# Algebraic Relativization and Arrow Logic

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# Algebraic Relativization and Arrow Logic

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door

Maarten Johannes Marx

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# Contents

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Amsterdam
December, 1994.

Maarten Marx

# Introduction

## Parsimony in applying logic

When one wants to use logic to formalize a problem in such diverse fields as philosophy, linguistics or computer science, the first difficulty one encounters is the following:

What is the *appropriate* logic which *fits* my specific problem?

There are two parameters in the above question which have to be made clear: appropriateness and fit. Assume that one has indeed good reasons to use *formal logic*[1].

It is easy to make the first criterion precise: the logic should contain enough expressive power to give an accurate description of the problem. A good example is Propositional Dynamic Logic (PDL) (cf. Harel [Har84]), which is expressive enough to formalize the regular programming constructions. Another example is first–order (FO) logic, which is expressive enough to capture most of mathematical reasoning.

Whether a logic fits a specific problem, is a question which seems harder to answer. While the appropriateness criterion asks for enough expressive power, a good fit could mean that there is not too much. In this view, the two criteria become necessary and sufficient conditions on the expressive power of a logic. Clearly, there is more to say about fit, but we stay with expressive power for a moment. The expressive power of a logic is closely connected to its *complexity*. In the article titled *Sources of Complexity: Content versus Wrapping*, J. van Benthem describes this situation as follows:

> Any description of a subject carries its own price in terms of complexity. To understand what is being described, one has to understand the mechanism of the language or logic employed, adding the complexity of the encoder to the subject matter being encoded. Put more succinctly, "complexity is a package of subject matter plus analytic tools". ([Ben94b], p.1)

Since one of the main advantages of a formalized problem is that one can make definite statements about the *complexity of the problem*, a close fit between the complexity of the encoder and that of the subject matter seems to be a sine qua non. As van Benthem describes:

> ...working logicians in linguistics or computer science often have the gut feeling that the styles of reasoning they are analyzing are largely decidable (...), but it is hard to give any mathematical underpinning of these working intuitions. ([Ben94b], p.9)

Could it be that the logic they used is appropriate, but does not fit in the sense that it contains an excess of expressive power? After all, when proving decidability, one proves decidability of the encoder, not of the subject matter.

[1] In the broad sense that a syntax and a semantics are precisely defined, so that a mathematical investigation of that logic itself becomes possible.

Two good examples of logics, carefully designed for the problem at hand, are the above mentioned PDL and the Lambek Calculus (cf. Lambek [Lam58], van Benthem [Ben88]). Both are decidable. Both[2] can be viewed as a fragment of Relation Algebra, which is *undecidable.* Besides being decidable, these two logics have found many applications besides their original purposes.

This brings us to a second point one should make about the fitness condition, and which might conflict with the previous one. How close should the fit between logical language and problem be? If we use only binary relations in the description of the problem, should we then abandon unary predicates in our logic? Should we always include all Booleans? One should be liberal here, and allow "natural" logics, which are rich enough to express the original problem, but which fit well with respect to complexity.

To conclude, when designing a logic for a specific problem, it seems to pay off to do one's *conceptual homework.* Instead of taking the well-driven paths and diving straight into deep water, assuming a beginner's mind seems to be more fruitful: What are the operations needed? What are the objects to be modeled and how much mathematical structure is needed to model them? What are the models? etc. If the conceptual homework is done well at this early stage of the work, there is reason to be confident in the complexity results later on.

In subsequent chapters, we will investigate decidable versions of relation algebra (cf. e.g., Jónsson [Jón91] or Maddux [Mad91b], for an introduction) and of first-order logic (with finitely or infinitely many variables). Stating and motivating the results, we will hardly refer to applications, letting the future decide whether these weakened logics can indeed play a rôle as sketched above.

The results presented here can be viewed as additions to the tool-box of the applied logician. Since these two logics are well-known and frequently used, it seems indispensable to have "computationally well-behaved" versions of them as well.

ARROW LOGIC. As is well known, the classical models for relation algebras are not finitely axiomatizable (Monk [Mon64], Andréka [And91a]), their equational theory is highly undecidable (Tarski-Givant [TG87], Andréka et al. [AKN$^+$94]), and the corresponding logic does not even have the weakest form of Craig Interpolation (Sain-Simon [SS94]). Almost the only positive thing one can mention is its enormous expressive power. (E.g., in Tarski-Givant [TG87] it is shown that this formalism is expressive enough to formalize set theory.)

The research project of looking for weakened decidable versions of relation algebra became known under the name of *Arrow Logic* (cf. van Benthem [Ben91a], Venema [Ven91], [Ven94]). If we take the name "arrow logic" in its original sense, it stands roughly for the whole landscape of possible semantic modelings for the "logic of transitions", using the language of relation algebras.

The ideas and techniques, we use for obtaining and investigating this landscape are not restricted to arrow logic in the sense described above, but can be used for almost every family of modal logics (including FO logic). For this reason, we will also use

[2] When forgetting for a moment the Kleene * in PDL.

arrow logic in a broader sense, namely as the process of "hunting for the *computational core*" of undecidable logics. (Hunting is, after all, what arrows are made for....)

COMPUTATIONAL CORE AND RELATIVIZATION. Fix a logical language $\mathcal{L}$. By the *computational core* of $\mathcal{L}$, we denote the class of all semantic modelings of $\mathcal{L}$ whose $\mathcal{L}$ theory is decidable. Given an undecidable semantically defined logic, we would like to indicate which features of the semantics are responsible for its undecidability. So, that part of the class of the decidable semantic modellings which is closest to the original is especially interesting. We will see that the well-investigated operation of *relativization* (cf. Henkin–Monk–Tarski [HMT71]) brings us in several cases to the computational core, while keeping most features of the original semantics.

By changing the semantics we get our first meta-logical question: is the new decidable logic finitely axiomatizable? Moreover, in Andréka–Németi–Sain [ANS94b] it is pointed out that *interpolation* has an important computational aspect. Besides decidability and finite axiomatizability, we will also study this third aspect of the computational core.

AGENDA. Summing up, we will study the computational core of two logics: Arrow Logic and First-Order Logic. Their algebraic counterparts are the classes of relation algebras and cylindric algebras, respectively. Both are classes of *algebras of relations*. We focus attention on three main aspects of the computational core: *decidability, finite axiomatizability and interpolation.*

We will state and prove our results in the framework of *algebraic logic* (cf. Andréka–Monk–Németi [AMN91]). We use techniques from both algebraic and modal logic. Most of the proofs and the theorems also have a very definite *modal-logical* character, making them understandable for readers with a modal-logical, but without an algebraic background.

ORGANIZATION. In the first chapter, we introduce the two families of logics –Arrow Logic and FO Logic/Cylindric Modal Logic– whose algebraic counterparts we will study in the subsequent chapters. For each logic, we will investigate all reducts systematically, and in some cases expansions with a strong operation (like the Kleene * or the *difference* operator) as well. For each logic treated here, we will investigate the three aspects of the computational core given above plus *Beth's definability property.* In chapter 2, we define these logics and their algebraic counterparts more precisely. Decidability of the algebraic counterparts of several versions of these logics is investigated in chapter 3. In chapter 4, we find axiomatizations for decidable, but still natural versions of relation algebra/arrow logic. In chapter 5, we focus on amalgamation and interpolation properties of the investigated algebraic varieties and logics. The last chapter is about arrow logic proper. We show how the earlier algebraic results give rise to equivalent statements at the logical level, and we investigate several strengthenings of the basic language.

APPLICATIONS IN ORGANIZATION AND MANAGMENT THEORY. This piece was developed while working at the *Center for Computer Science in Organization and Managment (CCSOM)*, a part of the faculty of humanities. An important line of research at CCSOM is the reconstruction of arguments in the theory of organization and managment, by formalizing them in (modal extensions of) first–order logic (cf, Péli et al. [PM94], [PBMN94] and Bruggeman [Bru94]). We had extensive discussions about which logics to use for this purpose. The final choice for first–order logic (with additional modalities for actions and preferences, cf, Huang [Hua94]) was made because 1) a classical "monotonic" consequence relation seemed to capture the argument structure, 2) large parts of the theory are concerned with (binary) relations, and 3) first–order logic is relatively well–known. While formally recovering the argument structure, we discovered that large parts were relatively simple (e.g., only "monotonicity reasoning" was involved). This led us to look at the weakened versions of first–order logic and arrow logic described here. An important part of the work remains to be done: see whether we can indeed use the weakened logics to capture the original arguments and to obtain further insights in (the complexity of) its structure.

# 1

# Main themes

## 1.1 Arrow logic is the modal logic of transitions

Arrow logic is a widely applicable system, being able to formalize many different notions from various disciplines like mathematics, computer science, linguistics and cognitive science (for an early application, see de Bakker–de Roever [BR73]). Its most highlighted application area is that of *dynamic semantics*, which unifies insights from all of the above mentioned fields. For more information on these applications of arrow logic, we refer the reader to Venema [Ven94]; for the dynamic perspective on semantics, see van Benthem [Ben91a].

In each of these fields, if one takes a "dynamic viewpoint", transitions (e.g., between cognitive states or between registers of a computer) become the basic object of interest. The intuitive idea of a transition between two states $A$ and $B$ is that of an *arrow* leading from $A$ to $B$. Dynamic meaning, as given by a set of transitions, can be described as a set of arrows. Arrow logic itself can be viewed as the modal logic of arrows.

The language of arrow logic is based on the language of relation algebras (that is, Booleans plus operators for *composition* and *converse* of binary relations, and a constant denoting the *identity* relation) allowing some modifications. While the language of arrow logic is relatively fixed, its semantics is highly dependent on the relevant application. We will see that there are at least two very natural semantics for arrow logic: directed graphs and directed multigraphs. From a mathematical point of view, arrow logic can be viewed as the enterprise of providing the language of relation algebras (cf. Jónsson–Tarski [JT52], Henkin–Monk–Tarski [HMT85]) with new semantics.

### Models for Arrow Logic

In the rest of this section, we describe a natural part of the landscape of arrow models. First we have to answer what an arrow is.

An *arrow* is a directed connection between two points.

Then we have two choices for drawing arrows as $\longrightarrow$.

- *Extensional view:* identify an arrow with the pair $\langle$*beginpoint, endpoint*$\rangle$. Then a set of arrows is nothing more than a *directed graph* or an ordinary binary relation.
- *Intensional view:* do not identify arrows with ordered pairs, but only require that an arrow has a unique begin– and end–point. Sets of arrows are then *directed multigraphs.*

So we can think of an arrow as an object equipped with two functions, say $l_0$ and $l_1$, providing the arrow with its begin– and endpoint (cf. Vakarelov [Vak92b]).

Having sketched what arrows are, we ask ourselves what constitutes an *arrow model.* Its domain should consist of arrows and, depending on the applications, there can be various existential conditions on the domain. This resembles common practice in modal logic: for different applications, there are different conditions on the accessibility relation(s). Here we give four examples of natural conditions on arrow models. Note that with the extensional view, the first three conditions taken together imply that the domain is an equivalence relation, i.e., a disjoint union of Cartesian squares. We state these conditions for models in which we identify arrows with pairs. Let $V \subseteq U \times U$ be the domain of an arrow model:

| | | |
|---|---|---|
| (1) | Reflexivity | $\langle u,v\rangle \in V \Rightarrow \langle u,u\rangle, \langle v,v\rangle \in V$ |
| (2) | Symmetry | $\langle u,v\rangle \in V \Rightarrow \langle v,u\rangle \in V$ |
| (3) | Transitivity | $\langle u,v\rangle, \langle v,w\rangle \in V \Rightarrow \langle u,w\rangle \in V$ |
| (4) | Classical/Square | $V$ is a full Cartesian product |

We gave two ways of modeling arrows, an extensional and an intensional one. There is a third viewpoint (cf. van Benthem [Ben91a], [Ben93]): arrows are just abstract objects which one can compose, take their inverse, and which might be "identity arrows". This view leads to the concept of an arrow frame, which is nothing more than a usual Kripke frame, where we call the worlds *arrows*, and there are accessibility relations which give meaning to the non–Boolean connectives (e.g., a ternary relation C which gives meaning to the binary connective which stands for composition). We will call models over these frames, *abstract arrow models.*

We call the whole range of modellings from abstract arrow models to "concrete" Cartesian square models, the *landscape of arrow logic.*

## Logical and Computational Core

Johan van Benthem, [Ben94b] stresses the difference between universal (Horn) conditions on the domains of the models of a logic and conditions with existential import. The purely universal (Horn) first–order requirements should, in his opinion, be viewed as the *logical core* of the semantics, while all conditions with existential import belong to some negotiable mathematical part. Thus, the core of the theory of transitions should be given by universal conditions. One is importing extra mathematical truth, if one asks for existential conditions as well. Note that we are not claiming that such existential conditions are forbidden, or anything like that, we only want to stress that such a theory will be about transitions *performed in a specific context.* The general theory of transitions should indeed be valid in all possible contexts.

Taking this into account, we can view arrow models without any existential conditions as the genuine logical core of arrow logic. We mentioned three aspects of the notion of computational core: decidability, finite axiomatizability and interpolation. One of the main results of this work is that we show that for arrow logics of directed graphs, the requirement of transitive domains forms a borderline in the landscape. If we consider the three conditions, reflexivity, symmetry and transitivity, then an arrow logic of directed graphs has any of the three aspects mentioned above if and only if the domains of the models are not neccessarily transitive relations (cf. section 6.2.1). So we can conclude that the "logical core" of the theory of transitions is part of its "computational core".

For this reason, we make a distinction between *weak* and *strong* existential conditions on models. A weak existential condition is a function with just one argument (e.g., the requirement of reflexive or symmetric universes), a strong condition is a function with more arguments (like transitivity of the universe).

## 1.2 CYLINDRIC MODAL LOGIC IS THE MODAL LOGIC OF ASSIGNMENTS

A similar analysis applies to first-order logic and assignments. When we perform this analysis, we see that, as was also the case with relational logic, classical FO models turn out to be the most restrictive class of models. Again, below this class we find a whole landscape of natural classes of models (cf. e.g., Németi [Ném92] and van Benthem [Ben94a]). Let us look at the basic declarative statement of FO logic: $\mathrm{M} \models \phi\ [\alpha]$, i.e., truth of a formula $\phi$ at a model M given an assignment $\alpha$. The most prominent citizens of FO logic are the quantifiers, whose meaning is defined as follows:

$$\mathrm{M} \models \exists v_i \phi \ \ [\alpha] \overset{\text{def}}{\Longleftrightarrow} (\exists d \in \mathrm{Dom}(\mathrm{M})) : \mathrm{M} \models \phi \ \ [\alpha^d_{v_i}]$$

Here, $\alpha^d_{v_i}$ is the assignment obtained from $\alpha$ by changing the value of $v_i$ to $d$, and leaving everything else fixed. There is an obvious "modal" view on this definition. Given a domain $D$ of a model M, let $A = {}^{\omega}D$ be the set of assignments, and define a set of (equivalence) relations $\equiv_{v_i}$ on $A \times A$ by $\alpha \equiv_{v_i} \beta \overset{\text{def}}{\Longleftrightarrow} (\forall v_j \neq v_i) : \alpha(v_j) = \beta(v_j)$. Moving the assignment to the front, we get a familiar modal pattern:

$$\mathrm{M}, \alpha \models \exists v_i \phi \overset{\text{def}}{\Longleftrightarrow} (\exists \beta \in A) : \alpha \equiv_{v_i} \beta \,\&\, \mathrm{M}, \beta \models \phi$$

So, just like we viewed relational logic as an instance of the modal logic of transitions, we can view FO logic as one particular instance of the *modal logic of assignments*. The appropriate name for the resulting family of logics, seems to be *cylindric modal logic*, originating with Yde Venema (cf. [Ven91], [Ven93]).

### MODELS FOR CYLINDRIC MODAL LOGIC

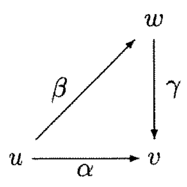
Assuming for the moment a beginners mind, we can redo the whole analysis of arrows, but now for assignments. Let us do that briefly. Suppose we are in $\mathcal{L}_\omega$, FO logic with countably many variables. Assignments should give meaning to each variable $v_i$ ($i \in \omega$). So, just like abstract arrows, assignments have to be objects equipped with, in this case $\omega$ many, functions $l_i$, each assigning a value of the domain to the variable $v_i$. Viewed from this perspective, the arrows from the previous section are assignments for $\mathcal{L}_2$, FO logic with 2 variables.

As with arrows, there is an intensional and an extensional viewpoint on assignments. Again, the classical view would be to identify an assignment $\alpha$ with the tuple $\langle l_i(\alpha) : i \in \omega \rangle$, but it is conceivable that one has an application in mind where this rigidity is not wanted[1]. As the area of intensional assignments is, to our knowledge, yet almost completely unexplored[2] we will concentrate here exclusively on the extensional ones.

As before, one faces a second ontological problem: which assignments should be present in the models? Should it, given a domain $D$ of discourse, be a full Cartesian power of $D$ as in classical FO logic? Again, it seems more reasonable to make this just one option out of many. An option which is justified when the universe of discourse is the realm of mathematical objects (indeed, the primary intended application of classical FO logic!), but an option which needs justification when applied to other areas. Apart from the question how many variables are needed for an application[3] there seem to be four basic options, completely analogous to the arrow logic case. Given a model M, let $A \subseteq {}^{\omega}\mathrm{Dom}(\mathrm{M})$ be the set of assignments.

(1) $A$ is closed under substitutions
(2) $A$ is closed under permutations
(3) $A$ is closed under "paths"
(4) $A$ is a full Cartesian product

We briefly explain what we mean with these requirements: (1) means that if $\alpha \in A$, then also $\alpha^{\alpha_i}_{v_j} \in A$ (i.e., if assignments are pairs, this means that $A$ is a reflexive relation). (2) means that if $\alpha \in A$ and $\pi$ is a permutation of the coordinates of $\alpha$, then also $\pi\alpha \in A$ (i.e., $A$ is a symmetric relation in the pair–case). (3) means that if $\alpha, \beta \in A \,\&\, \alpha \equiv_{v_i} \beta$, then $(\exists \gamma \in A) : \alpha \equiv_{v_j} \gamma \,\&\, l_j(\gamma) = l_i(\beta)$. (If assignments are pairs, this is the same as requiring that the set of assignments is an equivalence relation.) See the picture below for $i = 1$ and $j = 0$.




(1.1)

So we have a similar situation as before. There is a whole landscape of possible semantic modelings with at the bottom the models with abstract assignments, and somewhere

[1] In natural language, assignments can be seen as taking care of anaphoric binding of pronouns. One can easily imagine two examples of a written text with two ways of binding which have the same effect, but one is more difficult to understand than the other. Since anaphoric binding requires an active process from the reader, one could say that the "assignment instructions" differ, one being more difficult to *perform* than the other. This links up with the view that the existential quantifiers are a kind of program constructions (cf. Groenendijk–Stokhof [GS91]). There is no a priori reason to equate two programs which happen to have the same input/output behavior.

[2] An exception is the $n$–dimensional arrow logic of D. Vakarelov [Vak92a] which technically deals with intensional assignments but which seems to have other applications.

[3] That this is not a trivial issue is shown by Tarski–Givant [TG87], who show that one can build up set–theory, hence the whole of meta–mathematics, using only 3 variables.

near the top, the "drawable" assignment models, meeting more and more existential demands (see [Ben94a] for such a picture).

When one chooses not to have all assignments available for each model, one has to introduce an extra parameter –the set of available assignments– in the truth definition. The basic declarative statement then becomes

$$\mathrm{M}, A \models \phi\,[\alpha] \qquad \text{for } A \text{ an appropriate subset of } {}^{\omega}\mathrm{Dom(M)}$$

Clearly, this notion of truth is weaker than the classical one, since in the classical case there is only one appropriate set of assignments: $A = {}^{\omega}\mathrm{Dom(M)}$. The key validity which fails in the more general semantics is *commutativity of the quantifiers* $\exists v_i \exists v_j \phi \leftrightarrow \exists v_j \exists v_i \phi$. This is valid only if we make the (strong) requirement of closure under "paths". Németi [Ném86] showed that it is precisely this validity which is the cause of the undecidability of FO logic. The existential requirements given above lead to, at least, five natural classes of FO models in which assignments are $\alpha$–tuples. These models were introduced by Németi in [Ném86]. Let $\mathcal{L}_\alpha$ stand for FO logic with $\alpha$ many variables. Define $\mathsf{K}^\alpha$ as the class of FO models whose set of assignments can be any subset of the domain. Formally

$$\mathsf{K}^\alpha \stackrel{\text{def}}{=} \{\mathrm{M} = \langle\langle D, I\rangle, A\rangle : \langle D, I\rangle \text{ is a FO model and } A \subseteq {}^{\alpha}D\}$$

Let $\mathsf{K}^\alpha_S$ denote the subclass of $\mathsf{K}^\alpha$ in which the set of assignments is closed under taking substitutions, $\mathsf{K}^\alpha_P$ the subclass consisting of models which are closed under permutations, and $\mathsf{K}^\alpha_{SP}$ the subclass consisting of models which are closed under both. Let $\mathsf{K}^\alpha_{cubes}$ denote the classical FO models with $A = {}^{\alpha}D$. The models are related as given in figure 1.1. It is well known that for $\alpha \geq 3$, $\mathcal{L}_\alpha$ is undecidable when interpreted on the

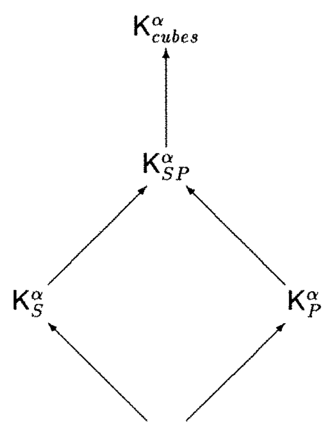



FIGURE 1.1: LANDSCAPE OF FO MODELS

classical models (cf. [HMT85]). But, as soon as we are willing to abandon the *strong existential conditions* on the assignments sets, we arrive in a more pleasant area. Then the logical core of FO logic is part of its computational core. István Németi showed that, for all $\alpha \leq \omega$, $\mathcal{L}_\alpha$ is *decidable* when interpreted on $\mathsf{K}^\alpha$ and $\mathsf{K}^\alpha_{SP}$, and for all $\alpha < \omega$, when interpreted on $\mathsf{K}^\alpha_S$ and $\mathsf{K}^\alpha_P$ ([Ném92], Thm 4.2).

## 1.3 RELATIVIZATION

In algebraic logic, both for the algebraic counterparts of arrow logic and of FO logic, the non–square and non–cubic extensional models have been studied relatively deeply, under the heading of *relativization* (cf. e.g., Henkin et al. [HMT+81], Maddux [Mad82] and Resek–Thompson [RT91]). The emphasis, however was different from what we have sketched above. As the name "relativization" indicates, the non–standard models were viewed as being derived from the standard classical ones, while in our set–up, the classical models appear as a very special case of the more basic (relativized) models. Recently, Andréka, van Benthem, Monk and Németi ([AT88], [Ben94a], [Mon91], [Ném91]) started promoting their study as structures which are interesting independently of their square or cube versions. Before, relativized algebras were not really studied in their own right[4], but as a tool to obtain results for the standard models[5]. Apart from that, one can view relativization as a way of *turning negative results into positive ones*, since several relativized versions of cylindric and relation algebras do have the nice properties of being decidable, finitely axiomatizable and having an interpolation theorem, which their classical counterparts lack.

## 1.4 FINE STRUCTURE OF DEFINABILITY

In one sense, we can still view the above logics with their more general semantics as being derived from the classical ones, because we used the *same logical language*. But just as we assumed a beginners mind when looking at the semantics, it pays to do the same at the syntactical level. Which "natural" operations on sequences do we want to express? Clearly, the existential quantifiers, changing one coordinate of a sequence, are among them. They are term–definable in arrow logic only once we have reflexive and symmetric domains, so it might be useful to add them in other cases too. We illustrate this point with the example of the quantifiers.

QUANTIFIERS AS PROGRAM INSTRUCTIONS. If we view the existential quantifier as a *program instruction* to change the value of a particular (set of) variables(s) –as is done in dynamic semantics (cf. [GS91])–, we can get insights by looking at the regular program constructions. Let the set $\{\exists x_i : i \in \omega\}$ be the atomic programs. We would like to have "programs" for *sequencing, choosing*, and *iterating* as well. Iteration, $\exists(x^*)$, can be expressed just by $\exists x$, since $\exists x \exists x \phi$ is equivalent to $\exists x \phi$. Sequencing, $\exists(x;y)$, is built into the language, because $(\exists(x;y)\phi \leftrightarrow \exists x \exists y \phi)$. The same holds for choice, since $(\exists(x \sqcup y)\phi \leftrightarrow \exists x \phi \vee \exists y \phi)$.

By the commutative law of the quantifiers, it doesn't matter in which order we

[4] As an illustration we mention the book [HMT+81], from which it turns out that the authors know much about cylindric relativized set algebras, but they treat this knowledge as secondary, hence they do not include a large part of this knowledge in the book.

[5] Because of this, the technique of relativization was studied quite extensively (cf. [HMT71] Chapter 2.2), but that is not what we are interested in now. We are interested in the relativized algebras in their own right.

change the variables, making $\exists x \exists y$ more something like *parallel execution.* So we can write $\exists(x \sqcap y)\phi$ as $\exists x \exists y \phi$ or $\exists y \exists x \phi$, because their order does not matter. But the commutative law only holds in models satisfying the "path" condition. In models without that condition, we cannot term–define this very natural operation of changing several places *simultaneously.* So it might be desirable to have them as primitive operators in the general case[6]. So, if we change the classical semantics, we have good reasons for changing the vocabulary as well. Natural operations which were term–definable before, and which we therefore "forget", need not be definable in the general case. The general–model analysis shows us the fine structure of the expressive power of the original vocabulary.

These observations give rise to a research area which is at present almost completely unexplored. Besides changing the semantics of well–known logics to "turn negative results into positive ones", one should reconsider the basic vocabulary as well. We could describe this field as follows:

> Relativize to obtain positive results, and then strengthen the expressive power as much as possible while keeping the positive results.

Besides adding what was term–definable before, one could also add *new* operations, like the Kleene $^*$ to relation algebras or the universal modality[7] to cylindric algebras of infinite dimension. We will encounter this theme in the chapter on arrow logic, in which we add several operations (e.g., Kleene $^*$) to the vocabulary, and in chapter 4, in which we expand the vocabulary of relation algebras with the difference operator.

## 1.5 BAO's AND GENERAL MODAL LOGIC

Both arrow logic and cylindric modal logic have intimate connections with algebraic logic. The classical models of arrow logic are given by the full (or square) relation set algebras and those of cylindric modal logic by the full (or cubic) cylindric set algebras. Both classes of algebras belong to the class of *Boolean Algebras with Operators* (BAO's). There is a very strong connection between the three concepts: BAO's, general modal logic and relational Kripke frames. This connection is given in figure 1.2, taken from Brink [Bri93].

For a brief explanation of the picture, we freely quote from Venema [Ven94]. The relation (a) between general modal logic and Boolean algebras with operators is very tight; for instance BAO's appear as the Lindenbaum–Tarski algebras of general modal logics. The relation (b) is closely connected to the work of Kripke, with relational Kripke frames providing a semantics for modal logics. The relation (c) was studied in

[6]S. Comer and D. Vakarelov independently suggested to study these operations, both with motivations from computer science (e.g. data–base theory) (cf. Comer [Com91] and Düntsch [Dün91]). These operations are known in the cylindric algebra literature as $C_\Gamma$, where $\Gamma$ can be any set of indices. The operators $C_\Gamma$ are studied in van Lambalgen–Simon [LS94] in the context of relativized cylindric algebras (see also Comer [Com91] and Simon [Sim90]).

[7]The universal modality is a special case of the above described parallel execution or the $C_\Gamma$'s. With the universal modality one changes all coordinates simultaneously. Clearly $C_\Gamma$, for infinite $\Gamma$, is not even definable on the classical models.

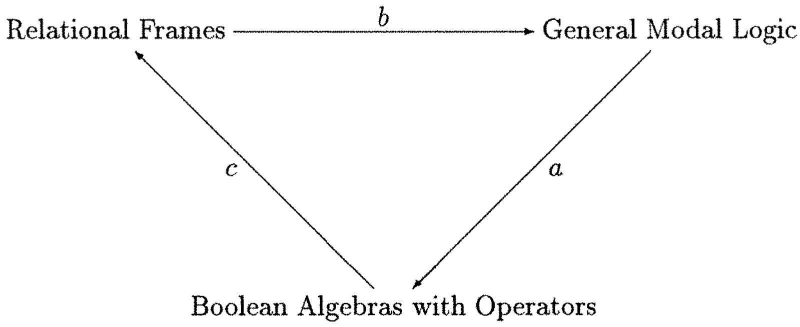



FIGURE 1.2: CONNECTIONS BETWEEN THREE FIELDS

Jónsson and Tarski's overview article on BAO's [JT52], long before the work of Kripke. In that paper they started, what is now known as, *the duality theory between BAO's and Relational Kripke Frames* (see Goldblatt [Gol88], [Gol93] for a recent overview).

From a mathematical perspective, both the development of arrow logic and of cylindric modal logic can be seen as filling in the modal part of the above picture, where the other two parts already existed. This work started explicitly with Venema's dissertation [Ven91]. Besides dealing with these two specific cases of BAO's and general modal logic, we will, when possible, treat them in a uniform manner, using the theory of BAO's as a unifying framework. In section 4.5 we show that every BAO can be represented as an algebra of relations. For modal logic this means that *every* modal logic can be viewed as a *multi-dimensional modal logic* (cf. [Ven91]).

# 2

# THE ALGEBRAS AND THE LOGICS

In this chapter, we define the cast of this piece, and fix notation for the coming chapters. We also provide a short review of that part of duality theory between BAO's and Kripke frames that will be used later on. In the first section, we deal in a general way with BAO's. In the second section, we provide some basic algebraic notions and some duality theory. In the next section, we explain why relativization is an important tool for studying the core of a logic. In the last two sections, we focus on two well-known classes of BAO's: relation algebras and cylindric algebras.

## 2.1 BAO'S, GENERAL MODAL LOGIC AND KRIPKE FRAMES

### 2.1.1 BOOLEAN ALGEBRAS WITH OPERATORS

BOOLEAN ALGEBRAS WITH OPERATORS (BAO'S). Almost all algebras in this work are *normal Boolean Algebras with Operators*. An algebra $\mathfrak{A} = \langle A, \vee, \wedge, -, 0, 1, \Diamond_i \rangle_{i \in I}$ is a Boolean Algebra with Operators (BAO) if $\langle A, \vee, \wedge, -, 0, 1 \rangle$ is a Boolean Algebra (BA), and every operation $\Diamond_i$ $(i \in I)$ is *additive* in each of its arguments. Here, additivity means that the operator distributes over join, as in (for a unary operator) $\Diamond(\tau_1 \vee \tau_2) = \Diamond\tau_1 \vee \Diamond\tau_2$. This property is also referred to as *distributivity*. An operation is called *normal*, if it equals 0, whenever one of its arguments equals 0. In algebraic logic, a (normal) distributive operation is called an *operator*, in modal logic, operators are called *modalities*. Note that by this definition every zero-ary operation is a normal operator. In the rest of this work, "operator" means "normal operator", and "BAO" means "normal BAO".

SIMILARITY TYPES. It is useful to introduce a special similarity type for BAO's. Let $O$ be a set of operation symbols, and $\rho : O \longrightarrow \omega$ a function assigning to each symbol in $O$ a finite rank or arity. By a *BAO type* $S$, we mean the pair $\langle O, \rho \rangle$. We usually assume that the rank of the operations is known, and identify $S$ with $O$. An algebra $\mathfrak{A}$ is of BAO-type $S = \langle O, \rho \rangle$ if and only if $\mathfrak{A}$ is a BA expanded with the operators in $O$, having their rank specified by $\rho$. As a variable ranging over operators we use $\Diamond$. If $S$ is given, we use $\mathsf{BA}^S$ for the class of all BAO's of type $S$. If $S$ consists of only one operator, say $\Diamond$, whose arity is clear from the context, we write $\mathsf{BA}^\Diamond$.

TERMS, EQUATIONS AND VALIDITY. Given a set of variables $X$ and a BAO-type $S = \langle O, \rho \rangle$, we use $\mathsf{Term}_S(X)$ to denote the set of terms constructed from variables in $X$ using the Booleans and the operation symbols in $O$. With $\mathsf{Eqlang}_S(X)$, we denote the set of equations $\{\tau_1 = \tau_2 : \tau_1, \tau_2 \in \mathsf{Term}_S(X)\}$. If we don't say anything about $X$, we assume it is countable. We use $\mathsf{x}, \mathsf{y}, \mathsf{z}$ to denote *algebraic variables*, and $\tau, \tau_1, \tau_2$ to denote arbitrary terms. The set of variables occuring in $\tau$ is denoted by $var(\tau)$. An *assignment* from the set of variables $X$ into an algebra $\mathfrak{A}$ is a map $h : X \longrightarrow A$. The pair $\langle \mathfrak{A}, h \rangle$ is called an *algebra–valuation pair*. An equation $\tau(\mathsf{x}_1, \ldots, \mathsf{x}_n) = \tau_1(\mathsf{x}_1, \ldots, \mathsf{x}_n)$ of BAO type $S$ is *valid* in an algebra $\mathfrak{A}$ of that type if for every assignment $h : \{\mathsf{x}_1, \ldots, \mathsf{x}_n\} \longrightarrow A$, $\tau(h(\mathsf{x}_1), \ldots, h(\mathsf{x}_n))$ equals $\tau_1(h(\mathsf{x}_1), \ldots, h(\mathsf{x}_n))$ in $\mathfrak{A}$. An equation $e$ is valid in a class of algebras $\mathsf{K}$ if $e$ is valid in every $\mathfrak{A} \in \mathsf{K}$. Both kinds of validity are denoted by $\models$. The set of equations valid in $\mathsf{K}$ is denoted by $\mathsf{Eq}(\mathsf{K})$. Using this notation we can give the following definition of specific classes of BAO's. For example, if $S = \langle \heartsuit, 2 \rangle$, then $\mathsf{BA}^S$ or $\mathsf{BA}^\heartsuit$ denotes the following class of algebras:

$$\begin{aligned} \mathsf{BA}^\heartsuit \stackrel{\text{def}}{=} \{\mathfrak{A} = \langle A, \vee, \wedge, -, 0, 1, \heartsuit \rangle : \ & \langle A, \vee, \wedge, -, 0, 1 \rangle \in \mathsf{BA} \ \& \ \mathfrak{A} \models \heartsuit(\mathsf{x}, 0) = \heartsuit(0, \mathsf{x}) = 0 \\ & \& \ \mathfrak{A} \models \heartsuit(\mathsf{x} \vee \mathsf{y}, \mathsf{z}) = \heartsuit(\mathsf{x}, \mathsf{z}) \vee \heartsuit(\mathsf{y}, \mathsf{z}) \\ & \& \ \mathfrak{A} \models \heartsuit(\mathsf{x}, \mathsf{y} \vee \mathsf{z}) = \heartsuit(\mathsf{x}, \mathsf{y}) \vee \heartsuit(\mathsf{x}, \mathsf{z})\} \end{aligned}$$

If $\Sigma$ is a set of equations in $\mathsf{Eqlang}_S(X)$, then $\mathsf{BA}^S(\Sigma)$ denotes the class $\{\mathfrak{A} \in \mathsf{BA}^S$ $\mathfrak{A} \models \Sigma\}$. We call such equationally defined classes "abstract classes".
A class $\mathsf{K}$ of algebras is called a *variety* iff it is definable by a set of equations. I.e., there exists a set of equations $\Sigma$, such that every algebra of the type of $\mathsf{K}$ validates $\Sigma$ if and only if it is a member of $\mathsf{K}$. We call a variety *finitely axiomatizable* if it is definable by a finite set of equations.

UNIVERSAL FORMULAS AND QUASI–EQUATIONS. Fix a BAO–type $S = \langle O, \rho \rangle$ and an equational language $\mathsf{Eqlang}_S(X)$. A *universal formula* of type $S$ is a FO formula build up from equations in $\mathsf{Eqlang}_S(X)$ using conjunction, disjunction and negation. A universal formula $\phi$ is valid in an algebra $\mathfrak{A}$ if $\phi$ is true for every assignment $h$ in $\mathfrak{A}$. The universal theory of a class of algebras $\mathsf{K}$, denoted by $\mathsf{Univ}(\mathsf{K})$, consists of the universal formulas which are valid in every algebra in $\mathsf{K}$.
A *quasi–equation* is a universal formula of the form $e_1 \& \ldots \& e_n \Rightarrow e_0$. Validity of quasi–equations is defined as for universal formulas. The quasi–equational theory of a class $\mathsf{K}$ is denoted by $\mathsf{Qeq}(\mathsf{K})$. A class of algebras is called a *quasi–variety* if it is definable by a set of quasi–equations. Clearly, every variety is a quasi–variety.

### 2.1.2 RELATIONAL KRIPKE FRAMES AND MODELS

As shown already in Jónsson–Tarski [JT52], to each class of BAO's of type $S = \langle O, \rho \rangle$ there belongs a class of *structures* or *frames* of a very similar type, which can be seen as a "semantics[1]" for the (abstract) class $\mathsf{BA}^S$.

CANONICAL FRAMES. Fix a type $S = \langle O, \rho \rangle$. Let $\mathfrak{A} \in \mathsf{BA}^S$. By $\mathfrak{Uf}\,\mathfrak{A}$, we denote the set of ultrafilters of $\mathfrak{A}$. $\mathfrak{A}_+$ denotes the frame $\langle \mathfrak{Uf}\,\mathfrak{A}, R^\diamond \rangle_{\diamond \in O}$, in which each $R^\diamond$ is a

[1]The word semantics is put between quotes, because it can be argued that this "semantics" is just as abstract as the equationally given class of BAO's. See the long quotation from Henkin–Monk–Tarski [HMT71] in the beginning of section 4.5 for such an argument.

$\rho\Diamond + 1$–ary relation on $\mathfrak{Uf}\,\mathfrak{A}$ defined as follows:

$$R^{\Diamond}yx_1\ldots x_{\rho\Diamond} \stackrel{\text{def}}{\Longleftrightarrow} (\forall \mathsf{x}_1\ldots\mathsf{x}_{\rho\Diamond} \in A) : ([\mathsf{x}_1 \in x_1 \& \ldots \& \mathsf{x}_{\rho\Diamond} \in x_{\rho\Diamond}] \Rightarrow \Diamond(\mathsf{x}_1,\ldots,\mathsf{x}_{\rho\Diamond}) \in y)$$

In the literature, the frame $\mathfrak{A}_+$ is called the *ultrafilter* or *canonical frame* or the *canonical atom structure* of the algebra $\mathfrak{A}$. Note that in this definition we follow the modal logical practice of putting the "result" as the first argument, whereas in [JT52] and in [Gol88] the result is the last argument.

Frames of type $S$. A frame $\mathcal{F}$ is of type $S = \langle O, \rho\rangle$ iff $\mathcal{F} = \langle W, R^{\Diamond}\rangle_{\Diamond\in O}$, $W$ is a set and, for every $\Diamond \in O$, the relation $R^{\Diamond}$ is a subset of ${}^{\rho\Diamond+1}W$. $\mathsf{K}^S$ denotes the class of all frames of type $S$. Since $\mathsf{K}^S$ is a class of FO structures, we can speak about $\mathsf{K}^S$ in the FO language with $(\rho\Diamond + 1)$–ary predicate symbols $R^{\Diamond}$, one for every $\Diamond$ in $O$. We call this language the *FO frame language* of type $S$. As a convention, the Roman capital $F$ corresponding to the script capital $\mathcal{F}$ denotes the domain of the frame $\mathcal{F}$.

Complex Algebras. Given a set $W$, $\mathcal{P}(W)$ denotes the *powerset* of $W$, and $\mathfrak{P}(W)$ the *Boolean powerset algebra* $\langle \mathcal{P}(W), \cup, \cap, -^W, \emptyset, W\rangle$ of $W$. Given a frame $\mathcal{F} = \langle W, R^{\Diamond}\rangle_{\Diamond\in O}$ of type $S$, we define its *complex algebra*, denoted by $\mathcal{F}^+$, as the algebra $\langle \mathfrak{P}(W), \Diamond\rangle_{\Diamond\in O}$. The operators are defined, using their corresponding frame relations, as follows. For $\mathsf{x}_1,\ldots,\mathsf{x}_{\rho\Diamond}$ subsets of $W$ and $\Diamond$ a $\rho\Diamond$–ary operator we define

$$\Diamond(\mathsf{x}_1,\ldots,\mathsf{x}_{\rho\Diamond}) \stackrel{\text{def}}{=} \{y \in W : (\exists x_1\ldots x_{\rho\Diamond} \in W)(R^{\Diamond}yx_1\ldots x_{\rho\Diamond} \& x_1 \in \mathsf{x}_1 \& \ldots \& x_{\rho\Diamond} \in \mathsf{x}_{\rho\Diamond})\}$$

If $\mathsf{K}$ is a class of frames, we use $\mathsf{K}^+$ to denote the class $\{\mathfrak{A} : \mathfrak{A} \cong \mathcal{F}^+ \text{ for some } \mathcal{F} \in \mathsf{K}\}$. Note that we defined $\mathsf{K}^+$ so that it is closed under isomorphisms. If $\mathsf{K}$ is a class of BAO's, $\mathbf{Cm}^{-1}\mathsf{K}$ denotes the class of frames $\{\mathcal{F} : \mathcal{F}^+ \in \mathsf{K}\}$.

Kripke Models. Let $\mathsf{K}$ be a class of frames of type $S$, let $\mathcal{F} \in \mathsf{K}$, and let $\mathsf{Term}_S(X)$ be the set of $S$ terms generated from a set of variables $X$. Let $\mathsf{v}$ be a function from $X$ to $\mathcal{P}(F)$. We call $\mathsf{v}$ the *valuation* of the variables. We call $\mathrm{M} = \langle \mathcal{F}, \mathsf{v}\rangle$ a *(Kripke) model* (over $\mathcal{F}$).
Given a model $\mathrm{M} = \langle \mathcal{F}, \mathsf{v}\rangle$, we define the *truth relation* $\Vdash$ between the elements of the domain of M and the terms from $\mathsf{Term}_S(X)$ inductively as follows. For $x \in F$, we define

| | | | |
|---|---|---|---|
| $\mathrm{M}, x \Vdash \mathsf{x}$ | $\stackrel{\text{def}}{\Longleftrightarrow}$ | $x \in \mathsf{v}(\mathsf{x})$ | for all variables $\mathsf{x} \in X$ |
| $\mathrm{M}, x \Vdash -\tau$ | $\stackrel{\text{def}}{\Longleftrightarrow}$ | $\mathrm{M}, x \nVdash \tau$ | |
| $\mathrm{M}, x \Vdash \tau_1 \vee \tau_2$ | $\stackrel{\text{def}}{\Longleftrightarrow}$ | $\mathrm{M}, x \Vdash \tau_1$ or $\mathrm{M}, x \Vdash \tau_2$ | |
| $\mathrm{M}, x \Vdash \Diamond(\tau_1,\ldots,\tau_{\rho\Diamond})$ | $\stackrel{\text{def}}{\Longleftrightarrow}$ | $(\exists y_1\ldots y_{\rho\Diamond} \in F) : R^{\Diamond}xy_1\ldots y_{\rho\Diamond} \,\&\, \mathrm{M}, y_1 \Vdash \tau_1 \,\&\ldots\&$ $\mathrm{M}, y_{\rho\Diamond} \Vdash \tau_{\rho\Diamond}$ | |

If M is clear from the context, we usually omit it from this relation. We equate the domain of a model with the domain of its underlying frame.

VALIDITY AT MODELS, FRAMES AND CLASSES OF FRAMES. We use $[\![\tau]\!]_M$ to denote the set $\{x \in \mathrm{Dom}(M) : M, x \Vdash \tau\}$, hence $[\![\tau]\!]_M$ gives us the meaning of $\tau$ in model M. In the following definition we define validity of equations at models, frames and classes of frames.

$$\begin{array}{lcl} M \models \tau_1 = \tau_2 & \overset{\text{def}}{\iff} & [\![\tau_1]\!]_M = [\![\tau_2]\!]_M \\ \mathcal{F} \models \tau_1 = \tau_2 & \overset{\text{def}}{\iff} & (\forall M = \langle \mathcal{F}, \mathsf{v} \rangle) : M \models \tau_1 = \tau_2 \\ \mathsf{K} \models \tau_1 = \tau_2 & \overset{\text{def}}{\iff} & (\forall \mathcal{F} \in \mathsf{K}) : \mathcal{F} \models \tau_1 = \tau_2 \end{array}$$

Let $\phi$ be a universal formula build up from equations in $\mathsf{Eqlang}_S(X)$. Validity of $\phi$ in a Kripke model $M = \langle \mathcal{F}, \mathsf{v} \rangle$ is defined recursively in the standard FO way using the clause for the atomic formulas (the equations) given above. We have the following connection between validity at models and validity at algebra–valuation pairs:

$$\langle \mathcal{F}, \mathsf{v} \rangle \models \phi \iff \mathcal{F}^+ \models \phi \quad \text{for the assignment } \mathsf{v}$$

Given a class $\mathsf{K}$ of frames of type $S$ and an equational language $\mathsf{Eqlang}_S(X)$ with $X$ an infinite set of variables, we use $\mathsf{Eq}(\mathsf{K})$ to denote the set of equations valid in $\mathsf{K}$. The next fact provides the connection between the defined validity relations on frames and on algebras.

FACT 2.1.1.

$$\begin{array}{lcl} M = \langle \mathcal{F}, \mathsf{v} \rangle \models \tau_1 = \tau_2 & \iff & \mathcal{F}^+ \models \tau_1 = \tau_2 \text{ for the assignment } \mathsf{v} \\ \mathcal{F} \models \tau_1 = \tau_2 & \iff & \mathcal{F}^+ \models \tau_1 = \tau_2 \\ \mathsf{K} \models \tau_1 = \tau_2 & \iff & \mathsf{K}^+ \models \tau_1 = \tau_2 \\ \mathsf{Eq}(\mathsf{K}) & = & \mathsf{Eq}(\mathsf{K}^+) \end{array}$$

### 2.1.3 GENERAL MODAL LOGIC

A general modal logic of BAO–type $S$ is a propositional logic expanded with the modalities from $S$. We will define logics semantically. In three steps we give the *language*, the *class of models* and the *meaning* of the formulas in each model.

FORMULAS "ARE" TERMS. What we call terms of type $S$ in algebra, are called well formed formulas (wff's) in modal logic. Algebraic variables correspond to logical propositional variables.

MODELS. The semantics of a general modal logic of type $S$ is given by Kripke models over frames of that type.

MEANING. The evaluation of wff's in a model is provided by the truth definition given above. A wff $\tau$ is true in a Kripke model $M = \langle \mathcal{F}, \mathsf{v} \rangle$ iff $[\![\tau]\!]_M = F$. This will be denoted by $M \models \tau$. Truth at a frame and truth at a class of frames is defined analogously as done above for equations. This is denoted by $\mathcal{F} \models \tau$ and $\mathsf{K} \models \tau$, respectively. If $\Gamma$ is a set of formulas, we use $[\![\Gamma]\!]_M$ to abbreviate $\bigcap\{[\![\tau]\!]_M : \tau \in \Gamma\}$.

EQUATIONS AND FORMULAS. There is a straightforward correspondence between *algebraic equations* and modal logical formulas. Because we always assume the Booleans, we can restrict ourselves to equations of the form $\tau = 1$. This restriction is warranted, because $\mathsf{BA} \models \tau_1 = \tau_2$ if and only if $\mathsf{BA} \models (\tau_1 \leftrightarrow \tau_2) = 1$. Here, $(\tau_1 \leftrightarrow \tau_2)$ abbreviates $(-\tau_1 \vee \tau_2) \wedge (-\tau_2 \vee \tau_1)$. If we fix a set of terms $\mathsf{Terms}_S(X)$ and a general modal-logical language $\mathsf{Fml}$ which use the same operations, then any bijection between the algebraic variables and the logical propositional variables can be extended to a translation function $^{\#} : \mathsf{Fml} \longrightarrow \mathsf{Terms}_S(X)$ such that for every frame $\mathcal{F}$ of type $S$ we have:

$$
\begin{array}{lllll}
(\forall \tau \in \mathsf{Fml}) : & \mathcal{F} \models \tau & \iff & \mathcal{F} \models \tau^{\#} = 1 \\
(\forall \tau \in \mathsf{Terms}_S(X)) : & \mathcal{F} \models \tau = 1 & \iff & \mathcal{F} \models \tau^{-\#} \\
(\forall \tau_1, \tau_2 \in \mathsf{Terms}_S(X)) : & \mathcal{F} \models \tau_1 = \tau_2 & \iff & \mathcal{F} \models \tau_1^{-\#} \leftrightarrow \tau_2^{-\#}
\end{array}
$$

The two languages are just *notational variants* of each other.

GENERAL MODAL LOGIC. Let a BAO type $S = \langle O, \rho \rangle$ be fixed, let $P$ be a countably infinite set of propositional variables, and let $\mathsf{K}$ be a class of frames of type $S$. A tuple $\langle \mathsf{Fml}_S(P), \mathsf{Mod}(\mathsf{K}), \Vdash_S \rangle$ is called a *general modal logic* $\mathcal{GML}(\mathsf{K})$ if

- $\mathsf{Fml}_S(P)$ is the smallest set such that
  - $P \subseteq \mathsf{Fml}_S(P)$,
  - if $\phi, \psi \in \mathsf{Fml}_S(P)$, then also $\neg\phi$ ("negation"), $\phi \wedge \psi$ ("conjunction") and $\phi \vee \psi$ ("disjunction") are in $\mathsf{Fml}_S(P)$,
  - for all $\Diamond \in O$, if $\phi_1, \ldots, \phi_{\rho\Diamond} \in \mathsf{Fml}_S(P)$, then also $\Diamond(\phi_1, \ldots, \phi_{\rho\Diamond}) \in \mathsf{Fml}_S(P)$
- $\mathsf{Mod}(\mathsf{K})$ denotes the class of all Kripke models over frames in $\mathsf{K}$.
- $\Vdash_S$ denotes the truth relation defined above.

When the type $S$ is clear from the context, we usually omit it as a subscript in the notions defined above. A wff $\phi$ is *valid* in the logic $\mathcal{GML}(\mathsf{K})$ if $\phi$ is true in every model from $\mathsf{Mod}(\mathsf{K})$. We denote this by $\mathsf{K} \models \phi$ or $\models_{\mathsf{K}} \phi$.

LOGICAL CONSEQUENCE. In logic, it is not the validity relation between models and wff's which plays the central rôle, but the consequence relation between (sets of) wff's. Consequence relations are also denoted by "$\models$". There are two standard ways of defining a consequence relation, a *local* one and a *global* one. Both make sense in their appropriate application domain; in modal logic the local one is most often used, while in algebraic logic the global one is dominant[2]. There is no difference in symbolism between the two notions, which is a common cause of confusion. Here we will make the distinction by a superscript $^{loc}$ or $^{glo}$. Let $\mathcal{GML}(\mathsf{K}) = \langle \mathsf{Fml}(P), \mathsf{Mod}(\mathsf{K}), \Vdash \rangle$ be a general modal logic. Let $\Gamma \subseteq \mathsf{Fml}(P)$ and $\phi \in \mathsf{Fml}(P)$. We define the consequence relations $\models^{loc}$ and $\models^{glo}$ as follows

$$
\begin{array}{lll}
\Gamma \models^{loc}_{\mathsf{K}} \phi & \stackrel{\text{def}}{\iff} & (\forall \mathrm{M} \in \mathsf{Mod}(\mathsf{K})) : [\![\Gamma]\!]_{\mathrm{M}} \subseteq [\![\phi]\!]_{\mathrm{M}} \\
\Gamma \models^{glo}_{\mathsf{K}} \phi & \stackrel{\text{def}}{\iff} & (\forall \mathrm{M} \in \mathsf{Mod}(\mathsf{K})) : \mathrm{M} \models \Gamma \Rightarrow \mathrm{M} \models \phi
\end{array}
$$

[2] For a comparison of the two see e.g., Venema [Ven91], Appendix B.

Note that if $\Gamma = \emptyset$, the two notions coincide. Also note that $\{\phi_1, \ldots, \phi_n\} \models^{glo}_{\mathsf{K}} \phi_0 \iff (\phi_1^{\#} = 1 \,\&\ldots\&\, \phi_n^{\#} = 1 \Rightarrow \phi_0^{\#} = 1) \in \mathsf{Qeq}(\mathsf{K}^+)$.

## 2.2 Review of basic duality theory

### Correspondence, canonicity and Sahlqvist forms

In this section, we indicate when we can reason at the frame level in order to obtain results at the algebraic level. It is attractive to reason at the frame level because it is usually easier. Everything in this section is built upon the following basic fact which can be found in [JT52]. Before we state it we recall some notions (see e.g., [HMT71] or [BS81]).

Homomorphisms and embeddings. If $\mathfrak{A}$ and $\mathfrak{B}$ are algebras of the same type, then a function $h : A \longrightarrow B$ is called a *homomorphism* from $\mathfrak{A}$ to $\mathfrak{B}$ if it commutes with all the operations. If $h$ is surjective, $\mathfrak{B}$ is called a *homomorphic image* of $\mathfrak{A}$. If $h$ is injective, it is called an *embedding*, and $\mathfrak{A}$ is isomorphic to a *subalgebra* of $\mathfrak{B}$. Embeddings are denoted by arrows with a tail "$\rightarrowtail$". If $\mathfrak{A}$ can be embedded in $\mathfrak{B}$, we write $\mathfrak{A} \leq \mathfrak{B}$ or $\mathfrak{A} \overset{h}{\rightarrowtail} \mathfrak{B}$.

Fact 2.2.1. Every BAO $\mathfrak{A}$ is embeddable in $(\mathfrak{A}_+)^+$ by the *canonical embedding function* $e_A : A \longrightarrow \mathcal{P}(\mathfrak{Uf}\,\mathfrak{A})$, which is defined as $e_A(\mathsf{x}) \overset{\text{def}}{=} \{x \in \mathfrak{Uf}\,\mathfrak{A} : \mathsf{x} \in x\}$. The algebra $(\mathfrak{A}_+)^+$ is called the *canonical embedding algebra* of $\mathfrak{A}$.

Correspondence. Correspondence theory (cf. van Benthem [Ben84]) is concerned with equations $e$ of BAO–type $S$ which correspond to a condition $\phi$ on type $S$ frames in the first order frame language of type $S$ such that

$$\mathcal{F} \models e \iff \mathcal{F} \models \phi$$

When $e$ and $\phi$ stand in this relation, we say that the equation $e$ *defines* the condition $\phi$, or that $e$ *corresponds to* $\phi$. We call condition $\phi$ the *frame correspondent* of $e$. Some well–known examples (see e.g., [JT52]) in BAO–type $\langle \Diamond, 1 \rangle$ are the equations $\mathsf{x} \leq \Diamond\mathsf{x}$, $\Diamond\Diamond\mathsf{x} \leq \Diamond\mathsf{x}$ and $\Diamond{-}\Diamond{-}\mathsf{x} \leq \mathsf{x}$ which correspond to reflexivity, transitivity and symmetry of the binary relation $R^{\Diamond}$, respectively.

Canonicity. We call an equation *canonical* if it is preserved under taking canonical embedding algebras. Hence an equation $e$ is canonical if

$$\mathfrak{A} \models e \iff (\mathfrak{A}_+)^+ \models e$$

(Note that the $\Leftarrow$ direction is trivially satisfied.) A variety which is closed under canonical embedding algebras is called a *canonical variety*. Hence every class of algebras which is defined by a set of canonical equations is a canonical variety. Especially interesting are canonical equations which define FO conditions on frames. The three equations given above are examples of these. Then the following holds. Let $e$ be a canonical equation which corresponds to condition $\phi$.

$$\mathfrak{A} \models e \overset{\text{canonicity}}{\iff} (\mathfrak{A}_+)^+ \models e \overset{\text{correspondence}}{\iff} \mathfrak{A}_+ \models \phi$$

REPRESENTING CANONICAL VARIETIES. Canonical equations which correspond to a FO condition give rise to easy representation theorems in the following way. For a class $\mathsf{K}$ of algebras, $\mathsf{SK}$ denotes the class of all its subalgebras. Note that if $\mathsf{K}$ is a class of frames, then $\mathsf{SK}^+$ is closed under isomorphisms.

FACT 2.2.2. Fix a BAO–type $S$. Let $e \in \mathsf{Eqlang}_S(X)$ be a canonical equation which corresponds to a frame condition $\phi$. Let $\mathsf{K}$ be the class of all frames of type $S$ which validate $\phi$. Then $\mathsf{SK}^+$ is a canonical variety. In fact $\mathsf{BA}^S(e) = \mathsf{SK}^+$.

PROOF. Because $e$ is a canonical equation, the class $\mathsf{BA}^S(e)$ is a canonical variety. We continue with the proof of the equality.

($\supseteq$) Let $\mathfrak{A} \in \mathsf{SK}^+$. Then $\mathfrak{A} \leq \mathcal{F}^+$ for some $\mathcal{F} \in \mathsf{K}$. But then, $\mathcal{F} \models \phi$, and, since $\phi$ corresponds to $e$, also $\mathcal{F}^+ \models e$. Since equations are preserved under taking subalgebras, also $\mathfrak{A} \models e$, whence $\mathfrak{A} \in \mathsf{BA}^S(e)$. Note that we only used the correspondence part.

($\subseteq$) Let $\mathfrak{A} \in \mathsf{BA}^S(e)$. By fact 2.2.1, $\mathfrak{A} \leq (\mathfrak{A}_+)^+$. Since $e$ is canonical, $(\mathfrak{A}_+)^+ \models e$. But then, by correspondence, $\mathfrak{A}_+ \models \phi$. Hence $\mathfrak{A}_+ \in \mathsf{K}$, so $\mathfrak{A} \in \mathsf{SK}^+$. QED

Clearly, this fact also holds for sets of canonical equations. If $\mathsf{K}$ is a class of frames and $\mathsf{SK}^+$ is a variety, we call it a *complex variety*[3].

POSITIVE AND SAHLQVIST EQUATIONS. We briefly review the correspondence theory of *Sahlqvist equations* (Sahlqvist [Sah75]), which is surveyed in de Rijke–Venema [RV91]. We do not recall the definition of Sahlqvist equations, since we will almost only deal with the easier and better-known *positive equations* (cf. [HMT71] p.440). A *term* $\tau$ is called positive if it does not contain any occurrence of the symbol $-$, for complementation; an *equation* is called positive if both its terms are positive.

Let an arbitrary BAO similarity type $S$ be fixed. The set of Sahlqvist equations of type $S$ is a strictly larger set than the set of positive equations (in the wider sense[4]) (cf. [RV91] Remark 3.6). An example of a Sahlqvist equation which is not positive is the equation $\Diamond{-}\Diamond{-}\mathsf{x} \leq \mathsf{x}$ given above. Another example is $\mathsf{x}^{\smile};-(\mathsf{x};\mathsf{y}) \wedge \mathsf{y} = 0$, which is equivalent to the last axiom of relation algebras ($RA_5$) (see 2.4.11).

The most interesting aspect of Sahlqvist equations (and hence of positive ones), is the following fact (for a proof, see [RV91] Thm 3.5).

FACT 2.2.3. Fix a type $S$. Let $e$ be a Sahlqvist equation in type $S$. Then there exists an effectively obtainable[5] sentence $\phi$ in the FO language of type $S$ such that (2.1) and

[3] This name was introduced by R. Goldblatt ([Gol88]).

[4] A term is positive in the wider sense if there is no subterm beginning with $-$ which contains an occurrence of a variable. It is assumed that the Boolean constants 0 and 1 are in the language. An *equation* is positive in the wider sense if both its terms are. See [HMT71] Remark 2.7.16.

[5] See the proof of Thm 3.3 in [RV91] for an algorithm.

(2.2) below hold. In particular, $e$ is canonical.

$$\mathcal{F} \models \phi \iff \mathcal{F}^+ \models e \tag{2.1}$$
$$\mathfrak{A} \models e \iff \mathfrak{A}_+ \models \phi \tag{2.2}$$

So, if $\Sigma$ is a set of Sahlqvist equations, and $\mathsf{K}_\Sigma$ is the class of all frames satisfying the frame correspondents of the equations in $\Sigma$, then the class $\mathsf{BA}^S(\Sigma)$ is a canonical variety which equals $\mathsf{SK}_\Sigma^+$.

REASONING IN FRAMES INSTEAD OF ALGEBRAS. There is another advantage of working with Sahlqvist equations. As Venema writes in [Ven91]:

> Maybe the nicest aspect of (2.1) and (2.2) is that it frees us from giving tedious algebraic derivations for Sahlqvist equations, allowing us to focus on reasoning in *atom structures.* ([Ven91] page 11)

This move from algebras and the algebraic description language to frames and their FO description language is justified by the following fact (for a proof, see [RV91] Prop 4.1).

FACT 2.2.4. Let $\mathsf{V}$ be a canonical variety, and $e_1$ and $e_2$ two Sahlqvist equations with first order correspondents $\phi_1$ and $\phi_2$. Then,

$$\mathbf{Cm}^{-1}\mathsf{V} \models \phi_1 \leftrightarrow \phi_2 \iff \mathsf{V} \models e_1 \leftrightarrow e_2$$

## CONSTRUCTIONS ON KRIPKE FRAMES

We recall the part of the duality theory between BAO's and relational Kripke frames which will be used later on. We rely on Goldblatt's overview article [Gol88]. When we use different terminology, we provide the terms used there in footnotes. The facts reported here can be found in Cor 3.2.5, Thm 3.3.1 and lemma 3.4.1 in [Gol88].

OPERATIONS ON CLASSES OF ALGEBRAS. Recall the following operations on classes of algebras from universal algebra (we use them in the sense of [HMT71]). Let K be a class of algebras.

| | |
|---|---|
| **I**K | class of all isomorphic copies of members of K |
| **S**K | class of all subalgebras of members of K |
| **P**K | class of algebras isomorphic to direct products of members of K |
| **Up**K | class of algebras isomorphic to ultraproducts of members of K |
| **H**K | class of all homomorphic images of members of K |
| **Sir**K | class of all subdirectly irreducible members of K |
| **Emb**K | class of all canonical embedding algebras of members of K |
| $\mathbf{Rd}_I$K | class of all $I$ reducts of members of K |

VARIETIES AND QUASI–VARIETIES. By Birkhoff's theorem a class of algebras is a *variety* if it is closed under taking **H**, **S** and **P**. In general, **HSPK** is a variety, and it is the smallest one containing K. A variety V is said to be *generated* by K if V = **HSPK**. A similar theorem states that a class of algebras is a *quasi–variety* if it is closed under taking **S**, **P** and **Up**. In general, **SPUpK** is a quasi–variety, it is the smallest one containing K, and it is said to be generated by K (see [BS81] Thm V.2.25).

OPERATIONS ON CLASSES OF FRAMES. We will use the following operations on frames.

| | |
|---|---|
| **Zig**K | class of all zigzagmorphic images of frames in K |
| **Du**K | class of all disjoint unions of systems of frames in K |
| **Gs**K | class of all generated subframes of members of K |
| **Gsp**K | class of all point generated subframes of K |
| **Ue**K | class of all ultrafilter extensions of members of K |
| $\mathbf{Rd}_I$K | class of all $I$ reducts of members of K |

ZIGZAGMORPHISMS. Let $\mathcal{F}$ and $\mathcal{G}$ be two frames of the same type. A function $f : F \twoheadrightarrow G$ is called a *zigzagmorphism*[6] if it is a surjective homomorphism[7], and it has the *zigzag property*, meaning that, for every $n+1$ ary relation $R$,

$$R^G f(x)y_1 \dots y_n \Rightarrow (\exists y'_1 \dots y'_n \in F) : R^F x y'_1 \dots y'_n \;\&\; f(y'_1) = y_1 \;\&\; \dots \;\&\; f(y'_n) = y_n$$

If $f : F \twoheadrightarrow G$ is a zigzagmorphism, $\mathcal{G}$ is called the zigzagmorphic image of $\mathcal{F}$. We denote this by $\mathcal{F} \overset{f}{\twoheadrightarrow} \mathcal{G}$. It is easy to see that, for two models $\mathrm{M} = \langle \mathcal{F}, \mathsf{v}_1 \rangle$ and $\mathrm{N} = \langle \mathcal{G}, \mathsf{v}_2 \rangle$, and a zigzagmorphism $f : F \twoheadrightarrow G$ which agrees on the valuations of the variables:[8]

$$\text{for every term } \tau, \text{ and for every } x \in F : \mathrm{M}, x \Vdash \tau \iff \mathrm{N}, f(x) \Vdash \tau$$

The operation of taking zigzagmorphic images corresponds to taking subalgebras in the following way. If $h : A \longrightarrow B$ and $X \subseteq B$, we use $h^{-1}[X]$ to denote the set $\{\mathsf{x} \in A : h(\mathsf{x}) \in X\}$. Let $\mathfrak{A}, \mathfrak{B}$ be BAO's of the same type. If $h : A \longrightarrow B$, we use $h_+$ to denote the function from $\mathfrak{Uf}\,\mathfrak{B}$ to $\mathfrak{Uf}\,\mathfrak{A}$, defined by $h_+(u) \overset{\text{def}}{=} h^{-1}[u]$. Let $\mathcal{F}, \mathcal{G}$ be frames of the same type. If $f : F \longrightarrow G$, then $f^+$ denotes the function from $\mathcal{P}(G)$ to $\mathcal{P}(F)$ defined by $f^+(\mathsf{x}) \overset{\text{def}}{=} f^{-1}[\mathsf{x}]$.

FACT 2.2.5. (i) If $h : A \rightarrowtail B$ is an embedding of $\mathfrak{A}$ into $\mathfrak{B}$, then $h_+ : \mathfrak{Uf}\,\mathfrak{B} \twoheadrightarrow \mathfrak{Uf}\,\mathfrak{A}$ is a zigzagmorphism from $\mathfrak{B}_+$ onto $\mathfrak{A}_+$.
(ii) If $f : F \twoheadrightarrow G$ is a zigzagmorphism from $\mathcal{F}$ onto $\mathcal{G}$, then $f^+ : \mathcal{P}(G) \rightarrowtail \mathcal{P}(F)$ is an embedding of $\mathcal{G}^+$ into $\mathcal{F}^+$.

---

[6]In [Gol88], these are called *bounded epimorphisms*.

[7]In the *model–theoretic sense*, hence for every relation $R$ in the similarity type: $R^F x_0 \dots x_n$ implies $R^G f(x_0) \dots f(x_n)$.

[8]i.e., for every variable $\mathsf{x}$, $f^*\mathsf{v}_1(\mathsf{x}) = \mathsf{v}_2(\mathsf{x})$.

DISJOINT UNIONS. Let $\langle \mathcal{F}_i \rangle_{i \in I}$ be a system of disjoint frames which are all of type $S$. Then the *disjoint union* $\sum_I \mathcal{F}_i$ is the frame $\langle \bigcup\{F_i : i \in I\}, \bigcup\{R_i^\diamond : i \in I\}\rangle_{\diamond \in S}$.
If $\langle \mathfrak{A}_i \rangle_{i \in I}$ is a system of algebras, $\prod_I \mathfrak{A}_i$ denotes their direct product.

FACT 2.2.6. $(\sum_I \mathcal{F}_i)^+$ is isomorphic to $\prod_I \mathcal{F}_i^+$.

(POINT–)GENERATED SUBFRAMES. The frame construction which corresponds to taking homomorphic images is that of taking *generated subframes*[9]. If $\mathcal{F}$ is a frame, then $\mathcal{F}'$ is a generated subframe of $\mathcal{F}$ if (i) $F' \subseteq F$, (ii) the relations $R'$ are the restrictions of $R$ to $F'$, and (iii) for every $n+1$ ary relation $R$ and for all $x \in F', y_1, \dots y_n \in F$, if $Rxy_1 \dots y_n$, then $y_1 \dots y_n \in F'$. If a subframe is generated by a singleton, it is called a *point–generated subframe* (i.e., $\mathcal{F}'$ is the smallest generated subframe containing that singleton).

FACT 2.2.7. (i) The complex algebra of a point–generated subframe is subdirectly irreducible.
(ii) For any class $\mathsf{K}$ of frames, $(\mathbf{Gsp}\mathsf{K} \subseteq \mathsf{K}\ \&\ \mathbf{Du}\mathsf{K} \subseteq \mathsf{K}) \Rightarrow \mathsf{K} = \mathbf{DuGsp}\mathsf{K}$.
(iii) If $\mathcal{F}$ is (isomorphic to) a generated subframe of $\mathcal{G}$, then $\mathcal{F}^+$ is a homomorphic image of $\mathcal{G}^+$.
(iv) If $\mathfrak{A}$ is a homomorphic image of $\mathfrak{B}$, then $\mathfrak{A}_+$ is isomorphic to a generated subframe of $\mathfrak{B}_+$.

ULTRAFILTER EXTENSIONS. If $\mathcal{F}$ is a frame, then the frame $(\mathcal{F}^+)_+$ is called its *ultrafilter extension* [10]. It is the dual "two–step construction" of taking canonical embedding algebras. We say that a class of frames $\mathsf{K}$ *reflects ultrafilter extensions* if, whenever $(\mathcal{F}^+)_+ \in \mathsf{K}$, also $\mathcal{F} \in \mathsf{K}$. We use $\overleftarrow{\mathbf{U}}\mathbf{e}\mathsf{K}$ to denote the class of frames $\{\mathcal{F} : \mathcal{F} \cong \mathcal{G}\ \&\ (\mathcal{G}^+)_+ \in \mathsf{K}\}$.

REDUCTS. Reducts of frames are defined in the usual model–theoretic sense.

CHARACTERIZATION OF EQUATIONALLY DEFINABLE FRAME–CLASSES. We say that a class $\mathsf{K}$ of frames is equationally definable if there exists a set of equations $\Delta$ such that $\mathsf{K}$ is the class of all frames which validate $\Delta$. For $\mathsf{V}$ a variety, we define $\mathsf{K}_\mathsf{V} \stackrel{\text{def}}{=} \{\mathcal{F} : \mathcal{F}^+ \in \mathsf{V}\}$, and, for $\mathsf{K}$ a class of frames, we define $\mathsf{V}_\mathsf{K} \stackrel{\text{def}}{=} \mathbf{HSP}\mathsf{K}^+$.

FACT 2.2.8. $\mathsf{K}$ is equationally definable iff $\mathsf{K} = \mathsf{K}_{\mathsf{V}_\mathsf{K}}$.

PROOF. $(\Rightarrow)$ $\mathcal{F} \in \mathsf{K} \iff \mathcal{F} \models \mathsf{Eq}(\mathsf{K}) \iff \mathcal{F} \models \mathsf{Eq}(\mathsf{V}_\mathsf{K}) \iff \mathcal{F}^+ \in \mathsf{V}_\mathsf{K} \iff \mathcal{F} \in \mathsf{K}_{\mathsf{V}_\mathsf{K}}$
$(\Leftarrow)$ $\mathcal{F} \in \mathsf{K} \iff \mathcal{F} \in \mathsf{K}_{\mathsf{V}_\mathsf{K}} \iff \mathcal{F}^+ \in \mathsf{V}_\mathsf{K} \iff \mathcal{F} \models \mathsf{Eq}(\mathsf{K})$ QED

The overall situation is described in the next theorem. The theorem shows that the four constructions discussed above are very fundamental. In a sense, this is the frame analogue of Birkhoff's theorem. The theorem in this generality can be found in [Gol88]

[9] Called *inner substructures* in [Gol88]
[10] Called *canonical extension* in [Gol88].

(Thm 3.7.7). For classes of frames of type $\langle\Diamond, 1\rangle$, the theorem is in van Benthem [Ben79]. (See also Goldblatt–Thomason [GT74], van Benthem [Ben83], [Ben91b].)

THEOREM 2.2.9 (VAN BENTHEM–GOLDBLATT). *For* $\mathsf{K}$ *any class of frames, the following are equivalent.*

(i) $\mathsf{K}$ *is closed under the formation of ultrafilter extensions, generated subframes, disjoint unions, zigzagmorphic images and it reflects ultrafilter extensions.*
(ii) $\mathsf{K} = \mathsf{K}_{\mathsf{V}_\mathsf{K}}$ *and* $\mathsf{V}_\mathsf{K}$ *is canonical.*
(iii) $\mathsf{K} = \mathsf{K}_\mathsf{V}$ *for some canonical variety* $\mathsf{V}$.

## 2.3 RELATIVIZATION AND THE LOGICAL CORE

In this small section, we introduce the operation of *relativization*, and show why it is so important when we want to find the *logical core* of a logic. We will restrict ourselves to general modal logics of the kind introduced above.

SUBFRAMES AND UNIVERSAL SENTENCES. In chapter 1, we argued that the logical core of a logic should not contain any existential import. For general modal logics $\mathcal{GML}(\mathsf{K})$, this means that $\mathsf{K}$ should be a universal class. The operation on classes of frames $\mathsf{K}$ which gives us the smallest universal class in which $\mathsf{K}$ is contained, is that of taking *substructures*. The notion of substructure is used in the FO model–theoretic sense. We recall the definition (cf. e.g., [CK90] or [Hod93]). Let $\mathcal{F} = \langle W, R_i\rangle_{i\in I}$ be a frame. A frame $\mathcal{F}' = \langle W', R_i'\rangle_{i\in I}$ is a *substructure* or *subframe* of $\mathcal{F}$ if $W' \subseteq W$, and the relations in $\mathcal{F}'$ are the restrictions of the relations in $\mathcal{F}$ to $W'$. We denote the subframe of $\mathcal{F}$ with domain $W'$ by $\mathcal{F}\restriction_{W'}$. If $\mathsf{K}$ is a class of frames, $\mathbf{Sub}\mathsf{K}$ denotes the class of all subframes of frames in $\mathsf{K}$.

