Model theory for extended modal languages

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Institute for Logic, Language and Computation
Universiteit van Amsterdam
Plantage Muidergracht 24
1018 TV Amsterdam
phone: +31-20-525 6051
fax: +31-20-525 5206
e-mail: illc@wins.uva.nl
homepage: http://www.illc.uva.nl/

Model theory for extended modal languages

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Balder David ten Cate

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Promotor: Prof.dr. J.A.G. Groenendijk
Faculteit Geesteswetenschappen
Universiteit van Amsterdam
Nieuwe Doelenstraat 15
1012 CP Amsterdam

Promotor: Prof.dr. J.F.A.K. van Benthem
Faculteit der Natuurwetenschappen, Wiskunde en Informatica
Universiteit van Amsterdam
Plantage Muidergracht 24
1018 TV Amsterdam

Co-promotor: Dr. P. Blackburn
INRIA Lorraine
615, rue du Jardin Botanique
54602 Villers ls Nancy Cedex, France

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Balder ten Cate
Drochia, January 2005

Chapter 1

Introduction

As an introduction, we will briefly discuss two themes that illustrate the context in which this thesis should be understood.

1.1 Generalized correspondence theory

It is well-known that modal logic can be viewed either as a fragment of first-order logic (when it is interpreted on models) or as a fragment of second-order logic (when it is interpreted on frames). In both cases, it is natural to ask exactly how the expressive power of modal logic and first-order (or second-order) logic compare.

From the first of the two perspectives, Van Benthem [11] proves that a first-order formula with one free variable is equivalent to (the standard translation of) a modal formula iff it is invariant for bisimulations. Put in the form of an equation:

$$\textit{modal logic} = \textit{first-order logic} \; / \; \textit{bisimulations} \tag{1.1}$$

While this result answers an important question, it also raises many questions. One set of questions is generated by fixing two parameters of the equation and asking for the correct solution. For instance,

$$x = \textit{monadic second-order logic} \; / \; \textit{bisimulations}$$

has the solution $x = \textit{modal } \mu\textit{-calculus}$ [67], and

$$\textit{tense logic} = \textit{first-order logic} \; / \; x$$

has the solution $x = \textit{two-way bisimulations}$. Likewise, one could ask for which x other than *first-order logic* the equation

$$\textit{modal logic} = x \; / \; \textit{bisimulations}$$

holds. In Chapter 12 of this thesis, it is proved that this equation also holds for x = *second-order propositional modal logic* (i.e., modal logic with propositional quantifiers).

A more interesting question, perhaps, is the following: just as (1.1) characterizes modal logic as a fragment of first-order logic, could we characterize first-order logic in terms of modal logic? In other words, does the equation

$$\textit{first-order logic} = \textit{modal logic} + x$$

have a natural solution? In this thesis, we give a positive answer. It is shown that first-order logic is the smallest extension of modal logic with nominals and the global modality that has interpolation. In other words:

$$\textit{first-order logic} = \textit{modal logic} + \textit{nominals} + \textit{global modality} + \textit{interpolation}$$

Likewise, it is shown that

$$\textit{first-order logic} = \textit{modal logic} + \textit{difference operator} + \textit{interpolation}$$

It is worth comparing these characterizations of first-order logic with Lindström's characterization [77, 8], which states that no proper extension of first-order logic has both compactness and the Löwenheim-Skolem property. Lindström's theorem characterizes first-order logic from above (i.e., as maximal with respect to certain properties), whereas our results characterizes it from below (i.e., as minimal with respect to certain properties). One may even combine the two, showing that first-order logic is the *unique* language that extends modal logic with *nominals* and the *global modality* and that has interpolation, compactness and the Löwenheim-Skolem property! [1]

A similar story can be told for modal formulas interpreted on frames. The celebrated Goldblatt-Thomason theorem [50] states that a first-order formula defines a modally definable frame class iff it is preserved under taking generated subframes, disjoint unions and bounded morphic images, and its negation is preserved under taking ultrafilter extensions. Again, this result raises many questions. To name a few:

Can the first-order formulas preserved under these frame constructions be characterized syntactically?

Van Benthem [11] gives a partial positive answer. For instance, he gives a syntactic characterization of the first-order formulas preserved under generated subframes, disjoint unions and bounded morphic images. However, an important

[1] Incidentally, the basic modal language itself can also be given a Lindström-style characterization, cf. [86].

question that has remained unanswered so far is whether the first-order formulas preserved under ultrafilter extensions can be syntactically characterized. In Chapter 2 of this thesis, we give a negative answer by showing that the first-order formulas preserved under ultrafilter extensions are not recursively enumerable.

Can we give similar characterizations for the frame classes definable in extensions of the modal language, such as with nominals or with propositional quantifiers?

Many results in this thesis can be seen as answers to this question. The frame definable power of several hybrid languages (i.e., extensions of the basic modal language involving nominals) is investigated, as well as that second order propositional modal logic (modal logic extended with propositional quantifiers). One of our results is, for instance, that an elementary frame class is definable in second order propositional modal logic iff it is closed under generated subframes and it reflects point-generatd subframes.

Conversely, an interesting line of questions is the following:

Can we find an extension of the modal language that can define precisely the elementary frame classes closed under generated subframes? Or that reflect ultrafilter extensions? ...

One answer is given in [56], where it is shown that the modal language with the global modality can define precisely the elementary frame classes closed under bounded morphic images that reflect ultrafilter extensions.

While this thesis does not contain any further answers to this question, some of its results can be seen as partial answers. In particular, our results suggest that $\mathcal{H}(\mathsf{E})$, the extension of modal logic with nominals and the global modality, comes close to defining all elementary frame classes that reflect ultrafilter extensions. Similarly, the language $\mathcal{H}(@,\downarrow)$ can define almost all elementary frame classes that are closed under generated subframes.

1.2 Hybrid logic

Given that modal logic is the bisimulation invariant fragment of a relational first-order language, one might ask what the bisimulation invariant fragment of a first-order language *with* constants is. In other words: what is the modal analogue of first-order constants? The answer is: *nominals*.

Nominals (denoted by $i, j, \ldots$) form a second sort of proposition letters, whose interpretation is required to be a singleton. In other words, nominals *name* worlds of the model. An example of a formula involving nominals is $\Diamond i \wedge \Box i$, which expresses that the world named by the nominal i is a successor of the current world, and that it is the only successor. The language obtained by adding nominals to the basic modal language, is called *the minimal hybrid language* $\mathcal{H}$.

In the presence of nominals, it is naturally to consider also another addition to the language, namely *satisfaction operators*. Satisfaction operators (denoted by $@_i, @_j, \ldots$) allow one to express that a formula holds at the world named by a nominal. For instance $@_i p$ expresses that p holds at the world named i, and $@_i \Diamond j$ expresses that the world named j is a successor of the world named i. The extension of the basic modal language with nominals and satisfaction operators is called the *basic hybrid language* $\mathcal{H}(@)$. As promised, $\mathcal{H}(@)$ is the bisimulation invariant fragment of a first-order language with constants. Of course, to make this precise one has to define bisimulations for languages containing constants. The details can be found in Chapter 4.

Besides $\mathcal{H}$ and $\mathcal{H}(@)$, a number of other hybrid languages will be studied in this thesis, most importantly $\mathcal{H}(\mathsf{E})$ and $\mathcal{H}(@, \downarrow)$. The largest part of this thesis can be seen as a detailed investigation into the model theory of these languages. We investigate expressivity, frame definability, axiomatizations, interpolation, and complexity.

Which properties of modal logics are preserved when the language is extended with nominals, satisfaction operators, etc.? And which techniques used for proving results about modal logics can still be used when facing hybrid logics?

We hope this thesis sheds light on these questions.

1.3 Overview of the thesis

With the exception of the first chapter, which discusses the basic modal language, the thesis is divided into two parts. Part I concerns the hybrid languages $\mathcal{H}, \mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$. Each chapter discusses a properties of these languages, such as expressivity, axiomatization, interpolation and complexity. Part II of the thesis discusses more expressive extensions of the basic modal language, namely the bounded fragment, the guarded fragment, relation algebra and second order propositional modal logic. Again, topics that are addressed include expressivity, axiomatization, interpolation and complexity. Figure 1.1 shows most of the languages, and how they relate in terms of expressivity.

Important topics that are not discussed in this thesis are proof theory, implementations, and real world applications.

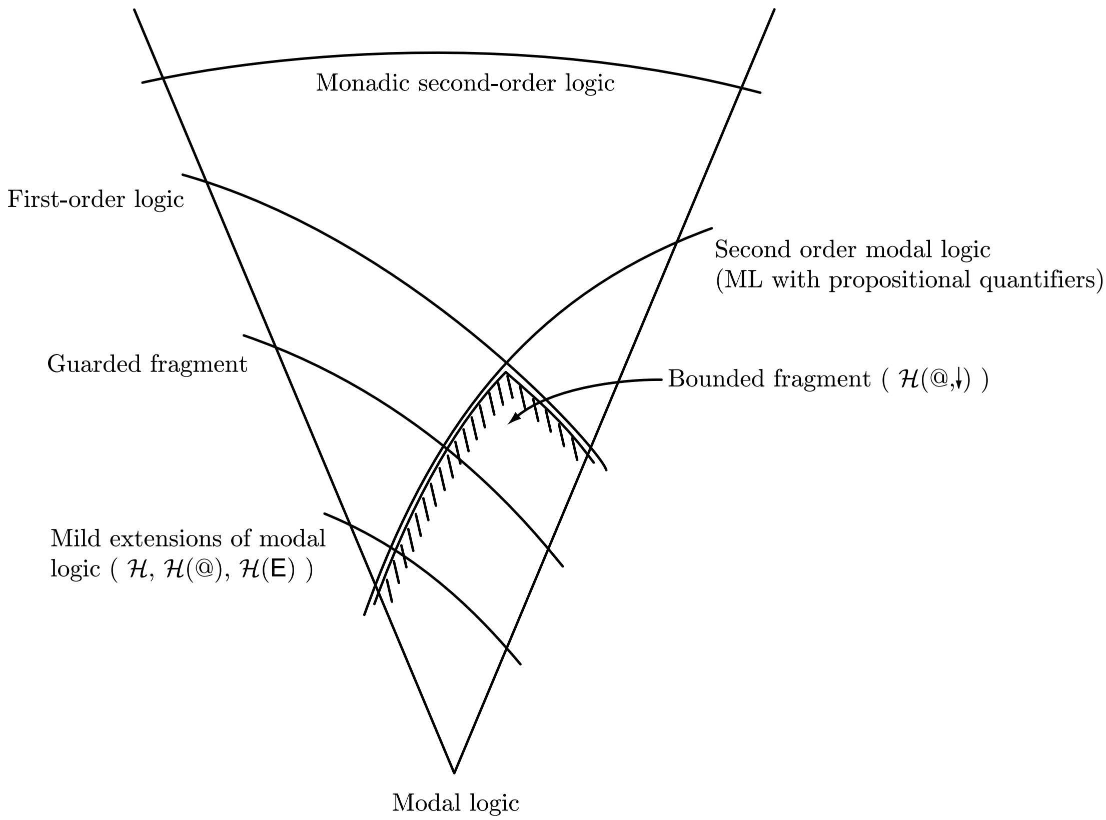

Figure 1.1: Extensions of the basic modal language

Chapter 2

Modal logic

This chapter serves two purposes. Firstly, it reviews the basic notions and results of modal logic, from a model theoretic perspective. Secondly, we prove the following new results: non-recursive enumerability of the first-order formulas preserved under ultrafilter extensions, an improvement of a general interpolation result for modal logics, and some results concernings hallow modal formulas (i.e., modal formulas in which no occurence of a proposition letter is in the scope of more than one modal operator).

2.1 Syntax and semantics

We will assume a countably infinite set of proposition letters PROP and a finite set of (unary) modalities MOD.[1] A *Kripke frame* is a pair $\mathfrak{F} = (W, (R_\diamond)_{\diamond \in \text{MOD}})$, where W is a set, called the domain of $\mathfrak{F}$, and each $R_\diamond$ is a binary relation over W. The elements of the domain of a frame are often called *worlds, states, points, nodes,* or simply *elements*. The relations $R_\diamond$ are often called *accessibility relations*. A Kripke model is a pair $(\mathfrak{F}, V)$, where $\mathfrak{F}$ is a Kripke frame, and $V : \text{PROP} \to \wp(W)$ is a *valuation* for $\mathfrak{F}$, i.e., a function that assigns to each proposition letter a subset of the domain of $\mathfrak{F}$. We will often drop the qualification "*Kripke*", and simply talk about frames and models.

The basic modal language $\mathcal{M}$ is a language that is used for describing models and frames. Its formulas are given by the following recursive definition.

$$\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi \wedge \psi \mid \diamond\varphi$$

The other connectives, such as $\Box$, will be considered shorthand notations. Given a model $\mathfrak{M} = (W, (R_\diamond)_{\diamond \in \text{MOD}}, V)$, a world $w \in W$ and a modal formula φ, truth or falsity of φ at w in $\mathfrak{M}$ is defined as follows, where $\mathfrak{M}, w \models \varphi$ expresses that φ

[1] In most parts of this thesis, we restrict attention to a finite set of unary modalities. This is only for presentational reasons, and all results we present can be generalized to infinitely many modalities and k-ary modalities ($k \geq 0$).

is true at w in $\mathfrak{M}$.

$$\begin{array}{lll} \mathfrak{M}, w \models \top & & \\ \mathfrak{M}, w \models p & \text{iff} & w \in V(p) \\ \mathfrak{M}, w \models \neg\varphi & \text{iff} & \mathfrak{M}, w \not\models \varphi \\ \mathfrak{M}, w \models \varphi \wedge \psi & \text{iff} & \mathfrak{M}, w \models \varphi \text{ and } \mathfrak{M}, w \models \psi \\ \mathfrak{M}, w \models \Diamond\varphi & \text{iff} & \text{there is a } v \in W \text{ such that } R_\Diamond(w, v) \text{ and } \mathfrak{M}, v \models \varphi \end{array}$$

We say that $\mathfrak{M}$ *globally satisfies* φ (notation: $\mathfrak{M} \models \varphi$) if $\mathfrak{M}, w \models \varphi$ for all $w \in W$. We say that φ is *valid* on a frame $\mathfrak{F}$ (notation: $\mathfrak{F} \models \varphi$) if $(\mathfrak{F}, V) \models \varphi$ for all valuations V for $\mathfrak{F}$. Dually, φ is *satisfiable* on a frame $\mathfrak{F}$ if there is a valuation V and a world w such that $\mathfrak{F}, V, w \models \varphi$. The *frame class defined by* φ is the class of all frames on which φ is valid. Finally, φ is said to be *valid* (notation $\models \varphi$) if φ is valid on all frames, and φ is said to be *satisfiable* if it is satisfiable on some frame.

The *modal depth* of a formula φ, denoted by $md(\varphi)$, is the maximal nesting of modal operators in φ. One can also give a proper inductive definition:

$$\begin{array}{lll} md(\top) & = & 0 \\ md(p) & = & 0 \\ md(\neg\varphi) & = & md(\varphi) \\ md(\varphi \wedge \psi) & = & \max\{md(\varphi), md(\psi)\} \\ md(\Diamond\varphi) & = & md(\varphi) + 1 \end{array}$$

In the remainder of this chapter, we review the model theory of the basic modal language $\mathcal{M}$, focusing on expressivity, frame definability, axiomatizations, interpolation, and decidability and complexity.

2.2 Bisimulations and expressivity on models

Bisimulation allow us to tell when two worlds in models can be distinguished by a modal formula.

2.2.1. Definition. *A bisimulation between models* $\mathfrak{M} = (W, (R_\Diamond)_{\Diamond \in \text{MOD}}, V)$ *and* $\mathfrak{N} = (W', (R'_\Diamond)_{\Diamond \in \text{MOD}}, V')$ *is a binary relation* $Z \subseteq W \times W'$ *satisfying the following conditions.*

Atom *If* wZv *then* $\mathfrak{M}, w \models p$ *iff* $\mathfrak{N}, v \models p$ *for all* $p \in \text{PROP}$

Zig *If* wZv *and* $wR_\Diamond w'$*, then there is a* $v' \in W'$ *such that* $vR'_\Diamond v'$ *and* $w'Zv'$*.*

Zag *If* wZv *and* $vR'_\Diamond v'$*, then there is a* $w' \in W$ *such that* $wR_\Diamond w'$ *and* $w'Zv'$*.*

We say that $\mathfrak{M}, w$ *and* $\mathfrak{N}, v$ *are bisimilar (notation:* $\mathfrak{M}, w \leftrightarroweq \mathfrak{N}, v$*) if there is a bisimulation* Z *between* $\mathfrak{M}$ *and* $\mathfrak{N}$ *such that* wZv*.*

Table 2.1: Standard translation from modal logic to $\mathcal{L}^1$

$$\begin{array}{lll} ST_x(\top) & = & \top \\ ST_x(p) & = & P_p(x) \\ ST_x(\neg\varphi) & = & \neg ST_x(\varphi) \\ ST_x(\varphi \wedge \psi) & = & ST_x(\varphi) \wedge ST_x(\psi) \\ ST_x(\Diamond\varphi) & = & \exists y(R(x,y) \wedge ST_y(\varphi)) \text{ for } y \text{ a variable distinct from } x \end{array}$$

Modal formulas cannot distinguish bisimilar points. In other words, if two points are bisimilar, they are modally equivalent. The converse does not hold in general, but it hold on ω-saturated models (cf. Appendix A). Let us write $\mathfrak{M}, w \equiv_{\mathcal{M}} \mathfrak{N}, v$ if for all modal formulas φ, $\mathfrak{M}, w \models \varphi$ iff $\mathfrak{N}, v \models \varphi$.

2.2.2. Theorem. *Let $\mathfrak{M}, \mathfrak{N}$ be models and w, v points in these models. If w and v are bisimilar then $\mathfrak{M}, w \equiv_{\mathcal{M}} \mathfrak{N}, v$. Conversely, if $\mathfrak{M}$ and $\mathfrak{N}$ are ω-saturated and $\mathfrak{M}, w \equiv_{\mathcal{M}} \mathfrak{N}, v$ then w and v are bisimilar.*

A proof can be found in [21].

