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Multiagent Resource Allocation with Sharable Items

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Abstract We study a particular multiagent resource allocation problem with indivisible, but sharable resources. In our model, the utility of an agent for using a bundle of resources is the difference between the value the agent would assign to that bundle in isolation and a congestion cost that depends on the number of agents she has to share each of the resources in her bundle with. The valuation function determining the value and the delay function determining the congestion cost can be agent-dependent. When the agents that share a resource also share control over that resource, then the current users of a resource will require some compensation when a new agent wants to join them using the resource. For this scenario of shared control, we study the existence of distributed negotiation protocols that lead to a social optimum. Depending on constraints on the valuation functions (mainly modularity), on the delay functions (such as convexity), and on the structural complexity of the deals between agents, we prove either the existence of a sequences of deals leading to a social optimum or the convergence of all sequences of deals to such an optimum. We also analyse the length of the path leading to such optimal allocations. For scenarios where the agents

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using a resource do not have shared control over that resource (i.e., where any agent can use any resource she wants), we study the existence of pure Nash equilibria, i.e., allocations in which no single agent has an incentive to add or drop any of the resources she is currently holding. We provide results for modular valuation functions, we analyse the length of the paths leading to a pure Nash equilibrium, and we relate our findings to results from the literature on congestion games.

Keywords Multiagent Resource Allocation, Congestion Games

1 Introduction

The generic problem of allocating a set of resources to a group of agents is a key problem in multiagent systems [5]. Some independent dimensions of a resource allocation problem are the type of resource (e.g., resources can be divisible or not, sharable or not), the allocation procedure (e.g., auctions or distributed mechanisms), and the criteria used to specify what makes an allocation optimal (e.g., maximising utilitarian or egalitarian social welfare, obtaining an envy-free division). Hence, there are many classes of problems to study in *multiagent resource allocation* (MARA).

Thus far, most work on distributed approaches to MARA has focussed on the case of indivisible *nonsharable* resources [6, 7, 9–12, 14, 22, 23]. In this setting, each resource is owned by one and only one agent, and each agent assigns a certain value to the set of resources she holds. To improve an allocation, agents can exchange resources (sometimes in combination with a monetary transfer). This line of research has identified protocols that converge to optimal allocations for certain classes of valuation functions. In some cases, simple protocols (e.g., involving only two agents and one resource at a time) are sufficient, while more complex protocols are required in others. Simple protocols exist, for instance, for maximising utilitarian social welfare [7] and for finding envy-free divisions [6]. The main goal of this paper is to extend this line of research to the case of indivisible resources that are *sharable*, as many resources are by their very nature sharable (e.g., roads or supercomputers).

The problem of sharing a set of resources is not new, and has received much attention in the game theory literature. In particular, *congestion games* [21] feature agents that share a set of resources and obtain utility for each resource they use. For a resource, the utility obtained is a function of the number of agents using that resource. This class of games is of particular interest, as congestion games have the property of possessing pure-strategy Nash equilibria. In the seminal paper by Rosenthal [21], agents using the same resource receive the same payoff, and this payoff depends only on the number of agents that use that resource. Milchtaich [16] extended the model by allowing the payoffs to be agent-dependent, but the existence of pure-strategy Nash equilibria is guaranteed only when each of the agents uses a single resource. More recently, Ackermann et al. [1], Byde et al. [4], and Voice et al. [26] extended the classes of games possessing Nash equilibria in pure strategies.

We introduce a model where the utility of an agent is the difference between the benefit derived from the set of resources she uses (defined in terms of a valuation function, as in the MARA framework for nonsharable items) and a cost that depends on the congestion of each resource (defined in terms of a delay function, as in congestion games). In doing so, we can study separately the effect of constraining the valuation or the congestion. For example, we will study a simple class of problems with *modular* valuation functions—which allows to consider a problem resource by resource—and we will present a variety of results that depend on the properties on the delay functions.

We distinguish between two types of scenarios. In the first one, we assume that the agents sharing a resource also share control over that resource. This means that they will ask for some (monetary) compensation when an additional agent wants to start using this resource, in return for the loss in utility they suffer as a result of the increased congestion. For this scenario we are interested in the design of simple negotiation protocols that agents might follow when agreeing on an allocation and we are interested in results that establish under what circumstances such a protocol will help the agents move towards an allocation with good social properties. Specifically, we are interested in *convergence results* similar to known results for the case of nonsharable resources [7, 14] that identify parameters for which any sequence of deals permitted by the protocol will eventually result in an allocation that is socially optimal (in the sense of maximising utilitarian social welfare, i.e., the sum of individual utilities). As we shall see, because of the increased expressive power of our framework with respect to the more widely studied case of nonsharable resources, such convergence results are harder to come by and the restrictions we need to impose to obtain positive results are more severe than they are for similar results for nonsharable resources. We also study the length of paths to a social optimum, i.e., the number of deals that will have to be implemented, in the worst case, to reach an optimal allocation under a given protocol, and we compare these results to similar results for the case of nonsharable resources [9, 12].

In the second scenario we consider, agents do not share control over the resources they use. Instead, any agent may start using any resource at any time. As long as a given resource has very few users, it will be attractive for other agents to join. Once it has a large number of users, for some agents the congestion cost they experience will outweigh the valuation they gain and they will decide to drop the resource from their bundle. The question then arises whether this kind of dynamic will terminate. In other words, for this scenario we are interested in the existence of Nash equilibria (in pure strategies), i.e., allocations in which no agent has an incentive to either add a new resource to its bundle or to drop a resource it currently holds.

The remainder of this paper is organised as follows. Section 2 defines the model of MARA we shall be working with and recalls a number of relevant results from the literature. In Section 3 we present our results regarding the reachability of allocations that maximise utilitarian social welfare when agents using a resource share control over that resource. These results show that for several simple but relevant instances of the general problem, it is possible to design simple negotiation protocols that are guaranteed to converge to a socially optimal allocation when individually rational agents negotiate. In other cases we are able to at least show that certain protocols permit a sequence of deals that will lead to an optimal allocation, even if not all sequences sanctioned by the protocol have that property. In Section 4 we then complement these results by providing bounds on the length of sequences of deals converging to an optimal allocation. In Section 5 we turn to allocation problems where agents need not seek each other's agreement before claiming a resource for their own use and we look for Nash equilibria for the congestion games induced by our model. In case agents have modular valuation functions, we are able to show that there always exists a pure Nash equilibrium, a result that is reminiscent of general equilibrium existence results for congestion games. We also discuss additional assumptions that ensure that agents will converge to such an equilibrium. Section 6 is a short review of related work and Section 7 concludes.

2 The Model

In this section, we introduce our model of MARA with sharable items; but first, we recall some details regarding the MARA framework with nonsharable items. Then we briefly discuss one specific issue arising in the context of sharable resources: the question of control of the resources.

2.1 MARA with Nonsharable Items

A MARA problem with indivisible nonsharable items [14] is defined as a triplet $(\mathcal{N}, \mathcal{R}, \mathcal{V})$, where $\mathcal{N} = \{1, 2, \ldots, n\}$ is a finite set of *agents*, $\mathcal{R}$ is a finite set of m *resources* (or *items*), and $\mathcal{V} = \langle v_1, \ldots, v_n \rangle$ is a profile of *valuation functions* with $v_i : 2^{\mathcal{R}} \to \mathbb{R}$ for each agent $i \in \mathcal{N}$. That is, each valuation function v_i is a mapping from bundles of resource agent i might obtain (subsets of $\mathcal{R}$) to real numbers reflecting their value to the agent. An *allocation* σ is a partition of the set of resources between the agents. We write σ_i for the bundle assigned to agent i under allocation σ.

A solution to a MARA problem is an allocation that satisfies certain properties. For example, we may want a solution to maximise utilitarian social welfare (i.e., the sum of the valuations of all agents), or to be envy-free (with no agent wanting to swap resources with any other agent). The question then arises of how to reach such a solution from a given initial allocation. A *deal* $\delta = (\sigma, \sigma')$ is a transformation from one allocation σ to another allocation σ'. A 1-*deal* is a deal involving the exchange of exactly one resource between two agents. Besides exchanging resources, we may also allow agents to make monetary side-payments. A *payment function* is a vector $p = \langle p_1, ..., p_n \rangle$ such that $\sum_{i \in \mathcal{N}} p_i = 0$. When $p_i > 0$, agent i must make a payment. When $p_i < 0$, agent i receives a payment. Now a deal $\delta = (\sigma, \sigma')$ is called *individually rational (IR)* if there exists a payment function p such that $v_i(\sigma'_i) - v_i(\sigma_i) > p_i$ for every agent $i \in \mathcal{N}$, except for agents i with $\sigma_i = \sigma'_i$ for whom $p_i = 0$ is also permitted.

Sandholm [23] showed that if agents only implement IR deals and if negotiation continues as long as there still are IR deals possible, then any such system will eventually converge to an optimal allocation:[1]

Theorem 1 *For allocation problems with nonsharable items, any sequence of IR deals will eventually result in an allocation with maximal utilitarian social welfare.*

[1] To be precise, Sandholm's work deals with the (mathematically equivalent) problem of *task* allocation. For a statement in the context of resource allocation and for a full proof, refer to Endriss et al. [14].

In other words, this framework does not permit any infinite sequence of IR deals and once an allocation that does not permit any follow-up deal that is IR has been reached, we can be certain that this allocation will be socially optimal. One drawback is that IR deals may be complex and involve many agents and resources. For *modular* valuation functions, satisfying $v(S \cup S') = v(S) + v(S') - v(S \cap S')$ for any sets $S, S' \subseteq \mathscr{R}$, however, much simpler deals are sufficient to reach a social optimum, as shown by the following theorem [7,14]:

Theorem 2 *For allocation problems with nonsharable items, if all valuation functions are modular, then any sequence of IR 1-deals will eventually result in an allocation with maximal utilitarian social welfare.*

2.2 MARA with Sharable Items

We now introduce a variant of the above MARA framework where resources are sharable. This is the framework we shall be working with for the remainder of this paper. A MARA problem with indivisible sharable items is defined as a tuple $(\mathscr{N}, \mathscr{R}, \mathscr{V}, \mathscr{D})$, where the set of agents $\mathscr{N}$, the set of resources $\mathscr{R}$, and the profile of valuation functions $\mathscr{V}$ are defined as before (and we shall turn to the definition of $\mathscr{D}$ in a moment). To simplify presentation, we shall assume that all valuation functions are *normalised*, i.e., $v_i(\emptyset) = 0$ for all agents $i \in \mathscr{N}$ (this assumption does not affect any of our results). For *modular* valuation functions (as defined in Section 2.1), we sometimes write $v_i(r)$ for $v_i(\{r\})$.

Rather than being a partition of the resources amongst the agents, an *allocation* σ now is merely a mapping of agents to subsets of $\mathscr{R}$. We again write $\sigma_i \subseteq \mathscr{R}$ for the bundle of resources used by agent $i \in \mathscr{N}$ under allocation σ. With reference to the literature on congestion games, we also call σ_i the *strategy* employed by agent i. In case $\mathscr{R}$ consists of only a single resource r, we shall sometimes identify σ with the set of agents using r.