LOGICAL CORE. Let $\mathcal{GML}(\mathsf{K}) = \langle \mathsf{Fml}, \mathsf{Mod}(\mathsf{K}), \Vdash\rangle$ be a general modal logic. The *logical core* of $\mathcal{GML}(\mathsf{K})$ is the general modal logic $\mathcal{GML}(\mathbf{Sub}\mathsf{K}) = \langle \mathsf{Fml}, \mathsf{Mod}(\mathbf{Sub}\mathsf{K}), \Vdash\rangle$.

RELATIVIZATION. In [HMT71] (Def 2.2.1) an operator $\mathfrak{Rl}_b$ of *relativizing* algebras of the cylindric type is introduced. For arbitrary BAO's it is defined as follows. Let $\mathfrak{A} = \langle A, \wedge, -, f_i\rangle_{i\in I}$ be a BAO, and suppose $b \in A$. Let $Rl_b\mathfrak{A} \stackrel{\text{def}}{=} \{\mathsf{x} \wedge b : \mathsf{x} \in A\}$. For all $\mathsf{x}_1, \ldots, \mathsf{x}_n \in Rl_b\mathfrak{A}$, let $\mathsf{x}_1 \wedge' \mathsf{x}_2 \stackrel{\text{def}}{=} \mathsf{x}_1 \wedge \mathsf{x}_2$, $-'\mathsf{x}_1 \stackrel{\text{def}}{=} -\mathsf{x}_1 \wedge b$, and $f_i'(\mathsf{x}_1, \ldots, \mathsf{x}_n) \stackrel{\text{def}}{=} f_i(\mathsf{x}_1, \ldots, \mathsf{x}_n) \wedge b$. Let $\mathfrak{Rl}_b\mathfrak{A} \stackrel{\text{def}}{=} \langle Rl_b\mathfrak{A}, \wedge', -', f_i'\rangle_{i\in I}$. We refer to $\mathfrak{Rl}_b\mathfrak{A}$ as the algebra obtained by *relativizing* the BAO $\mathfrak{A}$ to $b$. For $\mathsf{K}$ a class of algebras, $\mathbf{Rl}\mathsf{K} \stackrel{\text{def}}{=} \{\mathfrak{Rl}_b\mathfrak{B} : \mathfrak{B} \in \mathsf{K} \text{ and } b \in B\}$.

In sharp contrast with the operators $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{P}$, equations are in general not preserved under $\mathbf{Rl}$. Below, we characterize which *Sahlqvist* equations are preserved under $\mathbf{Rl}$. We end this paragraph with a technical lemma about relativization which will be useful later on.

LEMMA 2.3.1. (i) $\mathbf{Rl}$ *is a closure operator, i.e.,* $\mathsf{K} \subseteq \mathbf{Rl}\mathsf{K}$, $\mathbf{Rl}\mathbf{Rl}\mathsf{K} = \mathbf{Rl}\mathsf{K}$, *and* $\mathsf{K} \subseteq \mathsf{L} \Rightarrow \mathbf{Rl}\mathsf{K} \subseteq \mathbf{Rl}\mathsf{L}$.

(ii) $\mathbf{SRl} = \mathbf{SRlS} = \mathbf{RlSRl}$.
(iii) $\mathbf{Rl}$ *commutes with* $\mathbf{P}$ *and* $\mathbf{Up}$.
(iv) *If the class* $\mathsf{V}$ *is a quasi–variety, then* $\mathbf{SRl}\mathsf{V}$ *is a quasi–variety.*

PROOF. (i) and (ii). Straightforward.
(iii). Cf. [HMT71], Thm 2.2.8.
(iv). Since $\mathsf{V}$ is a quasi–variety, it is closed under $\mathbf{SPUp}$. By part (iii), the operator $\mathbf{PUp}$ commutes with $\mathbf{Rl}$. It follows from (ii) and the universal algebraic facts that $\mathbf{PS} \leq \mathbf{SP}$ and $\mathbf{UpS} \leq \mathbf{SUp}$ that $\mathbf{SRl}\mathsf{V}$ is closed under $\mathbf{SPUp}$. So it is a quasi–variety.
QED

CONVENTION. In the sequel, we will speak about the class of *relativized* relation algebras. In such a context, we use "relativized" as an abbreviation for *"subalgebras of relativized"*. We always take subalgebras as well, because –even if the original class is a variety– only relativizing can lead to a highly complex class of algebras (i.e., in general, they are not closed under subalgebras anymore). Examples of this are $\mathbf{SRl}\mathsf{CA}_\alpha \neq \mathbf{Rl}\mathsf{CA}_\alpha$ and $\mathbf{SRl}\mathsf{Cs}_\alpha \neq \mathbf{Rl}\mathsf{Cs}_\alpha$ when $\alpha > 2$ (cf. [HMT71], 5.5.6 and 5.5.7) and $\mathbf{SRl}\mathsf{RRA} \neq \mathbf{Rl}\mathsf{RRA}$ and $\mathbf{SRl}\mathsf{RA} \neq \mathbf{Rl}\mathsf{RA}$ (cf. Andréka [And88]). (These classes of algebras will be defined in the coming two sections.)

RELATIVIZATION AND SUBFRAMES. The following proposition establishes the connection between relativizations and subframes. It shows why *relativization* is a key tool for finding the core of a logic.

PROPOSITION 2.3.2. (i) *For any frame* $\mathcal{F}$ *and set* $W \subseteq F$, $(\mathcal{F}{\restriction_W})^+ = \mathfrak{Rl}_W\mathcal{F}^+$. *Hence* $\mathbf{Rl}\mathsf{K}^+ = (\mathbf{Sub}\mathsf{K})^+$.
(ii) *If* $\mathsf{V} = \mathbf{SP}\mathsf{K}^+$, *then* $\mathbf{SRl}\mathsf{V} = \mathbf{SP}(\mathbf{Sub}\mathsf{K})^+$.
(iii) *Let e be a canonical equation which defines a FO frame condition (e.g., a Sahlqvist equation). Then e is preserved under taking relativizations if and only if e corresponds to a universal sentence.*

PROOF. (i). By the definitions.
(ii). By (i) and lemma 2.3.1.
(iii). Let $\mathsf{V}$ be the variety defined by $e$. The conclusion follows immediately from the following statement:

$$\mathsf{V} = \mathbf{Rl}\mathsf{V} \iff \mathbf{Cm}^{-1}\mathsf{V} = \mathbf{SubCm}^{-1}\mathsf{V}$$

We prove the statement:
($\Rightarrow$). $\mathcal{F} \in \mathbf{SubCm}^{-1}\mathsf{V} \Rightarrow \mathcal{F} = \mathcal{G}{\restriction_{G'}}$ for some $\mathcal{G}^+ \in \mathsf{V}$ and $G' \subseteq G$. The algebra $\mathcal{F}^+$ is in $\mathsf{V}$, because $\mathcal{F}^+ = (\mathcal{G}{\restriction_{G'}})^+ = \mathfrak{Rl}_{G'}\mathcal{G}^+$ and the assumption. But then, $\mathcal{F} \in \mathbf{Cm}^{-1}\mathsf{V}$.
($\Leftarrow$). $\mathsf{V} = \mathbf{S}(\mathbf{Cm}^{-1}\mathsf{V})^+ = \mathbf{S}(\mathbf{SubCm}^{-1}\mathsf{V})^+ = \mathbf{SRl}(\mathbf{Cm}^{-1}\mathsf{V})^+ = \mathbf{RlSRl}(\mathbf{Cm}^{-1}\mathsf{V})^+ = \mathbf{RlS}(\mathbf{SubCm}^{-1}\mathsf{V})^+ = \mathbf{RlS}(\mathbf{Cm}^{-1}\mathsf{V})^+ = \mathbf{Rl}\mathsf{V}$ (the last step follows, because $\mathsf{V}$ is canonical).
QED

## 2.4 RELATION ALGEBRAS, ARROW LOGIC AND ARROW FRAMES

In this section, we concentrate on the similarity type of *relation algebras*, and deal exclusively with the extensional view of arrows. In chapter 6, we will return to the intensional view. We sketch a landscape of interesting classes of relation algebras which all contain the class RRA, and provide a menu of the properties we will investigate. To facilitate these investigations, we do some correspondence theory, which enables us to do most of our reasoning at the frame-level.

The study of algebras of binary relations goes back to de Morgan, Schröder and Peirce. Recent works which are both a good introduction to the field, and which also cover the history are Givant [Giv91], Jónsson [Jón91], Maddux [Mad91b] and Tarski–Givant [TG87]. For a mathematical introduction, cf. [HMT85]. For history, cf. also Maddux [Mad91a], Pratt [Pra92] and Annelis–Houser [AH91]. A gentle introduction designed for beginners, is the 1994 version of Németi [Ném91].

### 2.4.1 RELATION ALGEBRAS AND ARROW FRAMES

Define $\mathsf{BA}^{rel}$ as the class of all BAO's with one binary infix operator ";" (called "composition"), one unary postfix operator "$^\smile$" (called "converse") and one constant "id" (called "identity"). We will use $rel$ as an abbreviation for the BAO type $\{\langle ;, 2\rangle, \langle {}^\smile, 1\rangle, \langle \mathsf{id}, 0\rangle\}$. The abstract class $\mathsf{BA}^{rel}$ is obtained by an abstraction over the *concrete relations* (i.e., sets of pairs) of *concrete* relation algebras. Concrete relation algebras and their operations are defined as follows. For $s$ a sequence, we use $s_i$ to denote the $i$-th coordinate of $s$. For $V \subseteq U \times U$ a binary relation over some set $U$, define a ternary relation $\mathsf{C}_V$, a binary relation $\mathsf{F}_V$ and a unary one $\mathsf{I}_V$ on $V$ as follows:

$$\begin{aligned}
\mathsf{C}_V &\stackrel{\text{def}}{=} \{\langle x, y, z\rangle \in {}^3V : x_0 = y_0 \;\&\; y_1 = z_0 \;\&\; z_1 = x_1\} \\
\mathsf{F}_V &\stackrel{\text{def}}{=} \{\langle x, y\rangle \in {}^2V : x_0 = y_1 \;\&\; x_1 = y_0\} \\
\mathsf{I}_V &\stackrel{\text{def}}{=} \{x \in V : x_0 = x_1\}
\end{aligned}$$

An algebra $\mathfrak{A} = \langle \mathfrak{P}(V), \circ^V, {}^{-1^V}, \mathsf{Id}^V\rangle$ is called a *full relativized relation set algebra* if $\mathfrak{P}(V)$ is the Boolean powerset algebra with domain $\mathcal{P}(V)$, and the operators are defined as follows. For $\mathsf{x}, \mathsf{y} \subseteq V$ we define:

$$\begin{aligned}
\mathsf{x} \circ^V \mathsf{y} &\stackrel{\text{def}}{=} \{x \in V : (\exists yz \in V)(\mathsf{C}_V xyz \;\&\; y \in \mathsf{x} \;\&\; z \in \mathsf{y})\} \\
\mathsf{x}^{-1^V} &\stackrel{\text{def}}{=} \{x \in V : (\exists y \in V)(\mathsf{F}_V xy \;\&\; y \in \mathsf{x})\} \\
\mathsf{Id}^V &\stackrel{\text{def}}{=} \{x \in V : \mathsf{I}_V x\}
\end{aligned}$$

We attach a superscript $V$ to the operators because, since $V$ is only a *subset* of a Cartesian product $U \times U$, the meaning of the operators is dependent on $V$. We note that, for $\mathsf{x}, \mathsf{y} \subseteq V$, $\mathsf{x} \circ^V \mathsf{y} = (\mathsf{x} \circ \mathsf{y}) \cap V$, in which $\circ$ is the usual relation composition, and similarly for the other two operations. For this reason, we call the algebras *relativized.*

An equivalent definition of these algebras can be given using the notion of *pair-frames.* The notion of an *arrow-frame*, defined below, is known in the literature (cf. e.g., Maddux [Mad82]); the name is due to Johan van Benthem.

DEFINITION 2.4.1. (i) A structure $\mathcal{F} = \langle V, \mathsf{C}_V, \mathsf{F}_V, \mathsf{I}_V \rangle$ is called a *pair–frame* if $V \subseteq U \times U$ for some set $U$, and the relations are defined as given above.
(ii) If $V \subseteq U \times U$, then $\mathcal{F}_{pair}(V)$ denotes the pair–frame with domain $V$. The *base* of this pair–frame, denoted by $\mathsf{Base}(V)$, is defined as $\{u \in U : (\exists v \in U)(\langle u, v \rangle \in V \text{ or } \langle v, u \rangle \in V\}$.
(iii) A structure $\mathcal{F} = \langle W, \mathsf{C}, \mathsf{F}, \mathsf{I} \rangle$ is called an *arrow–frame* if $W$ is a set, $\mathsf{C} \subseteq {}^3W$, $\mathsf{F} \subseteq {}^2W$ and $\mathsf{I} \subseteq W$.
(iv) $\mathsf{K}^{rel}_{set}$ denotes the class of all pair–frames, and $\mathsf{K}^{rel}$ the class of all arrow–frames.

FACT 2.4.2. (i) If $\mathcal{F}$ is a pair–frame, then $\mathcal{F}^+$ is a full relativized relation set algebra.
(ii) If $\mathfrak{A}$ is a full relativized relation set algebra, then $\mathfrak{A} = (\mathcal{F}_{pair}(1^{\mathfrak{A}}))^+$.
(iii) $\mathsf{BA}^{rel} = \mathsf{S}(\mathsf{K}^{rel})^+$.

PROOF. (i) and (ii) follow from the definitions. (iii) follows from fact 2.2.1. QED

PAIR–FRAMES VS ARROW–FRAMES. We make the difference between complex algebras of arrow–frames and complex algebras of pair–frames clear by the notation for the operators. We use the notation for abstract (equationally defined) operators for complex algebras of arrow–frames, and the usual set–theoretic notation for the operators of *concrete* (i.e., set–theoretically defined) complex algebras of pair–frames. Our notation is summarized in table 2.1. If $V$ is clear from context, we sometimes forget about it in $\circ^V, {}^{-1^V}$ and $\mathsf{Id}^V$. For the relations $\mathsf{F}_V$ and $\mathsf{F}$ which give meaning to converse, we sometimes use the (partial) functions $\mathsf{f}_V$ and $\mathsf{f}$, respectively (see section 2.4.3 below).

| pair–frames | | arrow–frames | |
|---|---|---|---|
| operator | relation | operator | relation |
| $\circ$, $\circ^V$ | $\mathsf{C}_V$ | ; | $\mathsf{C}$ |
| $^{-1}$, $^{-1^V}$ | $\mathsf{F}_V, \mathsf{f}_V$ | $\smile$ | $\mathsf{F}$, $\mathsf{f}$ |
| $\mathsf{Id}$, $\mathsf{Id}^V$ | $\mathsf{I}_V$ | $\mathsf{id}$ | $\mathsf{I}$ |

TABLE 2.1: RELATION ALGEBRAIC OPERATORS AND THEIR FRAME RELATIONS

The great advantage of pair–frames is that we can draw pictures which immediately explain the meaning of the terms. In Kripke frames, one usually draws the elements of the domain (the "worlds") as points, and indicates the (accessibility) relations by arrows. In pair–frames, the "worlds" are pairs $\langle u, v \rangle$ which are drawn as an arrow going from $u$ to $v$, and the accessibility relations need not be drawn, since they are implicit in the arrows. One can say that in pair–frames the accessibility relations are coded inside the worlds. To avoid confusion, we draw abstract arrow–frames as Kripke frames. In figure 2.1, we establish our convention for drawing accessibility relations. At the left are the pair–frames, and at the right the arrow–frames.

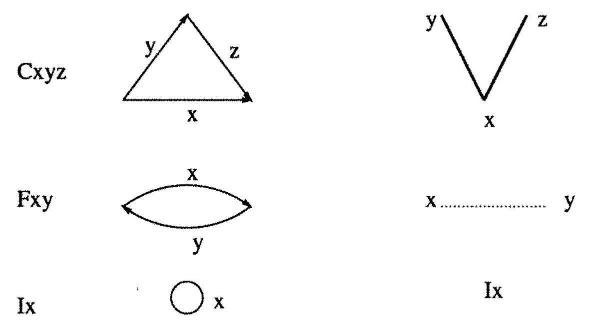



FIGURE 2.1: (ACCESSIBILITY) RELATIONS IN PAIR–FRAMES AND ARROW–FRAMES

### 2.4.2 A LANDSCAPE OF RELATION ALGEBRAS AND RELATIVIZED VERSIONS

In this section, we define a landscape of (relativized) relation algebras, and look at several properties of the classes which inhabit this landscape. We start with defining the classes.

DEFINITION 2.4.3. Let $R$ stand for *reflexive*, $S$ for *symmetric*, and $T$ for *transitive*. Let $H \subseteq \{R, S, T\}$. We use "$V$ is an $H$ relation", to abbreviate that $V$ has the properties mentioned in $H$. Define:

$$\mathsf{K}^{rel}_{setH} \stackrel{\text{def}}{=} \{\mathcal{F} \in \mathsf{K}^{rel}_{set} : F \text{ is an } H \text{ relation}\}$$
$$\mathsf{K}^{rel}_{setSQ} \stackrel{\text{def}}{=} \{\mathcal{F} \in \mathsf{K}^{rel}_{set} : F = U \times U \text{ for some set } U\}$$

$$\mathsf{SRlRRA} \stackrel{\text{def}}{=} \mathsf{S}(\mathsf{K}^{rel}_{set})^{+}$$
$$\mathbf{SRl}_H\mathsf{RRA} \stackrel{\text{def}}{=} \mathsf{S}(\mathsf{K}^{rel}_{setH})^{+}$$
$$\mathsf{RRA} \stackrel{\text{def}}{=} \mathsf{S}(\mathsf{K}^{rel}_{setRST})^{+}$$

We will refer to $\mathsf{K}^{rel}_{setSQ}$ as the *square* pair–frames , and to $\mathsf{K}^{rel}_{setRS}$ as the *locally square*[11] ones.

RELATIVIZATION. We explain our notation. In section 2.3, we introduced the operator **Rl** of relativization. In the definition above, we put constraints on the relativization in the subscript $_H$. For instance, $\mathbf{Rl}_R(\mathsf{K}^{rel}_{setSQ})^{+}$ means that we only relativize with *reflexive* relations. By writing out the definitions, one sees that $\mathsf{RRA} = \mathbf{SRl}_{RST}\mathsf{RRA} = \mathbf{SRl}_{RST}(\mathsf{K}^{rel}_{setSQ})^{+}$, and the names given to the relativized classes are correct. That is, we can start out with defining only the class RRA and obtain the others by relativizing and taking subalgebras. We gave a direct definition of the relativized classes to emphasize that the choice for a Cartesian product or an equivalence relation (as in the case of RRA) is just an option out of many, and that the relativized relation algebras

[11]The reason for this name is that $\mathsf{K}^{rel}_{setRS}$ equals the class $\{\mathcal{F} \in \mathsf{K}^{rel}_{set} : (\forall \langle u, v\rangle \in F)({}^2\{u, v\} \subseteq F)\}$.

have a natural definition on their own without referring to RRA. We have chosen for the "relativized names" to stay close to the literature.

### THE LANDSCAPE AND ITS PROPERTIES

The above defined classes of algebras can be ordered as given in figure 2.2 ($X \to Y$ denotes $X \supseteq Y$). It is not difficult to show that all the inclusions except the one labeled with $=$ are strict.

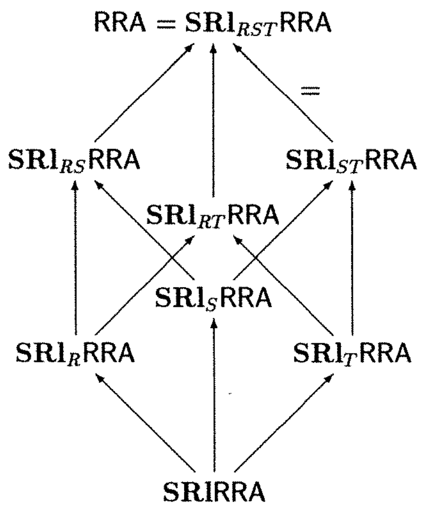



FIGURE 2.2: THE LANDSCAPE OF ALGEBRAS OF BINARY RELATIONS

One aim of this work is to study systematically all the classes in figure 2.2. For all of these classes, we will look at the following properties:

- whether they are varieties
- whether they are finitely axiomatizable
- whether their equational/universal theories are decidable
- whether they enjoy amalgamation, interpolation and definability properties

These notions will be made precise later on. Our intention here is to present an overview of what is known, and to indicate what we will add to this knowledge. We start with what is known about the pair–frames with a *transitive* domain.

THEOREM 2.4.4 (TARSKI). RRA *is a variety and* RRA $= \mathbf{SP}(\mathsf{K}^{rel}_{setSQ})^{+}$.

This theorem gives us the following corollary.

COROLLARY 2.4.5. *For all* $H \subseteq \{R, S, T\}$, *the class* $\mathbf{SRl}_H$RRA *is a quasi–variety.*

PROOF. Immediate by 2.3.1. QED

For $H \subset \{R, S, T\}$, the class $\mathbf{SRl}_H$RRA need not be a variety (it need not be closed under homomorphisms). Therefore, for any $H \subset \{R, S, T\}$ we will state whether the class $\mathbf{SRl}_H$RRA is a variety. Whenever $T \in H$, the answer is easy. Then we have a discriminator term (cf. [ANS94a]), and, by the universal algebraic fact that any quasi-variety with a discriminator term is a variety (cf. Németi [Ném91] Thm 9.2), we are done. When $T \in H$, this is the only property from the above list which is positive. Results (i) and (ii) in the next theorem are due to Andréka–Németi, the ones in (iii) and (iv) to Németi–Sain (cf. [ANS94a]).

THEOREM 2.4.6 (ANDRÉKA–NÉMETI–SAIN).
*Let* $\{T\} \subseteq H \subseteq \{R, S, T\}$. *Then* $\mathbf{SRl}_H\mathsf{RRA}$ *has the following negative properties:*

(i) *it is a variety, but not axiomatizable by finitely many equations,*
(ii) *its equational theory is undecidable,*
(iii) *amalgamation and interpolation fail, and*
(iv) *the Beth definability property of the corresponding (arrow) logic fails.*

Because of this theorem, we concentrate on the case with $T \notin H$. In table 2.2, we summarize the results of the above theorem, and we contrast them with the results we will find in the subsequent chapters for the cases when $T \notin H$. In the last column, we give the sections where we deal with these results.

| | $H \subseteq \{R, S\}$ $\mathbf{SRl}_H\mathsf{RRA}$ | $\{T\} \subseteq H \subseteq \{R, S, T\}$ $\mathbf{SRl}_H\mathsf{RRA}$ | section |
|---|---|---|---|
| • variety | yes | yes | 4.2 |
| • fin. axiomatizable by equations | yes | no | 4.2 |
| • decidable eq. theory | yes | no | 3.2 |
| • generated by its finite members | yes | no | 3.2 |
| • interpolation of inequalities | yes | no | 5.4 |

TABLE 2.2: RESULTS ABOUT THE LANDSCAPE BELOW RRA

We will prove the positive results in table 2.2 by working with arrow–frames. It will turn out that, for each class $\mathsf{K}^{rel}_{setH}$ with $H \subseteq \{R, S\}$, we can define –using finitely many canonical equations– a class of arrow–frames which has the same equational theory. These equations correspond to simple and intuitive FO conditions on arrow–frames. For this reason, we will hardly speak in the algebraic language about the classes of relation algebras. Instead we use the simpler FO language of arrow–frames. In the next subsection, we introduce the conditions we will work with, and get familiar with them. After that, we show how these conditions can be defined by means of equations.

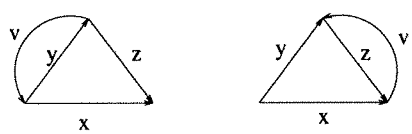
### 2.4.3 ARROW–FRAMES

CONDITIONS ON ARROW–FRAMES. Consider the conditions $(C_1) - (C_{15})$ on arrow–frames given below. Note that the first twelve conditions are all universal Horn sentences, and are *valid* on all pair–frames. Condition $(C_{13})$ is valid on *symmetric* pair–frames (i.e., pair–frames whose domain is a symmetric relation), and conditions $(C_{14})$ and $(C_{15})$ on *reflexive* ones.

| | | | |
|---|---|---|---|
| $(C_1)$ | $\forall xy(\mathsf{F}xy \Rightarrow \mathsf{F}yx)$ | $(C_{10})$ | $\forall xyz(\mathsf{F}xy \& \mathsf{F}xz \Rightarrow y = z)$ |
| $(C_2)$ | $\forall x(\mathsf{I}x \Rightarrow \mathsf{C}xxx)$ | $(C_{11})$ | $\forall xyz(\mathsf{C}xyz \& \mathsf{I}y \Rightarrow x = z)$ |
| $(C_3)$ | $\forall x(\mathsf{I}x \Rightarrow \mathsf{F}xx)$ | $(C_{12})$ | $\forall xyz(\mathsf{C}xyz \& \mathsf{I}z \Rightarrow x = y)$ |
| $(C_4)$ | $\forall xyzv(\mathsf{C}xyz \& \mathsf{F}yv \Rightarrow \mathsf{C}zvx)$ | $(C_{13})$ | $\forall x \exists y(\mathsf{F}xy)$ |
| $(C_5)$ | $\forall xyzv(\mathsf{C}xyz \& \mathsf{F}zv \Rightarrow \mathsf{C}yxv)$ | $(C_{14})$ | $\forall x \exists y(\mathsf{I}y \& \mathsf{C}xyx)$ |
| $(C_6)$ | $\forall xyz(\mathsf{C}xyz \; \& \; \mathsf{I}x \Rightarrow \mathsf{F}zy)$ | $(C_{15})$ | $\forall x \exists y(\mathsf{I}y \& \mathsf{C}xxy)$ |
| $(C_7)$ | $\forall xyzv(\mathsf{C}xyz \& \mathsf{C}xvx \& \mathsf{I}v \iff \mathsf{C}xyz \& \mathsf{C}yvy \& \mathsf{I}v)$ | | |
| $(C_8)$ | $\forall xyzv(\mathsf{C}xyz \& \mathsf{C}yyv \& \mathsf{I}v \iff \mathsf{C}xyz \& \mathsf{C}zvz \& \mathsf{I}v)$ | | |
| $(C_9)$ | $\forall xyzv(\mathsf{C}xyz \& \mathsf{C}zzv \& \mathsf{I}v \iff \mathsf{C}xyz \& \mathsf{C}xxv \& \mathsf{I}v)$ | | |

TABLE 2.3: CONDITIONS ON ARROW–FRAMES

The meaning of these conditions is most easily grasped by checking their validity using the proposed way of drawing pair–frames. We briefly go through the list. The meaning of $(C_1)$ and $(C_{10})$ is easy: every arrow has at most one converse and the converse relation is symmetric (i.e., if we look at the relation as a partial function $\mathsf{f}$, it says that if $\mathsf{f}x$ is defined, then so is $\mathsf{ff}x$ and it equals $x$. $(C_{13})$ says that every arrow has a converse. Conditions $(C_2)$ and $(C_3)$ state that an identity arrow is its own converse, and can be decomposed in itself. The meaning of $(C_4)$ and $(C_5)$ becomes clear by the following pictures:




Condition $(C_6)$ states that if an identity arrow is decomposed in $y$ and $z$, then $y$ and $z$ are converses. Conditions $(C_7)$ – $(C_9)$ express the fact that each arrow can have at most one identity arrow (its "domain") at its tail, and one at its head (its "range"). If an arrow is a pair $\langle u, v\rangle$, then its domain is $\langle u, u\rangle$, and its range is $\langle v, v\rangle$. So, if we can decompose $x$ into $y$ and $z$, then (if they are defined) the domain of $x$ equals the domain of $y$ $(C_7)$, the range of $y$ equals the domain of $z$ $(C_8)$, and the range of $z$ equals the range of $x$ $(C_9)$. Conditions $(C_{14})$ and $(C_{15})$ state that for every arrow, its domain and range are defined. The meaning of $(C_{11})$ and $(C_{12})$ is obvious.

PARTIAL FUNCTIONS FOR CONVERSE, DOMAIN AND RANGE.
By the conditions $(C_7), (C_9), (C_{10}) - (C_{12})$, we have three *partial* functions living in our frames. If one assumes $(C_{13})$ – $(C_{15})$ as well, they will be *total*. It is useful to make

them explicit, so define

$$\begin{array}{llll} \mathsf{f}x = y & \overset{\text{def}}{\Longleftrightarrow} & \mathsf{F}xy & \text{otherwise } \mathsf{f} \text{ is undefined} \\ x_l = y & \overset{\text{def}}{\Longleftrightarrow} & \mathsf{C}xyx \,\&\, \mathsf{I}y & \text{otherwise } (.)_l \text{ is undefined} \\ x_r = y & \overset{\text{def}}{\Longleftrightarrow} & \mathsf{C}xxy \,\&\, \mathsf{I}y & \text{otherwise } (.)_r \text{ is undefined} \end{array}$$

So, if they are defined, $\mathsf{f}x$ gives the converse arrow of $x$, and the functions $x_l$ and $x_r$ ($l$ for left and $r$ for right) give, what we called above, the domain and the range of $x$, respectively. In other words: the two functions give us the left and the right "end-point" of an arrow. It is convenient to have explicit symbols in our algebraic language corresponding to the two defined functions: define $\mathsf{s}_0^1\,\mathsf{x} \overset{\text{def}}{=} (\mathsf{id} \wedge \mathsf{x});1$ and $\mathsf{s}_1^0\,\mathsf{x} \overset{\text{def}}{=} 1;(\mathsf{id} \wedge \mathsf{x})$. Their meaning is given by the following three equations. This is easy to see by writing out the definitions.

$$\begin{array}{lll} \mathsf{s}_0^1\,\tau & = & \{x : x_l \in \tau\} \\ \mathsf{s}_1^0\,\tau & = & \{x : x_r \in \tau\} \\ \tau^{\smile} & = & \{x : \mathsf{f}x \in \tau\} \end{array}$$

The notation $\mathsf{s}_j^i$ comes from cylindric algebra theory, in which $\mathsf{s}_j^i$ is used as the *substitution* operator. Note that if the meaning[12] of $p$ is a binary relation, the meaning of $\mathsf{s}_1^0\,p$ is given by the set $\{\langle x, y\rangle : \langle y, y\rangle$ is in the meaning of $p\}$.

Note that, on reflexive and symmetric pair–frames, the three functions give us all permutations with repetitions of a pair $\langle u, v\rangle$ which are different from $\langle u, v\rangle$.

Unary operators whose accessibility relation is a total function have the nice property that they distribute over the Booleans (they are Boolean endomorphisms). If the relation is a partial function, then the operator distributes only over meet and join. For complementation of such an operator, only the weaker $-\Diamond\mathsf{x} \wedge \Diamond 1 = \Diamond{-\mathsf{x}}$ holds. Both sides of this equation say that a "world" has a "successor" and $\mathsf{x}$ is not true at that "successor".

Using the defined functions, we can formulate the following useful consequences of the frame conditions given above (see Prop.2.4.7 below)[13].

- $(T_0)$ $\mathsf{f}$, $(.)_l$ and $(.)_r$ are partial functions and $\mathsf{f}$ is idempotent
- $(T_1)$ $\mathsf{I}x \Rightarrow x = \mathsf{f}(x) = x_l = x_r$
  (in particular: if $\mathsf{I}x$ then all three functions are defined on $x$)
- $(T_2)$ $\mathsf{F}xy \Rightarrow x_l = y_r \,\&\, x_r = y_l$
- $(T_3)$ $\mathsf{C}xyz \Rightarrow x_l = y_l \,\&\, y_r = z_l \,\&\, z_r = x_r$
- $(T_4)$ $\mathsf{f}$, $(.)_l$ and $(.)_r$ are total

If $\mathsf{f}$ is a total function, we can write $(T_2)$ in a simpler way as $x_l = (\mathsf{f}x)_r$ and $x_r = (\mathsf{f}x)_l$. Since, by definition, $\mathsf{I}x_l$ and $\mathsf{I}x_r$, $(T_1)$ implies that $x_l = (x_l)_l = (x_l)_r$ and $x_r = (x_r)_r = (x_r)_l$.

[12] In FO logic $\mathsf{s}_1^0\,Pv_0v_1$ would be written as $\exists v_0(Pv_0v_1 \wedge v_0 = v_1)$ or equivalently $Pv_1v_1$. So the variable $v_1$ is substituted for $v_0$.

[13] If we use two partial functions $f$ and $g$, then $fx = gy$ means that either both $fx$ and $gy$ are undefined, or they are both defined and $fx = gy$. Idempotency of a partial function $f$ means that if $fx$ is defined, then $ffx = x$.

PROPOSITION 2.4.7. (i) $(C_1)-(C_{12}) \models (T_0)-(T_3)$
(ii) $(C_1)-(C_{15}) \models (T_0)-(T_4)$

PROOF. $(T_0)$: Functionality follows from $(C_7),(C_9),(C_{10})-(C_{12})$. Idempotency of f follows from $(C_1)$.
$(T_1)$: $\mathsf{I}x \overset{(C_2)\&(C_3)}{\Rightarrow} \mathsf{C}xxx \;\&\; \mathsf{F}xx \overset{\text{def}}{\Longleftrightarrow} x = \mathsf{f}x = x_l = x_r$.
$(T_3)$: This is just a short way of summarizing $(C_7)-(C_9)$.
$(T_2)$ : Suppose $\mathsf{F}xy$, and suppose $x_l$ is defined. Then $\mathsf{F}xy$ & $\mathsf{C}xx_lx$. Then, by $(C_5)$, we get $\mathsf{C}x_lxy$, hence, by $(T_3)$, $(x_l)_r = y_r$. Finally, by $(T_1)$, $(x_l)_r = x_l = y_r$. Now suppose $x_l$ is not defined: assume that $y_r$ is defined, and derive a contradiction in the same way as above using $(C_4)$. The other condition is proved in a similar way.
$(T_4)$: That these three functions are total, follows from $(C_{13})-(C_{15})$. QED

APPROXIMATIONS OF (REFLEXIVE OR SYMMETRIC) PAIR–FRAMES. Of course, conditions $(C_1)-(C_{15})$ are not arbitrarily chosen. They can be used to define classes of arrow–frames $\mathsf{K}^{rel}_{rlH}$ $(H \subseteq \{R,S\})$ which have the same equational theory as the classes of pair–frames $\mathsf{K}^{rel}_{setH}$. We prove this in chapter 4. These classes are given in the next definition. In figure 2.3, we present the inclusions between these classes of arrow–frames and pair–frames. All relations are strict. When an arrow is labeled by *eq*, this means that the two classes have the same equational theory (cf. theorem 4.2.1). In the diagram at the right, these relations are given for the varieties generated by these frame classes. (That these classes are actually varieties will be shown in the next subsection.)

DEFINITION 2.4.8.

$$\begin{array}{lcl}
\mathsf{K}^{rel}_{rl} & \overset{\text{def}}{=} & \{\mathcal{F} \in \mathsf{K}^{rel} : \mathcal{F} \models (C_1)-(C_{12})\} \\
\mathsf{K}^{rel}_{rlS} & \overset{\text{def}}{=} & \{\mathcal{F} \in \mathsf{K}^{rel}_{rl} : \mathcal{F} \models (C_{13})\} \\
\mathsf{K}^{rel}_{rlR} & \overset{\text{def}}{=} & \{\mathcal{F} \in \mathsf{K}^{rel}_{rl} : \mathcal{F} \models (C_{14}),(C_{15})\} \\
\mathsf{K}^{rel}_{rlRS} & \overset{\text{def}}{=} & \mathsf{K}^{rel}_{rlR} \cap \mathsf{K}^{rel}_{rlS}
\end{array}$$

## 2.4.4 ARROW CORRESPONDENCE

The next proposition shows how we can define the conditions $(C_1)$–$(C_{15})$ by canonical equations. It follows that the classes $\mathsf{S}(\mathsf{K}^{rel}_{rlH})^+$ $(H \subseteq \{R,S\})$ are finitely axiomatizable canonical varieties (cf. 2.4.10).

PROPOSITION 2.4.9. *For* $1 \le i \le 15$, *every equation* $(A_i)$ *given below is canonical, and it corresponds to the frame condition* $(C_i)$. *The correspondence of 2, and 7–9 hold*

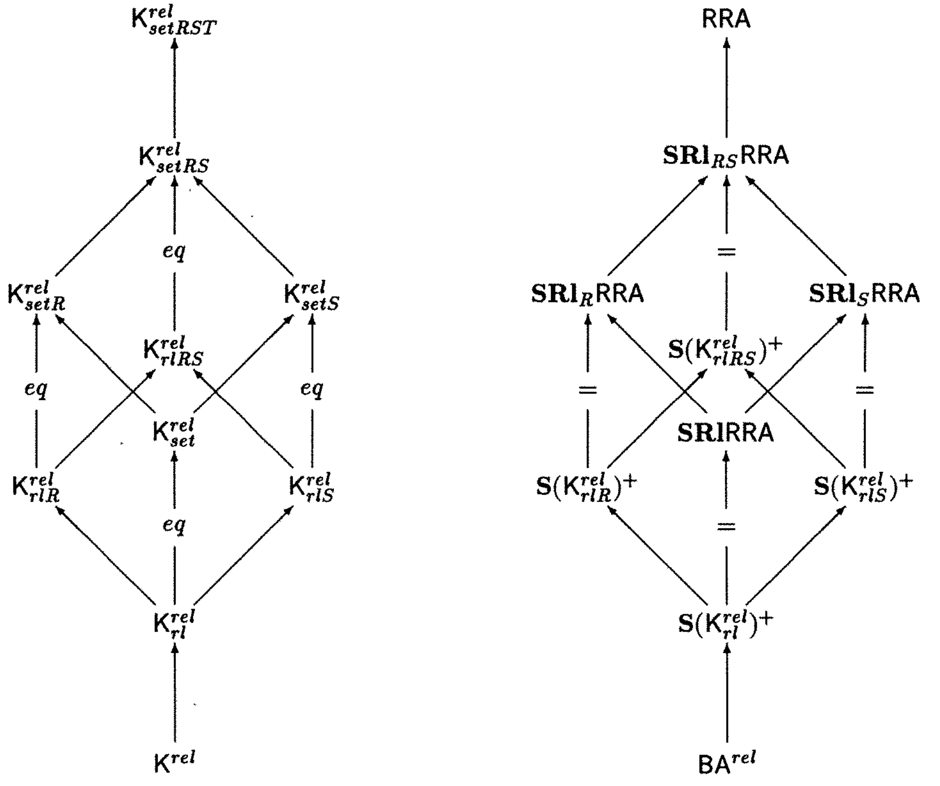



FIGURE 2.3: ARROW–FRAMES AND RELATION ALGEBRAS

*only when 11 and 12 are assumed.*

| | | | |
|---|---|---|---|
| $(C_1)$ | $\forall xy(\mathsf{F}xy \Rightarrow \mathsf{F}yx)$ | $(A_1)$ | $\mathsf{x} \wedge \mathsf{y}^{\smile} \leq (\mathsf{x}^{\smile} \wedge \mathsf{y})^{\smile}$ |
| $(C_2)$ | $\forall x(\mathsf{I}x \Rightarrow \mathsf{C}xxx)$ | $(A_2)$ | $\mathsf{id} = \mathsf{id} ; \mathsf{id}$ |
| $(C_3)$ | $\forall x(\mathsf{I}x \Rightarrow \mathsf{F}xx)$ | $(A_3)$ | $\mathsf{x} \wedge \mathsf{id} \leq \mathsf{x}^{\smile}$ |
| $(C_4)$ | $\forall xyzv(\mathsf{C}xyz \& \mathsf{F}yv \Rightarrow \mathsf{C}zvx)$ | $(A_4)$ | $\mathsf{x}^{\smile};-(\mathsf{x};\mathsf{y}) \leq -\mathsf{y}$ |
| $(C_5)$ | $\forall xyzv(\mathsf{C}xyz \& \mathsf{F}zv \Rightarrow \mathsf{C}yxv)$ | $(A_5)$ | $-(\mathsf{x};\mathsf{y});\mathsf{y}^{\smile} \leq -\mathsf{x}$ |
| $(C_6)$ | $\forall xyz(\mathsf{C}xyz \;\&\; \mathsf{I}x \Rightarrow \mathsf{F}zy)$ | $(A_6)$ | $\mathsf{x};-(\mathsf{x}^{\smile}) \leq -\,\mathsf{id}$ |
| $(C_7)$ | $\forall xyzv(\mathsf{C}xyz \& \mathsf{C}xvx \& \mathsf{I}v \iff \mathsf{C}xyz \& \mathsf{C}yvy \& \mathsf{I}v)$ | $(A_7)$ | $[(\mathsf{x} \wedge \mathsf{id});\mathsf{y}];\mathsf{z} = (\mathsf{x} \wedge \mathsf{id});[\mathsf{y};\mathsf{z}]$ |
| $(C_8)$ | $\forall xyzv(\mathsf{C}xyz \& \mathsf{C}yyv \& \mathsf{I}v \iff \mathsf{C}xyz \& \mathsf{C}zvz \& \mathsf{I}v)$ | $(A_8)$ | $[\mathsf{x};(\mathsf{y} \wedge \mathsf{id})];\mathsf{z} = \mathsf{x};[(\mathsf{y} \wedge \mathsf{id});\mathsf{z}]$ |
| $(C_9)$ | $\forall xyzv(\mathsf{C}xyz \& \mathsf{C}zzv \& \mathsf{I}v \iff \mathsf{C}xyz \& \mathsf{C}xxv \& \mathsf{I}v)$ | $(A_9)$ | $[\mathsf{x};\mathsf{y}];(\mathsf{z} \wedge \mathsf{id}) = \mathsf{x};[\mathsf{y};(\mathsf{z} \wedge \mathsf{id})]$ |
| $(C_{10})$ | $\forall xyz(\mathsf{F}xy \& \mathsf{F}xz \Rightarrow y = z)$ | $(A_{10})$ | $\mathsf{x}^{\smile} \wedge \mathsf{y}^{\smile} \leq (\mathsf{x} \wedge \mathsf{y})^{\smile}$ |
| $(C_{11})$ | $\forall xyz(\mathsf{C}xyz \& \mathsf{I}y \Rightarrow x = z)$ | $(A_{11})$ | $\mathsf{id} ;\mathsf{x} \leq \mathsf{x}$ |
| $(C_{12})$ | $\forall xyz(\mathsf{C}xyz \& \mathsf{I}z \Rightarrow x = y)$ | $(A_{12})$ | $\mathsf{x}; \mathsf{id} \leq \mathsf{x}$ |
| $(C_{13})$ | $\forall x \exists y(\mathsf{F}xy)$ | $(A_{13})$ | $1^{\smile} = 1$ |
| $(C_{14})$ | $\forall x \exists y(\mathsf{I}y \& \mathsf{C}xyx)$ | $(A_{14})$ | $\mathsf{x} \leq \mathsf{id} ;\mathsf{x}$ |
| $(C_{15})$ | $\forall x \exists y(\mathsf{I}y \& \mathsf{C}xxy)$ | $(A_{15})$ | $\mathsf{x} \leq \mathsf{x}; \mathsf{id}$ |

PROOF. Straightforward, since all equations are, or are equivalent to, *positive equations* (see the claim below).

CLAIM 1. All the axioms can be given using only Boolean meet, top, and the non Boolean operators (i.e., with *positive* equations). Axioms $(A_4)$ and $(A_5)$ are equivalent to $\mathsf{x} \wedge (\mathsf{y}^{\smile};\mathsf{z}) \leq \mathsf{y}^{\smile};(\mathsf{z} \wedge (\mathsf{y};\mathsf{x}))$ and $\mathsf{x} \wedge (\mathsf{y};\mathsf{z}^{\smile}) \leq (\mathsf{y} \wedge (\mathsf{x};\mathsf{z}));\mathsf{z}^{\smile}$, respectively. Axiom $(A_6)$ is equivalent to $\mathsf{id} \wedge (\mathsf{x};\mathsf{y}) \leq \mathsf{x};(\mathsf{y} \wedge \mathsf{x}^{\smile})$.

PROOF OF CLAIM. The equivalences are easy to show by using that $(A_4) - (A_6)$ are Sahlqvist equations and 2.2.4. ◀

As an example we show the correspondence of one side of $(C_7)$, which is a convenient reformulation of the "real" correspondent of $(A_7)$. This real correspondent is (2.3) below. The $(\Rightarrow)$ half of condition $(C_7)$ follows from (2.3) using condition $(C_{11})$.

$$\mathsf{C}xyz \;\&\; \mathsf{C}yvw \;\&\; \mathsf{I}v \Rightarrow (\exists u) : \mathsf{C}xvu \;\&\; \mathsf{C}uwz \tag{2.3}$$

Let $(A_7^{\leq})$ denote $[(\mathsf{x} \wedge \mathsf{id});\mathsf{y}];\mathsf{z} \leq (\mathsf{x} \wedge \mathsf{id});[\mathsf{y};\mathsf{z}]$.

CLAIM 2. $\mathcal{F} \models (A_7^{\leq}) \iff \mathcal{F} \models (2.3)$

PROOF OF CLAIM. Clearly, $(A_7^{\leq})$ holds on every frame satisfying (2.3). For the other side, suppose an arrow frame $\mathcal{F}$ validates $(A_7^{\leq})$. We have to show that $\mathcal{F}$ validates (2.3). Assume the antecedent of (2.3). Let $\mathsf{x}, \mathsf{y}, \mathsf{z}$ be variables, and define a valuation such that $\mathsf{v}(\mathsf{x}) = \{v\}, \mathsf{v}(\mathsf{y}) = \{w\}$ and $\mathsf{v}(\mathsf{z}) = \{z\}$. Then $\langle \mathcal{F}, \mathsf{v} \rangle, x \Vdash [(\mathsf{x} \wedge \mathsf{id});\mathsf{y}];\mathsf{z}$, which is the antecedent of $(A_7^{\leq})$. Because $\mathcal{F}$ validates $(A_7^{\leq})$, the point $x$ will also validate the consequent of this equation. But then, by the given valuation, there exists a $u$ such that the consequent of (2.3) holds. QED

Let $AX \subseteq \{(A_1) - (A_{15})\}$. Define $\mathsf{BA}^{rel}(AX) \stackrel{\text{def}}{=} \{\mathfrak{A} \in \mathsf{BA}^{rel} : \mathfrak{A} \models AX\}$.

THEOREM 2.4.10. *The four varieties given below are* canonical.

$$\begin{array}{lll} (i) & \mathsf{BA}^{rel}((A_1)-(A_{12})) & = \mathsf{S}(\mathsf{K}^{rel}_{rl})^+ \\ (ii) & \mathsf{BA}^{rel}((A_1)-(A_{13})) & = \mathsf{S}(\mathsf{K}^{rel}_{rlS})^+ \\ (iii) & \mathsf{BA}^{rel}((A_1)-(A_{12}),(A_{14}),(A_{15})) & = \mathsf{S}(\mathsf{K}^{rel}_{rlR})^+ \\ (iv) & \mathsf{BA}^{rel}((A_1)-(A_{15})) & = \mathsf{S}(\mathsf{K}^{rel}_{rlRS})^+ \end{array}$$

PROOF. Immediate by 2.4.9 and fact 2.2.2. QED

REMARK 2.4.11. Tarski proposed the following axioms in order to approximate the equational theory of the variety RRA. We list them, together with their FO correspondents on arrow–frames.

| | | |
|---|---|---|
| $(RA_0)$ | the $\mathsf{BA}^{rel}$ axioms[14] | |
| $(RA_1)$ | $(x;y);z = x;(y;z)$ | $\forall xyzuv((Cxyz \,\&\, Cyuv) \Rightarrow \exists w(Cxuw \,\&\, Cwvz))$ |
| | | $\forall xyzuv((Cxyz \,\&\, Czuv) \Rightarrow \exists w(Cxwv \,\&\, Cwyu))$ |
| $(RA_2)$ | $x;\mathsf{id} = x$ | $\forall x\exists z(\mathsf{I}z \,\&\, Cxxz)$ |
| | | $\forall xyz(Cxyz \,\&\, \mathsf{I}z \Rightarrow x = y)$ |
| $(RA_3)$ | $(x^{\smile})^{\smile} = x$ | $\forall x\exists! y(Fxy \,\&\, Fyx)$ |
| $(RA_4)$ | $(x;y)^{\smile} = y^{\smile};x^{\smile}$ | $\forall xyz(\exists w(Fxw \,\&\, Cwyz) \iff \exists uv(Fuy \,\&\, Fvz \,\&\, Cxvu))$ |
| $(RA_5)$ | $x^{\smile};-(x;y) \leq -y$ | $\forall xyzv((Cxyz \& Fyv) \Rightarrow Czvx)$ |

The variety $\{\mathfrak{A} \in \mathsf{BA}^{rel} : \mathfrak{A} \models (RA_0) - (RA_5)\}$ is called RA (the variety of *Relation Algebras*, cf. [HMT85] Def 5.3.1). The axioms $(RA_0)-(RA_5)$ are called the RA axioms.

All the RA axioms show up in a weakened form in our list $(A_1)-(A_{12})$. We deleted the existential import of the RA axioms by appropriate intersections. The associativity axiom returns as the three weakened forms of associativity $(A_7)-(A_9)$. The weakened form of axiom $(RA_2)$ is $x;\mathsf{id} = x \wedge (1;\mathsf{id})$, which is equivalent to axiom $(A_{12})$. Assuming $(A_{10})$, axiom $(A_1)$ is equivalent to the more appealing $(x^{\smile})^{\smile} = x \wedge 1^{\smile}$, which is a weakened form of $(RA_3)$. To delete the existential import from $(RA_4)$, one has to rewrite it as $((x \wedge 1^{\smile});(y \wedge 1^{\smile}))^{\smile} = (x^{\smile};y^{\smile}) \wedge 1^{\smile}$. By reasoning in frames, it is easily seen that this weakening follows from $(A_4)$ and $(A_5)$. $(RA_5)$ finally, is the only RA axiom without existential import; this is our $(A_4)$.

## 2.4.5 ARROW LOGIC

Arrow logic is the general modal logic –in the sense of section 2.1.3– of the type $rel = \{;,^{\smile},\mathsf{id}\}$ of relation algebras. In the literature, one can find many notations for these three operators. We will use the following: the composition operator is given by a binary infix operator "$\bullet$", converse is denoted by a unary prefix operator "$\otimes$" and for the identity constant we use "id" as before. We use an underline to denote the duals of the operators, so $\underline{\otimes}\phi \overset{\text{def}}{\iff} \neg\otimes\neg\phi$ and $\phi\underline{\bullet}\psi \overset{\text{def}}{\iff} \neg(\neg\phi \bullet \neg\psi)$. So, arrow logic is a propositional logic with countably many variables $\{p_i : i < \omega\}$, enriched with a binary modality "$\bullet$", an unary "$\otimes$" and a constant "id". Models are provided by *arrow-frames*

[14] I.e., a BA axiomatization plus equations which state that ; and $^{\smile}$ are normal operators.

$\mathcal{F} = \langle W, \mathsf{C}, \mathsf{F}, \mathsf{I}\rangle$ plus a valuation v. The modalities are interpreted on these models as follows:

$$\begin{array}{lll} \mathrm{M}, x \Vdash \mathsf{id} & \stackrel{\text{def}}{\Longleftrightarrow} & \mathsf{I}x \\ \mathrm{M}, x \Vdash \otimes\phi & \stackrel{\text{def}}{\Longleftrightarrow} & (\exists y \in W) : \mathsf{F}xy \ \& \ \mathrm{M}, y \Vdash \phi \\ \mathrm{M}, x \Vdash \phi \bullet \psi & \stackrel{\text{def}}{\Longleftrightarrow} & (\exists y, z \in W) : \mathsf{C}xyz \ \& \ \mathrm{M}, y \Vdash \phi \ \& \ \mathrm{M}, z \Vdash \psi \end{array}$$

In this way, an arrow logic $AL(\mathsf{K}) \stackrel{\text{def}}{=} \langle \mathsf{Fml}_{rel}(\{p_i : i < \omega\}), \mathsf{Mod}(\mathsf{K}), \Vdash_{rel}\rangle$ is completely determined by its specific class of arrow–frames. Yde Venema wrote a nice overview article about arrow logic ([Ven94]).

ALGEBRAIZATION OF ARROW LOGIC. If we algebraize an arrow logic of a class of frames $\mathsf{K}$ in the sense of algebraic logic (cf. e.g., Andréka et al. [ANSK94]), we get the class of algebras $\mathbf{SP}\mathsf{K}^+$. Statements about meta–logical properties of the general modal logic $\mathcal{GML}(\mathsf{K})$ translate into equivalent algebraic statements about the class $\mathbf{SP}\mathsf{K}^+$. (Cf. [ANSK94] for such equivalence theorems about completeness, compactness, decidability and interpolation.) In the coming three chapters, we will investigate algebraic counterparts of arrow logic of pair–frames. The algebraic counterparts of the arrow logics of the classes of pair–frames $\mathsf{K}^{rel}_{setH}$ are the classes of relativized relation algebras $\mathbf{SRl}_H\mathsf{RRA}$. In chapter 6, we transform the obtained results back into meta–logical statements about arrow logic.

## 2.5 CYLINDRIC ALGEBRAS, CYLINDRIC MODAL LOGIC AND ALPHA FRAMES

In this section, we introduce algebras and frames of the *cylindric* similarity type. The section is set up analogously to the previous one about the *relational* similarity type. In subsection 2.5.4, we compare the two types of algebras.

The standard work on cylindric algebras is the monograph Henkin–Monk–Tarski Parts I and II ([HMT71], [HMT85]). A book with a lot of information about *relativized* cylindric algebras is Henkin et al. [HMT⁺81]. A very extended survey, including the most recent developments, of this field is Németi [Ném91]. In our terminology and notation we follow [HMT71].

### 2.5.1 CYLINDRIC ALGEBRAS AND ALPHA FRAMES

Let $\alpha$ be any ordinal. Define $\mathsf{Bo}_\alpha$ as the class of all BAO's with $\alpha$ many unary operators "$\mathsf{c}_i$", one for each $i < \alpha$, and constants "$\mathsf{d}_{ij}$", one for each $i, j < \alpha$. The elements "$\mathsf{d}_{ij}$" are called *diagonals*, and the operators "$\mathsf{c}_i$" *cylindrifications*. We use a superscript $^{cyl\alpha}$ to denote this specific similarity type[15]. These algebras are the appropriate abstract version of the *cylindric set algebras* defined below. Let $V \subseteq {}^\alpha U$, for some set $U$; define

[15] The name $\mathsf{Bo}_\alpha$ comes from [HMT71], in which $\mathsf{Bo}_\alpha$ is called the class of all *Boolean algebras with operators of dimension* $\alpha$. Using our convention, we should have called this class $\mathsf{BA}^{cyl\alpha}$.

a set of binary relations $\equiv_i^V \subseteq V \times V$, and a set of unary relations $D_{ij}^V \subseteq V$ for each $i, j < \alpha$ as follows

$$\begin{array}{lcl} D_{ij}^V & \stackrel{\text{def}}{=} & \{s \in V : s_i = s_j\} \\ \equiv_i^V & \stackrel{\text{def}}{=} & \{\langle s, r\rangle \in V \times V : \text{ for all } j \neq i,\ s_j = r_j\} \end{array}$$

We call an algebra $\mathfrak{A} = \langle \mathfrak{P}(V), \mathsf{C}_i^V, \mathsf{D}_{ij}^V\rangle$ a *full cylindric relativized set algebra of dimension* $\alpha$, if $\mathfrak{P}(V)$ is the Boolean powerset algebra with domain $\mathcal{P}(V)$, and the operators are defined as follows:

$$\begin{array}{lcl} \mathsf{C}_i^V(\mathsf{x}) & \stackrel{\text{def}}{=} & \{s \in V : (\exists r) : s \equiv_i^V r \,\&\, r \in \mathsf{x}\} \\ \mathsf{D}_{ij}^V & \stackrel{\text{def}}{=} & \{s \in V : D_{ij}^V s\} \end{array}$$

When $V$ is clear from the context, we will suppress the superscript $V$.

The relations $\equiv_i$ and $D_{ij}$ give rise to the notion of *assignment–frames of dimension* $\alpha$. We call their abstract counterparts $\alpha$*–frames.* Venema [Ven93] calls assignment–frames in which the domain consists of a *full Cartesian product* $V = {}^\alpha U$, *cubes.* There is an obvious analogy with the *square* pair–frames.

DEFINITION 2.5.1. (i) A structure $\mathcal{F} = \langle V, \equiv_i^V, D_{ij}^V\rangle_{i,j<\alpha}$ is called an *assignment–frame of dimension* $\alpha$ if $V \subseteq {}^\alpha U$ for some set $U$, and the relations are defined as above.
(ii) If $V \subseteq {}^\alpha U$, then $\mathcal{F}_{as}(V)$ denotes the assignment–frame with domain $V$.
(iii) A structure $\mathcal{F} = \langle W, T^i, E^{ij}\rangle_{i,j<\alpha}$ is called an $\alpha$*–frame* if $W$ is a set, $T^i \subseteq W \times W$ and $E^{ij} \subseteq W$.
(iv) $\mathsf{K}_{set}^{cyl\alpha}$ denotes the class of all assignment–frames of dimension $\alpha$, and $\mathsf{K}^{cyl\alpha}$ the class of all $\alpha$–frames.

Clearly, if $\mathcal{F}$ is an assignment–frame, then $\mathcal{F}^+$ is a full cylindric relativized set algebra. If $\mathcal{F} = \langle W, T^i, E^{ij}\rangle_{i,j<\alpha}$ is an $\alpha$–frame, then $\mathcal{F}^+ = \langle \mathfrak{P}(W), \mathsf{c}_i, \mathsf{d}_{ij}\rangle_{i,j<\alpha}$ denotes the complex algebra of $\mathcal{F}$. The meaning of the $\mathsf{c}_i$'s and $\mathsf{d}_{ij}$'s are computed as above, but now using the abstract relations $T^i$ and $E^{ij}$. Fact 2.2.1 implies that $\mathsf{Bo}_\alpha = \mathsf{S}(\mathsf{K}^{cyl\alpha})^+$.

SUBSTITUTIONS. In order to define more restricted classes of assignment–frames, we define the notion of *substitutions* of sequences. For $s$ an $\alpha$ sequence, let $\mathbf{f}^{\,i}_{\,z}(s)$ denote that sequence $r$ which equals $z$ on its $i$–th coordinate, and agrees with $s$ on all other coordinates. We call $\mathbf{f}^{\,i}_{\,s_j}(s)$ the *substitution* of the $j$–th coordinate of $s$ for the $i$–th coordinate. Substitution functions are definable on assignment–frames of dimension $\alpha$ in the following way. For $i, j < \alpha$, define the (partial) *substitution* function $\mathsf{f}_j^i$ on $V \subseteq {}^\alpha U$ as follows. For $s, r \in V$, if $i \neq j$, then $\mathsf{f}_j^i s = r$ iff $(s \equiv_i r \,\&\, D_{ij} r)$. If $i = j$, then $\mathsf{f}_j^i(s) = s$. The function $\mathsf{f}_j^i$ corresponds to the operation $\mathsf{S}_j^i$ which is defined as $\mathsf{S}_j^i \mathsf{x} \stackrel{\text{def}}{=} \mathsf{C}_i(\mathsf{x} \wedge \mathsf{D}_{ij})$ if $i \neq j$, and $\mathsf{S}_j^i \mathsf{x} = \mathsf{x}$ if $i = j$. By these definitions, $\mathsf{S}_j^i \mathsf{x} = \{s \in V : \mathsf{f}_j^i s \in \mathsf{x}\}$.

NOTATION. Again, we make the difference between "abstract" and "concrete" operations clear in the notation. Table 2.4 summarizes it. The "abstract" versions of the operator $\mathsf{S}_j^i$ and the function $f_j^i$ are defined in the next subsection.

| assignment–frames | | α–frames | |
|---|---|---|---|
| operator | relation | operator | relation |
| $\mathsf{C}_i$ | $\equiv_i$ | $\mathsf{c}_i$ | $T^i$ |
| $\mathsf{D}_{ij}$ | $D_{ij}$ | $\mathsf{d}_{ij}$ | $E^{ij}$ |
| $\mathsf{S}^i_j$ | $\mathsf{f}^i_j$ | $\mathsf{s}^i_j$ | $f^i_j$ |

TABLE 2.4: CYLINDRIC ALGEBRAIC OPERATORS AND THEIR FRAME RELATIONS

### 2.5.2 CYLINDRIC RELATIVIZED SET ALGEBRAS

We define three special classes of assignment–frames by imposing one or more of the existential requirements mentioned in section 1.2.

DEFINITION 2.5.2.

$$\begin{array}{lcl} \mathsf{K}^{cyl\alpha}_{setD} & \stackrel{\text{def}}{=} & \{\mathcal{F} \in \mathsf{K}^{cyl\alpha}_{set} : (\forall s \in F)(\forall i,j < \alpha)(\mathsf{f}^j_i s \in F)\} \\ \mathsf{K}^{cyl\alpha}_{setG} & \stackrel{\text{def}}{=} & \{\mathcal{F} \in \mathsf{K}^{cyl\alpha}_{set} : (\forall s \in F)({}^{\alpha}\{s_i : i < \alpha\} \subseteq F)\} \\ \mathsf{K}^{cyl\alpha}_{cube} & \stackrel{\text{def}}{=} & \{\mathcal{F} \in \mathsf{K}^{cyl\alpha}_{set} : F = {}^{\alpha}U \text{ for some set } U\} \end{array}$$

The first class is the cylindric algebraic analogue of the *reflexive* pair–frames, the second of the *locally square* , and the third of the *square* pair–frames. In the next definition, we relate the classes of assignment–frames to classes of cylindric (relativized) set algebras which are known from the literature[16]. Note that, by definition, these classes are closed under isomorphisms (since $\mathbf{SK}^+ = \mathbf{ISK}^+$). Our notation differs from the convention introduced in [HMT85], because there the class $\mathsf{Crs}_\alpha$ is *not* closed under isomorphisms (i.e., $\mathsf{Crs}_\alpha$ is defined in [HMT85] as the class $\mathbf{S}\{\mathcal{F}^+ : \mathcal{F} \in \mathsf{K}^{cyl\alpha}_{set}\}$).

DEFINITION 2.5.3 (CYLINDRIC (RELATIVIZED) SET ALGEBRAS). Let $\alpha$ be any ordinal.

$$\begin{array}{lcllcl} \mathsf{Crs}_\alpha & \stackrel{\text{def}}{=} & \mathbf{S}(\mathsf{K}^{cyl\alpha}_{set})^+, & \mathsf{D}_\alpha & \stackrel{\text{def}}{=} & \mathbf{S}(\mathsf{K}^{cyl\alpha}_{setD})^+ \\ \mathsf{G}_\alpha & \stackrel{\text{def}}{=} & \mathbf{S}(\mathsf{K}^{cyl\alpha}_{setG})^+, & \mathsf{RCA}_\alpha & \stackrel{\text{def}}{=} & \mathbf{SP}(\mathsf{K}^{cyl\alpha}_{cube})^+ \end{array}$$

These four classes are related as follows: $\mathsf{RCA}_\alpha \subseteq \mathsf{G}_\alpha \subseteq \mathsf{D}_\alpha \subseteq \mathsf{Crs}_\alpha$. When $\alpha \leq 1$, all four classes become equal. When $\alpha > 1$, the inclusions are strict. This can be seen by the following equations:

$$\begin{array}{llllll} \mathsf{D}_2 & \models & \mathsf{c}_0\,\mathsf{d}_{01} = 1 & \text{but} & \mathsf{Crs}_2 & \not\models \; \mathsf{c}_0\,\mathsf{d}_{01} = 1 \\ \mathsf{G}_2 & \models & -\mathsf{d}_{01} \leq \mathsf{c}_0\,\mathsf{c}_1 - \mathsf{d}_{01} & \text{but} & \mathsf{D}_2 & \not\models \; -\mathsf{d}_{01} \leq \mathsf{c}_0\,\mathsf{c}_1 - \mathsf{d}_{01} \\ \mathsf{RCA}_2 & \models & \mathsf{c}_0\,\mathsf{c}_1\,\mathsf{x} = \mathsf{c}_1\,\mathsf{c}_0\,\mathsf{x} & \text{but} & \mathsf{G}_2 & \not\models \; \mathsf{c}_0\,\mathsf{c}_1\,\mathsf{x} = \mathsf{c}_1\,\mathsf{c}_0\,\mathsf{x} \end{array}$$

SURVEY OF PROPERTIES. In table 2.5, we present a survey of basic meta–mathematical properties of these four classes of algebras. The symbol "I" in the table stands for $2 < \alpha < \omega$, and the symbol "II" for $\alpha = \omega$. The sources of the theorems which are

16 $\mathsf{Crs}_\alpha$ (for Cylindric Relativized Set Algebras) and $\mathsf{RCA}_\alpha$ (for Representable Cylindric Algebras) are defined in [HMT85]. The class $\mathsf{D}_\alpha$ is defined in [AT88], and the class $\mathsf{G}_\alpha$ in [Ném92].

summarized in this table are provided in notes immediately below the table. When $0 \leq \alpha \leq 2$, all the properties except interpolation are positive for all four classes[17]. Interpolation holds for all classes when $\alpha \leq 1$, and for $\mathsf{Crs}_2$ and $\mathsf{D}_2$ (see 5.4.6). For $\mathsf{G}_2$, we don't know the answer, and the interpolation property fails for $\mathsf{RCA}_2$ (a result due to S. Comer [Com69]).

D. Resek and R. Thompson ([RT91]) provided axiomatizations of the varieties $\mathsf{Crs}_\alpha$ and $\mathsf{D}_\alpha$, for $\alpha$ any ordinal. H. Andréka ([AT88], [Mon91]) provided simpler axiom systems and proofs for these two theorems. The axiomatizations are recalled in the next section, because we need them later on. R. Thompson claims to have a finite axiomatization of the class $\mathsf{G}_\alpha$, but he did not reveal the axioms. Due to a result of D. Monk ([Mon69]), the variety $\mathsf{RCA}_\alpha$ is not finitely axiomatizable if $\alpha > 2$. Németi showed that for $\alpha > 2$, the variety $\mathsf{Crs}_\alpha$ is not finitely axiomatizable, not even by finitely many equation schemes (cf. [HMT85]). For related and strengthened results, see the remarks immediately below Thm 4 in [Ném91]. The decidability issues for these classes were settled by Tarski and Németi.

The trend in the table is similar to what we have seen with relation algebras, be it that the contrast is not so striking. When $\alpha > 2$, all properties fail for the class $\mathsf{RCA}_\alpha$ (the class of subalgebras of algebras whose domain is a disjoint union of full Cartesian products). As we have seen with relation algebras, properties tend to turn positive, once we abandon the requirement of full Cartesian products.

| | $\mathsf{Crs}_\alpha$ | | $\mathsf{D}_\alpha$ | | $\mathsf{G}_\alpha$ | | $\mathsf{RCA}_\alpha$ | |
|---|---|---|---|---|---|---|---|---|
| | I | II | I | II | I | II | I | II |
| • variety | *yes*[1] | *yes*[1] | *yes*[2] | *yes*[2] | *yes*[3] | *yes*[3] | *yes*[4] | *yes*[4] |
| • fin. (schema) axiomatizable | *no*[1] | *no*[1] | *yes*[2] | *yes*[2] | *yes*[3] | *yes*[3] | *no*[5] | *no*[5] |
| • decidable eq. theory | *yes*[6] | *yes*[6] | *yes*[6] | ? | *yes*[6] | *yes*[6] | *no*[7] | *no*[7] |
| • generated by its fin. members | ? | ? | *yes*[8] | ? | ? | ? | *no*[7] | *no*[7] |
| • interpolation of inequalities | *yes*[9] | *yes*[9] | *yes*[9] | *yes*[9] | ? | ? | *no*[10] | *no*[10] |

"I" stands for $2 < \alpha < \omega$ and "II" for $\alpha = \omega$.

1 Németi [HMT85] Thm 5.5.10, 5.5.12, 5.5.13
2 Resek–Thompson, Andréka [Mon91], [AT88]
3 Thompson unpublished
4 Henkin–Tarski [HMT85] Thm 3.1.103
5 Monk [HMT85] Thm 4.1.3
6 Németi [Ném92] Thm 10
7 Tarski [HMT85] Thm 4.2.18
8 Thm 3.3.1 in here
9 Thm 5.4.6 in here
10 Comer, Pigozzi [Pig72] Tab 2.4.1, 2.4.2

TABLE 2.5: PROPERTIES OF CYLINDRIC (RELATIVIZED) SET ALGEBRAS

[17] These results can be found in or derived from [HMT71] and [HMT85].

### 2.5.3 CORRESPONDENCE AND COMPLETENESS

We now recall the axiomatizations of $\mathsf{Crs}_\alpha$ and $\mathsf{D}_\alpha$. The axioms of $\mathsf{Crs}_\alpha$ and a soundness and completeness proof can be found in Monk [Mon91], but we note that, due to a misprint, axiom $(\mathbf{A}_{6\leq})$ given below is missing from his list. For a simpler reformulation and an easier soundness proof of the last axiom, we refer to van Lambalgen–Simon [LS94]. The $\mathsf{D}_\alpha$ axioms and a nice completeness proof can be found in Andréka–Thompson [AT88]. We provide a slight reformulation of the $\mathsf{D}_\alpha$ axioms, which makes the connection with the non–relativized axioms a bit clearer, and which will be helpful in the next chapter. In order to compare the axioms with the well investigated cylindric algebra axioms $\mathsf{CA}$, we list them here as well, together with their corresponding conditions on $\alpha$–frames (cf. [HMT71] Thm 2.7.40). The side–conditions in the definition apply to both the equation and the frame condition.

DEFINITION 2.5.4 ($\mathsf{CA}$ AXIOMS).

| | | | | |
|---|---|---|---|---|
| $(\mathbf{A}_1)$ | $\mathsf{x} \leq \mathsf{c}_i \mathsf{x}$ | $(\mathbf{C}_1)$ | $\forall x T^i xx$ | |
| $(\mathbf{A}_2)$ | $\mathsf{c}_i \mathsf{c}_i \mathsf{x} \leq \mathsf{c}_i \mathsf{x}$ | $(\mathbf{C}_2)$ | $\forall xyz(T^i xy \,\&\, T^i yz \Rightarrow T^i xz)$ | |
| $(\mathbf{A}_3)$ | $\mathsf{x} \leq -\mathsf{c}_i - \mathsf{c}_i \mathsf{x}$ | $(\mathbf{C}_3)$ | $\forall xy(T^i xy \Rightarrow T^i yx)$ | |
| $(\mathbf{A}_4)$ | $\mathsf{c}_i \mathsf{c}_j \mathsf{x} = \mathsf{c}_j \mathsf{c}_i \mathsf{x}$ | $(\mathbf{C}_4)$ | $\forall xyz(T^i xy \,\&\, T^j yz \Rightarrow \exists w(T^j xw \,\&\, T^i wz))$ | |
| $(\mathbf{A}_5)$ | $\mathsf{d}_{ii} = 1$ | $(\mathbf{C}_5)$ | $\forall x E^{ii} x$ | |
| $(\mathbf{A}_6)$ | $\mathsf{c}_k(\mathsf{d}_{ik} \wedge \mathsf{d}_{kj}) = \mathsf{d}_{ij}$ | $(\mathbf{C}_6)$ | $\forall x(E^{ij}x \Rightarrow \exists y(T^k xy \,\&\, E^{ik}y \,\&\, E^{kj}y))$ | if $k \notin \{i,j\}$ |
| | | | $\forall xy(T^k xy \,\&\, E^{ik}y \,\&\, E^{kj}y \Rightarrow E^{ij}x)$ | if $k \notin \{i,j\}$ |
| $(\mathbf{A}_7)$ | $\mathsf{d}_{ij} \wedge \mathsf{c}_i(\mathsf{d}_{ij} \wedge \mathsf{x}) \leq \mathsf{x}$ | $(\mathbf{C}_7)$ | $\forall xy(E^{ij}x \,\&\, T^i xy \,\&\, E^{ij}y \Rightarrow x = y)$ | if $i \neq j$ |

Define the class $\mathsf{CA}_\alpha$, of *cylindric algebras of dimension* $\alpha$, as the subclass of $\mathsf{Bo}_\alpha$ which validates all instances with indices smaller than $\alpha$ of the axioms $(\mathbf{A}_1) - (\mathbf{A}_7)$.

Note that all axioms, except number 3, are *positive equations*, and that axiom 3 is equivalent to the positive equation $\mathsf{x} \wedge \mathsf{c}_i \mathsf{y} \leq \mathsf{c}_i(\mathsf{c}_i \mathsf{x} \wedge \mathsf{y})$.

#### CORRESPONDENCE RESULTS

The correspondence results in this paragraph are all stated without proof. The latter are trivial, because all equations are positive.

SUBSTITUTIONS. Define the abstract substitution relation $f^i_j$ as follows: if $i = j$, then $f^i_j x = x$, and if $i \neq j$, then $f^i_j x = y \overset{\text{def}}{\Longleftrightarrow} T^i xy \,\&\, E^{ij} y$.