The *first-order correspondence language* $\mathcal{L}^1$ is the first-order language with equality that contains a unary predicate P_p for each proposition letter $p \in \text{PROP}$ and a binary relation $R_\Diamond$ for each modality $\Diamond \in \text{MOD}$. Any model $\mathfrak{M} = (W, (R_\Diamond)_{\Diamond\in\text{MOD}}, V)$ can be regarded as a model for the first-order correspondence language. The accessibility relations $R_\Diamond$ are used to interpret the binary relation $R_\Diamond$ and the unary predicates P_p are interpreted as the subsets that V assigns to the corresponding proposition letter. In what follows, we will not distinguish between Kripke models and models for the first-order correspondence language, and we will continue to use the notation $\mathfrak{M} = (W, (R_\Diamond)_{\Diamond\in\text{MOD}}, V)$.

Table 2.1 presents the standard translation ST_x from the modal language to the first-order correspondence language $\mathcal{L}^1$. This translation preserves truth, in the sense that for all modal formulas φ, models $\mathfrak{M}$, and worlds w of $\mathfrak{M}$, $\mathfrak{M}, w \models \varphi$ iff $\mathfrak{M} \models ST_x(\varphi)\ [x : w]$. In this way, the standard translation shows that modal logic is a fragment of first-order logic. Bisimulations allow one to characterize exactly *which* fragment. Call an $\mathcal{L}^1$-formula $\varphi(x_1, \ldots, x_n)$ *bisimulation invariant* if for all bisimulations Z between models $\mathfrak{M}$ and $\mathfrak{N}$ and for all $(w_1, v_1), \ldots, (w_n, v_n) \in Z$, $\mathfrak{M} \models \varphi\ [w_1, \ldots, w_n]$ iff $\mathfrak{N} \models \varphi\ [v_1, \ldots, v_n]$.

2.2.3. Theorem ([11]). *Let $\varphi(x)$ be a formula of the first-order correspondence language with at most one free variable. Then the following are equivalent:*

1. *$\varphi(x)$ is invariant under bisimulations*
2. *$\varphi(x)$ is equivalent to the standard translation of a modal formula.*

Rosen [87] proved that this result holds also on finite structures.

2.3 Frame definability

When interpreted on frames, modal formulas express second order frame conditions. For instance, the modal formula $p \to \Diamond p$ expresses the frame condition $\forall x.\forall P.(Px \to \exists y.(Rxy \wedge Py))$. At it happens, this particular second order formula is equivalent to the first-order formula $\forall x.Rxx$. However, this is in general not the case. For instance, the modal formula $\Box\Diamond p \to \Diamond\Box p$ expresses a frame condition that is not definable by first-order formulas.

To be a little more precise, given a set of modal formulas Σ, the *frame class defined by* Σ is the class of all frames on which each formula in Σ is valid. A frame class is *modally definable* if there is a set of modal formulas that defines it. A frame class is *elementary* if it is defined by a sentence of the *first order frame correspondence language* $\mathcal{L}^1_{fr}$, which is the first-order language with equality and binary relation symbol for each modality.[2]

In this section, we discuss a number of result concerning the relationship between modally definable frame classes and elementary frame classes. First, we will consider model theoretic characterizations. Then, we will review some attempts at syntactic characterizations.

Model theoretic characterizations

A famous result due to Goldblatt and Thomason characterizes the modally definable elementary frame classes in terms of four operations on frames.

2.3.1. Definition (Generated subframe). *A frame* $\mathfrak{F} = (W, (R_\Diamond)_{\Diamond \in \text{MOD}})$ *is a* generated subframe *of a frame* $\mathfrak{G} = (W', (R'_\Diamond)_{\Diamond \in \text{MOD}})$ *if* $W \subseteq W'$ *and for all* $(w, v) \in R'_\Diamond$ *(*$\Diamond \in \text{MOD}$*), if* $w \in W$ *then* $v \in W$*.*

2.3.2. Definition (Disjoint union). *Let* $\mathfrak{F}_i = (W_i, (R^i_\Diamond)_{\Diamond \in \text{MOD}})$ *(*$i \in I$*) be a set of frames with disjoint domains. The disjoint union of these frames, denoted by* $\biguplus_{i \in I} \mathfrak{F}_i$ *is the frame* $(\bigcup_{i \in I} W_i, (\bigcup_{i \in I} R^i_\Diamond)_{\Diamond \in \text{MOD}})$*.*

2.3.3. Definition (Bounded morphism). *A bounded morphism from a frame* $\mathfrak{F} = (W, (R_\Diamond)_{\Diamond \in \text{MOD}})$ *to a frame* $\mathfrak{G} = (W', (R'_\Diamond)_{\Diamond \in \text{MOD}})$ *is a function* $f : W \to W'$ *satisfying the following conditions.*

forth *for all* $w, v \in W$ *and* $\Diamond \in \text{MOD}$*, if* $R_\Diamond(w, v)$ *then* $R'_\Diamond(f(w), f(v))$

back *for all* $w \in W$*,* $v \in W'$ *and* $\Diamond \in \text{MOD}$*, if* $R'_\Diamond(f(w), v)$ *then there is a* $u \in W$ *such that* $R_\Diamond(w, u)$ *and* $f(u) = v$*.*

If there is a surjective bounded morphism from $\mathfrak{F}$ *to* $\mathfrak{G}$*, then we say that* $\mathfrak{G}$ *is a* bounded morphic image *of* $\mathfrak{F}$*.*

---

[2]Note that, in the literature, a class is sometimes called elementary if it is defined by a set of first-order formulas. Here, we call a class elementary if it is defined by a single first-order sentence.

In order to formulate the fourth operation on frames, we need to introduce a piece of notation. Given a frame $\mathfrak{F} = (W, (R_\diamond)_{\diamond \in \text{MOD}})$, $X \subseteq W$ and $\diamond \in \text{MOD}$, we will write $m_\diamond(X)$ for the set $\{w \in W \mid \exists v \in X.wR_\diamond v\}$. In other words, $m_\diamond(X)$ is the set of $\diamond$-predecessors of elements of X.

2.3.4. Definition (Ultrafilter extension). *Given a frame* $\mathfrak{F} = (W, (R_\diamond)_{\diamond \in \text{MOD}})$*, the* ultrafilter extension *of* $\mathfrak{F}$*, denoted by* $\mathfrak{ue}\mathfrak{F}$*, is the frame* $(\text{Uf}(W), (R^{\mathfrak{ue}}_\diamond)_{\diamond \in \text{MOD}})$*, where* $\text{Uf}(W)$ *is the set of ultrafilters over* W *(cf. Appendix A), and for* $u, v \in \text{Uf}(W)$*,* $R^{\mathfrak{ue}}_\diamond(u, v)$ *iff for all* $X \in v$*,* $m_\diamond(X) \in u$*.*

Every modally definable frame class is closed under disjoint unions, generated subframes and bounded morphic images. Furthermore, modally definable frame classes reflect ultrafilter extensions, meaning that whenever the ultrafilter extension of a frame is in the class, then the frame itself is in the class. Goldblatt and Thomason proved that the converse holds with respect to elementary frame classes.

2.3.5. Theorem (Goldblatt-Thomason[50]). *An elementary frame class is modally definable iff it is closed under generated subframes, disjoint unions and bounded morphic images, and reflects ultrafilter extensions.*

This tells us which elementary frame classes are modally definable. The opposite question, i.e., which modally definable frame classes are elementary, was answered by Van Benthem.

2.3.6. Theorem ([10]). *Let* K *be any modally definable frame class. The following are equivalent:*

1. K *is elementary*
2. K *is defined by a set of first-order sentences*
3. K *is closed under elementary equivalence*
4. K *is closed under ultrapowers.*

Syntactic characterizations

The above results do not tell us which modal formulas define an elementary frame class, nor which first-order formulas define a modally definable frame class.

As we will soon see (cf. Theorem 2.6.5), the problem whether a given modal formula defines an elementary frame class is highly undecidable. This implies that a syntactic characterization of the form "*a modal formula defines an elementary class iff it is equivalent to a formula of the form X*" with X a decidable class of formulas cannot be obtained. However, this still leaves open the question whether such a characterization exists if *equivalent* is replaced by *frame-equivalent.*

An important sufficient condition for elementarity was proved by Sahlqvist [88] and Van Benthem [11].

2.3.7. Definition (Sahlqvist formulas). *A modal formula is* positive *(*negative*) if every occurrence of a proposition letter is under the scope of an even (odd) number of negation signs.*

A Sahlqvist antecedent *is a formula built up from* $\top, \bot$*, boxed atoms of the form* $\Box_1 \cdots \Box_n p$ $(n \geq 0)$*, and negative formulas using conjunction, disjunction and diamonds.*

A Sahlqvist implication *is a formula of the form* $\varphi \to \psi$*, where* φ *is a Sahlqvist antecedent and* ψ *is positive.*

A Sahlqvist formula *is a formula that is obtained from Sahlvist implications by applying boxes and conjunction, and by applying disjunctions between formulas that do not share any proposition letters.*

2.3.8. Theorem ([88, 11]). *Every Sahlqvist formula defines an elementary class of frames.*

Likewise, Van Benthem [11] has shown that every modal formula that has modal depth at most one defines an elementary class of frames. Axioms of modal depth at most one were first considered by Lewis [76]. Van Benthem's result may be improved slightly, by considering the following class of formulas.

2.3.9. Definition (Shallow formulas). *A modal formula is* shallow *if every occurrence of a proposition letter is in the scope of at most one modal operator.*

2.3.10. Theorem. *Every shallow formula defines an elementary class of frames.*

Proof: The proof will be given in Section 2.4. □

Typical examples of shallow modal formulas are $p \to \Diamond p$, $\Diamond p \to \Box p$ and $\Diamond_1 p \to \Diamond_2 p$. Furthermore, every closed formula (i.e., formula containing no proposition letters) is shallow. The formula $\Box_1(p \vee q) \to \Diamond_2(p \wedge q)$ is an example of a shallow formula that is not a Sahlqvist formula.

Incidentally, correspondence results like these might also be obtained for languages other than the first-order correspondence language. Recently, [14] and [57] have independently found a generalization of the class of Sahlqvist formulas, with the property that every generalized Sahlqvist formula has a correspondent in LFP(FO), which is the extension of first-order logic with least fixed point operators. By results of [6], there are modal formulas that have no correspondent in LFP(FO), not even with respect to finite frames.

Next, let us address the question which first-order formulas define modally definable frame conditions. Again, no complete syntactic characterization is known.

Let a *p-formula* be a first-order formula obtained from atomic formulas (including equality statements) using conjunction, disjunction, existential and universal quantifiers, and bounded universal quantifiers of the form $\forall x(Rtx \to \cdot)$. A

Table 2.2: Formula that characterizes $(\mathbb{N}, <)$

$\forall x \forall y (x < y \rightarrow \forall z (y < z \rightarrow x < z))$	(transitivity)
$\forall xy (x < y \vee y < x \vee x = y)$	(trichotomy)
$\forall x \exists y (x < y)$	(unboundedness on the right)
$\exists x \forall y (y < x \rightarrow \bot)$	(boundedness on the left)
$\exists x (x < x) \rightarrow \exists xy (x < x \wedge x < y \wedge \neg (y < y))$	

p-sentence is a p-formula that is a sentence. An inductive argument shows that p-sentences are preserved under taking images of bounded morphisms. In fact, the converse holds as well, modulo logical equivalence.

2.3.11. Theorem (Feferman [39]). *A first-order sentence φ is preserved under surjective bounded morphisms iff φ is equivalent to a p-sentence.*

It follows that if a first-order sentence defines a modally definable frame class, then it is equivalent to a p-sentence. We can improve this a bit further. Let a *positive restricted formula* be a first-order formula built up from $\bot$ and atomic formulas, using conjunction, disjunction, and restricted quantification of the from $\exists y.(Rxy \wedge \cdot)$ and $\forall y.(Rxy \rightarrow \cdot)$, where x and y are distinct variables.

2.3.12. Theorem (Van Benthem [11]). *A first-order sentence φ is preserved under surjective bounded morphisms, generated subframes and disjoint unions iff φ is equivalent to $\forall x.\psi(x)$, for some positive restricted formula $\psi(x)$.*

Again, it follows that if a first-order sentence defines a modally definable frame class, it is equivalent to a sentence of the given form. What remains in order to obtain a complete characterization is to characterize anti-preservation under ultrafilter extensions. It is possible to give a preservation result similar to the above, that characterizes the first-order sentences (anti-)preserved under ultrafilter extensions? As we will now show, the answer is *No*.

Let $\vartheta_{(\mathbb{N},<)}$ be the conjunction of the formulas given in Table 2.2. Surprisingly, $\vartheta_{(\mathbb{N},<)}$ characterizes $(\mathbb{N}, <)$, in the sense that it is preserved under taking ultrafilter extensions precisely in case the original model is *not* isomorphic to $(\mathbb{N}, <)$.

2.3.13. Proposition. *For all models $\mathfrak{M}$, $\mathfrak{M} \cong (\mathbb{N}, <)$ iff $\mathfrak{M} \models \vartheta_{(\mathbb{N},<)}$ and $\mathfrak{ue}\mathfrak{M} \not\models \vartheta_{(\mathbb{N},<)}$.*

Proof: The left-to-right direction simply says that $(\mathbb{N}, <) \models \vartheta_{(\mathbb{N},<)}$ and $\mathfrak{ue}(\mathbb{N}, <) \not\models \vartheta_{(\mathbb{N},<)}$. That $(\mathbb{N}, <) \models \vartheta_{(\mathbb{N},<)}$ is clear. Now, consider the ultrafilter extension $\mathfrak{ue}(\mathbb{N}, <) = (\mathrm{Uf}(\mathbb{N}), <^{\mathfrak{ue}})$. As pointed out in [21, Example 2.58], this model consists of an isomorphic copy of the natural numbers, followed by an uncountable cluster containing all non-principal ultrafilters. In particular, for all non-principal ultrafilter u, $u <^{\mathfrak{ue}} u$. This implies that the antecedent of the fifth conjunct of $\vartheta_{(\mathbb{N},<)}$ is true in $\mathfrak{ue}(\mathbb{N},<)$. The consequent of this formula is clearly false (all non-principal ultrafilters are to the right of the principal ultrafilters). Hence, $\mathfrak{ue}(\mathbb{N},<) \not\models \vartheta_{(\mathbb{N},<)}$.

As for the right-to-left direction, suppose $\mathfrak{M} \models \vartheta_{(\mathbb{N},<)}$ and $\mathfrak{M} \not\cong (\mathbb{N},<)$. The first four conjuncts of $\vartheta_{(\mathbb{N},<)}$ express modally definable elementary frame properties (definable using the global modality and converse modalities, if needed), and hence, by a result of Van Benthem [11], are preserved under ultrafilter extensions. Hence, they are true in $\mathfrak{ue}\mathfrak{N}$. As for the fifth conjunct, we can distinguish two cases.

1. $\mathfrak{M} \models \exists xy.(x < x \wedge x < y \wedge y \not< y)$. Since this formula has no universal quantifiers, it is preserved under extensions. As $\mathfrak{ue}\mathfrak{M}$ is an extension of $\mathfrak{M}$, it follows that $\mathfrak{ue}\mathfrak{M} \models \exists xy.(x < x \wedge x < y \wedge \neg(y < y))$, and therefore $\mathfrak{ue}\mathfrak{M} \models \vartheta_{(\mathbb{N},<)}$.

2. $\mathfrak{M} \not\models \exists x.(x < x)$. Then $\mathfrak{M} = (D,<)$ for some set D and strict total order $<$ that is bounded on the left but unbounded on the right. If it would be the case that every point has only finitely many predecessors, $\mathfrak{M}$ would be isomorphic to $(\mathbb{N},<)$. By assumption, this is not the case. Hence, there is a point w for which there are infinitely many v such that $v < w$. Let S be the set of all predecessors of w.

   Now, consider the ultrafilter extension $\mathfrak{ue}\mathfrak{M} = (\mathrm{Uf}(D), <^{\mathfrak{ue}})$. Let π_w be the principal ultrafilter generated by w, and let u be a non-principal ultrafilter with $S \in u$ (such u exist since S is infinite). By construction, $u <^{\mathfrak{ue}} \pi_w$ and $\pi_w \not<^{\mathfrak{ue}} \pi_w$. Furthermore, $u <^{\mathfrak{ue}} u$. To see this, take any $X \in u$, and consider the set $Y = \{v \mid \exists x \in X.(v < x)\}$ It is easy to see that at most one element of X is not in Y, i.e., $|X \cap (D \setminus Y)| \leq 1$. Since u is non-principal, it follows that $D \setminus Y \notin u$, and therefore $Y \in u$.

   Thus, we have shown that $\mathfrak{ue}\mathfrak{M} \models \exists xy.(x < x \wedge x < y \wedge \neg(y < y))$, and thereby $\mathfrak{ue}\mathfrak{M} \models \vartheta_{(\mathbb{N},<)}$. □

We can still improve this result a bit. Consider the formula

$$\forall x \exists y.Sxy \ \wedge \ \forall x \forall y(Sxy \rightarrow x < y) \ \wedge \ \forall x \forall y(Sxy \rightarrow \forall z(x < z \rightarrow y = z \vee y < z))$$

This formula is preserved under ultrafilter extensions, and, on the natural numbers, it defines the successor relation (i.e., it expresses that Smn holds iff $n = m + 1$). Hence, if we let $\vartheta_{(\mathbb{N},<,Suc)}$ be the conjunction of $\vartheta_{(\mathbb{N},<)}$ and this formula, then we immediately obtain the following corollary and improvement of Proposition 2.3.13.

2.3.14. Proposition. *For all models $\mathfrak{M}$, $\mathfrak{M} \cong (\mathbb{N},<,Suc)$ iff $\mathfrak{M} \models \vartheta_{(\mathbb{N},<,Suc)}$ and $\mathfrak{ue}\mathfrak{M} \not\models \vartheta_{(\mathbb{N},<,Suc)}$.*

In fact, unary predicates $Zero$ and One and ternary relations $Plus$ and $Times$ (with the intended semantics) can be defined in a similar way, leading to a characterization à la Proposition 2.3.13 of the structure $(\mathbb{N}, <, Suc, 0, 1, +, \times)$. We will not give the details here. For present purposes, the following corollary of Proposition 2.3.14 is important.