$\mathscr{D}$, the fourth component of a MARA problem, is a set of functions $d_{i,r} : \{1,\ldots,n\} \to \mathbb{R}$, one for each pair of agents $i \in \mathscr{N}$ and resources $r \in \mathscr{R}$. For a given number k of agents (between 1 and n), $d_{i,r}(k)$ is the *delay* perceived (or the *cost* experienced) by agent i when she is one of k agents using resource r . Let $n_r(\sigma)$ be the number of agents that use resource r in allocation σ, i.e., $n_r(\sigma) = |\{i \in \mathscr{N} \mid r \in \sigma_i\}|$. That is, the delay of r experienced by agent i in allocation σ is $d_{i,r}(n_r(\sigma))$. We shall assume that the delay is a *nondecreasing* function in the number of agents using the resource, i.e., $d_{i,r}(k+1) \geqslant d_{i,r}(k)$ for all $k < n$. This models situations where an agent always prefers not to share a resource. A (delay) function d is *convex* if $d(k+2) - d(k+1) \geqslant d(k+1) - d(k)$ for all $k < n-1$; it is *concave* if $d(k+2) - d(k+1) \leqslant d(k+1) - d(k)$ for all $k < n-1$; and it is *linear* if it is both convex and concave. That is, if d is linear, then there exists an $\alpha \in \mathbb{R}$ such that $d(k) = k \cdot \alpha$ for all $k \leqslant n$.

The *utility* of agent $i \in \mathscr{N}$ under allocation σ is defined as

$$u_i(\sigma) = v_i(\sigma_i) - \sum_{r \in \sigma_i} d_{i,r}(n_r(\sigma)).$$

That is, agent i receives a benefit from using the resources of the bundle σ_i, but this benefit is reduced by the effects of the congestion.

We say that a MARA problem with sharable resources is *symmetric* when, for any given resource, all agents use the same delay function, i.e., $d_{i,r} = d_r$ for all agents $i \in \mathcal{N}$ (for some function d_r).

Observe that the original MARA framework can be simulated within the MARA framework with sharable items: using a delay function with a very high delay for $n \geqslant 2$, it will not be rational for any agent to share any item (for example, $d_{i,r}(1) = 0$ and $\forall k \geqslant 2$, $d_{i,r}(k) > \max_{\sigma_i \in 2^{\mathcal{R}}} v_i(\sigma_i)$, assuming positive valuation functions).

When the valuation functions are modular, we get a congestion game with the delay function of resource r for agent i being $d^{\star}_{i,r}(k) = v_i(\{r\}) - d_{i,r}(k)$, i.e., an agent-specific game as in the work of Milchtaich [16] and Ackermann et al. [1].

As before, a deal $\delta = (\sigma, \sigma')$ is a pair of allocations. We consider the following types of *simple deals:*

- ADD(i,r): agent i adds to its bundle a single resource she is not currently using. For $r \notin \sigma_i$, agent i will have $\sigma_i \cup \{r\}$ after the ADD(i,r) action.
- DROP(i,r): agent i drops a resource she is currently using, i.e., after the drop, agent i will use $\sigma_i \setminus \{r\}$.
- SWAP(i,j,r): agent i swaps the use of resource r with agent j, i.e., agent i drops the use of r and agent j adds the resource.[2]

As before, a *1-deal* is a deal that concerns a single item, but observe that now such a deal may involve more than just two agents. For instance, a deal in which ten agents each add the same resource to their bundle is a 1-deal. An ADD-deal, in which one agent adds a single resource to her bundle, is also a 1-deal. But note that it does not just involve that one agent, but also all the current users of the resource in question. DROP- and SWAP-deals are also 1-deals. Also note that, while with nonsharable resources, the utility of agents not actively taking part in a deal does not change, with sharable resources, the utility of agents currently using a resource that is part of a deal can be affected.

Deals may be coupled with monetary side payments. As in Section 2.1, a *payment function* is a vector $p = \langle p_1, \ldots, p_n \rangle$ such that $\sum_{i \in \mathcal{N}} p_i = 0$. A deal $\delta = (\sigma, \sigma')$ is *individually rational (IR)* if there exists a payment function p such that $u_i(\sigma') - u_i(\sigma) > p_i$ for every agent $i \in \mathcal{N}$, except for agents i unaffected by δ for whom $p_i = 0$ is also permitted. Here, an agent i is *unaffected* by a deal $\delta = (\sigma, \sigma')$ if $\sigma_i = \sigma'_i$ and $|\{j \in \mathcal{N} \mid r \in \sigma_j\}| = |\{j \in \mathcal{N} \mid r \in \sigma'_j\}|$ for all $r \in \sigma_i$. Note that an agent i that does not change her bundle may still receive a payment (from agents starting to use resources i uses) or may make a payment (to agents that stop using resources i uses). Side payments are important as they make it possible for a single agent to start using a resource without affecting the utility of other agents already using that resource, even when the bundles of those other agents remain the same.

[2] SWAP-deals should not to be confused with the *S(wap)-contracts* of Sandholm [23], which would correspond to the exchange of two resources between two agents.

2.3 Resource Control

With nonsharable resources, agents have complete control over the resources they own. For example, if agent i wants to use a particular resource owned by agent j, j must first agree to give up the item to i. With sharable resources, the notion of control is less clear. We can differentiate two variants:

(1) In the first variant, agents are free to use any resource they wish. This means that there is no mechanism to prevent an agent from starting to use a resource. This relates to strategic games with self-interested agents.
(2) In the second variant, agents must receive the consent of the agents using a resource before starting to use that resource. If the agents are rational, they will not accept that a new agent uses the resource if the delay function is strictly increasing. The only way to get access to a resource is either to compensate the current users with a side payment, or to perform a swap: to free other resources that are also used by the current users.

In this paper, we study both variants. In Sections 3 and 4, we assume that agents accept and allow deals only when they are beneficial. In particular, all agents owning a resource must agree before allowing another agent to use that resource. We study mechanisms that lead to allocations maximising utilitarian social welfare. In Section 5, we assume that agents are noncooperative and are free to use any resource they want. In that context, we investigate the problem of the existence of a pure-strategy Nash equilibrium.

Note that we could also assume that each resource has a single owner, who permits other agents to use the resource. This is the case studied in the work of Bachrach and Rosenschein [3], in which an owner knows the private production function of the resource and other agents can bid to use it. The goal of that work is to find protocols where no agent has an incentive to lie (e.g., the owner of a resource should not lie about the production function). In our work, we assume that the resources are initially allocated to the agents, and they have to find an optimal allocation to use them.

3 Convergence to an Optimal Allocation

We now investigate a MARA problem with the following properties: (1) the resources are indivisible and sharable; (2) agents using a resource also share the control of that resource; and (3) side payments between agents are allowed. In the following, we shall seek to identify protocols that lead to an allocation that maximises *utilitarian social welfare*, i.e., that maximise the function $sw(\sigma) = \sum_{i \in \mathcal{N}} u_i(\sigma_i)$ (in the remainder of the paper, we will mostly just write "social welfare").

We will first show that the convergence results for the framework with nonsharable items (Theorems 1 and 2) generalise to the case of sharable items. That is, we can guarantee convergence by means of arbitrarily complex deals and for modular valuation functions we can also guarantee convergence by means of deals involving just one resource (but possibly many agents). Then, we will present a series of results in which we allow only certain types of simple deals which also limit the number of agents involved. We will prove the existence of a path to a social optimum in some cases, and convergence of all paths in others.

3.1 General Convergence Results

We first show that Theorems 1 and 2 from Section 2.1 also apply to the framework with sharable resources. Closely following the approach familiar from the framework with nonsharable resources [14], we first establish an important lemma showing that side payments can be arranged in such a way that a given deal is beneficial for all the agents involved if and only if that deal increases social welfare.

Lemma 1 *A deal* $\delta = (\sigma, \sigma')$ *is IR iff* $sw(\sigma) < sw(\sigma')$.

Proof The proof of Lemma 1 of Endriss et al. [14] goes through: That an IR deal necessarily increases social welfare is shown by summing the inequalities $u_i(\sigma') - u_i(\sigma) > p_i$ over all agents and noting that the sum of the payments must be zero. To prove that a deal is IR when social welfare increases, one can check that the following function is a valid payment function: $p_i = u_i(\sigma') - u_i(\sigma) - (sw(\sigma') - sw(\sigma'))/n$. □

It is now easy to prove the counterpart of Theorem 1:

Theorem 3 *Any sequence of IR deals will eventually result in an allocation of resources with maximal utilitarian social welfare.*

Proof The proof is very close to the proof of Theorem 1 in the work of Endriss et al. [14]: the number of allocations is finite and, by Lemma 1, any IR deal increases social welfare and any improvement in social welfare corresponds to an IR deal; so we must eventually reach an allocation maximising social welfare. □

The significance of the theorem is that agents have no need to consider anything but their individual interests. Every single deal is bound to increase social welfare and there are no local optima the system could get stuck in. However, an IR deal may be quite complex as it may involve many agents and many resources at the same time. Finding such complex deals may turn out to be a difficult task. Indeed, as is well known, finding an allocation with maximal social welfare is NP-hard for the case of nonsharable items (see, e.g., [5]), and this result immediately extends to the more general framework of sharable items.

Under certain constraints, it is possible to reduce the complexity of the problem and work with simpler deals. Indeed, we can prove a counterpart of Theorem 2:

Theorem 4 *If all valuation functions are modular, then any sequence of IR 1-deals will eventually result in an allocation with maximal utilitarian social welfare.*

Proof We adapt the proof of Theorem 3 of Endriss et al. [14]. Note that for a normalised modular valuation function v, we have $v(R) = \sum_{r \in R} v(r)$ for any $R \subseteq \mathscr{R}$.

The set of resources and agents is finite, hence there is a finite number of allocations. Moreover, any IR deal strictly increases social welfare (see Lemma 1). Hence, the search for an allocation with maximal social welfare must terminate after a finite number of deals. As termination is guaranteed, we now must ensure there always exists an IR 1-deal from a suboptimal allocation.

Let σ and $\sigma^\star$ be two allocations such that $\sigma^\star$ maximises social welfare and $sw(\sigma) < sw(\sigma^\star)$. We denote by $\delta_{\sigma_i,r}$ the characteristic function that returns 1 when resource r is in σ_i and 0 otherwise. We can write the social welfare of σ as follows:

$$\begin{aligned} sw(\sigma) &= \sum_{i\in\mathcal{N}} \left(v_i(\sigma_i) - \sum_{r\in\sigma_i} d_{i,r}(n_r(\sigma)) \right) \\ &= \sum_{r\in\mathcal{R}} \sum_{i\in\mathcal{N}} \delta_{\sigma_i,r} \cdot (v_i(\{r\}) - d_{i,r}(n_r(\sigma))) \end{aligned}$$

This expression shows that the utility generated by at least one resource must increase for social welfare to increase. Hence, a deal involving that single resource must exist for improving social welfare. In addition, by Lemma 1, this deal will be IR, which concludes the proof of the theorem. □

Theorem 4 is independent of any assumptions regarding the delay functions; only the valuation functions are required to be modular. Under this condition, by means of a sequence of deals concerning a *single resource* each, it is possible to reach an allocation that maximises social welfare. However, each deal may involve *many agents* at the same time.

It is not always possible to decompose a complex deal into a sequence of only ADD- or only DROP-deals: SWAP-deals are sometimes needed. For example, consider the following resource allocation problem with two agents i and j and one resource r: the valuation functions are $v_i(r) = 4$ and $v_j(r) = 6$ and both agents have the same delay function defined by $d_r(1) = 2$ and $d_r(2) = 5$. Let us assume that agent i uses r, obtaining a utility of $4-2 = 2$. The action $\text{ADD}(j,r)$ is not rational as the utility of agent i would drop to $4-5 = -1$ and agent j would receive $6-5 = 1$, which is not sufficient to compensate the drop in utility of agent i. Only $\text{SWAP}(i,j,r)$ would be rational: agent j would get a utility of $6-2 = 4$, which is enough to compensate the drop in utility of agent i (who loses 2 units of utility).

