t_i from round j to l for welfare measure F as: $\mathcal{D}^{t_i,F}_{j\to l} = F(\boldsymbol{I}, \boldsymbol{\pi}, t_i, l) - F(\boldsymbol{I}, \boldsymbol{\pi}, t_i, j)$. We will omit the superscript F when F is clear from the context.*

2.3.2 Results

As mentioned above, one of our main goals is analysing to what extent we can obtain a fairer distribution of welfare by taking previous rounds into account. Work has been done on this by Lackner, Maly, and Rey (2021). They found a mixture of positive and negative results.

A first positive result states that with two agents, there will always exist a solution that converges to equal-satisfaction. For two agents, there will also exist a solution that converges to equal-relative satisfaction, but we can actually show the stronger result that such a solution will exist for any two arbitrary types of agents. We will therefore consider this as a separate result later on.

Proposition 2.3.14. Let $\boldsymbol{I}$ be any ∞-PPB instance with two agents and a bound $B^* \in \mathbb{N}$. Furthermore, assume that for every round $j \in \{1, \ldots, k\}$ and both agents that there is a project p with $c(p) \leq b_j$ that the agent approves of. Then there exists a non-empty feasible solution $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_k)$ that converges to equal-satisfaction.

However, this result cannot be generalised. Even for three agents we can find a k-PPB instance for which there does not exist any solution that converges to equal-satisfaction.

Example 2.3.15 (No convergence to equal-satisfaction for three agents)**.** Let $\boldsymbol{I}$ be a ∞-PPB instance with three agents 1, 2, 3 where agent 1 has type t_1 and agents 2 and 3 have type t_2. Assume $b_j = 1$ for every round $j \in \{1, \ldots, k\}$ and $c(p) = 1$ for all projects $p \in \mathcal{P}$. In every round, there are two projects and agent 1 approves of both, 2 approves of only one and 3 of the other one. Then, for every non-empty feasible solution $\boldsymbol{\pi}$ and every round j, we have $sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) = j$ and $sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2) = \frac{j}{2}$.

Hence

$$\lim_{j\to+\infty} \left(\frac{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2)}{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1)} \right) = \frac{1}{2} \neq 1.$$

Therefore, by definition of convergence to equal-F, it follows that there exists no solution $\boldsymbol{\pi}$ such that $\boldsymbol{\pi}$ converges to equal-satisfaction. $\triangle$

This example exploits the fact that if we don't put any restrictions on agents' ballots, some ballots might be strict supersets of other ballots. In Example 2.3.15, for instance, we have that $A_j(2) \subsetneq A_j(1)$ and $A_j(3) \subsetneq A_j(1)$ for any round $j \in \{1, \ldots, k\}$. However, given our definition of satisfaction, this implies that two agents' full ballots might be funded, while one agent is strictly more satisfied with this than the other agent.

There are several ways to limit the impact of this possibility. One of these is to consider *relative* satisfaction, which we will do later. Another one is to restrict agents to submit *knapsack* (that is, exhaustive) ballots. In this way, intuitively, if we fund some agents' full ballots, there might still be a difference between the satisfaction of the agents, but this difference will not be 'very big'. In particular, this avoids the possibility of ballots being strict supersets of other ballots.

Indeed, if we restrict agents' ballots to knapsack ballots, we find that for three agents there always exists a solution that converges to equal-satisfaction.

Before we prove this, we first require a very useful result about the ballots of agents.

Lemma 2.3.16. *Let $\boldsymbol{I} = (I_1, \ldots, I_k)$ be any k-PPB instance. Let $\mathcal{N}$ be the set of agents. And let $I_j = (\mathcal{P}_j, b_j, A_j) \in \boldsymbol{I}$, where $A_j(i)$ is exhaustive for all $i \in \mathcal{N}$. For any $i, i' \in \mathcal{N}$, if $A_j(i) \neq A_j(i')$, then there exist some projects $p, p^* \in \mathcal{P}_j$ such that $p \in A_j(i), p \notin A_j(i'), p^* \in A_j(i')$ and $p^* \notin A_j(i)$.*

Proof. Suppose that $A_j(i) \neq A_j(i')$ and suppose for contradiction that there doesn't exist a project $p \in \mathcal{P}_j$ with $p \in A_j(i)$ and $p \notin A_j(i')$. Thus for all projects $p \in \mathcal{P}_j$ we have that if $p \in A_j(i)$, then $p \in A_j(i')$. Thus, by definition of $\subseteq$, we have that $A_j(i) \subseteq A_j(i')$. Now there are two cases. Either $A_j(i) = A_j(i')$, contradicting our earlier assumption that $A_j(i) \neq A_j(i')$. Or $A_j(i) \subsetneq A_j(i')$, contradicting our assumption that ballots are exhaustive. The other case is symmetric. $\square$

Lemma 2.3.16 now enables us to prove that when there are exactly three agents with knapsack ballots, there exists a feasible solution that converges to equal-satisfaction. Before we prove this result, we will first give the main intuition behind the proof.

In order to show convergence, it suffices to show that the difference in satisfaction between any two types cannot get arbitrarily large.

There are three agents, and they can - by definition - be divided into at least one and at most three different types. The proof is structured according to the amount of types in which the agents are divided. If there is only one type, the result follows immediately. If there are two types, then we use Lemma 2.3.16 to pick an allocation that generates more marginal satisfaction for the worst-off type, ensuring that the difference between the two types does not get arbitrarily large. The final, and most complicated, case is when there are three types (meaning that every type contains exactly one agent) and when the agent of the worst-off type has exactly the same ballots as the best-off type's agent.

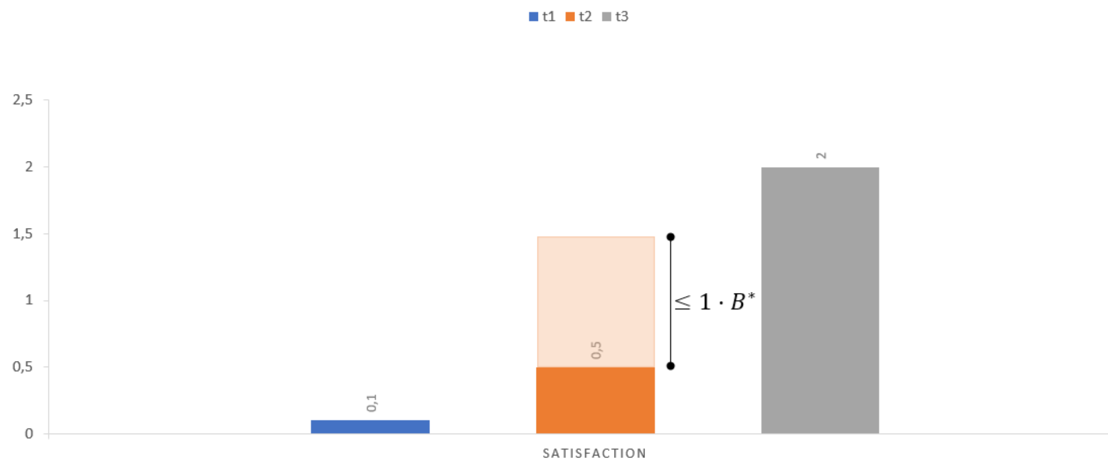

Figure 2.2: Ensuring a converging solution when there is a large difference between the satisfaction of the middle type and the satisfaction of the best-off type

This case is complicated, because we cannot naively pick the allocation that generates the most marginal satisfaction for the worst-off type. Instead, we need to pick an allocation by differentiating between two cases. If there is a relatively large difference between the satisfaction of the middle type and the best-off type, then we can pick an allocation that generates the most welfare for the middle type. This is illustrated by Figure 2.3.2. The transparant colours indicate the *marginal* satisfaction of the types, the opaque colours indicate the satisfaction of the types in some round.

If, however, there is a relatively small difference between the satisfaction of the middle type and the best-off type, then we can pick an allocation that generates no welfare for the middle type, and most welfare for both the worst-off and best-off type (since the ballots of their respective agents are identical). This is illustrated by Figure 2.3.2. In the figure, $sat^{\pi}_{t_-}$, $sat^{\pi}_{t_0}$ and $sat^{\pi}_{t_+}$ are shorthands, indicating the marginal satisfaction for the respective types t_-, t_0 and t_+ in some round due to allocation π.

Proposition 2.3.17. Consider an ∞-PPB instance $\boldsymbol{I} = (I_1, \ldots, I_k)$ with three agents where the ballot of each agent is exhaustive (i.e., is a knapsack ballot) in every round and there is a bound $B^* \in \mathbb{N}$. Then, there is a non-empty feasible solution that converges to equal-satisfaction.

Proof. Suppose w.l.o.g. that the agents are called 1,2 and 3. Since there are three agents, there can possibly be at most three types: either there is only one, there are two, or there are three. If there is only one type, then every agent belongs to the same type. In that case, convergence to equal-satisfaction is trivially satisfied. So now we prove the two other cases. First we prove the case in which there are only two types, then the case in which there are 3.

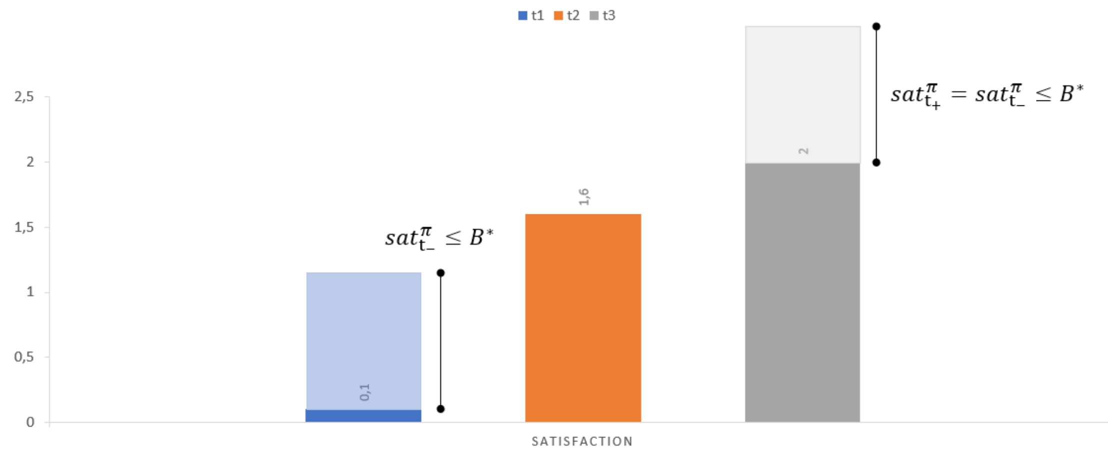

Figure 2.3: Ensuring a converging solution when there is a small difference between the satisfaction of the middle type and the satisfaction of the best-off type

Case 1: two types. So suppose that there are only two types: t_1 and t_2. Further, suppose w.l.o.g. that $T(1) = t_1$ and $T(2) = T(3) = t_2$. As for the general structure of the proof, we first show that there exists a solution $\boldsymbol{\pi}$ such that the difference in satisfaction between any two types is at most the bound B^* in any round $j \in \{1, \ldots, k\}$. After that we show how this implies the fact that the solution $\boldsymbol{\pi}$ converges to equal-satisfaction.

More precisely, we will show first that for any round $j \in \{1, \ldots, k\}$:

$$sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) - B^* \leq sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) + B^*. \tag{2.1}$$

We prove this by induction on the number of rounds j. For the base case where $j = 1$, by picking any allocation $\pi \subseteq \mathcal{P}_1$ at most B^* is spend on projects, by definition of B^*. Hence, by the definition of sat_j it holds that $0 \leq sat_1(\boldsymbol{I}, \boldsymbol{\pi}, t) \leq B^*$ for $t \in \{t_1, t_2\}$. And therefore it follows that $sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) - B^* \leq sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) + B^*$.

Now consider an arbitrary round j-1 and suppose that (1) holds for round j-1. We will show that it holds for round j too.

We have two cases: either $sat_{j-1}(\boldsymbol{I}_{j-1}, \boldsymbol{\pi}_{j-1}, t_1) \leq sat_{j-1}(\boldsymbol{I}_{j-1}, \boldsymbol{\pi}_{j-1}, t_2)$, or $sat_{j-1}(\boldsymbol{I}_{j-1}, \boldsymbol{\pi}_{j-1}, t_1) \geq sat_{j-1}(\boldsymbol{I}_{j-1}, \boldsymbol{\pi}_{j-1}, t_2)$.

Suppose first that $sat_{j-1}(\boldsymbol{I}_{j-1}, \boldsymbol{\pi}_{j-1}, t_1) \leq sat_{j-1}(\boldsymbol{I}_{j-1}, \boldsymbol{\pi}_{j-1}, t_2)$, i.e., suppose that in round j-1 type 1 has a lower satisfaction than type 2. Consider the budgeting problem $I_j = \langle \mathcal{P}_j, b_j, A_j \rangle \in \boldsymbol{I}$ for round j. Set $\pi_j = A_j(1)$. Note that this is possible since every agent $i \in \mathcal{N}$ has an exhaustive ballot.

Note then that $A_j(2) \cap \pi_j \subseteq A_j(1) \cap \pi_j = A_j(1)$ and that $A_j(3) \cap \pi_j \subseteq A_j(1) \cap \pi_j = A_j(1)$. Hence $sat^m_j(\boldsymbol{I}, \pi_j, 1) \geq sat^m_j(\boldsymbol{I}, \pi_j, 2)$ and $sat^m_j(\boldsymbol{I}, \pi_j, 1) \geq sat^m_j(\boldsymbol{I}, \pi_j, 2)$.

Recall now the definition of the marginal satisfaction for a type $t \in \mathcal{T}$ for round $j \in \{1, \dots, k\}$:

$$sat_j^m(\boldsymbol{I}, \pi_j, t) = \frac{1}{|t|} \sum_{i \in t} sat_j^m(\boldsymbol{I}, \pi_j, i).$$

From this definition and the inequalities above follows $sat_j^m(\boldsymbol{I}, \pi_j, t_1) \geq sat_j^m(\boldsymbol{I}, \pi_j, t_2)$.

Together with the assumption that $sat_{j-1}(\boldsymbol{I}_{j-1}, \boldsymbol{\pi}_{j-1}, t_1) \leq sat_{j-1}(\boldsymbol{I}_{j-1}, \boldsymbol{\pi}_{j-1}, t_2)$, it follows that $sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) - B^* \leq sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) + B^*$.

Now suppose, for the other case, that $sat_{j-1}(\boldsymbol{I}_{j-1}, \boldsymbol{\pi}_{j-1}, t_1) \geq sat_{j-1}(\boldsymbol{I}_{j-1}, \boldsymbol{\pi}_{j-1}, t_2)$. That is, suppose that in round j-1 type 1 has a higher satisfaction than type 2. Now consider again the budgting problem $I_j = \langle \mathcal{P}, b_j, A_j \rangle$ for round j. Either $A_j(1) = A_j(2) = A_j(3)$, or not. If $A_j(1) = A_j(2) = A_j(3)$, then clearly – regardless of what budget allocation π_j we select – the difference between the types' satisfaction doesn't change, so the result holds by the IH. If it is not the case, then, by definition, it follows that $A_j(1) \neq A_j(i)$ for some $i \in t_2$. By Lemma 2.3.16 there exists a project $p \in A_j(i)$ and $p \notin A_j(1)$ for some $i \in t_2$. Set $\pi_j = \{p\}$. Then $B^* \geq sat_j^m(\boldsymbol{I}, \pi_j, t_2) \geq sat_j^m(\boldsymbol{I}, \pi_j, t_1) = 0$. Together with the assumption that $sat_{j-1}(\boldsymbol{I}_{j-1}, \boldsymbol{\pi}_{j-1}, t_1) \geq sat_{j-1}(\boldsymbol{I}_{j-1}, \boldsymbol{\pi}_{j-1}, t_2)$ it follows that $sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) - B^* \leq sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) + B^*$.

So now we have a solution $\boldsymbol{\pi}$ such that for any round $j \in \{1, \dots, k\}$:

$$sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) - B^* \leq sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) + B^*.$$

Given this, it remains to be shown that $\boldsymbol{\pi}$ converges to equal-satisfaction. The intuitive argument is as follows. For both types it holds that after a finite amount of rounds, the type's satisfaction increases. Therefore, the more rounds proceed, the higher the total satisfaction for both types. However, the difference between both types is at most B^* in *any* round. So the difference in satisfaction between the two types does *not* increase the more rounds proceed. Therefore, the more rounds proceed, the smaller the difference in satisfaction becomes relative to the total satisfaction of both types. Thus when the number of rounds tends to infinity, the solution $\boldsymbol{\pi}$ converges to equal-satisfaction.

First observe that $sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) + sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) \geq \sum_{j'=1}^{j} c(\pi_{j'})$, by construction of $\pi_{j'}$. Together with our earlier result that $sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) - B^* \leq sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) + B^*$ it follows that $\lim_{j \to +\infty}(sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1)) = +\infty$ and $\lim_{j \to +\infty}(sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2)) = +\infty$. Hence we have

$$\lim_{j \to +\infty} \left(\frac{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) - B^*}{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1)} \right) = \lim_{j \to +\infty} \left(\frac{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2) + B^*}{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2)} \right) = 1.$$

By our previous result we have:

$$sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) - B^* \leq sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) + B^*$$

Dividing by $sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1)$ we get:

$$\frac{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) - B^*}{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1)} \leq \frac{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2)}{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1)} \leq \frac{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) + B^*}{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1)}$$

Since the inequality above holds for any round $j \in \{1, \ldots, k\}$ it follows that

$$\lim_{j \to +\infty} \left(\frac{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2)}{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1)} \right) = 1.$$

Case 2: three types. Now suppose that there are three types and suppose w.l.o.g. that $T(i) = t_i$ for all $i \in \mathcal{N}$. For this case we show that there exists some solution $\boldsymbol{\pi}$ such that the difference between any two types in any round $i \in \{1, \ldots, k\}$ is at most 2 times the bound B^*. So we show that there exists some solution $\boldsymbol{\pi}$ such that for any $i, i^* \in \{1, 2, 3\}$ we have that:

$$|sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_i) - sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_{i^*})| \leq 2B^*. \tag{2.2}$$

From this fact, convergence to equal-satisfaction will follow in the same way as in the previous case. Intuitively: if the difference between the satisfaction of the types remains inside this bound, then the more rounds proceed, the higher the total satisfaction will become, and the smaller the difference between the satisfaction of the types becomes relative to the total satisfaction.

We again prove this claim by induction on the number of rounds j. For the base case, suppose j=1. Then, as before, set $\pi_1 = \pi \subseteq \mathcal{P}_1$ for any $\pi \subseteq \mathcal{P}_1$. Then there exists a solution $\boldsymbol{\pi} = (\pi_1)$ such that for any $i, i^* \in \{1, 2, 3\}$: $|sat_1(\boldsymbol{I}, \boldsymbol{\pi}, t_i) - sat_1(\boldsymbol{I}, \boldsymbol{\pi}, t_{i^*})| \leq 2B^*$.

For the induction hypothesis, consider any $j - 1 \in \{1, \ldots, k\}$ and suppose that there exists a solution $\boldsymbol{\pi}$ such that
$|sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_i) - sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_{i^*})| \leq 2B^*$ for any $i, i^* \in \{1, 2, 3\}$. We will show that $|sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_i) - sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_{i^*})| \leq 2B^*$ for any $i, i^* \in \{1, 2, 3\}$. Also suppose w.l.o.g. that $t_1 = t_-^{j-1}$, $t_2 = t_0^{j-1}$ and $t_3 = t_+^{j-1}$.

Now consider two cases. Either $A_j(1) = A_j(3)$, or $A_j(1) \neq A_j(3)$. We show in both cases that the result follows. Suppose first that $A_j(1) = A_j(3)$. Now, in this case we make a further sub-case distinction. Either

$$sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_1) + B^* \tag{2.3}$$

or

$$sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_3) \leq sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_2) + B^* \tag{2.4}$$

Intuitively, it might be the case that t_2's satisfaction is more or equal (the first case) or less (the second case) than a full budget more than t_1's satisfaction.

Suppose first that (2.3) holds, i.e. that $sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_1) + B^*$, and suppose that $A_j(2) \neq A_j(1) = A_j(3)$ (if $A_j(1) = A_j(2) = A_j(3)$ the result follows immediately by the IH). By Lemma 2.3.16, there exists some project $p_2 \in \mathcal{P}$ s.t. $p_2 \in A_j(2)$ and $p_2 \notin A_j(1) = A_j(3)$. Then set $\pi_j = \{p_2\}$. By definition of sat^m_j it follows that $sat^m_j(\boldsymbol{I}, \pi_j, t_2) \geq sat^m_j(\boldsymbol{I}, \pi_j, t_1) = sat^m_j(\boldsymbol{I}, \pi_j, t_3) = 0$. Possibly, then t_2 'overtakes' t_3, or not. If $t_2 = t_0^{j-1} = t_+^j$ – i.e., if t_2 overtakes – then from our assumption that $sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_1) + B^*$, we have that $DIF_j = sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_0^{j-1}) - sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_-^j) \leq 1B^* + 1B^* = 2B^*$.

If $t_2 = t_0^{j-1} = t_0^j$ - i.e., if t_2 doesn't overtake – then clearly $DIF_j = DIF_{j-1} + DIF^{\pi}_j = DIF_{j-1} + 0 = DIF_{j-1}$, thus it follows by the IH that $|sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_i) - -sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_{i^*})| \leq 2B^*$.

Now suppose that (2.4) holds, i.e. that $sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_3) \leq sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_2) + B^*$. By Lemma 2.3.16 there exists some project $p_1 \in \mathcal{P}$ such that $p_1 \notin A_j(2)$ and $p_1 \in A_j(1) = A_j(3)$. Set $\pi_j = \{p_1\}$. Note that by assumption we have that $A_j(1) = A_j(3)$. Hence if $t_-^{j-1} = t_-^j$ it follows that $DIF_{j-1} = DIF_j$ and the result follows from the I.H..