2.3.15. Corollary. *Let φ by any relational first-order formula preserved under ultrafilter extensions (possibly containing relation symbols other than $<$ and S). The following are equivalent.*

1. *φ has a model that is an expansion of* $(\mathbb{N}, <, Suc)$
2. *$\varphi \wedge \vartheta_{(\mathbb{N},<,Suc)}$ is not preserved under ultrafilter extensions*

Again, we can improve this result slighly. Let $\vartheta^N_{(\mathbb{N},<,Suc)}$ be the result of relativising all quantifiers in $\vartheta_{(\mathbb{N},<,Suc)}$ with the unary predicate N (i.e., replacing subformulas of the form $\exists x.\psi$ by $\exists x.(Nx \wedge \psi)$ and subformulas of the form $\forall.\psi$ by $\forall x.(Nx \to \psi)$). It is not hard to see that a formula φ is preserved under ultrafilter extensions iff the relativisation φ^N is preserved under ultrafilter extensions, provided that N does not occur in φ. Hence, we obtain the following relativized version of Corollary 2.3.15.

2.3.16. Corollary. *Let φ by any relational first-order formula preserved under ultrafilter extensions (possibly containing relation symbols other than $<$, S and N). The following are equivalent.*

1. *φ has a model, of which the submodel defined by N is an expansion of* $(\mathbb{N}, <, Suc)$
2. *$\varphi \wedge \vartheta^N_{(\mathbb{N},<,Suc)}$ is not preserved under ultrafilter extensions*

Finally, we will use Corollary 2.3.16 to prove that the set of first-order formulas preserved under ultrafilter extensions is Π^1_1-hard.

2.3.17. Theorem. *Preservation of first-order formulas under ultrafilter extensions is Π^1_1-hard.*

Proof: We will make use of the Σ^1_1-complete recurrent tiling problem of Harel, cf. Appendix B. For any set of tiles $T = \{t_1, \ldots, t_n\}$ and designated tile $t_i \in T$, let $\varphi_{(T,t_i)}$ be the conjunction of formulas in Table 2.3, where $P_1, \ldots, P_n$ are unary predicates representing the tiles $t_1, \ldots, t_n$, and R_h, R_v as binary relation symbols. The following are equivalent.

1. $T_1, \ldots, T_n$ tile $\mathbb{N} \times \mathbb{N}$ such that t_i occurs infinitely often on the first row
2. $\varphi_{(T,t_i)}$ has a model, of which the submodel defined by N is an expansion of $(\mathbb{N}, <, Suc)$

Table 2.3: Encoding the the recurrent tiling problem

Two dimensional grid (modulo unwinding)

$$\forall x \exists y. R_h(x,y) \ \wedge \ \forall x \exists y. R_v(x,y)$$
$$\forall x \forall y (R_h(x,y) \to \forall z (R_h(x,z) \to y = z))$$
$$\forall x \forall y (R_v(x,y) \to \forall z (R_v(x,z) \to y = z))$$
$$\forall x \forall y (R_h(x,y) \to \forall z (R_v(x,z) \to \exists u. (R_v(y,u) \wedge R_h(z,u))))$$

Correct tiling

$$\forall x. \bigvee_{1 \leq k \leq n} \Big(P_k\, x \ \wedge \bigwedge_{\substack{1 \leq \ell \leq n \\ \ell \neq k}} \neg P_\ell\, x \Big)$$
$$\forall x \forall y. \Big(R_h(x,y) \to \bigvee_{(t_k)_{right} = (t_\ell)_{right}} (P_k\, x \wedge P_\ell\, y) \Big)$$
$$\forall x \forall y. \Big(R_v(x,y) \to \bigvee_{(t_k)_{top} = (t_\ell)_{bottom}} (P_k\, x \wedge P_\ell\, y) \Big)$$

Recurrence of tile t_i in the submodel defined by N

$$\forall x \forall y (R_h(x,y) \to (Nx \to (Ny \wedge Sxy)))$$
$$\forall x (Nx \to \exists y. (Ny \wedge x < y \wedge P_i\, y))$$

Table 2.4: Axioms and inference rules of $\mathbf{K}_{\mathcal{M}}$

(CT)	$\vdash \varphi$, for all classical tautologies φ
$(Dual)$	$\vdash \Diamond p \leftrightarrow \neg\Box\neg p$, for $\Box \in$ MOD
(K)	$\vdash \Box(p \to q) \to \Box p \to \Box q$, for $\Box \in$ MOD
(MP)	If $\vdash \varphi \to \psi$ and $\vdash \varphi$ then $\vdash \psi$
(Nec)	If $\vdash \varphi$ then $\vdash \Box\varphi$, for $\Box \in$ MOD
$(Subst)$	If $\vdash \varphi$ then $\vdash \varphi\sigma$, where σ is a substitution that uniformly replaces proposition letters by formulas.

3. $\varphi_{(T,t_i)} \wedge \vartheta^N_{(\mathbb{N},<,Suc)}$ is not preserved under ultrafilter extensions

The equivalence of (1) and (2) is relatively easy to see, and the equivalence between (2) and (3) follows from Corollary 2.3.16, since $\varphi_{(T,t_i)}$ is preserved under ultrafilter extensions (by the same reasoning as before: they express modally definable elementary frame properties). It follows that preservation of first-order formulas under ultrafilter extensions is Π^1_1-hard. □

In particular, it follows that the first-order sentences (anti-)preserved under ultrafilter extensions are not recursively enumerable, and cannot be characterized by means of a preservation theorem.

2.4 Completeness via general frames

Given a frame class K, one would like to describe the set of modal formulas valid on K ("the modal logic of K"). For the class of all frames, the axioms and inferences rules given in Table 2.4 constitute a sound and complete axiomatization. We will refer to this axiomatization as $\mathbf{K}_{\mathcal{M}}$. We will write $\vdash_{\mathbf{K}_{\mathcal{M}}} \varphi$ if φ is derivable in $\mathbf{K}_{\mathcal{M}}$.

2.4.1. Theorem (Basic Completeness). *For all modal formulas φ, $\models \varphi$ iff $\vdash_{\mathbf{K}_{\mathcal{M}}} \varphi$.*

Thus, $\mathbf{K}_{\mathcal{M}}$ axiomatizes the set of modal formulas valid on the class of all frames. In order to axiomatize more restricted frame classes, extra axioms (or rules) must be added to $\mathbf{K}_{\mathcal{M}}$. For any set Σ of modal formulas, we will use $\mathbf{K}_{\mathcal{M}}\Sigma$ to denote the axiomatization obtained by adding all formulas in Σ as axioms to $\mathbf{K}_{\mathcal{M}}$. One might hope that $\mathbf{K}_{\mathcal{M}}\Sigma$ completely axiomatizes the set of modal formulas valid on the frame class defined by Σ. Unfortunately, this is in general not the case. Nevertheless, there are natural classes of modal formulas, for which such a general completeness result can be obtained.

In order to facilitate the study of completeness and incompleteness, it is convenient to introduce a generalization of the notion of frames. A *general frame* consists a frame $\mathfrak{F} = (W, (R_\Diamond)_{\Diamond\in\text{MOD}})$ together with a set $\mathbb{A} \subseteq \wp(W)$ satisfying certain regularity conditions, to be spelled out below. The elements of $\mathbb{A}$ are called *admissible subsets*. A modal formula φ containing proposition letters $p_1, \ldots, p_n$ is said to be valid on a such a general frame if it is valid under any valuation that assigns admissible subsets to $p_1, \ldots, p_n$. Note that the ordinary frames, or *Kripke frames*, as we will refer to them in this section, are simply general frames for which the set of admissible subsets is the set of all subsets.

Recall that, given a frame $\mathfrak{F} = (W, (R_\Diamond)_{\Diamond \in \text{MOD}})$, $X \subseteq W$ and $\Diamond \in \text{MOD}$, $m_\Diamond(X) = \{w \in W \mid \exists v \in X.wR_\Diamond v\}$.

2.4.2. Definition (General frames). *A general frame is a pair* $(\mathfrak{F}, \mathbb{A})$, *where* $\mathfrak{F} = (W, (R_\Diamond)_{\Diamond \in \text{MOD}})$ *is a frame and* $\mathbb{A} \subseteq \wp(W)$, *such that* $W \in \mathbb{A}$ *and* $\mathbb{A}$ *is closed under complement, finite intersection and* $m_\Diamond$ *for* $\Diamond \in \text{MOD}$.

In addition, the general frame $(\mathfrak{F}, \mathbb{A})$ *is*

differentiated *if for all* $w, v \in W$ *with* $w \neq v$ *there is an* $A \in \mathbb{A}$ *such that* $w \in A$ *and* $v \notin A$

tight *if for all* $w, v \in W$ *and* $\Diamond \in \text{MOD}$ *such that* $(w, v) \notin R_\Diamond$ *there is an* $A \in \mathbb{A}$ *such that* $v \in A$ *and* $w \notin m_\Diamond(A)$

compact *if every* $\mathbb{A}' \subseteq \mathbb{A}$ *with the finite intersection property has a non-empty intersection*

refined *if it is differentiated and tight*

descriptive *if it is differentiated, tight and compact*

discrete *if for all* $w \in W$, $\{w\} \in \mathbb{A}$

atomless *if for no* $w \in W$, $\{w\} \in \mathbb{A}$

A valuation for a general frame $\mathfrak{F}$ is admissible if $V(p) \in \mathbb{A}$ for all $p \in \text{PROP}$. Validity with respect to general frames is defines as follows: $\mathfrak{F} \models \varphi$ if for all admissible valuations V and worlds w, $(\mathfrak{F}, V), w \models \varphi$. Every set Γ of modal formulas defines a class of general frames, namely the class consisting of those general frames on which each formula in Γ is valid.

Unlike Kripke frames, general frames offer a fully adequate semantics for modal logics, in the sense that for all sets Γ of modal formulas, $\mathbf{K}_\mathcal{M}\Gamma$ completely axiomatizes the set of modal formulas valid on the class of general frames defined by Γ. In fact, this holds even if we restrict attention to descriptive frames. Given a set of modal formulas Γ and a class K of general frames, we say that $\mathbf{K}_\mathcal{M}\Gamma$ is complete for K if $\mathbf{K}_\mathcal{M}\Gamma$ completely axiomatizes the set of modal formulas valid on K, i.e., for all φ, $\mathsf{K} \models \varphi$ iff $\vdash_{\mathbf{K}_\mathcal{M}\Gamma} \varphi$.

2.4.3. Theorem ([52]). *Let* Γ *be any set of modal formulas.* $\mathbf{K}_\mathcal{M}\Gamma$ *is complete for the class of descriptive general frames defined by* Γ.

Of course, our actual interest is not in general frames but in Kripke frames. Theorem 2.4.3 can be seen as an important first step towards proving Kripke completeness. The second step typically involves persistence, a notion that will be defined next.

2.4.4. Definition. *A modal formula* φ *is* persistent *with respect to a type of general frames (such as descriptive general frames, etc.) if for all general frames* $\mathfrak{F}$ *of the relevant type, if* $\mathfrak{F} \models \varphi$ *then* φ *is valid on the underlying Kripke frame of* $\mathfrak{F}$.

Persistence with respect to descriptive frame is also called *d-persistence*, or *canonicity*. Persistence with respect to discrete frames is often called *di-persistence*.

Recall the definition of Sahlqvist formulas on page 12. An important result in modal logic is the following.

2.4.5. Theorem ([88]). *Every modal Sahlqvist formula is persistent with respect to descriptive general frames.*

If we put Theorem 2.4.3 and Theorem 2.4.5 together, we obtain the following Kripke completeness result for Sahlqvist formulas.

2.4.6. Corollary ([88]). *If* Γ *is a set of Sahlqvist formulas, then* $\mathbf{K}_{\mathcal{M}}\Gamma$ *is complete for the class of Kripke frames defined by* Γ.

A similar result can be proved for shallow formulas. Recall that a modal formula is shallow if every occurrence of a proposition letter is under the scope of at most one modal operator.

2.4.7. Theorem. *Every shallow formula is persistent with respect to refined frames, and hence with respect to descriptive frames and with respect to discrete frames.*

Proof sketch: The proof proceeds by contraposition. Let $\mathfrak{F}$ be a refined general frame and suppose $\mathfrak{F}, V, w \not\models \varphi$, where φ is a shallow modal formula, V a not necessarily admissible valuation and w a world. We will construct an admissible valuation V' such that $\mathfrak{F}, V', w \not\models \varphi$, thus showing that $\mathfrak{F} \not\models \varphi$.

Let $\chi_1, \ldots, \chi_n$ be the closed subformulas of φ and let $p_1, \ldots, p_m$ be the proposition letters occurring in φ. In what follows, σ will always refer to a complete Boolean combination of $\chi_1, \ldots, \chi_n$, i.e., a formula of the form $(\neg)\chi_1 \wedge \cdots \wedge (\neg)\chi_n$, and τ will always refer to a complete Boolean combination of $p_1, \ldots, p_m$. We may in fact assume that φ is a Boolean combination of formulas of the form $\sigma \wedge \tau$ or $\Diamond(\sigma \wedge \tau)$. Let W_σ, W_τ and $W_{\sigma\tau}$ denote the subsets of the domain of $\mathfrak{F}$ defined by σ, τ and $\sigma \wedge \tau$, respectively, under the valuation V. Also, for $\Diamond$ one of the (finitely many) modalities occuring in φ, let $Suc_w^\Diamond$ denote the set of $\Diamond$-successors of w.

Fix any σ, and consider the set W_σ. Since σ is a closed formula, W_σ is admissible. The proposition letters $p_1, \ldots, p_m$ partition W_σ into 2^m disjoint (possibly empty and not necessarily admissible) subsets $W_{\sigma\tau}$ (with τ any complete Boolean combination of $p_1, \ldots, p_m$). For each such subset $W_{\sigma\tau}$, and for each modality $\Diamond$

with $W_{\sigma\tau} \cap Suc_w^\Diamond \neq \emptyset$, pick a witness of the non-emptyness of this intersection. Furthermore, if $w \in W_{\sigma\tau}$ for some τ, then add w as a witness. In this way, we pick finitely many witnesses for each $W_{\sigma\tau}$. By the differentiatedness of $\mathfrak{F}$, we can find for each τ and τ' an admissible set that separates the witnesses for $W_{\sigma\tau}$ from the witnesses for $W_{\sigma\tau'}$. Also, by the tightness of $\mathfrak{F}$, we can find for each τ and for each modality $\Diamond$ such that $W_{\sigma\tau} \cap Suc_w^\Diamond = \emptyset$ an admissible set that contains all witnesses of $W_{\sigma\tau}$ but that contains no $\Diamond$-successor of w. By taking appropriate intersections and unions of these admissible sets (and intersecting with W_σ), we obtain a new partition of W_σ into admissible subsets $W'_{\sigma\tau}$, such that each witness for a $W_{\sigma\tau}$ is still a member of $W'_{\sigma\tau}$. Hence,

- $W_{\sigma\tau} \cap Suc_w^\Diamond = \emptyset$ iff $W'_{\sigma\tau} \cap Suc_w^\Diamond = \emptyset$
- $w \in W_{\sigma\tau}$ iff $w \in W'_{\sigma\tau}$

Using these new partitions, we will now define a admissible valuation V'. For each proposition letter p_k ($k \leq m$), let $V'(p_k)$ be the union of all $W'_{\sigma\tau}$ with $\tau \models p_k$. By construction, V' is an admissible valuation, and $\mathfrak{F}, V, w$ and $\mathfrak{F}, V', w$ agree on φ. It follows that $\mathfrak{F}, V', w \not\models \varphi$, and hence $\mathfrak{F} \not\models \varphi$. □

Again, we obtain Kripke completeness as a corollary.

2.4.8. Corollary. *If Γ is a set of shallow formulas, then $\mathbf{K}_\mathcal{M}\Gamma$ is complete for the class of Kripke frames defined by Γ.*

In fact, combining Theorem 2.4.3, 2.4.5 and 2.4.7, we obtain completeness of $\mathbf{K}_\mathcal{M}\Gamma$ for all sets Γ consisting of shallow and/or Sahlqvist formulas.

Incidentally, every modal formula that is persistent with respect to refined frames defines an elementary frame class [73]. Hence, this also proves Theorem 2.3.10.

To finish this section, we briefly consider discrete general frames. Venema [98] proved the following persistence result with respect to discrete general frames.

2.4.9. Definition (Very simple Sahlqvist formulas). *A* very simple Sahlqvist antecedent *is a modal formula built up from $\top, \bot$ and proposition letters using conjunction and diamonds. A* very simple Sahlqvist formula *is an implication $\varphi \to \psi$, where φ is a very simple Sahlqvist antecedent and ψ is positive.*

2.4.10. Theorem ([98]). *Every very simple Sahlqvist formula is persistent with respect to discrete frames.*

This by itself does not imply completeness for logics axiomatized by very simple Sahlqvist formulas (even though this follows from Theorem 2.4.5). The reason is that $\mathbf{K}_\mathcal{M}\Gamma$ might not be complete for the class of discrete general frames defined by Γ. In other words, there is no analogue of Theorem 2.4.3 for discrete general frames. Indeed, Venema [98] proved the following strong incompleteness result.

2.4.11. THEOREM ([98]). *There is a modal formula φ such that $\mathbf{K}_{\mathcal{M}}\{\varphi\}$ is consistent and every general frame on which φ is valid is atomless.*

It follows that for the relevant formula φ, $\mathbf{K}_{\mathcal{M}}\{\varphi\}$ is incomplete with respect to the class of discrete frames defined by φ. Incidentally, the formula φ used by [98] contains more than one modality. This is necessarily so: an observation due to Makinson implies that, for all uni-modal formulas φ, if $\mathbf{K}_{\mathcal{M}}\{\varphi\}$ is consistent then it has a general frame whose domain is a singleton set. Clearly every such general frame is discrete.

2.5 Interpolation and Beth definability

Analogues of Craig's interpolation theorem have been proved for many modal logics. For any modal formula φ, let $\textsc{prop}(\varphi)$ is the set of proposition letters occurring in φ. Further, let us say that the basic modal language has interpolation on a frame class K, if for all modal formulas φ, ψ such that $\mathsf{K} \models \varphi \to \psi$, there is a modal formula ϑ such that $\mathsf{K} \models \varphi \to \vartheta$ and $\mathsf{K} \models \vartheta \to \psi$, and $\textsc{prop}(\vartheta) \subseteq \textsc{prop}(\varphi) \cap \textsc{prop}(\psi)$. Note that no restriction is made on the modalities occurring in ϑ. It would therefore be more appropriate to talk about *interpolation over proposition letters*, indicating that it is only the proposition letters in the interpolation that must occur both in the antecedent and in the consequent.