We now may ask whether Theorem 4 can be strengthened by only allowing certain types of 1-deals, in particular ADD-, DROP-, and SWAP-deals.

3.2 ADD-Deals only from Empty Allocation

Let us first consider the case of protocols that only permit ADD-deals. Clearly, for this case we cannot hope for a convergence theorem, even under the strongest assumptions on the delay functions, and even if the initial allocation is the empty allocation. A simple counterexample would be the case where an agent who has low (but above zero) valuation for a resource r claims that resource first, after which no sequence of ADD-deals could possibly still lead to an optimal allocation (assuming there are many slower agents who place a higher valuation on r).

For the case of MARA with nonsharable items, in the face of failure of convergence by means of simple IR deals, Dunne et al. [11] and Dunne and Chevaleyere [10] have studied the problem of checking whether it is at least the case that a sequence of deals of the desired type leading to an optimal allocation *exists* for a given scenario (the cited works analyse the computational complexity of this kind of problem). This is an interesting question also for our framework: For a given allocation problem, does there exist a sequence of IR ADD-deals leading from the initial allocation to an optimal allocation? Maybe somewhat surprisingly, we will be able to give a positive answer to this question whenever the initial allocation is the empty allocation and all delay functions are nondecreasing and convex (symmetry is not required). We first prove the following lemma:

Lemma 2 *For allocation problems with a single resource r, if all delay functions are nondecreasing and convex, and if $sw(\sigma) < sw(\sigma^\star)$ and $N \subset N^\star$ for two allocations σ and $\sigma^\star$ with corresponding sets $N = \{i \in \mathcal{N} \mid r \in \sigma_i\}$ and $N^\star = \{i \in \mathcal{N} \mid r \in \sigma_i^\star\}$, then there exists an agent $j \in N^\star \setminus N$ such that the deal* $\textsc{add}(j,r)$ *will be IR in allocation σ.*

Proof We will show that $\textsc{add}(j,r)$ is IR for any agent $j \in \operatorname{argmax}_i\{v_i(r) - d_{i,r}(|N|) \mid i \in N^\star \setminus N\}$. From $sw(\sigma^\star) > sw(\sigma)$ we get:

$$\sum_{i \in N^\star} v_i(r) - d_{i,r}(|N^\star|) > \sum_{i \in N} v_i(r) - d_{i,r}(|N|)$$

Let $\ell = |N^\star \setminus N|$. Simplifying above inequality, and dividing by ℓ yields:

$$\frac{1}{\ell} \sum_{i \in N^\star \setminus N} v_i(r) - d_{i,r}(|N|) > \frac{1}{\ell} \sum_{i \in N^\star} d_{i,r}(|N^\star|) - d_{i,r}(|N|)$$

Given our constraints on j, this entails:

$$v_j(r) - d_{j,r}(|N|) > \frac{1}{\ell} \sum_{i \in N^\star} d_{i,r}(|N^\star|) - d_{i,r}(|N|)$$

As each $d_{i,r}$ is *convex*, we have $\frac{1}{\ell}[d_{i,r}(|N^\star|) - d_{i,r}(|N|)] \geqslant d_{i,r}(|N|+1) - d_{i,r}(|N|)$ for any agent i; and thus:

$$v_j(r) - d_{j,r}(|N|) > \sum_{i \in N^\star} d_{i,r}(|N|+1) - d_{i,r}(|N|)$$

Now we subtract $d_{j,r}(|N|+1) - d_{j,r}(|N|)$ on either side of the inequality:

$$v_j(r) - d_{j,r}(|N|+1) > \sum_{i \in N^\star \setminus \{j\}} d_{i,r}(|N|+1) - d_{i,r}(|N|)$$

As each $d_{i,r}$ is *nondecreasing*, the term $d_{i,r}(|N|+1) - d_{i,r}(|N|)$ is nonnegative for all i, and we can subtract it from the righthand side any number of times. Note that $N \subseteq N^\star \setminus \{j\}$. Thus:

$$v_j(r) - d_{j,r}(|N|+1) > \sum_{i \in N} d_{i,r}(|N|+1) - d_{i,r}(|N|)$$

The lefthand side of this inequality is the utility gain of agent j for adding r to her bundle in allocation σ; the righthand side is the loss in utility of the agents already holding r. That is, the above inequality expresses that the deal $\textsc{add}(j,r)$ will be IR in allocation σ. □

The existence of an ADD-path from the empty allocation to an optimal allocation now follows almost immediately:

Theorem 5 *If all valuation functions are modular and all delay functions are nondecreasing and convex, then there exists a sequence of IR* ADD*-deals leading from the empty allocation to an allocation with maximal utilitarian social welfare.*

Proof We have seen earlier that in *modular* domains we can let agents negotiate over resources on an item-by-item basis. So it suffices to prove the claim for scenarios with just a single resource r.

Let $\sigma^\star$ be an optimal allocation and let $N^\star = \{i \in \mathcal{N} \mid r \in \sigma_i^\star\}$ be the set of agents holding r in that allocation. Now consider any suboptimal allocation σ with $N \subset N^\star$ for $N = \{i \in \mathcal{N} \mid r \in \sigma_i\}$. By Lemma 2, there exists an agent $j \in N^\star \setminus N$ such that the deal $\text{ADD}(j,r)$ is IR from σ. As the initial allocation (i.e., the empty allocation) satisfies the conditions required for Lemma 2 to apply and as any new allocation produced this way satisfies the same conditions, this shows that there always exists a finite sequence of IR ADD-deals leading from the initial allocation to $\sigma^\star$. □

Note that this result does *not* suggest any obvious protocol for *finding* such an optimal sequence. The reason is that it will be difficult for the agents to find out which agent j should claim r at any given stage: in the proof (of Lemma 2), j is defined as an agent belonging to the set $N^\star \setminus N$, which is unknown to the agents.

The restriction to convex delay functions in Theorem 5 is not redundant. If we omit this condition, the result may no longer hold. For example, if (some) agents have strictly concave delay functions, then we can construct examples where there exists no IR ADD-path from the empty to an optimal allocation. For instance, suppose there are a single resource r and three agents with the same valuation function v with $v(r) = 5$ and $v(\emptyset) = 0$, and the same concave delay function d_r with $d_r(1) = 0$ and $d_r(k) = 3$ for $k > 1$. Then, if no agent claims r, social welfare will be 0; if one agent claims r, social welfare will be 5; if two agents do, it will be $2 \cdot (5-3) = 4$; and if all three claim r, it will be $3 \cdot (5-3) = 6$ (maximal). But the full allocation cannot be reached from the empty allocation via an IR ADD-path, since adding the second agent would result in a loss of social welfare and thus not be IR (cf. Lemma 1). This situation is reminiscent of the *maximality theorems* of Chevaleyre et al. [7], who amongst other things show that no class of valuation functions strictly including the modular functions will permit convergence by means of IR 1-deals (for allocation problems with nonsharable items).

3.3 DROP-Deals only from Full Allocation

Next, we present a similar result for protocols that only allow for IR DROP-deals. Here we are able to establish a path-existence property if we start from the *full* (rather than the empty) allocation. Again, the core of the argument is in a technical lemma:

Lemma 3 *For allocation problems with a single resource r, if all delay functions are nondecreasing and convex, and if* $sw(\sigma) < sw(\sigma^\star)$ *and* $N \supset N^\star$ *for two allocations*

σ and $\sigma^\star$ with corresponding sets $N = \{i \in \mathcal{N} \mid r \in \sigma_i\}$ and $N^\star = \{i \in \mathcal{N} \mid r \in \sigma_i^\star\}$, then there exists an agent $j \in N \setminus N^\star$ such that the deal DROP(j,r) *will be IR in allocation σ.*

Proof The proof is similar to that of Lemma 2; so we only give a compressed version here. We will show that DROP(j,r) is IR for any agent $j \in \text{argmin}_i\{v_i(r) - d_{i,r}(|N|) \mid i \in N \setminus N^\star\}$. Let $\ell = |N \setminus N^\star|$. From $sw(\sigma^\star) > sw(\sigma)$, after some rewriting and dividing by ℓ, we get:

$$\frac{1}{\ell} \sum_{i \in N^\star} d_{i,r}(|N|) - d_{i,r}(|N^\star|) > \frac{1}{\ell} \sum_{i \in N \setminus N^\star} v_i(r) - d_{i,r}(|N|)$$

Given our constraints on j, this entails:

$$\frac{1}{\ell} \sum_{i \in N^\star} d_{i,r}(|N|) - d_{i,r}(|N^\star|) > v_j(r) - d_{j,r}(|N|)$$

As each $d_{i,r}$ is *convex*, we have $d_{i,r}(|N|) - d_{i,r}(|N|-1) \geqslant \frac{1}{\ell}[d_{i,r}(|N|) - d_{i,r}(|N^\star|)]$ for all i; and as each $d_{i,r}$ is *nondecreasing*, we have $d_{i,r}(|N|) - d_{i,r}(|N|-1) \geqslant 0$ and we can add this term any number of times on the lefthand side:

$$\sum_{i \in N} d_{i,r}(|N|) - d_{i,r}(|N|-1) > v_j(r) - d_{j,r}(|N|)$$

The righthand side of this inequality is the utility lost by agent j if she drops r; the lefthand side is the cost saved by the other agents holding r. That is, this inequality states that the deal DROP(j,r) is IR in σ. □

Theorem 6 *If all valuation functions are modular and all delay functions are nondecreasing and convex, then there exists a sequence of IR* DROP*-deals leading to an allocation with maximal utilitarian social welfare from the full allocation.*

Proof The claim follows from Lemma 3 in the same way as Theorem 5 did follow from Lemma 2. □

3.4 Mix of ADD/DROP/SWAP-Deals

We now turn our attention to more powerful protocols, with the aim of deriving convergence rather than just path-existence theorems. An important first result shows that if we allow all of our three simple types of deals (ADD, DROP, and SWAP), then we can get convergence from *any* initial allocation, albeit under stronger restrictions on the delay functions (namely, we now require symmetry):

Theorem 7 *If all valuation functions are modular and all delay functions are symmetric as well as nondecreasing and convex, then any sequence of IR* ADD-, DROP-, *and* SWAP*-deals will converge to an allocation with maximal utilitarian social welfare.*

Proof As we are operating in modular domains, it suffices to prove the claim for allocation problems with a single resource r. Let σ be any suboptimal allocation and let $N = \{i \in \mathcal{N} \mid r \in \sigma_i\}$. All we need to prove is that there exists an IR ADD-, DROP-, or SWAP-deal starting from σ. This will show that even when the protocol is restricted to these deal types, we can never get stuck in a suboptimal allocation; and as social welfare improves with every IR deal (cf. Lemma 1), we must eventually reach an optimal allocation.

To simplify the presentation, we shall assume that no two agents give the same value to r, i.e., $v_i(r) \neq v_j(r)$ whenever $i \neq j$, but the proof easily extends to the general case. Define for each $k \leqslant n$ the allocation σ^k as follows: $r \in \sigma_i^k$ if and only if $\#\{j \in \mathcal{N} \mid v_j(r) > v_i(r)\} < k$, i.e., this is the allocation where the k top agents (in terms of valuing r) obtain r. Observe that, since the delay functions are symmetric, amongst all allocations assigning exactly k agents to r, allocation σ^k has maximal social welfare.