PROPOSITION 2.5.5. *The axioms* $(\mathbf{A}_2), (\mathbf{A}_3), (\mathbf{A}_5), (\mathbf{A}_{6\geq})$ *and* $(\mathbf{A}_7)$ *are sufficient to make* $f^i_j$ *a total function. Without* $(\mathbf{A}_5)$ *and* $(\mathbf{A}_{6\geq})$, *it is a* partial *function.*

PROOF. Straightforward. QED

We also define the abstract substitution operator $\mathsf{s}^i_j$, in a similar way as we did above for the concrete one: if $i = j$, then $\mathsf{s}^i_j \mathsf{x} \overset{\text{def}}{=} \mathsf{x}$, and if $i \neq j$, then $\mathsf{s}^i_j \mathsf{x} \overset{\text{def}}{=} \mathsf{c}_i(\mathsf{x} \wedge \mathsf{d}_{ij})$. Using this operator, we can rewrite axioms 6 and 7 to the ones given below, making it clear that these axioms are about the relation between the substitutions and the diagonals.

$$(\mathbf{A}_{6'})\quad \mathsf{d}_{ij} = \mathsf{s}_i^k\,\mathsf{d}_{kj} \quad \text{if } k \neq j$$
$$(\mathbf{A}_{7'})\quad \mathsf{d}_{ij} \wedge \mathsf{s}_j^i \mathsf{x} \leq \mathsf{x}$$

AXIOMS FOR $\mathsf{D}_\alpha$. Consider the following two equations and FO frame conditions. The frame conditions correspond to these equations whenever the substitution functions are total.

| | | | |
|---|---|---|---|
| $(\mathbf{A}_{4+})$ | $\mathsf{s}_k^i \mathsf{s}_k^j \mathsf{x} = \mathsf{s}_k^j \mathsf{s}_k^i \mathsf{x}$ | $(\mathbf{C}_{4+})$ | $\forall x(f_k^j f_k^i x = f_k^i f_k^j x)$ |
| $(\mathbf{A}_{MGR})$ | $\mathsf{s}_i^k \mathsf{s}_j^i \mathsf{s}_m^j \mathsf{s}_k^m\, \mathsf{c}_k\, \mathsf{x} = \mathsf{s}_m^k \mathsf{s}_i^m \mathsf{s}_j^i \mathsf{s}_k^j\, \mathsf{c}_k\, \mathsf{x}$ if $k \notin \{i, j, m\}, m \notin \{i, j\}$ | $(\mathbf{C}_{MGR})$ | $\forall xy(T^k f_k^m f_m^j f_j^i f_i^k(x) y \iff T^k f_k^j f_j^i f_i^m f_m^k(x) y)$ |

Axiom 4+ is the weakened version of axiom 4. This becomes clear, when we write the definitions out, and get $\mathsf{c}_i(\mathsf{c}_j(\mathsf{x} \wedge \mathsf{d}_{jk}) \wedge \mathsf{d}_{ik}) = \mathsf{c}_j(\mathsf{c}_i\, \mathsf{x} \wedge \mathsf{d}_{ik}) \wedge \mathsf{d}_{jk})$. The second axiom is called the *merry go round equation.* The axioms $1-3, 4+, 5-7$ and $(\mathbf{A}_{MGR})$ are sufficient to axiomatize $\mathsf{D}_\alpha$ (see 2.5.7 below).

AXIOMS FOR $\mathsf{Crs}_\alpha$. The axioms which are sufficient to axiomatize the variety $\mathsf{Crs}_\alpha$ $(3 \leq \alpha \leq \omega)$ are far more complicated. We mentioned above that no finite axiomatization is possible. The infinite axiomatization consists of all CA axioms *without existential import*, plus the ones given immediately below[18] (cf. [Mon91]). We need some notation in order to formulate the infinite set of axioms (8) conveniently. We use $[i/j]$ for the function in ${}^\alpha\alpha$ which sends $i$ to $j$ and fixes all other elements.

| | | | |
|---|---|---|---|
| $(\mathbf{A}_{5a})$ | $\mathsf{d}_{ij} = \mathsf{d}_{ji}$ | $(\mathbf{C}_{5a})$ | $\forall x E^{ij} x \iff E^{ji} x$ |
| $(\mathbf{A}_{5b})$ | $\mathsf{d}_{ij} \wedge \mathsf{d}_{jk} \leq \mathsf{d}_{ik}$ | $(\mathbf{C}_{5b})$ | $\forall x(E^{ij} x \,\&\, E^{jk} x \Rightarrow E^{ik} x)$ |
| $(\mathbf{A}_{6\leq})$ | $\mathsf{c}_k(\mathsf{d}_{ik} \wedge \mathsf{d}_{kj}) \leq \mathsf{d}_{ij}$ | $(\mathbf{C}_{6\leq})$ | $\forall xy(T^k xy \,\&\, E^{ik} y \,\&\, E^{kj} y \Rightarrow E^{ij} x)$ if $k \notin \{i, j\}$ |

$(\mathbf{A}_8)$ $\quad \mathsf{s}_{j_n}^{i_n}\, \mathsf{c}_{k_n} \ldots \mathsf{s}_{j_1}^{i_1}\, \mathsf{c}_{k_1}\, \mathsf{x} \wedge \prod_{l \in K} \mathsf{d}_{l\nu(l)} \leq \mathsf{c}_i\, \mathsf{x}$
subject to condition $(*)$ below

Axiom $(\mathbf{A}_8)$ corresponds on $\alpha$–frames to $(\mathbf{C}_8)$ below[19], and is subject to the same condition $(*)$.

$$(\mathbf{C}_8)\ \forall x_0 \ldots x_n(\prod_{l \in K} E^{l\nu(l)} x_n \,\&\, T^{k_n} f_{j_n}^{i_n} x_n x_{n-1} \& \ldots \& T^{k_1} f_{j_1}^{i_1} x_1 x_0 \Rightarrow T^i x_n x_0)$$

$(*)$ where $K = \{i_1, \ldots, i_n, k_1, \ldots, k_n\} \backslash \{i\}$, $\nu = [i_n/j_n] \circ \ldots \circ [i_i/j_1]$ and $k_{m+1} \notin ([i_m/j_m] \circ \ldots \circ [i_1/j_1])^*(K)$ for all $m < n$.

[18] Note that $(\mathbf{A}_{5b})$ follows from $(\mathbf{A}_1), (\mathbf{A}_5)$ and $(\mathbf{A}_{6\leq})$. We leave it to be consistent with [Mon91].

[19] The condition should be read as follows: if all functions are defined and the antecedent holds then the consequent holds. Note that the functions do not contain any existential import in this way. The condition could be written equivalently as a universal Horn sentence without the substitution functions as follows: delete all occurrences of $f_j^i$'s for $i = j$, and, if $i \neq j$, rewrite $T^k f_j^i x, y$ as $T^i x, z \& E^{ij} z \& T^k z, y$ using new variables.

COMPLETENESS RESULTS

DEFINITION 2.5.6. Let $\alpha$ be any ordinal greater than 0.
(i) The class $\mathsf{K}_{rlD}^{cyl\alpha}$ is defined as the class of $\alpha$–frames which satisfy all instances with indices smaller than $\alpha$ of conditions $(\mathbf{C}_1) - (\mathbf{C}_3), (\mathbf{C}_{4+}), (\mathbf{C}_5) - (\mathbf{C}_7)$ plus $(\mathbf{C}_{MGR})$.
(ii) The class $\mathsf{K}_{rl}^{cyl\alpha}$ is that class of $\alpha$–frames which satisfies all instances with indices smaller than $\alpha$ of the conditions $(\mathbf{C}_1) - (\mathbf{C}_3), (\mathbf{C}_5), (\mathbf{C}_{5a}), (\mathbf{C}_{5b}), (\mathbf{C}_{6\leq}), (\mathbf{C}_7)$ and the set of axioms $(\mathbf{C}_8)$.

Let $AX$ be a set of equations. $\mathsf{Bo}_\alpha(AX)$ denotes the class $\{\mathfrak{A} \in \mathsf{Bo}_\alpha : \mathfrak{A} \models AX\}$.

THEOREM 2.5.7 (RESEK–THOMPSON). *Let $\alpha$ be any ordinal greater than 1. Then* $\mathsf{Crs}_\alpha$ *and* $\mathsf{D}_\alpha$ *are* canonical varieties. *In particular:*

$$\text{(i)}\quad \mathsf{Bo}_\alpha((\mathbf{A}_1) - (\mathbf{A}_3), (\mathbf{A}_5), (\mathbf{A}_{5a}), (\mathbf{A}_{5b}), (\mathbf{A}_{6\leq}), (\mathbf{A}_7), (\mathbf{A}_8)) = \mathsf{S}(\mathsf{K}_{rl}^{cyl\alpha})^+ = \mathsf{Crs}_\alpha$$
$$\text{(ii)}\quad \mathsf{Bo}_\alpha((\mathbf{A}_1) - (\mathbf{A}_3), (\mathbf{A}_{4+}), (\mathbf{A}_5) - (\mathbf{A}_7), (\mathbf{A}_{MGR})) = \mathsf{S}(\mathsf{K}_{rlD}^{cyl\alpha})^+ = \mathsf{D}_\alpha$$

PROOF. (i). Cf. [Mon91] Thm 9.4.
(ii). The axioms $1-3, 4*, 5-7$ and $MGR$ are sufficient to axiomatize $\mathsf{D}_\alpha$ (cf. [AT88] Thm 1).

$$(\mathbf{A}_{4*})\quad \mathsf{d}_{jk} \wedge \mathsf{c}_j\, \mathsf{c}_i\, \mathsf{x} \leq \mathsf{c}_i\, \mathsf{c}_j\, \mathsf{x} \quad \text{if } k \notin \{i, j\}$$

So the only difference between the two axiom systems is that 4∗ is replaced by 4+. But, assuming axioms 1–3 and 5–7, these two axioms are equivalent (cf. Thompson, [Tho90], Prop 1). QED

REMARK 2.5.8. Whenever $\alpha$ equals 0 or 1, the cases are much simpler. In case $\alpha = 0$, we have $\mathsf{Crs}_\alpha = \mathsf{D}_\alpha = \mathsf{RCA}_\alpha = \mathsf{BA}$. When $\alpha$ equals 1, $\mathsf{Crs}_\alpha = \mathsf{D}_\alpha = \mathsf{RCA}_\alpha = \mathsf{Bo}_\alpha((\mathbf{A}_1) - (\mathbf{A}_3), (\mathbf{A}_5))$. The class $\mathsf{Crs}_2$ can be axiomatized by the equations $(\mathbf{A}_1) - (\mathbf{A}_3), (\mathbf{A}_5), (\mathbf{A}_{5a}), (\mathbf{A}_7)$ (cf. [HMT85] Thm 5.5.5).

### 2.5.4 CYLINDRIC ALGEBRAS AND RELATION ALGEBRAS

We briefly compare the two types of algebras. Both are algebras of relations. Relation algebras are algebras of binary relations, and cylindric algebras of dimension $\alpha$ are algebras of $\alpha$–ary relations. Cylindric algebras of dimension $\alpha$ are the algebraic counterpart of FO logic with $\alpha$ many variables. The expressive power of RRA equals that of FO logic with 3 variables and only binary predicate symbols. (For a purely algebraic formulation of this result, see [HMT85] section 5.3.) We will compare the pair–frames and the assignment–frames of dimension 2. Recall the "path"–condition from section 1.2.

PROPOSITION 2.5.9. *Let $V$ be a binary relation. Then $V$ is transitive and symmetric iff $V$ is an equivalence relation iff $V$ satisfies the path–principles.*

PROOF. Straightforward. QED

The next proposition shows why the cylindrifications $\mathsf{C}_0$ and $\mathsf{C}_1$ are not taken as primitive operators in relation algebras.

PROPOSITION 2.5.10. (i) *Let $V$ be a reflexive and symmetric binary relation. Then* $\mathcal{F}_{pair}(V) \models \mathsf{C}_0^V \mathsf{x} = 1 \circ^V (\mathsf{Id}^V \wedge (1 \circ^V \mathsf{x}))$ *and* $\mathcal{F}_{pair}(V) \models \mathsf{C}_1^V \mathsf{x} = (\mathsf{Id}^V \wedge (\mathsf{x} \circ^V 1)) \circ^V 1$.
(ii) *Let $V$ be a binary relation satisfying the path-principles. Then* $\mathcal{F}_{pair}(V) \models \mathsf{C}_0^V \mathsf{x} = 1 \circ^V \mathsf{x}$ *and* $\mathcal{F}_{pair}(V) \models \mathsf{C}_1^V \mathsf{x} = \mathsf{x} \circ^V 1$.

PROOF. (i). Cf. 6.3.1. (ii). Immediate by the previous proposition. QED

## 2.5.5 CYLINDRIC MODAL LOGIC

An excellent exposition of *cylindric modal logic* (CML) can be found in Venema [Ven93] and [Ven91]. Here, we briefly sketch the basic idea of this system. The aim of CML is to study and devise a propositional modal formalism which is as expressive as first order logic. This can be done by restricting the syntax of FO logic in such a way that it behaves like a (multi)-modal propositional logic. We briefly describe this restriction. For a discussion of such a restricted syntax versus the usual syntax of FO logic (as well as their equivalence when we have infinitely many variables) we refer to [Ném91].

Suppose we have a language of FO logic with the constraint that there are $\alpha$ many variables (where $\alpha$ is a fixed but arbitrary ordinal), and that the only admissible atomic formulae are of the form $v_i = v_j$ $(i, j < \alpha)$ or $R(v_0 v_1 \ldots v_i \ldots)_{i<\alpha}$. Then the equalities $v_i = v_j$ can be seen as *constants* $\delta_{ij}$, and in writing the atomic relations we might as well leave out the variables since, due to their fixed order, they do not contain any information. But then, we are in a multi-modal propositional logic enriched with constants. As was explained in section 1.2, we can look at the quantifiers $\exists v_i$ as if they were modal operators $\Diamond_i$. So we can define the language of $\mathrm{CML}_\alpha$ as a multi-modal propositional language with a set of constants $\{\delta_{ij} : i, j < \alpha\}$ and $\alpha$ many modal operators $\Diamond_i$. The meaning of the formulas is naturally given in terms of *$\alpha$-frames*. Here are the key clauses. Let $\mathcal{F}$ be an $\alpha$-frame $\langle W, T^i, E^{ij} \rangle_{i,j<\alpha}$, $\mathrm{M} = \langle \mathcal{F}, \mathsf{v} \rangle$ a model and $w \in W$:

$$\begin{array}{lcl} \mathrm{M}, w \Vdash \delta_{ij} & \stackrel{\text{def}}{\Longleftrightarrow} & E^{ij} w \\ \mathrm{M}, w \Vdash \Diamond_i \phi & \stackrel{\text{def}}{\Longleftrightarrow} & (\exists v \in W) : T^i wv \;\&\; \mathrm{M}, v \Vdash \phi \end{array}$$

Note that if we use *$\alpha$-cubes* instead, we get the classical FO interpretation of the modalized FO formulas.

Using the equivalence theorems mentioned in section 2.4.5, and the fact that the algebraic counterpart of a cylindric modal logic of a class $\mathsf{K}$ of $\alpha$-frames is the class of algebras $\mathbf{SPK}^+$, all results in table 2.5 can be transformed into results about cylindric modal logic.

# 3

# DECIDABILITY

In this chapter, we focus on decidability for theories of relativized relation algebras $\mathbf{SRl}_H\mathsf{RRA}$ for $H \subseteq \{R, S, T\}$ (see 2.4.3) and of the variety of cylindric relativized set algebras $\mathsf{D}_\alpha$ (see 2.5.3). Our main results are that the universal theory of the class $\mathbf{SRl}_H\mathsf{RRA}$ is decidable if and only if $T \notin H$, and that for finite $\alpha$, the universal theory of $\mathsf{D}_\alpha$ is decidable.

This chapter is organized as follows. In the first section, we introduce the method of *filtration* which we will use for obtaining decidability results. This method is quite powerful: using it, we can show that the *universal theory* of a variety is decidable. In the second section, we apply this method to relativized relation algebras and in the third, to the variety $\mathsf{D}_\alpha$.

## 3.1 FILTRATIONS

We will use the well-known and widely applied filtration technique from modal logic (cf. Hughes–Creswell [HC84]), to show that, in favorable circumstances, the universal theory of a class of algebras is decidable. The idea of the filtration method can be described as follows. Given a class $\mathsf{K}$ of frames, a frame $\mathcal{F} \in \mathsf{K}$, a universal formula $\phi$ and a model $\mathrm{M} = \langle \mathcal{F}, \mathsf{v}\rangle$ which falsifies $\phi$, we use the set of subterms of terms occurring in equations in $\phi$ to create a finite model $\langle \mathcal{F}^*, \mathsf{v}^*\rangle$ such that $\mathcal{F}^*$ also belongs to $\mathsf{K}$, and $\phi$ still fails. The finite model is called the "filtration".

We will now make this idea precise. In the remainder of this section, an arbitrary BAO–type $S$ and a language $\mathsf{Eqlang}_S(X)$, for an arbitrary infinite set of variables $X$ are fixed. Terms are supposed to be $S$–terms constructed from variables in $X$.

DEFINITION 3.1.1 (FILTRATION). Let $\mathrm{M} = \langle \mathcal{F}, \mathsf{v}\rangle$ be a model of type $S$ and $\Sigma$ a set of $S$–terms which is closed[1] under taking subterms and under the Boolean operations. Define an equivalence relation $\equiv_\Sigma \subseteq F \times F$ as follows:

$$(\forall w, v \in F) : w \equiv_\Sigma v \overset{\text{def}}{\iff} (\forall \tau \in \Sigma)\, [\mathrm{M}, w \Vdash \tau \iff \mathrm{M}, v \Vdash \tau]$$

Let $\overline{w}$ denote the equivalence class $w/{\equiv_\Sigma}$, and identify equivalence classes $\overline{w}$ with the sets of terms $\{\tau \in \Sigma : \mathrm{M}, w \Vdash \tau\}$.
We call a model $\mathrm{M}^* = \langle \mathcal{F}^*, \mathsf{v}^*\rangle$ a *filtration* of M *through* $\Sigma$ if:

(i) $\mathcal{F}^*$ is of type $S$.
(ii) $F^* \overset{\text{def}}{=} \{\overline{w} : w \in F\}$

[1]We say that a set $X$ is closed under an $n$–ary operation $f$ if, whenever $\tau_1, \ldots, \tau_n \in X$, we have $f(\tau_1, \ldots, \tau_n) \in X$.

(iii) $\mathsf{v}^*(\mathsf{x}) \stackrel{\text{def}}{=} \{\overline{w} \in F^* : \mathsf{x} \in \overline{w}\}$, for all *variables* $\mathsf{x} \in \Sigma$.
(iv) **min** and **max** , given below, hold for every operator in $S$

The relations in the filtration are denoted by a superscript $^*$, like in $R^{\diamond *}$. For an $n$–ary operator $\diamond$ and $R^{\diamond} \subseteq {}^{n+1}F$, the relation in $\mathcal{F}$ corresponding to this operator, **min** and **max** are defined as follows:

**min** $R^{\diamond}x_0 \ldots x_n \Rightarrow R^{\diamond *}\overline{x_0} \ldots \overline{x_n}$
**max** $(\forall \diamond(\tau_1, \ldots, \tau_n) \in \Sigma) : ((R^{\diamond *}\overline{x}\,\overline{y_1} \ldots \overline{y_n} \,\&\, \tau_1 \in \overline{y_1} \,\&\ldots\&\, \tau_n \in y_n) \Rightarrow \diamond(\tau_1, \ldots, \tau_n) \in \overline{x})$

As their names indicate, **min** provides a lower–bound, and **max** an upper–bound for $R^{\diamond *}$. They are designed to make the following lemma true.

LEMMA 3.1.2 (TRUTH–LEMMA). *Let* $\mathrm{M}^* = \langle \mathcal{F}^*, \mathsf{v}^* \rangle$ *be a filtration of* $\mathrm{M} = \langle \mathcal{F}, \mathsf{v} \rangle$ *through* $\Sigma$. *Then* (i)–(iii) *below hold.*

(i) $(\forall \tau \in \Sigma)(\forall \overline{x} \in F^*) : \tau \in \overline{x} \iff \mathrm{M}^*, \overline{x} \Vdash \tau$
(ii) $(\forall \tau \in \Sigma)(\forall x \in F) : \mathrm{M}, x \Vdash \tau \iff \mathrm{M}^*, \overline{x} \Vdash \tau$
(iii) *Let* $\phi$ *be a Boolean combination of equations between terms in* $\Sigma$.
*Then* $\mathrm{M} \models \phi \iff \mathrm{M}^* \models \phi$.

PROOF. (i). A straightforward induction on the complexity of the terms. We show how **min** and **max** take care of the operators. Let $\diamond$ be a unary operator. Let $\diamond\tau \in \Sigma$. We compute:
($\Rightarrow$) $\diamond\tau \in \overline{x} \stackrel{\text{def}}{\iff} \mathrm{M}, x \Vdash \diamond\tau \stackrel{\text{def}}{\iff} (\exists y \in F) : R^{\diamond}xy \,\&\, \mathrm{M}, y \Vdash \tau \Rightarrow$ (by **min** and ind. hyp.) $(\exists \overline{y} \in F^*) : R^{\diamond *}\overline{x}\,\overline{y} \,\&\, \mathrm{M}^*, \overline{y} \Vdash \tau \stackrel{\text{def}}{\iff} \mathrm{M}^*, \overline{x} \Vdash \diamond\tau$.
($\Leftarrow$) $\mathrm{M}^*, \overline{x} \Vdash \diamond\tau \stackrel{\text{def}}{\iff} (\exists \overline{y} \in F^*) : R^{\diamond *}\overline{x}\,\overline{y} \,\&\, \mathrm{M}^*, \overline{y} \Vdash \tau \Rightarrow$ (by **max** and ind. hyp.) $\diamond\tau \in \overline{x}$.
(ii). Immediate by (i).
(iii). By induction, using (ii). QED

Note, that neither a filtration needs to be finite, nor that $\mathcal{F} \in \mathsf{K} \Rightarrow \mathcal{F}^* \in \mathsf{K}$ holds. It follows from the next proposition that for every model and for every set $\Sigma$, filtrations always exist. Call a filtration *minimal* for $R^{\diamond *}$ if it is defined minimally, that is

$$R^{\diamond *}\overline{x_0} \ldots \overline{x_n} \stackrel{\text{def}}{\iff} (\exists x'_0 \ldots x'_n) : \overline{x_0} = \overline{x'_0} \,\&\ldots\&\, \overline{x_n} = \overline{x'_n} \,\&\, R^{\diamond}x'_0 \ldots x'_n$$

and call it *maximal* for $R^{\diamond *}$ if it is defined maximally, as in

$$R^{\diamond *}\overline{x}\,\overline{y_1} \ldots \overline{y_n} \stackrel{\text{def}}{\iff} (\forall \diamond(\tau_1, \ldots, \tau_n) \in \Sigma) : [(\tau_1 \in \overline{y_1} \& \ldots \& \tau_n \in \overline{y_n}) \Rightarrow \diamond(\tau_1, \ldots, \tau_n) \in \overline{x}]$$

A filtration is called *minimal* if it is minimal for all the relations, and similarly for *"maximal"*.

PROPOSITION 3.1.3. *If* $R^{\diamond *}$ *is defined minimally or maximally, then it satisfies* **min** *and* **max** .

PROOF. This is a straightforward generalization of the corresponding statement in [HC84]. QED

FILTRATIONS LEAD TO DECIDABILITY

The following notion leads in favourable cases to strong decidability results.

DEFINITION 3.1.4 (ADMITS FILTRATIONS). Let $\mathsf{K}$ be a class of frames of type S. $\mathsf{K}$ *admits filtrations* if for any finite set of $S$–terms $X$, any $\mathcal{F} \in \mathsf{K}$ and any model $\mathrm{M} = \langle \mathcal{F}, \mathsf{v} \rangle$, there exists a set of $S$–terms $\Sigma \supseteq X$ and a filtration $\langle \mathcal{F}^*, \mathsf{v}^* \rangle$ of M through $\Sigma$ such that $\mathcal{F}^*$ *is finite and it belongs to* $\mathsf{K}$.

We call the set $\Sigma \supseteq X$ above, the *closure set* of $X$. It is, by definition 3.1.1, closed under taking subterms and under the Boolean operations. For $\mathsf{K}$ a class of frames and $\Sigma$ a set of terms, we call $\Sigma$ *finite modulo* $\mathsf{K}$ if there exists a finite subset $\Delta$ of $\Sigma$ such that $(\forall \tau \in \Sigma)(\exists \tau' \in \Delta) : \mathsf{K} \models \tau = \tau'$. For any class $\mathsf{K}$ of frames, the closure under the Boolean operations of every finite set of terms is finite modulo $\mathsf{K}$. If $\Sigma$ is finite modulo $\mathsf{K}$, then any filtration of a $\mathsf{K}$ frame through $\Sigma$ is finite.

Let $\mathsf{K}$ be a class of algebras or frames. With $\mathbf{Fin}\mathsf{K}$ we denote the class of all finite members of $\mathsf{K}$. Recall that a universal formula is a Boolean combination of equations, and that the universal theory of a class $\mathsf{K}$ of algebras is denoted by $\mathsf{Univ}(\mathsf{K})$.

LEMMA 3.1.5 (FILTRATION LEMMA). *Let* $\mathsf{K}$ *be a class of frames.*
(i) *If* $\mathsf{K}$ *admits filtrations, then* $\mathsf{Univ}(\mathsf{K}^+) = \mathsf{Univ}(\mathbf{Fin}\mathsf{K}^+)$.
(ii) *If* $\mathsf{K}$ *is basic elementary (i.e., definable by a single FO sentence) and admits filtrations, then the universal theory of* $\mathsf{K}^+$ *is* decidable.

PROOF. (i). Assume that $\mathsf{K}$ admits filtrations. Clearly, $\mathsf{Univ}(\mathsf{K}^+) \subseteq \mathsf{Univ}(\mathbf{Fin}\mathsf{K}^+)$. For the other side, suppose $\phi \notin \mathsf{Univ}(\mathsf{K}^+)$. Then there exists a frame $\mathcal{F} \in \mathsf{K}$ and a model $\mathrm{M} = \langle \mathcal{F}, \mathsf{v} \rangle$ such that $\mathrm{M} \not\models \phi$. Let $X$ be the set of subterms of all terms occurring in equations in $\phi$. Then $X$ is finite. Since $\mathsf{K}$ admits filtrations, there exists a frame $\mathcal{F}^* \in \mathbf{Fin}\mathsf{K}$ and a model $\mathrm{M}^* = \langle \mathcal{F}^*, \mathsf{v}^* \rangle$, which is a filtration of M through some set $\Sigma \supseteq X$. But then, 3.1.2.(iii) implies that $\mathrm{M}^* \not\models \phi$, whence $\phi \notin \mathsf{Univ}(\mathbf{Fin}\mathsf{K}^+)$.
(ii). Assume the antecedent. By an obvious change in the proof of Fact 1.4 in van Benthem [Ben84], we can recursively enumerate the set $\mathsf{Univ}(\mathsf{K}^+)$. Since $\mathsf{K}$ is definable by a single FO sentence, we can also recursively enumerate all the finite frames of $\mathsf{K}$. By part (i) of this lemma, $\mathsf{Univ}(\mathsf{K}^+) = \mathsf{Univ}(\mathbf{Fin}\mathsf{K}^+)$. So we have a recursive enumeration of the complement of $\mathsf{Univ}(\mathsf{K}^+)$ as well. Hence, $\mathsf{Univ}(\mathsf{K}^+)$ is decidable. QED

COROLLARY 3.1.6. *Let* $\mathsf{K}$ *be a basic elementary class of frames which admits filtrations. Then* (i)–(iii) *below hold.*

(i) $\mathsf{Univ}(\mathsf{SK}^+)$ *is decidable.*
(ii) $\mathsf{Univ}(\mathsf{SK}^+) = \mathsf{Univ}(\mathbf{Fin}\mathsf{SK}^+)$.
(iii) *The variety generated by* $\mathsf{K}^+$ *is generated by its finite members.*

PROOF. By the Filtration Lemma and preservation of universal formulas under S. QED

USING FILTRATIONS. Suppose we want to use filtrations for proving decidability of Univ(SK$^+$). The main difficulty in such a proof is to ensure that the filtration is both finite and belongs to the class K. For this last property, we have to ensure that the filtration satisfies the conditions which define that class. This problem will usually lead us to extend the closure set in such a way that we can still *control* these conditions in the filtration. So the "art" in filtration proofs is to make the closure set large enough to control the conditions, but at the same time small enough in order to end up with only a *finite* number of equivalence classes.

In general, *existential conditions* of the form $\forall\vec{x}(\phi \rightarrow \exists\vec{y}\psi)$, in which $\phi$ and $\psi$ are constructed using atoms, conjunction and disjunction (and maybe negation), are very difficult or impossible to control in filtrations. Ensuring the *existence* of elements satisfying the consequent of such a condition usually leads to a closure set which is too large. Positive FO formulas[2] are always preserved, because the filtration map is a homomorphism (by the **min** condition) (cf. [CK90] Thm 3.2.4). With universal (Horn) conditions we may have some hope that a filtration will work.

## LOCAL FINITENESS

The next lemma is a simple but important tool in filtration proofs. It provides us with a *semantic way* of showing that we can close a set of terms under some operations without losing finiteness. We recall the definition of locally finite (classes of) algebras from [BS81].

DEFINITION 3.1.7. (i) An algebra is *locally finite* if all its finitely generated[3] subalgebras are finite.
(ii) A class K of algebras is *locally finite* if every member of K is locally finite.
(iii) A frame is *locally finite* if all its finitely generated subframes are finite.
(iv) A class K of frames is *locally finite* if every member of K is locally finite.

LEMMA 3.1.8. (i) *Let* K *be a class of frames. If* K *is locally finite and* **Gsp**K *is finite, then the variety generated by* K$^+$ *is locally finite.*
(ii) *If a variety* V *is locally finite, then the closure of any finite set of terms under all operations of* V *is finite modulo* V.

PROOF. (i). Assume the antecedent. By duality theory, Eq(K) = Eq(**Gsp**K), hence **HSP**(K)$^+$ = **HSP**(**Gsp**K)$^+$. By the assumption, (**Gsp**K)$^+$ is a finite set of finite algebras. Then, by Thm II.10.16 in [BS81], **HSP**(**Gsp**K)$^+$ is locally finite.
(ii). This holds, because the finitely generated free algebras in V are finite. QED

So, if a class K of algebras is locally finite, then a filtration becomes trivial and will always work. More interesting classes of algebras are not locally finite, but they might have reducts which are. That means that we can close a finite set under the operations

[2] A formula is said to be *positive* iff it is built up from atomic formulas using only the connectives $\wedge$, $\vee$ and the quantifiers $\forall$, $\exists$.

[3] $\mathfrak{A}$ is a finitely generated subalgebra of $\mathfrak{B}$ iff $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$, and there exists a finite set $X \subseteq B$ such that $A = \bigcap\{Y : X \subseteq Y$, and $Y$ is a subuniverse of $\mathfrak{A}\}$.

of that reduct without losing finiteness. In the filtration proofs for $\mathbf{SRI}_H\mathsf{RRA}$ (for $H \subseteq \{R, S\}$) and $\mathsf{D}_\alpha$ this feature will be a crucial part.

## DIGRESSION: EXPANSIONS WITH THE UNIVERSAL MODALITY

In this small section, we show what happens to a class $\mathsf{K}$ of frames when we expand the language of $\mathsf{K}$ with the so called *universal modality.* This section is an aside and is not needed to understand the rest of this chapter. Let $\mathsf{K}$ be any class of frames of BAO–type $S$. Define

$$\mathsf{K}^\Box \stackrel{\text{def}}{=} \{\mathcal{F} = \langle W, R^i, U\rangle_{i\in S} : \langle W, R^i\rangle_{i\in S} \in \mathsf{K} \text{ and } U = W \times W\}$$

So, $\mathsf{K}^\Box$ is of BAO–type $S$ expanded with a unary operator $\Diamond$. Set $\Box\mathsf{x} \stackrel{\text{def}}{=} -\Diamond{-}\mathsf{x}$, and define $\Diamond$ in the standard way, given below, using the relation $U$. Let $\mathcal{F} = \langle W, R^i, U\rangle_{i\in S} \in \mathsf{K}^\Box$. For $\mathcal{F}^+ = \langle \mathfrak{P}(W), f^i, \Diamond\rangle_{f^i\in S}$ and $\mathsf{x} \subseteq W$ we define:

$$\Diamond\mathsf{x} \stackrel{\text{def}}{=} \{w \in W : (\exists x \in W)\ (Uwx \,\&\, x \in \mathsf{x})\}$$

The operator $\Diamond$ is called the *universal modality*, because it has the following behaviour: $\Diamond\mathsf{x} = 0$, if $\mathsf{x} = 0$, and $\Diamond\mathsf{x} = 1$ otherwise. It is easy to see that a class of BAO's has such an operator iff it has a *discriminator term.* Hence the variety generated by $(\mathsf{K}^\Box)^+$ is a *discriminator variety*, and we can use all the powerful techniques which are available for them (for definitions and applications, see e.g., [ANS94a]).

THEOREM 3.1.9. $\mathsf{K}$ *admits filtrations if and only if* $\mathsf{K}^\Box$ *admits filtrations.*

PROOF. From right to left is obvious. For the other side, suppose that $\mathsf{K}$ admits filtrations. Let $\mathcal{F}^\Box = \langle W, R^i, U\rangle_{i\in S} \in \mathsf{K}^\Box$, and let $X^\Box$ be a finite set of terms in the expanded language which is closed under taking subterms. Let $\mathrm{M}^\Box = \langle \mathcal{F}^\Box, \mathsf{v}\rangle$ be a model. We have to find a finite filtration of $\mathrm{M}^\Box$ through some set $\Sigma \supseteq X^\Box$, which belongs to the class $\mathsf{K}^\Box$.

We will use that, for every term $\tau$, $\mathrm{M}^\Box \models (\Diamond\tau = 1 \textit{ or } \Diamond\tau = 0)$. Create the set $X^{01}$ from the set $X^\Box$ by replacing every occurrence of a subterm of the form $\Diamond\tau$ in a term with 0 or 1. E.g., if $f(\Diamond\tau, \Diamond\tau_1) \in X^\Box$, then $\{f(1,1), f(1,0), f(0,1), f(0,0)\} \subset X^{01}$. Clearly, this is a finite set in the old language. Hence, by assumption, we can find a set $\Sigma \supseteq X^{01}$, a frame $\mathcal{F}^* = \langle W^*, R^{i*}\rangle_{i\in S} \in \mathsf{K}$ and a finite model $\mathrm{M}^* = \langle \mathcal{F}^*, \mathsf{v}^*\rangle$ which is a filtration of the $U$–free part of the model $\mathrm{M}^\Box$ through $\Sigma$.

Let $U^* = W^* \times W^*$. Define $\mathcal{F}^{\Box *}$ as $\langle W^*, R^{i*}, U^*\rangle_{i\in S}$, and $\mathrm{M}^{\Box *}$ as $\langle \mathcal{F}^{\Box *}, \mathsf{v}^*\rangle$. Clearly, $\mathcal{F}^{\Box *}$ is finite and belongs to $\mathsf{K}^\Box$. We claim that $\mathrm{M}^{\Box *}$ is a filtration of $\mathrm{M}^\Box$ through the set $\Sigma \cup X^\Box$. For the universal relation $U^*$, **min** and **max** are trivially satisfied. For the other operators, **min** is still true. So we have to show that 1) $\Sigma \cup X^\Box$ is closed under subterms and the Boolean operations, 2) $\equiv_\Sigma \,=\, \equiv_{(\Sigma\cup X^\Box)}$, and 3) **max** is satisfied for the old operators. These three claims follow from (3.1) below.

$$(\forall\tau \in X^\Box)(\exists\tau' \in X^{01}) : \mathrm{M}^\Box \models \tau = \tau' \qquad (3.1)$$

(3.1) follows from the definition of $X^{01}$ and the fact that $\mathrm{M}^\Box \models (\Diamond\tau = 1 \textit{ or } \Diamond\tau = 0)$.

QED

ASIDE ON COMPLETENESS. In favourable cases it is easy to find an axiomatization of the variety generated by $(\mathsf{K}^{\Box})^+$. Let $AX^{\Diamond}_S$ stand for the set consisting of

- equations which say that $\Diamond$ is an S5–diamond[4]
- an equation $f(\mathsf{x}_1,\ldots,\mathsf{x}_n) \leq \Diamond\mathsf{x}_1 \wedge \ldots \wedge \Diamond\mathsf{x}_n$ for every $n$–ary operator $f$ in $S$

THEOREM 3.1.10. *Let* $\mathsf{K}$ *be a class of frames of type* $S$. *Assume that* $\mathbf{S}\mathsf{K}^+$ *is a canonical variety which can be axiomatized by a set of equations* $AX$, *and that* $\mathbf{Zig}\mathsf{K} = \mathbf{Cm}^{-1}(\mathbf{S}\mathsf{K}^+)$. *Then the class* $\mathbf{SP}(\mathsf{K}^{\Box})^+$ *is a canonical variety which can be axiomatized by* $AX \cup AX^{\Diamond}_S$.

PROOF. Assume the condition of the theorem. The conclusion follows using straightforward duality computations from the following two facts, the proof of which only involves standard S5 arguments. Note that all the equations we add are positive, hence canonical.
(i) $AX \cup AX^{\Diamond}_S$ axiomatizes the variety $\mathbf{S}(\mathsf{K}^{\Box}\prime)^+$, in which $\mathsf{K}^{\Box}\prime$ is the class of frames from $\mathbf{Cm}^{-1}(\mathbf{S}\mathsf{K}^+)$ expanded with an equivalence relation $U$ which extends all other relations[5].
(ii) $\mathsf{K}^{\Box}\prime = \mathbf{Du}\{\mathcal{F} = \langle W, R^i, U\rangle_{i\in S} : \langle W, R^i\rangle_{i\in S} \in \mathbf{Zig}\mathsf{K} \;\&\; U = W \times W\} = \mathbf{DuZig}\mathsf{K}^{\Box} \subseteq \mathbf{ZigDu}\mathsf{K}^{\Box}$ QED

## 3.2 RELATIVIZED RELATION ALGEBRAS

Recall the set of equations $(A_1) - (A_{15})$ and their corresponding frame conditions $(C_1)-(C_{15})$ from 2.4.9. We show that varieties defined by several subsets of $(A_1)-(A_{15})$ are generated by their finite members, and that their universal theories are decidable. This fact can be applied to prove the same statement for the varieties of relativized relation algebras[6] $\mathbf{SRl}_H\mathsf{RRA}$ for $H \subseteq \{R, S\}$. The application has to wait until the next chapter in which we show that these varieties can be axiomatized by the given equations (cf. 4.2.5).

THEOREM 3.2.1. *Let* $\Sigma$ *be a set of equations such that* $\{(A_1) - (A_3), (A_7) - (A_{12})\} \subseteq \Sigma \subseteq \{(A_1) - (A_{15})\}$. *Then the variety* $\mathsf{BA}^{rel}(\Sigma)$ *has the following properties:*
(i) $\mathsf{Univ}(\mathsf{BA}^{rel}(\Sigma))$ *is decidable.*
(ii) $\mathsf{Univ}(\mathsf{BA}^{rel}(\Sigma)) = \mathsf{Univ}(\mathbf{Fin}\mathsf{BA}^{rel}(\Sigma))$.
(iii) $\mathsf{BA}^{rel}(\Sigma)$ *is generated by its finite members.*

The theorem follows from lemma 3.2.3. We prove it after that lemma. Contrast this theorem with the following result by Andréka et al. ([AKN+94]). The theorem states that associativity of composition leads to undecidability.

THEOREM 3.2.2 (ANDRÉKA ET AL.). *Let* $\Sigma$ *be a set of equations such that* $\{\mathsf{x};(\mathsf{y};\mathsf{z}) = (\mathsf{x};\mathsf{y});\mathsf{z}\} \subseteq \Sigma \subseteq \{e : \mathsf{RRA} \models e\}$. *Then the equational theory of* $\mathsf{BA}^{rel}(\Sigma)$ *is undecidable.*

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[4]They are: $\Diamond 0 = 0$, $\Diamond(\mathsf{x} \vee \mathsf{y}) = \Diamond\mathsf{x} \vee \Diamond\mathsf{y}$, $\mathsf{x} \leq \Diamond\mathsf{x}$, $\Diamond\Diamond\mathsf{x} = \Diamond\mathsf{x}$ and $\mathsf{x} \wedge \Diamond\mathsf{y} \leq \Diamond(\mathsf{y} \wedge \Diamond\mathsf{x})$.

[5]Formally: $\mathcal{F} = \langle W, R^i, U\rangle_{i\in I} \in \mathsf{K}^{\Box}\prime$ iff 1) $\langle W, R^i\rangle_{i\in I} \in \mathbf{Cm}^{-1}(\mathbf{S}\mathsf{K}^+)$, 2) $U$ is an equivalence relation, and 3) for every $n+1$ place relation $R^i$, $R^i y x_1 \ldots x_n \Rightarrow U y x_1 \;\&\; \ldots \;\&\; U y x_n$.

[6]That $\mathbf{SRl}_{RS}\mathsf{RRA}$ is generated by its finite members was shown in Németi [Ném87].

LEMMA 3.2.3. *Let $\Sigma$ be a set of frame conditions such that $\{(C_1) - (C_3), (C_7) - (C_{12})\} \subseteq \Sigma \subseteq \{(C_1) - (C_{15})\}$. Then the class $\mathsf{K}_\Sigma \stackrel{\text{def}}{=} \{\mathcal{F} \in \mathsf{K}^{rel} : \mathcal{F} \models \Sigma\}$ admits filtrations.*

PROOF. As indicated in section 2.4.3, we have three (partial) functions living in the frame classes for which we want to prove the lemma. The main difficulty in the proof will be to ensure that in the filtration these functions still behave correctly. To accomplish this, we have to close the closure set under the operators $\mathsf{id}, \mathsf{s}_0^1, \mathsf{s}_1^0$ and $\smile$. This is not dangerous, since our axioms are strong enough to ensure that such a closure set remains finite modulo $\mathsf{K}_\Sigma$. This last, crucial, part in the proof follows immediately from the next claim. Recall that the accessibility relations corresponding to $\mathsf{id}, \mathsf{s}_0^1, \mathsf{s}_1^0, \smile$ are $\mathsf{I}, (.)_l, (.)_r$ and $\mathsf{f}$, respectively (see section 2.4.3), and that (see the proof of 2.4.7) the conditions $\{(C_1) - (C_3), (C_7) - (C_{12})\}$ imply conditions $(T_0)$ and $(T_1)$ below.

$(T_0)$ $\mathsf{f}$, $(.)_l$ and $(.)_r$ are partial functions and $\mathsf{f}$ is idempotent
$(T_1)$ $\mathsf{I}x \Rightarrow x = \mathsf{f}(x) = x_l = x_r$

CLAIM 1. Let $\mathsf{K}$ be any class of arrow–frames which validates $(T_0)$ and $(T_1)$. Then the variety generated by the $\{\vee, \wedge, -, 0, 1, \mathsf{id}, \mathsf{s}_1^0, \mathsf{s}_0^1, \smile\}$–reduct of $\mathsf{K}^+$ is locally finite.

PROOF OF CLAIM. Let $\mathsf{K}$ be as stated in the claim. By 3.1.8, it is sufficient to show that every $\{\mathsf{I}, \mathsf{f}, (.)_l, (.)_r\}$–point–generated subframe of each member of $\mathsf{K}$ is finite (i.e., the $\{\mathsf{I}, \mathsf{f}, (.)_l, (.)_r\}$–reduct of $\mathsf{K}$ is locally finite), *and* that we have only finitely many of them. Let $\mathcal{F} \in \mathsf{K}$ and $x \in F$ be arbitrary. Since $\mathcal{F} \models (T_0), (T_1)$, we can write the frame as $\mathcal{F} = \langle W, \mathsf{C}, \mathsf{f}, (.)_l, (.)_r, \mathsf{I}\rangle$. By definition of the functions $(.)_l$ and $(.)_r$, it holds that if they are defined on $x$, then $\mathsf{I}x_l$ and $\mathsf{I}x_r$. Conditions $(T_0)$ and $(T_1)$ imply that the subframe which is $\{\mathsf{f}, (.)_l, (.)_r, \mathsf{I}\}$–generated by $\{x\}$ can be described as $\langle\{x, x_l, x_r, \mathsf{f}x, (\mathsf{f}x)_l, (\mathsf{f}x)_r\}, \mathsf{f}, {}_l, {}_r, \mathsf{I}\rangle$, so it is finite. Up to isomorphism, there are only finitely many such point–generated subframes. ◀

Let $\mathsf{K}_\Sigma$ be as in the lemma. Let $\mathcal{F} = \langle W, \mathsf{C}, \mathsf{F}, \mathsf{I}\rangle \in \mathsf{K}_\Sigma$, $\mathrm{M} = \langle \mathcal{F}, \mathsf{v}\rangle$ be a model, and let $X$ be a finite set of terms. Let the closure set $CL(X)$ be the smallest set containing $X \cup \{\mathsf{id}\}$ which is closed under taking subterms, $\mathsf{s}_1^0, \mathsf{s}_0^1, \smile$ and the Boolean operations. Let $\mathcal{F}^* = \langle W^*, \mathsf{C}^*, \mathsf{F}^*, \mathsf{I}^*\rangle$, and let $\langle \mathcal{F}^*, \mathsf{v}^*\rangle$ be the *minimal* filtration of M through $CL(X)$. Conditions (i) and (ii) of the next claim give us that $\mathsf{K}_\Sigma$ admits filtrations.

CLAIM 2. (i) $W^*$ is finite, and (ii) $\mathcal{F}^* \in \mathsf{K}_\Sigma$.

PROOF OF CLAIM. (i). The proof of 2.4.7 implies that $\mathsf{K}_\Sigma \models (T_0), (T_1)$. So, by claim 1, the closure set is finite modulo $\mathsf{K}_\Sigma^+$ (use 3.1.8.(ii)). So there are only finitely many equivalence classes, and $W^*$ is finite.
(ii). We need to show that $\mathcal{F}^* \models \Sigma$. We need another claim. From now on, we suppress the subscript ${}_{CL(X)}$ in $\equiv_{CL(X)}$.

CLAIM 3. The following statements hold for the above $\mathcal{F}$ and $\equiv$.

$$\mathsf{I}x \;\&\; x \equiv x' \;\Rightarrow\; \mathsf{I}x' \tag{3.2}$$
$$\mathsf{F}xy \;\&\; \mathsf{F}x'z \;\&\; x \equiv x' \;\Rightarrow\; y \equiv z \tag{3.3}$$

$$\mathsf{F}xy \,\&\, x \equiv x' \quad\Rightarrow\quad (\exists z) : \mathsf{F}x'z \tag{3.4}$$

$$\mathsf{C}xyx \,\&\, \mathsf{I}y \,\&\, x \equiv x' \quad\Rightarrow\quad (\exists z) : \mathsf{C}x'zx' \,\&\, \mathsf{I}z \,\&\, y \equiv z \tag{3.5}$$

$$\mathsf{C}xxy \,\&\, \mathsf{I}y \,\&\, x \equiv x' \quad\Rightarrow\quad (\exists z) : \mathsf{C}x'x'z \,\&\, \mathsf{I}z \,\&\, y \equiv z \tag{3.6}$$

PROOF OF CLAIM. (3.2). Immediate since $\mathsf{id} \in CL(X)$.
(3.3). Assume the antecedent. We compute:
$\tau \in \overline{y} \iff$ (since $CL(X)$ is closed under $^{\smile}$) $\tau^{\smile} \in \overline{x} \iff \tau^{\smile} \in \overline{x'} \iff \tau \in \overline{z}$.
(3.4). Assume the antecedent. Then $1^{\smile} \in \overline{x}$, whence $1^{\smile} \in \overline{x'}$, so $\exists z : \mathsf{F}x'z$.
(3.5). Assume the antecedent. Then $\mathsf{s}_0^1 1 = \mathsf{id}\,;1 \in \overline{x}$, hence also in $\overline{x'}$, so there exists a $z$ such that $\mathsf{C}x'zx' \,\&\, \mathsf{I}z$. To show that $y \equiv z$, suppose that $\tau \in \overline{y}$. We compute:
$\tau \in \overline{y} \iff$ (since $CL(X)$ is closed under taking $\mathsf{s}_0^1$) $\mathsf{s}_0^1\tau \in \overline{x} \iff \mathsf{s}_0^1\tau \in \overline{x'} \iff \tau \in \overline{z}$.
(3.6) is similar to (3.5), use here closure under $\mathsf{s}_1^0$. ◀

Now we are ready to prove that $\mathcal{F}^* \models \Sigma$. We first show that it satisfies the set $\{(C_1)-(C_3), (C_7)-(C_{12})\}$ of conditions which are always in $\Sigma$. Conditions $(C_1)-(C_3)$ are immediate because of the minimal filtration. Conditions $(C_7)-(C_9)$ are all similar; we show the $\Rightarrow$ side of $(C_8)$ as an example. Assume the antecedent. Then (by the *minimal* definition of $\mathsf{C}^*$) there exists $(x' \equiv x, y''' \equiv y'' \equiv y' \equiv y, z' \equiv z, v'' \equiv v' \equiv v)$ such that $\mathsf{C}x'y'z' \,\&\, \mathsf{C}y''y'''v' \,\&\, \mathsf{I}v''$. By (3.2) also $\mathsf{I}v'$, so by $(C_{12})$ we have $y''' = y''$, and by (3.6) and (3.2), $(\exists v''' \equiv v') : \mathsf{C}y'y'v''' \,\&\, \mathsf{I}v'''$. By $(C_8)$ we obtain $\mathsf{C}z'v'''z'$ and, by definition of $\mathsf{C}^*$, $\mathsf{C}^*\overline{z}, \overline{v}, \overline{z}$. Condition $(C_{10})$ is immediate by (3.3), and conditions $(C_{11})$ and $(C_{12})$ by (3.2).

The rest of the conditions in $\Sigma$ may vary. We show for each of the conditions $(C_4)-(C_6), (C_{13})-(C_{15})$ that, if $\mathcal{F}$ satisfies the condition, then $\mathcal{F}^*$ satisfies it too. For $(C_4)$, suppose $\mathsf{C}^*\overline{x}, \overline{y}, \overline{z} \,\&\, \mathsf{F}^*\overline{y}, \overline{v}$. Then there exists $(x' \equiv x, z' \equiv z, v' \equiv v \,\&\, y' \equiv y'' \equiv y)$ such that $\mathsf{C}x'y'z' \,\&\, \mathsf{F}y''v'$. By (3.3) and (3.4) we find $\mathsf{F}y'v'' \,\&\, v'' \equiv v'$ for some $v''$. Then, by $(C_4)$, $\mathsf{C}z'v''x'$, so, by the definition of $\mathsf{C}^*$, $\mathsf{C}^*\overline{z}, \overline{v}, \overline{x}$. Condition $(C_5)$ can be shown similarly. For $(C_6)$ use (3.2). Conditions $(C_{13})-(C_{15})$ are guaranteed because the filtration is minimal. QED

PROOF OF THEOREM 3.2.1. Let $\mathsf{BA}^{rel}(\Sigma)$ be a variety as stated in the theorem. Let $\Sigma'$ be the class of frame conditions corresponding to the equations in $\Sigma$. Then, by 2.4.9 and 2.2.2, $\mathsf{BA}^{rel}(\Sigma) = \mathsf{SK}^+_{\Sigma'}$. The conclusion of the theorem follows from 3.2.3 and 3.1.6. QED

## 3.3 RELATIVIZED CYLINDRIC ALGEBRAS

I. Németi has shown that the equational theories of the classes of cylindric relativized set algebras $\mathsf{Crs}_\alpha$ and $\mathsf{G}_\alpha$ are decidable, for all $\alpha \leq \omega$, and similar for $\mathsf{D}_\alpha$, provided that $\alpha$ is finite. (Cf. [Ném86] or Thm 10 in the updated version [Ném92]; for the definition of these classes see section 2.5.) In [Ném86][7] he raised the open problem whether, if $\alpha$ is finite, these varieties are generated by their finite members, and if their quasi-equational theories are decidable as well. We show that, for finite $\alpha$, the class of frames

[7] This is remark 12 in the 1992 version [Ném92].

corresponding to $\mathsf{D}_\alpha$ admits filtrations. Using this result, we can solve these two open problems for $\mathsf{D}_\alpha$. For the other classes, we refer to remark 3.3.4 below.

THEOREM 3.3.1. *Let $\alpha$ be any finite ordinal.*
(i) *The class $\mathsf{D}_\alpha$ is generated by its finite members.*
(ii) *The universal theory of $\mathsf{D}_\alpha$ is decidable, and* $\mathsf{Univ}(\mathsf{D}_\alpha) = \mathsf{Univ}(\mathbf{Fin}\mathsf{D}_\alpha)$.

These results follow in a straightforward way from the fact that the class of $\alpha$–frames $\mathsf{K}_{rlD}^{cyl\alpha}$ (see 2.5.6) admits filtrations. We will prove the theorem, having shown the next lemma.

LEMMA 3.3.2. *If $\alpha$ is finite, then the class $\mathsf{K}_{rlD}^{cyl\alpha}$ admits filtrations.*

As was the case with relation algebras, a filtration is possible when we close under taking substitutions. In the proof of the lemma, we need the following proposition.

PROPOSITION 3.3.3. *Let $\alpha \geq 1$ be finite and $\mathsf{K} \subseteq \mathsf{K}_{set}^{cyl\alpha}$. Then the variety generated by the $\{\vee, \wedge, -, 0, 1, \mathsf{d}_{ij}, \mathsf{s}_j^i\}_{i,j<\alpha}$–reduct of $\mathsf{K}^+$ is locally finite.*

PROOF. Let $F \subseteq {}^\alpha U$, for some finite $\alpha \geq 1$ and $\mathcal{F} \in \mathsf{K}_{set}^{cyl\alpha}$. To use 3.1.8, we must show that every $\{\mathsf{f}_j^i\}_{i,j<\alpha}$–point–generated subframe of $\mathcal{F}$ is finite, and there are only finitely many of them. (Note that the *unary* relations $D_{ij}$ do not generate any new elements, so we do not need to take them into account.) Let $x \in F$ be arbitrary. The domain of the subframe which is $\{\mathsf{f}_j^i\}_{i,j<\alpha}$–generated by $\{x\}$ equals the set $\{h(x) \in F : (\exists k)(h = \mathsf{f}_{j_1}^{i_1} \circ \ldots \circ \mathsf{f}_{j_k}^{i_k})$ for $(i_m, j_m < \alpha)\}$. Because $\mathcal{F} \in \mathsf{K}_{set}^{cyl\alpha}$, $h(x)$ is in ${}^\alpha\{x_0, \ldots, x_{\alpha-1}\}$. Hence, because $\alpha$ is finite, the frame generated by $\{x\}$ is finite. Up to isomorphism, there are only finitely many such point–generated subframes. QED

PROOF OF LEMMA 3.3.2. When $\alpha = 0$, we just have Boolean algebras, and the statement is trivial. When $\alpha = 1$, the class $\mathsf{K}_{rlD}^{cyl\alpha}$ forms the class of modal frames with one binary equivalence relation[8]. It is known that this last class admits filtrations (see e.g., [HC84], Thm 8.7). In the sequel, let $\alpha$ be any finite ordinal larger than 1.

Let $\mathcal{F} = \langle W, T^i, E^{ij}\rangle \in \mathsf{K}_{rlD}^{cyl\alpha}$, $\mathrm{M} = \langle \mathcal{F}, \mathsf{v}\rangle$ a model, and $X$ a finite set of terms. Let $CL(X)$ be the smallest set containing $X \cup \{\mathsf{d}_{ij} : i, j < \alpha\}$, which is closed under subterms, the Boolean operations and the $\mathsf{s}_j^i$'s. Define:

$$\begin{array}{lcl} W^* & \stackrel{\text{def}}{=} & \{\overline{w} : w \in W\} \\ E^{ij*}\overline{x} & \stackrel{\text{def}}{\Longleftrightarrow} & \mathsf{d}_{ij} \in \overline{x} \\ T^{i*}\overline{x}, \overline{y} & \stackrel{\text{def}}{\Longleftrightarrow} & [\mathsf{c}_i\tau \in \overline{x} \iff \mathsf{c}_i\tau \in \overline{y}] \end{array}$$

Let $\mathcal{F}^* = \langle W^*, T^{i*}, E^{ij*}\rangle$, and define $\mathsf{v}^* : var(CL(X)) \longrightarrow \mathcal{P}(W^*)$ in the standard way.

CLAIM 1. $\mathrm{M}^* = \langle \mathcal{F}^*, \mathsf{v}^*\rangle$ is a filtration of M through $CL(X)$.

PROOF OF CLAIM. We have to show that **min** and **max** hold. For $E^{ij}$, this is immediate. For $T^i$, this follows from the proof of Thm 8.7 in [HC84]. ◀

The next claim states that this filtration works.

[8] And a trivial unary relation $E^{00}$ which holds for every element of every frame.

CLAIM 2. (i) $W^*$ is finite, and (ii) $\mathcal{F}^* \in \mathsf{K}^{cyl\alpha}_{rlD}$.

PROOF OF CLAIM. (i). By 2.5.7, $\mathsf{S}(\mathsf{K}^{cyl\alpha}_{rlD})^+ = \mathsf{D}_\alpha = \mathsf{S}(\mathsf{K}^{cyl\alpha}_{setD})^+$. Hence $\mathsf{Eq}(\mathsf{K}^{cyl\alpha}_{rlD}) = \mathsf{Eq}(\mathsf{K}^{cyl\alpha}_{setD})$. Since $\mathsf{K}^{cyl\alpha}_{setD} \subseteq \mathsf{K}^{cyl\alpha}_{set}$, it follows from 3.3.3 that $CL(X)$ is finite modulo $\mathsf{K}^{cyl\alpha}_{setD}$. But then it is also finite modulo $\mathsf{K}^{cyl\alpha}_{rlD}$, whence $W^*$ is finite.
(ii). We have to show that $\mathcal{F}^*$ satisfies the conditions defining $\mathsf{K}^{cyl\alpha}_{rlD}$ (cf. 2.5.6). The first three conditions make the $T^i$ equivalence relations. This follows immediately from the definition of the $T^{i*}$. The other conditions, which deal with the substitutions and the diagonals, are handled in a uniform way using the next claim.

CLAIM 3.

$$E^{ij*}\overline{x} \iff E^{ij}x \tag{3.7}$$
$$\overline{x} = \overline{y} \Rightarrow \overline{f^i_j x} = \overline{f^i_j y} \tag{3.8}$$
$$\overline{f^i_j x} = \overline{y} \iff T^{i*}\overline{x},\overline{y} \,\&\, E^{ij*}\overline{y} \quad \text{if } i \neq j \tag{3.9}$$

PROOF OF CLAIM. The first two statements follow from the closure under the diagonals and the $\mathsf{s}^i_j$'s.
(3.9). The direction from left to right is immediate by **min** . For the other side, assume $i \neq j$ and $T^{i*}\overline{x},\overline{y} \,\&\, E^{ij*}\overline{y}$. We have to show that $\overline{y} = \overline{f^i_j x}$. We compute:
($\supseteq$): $\tau \in \overline{f^i_j x} \iff \mathsf{s}^i_j\tau \in \overline{x} \overset{\text{def}}{\iff} \mathsf{c}_i(\tau \wedge \mathsf{d}_{ij}) \in \overline{x} \overset{ass}{\iff} \mathsf{c}_i(\tau \wedge \mathsf{d}_{ij}) \wedge \mathsf{d}_{ij} \in \overline{y} \overset{(\mathbf{A}_7)}{\Rightarrow} \tau \in \overline{y}$.
($\subseteq$): $\tau \in \overline{y} \overset{ass}{\iff} \tau \wedge \mathsf{d}_{ij} \in \overline{y} \Rightarrow$ (by ($\mathbf{A}_1$), and because $CL(X)$ is closed under taking $\mathsf{s}^i_j$'s) $\mathsf{c}_i(\tau \wedge \mathsf{d}_{ij}) \overset{\text{def}}{=} \mathsf{s}^i_j\tau \in \overline{y} \iff$ (by $T^{i*}\overline{x},\overline{y}$) $\mathsf{s}^i_j\tau \in \overline{x} \iff \tau \in \overline{f^i_j x}$. ◀

Set $f^{i*}_j\overline{x} \overset{\text{def}}{=} \overline{f^i_j x}$. By (3.8) above, this is well defined. It follows from conditions ($\mathbf{C}_5$) and ($\mathbf{C}_6$) that, if $i \neq j$, we have $\mathcal{F} \models \forall x \exists y(T^i xy \,\&\, E^{ij}y)$. By **min**, this also holds in the filtration. But then, by (3.9), the $f^{i*}_j$ are *total functions* which are defined correctly. If the $f^i_j$ are total functions, we can rewrite conditions ($\mathbf{C}_6$) and ($\mathbf{C}_7$) to the following (note the correspondence with ($\mathbf{A}_{6'}$) and ($\mathbf{A}_{7'}$)):

($\mathbf{C}_{6'}$) $E^{ij}x \iff E^{kj}f^k_i x \quad \text{if } k \neq j$
($\mathbf{C}_{7'}$) $E^{ij}x \Rightarrow x = f^i_j x$

Now we can easily check the other conditions. ($\mathbf{C}_{4+}$) holds by definition of $f^{i*}_j$. Condition ($\mathbf{C}_5$) holds by (3.7). The others we spell out.
($\mathbf{C}_{6'}$): $E^{ij*}\overline{x} \overset{(3.7)}{\iff} E^{ij}x \overset{(\mathbf{C}_{6'})}{\iff} E^{kj}f^k_i x \overset{(3.7)}{\iff} E^{kj*}\overline{f^k_i x} \overset{\text{def}}{\iff} E^{kj*}f^{k*}_i\overline{x}$.
($\mathbf{C}_{7'}$): $E^{ij*}\overline{x} \overset{(3.7)}{\iff} E^{ij}x \overset{(\mathbf{C}_{7'})}{\Rightarrow} f^i_j x = x \Rightarrow \overline{f^i_j x} = \overline{x} \overset{\text{def}}{\iff} f^{i*}_j\overline{x} = \overline{x}$.
For ($\mathbf{C}_{MGR}$), assume the antecedent $T^{k*}f^{m*}_k f^{j*}_m f^{i*}_j f^{k*}_i(\overline{x}),\overline{y}$. By the definition of $T^{k*}$, the consequent holds iff $[\mathsf{c}_k\,\tau \in \overline{y} \iff \mathsf{c}_k\,\tau \in f^{j*}_k f^{i*}_j f^{m*}_i f^{k*}_m(\overline{x})]$. We compute:

$$
\begin{array}{lll}
\mathsf{c}_k\,\tau \in \overline{y} & \iff & (\text{by } T^{k*}f^{m*}_k f^{j*}_m f^{i*}_j f^{k*}_i(\overline{x}),\overline{y}) \\
\mathsf{c}_k\,\tau \in f^{m*}_k f^{j*}_m f^{i*}_j f^{k*}_i(\overline{x}) & \overset{\text{def}}{\iff} & \\
\mathsf{c}_k\,\tau \in \overline{f^m_k f^j_m f^i_j f^k_i(x)} & \iff & \\
\mathsf{s}^k_i\mathsf{s}^i_j\mathsf{s}^j_m\mathsf{s}^m_k\,\mathsf{c}_k\,\tau \in \overline{x} & \iff & (\text{by } (\mathbf{A}_{MGR})) \\
\mathsf{s}^k_m\mathsf{s}^m_i\mathsf{s}^i_j\mathsf{s}^j_k\,\mathsf{c}_k\,\tau \in \overline{x} & \iff & \\
\mathsf{c}_k\,\tau \in \overline{f^j_k f^i_j f^m_i f^k_m(x)} & \overset{\text{def}}{\iff} & \\
\mathsf{c}_k\,\tau \in f^{j*}_k f^{i*}_j f^{m*}_i f^{k*}_m(\overline{x}) & &
\end{array}
$$

The other side of $(\mathbf{C}_{MGR})$ is shown similarly. We checked all conditions, whence $\mathcal{F}^* \in \mathsf{K}_{rlD}^{cyl\alpha}$. QED

PROOF OF THEOREM 3.3.1. Let $\alpha$ be finite. The class of frames $\mathsf{K}_{rlD}^{cyl\alpha}$ is basic elementary, and admits filtrations by the previous lemma. By 2.5.7, the class $\mathsf{S}(\mathsf{K}_{rlD}^{cyl\alpha})^+$ is a variety which equals $\mathsf{D}_\alpha$. Then the theorem follows from 3.1.6. QED

REMARK 3.3.4. We briefly return to the two other (decidable) classes of cylindric relativized set algebras $\mathsf{Crs}_\alpha$ and $\mathsf{G}_\alpha$. For $\mathsf{G}_\alpha$, a similar proof as given above for $\mathsf{D}_\alpha$ awaits an axiomatization which is at present unknown to us. For $\mathsf{Crs}_\alpha$, a filtration proof is more difficult than the present one, since it is axiomatized by infinitely many axioms which are rather complicated. Note that for $\alpha \leq 2$ the above filtration proof goes through for $\mathsf{Crs}_\alpha$ (see 2.5.8 for the axiomatization of these classes).

CONJECTURE 3.3.5. The above given filtration proof goes through for $\mathsf{Crs}_\alpha$ and $\mathsf{G}_\alpha$.

REMARKS 3.3.6. When we compare the proof for the decidability of $\mathsf{Eq}(\mathsf{D}_\alpha)$ given here with the one of Németi, we can conclude that proving the stronger statement is easier, and has a larger pay–off. As the above remarks show however, using filtrations in a simple way needs a ("nice") axiomatization. Moreover, to show decidability the axiomatization should be *finite* as well. The more complicated, but powerful *mosaic method*, which is developed in [Ném92], does not need an axiomatization, and also works when a class *is not generated by its finite members.*

We now give an example where decidability cannot be shown by the filtration method. Define the variety $\mathsf{NCA}_\alpha$ of *Non–commutative Cylindric Algebras* as the class of those algebras of the cylindric type which satisfy all CA axioms except $(\mathbf{A}_4)$. That is,

$$\mathsf{NCA}_\alpha \stackrel{\text{def}}{=} \{\mathfrak{A} \in \mathsf{Bo}_\alpha : \mathfrak{A} \models (\mathbf{A}_1) - (\mathbf{A}_3), (\mathbf{A}_5) - (\mathbf{A}_7)\}$$

For $\alpha > 1$, by the completeness theorem for $\mathsf{D}_\alpha$, this class is not representable as subalgebras of complex algebras of assignment frames, but it clearly equals the class of subalgebras of complex algebras of $\alpha$–frames which satisfy the frame conditions corresponding to its axioms. Németi showed, using the mosaic method, that for finite $\alpha$, $\mathsf{Eq}(\mathsf{NCA}_\alpha) = \mathsf{Eq}(\mathbf{Fin}\mathsf{NCA}_\alpha)$, hence its equational theory is decidable (cf. [Ném92] Thm 5). He also showed, for $1 < \alpha$, that $\mathsf{Qeq}(\mathsf{NCA}_\alpha) \neq \mathsf{Qeq}(\mathbf{Fin}\mathsf{NCA}_\alpha)$. But then, the class of frames generating $\mathsf{NCA}_\alpha$ cannot admit filtrations[9], because that would imply that the quasi–equational theory of $\mathsf{NCA}_\alpha$ equals $\mathsf{Qeq}(\mathbf{Fin}\mathsf{NCA}_\alpha)$.

## 3.4 CONCLUDING REMARKS

The decidability results for relativized relation algebras lead to several decidable arrow logics. These logics might be used for applications where the complexity of the problem is low. The price we had to pay was the loss of associativity of composition (cf. 3.2.2). To get decidable versions of FO logic a similar price has to be paid: one should give

[9] Note that the problem is not in the conditions, but in the fact that the $\mathsf{s}^i_j$–reduct is not locally finite.

up the commutativity of the quantifiers (cf. [Ném92]).

We conclude this chapter with some questions.

1. It seems that in BAO's, or in general modal logic, decidability is indeed closely connected to the form of the FO theory of the class of frames. Existential quantifiers clearly form a dangerous point. Van Benthem [Ben93] conjectures that all modal logics (with one unary diamond) which are complete with respect to a class of frames defined by a finite Universal Horn theory are decidable. This problem is still open, and the results in Kracht [Kra93] show that one has to be careful, extending this conjecture to arbitrary similarity types.
   By the well-known translation (cf. van Benthem [Ben84]) of modal formulas to FO formulas, we know that every modal language is living inside a fragment of FO logic with finitely many variables. If the class of frames is elementary, one can derive the "modal validities" using a FO derivation system. One can view this translation as an "application" of FO logic to modal logic. Then the obvious question arises: *In which FO logic is modal logic living?* A possible way of proving van Benthem's conjecture would be to show that these modal logics are living in a FO logic whose consequence relation $\Gamma \models \phi$ is decidable for finite sets of sentences $\Gamma$. A possible candidate could be the FO logic corresponding to the variety $\mathsf{D}_\alpha$.
2. A question related to the previous one is the following. E. Orlowska [Orl91] shows that modal logics are interpretable in the class RA of relation algebras, and she defines a relational proof system for several modal logics. The purpose is to prove both theorems and meta-theorems of modal logic within the theory of relation algebras. This is nice, except that in several cases there is a mismatch in complexity: a decidable modal logic is interpreted in the undecidable logic of relation algebras. Would it be possible –using Orlowska's translation function– to interpret decidable modal logics in *decidable* weakened versions of relation algebras, and obtain results similar to hers?
3. We have shown that –for finite $\alpha$– the variety $\mathsf{D}_\alpha$ is generated by its finite members. What we do not know however, is if these finite algebras are isomorphic to ones with a *finite base*[10]. The problem whether every non $\mathsf{D}_\alpha$ valid equation can be refuted on an algebra with a finite base is still open. This property is sometimes called the *finite base property*.

[10] The two-element algebra with a universe consisting of the empty set and the set $\{\langle n, n\rangle : n < \omega\}$ is a finite $\mathsf{D}_2$ whose base $\omega$ is infinite.

# 4

# Representation & Axiomatization

To begin with, in this chapter, we show for several classes $\mathsf{K}$ of pair–frames that $\mathsf{SK}^+$ is a finitely axiomatizable variety. We obtain these results by working at the frame level, that is, we show that every frame which satisfies some specific finite set of equationally definable FO conditions is representable as a zigzagmorphic image of some (disjoint union of) frame(s) in $\mathsf{K}$. We start this chapter with a few general remarks about this proof strategy (section 4.1). Section 4.2 is about the classes of relativized relation algebras $\mathbf{SRl}_H\mathsf{RRA}$, for $H \subseteq \{R, S, T\}$. We show that these classes are finitely axiomatizable varieties if and only if $T \notin H$. The techniques we introduce here are used in the next section to obtain quick results about subreducts of $\mathbf{SRl}\mathsf{RRA}$. In section 4.4, we regain some of the expressive power of $\mathsf{RRA}$, which was lost by relativization, by adding the *difference operator* to the class $\mathbf{SRl}_{RS}\mathsf{RRA}$. We can still finitely axiomatize this expansion. In the last section, we generalize our results to arbitrary Boolean algebras with operators. We show that *every BAO* can be represented as an algebra of relations.

## 4.1 Axiomatizing BAO's by representing frames

In this section, we show how we can find an axiomatization of a class of representable BAO's by working solely with frames. Suppose we are given a class of algebras in which the operations are defined in a uniform set–theoretic manner. (Note that this implies that if two algebras are different, their universes are different.) We will refer to the closure under isomorphisms of such classes as *representable* or *concrete* classes. In [Ném91], it is explained that both taking subalgebras and taking direct products preserves the intuitive notion of representability. So, if a class of representable algebras is defined as $\mathsf{K}^+$, for $\mathsf{K}$ some class of frames, then $\mathbf{SP}\mathsf{K}^+$ is a representable class too. Thus we call a *frame* $\mathcal{F}$ representable as a $\mathsf{K}$ frame if $\mathcal{F}^+ \in \mathbf{SP}\mathsf{K}^+$. Duality theory then implies that $\mathcal{F}$ is representable as a $\mathsf{K}$ frame if the ultrafilter extension of $\mathcal{F}$ is a zigzagmorphic image of the ultrafilter extension of a disjoint union of frames in $\mathsf{K}$. If $\mathsf{K}$ is elementary and closed under disjoint unions, we can simplify this to the requirement that the ultrafilter extension of $\mathcal{F}$ is a zigzagmorphic image of a frame in $\mathsf{K}$ (use Thm 3.6.2 in [Gol88]).

We now restrict ourselves to elementary $\mathsf{K}$ which are closed under disjoint unions. Let $\mathsf{K}$ be such a class of frames. Lemma 3.6.5 and theorem 3.6.7 in [Gol88] imply that 1), $\mathsf{SK}^+ = \mathbf{SP}\mathsf{K}^+ = \mathbf{SPUp}\mathsf{K}^+$, hence $\mathsf{SK}^+$ is a *quasi–variety*, and 2), if $\mathsf{SK}^+$ is a variety, it is *canonical*. Now suppose that $\mathsf{SK}^+$ is a variety. Let $\mathsf{K}_V \stackrel{\text{def}}{=} \mathbf{Cm}^{-1}\mathsf{SK}^+$. Because $\mathsf{SK}^+$ is a canonical variety, it equals the class $\mathsf{SK}_V^+$. Hence every frame in $\mathsf{K}_V$

is representable as a K frame (i.e., its ultrafilter extension is a zigzagmorphic image of a K frame), so $\mathsf{K}_V = \overleftarrow{\mathbf{U}}\mathbf{eZig}\mathsf{K}$. This observation leads to the following fact.