So suppose that $t_-^{j-1} \neq t_-^j$. Since $sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_3) \leq sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_2) + B^*$, we have that $sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_3) - sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq B^*$. By definition of the marginal increase of satisfaction we have that $sat^m_j(\boldsymbol{I}, \pi_j, t_2) - sat_{j-1}(\boldsymbol{I}, \pi_j, t_3) \geq -1B^*$. Since by definition $sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2) = sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_2) + sat^m_j(\boldsymbol{I}, \pi_j, t_2)$ and $sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_3) = sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_3) + sat^m_j(\boldsymbol{I}, \pi_j, t_3)$ it follows that $|sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_3) - sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2)| \leq 2B^*$.

Now consider the case in which $A_j(1) \neq A_j(3)$. By Lemma 2.3.16, there exists some project $p \in A_j(1)$ s.t. $p \notin A_j(3)$. Set $\pi_j = \{p\}$. Then, clearly $sat^m_j(I_j, \pi_j, t_1) \geq sat^m_j(I_j, \pi_j, t_2)$ and $sat^m_j(I_j, \pi_j, t_1) > sat^m_j(I_j, \pi_j, t_3) = 0$.

Since $sat^m_j(I_j, \pi_j, t_3) = 0$, it follows that $DIF_j \leq DIF_{j-1}$. By the IH it follows that $|sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_i) - sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_{i^*})| \leq 2B^*$.

Through similar reasoning as for the case in which $|\mathcal{T}| = 2$, we find that this means that $\boldsymbol{\pi}$ satisfies convergence to equal-satisfaction. □

Even with the restriction to knapsack ballots, however, some ballots might be strictly more expensive than other ballots. By our definition of satisfaction, this implies again that funding two agents' full ballots might make one agent strictly more satisfied than the other. This possibility is again exploited in the following example, which shows that for eigth agents, there are k-PPB instances that, even when restricting the ballots to knapsack ballots, do not allow for a solution that converges to equal-satisfaction.

Example 2.3.18. Let $\boldsymbol{I}$ be a ∞-PPB instance. In every round j, we have $b_j = 10$, there are eight agents $1, \ldots, 8$ such that 1,2,3 have type t_1 and 4,5,6,7,8 have type t_2. Furthermore, there are six projects $p_1, \ldots, p_6$ such that $c(p_1) = c(p_2) = c(p_3) = 5$ and $c(p_4) = c(p_5) = c(p_6) = 3$. The ballots are such that, for every round j:

$$A_j(i) = \begin{cases} \{p_1, p_4\}, & \text{if } i = 1 \\ \{p_2, p_5\}, & \text{if } i = 2 \\ \{p_3, p_6\}, & \text{if } i = 3 \\ \{p_1, p_2\}, & \text{if } i = 4 \\ \{p_1, p_3\}, & \text{if } i = 5 \\ \{p_2, p_3\}, & \text{if } i = 6 \\ \{p_4, p_5, p_6\}, & \text{if } i = 7 \\ \{p_4, p_5, p_6\}, & \text{if } i = 8 \end{cases}$$

As can be checked, at any round j and for each project $p \in \mathfrak{P}$, we have that the marginal satisfaction for p for type t_2 is higher than for type t_1. This directly implies that there can be no non-empty solution converging to equal-satisfaction. We will refrain from giving the calculations here, since we will later prove the stronger result that even for seven agents there exists a k-PPB instance for which no such solution exists. $\triangle$

As mentioned before, the above counterexamples exploit the fact that some agents' ballots are more expensive than other ballots. As a result, a budget allocation that funds some agents' full ballot might generate strictly more welfare for this agent than another allocation would for another agent whose full ballot is funded. For instance, in Example 2.3.18, agent 4's full ballot is strictly more expensive than agent 3's ballot: $c(A_j(4)) = 10$, while $c(A_j(3)) = 8$. Restricting the ballots to knapsack ballots only partially solved this problem, since all ballots in Example 2.3.18 are knapsack ballots. Another possible solution is considering the notion of *relative* satisfaction, which makes the welfare relative to the total cost of the ballot.

Indeed, this gives us more positive results. First of all, we can show that for two arbitrary types of agents, there will always exist a solution $\boldsymbol{\pi}$ that converges to equal-relative satisfaction.

Theorem 2.3.19 (Convergence to equal-relative satisfaction for two types)**.** *Assume that $\boldsymbol{I}$ is an ∞-PPB-instance with non-empty knapsack ballots such that there are only two types and a bound $B^* \in \mathbb{N}$. Then, there is a non-empty feasible solution for $\boldsymbol{I}$ that converges to equal-relative satisfaction.*

We refrain from giving the proof of Theorem 2.3.19 here and instead refer the reader to the article of Lackner, Maly, and Rey (2021) for the proof.

2.4 Conclusions

In this chapter, we have layed the groundwork needed to understand the results in the coming chapters. We defined PB in a standard way, and then presented PPB as an extension of this framework. We presented several ways of formalising welfare and fairness in PPB. Some – though not much – research has already been conducted on the extent to which fair solutions can be guaranteed to exist and can be computed in PPB. We gave several of the results that this research has yielded.

Some of these results are positive, others negative. The extent to which fair solutions can be guaranteed to exist depends on the definitions of fairness and welfare that are used. We saw in Example 2.3.10 that defining fairness as equal-F is too strict, as even when there are four agents with knapsack ballots, a fair solution cannot be guaranteed to exist.

Defining fairness as convergence to equal-F yields more positive results. A solution converging to equal-satisfaction can be guaranteed to exist when there are two agents (Proposition 2.3.14), though this cannot be generalised to three agents (Example 2.3.15). However, if we restrict the ballots to knapsack ballots, convergence to equal-satisfaction *is* possible for three agents (Proposition 2.3.17), though not for more than eight agents (Example 2.3.18). Convergence to equal *relative* satisfaction is even more promising, since we can prove a fair solution always exists when there are at most two types (of arbitrary size) and we restrict the ballots to knapsack ballots (Theorem 2.3.19). Table 6.1 summarises all the results.

Chapter 3

Results about Satisfaction

In this chapter, we show two results about convergence to equal-satisfaction. As a first result, we show that when there are only four agents who are divided into at most three types, there always exists a solution that converges to equal-satisfaction. As was shown before, equal-satisfaction cannot, however, be guaranteed for an arbitrary amount of agents. As a second result, we show that even for seven agents it is not always possible to guarantee a solution that converges to equal-satisfaction.

3.1 Convergence to Equal-Satisfaction for Four Agents and at most Three Types

As mentioned above, in this section we will prove that when there are only four agents, who are divided in at most three different types, there always exists some non-empty feasible solution $\boldsymbol{\pi}$ that converges to equal-satisfaction. More precisely, in Section 3.1.3, we will prove the following theorem:

Theorem 3.1.1. *Consider an* ∞-PPB *instance* $\boldsymbol{I} = (I_1, I_2, \ldots)$ *with four agents where the ballot of each agent is exhaustive in every round,* $|\mathcal{T}| \leq 3$ *and there exists a constant* $B^* \in \mathbb{N}$ *with* $b_j \leq B^*$ *for every round j. Then, there is a non-empty feasible solution that converges to equal-satisfaction.*

We will first sketch the higher-level structure of the proof. To show convergence, we will show that the difference in satisfaction between any two types cannot get arbitrarily large, which will imply convergence analogously to the way explicated in the proof of Proposition 2.3.17. The proof of this fact consists in several case distinctions. The main case distinction is based on the size of the set of types $\mathcal{T}$. Based on this case distinction, we then make the further sub-case distinction based on how the agents are divided among the types.

In all of these cases we can show that there exists an allocation such that the total difference doesn't exceed $5 \cdot B^*$. In almost all cases, this can be shown by an induction

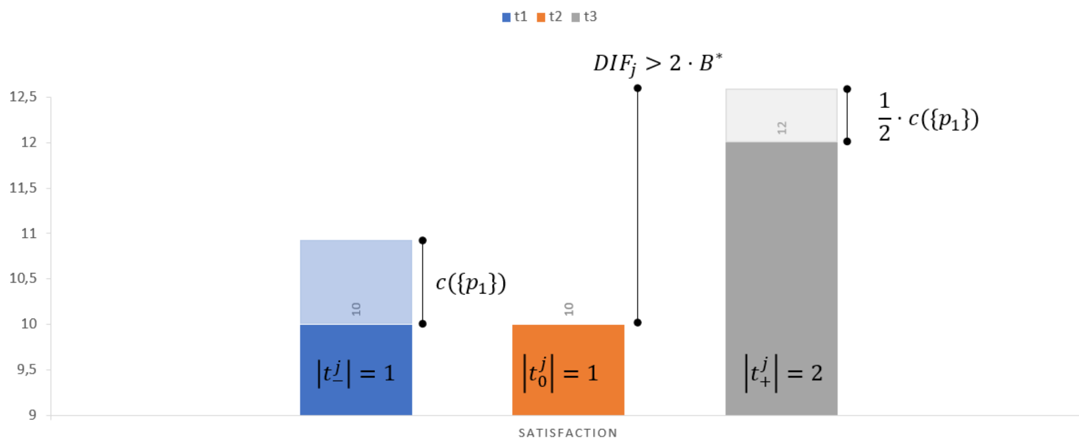

Figure 3.1: Illustrating a critical case for satisfaction

on the rounds, which shows that in all these cases either the total difference *decreases*, or – otherwise, if it increases – the total difference is extremely small (thereby certainly satisfying the $5 \cdot B^*$ bound).

There are, however, cases in which we *cannot* show this. In these cases, the total difference can *increase*, while not being extremely small. We will now give an example of such a 'critical' case.

Consider a k-PPB instance $\boldsymbol{I} = (I_1, \ldots, I_k)$ and some round $j \in \{1, \ldots, k\}$, with four agents and three types. And suppose that the best-off type t_+^j in round j has size 2, while the other types t_-^j and t_0^j have size 1. Furthermore, suppose that $sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_-^j) = sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_0^j) = 10 \cdot B^*$, while $sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_+^j) = 12 \cdot B^*$, i.e., types t_-^j and t_0^j have a satisfaction in round j of $10 \cdot B^*$, while type t_+^j has a satisfaction of $12 \cdot B^*$. By definition of DIF_j, we have that $DIF_j = 2 \cdot B^*$.

Now suppose that there are only two available projects p_1 and p_2, and that in round $j+1$ the ballots are as follows: the agent of type t_-^j votes for p_1, the agent of type t_0^j votes for p_2 and the two agents of type t_+^j vote mixed: one votes for p_1, while the other votes for p_2. Now we're forced to choose. We could either set $\pi_{j+1} = \{p_1\}$ or $\pi_{j+1} = \{p_2\}$. In both cases, however, one of the two lowest types gets 0 marginal satisfaction, and hence its satisfaction in round $j+1$ remains $10 \cdot B^*$. That is, either $sat_{j+1}(\boldsymbol{I}, \boldsymbol{\pi}, t_-^j) = sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_-^j) + sat_{j+1}^m(\boldsymbol{I}, \pi_{j+1}, t_-^j) = 10 \cdot B^*$ or $sat_{j+1}(\boldsymbol{I}, \boldsymbol{\pi}, t_0^j) = sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_0^j) + sat_{j+1}^m(\boldsymbol{I}, \pi_{j+1}, t_0^j) = 10 \cdot B^*$. However, clearly, we have that, regardless of π_{j+1}, t_+^j gets strictly more than 0 marginal satisfaction in round $j+1$. Hence $sat_{j+1}(\boldsymbol{I}, \boldsymbol{\pi}, t_+^j) > 12 \cdot B^*$. Therefore, $DIF_{j+1} > 2 \cdot B^*$ for all possible allocations π_{j+1}. Figure 3.1 visualises this example.

Thus, in such critical cases we cannot prevent the total difference from increasing, while it neither is extremely small. In order to prove that in all cases the total difference is bounded by $5 \cdot B^*$, we therefore need to ensure that a critical case cannot be a round j in which the total difference DIF_j is already $5 \cdot B^*$, for this would possibly force the total difference in the next round to be strictly higher: $DIF_{j+1} > 5 \cdot B^*$, violating the bound of $5 \cdot B^*$.

Intuitively, we will show this as follows. We suppose, for contradiction, that such a situation (which would force the total difference above the bound of $5 \cdot B^*$) *is* possible. As mentioned above, in all cases except for these complicated cases, we can show that either (1) the total difference *decreases*, or (2) it already is extremely small. This will imply that we must have encountered, in an earlier round, a critical case in which the total difference is close (or equal) to $2 \cdot B^*$ (also to e.g. $3 \cdot B^*$). We will prove two lemmas – we require *two* (respectively Lemma 3.1.7 and Lemma 3.1.8), because the situations that we require them for differ in the way the agents are divided among the three types – that show that if we start in such a round, there will always exist a solution that prevents the total difference from getting close to $5 \cdot B^*$. This then contradicts our earlier assumption that $DIF_{j+1} > 5 \cdot B^*$. Hence we will always be able to avoid critical cases in which the total difference is forced above the $5 \cdot B^*$ bound.

However, in order to prove these lemmas, and ultimately the theorem, we first require several definitions, which we give below.

3.1.1 Definitions

First, it will prove useful to be able to denote the difference in welfare between the middle type and the worst-off type. Note that since our notion of $t_0^{j,F}$ is only defined for three types, the notion of Γ_j^F – which is defined in terms of $t_0^{j,F}$ – is likewise defined only for three types.

Definition 3.1.2 (Γ_j^F)**.** *Given a set $\mathcal{N}$ of agents, a finite set $\mathfrak{P}$ of possible projects, a k-PPB instance $\boldsymbol{I} = (I_1, \ldots, I_k)$ with $k \in \mathbb{N} \cup \{\infty\}$, a solution $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_k)$ for $\boldsymbol{I}$, a cost function $c : \mathfrak{P} \to \mathbb{N}$, a set of types $\mathcal{T} \subseteq 2^{\mathcal{N}}$ of size 3, a round $j \in \{1, \ldots, k\}$ with $I_j = \langle \mathcal{P}_j, b_j, A_j \rangle$, a welfare measure $F : (\boldsymbol{I}, \boldsymbol{\pi}, t, j) \to \mathbb{R}$, a type $t_-^{j,F}$ and a type $t_0^{j,F}$, let $\Gamma_j^F := F(\boldsymbol{I}, \boldsymbol{\pi}, t_0^{j,F}, j) - F(\boldsymbol{I}, \boldsymbol{\pi}, t_-^{j,F}, j)$. We drop the superscript F if F is clear from the context.*

- *Given a budget allocation $\pi \subseteq \mathcal{P}_j$, we define the marginal increase of $\Gamma_j^{F,\pi}$ due to π as $\Gamma_j^{F,\pi} = \Gamma_j^F - \Gamma_{j-1}^F$.*

Similarly, it will prove useful to be able to denote the difference in welfare between the best-off type and the middle type. This notion is defined for any amount of types.

Definition 3.1.3 (Δ_j^F)**.** *Given a set $\mathcal{N}$ of agents, a finite set $\mathfrak{P}$ of possible projects, a k-PPB instance $\boldsymbol{I} = (I_1, \ldots, I_k)$ with $k \in \mathbb{N} \cup \{\infty\}$, a solution $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_k)$ for $\boldsymbol{I}$, a cost function $c : \mathfrak{P} \to \mathbb{N}$, a set of types $\mathcal{T} \subseteq 2^{\mathcal{N}}$, a round $j \in \{1, \ldots, k\}$ with*

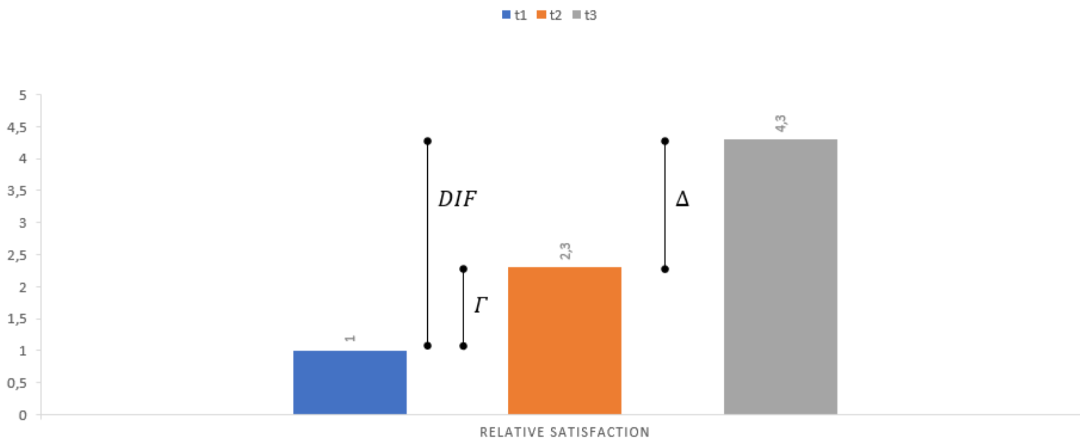

Figure 3.2: Illustrating *DIF*, Γ and Δ

$I_j = \langle \mathcal{P}_j, b_j, A_j \rangle$, a welfare measure $F : (\boldsymbol{I}, \boldsymbol{\pi}, t, j) \to \mathbb{R}$, a type $t_0^{j,F}$ and a type $t_+^{j,F}$, let $\Delta_j^F := F(\boldsymbol{I}, \boldsymbol{\pi}, t_+^{j,F}, j) - F(\boldsymbol{I}, \boldsymbol{\pi}, t_0^{j,F}, j)$. We drop the superscript F if F is clear from the context.

- *Given a budget allocation $\pi \subseteq \mathcal{P}_j$, we define the marginal increase of $\Delta_j^{F,\pi}$ due to π as $\Delta_j^{F,\pi} = \Delta_j^F - \Delta_{j-1}^F$.*

Figure 3.1.1 illustrates the concepts of *DIF*, Γ and Δ for some specific round and for the specific welfare function of relative satisfaction. Figure 3.3 illustrates the notions describing the marginal increase of *DIF*, Γ and Δ. The darker coloured boxes indicate the relative satisfaction of the three types in one specific round. The lighter coloured boxes indicate the marginal relative satisfaction of the types for some allocation π in the next round.

Note that by definition we have that $DIF_j^F = \Delta_j^F + \Gamma_j^F$. Next, we define the notion of a 'critical case'. A critical case intuitively refers to a case in which the total difference possibly increases.

Definition 3.1.4 (Critical case)**.** *Given a k-PPB instance $\boldsymbol{I} = (I_1, \ldots, I_k)$ with $k \in \mathbb{N} \cup \{\infty\}$, some round $j \in \{1, \ldots, k\}$, a set of types $\mathcal{T}$ of size 3 and a bound $B^* \in \mathbb{N}$, a round j is a critical case iff*

1. *$|t_-^j| = 1, |t_0^j| = 1, |t_+^j| = 2$ and $F(\boldsymbol{I}, \boldsymbol{\pi}, t_0^{j,F}, j) - F(\boldsymbol{I}, \boldsymbol{\pi}, t_-^{j,F}, j) < \frac{1}{2} \cdot B^*$, or*
2. *$|t_-^j| = 2, |t_0^j| = 1, |t_+^j| = 1$ and $F(\boldsymbol{I}, \boldsymbol{\pi}, t_+^{j,F}, j) - F(\boldsymbol{I}, \boldsymbol{\pi}, t_0^{j,F}, j) < \frac{1}{2} \cdot B^*$.*

Not only do we require the notion of a critical case, but we also require the notion of a critical case which is safe for some bound. The intuition behind this definition is that,

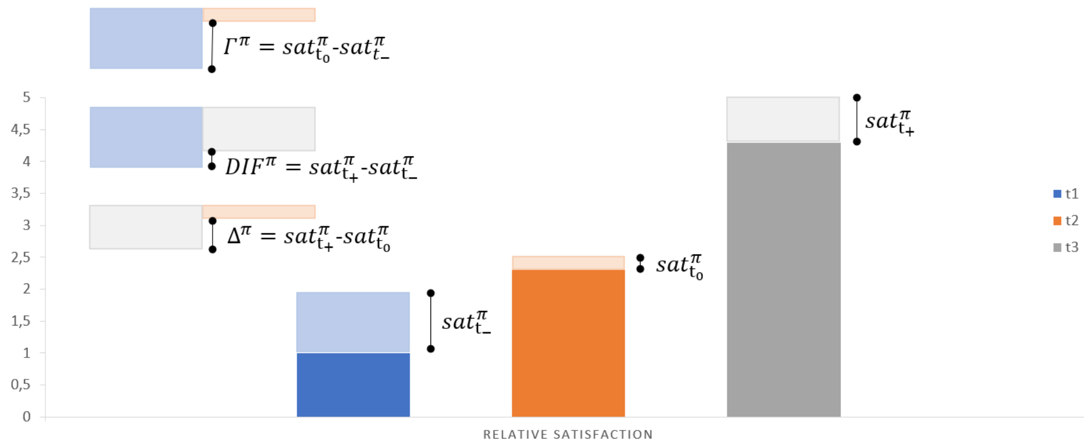

Figure 3.3: Marginal relative satisfaction

given some bound $x \cdot B^*$ with $x \in \mathbb{N}$, some critical cases are 'good' cases in the sense that the total difference is small enough with respect to Γ. For these cases, we say that they are 'safe' with respect to the bound.