Now, let $k = |N|$ be the number of agents holding r in the current allocation σ. We distinguish three cases:

(1) $\sigma \neq \sigma^k$: Then there exists an agent j with $r \notin \sigma_j$ such that $v_j(r) > v_i(r)$ for some agent i with $r \in \sigma_i$, i.e., the deal SWAP(i, j, r) will be IR (here we use the assumption that delay functions are symmetric).

(2) $\sigma = \sigma^k$ and there exists a $k^\star > k$ with $sw(\sigma^{k^\star}) > sw(\sigma)$: Then we must have $N \subset N^\star$ for $N^\star = \{i \in \mathcal{N} \mid r \in \sigma_i^{k^\star}\}$, because in both allocations the k top agents obtain r. Thus, as delay functions are nondecreasing and convex, Lemma 2 applies and we can infer that there exists an IR ADD-deal.

(3) $\sigma = \sigma^k$ and there exists a $k^\star < k$ with $sw(\sigma^{k^\star}) > sw(\sigma)$: Then we must have $N \supset N^\star$ (with $N^\star$ defined as before). Thus, as delay functions are nondecreasing and convex, Lemma 3 applies and there exists an IR DROP-deal.

There are no further cases, so we are done. □

This result is stronger than Theorem 4 in the sense that it relies on a simpler class of deals (never involving more than two agents at a time); it is weaker in the sense that it requires stronger (but not unreasonable) restrictions to the range of admissible delay functions. Compared to Theorems 5 and 6, Theorem 7 establishes again a convergence property, rather than just the existence of a path.

The symmetry assumption in Theorem 7 is not redundant. For example, if $v_1(r) = 10$ and $d_{1,r}(k) = 6k$, and $v_i(r) = 5$ and $d_{i,r}(k) = k$ for $i \in \{2, 3\}$, then the optimal allocation where agents 2 and 3 hold r is not reachable from the allocation where only agent 1 holds r by means of ADD-, DROP-, and SWAP-deals alone. Convexity is also a not a redundant condition (see the example at the end of Section 3.2).

3.5 Mix of ADD/SWAP-Deals with Control

Finally, we want to explore convergence for protocols using just two of our simple deals, namely ADD and SWAP. As we shall see, in this case we can prove convergence (from the empty allocation) if we add an additional "control component" that allows agents to avoid certain dead-ends. We shall suggest two such control mechanisms for this setting. Both results will heavily rely on the following technical lemma:

Lemma 4 *For allocation problems with a single resource r, if all delay functions are symmetric as well as nondecreasing and convex, then any sequence of IR* ADD- *and* SWAP-*deals starting from the empty allocation will converge to an allocation with maximal utilitarian social welfare, provided no* ADD-*deals are applied once $k^\star$ agents are holding r, where $k^\star$ is the maximum number of agents holding r in any allocation with maximal utilitarian social welfare.*

Proof Inspection of the proof of Theorem 7 shows that as long as the number of agents currently holding r is at most $k^\star$, either an IR ADD- or an IR SWAP-deal will be available (or an optimal allocation has already been reached). Provided we never apply an ADD-deal once $k^\star$ agents hold r, this condition will continue to be satisfied. The claim of the lemma follows. □

The next theorem shows that there are natural protocols for which the (seemingly cumbersome) precondition for the applicability of Lemma 4 is satisfied:

Theorem 8 *If all valuation functions are modular and all delay functions are symmetric as well as nondecreasing and convex, then any sequence of IR* ADD- *and* SWAP-*deals starting from the empty allocation will converge to an allocation with maximal utilitarian social welfare, provided* ADD-*deals are only applied when no* SWAP-*deal is IR.*

Proof Due to modularity, we can restrict attention to allocation problems with a single resource r and Lemma 4 becomes applicable. Let $k^\star$ be the maximal number of agents holding r in an optimal allocation. All we need to show is that once $k^\star$ agents do hold r, no ADD-deal will ever be applied. But this is clearly so if ADD-deals are only applied when no more SWAP-deals are IR. □

In allocation σ, we say that an IR deal $\delta = (\sigma, \sigma')$ is *greedy* with respect to a set Δ of deals applicable in σ, if it produces maximal social surplus of all the deals in Δ; that is, if $sw(\sigma') \geqslant sw(\sigma'')$ for all $\sigma'' \in \Delta$. A sequence of greedy deals of a given type is a sequence of deals for which the next deal is always the deal maximising social surplus over all applicable deals of the given type.

Theorem 9 *If all valuation functions are modular and all delay functions are symmetric as well as nondecreasing and convex, then any sequence of greedy IR* ADD- *and* SWAP-*deals starting from the empty allocation will converge to an allocation with maximal utilitarian social welfare.*

Proof Restricting once again attention to scenarios with a single resource r (permissible due to modularity), let $k^\star$ be the maximal number of agents holding r in an optimal allocation. We need to show that whenever a greedy protocol chooses an ADD-deal, then the number of agents currently holding r is still less than $k^\star$. By Theorem 8, the only critical case we need to account for is when there are both IR ADD- and SWAP-deals available.

To simplify presentation, assume $v_i(r) \neq v_j(r)$ whenever $i \neq j$ (this restriction is not crucial and the proof generalises easily). Let σ be the current allocation, let $N = \{i \in \mathcal{N} \mid r \in \sigma_i\}$, and let $k = |N|$. Let $j \in \text{argmin}_i \{v_i(r) \mid r \in \sigma_i\}$ be the agent placing the lowest value on r amongst those holding r in σ; and let $j' = \text{argmax}_i\{v_i(r) \mid r \notin \sigma_i\}$ be the agent putting the highest value on r of those *not* holding r.

Then the best possible SWAP-deal is SWAP(j, j', r). It increases social welfare by a margin of $v_{j'}(r) - v_j(r)$. The best possible ADD-deal is ADD(j', r). It increases social welfare by $v_{j'}(r) - (k+1)\cdot d_r(k+1) + k \cdot d_r(k)$. Hence, under a greedy protocol, an ADD-deal will only be chosen if the former quantity does not exceed the latter, i.e., if:

$$v_j(r) - (k+1)\cdot d_r(k+1) \geqslant -\, k \cdot d_r(k)$$

Now, let N^k be the set of the top k agents in terms of valuing r. As j valued r the least of all the $k+1$ agents in $N \cup \{j'\}$, we know that $j \notin N^k$, and we can rewrite above inequality as follows (by adding $\sum_{i \in N^k} v_i(r)$):

$$\sum_{i \in N^k \cup \{j\}} v_i(r) - d_r(k+1) \geqslant \sum_{i \in N^k} v_i(r) - d_r(k)$$

The lefthand side of this inequality is the social welfare generated if the $k+1$ agents in $N^k \cup \{j\}$ hold r; the righthand side is the social welfare for the best possible allocation in which k agents hold r. That is, there are allocations in which $k+1$ agents claim r that are at least as good as the best allocation in which k agents do. Hence, $k^\star > k$, which means that under a greedy protocol, an ADD-deal will only ever get applied if $k^\star$ has not yet been reached. The claim then follows from Lemma 4. □

The control mechanism of Theorem 9 (greediness) may be more relevant in practice than that of Theorem 8 (giving SWAP precedence over ADD) because it is reasonable to assume that agents will actively search for deals giving them high profit first and thereby indirectly implement a sequence of deals that will at least be approximately greedy.

3.6 Mix of DROP/SWAP-Deals with Control

It is possible to derive results corresponding to those in Section 3.5 for protocols allowing for DROP- and SWAP-deals only, starting from the full allocation. We only state the results here; the proofs are similar to those in Section 3.5.

Lemma 5 *For allocation problems with a single resource r, if all delay functions are symmetric as well as nondecreasing and convex, then any sequence of IR* DROP- *and* SWAP-*deals starting from the full allocation will converge to an allocation with maximal utilitarian social welfare, provided no* DROP-*deals are applied once only $k^\star$ agents are holding r, where $k^\star$ is the minimal number of agents holding r in any allocation with maximal utilitarian social welfare.*

Theorem 10 *If all valuation functions are modular and all delay functions are symmetric as well as nondecreasing and convex, then any sequence of IR* DROP- *and* SWAP-*deals starting from the full allocation will converge to an allocation with maximal utilitarian social welfare, provided* DROP-*deals are only applied when no* SWAP-*deal is IR.*

Theorem 11 *If all valuation functions are modular and all delay functions are symmetric as well as nondecreasing and convex, then any sequence of greedy IR* DROP-*and* SWAP-*deals starting from the full allocation will converge to an allocation with maximal utilitarian social welfare.*

4 Path Length to an Optimal Allocation

We have seen that when we do not impose any structural restrictions on deals, then any sequence of IR deals will culminate in an allocation that maximises social welfare (cf. Theorem 3). Actually, it is possible to reach the optimal allocation using a single complex deal (cf. Lemma 1), but such a deal may be difficult to compute. To ease this computational burden, we have focused on structurally simple IR deals, which are easier to identify and which are sufficient to ensure convergence if certain conditions are satisfied. However, so far we have not yet analysed the *number of deals* required to reach an optimal allocation for the various scenarios considered. This, the length of paths to an optimal allocation, is the topic of this section.

Results on the path length for MARA problems with nonsharable resources have previously been derived by Endriss and Maudet [12] and Dunne [9].

4.1 Paths of Deals without Restrictions

We first consider the case where there is no restrictions on the deals, except being IR. As pointed out above, in this case it is always possible to reach the optimal allocation by means of a single deal, i.e., the length of the *shortest path* is bounded from above by 1. But what about the *longest path*? To answer this question, we adapt the method used for the case of nonsharable resources [12]. We start with a lemma that will be useful for the coming results. Recall that $n = |\mathcal{N}|$ is the number of agents and $m = |\mathcal{R}|$ is the number of resources.

Lemma 6 *There exist utility functions such that any two distinct allocations have distinct utilitarian social welfare.*

Proof To prove this lemma, we provide an example of such a function. We first define a bijective function $v : 2^{\mathcal{R}} \to \{0, \ldots, 2^m - 1\}$ that maps each bundle of resources to a unique integer in $\{0, \ldots, 2^m - 1\}$. We then define the valuation function of agent i as $v_i : \sigma_i \mapsto v(\sigma_i) \cdot 2^{m \cdot i}$. We furthermore stipulate that there are no delays involved, i.e., $d_{i,r} = 0$ for all $i \in \mathcal{N}$ and all $r \in \mathcal{R}$.

The social welfare for an allocation σ is then $sw(\sigma) = \sum_{i \in \mathcal{N}} v(\sigma_i) \cdot 2^{m \cdot i}$. This sum can be thought of as the representation of $sw(\sigma)$ in a number system with base 2^m, $v(\sigma_i)$ playing the role of the digit, and i contributing to the position of that digit. Hence, for two distinct allocations σ and σ', $sw(\sigma)$ differs from $sw(\sigma')$ in at least one digit. Hence, they have different values. □

In the most general case, where there is no restriction on the valuation function or the delay, the number of deals required may be very large, as shown by the following theorem, which puts the general convergence result of Theorem 3 into perspective.

Theorem 12 *A sequence of IR deals can include up to (but no more than)* $2^{n \cdot m} - 1$ *deals.*

Proof First, observe that there are $2^{n \cdot m}$ distinct allocations: for each of the n agents and each of the m resources we have to choose whether the agent in question shall use the resource in question. Lemma 1 implies that no allocation can be visited twice on a path of IR deals. Hence $2^{n \cdot m} - 1$ certainly is an upper bound.