2.5.1. DEFINITION. *A bisimulation product of a set of frames $\{\mathfrak{F}_i \mid i \in I\}$ is a subframe $\mathfrak{G}$ of the cartesian product $\Pi_i \mathfrak{F}_i$ such that for each $i \in I$, the natural projection function $f_i : \mathfrak{G} \to \mathfrak{F}_i$ is a surjective bounded morphism.*

Bisimulation products are a special case of subdirect products (for the definition of cartesian products and subdirect products, see Appendix A). Their name is motivated by the following observation:

2.5.2. PROPOSITION ([80]). *Let $\mathfrak{H}$ be a submodel of the product $\mathfrak{F} \times \mathfrak{G}$. Then $\mathfrak{H}$ is a bisimulation product of $\mathfrak{F}$ and $\mathfrak{G}$ iff the domain of $\mathfrak{H}$ is a total frame bisimulation between $\mathfrak{F}$ and $\mathfrak{G}$.*

Here, with a total frame bisimulation between the frames $\mathfrak{F}$ and $\mathfrak{G}$ we mean a binary relation Z between the domains of $\mathfrak{F}$ and $\mathfrak{G}$ satisfying the *zig* and *zag* conditions of Definition 2.2.1, and such that for each world w of $\mathfrak{F}$ there is a world v of $\mathfrak{G}$ such that wZv, and vice versa.

We say that a class of frames K is *closed under bisimulation products* if for all $\mathfrak{F}, \mathfrak{G} \in \mathsf{K}$, all bisimulation products of $\mathfrak{F}$ and $\mathfrak{G}$ are in K. It was proved in [80] that if a frame class K is defined by a set of d-persistent modal formulas and closed under bisimulation products, then the basic modal language has interpolation relative to K. Here, we will slightly strengthen this result.[3]

[3]Strictly speaking, Theorem 2.5.3 is not a strengthening of the result of [80], since there are canonical modal formulas that define a non-elementary frame class [42].

2.5.3. Theorem (Interpolation for modal logics). *Let* K *be any elementary frame class closed under generated subframes and bisimulation products. Then the basic modal language has interpolation relative to* K.

Proof: Let K be any elementary frame class closed under generated subframes and bisimulation products, let $\mathsf{K} \models \varphi \rightarrow \psi$, and suppose for the sake of contradiction that there is no interpolant for this implication. Let $Cons(\varphi)$ be the set of modal formulas χ such that $\mathsf{K} \models \varphi \rightarrow \chi$ and $\textsc{prop}(\chi) \subseteq \textsc{prop}(\varphi) \cap \textsc{prop}(\psi)$.

> **Claim 1:** There is a model $\mathfrak{M}$ based on a frame in K, with a world w, such that $\mathfrak{M}, w \models Cons(\varphi) \cup \{\neg\psi\}$.
>
> **Proof of claim:** By Compactness, it suffices to show that every finite subset of $Cons(\varphi) \cup \{\neg\psi\}$ is satisfiable on K. Consider any $\chi_1, \ldots, \chi_n \in Cons(\varphi)$. If $\{\chi_1, \ldots, \chi_n, \neg\psi\}$ wouldn't be satisfiable on K, then $\chi_1 \wedge \cdots \wedge \chi_n$ would be an interpolant for $\varphi \rightarrow \psi$. By assumption, $\varphi \rightarrow \psi$ has no interpolant, and therefore, $\{\chi_1, \ldots, \chi_n, \neg\psi\}$ is satisfiable on K. $\dashv$

Since K is closed under generated subframes, we may assume that $\mathfrak{M}$ is generated by w. Let $Th(\mathfrak{M}, w)$ be the set of all modal formulas χ such that $\mathfrak{M}, w \models \chi$ and $\textsc{prop}(\chi) \subseteq \textsc{prop}(\varphi) \cap \textsc{prop}(\psi)$.

> **Claim 2:** There is a model $\mathfrak{N}$ based on a frame in K, with a world v, such that $\mathfrak{N}, v \models Th(\mathfrak{M}, w) \cup \{\varphi\}$.
>
> **Proof of claim:** By Compactness, it suffices to show that every finite subset of $Th(\mathfrak{M}, w) \cup \{\varphi\}$ is satisfiable on K. Consider any $\chi_1, \ldots, \chi_n \in Th(\mathfrak{M}, w)$. Suppose for the sake of contradiction that $\{\chi_1, \ldots, \chi_n, \varphi\}$ is not satisfiable on K. Then $\mathsf{K} \models \varphi \rightarrow \neg(\chi_1 \wedge \cdots \wedge \chi_n)$. Hence, $\neg(\chi_1 \wedge \cdots \wedge \chi_n) \in Cons(\varphi)$, and therefore, $\mathfrak{M}, w \models \neg(\chi_1 \wedge \cdots \wedge \chi_n)$. This contradicts the fact that $\chi_1, \ldots, \chi_n \in Th(\mathfrak{M}, w)$. $\dashv$

Again, we may assume that $\mathfrak{N}$ is generated by v. Let $\mathfrak{M}^+$ and $\mathfrak{N}^+$ be ω-saturated elementary extensions of $\mathfrak{M}$ and $\mathfrak{N}$. Since K is elementary, the underlying frames of $\mathfrak{M}^+$ and $\mathfrak{N}^+$ are in K. Define the binary relation Z between the domains of $\mathfrak{M}^+$ and $\mathfrak{N}^+$ by letting dZe if $\mathfrak{M}^+, d \models \chi \Leftrightarrow \mathfrak{N}^+, e \models \chi$ for all modal formulas χ with $\textsc{prop}(\chi) \subseteq \textsc{prop}(\varphi) \cap \textsc{prop}(\psi)$. In other words, dZe if d and e cannot be distinguished by a modal formula in the common language of φ and ψ. Note that, by construction, wZv.

> **Claim 3:** Z is a total bisimulation between $\mathfrak{M}^+$ and $\mathfrak{N}^+$ with respect to the common language of φ and ψ.

Proof of claim: We will show that Z satisfies the *zig* condition of Definition 2.2.1. The proof of the *zag* condition is similar, and that Z respects the proposition letters in $\textsc{prop}(\varphi) \cap \textsc{prop}(\psi)$ is immediate from its definition.

Suppose $w'Zv'$ and $w'R_\Diamond w''$. Let $\Gamma = \{ST_x(\chi) \mid \mathfrak{M}^+, w'' \models \chi \text{ and } \textsc{prop}(\chi) \subseteq \textsc{prop}(\varphi) \cap \textsc{prop}(\psi)\}$. We need to show that Γ is realized in $\mathfrak{N}^+$ by a $\Diamond$-successors of v'. By ω-saturatedness, it suffices to show that every finite subset of Γ is realized in $\mathfrak{N}^+$ by a $\Diamond$-successors of v'. But this is clearly the case: consider any $ST_x(\chi_1), \ldots, ST_x(\chi_n) \in \Gamma$. Then $\mathfrak{M}^+, w' \models \Diamond(\chi_1 \wedge \cdots \wedge \chi_n)$, and hence $\mathfrak{N}^+, v' \models \Diamond(\chi_1 \wedge \cdots \wedge \chi_n)$.

Finally, it needs to be shown that Z is a *total* bisimulation. Let $w' \in \mathfrak{M}^+$. Let $\Gamma = \{ST_x(\chi) \mid \mathfrak{M}^+, w' \models \chi \text{ and } \textsc{prop}(\chi) \subseteq \textsc{prop}(\varphi) \wedge \textsc{prop}(\psi)\}$. We need to show that Γ is realized in $\mathfrak{N}^+$. By ω-saturatedness, it suffices to show that every finite subset of Γ is realized in $\mathfrak{N}^+$. Let $ST_x(\chi_1), \ldots, ST_x(\chi_n) \in \Gamma$. Then $\exists x.(ST_x(\chi_1) \wedge \cdots \wedge ST_x(\chi_n))$ is true in $\mathfrak{M}^+$ and therefore also in $\mathfrak{M}$ (recall that $\mathfrak{M}^+$ is an elementary extension of $\mathfrak{M}$). Since $\mathfrak{M}$ is generated by w, there are $\Diamond_1, \ldots, \Diamond_m \in \textsc{mod}$ such that $\mathfrak{M}, w \models \Diamond_1 \cdots \Diamond_m(\chi_1 \wedge \cdots \wedge \chi_n)$. Hence, since wZv, we have that $\mathfrak{N}, v \models \Diamond_1 \cdots \Diamond_m(\chi_1 \wedge \cdots \wedge \chi_n)$. Since $\mathfrak{N}^+$ is an elementary extension of $\mathfrak{N}$, it follows that $\mathfrak{N}^+, v \models \Diamond_1 \cdots \Diamond_m(\chi_1 \wedge \cdots \wedge \chi_n)$. We conclude that there is a point v' such that $\mathfrak{N}^+, v' \models \chi_1 \wedge \cdots \wedge \chi_n$. ⊣

Let $\mathfrak{F}$ and $\mathfrak{G}$ be the underlying frames of $\mathfrak{M}^+$ and $\mathfrak{N}^+$. Then, in particular, Z is a total frame bisimulation between $\mathfrak{F}$ and $\mathfrak{G}$. Hence, by Proposition 2.5.2, there is a bisimulation product $\mathfrak{H} \in \mathsf{K}$ of $\mathfrak{F}$ and $\mathfrak{G}$ of which the domain is Z. By the definition of bisimulation products, the natural projections $f : \mathfrak{H} \to \mathfrak{F}$ and $g : \mathfrak{H} \to \mathfrak{G}$ are surjective bounded morphisms. For any proposition letter $p \in \textsc{prop}(\varphi)$, let $V(p) = \{u \mid \mathfrak{M}^+, f(u) \models p\}$, and for any proposition letter $p \in \textsc{prop}(\psi)$, let $V(p) = \{u \mid \mathfrak{N}^+, g(u) \models p\}$. The properties of Z guarantee that this V is well-defined for $p \in \textsc{prop}(\varphi) \cap \textsc{prop}(\psi)$. Finally, by a standard argument, the graph of f is a bisimulation between $(\mathfrak{H}, V)$ and $\mathfrak{M}^+$ with respect to $\textsc{prop}(\varphi)$, and the graph of g is a bisimulation between $(\mathfrak{H}, V)$ and $\mathfrak{N}^+$ with respect to $\textsc{prop}(\psi)$. It follows that $(\mathfrak{H}, V), \langle w, v\rangle \models \varphi \wedge \neg\psi$. This contradicts our initial assumption that $\mathsf{K} \models \varphi \to \psi$. □

This result cannot easily be strengthened. An example of an elementary frame class that is not closed under generated subframes but not under bisimulation products, on which the basic modal language lacks interpolation is the class defined by $\Diamond\Box p \to \Box\Diamond p$.[4]

An example of an elementary frame class closed under bisimulation products but not closed under generated subframes on which the basic modal language lacks

[4] To see that the basic modal language lacks interpolation on this frame class, consider the following implication.

$$\Big(\Box(s \to \Box(\neg p \to r)) \wedge \Box(t \to \Box(\neg p \to \neg r))\Big) \quad \to \quad \Big(\Diamond(s \wedge \Box(p \to q)) \to \Box(t \to \Diamond(p \wedge q))\Big)$$

interpolation is the class defined by $\forall x.(\forall y \exists z.R_1yz \to R_1xx) \wedge \forall x.(\exists y \forall z.(R_1yz \to \bot) \to R_2xx)$. It follows from Theorem 2.5.5 below that this first-order sentence is preserved under taking bisimulation products. Again, an easy bisimulation argument shows that there is no interpolant for the valid implication $p \wedge \neg\Diamond_1 p \to (q \to \Diamond_2 q)$. Note that this implication has an interpolant with global modality, namely $\mathsf{E}\Box_1\bot$. Indeed, a relatively straightforward adaptation of the proof of Theorem 2.5.3 shows that the modal language with global modality, $\mathcal{M}(\mathsf{E})$, has interpolation on any elementary frame class closed under bisimulation products.

The Beth property

Let $\models^{glo}$ denote the global entailment relation on models, i.e., $\Sigma \models^{glo} \varphi$ means that for all models $\mathfrak{M}$, if $\mathfrak{M}$ globally satisfies all formulas in Σ then $\mathfrak{M}$ globally satisfies φ. Global entailment relative to a class of frames, denoted by $\models^{glo}_{\mathsf{K}}$, is defined similarly. For a set of formulas $\Sigma(p)$ containing the proposition letter p (and possibly other proposition letters), we say that $\Sigma(p)$ *implicitly defines* p, relative to a frame class K, if $\Sigma(p) \cup \Sigma(p') \models^{glo}_{\mathsf{K}} p \leftrightarrow p'$. Here, p' is a proposition letter not occurring in Σ, and $\Sigma(p')$ is the result of replacing all occurrences of p by p' in $\Sigma(p)$. The basic modal language $\mathcal{M}$ is said to have the Beth property relative to a frame class K if whenever a set of modal formulas $\Sigma(p)$ implicitly defines a proposition letter p relative to K, then there is a modal formula ϑ in which p does not occur, such that $\Sigma \models^{glo}_{\mathsf{K}} p \leftrightarrow \vartheta$. The relevant formula ϑ is called an *explicit definition* of p, relative to Σ and K.

The Beth property is an important property. Intuitively, if a logic has it, this can be seen as evidence that its syntax and semantics match well. Tarski refers to the Beth property as completeness in the theory of definitions (as opposed to the theory of deductions).

By a standard argument, we obtain as a corollary of the above interpolation results the Beth property for the basic modal language, relative to every elementary frame class closed under bisimulation products and generated subframes.

2.5.4. THEOREM. *If K is a elementary frame class closed under generated subframes and bisimulation products, then the basic modal language has the Beth property relative to K.*

Proof: For ease of presentation we restrict attention to the uni-modal case. The proof generalizes easily to languages containing more modalities.

Let $\Sigma(p)$ be any set of modal formulas containing the proposition letter p (and possibly other proposition letters and nominals), and suppose $\Sigma(p)$ implicitly defines the proposition letter p, relative to K. Let p' be a new proposition letter,

---

This formula is valid on the given frame class, but a simple bisimulation argument shows that there is no interpolant. Note that, intuitively, an interpolant would have to express the fact that for every successor x satisfying s and for every successor y satisfying t, x and y have a common successor satisfying p.

and let $\Sigma(p')$ be the result of replacing all occurrences of p in Σ by p'. Then, by the definition of implicit definability, $\Sigma(p) \cup \Sigma(p') \models^{glo}_{\mathsf{K}} p \leftrightarrow p'$. Let $\Gamma(p) = \{\Box^n \varphi \mid \varphi \in \Sigma(p), n \in \omega\}$, and define $\Gamma(p')$ similarly.

Claim 1: $\Gamma(p) \cup \Gamma(p') \models_{\mathsf{K}} p \leftrightarrow p'$.

Proof of claim: Suppose $\mathfrak{M}, w \models \Gamma(p) \cup \Gamma(p')$ for some model $\mathfrak{M}$ based on a frame in K. Let $\mathfrak{M}_w$ be the submodel of $\mathfrak{M}$ generated by w. By closure under generated subframes, the underlying frame of $\mathfrak{M}_w$ is also in K. By construction, $\mathfrak{M}_w$ globally satisfies $\Sigma(p)$ and $\Sigma(p')$. It follows that $\mathfrak{M}_w$ globally satisfies $p \leftrightarrow p'$, hence, $\mathfrak{M}_w, w \models p \leftrightarrow p'$, hence $\mathfrak{M}, w \models p \leftrightarrow p'$. $\dashv$

By compactness, there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0(p) \cup \Gamma_0(p') \models_{\mathsf{K}} p \leftrightarrow p'$. It follows that $\models_{\mathsf{K}} (p \wedge \bigwedge \Gamma_0(p)) \to (\bigwedge \Gamma_0(p') \to p')$. Let ϑ be an interpolant for this implication. Then the following facts hold.

1. The proposition letters p and p' do not occur in ϑ.
2. $\models_{\mathsf{K}} (p \wedge \bigwedge \Gamma_0(p)) \to \vartheta$.
3. $\models_{\mathsf{K}} \vartheta \to (\bigwedge \Gamma_0(p') \to p')$, and hence, by uniform substitution, $\models_{\mathsf{K}} \vartheta \to (\bigwedge \Gamma_0(p) \to p)$.

We conclude that $\Gamma_0(p) \models_{\mathsf{K}} p \leftrightarrow \vartheta$, and hence $\Sigma(p) \models^{glo}_{\mathsf{K}} p \leftrightarrow \vartheta$. $\Box$

Here is a simple example of an elementary frame class on which the basic modal language *lacks* the Beth property. Let K be the class of frames satisfying $\exists x \forall yz.(Ryz \leftrightarrow y = x)$, and let $\Sigma = \{p \to \Box q, \neg p \to \Box \neg q\}$. Clearly, in models that are based on a frame in K and that globally satisfy Σ, q holds at a state iff p holds at the root, and hence, Σ implicitly defines q in terms of p, relative to K. However, a simple bisimulation argument shows that there is no explicit definition of q in terms of p, relative to Σ and K, in the basic modal language.

Preservation results for bisimulation products

One might ask for a syntactic characterization of the elementary frame properties that are preserved under taking bisimulation products. Such a preservation theorem can indeed be given. In what follows, we will characterize the first-order formulas that are preserved under bisimulation products, in the form of a preservation theorem. Recall the definition of p-formulas on page 12.

In the following proof we will refer to frames as models (models of the first-order frame correspondence language $\mathcal{L}^1_{fr}$, to be precise). This seems the more natural choice in the present context, since the theorem concerns first-order formulas.