Definition 3.1.5 (Critical case safe for some bound)**.** *Given a k-PPB instance $\boldsymbol{I} = (I_1, \ldots, I_k)$ with $k \in \mathbb{N} \cup \{\infty\}$, some round $j \in \{1, \ldots, k\}$, any natural number $x \in \mathbb{N}$, a set of types $\mathcal{T}$ of size 3 and any bound $B^* \in \mathbb{N}$, we say that round j is a critical case safe for $x \cdot B^*$ iff j is a critical case and DIF$_j \leq x + \frac{\Gamma_j}{2}$.*

3.1.2 Lemmas

We will now prove the lemmas that enable us to show the convergence-result.

As mentioned, in order to show convergence, we will show that there exists a way of selecting allocations so that the total difference doesn't exceed $5 \cdot B^*$. Recall that the definition of the total difference is based on the marginal satisfaction of types (in particular on the marginal satisfaction of the worst-off and best-off type). By definition, the marginal satisfaction of types is dependent on the marginal satisfaction of the agents of the type. And the marginal satisfaction of agents is, in its turn, dependent on the ballots of the agents.

We therefore require a general way of reasoning about the ballots of different agents. One of these ways is exemplified by Lemma 2.3.16, which we proved in the Preliminaries. It states that when two exhaustive (or 'knapsack') ballots are different, there will always be a project that is in one of the ballots, but not in the other, and vice versa. This will prove to be useful, for by selecting this project as an allocation, we know that one agent is not satisfied at all (i.e., gets 0 marginal satisfaction), which we can use to prevent – in some round i – the best-off type t^i_+ from gaining too much satisfaction.

Intuitively, Lemma 2.3.16 gives us information about the ballots of *agents*, and hence about how much marginal satisfaction certain allocations generate for agents. However, as mentioned above, we require information about how much marginal satisfaction allocations generate for *types* of agents. Given the fact that our set of agents has a limited size, Lemma 2.3.16 directly implies a fact about the marginal satisfaction for types of agents. The following lemma formalises this fact.

Lemma 3.1.6. *Let $\boldsymbol{I} = (I_1, \ldots, I_k)$ be any k-PPB instance with $k \in \mathbb{N} \cup \{\infty\}$ and four agents. Let $j \in \{1, \ldots, k\}$ be any round with $I_j = \langle \mathcal{P}_j, b_j, A_j \rangle \in \boldsymbol{I}$, where $A_j(i)$ is exhaustive for all $i \in \mathcal{N}$. Suppose that there are at most three types. Then for all types $t_i, t_{i'}$ with $i \in \{1, 2, 3\}$ and $i \neq i'$: if there exists agents $n \in t_i$ and $n' \in t_{i'}$ such that $A_j(n) \neq A_j(n')$, then there exists some $\pi_j \subseteq \mathcal{P}$ s.t. $\frac{1}{2} \cdot sat_j^m(\boldsymbol{I}, \pi_j, t_i) \geq sat_j^m(\boldsymbol{I}, \pi_j, t_{i'})$. Furthermore, if $|t_i| = 2$ and $|t_{i'}| = 1$, then there exists an allocation $\pi_j \subseteq \mathcal{P}$ s.t. $sat_j^m(\boldsymbol{I}, \pi_j, t_i) \geq \frac{1}{2} \cdot c(\pi_j)$ and $sat_j^m(\boldsymbol{I}, \pi_j, t_{i'}) = 0$.*

Proof. Consider an arbitrary round $j \in \{1, \ldots, k\}$, and consider two arbitrary types t_i and $t_{i'}$ with $i \in \{1, 2, 3\}$ and $i \neq i'$. We proceed by cases.

1) $|t_i| = 1$ and $|t_{i'}| = 1$. This case follows immediately from Lemma 2.3.16.

2) $|t_i| = 2$ and $|t_{i'}| = 1$. Suppose that there exists some agents $n \in t_i$ and $n' \in t_{i'}$ such that $A_j(n) \neq A_j(n')$. By Lemma 2.3.16, there exists some project p^* s.t. $p^* \in A_j(n)$ and $p^* \notin A_j(n')$. Set $\pi_j = \{p^*\}$. Let n^* be the agent $n^* \in t_i$ such that $n^* \neq n$. Either $p^* \in A_j(n^*)$ or $p^* \notin A_j(n^*)$. Suppose first that $p^* \in A_j(n^*)$. Then $sat_j^m(\boldsymbol{I}, \{p^*\}, t_i) = c(p^*)$, and $sat_j^m(\boldsymbol{I}, \{p^*\}, t_{i'}) = 0$. Hence there exists some $\pi_j \subseteq \mathcal{P}$ s.t. $\frac{1}{2} \cdot sat_j^m(\boldsymbol{I}, \pi_j, t_i) \geq sat_j^m(\boldsymbol{I}, \pi_j, t_{i'})$. And hence the claim holds. Suppose then that $p^* \notin A_j(n^*)$. Then $sat_j^m(\boldsymbol{I}, \{p^*\}, t_i) = \frac{1}{2} \cdot c(p)$. Since again, we have that $sat_j^m(\boldsymbol{I}, \{p^*\}, t_{i'}) = 0$, the claim follows.

3) $|t_i| = 1$ and $|t_{i'}| = 2$. Suppose that there exists some agents $n \in t_i$ and $n' \in t_{i'}$ such that $A_j(n) \neq A_j(n')$. By Lemma 2.3.16, there exists some project p^* s.t. $p^* \in A_j(n)$ and $p^* \notin A_j(n')$. Let n^* be the agent $n^* \in t_{i'}$ such that $n^* \neq n'$. We have that either $p^* \in A_j(n^*)$, or not. Suppose first that $p^* \notin A_j(n^*)$. Then $sat_j^m(\boldsymbol{I}, \{p^*\}, t_{i'}) = 0$. Clearly, then, the claim follows. Suppose now that $p^* \in A_j(n^*)$. Then $sat_j^m(\boldsymbol{I}, \{p^*\}, t_{i'}) = \frac{1}{2} \cdot c(\{p^*\})$, by definition of satisfaction. Since we have that $sat_j^m(\boldsymbol{I}, \{p^*\}, t_i) = c(\{p^*\})$ it follows that there exists some allocation $\pi_j \subseteq \mathcal{P}_j$ s.t. $\frac{1}{2} \cdot sat_j^m(\boldsymbol{I}, \pi_j, t_i) \geq sat_j^m(\boldsymbol{I}, \pi_j, t_{i'})$, and hence the claim holds. □

As mentioned above, we require two lemmas that say that, given that we start in a critical case in which the total difference is close to $2 \cdot B^*$, there exists a solution that prevents the total difference from getting close to $5 \cdot B^*$. Figure 3.1 visualises the round in which we start, i.e., a critical case in which the total difference is close to $2 \cdot B^*$. More precisely, we show that, given some round i, the total difference $\textit{DIF}_i$ is bounded by the following inequalities: $\textit{DIF}_i \leq 4 + \frac{\Gamma_i}{2}$ and $\textit{DIF}_i \leq 4.5 \cdot B^*$.

The reason that we require *two* lemmas is as follows. In the proof of the theorem we make a case distinction based on the way that the four agents are spread among the three types. As a result, there are *two* different cases (differing in the way the agents are divided among the types) in which we're forced to violate the $5 \cdot B^*$ bound, and of which we therefore need to argue that they can be avoided. Hence, we require two corresponding lemmas. The following lemma is the first of these two lemmas.

Lemma 3.1.7. *Let $\boldsymbol{I} = (I_1, \ldots, I_k)$ be any k-PPB instance. Suppose that there are four agents and three types. And let $I_j = \langle \mathcal{P}_j, b_j, A_j \rangle \in \boldsymbol{I}$, where $A_j(i)$ is exhaustive for all $i \in \mathcal{N}$. Let B^* be a bound for $\boldsymbol{I}$, i.e., let $B^* \in \mathbb{N}$ with $b_j \leq B^*$ for all $j \in \{1, \ldots, k\}$.*

Suppose that for some round $j \in \{1, \ldots, k\}$, we have that the following holds:[1]

- $sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_0^j) \leq sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_-^j) + \frac{1}{2} \cdot B^*$,
- $1.5 \cdot B^* < \textit{DIF}_j \leq 2 \cdot B^*$,
- $|t_+^j| = 2, |t_0^j| = 1, |t_-^j| = 1.$

Then there exists some solution $\boldsymbol{\pi}^* = (\pi_j, \ldots, \pi_k)$ for rounds $j, \ldots, k$ such that $\textit{DIF}_i \leq 4,5 \cdot B^*$ for any $i \in \{j, \ldots, k\}$.

Proof. We will prove by induction on the rounds that for any round $i \in \{j, \ldots, k\}$ the following holds:

- $\textit{DIF}_i \leq 4 \cdot B^* + \frac{\Gamma_i}{2}$, and
- $\textit{DIF}_i \leq 4,5 \cdot B^*$

Observe that we will prove a result that is strictly stronger than what we need to show. Not only do we show for any round $i \in \{1, \ldots, k\}$ that $\textit{DIF}_i \leq 4,5 \cdot B^*$ – which would suffice for our purposes – but we also show that $\textit{DIF}_i \leq 4 \cdot B^* + \frac{\Gamma_i}{2}$. The reason for this is that this will be helpful during our induction.

So we prove this by induction on the rounds. Suppose w.l.o.g. that $t_-^j = \{a\}$, $t_0^j = \{b\}$ and $t_+^j = \{c, d\}$.

For the base case, we consider round j. By assumption, we have that $1.5 \cdot B^* < \textit{DIF}_j \leq 2 \cdot B^*$. Hence, it immediately follows that $\textit{DIF}_j \leq 4 \cdot B^* + \frac{\Gamma_j}{2}$, and $\textit{DIF}_j \leq 4,5 \cdot B^*$.

So consider an arbitrary round $i \in \{j, \ldots, k\}$ such that $\textit{DIF}_i \leq 4,5 \cdot B^*$ and $\textit{DIF}_i \leq 4 \cdot B^* + \frac{\Gamma_i}{2}$. We proceed by cases. Cases are coloured in blue.

[1]These assumptions are a lot more stringent than that is required for the result to hold. The result holds for any round j that is safe for a bound of $B^* = 4$. These specific assumptions on j create a round that is, by definition, definitely safe for $B^* = 4$, but we could have started with any other round safe for $B^* = 4$ too. We make these extra stringent assumptions, because this is all that we need for Theorem 3.1.1.

Suppose first that $A_{i+1}(a) = A_{i+1}(b) = A_{i+1}(c) = A_{i+1}(d)$. By picking any $\pi \subseteq \mathcal{P}$ with $c(\pi) \leq B^*$ and $\pi \neq \emptyset$, it immediately follows that $\mathit{DIF}_i = \mathit{DIF}_{i+1}$ and $\Gamma_i = \Gamma_{i+1}$. Hence the claim follows by these facts and the IH.

As a second case, suppose that $A_{i+1}(a) = A_{i+1}(c) = A_{i+1}(d) \neq A_{i+1}(b)$. Now, in this case we make a further sub-case distinction. Either we have

$$sat_i(\boldsymbol{I}, \boldsymbol{\pi}, t_0^i) \leq sat_i(\boldsymbol{I}, \boldsymbol{\pi}, t_-^i) + B^*, \tag{A}$$

or

$$sat_i(\boldsymbol{I}, \boldsymbol{\pi}, t_+^i) \leq sat_i(\boldsymbol{I}, \boldsymbol{\pi}, t_0^i) + B^*, \tag{B}$$

or

$$sat_i(\boldsymbol{I}, \boldsymbol{\pi}, t_0^i) > sat_i(\boldsymbol{I}, \boldsymbol{\pi}, t_-^i) + B^* \text{ and } sat_i(\boldsymbol{I}, \boldsymbol{\pi}, t_+^i) > sat_i(\boldsymbol{I}, \boldsymbol{\pi}, t_0^i) + B^*. \tag{C}$$

1. Suppose (A) holds. Then by Lemma 2.3.16 there exists some project $p^* \in A_{i+1}(b)$ and $p^* \notin A_{i+1}(a) = A_{i+1}(c) = A_{i+1}(d)$. Set $\pi_{i+1} = \{p^*\}$. Then note that $\mathit{sat}^m_{i+1}(\boldsymbol{I}, \{p^*\}, t_+^i) = 0$.

   Now suppose that we have $t_+^i = t_+^{i+1}$. Then, since $t_+^i = t_+^{i+1}$ and $\mathcal{D}^{t_-^i}_{i \to i+1} = \mathcal{D}^{t_+^i}_{i \to i+1}$, we have that $\mathit{DIF}_i = \mathit{DIF}_{i+1}$. Hence we have that $\mathit{DIF}_{i+1} \leq 4 \cdot B^* + \frac{\Gamma_{i+1}}{2}$, which follows from this fact, the fact that $\Gamma_{i+1} \geq \Gamma_i$ and the IH. Since $\mathit{DIF}_i = \mathit{DIF}_{i+1}$ and $\mathit{DIF}_i \leq 4,5 \cdot B^*$ (by the IH), it also follows that $\mathit{DIF}_{i+1} \leq 4,5 \cdot B^*$.

   Suppose that $t_+^i \neq t_+^{i+1}$. By definition of B^*, $\mathit{sat}^m_{i+1}(\boldsymbol{I}, \{p^*\}, t_0^i) \leq B^*$. By this and our assumption $\mathit{sat}_i(\boldsymbol{I}, \boldsymbol{\pi}, t_0^i) \leq sat_i(\boldsymbol{I}, \boldsymbol{\pi}, t_-^i) + B^*$, it follows that $\mathit{DIF}_{i+1} \leq 4 \cdot B^* \leq 4 \cdot B^* + \frac{\Gamma_{i+1}}{2}$.

2. Suppose now that (B) ($\mathit{sat}_i(\boldsymbol{I}, \boldsymbol{\pi}, t_+^i) \leq sat_i(\boldsymbol{I}, \boldsymbol{\pi}, t_0^i) + B^*$) holds. Then by Lemma 2.3.16 there exists some project $p^* \in A_{i+1}(a) = A_{i+1}(c) = A_{i+1}(d)$ and $p^* \notin A_{i+1}(b)$. Set $\pi_{i+1} = \{p^*\}$. Now we make a case distinction.

   - First suppose that $t_0^i = t_0^{i+1}$. We first show that $\mathit{DIF}_{i+1} \leq 4,5 \cdot B^*$. Since by assumption $A_{i+1}(a) = A_{i+1}(c) = A_{i+1}(d)$, it follows that $\mathit{sat}^m_{i+1}(\boldsymbol{I}, \pi_{i+1}, t_+^i)$ $= sat^m_{i+1}(\boldsymbol{I}, \pi_{i+1}, t_-^i)$. Thus, it follows that $t_-^i = t_-^{i+1}$ and $t_+^i = t_+^{i+1}$. Thus, from this together with the fact that $\mathcal{D}^{t_-^i}_{i \to i+1} = \mathcal{D}^{t_+^i}_{i \to i+1}$, it follows that $\mathit{DIF}_i = \mathit{DIF}_{i+1}$. By this fact and the IH, we have that $\mathit{DIF}_{i+1} \leq 4,5 \cdot B^*$.

     Now we make a further case distinction to show that $\mathit{DIF}_{i+1} \leq 4 \cdot B^* + \frac{\Gamma_{i+1}}{2}$. Suppose first that $\mathit{DIF}_{i+1} < 4 \cdot B^*$. Then $\mathit{DIF}_{i+1} \leq 4 \cdot B^* + \frac{\Gamma_{i+1}}{2}$, since $\Gamma_{i+1} \geq 0$, by definition. Now suppose that $\mathit{DIF}_{i+1} \geq 4 \cdot B^*$. Note that $\Delta_i \leq 1 \cdot B^*$, by assumption (i.e., $\mathit{sat}_i(\boldsymbol{I}, \boldsymbol{\pi}, t_+^i) \leq sat_i(\boldsymbol{I}, \boldsymbol{\pi}, t_0^i) + B^*$). Therefore $\Delta_{i+1} \leq 2 \cdot B^*$, since $c(\pi_{i+1}) \leq 1 \cdot B^*$ by definition of π_{i+1}. Since by definition of DIF_{i+1}, $\mathit{DIF}_{i+1} = \Delta_{i+1} + \Gamma_{i+1}$, it follows that $\Gamma_{i+1} \geq 2 \cdot B^*$. Hence, $4 \cdot B^* + \frac{\Gamma_{i+1}}{2} \geq 5 \geq \mathit{DIF}_{i+1}$. Hence $\mathit{DIF}_{i+1} \leq 4 \cdot B^* + \frac{\Gamma_{i+1}}{2}$.

- Now suppose that $t_0^i \neq t_0^{i+1}$, i.e., $t_0^i = t_-^{i+1}$. It follows that $\Gamma_i \leq 1 \cdot B^*$ and $\Gamma_{i+1} \leq 1 \cdot B^*$. But, by assumption, we have that $\Delta_i \leq 1 \cdot B^*$. Since $\mathit{DIF}_i = \Gamma_i + \Delta_i$, we have that $\mathit{DIF}_i \leq 2 \cdot B^*$. Since by definition, $\mathit{DIF}_{i+1} \leq \mathit{DIF}_i + 1 \cdot B^*$, we have that $\mathit{DIF}_{i+1} \leq 3 \cdot B^* \leq 4,5 \cdot B^*$. And hence it also follows that $\mathit{DIF}_{i+1} \leq 4 \cdot B^* + \frac{\Gamma_{i+1}}{2}$, since $\Gamma_{i+1} \geq 0$ by definition of Γ_{i+1}.

3. Finally, suppose that (C) holds. Select $\pi_{i+1} = \{p^*\}$ with $p^* \in A_{i+1}(b)$ and $p^* \notin A_{i+1}(a) = A_{i+1}(c) = A_{i+1}(d)$. Then note that this case is similar to the cases above, except that it is impossible that $t_0^i \neq t_0^{i+1}$.

   So suppose that $t_0^i = t_0^{i+1}$. By definition of π_{i+1} it follows that $\mathit{DIF}_i = \mathit{DIF}_{i+1}$. By this fact and the IH, we have that $\mathit{DIF}_{i+1} \leq 4,5 \cdot B^*$. But clearly, since $\Gamma_{i+1} \geq \Gamma_i$, and $\mathit{DIF}_{i+1} \leq 4,5 \cdot B^*$, it follows that $\mathit{DIF}_{i+1} \leq 4 \cdot B^* + \frac{\Gamma_{i+1}}{2}$.

It might also be the case that $A_{i+1}(b) = A_{i+1}(c) = A_{i+1}(d) \neq A_{i+1}(a)$. By Lemma 2.3.16 there exists some project $p^* \in A_{i+1}(a)$ and $p^* \notin A_{i+1}(b) = A_{i+1}(c) = A_{i+1}(d)$. Set $\pi_{i+1} = \{p^*\}$.

We again make a case distinction. First suppose that $t_+^i \neq t_+^{i+1}$. By definition of π_{i+1} and the fact that $c(\pi_{i+1}) \leq 1 \cdot B^*$, it follows that $\mathit{DIF}_i \leq 1 \cdot B^*$, and hence that $\mathit{DIF}_{i+1} \leq 2 \cdot B^*$. But then it immediately follows that $\mathit{DIF}_{i+1} \leq 2 \cdot B^* \leq 4 \cdot B^* + \frac{\Gamma_{i+1}}{2}$.

Now suppose that we have $t_+^i = t_+^{i+1}$. But since $\mathcal{D}_{i\to i+1}^{t_-^i} > \mathcal{D}_{i\to i+1}^{t_+^i} = 0$, we have that $\mathit{DIF}_i \geq \mathit{DIF}_{i+1}$. So $\mathit{DIF}_{i+1} \leq 4,5 \cdot B^*$ follows immediately. Now we show that $\mathit{DIF}_{i+1} \leq 4 \cdot B^* + \frac{\Gamma_{i+1}}{2}$.

To show this, first suppose that $t_-^i = t_-^{i+1}$. By the IH we have that $\mathit{DIF}_i \leq 4 \cdot B^* + \frac{\Gamma_i}{2}$. Let $x = \mathcal{D}_{i\to i+1}^{t_-^i}$, and note that $\mathcal{D}_{i\to i+1}^{t_0^i} = \mathcal{D}_{i\to i+1}^{t_+^i} = 0$, by definition of π_{i+1}. But then it follows that $\mathit{DIF}_i - x \leq 4 \cdot B^* + \frac{\Gamma_i - x}{2}$ for any $x \in \mathbb{N}$. Note that $\mathit{DIF}_{i+1} = \mathit{DIF}_i - x$ and $\Gamma_{i+1} = \Gamma_i - x$. Hence $\mathit{DIF}_{i+1} \leq 4 \cdot B^* + \frac{\Gamma_{i+1}}{2}$.

Next, suppose that $t_-^i \neq t_-^{i+1}$. So note now that $\Delta_i \geq \Delta_{i+1}$. Note also that $\mathit{DIF}_{i+1} = \mathit{DIF}_i - \Gamma_i$. Hence $\mathit{DIF}_{i+1} \leq \mathit{DIF}_i$. By the IH it follows that $\mathit{DIF}_{i+1} \leq 4,5 \cdot B^*$. Hence, remains to show that $\mathit{DIF}_{i+1} \leq 4 \cdot B^* + \frac{\Gamma_{i+1}}{2}$. By the IH we have that $\mathit{DIF}_i \leq 4 \cdot B^* + \frac{\Gamma_i}{2}$. Note that $\mathit{DIF}_{i+1} = \mathit{DIF}_i - \Gamma_i$. We either have $\Gamma_{i+1} > \Gamma_i$ or $\Gamma_{i+1} \leq \Gamma_i$. Suppose first that $\Gamma_{i+1} > \Gamma_i$. The result that $\mathit{DIF}_{i+1} \leq 4 \cdot B^* + \frac{\Gamma_{i+1}}{2}$ then follows by the IH, the fact that $\mathit{DIF}_{i+1} \leq \mathit{DIF}_i$ and the fact that $\Gamma_{i+1} > \Gamma_i$.