Next we show that it is a *tight* upper bound. By Lemma 6, we know that there are utility functions such that all allocations have distinct social welfare. We can now build a sequence visiting all allocations: start from the allocation with minimal social welfare and each deal is the one leading to the next best allocation. Since social welfare increases for each deal, by Lemma 1, they all must be IR. Hence, we have a sequence of $2^{n \cdot m} - 1$ IR deals. □

That is, the longest path to an optimal allocation is bounded from above by $2^{n \cdot m} - 1$ and it is possible to exhibit paths of exactly that length. For comparison, the corresponding bound on the longest path for allocation problems with nonsharable resources is only $n^m - 1$ [12].

4.2 Paths of Simple Deals

Above we have looked at the general case of a sequence of IR deals, which may involve many resources and agents at the same time. We now turn to the case of simple deals, specifically those that involve only a single resource each (1-deals). We know from Theorem 4 that convergence by means of 1-deals is guaranteed when the valuation functions are modular. Under this condition, the following result is an upper bound for the longest 1-deal path.

Theorem 13 *For modular valuation functions, any sequence of IR 1-deals will have a length of at most* $O(2^n \cdot m)$.

Proof For a single resource, there are 2^n possible allocations. Since convergence is guaranteed, there cannot be any cycle of IR deals, so there are at most $2^n - 1$ deals to reach an optimal allocation when there is only a single resource in the system. Since the valuation functions are modular, at most $(2^n - 1) \cdot m$ 1-deals are necessary to reach the optimal allocation when there are m resources. □

Note that this bound is not claimed to be tight. Unlike for the scenario analysed for Theorem 12, here it may not be possible to fix the utility functions of the agents in such a way that there is a path of IR deals that goes through every possible allocation. The difficulty is that once an agent with high valuation starts using a resource, it may not be rational to stop using the resource. Proving a tighter bound is left as an open problem. The *shortest path* to an optimal allocation is never longer than m in this setting of 1-deals and modular valuations: using 1-deals we can move each resource to its optimal location in a single step.

If there are more severe restrictions on the type of deals allowed, reaching an optimal allocation can be considerably faster. This is the case, for instance, if either only ADD-deals or only DROP-deals are permitted (recall that we have considered this kind of scenario in our existence results Theorem 5 and Theorem 6).

Theorem 14 *A sequence of IR deals consisting only of* ADD*-deals (or only of* DROP*-deals) can include up to (but no more than) $n \cdot m$ deals.*

Proof The proof is immediate: For ADD-deals, in the worst case, the agents start with no resources and then each agent adds each resource, one at a time (an similarly, for DROP-deals, we can start with the allocation where all agents claim all resources and then drop them one at a time). □

Observe that these longest paths are also *shortest paths*: if we are only allowed ADD-deals then for some choice of utility functions and in case we start from the empty allocation, the shortest possible path to the social optimum may have length $n \cdot m$. This is in contrast to the scenario considered in Theorem 12, where, as we have seen, the *shortest path* always has length 1, i.e., is much shorter than the corresponding longest path.

We now consider sequences of deals containing ADD-, DROP- and SWAP-deals. We assume modular domains and symmetric convex delays, as our corresponding convergence result, Theorem 7, used those assumptions. Consequently, we can concentrate our study on a single resource r at a time. To simplify presentation, we assume that the agents are indexed in increasing order of the valuation function and that the agents have distinct values for the resource, i.e., we have $v_1(r) < v_2(r) < \cdots < v_n(r)$.

Let us consider a graph in which a node represents an allocation—i.e., the agents that use resource r—and the edges are the possible deals (of type ADD, DROP or SWAP). We can think of nodes as being located at one of n *levels*, where one level, say k, contains the allocations in which exactly k agents use r. Then the bottom level contains the empty allocation and the top level contains the full allocation. ADD- and DROP-deals involve allocations located at two consecutive levels, whereas SWAP-deals involve allocations at the same level. Since, by Lemma 1, social welfare must increase with every IR deal, we can use a directed graph: the direction of an edge corresponds to an increase in social welfare, i.e., an edge goes from the allocation with the lower social welfare to the allocation with the higher social welfare. A walk in the graph corresponds to a sequence of deals that improves social welfare. As a consequence, this directed graph must be acyclic. An example of such graph for the case of three agents is given in Figure 1.

We are now ready to provide upper bounds on the number of deals in a sequence of ADD-, DROP- and SWAP-deals. We start by investigating the maximal number of SWAP-deals that can occur within a single level. At level k, there are $\binom{n}{k}$ allocations, but the following result shows that, actually, only a quadratic number of allocations can be present in a sequence of SWAP-deals. This case is illustrated in Figure 2 with a population of $n = 6$ agents and for level $k = 2$. Although there are 15 possible allocations, the longest sequence of IR SWAP-deals is $k \cdot (n - k) = 8$.

Lemma 7 *For allocation problems with a single resource r, if all delay functions are symmetric as well as nondecreasing and convex, then any sequence of IR* SWAP*-deals at level k can include up to (but no more than) $k \cdot (n - k)$ deals.*

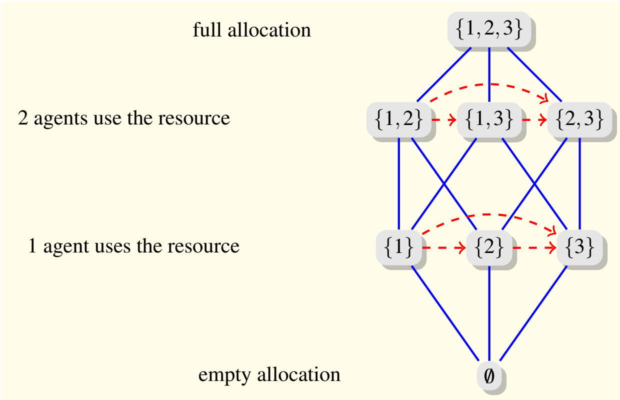

Fig. 1 Allocation graph for one resource shared between agents 1, 2 and 3, assuming $v_1(r) < v_2(r) < v_3(r)$; solid edges are ADD- or DROP-deals; dashed edges are SWAP-deals.

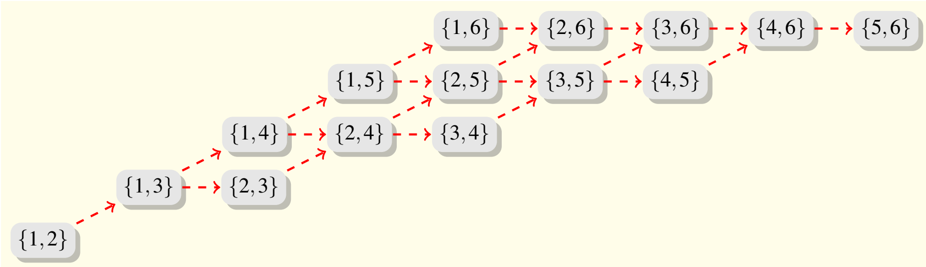

Fig. 2 Longest sequence of SWAP-deals at level $k = 2$ with $n = 6$ agents.

Proof At level k, k agents are using the resource r. An agent i can swap the use of r with an agent j if and only if $i < j$. To make a path of maximal length, all deals must be of the form where an agent i swaps with the next agent $i+1$ (otherwise, one transition is missed). Now consider the sum of the indices of the agents using r in two consecutive allocations along the path: it increases by exactly $+1$. Furthermore, the first allocation of the longest path must be $\sigma_k^{start} = \{1,\dots,k\}$ and the final allocation must be $\sigma_k^{end} = \{n-k+1,\dots,n\}$. Hence, the number of SWAP-deals along this longest path will be $\sum \sigma_k^{end} - \sum \sigma_k^{start} = \sum_{i=1}^{k}(n-k+i) - \sum_{i=1}^{k} i = k \cdot (n-k)$. □

Hence, the number of allocations visited on a pure SWAP-path is at most $k \cdot (n-k)+1$. When ADD-, DROP- and SWAP-deals are allowed, it is possible that the path goes through an allocation at level k and later comes back to a different allocation at level k. This might suggest that the number of allocations at level k visited on such a path could be higher than the number we can visit if we are forced to remain at level k throughout. Lemma 8 will show that this is not the case.

Lemma 8 *For allocation problems with a single resource r, if all delay functions are symmetric as well as nondecreasing and convex, then any sequence of IR* ADD-, DROP- *and* SWAP-*deals will visit at most $k \cdot (n-k)+1$ allocations at level k.*

Proof As we allow for sequences of IR ADD-, SWAP- and DROP-deals, it it possible that a path goes through an allocation σ_k^1 at level k, leaves it and comes back to a node σ_k^2 at level k. We shall prove that there exists a sequence of IR SWAP-deals that starts in σ_k^1 and ends in σ_k^2, i.e., it is also possible to reach σ_k^2 from σ_k^1 using a sequence of IR SWAP-deals alone (i.e., we can reach σ_k^2 from σ_k^1 without having to leave level k). The claim will then follow immediately from Lemma 7.

Observe that an ADD-deal followed by a SWAP-deal can be turned into either a SWAP-deal followed by an ADD-deal or into a single ADD-deal: if ADD(i,r)–SWAP(i',i'',r) is a sequence of IR deals leading from allocation σ to allocation σ', then either so is SWAP(i',i'',r)-ADD(i,r) (in case $i \neq i''$)[3] or so is ADD(i'',r) (in case $i = i''$). That is, any SWAP-deal to the right of an ADD-deal can either be eliminated or moved to the left of the ADD-deal. The same is true from DROP-SWAP sequences. Also observe that an ADD-DROP sequence can be reduced to a single SWAP-deal: if ADD(i,r)-DROP(i',r) is an IR path from σ to σ' (note that this implies $i \neq i'$), then SWAP(i',i,r) is an IR deal from σ to σ'. DROP-ADD sequences can be reduced in the same manner. To summarise, we can use the following simplification steps:

ADD-SWAP $\rightsquigarrow$ SWAP-ADD | ADD	DROP-SWAP $\rightsquigarrow$ SWAP-DROP | DROP
ADD-DROP $\rightsquigarrow$ SWAP	DROP-ADD $\rightsquigarrow$ SWAP

Now, by a simple inductive argument, any sequence of IR-deals of type ADD, DROP or SWAP can be transformed into a sequence of SWAP-deals followed by a sequence consisting either only of IR ADD-deals or only of IR DROP-deals. Furthermore, if the original sequence ends at the same level it started in, then this second sequence must in fact be empty, i.e., we are left with only SWAP-deals. This proves our claim that we can reach σ_k^2 from σ_k^1 by means of IR SWAP-deals alone and thereby completes the proof of the lemma. □

Lemma 8 allows us to provide an upper bound on the maximal number of deals required to reach an optimal allocation for the type of scenario covered by Theorem 7.

Theorem 15 *If all valuation functions are modular and all delay functions are symmetric as well as nondecreasing and convex, then any sequence of IR* ADD-, DROP- *and* SWAP-*deals will have a length of at most $O(n^3 \cdot m)$.*

Proof Due to modularity, we can consider each of the m resources independently. By Lemma 8, each sequence of deals involving r will visit at most $k \cdot (n-k)+1$ allocations at level k. Hence, $\sum_{k=0}^{n}[k \cdot (n-k)+1] = O(n^3)$ is an upper bound on the number of allocations visited across all levels when reassigning r. As we have to do this for each of the m resources, we obtain on overall upper bound of $O(n^3 \cdot m)$. □

The corresponding *shortest path* is $n \cdot m$. To see this, observe that for each of the m resources, the following sequence of deals will lead us to an optimal allocation: First, if too many agents currently use r, let those with the lowest valuation for r DROP it until the number of agents holding it is right (or, if there are too few agents holding

[3] Observe that at this point we are using our assumption that delays are symmetric, which entails that if a SWAP-deal is IR at one level it will also be IR at any other level.

r, let those with the highest valuation ADD it). Second, perform a sequence of SWAP-deals (asking the agent with the lowest valuation currently holding r to swap with the agent with the lowest valuation who *should* hold it but does not, and so forth). In case we need to go from the empty sequence to the full sequence (or *vice versa*), this amounts to a sequence of exactly $n \cdot m$ ADD-deals (or DROP-deals, respectively); in all other cases the sequence is shorter than that.