FACT 4.1.1. Let K be an elementary class of frames which is closed under disjoint unions. Then $\mathbf{S}\mathsf{K}^+$ is a finitely axiomatizable variety if and only if there exists a K valid canonical equation $e$ such that $\overleftarrow{\mathbf{U}}\mathbf{eZig}\mathsf{K} = \{\mathcal{F} : \mathcal{F} \models e\}$.

PROOF. ($\Rightarrow$) Assume the antecedent. We saw above that $\mathbf{S}\mathsf{K}^+$ is canonical. Because we have the Booleans, the class is axiomatizable by one equation $e$. But then $e$ is canonical. The rest follows from the earlier observations.
($\Leftarrow$) Assume the antecedent. We must prove that 1), $\mathbf{S}(\overleftarrow{\mathbf{U}}\mathbf{eZig}\mathsf{K})^+$ is a variety axiomatizable by $e$, and 2), $\mathbf{S}\mathsf{K}^+ = \mathbf{S}(\overleftarrow{\mathbf{U}}\mathbf{eZig}\mathsf{K})^+$. 1) holds, because $e$ is canonical. 2) follows from a straightforward duality computation. QED

AN EXAMPLE: PAIR–FRAMES. We illustrate the strategy which is implied by the last fact with the class of all pair–frames $\mathsf{K}^{rel}_{set}$. Clearly, this class is closed under disjoint unions. It is also easy to show that it is not closed under zigzagmorphic images (e.g., show that the $\mathsf{K}^{rel}_{set}$ valid frame condition ($\mathsf{F}_V xx \Rightarrow \mathsf{I}_V x$) is not preserved under zigzagmorphisms). Because $\mathsf{K}^{rel}_{set} = \mathbf{Sub}\mathsf{K}^{rel}_{setSQ}$ and $\mathsf{K}^{rel}_{setSQ}$ is elementary (cf. [Ven91]), $\mathsf{K}^{rel}_{set}$ is elementary[1]. So the class $\mathsf{K}^{rel}_{set}$ is elementary and closed under disjoint unions. We want to show that $\mathbf{S}(\mathsf{K}^{rel}_{set})^+$ is a finitely axiomatizable variety. By the last fact, it is necessary and sufficient to define a class K as the class of all arrow–frames satisfying some $\mathsf{K}^{rel}_{set}$ valid canonical equation, and show that $\mathsf{K} = \overleftarrow{\mathbf{U}}\mathbf{eZig}\mathsf{K}^{rel}_{set}$. It turns out that our task is even easier: in the next section we show that the class of arrow–frames $\mathsf{K}^{rel}_{rl}$, which is defined as all frames satisfying the set of canonical equations $(A_1) - (A_{12})$, equals the class $\mathbf{Zig}\mathsf{K}^{rel}_{set}$.

## 4.2 RELATIVIZED RELATION ALGEBRAS

We prove that, for $H \subseteq \{R, S\}$, the classes $\mathbf{SRl}_H\mathsf{RRA}$ are finitely axiomatizable canonical varieties. This section is organized as follows. First, we state our main results. Then we look at the reduct with only Booleans and composition, and show how to axiomatize that fragment. In the next subsection, we introduce the concept of a mosaic in order to deal with the additional difficulties coming from identity and converse, and adapt the easy proof for the "composition only"–reduct.

### 4.2.1 MAIN RESULTS

FINITE AXIOMATIZABILITY

The next theorem might look a bit clumsy, but it nicely shows the "route" we follow when we represent an abstract algebra. First, we embed it in a complex algebra over

[1] A FO definition of any class of pair–frames of the form $\langle V, Q\rangle$ with $\{\mathsf{C}_V\} \subseteq Q \subseteq \{\mathsf{C}_V, \mathsf{F}_V, \mathsf{I}_V\}$, and $V$ a binary $H$ relation ($H \subseteq \{R, S, T\}$) can be derived from Kuhler [Kuh94].

an arrow–frame (its canonical embedding algebra), and then we embed this second algebra in a complex algebra over a pair–frame. The axioms can be found on page 34.

THEOREM 4.2.1. *Let $H \subseteq \{R, S\}$. Then the class* $\mathbf{SRl}_H\mathsf{RRA}$ *is a finitely axiomatizable canonical variety. In particular:*

$$
\begin{array}{llllll}
\mathsf{BA}^{rel}((A_1)-(A_{12})) & = \mathsf{S}(\mathsf{K}^{rel}_{rl})^+ & = \mathsf{S}(\mathsf{K}^{rel}_{set})^+ & = \mathbf{SRl}\mathsf{RRA} \\
\mathsf{BA}^{rel}((A_1)-(A_{13})) & = \mathsf{S}(\mathsf{K}^{rel}_{rlS})^+ & = \mathsf{S}(\mathsf{K}^{rel}_{setS})^+ & = \mathbf{SRl}_S\mathsf{RRA} \\
\mathsf{BA}^{rel}((A_1)-(A_{12}),(A_{14}),(A_{15})) & = \mathsf{S}(\mathsf{K}^{rel}_{rlR})^+ & = \mathsf{S}(\mathsf{K}^{rel}_{setR})^+ & = \mathbf{SRl}_R\mathsf{RRA} \\
\mathsf{BA}^{rel}((A_1)-(A_{15})) & = \mathsf{S}(\mathsf{K}^{rel}_{rlRS})^+ & = \mathsf{S}(\mathsf{K}^{rel}_{setRS})^+ & = \mathbf{SRl}_{RS}\mathsf{RRA}
\end{array}
$$

PROOF. The first equality in each row is theorem 2.4.10, the second follows from lemma 4.2.3 below (use fact 2.2.5), and the third is the definition of the classes $\mathbf{SRl}_H\mathsf{RRA}$ . QED

The theorem has the following corollary.

COROLLARY 4.2.2. *Let $H \subseteq \{R, S, T\}$. The variety* $\mathbf{SRl}_H\mathsf{RRA}$ *is finitely axiomatizable if and only if $T \notin H$.*

PROOF. By theorems 4.2.1 and 2.4.6. QED

LEMMA 4.2.3. *Let $H \subseteq \{R, S\}$. Then* $\mathsf{K}^{rel}_{rlH} = \mathbf{Zig}\mathsf{K}^{rel}_{setH}$.

To warm up, we will prove a representation theorem for the reduct which only contains the composition operator and the Booleans. Rather surprisingly, no assumptions on the frames are needed in order to represent them as pair–frames. After that, we prove the above lemma in section 4.2.4.

REMARKS 4.2.4. Finite axiomatizability of $\mathbf{SRl}_{RS}\mathsf{RRA}$ was shown by R. Maddux ([Mad82], Thm 5.20). Maddux defined an axiomatic class WA (for weak associativity) by keeping all RA axioms, but replacing[2] the associativity axiom with the weaker $((\mathsf{id}\,;\tau);1);1 = (\mathsf{id}\,;\tau);(1;1)$, and showed that WA equals $\mathbf{SRl}_{RS}\mathsf{RRA}$. The WA axioms are a bit different from ours, because we get our result as a by–product of the axiomatization of $\mathbf{SRl}\mathsf{RRA}$. That this last class is finitely axiomatizable was shown by R. Kramer ([Kra91], Thm 5.4). The proof of Kramer is rather syntactic and complicated. The proof presented here uses a similar step–by–step construction which Maddux used in [Mad78] to prove[3] axiomatizability of $\mathbf{SRl}_{RS}\mathsf{RRA}$, together with the *mosaics* which Németi introduced in [Ném86] to prove decidability for relativized versions of cylindric (set) algebras. An advantage of this proof–method is that it gives us easy results for reducts as well.

[2]The situation is similar to the one with $\mathsf{D}_\alpha$, the relativized version of the cubic cylindric set algebras. To axiomatize $\mathsf{D}_\alpha$, one needs to weaken the axiom which makes the cylindrifications commute. However, with $\mathsf{D}_\alpha$, the extra merry–go–round equation is needed for a complete axiomatization.

[3]In [Mad82], he gives another proof, not using the step–by–step construction.

DECIDABILITY

By the last theorem, the decidability results of the previous chapter also apply to the classes $\mathbf{SRl}_H$RRA. Part (i) of the theorem below was proved by Németi ([Ném87]) for the class $\mathbf{SRl}_{RS}$RRA. All other results seem to be new.

THEOREM 4.2.5. *Let $H \subseteq \{R, S\}$. Then* (i)–(iii) *hold.*
(i) *The variety* $\mathbf{SRl}_H$RRA *is generated by its finite members.*
(ii) *The universal theory of* $\mathbf{SRl}_H$RRA *is decidable.*
(iii) Univ($\mathbf{SRl}_H$RRA) = Univ($\mathbf{FinSRl}_H$RRA).

PROOF. By theorems 4.2.1 and 3.2.1. QED

COROLLARY 4.2.6. *Let $H \subseteq \{R, S, T\}$. Let $P$ be any of the properties* (i)–(iii) *in the previous theorem. Then* $\mathbf{SRl}_H$RRA *has $P$ if and only if $T \notin H$.*

PROOF. By theorems 4.2.5 and 2.4.6. QED

## 4.2.2 WARM UP: BOOLEANS WITH COMPOSITION

We present one part of the technique –the step–by–step construction– which we will use to prove the above representation lemma (4.2.3), using a simple example. Define the following two classes of frames:

$$\begin{aligned} \mathsf{K}^{\mathsf{C}} &\stackrel{\text{def}}{=} \{\mathcal{F} = \langle W, \mathsf{C}\rangle : W \text{ is a set and } \mathsf{C} \subseteq W \times W \times W\} \\ \mathsf{K}^{\mathsf{C}}_{set} &\stackrel{\text{def}}{=} \{\mathcal{F} = \langle V, \mathsf{C}_V\rangle : V \subseteq U \times U \text{ for some set } U\} \end{aligned}$$

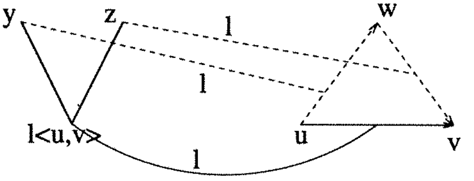
The next lemma is stated for finite frames only, because we wanted to make the example as simple as possible. In the proof of 4.2.3 (the analogous lemma for the whole language), we represent frames of any cardinality.

LEMMA 4.2.7. *Every finite $\mathcal{F} \in \mathsf{K}^{\mathsf{C}}$ is a zigzagmorphic image of some $\mathcal{G} \in \mathsf{K}^{\mathsf{C}}_{set}$.*

PROOF. Let $\mathcal{F} = \langle W, \mathsf{C}\rangle \in \mathsf{K}^{\mathsf{C}}$ be finite. Step by step, we will construct a set of pairs $V$ and a function $l : V \longrightarrow W$. It is convenient to think of $l$ as a labelling function, which labels each element of $V$ with an element of $W$. In each step $n+1$, we will add pairs to $V_n$, such that $l_{n+1}$ is a homomorphism and for all pairs in $V_n$, the function $l_{n+1}$ has the zigzag property. The function will not have the zigzag property for the pairs added in step $n+1$, but we repair that in the next step. Hence, after $\omega$ steps our zigzagmorphism is complete.

**Construction.**
Let $U$ be some countable set.
**step 0** In this step, we ensure that $l$ is surjective. Let $V_0 \subset U \times U$ such that (1), $|W| = |V_0|$, and (2), $V_0$ is irreflexive and disconnected, that is, $(\forall s, r \in V_0)(\forall i, j \leq 1)$ : $s_i = r_j \iff s = r \;\&\; i = j$. Let $l_0$ be any bijection between $V_0$ and $W$.
**step $n+1$** Let $X_n$ be the (finite) set of pairs which were added in the previous step. For each $\langle u, v\rangle \in X_n$ and $y, z \in W$ such that $\mathsf{C}l_n\langle u, v\rangle yz$, do the following:

Take an element $w \in U$ which was *not used before*, and add $\langle u, w\rangle$ and $\langle w, v\rangle$ to $V_n$. Then set $l_{n+1}(\langle u, w\rangle) = y$ and $l_{n+1}(\langle w, v\rangle) = z$ (see the picture below).
Define $V_{n+1}$ as the result of all these additions to $V_n$, and $l_{n+1}$ as the result of these additions to $l_n$.




**step** $\omega$ Set $V \stackrel{\text{def}}{=} \bigcup_{n<\omega} V_n$ and $l \stackrel{\text{def}}{=} \bigcup_{n<\omega} l_n$.
**End of construction**

CLAIM. $l : V \longrightarrow W$ is a zigzagmorphism from $\mathcal{G} = \langle V, \mathsf{C}_V\rangle$ onto $\mathcal{F} = \langle W, \mathsf{C}\rangle$.

PROOF OF CLAIM. The function $l$ is surjective by step 0. The zigzag property is immediate by the construction. To show that $l$ is a homomorphism we show by induction that it is a homomorphism after every step. This is clear for $V_0$, since $(\forall xyz \in V_0) : \neg \mathsf{C}_V xyz$. Suppose it holds for step $n$ and suppose $\mathsf{C}_V xyz$, and at least one of $\{x, y, z\}$ were added in the $n$+1–th step. Since we took a new point from $U$ for every "repair" we made, it follows from our construction that $(\exists y', z' \in W)$ : $\mathsf{C}l_n(x)y'z' \;\&\; l_{n+1}(y) = y' \;\&\; l_{n+1}(z) = z'$. But then $\mathsf{C}l_{n+1}(x)l_{n+1}(y)l_{n+1}(z)$. Hence, $l$ is a homomorphism after $n+1$ steps. Clearly, $l$ is a homomorphism after the limit step as well. QED

REMARK 4.2.8. The construction used above can be seen as a generalization of the *unraveling construction* from standard modal logic to binary modalities (cf. Sahlqvist [Sah75], de Rijke [Rij93]). In section 4.5, we will generalize the above construction to operators of arbitrary arity. In the spirit of Proposition 6.3.5 in [Rij93], we can also give a direct definition of the "unravelled pair–frame". Let $\mathcal{F}_a = \langle W, \mathsf{C}\rangle$ be a frame such that the subframe generated by $\{a\}$ is again $\mathcal{F}_a$. Define the set $B$ as the smallest set such that

- $\langle\langle a0, a1\rangle, a\rangle \in B$
- $s = \langle\langle u, v\rangle, x\rangle \in B \;\&\; \exists yz\mathsf{C}xyz \Rightarrow \{\langle\langle u, s{:}\mathsf{C}xyz\rangle, y\rangle, \langle\langle s{:}\mathsf{C}xyz, v\rangle, z\rangle\} \subseteq B$

Let $V \stackrel{\text{def}}{=} \{\langle u, v\rangle : (\exists x)(\langle\langle u, v\rangle, x\rangle \in B)\}$. Using the argument given in the above proof, it is straightforward to show that $B$ is a zigzagmorphism from the frame $\langle V, \mathsf{C}_V\rangle$ onto $\mathcal{F}_a$.

REMARK 4.2.9. To get an idea how we will prove a similar representation theorem for arrow–frames, the following might be useful. Think of the abstract frame as being built from little frames $\langle\{x, y, z\}, \mathsf{C}xyz\rangle$ ($x, y, z$ need not be different). Later, we call these little frames *mosaics*. Clearly, each such frame is representable by a triangle $\{\langle u, v\rangle, \langle u, w\rangle, \langle w, v\rangle\}$. If we needed to make a repair in the above construction, we added the representation of a mosaic to the partially constructed graph *using a fresh* *point.* The intuitive idea is that in the construction we play a kind of domino game in which the tiles ("represented mosaics") may want one or more tiles being laid next to them. If we play this game infinitely, we can fulfill the desires of each tile, and thereby create a zigzagmorphic pre-image of the frame which was to be represented. The function will be a homomorphism precisely because we always took fresh points.

REMARK 4.2.10. The construction does not depend on the finiteness of $\mathcal{F}$. A similar construction can be used to represent frames of any cardinality. The only difference is that, in general, we have to make infinitely many repairs in the inductive step. In the next section, we show how to change the construction to represent infinite frames as well. Lemma 4.5.3 generalizes the last lemma in two ways: it is about frames of any cardinality, and it represents frames where $\mathsf{C}$ can be any relation of rank higher than 2.

APPLICATION OF THE LEMMA. The above lemma leads to the following corollary. The argument which is used in its proof will be used in many places in this work.

COROLLARY 4.2.11. $\mathsf{BA}^{;} = \mathsf{S}(\mathsf{K}^{\mathsf{C}})^{+} = \mathsf{S}(\mathsf{K}^{\mathsf{C}}_{set})^{+}$ *is a canonical variety.*

PROOF. $\mathsf{S}(\mathsf{K}^{\mathsf{C}})^{+}$ is axiomatizable by the BAO–axioms, and obviously it is a canonical variety (use 2.2.1). Clearly, $\mathsf{K}^{\mathsf{C}}_{set} \subseteq \mathsf{K}^{\mathsf{C}}$. On the other hand, each frame from $\mathsf{K}^{\mathsf{C}}$ is a zigzagmorphic image of a frame from $\mathsf{K}^{\mathsf{C}}_{set}$, by 4.2.7 and 4.2.10 (or alternatively by 4.5.3)[4]. But then, by duality, $(\mathsf{K}^{\mathsf{C}})^{+} \subseteq \mathsf{S}(\mathsf{K}^{\mathsf{C}}_{set})^{+}$. Hence $\mathsf{S}(\mathsf{K}^{\mathsf{C}})^{+} = \mathsf{S}(\mathsf{K}^{\mathsf{C}}_{set})^{+}$. QED

### 4.2.3 REPRESENTATION BY MOSAICS

We now introduce the second concept of our method: mosaics. The next definition is a bit more general than needed for our present purposes, but this generality will be useful when dealing with reducts. We shall expand the similarity type of arrow–frames with the two partial functions $(.)_l$ and $(.)_r$. When a mosaic belongs to the class $\mathsf{K}^{rel}_{rl}$, we can delete these expansions again, because these functions are definable there (see 2.4.7).

DEFINITION 4.2.12 (MOSAICS). Let $\mathcal{F} = \langle W, \mathsf{C}, \mathsf{f}, (.)_l, (.)_r, \mathsf{I}\rangle$ be an arrow–frame expanded with partial functions $\mathsf{f}, (.)_l$ and $(.)_r$.
(i) $\mathcal{F}$ is an $\langle x,y,z\rangle$*–mosaic* iff $\{x,y,z\} \subseteq W$ ($x,y,z$ need not be different), $\mathsf{C}xyz$ holds, and there is no proper subset of $W$ which contains $\{x,y,z\}$ and which is closed under the functions $\mathsf{f}$, $(.)_l$ and $(.)_r$.
The elements $x,y,z$ are the *generators* of the $\langle x,y,z\rangle$–mosaic.
(ii) An $\langle x,y,z\rangle$–mosaic $\mathcal{F}$ is *repairable* if there exists a pair–frame $\langle\langle u,v\rangle,\langle u,w\rangle,\langle w,v\rangle\rangle$–mosaic $\mathcal{G}_{pair}(V)$ with base $\{u,v,w\}$ ($u,v,w$ need not be different), and a surjective function $l : V \longrightarrow F$ such that

- $l\langle u,v\rangle = x$, $l\langle u,w\rangle = y$ and $l\langle w,v\rangle = z$,

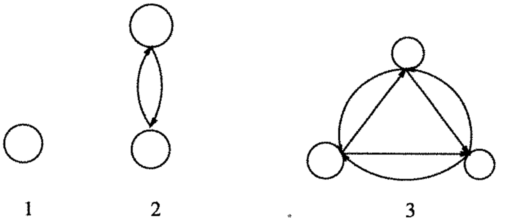
[4]Lemma 4.2.7 gives us only that $\mathsf{S}(\mathsf{K}^{\mathsf{C}})^{+} = \mathbf{HSP}(\mathsf{K}^{\mathsf{C}}_{set})^{+}$. It does so in the following way. It is obvious that $\mathsf{K}^{\mathsf{C}}$ allows filtrations (any filtration works), so $\mathsf{S}(\mathsf{K}^{\mathsf{C}})^{+} = \mathbf{HSP}(\mathbf{Fin}\mathsf{K}^{\mathsf{C}})^{+}$. Now apply the lemma (and use the fact that $\mathbf{SPK} = \mathbf{SPSK}$).

- $l$ is a homomorphism for $\mathsf{C}_V$,
- $(\forall s \in V) : \mathsf{l}_V s \iff \mathsf{l}l(s)$ and
- $l$ commutes with $\mathsf{f}, (.)_l$ and $(.)_r$ in the following strong sense: $l\mathsf{f}_V s = \mathsf{f}ls$ means that if one side is defined, then also the other, otherwise both sides are undefined, and similar for $(.)_l$ and $(.)_r$.

We call the tuple $\langle \mathcal{G}_{pair}(V), l, (u, v, w) \rangle$ a *repair* of $\mathcal{F}$. Sometimes we also call the pair–frame $\mathcal{G}_{pair}(V)$ a repair of $\mathcal{F}$.

REMARK 4.2.13. It is easy to see that in the class of locally square pair–frames $\mathsf{K}^{rel}_{setRS}$ every mosaic is one of the three *square* pair–frames in figure 4.1 below. In case 1, the mosaic is generated by one identity arrow; in case 2, one of the three generators is an identity arrow, and in case 3 none of the generators is an identity arrow. The set of $\mathsf{K}^{rel}_{set}$ mosaics can be described as follows:

$$\{\mathcal{F}_{pair}(\{\langle u,v\rangle, \langle u,w\rangle, \langle w,v\rangle\} \cup X\}) : \begin{array}{l} X \subseteq \{\langle u,u\rangle, \langle v,v\rangle, \langle w,w\rangle, \langle v,u\rangle, \langle w,u\rangle, \langle v,w\rangle\} \\ \text{and } u,v,w \text{ not necessarily different}\} \end{array}$$




(4.1)

Mosaics will be used in the step–by–step construction to repair a situation where we have a pair $s$ in the partially constructed graph, and $\mathsf{C}l_n(s)yz \,\&\, \neg\mathsf{l}y \,\&\, \neg\mathsf{l}z$ holds in the frame to be represented. Then we will add a repair of the $\langle l_n(s), y, z\rangle$–mosaic to the partially constructed graph. The definition of a repair ensures that we only have to repair these situations.

The following fact will be useful later on.

FACT 4.2.14. Let $\mathcal{F}$ be an $\langle x, y, z\rangle$–mosaic and $\langle \mathcal{G}_{pair}(V), l, (u, v, w)\rangle$ a repair of $\mathcal{F}$. Then
(i) For every point $x \in F$, and for every pair $s \in V$ such that $l(s) = x$, and for every function $\mathsf{f}, (.)_l, (.)_r$, the function is defined on $x$ if and only if it is defined on $s$.
(ii) If we view the partial functions as relations, then:

- $l$ is a surjective homomorphism from $\mathcal{G}_{pair}(V)$ onto $\mathcal{F}$,
- $l$ has the zigzag property for $\mathsf{l}, \mathsf{f}, (.)_l$ and $(.)_r$

PROOF. Immediate by the definitions. QED

REPAIRING MOSAICS

The next proposition contains the heart of the proof of lemma 4.2.3. We only need the axioms in the proof of this proposition. For gluing mosaics together, no additional axioms are needed, just as in the "composition–only" case. Recall that $\mathsf{K}^{rel}_{rl}$ is the class of arrow–frames which satisfy conditions $(C_1)-(C_{12})$ from 2.4.9.

PROPOSITION 4.2.15. (i) *Every $\langle x,y,z\rangle$–mosaic $\mathcal{F}\in\mathsf{K}^{rel}_{rl}$ is repairable.*
(ii) *Every $\langle x,y,z\rangle$–mosaic $\mathcal{F}\in\mathsf{K}^{rel}_{rl}$ is* uniquely *repairable up to isomorphism. Hence, given a base $\{u,v,w\}$ we can speak about* the repair *of $\mathcal{F}$.*
(iii) *Let $H\subseteq\{R,S\}$. If an $\langle x,y,z\rangle$–mosaic $\mathcal{F}$ belongs to $\mathsf{K}^{rel}_{rlH}$, then its repair belongs to $\mathsf{K}^{rel}_{setH}$.*

PROOF. (i). Let $\mathcal{F}\in\mathsf{K}^{rel}_{rl}$ be an $\langle x,y,z\rangle$–mosaic. In the proof, we use the $\mathsf{K}^{rel}_{rl}$ theorems $(T_0)$–$(T_3)$ from 2.4.7. Recall that these theorems should be read as if the functions are partial (e.g., $x_l=y_l$ means that either they are both undefined, or they are both defined and $x_l=y_l$).

We get the first insight by looking at the possible I "valuation" of the generators $x,y$ and $z$. The third column in the table below expresses the fact that if two of the three generators are identity arrows, then so is the third one. So we have only the cases with three, one or zero identity generators. All these results follow from conditions $(C_{11})$ and $(C_{12})$. The results in the last column follow from $(T_0)-(T_3)$ and $(C_6)$. This will become obvious if we look at the cases separately below.

| | $x$ | $y$ | $z$ | result | size of domain $F$ |
|---|---|---|---|---|---|
| *1* | I | I | I | $x{=}y{=}z$ | $\lvert F\rvert=1$ |
| *2* | I | I | | impossible | |
| *3* | I | | I | impossible | |
| *4* | I | | | $x\neq y, x\neq z$ | $2\leq\lvert F\rvert\leq 4$ |
| *5* | | I | I | impossible | |
| *6* | | I | | $x=z\neq y$ | $2\leq\lvert F\rvert\leq 4$ |
| *7* | | | I | $x=y\neq z$ | $2\leq\lvert F\rvert\leq 4$ |
| *8* | | | | | $1\leq\lvert F\rvert\leq 9$ |

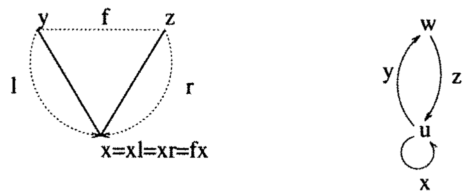
We will look at the remaining cases one by one.

**case 1**. In case 1, by $(T_1)$, the mosaic consists of just one element, and clearly that is isomorphic to the pair–frame $\mathcal{F}_{pair}(\{\langle u,u\rangle\})$.

**case 4**. In case 4, $(C_6)$ and $(C_1)$ imply $\mathsf{f}(y)=z$ & $\mathsf{f}(z)=y$. By $(T_3)$ and $(T_1)$, we have $x=x_l=x_r=\mathsf{f}x=y_l=z_r$. If $y_r$ is not defined, the domain of the mosaic equals $\{x,y,z\}$, and we repair it by the pair–frame at the right in figure 4.2.

These figures should be read as follows. At the left, we draw the mosaic which is to be repaired, and at the right the repair. An $x$ attached to an arrow $\langle u,v\rangle$ means that $x$ is represented by $\langle u,v\rangle$ (in other words $l\langle u,v\rangle=x$).

It follows from the argument given above that the $\langle\langle u,u\rangle,\langle u,w\rangle,\langle w,u\rangle\rangle$–mosaic and the function $l$ as given in figure 4.2 form a repair of the mosaic at the left. This is easily checked using the provided pictures.

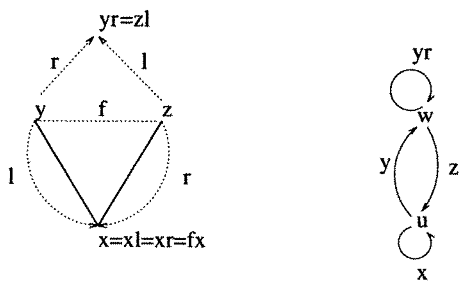



(4.2)

We continue with case 4. If $y_r$ is defined, $(C_4)$ (because $\mathsf{C}yyy_r$ & $\mathsf{F}yz$) implies that $\mathsf{C}y_rzy$. By $(T_1)$, $y_r = (y_r)_r = (y_r)_l = \mathsf{f}(y_r)$. Again by $(T_3)$, $z_l = y_r$, so we add $\langle w, w\rangle$ to the mosaic, and set $l\langle w, w\rangle = y_r$ (see figure 4.3).

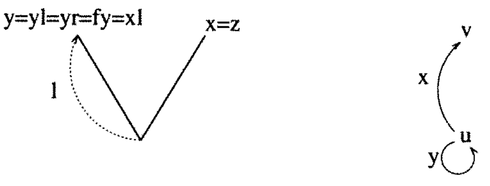
This picture really covers two cases: the one where $y \neq z$, and the one where $y = z$ (which implies that $x = y_r$). For the argument given above, this distinction does not matter: so we really covered both cases. All arguments in the sequel cover the cases when some of the points in the mosaic happen to be equal.

In a case 4 mosaic, the functions $\mathsf{f}, (.)_l$ and $(.)_r$ cannot generate further points, so we are done.




(4.3)

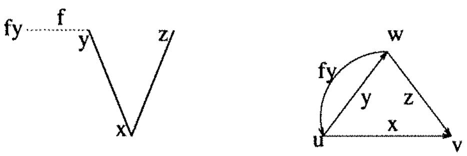
**cases 6 and 7**. If the functions are all defined, cases 6 and 7 are very similar to case 4. We treat case 6 only. Case 7 mosaics are handled similarly. If $x_r$ and $\mathsf{f}(x)$ are not defined, we represent $x$ by $\langle u, v\rangle$ $(u \neq v)$, and $y$ by $\langle u, u\rangle$, and we are done (see figure 4.4). If $x_r$ is defined, we add $\langle v, v\rangle$ as well, and set $l\langle v, v\rangle = x_r$. If $\mathsf{f}(x)$ is defined, $(C_5)$ implies $\mathsf{C}yx\mathsf{f}(x)$ and (if $x_r$ is defined as well) $\mathsf{C}x_r\mathsf{f}(x)x$, which lands us back in the two situations of case 4. We treated all possible case 6 mosaics.




(4.4)

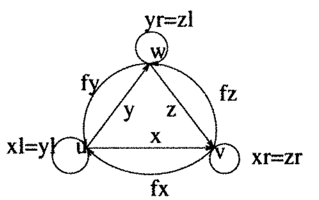
**case 8**. Case 8 finally will be repaired with a mosaic consisting of at least 3 non reflexive pairs. There are $64 = |\mathcal{P}(\{x_l, x_r, \mathsf{f}x, y_r, \mathsf{f}y, \mathsf{f}z\})|$ subcases. The heart of the representation is a triangle $\{\langle u, v\rangle, \langle u, w\rangle, \langle w, v\rangle\}$, with $l\langle u, v\rangle = x$, $l\langle u, w\rangle = y$ and

$l\langle w,v\rangle = z$, and $u, v, w$ are all different. Depending on the presence of other arrows in the mosaic, we have to add more pairs. First, suppose $x_l$ is defined. Then by $(T_3)$, $x_l = y_l$, and we can represent it by $\langle u,u\rangle$. Similarly for $y_r = z_l$ and $z_r = x_r$. If $\mathsf{f}(y)$ is defined, we need to represent that by $\langle w,u\rangle$. Then $\mathsf{C}_V\langle w,v\rangle, \langle w,u\rangle, \langle u,v\rangle$, but by $(C_4)$ also $\mathsf{C}z\mathsf{f}(y)x$ (see figure 4.5).




(4.5)

Use $(C_5)$ in the similar situation where $\mathsf{f}(z)$ is defined, and $(C_4)$ and $(C_5)$ when two or more of $\mathsf{f}(x), \mathsf{f}(y)$ and $\mathsf{f}(z)$ are defined. To see that the function $l$ behaves correctly on parts like $\{\langle u,u\rangle, \langle u,w\rangle, \langle w,u\rangle, \langle w,w\rangle\}$, reason as in case 4. If all functions are defined, the representation looks as in figure 4.6.




(4.6)

We covered all case 8 mosaics. So we covered all possible mosaics, and we have finished the proof of part (i) of the proposition. Parts (ii) and (iii) are immediate by the proof of part (i), in combination with fact 4.2.14(i). QED

MOSAICS OF AN ARROW–FRAME. Mosaics are very small arrow–frames which tend to live in bigger frames. For an arrow–frame $\mathcal{F} = \langle W, \mathsf{C}, \mathsf{f}, \mathsf{I}\rangle \in \mathsf{K}^{rel}_{rl}$, we define the notion of an $\langle x,y,z\rangle$*–mosaic of* $\mathcal{F}$ as follows. An $\langle x,y,z\rangle$–mosaic $\mathcal{G}$ is an $\langle x,y,z\rangle$–mosaic of $\mathcal{F}$ if $G \subseteq W$ and the relations in $\mathcal{G}$ are the restrictions of the relations in $\mathcal{F}$ to $G$. We can say the following things about these $\langle x,y,z\rangle$–mosaics of $\mathcal{F}$.

FACT 4.2.16. (i) Let $H \subseteq \{R, S\}$. If $\mathcal{F} \in \mathsf{K}^{rel}_{rlH}$, then every $\langle x,y,z\rangle$–mosaic of $\mathcal{F}$ is in $\mathsf{K}^{rel}_{rlH}$.
(ii) If $\mathcal{F} \in \mathsf{K}^{rel}_{rl}$, and $\{x,y,z\} \subseteq F$ such that $\mathsf{C}xyz$, then there exists a unique $\langle x,y,z\rangle$–mosaic of $\mathcal{F}$.
(iii) If $\mathcal{F} \in \mathsf{K}^{rel}_{rl}$, and $(.)_l$ or $(.)_r$ is a *total* function, then every point in $F$ belongs to some $\langle x,y,z\rangle$–mosaic of $\mathcal{F}$.
(iv) If $\mathcal{F} \in \mathsf{K}^{rel}_{rl}$, and both $(.)_l$ and $(.)_r$ are not *total* functions, then the only points which are not part of any $\langle x,y,z\rangle$–mosaic of $\mathcal{F}$ are those points $x$ such that $\neg\mathsf{I}x$, and which generate a subframe consisting only of $\{x\}$ or of $\{x, \mathsf{f}x\}$. Each such (point–)generated subframe is a zigzagmorphic image of the pair–frames $\mathcal{F}_{pair}(\{\langle u,v\rangle\})$ and $\mathcal{F}_{pair}(\{\langle u,v\rangle, \langle v,u\rangle\})$ $(u \neq v)$, respectively.

PROOF. By the definitions and 4.2.15. QED

### 4.2.4 Proof of lemma 4.2.3

We are ready for the proof of 4.2.3. We first give a sketch of the proof, after that we define the construction formally.

Proof–idea. Let $\mathcal{F} = \langle W, \mathsf{C}, \mathsf{F}, \mathsf{I}\rangle \in \mathsf{K}_{rl}^{rel}$ be arbitrary. Let $M$ be the set of all $\langle x, y, z\rangle$ mosaics of $\mathcal{F}$. We want to copy the step–by–step procedure from 4.2.7. There, our mosaics were much simpler, and all of them could be repaired in the same way. In the present proof, we repair whole mosaics at once. Then it is easy to see that we only need to repair situations where we have $\mathsf{C}l(x)yz \,\&\, \neg\mathsf{I}y \,\&\, \neg\mathsf{I}z$. (So we only need to repair case 4 and case 8 mosaics.) Since we know that we can repair each mosaic, we know that we can make any necessary reparation. So, by construction, we ensure that the function $l$ has the zigzag property. The more difficult question is whether the function $l$ is a homomorphism. In the simple situation of 4.2.7, we could prove that $l$ was a homomorphism, because we took fresh points every time we connected two mosaics in the representation. Since we took fresh points there, no other $\mathsf{C}_V$ relations came into existence in the representation, except the ones we explicitly constructed ourselves. But here we can always take fresh points too, because we only need to add mosaics with $\mathsf{C}l(x)yz \,\&\, \neg\mathsf{I}y \,\&\, \neg\mathsf{I}z$, whence we should represent $y$ and $z$ by non–identity pairs. But then, again the only new $\mathsf{C}_V$–relations are the ones of the added (represented) mosaic.

Proof of Lemma 4.2.3. Let $H \subseteq \{R, S\}$. We have to show that the class $\mathsf{K}_{rlH}^{rel}$ equals the class $\mathbf{Zig}\mathsf{K}_{setH}^{rel}$. The validity arguments in the beginning of section 2.4.3 show that $\mathsf{K}_{setH}^{rel} \subseteq \mathsf{K}_{rlH}^{rel}$. Since $\mathsf{K}_{rlH}^{rel}$ is closed under zigzagmorphism, this implies $\mathbf{Zig}\mathsf{K}_{setH}^{rel} \subseteq \mathsf{K}_{rlH}^{rel}$. The other side follows immediately from the next two statements.

(i) Every $\mathcal{F} \in \mathsf{K}_{rl}^{rel}$ is a zigzagmorphic image of some $\mathcal{G} \in \mathsf{K}_{set}^{rel}$.
(ii) Let $X \subseteq \{(C_{13}), (C_{14}), (C_{15})\}$. If in addition $\mathcal{F} \models X$, then also $\mathcal{G} \models X$.

(i). Let $\mathcal{F} = \langle W, \mathsf{C}, \mathsf{F}, \mathsf{I}\rangle \in \mathsf{K}_{rl}^{rel}$ be arbitrary. Let $M$ be the set of all $\langle x, y, z\rangle$–mosaics of $\mathcal{F}$. By 4.2.16, every mosaic in $M$ belongs to $\mathsf{K}_{rl}^{rel}$. So, by 4.2.15, all of them are repairable. We now define the necessary construction, interspersing it with the argument why it works.

**Construction**
Let $U$ be an infinite set such that $|U| \geq |W|$. We will use the elements of $U$ to construct the pairs of the representation. The condition ensures that $U$ is large enough for this purpose. Let $P \stackrel{\text{def}}{=} \{x \in W : (\forall yz)\neg(\mathsf{C}xyz \vee \mathsf{C}yxz \vee \mathsf{C}yzx)\}$ be the set of "loose" points (points which are not part of any mosaic). We represent the subframes generated by these loose points as stated in fact 4.2.16.(iv). I.e., by a subset $rep(P)$ of $U \times U$, such that all the representations are pairwise disjoint.

**step 0** Let $V_0 \subset U \times U$ be the *pairwise disjoint* union of $rep(P)$ and all representations of mosaics in $M$ such that $|U \setminus \mathsf{Base}(V_0)| = |U|$. Let $l_0$ be the function given by these representations.

CLAIM 1. $l_0$ is a surjective homomorphism which satisfies the zigzag property for $\mathsf{I}, \mathsf{F}$ and the substitution functions. This last claim is spelled out below.

$$\begin{array}{lll} (\forall x \in V_0) : \mathsf{F} l_0(x) y & \Rightarrow & (\exists y' \in V_0) : \mathsf{F}_V x y' \,\&\, l_0(y') = y \\ (\forall x \in V_0) : \mathsf{C} l_0(x) l_0(x) y \,\&\, \mathsf{I} y & \Rightarrow & (\exists y' \in V_0) : \mathsf{C}_V x x y' \,\&\, l_0(y') = y \\ (\forall x \in V_0) : \mathsf{C} l_0(x) y l_0(x) \,\&\, \mathsf{I} y & \Rightarrow & (\exists y' \in V_0) : \mathsf{C}_V x y' x \,\&\, l_0(y') = y \end{array}$$

PROOF OF CLAIM. $l_0$ is surjective, because we represented all loose points, and (see 4.2.16) all other points belong to at least one mosaic. By fact 4.2.16.(iv), $l_0$ is a zigzagmorphism for the part $rep(P)$. $l_0$ is a homomorphism, because, by 4.2.15 it is one for every mosaic separately, and all the representations of separate mosaics and $rep(P)$ are disjoint. The function $l_0$ satisfies the zigzag property for $\mathsf{I}, \mathsf{F}$ and the substitution functions, because we represented mosaics. ◀

**step n+1** Let $X_n$ be the set of pairs which were added in the previous step. Create for each $s \in X_n$ a function[5] $g_s : \{yz : \mathsf{C} l_n(s) yz \,\&\, \neg \mathsf{I} y \,\&\, \neg \mathsf{I} z\} \longrightarrow (U \setminus \mathsf{Base}(V_n))$ such that

- all $g_s$'s are injective,
- the ranges of the $g_s$'s are pairwise disjoint, and
- $|U \setminus \bigcup_{s \in X_n} (g_s^*)| = |U|$

Such a set of functions clearly exists. They guarantee that we use a brand new element from $U$ for every mosaic we add, and that the set $U \setminus \mathsf{Base}(V_{n+1})$ stays large enough to continue the construction using new elements.

For every $s \in X_n$, and for every $y, z \in W$ such that $\mathsf{C} l_n(s) yz \,\&\, \neg \mathsf{I} y \,\&\, \neg \mathsf{I} z$, we repair the $\langle l_n(s), y, z\rangle$–mosaic of $\mathcal{F}$ by the $\langle\langle s_0, s_1\rangle, \langle s_0, g_s(yz)\rangle, \langle g_s(yz), s_1\rangle\rangle$–mosaic, and add that representation to $V_n$, in this way creating $V_{n+1}$. Define $l_{n+1}$ as the extension of $l_n$ in which the new pairs are mapped as given by the repairs of the mosaics. This can be described formally as follows: define

$$REP_n \stackrel{\text{def}}{=} \{\langle \mathcal{F}_{pair}(S), h, (s_0, s_1, g_s(yz))\rangle : s \in X_n,\ \mathsf{C} l_n(s) yz \,\&\, \neg \mathsf{I} y \,\&\, \neg \mathsf{I} z \text{ and } \langle \mathcal{F}_{pair}(S), h, (s_0, s_1, g_s(yz))\rangle \text{ is the repair of the } \langle l_n(s), y, z\rangle\text{–mosaic of } \mathcal{F}\}$$

Then set:

$$\begin{array}{lllll} V_{n+1} & = & V_n & \cup & \{S : \langle \mathcal{F}_{pair}(S), h, (s_0, s_1, g_s(yz))\rangle \in REP_n\} \\ l_{n+1} & = & l_n & \cup & \{h : \langle \mathcal{F}_{pair}(S), h, (s_0, s_1, g_s(yz))\rangle \in REP_n\} \end{array}$$

$l_{n+1}$ is well defined, because for every repair $\langle \mathcal{F}_{pair}(S), h, (s_0, s_1, g_s(yz))\rangle \in REP_n$ and for every $r \in S \cap V_n$ it holds that $h(r) = l_n(r)$. This follows from the definition of repair of a mosaic.

CLAIM 2. Let $n < \omega$ be arbitrary. Then (i)–(iii) below hold.

(i) $l_n$ is a surjective homomorphism;

(ii) $l_n$ satisfies the zigzag property for $\mathsf{I}$ and $\mathsf{F}$;

(iii) Let $n > 0$. $l_n$ satisfies the zigzag property for $\mathsf{C}$ for all elements of $V_{n-1}$, i.e., $(\forall s \in V_{n-1})(\forall yz \in W) : (\mathsf{C} l_{n-1}(s) yz \Rightarrow (\exists y', z' \in V_n) : \mathsf{C}_V s y' z' \& l_n(y') = y \& l_n(z') = z)$.

[5]If $f : X \longrightarrow Y$ then we use $f^*(X)$ or –if the domain is clear by the context– $f^*$ to denote the range of $f$.

PROOF OF CLAIM. The proof is by induction on $n$. By the previous claim, (i), (ii) and (iii) restricted to the substitution functions hold for $l_0$ and $V_0$. Assume they hold for $l_n$ and $V_n$.
(i). Suppose a pair $s$ was added in the $n$+1–th step. Since $s$ is part of a mosaic, homomorphism is guaranteed for $\mathsf{I}_V$ and $\mathsf{F}_V$. For $\mathsf{C}_V$ we are in precisely the same simple situation as before. Since we always used fresh points, $s$ will only stand in $\mathsf{C}_V$ relations with elements of the mosaic it is a part of, and they are guaranteed by the mosaic proposition.
(ii). That $l_{n+1}$ has the zigzag property for $\mathsf{I}$ and $\mathsf{F}$ is, given that $l_n$ has, immediate because we represented whole mosaics.
(iii). Suppose $s$ was added in step $n$ and $\mathsf{C}l_n(s)yz$. If $\neg\mathsf{I}y$ and $\neg\mathsf{I}z$, we added the needed pre–images in step $n$+1. In the three other cases, the needed pre–images are either $s$ itself or $\langle s_0, s_0\rangle$ or $\langle s_1, s_1\rangle$, and since $s$ is part of a mosaic, these were –by induction hypothesis– already in $V_n$. ◀

**step** $\omega$. Set $V \stackrel{\text{def}}{=} \bigcup_{n<\omega} V_n$ and $l \stackrel{\text{def}}{=} \bigcup_{n<\omega} l_n$.
**End of construction**

CLAIM 3. $l$ is a zigzagmorphism from the pair–frame $\mathcal{F}_{pair}(V)$ onto the arrow–frame $\mathcal{F}$.

PROOF OF CLAIM. $l$ is surjective by step 0. $l_n$ is a homomorphism for every step, so $l$ is one too. $l$ is zigzag, because each added point is repaired in the next step. ◀

With the last claim we finished the proof of part (i) of the lemma. We are almost done.
(ii). If $\mathcal{F}$ satisfies one or more of the conditions $(C_{13}), (C_{14}), (C_{15})$, this means that the corresponding function is always defined. Hence, by 4.2.14.(i), in every representation of a mosaic the corresponding function is always defined, so the representation verifies these conditions. QED

## 4.3 REDUCTS OF RELATIVIZED RELATION ALGEBRAS

We look at subreducts of $\mathbf{SRl}_H\mathsf{RRA}$, for each $H \subseteq \{R, S, T\}$. For $Q \subseteq \{\mathsf{id}, \smile, ;\}$ and $H \subseteq \{R, S, T\}$, we denote the $(\{\vee, \wedge, -, 0, 1\} \cup Q)$–subreduct of the class $\mathbf{SRl}_H\mathsf{RRA}$ by $\mathbf{SRl}_H\mathbf{Rd}_Q\mathsf{RRA}$. This notation is warranted, because (cf. [HMT71]):

$$\mathbf{SRd}_Q\mathbf{SRl}_H\mathsf{RRA} = \mathbf{SRd}_Q\mathbf{Rl}_H\mathsf{RRA} = \mathbf{SRl}_H\mathbf{Rd}_Q\mathsf{RRA} \tag{4.7}$$

REMARK 4.3.1. Andréka–Németi [ANS94a] showed that whenever $T \in H$ and $; \in Q$, the class $\mathbf{SRl}_H\mathbf{Rd}_Q\mathsf{RRA}$ is neither finitely axiomatizable nor decidable. For this reason, we concentrate on the case without $T$. The case where $T \in H$ and $; \notin Q$ is uninteresting, because transitivity does not influence the behaviour of the operators except composition. Whenever $Q \subseteq \{\mathsf{id}, \smile\}$, 4.8 below holds (cf. [ANS94a], Thm 2.1.64).

$$\mathbf{SRl}_{RS}\mathbf{Rd}_Q\mathsf{RRA} = \mathbf{SRl}_{RST}\mathbf{Rd}_Q\mathsf{RRA} = \mathbf{SRd}_Q\mathsf{RRA} \tag{4.8}$$

AXIOMATIZABILITY

We start our investigation with an easy theorem.

THEOREM 4.3.2. *For any* $Q \subseteq \{\mathsf{id}, \smile, ;\}$ *and* $H \subseteq \{R, S, T\}$, *the class* $\mathbf{SRl}_H\mathbf{Rd}_Q\mathsf{RRA}$ *is a quasi-variety.*

PROOF. This follows from the universal algebraic facts (cf. [HMT71] and [HMT85]) that (1) RRA is a variety, hence it is closed under **SPUp**, (2) the operator **SPUp** commutes with **SRl** and **SRd**, (3) 4.7 above, and (4) every class closed under **SPUp** is a quasi-variety. QED

In virtue of the above theorem we will investigate for all interesting choices of $Q$ and $H$ whether the class $\mathbf{SRl}_H\mathbf{Rd}_Q\mathsf{RRA}$ is a variety. It turns out that, for every $Q$, the class $\mathbf{SRlRd}_Q\mathsf{RRA}$ is a finitely axiomatizable canonical variety. If we consider subreducts for other choices of $H$, the situation is not so uniform. Some are not varieties, some are, and for some, we don't know the answer. Table 4.1 lists the results we do have. As a contrast, we add the results for subreducts of RRA (recall that $\mathsf{RRA} = \mathbf{SRl}_{RST}\mathsf{RRA}$) in the fifth column. These results can be found in [ANS94a]. It is easy to show that the subreducts of RRA with composition are varieties, because they have a *discriminator term*. This is not the case when we do not have a transitive relation. For completeness' sake, we add the results for the full language at the bottom.

HOW TO READ TABLE 4.1. In the left column, we list the operators of the subreduct involved. The next four columns stand for the four different relativizations we have studied in the previous section. Each item in the table stands for the class $\mathbf{SRl}_H\mathbf{Rd}_Q\mathsf{RRA}$ in which $Q$ is given by the row and $H$ by the column. A V means that the class $\mathbf{SRl}_H\mathbf{Rd}_Q\mathsf{RRA}$ is a finitely axiomatizable canonical variety. We give the axiomatizations in table 4.2 below. With V* we denote that the class is a variety, but it is not axiomatizable by finitely many equations. A QV means that the class is not a variety, but it is a finitely axiomatizable quasi-variety, and a $\mathsf{QV}^?$ denotes that we only know that it is a quasi-variety (it might still be a variety).

| | any relation SRlRRA | reflexive relation $\mathbf{SRl}_R$RRA | symmetric relation $\mathbf{SRl}_S$RRA | reflexive and symmetric rel. $\mathbf{SRl}_{RS}$RRA | refl., symm. and transitive rel. $\mathbf{SRl}_{RST}\mathsf{RRA} = \mathsf{RRA}$ |
|---|---|---|---|---|---|
| id | V | QV | V | QV | QV |
| $\smile$ | V | QV | V | QV | QV |
| $\smile$, id | V | QV | V | QV | QV |
| ; | V | $\mathsf{QV}^?$ | $\mathsf{QV}^?$ | $\mathsf{QV}^?$ | V* |
| ;, $\smile$ | V | $\mathsf{QV}^?$ | V | $\mathsf{QV}^?$ | V* |
| ;, id | V | V | $\mathsf{QV}^?$ | $\mathsf{QV}^?$ | V* |
| ;, $\smile$, id | V | V | V | V | V* |

TABLE 4.1: SUBREDUCTS OF RELATIVIZED RELATION ALGEBRAS

We summarize the axiomatization results in the following theorem. We prove them in section 4.3.1. The axioms are listed in table 4.2.

THEOREM 4.3.3. *The following classes are all finitely axiomatizable canonical varieties:*

- $(\forall Q \subseteq \{;,\breve{}\,,\mathsf{id}\}) : \mathbf{SRlRd}_Q\mathsf{RRA}$
- $\mathbf{SRl}_S\mathbf{Rd}_{\{\mathsf{id}\}}\mathsf{RRA}$
- $(\forall \{\breve{}\} \subseteq Q \subseteq \{;,\breve{}\,,\mathsf{id}\}) : \mathbf{SRl}_S\mathbf{Rd}_Q\mathsf{RRA}$
- $\mathbf{SRl}_R\mathbf{Rd}_{\{;,\mathsf{id}\}}\mathsf{RRA}$

DECIDABILITY OF THE REDUCTS

Above, we have stated that whenever $T \in H$ and $; \in Q$, the equational theory of the resulting class is undecidable. For all other classes in table 4.1, the results are positive.

THEOREM 4.3.4. *Let* K *be any class occuring in table 4.1. The universal theory of* K *is decidable if and only if* K *is not marked with* $\mathsf{V}^*$.

PROOF. The undecidability results can be found in [ANS94a]. Here, we prove decidability for all classes not marked with $\mathsf{V}^*$. For the three quasi-varieties in the last column, the claim follows from (4.8) above and the argument given below. So we are left with the first four columns. By 4.2.5, we know that the universal theories of the first four classes in the bottom row are decidable. The following chain of reasoning shows that we can decide universal sentences in the reduct-language, using a decision procedure for the full language. Let $\phi$ be a universal sentence in the reduct $Q \subseteq \{\mathsf{id},\breve{}\,,;\}$. Then:

$$
\begin{array}{lcll}
\mathbf{SRl}_H\mathbf{Rd}_Q\mathsf{RRA} & \models \phi & \iff & \text{(by equality (4.7) above)} \\
\mathbf{SRd}_Q\mathbf{SRl}_H\mathsf{RRA} & \models \phi & \iff & \text{(universal sentences are preserved under } \mathbf{S}) \\
\mathbf{Rd}_Q\mathbf{SRl}_H\mathsf{RRA} & \models \phi & \iff & (\phi \text{ is in the language of } Q) \\
\mathbf{SRl}_H\mathsf{RRA} & \models \phi & &
\end{array}
$$

QED

There is also an opposite direction possible. Instead of taking reducts, we can expand the language with operators like the cylindrifications $\mathsf{c}_0$ and $\mathsf{c}_1$, or the universal modality $\Diamond$ which are term-definable in RRA (though not always when we relativize). We can also expand the language with operations which are not even RRA definable, such as the Kleene *. We will look at this direction of research in the next section, and in chapter 6.

## 4.3.1 AXIOMATIZING THE REDUCTS

We investigate the reducts in table 4.1, following the order of that table. Here is a sketch for the first four cases.

### REDUCTS WITHOUT COMPOSITION

ONLY IDENTITY. It is straightforward to see that this subreduct is axiomatizable by the BA axioms whenever $H = \emptyset$ or $H = S$. Relativizing with a reflexive (and symmetric) relation does not yield a variety. To see this, take the complex algebra of the square pair–frame $\mathcal{F}_{pair}(^2\{u,v\})$ $(u \neq v)$. It is easy to check that relativizing with $\{\langle u,v\rangle, \langle v,u\rangle\}$ is a homomorphism onto the 4–element BA in which $\mathsf{id} = 0$. Clearly, this algebra is not representable as a (subalgebra of a) complex algebra of a reflexive (and symmetric) pair–frame. Hence, for $H = R$ or $H = RS$, the class $\mathbf{SRl}_H\mathbf{Rd}_{\{\mathsf{id}\}}\mathsf{RRA}$ is not closed under homomorphisms: so it is not a variety. It is a finitely axiomatizable quasi–variety, however. If we add the quasi–equation $(\mathsf{id} = 0 \Rightarrow 0 = 1)$ to the BA axioms, then every non–trivial algebra contains a non–zero identity. This quasi–equation defines the frame condition $\exists x(x = x) \Rightarrow \exists x(\mathsf{I}x)$.

ONLY CONVERSE. The subreduct with only converse is axiomatizable by axioms $(A_1)$ and $(A_{10})$. This follows from 4.2.16.(iv). If we want a symmetric relativization, we add axiom $(A_{13})$. The counterexample given in the previous paragraph shows, here as well, that relativizing with a *reflexive* relation is not a variety. But this is repairable by adding the quasi–equation $(\tau^{\smile} = -\tau \Rightarrow 1 = 0)$ which defines the frame condition $\exists x(x = x) \Rightarrow \exists x(\mathsf{f}x = x)$.

CONVERSE AND IDENTITY. Clearly, it is sufficient to add axiom $(A_3)$ to the axioms given above to get axiomatizations for $\mathbf{SRlRd}_{\{\mathsf{id},\smile\}}\mathsf{RRA}$ and $\mathbf{SRl}_S\mathbf{Rd}_{\{\mathsf{id},\smile\}}\mathsf{RRA}$, respectively. Again, the above counterexample kills the reflexive relativization. Adding the quasi–equation $(\mathsf{id} = 0 \Rightarrow 0 = 1)$ helps us out here.

CONVERSE, IDENTITY AND THE SUBSTITUTIONS. If we add the substitution operators $\mathsf{s}^0_1$ and $\mathsf{s}^1_0$ to the similarity type of converse and identity, then *all* relativizations become finitely axiomatizable canonical varieties. Because transitivity also does not influence the behaviour of the substitutions, it holds that $\mathbf{SRl}_{RS}\mathbf{Rd}_{\{\mathsf{id},\smile,\mathsf{s}^1_0,\mathsf{s}^0_1\}}\mathsf{RRA} = \mathbf{SRd}_{\{\mathsf{id},\smile,\mathsf{s}^1_0,\mathsf{s}^0_1\}}\mathsf{RRA}$. These results hold, because the proof for the reflexive and symmetric case in [ANS94a] (Thm 2.1.64) goes through.

### REDUCTS WITH COMPOSITION

The subreducts considered so far were extremely simple, because there were only finitely many finite subdirect irreducible algebras to consider. If we add composition, we get infinitely many subdirect irreducible algebras (e.g., every complex algebra of a connected directed graph is one). In order to axiomatize these subreducts, we use the step–by–step construction and the mosaic idea again. In the construction, no axioms are needed, so it again suffices to see whether we can repair mosaics. The proofs are simple adaptations of the proof for the full language. We sketch the changes involved.

ONLY COMPOSITION. For arbitrary relativizations, we treated this subreduct already in 4.2.11. For the others, we do not know the answer.

COMPOSITION AND CONVERSE. If we consider both converse and composition, the representation is a simple adaptation of the case with only composition. Define mosaics as before, but forget about the functions for $s^1_0$ and $s^0_1$ (so they are only closed under "converse arrows"). Represent each mosaic using non–identity pairs only. It is straightforward to see that we only need $(A_4)$ and $(A_5)$ besides the converse axioms $(A_1)$ and $(A_{10})$. For a symmetric relativization, it is enough to add $(A_{13})$. We do not know the answer for reflexive and for reflexive–and–symmetric relativizations.

COMPOSITION AND IDENTITY. This subreduct is the most interesting one. In several applications of arrow logic, the converse operator does not have a natural interpretation, so it would be nice if it is not needed. It is also interesting, because we do need an extra axiom besides the ones mentioning composition and identity we had already. In a sense, this subreduct shows the "hidden power" of the axiom $(A_4)$ (recall that $(A_4)$ is the RA axiom $(RA_5)$). We do not know whether relativizing with symmetric or reflexive–and–symmetric relations is a variety.

We use the same construction as before. Redefine the concept of a mosaic by forgetting the function for converse. We have to repeat the analysis of 4.2.15, and check carefully whether we used axioms mentioning converse to prove something concerning identity and composition only. Clearly, we need all axioms which do not mention converse. Having them, we get the same five possible situations as in the proof of 4.2.15. For case 1, we do not need converse. In case 4, we used condition $(C_4)$, which mentions the frame relation F for converse, to show that, if $y_1$ was defined, we had $\mathsf{C}y_1zy$. So here we need a new axiom, namely

$$(C_{22})\quad \mathsf{C}xyz \;\&\; \mathsf{I}x \;\&\; \mathsf{C}yyv \;\&\; \mathsf{I}v \Rightarrow \mathsf{C}vzy$$

In cases 6, 7 and 8, there are no "converse arrows". So, if we find a canonical equation defining $(C_{22})$, we are done. We propose the following:

$$(A_{22})\quad (\mathsf{x};(\mathsf{id}\wedge-(\mathsf{y};\mathsf{x})));\mathsf{y} \leq -\,\mathsf{id}$$

PROPOSITION 4.3.5. *Assume condition $(C_{12})$. Then for any arrow frame $\mathcal{F} = \langle W, \mathsf{C}, \mathsf{I}\rangle$, $\mathcal{F} \models (C_{22}) \iff \mathcal{F} \models (A_{22})$. The equation $(A_{22})$ is canonical.*

PROOF. $(A_{22})$ is a Sahlqvist equation. A positive equation which does the same job was found by András Simon. It is $\mathsf{id}\wedge((\mathsf{x};(\mathsf{y}\wedge\mathsf{id}));\mathsf{z}) \leq (\mathsf{x};(\mathsf{y}\wedge\mathsf{id}\wedge(\mathsf{z};\mathsf{x})));\mathsf{z}$. QED

Clearly, if we add axioms $(A_{14})$ and $(A_{15})$ to the ones above, we get an axiomatization for the reflexive relativization.

SUMMARY OF THE RESULTS. We summarize these results about subreducts in table 4.2. We do not mention the BAO axioms in this table, but of course they are assumed as well.

| Variety | Axiomatizable by axioms |
|---|---|
| $\mathbf{SRlRd}_{\{\mathsf{id}\}}\mathsf{RRA}$ | $\emptyset$ |
| $\mathbf{SRl}_S\mathbf{Rd}_{\{\mathsf{id}\}}\mathsf{RRA}$ | $\emptyset$ |
| $\mathbf{SRlRd}_{\{\smile\}}\mathsf{RRA}$ | $(A_1), (A_{10})$ |
| $\mathbf{SRl}_S\mathbf{Rd}_{\{\smile\}}\mathsf{RRA}$ | $(A_1), (A_{10}), (A_{13})$ |
| $\mathbf{SRlRd}_{\{\smile,\mathsf{id}\}}\mathsf{RRA}$ | $(A_1), (A_3), (A_{10})$ |
| $\mathbf{SRl}_S\mathbf{Rd}_{\{\smile,\mathsf{id}\}}\mathsf{RRA}$ | $(A_1), (A_3), (A_{10}), (A_{13})$ |
| | |
| $\mathbf{SRlRd}_{\{;\}}\mathsf{RRA}$ | $\emptyset$ |
| $\mathbf{SRlRd}_{\{;,\smile\}}\mathsf{RRA}$ | $(A_1), (A_4), (A_5), (A_{10})$ |
| $\mathbf{SRl}_S\mathbf{Rd}_{\{;,\smile\}}\mathsf{RRA}$ | $(A_1), (A_4), (A_5), (A_{10}), (A_{13})$ |
| $\mathbf{SRlRd}_{\{;,\mathsf{id}\}}\mathsf{RRA}$ | $(A_2), (A_7) - (A_9), (A_{11}), (A_{12}), (A_{22})$ |
| $\mathbf{SRl}_R\mathbf{Rd}_{\{;,\mathsf{id}\}}\mathsf{RRA}$ | $(A_2), (A_7) - (A_9), (A_{11}), (A_{12}), (A_{14}), (A_{15}), (A_{22})$ |

TABLE 4.2: AXIOMATIZATIONS OF SUBREDUCTS OF RELATIVIZED RELATION ALGEBRAS.

## 4.4 ADDING THE DIFFERENCE OPERATOR

We add the *difference operator* to the similarity type of relation algebras. This operator adds quite some expressive power to the language (see e.g., de Rijke [Rij93], Sain [Sai88], Venema [Ven91]). A variety with this operator becomes a discriminator variety, so all the powerful tools and techniques of these varieties become available. The difference operator is a natural operator to add to this similarity type. As is shown in [Ven91], it is term–definable[6] on the square algebras of relations. It is however not term–definable on the relativized ones we studied in this chapter[7]. It will turn out that if we add the difference operator to the variety $\mathbf{SRl}_{RS}\mathsf{RRA}$, we keep decidability and finite axiomatizability (theorems 4.4.4 and 4.4.3).

This section is organized as follows. We start with the main results. In the following section, we gently introduce the difference operator by showing what happens if we add it to Boolean Algebras. In section 4.4.3, we do some correspondence theory and prove the finite axiomatizability theorem. In an appendix to this section, we show how one can obtain an easy finite axiomatizability result if one allows the so called *irreflexivity rule*.

### 4.4.1 INTRODUCTION AND RESULTS

DEFINITION 4.4.1. Let $W$ be a set. We define the following unary operations from

[6] Define it as $\mathsf{D}\,\tau \stackrel{\text{def}}{=} (1;\tau;-\mathsf{id}) \vee (-\mathsf{id}\,;\tau;1)$. See Def 3.3.7 and Prop 3.3.8 in [Ven91].

[7] This follows from the fact that these relativized varieties are not *discriminator varieties* ([ANS94a]), and any variety containing the difference operator has a discriminator term.

$\mathcal{P}(W)$ to $\mathcal{P}(W)$. For $\mathsf{x} \subseteq W$, set:

$$\mathsf{D}^W \mathsf{x} \stackrel{\text{def}}{=} \{w \in W : (\exists v \in W) : v \neq w \ \&\ v \in \mathsf{x}\}$$
$$\Diamond^W \mathsf{x} \stackrel{\text{def}}{=} \{w \in W : (\exists v \in W) : \ v \in \mathsf{x}\}$$

If there is no danger of confusion, we suppress the superscript $^W$ with these operations (but it should be noted that their behaviour depends on the choice of $W$). For obvious reasons, the operation $\mathsf{D}$ is called the *difference operator.* $\Diamond$ is the universal modality we saw before. It is easy to see that we can term–define the universal modality as $\Diamond \mathsf{x} \stackrel{\text{def}}{=} \mathsf{D}\,\mathsf{x} \cup \mathsf{x}$. Since $\mathsf{D}$ is a unary operator, it's meaning is given on frames by a binary relation. On the intended "concrete" frames this will be the *inequality* relation, on "abstract" frames we will use the letter $\mathsf{R}$ for this relation. We will also use the notation given above for abstract operators.

For $H \subseteq \{R, S, T\}$, we define the following classes of frames and algebras.

$$\mathsf{KD}^{rel}_{setH} \stackrel{\text{def}}{=} \{\mathcal{F} = \langle V, \mathsf{C}_V, \mathsf{f}_V, \mathsf{I}_V, \neq\rangle : \langle V, \mathsf{C}_V, \mathsf{f}_V, \mathsf{I}_V\rangle \in \mathsf{K}^{rel}_{setH}\}$$
$$\mathbf{SRl}_H\mathsf{RRA{+}D} \stackrel{\text{def}}{=} \mathbf{SP}(\mathsf{KD}^{rel}_{setH})^+ \stackrel{\text{def}}{=} \mathbf{SP}\{\langle \mathfrak{P}(V), \circ^V, {}^{-1^V}, \mathsf{Id}^V, \mathsf{D}^V\rangle : V \text{ is an } H \text{ relation}\}$$
$$\mathsf{RRA{+}D} \stackrel{\text{def}}{=} \mathbf{SP}\{\mathcal{F} = \langle V, \mathsf{C}_V, \mathsf{F}_V, \mathsf{I}_V, \neq\rangle : V = U \times U \text{ for some set } U\}^+$$

REMARK 4.4.2. As we said, RRA is term–definably equivalent with RRA+D (cf. [Ven91] Prop 3.3.8). On the other hand, RRA $= \mathbf{SRl}_{RST}$RRA, but RRA+D is a strict subvariety of $\mathbf{SRl}_{RST}$RRA+D. It is strict, because the equality $\mathsf{D}\,\mathsf{x} = (1;\mathsf{x};-\mathsf{id}) \vee (-\mathsf{id}\,;\mathsf{x};1)$ (i.e., the definition of D on the squares) does not hold in the latter class. In this section, we concentrate on the class $\mathbf{SRl}_{RS}$RRA+D.

AXIOMATIZABILITY. The next theorem is a joint result with Szabolcs Mikulás, István Németi and András Simon.

THEOREM 4.4.3. $\mathbf{SRl}_{RS}$RRA+D *is a finitely axiomatizable canonical variety.*

A purely algebraic proof, without any correspondence results can be found in [MMNS94]. Here we will follow another road[8]. We show that the class $\mathbf{SRl}_{RS}$RRA+D equals a finite axiomatizable canonical variety $\mathsf{S(K)}^+$ for K some class of arrow–frames expanded with a binary relation R. We prove the theorem in section 4.4.3.

DECIDABILITY. Since we enlarged the expressive power of our language considerably, getting closer to the undecidable variety RRA, it becomes an interesting question whether or not the equational theory of the class $\mathbf{SRl}_{RS}$RRA+D is *decidable*, or even generated by its finite members. The first question is answered positively, the second is still open.

[8]It is quite interesting to see, how the two different styles of proof lead to two quite different axiomatizations. The axioms given here are the equations corresponding to the (intuitive) frame conditions, which we found in the representation proof. The axioms given in [MMNS94] are a direct description of the behaviour of the difference operator, and are maybe more intuitive as (quasi–)equations themselves.

THEOREM 4.4.4 (ANDRÉKA, MIKULÁS, NÉMETI).
*The equational theory of* $\mathbf{SRl}_{RS}\mathsf{RRA}+\mathsf{D}$ *is decidable.*

PROOF. Cf. [AMN94]. QED

### 4.4.2 FIRST EXERCISE: BOOLEAN ALGEBRAS WITH D

To get some intuition about the difference operator, we briefly sketch how it behaves when added to BA. The difficulties we encounter later are very similar to those in this simple case. A good overview article on D is chapter 3 in de Rijke [Rij93]. Theorems 4.4.5 and 4.4.6 below were proved independently by several authors; for their history, see [Rij93].

Define $\mathsf{KD}_{set}$ as the class of all frames $\mathcal{F} = \langle W, \neq \rangle$. Recall that BA can be defined as $\mathbf{IS}\{\mathfrak{P}(W) : W \text{ is a set}\}$. We define the class BA+ D as follows:

$$\mathsf{BA}+\mathsf{D} \stackrel{\text{def}}{=} \mathbf{SP}(\mathsf{KD}_{set})^{+} \stackrel{\text{def}}{=} \mathbf{SP}\{\langle \mathfrak{P}(W), \mathsf{D}^{W}\rangle : W \text{ is some set}\}$$

Define the class KD as the class of all frames $\mathcal{F} = \langle W, \mathsf{R}\rangle$ satisfying conditions $(C_{16})$ and $(C_{17})$ below. $\mathsf{BA}^{\mathsf{D}}((A_{16}) - (A_{17}))$ denotes the class of all BAO's of BAO–type $\langle \mathsf{D}, 1\rangle$ which satisfy equations $(A_{16})$ and $(A_{17})$ below.

$$
\begin{array}{llll}
(C_{16}) & \mathsf{R}xy \Rightarrow \mathsf{R}yx & (A_{16}) & \mathsf{x} \wedge \mathsf{D}\,\mathsf{y} \leq \mathsf{D}(\mathsf{y} \wedge \mathsf{D}\,\mathsf{x}) \\
(C_{17}) & \mathsf{R}xy \;\&\; \mathsf{R}yz \Rightarrow x = z \vee \mathsf{R}xz & (A_{17}) & \mathsf{D}\,\mathsf{D}\,\mathsf{x} \leq (\mathsf{x} \vee \mathsf{D}\,\mathsf{x})
\end{array}
$$

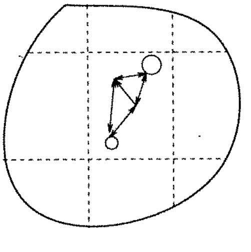
Note that these two equations ensure that the defined $\Diamond$ becomes a *complemented closure operator* (in modal–logical terms: an S5–type modality).

THEOREM 4.4.5. $\mathsf{BA}^{\mathsf{D}}((A_{16}) - (A_{17})) = \mathsf{S}(\mathsf{KD})^{+} = \mathsf{BA}+\mathsf{D}$

PROOF. The first equality follows from the positive form of the equations. Because BA+ D is defined as $\mathbf{SP}(\mathsf{KD}_{set})^{+}$, it is sufficient to show that 1) $\mathsf{KD}_{set} \subseteq \mathsf{KD}$, and 2) every frame in KD is a disjoint union of zigzagmorphic images of frames from $\mathsf{KD}_{set}$. Clearly $(C_{16})$ and $(C_{17})$ hold for the inequality relation, so 1) holds. To show the second equality, we reason at the frame level. To pin down the inequality relation, we need that it is *irreflexive* and *almost universal.*

$$
\begin{array}{ll}
(Irr) & (\forall x) : \neg \mathsf{R}xx \\
(AU) & (\forall xy) : x \neq y \Rightarrow \mathsf{R}xy
\end{array}
$$

Because irreflexivity is not preserved under zigzagmorphisms, it is not definable by an equation. $(AU)$ is not definable either, because it is not preserved under disjoint unions. The next picture shows an average KD frame. It consists of a disjoint union of frames in which R is almost universal.



So we know that the problems are caused by disjoint unions and zigzagmorphisms. As happened before however, they can be solved by them as well.

CLAIM. Every $\mathcal{F} \in \mathsf{KD}$ is a disjoint union of zigzagmorphic images of frames from $\mathsf{KD}_{set}$.

PROOF OF CLAIM. The disjoint union part is clear by the picture. Let $\mathcal{F} = \langle W, \mathsf{R}\rangle \models (AU)$. In order to show that $\mathcal{F}$ is a zigzagmorphic image of a frame $\langle W', \neq\rangle$, we just have to *copy the* R*-reflexive points*. So, let $W' \stackrel{\text{def}}{=} W \cup \{x' : \mathsf{R}xx\}$, and define $\mathcal{G} = \langle W', \neq\rangle$. Define the obvious zigzagmorphism $l : W' \longrightarrow W$, as $l(x) = x$ and $l(x') = x$. ◀

The theorem follows from the claim using 2.2.5, 2.2.6 and the universal algebraic fact that $\mathbf{PS} \leq \mathbf{SP}$. QED

THEOREM 4.4.6. $\mathsf{Eq}(\mathsf{BA}+\mathsf{D})$ *is decidable.*

PROOF. Take a minimal filtration of a $\mathsf{KD}_{set}$ model. It is straightforward to show that the filtration satisfies $(AU)$. Hence it satisfies $(C_{16})$ and $(C_{17})$. By 4.4.5 this is enough. QED

### 4.4.3 CORRESPONDENCE AND REPRESENTATION

We prove the finite axiomatizability result for $\mathbf{SRl}_{RS}\mathsf{RRA}+\mathsf{D}$ in the same spirit as we did without the $\mathsf{D}$ operator. First we define a finitely axiomatizable class of arrow-frames expanded with a binary relation R. Then we show that each frame of that class can be represented as a member of $\mathsf{KD}^{rel}_{setRS}$.