So suppose that $\Gamma_{i+1} \leq \Gamma_i$. Then $\Gamma_{i+1} = \Gamma_i - x$ for some $x \in \mathbb{N}$ with $0 \leq x \leq 1 \cdot \Gamma_i$, for $\Gamma_i \not< 0$, by definition of Γ_i. But then note that it follows that $\mathit{DIF}_i - \Gamma_i \leq 4 \cdot B^* + \frac{\Gamma_i}{2} - \frac{x}{2}$ for $x \in \mathbb{N}$ with $0 \leq x \leq 1 \cdot \Gamma_i$, since $\Gamma_i \geq \frac{x}{2}$. Hence we have $\mathit{DIF}_i - \Gamma_i \leq 4 \cdot B^* + \frac{\Gamma_i - x}{2}$. But since $\Gamma_{i+1} = \Gamma_i - x$, and since $\mathit{DIF}_{i+1} = \mathit{DIF}_i - \Gamma_i$, it follows that $\mathit{DIF}_{i+1} \leq 4 \cdot B^* + \frac{\Gamma_{i+1}}{2}$.

As a final case, we have that for $c, d \in t_+^i$, it holds that both $A_{i+1}(c) \neq A_{i+1}(a)$ or $A_{i+1}(d) \neq A_{i+1}(a)$, and $A_{i+1}(d) \neq A_{i+1}(b)$ or $A_{i+1}(c) \neq A_{i+1}(b)$. Now, by Lemma 3.1.6 there exists some $\{p^*\} = \pi^*$ s.t. $\frac{1}{2} \cdot sat_{i+1}^m(\boldsymbol{I}, \pi^*, t_-^i) \geq sat_{i+1}^m(\boldsymbol{I}, \pi^*, t_+^i)$. Set $\pi_{i+1} = \pi^* = \{p^*\}$.

- To show our result, first suppose that $t^i_- = t^{i+1}_-$. There are two sub-cases.
  - First suppose that $t^i_+ \neq t^{i+1}_+$. Since $t^i_- = t^{i+1}_-$, it follows that $t^i_0 = t^{i+1}_+$. But since by definition of π^* it follows that $\mathcal{D}^{t^{i+1}_-}_{i \to i+1} \geq \mathcal{D}^{t^i_0}_{i \to i+1} = \mathcal{D}^{t^{i+1}_+}_{i \to i+1}$, we have $\mathit{DIF}_{i+1} \leq \mathit{DIF}_i$. Since $\mathit{DIF}_{i+1} \leq \mathit{DIF}_i$, by the IH it follows that $\mathit{DIF}_{i+1} \leq 4,5 \cdot B^*$. Since $t^i_+ \neq t^{i+1}_+$, it follows that $\Delta_i \leq 1 \cdot B^*$ and $\Delta_{i+1} \leq 1 \cdot B^*$. Now either $\Gamma_{i+1} \leq 3 \cdot B^*$ or $\Gamma_{i+1} > 3 \cdot B^*$. First suppose that $\Gamma_{i+1} \leq 3 \cdot B^*$. Since $\mathit{DIF}_{i+1} = \Gamma_{i+1} + \Delta_{i+1}$, it follows that $\mathit{DIF}_{i+1} \leq 4 \cdot B^*$. Hence $\mathit{DIF}_{i+1} \leq 4 \cdot B^* + \frac{\Gamma_{i+1}}{2}$ follows immediately, for $\Gamma_{i+1} \geq 0$, by definition of Γ_{i+1}. Now suppose that $\Gamma_{i+1} > 3 \cdot B^*$. Since $\mathit{DIF}_{i+1} \leq 4,5 \cdot B^*$, the result follows immediately from the fact that $4,5 \cdot B^* \leq 4 \cdot B^* + \frac{3}{2}$.
  - Now suppose that $t^i_+ = t^{i+1}_+$. First note that from our choice of π^* follows $\mathcal{D}^{t^{i+1}_-}_{i \to i+1} \geq \mathcal{D}^{t^{i+1}_+}_{i \to i+1}$. By the IH then follows that $\mathit{DIF}_{i+1} \leq 4,5 \cdot B^*$. So remains to show that $\mathit{DIF}_{i+1} \leq 4 \cdot B^* + \frac{\Gamma_{i+1}}{2}$. Observe that by the IH we have that $\mathit{DIF}_i \leq 4 \cdot B^* + \frac{\Gamma_i}{2}$. Note that from our choice of allocation $\pi^* = \{p^*\}$ follows $\mathit{DIF}^{\pi^*}_{i+1} \leq -\frac{1}{2} \cdot c(\{p^*\})$. And note that $\Gamma^{\pi^*}_{i+1} \geq -c(\{p^*\})$. Then $\mathit{DIF}_{i+1} = \mathit{DIF}_i + \mathit{DIF}^{\pi^*}_{i+1} \leq 4 \cdot B^* + \frac{\Gamma_i + \Gamma^{\pi^*}_{i+1}}{2} = 4 \cdot B^* + \frac{\Gamma_{i+1}}{2}$.
- Now suppose that $t^i_- \neq t^{i+1}_-$. There are again two sub-cases.
  - $t^i_+ = t^{i+1}_+$. We first show that $\mathit{DIF}_{i+1} \leq 4 \cdot B^* + \frac{\Gamma_{i+1}}{2}$. By the IH we have that $\mathit{DIF}_i \leq 4 \cdot B^* + \frac{\Gamma_i}{2}$. Now we will show that from this fact and the fact that $\mathit{DIF}_i = \Delta_i + \Gamma_i$ it follows that $\Delta_i + \frac{\Gamma_i}{2} \leq 4 \cdot B^*$, which we will later use to show our result.

$$\mathit{DIF}_i = \Delta_i + \Gamma_i$$
$$\Delta_i + \Gamma_i \leq 4 \cdot B^* + \frac{\Gamma_i}{2}$$
$$\Delta_i + \Gamma_i - \frac{\Gamma_i}{2} \leq 4 \cdot B^*$$
$$\Delta_i + \frac{\Gamma_i}{2} \leq 4 \cdot B^*.$$

    Now, we first show that $\Delta_{i+1} \leq \Delta_i + \frac{\Gamma_i}{2} + \frac{\Gamma_{i+1}}{2} - \Gamma_{i+1}$. First note that by assumption we have $t^{i+1}_0 \neq t^i_-$, $t^{i+1}_- \neq t^i_0$ and $t^{i+1}_+ = t^i_+$. For sake of brevity, let $\mathcal{D}^{t^i_-}_{i \to i+1} = x$, $\mathcal{D}^{t^i_0}_{i \to i+1} = y$ and $\mathcal{D}^{t^i_+}_{i \to i+1} = z$. By definition of π_{i+1} we have that $x \geq y$ and $\frac{1}{2}x \geq z$. Note also that $x \geq \Gamma_i$, since $t^{i+1}_- = t^i_0$. By definition, then, $x = \Gamma_i + \Gamma_{i+1}$. Hence $z \leq \frac{1}{2}x = \frac{1}{2}(\Gamma_i + \Gamma_{i+1})$. Thus $z \leq \frac{\Gamma_i}{2} + \frac{\Gamma_{i+1}}{2}$. Note that $\Delta_{i+1} = \Delta_i + z - \Gamma_{i+1}$. Since $z \leq \frac{\Gamma_i}{2} + \frac{\Gamma_{i+1}}{2}$, we have that $\Delta_{i+1} \leq \Delta_i + \frac{\Gamma_i}{2} + \frac{\Gamma_{i+1}}{2} - \Gamma_{i+1}$, which is what we needed to show.

But now note that by definition we have that:

$$\mathit{DIF}_{i+1} = \Delta_{i+1} + \Gamma_{i+1}$$

Hence by the result above and the above definition, the following inequality holds:

$$\Delta_{i+1} + \Gamma_{i+1} \leq \Delta_i + \frac{\Gamma_i}{2} + \frac{\Gamma_{i+1}}{2} - \Gamma_{i+1} + \Gamma_{i+1}$$

$$= \Delta_i + \frac{\Gamma_i}{2} - \frac{\Gamma_{i+1}}{2} + \Gamma_{i+1}$$

$$= \Delta_i + \frac{\Gamma_i}{2} + \frac{\Gamma_{i+1}}{2}$$

And since, as shown above, $\Delta_i + \frac{\Gamma_i}{2} \leq 4 \cdot B^*$, it follows that $\mathit{DIF}_{i+1} \leq 4 \cdot B^* + \frac{\Gamma_{i+1}}{2}$. Remains to show that $\mathit{DIF}_{i+1} \leq 4,5 \cdot B^*$. Suppose that $\mathit{DIF}_{i+1} \geq \mathit{DIF}_i$, for if not then the result follows immediately by the IH.

To show that $\mathit{DIF}_{i+1} \leq 4,5 \cdot B^*$, note that by assumption $t^i_- \neq t^{i+1}_-$ and $t^i_+ = t^{i+1}_+$, hence $t^i_- = t^{i+1}_0$. Therefore, it follows that $\Gamma_{i+1} \leq 1 \cdot B^*$. Since $\mathit{DIF}_{i+1} \leq 4 \cdot B^* + \frac{\Gamma_{i+1}}{2}$ with $\Gamma_{i+1} \leq 1 \cdot B^*$, it follows immediately that $\mathit{DIF}_{i+1} \leq 4,5 \cdot B^*$.

– $t^i_+ \neq t^{i+1}_+$. Since by assumption both $t^i_+ \neq t^{i+1}_+$ and $t^i_- \neq t^{i+1}_-$, we have that either $t^i_- = t^{i+1}_0$ or $t^i_- = t^{i+1}_+$. In both cases, we clearly have $\Gamma_i, \Gamma_{i+1}, \Delta_i, \Delta_{i+1} \leq 1 \cdot B^*$. Therefore it follows that $\mathit{DIF}_{i+1} \leq 2 \cdot B^*$. And since $\Gamma_{i+1} \geq 0$, we also have $\mathit{DIF}_{i+1} \leq 4 + \frac{\Gamma_{i+1}}{2}$. □

Lemma 3.1.7 is, as mentioned, designed to handle the first complicated scenario. It states that while the total difference in this complicated scenario might increase, it will never increase by an arbitarily high amount. More precisely, the total difference in some round i is described by the following inequalities: $\mathit{DIF}_i \leq 4 + \frac{\Gamma_i}{2}$ and $\mathit{DIF}_i \leq 4.5 \cdot B^*$. As will be apparent in the proof of the result on convergence, there is another – though closely related – complicated scenario. Lemma 3.1.8 is designed to handle this scenario. Since these scenario's are so closely related, we are able to use techniques in the proof of Lemma 3.1.8 that are similar to those that we used in the proof of Lemma 3.1.7. Hence, we refrain from writing out the exact same techniques again, and instead mention in what cases techniques from Lemma 3.1.7 can be applied.

As mentioned, we require two lemmas because of the way the agents can be divided among the three types. Figure 3.1 visualises the critical case that we started with in the *first* lemma. Figure 3.1.2 visualises the critical case that we start with in the *second* lemma.

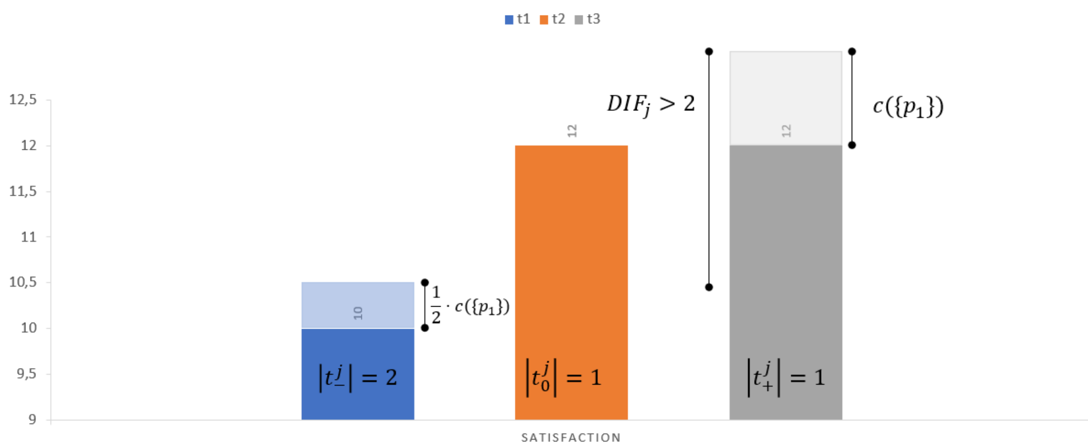

Figure 3.4: Illustrating a second critical case for satisfaction

Lemma 3.1.8. *Let $\boldsymbol{I} = (I_1, \ldots, I_k)$ be any k-PPB instance. Suppose that there are four agents and three types. And let $I_j = \langle \mathcal{P}_j, b_j, A_j \rangle \in \boldsymbol{I}$, where $A_j(i)$ is exhaustive for all $i \in \mathcal{N}$. And let B^* be the bound $B^* \in \mathbb{N}$ with $b_j \leq B^*$ for all $j \in \{1, \ldots, k\}$.*

Suppose that for some round $j \in \{1, \ldots, k\}$, we have that the following holds:

- $|t_+^j| = 1, |t_0^j| = 1, |t_-^j| = 2$
- $1.5 \cdot B^* \leq \mathit{DIF}_j \leq 2 \cdot B^*$.
- $\mathit{sat}_j(\boldsymbol{I}, \boldsymbol{\pi}, t_0^j) > \mathit{sat}_j(\boldsymbol{I}, \boldsymbol{\pi}, t_+^j) - \frac{1}{2} \cdot B^*$.

Then there exists some solution $\boldsymbol{\pi}^* = (\pi_j, \ldots, \pi_k)$ for rounds $j, \ldots, k$ such that $\mathit{DIF}_i \leq 4,5 \cdot B^*$ for any $i \in \{j, \ldots, k\}$.

Proof. The proof is similar to the proof of Lemma 3.1.7. We briefly sketch the outlines of the proof, but refrain from filling in the details for the sake of brevity.

Suppose w.l.o.g. that $t_-^i = \{a, b\}, t_0^i = \{c\}, t_+^i = \{d\}$.

We will again prove by induction on the rounds that for any round $i \in \{j, \ldots, k\}$ the following holds:

- $\mathit{DIF}_i \leq 4 \cdot B^* + \frac{\Gamma_i}{2}$, and
- $\mathit{DIF}_i \leq 4,5 \cdot B^*$

The base case is again trivial, so for the inductive step consider an arbitrary round $i \in \{j, \ldots, k\}$ such that $\mathit{DIF}_i \leq 4,5 \cdot B^*$ and $\mathit{DIF}_i \leq 4 \cdot B^* + \frac{\Gamma_i}{2}$. We again make the same case distinction based on the similarity of the agents' ballots.

If $A_{i+1}(c) = A_{i+1}(d)$, the result follows straightforwardly. For suppose that there exists some agents $x, y \in t_-^i$ such that $A_{i+1}(x) = A_{i+1}(c) = A_{i+1}(d)$ and $A_{i+1}(y) \neq$

$A_{i+1}(c) = A_{i+1}(d)$. Then by Lemma 2.3.16 there exists a project $p^* \in A_{i+1}(y)$ such that $p^* \notin A_{i+1}(x) = A_{i+1}(c) = A_{i+1}(d)$. Set $\pi^* = \{p^*\}$. Then by definition of *sat*, we have that $sat^m_{i+1}(\boldsymbol{I}, \pi^*, t^i_-) = \frac{1}{2}c(\pi^*)$ and $sat^m_{i+1}(\boldsymbol{I}, \pi^*, t^i_0) = sat^m_{i+1}(\boldsymbol{I}, \pi^*, t^i_+) = 0$. The result then follows analogously to the case in which $A_{i+1}(b) = A_{i+1}(c) = A_{i+1}(d) \neq A_{i+1}(a)$ for Lemma 3.1.7.

And suppose that that there does not exist some agents $x, y \in t^i_-$ such that $A_{i+1}(x) = A_{i+1}(c) = A_{i+1}(d)$ and $A_{i+1}(y) \neq A_{i+1}(c) = A_{i+1}(d)$. Then possibly $A_{i+1}(x) = A_{i+1}(y) = A_{i+1}(c) = A_{i+1}(d)$, in which case the result follows similarly to the case when $A_{i+1}(a) = A_{i+1}(b) = A_{i+1}(c) = A_{i+1}(d)$ from Lemma 3.1.7. Or $A_{i+1}(x) = A_{i+1}(y) \neq A_{i+1}(c) = A_{i+1}(d)$, in which case the result follows analogously to the case in which $A_{i+1}(a) \neq A_{i+1}(b) = A_{i+1}(c) = A_{i+1}(d)$.

And if $A_{i+1}(x) \neq A_{i+1}(y)$, then it follows that $A_{i+1}(x) \neq A_{i+1}(c) = A_{i+1}(d)$ and $A_{i+1}(y) \neq A_{i+1}(c) = A_{i+1}(d)$. Then by Lemma 2.3.16 there exists a project $p^* \in A_{i+1}(y)$ such that $p^* \notin A_{i+1}(c) = A_{i+1}(d)$. Set $\pi^* = \{p^*\}$. Then by definition of *sat*, we have that $sat^m_{i+1}(\boldsymbol{I}, \pi^*, t^i_-) \geq \frac{1}{2}c(\pi^*)$ and $sat^m_{i+1}(\boldsymbol{I}, \pi^*, t^i_0) = sat^m_{i+1}(\boldsymbol{I}, \pi^*, t^i_+) = 0$. The result then follows analogously to the case in which $A_{i+1}(b) = A_{i+1}(c) = A_{i+1}(d) \neq A_{i+1}(a)$ for Lemma 3.1.7.

So suppose that $A_{i+1}(c) \neq A_{i+1}(d)$. We can again make a case distinction.

Suppose first that $A_{i+1}(a) = A_{i+1}(b)$. Since $A_{i+1}(c) \neq A_{i+1}(d)$, possibly:

1. $A_{i+1}(a) = A_{i+1}(b) = A_{i+1}(c) \neq A_{i+1}(d)$
2. $A_{i+1}(a) = A_{i+1}(b) = A_{i+1}(d) \neq A_{i+1}(c)$
3. $A_{i+1}(a) = A_{i+1}(b) \neq A_{i+1}(c)$ and $A_{i+1}(a) = A_{i+1}(b) \neq A_{i+1}(d)$.

By assumption $c \in t^i_0$ and $d \in t^i_+$. If (2) holds and $sat_i(\boldsymbol{I}, \boldsymbol{\pi}, t^i_0) \leq sat_i(\boldsymbol{I}, \boldsymbol{\pi}, t^i_+) - B^*$, then set $\pi_{i+1} = \{p^*\}$ for a project $p^* \in A_{i+1}(c)$ and $p^* \notin A_{i+1}(a) = A_{i+1}(b) = A_{i+1}(d)$, which exists by Lemma 2.3.16. The result follows then analogously to the case when $A_{i+1}(a) = A_{i+1}(c) = A_{i+1}(d) \neq A_{i+1}(b)$ in Lemma 3.1.7.

If (2) holds and $sat_i(\boldsymbol{I}, \boldsymbol{\pi}, t^i_0) > sat_i(\boldsymbol{I}, \boldsymbol{\pi}, t^i_+) - B^*$, then set $\pi^* = \{p\}$ for some project $p \in A_{i+1}(a) = A_{i+1}(b)$. The result then follows analogously to the case when $A_{i+1}(a) = A_{i+1}(c) = A_{i+1}(d) \neq A_{i+1}(b)$ in Lemma 3.1.7.

If (3) holds, then the result follows similarly to the case when $A_{i+1}(a) \neq A_{i+1}(b) = A_{i+1}(c) = A_{i+1}(d)$ in Lemma 3.1.7. If (1) holds, the result follows analogously to the final case of Lemma 3.1.7 in which $A_{i+1}(c) \neq A_{i+1}(a)$ or $A_{i+1}(d) \neq A_{i+1}(a)$, and $A_{i+1}(d) \neq A_{i+1}(b)$ or $A_{i+1}(c) \neq A_{i+1}(b)$.

Now suppose that $A_{i+1}(a) \neq A_{i+1}(b)$. The most interesting case is the case in which for each $x \in t^i_-$ either $A_{i+1}(x) = A_{i+1}(c)$ or $A_{i+1}(x) = A_{i+1}(d)$. This case follows analogously to the case in which $A_{i+1}(c) \neq A_{i+1}(a)$ or $A_{i+1}(d) \neq A_{i+1}(a)$, and $A_{i+1}(d) \neq A_{i+1}(b)$ or $A_{i+1}(c) \neq A_{i+1}(b)$ for Lemma 3.1.7.

□

3.1.3 Proof of the Theorem

Now that we have set up the appropriate definitions and that we have given the right lemmas, we are able to prove the theorem.

Theorem 3.1.1. *Consider an* ∞-PPB *instance* $\boldsymbol{I} = (I_1, I_2, \ldots)$ *with four agents where the ballot of each agent is exhaustive in every round,* $|\mathcal{T}| \leq 3$ *and there exists a constant* $B^* \in \mathbb{N}$ *with* $b_j \leq B^*$ *for every round j. Then, there is a non-empty feasible solution that converges to equal-satisfaction.*

Proof. Let $\mathcal{N} = \{1, 2, 3, 4\}$. Suppose in every round the agents submit non-empty knapsack ballots. In order to prove the theorem, we make the following case distinction, depending on $|\mathcal{T}|$. We first show the result for the case when $|\mathcal{T}| = 1$, then $|\mathcal{T}| = 2$ and then $|\mathcal{T}| = 3$.

Case 1: 1 type.