Clearly, the upper bound of Theorem 15 is also valid for sequences of ADD/SWAP-deals and DROP/SWAP-deals with control (as analysed in Sections 3.5 and 3.6).

However, we can provide better bounds for protocols that not only restrict the type of deals available but also control the order of deals. For Theorem 8, the protocol used suggests that, starting from an empty allocation, we first make all the possible SWAP-deals before making an ADD-deal. In the worst case, the sequence will visit all the allocations at level 1 and end up with the allocation $\{n\}$, thus using $n-1$ deals at level 1. Then, in the worst case, the agent that values the resource the least (i.e., agent 1) also starts using the resource. From this allocation, $(\{1,n\})$, $n-2$-swaps are left to reach the allocation $\{n-1,n\}$ (since it is not possible that another agent performs a SWAP-deal with agent n). Again, in the worst case, agent 1 performs a further ADD-deal. Generalising, after moving to level k, only $n-k$ SWAP-deals are possible at level k. Hence, there are at most n ADD-deals and $(n-1)+(n-2)+(n-3)+\cdots+1 = \frac{n\cdot(n-1)}{2}$ SWAP-deals. That is, there are a maximum number of $\frac{n\cdot(n+1)}{2} = O(n^2)$ deals possible involving a given resource r, and thus at most $O(n^2 \cdot m)$ deals overall.

For Theorem 9, the protocol starts from an empty allocation and the next deal is the one with maximal social surplus. Hence, the first deal will be agent n performing an ADD-deal. Then, the only possible deal is $n-1$ performing an ADD-deal. Thus, in the worst case, all agents add the resource in reverse order of their index, and hence, the maximum number of ADD-deals involving r is n. Observe that this upper bound does not change when we start from an allocation that is different from the empty allocation. Hence, in this case we obtain an upper bound of $n \cdot m$. That is, the greedy policy leads to shorter paths to the optimum than the precedence policy (at least in the worst case). On the other hand, identifying the next deal to implement is more costly under the greedy policy.

The same bounds apply to the scenarios of Theorems 10 and 11, which deal with the combination of DROP- and SWAP-deals with control.

5 Nash Equilibria in Noncooperative MARA

Our results on convergence and path length all pertain to scenarios where the users of a resource also share control of that resource: a deal is acceptable when no agent involved in that deal is worse off and at least one agent involved in the deal benefits, and we said that such a deal is IR. We now consider situations where no such control exists: an agent is free to use any resource she likes. In particular, if an agent can gain utility by using an additional resource, she will do so, regardless of the consequences for other agents using the resources. Hence, agents are selfish and do not take into account the utility of other agents. This situation can be modelled using standard concepts from (noncooperative) game theory [18]. For an agent the bundle of resources σ_i she claims now corresponds to the *strategy* (or *action*) she chooses to take. An allocation $\sigma = \langle \sigma_1, \ldots, \sigma_n \rangle$ corresponds to a *strategy profile* (also known as *joint action*) and each agent receives a *payoff* corresponding to her utility for σ. More concretely, our allocation problems may be interpreted as a class of *congestion games* [21, 16, 1, 4, 26]: the payoff of each agent depends, in part, on the congestion created as a result of several agents claiming the same resources (that is, each agent's utility function is, in part, defined in terms of its delay function).

In our model, agents can only play *pure* (rather than *mixed*) strategies (as in our model an agent uses a resource or not, but we do not model partial use of the resource or probabilities of using a resource). A natural question that arises in this context is under what circumstances an *allocation game* (i.e., an allocation problem interpreted as a game) has a (pure) *Nash equilibrium* (NE), i.e., an allocation (strategy profile) such that no agent has an incentive to change the bundle she claims (the strategy she plays). A second question that arises is whether we can find such a NE by means of a sequence of moves that are in the interest of the agents performing them. This section will be devoted to the discussion of these two questions.

Formally, a strategy profile $\sigma = \langle \sigma_1, \ldots, \sigma_n \rangle$ is called a *Nash equilibrium in pure strategies* (or *pure NE)* if for no agent $i \in \mathcal{N}$ there exists an alternative strategy $\sigma'_i \subseteq \mathcal{R}$ such that $u_i(\sigma_{-i}, \sigma'_i) > u_i(\sigma)$, where $(\sigma_{-i}, \sigma'_i) = \langle \sigma_1, \ldots, \sigma_{i-1}, \sigma'_i, \sigma_{i+1}, \ldots, \sigma_n \rangle$. Equilibria are important, because if no NE exists, then we should not expect agents to ever agree on an allocation. And in case a NE does exist, the question is whether the agents will be able to (easily) find this allocation.

In this section, after presenting an instance of the allocation problem that does not admit a pure NE, we discuss classes of games that always possess pure equilibria. In particular, this is the case for allocation games with modular valuation functions and nondecreasing delay functions. This may be shown by proving that this class of games is a special case of the class of so-called *congestion-averse* games, for which existence of a pure NE is a known result [4, 26]. However, in the sequel, we do not rely on this more general class of games, but instead present a simple and direct proof. We also prove convergence to a NE for this class of games by means of simple moves and provide an upper bound on the number of such moves before convergence.

5.1 Examples

Let us first demonstrate that in the most general case for a MARA problem with sharable resources, there may exist no pure NE. Consider the example in Figure 3, adapted from Milchtaich [17], with two agents and six resources. Agents have the same delay when they share a resource, and a different one when they use it on their own. For any bundle other than the ones indicated in the table the agents do not get any valuation. One can check that there is a cycle of best responses: when agent 2 uses $\{b,d\}$, the best response of agent 1 is to use f; if 1 uses f, 2's best response is to use $\{a,c\}$; when 2 uses $\{a,c\}$, 1 should in turn use $\{a,d,e\}$; and finally, when 1 uses $\{a,d,e\}$, 2 should use $\{b,d\}$. Hence, there is no pure NE. This is depicted in Figure 3.

$r =$	a	b	c	d	e	f	$v_1(\{a,d,e\}) = 100$
$d_{1,r}(1) =$	20	45	48	20	16	65	$v_1(\{f\}) = 100$
$d_{2,r}(1) =$	24	45	48	28	32	130	$v_2(\{b,d\}) = 100$
$d_{i,r}(2) =$	28	45	48	30	48	195	$v_2(\{a,c\}) = 100$

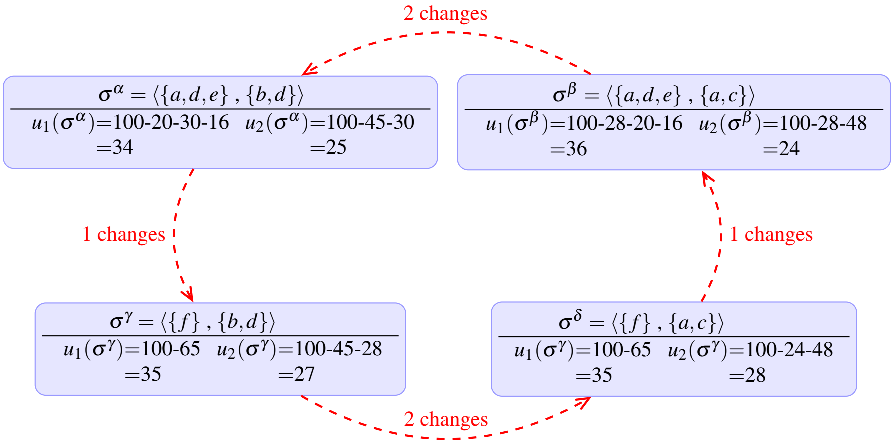

Fig. 3 Example of an allocation game with no pure NE.

Furthermore, when a pure NE does exist, the corresponding allocation need not have maximal social welfare. For example, consider the following symmetric problem with two agents, called 1 and 2, and a single resource r: $v_1(r) = 4$, $v_2(r) = 3$, $d(1) = 0$ and $d(2) = 2$. The social optimum is when agent 1 uses the resource, with a value of $4 - 0 = 4$. However, agent 2 has an incentive to also use r as it can get utility $3 - 2 = 1$ instead of 0. In that situation, however, the social welfare drops to $4 - 2 + 3 - 2 = 3$, and neither agent has an incentive to drop the resource. Hence, in this case a suboptimal allocation is a NE.

5.2 Existence of Pure Nash Equilibria

Next, we focus on identifying classes of MARA problems possessing pure Nash equilibria. We begin with a simple observation regarding games where the cost of congestion is smaller than the marginal valuation of bundles: in this case the allocation where all agents use all resources is a pure NE.

Fact 1 *Every allocation game in which marginal valuation always exceeds delay, i.e., in which $v_i(S \cup \{r\}) - v_i(\sigma) > d_{i,r}(k)$ for any $k \leqslant n$ (for all $i \in \mathscr{N}$, $S \subseteq \mathscr{R}$, $r \in \mathscr{R} \setminus S$), has got a pure NE.*

Proof The allocation where every agent claims every resource is a NE in this kind of game. (In fact, above inequality only needs to hold for $k = n$.) □

This result is interesting only in so far as it shows that existence results are achievable in principle, even if we do not constrain the range of valuation functions (of course, this freedom is gained at the expense of a very severe restriction on the delay function).

To obtain more interesting results, we may consider games that are *congestion-averse* in the sense of Byde et al. [4]. Specifically, Voice et al. [26] have identified three axioms that together provide a sufficient condition for the existence of a NE. One can show that modular games with nondecreasing delay functions satisfy these axioms, and so are guaranteed to possess a pure NE. Instead of introducing the definition of congestion-averse games and of proving that this particular class of allocation games is also congestion-averse, we provide a simple and direct proof.

Before doing so, we momentarily focus on allocation games with just a single resource r and show that when delay functions are nondecreasing, a pure NE always exists. Let us introduce some notation that will be useful for our proofs. For $k \geqslant 1$, we call A_k the set of agents that have nonnegative utility when k agents are using the resource r, i.e., $A_k = \{i \in \mathscr{N} \mid v_i(r) - d_{i,r}(k) \geqslant 0\}$. Let furthermore $A_0 = \mathscr{N}$, which is a reasonable choice as all agents get the nonnegative utility 0 when 0 agents use r. Since the delay functions are nondecreasing by assumption, we have the following inclusions: $A_n \subseteq A_{n-1} \subseteq \ldots \subseteq A_2 \subseteq A_1 \subseteq A_0$. If all agents enjoy nonnegative utility if they use r on their own, then we have $A_1 = \mathscr{N}$; otherwise $A_1 \subset \mathscr{N}$. Note that if we have $|A_k| < k$, then it is not possible that k agents use the resource and all derive a nonnegative utility from doing so. Finally, let $k^\star = \max\{k \in \{0,\ldots,n\} \mid |A_k| \geqslant k\}$. Observe that $k^\star$ is well-defined: at least $k = 0$ will be an element of the set (that $k^\star$ is the maximum of), because $|A_0| = |\mathscr{N}| \geqslant 0$ for any game. As we shall see, in a NE there will be $k^\star$ agents using the resource; it is not possible that more agents use the resource (otherwise some agents would get a negative utility by definition of $k^\star$), and the agents using the resource do not have an incentive to drop as they enjoy nonnegative utility.