2.5.5. Theorem. *A first-order sentence φ is preserved under bisimulation products iff φ is equivalent to a conjunction of sentences of the form $\forall\vec{x}(\psi \to \chi)$, where ψ is a p-formula and χ is either an atomic formula or $\bot$.*

Proof: [$\Leftarrow$] The right-to-left direction is easy to prove: consider any formula of the form $\forall\vec{x}(\psi \to \chi)$ with ψ and χ as specified above, and let $\mathfrak{G}$ be a bisimulation product of frames $(\mathfrak{F}_i)_{i\in I}$. For each $i \in I$, let $f_i : \mathfrak{G} \to \mathfrak{F}_i$ be the natural projection. By definition, each f_i is a surjective bounded morphism. Next, suppose by contraposition that $\mathfrak{G} \not\models \forall\vec{x}(\psi \to \chi)$. Then there are $d_1, \ldots, d_n$ such that $\mathfrak{G} \models \psi \wedge \neg\chi\ [d_1, \ldots, d_n]$. Since χ is an atomic formula, and by the definition of bisimulation products, $\mathfrak{F}_i \models \neg\chi\ [f_i(d_1), \ldots, f_i(d_i)]$ for some $i \in I$. Furthermore, by preservation under surjective bounded morphisms, $\mathfrak{F}_i \models \psi\ [f_i(d_1), \ldots, f_i(d_n)]$. It follows that $\mathfrak{F}_i \not\models \forall\vec{x}(\psi \to \chi)$.

[$\Rightarrow$] Call an *basic p-Horn sentence* a sentence of the form $\forall\vec{x}(\psi \to \chi)$, where ψ is a p-formulas and χ is an atom or $\bot$. Let L_0 be the vocabulary of φ, and for any vocabulary L, let $PCons_L(\varphi)$ be the set of basic p-Horn sentences in L entailed by φ. Suppose $\mathfrak{M}_0 \models PCons_{L_0}(\varphi)$. We will show that $\mathfrak{M}_0 \models \varphi$. It then follows by compactness that φ is equivalent to a conjunction of finitely many basic p-Horn sentences.

We will perform a sort of step by step construction. Call an *approximation* a triple $A = (L, \mathfrak{M}, S)$, where $L \supseteq L_0$ is a (not necessarily countable) vocabulary, $\mathfrak{M}$ is an L-model satisfying $PCons_L(\varphi)$ and S is a set of L-models satisfying φ, such that every p-sentence true in $\mathfrak{M}$ is true in all models in S. In particular, let A_0 be the approximation $(L_0, \mathfrak{M}_0, \emptyset)$. We call an approximation $A = (L, \mathfrak{M}, S)$ *perfect* if it satisfies the following additional properties.

1. Every element of $\mathfrak{M}$ or of some model $\mathfrak{N} \in S$ is named by a constant.
2. For constant c and every point w in some model $\mathfrak{N} \in S$, if $\mathfrak{N} \models Rcx\ [w]$ then there is a constant k_{cw} naming w such that $\mathfrak{M} \models Rck_{cw}$.
3. For every atomic sentence α (including equality statements), if $\mathfrak{M} \not\models \alpha$ then there is an $\mathfrak{N} \in S$ such that $\mathfrak{N} \not\models \alpha$

We are interested in a perfect approximation, for the following reason.

Claim 1: If $(L, \mathfrak{M}, S)$ is a perfect approximation, then $\mathfrak{M}$ is isomorphic to a bisimulation product of the models in S, and hence $\mathfrak{M} \models \varphi$.

Proof of claim: For each $\mathfrak{N} \in S$, let $f_{\mathfrak{N}} : \mathfrak{M} \to \mathfrak{N}$ be the natural function induced by the constants, and let $g : \mathfrak{M} \to \Pi_{\mathfrak{N}\in S}\mathfrak{N}$ such that $g(x) = \langle f_{\mathfrak{N}}(x)\rangle_{\mathfrak{N}\in S}$. From the fact that $(L, \mathfrak{M}, S)$ is a perfect approximation, it follows that $f_{\mathfrak{N}}$ is a surjective bounded morphism for each $\mathfrak{N} \in S$, and that g is an embedding, i.e., an injection that preserves truth and falsity of atomic formulas (note that, since every atomic sentence α is a p-formula, $\mathfrak{M} \models \alpha$ iff each $\mathfrak{N} \in S$ satisfies

α). It follows that $\mathfrak{N}$ is isomorphic to a bisimulation product of the models in S. ⊣

We also need some other lemmas on approximations. In what follows, we will use the notation $\mathfrak{M} \preceq_L \mathfrak{N}$ to say that $\mathfrak{N}$ is an elementary extension of $\mathfrak{M}$, relative to the vocabulary L. We write $(L, \mathfrak{M}, S) \preceq (L', \mathfrak{M}', S')$ if the following holds: $L \subseteq L'$, $\mathfrak{M} \preceq_L \mathfrak{M}'$ and there is an injection $f : S \to S'$ such that for all $\mathfrak{N} \in S$, $\mathfrak{N} \preceq_L f(\mathfrak{N})$. We will write $(L, \mathfrak{M}, S) \preceq_f (L', \mathfrak{M}', S')$ if we wish to indicate the injection.

Claim 2: For each approximation $(L, \mathfrak{M}, S)$ there is an approximation $(L', \mathfrak{M}', S')$ such that $(L, \mathfrak{M}, S) \preceq (L', \mathfrak{M}', S')$ and every element of $\mathfrak{M}'$ is named by a constant.

Proof of claim: Let L' extend L with a constant c_w for each world w of $\mathfrak{M}$, and let $\mathfrak{M}'$ be the natural L'-expansion of $\mathfrak{M}$. Then $\mathfrak{M}' \models PCons_{L'}(\varphi)$. This holds, for consider any $\psi \in PCons_{L'}(\varphi)$, let $c_1, \ldots, c_n$ be the constants of $L' \setminus L$ occurring in ψ, and let $x_1, \ldots, x_n$ be corresponding new variables. Then $\forall x_1, \ldots, x_n.\psi[\vec{x}/\vec{c}] \in PCons_L(\varphi)$, hence $\mathfrak{M} \models \forall x_1, \ldots, x_n.\psi[\vec{x}/\vec{c}]$, hence, $\mathfrak{M}' \models \forall x_1, \ldots, x_n.\psi[\vec{x}/\vec{c}]$, hence $\mathfrak{M}' \models \psi$.

Let $PTh(\mathfrak{M}')$ for the set of p-sentences true in $\mathfrak{M}'$.

Next, consider any $\mathfrak{N} \in S$. Every finite set of p-sentences $\psi_1, \ldots, \psi_n \in PTh(\mathfrak{M}')$ is true in some expansion of $\mathfrak{N}$. For, let $c_1, \ldots, c_m$ be the constants of $L' \setminus L$ occurring in $\psi_1, \ldots, \psi_n$, and let $x_1, \ldots, x_m$ be corresponding new variables. Then the p-sentence $\exists x_1 \cdots x_m. \bigwedge_{k=1\ldots n} \psi_k[\vec{c}/\vec{x}]$ is true in $\mathfrak{M}$ and hence also in $\mathfrak{N}$. In other words, $\mathfrak{N}$ has an expansion satisfying $\psi_1, \ldots, \psi_n$. It follows by a well-known model theoretic argument that $\mathfrak{N}' \models PTh(\mathfrak{M}')$ for some L'-model $\mathfrak{N}'$, with $\mathfrak{N} \preceq_L \mathfrak{N}'$. It follows that $\mathfrak{N}' \models \varphi$.

Finally, let $S' = \{\mathfrak{N}' \mid \mathfrak{N} \in S\}$. Then, by the above considerations, $(L', \mathfrak{M}', S')$ is an approximation, and $(L, \mathfrak{M}, S) \preceq (L', \mathfrak{M}', S')$. Moreover, every element of $\mathfrak{M}'$ is named by a constant. ⊣

Claim 3: For each approximation $(L, \mathfrak{M}, S)$ there is an approximation $(L', \mathfrak{M}', S')$ such that $(L, \mathfrak{M}, S) \preceq_f (L', \mathfrak{M}', S')$ for some f and such that the following holds for each $\mathfrak{N} \in S$:

1. For every element w of $\mathfrak{N}$ there is a constant c_w that names w in $f(\mathfrak{N})$.
2. If $\mathfrak{N} \models Rcx\ [w]$ for some constant $c \in L$ and element w, then there is a constant $k_{cw} \in L'$ that names w in $f(\mathfrak{N})$, such that $\mathfrak{M}' \models Rck_{cw}$.

Proof of claim: For each $\mathfrak{N} \in S$, let the extension $L^{(\mathfrak{N})}$ of L be defined as follows. For each element w of $\mathfrak{N}$ pick a new constant c_w. Furthermore, for each constant $c \in L$ and element w of $\mathfrak{N}$ such that $\mathfrak{N} \models Rcx\ [w]$, pick a new constant k_{cw}. Let $L^{(\mathfrak{N})}$ be the extension of L with these constants, and let $\mathfrak{N}_1$ be the natural expansion of $\mathfrak{N}$ to $L^{(\mathfrak{N})}$. Let $\Delta_{\mathfrak{N}_1} = \{\neg\varphi \mid \varphi \text{ is a p-sentence of } L^{(\mathfrak{N})} \text{ and } \mathfrak{N}_1 \not\models \varphi\} \cup \{Rck_{cw} \mid c \in L \text{ and } k_{cw} \in L^{(\mathfrak{N})} \setminus L\}$.

As a first step, we claim that, for each $\mathfrak{N} \in S$, every finite subset of $\Delta_{\mathfrak{N}_1}$ is true in some expansion of $\mathfrak{M}$. For, let $\psi_1, \ldots, \psi_n, Rc_1k_{c_1w_1}, \ldots, Rc_mk_{c_mw_m} \in \Delta_{\mathfrak{N}_1}$, where $\psi_1, \ldots, \psi_n$ are negated p-sentences. Let $c_{w_1}, \ldots, c_{w_k}$ be the constants of $L^{(\mathfrak{N})} \setminus L$ occurring in $\psi_1, \ldots, \psi_n$, other than $k_{c_1w_1}, \ldots, k_{c_mw_m}$. Let χ be the L-sentence

$$\exists c_{w_1} \ldots c_{w_n} \exists k_{c_1w_1}(Rc_1k_{c_1w_1} \wedge \cdots \exists k_{c_mw_m}(Rc_mk_{c_mw_m} \wedge \bigwedge_{i\leq n} \psi_i) \cdots)$$

Then $\mathfrak{N} \models \chi$, and hence, since the negation of χ is equivalent to a p-sentence, $\mathfrak{M} \models \chi$. It follows that some expansion of $\mathfrak{M}$ satisfies $\psi_1, \ldots, \psi_n, Rc_1k_{c_1w_1}, \ldots, Rc_mk_{c_mw_m} \in \Delta_{\mathfrak{N}_1}$.

Next, let $L' = \bigcup_{\mathfrak{N}\in S} L^{(\mathfrak{N})}$, and let Δ be the set of L' formulas $\bigcup_{\mathfrak{N}\in S} \Delta_{\mathfrak{N}_1}$. Since $L^{(\mathfrak{N})} \setminus L$ is disjoint from $L^{(\mathfrak{K})} \setminus L$ for $\mathfrak{N} \neq \mathfrak{K}$ ($\mathfrak{N}, \mathfrak{K} \in S$), it follows from the above considerations that every finite subset of Δ is true in some expansion of $\mathfrak{M}$, and hence, by a familiar model theoretic argument, Δ has model $\mathfrak{M}'$ such that $\mathfrak{M} \preceq_L \mathfrak{M}'$. By construction, every p-sentence of $L^{(\mathfrak{N})}$ true in $\mathfrak{M}'$ is true in $\mathfrak{N}_1$, for $\mathfrak{N} \in S$.

Next, we claim that $\mathfrak{M}' \models PCons_{L'}(\varphi)$. This is quite easily seen: let $\psi \in PCons_{L'}(\varphi)$, and let $c_1, \ldots, c_m$ be the constants of $L' \setminus L$ occurring in ψ. Pick corresponding variables $x_1, \ldots, x_m$. Then $\forall x_1 \ldots x_n.\psi[c_1/x_1, \ldots, c_m/x_m] \in PCons_L(\varphi)$, hence $\mathfrak{M} \models \forall x_1 \ldots x_n.\psi[c_1/x_1, \ldots, c_m/x_m]$, hence $\mathfrak{M}' \models \forall x_1 \ldots x_n.\psi[c_1/x_1, \ldots, c_m/x_m]$, hence $\mathfrak{M} \models \psi$.

Finally, we apply the same technique as in the proof of Claim 2 to obtain an L'-model $\mathfrak{N}'$ with $\mathfrak{N}_1 \preceq_{L^{(\mathfrak{N})}} \mathfrak{N}'$, and we set $S' = \{\mathfrak{N}' \mid \mathfrak{N} \in S\}$, and take $f : S \to S'$ such that $f(\mathfrak{N}) = \mathfrak{N}'$. Then $(L', \mathfrak{M}', S')$ is an approximation, $(L, \mathfrak{M}, S) \preceq_f (L', \mathfrak{M}', S')$ and all other requirements are fulfilled. $\dashv$

Claim 4: For each approximation $(L, \mathfrak{M}, S)$ there is an approximation $(L', \mathfrak{M}', S')$ such that $(L, \mathfrak{M}, S) \preceq (L', \mathfrak{M}', S')$ and for each atomic L'-sentence α with $\mathfrak{M}' \not\models \alpha$ (including equality statements), there is a model $\mathfrak{N} \in S'$ such that $\mathfrak{N} \not\models \alpha$.

Proof of claim: For each atomic L-sentence α with $\mathfrak{M} \not\models \alpha$, there is a model $\mathfrak{N}_{\neg\alpha}$ such that $\mathfrak{N}_{\neg\alpha} \models PTh(\mathfrak{M}) \cup \{\varphi, \neg\alpha\}$, where $PTh(\mathfrak{M})$ is the set of p-sentences true in $\mathfrak{M}'$. For, suppose not. Then by compactness, there are $\psi_1, \ldots, \psi_n \in PTh(\mathfrak{M})$ such that $\psi_1 \wedge \cdots \wedge \psi_n \wedge \varphi \wedge \neg\alpha$ is not satisfiable, and hence $(\psi_1 \wedge \cdots \wedge \psi_n \to \alpha) \in PCons_L(\varphi)$. This contradicts the assumption that $\mathfrak{M} \models PCons_L(\varphi)$.

Let $S' = S \cup \{\mathfrak{N}_{\neg\alpha} \mid \mathfrak{M} \not\models \alpha\}$. Then $(L, \mathfrak{M}, S')$ is an approximation, and, by construction, for each atomic L-sentence α with $\mathfrak{M} \not\models \alpha$ there is a model $\mathfrak{N} \in S'$ such that $\mathfrak{N} \not\models \alpha$. ⊣

We will now construct an infinite sequence of approximations and, as the limit of that sequence, a perfect approximation. Recall that A_0 is the approximation $= (L_0, \mathfrak{M}_0, \emptyset)$. Now, for given A_k, apply one of the Claims 2, 3, 4 (depending on k mod 3) to obtain A_{k+1}. In this way, we obtain a sequence of approximations $A_0 \preceq_{f_0} A_1 \preceq_{f_1} A_2 \preceq_{f_2} \ldots$ as in Figure 2.1. The limit of this sequence is a perfect approximation. More precisely, let $L_\omega = \bigcup_k L_k$, let $\mathfrak{M}_\omega = \bigcup_k \mathfrak{M}_k$ be the union of the elementary chain $\mathfrak{M}_0 \preceq_{L_0} \mathfrak{M}_1 \preceq_{L_1} \ldots$, and, finally, let S_ω be defined as follows. Each model $\mathfrak{N} \in S_k$ ($k \in \omega$) is the start of an elementary chain $\mathfrak{N} \preceq_{L_k} f_k(\mathfrak{N}) \preceq_{L_{k+1}} f_{k+1}(f_k(\mathfrak{N})) \preceq_{L_{k+2}} \cdots$. Let S_ω be the set of unions of such elementary chains. By construction, $(L_\omega, \mathfrak{M}_\omega, S_\omega)$ is a perfect approximation, and hence, by Claim 1, $\mathfrak{M}_\omega \models \varphi$. Since $\mathfrak{M}_0 \preceq_{L_0} \mathfrak{M}_\omega$, we obtain that $\mathfrak{M}_0 \models \varphi$. □

Incidentally, the above proof is somewhat reminiscent to that of Van Benthem [14] for first-order formulas preserved under predicate intersection.

A similar characterization can be given for the first-order sentences that are preserved under bisimulation products and generated subframes. Call a strict p-sentence one that contains no unbounded universal quantifiers. In other words: bounded universal quantifiers, unbounded existential quantifiers, positive atoms.

2.5.6. THEOREM. *A first-order sentence is preserved under bisimulation products and generated subframes iff it is equivalent to a conjunction of formulas of the form $\forall\vec{x}(\varphi \rightarrow \psi)$ where φ is a strict p-formula and ψ is atomic or $\bot$.*

2.6 Decidability and complexity

Many decision problems can be formulated in the context of modal logic. We will mention a few. The *model checking* problem: given $\mathfrak{M}, w$ and φ, check if $\mathfrak{M}, w \models \varphi$.

2.6.1. THEOREM ([61]). *The model checking problem for modal formulas can be solved in polynomial time.*

The *frame checking* problem: given $\mathfrak{F}$ and φ, check if $\mathfrak{F} \models \varphi$.

2.6.2. THEOREM. *The frame checking problem for modal formulas is co*-NP-*complete.*

The *modal equivalence* problem: given $\mathfrak{M}, w$ and $\mathfrak{N}, v$, check if there is a modal formula that distinguishes w from v.

2.6.3. THEOREM ([82]). *The modal equivalence problem can be solved in polynomial time.*

	stage 0			stage k		stage $k+1$		stage $k+2$			limit
vocabulary	L_0	$\cdots$	$\cdots$	L_k	$\subseteq$	L_{k+1}	$\subseteq$	L_{k+2}	$\subseteq$	$\cdots$	$L_\omega = \bigcup_k L_k$
model	$\mathfrak{M}_0$	$\cdots$	$\cdots$	$\mathfrak{M}_k$	$\preceq$	$\mathfrak{M}_{k+1}$	$\preceq$	$\mathfrak{M}_{k+2}$	$\preceq$	$\cdots$	$\mathfrak{M}_\omega = \bigcup_k \mathfrak{M}_k$
factors	$S_0 = \emptyset$	$\cdots$	$\cdots$	$S_k = \{$		$S_{k+1} = \{$		$S_{k+2} = \{$		$\cdots$	$S_\omega = \{$
				$\vdots$		$\vdots$		$\vdots$			$\vdots$
				$\mathfrak{N}$,	$\preceq$	$\mathfrak{N}'$,	$\preceq$	$\mathfrak{N}''$,	$\preceq$	$\cdots$	$\mathfrak{N}_\omega$,
				$\vdots$		$\vdots$		$\vdots$			$\vdots$
				$\}$		$\}$		$\}$			$\}$

Figure 2.1: Sequence of approximations, with as its limit a perfect approximation

The *frame satisfiability* problem: given a formula φ, check if there is a frame on which φ is valid.