This case is trivial, for

$$\lim_{j \to \infty} \frac{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t)}{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t)} = 1 \tag{3.1}$$

for any solution $\boldsymbol{\pi}$ and for any type $t \in \mathcal{T}$.

Case 2: two types.

Now suppose that $|\mathcal{T}| = 2$.

We will show the following.

$$sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) - 2B^* \leq sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) + 2B^*. \tag{3.2}$$

We prove this by induction on the round j. For the base case where $j = 1$, at most B^* is spend on projects. Hence, by the definition of sat_j it holds that $0 \leq sat_1(\boldsymbol{I}, \boldsymbol{\pi}, t) \leq B^*$ for $t \in \{t_1, t_2\}$. It follows that $sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) - 2B^* \leq sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) + 2B^*$.

Now suppose that (3.2) holds for round j-1. We will show that it holds for round j too.

Suppose that there exists some agent $x \in t_1$ and some agent $y \in t_2$ such that $A_j(x) \neq A_j(y)$ (If not, set any allocation $\pi_j = \pi^*$ and the result follows immediately). By Lemma 2.3.16, there exists some project $p \subseteq \mathcal{P}_j$ s.t. $p \in A_j(x)$ and $p \notin A_j(y)$. Set $\pi_j = \{p\}$. The result then follows from the fact that $sat_j^m(\boldsymbol{I}, \{p\}, t_1) \geq sat_j^m(\boldsymbol{I}, \{p\}, t_2)$ and the IH.

Convergence follows from (3.2) analogously to the way explicated in the proof of Proposition 2.3.17.

Case 3: three types

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We show that for every round $j \in \{1, \ldots, k\}$ and for any two types $t_i, t_{i'}$ we have:

$$sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_i) - 5 \cdot B^* \leq sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_{i'}) \leq sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_i) + 5 \cdot B^*. \tag{3.3}$$

We prove this by induction on the number of rounds j.

The base case is trivial.

So consider an arbitrary round $j \in \{1, \ldots, k\}$. We suppose that there exists some agents $i \in t_-^j$ and $i' \in t_+^j$ such that $A_{j+1}(i) \neq A_{j+1}(i')$. If not, then this means that $A_{j+1}(i) = A_{j+1}(i')$ for all $i \in t_-^j$ and for all $i' \in t_+^j$. But then there are only two possibilities. Either $A_{j+1}(i'') = A_{j+1}(i)$ for all $i'' \in t_0^j$, or not.

If the former, then clearly $DIF_{j+1} = DIF_j$ for any allocation $\pi \subseteq \mathcal{P}$. Hence $DIF_{j+1} \leq DIF_j$ for any allocation $\pi \subseteq \mathcal{P}$. And then the result follows from the IH.

If the latter, then there exists some $i'' \in t_0^j$ such that $A_{j+1}(i'') \neq A_{j+1}(i) = A_{j+1}(i')$ for all $i \in t_-^j$ and $i' \in t_0^j$. Distinguish then two cases. Either $sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_0^j) \geq sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_+^j) - 1 \cdot B^*$, or $sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_0^j) < sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_+^j) - 1 \cdot B^*$.

- Suppose first that $sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_0^j) \geq sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_+^j) - 1 \cdot B^*$. Then by Lemma 2.3.16 there exists some project $p^* \in A_{j+1}(i) = A_{j+1}(i')$ such that $p^* \notin A_{j+1}(i'')$ for all $i \in t_-^j$ and $i' \in t_+^j$. Then, set $\pi_{j+1} = \{p^*\}$. We can distinguish two cases. If $t_-^j = t_-^{j+1}$, then $DIF_j = DIF_{j+1}$. If $t_-^j \neq t_-^{j+1}$, then $\Gamma_i \leq 1 \cdot B^*$. Since by assumption $\Delta_i \leq 1 \cdot B^*$, it follows that $DIF_i = \Gamma_i + \Delta_i \leq 2 \cdot B^*$. Thus $DIF_{i+1} \leq 3 \cdot B^*$, hence the claim follows.

- Suppose now that $sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_0^j) < sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_+^j) - 1 \cdot B^*$. Then, by Lemma 2.3.16 there exists some project $p^* \in A_{j+1}(i'')$ such that $p^* \notin A_{j+1}(i)$ and $p^* \notin A_{j+1}(i')$ for all $i \in t_-^j$ and $i' \in t_+^j$. Then, set $\pi_{j+1} = \{p^*\}$. Then we have $sat_{j+1}^m(\boldsymbol{I}, \pi_{j+1}, t_-^j) = sat_{j+1}^m(\boldsymbol{I}, \pi_{j+1}, t_+^j) = 0$. By assumption $\Delta_j > 1 \cdot B^*$, thus $t_+^{j+1} = t_+^j$. From this fact and our assumption that $A_{j+1}(i) = A_{j+1}(i')$ for all $i \in t_-^j$ and all $i' \in t_+^j$, it follows that $DIF_j = DIF_{j+1}$. Hence the claim follows by the IH.

So from now on we will assume that there exists some agents $i \in t_-^j$ and $i' \in t_+^j$ such that $A_{j+1}(i) \neq A_{j+1}(i')$.

We make the following case distinction.

Case 1: $|t_-^j| = 1, |t_0^j| = 1, |t_+^j| = 2$. Suppose w.l.o.g. that $t_-^j = \{a\}, t_0^j = \{b\}, t_+^j = \{c, d\}$.

Sub-case 1: $sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_0^j) > sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_-^j) + \frac{1}{2} \cdot B^*$. Then by our assumption that there exists some agents $i \in t_-^j$ and $i' \in t_+^j$ such that $A_{j+1}(i) \neq A_{j+1}(i')$ and Lemma 3.1.6 there exists an allocation π_{j+1} s.t. $\frac{1}{2} \cdot sat_{j+1}^m(\boldsymbol{I}, \pi_{j+1}, t_-^j) \geq sat_{j+1}^m(\boldsymbol{I}, \pi_{j+1}, t_+^j)$. Now, distinguish again two cases.

- $t_-^j = t_-^{j+1}$. Then, since $\mathcal{D}_{j\to j+1}^{t_-^j} > \mathcal{D}_{j\to j+1}^{t_+^j}$, it follows that $sat_{j+1}(\boldsymbol{I}, \boldsymbol{\pi}, t_-^{j+1}) - sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_-^j) > sat_{j+1}(\boldsymbol{I}, \boldsymbol{\pi}, t_+^{j+1}) - sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_+^j)$. Hence $sat_{j+1}(\boldsymbol{I}, \boldsymbol{\pi}, t_+^{j+1}) - sat_{j+1}(\boldsymbol{I}, \boldsymbol{\pi}, t_-^{j+1}) \leq sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_+^j) - sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_-^j)$. Hence $DIF_{j+1} \leq DIF_j$. From this and the induction hypothesis the claim follows.

- $t_-^j \neq t_-^{j+1}$. Then $sat_{j+1}(\boldsymbol{I}, \boldsymbol{\pi}, t_-^j) \geq sat_{j+1}(\boldsymbol{I}, \boldsymbol{\pi}, t_-^{j+1}) \geq sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_-^j) + \frac{1}{2} \cdot B^*$. Since $\mathcal{D}_{j\to j+1}^{t_+^j} \leq \frac{1}{2} \cdot B^*$, we have that $DIF_{j+1} \leq DIF_j$. From this and the induction hypothesis, the claim follows.

Sub-case 2: $DIF_j \leq 4.5 \cdot B^*$. By Lemma 3.1.6 we have an allocation $\pi^* = \{p^*\}$ s.t. $\mathcal{D}_{j\to j+1}^{t_+^j} \leq \frac{1}{2} \cdot B^*$. Set $\pi_{j+1} = \pi^*$. We make a case distinction. Suppose that $t_+^j = t_+^{j+1}$. Then $\mathcal{D}_{j\to j+1}^{t_+^j} \leq \frac{1}{2} \cdot B^*$ and $sat_{j+1}(\boldsymbol{I}, \boldsymbol{\pi}, t_-^{j+1}) - sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_-^j) \geq 0$. Hence $DIF_{j+1} \leq DIF_j + \frac{1}{2} \cdot B^*$. Thus $DIF_{j+1} \leq 5 \cdot B^*$.

Suppose now that $t_+^j \neq t_+^{j+1}$. If $sat_{j+1}(\boldsymbol{I}, \boldsymbol{\pi}, t_-^j) \geq sat_{j+1}(\boldsymbol{I}, \boldsymbol{\pi}, t_+^j)$, then clearly $DIF_{j+1} \leq 1 \cdot B^* \leq 5 \cdot B^*$. If $sat_{j+1}(\boldsymbol{I}, \boldsymbol{\pi}, t_-^j) < sat_{j+1}(\boldsymbol{I}, \boldsymbol{\pi}, t_+^j)$, then by our definition of π^* it follows that $\mathcal{D}_{j\to j+1}^{t_-^j} \geq \mathcal{D}_{j\to j+1}^{t_0^j}$, and hence $DIF_{j+1} \leq DIF_j$, and therefore since $DIF_j \leq 4,5 \cdot B^*$ we have that $DIF_{j+1} \leq 5 \cdot B^*$.

Sub-case 3: So consider the final case in which $4.5 \cdot B^* < DIF_j \leq 5 \cdot B^*$, and in which we have that $sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_0^j) \leq sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_-^j) + \frac{1}{2} \cdot B^*$. We call this the 'critical case', and argue that this situation/case is impossible.

Suppose, for contradiction, that it *is* possible. Since, by assumption, we have that $DIF_j > 4.5 \cdot B^*$, there must exist some round $l + 1 \in \{1, \ldots, j\}$ s.t. $DIF_{l+1} \geq 2 \cdot B^*$ and $DIF_i < 2 \cdot B^*$ for all other rounds $i \in \{1, \ldots, j\}$ s.t. $1 \leq i \leq l$. Consider this round.

Note that the only situation in which the difference DIF_i can increase (i.e., in which $DIF_{i+1} > DIF_i$) is when $DIF_i > 1.5 \cdot B^*$ and $sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_0^j) \leq sat_i(\boldsymbol{I}, \boldsymbol{\pi}, t_-^j) + \frac{1}{2} \cdot B^*$. This follows by reasoning analogous to the cases above. Therefore, we know that $1.5 \cdot B^* < DIF_l \leq 2 \cdot B^*$ and $sat_l(\boldsymbol{I}, \boldsymbol{\pi}, t_0^l) \leq sat_l(\boldsymbol{I}, \boldsymbol{\pi}, t_-^l) + \frac{1}{2} \cdot B^*$.

Hence, we're now in a position to apply Lemma 3.1.7 (i.e., the conditions for Lemma 3.1.7 apply now). According to Lemma 3.1.7 there exists some solution $\boldsymbol{\pi}^* = (\pi_l, \ldots, \pi_k)$ for rounds $l, \ldots, k$ such that $DIF_i \leq 4,5 \cdot B^*$ for any $i \in \{l, \ldots, k\}$ and with $j \in \{l, \ldots, k\}$. This contradicts the fact that $DIF_j > 4.5 \cdot B^*$.

Case 2: $|t_-^j| = 1, |t_0^j| = 2, |t_+^j| = 1$. Suppose w.l.o.g. that $t_-^j = \{a\}, t_0^j = \{b, c\}, t_+^j = \{d\}$.

Sub-case 1: $sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_0^j) < sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_+^j) - \frac{1}{2} \cdot B^*$. By Lemma 3.1.6, we have an allocation $\pi^* = \{p^*\}$ s.t. $sat_{j+1}^m(\boldsymbol{I}, \pi^*, t_-^j) \geq \frac{1}{2} \cdot c(\pi^*)$ and $sat_{j+1}^m(\boldsymbol{I}, \boldsymbol{\pi}, t_+^j) = 0$. Set $\pi_{j+1} = \pi^*$. If $t_+^j = t_+^{j+1}$, the result is immediate by our selection of π^*. If $t_+^j \neq t_+^{j+1}$, then the result follows from the fact that $sat_{j+1}^m(\boldsymbol{I}, \pi^*, t_-^j) \geq sat_{j+1}^m(\boldsymbol{I}, \pi^*, t_0^j)$, that $sat_{j+1}^m(\boldsymbol{I}, \pi^*, t_0^j) \leq 1 \cdot B^*$ and our assumption that $sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_0^j) < sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_+^j) - \frac{1}{2} \cdot B^*$.

Sub-case 2: $sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_0^j) \geq sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_+^j) - \frac{1}{2} \cdot B^*$. By Lemma 3.1.6, we have an allocation $\pi^* = \{p^*\}$ s.t. $sat_{j+1}^m(\boldsymbol{I}, \pi^*, t_-^j) = c(\pi^*)$ and $sat_{j+1}^m(\boldsymbol{I}, \pi^*, t_+^j) = 0$. And $sat_{j+1}^m(\boldsymbol{I}, \pi^*, t_0^j) \leq c(\pi^*)$. Set $\pi_{j+1} = \pi^*$. Then, possibly $t_+^j \neq t_+^{j+1}$, or not. If the former, then $DIF_j \geq DIF_{j+1}$ follows from the fact that $sat_{j+1}^m(\boldsymbol{I}, \boldsymbol{\pi}, t_0^j) \leq sat_{j+1}^m(\boldsymbol{I}, \boldsymbol{\pi}, t_-^j)$. If the latter, then $DIF_j \geq DIF_{j+1}$ follows immediately. Hence, in both cases, the claim follows.

Case 3: $|t_-^j| = 2, |t_0^j| = 1, |t_+^j| = 1$. Suppose w.l.o.g. that $t_-^j = \{a, b\}, t_0^j = \{c\}, t_+^j = \{d\}$.

Sub-case 1: $sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_0^j) \leq sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_+^j) - \frac{1}{2} \cdot B^*$. By Lemma 3.1.6, we have an allocation $\pi^* = \{p^*\}$ s.t. $sat_{j+1}^m(\boldsymbol{I}, \pi^*, t_-^j) \geq \frac{1}{2} \cdot c(\pi^*)$ and $sat_{j+1}^m(\boldsymbol{I}, \boldsymbol{\pi}, t_+^j) = 0$. Set $\pi_{j+1} = \pi^*$. Since $\pi^* = \{p^*\}$, by definition of π^*, we have possibly that $p^* \in A_{j+1}(c)$ for $c \in t_0^j$, or $p^* \notin A_{j+1}(c)$ for $c \in t_0^j$. Hence we have either that $t_0^j = t_0^{j+1}$, or $t_0^j \neq t_0^{j+1}$. Suppose that $t_0^j = t_0^{j+1}$. Then together with the fact that $\mathcal{D}_{j \to j+1}^{t_-^{j+1}} \geq \mathcal{D}_{j \to j+1}^{t_+^j}$, it follows that $DIF_j > DIF_{j+1}$. Hence, together with the IH, the claim follows.

Now suppose, for the other case, that $t_0^j \neq t_0^{j+1}$. Consider then two cases. Either $t_0^j = t_-^{j+1}$ or $t_0^j = t_+^{j+1}$. Suppose that $t_0^j = t_+^{j+1}$. Hence, we have that $sat_{j+1}(\boldsymbol{I}, \boldsymbol{\pi}, t_0^j) = sat_{j+1}(\boldsymbol{I}, \boldsymbol{\pi}, t_+^{j+1}) \leq sat_{j+1}(\boldsymbol{I}, \boldsymbol{\pi}, t_+^j) + \frac{1}{2} \cdot c(\pi^*)$. However, we also have that $\mathcal{D}_{j \to j+1}^{t_-^j} = c(\pi^*)$, by assumption. Hence $DIF_j \geq DIF_{j+1}$, and then the claim follows from the IH. Now suppose that $t_0^j = t_-^{j+1}$. Then clearly $DIF_j \geq DIF_{j+1}$ and the result follows from this fact and the IH.

Sub-case 2: $DIF_j \leq 4.5 \cdot B^*$. Again, by Lemma 3.1.6, we have an allocation $\pi^* = \{p^*\}$ s.t. $sat_{j+1}^m(\boldsymbol{I}, \pi^*, t_-^j) \geq \frac{1}{2} \cdot c(\pi^*)$ and $sat_{j+1}^m(\boldsymbol{I}, \pi^*, t_+^j) = 0$. Set $\pi_{j+1} = \pi^*$. Possibly, we have $sat_{j+1}^m(\boldsymbol{I}, \pi^*, t_0^j) = c(\pi^*) \leq B^*$. By analogous reasoning to sub-case 1, we have that $DIF_j > DIF_{j+1}$. Then, the claim again follows from the IH.

Sub-case 3: So we're left with the final case in which $DIF_j > 4,5 \cdot B^*$ and $sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_0^j) > sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_+^j) - \frac{1}{2} \cdot B^*$. We call this again the 'critical case', and again argue that this situation/case is impossible.

Suppose, for contradiction, that it *is* possible. Since, by assumption, we have that $DIF_j > 4,5 \cdot B^*$, there must exist some round $l + 1 \in \{1, \ldots, j\}$ s.t. $DIF_{l+1} \geq 2 \cdot B^*$ and $DIF_i < 2 \cdot B^*$ for all other rounds $i \in \{1, \ldots, j\}$ s.t. $1 \leq i \leq l$. Consider this round.

Note that the only situation in which the difference DIF_l can increase (i.e., in which $DIF_{l+1} > DIF_l$) is when $DIF_l > 1.5 \cdot B^*$ and $sat_l(\boldsymbol{I}, \boldsymbol{\pi}, t_0^l) > sat_l(\boldsymbol{I}, \boldsymbol{\pi}, t_+^l) - \frac{1}{2} \cdot B^*$.

This follows by reasoning analogous to the cases above, which show that in all cases $\mathit{DIF}_j > \mathit{DIF}_{j+1}$. Thus, suppose that $1.5 \cdot B^* < \mathit{DIF}_l \leq 2 \cdot B^*$ and $sat_l(\boldsymbol{I}, \boldsymbol{\pi}, t_0^l) > sat_l(\boldsymbol{I}, \boldsymbol{\pi}, t_+^l) - \frac{1}{2} \cdot B^*$.

Hence, we're now in a position to apply Lemma 3.1.8 (i.e., the conditions for Lemma 3.1.8 apply now). According to Lemma 3.1.8 there exists some solution $\boldsymbol{\pi}^* = (\pi_l, \ldots, \pi_k)$ such that $\mathit{DIF}_i \leq 4,5 \cdot B^*$ for any $i \in \{l, \ldots, k\}$. Note that $j \in \{l, \ldots, k\}$.

This contradicts the fact that $\mathit{DIF}_j > 4.5 \cdot B^*$.

From (3.3), the convergence result now follows analogously to the way explicated in the proof of Proposition 2.3.17. □

3.2 No Convergence for Seven Agents

Theorem 3.1.1 shows that if we have four agents, who are divided among at most three types, there always exists some solution $\boldsymbol{\pi}$ that converges to equal-satisfaction. However, as was already apparent from Example 2.3.18, this result cannot be generalised to an arbitrary amount of agents. For example, the result fails when we have eigth agents. We can prove the stronger result that even for seven agents, the result already fails.

Proposition 3.2.1 (No convergence for seven agents)**.** There exists some k-PPB instance $\boldsymbol{I} = (I_1, \ldots, I_k)$ with seven agents who submit knapsack ballots such that there exists no solution $\boldsymbol{\pi}$ for $\boldsymbol{I}$ that converges to equal-satisfaction.

Proof. Let $\boldsymbol{I} = (I_1, \ldots I_k)$ be a ∞-PPB instance with seven agents $\mathcal{N} = \{1, \ldots, 7\}$ who submit knapsack ballots. In every round j, we have $b_j = 10$. In every round we have that agents 1 and 2 have type t_1 and agents 3,4,5,6,7 have type t_2. Furthermore, there are six projects $p_1, \ldots, p_6$ such that $c(p_1) = c(p_2) = c(p_3) = 1$, $c(p_4) = 3$ and $c(p_5) = c(p_6) = 5$. The approval function $A : \mathcal{N} \to 2^{\mathcal{P}}$ is such that, for every round j:

- $A_j(1) = \{p_1, p_2, p_3, p_4\}$,
- $A_j(2) = \{p_5, p_6\}$,
- $A_j(3) = \{p_5, p_1, p_2, p_4\}$,
- $A_j(4) = \{p_6, p_1, p_3, p_4\}$,
- $A_j(5) = \{p_5, p_2, p_3, p_4\}$,
- $A_j(6) = \{p_5, p_6\}$,
- $A_j(7) = \{p_6, p_1, p_2, p_3\}$.

A graphical representation of the proof is provided immediately after the end of the proof.

We now check that for each project the marginal satisfaction for type t_2 is higher than for type t_1. It then follows that for each round j with $I_j = \langle \mathcal{P}_j, 10, A_j \rangle$ and for any *allocation* $\pi_j \subseteq \mathcal{P}_j$ the marginal satisfaction for type t_2 is higher than for type t_1. This then directly implies that there can be no non-empty solution converging to equal-satisfaction.

First, recall the definition of the marginal satisfaction for a voter. The marginal satisfaction of agent $i \in \mathcal{N}$ for round $j \in \{1, \ldots, k\}$ is defined as $sat^m_j(\boldsymbol{I}, \pi_j, i) = c(\pi_j \cap A_j(i))$. The marginal satisfaction of a *type* $t \in \mathcal{T}$ is defined as: $sat^m_j(\boldsymbol{I}, \pi_j, t) = \frac{1}{|t|} \sum_{i \in t} sat^m_j(\boldsymbol{I}, \pi_j, i)$.

We show that in every round $j \in \{1, \ldots, k\}$, for some $i \in \mathbb{N}$ and for all projects $p \in \mathfrak{P}$ with $\mathfrak{P} = \{p_i | 1 \leq i \leq 7\}$, we have that $sat^m_j(\boldsymbol{I}, p, t_1) < sat^m_j(\boldsymbol{I}, p, t_2)$.