Lemma 9 *Every allocation game with a single resource and nondecreasing delay functions has got a pure NE.*

Proof Let $k^\star$ be defined as above and let A be a set of $k^\star$ agents such that $A_{k^\star+1} \subseteq A \subseteq A_{k^\star}$. Such a set A exists, because $A_{k^\star+1} \subseteq A_{k^\star}$ by nondecreasingness of the delay functions and $|A_{k^\star+1}| \leqslant k^\star$ by maximality of $k^\star$. We claim that the allocation σ where all agents in A use the resource and all other agents do not use it (i.e., $\forall i \in A$, $\sigma_i = \{r\}$ and $\forall i \in \mathscr{N} \setminus A$, $\sigma_i = \emptyset$) is a pure NE. If $i \in A$, agent i gets a nonnegative utility; hence, i has no incentive to drop the resource. If $i \notin A$, then $i \notin A_{k^\star+1}$. Consequently, agent i would not get a (strictly) positive utility if it were to add the resource. □

This result is not surprising given the result of Milchtaich [16]. Our next result introduces a difference with respect to the model of Milchtaich, in which each agent can only use a single resource. For our result, on the other hand, agents may use several resources, although the valuation function is required to be modular.

Theorem 16 *Every allocation game with modular valuation functions and nondecreasing delay functions has got a pure NE.*

Proof For each resource $r \in \mathscr{R}$, Lemma 9 guarantees the existence of a pure NE σ^r. Let σ be the allocation where the strategy of each agent i is the union of strategies in σ_i^r. Given that for modular valuation functions we can treat the problem item-by-item, the allocation σ is a pure NE. □

5.3 Convergence to Pure Nash Equilibria

Existence of a NE alone does not ensure that agents will actually settle on a NE. Therefore, we now explore protocols under which we can ensure convergence to a NE. That is, we are now turning to a similar question as we have investigated in Section 3, except that now we do not focus on IR deals but rather on moves that each agent can perform on her own, without consideration for the payoff of other agents. We use the same terminology and notation as before: ADD(i,r) is the move of agent i adding resource r to her bundle and DROP(i,r) is the move of agent i dropping r from her bundle (we avoid the term "deal", as now consent from other agents is not required). Amongst those moves that only involve a single resource at a time, these are the only moves an agent can execute on her own. We call a move *profitable* if it increases the utility of the agent executing it.

Suppose all valuation functions are modular and all delay functions are nondecreasing, i.e., there exists at least one pure NE (cf. Theorem 16). If we start with some arbitrary initial allocation and allow agents to implement any profitable ADD- or DROP-move they wish, can we be sure to end up in a NE? The following result provides a positive answer to this question.

Theorem 17 *For allocation games with modular valuation functions and nondecreasing delay functions, any sequence of profitable* ADD- *and* DROP-*moves will converge to a pure NE.*

Proof W.l.o.g., due to the modularity of the valuation functions and due to the fact that we are only considering moves involving a single resource at a time, we can focus on the case where there only is one resource r in the system. In this case an allocation σ is a pure NE if and only if there exists no profitable ADD- or DROP-move starting in σ. That is, we have convergence if and only if we can show that any sequence of profitable moves must eventually terminate, i.e., reach an allocation where there are no more profitable moves possible. As the set of allocations if finite, there can be no acyclic infinite path. Hence, any sequence of profitable moves must either terminate or amount to a cycle. That is, if we can show that assuming there is a cycle leads to a contradiction, then we are done.

So suppose there exists a sequence of profitable ADD- or DROP-moves that is a cycle. This cycle must include an allocation $\sigma^\star$ that maximises the number of agents claiming r amongst all allocations along the cycle (there may be more than one allocation that is maximal in this sense; let $\sigma^\star$ be one of them). By definition, the move leading to $\sigma^\star$ must be an ADD-move. Let $i^\star$ be the agent executing that move. That is, agent $i^\star$ must derive a strictly positive utility from holding r in $\sigma^\star$. Given that we assume her delay function to be nondecreasing, she also must derive a strictly positive utility from any of the other allocations on our cyclic path in which she holds r (because they all involve equally many or fewer agents claiming r). But this means that at no point on this path it would be profitable for $i^\star$ to DROP r again. This in turn means that the move ADD$(i^\star, r)$ actually cannot have been part of the cycle in the first place, and we have derived our contradiction. □

Our proof may suggest that the games we consider have the *finite improvement property* (FIP). This property says that sequences of profitable moves, implemented by one agent at a time, are *finite*. When a game has the FIP, it is guaranteed to have a pure NE. We stress that our result does *not* imply that the resource allocation games considered here have the FIP. The reason is that an agent could in principle perform what we will call a SWITCH-move: the same agent drops one or more resources and adds one or more other resources at the same time.[4] With such SWITCH-moves, it is possible to build a cycle of allocations where one agent at a time changes her allocation and improves her utility. This is demonstrated by the example in Figure 4, which we have adapted from an example due to Milchtaich [16]. Hence, our class of allocation games does not have the FIP (even under the assumptions of Theorem 17).

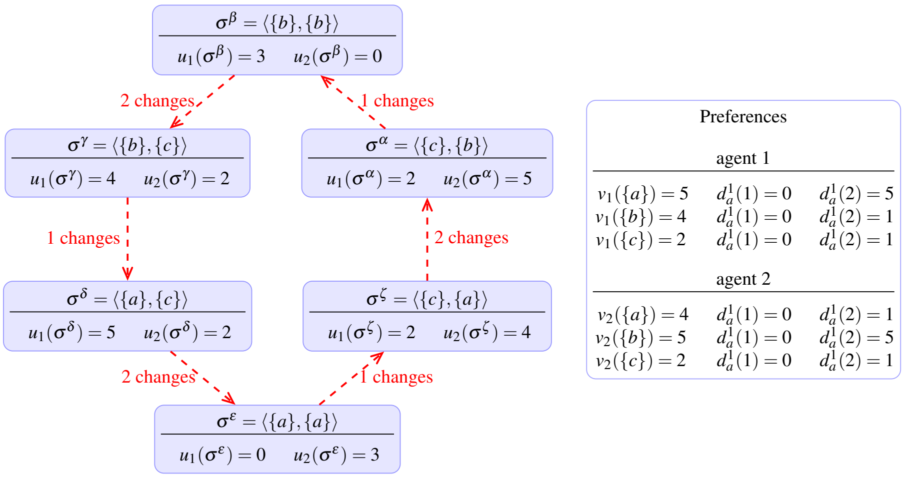

Fig. 4 Example of a cycle of a sequence of improving IR moves.

As we have seen, while there is no convergence to a NE in general, convergence can be guaranteed if agents only implement moves that concern a single resource each. It is reasonable to assume that agents will do so. In particular, the moves used in our example for a cycle are artificial in the sense that they combine two actions in one move—one that decreases the agent's payoff (or keeps it at the same level) and another that increases her payoff by a larger margin, which together is not a best-response move. The agent could increase her immediate payoff by only implementing the second part of this move. The game in the example is also *congestion-averse* and the convergence result of Voice et al. [26] still holds as it relies on best-response dynamics. Note that Byde et al. had already remarked that the larger class of *congestion-averse* games may not have the FIP.

[4] Note that this is different from the SWAP-deal in which an agent drops a resource and *another agent* starts using the same resource.

Finally, in analogy to the results on path length we have established for the cooperative case, we may ask *how many* profitable moves the agents must implement in the worst case before reaching a NE.

Theorem 18 *For allocation games with modular valuation functions and nondecreasing delay functions, any sequence of profitable* ADD- *and* DROP-*moves will have a length of at most* $O(n^2 \cdot m)$.

Proof First, observe that due to the modularity assumption, any upper bound on the length of a path of profitable moves must be linear in m. In fact, w.l.o.g., we may assume that initially all moves concerning the first resource are implemented, then all moves regarding the second resource, and so forth. That is, it is sufficient to consider the case of a single resource r to see how the number of agents n affects the overall bound.

Note the following important effect of our assumption of the delay functions being nondecreasing. This assumption entails that each agent i has a (personal) tolerance value t_i: if the current number of agents claiming r (including i herself) is strictly above t_i, then i will want to drop r at the next opportunity; if the current number of agents claiming r (now excluding i) is strictly below t_i, then she will want to add r at the next opportunity.

Now consider the following equivalence relation on moves (recall that a move is a pair of allocations): two moves are equivalent if (a) they either are both an ADD-move or both a DROP-move (two possibilities), if (b) it is the same agent performing both moves (n possibilities), and if (c) the congestion level immediately before the move is the same for both of them (n possibilities: it can be anything from 0 to $n-1$ for an ADD-move and anything from 1 to n for a DROP-move). That is, we obtain $2n^2$ equivalence classes of moves. For example, two ADD-moves performed by agent i are in the same equivalence class if the number of agents holding r before the move is the same for both of them.

We shall now argue that after every move we observe, we can completely exclude one of the above equivalence classes from further consideration: no move from that class can ever be performed again later on.

Suppose that at the very beginning of the process we observe agent i perform, say, a DROP-move, with the prior level of congestion being x. Then we know that $t_i < x$. We then also know that she will certainly never be observed to perform an ADD-move when the current level of congestion is $n-1$ (because of $x \leqslant n$ and $t_i < x$ we know $t_i \leqslant n-1$). That is, we can exclude the equivalence class of ADD-moves by i from level $n-1$ from further consideration. But this means that the level of congestion will never again be n. The reason is that this would require agent i to perform an ADD at some point, which in turn would require the level of congestion to first fall below t_i. This, in turn, would require a second agent to first perform a DROP. But then that agent would require congestion to drop even further, before she would be willing to join again, and so forth. To summarise: observing a DROP-move at the start of the process means that the the maximally achievable congestion level goes down to $n-1$.

Now suppose we observe a second agent j perform a DROP-move at some later point in time, when congestion is $y \leqslant n-1$. Then j will never again perform an ADD-move when congestion is $n-2$ (because $t_j < y$ and $y \leqslant n-1$ entail $t_j \leqslant n-2$). And so forth: every time we observe a DROP-move by a new agent, we can lower the maximal congestion achievable later on by 1 (if an agent performs a DROP-move who has done so before, the maximal level of congestion achievable later on is not directly affected). Thus, every time an agent performs a DROP (whether she does so for the first time or not) we can exclude one more type of ADD-move (the one starting from the level of congestion immediately below the current maximal level of congestion, for that particular agent). Of course, the converse also holds: every observed ADD (by an agent who did not ADD before) reduces the range of available types of DROP-moves by 1.

To summarise, after every move we can exclude one of the $2n^2$ equivalence classes. Hence, the number of moves involving a given resource r must be bounded from above by $2n^2 \in O(n^2)$. This in turn means that the number of all moves must be bounded from above by $O(n^2 \cdot m)$. □

In related work on *congestion-averse* games, Voice et al. [26] provide an algorithm that converges to a NE in time polynomial in the number of players and resources. However, the elementary changes considered in their work also include a SWITCH-move that consists of adding a resource while dropping another at the same time, while we focus on ADD- and DROP-moves only. We used such SWITCH-moves in the example of Figure 4 showing that a sequence of improving moves could be infinite.