2.6.4. THEOREM. *The frame satisfiability problem for modal formulas is highly undecidable, in fact not analytical.*

Proof: By Theorem B.0.3, the satisfiability problem for monadic second order formulas in one binary relation is non-analytical. Thomason [95] reduced this problem to the following problem:

> Given uni-modal formulas φ, ψ of the basic modal language, such that ψ is closed (i.e., contains no proposition letters). Does φ entail ψ on frames (i.e., is ψ valid on every frame on which φ is valid)?

This problem can again be reduced to the frame satisfiability problem: it suffices to note that, for a modal operator $\Diamond$ not occurring in φ and ψ, φ entails ψ on frames iff $\varphi \wedge \Diamond\neg\psi$ has no frame (here, we use the fact that ψ is a closed formula). □

Incidentally, the frame satisfiability problem for uni-modal formulas is trivially decidable in co-non-deterministic polynomial time, due to the fact that every frame satisfiable uni-modal formula has a singleton frame.

The *elementarity* problem: given a formula φ, does φ define an elementary frame class?

2.6.5. THEOREM. *The problem whether a given modal formula defines an elementary frame class is highly undecidable, in fact not analytical.*

Proof: Let φ be a modal formula, and let $\Diamond$ be a modal operator not occurring in φ. Then φ is frame satisfiable iff $\varphi \wedge (\Box\Diamond p \to \Diamond\Box p)$ is not elementary. It follows by Theorem 2.6.4 that the elementarity problem is not analytical. □

Finally, the decision problem that will receive most attention in this thesis is the *satisfiability problem.* For a given frame class K, the problem is to test if a modal formula is satisfiable on K or not. For different classes K, and for different extensions of the basic modal language, we will address the question if this problem is decidable, and if so, what is its complexity.

Let us say that a frame class K has the finite model property if whenever a modal formula is satisfiable on a frame in K, then it is satisfiable on a finite frame in K. If K has the finite model property, and if membership of a frame with respect to K can be tested effectively, then the modal formulas that are satisfiable on K can be enumerated: simply enumerate all triples $(\mathfrak{M}, w, \varphi)$, where $\mathfrak{M}$ is a finite model, w is a world of $\mathfrak{M}$ and φ is a modal formula, and check for each such triple if $\mathfrak{M}, w \models \varphi$ and if the underlying frame of $\mathfrak{M}$ is in K.

Dually, if $\mathbf{K}_{\mathcal{M}}\{\varphi\}$ is complete with respect to K, for some φ, then we can use this in order to enumerate the formulas that are not satisfiable with respect to K: simply enumerate all negations of formulas derivable in $\mathbf{K}_{\mathcal{M}}\{\varphi\}$.

If both the satisfiable and the non-satisfiable formulas can be enumerated, then the satisfiability problem is decidable: the decision procedure simply performs both enumerations in parallel, and stops as soon as the input formula occurs in one of the two enumerations. Since every formula is either satisfiable or non-satisfiable, the algorithm will stop after a finite amount of time. Note that while decidability might be shown in this way, no concrete bounds on the amount of time, or space, needed to solve the problem can be derived.

A useful method for proving the decidability and the finite model property is using *filtrations*. Let $\mathfrak{M}$ be a model based on a frame $\mathfrak{F} = (W, R)$ and let Σ be a set of formulas closed under subformulas. Define an equivalence relation $\sim_\Sigma$ on W such that for every $w, v \in W$:

$$w \sim_\Sigma v \ \text{ iff } \ \text{for every } \psi \in \Sigma,\ \mathfrak{M}, w \models \psi \text{ iff } \mathfrak{M}, v \models \psi$$

Denote by $[w]$ the $\sim_\Sigma$-equivalence class containing w and let $W/_{\sim_\Sigma}$ be the set of all $\sim_\Sigma$-equivalence classes of W. Define a valuation V_Σ on $W/_{\sim_\Sigma}$ such that $V_\Sigma(p) = \{[w] \mid w \in V(p)\}$. The model $\mathfrak{M}/_{\sim_\Sigma} = (W/_{\sim_\Sigma}, R_\Sigma, V_\Sigma)$ is called a *filtration* of $\mathfrak{M}$ through Σ if R_Σ is a binary relation on $W/_{\sim_\Sigma}$ such that for any $\psi \in \Sigma$ and $w \in W$, $\mathfrak{M}, w \models \psi$ iff $\mathfrak{M}/_{\sim_\Sigma}, [w] \models \psi$. This notion can be generalized to multi-modal languages as well.

2.6.6. Definition (Filtrations). *A frame class* K admits filtration *if for every modal formula φ there is a finite set of formulas Σ_φ containing all subformulas of φ, such that whenever $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}$ based on a frame in K, there is a filtration of $\mathfrak{M}$ over Σ_φ whose underlying frame is in K.*

We say that K admits polynomial filtration *if it admits filtration and the size of Σ_φ is polynomial in the length of φ. We say that* K admits simple *filtration if it admits filtration and for every formula φ, Σ_φ is the set of subformulas of φ.*

Since $|W/_{\sim_\Sigma}| \leq 2^{|\Sigma|}$, if K admits filtration then it has the finite model property. Since the number of subformulas of φ is polynomial in the length of φ, every frame class that admits simple filtration admits polynomial filtration.

2.6.7. Theorem. *Let K be any elementary frame class. If K admits polynomial filtration then satisfiability of modal formulas with respect to K can be decided in* NExpTime.

Proof: This can be considered a folklore result.

If K admits polynomial filtration, then every satisfiable formula φ has a model whose size can be bounded by an exponential in the length of φ. It therefore suffices to guess such a model and check if it satisfies φ and if the underlying frame is in K. Both of these checks can be performed in polynomial time (note that the model checking problem for a fixed first order formula can be solved in polynomial time). □

Frame classes defined by shallow formulas give us a nice example for the use of the filtration method.

2.6.8. THEOREM. *Every frame class defined by a finite set of shallow modal formulas admits polynomial filtration, hence has the finite model property and has a satisfiability problem that can be solved in* NEXPTIME.

Proof: Lewis [76] proved a restricted version of this result, for frame classes defined by modal formulas with modal depth at most 1. The same proof can be used to prove our more general result, with a small modification. Let K be a frame class defined by a finite set Γ of shallow modal formulas. For any modal formula φ, define Σ_φ to be the union of the set of subformulas of φ with the set of closed subformulas of formulas in Γ (recall that a formula is closed if it contains no proposition letters). Proceeding as in [76] using Σ_φ as the filtration set for φ, one can construct for every model $\mathfrak{M}$ based on a frame in K a filtration $\mathfrak{M}'$ with respect to Σ_φ, such that the underlying frame of $\mathfrak{M}'$ is in K, and φ is satisfied at some world in $\mathfrak{M}'$.

Alternatively, a proof of this result can be extracted from the proof of Theorem 7.4.2. □

Part I

Hybrid logics

Chapter 3

Introduction to hybrid languages

Part I of this thesis concerns *hybrid languages*. These are extensions of the basic modal language involving *nominals*. Syntactically, nominals act as a second sort of proposition letters. However, semantically, their interpretation is restricted to singleton sets. In other words, nominals act as names for elements of the model, much like constants in first-order logic. Examples of modal formulas containing nominals are $i \wedge \neg\Diamond i$ ("the current world is named by the nominal i, and it not a successor of itself") and $\Diamond i \wedge \Box i$ ("the world named i is a successor of the current world, and it is the only successor").

Hybrid languages have a long history: the use of nominals can be traced back Prior and Bull's work in the sixties [84, 25]. Nominals were reinvented at several occasions. The history of hybrid languages and the many motivations for studying them will not be discussed further here, but the reader is referred to [83, 18] for two excellent expositions.

This chapter introduce three hybrid languages, $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$. Different apects of these languages, including expressivity, axiomatizations, interpolation and complexity, will be studied in Chapter 4–8. Besides giving the syntax and semantics of $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$, the present chapter also contains a number of syntactic normal form results that will be used later on.

Incidentally, one hybrid language has not yet been mentioned, namely $\mathcal{H}(@,\downarrow)$. For reasons that will become clear later on, the study of $\mathcal{H}(@,\downarrow)$ will be postponed until Chapter 9 of this thesis. For now, it suffices to say that, in many respects, $\mathcal{H}(@,\downarrow)$ is more similar to first-order logic than to the languages studied here.

3.1 Syntax and semantics of $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$

As was mentioned already, nominals are simply proposition letters whose interpretation is always a singleton set. In other words, nominals name elements of the domain.

Besides nominals, we will also consider *satisfaction operators*. Satisfaction operators are operators that make it possible to express that a formula holds at a world named by a nominal. An example of a formula containing a nominal and a satisfaction operator is $@_i\Diamond p$, which states that the world named i has a successor satisfying p.

The last addition to the language that we will consider is the *global modality*, E. It is a special modal operator that has as its accessibility relation the total relation. In other words, $\mathsf{E}\varphi$ holds at a world if there is a world (any world) in the model satisfying φ. The dual of E, denoted by A expresses global truth: $\mathsf{A}\varphi$ holds at a world if φ holds at every world in the model. Note that satisfaction operators can be defined using the global modality: $@_i\varphi$ is equivalent to $\mathsf{E}(i \wedge \varphi)$ and $\mathsf{A}(i \to \varphi)$.

Formally, let $\textsc{prop}$ be a countably infinite set of proposition letters, $\textsc{nom}$ a countably infinite set of nominals, and let $\textsc{mod}$ be a finite set of unary modalities (most of our results generalize to the case with infinitely many modalities, and to modality with arbitrary arity). Then the syntax of the languages $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$ is defined as follows.

$$\varphi ::= \top \mid p \mid i \mid \neg\varphi \mid \varphi \wedge \psi \mid \Diamond\varphi \qquad (\mathcal{H})$$

$$\varphi ::= \top \mid p \mid i \mid \neg\varphi \mid \varphi \wedge \psi \mid \Diamond\varphi \mid @_i\varphi \qquad (\mathcal{H}(@))$$

$$\varphi ::= \top \mid p \mid i \mid \neg\varphi \mid \varphi \wedge \psi \mid \Diamond\varphi \mid \mathsf{E}\varphi \qquad (\mathcal{H}(\mathsf{E}))$$

where $p \in \textsc{prop}$, $i \in \textsc{nom}$ and $\Diamond \in \textsc{mod}$. We use $\textsc{nom}(\varphi)$, $\textsc{mod}(\varphi)$ and $\textsc{prop}(\varphi)$ to refer to the respective symbols occurring in φ. We employ the usual abbreviations. In particular, $\Box\varphi$ is shorthand for $\neg\Diamond\neg\varphi$ and $\mathsf{A}\varphi$ is shorthand for $\neg\mathsf{E}\neg\varphi$.

The frames we work with are the same as for plain modal logic: they are of the form $\mathfrak{F} = (W, (R_\Diamond)_{\Diamond\in\textsc{mod}})$, where W is a set of worlds and each $R_\Diamond$ is a binary relation over W. Models for hybrid languages are pairs $(\mathfrak{F}, V)$, where $\mathfrak{F}$ is a frame and V is a valuation function for the proposition letters and nominals, such that $|V(i)| = 1$ for $i \in \textsc{nom}$. In other words, nominals are true at exactly one point in the model. Relative to such models, the formulas of our hybrid languages are evaluated as follows.

$$\begin{array}{lll}
(\mathfrak{M}, w) \models \top & & \\
(\mathfrak{M}, w) \models p & \text{iff} & w \in V(p) \\
(\mathfrak{M}, w) \models i & \text{iff} & w \in V(i) \\
(\mathfrak{M}, w) \models \neg\varphi & \text{iff} & (\mathfrak{M}, w) \not\models \varphi \\
(\mathfrak{M}, w) \models \varphi \wedge \psi & \text{iff} & (\mathfrak{M}, w) \models \varphi \text{ and } (\mathfrak{M}, w) \models \psi \\
(\mathfrak{M}, w) \models \Diamond\varphi & \text{iff} & \text{there is a } v \in W \text{ such that } wR_\Diamond v \text{ and } (\mathfrak{M}, v) \models \varphi \\
(\mathfrak{M}, w) \models @_i\varphi & \text{iff} & (\mathfrak{M}, v) \models \varphi \text{ where } V(i) = \{v\} \\
(\mathfrak{M}, w) \models \mathsf{E}\varphi & \text{iff} & \text{there is a } v \in W \text{ such that } (\mathfrak{M}, v) \models \varphi
\end{array}$$

where $p \in \textsc{prop}$, $i \in \textsc{nom}$ and $\Diamond \in \textsc{mod}$.

Validity and satisfiability with respect to a frame or class of frames is defined as for modal formulas. The frame class defined by a hybrid formula is simply the class consisting of the frames on which the formula is valid.

The modal depth of a hybrid formula φ is defined as on page 8, not counting occurences of satisfaction operators or the global modality, i.e., $md(@_i\varphi) = md(\mathsf{E}\varphi) = md(\varphi)$.

A hybrid formula is said to be *pure* if it contains no proposition letters (nominals are allowed).

3.2 First-order correspondence languages

The *first-order correspondence language* for our hybrid languages, $\mathcal{L}^1$, is the first-order language with equality over the vocabulary containing a constant c_i for each $i \in \text{NOM}$, a unary predicate symbol P_p for each $p \in \text{PROP}$, and a binary relation symbol $R_\Diamond$ for each $\Diamond \in \text{MOD}$. A model $\mathfrak{M} = (W, (R_\Diamond)_{\Diamond\in\text{MOD}}, V)$ can be seen as a model for the first-order language $\mathcal{L}^1$: the interpretation of the constants c_i is given by $V(i)$, the interpretation of the unary predicate symbols P_p is given by $V(p)$ and the interpretation of the binary relation symbols $R_\Diamond$ is the relations $R_\Diamond$ of $\mathfrak{M}$.

Note that we use the same notation, $\mathcal{L}^1$, to refer to the correspondence language for the basic modal language, and the correspondence language for hybrid languages. These two languages differ, in that the latter contains a constant for each nominal. It will always be clear from context which language we are referring to.

3.2.1. Definition. *The standard translation $ST_x(\cdot)$ maps formulas of $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$ to formulas of $\mathcal{L}^1$ with at most one free variable. It is defined as follows, where x and y are distinct first-order variables.*

$$\begin{array}{lcl}
ST_x(\top) & = & \top \\
ST_x(p) & = & P_p\, x \\
ST_x(i) & = & x = c_i \\
ST_x(\neg\varphi) & = & \neg ST_x(\varphi) \\
ST_x(\varphi \wedge \psi) & = & ST_x(\varphi) \wedge ST_x(\psi) \\
ST_x(\Diamond\varphi) & = & \exists y.(R_\Diamond xy \wedge ST_y(\varphi)) \\
ST_x(@_i\varphi) & = & \exists y.(y = c_i \wedge ST_y(\varphi)) \\
ST_x(\mathsf{E}\varphi) & = & \exists y.ST_y(\varphi)
\end{array}$$

3.2.2. Theorem ([47, 46, 16]). *For all hybrid formulas φ, models $\mathfrak{M}$ and worlds w, $\mathfrak{M}, w \models \varphi$ iff $\mathfrak{M} \models ST_x(\varphi)\ [x : w]$.*

When interpreted on frames, hybrid formulas express second order properties of frames. In this case, it is more appropriate to consider a first-order correspondence language that contains only the relation symbols interpreted by the frame, not the unary predicates and constants that are interpreted by the valuation function. Recall that the *first-order frame correspondence language*, $\mathcal{L}^1_{fr}$, is the first-order language with equality over the vocabulary containing a binary relation symbol $R_\Diamond$ for each $\Diamond \in \textsc{mod}$. Also recall that a frame $\mathfrak{F} = (W, (R_\Diamond)_{\Diamond\in\textsc{mod}})$ can be seen as a model for $\mathcal{L}^1_{fr}$: the interpretation of the binary relation symbols $R_\Diamond$ is the relations $R_\Diamond$ of the frame. While not every hybrid formula has an $\mathcal{L}^1_{fr}$-correspondent, some formulas do. For instance, $p \to \Diamond p$ defines the same class of frames as $\forall x.R_\Diamond xx$, and $i \to \neg\Diamond i$ defines the same class of frames as $\forall x.\neg R_\Diamond xx$. The next chapter is devoted to a comparison of the expressivity of hybrid formulas on the one hand and $\mathcal{L}^1$-formulas and $\mathcal{L}^1_{fr}$-formulas on the other hand.

3.3 Syntactic normal forms for hybrid formulas

This section contains results on syntactic normal forms for formulas of $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$. These results will be used in later parts of this thesis.

3.3.1. DEFINITION. *An $\mathcal{H}(@)$ formula φ is in* @-normal form *if no satisfaction operator occurs in the scope of a modal operator or of another satisfaction operator. If, in addition, φ is a Boolean combination of @-prefixed formulas, then φ is in* strong @-normal form*.*

With an @-prefixed formula, we mean a formula of the form $@_i\psi$. It is easy to see that the $\mathcal{H}(@)$-formulas in (strong) @-normal form are precisely the formulas generated by the following recursive definition, where χ is an $\mathcal{H}$-formula, and $i \in \textsc{nom}$.

$$\varphi \;::=\; \chi \mid @_i\chi \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi \quad \text{(@-normal form)}$$
$$\varphi \;::=\; \top \mid @_i\chi \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi \quad \text{(strong @-normal form)}$$

3.3.2. THEOREM. *Every $\mathcal{H}(@)$-formula is equivalent to an $\mathcal{H}(@)$-formula in @-normal form. Moreover, every $\mathcal{H}(@)$-formula of the form $@_i\psi$ is equivalent to an $\mathcal{H}(@)$-formula in strong @-normal form.*

Proof: If φ is a $\mathcal{H}(@)$-formula containing a subformula of the form $@_i\psi$, then φ is equivalent to $(@_j\psi\wedge\varphi[@_j\psi/\top])\vee(\neg @_j\psi\wedge\varphi[@_j\psi/\bot])$. By repeated application of this equivalence, any $\mathcal{H}(@)$ formula may be turned into a formula that is in @-normal form. Moreover, if the original formula was itself of the form $@_i\psi$ then the resulting formula will be in strong @-normal from. □

The exponential blowup involved in the proof of Theorem 3.3.2 cannot be avoided.