(p_1) By assumption $c(p_1) = 1$. We have that $p_1 \in A_j(1)$, hence $sat^m_j(\boldsymbol{I}, p_1, 1) = c(\{p_1\} \cap A_j(1)) = c(p_1) = 1$. Since $p_1 \notin A_j(2)$, we have that $sat^m_j(\boldsymbol{I}, p_1, 2) = 0$. Hence, by definition, we have $sat^m_j(\boldsymbol{I}, p_1, t_1) = \frac{1}{2} \sum_{i \in t_1} sat^m_j(\boldsymbol{I}, p_1, i) = 0.5$.

We have that $p_1 \in A_j(3)$, $p_1 \in A_j(4)$ and $p_1 \in A_j(7)$. Hence $sat^m_j(\boldsymbol{I}, p_1, 3) = sat^m_j(\boldsymbol{I}, p_1, 4) = sat^m_j(\boldsymbol{I}, p_1, 7) = c(\{p_1\} \cap A_j(3)) = c(\{p_1\} \cap A_j(4)) = c(\{p_1\} \cap A_j(7)) = c(p_1) = 1$. Since $p_1 \notin A_j(5)$ and $p_1 \notin A_j(6)$, we have that $sat^m_j(\boldsymbol{I}, p_1, 5) = sat^m_j(\boldsymbol{I}, p_1, 6) = 0$. Hence, by definition, we have $sat^m_j(\boldsymbol{I}, p_1, t_2) = \frac{1}{5} \sum_{i \in t_2} sat^m_j(\boldsymbol{I}, p_1, i) = \frac{3}{5} = 0.6 > 0.5$. So the marginal satisfaction for project p_1 for type t_2 is strictly higher than for type t_1.

(p_2) By assumption $c(p_2) = 1$. We have that $p_2 \in A_j(1)$, hence $sat^m_j(\boldsymbol{I}, p_2, 1) = c(\{p_2\} \cap A_j(1)) = c(p_2) = 1$. Since $p_2 \notin A_j(2)$, we have that $sat^m_j(\boldsymbol{I}, p_2, 2) = 0$. Hence, by definition, we have $sat^m_j(\boldsymbol{I}, p_2, t_1) = \frac{1}{2} \sum_{i \in t_1} sat^m_j(\boldsymbol{I}, p_2, i) = 0.5$.

We have that $p_2 \in A_j(3)$, $p_2 \in A_j(5)$ and $p_2 \in A_j(7)$. Hence $sat^m_j(\boldsymbol{I}, p_2, 3) = sat^m_j(\boldsymbol{I}, p_2, 5) = sat^m_j(\boldsymbol{I}, p_2, 7) = c(\{p_2\} \cap A_j(3)) = c(\{p_2\} \cap A_j(5)) = c(\{p_2\} \cap A_j(7)) = c(p_2) = 1$. Since $p_2 \notin A_j(4)$ and $p_2 \notin A_j(6)$, we have that $sat^m_j(\boldsymbol{I}, p_2, 4) = sat^m_j(\boldsymbol{I}, p_2, 6) = 0$. Hence, by definition, we have $sat^m_j(\boldsymbol{I}, p_2, t_2) = \frac{1}{5} \sum_{i \in t_2} sat^m_j(\boldsymbol{I}, p_2, i) = \frac{3}{5} = 0.6 > 0.5$. So the marginal satisfaction for project p_2 for type t_2 is strictly higher than for type t_1.

(p_3) By assumption $c(p_3) = 1$. We have that $p_3 \in A_j(1)$, hence $sat^m_j(\boldsymbol{I}, p_3, 1) = c(\{p_3\} \cap A_j(1)) = c(p_3) = 1$. Since $p_3 \notin A_j(2)$, we have that $sat^m_j(\boldsymbol{I}, p_3, 2) = 0$. Hence, by definition, we have $sat^m_j(\boldsymbol{I}, p_3, t_1) = \frac{1}{2} \sum_{i \in t_1} sat^m_j(\boldsymbol{I}, p_3, i) = 0.5$.

We have that $p_3 \in A_j(4)$, $p_3 \in A_j(5)$ and $p_3 \in A_j(7)$. Hence $sat^m_j(\boldsymbol{I}, p_3, 4) = sat^m_j(\boldsymbol{I}, p_3, 5) = sat^m_j(\boldsymbol{I}, p_3, 7) = c(\{p_3\} \cap A_j(4)) = c(\{p_3\} \cap A_j(5)) = c(\{p_3\} \cap$

$A_j(7)) = c(p_3) = 1$. Since $p_3 \notin A_j(3)$ and $p_3 \notin A_j(6)$, we have that $sat_j^m(\boldsymbol{I}, p_3, 3) = sat_j^m(\boldsymbol{I}, p_3, 6) = 0$. Hence, by definition, we have $sat_j^m(\boldsymbol{I}, p_3, t_2) = \frac{1}{5}\sum_{i\in t_2} sat_j^m(\boldsymbol{I}, p_3, i) = \frac{3}{5} = 0.6 > 0.5$. So the marginal satisfaction for project p_3 for type t_2 is strictly higher than for type t_1.

(p_4) By assumption $c(p_4) = 3$. We have that $p_4 \in A_j(1)$, hence $sat_j^m(\boldsymbol{I}, p_4, 1) = c(\{p_4\} \cap A_j(1)) = c(p_4) = 3$. Since $p_4 \notin A_j(2)$, we have that $sat_j^m(\boldsymbol{I}, p_4, 2) = 0$. Hence, by definition, we have $sat_j^m(\boldsymbol{I}, p_4, t_1) = \frac{1}{2}\sum_{i\in t_1} sat_j^m(\boldsymbol{I}, p_4, i) = 1.5$.

We have that $p_4 \in A_j(3)$, $p_4 \in A_j(4)$ and $p_4 \in A_j(5)$. Hence $sat_j^m(\boldsymbol{I}, p_4, 3) = sat_j^m(\boldsymbol{I}, p_4, 4) = sat_j^m(\boldsymbol{I}, p_4, 5) = c(\{p_4\} \cap A_j(3)) = c(\{p_4\} \cap A_j(4)) = c(\{p_4\} \cap A_j(5)) = c(p_4) = 3$. Since $p_4 \notin A_j(6)$ and $p_4 \notin A_j(7)$, we have that $sat_j^m(\boldsymbol{I}, p_4, 6) = sat_j^m(\boldsymbol{I}, p_4, 7) = 0$. Hence, by definition, we have $sat_j^m(\boldsymbol{I}, p_4, t_2) = \frac{1}{5}\sum_{i\in t_2} sat_j^m(\boldsymbol{I}, p_4, i) = \frac{9}{5} = 1.8 > 1.5$. So the marginal satisfaction for project p_4 for type t_2 is strictly higher than for type t_1.

(p_5) By assumption $c(p_5) = 5$. We have that $p_5 \in A_j(2)$, hence $sat_j^m(\boldsymbol{I}, p_5, 2) = c(\{p_5\} \cap A_j(2)) = c(p_5) = 5$. Since $p_5 \notin A_j(1)$, we have that $sat_j^m(\boldsymbol{I}, p_5, 1) = 0$. Hence, by definition, we have $sat_j^m(\boldsymbol{I}, p_5, t_1) = \frac{1}{2}\sum_{i\in t_1} sat_j^m(\boldsymbol{I}, p_5, i) = 2.5$.

We have that $p_5 \in A_j(3)$, $p_5 \in A_j(5)$ and $p_5 \in A_j(6)$. Hence $sat_j^m(\boldsymbol{I}, p_5, 3) = sat_j^m(\boldsymbol{I}, p_5, 5) = sat_j^m(\boldsymbol{I}, p_5, 6) = c(\{p_5\} \cap A_j(3)) = c(\{p_5\} \cap A_j(5)) = c(\{p_5\} \cap A_j(6)) = c(p_5) = 5$. Since $p_5 \notin A_j(4)$ and $p_5 \notin A_j(7)$, we have that $sat_j^m(\boldsymbol{I}, p_5, 4) = sat_j^m(\boldsymbol{I}, p_5, 7) = 0$. Hence, by definition, we have $sat_j^m(\boldsymbol{I}, p_5, t_2) = \frac{1}{5}\sum_{i\in t_2} sat_j^m(\boldsymbol{I}, p_5, i) = \frac{15}{5} = 3 > 2.5$. So the marginal satisfaction for project p_5 for type t_2 is strictly higher than for type t_1.

(p_6) By assumption $c(p_6) = 5$. We have that $p_6 \in A_j(2)$, hence $sat_j^m(\boldsymbol{I}, p_6, 2) = c(\{p_6\} \cap A_j(2)) = c(p_6) = 5$. Since $p_6 \notin A_j(1)$, we have that $sat_j^m(\boldsymbol{I}, p_6, 1) = 0$. Hence, by definition, we have $sat_j^m(\boldsymbol{I}, p_6, t_1) = \frac{1}{2}\sum_{i\in t_1} sat_j^m(\boldsymbol{I}, p_6, i) = 2.5$.

We have that $p_6 \in A_j(4)$, $p_6 \in A_j(6)$ and $p_6 \in A_j(7)$. Hence $sat_j^m(\boldsymbol{I}, p_6, 4) = sat_j^m(\boldsymbol{I}, p_6, 6) = sat_j^m(\boldsymbol{I}, p_6, 7) = c(\{p_6\} \cap A_j(4)) = c(\{p_6\} \cap A_j(6)) = c(\{p_6\} \cap A_j(7)) = c(p_6) = 5$. Since $p_6 \notin A_j(3)$ and $p_6 \notin A_j(5)$, we have that $sat_j^m(\boldsymbol{I}, p_6, 3) = sat_j^m(\boldsymbol{I}, p_6, 5) = 0$. Hence, by definition, we have $sat_j^m(\boldsymbol{I}, p_6, t_2) = \frac{1}{5}\sum_{i\in t_2} sat_j^m(\boldsymbol{I}, p_6, i) = \frac{15}{5} = 3 > 2.5$. So the marginal satisfaction for project p_6 for type t_2 is strictly higher than for type t_1. □

The following is a graphical representation of the proof of Proposition 3.2.1. We can represent the defined approval function $A : \mathcal{N} \to 2^{\mathcal{P}}$ as follows, where the ith column indicates the projects that are approved by agent $i \in \mathcal{N}$.

p_4	
p_3	
p_2	p_6
p_1	p_5

t_1

p_4	p_4	p_4		p_3
p_2	p_3	p_3		p_2
p_1	p_1	p_2	p_6	p_1
p_5	p_6	p_5	p_5	p_6

t_2

Our claim is that for each project $p \in \mathcal{P}$ the marginal satisfaction is higher for type t_2 than for type t_1. The marginal satisfaction of a type is defined to be the average marginal satisfaction of the type's agents. To prove our claim, we only need to look at the marginal satisfaction of agents due to some singleton allocation $\{p\} \subseteq \mathcal{P}$. Hence it suffices to check for each project whether – on average – it is contained in strictly more approval sets of agents in t_2 than in those of agents in t_1. This implies that no project p can occur in *all* approval sets of t_1's agents, for this would imply that strictly more agents of t_2 approve of p than that t_2 has agents, which is impossible. Hence all projects must be approved by at most one agent of type t_1. The claim is then implied by the fact that all projects occur in at least three of t_2's agents' approval sets, which can easily be seen by re-ordering the representation above to the one below.

p_6	
p_5	
p_4	
p_3	
p_2	
p_1	

t_1

p_6	p_6	p_6		
p_5	p_5	p_5		
p_4	p_4	p_4		
p_3	p_3	p_3		
p_2	p_2	p_2		
p_1	p_1	p_1		

t_2

3.3 Discussion

In this chapter we examined the extent to which we can guarantee fair solutions when we define a fair solution to be a solution that converges to equal-satisfaction. We can guarantee a converging solution to exist when there at most four agents (divided into at most three types), though we cannot do this when there are more than seven agents.

As mentioned before, intuitively, one main reason for why we cannot guarantee a solution that converges to equal-satisfaction when there are seven agents (or more) is that some agents' ballots have a strictly higher cost than other agents' ballots. For example, the cost of agent 1's ballot is strictly lower than agents 4's ballot (i.e., $c(A_j(1)) = 6 < c(A_j(4)) = 10$). In this sense, less projects 'fit' in the ballots of type 1's agents than in the ballots of type 2's agents. In Example 3.2.1, this fact explains why it is possible that *all* projects generate a higher marginal satisfaction for t_2 than for t_1.

Hence, in general, in order to guarantee converging solutions for more than seven agents, we should at least diminish the possible discrepancy in the cost of agents' full ballots. One promising way to do this is to consider the welfare measure of *relative* satisfaction, in which the satisfaction of an agent is made relative to her maximal possible satisfaction. This, indeed, yields more positive results. We will prove a main result in Chapter 5.

Chapter 4

Complexity of Equal-Share

As was mentioned before, there are multiple ways to define the welfare of agents. In the results above, we defined the welfare of a type as *satisfaction*, which intuitively corresponds to how much budget is spent on average on the preferences of the agents of the type. There are, however, other possible welfare measures. In the preliminaries, we identified *share* as another possible conception of welfare, corresponding roughly to the amount of effort that has been made to satisfy agents' preferences.

We cannot always guarantee, even in limited circumstances, that an outcome exists that fairly divides the share among the agents – let alone one that fairly divides the share among the types (Lackner, Maly, and Rey, 2021). Sometimes such a fair outcome exists, but sometimes it doesn't. In this chapter we will show that we cannot efficiently compute whether or not an outcome satisfying equal-share exists.

Lackner, Maly, and Rey (2021) already proved that the EQUAL-SHARE problem, which we will define below, is weakly **NP**-complete. We will show that the EQUAL-SHARE problem is strongly **NP**-complete. First, we give a definition of the EQUAL-SHARE problem and the X3C problem, as we will show that X3C reduces to EQUAL-SHARE.

X3C
Input: A set X with $|X| = 3q$ with $q \in \mathbb{N}$. A collection $C \subseteq \mathcal{P}(X)$ of 3-element subsets of X.
Question: Is there a subset $C' \subseteq C$ where every element of X occurs in exactly one member of C'?

EQUAL-SHARE
Input: A k-PPB instance $\boldsymbol{I} = (I_1, \ldots, I_k)$ and a solution $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_{k-1})$.
Question: Is there a non-empty and feasible budget allocation π_k for I_k such that $(\pi_1, \ldots, \pi_{k-1}, \pi_k)$ provides equal-share at round k?

Theorem 4.0.1. *EQUAL-SHARE is strongly* **NP***-complete.*

Proof. Membership of EQUAL-SHARE in NP is clear, the certificate being the solution itself. Hence, we only need to show that EQUAL-SHARE is NP-hard. We do this by reducing from X3C, which is known to be strongly NP-complete (Garey and Johnson, 1979).

Let X3C $= \{\langle X, C\rangle : C' \subseteq C$ is an exact cover of $X\}$ and let EQUAL-SHARE $= \{\langle \boldsymbol{I}, \boldsymbol{\pi}\rangle : \pi_k$ provides equal-share at round $k\}$ with $\boldsymbol{I} = (I_1, \dots, I_k)$ and $\boldsymbol{\pi} = (\pi_1, \dots, \pi_{k-1})$.

Consider any set X with $|X| = 3q$ with $q \in \mathbb{N}$, and a collection $C \subseteq \mathcal{P}(X)$ of 3-element subsets of X. We show a way to map every collection $C \subseteq \mathcal{P}(X)$ into a k-PPB instance $\boldsymbol{I}$ such that there exists an exact cover $C' \subseteq C$ of X iff there exists a budget allocation π_k for I_k that provides equal-share at round k.

We construct a 1-PPB instance $\boldsymbol{I} = (I)$ with $I = \langle \mathcal{P}, b, A\rangle$ as follows.

- Firstly, we define the set of agents. Let $\mathcal{N}$ be the set of agents such that $\mathcal{N} = \{i_x : x \in X\} \cup \{i^*, i^{**}, i^{***}\}$. I.e., $|\mathcal{N}| = |X| + 3$. We thus associate an agent with each $x \in X$, and in addition we have three 'special' agents. Each agent has its own unique type.

- Next, we define the set of available projects $\mathcal{P}$. Let $\mathcal{P} = \{p_y : y \in C\} \cup \{p^*\}$, i.e., with each element $y = \{u, v, w\} \in C$ with $u, v, w \in X$, which is by definition a 3-element subset of X, we associate a project $p_y = p_{\{u,v,w\}}$. In addition, we have a 'special project' p^*. Let $c(p) = 1$ for all $p \in \mathcal{P}$.

- Let our budget limit $b = \frac{|X|}{3} + 1$.

- Finally, we define the approval function A. First we define the approval sets of the non-special agents, then we define the approval sets of the special agents. Let $A(i_x) = \{p_y : x \in y\}$ for all agents $i_x \notin \{i^*, i^{**}, i^{***}\}$. That is, we define the approval sets of the non-special agents as follows. Each agent i_x is associated with an element x of X. Every project is associated with a 3-element subset $y = \{u, v, w\} \in C$ of X. We say that an agent i_x approves of $y = \{u, v, w\}$ if the element x of X that the agent is associated with is an element of the 3-element subset y that the project is associated with.

  Next, we define the approval sets of the three special agents. Let $A(i^*) = A(i^{**}) = A(i^{***}) = \{p^*\}$. We thus say that each special agent approves of a special project p^* such that no other agent approves of that project.

This reduction can clearly be done in polynomial time.
Next we show that there exists an exact cover $C' \subseteq C$ of X iff there exists a budget allocation π_k for I_k that provides equal-share at round k.

$\langle X, C\rangle \in$ X3C $\Longleftarrow \langle \boldsymbol{I}, \boldsymbol{\pi}\rangle \in$ EQUAL-SHARE.

Suppose that there exists a non-empty and feasible budget allocation π for $\boldsymbol{I} = (I)$ that satisfies equal-share. Note that π is non-empty. Thus, there exists some project $p \in \mathcal{P}$ s.t. $p \in \pi$. By our definition of A it follows that $p \in A(i)$ for some $i \in \mathcal{N}$. By definition of share, that means that $\mathit{share}_1^m(\boldsymbol{I}, \pi, i) > 0$. From equal-share, this implies that every agent must have strictly more than 0 share.

It thus follows also that the three special agents i^*, i^{**} and i^{***} must have strictly more than 0 share. Note now that the special agents approve of only one project. By definition of share, it follows that

$$\mathit{share}_1^m(\boldsymbol{I}, \pi, i^*) = \sum_{p \in \pi \cap A(i^*)} \frac{c(p)}{|\{i' \in \mathcal{N} | p \in A(i')\}|} = \frac{1}{3}.$$

By equal-share it follows that all types have a share of $\frac{1}{3}$. Since each agent has its unique type, it follows by definition of the marginal share of a type in a round that all agents have share $\frac{1}{3}$. By construction it follows that all projects $p \in \pi$ are approved by exactly three agents.

But then it follows that every agent approves of only one project $p \in \pi$ of the budget allocation. For suppose not. Either there exists an agent i that approves of no project of π, or there exists an agent i that approves of more than one project of π. If i approves of no project $p \in \pi$, then $\mathit{share}_1^m(\boldsymbol{I}, \pi, i) = 0$, which contradicts the fact that each agent has $\frac{1}{3}$ share. So suppose the agent approves of more than one project $p \in \pi$. Since every project is approved by exactly three agents, it follows that:

$$\mathit{share}_1^m(\boldsymbol{I}, \pi, i) = \sum_{p \in \pi \cap A(i)} \frac{c(p)}{|\{i' \in \mathcal{N} | p \in A(i')\}|} = \frac{1}{3} \cdot x$$

with $x \in \mathbb{N}$ and $x \geq 2$, which contradicts the fact that each agent has $\frac{1}{3}$ share.

Thus, for every agent $i \in \mathcal{N}$ it holds that $|A(i) \cap \pi| = 1$.

Now we pick as exact cover C' all the sets $y \in C$ such that $p_y \in \pi$, i.e., $C' = \{y \in C : p_y \in \pi\}$. It follows from the fact that $|A(i) \cap \pi| = 1$ for all $i \in \mathcal{N}$ and the fact that every $p \in \mathcal{P}$ is approved by exactly three agents that C' is an exact cover of X.

$\langle X, C \rangle \in$ X3C $\Longrightarrow \langle \boldsymbol{I}, \boldsymbol{\pi} \rangle \in$ EQUAL-SHARE.

Suppose that there exists an exact cover $C' \subseteq C$ of X. We show that there exists a budget allocation π for $\boldsymbol{I} = (I)$ that satisfies equal-share. By definition of an exact cover, we have that for all $x \in X$, $x \in y$ for some $y \in C' \subseteq C$ and $x \notin y'$ for any y' such that $y' \neq y$ and $y' \in C'$. Hence, by construction, by setting $\pi = \{p_y : y \in C'\}$, all the non-special agents have share $\frac{1}{3}$. To satisfy equal-share, we need to in addition ensure $\frac{1}{3}$ share for the special agents, which we do by including p^* in the allocation. Hence, the allocation $\pi = \{p_y : y \in C'\} \cup \{p^*\}$ for $\boldsymbol{I} = (I)$ satisfies equal-share. □

Chapter 5

Results about Convergence to Equal-Relative Satisfaction

In the previous chapter, we have seen results on two of the three ways of defining welfare that we stated in the preliminaries: satisfaction and share. In this chapter, we focus on the third definition of welfare, which is *relative* satisfaction. We have seen that this definition of welfare allows for more positive results with respect to realising fair solutions. For example, Theorem 2.3.19 entails that if there are at most 2 *types* of agents, then a solution converging to equal-relative satisfaction can always be guaranteed to exist.