#### CORRESPONDENCE

Define the following abbreviations[9]: $\mathsf{dom}\,\mathsf{x} \stackrel{\text{def}}{=} \mathsf{id} \wedge (\mathsf{x};1)$ and $\mathsf{ran}\,\mathsf{x} \stackrel{\text{def}}{=} \mathsf{id} \wedge (1;\mathsf{x})$. Recall that $\Diamond\mathsf{x}$ was defined using the difference operator as $\mathsf{x} \vee \mathsf{D}\,\mathsf{x}$. We abbreviate $(x = y \vee \mathsf{R}xy)$ by $\mathsf{U}x$. Consider the conditions $(C_{18}) - (C_{21})$ in table 4.3 below on arrow-frames $\mathcal{F} = \langle W, \mathsf{C}, \mathsf{F}, (.)_l, (.)_r, \mathsf{I}, \mathsf{R}\rangle$ expanded with a binary relation R, which interprets the difference operator. We assume that $(.)_l$ and $(.)_r$ are total functions. Conditions $(C_{18})$ and $(C_{19})$ express that the frame relation for the universal modality $\Diamond$ contains all other frame relations. We will use $(C_{20})$ in its equivalent form $x_l = y_l \,\&\, y_r = z_l \,\&\, \neg\mathsf{R}x_r z_r \Rightarrow \mathsf{C}xyz$. *If* R *would be the inequality relation*, then $(C_{21})$ states that an arrow is uniquely determined by its domain and range, and $(C_{20})$ together with its inverse $\mathsf{C}xyz \Rightarrow x_l = y_l, y_r = z_l \,\&\, z_r = x_r$ (i.e., $(T_3)$), states that composition can be characterized in terms of the domain and range functions. Clearly, whenever R is the inequality relation, these four conditions are valid on locally square pair-frames.
Define the following class of frames:

$$\mathsf{KD}^{rel}_{rlRS} \stackrel{\text{def}}{=} \{\mathcal{F} = \langle W, \mathsf{C}, \mathsf{f}, \mathsf{I}, \mathsf{R}\rangle : \langle W, \mathsf{C}, \mathsf{f}, \mathsf{I}\rangle \in \mathsf{K}^{rel}_{rlRS} \,\&\, \mathcal{F} \models (C_{16}) - (C_{21})\}$$

[9] Note that $\mathsf{dom}$ is the conjugate of $\mathsf{s}^1_0$, and $\mathsf{ran}$ the conjugate of $\mathsf{s}^0_1$ in the sense of [JT52]. I.e., we can describe the meaning of $\mathsf{dom}\,\mathsf{x}$ as the set $\{\mathsf{f}^1_0 x : x \in \tau\}$.

| | | | |
|---|---|---|---|
| $(C_{18})$ | $\mathsf{F}xy \Rightarrow \mathsf{U}xy$ | $(A_{18})$ | $\mathsf{y}^{\smile} \leq \Diamond\mathsf{y}$ |
| $(C_{19})$ | $\mathsf{C}xyz \Rightarrow \mathsf{U}xy \;\&\; \mathsf{U}xz$ | $(A_{19})$ | $\mathsf{y;z} \leq \Diamond\mathsf{y} \wedge \Diamond\mathsf{z}$ |
| $(C_{20})$ | $x_l = y_l \;\&\; y_r = z_l \Rightarrow \mathsf{C}xyz \vee \mathsf{R}x_r z_r$ | $(A_{20})$ | $\mathsf{s}_0^1\,\mathsf{dom}(\mathsf{y};\mathsf{dom}\,\mathsf{z}) \leq \mathsf{y;z} \vee \mathsf{s}_1^0\,\mathsf{D}\,\mathsf{ran}\,\mathsf{z}$ |
| $(C_{21})$ | $\mathsf{R}xy \iff \mathsf{R}x_l y_l \vee \mathsf{R}x_r y_r$ | $(A_{21})$ | $\mathsf{D}\,\mathsf{y} = \mathsf{s}_0^1\,\mathsf{D}\,\mathsf{dom}\,\mathsf{y} \vee \mathsf{s}_1^0\,\mathsf{D}\,\mathsf{ran}\,\mathsf{y}$ |

TABLE 4.3: CONDITIONS AND EQUATIONS FOR ARROW–FRAMES WITH D

The next proposition shows that we can indeed define these frame conditions by the given canonical equations[10]. Recall that every arrow–frame satisfying conditions $(C_1)$ – $(C_{15})$ is a zigzagmorphic image of a locally square pair–frame (see 4.2.3).

PROPOSITION 4.4.7. (i) *Equations* $(A_{18}) - (A_{21})$ *are canonical;*
(ii) *Assume that an expanded arrow–frame* $\mathcal{F} = \langle W, \mathsf{C}, \mathsf{f}, \mathsf{I}, \mathsf{R}\rangle$ *satisfies conditions* $(C_1)$ – $(C_{15})$. *Then, for* $18 \leq i \leq 21$, $\mathcal{F} \models (C_i) \iff \mathcal{F} \models (A_i)$.

PROOF. (i). $(A_{18}) - (A_{21})$ are positive equations.
(ii). This follows from a straightforward Sahlqvist computation. QED

SOME CONSEQUENCES OF $(C_1) - (C_{20})$. Conditions $(D_1) - (D_4)$ below are derivable from $(C_1) - (C_{20})$. $(D_1)$ and $(D_2)$ are just variants of $(C_{20})$. $(D_3)$ and $(D_4)$ express that if $x_l = y_l$ and $x_r = y_r$ and one of the two pairs is R irreflexive (i.e., an abstract singleton), then $x$ equals $y$.

PROPOSITION 4.4.8. *The following theorems follow from conditions* $(C_1) - (C_{20})$:

| | | | |
|---|---|---|---|
| $(D_1)$ | $\neg\mathsf{R}x_l y_l \;\&\; y_r = z_l \;\&\; z_r = x_r \Rightarrow \mathsf{C}xyz$ | $(D_3)$ | $x_l = y_l \;\&\; \neg\mathsf{R}x_r y_r \Rightarrow x = y$ |
| $(D_2)$ | $x_l = y_l \;\&\; \neg\mathsf{R}y_r z_l \;\&\; z_r = x_r \Rightarrow \mathsf{C}xyz$ | $(D_4)$ | $\neg\mathsf{R}x_l y_l \;\&\; x_r = y_r \Rightarrow x = y$ |

PROOF. Recall conditions $(T_0) - (T_3)$ from 2.4.7. $(D_1)$ and $(D_2)$ follow from $(C_{20})$, $(C_4)$, $(C_5)$ and $(T_2)$. For $(D_3)$, assume its antecedent. Use $(T_1)$ and $(T_2)$ to derive that $(x_l)_l = x_l, \neg\mathsf{R}x_r(\mathsf{f}y)_l \;\&\; (\mathsf{f}y)_r = (x_l)_r$. Then $(D_2)$ implies that $\mathsf{C}x_l x\mathsf{f}y$, and $(C_6)$ that $\mathsf{ff}y = x$. But then, idempotence of f implies $x = y$. $(D_4)$ follows easily from $(D_3)$. QED

$(D_3)$ and $(D_4)$ are similar to $(C_{21})$, but $(C_{21})$ does not follow from $(C_1) - (C_{20})$.

PROPOSITION 4.4.9. $(C_1) - (C_{20}) \not\models (C_{21})$.

PROOF. Take the two–element $\langle x_l, x, x\rangle$–mosaic $\mathcal{F} \in \mathsf{K}^{rel}_{rlRS}$ (here $x_r = x_l$) expanded with R which validates $(\mathsf{R}xx \;\&\; \mathsf{R}xx_l \;\&\; \neg\mathsf{R}x_l x_l)$ (hence also $\neg\mathsf{R}x_r x_r$). It satisfies $(C_1)$ – $(C_{20})$, but not $(C_{21})$. QED

[10] The assumption that a frame should satisfy all the axioms $(C_1) - (C_{15})$ is unnecessarily strong.

REPRESENTATION (PROOF OF THEOREM 4.4.3)

The finite axiomatizability theorem follows in a straightforward way from the next representation lemma. We prove the theorem after the lemma. Recall that $(AU)$ denotes $(x \neq y \Rightarrow \mathsf{R}xy)$.

LEMMA 4.4.10. (i) *Each* $\mathcal{F} \in \mathsf{KD}^{rel}_{rlRS}$ *consists of a disjoint union of frames satisfying* $(AU)$.
(ii) *Each* $\mathcal{F} \in \mathsf{KD}^{rel}_{rlRS}$ *which satisfies* $(AU)$ *is a zigzagmorphic image of some* $\mathcal{G} \in \mathsf{KD}^{rel}_{setRS}$.

PROOF. (i). Let $\mathcal{F} = \langle W, \mathsf{C}, \mathsf{f}, \mathsf{I}, \mathsf{R}\rangle \in \mathsf{KD}^{rel}_{rlRS}$. Define a binary relation $\equiv$ on $W$ as $(x \equiv y \stackrel{\text{def}}{\Longleftrightarrow} x = y \vee \mathsf{R}xy)$. Conditions $(C_{16})$ and $(C_{17})$ imply that $\equiv$ is an equivalence relation. We denote the equivalence class of $x$ by $\overline{x} \stackrel{\text{def}}{=} \{y \in W : x \equiv y\}$. Define for each equivalence class a frame $\mathcal{F}_{\overline{x}} \stackrel{\text{def}}{=} \langle \overline{x}, \mathsf{C}', \mathsf{f}', \mathsf{I}', \mathsf{R}'\rangle$ in which the relations are the restrictions to $\overline{x}$. We claim that each $\mathcal{F}_{\overline{x}} \models (AU)$, and $\mathcal{F}$ is a disjoint union of the system of frames $\{\mathcal{F}_{\overline{x}} : x \in F\}$.This proves part (i). The first part of the claim is immediate. For the second, it suffices to show that each $\mathcal{F}_{\overline{x}}$ is a subframe of $\mathcal{F}$ generated by $\overline{x}$, which is precisely the point of conditions $(C_{18})$ and $(C_{19})$.

(ii). The proof of part (ii) consists of two steps, corresponding to the two things which can go wrong with the accessibility relation of the difference operator. First we show that $\mathcal{F}$ is a zigzagmorphic image of a pair–frame expanded with a relation $\mathsf{R}$ which satisfies $(AU)$. In the second step, we make this relation irreflexive, thereby turning it into the inequality relation. These two steps are given in the schema below.

| | | step I | | step II | |
|---|---|---|---|---|---|
| full language | $\mathcal{F} \in \mathsf{KD}^{rel}_{rlRS}$ | $\overset{l}{\twoheadleftarrow}$ | $\mathcal{G}_{pair}(V) \in \mathsf{KD}^{rel}_{rlRS}$ | $\overset{p}{\twoheadleftarrow}$ | $\mathcal{H}_{pair}(H) \in \mathsf{KD}^{rel}_{setRS}$ |
| | $\vdots$ | | $\vdots$ | | |
| D–free reduct | $\mathcal{F}^* \in \mathsf{K}^{rel}_{rlRS}$ | $\overset{l^*}{\twoheadleftarrow}$ | $\mathcal{G}^*{}_{pair}(V^*) \in \mathsf{K}^{rel}_{setRS}$ | | |

Let $\mathcal{F} = \langle W, \mathsf{C}, \mathsf{f}, \mathsf{I}, \mathsf{R}\rangle \in \mathsf{KD}^{rel}_{rlRS}$ satisfy $(AU)$. By lemma 4.2.3, we may assume that the R–free reduct $\mathcal{F}^*$ of $\mathcal{F}$ is a zigzagmorphic image, say by function $l^*$, of a pair–frame $\mathcal{G}^*{}_{pair}(V^*) = \langle V^*, \mathsf{C}_{V^*}, \mathsf{f}_{V^*}, \mathsf{I}_{V^*}\rangle$, for some reflexive and symmetric relation $V^*$ with base $U^*$.

STEP I. The problem with the representation $\mathcal{G}^*{}_{pair}(V^*)$ is that it may contain two different pairs $x$ and $y$ which get mapped to one R–irreflexive point in $\mathcal{F}$. This will prevent extending the zigzagmorphism $l^*$ to one for $\mathsf{R}$ as well. To eliminate this problem we create a new pair–frame $\mathcal{G}_{pair}(V)$. Define an equivalence relation $\equiv$ on the base $U^*$ as follows:

$$(\forall u, v \in U^*) : u \equiv v \stackrel{\text{def}}{\Longleftrightarrow} u = v \text{ or } \neg \mathsf{R}l^*\langle u, u\rangle, l^*\langle v, v\rangle$$

CLAIM 1. (i) $\equiv$ is an equivalence relation;
(ii) $u \equiv v \Rightarrow l^*\langle u, u\rangle = l^*\langle v, v\rangle$.

PROOF OF CLAIM. Because $\mathcal{F} \models (AU)$. ◀

Set,

$$U \stackrel{\text{def}}{=} U^*/\equiv$$
$$V \stackrel{\text{def}}{=} \{\langle u/\equiv, v/\equiv\rangle \in U \times U : \langle u, v\rangle \in V^*\}$$

Define a function $l : V \longrightarrow F$ as $l\langle u/\equiv, v/\equiv\rangle \stackrel{\text{def}}{=} l^*\langle u', v'\rangle$, for some $u' \in u/\equiv$ and $v' \in v/\equiv$. Note that, by the definition of $V$, for every pair $\langle u/\equiv, v/\equiv\rangle \in V$, there exist a pair $\langle u', v'\rangle \in V^*$ such that $u \equiv u'$ and $v \equiv v'$. Hence, $l$ is defined for every element in $V$. The next claim states that this is a real definition.

CLAIM 2. $l$ is well defined. That is, for every $\langle u, v\rangle, \langle u', v'\rangle \in V^*$, if $u \equiv u'$ and $v \equiv v'$, then $l^*\langle u, v\rangle = l^*\langle u', v'\rangle$.

PROOF OF CLAIM. Suppose $\langle u, v\rangle, \langle u', v'\rangle \in V^*$, $u \equiv u'$ and $v \equiv v'$. We have four cases, according to whether $u = u'$ and $v = v'$. If $u = u'$ and $v = v'$, the statement is trivial. So assume otherwise:

Case 2: $[u \neq u' \,\&\, v \neq v']$. Then the definition of $\equiv$ implies that $\neg \mathsf{R} l^*\langle u, u\rangle, l^*\langle u', u'\rangle$ and $\neg \mathsf{R} l^*\langle v, v\rangle l^*\langle v', v'\rangle$ hold. Since $l^*$ is a zigzagmorphism for the relational operators, this means that $\neg \mathsf{R}(l^*\langle u, v\rangle)_l, (l^*\langle u', v'\rangle)_l$ and $\neg \mathsf{R}(l^*\langle u, v\rangle)_r, (l^*\langle u', v'\rangle)_r$ hold. Then condition $(C_{21})$ implies that $\neg \mathsf{R} l^*\langle u, v\rangle, l^*\langle u', v'\rangle$. So, by $(AU)$, $l^*\langle u, v\rangle = l^*\langle u', v'\rangle$.

Case 3 and 4: $[u = u' \,\&\, \neg \mathsf{R} l^*\langle v, v\rangle, l^*\langle v', v'\rangle\,]$ and $[\neg \mathsf{R} l^*\langle u, u\rangle, l^*\langle u', u'\rangle \,\&\, v = v']$. These cases are solved in a similar way, but now using conditions $(D_3)$ and $(D_4)$ from 4.4.8. ◀

To finish the first step of the proof, define an accessibility relation $\mathsf{R}_V$ on the pair–frame $\mathcal{G}_{pair}(V)$ as $\mathsf{R}_V xy \stackrel{\text{def}}{\Longleftrightarrow} \mathsf{R} l(x) l(y)$. Call this frame $\mathcal{G} = \langle V, \mathsf{C}_V, \mathsf{f}_V, \mathsf{I}_V, \mathsf{R}_V\rangle$. The next claim states that we have accomplished our first goal.

CLAIM 3. (i) $V$ is a reflexive and symmetric relation;

(ii) $\mathcal{G} \models x \neq y \Rightarrow \mathsf{R}_V xy$;

(iii) The function $l$ is a zigzagmorphism from $\mathcal{G}$ onto the frame $\mathcal{F}$.

PROOF OF CLAIM. (i). Obvious.
(ii). We will denote $u/\equiv$ by $\overline{u}$. We prove the claim by contraposition:

| | | |
|---|---|---|
| $\neg \mathsf{R}_V\langle \overline{u}, \overline{v}\rangle, \langle \overline{u'}, \overline{v'}\rangle$ | $\Longleftrightarrow$ | (using well–definedness of $l$) |
| $\neg \mathsf{R} l^*\langle u, v\rangle, l^*\langle u', v'\rangle$ | $\stackrel{(C_{21})}{\Longleftrightarrow}$ | (using that $l^*$ is a zigzagmorphism) |
| $\neg \mathsf{R} l^*\langle u, u\rangle, l^*\langle u', u'\rangle \,\&\, \neg \mathsf{R} l^*\langle v, v\rangle, l^*\langle v', v'\rangle$ | $\Rightarrow$ | (definition of $\equiv$) |
| $\overline{u} = \overline{u'}$ and $\overline{v} = \overline{v'}$ | | |

(iii). All steps in this proof except homomorphism for $\mathsf{C}_V$ are straightforward by claim 2. To show that $l$ is a homomorphism for $\mathsf{C}_V$, suppose $\{\langle \overline{u}, \overline{v}\rangle, \langle \overline{u}, \overline{w}\rangle, \langle \overline{w}, \overline{v}\rangle\} \subseteq V$. We have to show that $\mathsf{C} l\langle \overline{u}, \overline{v}\rangle, l\langle \overline{u}, \overline{w}\rangle, l\langle \overline{w}, \overline{v}\rangle$ holds. By definition of $V$, we have $u, u', v, v', w, w' \in U^*$, $\{\langle u, v\rangle, \langle u', w\rangle, \langle w', v'\rangle\} \subseteq V^*$, and $u \equiv u', w \equiv w'$ and $v \equiv v'$. By the definition of $l$, it is sufficient to show that $\mathsf{C} l^*\langle u, v\rangle, l^*\langle u', w\rangle, l^*\langle w', v'\rangle$ holds.

There are several cases, depending on why the points are equivalent. One easy case is this. If $u = u', w = w'$ and $v = v'$, then, since $l^*$ is a homomorphism, we have $\mathsf{C} l^*\langle u, v\rangle, l^*\langle u, w\rangle, l^*\langle w, v\rangle$. In all other cases, for at least one of the three pairs of equivalent points, the reflexive pairs at those points are mapped to an R–irreflexive arrow. For these cases, we need condition $(C_{20})$ and the fact that $\mathcal{F} \models (AU)$. The next claim helps.

CLAIM 4. If $\mathcal{F} \in \mathsf{KD}^{rel}_{rlRS}$ and $\mathcal{F} \models (AU)$, then

$$\mathcal{F} \models \quad [x_l = y_l \,\&\, y_r = z_l \,\&\, z_r = x_r \,\&\, (\neg \mathsf{R} x_l y_l \vee \neg \mathsf{R} y_r z_l \vee \neg \mathsf{R} z_r x_r)] \Rightarrow \mathsf{C}xyz$$

In words: if $x_l = y_l \& y_r = z_l \& z_r = x_r$ and at least one of the pairs $\langle x_l, y_l\rangle, \langle y_r z_l\rangle, \langle z_r, x_r\rangle$ is R irreflexive (i.e., an "abstract singleton"), then $x$ can be decomposed into $y$ and $z$.

PROOF OF CLAIM. This follows from $(C_{20}), (D_1), (D_2)$ and $(AU)$. ◀

We show with an example how this claim helps us out. Suppose $\neg \mathsf{R} l^*\langle u, u\rangle l^*\langle u', u'\rangle$ and $w = w'$ and $v = v'$. Because $l^*$ is a zigzagmorphism we have $l^*\langle u, u\rangle = (l^*\langle u, v\rangle)_l$, and similarly for the others. This implies that

$$\neg \mathsf{R}(l^*\langle u, v\rangle)_l (l^*\langle u', w\rangle)_l \,\&\, (l^*\langle u', w\rangle)_r = (l^*\langle w', v'\rangle)_l \,\&\, (l^*\langle w', v'\rangle)_r = (l^*\langle u, v\rangle)_r$$

So by the above claim, $\mathsf{C} l^*\langle u, v\rangle, l^*\langle u', w\rangle, l^*\langle w', v'\rangle$, whence also $\mathsf{C} l\langle \overline{u}, \overline{v}\rangle, l\langle \overline{u}, \overline{w}\rangle, l\langle \overline{w}, \overline{v}\rangle$, which is what we had to prove. ◀

The general situation is sketched in figure 4.1 below. At the top we draw the situation in $V^*$, and at the bottom the situation in $\mathcal{F}$. The dotted arrows denote the function $l^*$. The dashed arrows denote the functions $(.)_l$ and $(.)_r$ in the frame $\mathcal{F}$. By Claim 1.(ii), the reflexive pairs at two equivalent points are mapped to the same place (e.g., $l^*\langle u, u\rangle = l^*\langle u', u'\rangle$).

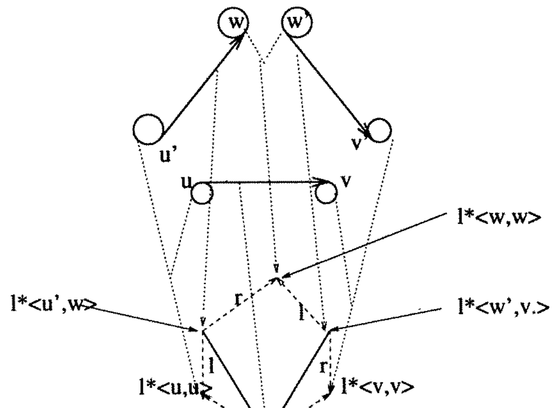



FIGURE 4.1: $l$ IS A HOMOMORPHISM FOR THE RELATION C

STEP II. Since the frame $\mathcal{G}$, constructed in the previous step, is a pair–frame, we only have to make $\mathsf{R}_V$ the inequality. Since $\mathcal{G} \models (AU)$, it suffices to make $\mathsf{R}_V$ *irreflexive.* Define the following two sets:

$$BAD \stackrel{\text{def}}{=} \{u \in U : \mathsf{R}_V\langle u,u\rangle\langle u,u\rangle\}$$
$$COPIES \stackrel{\text{def}}{=} \{\langle u',u'\rangle : u \in BAD\} \cup \{\langle v,u'\rangle,\langle u',v\rangle : \langle u,v\rangle \in V, u \in BAD \,\&\, u \neq v\}$$

Without loss of generality we may assume that $COPIES$ is disjoint from $V$. Let $\mathcal{H} = \langle H, \mathsf{C}_H, \mathsf{f}_H, \mathsf{I}_H, \neq\rangle \in \mathsf{KD}^{rel}_{setRS}$ be given by the set $H \stackrel{\text{def}}{=} V \cup COPIES$. Define $p : H \longrightarrow V$ as the unique function such that

- $p \restriction V$ is the identity function,
- $p(\langle u',u'\rangle) \stackrel{\text{def}}{=} \langle u,u\rangle$ if $u \in BAD$, and
- $p(\langle u',v\rangle) \stackrel{\text{def}}{=} \langle u,v\rangle$ and $p(\langle v,u'\rangle) \stackrel{\text{def}}{=} \langle v,u\rangle$ if $u \neq v$ and $u \in BAD$.

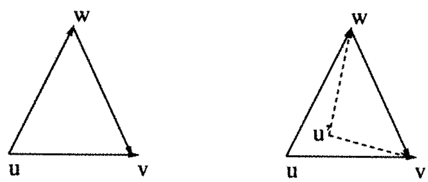
The next claim states that, for $\mathsf{R}_V$ we did enough. That is, we *only* copied $\mathsf{R}_V$ reflexive arrows.

CLAIM 5. (i) $(\forall x \in V) : (\mathsf{R}_V xx \iff$ there exists a copy of $x$ in $COPIES)$;
(ii) $(\forall x,y \in H) : ((x \neq y \,\&\, p(x) = p(y)) \Rightarrow \mathsf{R}_V p(x)p(y))$.

PROOF OF CLAIM. (i). Suppose $\mathsf{R}_V\langle u,v\rangle\langle u,v\rangle$ for some $\langle u,v\rangle \in V$. If $u = v$, then the claim holds by definition. So, suppose $u \neq v$. Then: $\mathsf{R}_V\langle u,v\rangle\langle u,v\rangle \stackrel{(C_{21})}{\iff} \mathsf{R}_V\langle u,u\rangle\langle u,u\rangle \vee \mathsf{R}_V\langle v,v\rangle\langle v,v\rangle \iff u \in BAD$ or $v \in BAD \iff \langle u',v\rangle \in COPIES$ or $\langle u,v'\rangle \in COPIES$.
(ii) follows from (i), since two pairs of $H$ can only be mapped to the same pair in $V$, when they are copies of each other. ◀

CLAIM 6. $p$ is a zigzagmorphism from $\mathcal{H}$ onto $\mathcal{G}$.

PROOF OF CLAIM. Clearly $p$ is surjective. That $p$ is a zigzagmorphism for $\mathsf{R}_V$ is immediate by claim 5. For $\mathsf{I}$ and $\mathsf{f}$ this is straightforward to check For $\mathsf{C}$ observe that, if $\{\langle u,v\rangle,\langle u,w\rangle,\langle w,v\rangle\} \subseteq H$, then either they all are in $V$, or one pair is in $V$ and the other two are in $COPIES$. The next picture might be helpful. At the left is the situation in $\mathcal{G}$ with $u \in BAD$ (so, $\{\langle u',w\rangle,\langle u',v\rangle\} \subseteq COPIES$), and at the right its representation in $\mathcal{H}$.




◀

With these two steps we have finished the proof, because our original frame $\mathcal{F}$ will be a zigzagmorphic image of the frame $\mathcal{H}$ by the function $l \circ p$. QED

We finish this section with the proof of the finite axiomatizability theorem.

PROOF OF THEOREM 4.4.3. Proposition 4.4.7 implies that the class $\mathsf{S}(\mathsf{KD}^{rel}_{rlRS})^+$ is a finitely axiomatizable canonical variety. So it is sufficient to show that $\mathbf{SP}(\mathsf{KD}^{rel}_{setRS})^+ = \mathsf{S}(\mathsf{KD}^{rel}_{rlRS})^+$. By soundness, $\mathsf{KD}^{rel}_{setRS} \subseteq \mathsf{KD}^{rel}_{rlRS}$. By the previous lemma, $\mathsf{KD}^{rel}_{rlRS} \subseteq \mathbf{DuZig}\mathsf{KD}^{rel}_{setRS}$. So by duality and $\mathbf{PS} \leq \mathbf{SP}$, $(\mathsf{KD}^{rel}_{rlRS})^+ \subseteq \mathbf{SP}(\mathsf{KD}^{rel}_{setRS})^+$. Hence the two varieties are equal. QED

FURTHER ROADS: COUNTING MODALITIES. Define the following operations from $\mathcal{P}(W)$ to $\mathcal{P}(W)$:

$$\Diamond^{nW}\mathsf{x} \overset{\text{def}}{=} W \text{ if } |\mathsf{x}| \geq n, \text{ else } \Diamond^{nW}\mathsf{x} = \emptyset$$

The *"counting modalities"* $\Diamond^n$ are also known under the name of graded modalities (cf. e.g., Fine [Fin72], van der Hoek [Hoe92]). The difference operator is term-definably equivalent with the two "counting modalities" $\Diamond^1$ and $\Diamond^2$, by the following definitions: (we use $\tau \rightarrow \tau_1$ to abbreviate $-\tau \vee \tau_1$)

$$\Diamond^1\mathsf{x} \overset{\text{def}}{=} \mathsf{x} \vee \mathsf{D}\,\mathsf{x}, \quad \Diamond^2\mathsf{x} \overset{\text{def}}{=} \mathsf{D}(\mathsf{x} \wedge \mathsf{D}\,\mathsf{x}), \quad \mathsf{D}\,\mathsf{x} \overset{\text{def}}{=} \Diamond^1\mathsf{x} \wedge (\mathsf{x} \rightarrow \Diamond^2\mathsf{x})$$

A well investigated class is $\mathsf{BA}+n\text{-}times$:

$$\mathsf{BA}+n\text{-}times \overset{\text{def}}{=} \mathbf{SP}\{\mathfrak{A} = \langle \mathfrak{P}(W), \Diamond^{nW}\rangle_{0<n<\omega} : W \text{ is some set}\}$$

Mikulás–Németi [MN94] showed that the equational theory of the expansion[11] of the class $\mathbf{SRl}_{RS}\mathsf{RRA}$ with the set of operations $\{\Diamond^n : 0 < n < \omega\}$ is decidable. They also showed that it is a variety, axiomatizable by finitely many schemas.

## APPENDIX. COMPLETENESS WITH THE IRREFLEXIVITY RULE

A finite axiomatization of the equational theory of the variety $\mathbf{SRl}_{RS}\mathsf{RRA}+\mathsf{D}$ is easily obtained, using the *irreflexivity rule*. We show this, using the similar theorem of Venema with respect to the class RRA (cf. [Ven91] Thm 3.3.37). He showed that, if we add the *irreflexivity rule* for the (term definable!) difference operator to the RA axioms (see 2.4.11), we get an axiomatization of RRA. The irreflexivity rule is defined as follows: Let $t(\mathsf{x}_1, \ldots, \mathsf{x}_n)$ denote an arbitrary term generated from variables $\mathsf{x}_1, \ldots, \mathsf{x}_n$. Define $\mathsf{O}\,\mathsf{x} \overset{\text{def}}{=} \mathsf{x} \wedge -\mathsf{D}\,\mathsf{x}$.

**Irreflexivity Rule** $\quad \dfrac{\mathsf{O}\,\mathsf{x}_0 \leq t(\mathsf{x}_1, \ldots, \mathsf{x}_n)}{t(\mathsf{x}_1, \ldots, \mathsf{x}_n) = 1}$ if $\mathsf{x}_0$ does not occur among $\mathsf{x}_1, \ldots, \mathsf{x}_n$

When added to the axioms for the difference operator, this rule makes the frame relation for the difference operator irreflexive (see [Ven91] for details). Let $\Sigma$ denote

[11] We define this and similar classes by adding the new modalities to the full complex algebras of the old class and take the **SP** closure of that, i.e.

$$\mathbf{SRl}_H\mathsf{RRA}+n\text{-}times \overset{\text{def}}{=} \mathbf{SP}\{\mathfrak{A} = \langle \mathfrak{P}(V), \circ^V, ^{-1^V}, \mathsf{Id}^V, \Diamond^{nV}\rangle_{0<n<\omega} : V \text{ is an } H \text{ relation}\}$$

the derivation system consisting of the axioms and rules of equational logic, the BAO–axioms for the similarity type of $\mathbf{SRl}_{RS}\mathsf{RRA}+\mathsf{D}$ and axioms $(A_1)$ – $(A_{20})$. We want to use Venema's powerful **SD**–theorem ([Ven91], Thm 2.7.7) to prove that $\Sigma$ plus the irreflexivity rule enumerates $\mathsf{Eq}(\mathbf{SRl}_{RS}\mathsf{RRA}+\mathsf{D})$. In order to use that theorem, the derivation system has to satisfy the following requirements:

1. All its axioms are in Sahlqvist form.
2. It is a *versatile* similarity type (meaning that for every operator all its conjugates are term–definable).
3. $\Sigma$ contains the axioms $(A_{16})$ and $(A_{17})$.
4. For every $n$–ary non–Boolean operator $f$, $\Sigma$ contains $f(\mathsf{x}_1,\ldots,\mathsf{x}_n) \leq \Diamond\mathsf{x}_1 \wedge \ldots \wedge \Diamond\mathsf{x}_n$.

PROPOSITION 4.4.11. *$\Sigma$ satisfies the requirements 1–4 above.*

PROOF. Requirements 1, 3 and 4 are obvious. For 2, note that $\mathsf{D}$ and $^{\smile}$ are *self–conjugate*, and that the two conjugates of ";" can be defined as follows: $\mathsf{x} \rhd \mathsf{y} \stackrel{\text{def}}{=} \mathsf{x}^{\smile};\mathsf{y}$ and $\mathsf{x} \lhd \mathsf{y} \stackrel{\text{def}}{=} \mathsf{x};\mathsf{y}^{\smile}$ (cf. Prop 6.3.6 here or Def 3.3.35 in [Ven91]). QED

Let $\Sigma^+$ be the derivation system which is obtained by adding the irreflexivity rule to $\Sigma$. We are ready to formulate the completeness theorem.

THEOREM 4.4.12. $\Sigma^+ \vdash \tau = \sigma \iff \mathbf{SRl}_{RS}\mathsf{RRA}+\mathsf{D} \models \tau = \sigma$

PROOF. Define the class $\mathsf{KD}^{rel\neq}_{rlRS-}$ as $\{\mathcal{F} = \langle W, \mathsf{C}, \mathsf{F}, \mathsf{I}, \neq\rangle : \mathcal{F} \models (C_1) - (C_{20})\}$. Using Venema's **SD**–theorem and 4.4.7 and 4.4.11, we find that

$$\Sigma^+ \vdash \tau = \sigma \iff \mathsf{KD}^{rel\neq}_{rlRS-} \models \tau = \sigma \tag{4.9}$$

The theorem then follows from the next claim.

CLAIM 1. $\mathsf{KD}^{rel\neq}_{rlRS-} = \mathbf{I}\mathsf{KD}^{rel}_{setRS}$

PROOF OF CLAIM. The inclusion from right to left is immediate since all conditions listed are valid on pair–frames. For the other side, we make use of the following observation. Every frame $\mathcal{F} \in \mathsf{KD}^{rel\neq}_{rlRS-}$ satisfies the following conditions (by propositions 2.4.7, 4.4.8 and the fact that $\mathsf{R}$ is the inequality relation)

$(T_0)$ $(.)_l, (.)_r, \mathsf{f}$ are total functions
$(T_1)$ $\mathsf{I}x \Rightarrow x = \mathsf{f}(x) = x_l = x_r$
$(T_2)$ $x_l = (\mathsf{f}x)_r$ and $x_r = (\mathsf{f}x)_l$
$(T_3)$ $\mathsf{C}xyz \Rightarrow x_l = y_l \;\&\; y_r = z_l \;\&\; z_r = x_r$

$(S_1)$ $x_l = y_l \;\&\; x_r = y_r \Rightarrow x = y$
$(S_2)$ $x_l = y_l \;\&\; y_r = z_l \;\&\; x_r = z_r \Rightarrow \mathsf{C}xyz$

We will adopt the representation for *square* pair–frames in [Ven91] (proof of Thm 3.3.26) to our situation. The idea of the proof comes from the fact that, once the domain of a concrete pair–frame is a reflexive relation, we can identify the elements $u$ of the base set of that relation with the identity pairs $\langle u, u\rangle$. Thus we will represent each abstract arrow $x$ by the pair $\langle x_l, x_r\rangle$. Let $\mathcal{F} \doteq \langle W, \mathsf{C}, \mathsf{f}, \mathsf{I}, \neq\rangle \in \mathsf{KD}^{rel\neq}_{rlRS-}$. Define a pair–frame $\mathcal{G}_{pair}(V) = \langle V, \mathsf{C}_V, \mathsf{f}_V, \mathsf{I}_V, \neq\rangle$ over the set

$$V \stackrel{\text{def}}{=} \{\langle y, z\rangle \in (\mathsf{I} \times \mathsf{I}) : (\exists x \in W)(x_l = y \,\&\, x_r = z)\}$$

CLAIM 2. (i) $V$ is a reflexive and symmetric relation;
(ii) The function $l : W \longrightarrow V$ defined as $l(x) \stackrel{\text{def}}{=} \langle x_l, x_r\rangle$ is a frame isomorphism between $\mathcal{F}$ and $\mathcal{G}_{pair}(V)$.

PROOF OF CLAIM. (i). For reflexivity, let $\langle u, v\rangle \in V$. Then $(\exists x \in W) : x_l = u \& x_r = v$. By $(T_1) : u = x_l = (x_l)_l = (x_l)_r$, whence $\langle u, u\rangle \in V$. Similarly, $\langle v, v\rangle \in V$. For symmetry use $(T_2)$ and $(T_0)$.
(ii). $l$ is a bijection.
Surjectivity is immediate by the definition of $V$, and injectivity follows from $(S_1)$.
$l$ is a homomorphism.
For $\mathsf{I}$ by $(T_1)$, for $\mathsf{f}$ by $(T_2)$, for $\mathsf{C}$ by $(T_3)$, and for $\neq$ because $l$ is a bijection.
$l^{-1}$ is a homomorphism too.
Because $l$ is a bijection, and, by $(T_0)$, $\mathsf{f}$ is a function, the part for $\mathsf{f}$ was proved in the previous claim. For $\mathsf{I}_V$, if $\mathsf{I}_V lx$ then $x_l = x_r$. Use $(T_1)$ and $(S_1)$ to derive that $x_l = x$. But then $\mathsf{I}x$. For $\mathsf{C}_V$, suppose $\mathsf{C}_V lxlylz$. This holds if and only if (by definition) $x_l = y_l, y_r = z_l$ and $z_r = x_r$. But then, by $(S_2)$, we have $\mathsf{C}xyx$. ◀

We proved that $\mathsf{KD}^{rel\neq}_{rlRS-} = \mathbf{I}\mathsf{KD}^{rel}_{setRS}$. ◀

So, $\mathsf{KD}^{rel\neq}_{rlRS-} = \mathbf{I}\mathsf{KD}^{rel}_{setRS}$. The complex algebras of this last class generate the variety $\mathbf{SRI}_{RS}\mathsf{RRA}+\mathsf{D}$. Hence we are done. QED

REMARK 4.4.13. We did not need axiom $(A_{21})$ in the presence of the irreflexivity rule, because it became derivable. From the above completeness result, and the fact that $(A_{21})$ is independent of the other axioms (cf. 4.4.9), it follows that the irreflexivity rule is not conservative when added to the axioms $(A_1) - (A_{20})$.

## 4.5 REPRESENTING BAO'S AS ALGEBRAS OF RELATIONS

In this section we leave the algebras of binary relations, and go to algebras of relations of arbitrary rank. We generalize the notion of composition to relations of any finite rank, and show that for this generalization, we have a representation theorem similar to the one for the "composition only" reduct of $\mathbf{SRI}\mathsf{RRA}$ (4.2.11).

### 4.5.1 INTRODUCTION AND MOTIVATION

INTRODUCTION. In chapter 2 and in the previous sections of this chapter, we have seen two notions of representability for "abstract" (equationally defined) BAO's, correlated with two notions of semantics for the corresponding modal logics. The first, easy one, was a representation using Kripke frames. In the sequel, this will be called a *relational representation*, or relational semantics. Recall that, by fact 2.2.1, every BAO can be represented as a subalgebra of a complex algebra of a Kripke frame.

In the previous three sections, we did quite some work to obtain also a second representation, using pair–frames. These frames are completely determined by their universes. In these frames, the accessibility relations are already present ("coded") in the worlds. Our motivation for studying pair–frames was that we wanted to draw arrows as real arrows. In [HMT71], this second kind of representation is called, in the context of cylindric algebras, a *geometrical representation*. We will use the same term here (and also geometrical semantics). A geometrical representation is "concrete" in the sense that 1) if two algebras are different then their universes are different, and even 2) the operations are set–theoretically defined.

MAIN RESULT. In this section, we will show that *every BAO* has, besides a relational representation, also a *concrete geometrical representation* as a subalgebra of an algebra whose universe consists of sets of sequences and whose operations are defined in set–theoretic terms (theorem 4.5.6). So, every BAO can be represented as an algebra of relations. In modal–logical terms, this means that every general modal logic can be viewed as a *multi–dimensional* modal logic (cf. [Ven91]). A concrete example of this result is 4.2.11. There we showed that every BAO of type $\langle ;, 2\rangle$ is isomorphic to a subalgebra of an algebra $\mathfrak{A} = \langle \mathfrak{P}(V), \circ^V\rangle$ for some binary relation $V$ ($\circ^V$ is ordinary relation composition relativized to $V$). Theorem 4.5.2 below generalizes this result to all operators of rank higher than 1.

MOTIVATION. To motivate this representation, we quote a part from [HMT71], in which the two representations are compared in the context of cylindric algebras. Clearly, these remarks have a very general character. Theorem 2.7.43(ii) in the quotation is the theorem that for any $\alpha$, $\mathsf{CA}_\alpha = \mathsf{S}(\mathsf{K}_{\mathsf{CA}}^{cyl\alpha})^+$. (Here $\mathsf{K}_{\mathsf{CA}}^{cyl\alpha}$ is the class of $\alpha$–frames which satisfy the frame conditions corresponding to the CA axioms.)

> The results known in modern mathematics as representation theorems have as a rule the following character: in each of them a class K of "abstractly" defined mathematical structures is considered, a subclass L of this class is singled out, and it is shown that every structure in K is isomorphic to some structure in L; the proof frequently consists in effectively correlating, with any given structure $\mathfrak{S}$ in K, its isomorphic image in L –the representative of $\mathfrak{S}$. The value of a representation theorem depends both on the scope of the class K and on such properties of members of L as simplicity of structure and "concreteness" of notions involved in their construction.
>
> In the case of relational representability the scope of the representation theorem 2.7.43(ii) is wide: the class K consists of all CA's. The class L is formed by the complex algebras of cylindric atom structures (*authors note:* $\mathsf{K}_{\mathsf{CA}}^{cyl\alpha}$ is the class of, what is here called, "cylindric atom structures") and their subalgebras. The Boolean operations in these representatives are "concrete", well–determined set–theoretical notions used in constructing Boolean set algebras. On the other hand, the extra–Boolean operations are defined in terms of the fundamental relations of cylindric atom structures (*authors note:* the relations $T^i$ and $E^{ij}$)

> and have therefore as "abstract" a character as the corresponding notions in arbitrary CA's. *Hence 2.7.43(ii) is not what could be regarded as a satisfactory representation theorem for arbitrary cylindric algebras.* (emphasis by MM). [...]
>
> ...the notion of geometrical representability is intuitively much more satisfactory and valuable than that of relational representability. The generalized cylindric set algebras, which serve as representatives under this notion, are in a sense "concrete" algebraic structures (or, at any rate, structures which are much more "concrete" than arbitrary CA's). All the fundamental operations and distinguished elements of these algebras are defined in straightforward set-theoretical terms; the definitions are uniform for all algebras involved, and, as a consequence, each of the algebras is uniquely determined by its universe. ([HMT71], remark 2.7.46)

Besides the advantages described above, there is also an important methodological application of the geometrical semantics of BAO's. Namely, as it turns out from Németi [Ném91], algebras of relations have a powerful methodology, and it does not matter too much what the basic operations are (from the point of view of the applicability of that theory). Therefore, efforts have been made by many researchers to base as many logics on algebras of relations as possible. The present result is a considerable step forward in this program. It shows that the algebraic–logical counterpart of general modal logics can be chosen to be a kind of algebras of relations.

ORGANIZATION OF THIS SECTION. Except for some standard definitions from chapter 2, section 4.5 can be read independently from the rest of this work. It is organized as follows. In section 4.5.2, we introduce a generalization of binary composition to $n$–ary composition of $n$–ary relations. We show that BAO's with one $n$–ary operator ($n \geq 2$) can be represented as algebras of $n$–ary relations. The operator is represented as $n$–ary composition. Moreover, if we add the conjugates of that operator, we can keep this rather natural representation. In section 4.5.3, we show how we can represent every BAO as an algebra of relations.

NOTATION. With BAO we always mean *normal* BAO. We use $\vec{u}$ to denote a sequence of variables $u_0, \ldots, u_n$ for arbitrary $n$, as well as the product notation $\vec{u}(i) = u_i$. As variables ranging over $n$–tuples we use $s$ and $r$; $s_i$ denotes the $i$–th element of $s$. Besides $\Diamond$, we will use $\mathsf{f}^i$ to denote equationally defined non–Boolean operators; for their represented counterparts we sometimes use $\mathsf{F}^i$. As before, $\rho$ provides each operator with its arity. We use $I$ as an index set for the operators of an algebra.

## 4.5.2 $n$-ARY COMPOSITION AND ITS CONJUGATES

In this section, we study algebras of relations with rather special operators. Let $V \subseteq {}^{\alpha}U$ for some set $U$ and $\alpha \leq \omega$, and let $\mathfrak{A}$ be an algebra with universe $\mathcal{P}(V)$. The simplest operations we use are the *diagonals* $\mathsf{D}^V_{ij} \stackrel{\text{def}}{=} \{s \in V : s_i = s_j\}$.

$n$–ARY COMPOSITION. The major operator in this section is a generalization of binary composition to $n$–ary relations. We use $\bullet^V$ to denote this operator (the context provides its specific arity). The $n$–ary operator $\bullet^V$ has the following definition. Suppose that $\alpha = n$ for some finite $n \geq 1$, and $\mathsf{x}_0, \ldots, \mathsf{x}_{n-1}$ are subsets of $V \subseteq {}^\alpha U$. Then:

$$\begin{array}{rcll} \langle x_0, x_1, \ldots, x_{n-1}\rangle \in \bullet^V(\mathsf{x}_0, \mathsf{x}_1, \ldots, \mathsf{x}_{n-1}) & \overset{\text{def}}{\Longleftrightarrow} & \langle x_0, x_1, \ldots, x_{n-1}\rangle \in V & \& \\ & (\exists \mathbf{z}) : ( & \langle \mathbf{z}, x_1, \ldots, x_{n-1}\rangle \in \mathsf{x}_0 & \& \\ & & \langle x_0, \mathbf{z}, x_2, \ldots, x_{n-1}\rangle \in \mathsf{x}_1 & \& \\ & & \vdots & \\ & & \langle x_0, \ldots, x_{n-2}, \mathbf{z}\rangle \in \mathsf{x}_{n-1}) & \end{array}$$

For $\alpha = 1$, we get the well–known operation of cylindrification; for $\alpha = 2$, we get ordinary composition of binary relations (in reverse order). For convenience we use the substitution function $\mathbf{f}^{\,i}_{\,z}$ (cf. section 2.5). Then we can define $\bullet^V$ easily as follows:

$$\bullet^V(\mathsf{x}_0, \ldots, \mathsf{x}_{n-1}) \overset{\text{def}}{=} \{s \in V : (\exists z)(\mathbf{f}^{\,0}_{\,z}\, s \in \mathsf{x}_0 \,\&\ldots\&\, \mathbf{f}^{\,n-1}_{\,z}\, s \in \mathsf{x}_{n-1})\}$$

Note that $\bullet^V$ is dependent on the universe $V$. Since all our operators will be *relativized*, we usually suppress this superscript.

$n$–ARY COMPOSITION OF RELATIONS OF HIGHER RANK. In the next section, we use $n$–ary composition operators on sets of relations with rank higher than $n$, say $\alpha$. The idea uses that the operator works only on a specific $n$–long part. On that part, it behaves just like $n$–ary composition. We define these operators as follows. Let $V \subseteq {}^\alpha U$, $j, j + (n-1) < \alpha$, and let $\Gamma = \langle j, j+1, \ldots, j + (n-1)\rangle$ be a sequence of consecutive numbers:

$$\bullet_\Gamma(\mathsf{x}_0, \ldots, \mathsf{x}_{n-1}) \overset{\text{def}}{=} \{s \in V : (\exists z)(\mathbf{f}^{\,\Gamma_0}_{\,z}\, s \in \mathsf{x}_0 \,\&\ldots\&\, \mathbf{f}^{\,\Gamma_{n-1}}_{\,z}\, s \in \mathsf{x}_{n-1})\}$$

Again, $\bullet_\Gamma$ is relativized to $V$. The definition of $\bullet_{\langle 2,3,4\rangle}$ on sets of 7-ary relations is given schematically below: (a "–" indicates that any element is allowed at this place)

$$\begin{array}{rrcl} & \langle x, y, a, b, c, v, w\rangle & \in & \bullet_{\langle 2,3,4\rangle}(\mathsf{x}_0, \mathsf{x}_1, \mathsf{x}_2) \\ & & \Updownarrow & \\ & \langle x, y, a, b, c, v, w\rangle & \in & V \,\& \\ (\exists \mathbf{z}) : ( & \langle -, -, \mathbf{z}, b, c, -, -\rangle & \in & \mathsf{x}_0 \,\& \\ & \langle -, -, a, \mathbf{z}, c, -, -\rangle & \in & \mathsf{x}_1 \,\& \\ & \langle -, -, a, b, \mathbf{z}, -, -\rangle & \in & \mathsf{x}_2) \end{array}$$

CONJUGATES OF $n$–ARY COMPOSITION. Conjugates of operators were studied in Jónsson–Tarski [JT52]. There the emphasis is on conjugates of unary operators. Conjugates of binary (relation) composition ("residuals") occur in Birkhoff [Bir67]. Recent papers which are largely devoted to conjugates of relation composition are Jónsson–Tsinakis [JT93], Jipsen [Jip92], Jónsson–Jipsen–Rafter [JJR] and Andréka–Németi [AN]. Pratt [Pra90a] has applications in computer science.

DEFINITION 4.5.1. Let $\mathfrak{A} = \langle A, \vee, -, \mathsf{f}^0, \ldots, \mathsf{f}^n \rangle$ be a BAO, in which all operators have rank $n$. We call $\mathfrak{A}$ *fully conjugated* if $\mathfrak{A}$ satisfies the following equations (for all $1 \leq i \leq n$):

$$\begin{aligned} \mathsf{f}^0(\mathsf{x}_1, \ldots, \mathsf{x}_{i-1}, -\mathsf{f}^i(\mathsf{x}_1, \ldots, -\mathsf{x}_i, \ldots, \mathsf{x}_n), \mathsf{x}_{i+1}, \ldots, \mathsf{x}_n) &\leq \mathsf{x}_i \\ \mathsf{f}^i(\mathsf{x}_1, \ldots, \mathsf{x}_{i-1}, -\mathsf{f}^0(\mathsf{x}_1, \ldots, -\mathsf{x}_i, \ldots, \mathsf{x}_n), \mathsf{x}_{i+1}, \ldots, \mathsf{x}_n) &\leq \mathsf{x}_i \end{aligned}$$

$\mathsf{CBAO}^n$ denotes the class of all fully conjugated BAO's with operators of rank $n$.

We tend to give $\mathsf{f}^0$ a special rôle, and will usually denote it by $\Diamond$. We call $\mathsf{f}^i$ the $i$–th conjugate of $\Diamond$. We use $\heartsuit^i$ to denote the conjugates of $\bullet^V$. On $V \subseteq {}^nU$, they are defined as follows:

$$\heartsuit^i(\mathsf{x}_0, \ldots, \mathsf{x}_{n-1}) \stackrel{\text{def}}{=} \{s \in V : (\exists z \in U)(\mathbf{f}\,^{i-1}_{z}\,\mathbf{f}\,^{0}_{s_{i-1}}\, s \in \mathsf{x}_0 \,\&\ldots\&\, \mathbf{f}\,^{i-1}_{z}\,\mathbf{f}\,^{n-1}_{s_{i-1}}\, s \in \mathsf{x}_{n-1})\}$$

$\mathbf{f}\,^i_z\,\mathbf{f}\,^j_{s_i}\, s$ is the result of replacing $s_j$ by $s_i$, and $s_i$ by $z$. As an example, consider a 4–ary operator $\heartsuit^2$, which is the second conjugate of $\bullet^V$:

$$\begin{aligned} \langle x_0, x_1, x_2, x_3 \rangle \in \heartsuit^2(\mathsf{x}_0, \mathsf{x}_1, \mathsf{x}_2, \mathsf{x}_3) \stackrel{\text{def}}{\Longleftrightarrow} \quad & \langle x_0, x_1, x_2, x_3 \rangle \in V \,\& \\ (\exists \mathbf{z}) : ( \quad & \langle x_1, \mathbf{z}, x_2, x_3 \rangle \in \mathsf{x}_0 \,\& \\ & \langle x_0, \mathbf{z}, x_2, x_3 \rangle \in \mathsf{x}_1 \,\& \\ & \langle x_0, \mathbf{z}, x_1, x_3 \rangle \in \mathsf{x}_2 \,\& \\ & \langle x_0, \mathbf{z}, x_2, x_1 \rangle \in \mathsf{x}_3) \end{aligned}$$

For $n = 1$, we get cylindrification again (cylindrification is *self-conjugate*). For $n = 2$, we get the conjugates of binary $\circ$, which are usually denoted by $\lhd$ and $\rhd$. The familiar left and right *residuals* $\backslash$ and $/$ of $\circ$ (see e.g. Jónsson [Jón91]) can be defined by $x \backslash y \stackrel{\text{def}}{=} -(x \rhd -y)$ and $x/y \stackrel{\text{def}}{=} -(-x \lhd y)$. (For a direct definition of these operations see section 6.3.2.) Equations 4.10–4.13 below, which are precisely the conjugate conditions from 4.5.1 for binary operators, are valid[12] on every (relativized) algebra of binary relations.

$$\mathsf{x} \circ (\mathsf{x} \backslash \mathsf{y}) \leq \mathsf{y} \tag{4.10}$$

$$\mathsf{y} \leq \mathsf{x} \backslash (\mathsf{x} \circ \mathsf{y}) \tag{4.11}$$

$$(\mathsf{x}/\mathsf{y}) \circ \mathsf{y} \leq \mathsf{x} \tag{4.12}$$

$$\mathsf{x} \leq (\mathsf{x} \circ \mathsf{y})/\mathsf{y} \tag{4.13}$$

## GEOMETRICAL REPRESENTATION OF BAO'S WITH ONE OPERATOR

We generalize the result (4.2.11) for binary composition from the beginning of this chapter. We show that every BAO with one $n$–ary operator can be represented as an algebra of $n$–ary relations, in which the operator is $n$–ary composition. With $n = 1$, we

[12] Once we see that $\mathsf{x} \rhd \mathsf{y} = \mathsf{x}^{-1} \circ \mathsf{y}$, in which $^{-1}$ is the operation of taking converses, we notice that (2) is just another way of writing the last axiom of relation algebras $\mathsf{x}^{-1} \circ [-(\mathsf{x} \circ \mathsf{y})] \leq -\mathsf{y}$. (cf. [HMT85] definition 5.3.1) These equations are also very familiar from the Lambek Calculus (see Lambek [Lam58] or van Benthem [Ben88]).

are in (diagonal–free) cylindric set algebras of dimension 1, and that axiomatization is well–known: the operator has to be a complemented closure operator. Once $n > 1$, the situation changes radically: only the BAO axioms are needed for representation. We say that an algebra $\mathfrak{A} = \langle A, \vee, -, \mathsf{f}^i\rangle_{i\in I}$ *satisfies the BAO axioms* iff $\mathfrak{A}$ satisfies the BA axioms plus equations which state that every $\mathsf{f}^i$ is normal and additive in each of its arguments.

THEOREM 4.5.2. *Let* $\mathsf{K} \stackrel{\text{def}}{=} \{\mathfrak{A} = \langle A, \vee, -, \Diamond\rangle : \mathfrak{A}$ *satisfies the BAO axioms*$\}$. *If* $\Diamond$ *is an* $n$*–ary* ($n \geq 2$) *operator, then* $\mathsf{K} = \mathbf{IS}\{\langle\mathfrak{P}(V), \bullet^V\rangle : V \subseteq {}^nU$ *for some set* $U\}$. *Here* $\bullet^V$ *is* $n$*–ary composition relativized to* $V$.

PROOF. The theorem follows from lemma 4.5.3 below, in the same way as 4.2.11 followed from 4.2.7. The proof of 4.5.3 is a straightforward generalization of the step–by–step construction introduced in the proof of 4.2.7. QED

LEMMA 4.5.3. *Let* $n \geq 2$. *Every frame* $\mathcal{F} = \langle W, R\rangle$ *with* $R \subseteq {}^{n+1}W$ *is a zigzagmorphic image of a frame* $\mathcal{G} = \langle V, \mathbf{R}\rangle$, *in which* $V \subseteq {}^nU$ *for some set* $U$, *and* $\mathbf{R}$ *is defined as follows:*

$$(\forall y, x_1, \ldots, x_n \in V) : \mathbf{R}yx_1\ldots x_n \stackrel{\text{def}}{\Longleftrightarrow} \{y\} = \bullet^V(\{x_1\}, \ldots, \{x_n\})$$

PROOF. Fix some $n \geq 2$, and a frame $\mathcal{F} = \langle W, R\rangle$ with $R \subseteq {}^{n+1}W$. Step by step we create the frame $\mathcal{G}$ and a zigzagmorphism $l : V \longrightarrow W$, just as we did before with binary composition. To make the proof more perspicuous, we use a different scheduling of the repairs. Instead of repairing all "zigzag faults" of all sequences which were added in the previous step, we only repair one sequence at each step. Using a suitable scheduling function for the construction, this will have the same effect as our earlier construction.

Choose an infinite ordinal $\kappa$ such that $|W| \leq \kappa$. Let $P$ be a set of cardinality $|\kappa|$; we will use this set to create $V$. In this new setup we need a function which directs the construction process. So, let $\sigma : \kappa \longrightarrow {}^nP$ be a function such that

$$(\#) \qquad (\forall s \in {}^nP)(\forall\lambda < \kappa)(\exists\nu < \kappa) : \lambda \leq \nu \;\&\; \sigma(\nu + 1) = s$$

In Andréka–Mikulás [AM94a], it is shown that such a function always exists. Condition # ensures that we will make every necessary repair. At each step, we construct a tuple $G_\alpha = \langle U_\alpha, V_\alpha, l_\alpha\rangle$ such that for all $\alpha \leq \kappa$ :

- $U_\alpha \subseteq P$
- $V_\alpha \subseteq {}^nU_\alpha$
- $l_\alpha : V_\alpha \longrightarrow W$ is a surjective homomorphism

After the $\kappa$–th step, all necessary repairs will have been made, whence $l$ has the zigzag property as well. This will prove the claim.

**Construction**

**step 0** Take $U_0 \subset P$ such that $|P \setminus U_0| = \kappa$, and take $V_0 \subset {}^nU_0$ such that $|V_0| = |W|$ and $(\forall s, r \in V_0)(\forall i, j < n) : s_i = r_j \iff s = r \;\&\; i = j$. Let $l_0$ be any bijection between $V_0$ and $W$.

**step** $\alpha+1$ Let $\sigma(\alpha+1)=s$.

If $s \notin V_\alpha$, then $U_{\alpha+1} \stackrel{\text{def}}{=} U_\alpha$, $l_{\alpha+1} \stackrel{\text{def}}{=} l_\alpha$ and $V_{\alpha+1} \stackrel{\text{def}}{=} V_\alpha$.

Else, do the following. Define a function[13] $g_{\alpha+1} : \{\vec{u} : Rl_\alpha s\vec{u}\} \longrightarrow (P \setminus U_\alpha)$ such that $g_{\alpha+1}$ is injective and $|P \setminus g^*_{\alpha+1}| = \kappa$. Since we have chosen $\kappa$ and $P$ large enough, such a function will always exist. Now set :

$$\begin{array}{lcll} U_{\alpha+1} & \stackrel{\text{def}}{=} & U_\alpha & \cup \ g^*_{\alpha+1} \\ V_{\alpha+1} & \stackrel{\text{def}}{=} & V_\alpha & \cup \ \bigcup_{i<n}\{\mathbf{f}^{\,i}_{g_{\alpha+1}(\vec{u})}\, s \in {}^nP : Rl_\alpha s, \vec{u}\} \\ l_{\alpha+1} & \stackrel{\text{def}}{=} & l_\alpha & \cup \ \bigcup_{i<n}\{\langle \mathbf{f}^{\,i}_{g_{\alpha+1}(\vec{u})}\, s, \vec{u}(i)\rangle : Rl_\alpha s\vec{u}\} \end{array}$$

**Limit Step** If $\alpha$ is a limit ordinal, set

$$U_\alpha \stackrel{\text{def}}{=} \bigcup_{\beta<\alpha} U_\beta, \quad V_\alpha \stackrel{\text{def}}{=} \bigcup_{\beta<\alpha} V_\beta, \quad l_\alpha \stackrel{\text{def}}{=} \bigcup_{\beta<\alpha} l_\beta$$

Note that $G_\kappa$ is created with a limit step.

**End of Construction**

Finally, define $\mathcal{G} = \langle V_\kappa, \mathbf{R}\rangle$, in which $\mathbf{R}$ is defined as above, and set $l = l_\kappa$.

CLAIM. The function $l : V_\kappa \longrightarrow W$ is zigzagmorphism from $\mathcal{G}$ onto $\mathcal{F}$.

PROOF OF CLAIM. The function $l$ is surjective by step 0. For the zigzag property, suppose $Rl(s)\vec{u}$. Suppose $s$ was added in the $\lambda$–th step. Then, by condition # on the scheduling function $\sigma$, there exists an ordinal $\nu+1$ such that $\lambda < \nu+1 < \kappa$ and $\sigma(\nu+1) = s$. But then, $r_0, \ldots, r_{n-1}$ were added in this step, such that $r_i = \mathbf{f}^{\,i}_{g_{\nu+1}(\vec{u})}\, s$ and $l_{\nu+1}(r_i) = \vec{u}(i)$. Hence, by definition, we have $\mathbf{R}sr_0 \ldots r_{n-1}$ and $l(r_i) = \vec{u}(i)$.

The function $l$ is a homomorphism, if it is one for every step. We show this by induction on the construction. $l_0$ is a homomorphism, because in step 0, all sequences are disconnected. Suppose $l_\alpha$ is a homomorphism. Let $\mathbf{R}sr_0 \ldots r_{n-1}$, with $s, r_0, \ldots, r_{n-1}$ all in $V_{\alpha+1}$, and at least one of them is in $(V_{\alpha+1} \setminus V_\alpha)$. Because the conditions on $g_{\alpha+1}$ ensure that we use a brand new element from $P$ for every repair, $s \in V_\alpha$, $r_0, \ldots, r_{n-1} \in V_{\alpha+1} \setminus V_\alpha$, and $Rl_\alpha s l_{\alpha+1} r_0 \ldots, l_{\alpha+1} r_{n-1}$. Hence $l_{\alpha+1}$ is a homomorphism. Finally, in limit steps, nothing can go wrong. So $l$ is a homomorphism. QED

## GEOMETRICAL REPRESENTATION OF CONJUGATED BAO'S

Now we turn to the class of conjugated BAO's, and show how to adjust the preceding proof. Let $\mathsf{K}^n_{con}$ denote the class of all frames $\mathcal{F} = \langle W, R^\diamond, R^1, \ldots, R^n\rangle$, in which all the relations are $n+1$–ary, and which satisfy condition **con** for all $i$ with $1 \le i \le n$:

$$(\mathbf{con}) \quad R^\diamond(y, x_1, \ldots, x_i, \ldots, x_n) \iff R^i(x_i, x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n)$$

THEOREM 4.5.4. *Let $2 \le n < \omega$. Then:*

$$\mathsf{CBAO}^n = \mathsf{S}(\mathsf{K}^n_{con})^+ = \mathbf{IS}\{\langle \mathfrak{P}(V), \bullet^V, \heartsuit^1, \ldots, \heartsuit^n\rangle : V \subseteq {}^nU \textit{ for some set } U\}$$

[13] Recall that if $f : X \longrightarrow Y$, then we use $f^*(X)$ or $f^*$ to denote the range of $f$.

PROOF. The first equality follows immediately from the Sahlqvist form of the equations. The second equality follows, by a now very familiar argument, from the next claim.

CLAIM 1. Every $\mathcal{F} \in \mathsf{K}^n_{con}$ is a zigzagmorphic image of a frame $\mathcal{G} = \langle V, \mathbf{R}^\diamond, \mathbf{R}^1, \ldots, \mathbf{R}^n \rangle$, in which $V \subseteq {}^nU$ for some set $U$, $\mathbf{R}^\diamond$ is defined as in 4.5.3, and the relations for the conjugates in $\mathcal{G}$ are defined as follows:

$$\mathbf{R}^i(y, x_1, \ldots, x_n) \stackrel{\text{def}}{\Longleftrightarrow} \{y\} = \heartsuit^i(\{x_1\}, \ldots, \{x_n\})$$

PROOF OF CLAIM. We use a similar construction as in the previous proof except for a change in the inductive step. Note that, by the definitions of the $\mathbf{R}^i$, **con** holds in $\mathcal{G}$. Here is the modified inductive step. Instead of one single function $g_{\alpha+1}$, we create such a function for each of the operators (we omit the subscript $_{\alpha+1}$ from now on). Define $g^0 : \{\vec{u} : R^\diamond l_\alpha s \vec{u}\} \longrightarrow (P \setminus U_\alpha)$, and for $1 \leq i \leq n$, $g^i : \{\vec{u} : R^i l_\alpha s \vec{u}\} \longrightarrow (P \setminus U_\alpha)$ such that

- all $g^i$ are one to one
- the ranges of the $g^i$ are pairwise disjoint
- $|P \setminus (\bigcup_{0 \leq i \leq n} (g^i)^*)| = \kappa$

Now set,

$$\begin{array}{rclcrl}
U_{\alpha+1} & = & U_\alpha & \cup & \bigcup_{0\leq i \leq n} & (g^i)^* \\
V_{\alpha+1} & = & V_\alpha & \cup & \bigcup_{j<n} & \{\mathbf{f}^{\,j}_{g^0(\vec{u})}\, s : R^\diamond l_\alpha s \vec{u}\} \\
 & & & \cup & \bigcup_{1 \leq i \leq n,\, j<n} & \{\mathbf{f}^{\,i-1}_{g^i(\vec{u})}\, \mathbf{f}^{\,j}_{s_{i-1}}\, s : R^i l_\alpha s \vec{u}\} \\
l_{\alpha+1} & = & l_\alpha & \cup & \bigcup_{j \leq n} & \{\langle \mathbf{f}^{\,j}_{g^0(\vec{u})}\, s, \vec{u}(j)\rangle : R^\diamond l_\alpha s \vec{u}\} \\
 & & & \cup & \bigcup_{1 \leq i \leq n,\, j<n} & \{\langle \mathbf{f}^{\,i-1}_{g^i(\vec{u})}\, \mathbf{f}^{\,j}_{s_{i-1}}\, s, \vec{u}(j)\rangle : R^i l_\alpha s \vec{u}\}
\end{array}$$

CLAIM 2. $l$ is a zigzagmorphism from $\mathcal{G}$ onto $\mathcal{F}$.

PROOF OF CLAIM. The same argument as before shows that $l$ is surjective and has the zigzag property, given the new inductive step. Next, we show by induction that $l$ is a homomorphism for $\mathbf{R}^\diamond$. In step 0, this is obvious. Suppose $l$ is homomorphic for $\mathbf{R}^\diamond$ for all elements in $V_\alpha$. By the conditions on the $g^i$, there is only one way in which new elements can come to stand in the $\mathbf{R}^\diamond$ relation, and that is when $R^i l_\alpha s \vec{u}$ holds for some $s, i$ and $\vec{u}$ with $\sigma(\alpha+1) = s$. Then we added $r_1, \ldots, r_n$ such that $l_{\alpha+1}(r_j) = \vec{u}(j)$. If $i = 0$, then $\mathbf{R}^\diamond s, r_1, \ldots, r_n$. But, then also $R^\diamond l_{\alpha+1} s, l_{\alpha+1}(r_1), \ldots, l_{\alpha+1}(r_n)$. If $i \neq 0$, then $\mathbf{R}^i s, r_1, \ldots, r_n$. Thus, by **con**, $\mathbf{R}^\diamond r_i, r_1, \ldots, r_{i-1}, s, r_{i+1}, \ldots, r_n$. But, since these $r_j$ were added for $R^i$, we have $R^i l_\alpha(s), l_{\alpha+1}(r_1), \ldots, l_{\alpha+1}(r_n)$. By **con** in $\mathcal{F}$, we then have $R^\diamond l(r_i), l(r_1), \ldots, l(r_{i-1}), l(s), l(r_{i+1}), \ldots, l(r_n)$. So we find

$$(**) \quad \mathbf{R}^\diamond y, x_1, \ldots, x_n \Rightarrow R^\diamond l(y), l(x_1), \ldots, l(x_n)$$

Therefore, $l_{\alpha+1}$ is a homomorphism for $\mathbf{R}^\diamond$. To show the same for the other relations, suppose $\mathbf{R}^i(y, x_1, \ldots, x_n)$ and compute:

$$\begin{array}{lcl}
\mathbf{R}^i(y, x_1, \ldots, x_n) & \Longleftrightarrow & \text{by } \mathbf{con} \\
\mathbf{R}^\diamond x_i, x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n & \Rightarrow & (**) \\
R^\diamond l(x_i), l(x_1), \ldots, l(x_{i-1}), l(y), l(x_{i+1}), \ldots, l(x_n) & \Longleftrightarrow & \text{by } \mathbf{con} \\
R^i l(y), l(x_1), \ldots, l(x_n) & &
\end{array}$$

Therefore, $l_{\alpha+1}$ is a homomorphism in general. Thus, $l$ is a homomorphism. Hence, it is a zigzagmorphism. ◀

We finished the proof. QED

### 4.5.3 Geometrical Representation of Arbitrary BAO's

Now we generalize 4.5.2 to algebras with arbitrarily many operators. The work presented in this subsection is the result of a cooperation with István Németi and Ildikó Sain. We know already how to represent BAO's with one operator of arity higher than 1. Before we prove the general theorem, we show how to represent BAO's with a unary operator or a constant.

LEMMA 4.5.5. *Let* $\mathsf{K} \stackrel{\text{def}}{=} \{\mathfrak{A} = \langle A, \vee, -, \Diamond\rangle : \mathfrak{A}$ *satisfies the BAO axioms*$\}$.

(i) *If* $\Diamond$ *is a constant, then* $\mathsf{K} = \mathbf{IS}\{\mathfrak{B} = \langle\mathfrak{P}(V), \mathsf{D}_{01}\rangle : V \subseteq U \times U$ *for some set* $U\}$.
(ii) *If* $\Diamond$ *is a unary operator, then* $\mathsf{K} = \mathbf{IS}\{\mathfrak{B} = \langle\mathfrak{P}(V), \mathsf{F}\rangle : V \subseteq U \times U$ *for some set* $U\}$, *in which* $\mathsf{F}(\mathsf{x}) \stackrel{\text{def}}{=} \mathsf{x}\bullet^V\mathsf{x}$.

PROOF. (i). Let $\mathfrak{A} \in \mathsf{K}$, and let $\Diamond$ be a constant. Its canonical frame is $\mathfrak{A}_+ = \langle W, R\rangle$, with $R \subseteq W$. Let $U$ be an infinite set such that $|U \times U| \geq |W|$. Define a function $l : W \longrightarrow U \times U$ such that $l$ is injective, and $(\forall u \in W) : Ru \iff (lu)_0 = (lu)_1$. Define $\mathcal{F} = \langle l^*(W), \{s \in l^*(W) : s_0 = s_1\}\rangle$. It is immediate that $l$ is a frame isomorphism. Using the argument given before, this is sufficient to prove the lemma.
(ii). Clearly, $\mathsf{F}(\mathsf{x})$ is normal and additive. For the other side, suppose that $\mathfrak{A} = \langle A, \vee, -, \Diamond\rangle \in \mathsf{K}$, and that $\Diamond$ is unary. Let $\mathfrak{A}' = \langle A, \vee, -, \mathsf{f}\rangle$, with $\mathsf{f}$ binary, be defined from $\mathfrak{A}$ by $\mathsf{f}(\tau, \tau_1) \stackrel{\text{def}}{=} \Diamond(\tau \wedge \tau_1)$. Then $\mathfrak{A}'$ is term–definably equivalent to $\mathfrak{A}$ by $\Diamond(\tau) \stackrel{\text{def}}{=} \mathsf{f}(\tau, \tau)$.

Clearly, $\mathsf{f}$ is normal, and it is additive because of distribution of $\vee$ over $\wedge$. Hence we can apply 4.5.2, and represent $\mathfrak{A}'$ as a subalgebra of $\mathfrak{B}' = \langle\mathfrak{P}(V), \bullet^V\rangle$, in which $V \subseteq U \times U$ for some set $U$, and $\bullet^V$ is binary. Define $\mathfrak{B} = \langle\mathfrak{P}(V), \mathsf{F}\rangle$, using $\mathsf{F}(\tau) \stackrel{\text{def}}{=} \tau\bullet^V\tau$ as above, and we get the desired algebra. QED

Now we know how to represent every operator separately as an operation on relations, we are ready for the general theorem.