3.3.3. PROPOSITION. *There is no polynomial translation from $\mathcal{H}(@)$-formulas to $\mathcal{H}(@)$-formulas in @-normal form.*

Proof: Consider the sequence of $\mathcal{H}(@)$-formulas $\varphi_n = \Diamond \bigwedge_{k=1,\ldots,n}(p_k \leftrightarrow @_i p_k)$, with $n \in \omega$. Each φ_n has length polynomial in n, even if the bi-implication sign is treated as a defined connective. Now, take any sequence ψ_n $(n \in \omega)$ of formulas in @-normal form, such that the length of ψ_n is bounded by a polynomial in n. We will show that $\varphi_n \not\equiv \psi_n$ for some $n \in \omega$.

For $n \in \omega$, let F_n be the set of all functions $f : \{1 \ldots, n\} \to \{0, 1\}$. For each subset $G \subseteq F_n$, define a model $\mathfrak{M}_G = (W, R, V)$ as follows. The domain W consists of all $f \in G$, together with two extra worlds, w, v. The relation R connects w to each function $f \in G$. The valuation function V is such that $V(p_k) = \{f \in G \mid f(k) = 1\}$, for $k = 1, \ldots, n$, and $V(i) = \{v\}$.

Since the number of subsets of F_n is doubly exponential in n, and the number of subformulas of ψ_n is polynomial in n, for large enough n there must exist $G_1, G_2 \subseteq F_n$ such that $G_1 \neq G_2$ and such that $(\mathfrak{M}_{G_1}, w)$ and $(\mathfrak{M}_{G_2}, w)$ agree on the truth of all subformulas of ψ_n. Without loss of generality, we may assume that $G_1 \setminus G_2 \neq \emptyset$. Let $g \in G_1 \setminus G_2$. As a final step, let the models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be identical to $\mathfrak{M}_{G_1}$ and $\mathfrak{M}_{G_2}$, respectively, except that in both cases $v \in V(p_k)$ for all $k \leq n$ with $g(k) = 1$. A simple inductive argument shows that $\mathfrak{M}_1, w \models \psi_n$ iff $\mathfrak{M}_2, w \models \psi_n$. However, by construction, $\mathfrak{M}_1 \models \varphi_n$ and $\mathfrak{M}_2 \not\models \varphi_n$. We conclude that $\psi_n \neq \varphi_n$. □

For many purposes, rather than having a truth preserving translation from $\mathcal{H}(@)$-formulas to $\mathcal{H}(@)$-formulas in @-normal form, it is enough to have a translation that preserves satisfiability with respect to arbitrary frame classes. Translations of the latter kind are often enough for deriving complexity results or frame definability results. Fortunately, there is a polynomial time satisfiability preserving translation from $\mathcal{H}(@)$-formulas to $\mathcal{H}(@)$-formulas in @-normal form.

3.3.4. Theorem. *There is a polynomial time translation from $\mathcal{H}(@)$-formulas to $\mathcal{H}(@)$-formulas in strong @-normal form that preserves satisfiability on any frame class.*

Proof: Let any $\mathcal{H}(@)$-formula φ be given. Pick a new nominal i, and for every subformula of φ of the form $@_j\psi$ introduce a new proposition letter $p_{@_j\psi}$. Now, define the mapping $(.)^*$ on subformulas of φ as follows:

$$
\begin{aligned}
p^* &= p \\
i^* &= i \\
(\psi_1 \wedge \psi_2)^* &= \psi_1^* \wedge \psi_2^* \\
(\psi_1 \vee \psi_2)^* &= \psi_1^* \vee \psi_2^* \\
(\neg\psi)^* &= \neg\psi^* \\
(\Diamond\psi)^* &= \Diamond\psi^* \\
(@_j\psi)^* &= p_{@_j\psi}
\end{aligned}
$$

Note that $(.)^*$ maps subformulas of φ to formulas (of the extended language) in which no satisfaction operator occurs. Finally, we translate φ as $\varphi' =$

$$@_i(\varphi^*) \wedge \bigwedge_{\substack{@_j\psi \in Sub(\varphi) \\ m \leq md(\varphi) \\ k \in \text{NOM}(\varphi)\cup\{i\}}} (\neg @_j(\psi^*) \rightarrow @_k \Box^m \neg p_{@_j\psi}) \wedge (@_j(\psi^*) \rightarrow @_k \Box^m p_{@_j\psi})$$

where $md(\varphi)$ is the modal depth of φ. Note that φ' is in strong @-normal form, and that the length of φ' is polynomial in the length of φ. We claim that φ is satisfiable on a frame $\mathfrak{F}$ iff φ' is satisfiable on $\mathfrak{F}$. We prove both directions.

[$\Rightarrow$] Suppose $(\mathfrak{F}, V), w \Vdash \varphi$. Let V' be the valuation that extends V such that $V'(i) = \{w\}$ and such that $v \in V'(p_{@_j\psi})$ iff $(\mathfrak{F}, V), v \Vdash @_j\psi$, for all $v \in \mathfrak{F}$ and subformulas $@_j\psi$ of φ. A straightforward induction argument shows that $\mathfrak{M}, v \models \psi$ iff $\mathfrak{M}, v \models \psi^*$ for all worlds v and subformulas ψ of φ. From this, it follows that $\mathfrak{M}, w \models @_i\varphi^*$ and also (by definition of V') that all the other conjuncts of φ' are true at w. Hence, $\mathfrak{M}, w \models \varphi'$.

[$\Leftarrow$] Suppose $(\mathfrak{F}, V), w \Vdash \varphi'$. Let $V(i) = \{v\}$. Our task is to show that $(\mathfrak{F}, V), v \Vdash \varphi$.

For any point $u \in \mathfrak{F}$, let $d(u)$ be the minimal number of transitions needed to reach u from v or from some other point of $\mathfrak{F}$ denoted by one of the nominals occurring in φ (let $d(u) = \infty$ if u is not reachable from v nor from any other point denoted by a nominal occurring in φ). By construction (cf. the second conjunct of φ'), we have the following:

> for all $u \in \mathfrak{F}$ and for all subformulas $@_j\psi$ of φ, if $d(u) \leq md(\varphi^*)$ then $\mathfrak{M}, u \Vdash p_{@_j\psi}$ iff $\mathfrak{M}, u \Vdash @_j(\psi^*)$.

It follows by induction on ψ that

> for all $u \in \mathfrak{F}$ and for all subformulas ψ of φ, if $d(u) + md(\psi^*) \leq md(\varphi)$, then $\mathfrak{M}, u \models \psi^*$ iff $\mathfrak{M}, u \models \psi$.

Finally, we conclude that $\mathfrak{M}, v \Vdash \varphi$. □

It follows that the satisfiability problem for $\mathcal{H}(@)$-formulas, relative to any frame class, is polynomially reducible to the satisfiability problem of $\mathcal{H}(@)$-formulas in @-normal form, with respect to the same frame class. It also follows that every frame class definable by $\mathcal{H}(@)$-formulas is definable by $\mathcal{H}(@)$-formulas in @-normal form.

Next, let us consider the language $\mathcal{H}(\mathsf{E})$.

3.3.5. Definition. *An $\mathcal{H}(\mathsf{E})$ formula φ is in* E-normal form *if no occurrence of* E *is in the scope of a modal operator or of another occurrence of* E*. If, in addition, φ is a Boolean combination of* E*-prefixed formulas, then φ is in* strong E-normal form.

Again, with an E-prefixed formula, we mean a formula of the form $\mathsf{E}\psi$. Keep in mind that $\mathsf{A}\psi$ is shorthand for $\neg\mathsf{E}\neg\psi$. It is easy to see that the E-formulas in (strong) @-normal form are precisely the formulas generated by the following recursive definition, where χ is an $\mathcal{H}$-formula.

$$\begin{array}{rcll} \varphi & ::= & \chi \mid \mathsf{E}\chi \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi & (\mathsf{E}\text{-normal form}) \\ \varphi & ::= & \top \mid \mathsf{E}\chi \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi & (\text{strong } \mathsf{E}\text{-normal form}) \end{array}$$

By similar arguments as in the proof of Theorem 3.3.2, we obtain the following.

3.3.6. THEOREM ([56]). *Every $\mathcal{H}(\mathsf{E})$-formula is equivalent to an $\mathcal{H}(\mathsf{E})$-formula in E-normal form. Moreover, every $\mathcal{H}(\mathsf{E})$-formula of the form $@_i\psi$ is equivalent to an $\mathcal{H}(\mathsf{E})$-formula in strong E-normal form.*

Again, the polynomial translation can be found that turns every formula into an equisatisfiable formula in normal form.

3.3.7. THEOREM. *There is a polynomial time translation from $\mathcal{H}(\mathsf{E})$-formulas to $\mathcal{H}(\mathsf{E})$-formulas in strong E-normal form that preserves satisfiability on any frame class.*

Proof: Let any $\mathcal{H}(\mathsf{E})$-formula φ be given. For every subformula of φ of the form $\mathsf{E}\psi$ introduce a new proposition letter $p_{\mathsf{E}\psi}$. Now, define the mapping $(.)^*$ on subformulas of φ as follows:

$$\begin{array}{rcl} p^* & = & p \\ i^* & = & i \\ (\psi_1 \wedge \psi_2)^* & = & \psi_1^* \wedge \psi_2^* \\ (\psi_1 \vee \psi_2)^* & = & \psi_1^* \vee \psi_2^* \\ (\neg\psi)^* & = & \neg\psi^* \\ (\Diamond\psi)^* & = & \Diamond\psi^* \\ (\mathsf{E}\psi)^* & = & p_{\mathsf{E}\psi} \end{array}$$

Note that $(.)^*$ maps subformulas of φ to formulas (of the extended language) in which no satisfaction operator occurs. Finally, we translate φ as $\varphi' =$

$$\mathsf{E}\varphi^* \wedge \bigwedge_{\mathsf{E}\psi \in Sub(\varphi)} (\mathsf{E}\psi^* \to \mathsf{A}p_{\mathsf{E}\psi}) \wedge (\neg\mathsf{E}\psi^* \to \mathsf{A}\neg p_{\mathsf{E}\psi})$$

Note that φ' is in strong E-normal form, and that its length is polynomial in the lenght of φ. A similar argument as in the proof of Theorem 3.3.4 shows that φ is satisfiable on a frame $\mathfrak{F}$ iff φ' is satisfiable on $\mathfrak{F}$. □

Here finishes the introductory chapter of Part I. The following chapters will study different aspects of the languages $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$.

Chapter 4

Expressivity and definability

One of the main reasons why hybrid languages have gained popularity in the last decades is that many properties of frames that are not modally definable can be defined using nominals. Typical examples include irreflexivity ($i \to \neg\Diamond i$) and anti-symmetry ($(i \wedge \Diamond(j \wedge \Diamond i)) \to j$). These formulas are pure, meaning that they do not contain any proposition letters. A second important reason for the growing popularity of hybrid languages is a general completeness result for logics axiomatized by pure formulas.

Surprisingly little is known about the precise expressivity of hybrid languages. Ideally, one would like to have a Goldblatt-Thomason-style characterization of the frame classes definable by (sets of) (pure) formulas of $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$. The only known result in this direction is a characterization of the elementary frame classes definable in $\mathcal{H}(\mathsf{E})$, due to Gargov and Goranko [46]. Their proof is essentially algebraic in nature, and relies on a connection between $\mathcal{H}(\mathsf{E})$ and $\mathcal{M}(\mathsf{D})$ (i.e., the extension of the basic modal language with the difference operator, cf. Chapter 11). It is not clear how to generalize the proof to other hybrid languages.

In this chapter, we will characterize the elementary frame classes definable in $\mathcal{H}(@)$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$, as well as the elementary frame classes definable by pure formulas of these languages. Our proofs will be based on Van Benthem's model theoretic proof of the Goldblatt Thomason theorem [12] In order to state the characterizations, we will introduce two new types of morphisms between frames, which we will call *ultrafilter morphisms* and *bisimulation systems*. Our main results are summarized in Table 4.1.

Before we start, we would like to take note of the following curious fact concerning frame classes definable in $\mathcal{H}(@)$, the proof of which is straightforward. This result will not play any role in the remainder of this chapter but is interesting in its own right. Note that a similar result does not hold for the basic modal language.

Table 4.1: Elementary frame classes definable in $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$

	frame classes defined by arbitrary formulas	*frame classes defined by pure formulas*
$\mathcal{H}$	closed under ultrafilter morphic images, generated subframes, and (*)	closed under images of bisimulation systems, generated subframes, and (*)
$\mathcal{H}(@)$	closed under ultrafilter morphic images and generated subframes	closed under images of bisimulation systems and generated subframes
$\mathcal{H}(\mathsf{E})$	closed under ultrafilter morphic images	closed under images of bisimulation systems

(*) If every point-generated subframe of $\mathfrak{F}$ is a proper generated subframe of a frame in the class, then $\mathfrak{F}$ is in the class.

For *nominal bounded* $\mathcal{H}$-formulas (a notion that will be defined in Section 4.2), the condition (*) can be simplified to

(*′) If every point-generated subframe of $\mathfrak{F}$ is in the class, then $\mathfrak{F}$ is in the class.

4.0.8. PROPOSITION. *For all $\mathcal{H}(@)$ formulas φ, ψ that do not share any proposition letters, and for all distinct nominals i, j not occurring in φ and ψ, $@_i\varphi \vee @_j\psi$ defines the union of the frame classes defined by φ and ψ.*

4.1 Bisimulations and expressivity on models

Recall Theorem 2.2.3, which states that a formula $\varphi(x)$ of the first-order correspondence language for modal logic[1] with at most one free variable is equivalent to the standard translation of a modal formula iff $\varphi(x)$ is invariant under bisimulations. This result can be extended without much effort to the hybrid languages $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$, by slightly varying the definition of bisimulations.

4.1.1. DEFINITION. *An $\mathcal{H}$-bisimulation between models $\mathfrak{M} = (W, (R_\Diamond)_{\Diamond\in\text{MOD}}, V)$ and $\mathfrak{N} = (W', (S_\Diamond)_{\Diamond\in\text{MOD}}, V')$ is a binary relation $Z \subseteq W \times V$ satisfying the following conditions:*

> **Atom** *If wZv then $w \in V(p)$ iff $v \in V'(p)$, for all $p \in \text{PROP} \cup \text{NOM}$.*
>
> **Zig** *If wZv and $wR_\Diamond w'$, then there is a v' such that $vS_\Diamond v'$ and $w'Zv'$*
>
> **Zag** *If wZv and $vS_\Diamond v'$, then there is a w' such that $wR_\Diamond w'$ and $w'Zv'$*

An $\mathcal{H}(@)$-bisimulation is a $\mathcal{H}$-bisimulation Z satisfying in addition

> **Nom** *If $w \in V(i)$ and $v \in V'(i)$ for some $i \in \text{NOM}$, then wZv.*

An $\mathcal{H}(\mathsf{E})$-bisimulation is a total $\mathcal{H}$-bisimulation, i.e., a $\mathcal{H}$-bisimulation Z such that $\forall w \in W\ \exists v \in W'.wZv$ and $\forall v \in W'\ \exists w \in W.wZv$. Note that every $\mathcal{H}(\mathsf{E})$-bisimulation is a $\mathcal{H}(@)$-bisimulation.

These bisimulation notions capture, the indistinguishability relation for the languages $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$, in the same way that potential isomorphisms capture the indistinguishability relation for first-order logic. Let $\mathcal{L}$ be one of the languages $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$. Given two models $\mathfrak{M}, \mathfrak{N}$ with points w, v, we say that $\mathfrak{M}, w$ and $\mathfrak{N}, v$ are *$\mathcal{L}$-indistinguishable*, notation $\mathfrak{M}, w \equiv_\mathcal{L} \mathfrak{N}, v$, if for all $\mathcal{L}$-formulas φ, $\mathfrak{M}, w \models \varphi$ iff $\mathfrak{N}, v \models \varphi$. We say that $\mathfrak{M}, w$ and $\mathfrak{N}, v$ are *$\mathcal{L}$-bisimilar* if there is an $\mathcal{L}$-bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$ connecting w to v. When no ambiguity can arise, we will often not specify the models explicitly, and say that two worlds, w and v, are $\mathcal{L}$-bisimilar. A formula $\varphi(x_1, \ldots, x_n)$ of the first-order correspondence language is said to be *invariant under $\mathcal{L}$-bisimulations*, if for all models $\mathfrak{M}, \mathfrak{N}$, elements $d_1, \ldots, d_n$ of the domain of $\mathfrak{M}$ and elements $e_1, \ldots, e_n$ of the domain of $\mathfrak{N}$, if d_i and e_i are $\mathcal{L}$-bisimilar for $i = 1 \ldots n$, then $\mathfrak{M} \models \varphi\ [d_1, \ldots, d_n]$ iff $\mathfrak{N} \models \varphi\ [e_1, \ldots, e_n]$.

---

[1] The first-order correspondence language for modal logic differs from the first-order correspondence language for hybrid logic in that the latter has a constant for each nominal.

4.1.2. Theorem. *Let $\mathfrak{M}, \mathfrak{N}$ be models and w, v points in these models. Let $\mathcal{L}$ be one of the languages $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$. If w and v are $\mathcal{L}$-bisimilar then $\mathfrak{M}, w \equiv_{\mathcal{L}} \mathfrak{N}, v$. Conversely, if $\mathfrak{M}$ and $\mathfrak{N}$ are ω-saturated and $\mathfrak{M}, w \equiv_{\mathcal{L}} \mathfrak{N}, v$ then w and v are $\mathcal{L}$-bisimilar.*

The proof of Theorem 4.1.2 is a straightforward generalization of the one for modal logic, and the nominals do not give rise to additional complications. Using a standard argument, one obtains from this the following analogue of Theorem 2.2.3.

4.1.3. Theorem. *Let $\varphi(x)$ be an $\mathcal{L}^1$-formula with at most one free variable. Let $\mathcal{L}$ be one of the languages $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$. Then the following are equivalent:*

1. *$\varphi(x)$ is equivalent to the standard translation of an $\mathcal{L}$-formula*
2. *$\varphi(x)$ is invariant under $\mathcal{L}$-bisimulations.*

In other words, Van Benthem's bisimulation characterization for the basic modal language can be adapted without any problems to the hybrid languages $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$. If we consider frames rather than models, the situation will be quite different. While Goldblatt and Thomason's characterization of the modally definable elementary frame classes has hybrid analogues, obtaining these analogues requires more creativity, as we will see in the next section.