6 Related Work

There is a vast literature on solving allocation problems using different approaches [5], ranging from centralised procedures such as combinatorial auctions [8] to decentralised ones such as exchange economies (examples for which include work on Walrasian equilibria [15] for divisible goods as well as applications of concepts from cooperative game theory [25]). Our focus has been on decentralised mechanisms for allocating indivisible resources the use of which can be shared amongst several agents. Our mechanisms allow agents to implement a sequence of simple agents that improve their immediate payoff, i.e., we have focussed on agents that are myopic.

In this section, we briefly review related work regarding both the "cooperative" and the "noncooperative" instance of our framework. We start with the topic of convergence to a socially optimal allocation by means of negotiation using IR deals (including the length of paths exhibited by these convergence results) and then move on to the existence of Nash equilibria for the congestion games induced by allocation problems with sharable resources.

6.1 Negotiating Socially Optimal Allocations

While our focus on sharable resources is new, there has been a significant amount of work on negotiating socially optimal allocations of nonsharable resources. The first convergence result, formulated as Theorem 1 in this paper, is due to Sandholm [23]. Endriss et al. [13, 14] later showed how this convergence result can be reduced to an equivalence between individual rationality and social welfare improvements (by means of a result similar to our Lemma 1) and broadened the research agenda on convergence by also considering negotiation without monetary side payments, optimality criteria not defined in terms of utilitarian social welfare, and the analysis of scenarios with valuations drawn from certain restricted classes. Regarding alternative notions of optimality, both egalitarian social welfare [14] and envy-freeness [6] have been considered in this literature. Dunne et al. [11] initiated the study of the computational complexity of problems that arise in the context of considerations pertaining to convergence, such as the question of deciding whether reaching an optimal allocation is possible by means of 1-deals. Others, e.g., Saha and Sen [22] and Bachrach and Rosenschein [3], have focussed on the design of practical negotiation protocols.

Most of our convergence results concern very simple types of deals, each involving only a single resource. Sandholm [23] already distinguished different types of deals. His "original contracts" correspond to our 1-deals. He also considered, amongst others, "cluster-contracts" (involving the reallocation of a set of resources from one agent to another) and "swap-contracts" (which, unlike our SWAP-deals, concern the exchange of *two* resources between two agents). Because of the fact that we deal with sharable resources, the space of simple deals is much richer than for the framework with nonsharable resources. Indeed, while for nonsharable resources, 1-deals are the simplest possible type of deal, for sharable resources we have been able to introduce a number of different types of deal (ADD, DROP and SWAP) that are all special cases of the class of 1-deals. For sharable resources, Chevaleyre et al. [7] have studied convergence by means of simple deals in depth. One of their main results, for instance, shows that the class of modular valuation functions is in fact maximal amongst those classes that can ensure the kind of convergence results for 1-deals given by Theorem 2. There are no results in the literature that closely resemble our results for convergence by means of ADD-, DROP- and SWAP-deals, simply because this fine-grained ontology of deals does not exist for the standard framework with nonsharable resources.

Our first result on the longest path to an optimal allocation (Theorem 12) mimics a similar result for nonsharable resources by Endriss and Maudet [12]. In related work, Dunne [9] has established bounds on the number of simple deals (particularly 1-deals) required to reach an optimal allocation in settings where convergence by such deals is not guaranteed in general. Our results on path length for various combinations of ADD-, DROP- and SWAP-deals are not closely related known results in the literature. Both the results (with relatively low bounds) and the techniques employed differ from previous work.

6.2 Congestion Games

In the model proposed by Rosenthal in his seminal paper on congestion games [21], each agent must choose from a fixed set of strategies available to that agent, with each strategy corresponding to a particular bundle of resources. The available strategies in our model are all possible sets of resources while Rosenthal's model is more general as one can restrict the set of strategies. Each agent derives a cost (i.e., negative utility) from the congestion of the resources in her bundle. Her overall cost is the sum of the costs of using each of her resources (i.e., utility is additive with respect to delays, as in our model). Two important restrictions are that (1) all agents are assumed to use the same delay function (i.e., delays are symmetric) and that (2) costs (i.e., utilities) only depend on the number of agents using the same resources, not on their identities. We can embed Rosenthal's model into ours as follows: First, for each σ_i in the fixed set of strategies available to agent i, let $v_i(\sigma_i) = \Omega$ for some sufficiently large number Ω and let v_i be equal to 0 for all other bundles. Second, pick a single delay function d that not only is independent of the agent (i.e., that is symmetric) but that is also independent of the resource it is applied to. To be precise, we require Ω to be strictly larger than m multiplied with the maximal value of d. Now any bundle that is part of agent i's "allowed" set will result in a strictly positive utility (even under maximal congestion), while all other bundles will result in negative utility or at best in utility 0.This ensures that rational agents will only pick allowed bundles. Furthermore, once every agent chooses an allowed bundle, their utility will only depend on congestion in exactly the same way as in Rosenthal's model.

Rosenthal showed that every game belonging to the class of congestion games he defined must possess a pure NE. He proved this result using potential functions. Later Monderer and Shapley [24] showed that the class of finite potential games coincides with the class of Rosenthal's congestion games. Our Theorem 16 neither implies nor is implied by Rosenthal's result. On the one hand, our result applies to modular valuation functions only, while the dichotomous valuation functions (assigning either 0 or Ω to any given bundle) we have used to model Rosenthal's restricted strategy space are not modular. On the other hand, the class of games covered by our theorem is richer in other respects: it allows for arbitrary valuations given to individual resources and it allows for delay functions that are both agent- and resource-dependent.

The basic model of Rosenthal can be made more realistic in many ways. One option is to assume that each agent has a weight, and that the congestion of a resource is the weight of all agents using it (weighted congestion games). Rosenthal showed that in general, such a game need not have a pure NE. Another option is to allow for agent-specific payoffs. Milchtaich [16] introduced a model where the payoff is agent-specific. To prove that any game in that class also posses a pure NE, Milchtaich however had to restrict the strategy space of the agents to the use of a single resource (and he also required the payoff to be nonincreasing in the congestion, i.e., the delay function to be nondecreasing). We can embed Milchtaich's model into our model in the same way in which we have accommodated Rosenthal's model, namely by working with dichotomous valuations functions that map ever singleton to Ω and all other bundles to 0. Clearly, our Theorem 16 (allowing agents to claim several resources, but not applying to dichotomous valuation functions) neither implies nor is implied by Milchtaich's existence result. Later, Ackermann et al. [1] further generalised Milchtaich's model by allowing agents to use certain sets of resources (sets forming the base of a matroid), and also by allowing for weighted congestion.

Another generalisation is to consider that resources execute tasks that are sent by users. In this setting, the delay of a user may not only depend on the number of agents using that resource, but also on the order at which the tasks are executed. To minimise the delay, a user may want to send her task to multiple resources. The user may behave similarly when resources can fail, but this time to improve the likelihood that the task is performed. Each of these situations were modelled in separate games (asynchronous congestion games in [19] and congestion games with load-dependent failures [20]).

An even more general class of games possessing a pure NE is the class of games with congestion-averse utilities introduced by Byde et al [4]. This class of games is defined axiomatically and contains all the classes of games we have just described. When the valuation functions are modular, our games are also congestion-averse, although this may no longer be the case when valuations are not modular. The main tool used are two types of sequences of simple moves: the first type is called a *drop ladder* and is composed of one DROP-move followed by successive SWITCH-moves; the second type is a drop ladder followed by by a maximally profitable ADD-move. Byde et al. use these ladders both in the proof of the existence of a pure NE and in the corresponding algorithm to find those equilibria. These ladders are reminiscent of our protocols with simple moves. In a later paper, Voice et al. [26] generalised this class of games further by defining matroid congestion-averse games. The conditions of their model are shown to be necessary for guaranteeing the existence of pure NE, as relaxing some constraints can lead to games without a pure NE. This is interesting as this provides some boundary for investigating extensions of our model that are guaranteed to possess a pure NE.

7 Conclusion

We have introduced a powerful and flexible model of multiagent resource allocation with sharable items. The model integrates features from models developed in two different strands of the literature: the distributed approach to resource allocation in multiagent systems and congestion games studied in game theory. Most of our technical contributions focus on specific instances of the general model, particularly (but not exclusively) the case of allocation problems with agents that have modular valuation functions.

Our first set of results all concern conditions for the *convergence* to a social optimum by means of simple negotiation protocols. As for the previously studied case of nonsharable resources, we have seen that convergence can always be guaranteed when arbitrarily complex deals are available, and that deals involving just one resource suffice in modular domains. Unlike for nonsharable resources, in our scenario the latter type of deal may involve more than two agents, which calls for a finer analysis. We have been able to show that three simple types of deals (one in which one agent is starting using a resource; another one in which one agent is stopping using a resource; and a last one in which one agent is stopping using a resource that a different agent is starting to use) suffice when the delay functions meet certain conditions. We also have shown that the protocols can be further simplified by assuming that agents are greedy in the sense of making the most profitable deal first. We have proved the *existence* of a path to an optimum under weaker conditions. These results, summarised in Table 1, complement existing ones on convergence for different MARA scenarios and deepen our understanding of the area as a whole.

Thm	Result	Path	Val.	Delay	Symm.	Deals	Init. Alloc.	Control
3	convergence	$2^{n\cdot m}-1$	any	any	no	all	any	none
4	convergence	$O(2^n \cdot m)$	modular	any	no	1-deals	any	none
5	existence	$n \cdot m$	modular	n.d.+convex	no	ADD	empty	none
6	existence	$n \cdot m$	modular	n.d.+convex	no	DROP	full	none
7	convergence	$O(n^3 \cdot m)$	modular	n.d.+convex	yes	ADD-DROP-SWAP	any	none
8	convergence	$O(n^2 \cdot m)$	modular	n.d.+convex	yes	ADD-SWAP	empty	precedence
9	convergence	$n \cdot m$	modular	n.d.+convex	yes	ADD-SWAP	empty	greedy
10	convergence	$O(n^2 \cdot m)$	modular	n.d.+convex	yes	DROP-SWAP	full	precedence
11	convergence	$n \cdot m$	modular	n.d.+convex	yes	DROP-SWAP	full	greedy

Table 1 Reachability results and corresponding bounds on the longest path (for n agents and m resources).

Table 1 also covers our results on the maximal length of a path of deals leading to a socially optimal allocation. These bounds put our convergence results into perspective by demonstrating that, while convergence is guaranteed for general allocation problems, in practice it might require a prohibitively long sequence of deals. On the positive side of things, our convergence results for simpler instances of the framework are complemented by significantly better upper bounds on the length of paths to the optimum.

In the final part of the paper, we have focussed on a noncooperative variant of our framework, where each agent can claim and release resources at will. Allocation problems then become allocation games that are similar to congestion games studied in the literature. We have been able to show that when valuation functions are modular (a strong condition) and when delay functions are nondecreasing (a very common and unproblematic assumption), then such a game always has a Nash equilibrium in pure strategies. This ties in nicely with existing results in the literature on congestion games. We have also been able to show that any sequence of moves in which a single purely self-interested agent either claims or releases a resource will always converge to such a Nash equilibrium, which provides a connection to the convergence results for the cooperative variant of our framework studied in the main part of the paper.

Most of our results apply to particular instances of the general model for multiagent resource allocation with sharable items, by imposing relevant restrictions on valuation functions, delay functions, or both. Future work should seek to explore further such instances. For instance, we may ask what types of protocols can guarantee convergence to a social optimum if the class of potential valuation functions is neither the class of modular functions nor the class of all set functions. We may also investigate conditions for convergence to allocations that are optimal in the sense of maximising the utility of the weakest agent (egalitarian social welfare) or in the sense of being envy-free.

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