THEOREM 4.5.6. *Let* $\mathsf{K} \stackrel{\text{def}}{=} \{\mathfrak{A} = \langle A, \vee, -, \mathsf{f}^i\rangle_{i\in I} : \mathfrak{A}$ *satisfies the BAO axioms*$\}$, *for* $I \subseteq \omega$. *Then,*

$$\mathsf{K} = \mathbf{IS}\{\mathfrak{B} = \langle\mathfrak{P}(V), \mathsf{F}^i\rangle_{i\in I} : V \subseteq {}^{\alpha}U \textit{ for some set } U\},$$

*where*

1. $\alpha = (2 \cdot |\{\mathsf{f}^i : \rho(\mathsf{f}^i) \leq 1\}|) + \Sigma\,(n \cdot |\{\mathsf{f}^i : \rho(\mathsf{f}^i) = n\}|)$
2. *the nullary operators* $\mathsf{F}^i$ *are* $\mathsf{D}_{ij}$
3. *the unary operators* $\mathsf{F}^i$ *are* $\bullet_\Gamma(\mathsf{x}, \mathsf{x})$, *for some* $\Gamma \in {}^2\alpha$
4. *the n-ary operators* $\mathsf{F}^i$ *are n–ary* $\bullet_\Gamma$, *for some* $\Gamma \in {}^n\alpha$

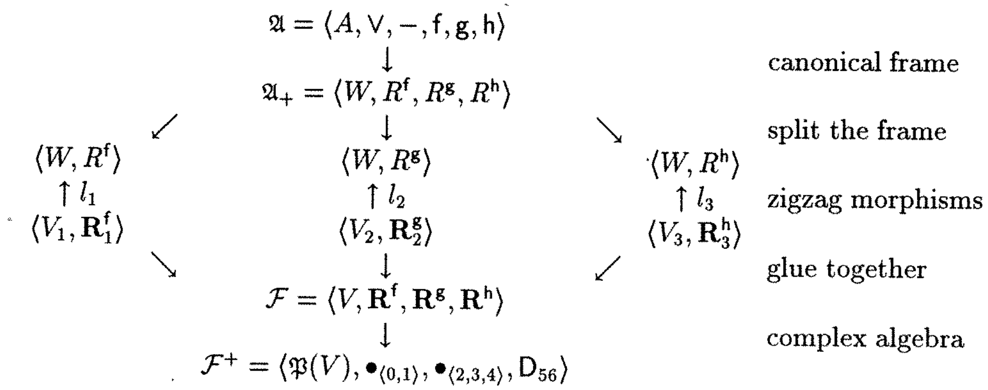



FIGURE 4.2: ROAD–MAP OF THE PROOF OF 4.5.6

PROOF. It is easy to see that the relevant operators are normal and additive. We continue with the representation part of the theorem. Let $\mathfrak{A} = \langle A, -, \vee, \mathsf{f}^i\rangle_{i\in I} \in \mathsf{K}$, and let $\mathfrak{A}_+ = \langle W, R^i\rangle_{i\in I}$. Again, we create a frame $\mathcal{F} = \langle V, \mathbf{R}^i\rangle_{i\in I}$, and a zigzagmorphism $l : V \longrightarrow W$. The proof consist of two parts. In the first, we split $\mathfrak{A}_+$ into frames $\langle W, R^i\rangle$, one for each operator. Then we apply 4.5.2 and 4.5.5, obtaining zigzagmorphic pre–images for each of these frames. In the second part, we glue these pre–images together, and obtain the desired frame $\mathcal{F}$.

We describe the proof for the case of three operators f, g and h with $\rho(\mathsf{f}) = 2$, $\rho(\mathsf{g}) = 3$, and $\rho(\mathsf{h}) = 0$. A "road–map" of this proof is given in figure 4.2. It will be clear from the proof how to extend it to any set of operators.

Applying 4.5.2 and 4.5.5 to the three frames $\langle W, R^{\mathsf{f}}\rangle$, $\langle W, R^{\mathsf{g}}\rangle$ and $\langle W, R^{\mathsf{h}}\rangle$ one gets three frames $\mathcal{F}_1 = \langle V_1, \mathbf{R}_1^{\mathsf{f}}\rangle$, $\mathcal{F}_2 = \langle V_2, \mathbf{R}_2^{\mathsf{g}}\rangle$ and $\mathcal{F}_3 = \langle V_3, \mathbf{R}_3^{\mathsf{h}}\rangle$, in which $V_1$ and $V_3$ are binary relations on sets $U_1$ and $U_3$, respectively, and $V_2$ is a ternary relation on some set $U_2$. The relations $\mathbf{R}_i$ are defined as stated before. The frames $\langle W, R^{\mathsf{f}}\rangle$, $\langle W, R^{\mathsf{g}}\rangle$, and $\langle W, R^{\mathsf{h}}\rangle$ are zigzagmorphic images of $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_3$ by the functions $l_1, l_2$, and $l_3$, respectively.

We now describe the "gluing" part. Define $\mathcal{F} = \langle V, \mathbf{R}^{\mathsf{f}}, \mathbf{R}^{\mathsf{g}}, \mathbf{R}^{\mathsf{h}}\rangle$ as follows:

$$
\begin{array}{lcl}
V & \stackrel{\text{def}}{=} & \{s \in {}^7(U_1 \cup U_2 \cup U_3) : \langle s_0, s_1\rangle \in V_1 \ \&\ \langle s_2, s_3, s_4\rangle \in V_2 \ \& \\
 & & \langle s_5, s_6\rangle \in V_3 \ \&\ l_1(\langle s_0, s_1\rangle) = l_2(\langle s_2, s_3, s_4\rangle) = l_3(\langle s_5, s_6\rangle)\} \\
\mathbf{R}^{\mathsf{f}}(x, y, z) & \stackrel{\text{def}}{\Longleftrightarrow} & \{x\} = \bullet_{\langle 0,1\rangle}(\{y\}, \{z\}) \\
\mathbf{R}^{\mathsf{g}}(x, y, z, v) & \stackrel{\text{def}}{\Longleftrightarrow} & \{x\} = \bullet_{\langle 2,3,4\rangle}(\{y\}, \{z\}, \{v\}) \\
\mathbf{R}^{\mathsf{h}}(x) & \stackrel{\text{def}}{\Longleftrightarrow} & x_5 = x_6
\end{array}
$$

By writing out definitions, we see that $\mathbf{R}^{\mathsf{f}}(x, y, z)$ iff $\mathbf{R}_1^{\mathsf{f}}(\langle x_0, x_1\rangle, \langle y_0, y_1\rangle, \langle z_0, z_1\rangle)$, and similarly for the two other relations. Now, define a function $p : V \longrightarrow W$ as $p(s) = l_1(\langle s_0, s_1\rangle)$.

CLAIM. $p$ is a zigzagmorphism from $\mathcal{F}$ onto $\mathfrak{A}_+$.

PROOF OF CLAIM. *p is surjective.* Let $u \in W$, then (because $l_1, l_2, l_3$ are surjective) there exists $x \in V_1$, $y \in V_2$ and $z \in V_3$, such that $l_1(x) = l_2(y) = l_3(z) = u$. Thus, $\langle x_0, x_1, y_0, y_1, y_3, z_0, z_1 \rangle$ is in $V$, and its $p$–image equals $u$.
*p is a homomorphism.* Suppose that $\mathbf{R}^{\mathsf{f}}(x,y,z)$ holds:

$$\begin{array}{lcl} \mathbf{R}^{\mathsf{f}}(x,y,z) & \Longleftrightarrow & \text{(by definition of } V \text{ and } \mathbf{R}^{\mathsf{f}}) \\ \mathbf{R}_1^{\mathsf{f}}(\langle x_0,x_1\rangle,\langle y_0,y_1\rangle,\langle z_0,z_1\rangle) & \Rightarrow & (l_1 \text{ is a homomorphism}) \\ R^{\mathsf{f}}(l_1(\langle x_0,x_1\rangle), l_1(\langle y_0,y_1\rangle), l_1(\langle z_0,z_1\rangle)) & \overset{\text{def}}{\Longleftrightarrow} & \\ R^{\mathsf{f}}(p(x),p(y),p(z)) & & \end{array}$$

Because $p(s) = l_1(\langle s_0, s_1\rangle) = l_2(\langle s_2, s_3, s_4\rangle) = l_3(\langle s_5, s_6\rangle)$, the proofs for $\mathbf{R}^{\mathsf{g}}$ and $\mathbf{R}^{\mathsf{h}}$ are similar.
*p is zigzag.* Suppose $R^{\mathsf{g}}(p(s), y_1, y_2, y_3)$ holds. Then, since $l_2$ is zigzag and $p(s) = l_2(\langle s_2, s_3, s_4\rangle)$, we find $y_1\prime, y_2\prime, y_3\prime \in V_2$, such that $\mathbf{R}_2^{\mathsf{g}}(\langle s_2, s_3, s_4\rangle, y_1\prime, y_2\prime, y_3\prime)$ & $l_2(y_j\prime) = y_j$. Choose $r, t, v \in V$ which agree on the second, third and fourth coordinate with $y_1\prime, y_2\prime, y_3\prime$, respectively. Since all labelling functions are surjective we can find such $r, t, v$. By definition of $\mathbf{R}^{\mathsf{g}}$ and $p$, we have $\mathbf{R}^{\mathsf{g}} s, r, t, v$ and $p(r) = y_1$, $p(t) = y_2$ & $p(v) = y_3$. The proofs for $\mathbf{R}^{\mathsf{f}}$ and $\mathbf{R}^{\mathsf{h}}$ are similar. ◀

We have proven the theorem for this special case. Note that $\alpha = 7 = 2 \cdot |\{\mathsf{h}\}| + 2 \cdot |\{\mathsf{f}\}| + 3 \cdot |\{\mathsf{g}\}|$. Looking at the road–map of this proof, we see immediately that it can be extended to any set of operators. QED

## 4.6 CONCLUDING REMARKS

Combination of the mosaic–idea and the step–by–step construction led to simple axiomatizations for relativized relation algebras. The given proof also gave us easy characterizations for reducts of **SRIRRA**. Relativization moreover, still gives us positive results if we add the powerful difference operator. The last section showed that the step–by–step proof is quite widely applicable, and so are algebras of relations. We conclude with some questions.

1. Are the reducts labelled with $\mathsf{QV}^?$ in table 4.1, finitely axiomatizable? Are they varieties?
2. There exists a nice duality theory between BAO's and relational Kripke frames. Is there something similar between BAO's and the concrete frames we obtained in section 4.5?
3. The axiomatization for **SRIRRA** was obtained by showing that we could FO axiomatize the class of all zigzagmorphic images of all pair–frames, and then finding canonical equations which characterized these FO axioms. It seems possible to obtain this result in a purely FO proof–theoretical manner. This would go as follows. It follows from the preservation arguments in van Benthem [Ben83] that, if a class K is defined by a FO theory $\Gamma$, then **ZigK** is defined by all FO consequences of $\Gamma$ which can be written in the following form:

   construct from atoms and falsum, and use only $\wedge, \vee, \forall$ and $\exists$, plus restricted universal quantification.

So the class of all zigzagmorphic images of an elementary frame class is itself elementary. Let $\Sigma$ be the above given FO definition of **ZigK**. The task is to find a (finite) set of **ZigK** valid FO sentences $\Sigma'$ such that $\Sigma' \models \Sigma$, and $\Sigma'$ is definable by canonical equations. Is this a feasible strategy? How general is it?

# 5

# AMALGAMATION & INTERPOLATION

In this chapter, we look at a third fundamental aspect of core logics: *interpolation properties*. In [ANS94b], it is shown that Craig interpolation has a strong computational aspect (cf. also Rodenburg [Rod91b], [Rod91a], [Rod92]). A theorem–prover for a logic with the interpolation property can be set up in a modular way, with channels between different databases through which only limited information (the interpolants) can float. In many situations, this modular set–up will reduce the search space, and make the theorem–prover more efficient. We will also study the related notion of Beth definability. For the two main logics under investigation in this work, we have the following results:

- For $H \subseteq \{R, S, T\}$, the arrow logic of $H$ pair–frames (i.e., $\mathcal{GML}(\mathsf{K}^{rel}_{setH})$) has interpolation and Beth definability iff $T \notin H$.
- The cylindric modal logics of the classes of assignment frames $\mathsf{K}^{cyl\alpha}_{set}$ and $\mathsf{K}^{cyl\alpha}_{setD}$ ($\alpha$ any ordinal) have interpolation and Beth definability.

The crucial step in an interpolation argument is the construction of a model out of two other models (cf. e.g., Hodges [Hod93], Thm 6.6.3, or van Benthem [Ben94a], appendix 12). This construction –known as *amalgamation*– is of interest on its own, because it has further applications. (E.g., the step–by–step constructions in the previous chapter can be seen as a repeated process of amalgamating algebras; see also [Hod93] Sec.6.4ff.) In this chapter, we concentrate on algebraic amalgamation, and then derive interpolation properties on the logic side. There is a long tradition of connecting interpolation properties of logics with amalgamation properties of the corresponding classes of algebras (cf. e.g., Pigozzi [Pig72], Czelakowski [Cze81], Németi [Ném83], Sain [Sai90], Maksimova [Mak91a], Andréka et.al. [ANSK94], [ANS94c], [AN94]).

ORGANIZATION. In the first section, we state the definitions of interpolation and amalgamation, and summarize their connections. A summary of these connections is given in table 5.1. In section 5.2, we introduce a new operation on frames, called *zigzag products*, and use it to give a structural description of a large class of BAO's which allow a very strong form of amalgamation (lemma 5.2.6). In the next section, we study preservation of FO sentences under taking zigzag products. We give a syntactic description of a large class of BAO's with amalgamation, and of a large class of general modal logics with interpolation and Beth definability (theorems 5.3.5 and 5.3.6). In section 5.4, we apply these general results to the classes of BAO's and logics that we have studied before. In appendix 5.6, we give a reformulation of amalgamation in terms of frames, and show how this formulation can be used to find quick proofs of failure of amalgamation.

**Convention**
Throughout this chapter, we always assume that we have the Boolean 0 and 1.

# 5.1 AMALGAMATION, INTERPOLATION AND DEFINABILITY

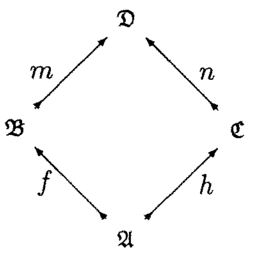
We state the definitions of the three notions in the title, and present the connections between them which are known from the literature. A summary can be found in table 5.1.

## 5.1.1 AMALGAMATION

We start with several kinds of amalgamation properties. Note that *super-amalgamation* (SUPAP)[1] requires a partial ordering on the algebras.

DEFINITION 5.1.1 (AMALGAMATION). Let $\mathsf{K}$ be a class of algebras.

1. $\mathsf{K}$ has the *Embedding Property (EP)* if, for any $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in \mathsf{K}$, and embeddings $f, h$ such that $\mathfrak{B} \overset{f}{\leftarrowtail} \mathfrak{A} \overset{h}{\rightarrowtail} \mathfrak{C}$, there exists $\mathfrak{D} \in \mathsf{K}$, and embeddings $m, n$ such that $\mathfrak{B} \overset{m}{\rightarrowtail} \mathfrak{D} \overset{n}{\leftarrowtail} \mathfrak{C}$.



(5.1)

2. $\mathsf{K}$ has the *Amalgamation Property (AP)* if 1 can be strengthened[2] by requiring that $m \circ f = n \circ h$.
3. $\mathsf{K}$ has the *Strong Amalgamation Property (SAP)* if 2 can be strengthened by requiring[3] that $m^*(B) \cap n^*(C) = (m \circ f)^*(A)$.
4. $\mathsf{K}$ has the *Super Amalgamation Property (SUPAP)* if 2 can be strengthened by requiring that for all $\mathsf{x} \in B, \mathsf{y} \in C$,

$$m(\mathsf{x}) \leq n(\mathsf{y}) \;\Rightarrow\; (\exists \mathsf{z} \in A) : \mathsf{x} \leq f(\mathsf{z}) \;\&\; h(\mathsf{z}) \leq \mathsf{y}$$
$$n(\mathsf{y}) \leq m(\mathsf{x}) \;\Rightarrow\; (\exists \mathsf{z} \in A) : \mathsf{y} \leq h(\mathsf{z}) \;\&\; f(\mathsf{z}) \leq \mathsf{x}$$

[1] SAP seems to be the common abbreviation in the literature for Strong Amalgamation. Unfortunately, Maksimova uses SAP to abbreviate *Super Amalgamation* in [Mak91a].

[2] We use this to abbreviate that, if we strengthen 1 by adding the requirement $m \circ f = n \circ h$ to the conclusion, then the stronger version of 1 will still hold in $\mathsf{K}$.

[3] An equivalent formulation of this strengthening is: $m^*(B \setminus f^*(A)) \cap n^*(C \setminus h^*(A)) = \emptyset$.

REMARKS 5.1.2. Kiss et al. [KMPT83] contains a vast amount of information about EP, AP, SAP and other related concepts, together with a very extended bibliography of the field. The definition of SUPAP is due to L. Maksimova (cf. [Mak91a]).

The amalgamation property (AP) speaks about amalgamating algebras in such a way that the amalgam agrees on the common subalgebra. If the amalgamation is strong, the common subalgebra is the only overlap between the two algebras in the amalgam. Every super amalgamation is also strong: as is easy to see by rewriting the extra condition for SAP to the equivalent statement (5.2).

$$(\forall \mathsf{x} \in B, \forall \mathsf{y} \in C) : m(\mathsf{x}) = n(\mathsf{y}) \Rightarrow (\exists \mathsf{z} \in A) : \mathsf{x} = f(\mathsf{z}) \; \& \; \mathsf{y} = h(\mathsf{z}) \tag{5.2}$$

We call the element z, as it occurs in condition (5.2) above and in the condition for SUPAP, an *interpolant*. Figure 5.1 contains a simple example, in Boolean algebras, of a strong amalgamation which is not super. To have a super amalgamation in this case, not only $b$ and $c$ must be atoms, but $-b$ and $-c$ as well.

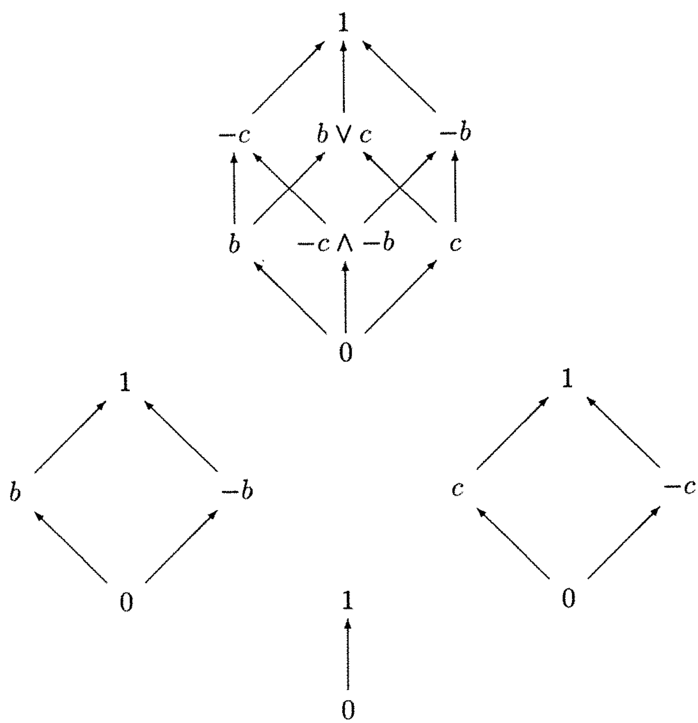



FIGURE 5.1: STRONG-, BUT NOT SUPER–AMALGAMATION

For classes of (ordered) algebras, the differences between AP, SAP and SUPAP are illustrated by the following examples. The variety of distributive lattices has AP but not SAP. Maksimova has shown the existence of a variety V of BAO–type $\langle \Diamond, 1 \rangle$ (V is a variety of closure algebras) with SAP that lacks SUPAP ([Mak91b], Thm 1).

SURJECTIVENESS OF EPIMORPHISMS. A notion which is often studied together with amalgamation is *surjectiveness of epimorphisms* (ES) (cf. [KMPT83]). This notion is closely related to Beth's definability property (see below). We recall the definition from [KMPT83]. Let $\mathsf{K}$ be a class of algebras, $\mathfrak{A}, \mathfrak{B} \in \mathsf{K}$ and $f : A \longrightarrow B$ a homomorphism. The function $f$ is called an *epimorphism* of $\mathsf{K}$ (*epi* for short) iff for all $\mathfrak{C} \in \mathsf{K}$ and all $h, g \in Hom(\mathfrak{B}, \mathfrak{C})$ we have $(hf = gf \Rightarrow h = g)$. The class $\mathsf{K}$ has ES iff every epimorphism $f$ as above is onto $\mathfrak{B}$.

CONNECTIONS. The above defined notions interact in the following way.

PROPOSITION 5.1.3. *Let* $\mathsf{K}$ *be a class of algebras.*
(i) $\mathsf{K}$ *has SUPAP* $\Rightarrow$ $\mathsf{K}$ *has SAP* $\Rightarrow$ $\mathsf{K}$ *has AP* $\Rightarrow$ $\mathsf{K}$ *has EP.*
(ii) $\mathsf{K}$ *has SAP* $\Rightarrow$ $\mathsf{K}$ *has ES*
(iii) *If* $\mathsf{K}$ *is a* quasi–variety, *then* $\mathsf{K}$ *has SAP if and only if* $\mathsf{K}$ *has AP and ES*

PROOF. (i). By the definitions and (5.2) above. (ii)–(iii). Cf. [KMPT83] Prop 1.10 and 6.3. QED

### 5.1.2 INTERPOLATION AND DEFINABILITY

CRAIG INTERPOLATION. W. Craig proved the interpolation theorem for first order logic in 1957 ([Cra57]). Since then, many papers appeared on (failure of) interpolation in other logics. Craig's interpolation theorem can be formulated for general modal logics in two different ways[4]. Let $\mathcal{GML}(\mathsf{K}) = \langle \mathsf{Fml}(P), \mathsf{Mod}(\mathsf{K}), \Vdash \rangle$ be a general modal logic in the sense of section 2.1.3. We say that $\mathcal{GML}(\mathsf{K})$ has the *Strong Craig Interpolation property* (SCI) if, for any two formulas $\phi, \psi \in \mathsf{Fml}(P)$, if $\models_{\mathsf{K}} \phi \rightarrow \psi$, then there is a formula $\theta \in \mathsf{Fml}(P)$ such that $\models_{\mathsf{K}} (\phi \rightarrow \theta) \wedge (\theta \rightarrow \psi)$, and $\theta$ is constructed from variables in both $\phi$ and $\psi$. The formula $\theta$ is called an *interpolant.* We say that $\mathcal{GML}(\mathsf{K})$ has the *Weak Craig Interpolation property* (WCI) if, for any two formulas $\phi$ and $\psi$, if $\phi \models^{glo}_{\mathsf{K}} \psi$, then there is a $\theta$ such that $\phi \models^{glo}_{\mathsf{K}} \theta$ and $\theta \models^{glo}_{\mathsf{K}} \psi$, and $\theta$ is constructed from variables in both $\phi$ and $\psi$.

ALGEBRAIC INTERPOLATION. It is straightforward to give "translations" of interpolation properties at the algebraic level. These translations can be found for instance in [Pig72] or [Mak91a]. We only recall the translation of SCI. Let $\mathsf{K}$ be a class of algebras with a partial ordening $\leq$. We say that $\mathsf{K}$ has the *interpolation property of inequalities* (IPI) if, for any terms $\tau, \tau_1$ such that $\mathsf{K} \models \tau \leq \tau_1$, there is a term $\tau_2$ with $var(\tau_2) \subseteq var(\tau) \cap var(\tau_1)$ and $\mathsf{K} \models \tau \leq \tau_2 \leq \tau_1$.

PROPOSITION 5.1.4. *Let* $\mathsf{K}$ *be a class of frames. The class of algebras* $\mathbf{SPK}^+$ *has IPI if and only if the logic* $\mathcal{GML}(\mathsf{K})$ *has SCI.*

PROOF. This follows using the translation between modal–logical formulas and algebraic terms. QED

[4]They only differ with the *global consequence*, because with the local consequence we have $\phi \models^{loc} \psi \iff \models^{loc} \phi \rightarrow \psi$.

BETH'S DEFINABILITY PROPERTY. A meta-logical property which is often studied in conjunction with interpolation is *Beth definability* (BD). A logic has this property if, for every *implicit* definition there is an *explicit* definition. These notions are defined as follows (cf. [CK90] p.90). Let $\mathcal{GML}(\mathsf{K})$ be an arbitrary, but fixed general modal logic, with a set of formulas $\mathsf{Fml}(P)$. Let $p$ and $p'$ be two new propositional variables not in $P$. Let $\Sigma(p)$ be a set of formulas in the language $\mathsf{Fml}(P \cup \{p\})$, and let $\Sigma(p')$ be the corresponding set in $\mathsf{Fml}(P \cup \{p'\})$, formed by replacing $p$ everywhere by $p'$. We say that $\Sigma(p)$ defines $p$ *implicitly* iff

$$\Sigma(p) \cup \Sigma(p') \models^{glo}_{\mathsf{K}} p \leftrightarrow p'$$

$\Sigma(p)$ is said to define $p$ *explicitly* iff there exists a formula $\theta \in \mathsf{Fml}(P)$ such that

$$\Sigma(p) \models^{glo}_{\mathsf{K}} \theta \leftrightarrow p$$

DEFINABILITY, DEDUCTION TERMS, AND INTERPOLATION

There is a strong connection between the notions of strong and weak interpolation and definability. In Andréka–Németi [AN94] it is shown that in general neither strong interpolation implies the weak one, nor strong interpolation implies definability. In many cases however, the "expected" implications hold. The counterexamples constructed in [AN94] show that both assumptions on the logic in the next theorem are really needed. Below we give a simple counterexample (5.1.8) to the implication SCI $\Rightarrow$ BD.

DEFINITION 5.1.5. A general modal logic $\mathcal{GML}(\mathsf{K})$ has a *local deduction term* if, for any two formulas $\phi, \psi$, there exists a formula $f(\phi)$, built up from propositional variables in $\phi$, such that

(i) $\phi \models^{glo}_{\mathsf{K}} \psi \iff \models_{\mathsf{K}} f(\phi) \rightarrow \psi$,

(ii) For all $\theta$, $\{\phi, f(\phi) \rightarrow \theta\} \models^{glo}_{\mathsf{K}} \theta$.

THEOREM 5.1.6. *Let $\mathcal{GML}(\mathsf{K})$ be a general modal logic with a local deduction term.*

(i) *If $\mathcal{GML}(\mathsf{K})$ has strong interpolation, it also has weak interpolation.*

(ii) *If $\mathcal{GML}(\mathsf{K})$ has strong interpolation and is compact, it has Beth definability.*

PROOF. (i). Obvious. (ii). Because the proof of Beth definability from interpolation for FO logic in [CK90] (Thm 2.2.20) goes through. QED

The assumptions of compactness and a local deduction term are very often fulfilled in general modal logics. It follows from the next theorem that they are satisfied for every general modal logic which is strongly sound and complete with respect to an extension of the standard **K** derivation system[5]. This result is stated in Czelakowski [Cze81] p.339 for modal logics of type $\langle \Diamond, 1 \rangle$ and attributed to Perzanowski [Per73].

THEOREM 5.1.7. *Let $\mathsf{K}$ be a class of frames. If $\mathbf{SPK}^+$ is a variety, then $\mathcal{GML}(\mathsf{K})$ is compact and has a local deduction term.*

[5]That is, a derivation system consisting of all propositional tautologies, distribution axioms for all modalities, as rules modus ponens, universal generalization and substitution, plus additional axioms.

PROOF. Suppose the antecedent. Thm 3.2.20 in [ANSK94] implies that $\mathcal{GML}(\mathsf{K})$ is compact iff $\mathbf{SPK}^+$ is a quasi–variety. Hence $\mathcal{GML}(\mathsf{K})$ is compact. We continue with showing that $\mathcal{GML}(\mathsf{K})$ has a local deduction term. We prove the claim for a finite similarity type, since it is more instructive. The same proof goes through for infinite types as well. Let $\mathsf{K}$ be of finite BAO type $S$. We define the following abbreviations:

$$\begin{array}{lcl}\langle S\rangle\tau & \stackrel{\text{def}}{=} & \bigvee\{f(\tau,1,\ldots,1), f(1,\tau,1,\ldots,1)\ldots f(1,\ldots,1,\tau) : f\in S\}\\ [S]\tau & \stackrel{\text{def}}{=} & -\langle S\rangle-\tau\\ \tau^0 & \stackrel{\text{def}}{=} & \tau\\ \tau^{n+1} & \stackrel{\text{def}}{=} & \tau^n\wedge[S]\tau^n\end{array}$$

Because we assumed that the similarity type is finite, $\langle S\rangle\tau$ is well defined. In the terminology of van Benthem [Ben83], $w \Vdash \tau^n$ iff $\tau$ holds everywhere in the $n$–th hull around $w$. We use the same definitions for the logical language. Using a similar argument as in Prop 2.33 in [Ben83] we find that $\phi \models^{glo}_{\mathsf{K}} \psi \iff \{\phi^n : n<\omega\} \models^{loc}_{\mathsf{K}} \psi$. Because $\mathbf{SPK}$ is variety, also $\models^{loc}_{\mathsf{K}}$ is compact. But then there exists a $n$ such that $\phi \models^{glo}_{\mathsf{K}} \psi \iff \models_{\mathsf{K}} \phi^n \to \psi$. Clearly, $\phi^n$ also satisfies the second condition of a deduction term. QED

EXAMPLE 5.1.8. We give an example of a logic which has strong interpolation, but doesn't have Beth definability. Stronger examples (e.g., in which the logic is also compact and has WCI) based on the same logic can be found in [AN94]. This logic is known from temporal logic and the theory of program specifications, and described for instance in Andréka et al. [AGM+94]. We define the logic $\mathsf{Tp}$ as a tuple $\langle \mathsf{Fml}, \mathsf{Mod}(\mathsf{K}), \Vdash\rangle$ in which

- $\mathsf{Fml}$ is the smallest set containing countably many variables, and as connectives it has the Booleans and two unary modalities $\mathsf{Fi}$ and $\mathsf{N}$.
- $\mathsf{K} = \{\langle\omega, 0, succ\rangle\}$
- $\Vdash$ is defined for the modalities as:

$$\begin{array}{llcl}(\forall n\in\omega): & n\Vdash \mathsf{Fi}\phi & \stackrel{\text{def}}{\iff} & 0\Vdash\phi\\ (\forall n\in\omega): & n\Vdash \mathsf{N}\phi & \stackrel{\text{def}}{\iff} & succ(n)\Vdash\phi\end{array}$$

In [AGM+94] (Thm 2.2.4), a weakly complete axiomatization is given for this logic. On top of the basic $\mathbf{K}$ derivation system, the following four axioms are needed.

$$\begin{array}{llll}(1) & \mathsf{Fi}\neg p\leftrightarrow\neg\mathsf{Fi}p & (2) & \mathsf{N}\neg p\leftrightarrow\neg\mathsf{N}p\\ (3) & \mathsf{Fi}p\leftrightarrow\mathsf{FiFi}p & (4) & \mathsf{Fi}p\leftrightarrow\mathsf{NFi}p\end{array}$$

By their Sahlqvist form, it is easy to see that these axioms characterize the class of all frames $\mathcal{F} = \langle W, f, n\rangle$, with $f$ and $n$ total functions, $fx = ffx$ and $fx = fnx$. Call this class $\mathsf{L}$. Then $\mathcal{GML}(\mathsf{L})$ has strong interpolation by 5.3.6 way below. The logics $\mathcal{GML}(\mathsf{L})$ and $\mathsf{Tp}$ have the same validities, hence $\mathsf{Tp}$ has SCI. In [AN94] (Thm 2), it is shown that $\mathsf{Tp}$ lacks BD. An implicit definition –of the point 0– is given which cannot be made explicit. The definition is $\{\mathsf{Fi}p, \mathsf{N}\neg p\}$.

### 5.1.3 CONNECTIONS: AMALGAMATION, INTERPOLATION AND DEFINABILITY

We present the connections between amalgamation properties of a class of BAO's $\mathbf{SPK}^+$ on the one hand, and interpolation and definability properties of a general modal logic $\mathcal{GML}(\mathsf{K})$ on the other. All results are known, or derived easily from the literature.

THEOREM 5.1.9. *Let* $\mathsf{K}$ *be a class of frames.*
(i) *If* $\mathbf{SPK}^+$ *has SUPAP, then* $\mathcal{GML}(\mathsf{K})$ *has SCI.*
(ii) (MAKSIMOVA, MADARASZ) *If* $\mathbf{SPK}^+$ *is a variety, then SUPAP of* $\mathbf{SPK}^+$ *is equivalent with SCI of* $\mathcal{GML}(\mathsf{K})$.
(iii) *If* $\mathbf{SPK}^+$ *has AP, then* $\mathcal{GML}(\mathsf{K})$ *has WCI.*
(iva) *If* $\mathbf{SPK}^+$ *is a variety, then AP of* $\mathbf{SPK}^+$ *is equivalent with WCI of* $\mathcal{GML}(\mathsf{K})$.
(ivb) (CZELAKOWSKI) *If* $\mathcal{GML}(\mathsf{K})$ *is compact and has a local deduction term, then AP of* $\mathbf{SPK}^+$ *is equivalent with WCI of* $\mathcal{GML}(\mathsf{K})$.
(v) (NÉMETI) $\mathbf{SPK}^+$ *has ES if and only if* $\mathcal{GML}(\mathsf{K})$ *has BD.*

The results in this theorem, together with those of the previous two subsections, are summarized in table 5.1. Left of the dotted line in the middle, are the properties of the class of algebras $\mathbf{SPK}^+$. At the right are the properties of the general modal logic $\mathcal{GML}(\mathsf{K})$. The numbers attached to the implications provide the reference to the theorems used. The implications written with a black arrow hold always. The dashed arrows denote implications which hold only when the conditions mentioned in the theorem are met. If $\mathbf{SPK}^+$ is a variety, then all implications hold. The next corollary provides the logical counterpart of the strong amalgamation property.

COROLLARY 5.1.10. *Let* $\mathsf{K}$ *be a class of frames and* $\mathbf{SPK}^+$ *a variety. Then* $\mathbf{SPK}^+$ *has SAP if and only if* $\mathcal{GML}(\mathsf{K})$ *has BD and WCI.*

PROOF. By 5.1.3.(iii) and 5.1.9.(iva) and (v). QED

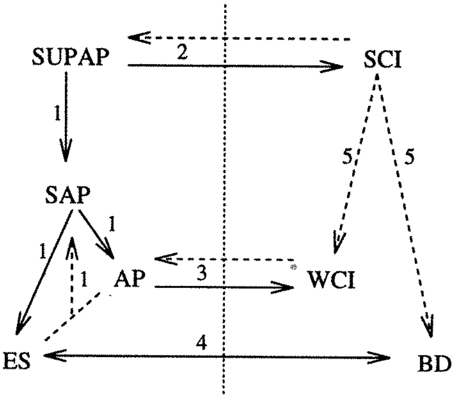
PROOF OF THEOREM 5.1.9. (i). Suppose $\mathsf{L} \stackrel{\text{def}}{=} \mathbf{SPK}^+$ has SUPAP. The conclusion follows from the following claim and 5.1.4.

CLAIM. $\mathsf{L}$ has IPI.

PROOF OF CLAIM. Suppose $\mathsf{L} \models \tau \leq \tau_1$. Create the following three $\mathsf{L}$–free algebras: $\mathfrak{F}_{\mathsf{L}}(var(\tau)), \mathfrak{F}_{\mathsf{L}}(var(\tau_1))$ and $\mathfrak{F}_{\mathsf{L}}(var(\tau) \cap var(\tau_1))$. All three belong to $\mathsf{L}$. $\mathfrak{F}_{\mathsf{L}}(var(\tau) \cap var(\tau_1))$ can be embedded into the other two by the identity mappings $f$ and $h$. So we have

$$\mathfrak{F}_{\mathsf{L}}(var(\tau)) \stackrel{f}{\leftarrowtail} \mathfrak{F}_{\mathsf{L}}(var(\tau) \cap var(\tau_1)) \stackrel{h}{\rightarrowtail} \mathfrak{F}_{\mathsf{L}}(var(\tau_1))$$

Since $\mathsf{L}$ has SUPAP, there exists an algebra $\mathfrak{A} \in \mathsf{L}$ which is a super–amalgam, say with functions $m, n$ such that $\mathfrak{F}_{\mathsf{L}}(var(\tau)) \stackrel{m}{\rightarrowtail} \mathfrak{A} \stackrel{n}{\leftarrowtail} \mathfrak{F}_{\mathsf{L}}(var(\tau_1))$. But then (because $\mathsf{L} \models \tau \leq \tau_1$), $\mathfrak{A} \models m(\tau) \leq n(\tau_1)$. Hence, by the super condition, there exists an element $\tau_2 \in \mathrm{Dom}(\mathfrak{F}_{\mathsf{L}}(var(\tau) \cap var(\tau_1)))$ such that $\mathfrak{F}_{\mathsf{L}}(var(\tau)) \models \tau \leq f(\tau_2)$ and $\mathfrak{F}_{\mathsf{L}}(var(\tau_1)) \models h(\tau_2) \leq \tau_1$. Because $f$ and $h$ are identity mappings, the bottom algebra is generated by the common variables, and the algebras are $\mathsf{L}$–*free*, we have $\mathsf{L} \models \tau \leq \tau_2 \leq \tau_1$, and $var(\tau_2) \subseteq var(\tau) \cap var(\tau_1)$. ◀




| | | | |
|---|---|---|---|
| 1) | Proposition 5.1.3 | 4) | Theorem 5.1.9.(v) |
| 2) | Theorem 5.1.9.(i) and (ii) | 5) | Theorem 5.1.6. |
| 3) | Theorem 5.1.9.(iii) and (iv) | | |

TABLE 5.1: SUMMARY OF CONNECTIONS BETWEEN AMALGAMATION, INTERPOLATION AND DEFINABILITY

(ii). For unary similarity types, this equivalence may be proved in the same way as done for varieties with one unary operator in [Mak92] ([Mak94]). The general result can be found in [Mad94].
(iii). Cf. Pigozzi [Pig72] remarks 1.2.15. For a history of this result, we refer to these remarks.
(iv). Part (b) follows from [Cze81] Thm 3. The requirements on the deduction term in that theorem are a bit different. But assuming compactness, they follow from the formulation given here. Part (a) is a consequence of part (b) by 5.1.7.
(v). Cf. [Ném83]. QED

## 5.2 ZIGZAG PRODUCTS

We introduce a new operation on frames, called *zigzag products*. This notion has close connections with both amalgamation and bisimulation. We use it to give a structural description of a large class of varieties which have super–amalgamation (lemma 5.2.6).

In what follows we use the following operations on frames: taking subframes, products and subdirect products. All these notions are used in the FO model–theoretic sense. We recall the definitions (cf. e.g., [CK90] or [Hod93]). The notion of a *subframe* was defined in section 2.3. Let $\langle \mathcal{F}_i \rangle_{i \in I}$ be a system of frames of the same type. The frame $\mathcal{G} = \prod_{i \in I} \mathcal{F}_i$ is the *direct product* of the frames $\langle \mathcal{F}_i \rangle_{i \in I}$, if $G = \{\langle x_i : i \in I \rangle : x_i \in F_i\}$ and the relations are defined coordinate–wise. A frame $\mathcal{G}$ is a *subdirect product* of a system of frames $\langle \mathcal{F}_i \rangle_{i \in I}$, if it is a subframe of the direct product $\prod_{i \in I} \mathcal{F}_i$, and the projections are *surjective*.

DEFINITION 5.2.1 (ZIGZAG PRODUCTS). Let $\langle \mathcal{F}_i \rangle_{i \in I}$ be a system of frames of the same type. Any substructure of the direct product $\prod_{i \in I} \mathcal{F}_i$ from which the projections are *zigzagmorphisms*, is called a *zigzag product* of $\langle \mathcal{F}_i \rangle_{i \in I}$. A zigzag product of a finite system of frames is called a *finite zigzag product.*

So, a zigzag product is a *subdirect product* with the additional condition that the projections have the zigzag property.

### 5.2.1 ZIGZAG PRODUCTS, BISIMULATION AND AMALGAMATION

We show that the three concepts mentioned in the title are very closely connected.

DEFINITION 5.2.2 (BISIMULATION). Let $\mathcal{F}$ and $\mathcal{G}$ be frames of BAO type $S$. Let $B \subseteq F \times G$.
(i) We call $B$ a *bisimulation* between $\mathcal{F}$ and $\mathcal{G}$ if for any relation $R$ of type $S$:

(1) if $xBx'$ & $R^{\mathcal{F}}xy_1 \dots y_n$, then $(\exists y'_1, \dots, y'_n \in G) : y_iBy'_i$ & $R^{\mathcal{G}}x'y'_1 \dots y'_n$
(2) similarly in the other direction

(ii) A bisimulation $B$ between $\mathcal{F}$ and $\mathcal{G}$ is called a *zigzag connection* between $\mathcal{F}$ and $\mathcal{G}$ if the domain of $B$ equals $F$ and its range equals $G$.

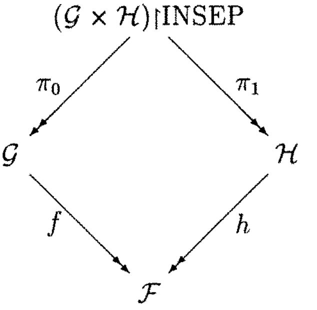
A thorough treatment of bisimulations can be found in de Rijke [Rij93]. Note that if $B$ is a zigzag connection and $B$ is a function, then $B$ is a zigzagmorphism from $\mathcal{F}$ onto $\mathcal{G}$. The next proposition states the connection between bisimulations and zigzag products. It shows that binary zigzag products form an elegant tool to describe all zigzag connections between two frames.

PROPOSITION 5.2.3. *Let $\mathcal{F}$ and $\mathcal{G}$ be frames of the same type and $B \subseteq F \times G$. Then $B$ is a zigzag connection* if and only if *$(\mathcal{F} \times \mathcal{G}){\restriction}B$ is a zigzag product.*

PROOF. Because the relations in the product are defined coordinate–wise, and $\langle x, x' \rangle$ is an element of the domain of the zigzag product iff $xBx'$. QED

If $(\mathcal{F} \times \mathcal{G}){\restriction}B$ is a zigzag product, we use $\pi_0$ and $\pi_1$ to denote the *projections.* The next two propositions indicate the connections between zigzag products and amalgamation.

PROPOSITION 5.2.4. *If $\mathcal{G} \overset{f}{\twoheadrightarrow} \mathcal{F} \overset{h}{\twoheadleftarrow} \mathcal{H}$, then* INSEP $\overset{\text{def}}{=} \{\langle x, y \rangle \in G \times H : f(x) = h(y)\}$ *is a zigzag connection, the frame $(\mathcal{G} \times \mathcal{H}){\restriction}$*INSEP *is a zigzag product of $\mathcal{G}$ and $\mathcal{H}$, and diagram (5.3) commutes:*

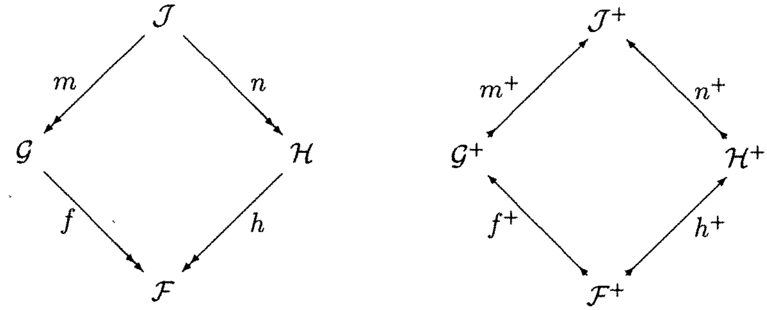



(5.3)

PROOF. Straightforward. QED

If $\mathcal{G} \overset{f}{\twoheadrightarrow} \mathcal{F} \overset{h}{\twoheadleftarrow} \mathcal{H}$, we call elements $x \in G, y \in H$ with $f(x) = h(y)$, *inseparable*[6], because the common frame $\mathcal{F}$ cannot separate the two. By the above proposition, inseparable $x$ and $y$ bisimulate.

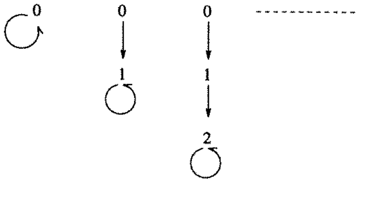
PROPOSITION 5.2.5. *Let $\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{J}$ be frames of the same type.*
(i) *If the left diagram of (5.4) commutes, then the right diagram of (5.4) commutes (it is an amalgamation).*
(ii) *If the left diagram of (5.4) commutes, $J = \{\langle x, y\rangle \in G \times H : f(x) = h(y)\}$, and $m$ and $n$ are the two projections, then the amalgamation at the right is super.*




(5.4)

PROOF. (i). Straightforward.
(ii). Assume the antecedent. We show the super condition only. That is, if $m^+(\mathsf{x}) \leq n^+(\mathsf{y})$, then there exist an interpolant $\mathsf{z}$. (The second condition is analogous.) Suppose that $m^+(\mathsf{x}) \leq n^+(\mathsf{y})$. Let $\mathsf{z} \overset{\text{def}}{=} \{f(x) \in F : x \in \mathsf{x}\}$. The definition of $f^+$ implies that $\mathsf{x} \leq f^+(\mathsf{z})$. In order to show that $h^+(\mathsf{z}) \leq \mathsf{y}$, suppose that $y \in h^+(\mathsf{z})$. Then for some $x \in \mathsf{x}$, $h(y) = f(x)$. So $x$ and $y$ are inseparable. Since $J$ contains all inseparable pairs, $\langle x, y\rangle \in m^+(\mathsf{x})$. But then, by assumption, $\langle x, y\rangle \in n^+(\mathsf{y})$, whence $y \in \mathsf{y}$. QED

ZIGZAG PRODUCTS AND UNRAVELING. Zigzag products are also connected to *unravelling*. For instance, the well-known unravelling of the frame $\langle\{x\}\{\langle x, x\rangle\}\rangle$ to the frame $\langle\omega, succ\rangle$ can be described as an infinite zigzag product of the frames presented in the figure below. Take that substructure of the product which contains $\langle 0, 0, \ldots\rangle, \langle 0, 1, 1 \ldots\rangle, \langle 0, 1, 2, 2 \ldots\rangle \ldots$. It is easy to see that this is a zigzag product which is isomorphic to the frame $\langle\omega, succ\rangle$.




(5.5)

Note that the frames are precisely what one gets via a step–by–step unraveling procedure of $\langle\{x\}\{\langle x, x\rangle\}\rangle$.

[6]The term *inseparable* comes from Chang–Keisler [CK90]. It is used for a similar purpose in the proof of Craig's Interpolation theorem for FO logic.

### 5.2.2 THE ZIGZAG PRODUCT LEMMA

The next lemma is an improvement of lemma 3 in Németi [Ném85]. Németi used this lemma to prove SAP for the class $\mathsf{Crs}_\alpha$ ($\alpha$ any ordinal). At the end of this section, we compare the two results.

LEMMA 5.2.6 (ZIGZAG PRODUCT LEMMA). *Let* $\mathsf{K} \subseteq \mathsf{BAO}$ *and* $\mathsf{L}$ *a class of structures, both of BAO type S. Assume that* (i)–(iii) *below hold.*

(i) $\mathsf{L}$ *is closed under taking finite zigzag products*[7]

(ii) $\mathcal{F} \in \mathsf{L} \Rightarrow \mathcal{F}^+ \in \mathsf{K}$

(iii) $\mathfrak{A} \in \mathsf{K} \Rightarrow \mathfrak{A}_+ \in \mathsf{L}$

*Then* $\mathsf{K}$ *has SUPAP.*

Conditions (ii) and (iii) always hold when $\mathsf{K}$ is a *canonical variety* and $\mathsf{L} = \mathbf{Cm}^{-1}\mathsf{K}$. The next proposition states some implications of conditions (ii) and (iii), and shows its range of applicability.

PROPOSITION 5.2.7. *Let* $\mathsf{K} \subseteq \mathsf{BAO}$ *and* $\mathsf{L}$ *a class of structures, both of BAO type S. Assume conditions* (ii) *and* (iii) *of the previous lemma. Then*

(i) $\mathsf{K}$ *is closed under taking canonical embedding algebras, whence* $\mathsf{SK} = \mathsf{SL}^+$.

(ii) *If* $\mathsf{K}$ *is a variety, then it is a* canonical, complex variety $\mathsf{K} = \mathsf{SL}^+$.

(iii) *If* $\mathsf{L}$ *reflects ultrafilter extensions, then* $\mathsf{L} = \mathbf{Cm}^{-1}\mathsf{K}$.

PROOF. Straightforward. QED

PROOF OF LEMMA 5.2.6. Assume conditions (i)–(iii) of the lemma. Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in \mathsf{K}$ such that $\mathfrak{B} \overset{f}{\leftarrowtail} \mathfrak{A} \overset{h}{\rightarrowtail} \mathfrak{C}$. We have to show that there exists a $\mathfrak{D} \in \mathsf{K}$ which is a super–amalgam for $\mathfrak{B} \overset{f}{\leftarrowtail} \mathfrak{A} \overset{h}{\rightarrowtail} \mathfrak{C}$. Instead of amalgamating directly, we first embed these algebras in their canonical embedding algebras. Condition (iii) implies that $\mathfrak{A}_+, \mathfrak{B}_+, \mathfrak{C}_+ \in \mathsf{L}$. By 2.2.5: $\mathfrak{B}_+ \overset{f_+}{\twoheadrightarrow} \mathfrak{A}_+ \overset{h_+}{\twoheadleftarrow} \mathfrak{C}_+$. By 5.2.4, the set INSEP $\overset{\text{def}}{=} \{\langle x, y\rangle : f_+(x) = h_+(y)\}$ is a zigzag connection. Let $\mathcal{F}$ be the zigzag product $(\mathfrak{A}_+ \times \mathfrak{B}_+)\restriction$INSEP. By 5.2.4, the projections are zigzagmorphisms which commute with $f_+$ and $h_+$. To continue, we need claim 1 below. Here, we use $f^\#$ to abbreviate $(f_+)^+$, and $e_A$ to denote the canonical embedding function from $A$ to $\mathcal{P}(\mathfrak{Uf}\,\mathfrak{A})$ (which is $\mathrm{Dom}((\mathfrak{A}_+)^+)$).

CLAIM 1. *In figure* (5.2) *below, the following are equivalent:*

$$f_+ \circ \pi_0 = h_+ \circ \pi_1 \tag{5.6}$$

$$\pi_0^+ \circ f^\# = \pi_1^+ \circ h^\# \tag{5.7}$$

$$\pi_0^+ \circ e_B \circ f = \pi_1^+ \circ e_C \circ h \tag{5.8}$$

PROOF OF CLAIM. (5.6 $\Rightarrow$5.7): by 5.2.5(i).

(5.7 $\Rightarrow$ 5.8): by the fact that $e_B \circ f = f^\# \circ e_A$ and $e_C \circ h = h^\# \circ e_A$.

(5.8 $\Rightarrow$ 5.6): by writing out the definitions. ◀

[7] In general, a system of frames has many zigzag products; closure under zigzag products means that each of them is a member of the class.

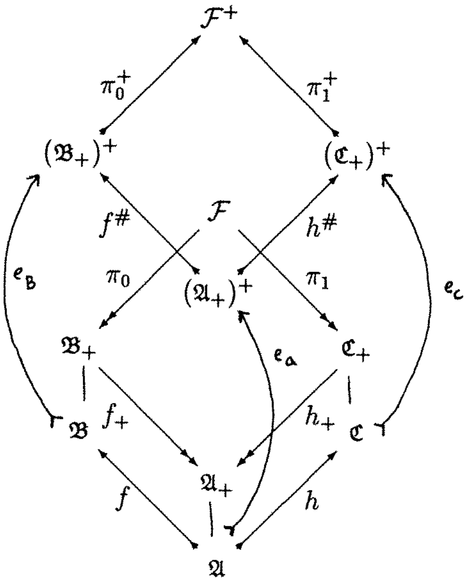



FIGURE 5.2: AMALGAMATING $\mathfrak{B} \overset{f}{\leftarrowtail} \mathfrak{A} \overset{h}{\rightarrowtail} \mathfrak{C}$

So, by claim 1, the algebras $\mathfrak{B}$ and $\mathfrak{C}$ can be embedded into $\mathcal{F}^+$ by $\pi_0^+ \circ e_B$ and $\pi_1^+ \circ e_C$, respectively, and the diagram commutes (i.e., $(\pi_0^+ \circ e_B) \circ f = (\pi_1^+ \circ e_C) \circ h$). So $\mathfrak{B} \overset{f}{\leftarrowtail} \mathfrak{A} \overset{h}{\rightarrowtail} \mathfrak{C}$ is amalgamated in $\mathcal{F}^+$, which is a member of $\mathsf{K}$ by conditions (i) and (ii).

To conclude, we have to show that the amalgamation is super. That is, condition (5.9) below holds (the second condition is analogous).

$$(\forall \mathsf{x} \in B)(\forall \mathsf{y} \in C) : \pi_0^+ e_B(\mathsf{x}) \leq \pi_1^+ e_C(\mathsf{y}) \Rightarrow (\exists \mathsf{z} \in A) : \mathsf{x} \leq f(\mathsf{z}) \,\&\, h(\mathsf{z}) \leq \mathsf{y} \qquad (5.9)$$

Assume the antecedent of (5.9) for arbitrary $\mathsf{x} \in B$ and $\mathsf{y} \in C$. First suppose there exists some $\mathsf{z} \in A$ with $\mathsf{x} = f(\mathsf{z})$. (The argument for $\mathsf{y} = h(\mathsf{z})$ is the same.) Then, $\pi_0^+ e_B(\mathsf{x}) = \pi_0^+ e_B f(\mathsf{z}) =$ (because the diagram commutes) $\pi_1^+ e_C h(\mathsf{z}) \leq$ (assumption) $\pi_1^+ e_C \mathsf{y}$. But then, since the functions are homomorphisms, $h(\mathsf{z}) \leq \mathsf{y}$. Now assume $\mathsf{x} \in B \setminus f^*(A)$ and $\mathsf{y} \in C \setminus h^*(A)$. Suppose to the contrary that there is *no* interpolant. We have to show that $\pi_0^+ e_B(\mathsf{x}) \not\leq \pi_1^+ e_C(\mathsf{y})$. Since the amalgam is a complex algebra of a frame which contains *all* inseparable pairs from $\mathfrak{B}_+$ and $\mathfrak{C}_+$, it suffices to show the existence of a pair of inseparable ultrafilters $\langle u, v\rangle$ from $(\mathfrak{Uf}\,\mathfrak{B} \times \mathfrak{Uf}\,\mathfrak{C})$ such that $\mathsf{x} \in u$ and $\mathsf{y} \notin v$. Create the set $\{\mathsf{z} \in A : \mathsf{x} \leq f(\mathsf{z})\} \cup \{-\mathsf{z} \in A : h(\mathsf{z}) \leq \mathsf{y}\}$. By the assumptions that $\mathsf{x} \notin f^*(A)$, $\mathsf{y} \notin h^*(A)$ and there is no interpolant, this set has the finite intersection property. So it can be extended to an ultrafilter $w \in \mathfrak{Uf}\,\mathfrak{A}$. Since $\mathsf{x}$ and $\mathsf{y}$ do not belong to the images of $\mathfrak{A}$, $f^*(w)$ can be extended to an ultrafilter $u \in \mathfrak{Uf}\,\mathfrak{B}$ containing $\mathsf{x}$, and $h^*(w)$ to an ultrafilter $v \in \mathfrak{Uf}\,\mathfrak{C}$ containing $-\mathsf{y}$. Because $f_+(u) = w = h_+(v)$, the ultrafilters $u$ and $v$ are inseparable. We are done. QED

REMARKS 5.2.8. Looking at the above proofs, the antecedent of the Zigzag Product Lemma might have done with the weaker condition of being closed under "INSEP products", instead of finite zigzag products. Here, by the *INSEP product* of $\mathcal{G} \overset{f}{\twoheadrightarrow} \mathcal{F} \overset{h}{\twoheadleftarrow} \mathcal{H}$ we mean the frame $(\mathcal{G} \times \mathcal{H}){\restriction}$INSEP, with INSEP as above. Call $\mathsf{K}$ *closed under INSEP products* if, for every $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathsf{K}$ such that $\mathcal{G} \overset{f}{\twoheadrightarrow} \mathcal{F} \overset{h}{\twoheadleftarrow} \mathcal{H}$, their INSEP product also belongs to $\mathsf{K}$. We have chosen to work with finite zigzag products for the following two reasons. First, we wanted a natural notion which might have other applications. Moreover, it allows us to use the FO model theory, developed about *subdirect products*. Second, closure under INSEP products is still stronger than SUPAP of the corresponding variety. In the appendix to this chapter, we show how to adjust INSEP products in order to get an equivalent formulation of (S)AP in terms of frames.

Using the implication SUPAP $\Rightarrow$ SCI, the Zigzag Product Lemma leads to a description of a large class of general modal logics with SCI. Van Benthem [Ben94a] shows how the binary zigzag product construction can be used for a direct, model–theoretic, proof of SCI.

Several persons observed the similarity between the INSEP product construction and the pull–back construction in category theory. We show that, in general, there is no connection between the two notions. Let $\mathsf{K}$ be the class of frames $\{\mathcal{F} = \langle W, R\rangle : R \subseteq W \times W\}$, and $Zig \overset{\text{def}}{=} \{f : (\exists \mathcal{F}, \mathcal{G} \in \mathsf{K})(\mathcal{F} \overset{f}{\twoheadrightarrow} \mathcal{G})\}$. Let $C$ be the category $\langle \mathsf{K}, Zig\rangle$. We claim that

(i) $\mathsf{K}$ is closed under INSEP products, but

(ii) $C$ is not closed under pull–backs.

(i) is obvious. For (ii), let $\mathcal{G} = \langle \{a, b\}, {}^2\{a, b\}\rangle$ and $\mathcal{F} = \langle \{a\}, {}^2\{a\}\rangle$. Clearly $\mathcal{F}$ is a zigzagmorphic image of $\mathcal{G}$. The INSEP product of $\mathcal{G} \twoheadrightarrow \mathcal{F} \twoheadleftarrow \mathcal{G}$ is the frame $\mathcal{G} \times \mathcal{G}$. There are several zigzag products of $\mathcal{G}$ with itself which also produce a commuting diagram (e.g., $\mathcal{G}$ itself and the disjoint union of $\mathcal{G}$ and a copy of $\mathcal{G}$). Now, suppose $\mathcal{H}$ is a pull–back for $\mathcal{G} \twoheadrightarrow \mathcal{F} \twoheadleftarrow \mathcal{G}$. Then, because a pull–back is minimal, $\mathcal{H}$ must be isomorphic to $\mathcal{G}$. But $\mathcal{G}$ is not a zigzagmorphic image of the disjoint union of $\mathcal{G}$ and a copy of $\mathcal{G}$. Hence $\mathcal{H}$ is not a pull–back.

## COMPARISON WITH AN EARLIER RESULT

Németi ([Ném85]) proved that the class $\mathsf{Crs}_\alpha$ (for any ordinal $\alpha$) has SAP, using lemma 5.2.9 below. The zigzag product lemma is an improvement of this result. We briefly compare the two lemmas. Recall that $\mathsf{Bo}_\alpha$ denotes the restriction of the class $\mathsf{BAO}$ to all algebras of the cylindric similarity type of dimension $\alpha$. If $\mathfrak{A} \in \mathsf{Bo}_\alpha$ and $\mathcal{F}$ is an $\alpha$–frame, then $\mathfrak{A} \subseteq^{cs} \mathcal{F}^+$ states that $\mathfrak{A}$ is *compact* and $\mathcal{F}$ is a *saturated representation* for $\mathfrak{A}$. For their precise definitions, we refer to Definition 2 in [Ném85].

LEMMA 5.2.9 (NÉMETI). *For any $\alpha$, let $\mathsf{K} \subseteq \mathsf{Bo}_\alpha$ and let $\mathsf{L}$ denote a class of $\alpha$–frames. Assume that* (i')–(iii') *hold.*

(i') $\mathsf{L}$ *is closed under taking substructures of finite products of members of* $\mathsf{L}$

(ii') $\mathcal{F} \in \mathsf{L} \Rightarrow \mathcal{F}^+ \in \mathsf{K}$

(iii') $\mathsf{K} \subseteq \mathbf{I}\{\mathfrak{A} \in \mathsf{K} : (\exists \mathcal{F} \in \mathsf{L})(\mathfrak{A} \subseteq^{cs} \mathcal{F}^+)\}$.

*Then* $\mathsf{K}$ *has SAP.*

Besides the fact that Németi's lemma is about strong amalgamation and the Zigzag Product Lemma about the stronger *super amalgamation*, there are three further differences. The first is that 5.2.6 holds for all BAO's, while 5.2.9 is only stated for the BAO's of the cylindric type. This is not a serious difference, as is pointed out in Sain [Sai90]. The second difference lies in condition (iii). In the proof of lemma 1.2 in [Sai90], it is shown how condition (iii') follows from the fact that K is a canonical variety. If K is a variety, then conditions (ii) and (iii) imply that it is canonical. So, if K is a variety, condition (iii') is weaker than (iii), hence in this respect 5.2.9 is more general. However, for many applications (including $\mathsf{Crs}_\alpha$), the easier condition (iii) already works.

The important difference between the two lemmas lies in condition (i). Ildikó Sain pointed out that (i') is too strong. There are several examples of canonical varieties satisfying (i), (ii) and (iii) but not (i') (e.g., the classes S4.1, $\mathbf{SRL}_{RS}\mathsf{RRA}$ and $\mathsf{D}_\alpha$, as we will see later). So they enjoy SAP (and even SUPAP) by the Zigzag Product Lemma, but not by 5.2.9[8]. In [Sai90] (Problem 1.4), Sain asks for a stronger version of 5.2.9 which is still natural but has wider applicability. Lemma 5.2.6, and its consequence 5.3.5 below, can be seen as an answer to this problem.

## 5.3 PRESERVATION

In general, validity is not preserved under taking (finite) zigzag products. The following (canonical) equations are examples of this phenomenon (cf. 5.4.12 and 5.6.6). We state them, together with their frame correspondents.

$$\begin{array}{ll} \mathsf{D}\,\mathsf{D}\,\mathsf{x} \leq \mathsf{x} \vee \mathsf{D}\,\mathsf{x} & \forall xyz((\mathsf{R}xy \,\&\, \mathsf{R}yz) \Rightarrow (x = z \vee \mathsf{R}xz)) \\ \mathsf{c}_0\,\mathsf{c}_1\,\mathsf{x} \leq \mathsf{c}_1\,\mathsf{c}_0\,\mathsf{x} & \forall xyz((T^0xy \,\&\, T^1yz) \Rightarrow \exists w(T^1xw \,\&\, T^0wz)) \\ (\mathsf{x};\mathsf{y});\mathsf{z} \leq \mathsf{x}(;\mathsf{y};\mathsf{z}) & \forall xyzuv((\mathsf{C}xyz \,\&\, \mathsf{C}yuv) \Rightarrow \exists w(\mathsf{C}xuw \,\&\, \mathsf{C}wvz)) \end{array}$$

In this section, we give a partial answer to the question which FO sentences are preserved under (finite) zigzag products. This leads to versions of the zigzag product lemma which are particularly easy to apply (theorems 5.3.5 and 5.3.6). We call a FO sentence a *(finite) zigzag product sentence* if it is preserved under (finite) zigzag products.

A FO sentence $\phi$ is a *special Horn sentence* iff $\phi$ is a conjunction of sentences of the form $(\forall\vec{x})(\psi \rightarrow \theta)$ with $\theta$ atomic, and $\psi$ a positive formula. Every *universal Horn* sentence is a special Horn sentence. The special Horn sentences are precisely the sentences preserved under *subdirect products* (cf. [CK90], exercise 6.2.10). Since zigzag products are subdirect products, every special Horn sentence is a zigzag product sentence. The set of finite zigzag product sentences turns out to be larger. To illustrate what sentences can be preserved (and how) we look at the modal variety S4.1. The

[8] This does not follow from 5.2.9, but in the case of $\mathbf{SRl}_{RS}\mathsf{RRA}$ and $\mathsf{D}_\alpha$, SUPAP follows from the *proof* of that lemma. We have to use the following easy fact. If a variety V has SUPAP, then any strengthening of V with equations which do not contain variables has SUPAP too. The class $\mathsf{D}_\alpha$ can be obtained by adding the equations ($\mathsf{C}_6$) to $\mathsf{Crs}_\alpha$. $\mathbf{SRl}_{RS}\mathsf{RRA}$ is obtained from the variety $\mathbf{SRl}\mathsf{RRA}$ by adding the equations $\mathsf{id}\,;1 = 1$ and $1^{\smile} = 1$.

example also shows that the distinction between finite and arbitrary zigzag products is important.

### 5.3.1 ILLUSTRATION: THE MODAL VARIETY S4.1

Define the variety S4.1 as the class of all BAO's of type $\langle\Diamond, 1\rangle$ which satisfy the equalities **T**, 4, and **M** below. Here, $\Box x \stackrel{\text{def}}{=} -\Diamond - x$ as usual.

$$\begin{array}{ll} \mathbf{T} & x \leq \Diamond x \\ 4 & \Diamond\Diamond x \leq \Diamond x \\ \mathbf{M} & \Box\Diamond x \leq \Diamond\Box x \end{array}$$

Define $\mathsf{K}_{S4.1}$ as the class of all frames $\mathcal{F} = \langle W, R\rangle$ in which $W$ is a set, and $R \subseteq W \times W$ is a transitive and reflexive relation which satisfies

$$\forall x \exists y (Rxy \;\&\; \forall z (Ryz \Rightarrow y = z)) \tag{5.10}$$

Condition (5.10) expresses that each point $x$ has an $R$–last point $y$ after it. Typical examples of $\mathsf{K}_{S4.1}$ frames are transitive reflexive trees of finite depth. A typical non–example is the frame $\langle \mathsf{N}, \leq\rangle$ of the natural numbers.

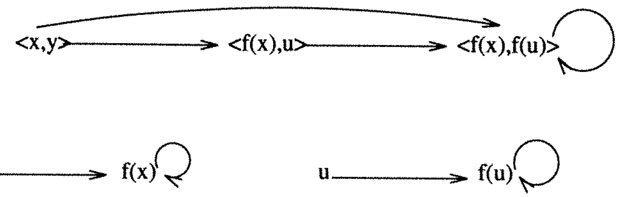
THEOREM 5.3.1 (LEMMON). $\mathsf{S4.1} = \mathbf{S}(\mathsf{K}_{S4.1})^+$ *is a canonical variety, and* $\mathbf{Cm}^{-1}\mathsf{S4.1} = \mathsf{K}_{S4.1}$.

PROOF. Cf. Bull–Segerberg [BS84] section 14. QED

The next proposition implies –by the Zigzag Product Lemma– that S4.1 has the super amalgamation property.

PROPOSITION 5.3.2. (i) $\mathsf{K}_{S4.1}$ *is closed under finite zigzag products.*
(ii) $\mathsf{K}_{S4.1}$ *is not closed under infinite zigzag products.*

PROOF. (i). We give the proof for a binary zigzag product. This is sufficient, since every *finite* zigzag product is isomorphic to a repetition of binary zigzag products. Let $\mathcal{G} = \langle W^G, R^G\rangle$ and $\mathcal{H} = \langle W^H, R^H\rangle$ be in $\mathsf{K}_{S4.1}$, and let $\mathcal{F} = \langle W^F, R^F\rangle$ be a zigzag product of $\mathcal{G}$ and $\mathcal{H}$. We have to show that $\mathcal{F} \in \mathsf{K}_{S4.1}$. Since reflexivity and transitivity are expressed by universal Horn sentences, $\mathcal{F}$ will satisfy them. We now show that it satisfies (5.10) as well. For convenience, we introduce a (Skolem) function $f$ which provides each point with an $R$–last point. Let $\langle x, y\rangle \in W^F$. We have to show that it is $R$–related to an endpoint. The argument is illustrated in figure 5.11 below. Since $R^G x f^G x$, there must be a pair $\langle f^G x, u\rangle \in F$ such that $R^F\langle x, y\rangle\langle f^G x, u\rangle$. Also, since $R^H u, f^H u$, we find $R^F\langle f^G x, u\rangle\langle v, f^H u\rangle$, for some $v$. Because $f^G x$ is an endpoint, $f^G x$ must equal $v$. Then, $R^F\langle x, y\rangle\langle f^G x, f^H u\rangle$ by transitivity. Because the projections are homomorphisms, $\langle f^G x, f^H u\rangle$ is an endpoint.




(5.11)

(ii). We show the existence of a frame which is a zigzag product of $\omega$ many frames in $\mathsf{K}_{S4.1}$, but which itself does *not* belong to $\mathsf{K}_{S4.1}$. We will use von Neumann notation for ordinals (e.g., $n = \{0, 1, \ldots, n-1\}$). We claim that the frame $\langle \omega, \subseteq \rangle \notin \mathsf{K}_{S4.1}$ is an infinite zigzag product of the system of frames $\langle\langle n, \subseteq\rangle\rangle_{0<n<\omega}$. All these finite frames belong to $\mathsf{K}_{S4.1}$. Define a system of morphisms $f_n : \omega \longrightarrow n$ via $\forall i < n : f_n(i) = i$ and $\forall i \geq n : f_n(i) = n-1$. Clearly, all these functions are zigzagmorphisms from $\langle \omega, \subseteq \rangle$ onto $\langle n, \subseteq \rangle$. To bring this into the shape of a zigzag product, define a function $g : \omega \longrightarrow {}^{\omega}\omega$ by $g(n) = \langle 0, 1, 2, \ldots, n-1, n, n, \ldots \rangle$. Define a binary relation $\preccurlyeq$ on $g^*(\omega)$ by $x \preccurlyeq y$ iff $(\forall i)(x_i \leq y_i)$. The frame $\langle g^*(\omega), \preccurlyeq \rangle$ is isomorphic to the frame $\langle \omega, \subseteq \rangle$, and it is an infinite zigzag product of the system of frames $\langle\langle n, \subseteq\rangle\rangle_{0<n<\omega}$. QED

## 5.3.2 A PARTIAL SOLUTION

We syntactically describe a part of the finite zigzag product sentences. In the next definition and lemma, we temporarily use $\mathsf{Mod}(\Sigma)$ in its FO model–theoretic sense: for a set $\Sigma$ of FO sentences of type $S$, $\mathsf{Mod}(\Sigma)$ denotes the class of all structures of type $S$ which validate $\Sigma$.

DEFINITION 5.3.3. Let $\Sigma$ be a FO theory in BAO type $S$. We call $\Sigma$ *clausifiable* if there exists a set of function symbols $\Psi$, and a FO theory $\Gamma$ in the language $S \cup \Psi$ such that:

(i) $\mathbf{Rd}_S \mathsf{Mod}(\Gamma) \subseteq \mathsf{Mod}(\Sigma)$,
(ii) $\Gamma$ consists of special Horn sentences (e.g., of universal Horn sentences),
(iii) For every $\mathcal{F}, \mathcal{G} \in \mathsf{Mod}(\Sigma)$, and every zigzag connection $B$ between $\mathcal{F}$ and $\mathcal{G}$, there exists expansions $\mathcal{F}^*, \mathcal{G}^* \in \mathsf{Mod}(\Gamma)$ such that $\mathbf{Rd}_S\mathcal{F}^* = \mathcal{F}$ and $\mathbf{Rd}_S\mathcal{G}^* = \mathcal{G}$, and for every $n$–place function $f \in \Psi$: $x_1By_1, \ldots, x_nBy_n \Rightarrow f(x_1, \ldots, x_n)Bf(y_1, \ldots, y_n)$.

Examples of obviously clausifiable theories are the sets of conditions defining the classes $\mathsf{K}^{rel}_{rlH}$ for $H \subseteq \{R, S\}$, and $\mathsf{K}^{cyl\alpha}_{rl}$ and $\mathsf{K}^{cyl\alpha}_{rlD}$ (see sections 2.4 and 2.5). Another example is the class $\mathsf{K}_{S4.1}$ from the previous subsection (the function to be added is a *Skolem* function).

LEMMA 5.3.4. *Let $\Sigma$ be a FO theory in BAO type $S$. If $\Sigma$ is* clausifiable, *then* $\mathsf{Mod}(\Sigma)$ *is closed under finite zigzag products.*

PROOF. Suppose $\Sigma$ is clausifiable. Let $\Gamma$ be as in 5.3.3. Let $\mathcal{G}$ and $\mathcal{H}$ be in $\mathsf{Mod}(\Sigma)$, and let $\mathcal{J}$ be a zigzag product (in type $S$) of $\mathcal{G}$ and $\mathcal{H}$. We have to show that $\mathcal{J} \in \mathsf{Mod}(\Sigma)$. (Again we only need to look at *binary* zigzag products.) It follows from 5.2.3

that $\mathrm{Dom}(\mathcal{J})$ is a zigzag connection in type $S$ between $\mathcal{G}$ and $\mathcal{H}$. Hence, by clause (iii) of 5.3.3, we can expand $\mathcal{G}$ and $\mathcal{H}$ to $\mathcal{G}^*, \mathcal{H}^* \in \mathsf{Mod}(\Gamma)$ satisfying the conditions of that clause. Define the new functions on $\mathcal{J}$ coordinate–wise (i.e., $f^{\mathcal{J}}(x_1, \ldots, x_n) = \langle f^{\mathcal{G}^*}(\pi_0 x_1, \ldots, \pi_0 x_n), f^{\mathcal{H}^*}(\pi_1 x_1, \ldots, \pi_1 x_n)\rangle$ ), and call this frame $\mathcal{J}^*$. Now $\mathcal{J}^*$ is closed under these functions, and $\mathcal{J}^*$ is a zigzag product of $\mathcal{G}^*$ and $\mathcal{H}^*$ in the expanded language. Since $\Gamma$ consists of special Horn sentences, $\mathsf{Mod}(\Gamma)$ is closed under zigzag products. Hence $\mathcal{J}^*$ is in $\mathsf{Mod}(\Gamma)$. But then, by clause (i), $\mathcal{J}$ is in $\mathsf{Mod}(\Sigma)$. QED

Using this lemma, we can provide a user–friendly version of the zigzag product lemma.