In this chapter, we will show that – given some additional assumptions – we can always guarantee the existence of a solution that converges to equal-relative satisfaction when there are three types of agents. The proof of this fact will be constructive, meaning that the proof does not only show *that* a fair solution exists, but also provides a way of computing this solution.

The rest of the chapter is structured as follows. We will first make some assumptions about the input of the PPB-instance. They differ in how stringent they are. These assumptions give rise to a symmetry in the ballots of the agents, which we explicate in the subsequent section. This symmetry-result then allows us to prove the guaranteed existence of a converging solution in the next section.

5.1 Assumptions

First, we explicate the assumptions that we will make during this chapter. We make these assumptions so that it is mathematically and conceptually easier to reason about convergence for three types of agents. More precisely, we make three assumptions: we will assume that the three types have an equal size, that there are 2 candidates in each round and that each agent votes for one candidate.

Remark 5.1.1. *Given a k-PPB instance $\boldsymbol{I}$, a solution $\boldsymbol{\pi}$ for $\boldsymbol{I}$, a set of three types $\mathcal{T}$ with $t_-^j, t_0^j, t_+^j \in \mathcal{T}$ for any $j \in \{1, \dots, k\}$, we suppose:*

1. *that the types have equal size: $|t_-^j| = |t_0^j| = |t_+^j|$ for each $j \in \{1, \dots, k\}$.*
2. *that there are only 2 candidates in each round: $|\mathcal{P}_j| = 2$ for each $j \in \{1, \dots, k\}$. Possibly, however, $\mathcal{P}_j \neq \mathcal{P}_i$ for some rounds $j, i \in \{1, \dots, k\}$,*
3. *that each agent votes for one candidate: $|A_j(i)| = 1$ for each $j \in \{1, \dots, k\}$ and for each agent $i \in \mathcal{N}$.*

We will in the following sometimes refer to k-PPB instances that satisfy these assumptions as restricted *k-PPB instances.*

5.2 Symmetry

In order to prove the theorem, we will show that due to the Assumptions 5.1.1, there exists a symmetry in the marginal Γ, *DIF* and Δ. This symmetry is caused by the structure of the ballots. We require the notions of multisets and multiplicity to reason about this structure. Given some round j, the multiset that we define intuitively corresponds to the collection of all projects that the agents of some type approve of in j, where each project is treated as a unique element. That is, if project p is approved of by two agents, then p will be in the collection twice. Given a multiset, the multiplicity of p corresponds to the amount of times that it appears in the multiset. Hence, if all agents of a type t vote unanimously for p, then the multiset will contain the singleton $\{p\}$, of which the multiplicity is equal to $|t|$.

Definition 5.2.1 (Multisets and multiplicity)**.** *Given a restricted k-PPB instance $\boldsymbol{I}$ with $k \in \mathbb{N} \cup \{\infty\}$, a solution $\boldsymbol{\pi}$ for $\boldsymbol{I}$, a type $t \in \mathcal{T}$ with $t = \{l, \dots, n\}$ for some agents $l, n \in \mathcal{N}$, a round $j \in \{1, \dots, k\}$ with $I_j = \langle \mathcal{P}_j, b_j, A_j \rangle$, and given two projects $p_1, p_2 \in \mathcal{P}_j$, we define the multiset $\overline{X}_j^t$ of type t in round j to be $\overline{X}_j^t = [A_j(l), \dots, A_j(n)]$. Further, we denote by $\overline{A}_j^t(p_1)$ the multiplicity of $\{p_1\}$ in $\overline{X}_j^t$, and refer to it as the* multiplicity of *p_1 if the type and the round are clear from the context.*

This definition now enables us to prove the following lemma.

Lemma 5.2.2 (Symmetry)**.** *Given a restricted k-PPB instance $\boldsymbol{I}$, a set of types $\mathcal{T}$ with $|\mathcal{T}| = 3$, some round $j \in \{1, \dots, k\}$ with $I_j = \langle \mathcal{P}_j, b_j, A_j \rangle$, then there are two budget allocations $\pi_1, \pi_2 \subseteq \mathcal{P}_j$ such that:*

- *$\Gamma_j^{\pi_1} = -\Gamma_j^{\pi_2}$, and*
- *$\mathit{DIF}_j^{\pi_1} = -\mathit{DIF}_j^{\pi_2}$.*

Proof. Let $\boldsymbol{I} = (I_1, \ldots, I_k)$ be a restricted k-PPB instance. Suppose that we have three types t_1, t_2, t_3. Consider an arbitrary round $j \in \{1, \ldots, k\}$ with $I_j = \langle \mathcal{P}_j, b_j, A_j \rangle$. By assumption $|\mathcal{P}_j| = 2$. Suppose w.l.o.g. that $\mathcal{P}_j = \{A, B\}$. We then have three possible allocations: $\pi_1 = \{A\}, \pi_2 = \{B\}, \pi_3 = \{A, B\}$.

First note that since, by assumption, we have $|t^j_-| = |t^j_0| = |t^j_+|$ and $A_j(i) = \{A\}$ or $A_j(i) = \{B\}$, it follows that the multiplicity of A can be defined in terms of the multiplicity of B. That is: $\overline{A}^t_j(A) = |t| - \overline{A}^t_j(B)$, for any type $t \in \mathcal{T}$. And, similarly: $\overline{A}^t_j(B) = |t| - \overline{A}^t_j(A)$, for any type $t \in \mathcal{T}$.

First, we show that $\mathit{DIF}^{\pi_1}_j = -\mathit{DIF}^{\pi_2}_j$. By definition of $\mathit{DIF}^{\{A\}}_j$, we have: $\mathit{DIF}^{\{A\}}_j = z - x$ with $x = \mathcal{D}^{t^j_-}_{j-1\to j}$ and $z = \mathcal{D}^{t^j_+}_{j-1\to j}$. By definition of relative satisfaction and the assumptions of 5.1.1 it follows that $x = \mathcal{D}^{t^j_-}_{j-1\to j} = \frac{\overline{A}^{t^j_-}_j(A)}{|t|}$ and $z = \mathcal{D}^{t^j_+}_{j-1\to j} = \frac{\overline{A}^{t^j_+}_j(A)}{|t|}$. Hence, $\mathit{DIF}^{\{A\}}_j$ can be written as: $\mathit{DIF}^{\{A\}}_j = z - x = \frac{\overline{A}^{t^j_+}_j(A)}{|t|} - \frac{\overline{A}^{t^j_-}_j(A)}{|t|}$.

By substituting $\overline{A}^t_j(A)$ for $|t| - \overline{A}^t_j(B)$ we get:

$$\mathit{DIF}^{\{A\}}_j = \frac{|t| - \overline{A}^{t^j_+}_j(B) - |t| + \overline{A}^{t^j_-}_j(B)}{|t|} = \frac{\overline{A}^{t^j_-}_j(B) - \overline{A}^{t^j_+}_j(B)}{|t|}$$

Hence

$$-\mathit{DIF}^{\{A\}}_j = -\frac{\overline{A}^{t^j_-}_j(B) - \overline{A}^{t^j_+}_j(B)}{|t|} = \frac{\overline{A}^{t^j_+}_j(B) - \overline{A}^{t^j_-}_j(B)}{|t|}$$

$$= \frac{\overline{A}^{t^j_+}_j(B)}{|t|} - \frac{\overline{A}^{t^j_-}_j(B)}{|t|} = \mathit{DIF}^{\{B\}}_j.$$

Next we show that $\Gamma^{\pi_1}_j = -\Gamma^{\pi_2}_j$. By definition, we have that $\Gamma^{\{A\}}_j = y - x$ with $\mathcal{D}^{t^j_0}_{j-1\to j} = y$ and $\mathcal{D}^{t^j_-}_{j-1\to j} = x$. From the assumptions of 5.1.1 and the definition of relative satisfaction it follows that $x = \frac{\overline{A}^{t^j_-}_j(A)}{|t|}$ and $y = \frac{\overline{A}^{t^j_0}_j(A)}{|t|}$. Hence we have:

$$\Gamma^{\{A\}}_j = \frac{\overline{A}^{t^j_0}_j(A)}{|t|} - \frac{\overline{A}^{t^j_-}_j(A)}{|t|} = \frac{\overline{A}^{t^j_0}_j(A) - \overline{A}^{t^j_-}_j(A)}{|t|}.$$

Substituting $|t| - \overline{A}^{t^j}_j(B)$ for $\overline{A}^{t^j}_j(A)$, we get:

$$\Gamma^{\{A\}}_j = \frac{(|t| - \overline{A}^{t^j_0}_j(B)) - (|t| - \overline{A}^{t^j_-}_j(B))}{|t|} = \frac{|t| - \overline{A}^{t^j_0}_j(B) - |t| + \overline{A}^{t^j_-}_j(B)}{|t|}$$

$$= \frac{\overline{A}_j^{t_-^j}(B) - \overline{A}_j^{t_0^j}(B)}{|t|}.$$

Thus

$$-\Gamma_j^{\{A\}} = \frac{\overline{A}_j^{t_0^j}(B) - \overline{A}_j^{t_-^j}(B)}{|t|} = \Gamma_j^{\{B\}}. \qquad \square$$

5.3 Proving Convergence for Three Types

In this section, we prove the following theorem that states that, given our assumption, we can guarantee the existence of a solution that converges to equal-relative satisfaction:

Theorem 5.3.1. *Given a restricted ∞-PPB instance $\boldsymbol{I}$ and a bound $B^* \in \mathbb{N}$, there is a non-empty feasible solution for $\boldsymbol{I}$ that converges to equal-relative satisfaction.*

Proof. The high-level structure of the proof is as follows. In order to show that we can always guarantee the existence of a converging solution, we show – as we did before – that the total difference is bounded. This implies convergence in the way that we explicated in the proof of Proposition 2.3.17. In particular, we will show that the total difference is bounded by a specific step-function f, which has a maximum. This step-function is based on a certain sequence, which we call the step-sequence, which we can construct based on the size of the types.

In the order of which they occur in the proof, the proof consists of four steps:

1. Constructing a 'step'-sequence
2. Constructing a 'step'-function based on the step-sequence
3. Showing that PPB-instances are bounded by the step-function
4. Showing that the step-function has a maximum, and concluding that there is therefore convergence.

5.3.1 Constructing a 'Step'-Sequence

We generate an infinite sequence $(x_0, \ldots)$ which we call a step-sequence. We will show that it satisfies the following property.

For any entry $x_k \in (x_0, \ldots)$, we have that:

$$\sum_{i=0}^{n} x_{k-i} \leq 1 \text{ implies } \sum_{i=0}^{n} x_{k+i} \leq \sum_{i=0}^{n} x_{k-1-i} + \frac{1}{|t|}.$$

First, let $k = \begin{cases} \frac{|t|}{2}, & \text{if } k \text{ is even} \\ \frac{|t|+1}{2}, & \text{if } k \text{ is odd.} \end{cases}$
Then set $x_i = 2 \cdot \frac{1}{|t|}$ for each $i \in \{x_0, \ldots, x_k\}$.

- If $(x_j, \ldots, x_l)$ is a sub-sequence s.t. $\sum_{i=j}^{l} x_i \geq 1$ and $x_i = x_{i'}$ for each $x_i, x_{i'} \in (x_j, \ldots, x_l)$, then set $x_{l+1} = x_l + \frac{1}{|t|}$.
- Let $(x_z, \ldots, x_{z'})$ be the first further sub-sequence that satisfies this condition, i.e., s.t. $z > l$, $\sum_{i=z}^{z'} x_i \geq 1$ and $x_i = x_{i'}$ for each $x_i, x_{i'} \in (x_z, \ldots, x_{z'})$ and there exists no $x_{l'}$ s.t. $x_l < x_{l'} < x_z$.
- Call any $x_i \in (x_l, \ldots, x_{z'})$ s.t. $x_i = x_l$ and $l \leq i \leq z'$ 'non-raised' and any x_i s.t. $x_i = x_l + \frac{1}{|t|}| = x_{l+1}$ and $l \leq i \leq z'$ 'raised'.
- For all raised $x_r \in (x_l, \ldots, x_{z'})$ and for each k s.t. $\sum_{i=1}^{k} x_{r-i} \leq 1$, set $x_{r+k} = x_{r-1-k}$. Call this process 'forcing'. We say that an entry x_{r+k} is *forced* by x_r iff x_{r+k} is non-raised, $x_{r+k} = x_{r-1-k}$ and $\sum_{i=1}^{k} x_{r-k} \leq 1$. Set $x_{r+k+1} = x_{l+1}$ (i.e., raise x_{r+k+1}) when x_{r+k+1} is not forced.
- *Exception.*[1] If for some raised entry x_r there exists some entry x_{r-1-i^*} s.t. x_{r-1-i^*} is raised and $\sum_{i=1}^{i^*} x_{r-i} \leq 1$, while x_{r+i^*} was forced (by some entry x_b with $b < r$), then set x_{r+i^*} to be non-raised. That is, set $x_{r+i^*} = x_l$. Let x_{r+f} be the first entry with $r + f > r + i^*$ that is not forced and s.t. $\sum_{i=1}^{f} x_{r-f} \leq 1$. Then set x_{r+f} to be raised. That is, set $x_{r+f} = x_l + \frac{1}{|t|}$.
- When you reach the sub-sequence $x_z, \ldots, x_{z'}$, iterate this procedure.
- Whenever $x_n = 1$ and $x_{n-i} < 1$ for all $1 \leq i \leq n$, set $x_{n+l} = 1$ for all $l \in \mathbb{N}$.

To see why this sequence satisfies the desired property, consider any entry $x_k \in (x_0, \ldots)$ and some $n \in \mathbb{N}$ such that $\sum_{i=1}^{n} x_{k-i} \leq 1$. By construction, there is at most one entry $x_{k+i} \in (x_k, \ldots, x_{k+n})$ that is forced, while $x_{k-i} \in (x_{k-1}, \ldots, x_{k-1-n})$ is not forced. Hence it immediately follows that the property is satisfied.

We will now illustrate the algorithm generating the right sequence.

Example 5.3.2 (Illustrating the step-sequence)**.** We will illustrate the way that the algorithm above generates the step-sequence based on the size of the types, and why this sequence satisfies our desired property.

[1]The intuition is the following. You copy and paste in the naieve way specified above. But sometimes an *earlier* raised entry already forced some number. Then we clearly cannot raise these numbers anymore. We should leave them the way they are, but this leaves you with the opportunity to raise the next un-raised (and unforced) entry.

Suppose that the three types have size 10, i.e. that $|t| = 10$. Now consider the following sub-sequence:

$$(0.3, 0.3, 0.3, 0.3) \tag{5.1}$$

This is a sub-sequence $(x_j, \ldots, x_l)$ s.t. $\sum_{i=j}^{l} x_i \geq 1$ and $x_i = x_{i'}$ for each $x_i, x_{i'} \in (x_j, \ldots, x_l)$, and hence is an appropriate sub-sequence for the first step of the algorithm.

In the algorithm, we also consider the 'first further sub-sequence that satisfies this condition'. It is the sub-sequence $(x_z, \ldots, x_{z'})$ s.t. $z > l$, $\sum_{i=z}^{z'} x_i \geq 1$ and $x_i = x_{i'}$ for each $x_i, x_{i'} \in (x_z, \ldots, x_{z'})$ and there exists no $x_{l'}$ s.t. $x_l < x_{l'} < x_z$. The sub-sequence satisfying these conditions is $(0.4, 0.4, 0.4)$. Intuitively, it corresponds to the next sub-sequence of length more than 1 in which all the entries are exactly $\frac{1}{|t|}$ higher than in the previous sub-sequence.

Starting with $(0.3, 0.3, 0.3, 0.3)$, the algorithm then tells us: set $x_{l+1} = x_l + \frac{1}{|t|}$. Hence we now get the sub-sequence:

$$(0.3, 0.3, 0.3, 0.3, 0.4) \tag{5.2}$$

where the entry $x_{l+1} = 0.3 + \frac{1}{10} = 0.4$. We call the entries with value 0.3 'non-raised' and the entries with value 0.4 'raised'.

The 5th entry in our sub-sequence is raised. The algorithm then tells us to force the further entries. It states that for each $k \in \mathbb{N}$ and some raised entry x_r s.t. $\sum_{i=1}^{k} x_{r-i} \leq 1$, we should set $x_{r+k} = x_{r-1-k}$. For example, if we denote the 5th entry in our sub-sequence by x_5, we set $x_{5+1} = x_{5-1-1}$. Since $x_{5-1-1} = x_3 = 0.3$, we set $x_6 = 0.3$. Similarly, $x_{5+2} = x_{5-3} = 0.3$ and $x_{5+3} = x_{5-4} = 0.3$. We don't force the $4 + 4 = 8$'th entry, for $\sum_{i=1}^{4} x_{5-i} > 1$.

Hence we get the following sub-sequence:

$$(0.3, 0.3, 0.3, 0.3, 0.4, 0.3, 0.3) \tag{5.3}$$

Since the 8'th entry of Sequence 5.3 is not forced, we raise it. Hence we get:

$$(0.3, 0.3, 0.3, 0.3, 0.4, 0.3, 0.3, 0.4) \tag{5.4}$$

Since the 8th entry is raised, the algorithm again tells us to force the further entries as follows: $x_{8+1} = x_{8-1-1} = x_6 = 0.3$, $x_{8+2} = x_{8-1-2} = x_5 = 0.4$. We don't force the 11th entry, since $\sum_{i=1}^{4} x_{8-i} > 1$.

Hence we get

$$(0.3, 0.3, 0.3, 0.3, 0.4, 0.3, 0.3, 0.4, 0.3, 0.4) \tag{5.5}$$

Similarly, the 10th entry forces:

$$(0.3, 0.3, 0.3, 0.3, 0.4, 0.3, 0.3, 0.4, 0.3, 0.4, 0.4, 0.3) \tag{5.6}$$

And finally we get:

$$(0.3, 0.3, 0.3, 0.3, 0.4, 0.3, 0.3, 0.4, 0.3, 0.4, 0.4, 0.3, 0.4, 0.4, 0.4) \tag{5.7}$$

Now we've reached the sub-sequence $(0.4, 0.4, 0.4)$, which is again a sub-sequence which has a length that is larger than 1 and in which all the entries have the same length. Then we can iterate the same procedure starting from this sub-sequence. △

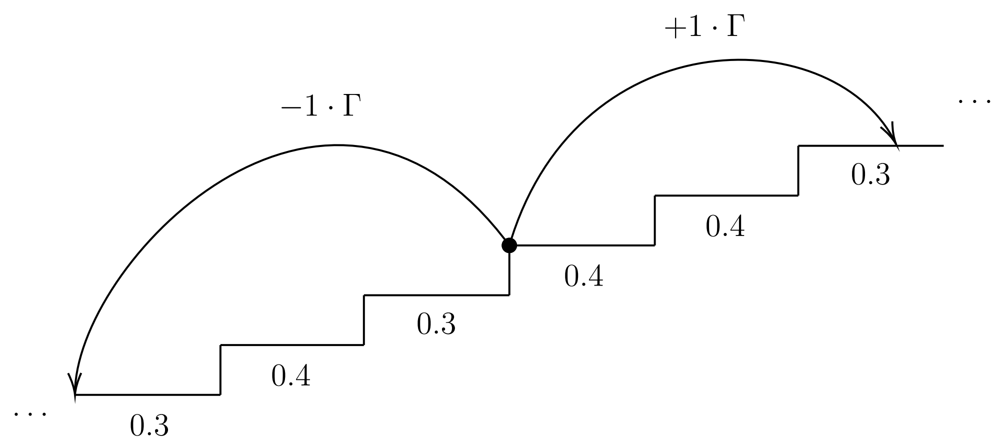
The specified sequence clearly always exists. Furthermore, it satisfies the property mentioned above, i.e., if $\sum_{i=1}^{n} x_{k-i} \leq 1$, then: $\sum_{i=0}^{n} x_{k+i} \leq \sum_{i=0}^{n} x_{k-1-i} + \frac{1}{|t|}$.

5.3.2 Constructing a 'Step'-Function Based on the Step-Sequence

We now define a function $f(\Gamma)$ based on the step-sequence $(x_0, \ldots)$. We say

$$f(\Gamma) = \frac{y}{|t|} \text{ iff } \sum_{i=0}^{y-1} x_i \leq \Gamma < \sum_{i=0}^{y} x_i.$$

The step-function is illustrated below. The idea behind the function is to take the value of the entries of the step-sequence as the lengths of the steps in the function. Note that this is a specific fragment of the sequence given in Example 5.3.2.



5.3.3 Showing that PPB-instances are Bounded by the Step Function

Given some restricted k-PPB instance $\boldsymbol{I} = (I_1, \ldots, I_k)$ and some round $j \in \{1, \ldots, k\}$, we claim that

$$f(\Gamma_j) \geq \mathit{DIF}_j.$$

By induction on the rounds. The base case is trivial. So consider any round j and suppose that:

$$f(\Gamma_j) \geq \mathit{DIF}_j. \tag{5.8}$$

We show that $f(\Gamma_{j+1}) \geq \mathit{DIF}_{j+1}$. By definition, $\mathit{DIF}_j = \frac{x}{|t|}$ for some $x \in \mathbb{N}$ and $\Gamma_j = \frac{y}{|t|}$ for some $y \in \mathbb{N}$ (similar for the marginal increase in Γ^{π}_{j+1} and DIF^{π}_{j+1}). By

Assumptions 5.1.1, we have two allocations π_1 and π_2. We assume w.l.o.g. that π_1 increases Γ_j (i.e., that $\Gamma_{j+1}^{\pi_1} \geq 0$), while π_2 decreases Γ_j (see Symmetry 5.2.2).

Now, let $X = f(\Gamma_j + \Gamma_{j+1}^{\pi_1}) - f(\Gamma_j)$ and let $Y = f(\Gamma_j) - f(\Gamma_j - \Gamma_{j+1}^{\pi_1})$. By definition of f, we have that

$$\sum_{i=0}^{y-1} x_i \leq \Gamma_j < \sum_{i=0}^{y} x_i \tag{5.9}$$

for some $y \in \mathbb{N}$.