THEOREM 5.3.5. *Let* $\mathsf{V} \subseteq \mathsf{BAO}$ *be a* canonical *variety, defined by a set of equations which correspond to a FO theory* $\Sigma$. *If* $\Sigma$ *is clausifiable, then* $\mathsf{V}$ *has SUPAP.*

PROOF. By the Zigzag Product Lemma and 5.3.4. QED

For general modal logics, this theorem can be formulated as follows.

THEOREM 5.3.6. *Let* $\mathcal{GML}(\mathsf{K})$ *be a canonical general modal logic. If* $\mathsf{K}$ *can be defined by a clausifiable set of FO sentences, then* $\mathcal{GML}(\mathsf{K})$ *has both strong and weak interpolation, as well as Beth definability.*

PROOF. By the previous theorem and the results in table 5.1. QED

## 5.4 APPLICATIONS TO RELATION AND CYLINDRIC ALGEBRAS

In this section, we apply the general results obtained so far to the classes of relativized relation and cylindric algebras and their logical counterparts: arrow logic and cylindric modal logic/FO logic.

### 5.4.1 RELATION ALGEBRA AND ARROW LOGIC

THEOREM 5.4.1. (i) *Let* $H \subseteq \{R, S\}$. *The variety of relativized relation algebras* $\mathbf{SRl}_H\mathsf{RRA}$ *enjoys super amalgamation and interpolation of inequalities.*
(ii) *All reducts of relativized relation algebras considered in table 4.2 have SUPAP and IPI.*

PROOF. SUPAP follows from the axiomatizations (4.2.1 and 4.3.3) by 5.3.5. Then IPI follows, using 5.1.9.(ii). QED

COROLLARY 5.4.2. *Let* $P$ *be any of the properties ES, AP, SAP, SUPAP, or IPI. Let* $H \subseteq \{R, S, T\}$. *Then* $\mathbf{SRl}_H\mathsf{RRA}$ *has* $P$ *if and only if* $T \notin H$.

PROOF. By 2.4.6, the previous theorem and the implications in table 5.1. QED

STRONGER FORMS OF INTERPOLATION

The requirement on the interpolant is precisely the same as in Craig's theorem for FO logic. Lyndon strengthened Craig's theorem for FO logic by adding more requirements[9] (cf. [CK90] Thm 2.2.24). Johan van Benthem suggested another strengthening of the interpolation property (cf. [Ben94a]). We say that a class K has the *strengthened "and" ("or")* IPI if it has IPI with the additional requirement that the interpolant must be constructed from Booleans using only non-Boolean operators which occur in the antecedent *and* (*or*) the consequent. The strengthened forms of Craig interpolation are defined similarly.

THEOREM 5.4.3. *The variety* **SRlRRA** *has the strengthened "or" IPI, but it lacks the strengthened "and" IPI.*

PROOF. We first show that "and" IPI fails. **SRlRRA** $\models \mathsf{id} \leq (-\mathsf{x} \vee \mathsf{x}^{\smile})$. The antecedent and the consequent do not share any variables nor operators, so the only interpolants are 0 and 1. But neither qualifies. By 5.4.1.(ii), for every $Q \subseteq \{;, ^{\smile}, \mathsf{id}\}$, the $Q$-subreduct of **SRlRRA** has IPI. To show that **SRlRRA** has "or" IPI, we use that we can find the interpolant in the appropriate subreduct. This follows because, for every equation $e$ in the $Q$-subreduct, $\mathbf{SRlRd}_Q\mathsf{RRA} \models e \iff$ **SRlRRA** $\models e$ (cf., the proof of 4.3.4). QED

REMARK 5.4.4. Constants play a special rôle in interpolation, witness the easy failure of "and" IPI above, and also below in 5.4.9. A reasonable weakening of strengthened "and" IPI is to allow the interpolant to be constructed from Booleans, all constants in the similarity type, and non-constant non-Boolean operators which occur in the antecedent and in the consequent. It is not unlikely that this weakened form of "and" IPI holds for **SRlRRA**. The technique presented in van Benthem [Ben94a] to prove "and" IPI for the diagonal-free reduct of $\mathsf{Crs}_\alpha$ is applicable here as well. This technique may also be used to prove stronger forms of IPI for the other classes described in theorem 5.4.1.

ARROW LOGIC. For arrow logics, these results give rise to the following theorem.

THEOREM 5.4.5. (i) *Let* $H \subseteq \{R, S\}$. *The arrow logic of the class of pair-frames* $\mathsf{K}^{rel}_{setH}$ *has SCI, WCI, and BD.*
(ii) *The arrow logic of the class of all pair-frames* $\mathsf{K}^{rel}_{set}$ *has strengthened "or" SCI.*

PROOF. By the last two theorems, using the implications in table 5.1. QED

## 5.4.2 CYLINDRIC ALGEBRA AND FIRST ORDER LOGIC

THEOREM 5.4.6. *Let* $\alpha$ *be any ordinal. The varieties of cylindric relativized set algebras* $\mathsf{Crs}_\alpha$ *and* $\mathsf{D}_\alpha$ *have SUPAP and IPI.*

---

[9] "Every relation symbol (excluding identity) which occurs positively in the interpolant occurs positively in both the antecedent and the consequent, and similar for negative occurence".

PROOF. By the same argument as used in 5.4.1. Now use the axiomatizations provided in 2.5.7. QED

For $\mathsf{Crs}_\alpha$, we can do a bit better. We need to define some notions. Let $0 \leq \alpha \leq \omega$ and let $\tau$ be a term of the cylindric type $cyl\alpha$. By $ind(\tau)$, we denote the smallest ordinal $\gamma \leq \alpha$ such that all indices occurring in $\mathsf{d}_{ij}$'s and $\mathsf{c}_i$'s in $\tau$ are smaller than $\gamma$.

DEFINITION 5.4.7. Let $\mathsf{V}_\alpha \subseteq \mathsf{Bo}_\alpha$ be a variety and $0 \leq \alpha \leq \omega$. We say that $\mathsf{V}_\alpha$ has the *variable-restricted IPI* if for all $\tau, \tau_1$ such that $\mathsf{V}_\alpha \models \tau \leq \tau_1$, there exists an interpolant $\tau_2$ such that

1. $var(\tau_2) \subseteq var(\tau) \cap var(\tau_1)$
2. $ind(\tau_2) \leq max(ind(\tau), ind(\tau_1))$
3. $\mathsf{V}_\alpha \models \tau \leq \tau_2 \leq \tau_1$

We say that $\mathsf{V}_\alpha$ has the *very variable-restricted IPI* if condition 2 can be strengthened to $ind(\tau_2) \subseteq ind(\tau) \cap ind(\tau_1)$.

If we only consider the cylindrifications, "very variable-restricted IPI" is "strengthened "and" IPI". The (very) variable-restricted SCI is defined for cylindric modal logic in a similar way.

THEOREM 5.4.8. *For any $\alpha$ with $0 \leq \alpha \leq \omega$, $\mathsf{Crs}_\alpha$ has the variable-restricted IPI.*

PROOF. By the same argument as given above for **SRIRRA**. One needs the following result, which is part of lemma 10.10 in [Ném92]: Let $\gamma \leq \alpha$ be finite and let $\gamma = max(ind(\tau), ind(\tau_1))$. Then, $\mathsf{Crs}_\alpha \models \tau \leq \tau_1 \iff \mathsf{Crs}_\gamma \models \tau \leq \tau_1$. QED

The next theorem shows that, by dropping the diagonals, one can strengthen this result to very variable-restricted IPI (a result by van Benthem). $\mathsf{Crs}_\alpha$ itself does not have this property.

THEOREM 5.4.9. (i) (VAN BENTHEM) *Let $0 \leq \alpha \leq \omega$. The diagonal-free reduct of $\mathsf{Crs}_\alpha$ has the very variable-restricted IPI.*
(ii) *Let $3 \leq \alpha \leq \omega$. $\mathsf{Crs}_\alpha$ does not have the very variable-restricted IPI.*

PROOF. (i). Cf. [Ben94a] appendix 12.
(ii). Let $\tau = -\mathsf{c}_1 - \mathsf{d}_{01}$ and $\sigma = (-\mathsf{c}_1 - \mathsf{d}_{12} \vee - \mathsf{c}_1\, \mathsf{d}_{12})$. Then[10] $\mathsf{Crs}_3 \models \tau \leq \sigma$. Neither $\tau$ nor $\sigma$ uses variables, and 1 is the only common index. So for very variable-restricted interpolation, we would need a term $\gamma$ which uses only the index 1, and which does not use any variable. Then $\gamma$ must be built up from Boolean 0 and 1, using only the operator $\mathsf{c}_1$ and Booleans. The value of any such term is 0 or 1. But neither 0 nor 1 is an interpolant for the above $\tau$ and $\sigma$. QED

CYLINDRIC MODAL LOGIC. We show what these results mean for *cylindric modal logic* (and hence for FO logic).

THEOREM 5.4.10. *Let $0 \leq \alpha \leq \omega$.*
(i) *The cylindric modal logics of the classes $\mathsf{K}^{cyl\alpha}_{set}$ and $\mathsf{K}^{cyl\alpha}_{setD}$ of $\alpha$-dimensional assignment frames have SCI, WCI and BD.*

[10] In FO logic, $\tau \leq \sigma$ would be written as $(\forall v_1 v_0 = v_1) \to ((\exists v_1 v_1 = v_2) \to (\forall v_1 v_1 = v_2))$.

(ii) *The cylindric modal logic of the class* $\mathsf{K}_{set}^{cyl\alpha}$ *has the variable–restricted SCI; but if* $\alpha > 2$*, not the very variable–restricted SCI.*
(iii) (VAN BENTHEM) *The* $\mathsf{d}_{ij}$*–free reduct of the cylindric modal logic of* $\mathsf{K}_{set}^{cyl\alpha}$ *has the very variable–restricted SCI.*

PROOF. By theorems 5.4.6, 5.4.8 and 5.4.9. QED

In terms of FO logic, part (ii) of the above theorem implies that an interpolant in restricted[11] FO logic with the generalized $\mathsf{K}^\alpha$ semantics from section 1.2, does not need more variables than occur in contexts as $\exists v_i$ or $v_i = v_j$ in the antecedent or the consequent. One can contrast this result with the following fact about classical FO logic. Let $\mathcal{L}_n$ stand for restricted classical FO logic with no function symbols and only $n$ variables. Hajnal Andréka [AvBN93] proved that, for finite $n \geq 2$, $\mathcal{L}_n$ lacks SCI. But, since $\mathcal{L}_\omega$ has that property (for sentences), we can find an interpolant in some larger $\mathcal{L}_{n+k}$. D. Gabbay has asked whether there exists a bound on $k$ which would depend only on $n$. The theorem says there isn't.

THEOREM 5.4.11 (ANDRÉKA). *Let* $n \geq 2$*. Then* $\mathcal{L}_n$ *lacks SCI in the following strong sense. There are* $\mathcal{L}_2$ *sentences* $\phi, \psi$ *such that* $\models \phi \to \psi$*, but for no* $\mathcal{L}_n$ *formula* $\theta$ *in the common vocabulary of* $\phi$ *and* $\psi$*,* $\models \phi \to \theta$ *and* $\models \theta \to \psi$ *hold.*

### 5.4.3 EXPANSIONS WITH THE DIFFERENCE OPERATOR AND COUNTING MODALITIES

If we add the difference operator to the above logics, then amalgamation, interpolation and definability disappear. The problem is that the difference operator gives us a limited way of counting (recall that –assuming the Booleans– the difference operator is term–definably equivalent with the counting modalities $\{\Diamond^1, \Diamond^2\}$). Andréka–Németi–Sain [ANS94c] showed that, if we add *all* counting modalities to BA+ D, the positive properties reappear. They also showed that, with relativized relation and cylindric algebras, this strategy need not work.

The next theorem is a joint result with Ildikó Sain. Define the class $\mathsf{Crs}_\alpha$+ D as follows

$$\mathsf{Crs}_\alpha\text{+ D} \stackrel{\text{def}}{=} \mathbf{SP}\{\mathcal{F} = \langle V, \equiv_i^V, D_{ij}^V, \neq\rangle_{i,j<\alpha} : V \subseteq {}^\alpha U \text{ for some set } U\}^+$$

The expansions with D of $\mathsf{D}_\alpha, \mathsf{G}_\alpha$ and $\mathsf{RCA}_\alpha$ are defined in a similar way, following definition 2.5.3.

THEOREM 5.4.12. (i) BA+ D *lacks the embedding property (EP).*
(ii) RRA+ D *lacks EP.*
(iii) *Let* $H \subseteq \{R, S, T\}$*.* $\mathbf{SRl}_H$RRA+ D *lacks EP.*
(iv) *Let* $\alpha$ *be finite.* $\mathsf{RCA}_\alpha$+ D *lacks EP.*
(v) *Let* $\alpha$ *be arbitrary.* $\mathsf{Crs}_\alpha$+ D, $\mathsf{D}_\alpha$+ D *and* $\mathsf{G}_\alpha$+ D *lack EP.*

[11] Meaning that there are no function symbols, and that every atomic sentence is of the form $Rv_0, v_1, \ldots, v_{\alpha-1}$ (see section 2.5.5).

It follows from this theorem that the universal FO (Sahlqvist) sentence $\forall xyz((Rxy \& Ryz) \Rightarrow (x = z \vee Rxz))$ (i.e., pseudo-transitivity of D) is not a finite zigzag product sentence.

PROOF. (i). Take the following three algebras from BA+ D: $\mathfrak{A} = \langle\{\emptyset, \{a,b\}\}, \cup, -, \mathsf{D}\rangle$, $\mathfrak{B} = \langle\mathfrak{P}(\{a,b,c\}), \mathsf{D}\rangle$ and $\mathfrak{C} = \langle\mathfrak{P}(\{a,b\}), \mathsf{D}\rangle$ ($a, b, c$ are all different). Clearly, $\mathfrak{A}$ can be embedded into $\mathfrak{B}$ and $\mathfrak{C}$. Suppose that $\mathfrak{B}$ and $\mathfrak{C}$ can be embedded into an algebra $\mathfrak{D} \in$ BA+ D, by functions $m$ and $n$, respectively. Then $\mathfrak{D}$ should contain three atoms as images of the singletons of $\mathfrak{B}$. But then, one singleton, say $\{a\}$, of $\mathfrak{C}$ has to be mapped to a "non-singleton" in $\mathfrak{D}$. So $\mathfrak{C} \models \Diamond^2\{a\} = 0$ and $\mathfrak{D} \models \Diamond^2 n(\{a\}) = 1$, which leads to a contradiction.
(ii). RRA+ D is term-definably equivalent with RRA (cf. [Ven91] Prop 3.3.). But RRA does not have EP (cf. McKenzie [McK66]).
(iii). The same counterexample as in (i) works here as well. Expand the algebras $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ with the operators of the relational type. Let all singletons be below the identity (i.e., for instance $a = \langle a, a\rangle$, $b = \langle b, b\rangle$, $c = \langle c, c\rangle$). Then the three algebras are in $\mathbf{SRl}_{RST}$RRA+ D. $\mathfrak{A}$ can again be embedded into $\mathfrak{B}$ and $\mathfrak{C}$. But again, we can not embed $\mathfrak{B}$ and $\mathfrak{C}$ into one algebra.
(iv). For $\alpha \leq 1$, this follows from (i), since then $\mathsf{RCA}_\alpha$+ D is term-definably equivalent with BA+ D. For finite $\alpha$ larger than 1, $\mathsf{RCA}_\alpha$+ D is term-definably equivalent with $\mathsf{RCA}_\alpha$ (cf. [Ven91] Prop 4.2.15). But $\mathsf{RCA}_\alpha$ does not have EP for $2 \leq \alpha < \omega$ (cf. Comer [Com69]).
(v). Use the same counterexample as in (i), and change it as in (iii). QED

The above theorem has the following consequences in the realm of logical calculi.

THEOREM 5.4.13. *The following logics with the difference operator lack WCI and SCI.*

- *propositional logic with* D
- *the* D*-expansion of the arrow logic* $\mathcal{GML}(\mathsf{K}^{rel}_{setH})$, *for all* $H \subseteq \{R, S, T\}$ *or* $H = SQ$
- *the* D*-expansion of the cylindric modal logic* $\mathcal{GML}(\mathsf{K})$, *for* K *one of* $\mathsf{K}^{cyl\alpha}_{set}$, $\mathsf{K}^{cyl\alpha}_{setD}$, $\mathsf{K}^{cyl\alpha}_{setG}$ *for any* $\alpha$, *or* $\mathsf{K} = \mathsf{K}^{cyl\alpha}_{cube}$ *for finite* $\alpha$.

PROOF. The difference operator makes all classes which occur in theorem 5.4.12 discriminator varieties. Then the theorem follows from the previous one by the implications in table 5.1. QED

I. Sain [Sai94] settled the question of the definability property (BD) for these logics.

THEOREM 5.4.14 (SAIN). (i) *Propositional logic with* D *has BD*
(ii) *Let* $H \subseteq \{R, S, T\}$ *or* $H = SQ$. *The* D*-expansion of* $\mathcal{GML}(\mathsf{K}^{rel}_{setH})$ *lacks BD.*
(iii) *Let* $2 \leq \alpha \leq \omega$, *and* $\mathsf{K}^{cyl\alpha}_{cube} \subseteq \mathsf{K} \subseteq \mathsf{K}^{cyl\alpha}_{set}$. *The* D*-expansion* $\mathcal{GML}(\mathsf{K})$ *lacks BD.*

PROOF. The proof goes by showing that ES does not hold in the **SP** closure of the complex algebras of the frame classes. The proof of this last statement is an adaptation of Sain's counterexample given in the proof of [Sai90], Thm 2. QED

EXPANSIONS WITH COUNTING MODALITIES. Andréka–Németi–Sain showed that, if we add all counting modalities to propositional logic with D, interpolation and definability reappear ([ANS94c], Thm 15). Németi–Sain [Sai94] showed that these properties still fail when we add all counting modalities to the arrow logic of the class of pair-frames $\mathsf{K}^{rel}_{setH}$, for $H = SQ$ or $H \subseteq \{R, S, T\}$, or to the cylindric modal logic of the class of all assignment frames $\mathsf{K}^{rel}_{set}$, for $2 \leq \alpha \leq \omega$. This can also be shown by an adaptation of Sain's counterexample. One kills this proof, however, by adding all of the following operations. For $V \subseteq {}^{\alpha}U$, $i < \alpha$, $n$ some finite ordinal and $\tau \subseteq V$, define

$$\Diamond_i^n \tau \stackrel{\text{def}}{=} \{s \in V : \ |\{r \in \tau : s \equiv_i r\}| \geq n\}$$

It is an open problem, whether we retrieve BD in this logic by adding all *these* counting modalities[12].

## 5.5 CONCLUDING REMARKS

In section 2.3 we defined the logical core of a general modal logic $\mathcal{GML}(\mathsf{K})$ as the logic of the class which satisfies all universal FO conditions that are valid in K. What we have shown here is that the logical core in a stricter sense –that part which satisfies all universal *Horn* definable conditions– behaves as it should with respect to interpolation. We saw that universal non–Horn conditions, like pseudo–transitivity of the difference operator, can kill the interpolation property. This might lead us to conclude that the logical core of a logic should be its universal *Horn* part (allowing extra function symbols as in 5.3.3). In other words, instead of obtaining the core by adding all subframes (as in section 2.3), we add all finite zigzag products. Then we always obtain interpolation and Beth definability. We end with two questions.

1. Is there a syntactic characterization of the finite zigzag product sentences?
2. All arrow logics with interpolation had the finite model property (fmp). W. Rautenberg ([Rau83] Problem 2) asked whether there is a modal logic with strong interpolation, but without the fmp. L. Maksimova gave a positive answer to this question ([Mak91a] Thm 7). Is there also a general modal logic whose class of frames is closed under finite zigzag products, but which lacks the fmp?

## 5.6 APPENDIX: REFORMULATION OF (S)AP WITH APPLICATIONS

One can weaken the INSEP product construction from the proof of the zigzag product lemma to obtain an equivalent formulation of (S)AP using frames. We will use this

[12] Note that these counting modalities have a *local* character: they count often we can change a sequence at a coordinate $i$. In FO logic, they can be defined as follows (let $R$ be a binary predicate):

$$\Diamond_0^n Rv_0v_1 \stackrel{\text{def}}{\iff} \exists v_2, \ldots, v_{n+2}(\bigwedge\{v_i \neq v_j : 2 \leq i, j \leq n+2 \,\&\, i \neq j\} \wedge Rv_2v_1 \wedge \ldots \wedge Rv_{n+2}v_1)$$

fact for simple proofs that certain classes of algebras lack AP. A similar analysis is possible for the embedding property and super-amalgamation.

### 5.6.1 THE EQUIVALENCE LEMMA

LEMMA 5.6.1. *Let* $\mathsf{K} \subseteq \mathsf{BAO}$ *and* $\mathsf{L}$ *a class of frames, both of BAO type S. Assume that*

(i) $\mathsf{L}$ *is closed under zigzagmorphic images*
(ii) $\mathcal{F} \in \mathsf{L} \Rightarrow \mathcal{F}^+ \in \mathsf{K}$
(iii) $\mathfrak{A} \in \mathsf{K} \Rightarrow \mathfrak{A}_+ \in \mathsf{L}$

*Then* (1) *and* (2) *are equivalent:*

(1) $\mathsf{K}$ *has AP*
(2) $(\forall \mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in \mathsf{K})(\forall f, h)$ *if* $\mathfrak{B} \overset{f}{\leftarrowtail} \mathfrak{A} \overset{h}{\rightarrowtail} \mathfrak{C}$, *then there exists* $\mathcal{F} \in \mathsf{L}$ *satisfying:*[13]
(a) $\mathrm{Dom}(\mathcal{F}) \subseteq (\mathrm{Dom}(\mathfrak{B}_+) \times \mathrm{Dom}(\mathfrak{C}_+)) \restriction \mathrm{INSEP}$ *(this makes the diagram commute)*
(b) *the projections* $\pi_0$ *and* $\pi_1$ *are zigzagmorphisms from* $\mathcal{F}$ *onto* $\mathfrak{B}_+$ *and* $\mathfrak{C}_+$, *respectively.*

LEMMA 5.6.2. *Let* $\mathsf{K}$, $\mathsf{L}$ *and conditions* (i)–(iii) *be as in the previous lemma. Then* (1) *and* (2) *below are equivalent.*

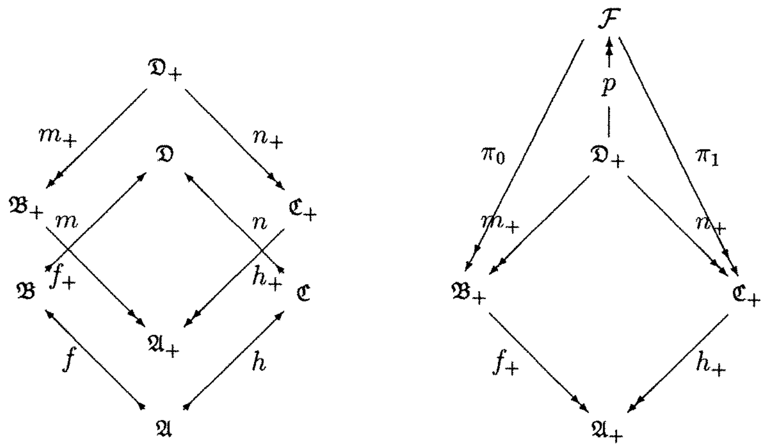
(1) $\mathsf{K}$ *has SAP*
(2) *Condition* (2) *from the previous lemma strengthened with* (c):
(c) $(\forall x \in (B \setminus f^*(A))(\forall y \in (C \setminus h^*(A))(\exists u \in \mathrm{Dom}(\mathcal{F})) : x \in \pi_0(u) \iff y \notin \pi_1(u)$

Before we prove these lemmas, we sketch how the frame $\mathcal{F}$ in condition (2) relates to the INSEP product construction which was defined in 5.2.8, and used in the proof of the zigzag product lemma. By definition, that construction satisfies conditions (a) and (b), and it is easy to show that it also satisfies condition (c). On the other hand, the frame $\mathcal{F}$ from (2) need not be an INSEP product of $\mathfrak{B}_+$ and $\mathfrak{C}_+$. Its domain is a subset of $(B_+ \times C_+)$ restricted to INSEP, and the projections are surjective, but it need not be the whole of INSEP. The second difference is that $\mathcal{F}$ validates only a subset of the relations that a subdirect product validates (because the projection functions are homomorphisms). It has to validate just enough, to make the projections zigzag as well. In 5.6.4 below, we give an example of BAO variety $\mathsf{V}$ which has SUPAP, but whose class of frames $\mathbf{Cm}^{-1}\mathsf{V}$ is neither closed under INSEP–, nor under zigzag–products.

PROOF OF LEMMA 5.6.1. Assume conditions (i)–(iii).
(2) $\Rightarrow$ (1). Assume (2). Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in \mathsf{K}$ such that $\mathfrak{B} \overset{f}{\leftarrowtail} \mathfrak{A} \overset{h}{\rightarrowtail} \mathfrak{C}$. By assumption, we find a frame $\mathcal{F} \in \mathsf{L}$ satisfying conditions (a) and (b). The restriction to INSEP implies that $f_+ \circ \pi_0 = h_+ \circ \pi_1$. So, by claim 1 in the proof of 5.2.6, the algebras $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ can be amalgamated in $\mathcal{F}^+$ by the embeddings $\pi_0^+ \circ e_B$ and $\pi_1^+ \circ e_C$. Since $\mathcal{F} \in \mathsf{L}$, by condition (ii), $\mathcal{F}^+ \in \mathsf{K}$. Thus $\mathsf{K}$ has AP.
(1) $\Rightarrow$ (2). Assume (1). Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in \mathsf{K}$ such that $\mathfrak{B} \overset{f}{\leftarrowtail} \mathfrak{A} \overset{h}{\rightarrowtail} \mathfrak{C}$. By assumption, $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ can be amalgamated in an algebra $\mathfrak{D} \in \mathsf{K}$, by embeddings $m$ and $n$. By writing out

[13] Again we let $\mathrm{INSEP} \overset{\text{def}}{=} \{\langle x, y\rangle \in \mathrm{Dom}(\mathfrak{B}_+) \times \mathrm{Dom}(\mathfrak{C}_+) : f_+(x) = h_+(y)\}$.

the definitions one sees that the upper diagram at the left in (5.12) below commutes. We will define a frame $\mathcal{F}$ satisfying (a) and (b) which is a zigzagmorphic image of $\mathfrak{D}_+$ (see the diagram at the right). This is enough since, by assumption, $\mathfrak{D} \in \mathsf{K}$, by (iii), $\mathfrak{D}_+ \in \mathsf{L}$, and then by (i), $\mathcal{F} \in \mathsf{L}$.




(5.12)

Define a function $p : \mathrm{Dom}(\mathfrak{D}_+) \longrightarrow \{\langle m_+(x), n_+(x)\rangle : x \in \mathrm{Dom}(\mathfrak{D}_+)\}$ by $p(x) \stackrel{\text{def}}{=} \langle m_+(x), n_+(x)\rangle$. Define a frame $\mathcal{F}$ of the type of $\mathfrak{D}_+$ by $\mathrm{Dom}(\mathcal{F}) \stackrel{\text{def}}{=} \{\langle m_+(x), n_+(x)\rangle : x \in \mathrm{Dom}(\mathfrak{D}_+)\}$ and $R^{\mathcal{F}} p(y) x_1 \ldots x_n \stackrel{\text{def}}{\Longleftrightarrow} (\exists x_1' \ldots x_n') : p(x_i') = x_i \;\&\; R^{\mathfrak{D}+} y x_1' \ldots x_n'$.

CLAIM. (i) $p$ is a zigzagmorphism from $\mathfrak{D}_+$ onto $\mathcal{F}$, and $m_+ = \pi_0 \circ p$, $n_+ = \pi_1 \circ p$; (ii) $\mathcal{F}$ satisfies conditions (a) and (b) from the lemma.

PROOF OF CLAIM. (i). Immediate by the definitions of $p$ and $\mathcal{F}$.
(ii). (a) Suppose $x \in \mathrm{Dom}(\mathcal{F})$. Then, $(\exists z \in \mathrm{Dom}(\mathfrak{D}_+)) : x = \langle m_+(z), n_+(z)\rangle$. Then $f_+ m_+(z) = h_+ n_+(z)$ implies that $x \in$ INSEP.
(b) Because $m_+$ is a zigzagmophism and $m_+ = \pi_0 \circ p$, $\pi_0$ is a zigzagmorphism from $\mathcal{F}$ onto $\mathfrak{B}_+$. The argument for $\pi_1$ is similar. QED

PROOF OF LEMMA 5.6.2. The proof is the same as the preceding one, except for the extra condition of strong amalgamation.
(2) ⇒ (1). For the strengthening to SAP, we must show that $(\forall \mathsf{x} \in (B \setminus f^*(A))(\forall \mathsf{y} \in (C \setminus h^*(A)) : \pi_0^+ e_B(\mathsf{x}) \neq \pi_1^+ e_C(\mathsf{y})$. But that is immediate by condition (c).
(1) ⇒ (2). The next claim suffices.

CLAIM. If $\mathfrak{D}$ is a strong amalgamation, then $\mathcal{F}$ satisfies condition (c).

PROOF OF CLAIM. If $\mathfrak{D}$ is a strong amalgamation, then $(\forall \mathsf{x} \in (B \setminus f^*(A))(\forall \mathsf{y} \in (C \setminus h^*(A)) : m(\mathsf{x}) \neq n(\mathsf{y})$. But then, there is an ultrafilter $u \in \mathfrak{Uf}\,\mathfrak{D}$ such that $m(\mathsf{x}) \in u \iff n(\mathsf{y}) \notin u$. Hence $\mathsf{x} \in m_+(u) \iff \mathsf{y} \notin n_+(u)$. Thus, there exists a $z = \langle m_+(u), n_+(u)\rangle \in \mathrm{Dom}(\mathcal{F})$ and $\mathsf{x} \in \pi_0(z) \iff \mathsf{y} \notin \pi_1(z)$. This is precisely condition (c). QED

To obtain positive results, lemma 5.6.1 is more difficult to apply than the zigzag product lemma. Especially, when proving SAP, condition (c) is awkward. Since this lemma states an equivalence however, we can apply it to obtain negative results as well. Especially with finite frames, the conditions in (2) are easy to handle, and they give rise to simple counterexamples. We will give several of these counterexamples in the next section. For a streamlined application of 5.6.1, we use the following result.

LEMMA 5.6.3. *Let* $\mathsf{K} \subseteq \mathsf{BAO}$ *and* $\mathsf{L}$ *a class of frames, both of BAO type* $S$. *Assume conditions* (i)–(iii) *of lemma 5.6.1. Then we have:*

1. *If there exist* finite *frames* $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathsf{L}$ *and zigzagmorphisms* $f, h$ *such that* $\mathcal{G} \overset{f}{\twoheadrightarrow} \mathcal{F} \overset{h}{\twoheadleftarrow} \mathcal{H}$, *and there exists no* $\mathcal{J} \in \mathsf{L}$ *satisfying* (a) *and* (b) *below, then* $\mathsf{K}$ does not have AP.
2. *If* (1) *can be strengthened by replacing the subset relation in* (a) *by an equality, then* $\mathsf{K}$ does not have SAP.
   (a) $J \subseteq (G \times H) \restriction \text{INSEP}$
   (b) *the projections are zigzagmorphisms.*

PROOF. Assume conditions (i)–(iii).
(1). Assume the antecedent of (1). By condition (ii), $\mathcal{F}^+, \mathcal{G}^+, \mathcal{H}^+ \in \mathsf{K}$ and $\mathcal{G}^+ \overset{f^+}{\leftarrowtail} \mathcal{F}^+ \overset{h^+}{\rightarrowtail} \mathcal{H}^+$. Since the frames (and hence their complex algebras) are finite, they are isomorphic to the canonical frames of their complex algebras (i.e., e.g., $\mathcal{F} \cong (\mathcal{F}^+)_+$). But then, by 5.6.1, $\mathsf{K}$ does not have AP.
(2) follows from the observation that (a), together with (c) of 5.6.2, ensures that with finite frames the domain of $\mathcal{J}$ equals INSEP. (Because on finite algebras, atoms correspond one to one with ultrafilters.) QED

SUPAP WITHOUT ZIGZAG PRODUCTS. We give an example of a canonical variety of BAO's which has SUPAP, although this cannot be shown by the zigzag product lemma. We define the following classes:

$$\mathsf{K}_{grad} \overset{\text{def}}{=} \{\mathcal{F} = \langle W, R^i \rangle_{0<i<\omega} : R^i \subseteq {}^{i+1}W \;\&\; (R^i y x_1 \ldots x_i \iff x_1 \ldots x_i \text{ distinct})\}$$
$$\mathsf{V}_{grad} \overset{\text{def}}{=} \mathbf{SP}(\mathsf{K}_{grad})^+$$

Note that each frame in $\mathsf{K}_{grad}$ is uniquely determined by its universe. In [ANS94c] Thm 15, we find the following facts about $\mathsf{V}_{grad}$: it is a canonical discriminator variety which has SUPAP, and it is term-definably equivalent to the variety $\mathsf{BA} + n$*-times* of Boolean algebras expanded with all counting modalities $\Diamond^n$. The general modal logic of the class $\mathsf{K}_{grad}$ has SCI, WCI, and BD. Because $\mathsf{V}_{grad}$ is a canonical variety, conditions (ii) and (iii) of the zigzag product lemma are fulfilled for $\mathsf{V}_{grad}$ and $\mathbf{Cm}^{-1}\mathsf{V}_{grad}$. We show that SUPAP cannot be proved using the zigzag product lemma, because condition (i) is not satisfied.

THEOREM 5.6.4. $\mathbf{Cm}^{-1}\mathsf{V}_{grad}$ *is neither closed under zigzag products nor under INSEP products.*

PROOF. We use the following fact, which is easy to check: $\mathsf{V}_{grad} \models (\Diamond^1 \mathsf{x} \wedge \Diamond^1 \mathsf{y}) \leq (\Diamond^2(\mathsf{x}, \mathsf{y}) \vee \Diamond^1(\mathsf{x} \wedge \mathsf{y}))$. This positive equation corresponds on frames of $\mathbf{Cm}^{-1}\mathsf{V}_{grad}$ to the condition $\forall xyz((R^1xy \,\&\, R^1xz) \Rightarrow (y = z \vee R^2xyz)$. So, every frame in $\mathbf{Cm}^{-1}\mathsf{V}_{grad}$ satisfies this condition. Take a $\mathcal{F} \in \mathsf{K}_{grad}$ with $F = \{a, b\}$ ($a$ and $b$ are different). It is not difficult to see that $\mathcal{F} \times \mathcal{F}$ is a zigzag product of $\mathcal{F}$ with itself (see figure 5.3). In $\mathcal{F} \times \mathcal{F}$, the binary relation $R^1$ is the universal relation, the relation $R^2$ is as in the picture, and all other relations are empty. The frame $\mathcal{F} \times \mathcal{F}$ can not belong to $\mathbf{Cm}^{-1}\mathsf{V}_{grad}$, because it does not validate the above condition. This shows that $\mathbf{Cm}^{-1}\mathsf{V}_{grad}$ is not closed under zigzag products. The same example shows that it is not closed under INSEP products either. For, the frame $\mathcal{G} = \langle\{w\}, R^i\rangle_{0<i<\omega}$ with $R^1xx, R^2xxx$, and, for all $i > 2$, $R^i = \emptyset$, is a zigzagmorphic image of $\mathcal{F}$, and $\mathcal{F} \times \mathcal{F}$ is an INSEP product of the frames $\mathcal{F}, \mathcal{G}$, and $\mathcal{F}$. QED

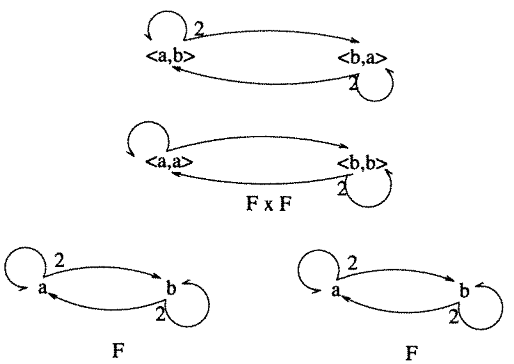



FIGURE 5.3: COUNTEREXAMPLE FOR THEOREM 5.6.4.

### 5.6.2 APPLICATIONS OF THE EQUIVALENCE LEMMA

We now show how to apply 5.6.1 in order to obtain negative results. We give five Sahlqvist equations which will typically kill the amalgamation property, all well-known from algebraic logic or modal logic. Consider the equations in table 5.2. (1) is usually referred to as *density*, (2) as the *Church–Rosser property*, (3) is the *not-branching axiom* from temporal logic, (4), *commutativity of the* $\mathsf{c}_i$*'s* comes from cylindric algebra theory, and corresponds to commutativity of FO quantifiers, and (5) is one half of the associativity axiom from relation algebras. Define five classes of algebras $\mathsf{K}_i \subset \mathsf{BAO}$, with $\mathsf{K}_1, \mathsf{K}_2$ and $\mathsf{K}_3$ of type $\{\langle \Diamond, 1\rangle\}$, $\mathsf{K}_4$ of type $\{\langle \mathsf{c}_0, 1\rangle, \langle \mathsf{c}_1, 1\rangle\}$, and $\mathsf{K}_5$ of type $\{\langle ;, 2\rangle\}$, and five classes of frames $\mathsf{L}_i$ of the corresponding type (here $i$ denotes either the equation or its FO correspondent from table 5.2):

$$\begin{aligned} \mathsf{K}_i &\stackrel{\text{def}}{=} \{\mathfrak{A} \in \mathsf{BAO} : \mathfrak{A} \models i\} \\ \mathsf{L}_i &\stackrel{\text{def}}{=} \{\mathcal{F} : \mathcal{F} \models i\} \end{aligned}$$

PROPOSITION 5.6.5. *For all* $i$ $(1 \leq i \leq 5)$, *conditions* (i)–(iii) *from lemma 5.6.1 hold. Hence these classes are canonical varieties* $\mathsf{K}_i = \mathbf{S}(\mathsf{L}_i)^+$.

| | Equation | FO correspondent |
|---|---|---|
| (1) | $\Diamond x \leq \Diamond\Diamond x$ | $\forall xy Rxy \Rightarrow \exists z Rxz \;\&\; Rzy$ |
| (2) | $\Diamond\Box x \leq \Box\Diamond x$ | $\forall xyz Rxy \;\&\; Rxz \Rightarrow \exists w Ryw \;\&\; Rzw$ |
| (3) | $\Diamond x \wedge \Diamond y \leq \Diamond(x \wedge \Diamond y) \vee \Diamond(y \wedge \Diamond x)$ | $\forall xyz(Rxy \;\&\; Rxz \Rightarrow (Ryz \vee Rzy))$ |
| (4) | $c_i c_j x \leq c_j c_i x$ | $\forall xyz T^i xy \;\&\; T^j yz \Rightarrow \exists w T^j xw \;\&\; T^i wz$ |
| (5) | $(x;y);z \leq x;(y;z)$ | $\forall xyzuv Cxyz \;\&\; Cyuv \Rightarrow \exists t Cxut \;\&\; Ctvz$ |

TABLE 5.2: EQUATIONS WHICH CAN KILL AP

PROOF. By the Sahlqvist form of the equations. QED

THEOREM 5.6.6. *None of the above* $K_i$ *has the amalgamation property.*

In the counterexamples to come, the whole argument is given in the pictures. We explain how they work. Zigzagmorphisms are given by dotted lines, accessibility relations in frames by fat arrows (an arrow leading from $x$ to $y$ means that $Rxy$). A possible amalgam is always given at the top of such a picture. We do not draw the projections, since they are coded in the names of the points in the amalgam.

PROOF. Each proof has the following ingredients. We construct three finite frames $\mathcal{F}, \mathcal{G}, \mathcal{H} \in L_i$, and zigzagmorphisms $f, h$ such that $\mathcal{G} \overset{f}{\twoheadrightarrow} \mathcal{F} \overset{h}{\twoheadleftarrow} \mathcal{H}$. Then we show that there cannot be a $\mathcal{J} \in L_i$, satisfying conditions (a) and (b) from lemma 5.6.3. By that lemma and 5.6.5, this is enough to prove the theorem. If $\mathcal{F}$ is a frame, then $F$ denotes its domain and its relations are given by $R^F$.

CLAIM 1. $K_1$ does not have AP.

PROOF OF CLAIM. Define $\mathcal{F}, \mathcal{G}, \mathcal{H}$ as in figure 5.4. Clearly, all of them belong to $L_1$. Define functions $f : G \longrightarrow F$ by $f(x) = x^{\#}$ and $f(b^*) = b^{\#}$, and $h : H \longrightarrow F$ by $h(x\prime) = x^{\#}$ and $f(b^*\prime) = b^{\#}$. It is easy to see that $f$ and $h$ are zigzagmorphisms. Since $\mathcal{G}$ and $\mathcal{H}$ are isomorphic, it is enough to check one of the two. The other counterexamples are set up in the same way. Now suppose there exists $\mathcal{J} \in L_1$ satisfying (a) and (b). Then $R^{\mathcal{J}}\langle a, a\prime\rangle, \langle b, b\prime\rangle$, so there should be a $z \in J$ with $R^{\mathcal{J}}\langle a, a\prime\rangle, z \;\&\; R^{\mathcal{J}} z, \langle b, b\prime\rangle$. But then, by condition (b), $z = \langle c, d\prime\rangle$: whence $z \notin$ INSEP, a contradiction with (a). ◀

CLAIM 2. $K_2$ does not have AP.

PROOF OF CLAIM. Define $\mathcal{F}, \mathcal{G}, \mathcal{H} \in L_2$ as in figure 5.5. Define $f : G \longrightarrow F$ by $f(x) = x^{\#}$ and $f(e^*) = e^{\#}$, and $h : H \longrightarrow F$ by $h(x\prime) = x^{\#}$ and $f(d^*\prime) = d^{\#}$. Then $f$ and $h$ are zigzagmorphisms. Again, suppose there exists $\mathcal{J} \in L_1$ satisfying (a) and (b). Then $R^{\mathcal{J}}\langle a, a\prime\rangle, \langle b, b\prime\rangle \;\&\; R^{\mathcal{J}}\langle a, a\prime\rangle, \langle c, c\prime\rangle$. So, there should be a $z \in J$ such that $R^{\mathcal{J}}\langle b, b\prime\rangle, z \;\&\; R^{\mathcal{J}}\langle c, c\prime\rangle, z$. But then, by (b): $z = \langle d, e'\rangle$, which contradicts (a). ◀

CLAIM 3. $K_3$ does not have AP.

PROOF OF CLAIM. This example is a bit different from the others. Here, it is not an existential quantifier, but a disjunction which causes trouble. Figure 5.6 speaks for itself. ◀

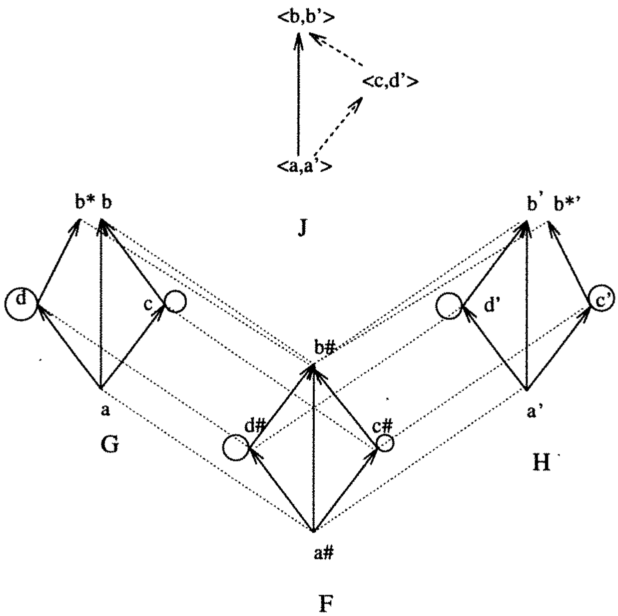



FIGURE 5.4: COUNTEREXAMPLE FOR $\mathsf{K}_1$

CLAIM 4. $\mathsf{K}_4$ does not have AP.

PROOF OF CLAIM. We only give the picture (figure 5.7). We use dotted lines for $T^0$ and black ones for $T^1$. The argument is the same as for $\mathsf{K}_1$. ◀

CLAIM 5. $\mathsf{K}_5$ does not have AP.

PROOF OF CLAIM. Drawings are more difficult to make with ternary relations. We will denote elements by arrows ⟨head, tail⟩, and define $\mathsf{C}xyz \iff x(0) = y(0)\,\&\,y(1) = z(0)\,\&\,x(1) = z(1)$, staying close to the intuition that $\mathsf{C}$ denotes composition. But we cannot equate an arrow with an ordered pair, whence $\mathsf{C}$ does not give relational composition, but "composition in multigraphs". We will exploit that $\mathsf{C}$ is not a function. For clarity, we also give the relations depicted in figure 5.8:

$$\mathsf{C}^{\mathcal{G}} \stackrel{\text{def}}{=} \{\langle a,b,c\rangle, \langle b,d,e\rangle, \langle a,d,x\rangle, \langle x,e,c\rangle, \langle a,d^*,x^*\rangle, \langle x^*,e^*,c^*\rangle\}$$
$$\mathsf{C}^{\mathcal{H}} \stackrel{\text{def}}{=} \{\langle a',b',c'\rangle, \langle b',d',e'\rangle, \langle a',d',x'\rangle, \langle x',e',c'\rangle, \langle a',d^{*\prime},x^{*\prime}\rangle, \langle x^{*\prime},e^{*\prime},c^{*\prime}\rangle\}$$
$$\mathsf{C}^{\mathcal{F}} \stackrel{\text{def}}{=} \{\langle a^\#,b^\#,c^\#\rangle, \langle b^\#,d^\#,e^\#\rangle, \langle a^\#,d^\#,x^\#\rangle, \langle a^\#,d^\#,x^{*\#}\rangle, \langle x^\#,e^\#,c^\#\rangle, \langle x^{*\#},e^\#,c^\#\rangle\}$$

Define $f$ and $h$ such that $(\forall y \in \{a,b,c,d,e\}) : f(y) = f(y^*) = y^\# = h(y') = h(y^{*\prime})$, $f(x) = x^\# = h(x^{*\prime})$, and $f(x^*) = x^{*\#} = h(x')$. Then $f$ and $h$ are zigzagmorphisms. Now, suppose there exists $\mathcal{J} \in \mathsf{L}_1$ satisfying (a) and (b). Then $\mathcal{J}$ is as in the figure, so there must be a $z$ as denoted by the dashed arrow. But then by (b), $z = \langle x, x'\rangle$, which contradicts (a). QED

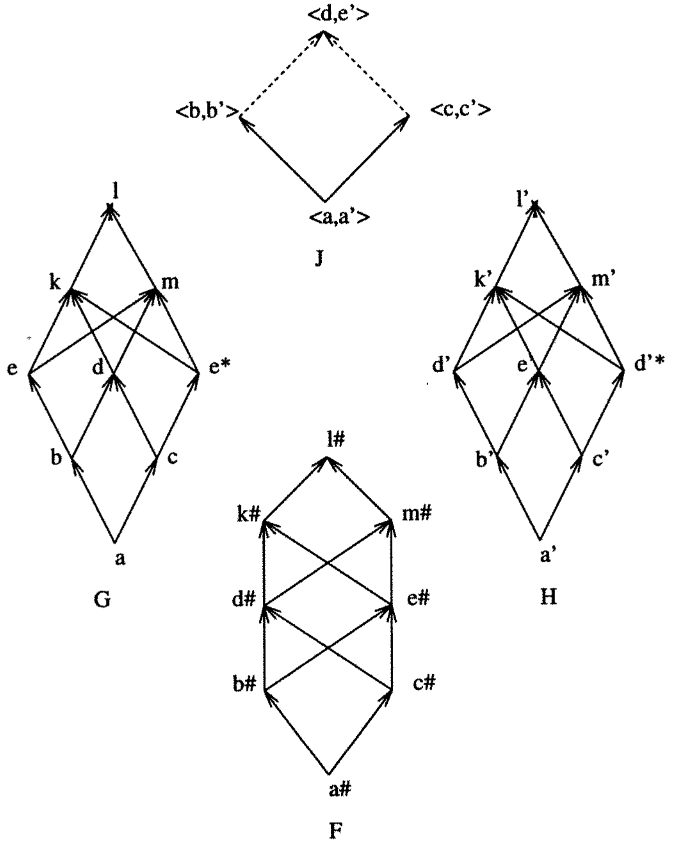



FIGURE 5.5: COUNTEREXAMPLE FOR $K_2$

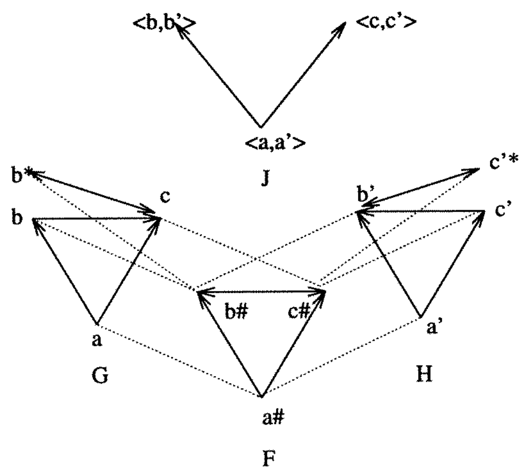



FIGURE 5.6: COUNTEREXAMPLE FOR $K_3$

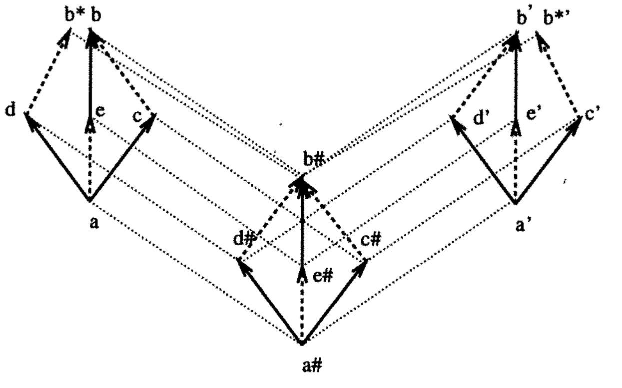



FIGURE 5.7: COUNTEREXAMPLE FOR $K_4$

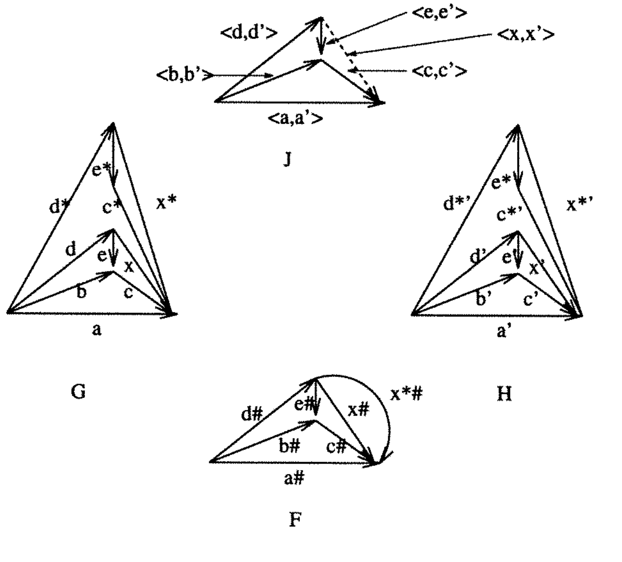



FIGURE 5.8: COUNTEREXAMPLE FOR $K_5$

# 6

# Applications to Arrow Logic

Having finished our algebraic–logical investigations, we return to arrow logic proper, and see what we have learnt. Besides transforming earlier results into "arrow language", we give several expansions of the latter (e.g., with Kleene star, slashes and cylindrifications). We end the chapter with a two–sorted version of arrow logic (proposed by Johan van Benthem [Ben93]).

This chapter can be read independently from the previous ones. We do not repeat earlier definitions, but they can easily be found using the index.

**Convention**

We will only use the *local consequence relation* defined in section 2.1.3, except for Beth's definability property, which is meant in the global sense. Hence we omit the superscript, and merely write $\models$, reserving $\models^{glo}$ for the exceptions.

## 6.1 Introduction

Arrow logic was introduced intuitively in section 1.1, and defined more precisely in section 2.4.5.

Models for arrow logic. There are at least three classes of models available for arrow logic:

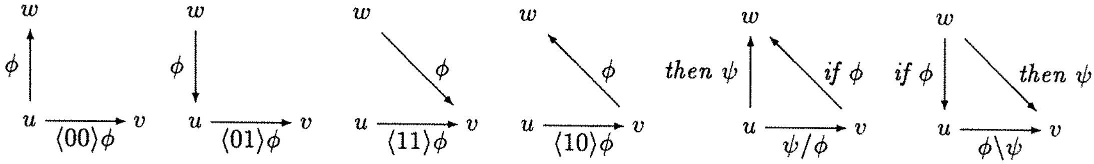
- (abstract) Kripke models (i.e., arrow–frames)
- directed graphs (i.e., pair–frames)
- directed multigraphs

We argued that intuitively, graphs and multigraphs are preferable to Kripke models, because of their more concrete "pictorial" character. Within directed (multi)graphs we can make further distinctions, depending on the availability of arrows. Thus, we find a landscape ranging from the class of all directed (multi)graphs to the class of "square" (multi)graphs (where the set of arrows is a full Cartesian product $U \times U$, for some set $U$ of "states"). In between the two ends, we investigate any combination of the following three requirements on universes of pair–frames: reflexivity, symmetry and transitivity (cf. figure 2.2). In what follows, we focus mainly on directed graphs. With these models, the requirement of transitive domains forms the borderline between positive and negative meta–logical properties (cf. section 6.2.1). In section 6.2.2, we summarize the known results about multigraphs.

LANGUAGES FOR ARROW LOGIC. The propositional language with extra connectives for composition, converse and identity is the core language of arrow logic (mainly for historical reasons, because this is the language of relation algebras). Clearly, these are not all connectives which are interesting when reasoning about arrows. In the literature, we can find the following extras: the $\langle ij\rangle$ modalities from Vakarelov's work on arrow logic (cf. [Vak92b] and also Venema [Ven89], [Ven91]), which correspond to the cylindric diamonds $\Diamond_0$, $\Diamond_1$ and to the two "domino" modalities; the slashes ("residuals"), which are the semi-duals of the conjugates of the composition modality (see section 4.5.2), and the Kleene star. For a pair-frame $\mathcal{F}$, a model $\mathrm{M} = \langle \mathcal{F}, \mathsf{v}\rangle$, and $x \in F$ we define:

$$\begin{array}{lcl}
\mathrm{M}, x \Vdash \langle ij\rangle\phi & \stackrel{\text{def}}{\Longleftrightarrow} & (\exists y) : x_i = y_j \;\&\; \mathrm{M}, y \Vdash \phi \;\; (i,j \in \{0,1\}) \\
\mathrm{M}, x \Vdash \phi\backslash\psi & \stackrel{\text{def}}{\Longleftrightarrow} & (\forall yz) : (x_0 = y_1, y_0 = z_0, z_1 = x_1 \;\&\; \mathrm{M}, y \Vdash \phi) \Rightarrow \mathrm{M}, z \Vdash \psi \\
\mathrm{M}, x \Vdash \psi/\phi & \stackrel{\text{def}}{\Longleftrightarrow} & (\forall yz) : (x_0 = y_0, x_1 = z_0, z_1 = y_1 \;\&\; \mathrm{M}, z \Vdash \phi) \Rightarrow \mathrm{M}, y \Vdash \psi \\
\mathrm{M}, x \Vdash \phi^* & \stackrel{\text{def}}{\Longleftrightarrow} & \mathrm{M}, x \Vdash \phi \textit{ or } x \textit{ can be finitely decomposed into arrows where } \phi \textit{ holds}
\end{array}$$

Here is the meaning of the new operators in pictures:




Other connectives which we might add –without a specific "arrow-behaviour"– are the universal modality (cf. sec 3.1), the difference operator, and counting or graded modalities (cf. sec 4.4)[1]. We repeat their definitions for convenience:

$$\begin{array}{lcl}
\mathrm{M}, x \Vdash \Diamond\phi & \stackrel{\text{def}}{\Longleftrightarrow} & (\exists y) : \mathrm{M}, y \Vdash \phi \\
\mathrm{M}, x \Vdash \mathsf{D}\,\phi & \stackrel{\text{def}}{\Longleftrightarrow} & (\exists y) : x \neq y \;\&\; \mathrm{M}, y \Vdash \phi \\
\mathrm{M}, x \Vdash \Diamond^n\phi & \stackrel{\text{def}}{\Longleftrightarrow} & |\{w \in F : \mathrm{M}, w \Vdash \phi\}| \geq n
\end{array}$$

The reason that these operators are less well-studied is that most of them become term-definable in the core language, when interpreted on the square pair-frames.

PROPOSITION 6.1.1. *In the arrow logic of square pair-frames $AL(\mathsf{K}^{rel}_{setSQ})$, the following connectives are definable:* $\langle ij\rangle$ $(i,j \in \{0,1\}), \backslash, /, \Diamond, \mathsf{D}, \Diamond^1, \Diamond^2, \Diamond^3$.

PROOF. The expressive power of this arrow logic equals that of first order logic with three variables and only binary predicate symbols (cf. [HMT85] Thm 5.3.16). All these connectives can be defined using only three variables. QED

These connectives are no longer definable over larger classes of models. So it is interesting to see what happens with them on non-square models. We look at several such expansions in section 6.3. For each one, we investigate the complete landscape that was drawn above.

[1] Sain [Sai88] argues that the difference operator is well-suited to study program verification. This might be a reason to add D to arrow logic, when applied in computer science.

TWO–SORTED ARROW LOGIC. Arrow logic is similar to Propositional Dynamic Logic (PDL; cf. Harel [Har84]), in the sense that both logics are designed to reason about transitions or programs. The difference is that in PDL we can also reason about the *input/output* behavior of programs. Program expressions are interpreted at *transitions*; but in addition, we have state formulas that can be interpreted at *states*. In arrow logic, we can only interpret at *transitions*. Johan van Benthem ([Ben93]) introduced a two–sorted version of arrow logic in which we can reason about states as well. In section 6.4, we study such a system with a pair–frame semantics. We compare this system with PDL, and with the related "Peirce Algebras" (cf. Brink et al. [BBS94]).

PRELIMINARIES

Recall the definition of arrow logic $AL(\mathsf{K})$ from section 2.4.5. We need to define a derivation system for arrow logic. The definition given below (cf., [Ven94]), is a straightforward generalization of the well–known **K** axiomatization for unary modal logic. For every general modal logic, such a derivation system exists. A formula $\phi$ is derivable from the K axioms if and only if the equation[2] $\phi^{\#} = 1$ is derivable from the BAO axioms in equational logic. If we add *axioms* to the **K** derivation system, and their corresponding equations to the BAO axioms, this correspondence remains.

DEFINITION 6.1.2. A *derivation system* for arrow logic is a pair $\langle A, R\rangle$ with $A$ a set of axioms and $R$ a set of derivation rules. A derivation system is called *normal* if $A$ contains the following axioms:

$(CT)$ all classical tautologies[3]

$(DB)$ $((p \vee p') \bullet q) \leftrightarrow (p \bullet q \vee p' \bullet q)$, $(p \bullet (q \vee q')) \leftrightarrow (p \bullet q \vee p \bullet q')$
$\otimes(p \vee q) \leftrightarrow \otimes p \vee \otimes q$

and $R$ contains the rules of *Modus Ponens, Universal Generalization* and *Substitution*:

$(MP)$ $\phi, \phi \to \psi \ / \ \psi$

$(UG)$ $\phi \ / \ \phi \underline{\bullet} \psi, \psi \underline{\bullet} \phi$
$\phi \ / \underline{\otimes} \phi$

$(SUB)$ $\phi \ / \ \sigma\phi$ for $\sigma$ a map uniformly substituting formulas for propositional variables in formulas.

A normal derivation system is called *orthodox* if $(MP), (UG)$ and $(SUB)$ are the only derivation rules of the system. For any set $\Sigma$ of formulas, $\Omega(\Sigma)$ denotes the orthodox derivation system having axioms $\Sigma, (CT)$ and $(DB)$. A formula is a *theorem* of the derivation system $\Delta = \langle A, R\rangle$, notation $\Delta \vdash \phi$, if $\phi$ is the last item $\phi_n$ of a sequence $\phi_0, \ldots, \phi_n$ of formulas such that each $\phi_i$ is either an axiom or the result of applying a rule to formulas $\{\phi_0, \ldots, \phi_{i-1}\}$. A formula $\phi$ is *derivable* from a set of formulas $\Gamma$, notation: $\Gamma \vdash_\Delta \phi$, if there are $\gamma_1, \ldots, \gamma_n$ in $\Gamma$ such that $\Delta \vdash (\gamma_1 \wedge \ldots \wedge \gamma_n) \to \phi$. A derivation system $\Delta$ is *sound* with respect to a class $\mathsf{K}$ of (arrow) frames, if every theorem of $\Delta$ is valid in $\mathsf{K}$, *complete* if every $\mathsf{K}$–valid formula is a theorem of $\Delta$.

[2]The function $\#$ denotes the trivial translation from logical formulas to algebraic terms. See section 2.1.3.

[3]Or any finite axiomatization of the Boolean (propositional) part.

Notation: $\Delta \vdash \phi \iff \mathsf{K} \models \phi$. It is *strongly* sound and strongly complete, if $\Gamma \vdash_\Delta \phi \iff \Gamma \models_\mathsf{K} \phi$ ($\models_\mathsf{K}$ is the *local* consequence).

In the next definition, we formulate the meta-logical properties we will study.

DEFINITION 6.1.3. Let $\mathcal{GML}(\mathsf{K}) = \langle \mathsf{Fml}(P), \mathsf{Mod}(\mathsf{K}), \Vdash \rangle$ be a general modal logic of some fixed type $S$.
(i) $\mathcal{GML}(\mathsf{K})$ is *decidable* if the set of validities $\{\phi \in \mathsf{Fml}(P) : \models_\mathsf{K} \phi\}$ is a decidable set.
(ii) $\mathcal{GML}(\mathsf{K})$ has the *finite frame property* (ffp) if every formula $\phi$ which is not $\mathcal{GML}(\mathsf{K})$ valid can be falsified on a finite frame from $\mathsf{K}$.
(iii) $\mathcal{GML}(\mathsf{K})$ *admits a (strongly) sound and complete finite axiomatization* by an orthodox derivation system, if there exists a finite set of formulas $\Sigma \subset \mathsf{Fml}(P)$ such that $\Omega(\Sigma)$ is a (strongly) sound and complete derivation system for $\mathcal{GML}(\mathsf{K})$.
(iv) A logic is *canonical* if the canonical frame of the Lindenbaum–Tarski algebra[4] of $\mathcal{GML}(\mathsf{K})$ is a member of $\mathsf{K}$.
(v) Strong and weak *Craig interpolation* and *Beth definability* are as defined in section 5.1.2.

## 6.2 THE CORE LANGUAGE

### 6.2.1 ARROW LOGIC OF DIRECTED GRAPHS

The preceding chapters already drew a landscape of arrow logics for pair–frames. We considered the class of square pair–frames $\mathsf{K}^{rel}_{setSQ}$, and the classes $\mathsf{K}^{rel}_{setH}$ in which the universe of the frames is an $H$ relation. Here $H \subseteq \{R, S, T\}$, and $R$ stands for "reflexive", $S$ for "symmetric" and $T$ for "transitive". $\mathsf{K}^{rel}_{setSQ}$ equals the closure under disjoint unions of the class $\mathsf{K}^{rel}_{setRST}$. Hence the logics $AL(\mathsf{K}^{rel}_{setSQ})$ and $AL(\mathsf{K}^{rel}_{setRST})$ are equivalent. For this reason, we do not mention the arrow logic $AL(\mathsf{K}^{rel}_{setSQ})$ explicitly. Thus, the three preceding chapters lead to the following general description of arrow logics.

THEOREM 6.2.1. *The arrow logic $AL(\mathsf{K}^{rel}_{setH})$ is decidable iff $T \notin H$.*

THEOREM 6.2.2. *The arrow logic $AL(\mathsf{K}^{rel}_{setH})$ admits a strongly sound and complete finite axiomatization by an orthodox derivation system iff $T \notin H$.*

Let $H \subseteq \{R, S\}$. Recall that $^{-\#}$ denotes the translation from algebraic terms into logical formulas (e.g., $(\mathsf{x} \wedge \mathsf{id} \leq \mathsf{x}^\smile)^{-\#} = (p \wedge \mathsf{id}) \to \otimes p$). Recall the algebraic equations $(A_1) - (A_{15})$ from 2.4.9. Define $\Sigma_\emptyset \stackrel{\text{def}}{=} \{(A_1)^{-\#} - (A_{12})^{-\#}\}$, $\Sigma_S \stackrel{\text{def}}{=} \Sigma_\emptyset \cup \{(A_{13})^{-\#}\}$, $\Sigma_R \stackrel{\text{def}}{=} \Sigma_\emptyset \cup \{(A_{14})^{-\#}, (A_{15})^{-\#}\}$ and $\Sigma_{RS} \stackrel{\text{def}}{=} \Sigma_R \cup \Sigma_S$.

THEOREM 6.2.3. *Let $H \subseteq \{R, S\}$. The derivation system $\Omega(\Sigma_H)$ is strongly sound and complete with respect to the arrow logic $AL(\mathsf{K}^{rel}_{setH})$.*

[4] The Lindenbaum–Tarski algebra of a logic $\mathcal{GML}(\mathsf{K})$ is the term algebra $\mathfrak{A}$ generated by the set of propositional variables, factored out by $\mathcal{GML}(\mathsf{K})$ equivalence. Its canonical frame is $\mathfrak{A}_+$. The *canonical model* of a logic is the model $\langle \mathfrak{A}_+, \mathsf{v} \rangle$, where $\mathsf{v}$ is defined by $\mathsf{v}(p) = \{w \in \mathrm{Dom}(\mathfrak{A}_+) : p \in w\}$.

THEOREM 6.2.4. *Let $P$ be any of strong interpolation, weak interpolation or Beth definability. The arrow logic $AL(\mathsf{K}^{rel}_{setH})$ has $P$ iff $T \notin H$.*

We prove all theorems at once, using our earlier results about the varieties $\mathbf{S}(\mathsf{K}^{rel}_{setH})^+$.

PROOF. Recall that, for any general modal logic $\mathcal{GML}(\mathsf{K})$ , and for any formula $\phi$ in the language of that logic:

$$\models_{\mathsf{K}} \phi \iff \mathsf{K}^+ \models \phi^\# = 1 \iff \mathbf{HSPK}^+ \models \phi^\# = 1$$

(Thm 6.2.1) By 4.2.6.
(Thm 6.2.2) This follows from 4.2.2, which stated that the canonical varieties $\mathbf{S}(\mathsf{K}^{rel}_{setH})^+$ are finitely axiomatizable iff $T \notin H$.
(Thm 6.2.3) By 4.2.1. We also give a direct proof. Let $\Delta = \Omega(\Sigma_H)$ be the derivation system for $AL(\mathsf{K}^{rel}_{setH})$. Assume $\Gamma \not\vdash_\Delta \phi$. Let $\mathrm{M} = \langle \mathcal{F}, \mathsf{v} \rangle$ be the canonical model of the derivation system[5]. Then $[\![\Gamma]\!]_{\mathrm{M}} \not\leq [\![\phi]\!]_{\mathrm{M}}$. It follows from 2.4.9 that $\mathcal{F} \in \mathsf{K}^{rel}_{rlH}$, and from 4.2.3, that $\mathcal{F}$ is a zigzagmorphic image of some frame $\mathcal{G} \in \mathsf{K}^{rel}_{setH}$. But then, there is a model $\mathrm{M}' = \langle \mathcal{G}, \mathsf{v}' \rangle$ in which $[\![\Gamma]\!]_{\mathrm{M}'} \not\leq [\![\phi]\!]_{\mathrm{M}'}$, whence $\Gamma \models \phi$ is not valid in $AL(\mathsf{K}^{rel}_{setH})$.
(Thm 6.2.4) The positive side is 5.4.5. The negative side follows from 5.4.2, using the implications in table 5.1. QED

PROBLEM 6.2.5. It is unknown whether the decidable arrow logics $AL(\mathsf{K}^{rel}_{setH})$ for $H \subseteq \{R, S\}$ also have the *finite frame property*. By 3.2.3, we do know that the logics of the strictly larger classes of abstract arrow–frames $AL(\mathsf{K}^{rel}_{rlH})$, which are logically equivalent to $AL(\mathsf{K}^{rel}_{setH})$, have the finite frame property.

## THE BORDER OF "COMPUTATIONAL NICE BEHAVIOUR"

We argued that there are at least three important aspects to the notion of the computational core of a logic: decidability, finite axiomatizability and interpolation. The above theorems show that, for the arrow logics of pair–frames, the border of "computational nice behaviour" is transitivity of the universe.

In the next section, we consider expansions of the core language. The next two theorems show that, for these expansions, transitivity also leads to negative results. Let $Q$ denote any subset of $\{\langle ij \rangle (i, j \in \{0, 1\}), \backslash, /, \Diamond, \mathsf{D}, \Diamond^n (0 < n < \omega)\}$. $AL(\mathsf{K}^{rel}_{setH})+Q$ denotes the expansion of the arrow logic $AL(\mathsf{K}^{rel}_{setH})$ with connectives in $Q$.

THEOREM 6.2.6. *Let $\{T\} \subseteq H \subseteq \{R, S, T\}$. $AL(\mathsf{K}^{rel}_{setH})+Q$ is undecidable.*

PROOF. By 6.2.1 and the fact that all the expansions are conservative. QED

THEOREM 6.2.7 (ANDRÉKA). *Let $\{T\} \subseteq H \subseteq \{R, S, T\}$. $AL(\mathsf{K}^{rel}_{setH})+Q$ does not admit a finite sound and complete axiomatization by an orthodox derivation system.*

PROOF. Cf. [And91a]. QED

[5] So $\mathcal{F}$ is the ultrafilter frame of the term–algebra, factored out by the congruence $\{\langle \phi, \psi \rangle : \Omega(\Sigma_H) \vdash \phi \leftrightarrow \psi\}$, and the valuation of the propositional variables is given by $\mathsf{v}(p) \stackrel{\text{def}}{=} \{u \in F : p \in u\}$.

Such general results are not available for interpolation and definability. We inspected the proofs for failure of these properties, and they all seem to go through with the new connectives added.

CONJECTURE 6.2.8. Let $\{T\} \subseteq H \subseteq \{R, S, T\}$. Let $P$ be any of SCI, WCI or BD. $AL(\mathsf{K}^{rel}_{setH})+Q$ does not have $P$.

### 6.2.2 ARROW LOGIC OF DIRECTED MULTIGRAPHS

We now turn to directed multigraphs. Arrow logic for directed multigraphs was first studied by D. Vakarelov ([Vak92b]), using a propositional language with the four $\langle ij \rangle$ modalities. A. Arsov added the operators from relation algebras to this type (see Arsov et al. [AM94b] and below). A. Kuhler ([Kuh94]) provided a finite axiomatization of arrow logic in the relational type of directed multigraphs and of locally square (i.e., reflexive and symmetric) directed multigraphs. She used the same combination of mosaics and step–by–step construction as ours in section 4.2. Thus, this combination of ideas can be fruitfully applied in different situations.

With multigraphs, there are several reasonable ways of defining composition and converse. (For identity, there is obviously just one definition.) Kuhler and Arsov used the following definition. A directed multigraph is a tuple $\langle Ar, Po, 0, 1\rangle$ with $Ar$ a set of arrows ("edges"), $Po$ a set of points ("nodes"), and 0 and 1 two surjective functions from $Ar$ to $Po$, providing each arrow $a$ with its head "$a(0)$" and its tail "$a(1)$", respectively. (Vakarelov calls these tuples *arrow structures*.) We call a directed multigraph *locally square* if $(\forall x \in Ar)(\exists yzw \in Ar) : x(0) = y(1), x(1) = y(0), x(0) = z(0) = z(1) \& x(1) = w(0) = w(1)$. Arsov and Kuhler define the following arrow–frames from directed multigraphs. Let $G = \langle Ar, Po, 0, 1\rangle$ be a directed multigraph. Then the frame $\mathcal{F}_G = \langle Ar, \mathsf{C}_G, \mathsf{F}_G, \mathsf{I}_G\rangle$ is a *multigraph arrow–frame* if

$$\begin{array}{lcl} \mathsf{I}_G x & \stackrel{\text{def}}{\Longleftrightarrow} & x(0) = x(1) \\ \mathsf{F}_G xy & \stackrel{\text{def}}{\Longleftrightarrow} & x(0) = y(1) \;\&\; x(1) = y(0) \\ \mathsf{C}_G xyz & \stackrel{\text{def}}{\Longleftrightarrow} & x(0) = y(0), y(1) = z(0) \;\&\; z(1) = x(1) \end{array}$$

Define,

$$\begin{array}{lcl} AK & \stackrel{\text{def}}{=} & \{\mathcal{F}_G : G \text{ is a directed multigraph}\} \\ AK_{RS} & \stackrel{\text{def}}{=} & \{\mathcal{F}_G : G \text{ is a locally square directed multigraph}\} \end{array}$$

THEOREM 6.2.9 (KUHLER). *$AK$ and $AK_{RS}$ admit strongly complete finite axiomatizations by orthodox derivation systems.*

PROOF. Cf. [Kuh94]. QED

THEOREM 6.2.10. *$AK$ and $AK_{RS}$ enjoy Craig interpolation and Beth definability.*

PROOF. This follows from 5.3.6. The given axioms are Sahlqvist, and correspond to (clausifiable) Horn theories. QED

The above-mentioned result by Arsov is the following. The $\langle ij\rangle$ modalities are defined on multigraphs, using the functions 0 and 1, as one would expect. For instance, if M is a model over a multigraph frame, $\mathrm{M}, x \Vdash \langle 00\rangle\phi \overset{\text{def}}{\Longleftrightarrow} (\exists y) : x(0) = y(0) \,\&\, \mathrm{M}, y \Vdash \phi$.

THEOREM 6.2.11 (ARSOV). *The expansions of $AK$ and $AK_{RS}$ with the $\langle ij\rangle$ modalities admit strongly complete finite axiomatizations by orthodox derivation systems.*

PROOF. Cf. [AM94b]. QED

THEOREM 6.2.12. *The expansions of $AK$ and $AK_{RS}$ with the $\langle ij\rangle$ modalities enjoy interpolation and definability.*

PROOF. All axioms are in Sahlqvist form, and correspond to universal Horn sentences. Again, use 5.3.6. QED

PROBLEM 6.2.13. It is unknown whether these arrow logics of directed multigraphs are *decidable.*

## 6.3 EXPANSIONS OF THE CORE LANGUAGE

In this section, we expand the core language with the modalities discussed in the introduction:

- $\langle ij\rangle$ modalities (sec 6.3.1)
- slashes (without converse) (sec 6.3.2)
- universal modality, difference operator and counting modalities (sec 6.3.3)
- Kleene star (plus the universal modality) (sec 6.3.4)

We denote expansions of an arrow logic $AL(\mathsf{K})$ with a set of operators $Q$ as $AL(\mathsf{K}){+}Q$.

### 6.3.1 CYLINDRIC DIAMONDS AND DOMINOS

The four $\langle ij\rangle$ modalities were studied by Y. Venema ([Ven89], [Ven91]) in two-dimensional modal logic (i.e., directed graphs), and by D. Vakarelov and A. Arsov in the context of multigraphs ([AM94b], [Vak92b]).

PROPOSITION 6.3.1. *The four $\langle ij\rangle$ modalities are term-definable in the arrow logic $AL(\mathsf{K}^{rel}_{setH})$ if $\{R, S\} \subseteq H$:*

$$\begin{array}{lllllll}\langle 00\rangle\phi & \leftrightarrow & (\mathsf{id} \wedge (\phi \bullet \top)) \bullet \top & \qquad & \langle 11\rangle\phi & \leftrightarrow & \top \bullet (\mathsf{id} \wedge (\top \bullet \phi)) \\ \langle 01\rangle\phi & \leftrightarrow & (\mathsf{id} \wedge (\top \bullet \phi)) \bullet \top & & \langle 10\rangle\phi & \leftrightarrow & \top \bullet (\mathsf{id} \wedge (\phi \bullet \top))\end{array}$$

PROOF. We show one example:

$$\begin{array}{lll} & \langle u, v\rangle \Vdash \langle 00\rangle\phi & \overset{\text{def}}{\Longleftrightarrow} \\ (\exists v') : & \langle u, v'\rangle \Vdash \phi & \Longleftrightarrow \text{(by reflexivity and symmetry)} \\ & \langle u, u\rangle \Vdash \mathsf{id} \wedge (\phi \bullet \top) & \overset{\text{def}}{\Longleftrightarrow} \\ & \langle u, v\rangle \Vdash (\mathsf{id} \wedge (\phi \bullet \top)) \bullet \top & \end{array}$$

QED

Let us see what happens if we add them to the arrow logic of non locally–square pair–frames. Let $AL(\mathsf{K}^{rel}_{setH})+\langle ij\rangle$ be the arrow logic of $H$ pair–frames, expanded with the four $\langle ij\rangle$ modalities.

AXIOMS FOR $AL(\mathsf{K}^{rel}_{setH})+\langle ij\rangle$. We provide complete axioms for $AL(\mathsf{K}^{rel}_{setH})+\langle ij\rangle$, together with their frame correspondents[6]. Note that $(V_1)-(V_8)$ are valid in multigraphs as well. Let $H \subseteq \{R, S\}$, and let $\Delta^{\langle ij\rangle}_H$ be the following derivation system:

(i) The $AL(\mathsf{K}^{rel}_{setH})$ derivation system

(ii) Distribution axioms and $(UG)$ rules for the four new modalities.

(iiii) A complete axiomatization for the $\langle ij\rangle$ reduct (cf. [Vak92b]). For $i, j, k \in \{0, 1\}$:

| | | |
|---|---|---|
| $(V_1)$ | $p \to \langle ii\rangle p$ | $R^{ii}xx$ |
| $(V_2)$ | $(p \wedge \langle ij\rangle q) \to \langle ij\rangle(q \wedge \langle ji\rangle p)$ | $R^{ij}xy \Rightarrow R^{ji}yx$ |
| $(V_3)$ | $\langle ij\rangle\langle jk\rangle p \to \langle ik\rangle p$ | $R^{ij}xy \,\&\, R^{jk}yz \Rightarrow R^{ik}xz$ |

(iv) Axioms[7] governing the interplay between the old and the new connectives:

| | | | |
|---|---|---|---|
| $(V_4)$ | $\mathsf{id} \wedge p \to \langle ij\rangle p$ | $\mathsf{I}x \Rightarrow R^{ij}xx$ | |
| $(V_5)$ | $\otimes p \to \langle 01\rangle p \wedge \langle 10\rangle p$ | $\mathsf{F}xy \Rightarrow R^{01}xy \,\&\, R^{10}xy$ | |
| $(V_6)$ | $p \bullet q \to \langle 00\rangle(p \wedge \langle 10\rangle q) \wedge \langle 11\rangle q$ | $\mathsf{C}xyz \Rightarrow R^{00}xy \,\&\, R^{10}yz \,\&\, R^{11}xz$ | |
| $(V_7)$ | $\langle 0i\rangle(\mathsf{id} \wedge p) \to (\mathsf{id} \wedge p) \bullet \top$ | $R^{0i}xy \,\&\, \mathsf{I}y \Rightarrow \mathsf{C}xyx$ | for $i \in \{0, 1\}$ |
| $(V_8)$ | $\langle 1i\rangle(\mathsf{id} \wedge p) \to \top \bullet (\mathsf{id} \wedge p)$ | $R^{1i}xy \,\&\, \mathsf{I}y \Rightarrow \mathsf{C}xxy$ | for $i \in \{0, 1\}$ |

THEOREM 6.3.2. *Let $H \subseteq \{R, S\}$. The derivation system $\Delta^{\langle ij\rangle}_H$ is strongly sound and complete with respect to $AL(\mathsf{K}^{rel}_{setH})+\langle ij\rangle$.*

PROOF. For $AL(\mathsf{K}^{rel}_{setRS}) + \langle ij\rangle$, the theorem follows from the results for the core language plus 6.3.1. We continue with the other cases. Strong completeness follows, because every expanded arrow–frame $\mathcal{F} = \langle W, \mathsf{C}, \mathsf{F}, \mathsf{I}, R^{ij}\rangle_{i,j\leq 1}$ which satisfies $(V_1)$ – $(V_8)$, is a zigzagmorphic image of a pair–frame in which the accessibility relations for the $\langle ij\rangle$'s are defined as one would expect. (For example, $R^{00}_V xy \overset{\text{def}}{\Longleftrightarrow} x_0 = y_0$.) A straightforward adaptation of the graph–construction in chapter 4 will show this. This result also follows from Arsov's proof ([AM94b]). He also used a step–by–step construction to build a directed multigraph. The latter is a bit more involved, because of the abundance of arrows in multigraphs. If we add the (not multigraph valid) axioms $(A_{10}) - (A_{12})$ to his list, they collapse into the set of $\Delta^{\langle ij\rangle}_H$ axioms, and the constructed multigraph will be a graph. QED

THEOREM 6.3.3. *Let $H \subseteq \{R, S\}$. $AL(\mathsf{K}^{rel}_{setH})+\langle ij\rangle$ is decidable.*

PROOF. To show decidability of $AL(\mathsf{K}^{rel}_{setH})+\langle ij\rangle$, filtrate a frame satisfying the $\Delta^{\langle ij\rangle}_H$ axioms as was done without the $\langle ij\rangle$'s (cf. the proof of 3.2.3), using the technique

[6]The correspondence between the axioms and the frame conditions $(V_7)$ and $(V_8)$ holds only assuming the $AL(\mathsf{K}^{rel}_{set})$ axioms. Without them, we should change $(V_7)$ to $q \wedge \langle 0i\rangle(\mathsf{id} \wedge p) \to (\mathsf{id} \wedge p) \bullet q$, and make a similar change for $(V_8)$.

[7]Assuming the $AL(\mathsf{K}^{rel}_{set})$ axioms, axiom 4 follows from 5 and 6. We leave it for aesthetic reasons.

described in [Vak92b] to handle the new operators. Vakarelov proves that the logic with only the $\langle ij\rangle$ modalities admits filtrations using the following trick (see [Vak92b] Thm 4.4). Ensure that the closure set $CL(X)$ satisfies the following condition:

$$\text{if for some } i,j \in \{0,1\}, \langle ij\rangle\phi \in CL(X), \text{ then for any } i,j \in \{0,1\} : \langle ij\rangle\phi \in CL(X)$$

Define the $R^{ij}$ relations in the filtration as follows:

$$R^{ij*}\overline{x},\overline{y} \stackrel{\text{def}}{\iff} (\forall\langle ij\rangle\phi \in CL(X))(\forall k \in \{0,1\}) : \mathrm{M}, x \Vdash \langle ik\rangle\phi \iff \mathrm{M}, y \Vdash \langle jk\rangle\phi$$

In [Vak92b] (Thm 4.4), it is shown that **min** and **max** are guaranteed, and that the filtration satisfies $(V_1) - (V_3)$. In the proof for the combined language, it is easy to show that conditions $(V_4) - (V_8)$ hold as well[8]. QED

THEOREM 6.3.4. *Let $H \subseteq \{R,S\}$. $AL(\mathsf{K}^{rel}_{setH})$+$\langle ij\rangle$ enjoy interpolation and Beth definability.*

PROOF. Immediate from the given axiomatizations and 5.3.6. QED

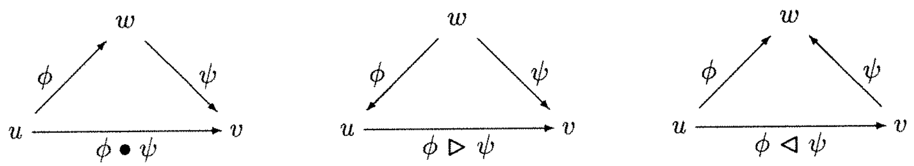
COROLLARY 6.3.5. *Let $H \subseteq \{R,S,T\}$.*
(i) *$AL(\mathsf{K}^{rel}_{setH})$+$\langle ij\rangle$ admits a finite axiomatization iff $T \notin H$.*
(ii) *$AL(\mathsf{K}^{rel}_{setH})$+$\langle ij\rangle$ is decidable iff $T \notin H$.*

## 6.3.2 SLASHED ARROW LOGIC

We now study a version of arrow logic in which we drop the converse operator, and have the slashes instead. We take the conjugates[9] of $\bullet$ as primitive operators. The conjugates $\lhd$ and $\rhd$ of $\bullet$ are the semi–duals of the slashes, as explained in section 4.5.2. Their meaning is given by the accessibility relation $\mathsf{C}$ of the $\bullet$:

$$\mathrm{M}, x \Vdash \phi \rhd \psi \stackrel{\text{def}}{\iff} \exists y,z : \mathsf{C}zyx \;\&\; \mathrm{M}, y \Vdash \phi \;\&\; \mathrm{M}, z \Vdash \psi$$
$$\mathrm{M}, x \Vdash \phi \lhd \psi \stackrel{\text{def}}{\iff} \exists y,z : \mathsf{C}yxz \;\&\; \mathrm{M}, y \Vdash \phi \;\&\; \mathrm{M}, z \Vdash \psi$$

On pair–frames, these definitions work as follows:




PROPOSITION 6.3.6. *On locally square pair–frames, the similarity types $\{\bullet,\otimes,\mathsf{id}\}$, $\{\bullet,\lhd,\mathsf{id}\}$, $\{\bullet,\rhd,\mathsf{id}\}$ and $\{\bullet,\lhd,\rhd,\mathsf{id}\}$ are interdefinable.*

---

[8] The only non-trivial part is to show $(V_7)$ and $(V_8)$. Here one should realize that, using $(V_6)$, the implication in $(V_7)$ and $V_8)$ can be strengthened to a biconditional. Then use the fact that the closure set is closed under taking $\mathsf{s}^i_j\,\phi$ (i.e., $(\mathsf{id} \wedge \phi) \bullet \top$ and $\top \bullet (\mathsf{id} \wedge \phi)$).

[9] The intuitive explanation of this notion is that the conjugates are the "backward looking" versions of the operator $\bullet$, in a similar way as $P$ is the backward looking version of $F$ in temporal logic.

PROOF. Define the triangles from converse: as in $\phi \lhd \psi \leftrightarrow \phi \bullet \otimes\psi$. To show that this works, one only needs symmetry. On the other hand, define converse from the triangles: as in $\otimes\phi \leftrightarrow (\mathsf{id} \lhd \phi)$. To show that this definition works, one only needs reflexivity. QED

So, on symmetric frames, we can choose the relational type, since it is easier to handle. For applications however, the "slashed" version might be more natural. See, for instance, Pratt's action logic, with slashes as dynamic implication [Pra90b], or Roorda's extension of the Lambek Calculus with Booleans ([Roo91]). The motivation for converse is usually weaker than for composition and identity. Note, in this respect, that the converse–free reduct of arrow logic behaves just as nicely as arrow logic with converse (cf. section 4.3 on reducts).

Let $SAL_H$ (for *slashed arrow logic*) stand for the arrow logic of type $\{\bullet, \lhd, \rhd, \mathsf{id}\}$ over all $H$ pair–frames ($H \subseteq \{R, S, T\}$). The only interesting new cases are those with $H = \emptyset$ or $H = R$. (The same similarity type was studied under the name "CARL" by Sz. Mikulás in [Mik94]. In that paper only abstract arrow–frames were studied.) In section 4.5.2, we have seen that, without the identity constant, the four axioms $(L_1) - (L_4)$ below, are sufficient for a complete axiomatization (cf. 4.5.4). These are well–known from the Lambek Calculus: they say that the three operations are conjugates. If we add the identity constant, we find a complete axiomatization by a straightforward implementation[10] of the mosaic idea into the graph–construction for the identity–free case. Here is the resulting derivation system.

AXIOMS FOR $SAL_H$. Let $\Delta^{SAL}$ be the derivation system consisting of

(i) Distribution laws and $UG$–rules for the modalities $\{\bullet, \lhd, \rhd\}$

(ii) Translations of the equations which axiomatize the converse–free reduct of the pair–frames (i.e., the axiomatization of the variety $\mathbf{SRlRd}_{\{;,\mathsf{id}\}}\mathsf{RRA}$ given in table 4.2).

(iii) The conjugate axioms:

$$(L_1)\quad p \bullet (p\backslash q) \rightarrow q, \quad (L_2)\quad q \rightarrow p\backslash(p \bullet q)$$
$$(L_3)\quad (p/q) \bullet q \rightarrow p, \quad (L_4)\quad p \rightarrow (p \bullet q)/q$$

(iv) Axioms which give $\lhd$ and $\rhd$ their "converse" behaviour. Assuming all the other axioms, the latter correspond to the given frame conditions.

$$(L_5)\quad (p \rhd \mathsf{id}) \wedge (q \rhd \mathsf{id}) \rightarrow (p \wedge q) \rhd \mathsf{id} \qquad \mathsf{I}x \;\&\; \mathsf{C}xyz \;\&\; \mathsf{C}xy'z \Rightarrow y = y'$$
$$(L_6)\quad (\mathsf{id} \lhd p) \wedge (\mathsf{id} \lhd q) \rightarrow \mathsf{id} \lhd (p \wedge q) \qquad \mathsf{I}x \;\&\; \mathsf{C}xyz \;\&\; \mathsf{C}xyz' \Rightarrow z = z'$$

The derivation system $\Delta_R^{SAL}$ is obtained by adding the axiom $(\mathsf{id} \bullet \top \wedge \top \bullet \mathsf{id})$ to $\Delta^{SAL}$.

THEOREM 6.3.7. *Let $H = \emptyset$ or $H = R$. The derivation system $\Delta_H^{SAL}$ is strongly sound and complete with respect to $SAL_H$.*

[10] The trick is that one closes mosaics under "converses" as defined by $\lhd$ and $\rhd$. For instance, define a (partial) function $f^{\lhd}x = y$ if and only if $\exists z(\mathsf{C}zxy \;\&\; \mathsf{I}z)$, and close mosaics under this function (and its natural dual $f^{\rhd}$).

PROOF. As mentioned above, this is a straightforward adaptation of earlier ideas. QED

THEOREM 6.3.8. *$SAL$ and $SAL_R$ enjoy interpolation and Beth definability.*

PROOF. Immediate from the given axiomatization and 5.3.6. QED

CONJECTURE 6.3.9. We conjecture that $SAL$ admits filtrations, hence that it is also *decidable.*

Since the slashes are definable in $AL(\mathsf{K}^{rel}_{setH})$ if $S \in H$, we have the following corollary.

COROLLARY 6.3.10. *Let $H \subseteq \{R, S, T\}$. $SAL_H$ admits a finite axiomatization iff $T \notin H$.*

## 6.3.3 UNIVERSAL MODALITY, DIFFERENCE OPERATOR AND COUNTING MODALITIES

We quickly mention some relevant results which can be extracted from our previous analysis.

### UNIVERSAL MODALITY

PROPOSITION 6.3.11. *Let $H \subseteq \{R, S, T\}$. The universal modality $\Diamond$ is definable in $AL(\mathsf{K}^{rel}_{setH})$ iff $T \in H$.*

PROOF. Because $\mathbf{SP}(\mathsf{K}^{rel}_{setH})^+$ is a discriminator variety iff $T \in H$ (cf. [ANS94a]). QED

$AL(\mathsf{K}^{rel}_{setH})$+$\Diamond$ denotes the expansion of $AL(\mathsf{K}^{rel}_{setH})$ with the universal modality $\Diamond$.

THEOREM 6.3.12. *Let $H \subseteq \{R, S\}$. Then $AL(\mathsf{K}^{rel}_{setH})$+$\Diamond$ is decidable, admits a strongly complete finite axiomatization, and enjoys interpolation and Beth definability.*

PROOF. Strong completeness follows from 3.1.10. This theorem can be applied, because $\mathbf{Cm}^{-1}\mathbf{S}(\mathsf{K}^{rel}_{setH})^+ = \mathsf{K}^{rel}_{rlH} = \mathbf{Zig}\mathsf{K}^{rel}_{setH}$ (cf. 4.2.3). All the axioms we add are canonical and correspond to universal Horn sentences, so clausifiability, whence SCI, WCI and BD are preserved. Decidability follows from 3.1.9 and the fact that the class $\mathsf{K}^{rel}_{rlH}$ admits filtrations (cf. 3.2.3). QED

COROLLARY 6.3.13. *Let $P$ be any of the properties mentioned in theorem 6.3.12. Then $AL(\mathsf{K}^{rel}_{setH})$+$\Diamond$ has $P$ iff $T \notin H$.*

### DIFFERENCE OPERATOR

The difference operator D was extensively discussed in chapter 4. We added this operator to the variety of locally square relation algebras, and saw that a finite *equational* axiomatization was possible. The price paid for this extra expressive power was that the variety lost the amalgamation property. The corresponding logic does not have the interpolation property (a counterexample is given below). We even lost Beth's definability property, by a result of I. Sain (see 5.4.14). The logic is still decidable, by a result of Andréka, Mikulás and Németi ([AMN94]); but we do not know whether this can be shown by *filtration*.

THEOREM 6.3.14. *$AL(\mathsf{K}^{rel}_{setRS})+\mathsf{D}$ admits a finite axiomatization by an orthodox derivation system, and it is decidable. The logic lacks Beth definability and interpolation.*

PROOF. Finite axiomatizability follows from 4.4.3. In addition, we show how interpolation fails. Formula (6.1) below is a counterexample. Recall that $\Diamond^n\phi$ stands for "there are at least $n$ $\phi$–worlds", and that, assuming the Booleans, $\{\Diamond^1, \Diamond^2\}$ and the $\mathsf{D}$ operator are inter–definable.

$$[\Diamond^2(p \wedge \mathsf{id}) \wedge \Diamond^1(\neg p \wedge \mathsf{id})] \rightarrow [(\Diamond^1 q \wedge \Diamond^1 \neg q) \rightarrow (\Diamond^2 q \vee \Diamond^2 \neg q)] \tag{6.1}$$

The intuitive meaning is the obvious truth: "if there are three identity worlds, then there are three worlds". An interpolant can only be constructed using the constants $\top$ and $\mathsf{id}$. But then, we cannot express that there are three identity worlds (we can only count up to two with the difference operator). QED

REMARK 6.3.15. For stronger negative results with respect to interpolation and definability, we refer to 5.4.13 and 5.4.14. $\mathsf{D}$–expansions for classes of pair–frames which are larger than $\mathsf{K}^{rel}_{setRS}$ have not yet been investigated.

COUNTING MODALITIES. The counting modalities were briefly discussed in sections 4.4 and 5.4.3. We state some known results about their addition to arrow logic of pair–frames.

THEOREM 6.3.16 (MIKULÁS–NÉMETI). *Let $H \subseteq \{R, S, T\}$. $AL(\mathsf{K}^{rel}_{setH})+\{\Diamond^n : 0 < n < \omega\}$ is decidable iff $T \notin H$.*

PROOF. Cf. [MN94]. QED

THEOREM 6.3.17 (MIKULÁS–NÉMETI). *$AL(\mathsf{K}^{rel}_{setRS})+\{\Diamond^n : 0 < n < \omega\}$ is strongly completely axiomatizable by an orthodox derivation system with finitely many axiom schemas.*

PROOF. Cf. [MN94]. QED

The last theorem is an unpublished result by Németi–Sain.

THEOREM 6.3.18 (NÉMETI–SAIN). *Let $H \subseteq \{R, S, T\}$ or $H = SQ$. $AL(\mathsf{K}^{rel}_{setH})+\{\Diamond^n : 0 < n < \omega\}$ lacks interpolation and Beth definability.*

### 6.3.4 THE KLEENE STAR

We start by defining the meaning of Kleene * on arrow–frames, using a special accessibility relation. After that, we prove our main result, roughly saying that, if a finitely axiomatizable canonical arrow logic admits filtrations, we have a weakly complete finite derivation system for its expansion with *. Using our filtration and axiomatization results from previous chapters, this gives us several complete and decidable *–expansions of pair arrow logics.

REMARK 6.3.19. Some of our results could be derived using the ideas and results in van Benthem [Ben93]. The disadvantage of the type of proof presented there is that for every different arrow logic, one has to adjust the argument considerably. Moreover, most of its work is in proving the finite frame property for the logic without $^*$. Since we know this already for several arrow logics, we tried to find a proof which separates the latter issue from the difficulties arising from adding $^*$. The result (lemma 6.3.20) is widely applicable, and provides a means of reducing the problem of axiomatizing an arrow logic with $^*$ to the problem of finding an appropriate filtration for that logic without $^*$.

MEANING OF THE KLEENE STAR. The intuitive meaning of the Kleene star on arrow-frames is given as follows:

$$\mathrm{M}, x \Vdash \phi^* \iff x \text{ is a } \phi \text{ arrow, or } x \text{ can be finitely C–decomposed into } \phi \text{ arrows}$$

Note that the star is not a modality, because it does not distribute over disjunction. It is *normal* and *monotonic*. We make its meaning precise, using an accessibility relation between points and sets of points. Here, we will use the concept of a *mountain*. Let $\mathcal{F} = \langle W, \mathsf{C}, \mathsf{F}, \mathsf{I}\rangle$ be an arrow–frame. $\mathcal{F}$–mountains are defined inductively:

- for all $x \in W$, the tuple $\langle x, \emptyset, \{x\}\rangle$ is an $\mathcal{F}$–mountain
- if $\mathsf{C}xyz$, and $\langle y, Y_1, B_1\rangle$ and $\langle z, Y_2, B_2\rangle$ are $\mathcal{F}$–mountains, then $\langle x, (\{\langle xyz\rangle\} \cup Y_1 \cup Y_2), (B_1 \cup B_2)\rangle$ is an $\mathcal{F}$–mountain
- these are all the $\mathcal{F}$–mountains.

Every mountain $\langle x, Y, B\rangle$ represents a finite decomposition of $x$ into arrows from $B$. We define the accessibility relation for $^*$ on a frame $\mathcal{F} = \langle W, \mathsf{C}, \mathsf{F}, \mathsf{I}\rangle$ as a relation $S^{\mathcal{F}} \subseteq W \times \mathcal{P}(W)$:

$$S^{\mathcal{F}}(x, B) \stackrel{\text{def}}{\iff} (\exists Y \subseteq \mathsf{C}) : \langle x, Y, B\rangle \text{ is an } \mathcal{F}\text{–mountain}$$

This relation is completely determined by the domain of $\mathcal{F}$ and the relation C. Now the meaning of $\phi^*$ can be defined as follows:

$$\mathrm{M} = \langle \mathcal{F}, \mathsf{v}\rangle, x \Vdash \phi^* \stackrel{\text{def}}{\iff} (\exists B \subseteq W) : S^{\mathcal{F}}(x, B) \;\&\; B \subseteq [\![\phi]\!]_{\mathrm{M}}$$

If $AL(\mathsf{K})$ is an arrow logic of any type containing at least the composition operator, $AL(\mathsf{K})+^*$ denotes its expansion with $^*$. Note that we do not change the class of models, only the set of formulas and the truth definition.

COMPACTNESS. Let $\phi^n$ stand for the disjunction of all formulas containing $n$ copies of $\phi$ separated by $\bullet$'s and brackets. (E.g., $\phi^3 = \phi \bullet (\phi \bullet \phi) \vee (\phi \bullet \phi) \bullet \phi$.) Intuitively, $\phi^*$ is equivalent to the infinite disjunction of all $\phi^n$. Clearly, $\{\neg\phi^n : n < \omega\} \models \neg\phi^*$, but $\phi^*$ is always satisfiable together with any finite set of $\neg\phi^n$'s. This shows that the logic is *not compact*, and that means that we can only hope for weak completeness in the presence of the star.

OBTAINING COMPLETENESS. Van Benthem [Ben93] proposed the following axioms and (unorthodox) rule for $^*$, which are easily seen to be valid[11].

(1*) $p \rightarrow p^*$
(2*) $p^* \bullet p^* \rightarrow p^*$

**∗–rule** $\dfrac{\vdash \phi \rightarrow \psi,\ \vdash \psi \bullet \psi \rightarrow \psi}{\vdash \phi^* \rightarrow \psi}$

The next lemma shows that these often suffice for weakly axiomatizing $^*$–expansions of arrow logics.

LEMMA 6.3.20 (STAR LEMMA). *Let $AL(\mathsf{K})$ be an arrow logic of any type containing at least the composition operator. Suppose $AL(\mathsf{K})$ is canonical and axiomatizable by a derivation system $\Omega(\Sigma)$. If $AL(\mathsf{K})$ admits filtrations, with the restriction that the filtration for the accessibility relation $\mathsf{C}$ for the composition operator is minimal, then $AL(\mathsf{K})+^*$ is weakly completely axiomatizable by the derivation system obtained from $\Omega(\Sigma)$ by adding the axioms $(1^*), (2^*)$ and the $*$–rule. Moreover, if $AL(\mathsf{K})$ is* finitely axiomatizable, *then $AL(\mathsf{K})+^*$ is decidable.*

REMARK 6.3.21. The restriction on the relation $\mathsf{C}$ is not crucial. If we do not know that the filtration for $\mathsf{C}$ is minimal, it is sufficient that the closure set $CL(X)$ (through which we filtrate) satisfies $(\phi^* \in CL(X) \Rightarrow \phi^* \bullet \phi^* \in CL(X))$. Since the filtrations we used in chapter 3 for arrow–frames are minimal for $\mathsf{C}$, this weaker version of the lemma is sufficient here.

PROOF OF LEMMA 6.3.20. Let $AL(\mathsf{K})$ be a canonical arrow logic which admits filtrations with the mentioned restriction, and which is axiomatizable by $\Omega(\Sigma)$. Let $\Omega(\Sigma^*)$ be the derivation system obtained from $\Omega(\Sigma)$ by adding the $^*$ axioms and rule. We have to show that $\Omega(\Sigma^*)$ forms a weakly complete axiomatization of $AL(\mathsf{K})+^*$. So assume $\Omega(\Sigma^*) \not\vdash \phi$. Let $\mathrm{M} = \langle \mathcal{F}, \mathsf{v} \rangle$ be the canonical model[12] of the derivation system $\Omega(\Sigma^*)$. Because $AL(\mathsf{K})$ is canonical, and we assumed that the derivation system is complete for $AL(\mathsf{K})$, the frame $\mathcal{F}$ is a member of $\mathsf{K}$. The problem with this model (due to the failure of compactness) is that it includes ultrafilters containing a formula $\psi^*$ without any "witness" $\psi^n$. Hence, such an ultrafilter cannot be "finitely $\mathsf{C}$–decomposed into $\psi$ arrows". We will solve this problem by creating a finite model.

Let $X^\phi$ be the set of subformulas of $\phi$. This set will contain formulas of the form $\psi^*$. From the point of view of the old logic, these mean nothing, so we can just regard them as propositional variables[13]. Then $X^\phi$ can be viewed as a (finite) set of formulas in

[11] At the frame level, (1*) and (2*) correspond to conditions $\forall x S^{\mathcal{F}} x\{x\}$ and $\mathsf{C}xyz \ \&\ S^{\mathcal{F}} y B_1 \ \&\ S^{\mathcal{F}} z B_2 \Rightarrow S^{\mathcal{F}} x(B_1 \cup B_2)$, respectively. The following version of the rule is valid for *global* consequence.

$$\frac{\phi \models^{glo} \psi,\ \psi \bullet \psi \models^{glo} \psi}{\phi^* \models^{glo} \psi}$$

[12] See footnote 5 for a definition.

[13] Note that the only important thing is the outermost $^*$ subformula (i.e., $(p \bullet q^*)^*$ is regarded as a propositional variable).

the old (*–free) language. By assumption, we can filtrate the canonical model through a set $CL(\phi) \supseteq X^\phi$, obtaining a finite model $\mathrm{M}^* = \langle \mathcal{F}^*, \mathsf{v}^* \rangle$, in which the relation $\mathsf{C}$ is defined *minimally*. Define $\mathsf{v}^*$ only on the "real" propositional variables. Recall that we used $F$ to denote the domain of the frame $\mathcal{F}$, and that a "minimal filtration" meant that

$$\mathsf{C}^*\overline{x}, \overline{y}, \overline{z} \overset{\text{def}}{\Longleftrightarrow} (\exists uvw \in F) : \mathsf{C}uvw \,\&\, \overline{x} = \overline{u} \,\&\, \overline{y} = \overline{v} \,\&\, \overline{z} = \overline{w}$$

Note that we also use a superscript * to denote the filtration. (In this proof, a superscript * *only* denotes the Kleene star if it is attached to a formula.) By assumption, $\mathcal{F}^* \in \mathsf{K}$ and $\mathcal{F}^*$ is finite. We are almost ready if we can prove the truth–lemma:

CLAIM 1 (TRUTH LEMMA). (T) $(\forall \psi \in CL(\phi))(\forall w \in F) : \psi \in \overline{w} \iff \mathrm{M}^*, \overline{w} \Vdash \psi$

PROOF OF CLAIM. By induction on the complexity of $\psi$. By assumption, we can perform the inductive proof for all connectives except * (because **min** and **max** are satisfied for all the "old" modalities). For *, we need additional work. In the sequel, we use $\delta^w$ to denote the formula which uniquely describes the point $\overline{w}$ in the filtration[14]. For every formula $\psi^*$ in the closure set, we define a formula which describes the set of points in the filtration where $\psi^*$ holds by the truth definition. Define this formula as:

$$\psi^{\#} \overset{\text{def}}{=} \bigvee \{\delta^w : \mathrm{M}^*, \overline{w} \Vdash \psi^*\}$$

Because $F^*$ is finite, $\psi^{\#}$ is a formula.

CLAIM 2. $(\forall \psi^* \in CL(\phi)) : \Omega(\Sigma^*) \vdash \psi^* \leftrightarrow \psi^{\#}$.

PROOF OF CLAIM. Suppose that $\psi^* \in CL(\phi)$. We start with the easy side.
$\Omega(\Sigma^*) \vdash \psi^{\#} \to \psi^*$. Suppose that $\psi^{\#} \in w$, for an arbitrary $w \in F$. We have to show that $\psi^* \in w$. By the definition of $\psi^{\#}$ and the truth definition, we should show that (6.2) below holds (this is sufficient, because $\psi^* \in \overline{w} \iff \psi^* \in w$).

$$[(\exists Y \subseteq \mathsf{C}^*) : \langle \overline{w}, Y, B \rangle \text{ is an } \mathcal{F}^*\text{–mountain}\,] \,\&\, B \subseteq [\![\psi]\!]_{\mathrm{M}^*} \Rightarrow \psi^* \in \overline{w} \tag{6.2}$$

We do an induction on the "height of the mountains", measured by the cardinality of the set $Y$. So we do a double induction. Call the induction hypothesis for the truth–lemma IH1, and the one for this claim IH2.

$|Y| = 0$. Suppose $\langle \overline{w}, \emptyset, B \rangle$ is an $\mathcal{F}^*$–mountain and $B \subseteq [\![\psi]\!]_{\mathrm{M}^*}$. Then, by definition, $B = \{\overline{w}\}$, whence $\mathrm{M}^*, \overline{w} \Vdash \psi$. But then, by IH1, $\psi \in \overline{w}$, and by axiom (1*), $\psi^* \in \overline{w}$.

$|Y| = n+1$. We may assume that (6.2) holds for all smaller $Y$. Suppose $\langle \overline{w}, Y, B \rangle$ is an $\mathcal{F}^*$–mountain and $B \subseteq [\![\psi]\!]_{\mathrm{M}^*}$. By definition of mountains, we can find $\overline{y}, \overline{z}$ such that $\mathsf{C}^*\overline{w}, \overline{y}, \overline{z}$, and $\langle \overline{y}, Y_1, B_1 \rangle$ and $\langle \overline{z}, Y_2, B_2 \rangle$ are $\mathcal{F}^*$–mountains with $Y = (Y_1 \cup Y_2 \cup \langle \overline{w}, \overline{y}, \overline{z} \rangle)$, $B = (B_1 \cup B_2)$, and $Y_1$ and $Y_2$ are strictly smaller than $Y$. So IH2 implies that $\psi^* \in \overline{y}$

[14] See [HC84] p.137 how one can define such a formula. Note that, $u \equiv_{CL(\phi)} v \iff \vdash \delta^u \leftrightarrow \delta^v$

and $\psi^* \in \overline{z}$. Now we use our assumption of a minimal definition for C and compute

$\mathsf{C}^*\overline{w}, \overline{y}, \overline{z} \,\&\, \psi^* \in \overline{y} \,\&\, \psi^* \in \overline{z} \iff$ (assumption)
$(\exists u, v, x) : \mathsf{C}uvx \,\&\, \overline{u} = \overline{w}, \overline{v} = \overline{y} \,\&\, \overline{x} = \overline{z} \,\&\, \psi^* \in \overline{y} \,\&\, \psi^* \in \overline{z} \Rightarrow$ (because $\psi^* \in CL(\phi)$)
$\psi^* \in v \,\&\, \psi^* \in x \Rightarrow$ (by definition of C)
$\psi^* \bullet \psi^* \in u \Rightarrow$ (axiom ($2^*$))
$\psi^* \in u \Rightarrow$ (because $\psi^* \in CL(\phi)$ and $\overline{u} = \overline{w}$)
$\psi^* \in \overline{w}$

This finishes the first half of the claim. The crucial step for the other half is the next claim.

CLAIM 3. For all $\delta^w, \delta^v$ belonging to $\psi^\#$, $\Omega(\Sigma^*) \vdash \delta^w \bullet \delta^v \to \psi^\#$.

PROOF OF CLAIM. Suppose otherwise. Then $\delta^w \bullet \delta^v \wedge \neg\psi^\#$ is consistent. Let $u$ be a maximally consistent extension of $\delta^w \bullet \delta^v \wedge \neg\psi^\#$ (that is, an ultrafilter containing $\delta^w \bullet \delta^v \wedge \neg\psi^\#$). Then the canonical frame $\mathcal{F}$ has $w', v'$ with $\mathsf{C}uw'v' \,\&\, \delta^w \in w' \,\&\, \delta^v \in v'$. Hence in the filtration, $\mathsf{C}^*\overline{u}, \overline{w}, \overline{v}$. By definition of $\psi^\#$, $\overline{w} \Vdash \psi^* \& \overline{v} \Vdash \psi^*$, hence $\overline{u} \Vdash \psi^* \bullet \psi^*$. Thus, $\overline{u} \models \psi^*$ and $\delta^u$ belongs to $\psi^\#$, whence $\vdash \delta^u \to \psi^\#$ and $\psi^\# \in u$, a contradiction. ◀

To prove $\Omega(\Sigma^*) \vdash \psi^* \to \psi^\#$, we now proceed as follows. We have $\vdash \psi \to \psi^\#$, since $\vdash \psi \leftrightarrow \bigvee\{\delta^w : \psi \in w\}$ by propositional logic and the fact that $\psi \in CL(\phi)$. By claim 2, using distribution of $\bullet$ over $\vee$: $\vdash \psi^\# \bullet \psi^\# \to \psi^\#$. So, by the $*$-rule, $\vdash \psi^* \to \psi^\#$. ◀

Now we are ready to finish the proof of the truth–lemma with the inductive step for $*$. Suppose that $\psi^* \in CL(\phi)$. Then $\psi^* \in \overline{w} \stackrel{\text{def}}{\iff} \psi^* \in w \iff$ (claim 2) $\psi^\# \in w \iff \delta^{\overline{w}} \in w \,\&\, \mathrm{M}^*, \overline{w} \Vdash \psi^* \stackrel{\text{def}}{\iff} \mathrm{M}^*, \overline{w} \Vdash \psi^*$. ◀

We now conclude our main proof. Henceforth, we may assume that (**T**) holds for all formulas in $CL(\phi)$. Now, since $\Omega(\Sigma^*) \not\vdash \phi$, there exists a $w \in F$ (the universe of the canonical frame) such that $\phi \notin w$: whence, by (**T**), $\mathrm{M}^*, \overline{w} \not\Vdash \phi$. Since $\mathrm{M}^*$ is an $AL(\mathsf{K})+^*$ model, we have $AL(\mathsf{K})+^* \not\models \phi$. This proves weak completeness. Because $\mathrm{M}^*$ is finite, we have also shown that $AL(\mathsf{K})+^*$ has the finite frame property. If the old logic is finitely axiomatizable, the expansion is finitely axiomatizable, so it is also decidable.

QED

## APPLICATIONS OF THE STAR LEMMA

THEOREM 6.3.22. *Let $H \subseteq \{R, S\}$. The logic $AL(\mathsf{K}^{rel}_{setH})+^*$ admits a weakly complete axiomatization by the $AL(\mathsf{K}^{rel}_{setH})$ derivation system extended with axioms ($1^*$) and ($2^*$) and the $*$-rule.*

THEOREM 6.3.23. *Let $H \subseteq \{R, S\}$. The logic $AL(\mathsf{K}^{rel}_{setH})+^*$ is decidable.*

We prove the two theorems together.

PROOF. Let $H \subseteq \{R, S\}$. Lemma 4.2.3 states that the class of arrow–frames $\mathsf{K}^{rel}_{rlH}$ (cf. 2.4.8) equals the class of zigzagmorphic images of the class of pair–frames $\mathsf{K}^{rel}_{setH}$. It follows that $AL(\mathsf{K}^{rel}_{setH})+^*$ is equivalent to $AL(\mathsf{K}^{rel}_{rlH})+^*$, because the meaning of $^*$ is determined by the accessibility relation C of the composition operator. But then, it suffices to prove the theorem for the arrow logics $AL(\mathsf{K}^{rel}_{rlH})+^*$. These logics are canonical, finitely axiomatizable, and –by the proof of 3.2.3– they allow filtrations in the restricted sense of lemma 6.3.20. Now apply that lemma. QED

Once $T \in H$, the old arrow logic becomes undecidable, hence its corresponding class of frames cannot allow filtrations, and the above proof does not go through. We have our usual corollary. Interpolation and definability must await further investigations.

COROLLARY 6.3.24. *Let* $H \subseteq \{R, S, T\}$.
(i) $AL(\mathsf{K}^{rel}_{setH})+^*$ *admits a weakly complete finite axiomatization iff* $T \notin H$.
(ii) $AL(\mathsf{K}^{rel}_{setH})+^*$ *is decidable iff* $T \notin H$.

EXPANSION WITH THE UNIVERSAL MODALITY. A debatable aspect of the above theorem is that its derivation system is unorthodox, because we added an *inference rule*. With the universal modality $\Diamond$ in the language, we can replace the $*$–rule with axiom (3*):

$$(3^*) \quad \Box((p \rightarrow q) \wedge (q \bullet q \rightarrow q)) \rightarrow (p^* \rightarrow q)$$

If $\Box$ is the dual of the universal modality, this axiom is valid on pair–frames, and we can derive the $*$–rule from it. The next theorem states that we can also axiomatize this expansion of $AL(\mathsf{K}^{rel}_{setH})+^*$.

THEOREM 6.3.25. *Let* $H \subseteq \{R, S\}$. $AL(\mathsf{K}^{rel}_{setH})+\{^*, \Diamond\}$ *admits a weakly complete axiomatization by the following derivation system:*

(i) *The* $AL(\mathsf{K}^{rel}_{setH})$ *derivation system*
(ii) *An* S5 *axiomatization for* $\Diamond$ *(cf. e.g.,* [HC84])
(iii) *Axioms* $\otimes p \rightarrow \Diamond p$ *and* $p \bullet q \rightarrow \Diamond p \wedge \Diamond q$ *for the interaction of* $\Diamond$ *with the old connectives*[15]
(iv). *Axioms* $(1^*), (2^*)$ *and* $(3^*)$ *for the Kleene star.*

THEOREM 6.3.26. *Let* $H \subseteq \{R, S\}$. $AL(\mathsf{K}^{rel}_{setH})+\{^*, \Diamond\}$ *is decidable.*

We prove the two theorems together.

PROOF. We give a proof–sketch. Take the canonical model $\mathrm{M} = \langle \mathcal{F}, \mathsf{v} \rangle$ of the derivation system given in the theorem. Filtrate it as described in the proof of 3.2.3. Define the relation $U^*$ for the $\Diamond$ in the filtration by $U^*\overline{x}, \overline{y} \overset{\text{def}}{\Longleftrightarrow} (\Diamond\psi \in \overline{x} \iff \Diamond\psi \in \overline{y})$. Then (by Thm 8.7 in [HC84]), this is a filtration and $U^*$ is an equivalence relation. Since both the relations for composition and converse are defined minimally, the axioms in (iii) hold. Define the star in the filtration, and use the Star Lemma to prove the truth–lemma. Suppose the point which falsified the non-derivable formula was $\overline{w}$. Generate the subframe from $\overline{w}$. Here the formula is still falsified, and $U^*$ will be the universal

[15] Note that $p^* \rightarrow \Diamond p$ is derivable.

relation. Clearly this frame is finite. Because its frame validates all the arrow–logical axioms, it is a zigzagmorphic image of a pair–frame in which $\Diamond$ is the universal modality. Since the meaning of star is determined by the relation $C_V$ for the composition operator, the formula will still fail in this pair–frame. QED

## 6.4 Two–Sorted arrow logic

One can also develop a two sorted arrow logic of pair–frames, as proposed by van Benthem ([Ben93]) in the context of abstract arrow–frames. This logic can reason about two domains: both states and transitions. The new language has appropriate modalities to reason within the two domains, and to reason about connections between them. In de Rijke [Rij93], the importance of many–sorted modal logics is stressed. All applications mentioned there can also be performed in our framework, without occurring undecidability.

The connectives. The proposed logic has a two-sorted language of state assertions (with meta–variables $\phi, \psi, \ldots$) and transition formulas or *programs* (with meta–variables $\pi_1, \pi_2, \ldots$). Two new connectives, taken from propositional dynamic logic, provide the *connection* between the two sorts:

$$\begin{array}{lcl} \mathrm{M}, x \Vdash \langle\pi\rangle\phi & \stackrel{\text{def}}{\Longleftrightarrow} & (\exists y) : \mathrm{M}, \langle x, y\rangle \Vdash \pi \ \&\ \mathrm{M}, y \Vdash \phi \\ \mathrm{M}, \langle x, y\rangle \Vdash \phi? & \stackrel{\text{def}}{\Longleftrightarrow} & x = y \ \&\ \mathrm{M}, x \Vdash \phi \end{array}$$

Van Benthem proposed three simpler connectives[16] from which these two can be defined (see 6.4.3 below). Note the similarity of **L** and **R** with the operators $s_0^1$ and $s_1^0$, respectively[17].

$$\begin{array}{lcl} \mathrm{M}, \langle x, y\rangle \Vdash \mathbf{L}\phi & \stackrel{\text{def}}{\Longleftrightarrow} & \mathrm{M}, x \Vdash \phi \\ \mathrm{M}, \langle x, y\rangle \Vdash \mathbf{R}\phi & \stackrel{\text{def}}{\Longleftrightarrow} & \mathrm{M}, y \Vdash \phi \\ \mathrm{M}, x \Vdash \mathbf{D}\pi & \stackrel{\text{def}}{\Longleftrightarrow} & \mathrm{M}, \langle x, x\rangle \Vdash \pi \end{array}$$

Schematically, we can represent the language as follows (see van Benthem [Ben91a], for an explanation of the concepts *mode* and *projection*):

$\longrightarrow$ modes $\longrightarrow$
$\mathbf{L}, \mathbf{R}, ?$

| propositional logic interpreted on states | | arrow logic interpreted on transitions |
|---|---|---|

$\longleftarrow$ projections $\longleftarrow$
$\mathbf{D}, \langle . \rangle\phi$

[16] In the terminology of [Ben91a], **D** is the only permutation–invariant projection which is a Boolean homomorphism, and **L** and **R** are the only such modes which are Boolean homomorphisms.

[17] Please do *not* confuse **D** with the difference operator D. This **D** stands for *diagonal*, and **L** and **R** for *left* and *right*, respectively.

The $\mathbf{D}$ and ? connectives make most sense in reflexive pair–frames. If we have a symmetric domain as well, $\mathbf{L}\phi$ and $\mathbf{R}\phi$ become interdefinable. Hence, from now on we will work in this class of frames.

DEFINITION 6.4.1. State assertions ST and programs PR are the smallest sets satisfying:

- $\{q_i : i < \omega\} \subseteq \mathsf{ST}$ and, if $\phi, \psi \in \mathsf{ST}$ and $\pi \in \mathsf{PR}$, then $\neg\phi, (\phi \vee \psi), \mathbf{D}\pi \in \mathsf{ST}$
- $\{p_i : i < \omega\} \cup \{\mathsf{id}\} \subseteq \mathsf{PR}$, and if $\pi_1, \pi_2 \in \mathsf{PR}$ and $\phi \in \mathsf{ST}$, then $-\pi_1, (\pi_1 \cup \pi_2), \otimes\pi_1, (\pi_1 \bullet \pi_2), \mathbf{L}\phi \in \mathsf{PR}$

Here, "$-$" denotes the negation of a program, "$\cup$" the disjunction of two programs (for conjunction, we use "$\cap$"). For the Boolean top, we use "$\top$", for the arrow logical one, we use "1". We will use "$\rightarrow$" for material implication in both sorts. Now we can define *propositional dynamic arrow logic* of locally square pair–frames.

DEFINITION 6.4.2. $DAL_{pair}$ is a triple $\langle \mathsf{Fml}, \mathsf{Mod}, \Vdash \rangle$ in which:

- $\mathsf{Fml} = \mathsf{ST} \cup \mathsf{PR}$
- $\mathrm{M} = \langle Ar, Po, \mathsf{v}^{\mathsf{PR}}, \mathsf{v}^{\mathsf{ST}} \rangle$ is a $DAL_{pair}$ model if $Ar$ is a reflexive and symmetric binary relation with base $Po$, and $\mathsf{v}^{\mathsf{PR}} : \{p_i : i < \omega\} \longrightarrow \mathcal{P}(Ar)$ and $\mathsf{v}^{\mathsf{ST}} : \{q_i : i < \omega\} \longrightarrow \mathcal{P}(Po)$ are valuation functions for the propositional variables in PR and ST, respectively. Mod is the class of all such models.
- $\Vdash$ gives meaning to the formulas in every model. ST–formulas are interpreted on the set $Po$ (of states) and PR–formulas on the set $Ar$ (of arrows) as one would expect for the given connectives. For the new connectives, $\Vdash$ was defined above.

The next proposition shows that $DAL_{pair}$ is strong enough to capture the mode and projection from PDL.

PROPOSITION 6.4.3. *On reflexive and symmetric pair–frames, the languages* $\{\bullet, \otimes, \mathsf{id}, \mathbf{L}, \mathbf{D}\}$ *and* $\{\bullet, \otimes, \mathsf{id}, ?, \langle . \rangle .\}$ *are equally expressive.*

PROOF. First, we express $\mathbf{R}$, ? and $\langle . \rangle .$ in $DAL_{pair}$, just as in [Ben93]. We need symmetry of the universe for the first, and reflexivity and symmetry for the last clause.

$$\begin{array}{lcl} \mathbf{R}\phi & \leftrightarrow & \otimes(\mathbf{L}\phi) \\ \phi? & \leftrightarrow & \mathsf{id} \cap \mathbf{L}\phi \\ \langle \pi \rangle \phi & \leftrightarrow & \mathbf{D}(\pi \bullet \mathbf{L}\phi) \end{array}$$

On the other hand, with ? and $\langle . \rangle .$ as primitives, one can express $\mathbf{L}$ and $\mathbf{D}$ as follows:

$$\begin{array}{lcl} \mathbf{L}\phi & \leftrightarrow & \phi? \bullet 1 \\ \mathbf{D}\pi & \leftrightarrow & \langle \mathsf{id} \cap \pi \rangle \top \end{array}$$

For these definitions, only reflexivity is needed. QED

DECIDABILITY

THEOREM 6.4.4. *$DAL_{pair}$ is decidable.*

PROOF. We can give a direct proof, adjusting our previous filtration. An easier way is provided by the fact that in $AL(\mathsf{K}^{rel}_{setRS})$, one can encode all the *state assertions* of $DAL_{pair}$, viewed as programs which hold only at identity arrows. Define the following inductive translation function $^\circ$ from $DAL_{pair}$ formulas to arrow logical formulas:

$$\begin{array}{rcllcl}
p_i^\circ & = & p_i & q_i^\circ & = & q_i \cap \mathsf{id} \\
(-\pi)^\circ & = & -(\pi^\circ) & (\neg\phi)^\circ & = & -(\phi^\circ) \cap \mathsf{id} \\
(\pi_1 \cup \pi_2)^\circ & = & \pi_1^\circ \cup \pi_2^\circ & (\phi \vee \psi)^\circ & = & (\phi^\circ \cup \psi^\circ) \cap \mathsf{id} \\
(\mathbf{L}\phi)^\circ & = & (\mathsf{id} \cap \phi^\circ) \bullet 1 & (\mathbf{D}\pi)^\circ & = & \pi^\circ \cap \mathsf{id} \\
\mathsf{id}^\circ & = & \mathsf{id} & & & \\
(\otimes\pi)^\circ & = & \otimes(\pi^\circ) & & & \\
(\pi_1 \bullet \pi_2)^\circ & = & (\pi_1^\circ \bullet \pi_2^\circ) & & &
\end{array}$$

An easy induction shows that a formula is $DAL_{pair}$ valid if and only if its translation is $AL(\mathsf{K}^{rel}_{setRS})$ valid. (Cf. Brink et al. [BBS94] for a similar translation). But then, we can decide $DAL_{pair}$ formulas in the decidable logic $AL(\mathsf{K}^{rel}_{setRS})$. QED

COMPLETENESS

Next, we provide a complete axiomatization for $DAL_{pair}$. As in chapter 4, we first define abstract $DAL$ frames, then we restrict that class to a suitable class $\mathsf{K}_{dal}$, and show that every frame from $\mathsf{K}_{dal}$ is a zigzagmorphic image of a $DAL_{pair}$ frame.

DEFINITION 6.4.5. (i) $\mathcal{F} = \langle Ar, Po, \mathsf{C}, \mathsf{F}, \mathsf{I}, l, d\rangle$ is a *$DAL$ frame* if (1) $Ar$ is a set (of arrows), (2) $Po$ is a set (of begin and end points of arrows), (3) $\langle Ar, \mathsf{C}, \mathsf{F}, \mathsf{I}\rangle$ is an arrow–frame, (4) $l : Ar \longrightarrow Po$ is a function (providing each arrow with its starting point), and (5) $d : Po \longrightarrow Ar$ is a function (providing each point with the identity arrow on that point). The meaning of the two new connectives on these frames is:

$$\begin{array}{llcl}
(\forall x \in Ar): & \mathrm{M}, x \Vdash \mathbf{L}\phi & \stackrel{\text{def}}{\Longleftrightarrow} & \mathrm{M}, l(x) \Vdash \phi \\
(\forall w \in Po): & \mathrm{M}, w \Vdash \mathbf{D}\pi & \stackrel{\text{def}}{\Longleftrightarrow} & \mathrm{M}, d(w) \Vdash \pi
\end{array}$$

(ii) $\mathsf{K}_{dal}$ is the class of all $DAL$ frames which satisfy:

- $(D_0)$ conditions $(C_1) - (C_{15})$ from section 2.4.3[18]
- $(D_1)$ $l$ and $d$ are total functions
- $(D_2)$ $(\forall w \in Po) : \mathsf{I}(d(w))$
- $(D_3)$ $(\forall w \in Po) : w = l(d(w))$
- $(D_4)$ $(\forall x \in Ar) : \mathsf{I}x \Rightarrow x = d(l(x))$
- $(D_5)$ $(\forall xy \in Ar) : \mathsf{C}xyx \;\&\; \mathsf{I}y \Rightarrow l(x) = l(y)$

[18]These are the requirements on $\langle Ar, \mathsf{C}, \mathsf{F}, \mathsf{I}\rangle$ which suffices for a representation as a reflexive and symmetric pair–frame.

(iii) Let $V \subseteq U \times U$ be a symmetric and reflexive relation. Define $\mathcal{FR}(V) \stackrel{\text{def}}{=} \langle V, V_0, \mathsf{C}_V, \mathsf{F}_V, \mathsf{I}_V, l_V, d_V \rangle$, in which $V_0 = \mathsf{Base}(V)$, the relations $\mathsf{C}_V, \mathsf{F}_V, \mathsf{I}_V$ are defined as in section 2.4, and $(\forall \langle u, v \rangle \in V) : l_V(\langle u, v \rangle) = u$ and $(\forall w \in V_0) : d_V(w) = \langle w, w \rangle$. These frames are called $DAL_{pair}$ frames.

THEOREM 6.4.6. *$DAL_{pair}$ is strongly completely axiomatizable by adding the following axioms to the basic derivation system:*[19]

- $(DA_0)$ *all $AL(\mathsf{K}^{rel}_{setRS})$ axioms*
- $(DA_1)$ **L** *and* **D** *distribute over negation*
- $(DA_2)$ $\mathbf{D}\,\mathsf{id}$
- $(DA_3)$ $\mathbf{DL}\phi \leftrightarrow \phi$
- $(DA_4)$ $\mathsf{id} \rightarrow (\mathbf{LD}\pi \leftrightarrow \pi)$
- $(DA_5)$ $(\mathsf{id} \cap \mathbf{L}\phi) \bullet 1 \leftrightarrow \mathbf{L}\phi$

PROOF. Soundness is easy to check. As for completeness, it is easy to see that any $DAL$ frame satisfies the axioms iff it satisfies conditions $(D_0) - (D_5)$ (because all axioms are Sahlqvist formulas). So, by the now familiar argument, it suffices to show that any frame $\mathcal{F} \in \mathsf{K}_{dal}$ is a zigzagmorphic image of some frame $\mathcal{FR}(V)$, for $V$ a reflexive and symmetric relation. The relevant "two–sorted zigzagmorphism" works as follows: Let $\mathcal{F} = \langle Ar^F, Po^F, \mathsf{C}^F, \mathsf{F}^F, \mathsf{I}^F, l^F, d^F \rangle$ and $\mathcal{G} = \langle Ar^G, Po^G, \mathsf{C}^G, \mathsf{F}^G, \mathsf{I}^G, l^G, d^G \rangle$ be in $\mathsf{K}_{dal}$. The functions $p : Ar^F \longrightarrow Ar^G$ and $p^* : Po^F \longrightarrow Po^G$ constitute a *zigzagmorphism* if (1) $p$ is a zigzagmorphism for the $\mathsf{C}, \mathsf{F}, \mathsf{I}$ part, (2) $p^*$ is surjective, and (3) $(\forall w \in Po^F) : p(d^F(w)) = d^G(p^*(w))$ and $(\forall x \in Ar^F) : p^*(l^F(x)) = l^G(p(x))$.

Let $\mathcal{F} = \langle Ar, Po, \mathsf{C}, \mathsf{F}, \mathsf{I}, l, d \rangle \in \mathsf{K}_{dal}$. By lemma 4.2.3, the reduct $\langle Ar, \mathsf{C}, \mathsf{F}, \mathsf{I} \rangle$ is a zigzagmorphic image of some locally square pair–frame $\langle V, \mathsf{C}_V, \mathsf{F}_V, \mathsf{I}_V \rangle$, say by the function $p : V \longrightarrow Ar$. Take the frame $\mathcal{FR}(V)$ and define $p^* : V_0 \longrightarrow Po$ as $p^*(w) = l(p(d_V(w)))$.

CLAIM. The functions $p$ and $p^*$ form a zigzagmorphism from $\mathcal{FR}(V)$ onto $\mathcal{F}$.

PROOF OF CLAIM. We have to show that (1) $p^*$ is surjective, (2) $p(d_V(w)) = d(p^*(w))$ and (3) $p^*(l_V(x)) = l(p(x))$. Let us compute.

(1) Suppose $w \in Po$, then $d(w) \in Ar$. Because $p$ is surjective and (by condition $(D_2)$) $\mathsf{I}(d(w))$, there is some $x \in V$ such that $p(x) = d(w)$ & $\mathsf{I}_V(x)$. Since $\mathsf{I}_V(x)$, also $x = d_V(x_0)$, whence $p^*(x_0) \stackrel{\text{def}}{=} l(p(d_V(x_0))) = l(p(x)) = l(d(w)) \stackrel{(D_3)}{=} w$.

(2) Let $w \in V_0$. Then, by definition, $\mathsf{I}_V(d_V(w))$, whence by assumption, $\mathsf{I}(p(d_V(w)))$, so by $(D_4)$: $p(d_V(w)) = d(l(p(d_V(w)))) \stackrel{\text{def}}{=} d(p^*(w))$.

(3) Let $x \in V$. Then, by definition, $\mathsf{C}_V x, d_V(l_V(x)), x$ and $\mathsf{I}_V d_V(l_V(x))$. Because $p$ is a zigzagmorphism, $\mathsf{C}p(x), p(d_V(l_V(x))), p(x)$ and $\mathsf{I}p(d_V(l_V(x)))$. Then, $(D_5)$ implies that $l(p(x)) = l(p(d_V(l_V(x)))) \stackrel{\text{def}}{=} p^*(l_V(x))$. QED

DEFINABILITY AND INTERPOLATION

THEOREM 6.4.7. *$DAL_{pair}$ enjoys Craig interpolation and Beth definability.*

[19] I.e., we add $(UG)$ rules for **L** and **D** and distribution axioms $\mathbf{L}(q \vee q') \leftrightarrow (\mathbf{L}q \cup \mathbf{L}q')$ and $\mathbf{D}(p \cup p') \leftrightarrow (\mathbf{D}p \vee \mathbf{D}p')$ to the basic derivation system for arrow logic defined in 6.1.2.

PROOF. Immediate by the given axiomatization, the frame correspondents and 5.3.6.
QED

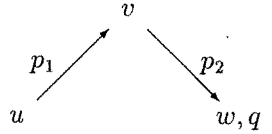
CONCLUSION. We have seen that making arrow logic two–sorted can be done without losing any of the positive properties of the one–sorted system. This is also the conclusion of [Ben93] in the context of Kripke frames. What is new here is that the logic can be given a natural pair–frame semantics, which is finitely axiomatizable. This logic also behaves well from the computational point of view: it is decidable and it has the interpolation property.

### 6.4.1 CONNECTIONS WITH OTHER SYSTEMS

PROPOSITIONAL DYNAMIC LOGIC. We briefly compare $DAL_{pair}$ with propositional dynamic logic (PDL). As we have seen, the Kleene star can be added to the arrow logical part of $DAL_{pair}$, which yields a PDL–like system over locally square pair–frames. For this comparison, define that subclass of $\mathsf{K}_{dal}$ in which *composition is associative*, namely: $\mathsf{K}_{dal}^{ass} \stackrel{\text{def}}{=} \{\mathcal{F} \in \mathsf{K}_{dal} : \mathcal{F} \models \pi_1 \bullet (\pi_2 \bullet \pi_3) \leftrightarrow (\pi_1 \bullet \pi_2) \bullet \pi_3\}$. (Remark 2.4.11 gives the frame correspondent of this axiom.) Note that $\mathsf{K}_{dal}^{ass}$ validates all RA axioms, so it inherits all negative properties of RA.

PROPOSITION 6.4.8. (i) *Every* $^*$*-free PDL formula which is valid in PDL, is also valid in* $\mathsf{K}_{dal}^{ass}$.
(ii) *All* $^*$*-free PDL axioms, except* $\langle\pi_1\rangle\langle\pi_2\rangle\phi \rightarrow \langle\pi_1 \bullet \pi_2\rangle\phi$, *are valid in* $\mathsf{K}_{dal}$.

PROOF. The validities follow from a straightforward computation. The following $DAL_{pair}$ model is a counterexample for $\langle\pi_1\rangle\langle\pi_2\rangle\phi \rightarrow \langle\pi_1 \bullet \pi_2\rangle\phi$. Its domain consists of the set $({}^2\{u,v\} \cup {}^2\{v,w\})$. Let $v^{\mathsf{ST}}(q) = \{w\}$, $v^{\mathsf{PR}}(p_1) = \{\langle u,v\rangle\}$, and $v^{\mathsf{PR}}(p_2) = \{\langle v,w\rangle\}$ Then $u \Vdash \langle p_1\rangle\langle p_2\rangle q$, but $u \nVdash \langle p_1 \bullet p_2\rangle q$.




(6.3)

QED

WEAK PEIRCE ALGEBRAS

Peirce Algebras are discussed in Brink et al. [BBS94] and in de Rijke [Rij93]. They have several applications in computer science and knowledge engineering. De Rijke ([Rij93] p.104) asks for an "arrow version of Peirce algebras which is sufficiently expressive for applications but is still decidable". The logic $DAL_{pair}$ might be an answer to this question[20]. To conclude, we present a weakened version of Peirce Algebras with some nice properties that Peirce Algebras lack.

[20] If $DAL_{pair}$ is still not expressive enough, we can add for instance the difference operator or the Kleene star. It follows from the previous results that, except if we add both, decidability remains.

PEIRCE ALGEBRAS. We copy the definition from [Rij93]. A *Peirce Algebra* is a two sorted algebra $\langle \mathfrak{R}, \mathfrak{B}, :, ^c\rangle$ in which $\mathfrak{R} \in \mathsf{RA}$ and $\mathfrak{B} \in \mathsf{BA}$. The binary operator ":" is a function from $R \times B$ to $B$, called *Peirce product*, and the unary operator "$c$" is a function from $B$ to $R$. The operators which form the connections between the two sorts have to satisfy $(P_1) - (P_8)$ below.

$$\begin{array}{llll}
(P_1) & \pi : (\phi \vee \psi) = (\pi : \phi) \vee (\pi : \psi) & (P_5) & 0 : \phi = 0 \\
(P_2) & (\pi_1 \vee \pi_2) : \phi = (\pi_1 : \phi) \vee (\pi_2 : \phi) & (P_6) & \pi^{\smile} : -(\pi : \phi) \leq -\phi \\
(P_3) & \pi_1 : (\pi_2 : \phi) = (\pi_1 ; \pi_2) : \phi & (P_7) & \phi^c : 1 = \phi \\
(P_4) & \mathsf{id} : \phi = \phi & (P_8) & (\pi : 1)^c = \pi ; 1
\end{array}$$

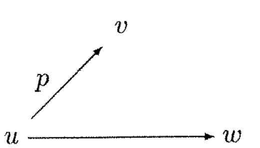
The intended models are subalgebras of direct products of two–sorted algebras $\langle (\mathcal{F}_{pair}(U \times U))^+, \mathfrak{P}(U), \langle . \rangle ., \mathbf{L}\rangle$ for some set $U$.

PEIRCE ALGEBRAS AND $DAL_{pair}$. As we have seen, $\langle . \rangle .$ and $\mathbf{D}$ are interdefinable in $DAL_{pair}$. So the intended models of Peirce Algebras are that subclass of $DAL_{pair}$ models in which the set of pairs is a full Cartesian square. Axiomatically, the only difference between *the relational part* of Peirce Algebras and that of $DAL_{pair}$ is that in Peirce Algebras composition is associative, while in $DAL_{pair}$ it is only *weakly associative*. The next proposition tells us that this is the only important difference between $DAL_{pair}$ and Peirce Algebras. Here are the trivial translations from the above axioms into the $DAL_{pair}$ language.

$$\begin{array}{llll}
(P_1') & \langle \pi \rangle (\phi \vee \psi) \leftrightarrow \langle \pi \rangle \phi \vee \langle \pi \rangle \psi & (P_5') & \langle 0 \rangle \phi \leftrightarrow \bot \\
(P_2') & \langle \pi_1 \cup \pi_2 \rangle \phi \leftrightarrow \langle \pi_1 \rangle \phi \vee \langle \pi_2 \rangle \phi & (P_6') & \langle \otimes \pi \rangle \neg \langle \pi \rangle \phi \rightarrow \neg \phi \\
(P_3') & \langle \pi_1 \rangle \langle \pi_2 \rangle \phi \leftrightarrow \langle \pi_1 \bullet \pi_2 \rangle \phi & (P_7') & \langle \mathbf{L} \phi \rangle \top \leftrightarrow \phi \\
(P_4') & \langle \mathsf{id} \rangle \phi \leftrightarrow \phi & (P_8') & \mathbf{L} \langle \pi \rangle \top \leftrightarrow \pi \bullet 1
\end{array}$$

PROPOSITION 6.4.9. (i) $DAL_{pair} \models (P_1'), (P_2'), (P_{3\leftarrow}'), (P_4'), (P_5'), (P_6'), (P_7'), (P_{8\leftarrow}')$.
(ii) $DAL_{pair} \not\models (P_{3\rightarrow}'), (P_{8\rightarrow}')$.
(iii) $\mathsf{K}_{dal}^{ass} \models (P_1') - (P_8')$.

PROOF. (i). By direct calculation. (ii). For $(P_{3\rightarrow}')$, this was proved in 6.4.8. We give a counterexample to $\mathbf{L}\langle \pi \rangle \top \rightarrow \pi \bullet 1$. Let the domain of model M be the reflexive and symmetric closure of $\{\langle u, v\rangle, \langle u, w\rangle\}$ and set $v^{\mathsf{PR}}(p) = \{\langle u, v\rangle\}$. Then $\langle u, w\rangle \Vdash \mathbf{L}\langle p \rangle \top$, but $\langle u, w\rangle \nVdash p \bullet 1$.




(6.4)

(iii). By direct calculation. QED

Thus, $\mathsf{K}_{dal}^{ass}$ is at least as strong as the logic of Peirce Algebras. Conversely, consider the class of *Representable Weak Peirce Algebras* (RWPA), whose relational component consists only of subalgebras of complex algebras of locally square pair–frames. (I.e., an RWPA is an algebra of the form $\langle \mathfrak{R}, \mathfrak{B}, \langle . \rangle ., \mathbf{L} \rangle$ with $\mathfrak{R} \leq (\mathcal{F}_{pair}(V))^+$ and $\mathfrak{B} \leq \mathfrak{P}(V_0)$, for a reflexive and symmetric relation $V$ with base $V_0$.) By proposition 6.4.3, the algebraic version of $DAL_{pair}$ is (term–definably equivalent to) the class RWPA. All earlier positive results for $DAL_{pair}$, which do not hold in Peirce Algebras, carry over to RWPA. So the strategy of obtaining positive results by widening the class of models to the "logical core", also works for Peirce Algebras.

## 6.5 Concluding remarks

We did two things to obtain computationally well–behaving versions of arrow logic of pair–frames. First, we weakened the requirements on the universes. We saw that transitivity of the universe forms the borderline between positive and negative behaviour. Second, we strengthened the expressive power of the well–behaving arrow logics. We saw that in most cases it is possible to add connectives while keeping the positive results, and that for these strengthened logics the borderline is again transitivity. For further fine–tuning, we learned the following. First, strengthening the derivation system $\Omega(\{p \bullet (q \bullet r) \leftrightarrow (p \bullet q) \bullet r\})$ with axioms valid on square pair–frames, leads to undecidability (Thm 3.2.2). Second, adding operators like D, which lead to a limited way of counting, result in the loss of interpolation and Beth definability. Further research in this area could consider the following questions:

1. We only focused on decidability vs. undecidability. What are the changes in *complexity* if we change the models or the vocabulary?
2. We always assumed all Booleans. What is the precise rôle of them in the negative results? (For some answers, see [AKN+94] (on (un)decidability) and Andréka [And89], [And91b] (for finite axiomatizability).)
3. How far can we go with strengthening the vocabulary? I.e., which expansions lead to undecidability, or lack of finite axiomatizability?
4. In which areas of, for instance, computer science can we fruitfully apply the decidable non–associative arrow logics instead of the undecidable relation algebras?
5. Where can we fruitfully apply the decidable non–commutative FO logics? Is it possible to explain the abundance of decidability in modal logic by showing that all decidable logics can be interpreted in a fragment of a decidable FO logic?

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# Index

# List of Symbols

# SAMENVATTING

We onderzoeken verscheidene verzwakte vormen van eerste orde logica en van de logica van binaire relaties die wordt gegeven door representeerbare relatie algebras. De belangrijkste reden om de twee welbekende en veel gebruikte logicas te verzwakken is hun complexiteit: de theorie van beide systemen is onbeslisbaar. Deze logicas worden niet alleen toegepast in gebieden waar deze complexiteit nodig is, zoals in de wiskunde, maar ook in tal van andere disciplines (informatica, linguistiek, sociale wetenschappen) waar de problemen vaak eenvoudiger (i.e., beslisbaar) zijn. Om deze reden is het gewenst om nieuwe versies te ontwikkelen die wat betreft semantiek en uitdrukkingskracht zo dicht mogelijk bij het origineel staan, maar die een beter computationeel gedrag vertonen.

In eerste orde logica komt de verandering er op neer dat, gegeven een model $M = \langle D, I \rangle$, de verzameling bedelingen slechts een deelverzameling is van ${}^{\omega}D$, en niet, zoals in het klassieke geval, gelijk is aan ${}^{\omega}D$. In hoofdstuk 3 tonen we beslisbaarheid aan van zo'n verzwakte eerste orde logica met behulp van filtratie. In hoofdstuk 5 kijken we naar de Craig interpolatie en Beth definieerbaarheid van de zwakkere logicas.

De verzwakking van de logica van binaire relaties is door J. van Benthem "arrow logic", of "pijl–logica", gedoopt. Dit is een modale logica, geïnterpreteerd op een verzameling pijlen, met modaliteiten voor compositie en converse van pijlen, en een (constante) modaliteit die aangeeft dat het begin- en eindpunt van een pijl dezelfde is. In dit werk identificeren we pijlen met het paar ⟨*beginpunt, eindpunt*⟩. We onderzoeken het gehele spectrum van pijl–logicas waar het domein van de modellen een binaire relatie is die voldoet aan een combinatie van de eisen {reflexiviteit, symmetrie, transitiviteit, Cartesisch product}. In hoofdstuk 3 kijken we naar beslisbaarheid, in het volgende hoofdstuk naar eindige axiomatiseerbaarheid, en in hoofdstuk 5 naar interpolatie en Beth definieerbaarheid. Onze resultaten zijn in één zin samen te vatten: een pijl–logica heeft één of meer van deze positieve eigenschappen dan en slechts dan alleen als de domeinen van de modellen niet noodzakelijk transitieve relaties zijn.

Het onderzoek wordt uitgevoerd binnen het raamwerk van de algebraische logica. Dit betekent dat we eerst de algebraische tegenhangers van de logische systemen bestuderen, en dan de resultaten vertalen naar de logische kant. In hoofdstuk 6 plukken we de vruchten voor pijl–logica van de voorafgaande algebraische studie. Daarnaast onderzoeken we enige uitbreidingen van de taal. Daar zien we hetzelfde beeld als voorheen: transitiviteit vormt de grens tussen positieve en negatieve computationele eigenschappen.

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