By definition of $\Gamma_{j+1}^{\pi_1}$ it follows that:

$$\sum_{i=0}^{k} x_{y-1+i} \leq \Gamma_{j+1}^{\pi_1} < \sum_{i=0}^{k} x_{y+i} \tag{5.10}$$

for some $k \in \mathbb{N}$.

Similarly, for π_2 we have that:

$$\sum_{i=0}^{k^*} x_{y-1-i} \leq \Gamma_{j+1}^{\pi_2} < \sum_{i=0}^{k^*} x_{y-i} \tag{5.11}$$

for some $k^* \in \mathbb{N}$.

We will first show that $Y \leq X + \frac{1}{|t|}$ if $\sum_{i=1}^{k} x_{y-i} \leq 1$.

By (5.10) we have that $\Gamma_{j+1}^{\pi_1} < \sum_{i=0}^{k} x_{y+i}$. Note that by definition of Γ and *DIF* we have that $\Gamma_j = \frac{x}{|t|}$ and $DIF_j = \frac{y}{|t|}$ for some $x, y \in \mathbb{N}$ (and similarly for Γ_{j+1}^{π} and DIF_{j+1}^{π}). Furthermore $\sum_{i=0}^{k} x_{y+i} = \frac{z}{|t|}$ for some $z \in \mathbb{N}$, by definition of the step-sequence. Hence, from $\Gamma_{j+1}^{\pi_1} < \sum_{i=0}^{k} x_{y+i}$ it follows that:

$$\Gamma_{j+1}^{\pi_1} \leq \sum_{i=0}^{k} x_{y+i} - \frac{1}{|t|}. \tag{5.12}$$

By assumption, we have that $\sum_{i=1}^{k} x_{y-i} \leq 1$. From the property of the step-sequence it then follows that:

$$\sum_{i=0}^{k} x_{y+i} \leq \sum_{i=0}^{k} x_{y-1-i} + \frac{1}{|t|} \tag{5.13}$$

Subtracting $\frac{1}{|t|}$ on both sides of the equation of (5.13) gives: $\sum_{i=0}^{k} x_{y+i} - \frac{1}{|t|} \leq \sum_{i=0}^{k} x_{y-1-i}$. From (5.12) we have that $\Gamma_{j+1}^{\pi_1} \leq \sum_{i=0}^{k} x_{y+i} - \frac{1}{|t|}$. Hence we get $\Gamma_{j+1}^{\pi_1} \leq \sum_{i=0}^{k} x_{y+i} - \frac{1}{|t|} \leq \sum_{i=0}^{k} x_{y-1-i}$. Thus $\Gamma_{j+1}^{\pi_1} \leq \sum_{i=0}^{k} x_{y-1-i}$. By Symmetry 5.2.2 it follows that $\Gamma_{j+1}^{\pi_2} = -\Gamma_{j+1}^{\pi_1} \geq -\sum_{i=0}^{k} x_{y-1-i}$.

Therefore we have that $\Gamma_j - \Gamma^{\pi_1}_{j+1} \geq \sum_{i=0}^{y-1} x_i - \sum_{i=0}^{k} x_{y-1-i} = \sum_{i=0}^{y-2-k} x_i$. By definition of f, we have that $f(\Gamma_j - \Gamma^{\pi_1}_{j+1}) = f(\Gamma_j + \Gamma^{\pi_2}_{j+1}) \geq \frac{y-2-k}{|t|}$. Since by assumption $f(\Gamma_j) = \frac{y-1}{|t|}$, it follows that:

$$
\begin{aligned}
Y &= f(\Gamma_j) - f(\Gamma_j - \Gamma^{\pi_1}_{j+1}) \\
Y &\leq \frac{(y-1) - (y-2-k)}{|t|} \\
Y &\leq \frac{y-1-y+2+k}{|t|} \\
Y &\leq \frac{k+1}{|t|} \\
Y &\leq \frac{k}{|t|} + \frac{1}{|t|}.
\end{aligned}
$$

And since by (5.10) we have that $f(\Gamma^{\pi_1}_{j+1}) = \frac{y-1-k}{|t|}$, it follows that:

$$
\begin{aligned}
X &= f(\Gamma_j + \Gamma^{\pi_1}_{j+1}) - f(\Gamma_j) \\
X &= \frac{(y-1+k) - (y-1)}{|t|} \\
X &= \frac{y-1+k-y+1}{|t|} \\
X &= \frac{k}{|t|}.
\end{aligned}
$$

Hence $Y \leq X + \frac{1}{|t|}$ follows if $\sum_{i=1}^{k} x_{y-i} \leq 1$.

Now suppose that $\sum_{i=1}^{k} x_{y-i} > 1$. By definition of relative satisfaction, it follows that $\Gamma^{\pi_1}_j \leq 1$, hence $\sum_{i=1}^{k-1} x_{y-i} \leq 1$. Consider any $\Gamma^{\pi'}_{j+1}$ s.t. $\sum_{i=0}^{k-1} x_{y-1+i} \leq \Gamma^{\pi'}_{j+1} < \sum_{i=0}^{k-1} x_{y+i}$.

By the proof of the claim above, it follows that $\Gamma_j - \Gamma^{\pi'}_{j+1} \geq \sum_{i=0}^{y-1-k} x_i$ and $\Gamma_j + \Gamma^{\pi'}_{j+1} \geq \sum_{i=0}^{y+k-2} x_i$. But now it follows by definition of $\Gamma^{\pi'}_{j+1}$ that $\Gamma_j - \Gamma^{\pi_1}_{j+1} \geq \sum_{i=0}^{y-2-k} x_i$ and $\Gamma_j + \Gamma^{\pi_1}_{j+1} \geq \sum_{i=0}^{y+k-1} x_i$. And now the result follows similarly to the case above.
Thus we get that $Y \leq X + \frac{1}{|t|}$. We will now show how our result follows from this fact.
First note that $f(\Gamma_j) \geq \mathit{DIF}_j$, by the I.H..

Consider now 2 cases. Either

$$f(\Gamma_j + \Gamma^{\pi_1}_{j+1}) \geq \mathit{DIF}_j + \mathit{DIF}^{\pi_1}_{j+1} \text{ , or} \tag{5.14}$$

$$f(\Gamma_j + \Gamma^{\pi_1}_{j+1}) < \mathit{DIF}_j + \mathit{DIF}^{\pi_1}_{j+1}. \tag{5.15}$$

If (5.14) holds, then we are done. So suppose that $f(\Gamma_j + \Gamma^{\pi_1}_{j+1}) < \mathit{DIF}_j + \mathit{DIF}^{\pi_1}_{j+1}$. By definition of f, it follows that $f(\Gamma_j + \Gamma^{\pi_1}_{j+1}) \leq \mathit{DIF}_j - \mathit{DIF}^{\pi_1}_{j+1} - \frac{1}{|t|}$.

By definition of X, we have that

$$X = f(\Gamma_j + \Gamma^{\pi_1}_{j+1}) - f(\Gamma_j) \text{ . Hence}$$

$$f(\Gamma_j) + X = f(\Gamma_j + \Gamma^{\pi_1}_{j+1}).$$

$$\text{Thus, by (5.14): } f(\Gamma_j) + X = f(\Gamma_j + \Gamma^{\pi_1}_{j+1}) \leq \mathit{DIF}_j - \mathit{DIF}^{\pi_1}_{j+1} - \frac{1}{|t|}.$$

Since by the I.H. we have $\mathit{DIF}_j \leq f(\Gamma_j)$, it follows that

$$\mathit{DIF}_j + X \leq f(\Gamma_j) + X = f(\Gamma_j + \Gamma^{\pi_1}_{j+1}) \leq \mathit{DIF}_j - \mathit{DIF}^{\pi_1}_{j+1} - \frac{1}{|t|}.$$

Thus $\mathit{DIF}_j + X \leq \mathit{DIF}_j + \mathit{DIF}^{\pi_1}_{j+1} - \frac{1}{|t|}$, and hence

$$X \leq \mathit{DIF}^{\pi_1}_{j+1} - \frac{1}{|t|} \tag{5.16}$$

But now note that since $Y \leq X + \frac{1}{|t|}$, we have $Y - \frac{1}{|t|} \leq X$. By (5.16), we have $Y - \frac{1}{|t|} \leq X \leq \mathit{DIF}^{\pi_1}_{j+1} - \frac{1}{|t|}$. Note that by Symmetry 5.2.2 and by definition of Y we have that

$$f(\Gamma_j + \Gamma^{\pi_2}_{j+1}) = f(\Gamma_j - \Gamma^{\pi_1}_{j+1}) = f(\Gamma_j) - Y. \tag{5.17}$$

We know that $f(\Gamma_j) - Y \geq \mathit{DIF}_j - \mathit{DIF}^{\pi_1}_{j+1} + \frac{1}{|t|} = \mathit{DIF}_j + \mathit{DIF}^{\pi_2}_{j+1} + \frac{1}{|t|}$. Hence by (5.17):

$$f(\Gamma_j + \Gamma^{\pi_2}_{j+1}) \geq \mathit{DIF}_j + \mathit{DIF}^{\pi_2}_{j+1}.$$

Since by definition of Γ and *DIF* we have that $\Gamma_{j+1} = \Gamma_j + \Gamma^{\pi_2}_{j+1}$ and $\mathit{DIF}_{j+1} = \mathit{DIF}_j + \mathit{DIF}^{\pi_2}_{j+1}$, it follows that $f(\Gamma_{j+1}) \geq \mathit{DIF}_{j+1}$.

The property of the step-function that we proved above is illustrated by Figure 5.1, considering again the case in which $|t| = 10$. The intuitive idea is as follows. Start at any place on the step-function. By Symmetry 5.2.2 there are two allocations π_1 and π_2 that either *increase* or *decrease* Γ and the *DIF*. Suppose – the other cases are easier – that the allocation π that *increases* Γ is also the allocation increasing the *DIF* (as is the case in the figure). Suppose w.l.o.g. that π_1 is the allocation increasing Γ and the *DIF*. By definition, any *increase* in Γ by π_1 ensures a difference of at least $\frac{1}{|t|}$ from the next step.

This is illustrated by Figure 5.1: the green and black arrows indicate four possible allocations (black arrows indicate one pair of allocations, green arrows indicate another).

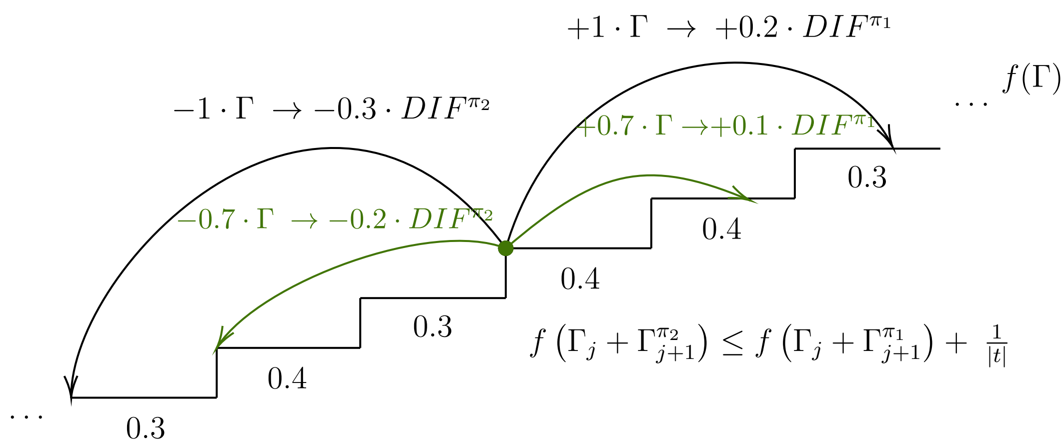


Figure 5.1: $f(\Gamma)$ bounds PPB-instances

Consider for example the green 'up'-arrow (increasing Γ). The next step begins at exactly $\frac{8}{10}$ from the starting point and there is 0.7 increase in Γ. So the difference to the next step is exactly $\frac{1}{|t|}$.

Together with the way the sequence was constructed, this fact guarantees that the same *decrease* in Γ by π_2 prevents us from 'falling from the step'. To illustrate, consider again the green arrows. As the figure shows, we increase Γ by 0.7. From our starting point, the two subsequent steps both have a length of 0.4. Therefore, we 'climb one step'. The two preceding steps, however, are a bit shorter; being respectively of length 0.3 and 0.4. The step-sequence now guarantees us that this difference in length is at most $\frac{1}{|t|} = 0.1$. Therefore, by going backwards, we 'do not fall off the step', and thus fall at most one more step than we climb.

That is, every time we decrease Γ, we decrease the *DIF* by at most $\frac{1}{|t|}$ more than we *increase* the *DIF* by increasing Γ.

5.3.4 Showing that the Step Function has a Maximum, and Concluding that there is Therefore Convergence Possible

By definition of the step-function, the step-function has a maximum value. Hence the total difference is bounded. Therefore convergence follows similarly to the way explicated in Proposition 2.3.17. □

5.4 Discussion

We end with two small notes on this chapter.

First, as was mentioned before, the proof of Theorem 5.3.1 is constructive, meaning that it does not only show *that* a converging solution exists, but also shows how to compute this solution.

Second, Theorem 5.3.1 restricts the PB setting to a setting that is resemblant to that of multi-winner voting, as the projects do not have a different cost and the voters approve of only one project. It is not immediately clear how the current proof could be generalised to the case in which these assumptions are dropped. The final assumption, however, which states that the size of the types should be equal, is easier to drop. The main reason for introducing this assumption is that it makes it conceptually easier to reason about convergence. However, the proof does not rely on this assumption to any essential degree.

Chapter 6

Conclusion

6.1 Summary of Results

This thesis analysed the extent to which fair solutions can be realised in PPB. In Table 6.1 all the results that we have discussed are summarised. In general, we found that the extent to which fair solutions can be realised depends heavily on the used definitions of welfare and fairness, as also on the amount of agents and types that are involved in the PPB process. In Chapter 2, we analysed several of the results on realising fairness in PPB that have been found in recent work (in particular by Lackner, Maly, and Rey (2021)). We saw, for example, that equal-F is a stringent notion of fairness. Given some intuitive welfare measures, there exist k-PPB instances for which no solution exists that satisfies equal-F, even for a small amount of agents voting with knapsack ballots (Example 2.3.15). *Convergence* to equal-F seemed more promising, as we can guarantee the existence of solutions that converge to equal-F, even for two *arbitrary* types of agents (Theorem 2.3.19).

In Chapter 3, we found that one reason for the impossibility of guaranteeing solutions satisfying (convergence to) equal-F is the fact that some agents' ballots can be significantly more expensive than those of other agents. Restricting the ballots to *knapsack* ballots is one way to limit the size of this possible dissimilarity. This allowed us to prove a positive result for equal-F, stating that for 4 agents with knapsack ballots, we can guarantee the existence of a solution converging to equal-satisfaction (Theorem 3.1.1). However, we showed that we could not generalise this result to an arbitrary amount of agents, as for seven agents with knapsack ballots, a converging solution cannot be guaranteed to exist (Proposition 3.2.1).

In Chapter 4, we found that computing whether a solution exists that satisfies equal-share is strongly **NP**-complete. And, finally, in Chapter 5, we found that given several assumptions (Assumptions 5.1.1), we can guarantee a solution converging to equal-relative satisfaction (Theorem 5.3.1) for three types of *arbitrary* size.

Convergence to equal-sat	$\leq$ 2 agents ✓(Prop. 2.3.14)	>3 agents ✗ (Ex. 2.3.15)	4 agents (knapsack) ✓(Thm. 3.1.1)	$>$ 7 agents (knapsack) ✗ (Prop. 3.2.1)
Convergence to equal-rsat	Two types ✓(Thm. 2.3.19)	Three types (Ass. 5.1.1) ✓(Thm. 5.3.1)		
Equal-share	Strongly **NP**-complete (Thm. 4.0.1)			
Equal-sat/rsat	>3 agents ✗ (Ex. 2.3.10)			

Table 6.1: Summary of the results discussed in the current thesis

6.2 Discussion of Possible Applications

We will now discuss some implications and possible applications of the results of the thesis.

First of all, note that the PPB model and the results that we have found to apply in it are more general than the way that we have so far interpreted the model and the results. We interpreted them to be about the real-world process of participatory budgeting. For example, we stated that in the model there are *agents* who vote on some *public projects*, such that these projects have a specific *cost*. And in each round (usually corresponding to a *year*), we dedicate some of the available *budget* to realising some of the projects. These are all specific interpretations that are not forced by the model itself. The model itself is more general than this. For example, it does not necessarily describe ‘agents having preferences over public projects’, but, instead, merely describes ‘some things having preferences over some other things.’

This generality of the model (and therefore of the results that pertain to it) opens possibilities for other real-world applications than participatory budgeting. We’ll briefly sketch two of these.

Everyday collective decision-making. First of all, these results could be used in mundane everyday life decision-making. For example, consider a family with two parents and two children who have to decide every evening on what Netflix-show to watch. The family wants to choose the shows in a fair way. Example 2.3.10 shows that we cannot guarantee a perfectly fair solution every single night. However, Theorem 3.1.1 provides us with an algorithm that generates a fair solution on the long run.[1] A simple implementation of this algorithm would tell them every night what show to watch. This even holds when the preferences of the family members change, as also when the available shows differ per night.

Political decision making. Secondly, consider a student council with members from three different political student parties. Each year, they can vote for the thing in which the university should, according to them, invest some amount of money. After initial shortlisting, they can choose from two available options. The university wants to select options that are fair with respect to the three parties. Note that these parties might have any number of members and that these members might vote in any possible way (i.e., the members of a party do not have to vote unanimously for the same option). Theorem 5.3.1 provides us with an algorithm that tells the university, for any given year, what option to invest in to ensure a fair solution on the long run.

One final note on the practical application of these results: they are useful whenever making a collective decision is primarily a matter of personal preference (or 'taste'), but not so much when the issue at hand is complicated, bearing on many factual considerations. This interesting point can be illustrated by considering Example 1.3 again. Suppose that we do not let Jessy, Maureen, Paul, John and George vote on some *public projects*, but that we let them vote on certain thematic issues. For example, we could let them vote – each year – on issues ranging from the way to reform the council tax to the way that the municipality should shape its sustainability policy. Then we apply the results of the current thesis to generate a solution that is fair: in year one the municipality might adopt aspects of a sustainability policy that is in line with the preferences of Jessy and Maureen, while in year two it adopts a reform of the council tax that is in line with the preferences of Paul, and so on and so forth.

This system, where agents vote on thematic issues (also called 'thematic PB' (A. Shah, 2007)), seems inadequate because the citizens might not be fully informed about all of the factual considerations that are involved in these issues (A. Shah, 2007). For example, Paul might not be aware of the need for a sustainability policy in the first place, or might not know what the *current* policy is and how that currently works out. However, information about these questions is important in order to make the 'right' decision.

To conclude, whenever we make a collective decision, we aim to make the 'right' decision. Sometimes, what is the right decision depends only (or mostly) on the personal preferences of the agents involved, as is the case, for example, in the Netflix-situation discussed above. To a large extent, this also holds when citizens need to decide on what public projects to fund (as is the case in participatory budgeting). However, in some other scenario's, making the right decision depends *not only* on people's preferences, but also on the facts that people believe to hold. While people's preferences should be considered

[1] We model the children as having their own unique type, while the parents share their type.

to be equally valuable, people's factual estimates are not equally valuable. After all, some people have more expertise (are more well-informed) than others on certain topics. Therefore, the results of the current thesis – aiming to satisfy everybody's preferences to the same extent – do not naturally extend to collective decisions on issues that concern many factual considerations, since we should not want to satisfy everybody's factual considerations equally much.

6.3 Future Work

Since the work on perpetual voting in general, and on PPB in particular, is recent, there are a lot of opportunities for future research. First, one could investigate whether more general convergence results can be attained for convergence to equal-relative satisfaction (being a particularly suitable notion of fairness and welfare). For example, it would be interesting to investigate whether the proof of Theorem 5.3.1 can be adapted so as to drop the assumptions of the theorem. Furthermore, it would be interesting to see whether a converging solution is guaranteed to exist (using Assumptions 5.1.1, or not) for more than three types of agents. Finally, one could try to prove an upper bound on the amount of types for which you can guarantee a solution that converges to equal-relative satisfaction. For example, can such a solution always be guaranteed for twenty types? Clearly, the more types that such a solution can be guaranteed for, the more practical relevance these results have.

Second, one could investigate the used notions of welfare and fairness from a philosophical and psychological perspective. For example, is satisfaction a correct way of measuring the welfare that an allocation generates for an agent? As mentioned, this measure is based on the assumption that there is a correlation between the cost of the project and the happiness that funding this project generates for an agent. However, one could wonder whether this assumption is correct and, if it is correct, how strong this correlation is. One could also wonder whether the fairness notion of convergence to equal-F is in line with our intuitions about fairness. Though a solution converging to equal-F is fair *in the limit*, there might still be situations in this process in which the total difference is large.

Third, and related to the remark above, one could investigate new notions of fairness and welfare and analyse the extent to which fair solutions can be guaranteed to exist w.r.t. these notions. As mentioned above, the extent to which we can realise fair solutions depends heavily on the notions of fairness and welfare that are used.

Lastly, in the current thesis we mainly investigated the extent to which solutions exist that converge to equal-F. However, as stated above, a converging solution does not guarantee that the total difference is small during all rounds. This is especially relevant from a practical perspective, since – for example in participatory budgeting – people will not be involved in the process for an infinite amount of rounds. It would therefore be interesting to investigate tight upper bounds on the total difference.

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