Before we go on the discuss frame definability, it is useful to introduce the notion of a generated submodel.

4.1.4. Definition (Generated submodel). *$\mathfrak{M} = (W, (R_\Diamond)_{\Diamond \in \text{MOD}}, V)$ is a* generated submodel *of $\mathfrak{N} = (W', (R'_\Diamond)_{\Diamond \in \text{MOD}}, V')$ if $\mathfrak{M}$ is a submodel of $\mathfrak{N}$ and for all $(w, v) \in R'_\Diamond$ ($\Diamond \in$ MOD), if $w \in W$ then $v \in W$.*

In other words, a generated submodel is a submodel whose domain is closed under the relations (cf. Appendix A for a definition of submodels). Clearly, if $\mathfrak{M}$ is a generated submodel of $\mathfrak{N}$ then $\mathfrak{M}$ must contain all elements of $\mathfrak{N}$ that are named by a nominal. For any model $\mathfrak{M}$ and for any subset X of the domain of $\mathfrak{M}$, the *submodel generated by* X is the smallest generated submodel of $\mathfrak{M}$ whose domain contains all elements of X. It is not hard to see that this is well-defined. In fact, the submodel of $\mathfrak{M}$ generated by X is precisely the submodel of $\mathfrak{M}$ whose domain consists of all worlds reachable from a world in X or from a world named by a nominal, in finitely many steps along the union of all relations. It is easy to see that if $\mathfrak{M}$ is a generated submodel of $\mathfrak{N}$, then the natural inclusion function, which is the identity function on the domain of $\mathfrak{M}$, is a $\mathcal{H}(@)$-bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$. It follows by Theorem 4.1.2 that $\mathcal{H}(@)$-formulas are invariant for generated submodels: for all worlds w of $\mathfrak{M}$ and $\mathcal{H}(@)$-formulas φ, $\mathfrak{M}, w \models \varphi$ iff $\mathfrak{N}, w \models \varphi$.

4.2 Operations on frames and formulas they preserve

In this section, we review and introduce several operations on frames, and we discuss to what extent they preserve validity of hybrid formulas. We focus on the three hybrid languages $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$. The frame operations discussed here will be put to use in the next sections, where we characterize the elementary frame classes definable in these hybrid languages.

Bounded morphisms

Frame validity of hybrid formulas is not preserved under taking images of bounded morphisms. Consider for instance the formula $i \to \neg\Diamond i$, which defines irreflexivity, and let $\mathfrak{F} = (\{0,1\}, \{(0,1),(1,0)\})$ and $\mathfrak{G} = (\{0\}, \{(0,0)\})$. Then $\mathfrak{G}$ is a bounded morphic image of $\mathfrak{F}$ and $\mathfrak{F}$ is irreflexive, but $\mathfrak{G}$ is not.

Nevertheless, in a restricted form bounded morphisms are of relevance for hybrid logic, as will become clear in Proposition 4.2.6.

Generated subframes

It is well known that validity of modal formulas containing the global modality is in general not preserved under taking generated subframes. A typical example is the formula $\mathsf{E}\Diamond\top$, which defines non-emptiness of the accessibility relation. Clearly, validity of $\mathcal{H}(\mathsf{E})$-formulas is also not preserved under taking generated subframes. However, taking generated subframes does preserve validity of $\mathcal{H}(@)$-formulas.

4.2.1. PROPOSITION. *$\mathcal{H}(@)$-definable frame classes are closed under generated subframes.*

Proof: The proof is the analogous to the one for the basic modal language: let $\mathfrak{F}$ be a generated subframe of $\mathfrak{G}$, and let V be any valuation for $\mathfrak{F}$. V can be conceived of as a valuation for $\mathfrak{G}$, by considering all proposition letters and nominals to be false at points outside of $\mathfrak{F}$. It is easily seen that the identity relation on $\mathfrak{F}$ is an $\mathcal{H}(@)$-bisimulation between the models $(\mathfrak{F}, V)$ and $(\mathfrak{G}, V)$. It follows that whenever a $\mathcal{H}(@)$-formula is refuted on $\mathfrak{F}$, it is refuted on $\mathfrak{G}$ under the same valuation and at the same point. □

Disjoint unions

The formula $\Diamond i$, which defines the class of frames in which $\Diamond$ is the global modality, nicely exemplifies the fact that validity of hybrid formulas is not preserved under taking disjoint unions. Nevertheless, a weak form of preservation under disjoint unions holds for the language $\mathcal{H}$: validity of $\mathcal{H}$-formulas is preserved under taking disjoint unions of frames that are not point generated. Generalizing this a bit further, we obtain the following preservation result.

4.2.2. PROPOSITION. *Let φ be an $\mathcal{H}$-formula, and $\mathfrak{F}$ a frame such that every point-generated subframe of $\mathfrak{F}$ is a proper generated subframe of a frame on which φ is valid. Then $\mathfrak{F} \models \varphi$.*

Proof: Let V be any valuation on $\mathfrak{F}$, and let w any point in $\mathfrak{F}$. By assumption, the point-generated subframe $\mathfrak{F}_w$ is a proper generated subframe of a frame $\mathfrak{G}$ with $\mathfrak{G} \models \varphi$. Let v be any point in $\mathfrak{G}$ that is not in $\mathfrak{F}_w$, and let V' be the valuation for $\mathfrak{G}$ defined as follows. For $p \in \text{PROP}$, let $V'(p)$ is the restriction of $V(p)$ to $\mathfrak{F}_w$. For $i \in \text{NOM}$, if $V(i)$ is in $\mathfrak{F}_w$ then let $V'(i) = V(i)$, otherwise let $V'(i) = \{v\}$. As is easily seen, the identity relation on $\mathfrak{F}_w$ is an $\mathcal{H}$-bisimulation between the models $(\mathfrak{F}, V)$ and $(\mathfrak{G}, V')$. Since $\mathfrak{G}, V', w \models \varphi$, we conclude that $\mathfrak{F}, V, w \models \varphi$. □

While validity of $\mathcal{H}$-formulas is in general not preserved under taking disjoint unions, there is a natural fragment of $\mathcal{H}$ that does satisfy this condition. Call an $\mathcal{H}$-formula *nominal bounded* if it is a conjunction of formulas of the form

$$\Big(\bigwedge_{k=1,\ldots,m} (\Diamond_{k,1} \cdots \Diamond_{k,n_k} i_k) \Big) \quad \to \quad \psi$$

where $\Diamond_{1,1}, \ldots, \Diamond_{m,n_m} \in \text{MOD}$ $(n_1, \ldots, n_m \geq 0)$ and ψ contains no nominals besides $i_1, \ldots, i_m$. Notice how the antecedent requires that all nominals occurring in ψ denote a point within the generated subframe. It is not hard to show that validity of nominal bounded formulas is preserved under taking disjoint union (cf. also Theorem 2.1 in [17]). In fact, something stronger holds. For any frame $\mathfrak{F}$ and world w of $\mathfrak{F}$, let $\mathfrak{F}_w$ denote the subframe of $\mathfrak{F}$ generated by $\mathfrak{F}$. $\mathfrak{F}_w$ is called a *point-generated subframe* of $\mathfrak{F}$, because it is generated by a single point. We say that a frame class K *reflects point-generated subframes* if for all frames $\mathfrak{F}$, if every point-generated subframe of $\mathfrak{F}$ is in K then $\mathfrak{F} \in \mathsf{K}$.

4.2.3. PROPOSITION. *Let φ be a nominal bounded $\mathcal{H}$-formula. Then the class of frames defined by φ reflects point-generated subframes.*

Proof: We reason by contraposition. Suppose $\mathfrak{F} \not\models \varphi$, i.e, one of the conjuncts of φ is falsified on $\mathfrak{F}$ at some point w under some valuation. By the truth of its antecedent, the nominals involved all denote points in the generated subframe $\mathfrak{F}_w$. Hence, the same conjunct of φ can be falsified on the point-generated subframe $\mathfrak{F}_w$. □

4.2.4. COROLLARY. *Let φ be a nominal bounded $\mathcal{H}$-formula and let $\{\mathfrak{F}_i \mid i \in I\}$ be a set of frames. If $\mathfrak{F}_i \models \varphi$ for all $i \in I$, then $\biguplus_{i \in I} \mathfrak{F}_i \models \varphi$.*

Ultrafilter extensions and ultrafilter morphisms

Unlike the frame operations discussed above, ultrafilter extensions anti-preserve validity of hybrid formulas in exactly the same way as modal formulas. If a $\mathcal{H}(\mathsf{E})$-formula φ is valid on the ultrafilter extension $\mathfrak{ue}\mathfrak{F}$ of a frame $\mathfrak{F}$, then φ is also valid on $\mathfrak{F}$ itself. In fact, something stronger holds: hybrid formulas are preserved under taking *ultrafilter morphic images*, to be defined below.

4.2.5. DEFINITION. *Let $\mathfrak{F}$ and $\mathfrak{G}$ be frames. $\mathfrak{G}$ is an ultrafilter morphic image of $\mathfrak{F}$ if there is a surjective bounded morphism $f : \mathfrak{F} \to \mathfrak{ue}\mathfrak{G}$ such that $|f^{-1}(u)| = 1$ for all principal ultrafilters $u \in \mathfrak{ue}\mathfrak{G}$.*

Since this construction will play an important role in the next section, we will try to provide some intuition for it. First of all, note that whenever $\mathfrak{G}$ is an ultrafilter morphic image of a frame $\mathfrak{F}$, $\mathfrak{ue}\mathfrak{G}$ is a bounded morphic image of $\mathfrak{F}$. It follows that the validity of modal formulas is preserved under taking ultrafilter morphic images. The same holds for $\mathcal{H}(\mathsf{E})$-formulas, even though the latter are not preserved under taking bounded morphic images, as we noticed before.

4.2.6. PROPOSITION. *Validity of $\mathcal{H}(\mathsf{E})$-formulas is preserved under taking ultrafilter morphic images.*

Proof: Let φ be an $\mathcal{H}(\mathsf{E})$-formula, let $f : \mathfrak{F} \to \mathfrak{ue}\mathfrak{G}$ be a surjective bounded morphism that is injective with respect to principal ultrafilters, and suppose $\mathfrak{G} \not\models \varphi$. We will show that $\mathfrak{F} \not\models \varphi$.

Let V be a valuation and w a world such that $\mathfrak{G}, V, w \not\models \varphi$. Define the valuation $V^{\mathfrak{ue}}$ on $\mathfrak{ue}\mathfrak{G}$ such that $V^{\mathfrak{ue}}(p) = \{u \mid V(p) \in u\}$ for all proposition letters p and $V^{\mathfrak{ue}}(i) = \{u \mid V(i) \in u\}$ for all nominals i. It is easily seen that $V^{\mathfrak{ue}}$ assigns to each nominal a singleton set consisting of a principal ultrafilter, and hence $V^{\mathfrak{ue}}$ is a well-defined hybrid valuation. Moreover, a standard argument [21, Proposition 2.59] shows that for all worlds v and formulas ψ, $(\mathfrak{G}, V), v \models \psi$ iff $(\mathfrak{ue}\mathfrak{G}, V^{\mathfrak{ue}}), \pi_v \models \psi$, where π_v is the principal ultrafilter generated by v. It follows that $\mathfrak{ue}\mathfrak{G}, V^{\mathfrak{ue}}, \pi_w \not\models \varphi$.

Next, define the valuation V' for $\mathfrak{F}$ such that $V'(p) = \{v \mid f(v) \in V^{\mathfrak{ue}}(p)\}$ for all proposition letters p and $V'(i) = \{v \mid f(v) \in V^{\mathfrak{ue}}(i)\}$ for all nominals i. Since f is injective on principal ultrafilters and nominals denote principal ultrafilters in $\mathfrak{ue}\mathfrak{G}$, $V'(i)$ is a singleton, for all nominals i, and hence $(\mathfrak{F}, V')$ is a well-defined hybrid model. Furthermore, a standard argument shows that, since f is a surjective bounded morphism, the graph of f is an $\mathcal{H}(\mathsf{E})$-bisimulation between $\mathfrak{ue}\mathfrak{G}$ and $\mathfrak{F}$. Since f is surjective, there is a $v \in \mathfrak{F}$ such that $f(v) = \pi_w$. By invariance under $\mathcal{H}(\mathsf{E})$-bisimulations, $\mathfrak{F}, V', u \not\models \varphi$, and hence $\mathfrak{F} \not\models \varphi$ □

Next, observe that every frame is an ultrafilter morphic image of its ultrafilter extension. It follows that, in general, if a property of frames is preserved under taking ultrafilter morphic images, then it is anti-preserved under taking ultrafilter extensions (i.e., its complement is preserved under taking ultrafilter extensions). The converse of this fact does not hold, as is shown by the following proposition.

4.2.7. PROPOSITION. *The frame condition $\forall x \exists y (R_2 xy \wedge \exists z.(R_1 xy \wedge y \neq z))$ is preserved under generated subframes and disjoint unions and anti-preserved under ultrafilter extensions, but it is not preserved under taking ultrafilter morphic images.*

Proof: It is easy to see that the given frame condition φ is preserved under taking generated subframes and disjoint unions. That it is anti-preserved under ultrafilter extensions can be seen as follows: the negation of φ is (modulo simple syntactic manipulations) a p-sentence, and is therefore, by Theorem 2.3.11, preserved under taking bounded morphic images. Furthermore, every first-order sentence preserved under taking bounded morphic images is preserved under taking ultrafilter extensions [21, Theorem 3.17]. It follows that φ itself is *anti*-preserved under taking ultrafilter extensions.

Next, we will show that φ is not preserved under taking ultrafilter morphic images. Consider the frame $\mathfrak{F} = (\omega, Id_\omega, \omega \times \omega)$. As is not hard to see, $\mathfrak{ue}\mathfrak{F} = (\mathrm{Uf}(\omega), Id_{\mathrm{Uf}(\omega)}, \mathrm{Uf}(\omega) \times \mathrm{Uf}(\omega))$. Let $\mathfrak{G} = (W, R, W \times W)$, where $W = \{u \in \mathrm{Uf}(\omega) \mid u \text{ is principal}\} \cup \{\langle u, 0\rangle, \langle u, 1\rangle \mid u \in \mathrm{Uf}(\omega) \text{ is non-principal}\}$ and $R = Id_W \cup \{(\langle u, 0\rangle, \langle u, 1\rangle), (\langle u, 1\rangle, \langle u, 0\rangle) \mid u \in \mathrm{Uf}(\omega) \text{ is non-principal}\}$. As one can easily see, the natural map from $\mathfrak{G}$ to $\mathfrak{ue}\mathfrak{F}$ is a surjective bounded morphism and is injective with respect to principal ultrafilters. However, $\mathfrak{G}$ satisfies φ whereas $\mathfrak{F}$ does not. □

Finally, let us spend some words on preservation under ultrafilters (as opposed to anti-preservation). It is known that validity of modal formulas that define first-order frame conditions is preserved under passage from a frame to its ultrafilter extension [21, Corollary 3.18]. The question arises if a similar result hold for hybrid logic. The answer is negative. Consider the formula $i \to \Box \neg i$, which expresses the first-order property of irreflexivity. The natural numbers with their strict ordering clearly form an irreflexive frame, and its ultrafilter extension contains reflexive points (in fact, every non-principal ultrafilter forms an reflexive point).

Bisimulation systems

If a modal formula contains no proposition letters, its validity on a frame is preserved under total bisimulations. This fact is well-known, and follows immediately from Theorem 2.2.3. Note that a bisimulation between frames $\mathfrak{F}$ and $\mathfrak{G}$ is just what one could expect: a binary relation between the domains of $\mathfrak{F}$ and $\mathfrak{G}$ satisfying the *zig* and *zag* clauses of Definition 4.1.1. Also recall that a bisimulation is total if its domain includes every point of $\mathfrak{F}$ and its range includes every point in $\mathfrak{G}$.

A similar result can be obtained for hybrid logic. Recall that in hybrid logic, pure formulas are the ones that contain no proposition letters, though possibly nominals. In general, validity of pure hybrid formulas is clearly not preserved under total bisimulations. It is however preserved under *bisimulation systems*, as defined below.

4.2.8. DEFINITION. *Given a bisimulation Z between frames $\mathfrak{F}$ and $\mathfrak{G}$, and a subset X of the domain of $\mathfrak{G}$, we say that Z* respects *X if the following two conditions hold for all $x \in X$:*

1. *There exists exactly one w such that wZx.*

2. *For all w, v, if wZx and wZv then $v = x$.*

4.2.9. DEFINITION. *A bisimulation system from $\mathfrak{F}$ to $\mathfrak{G}$ is a function $\mathcal{Z}$ that assigns to each finite subset $X \subseteq \mathfrak{G}$ a total bisimulation $\mathcal{Z}(X) \subseteq \mathfrak{F} \times \mathfrak{G}$ respecting X.*

4.2.10. THEOREM. *Validity of pure $\mathcal{H}(\mathsf{E})$-formulas is preserved under taking images of bisimulation systems.*

Proof: Let $\mathcal{Z}$ be a bisimulation system between $\mathfrak{F}$ and $\mathfrak{G}$, and suppose $\mathfrak{G} \not\models \varphi$, for some pure $\mathcal{H}(\mathsf{E})$-formula φ. We will show that $\mathfrak{F} \not\models \varphi$. Let $i_1, \ldots, i_n$ be the nominals occurring in φ. Let V be an assignment for these nominals and $v \in \mathfrak{G}$ a world such that $(\mathfrak{G}, V), v \not\models \varphi$. Let $v_1, \ldots, v_n$ be the worlds named by the nominals $i_1, \ldots, i_n$. Let $Z = \mathcal{Z}(\{v_1, \ldots, v_n\})$ be a bisimulation respecting $v_1, \ldots, v_n$, and define V' to be the valuation for $\mathfrak{F}$ that sends every nominal i_k to the unique point w_k such that $w_k Z v_k$. Then Z is easily seen to be a total bisimulation between $(\mathfrak{F}, V')$ and $(\mathfrak{G}, V)$. Hence, φ is falsified somewhere in the model $(\mathfrak{F}, V')$, and therefore $\mathfrak{F} \not\models \varphi$. □

Bisimulation systems are, intuitively speaking, a cross-over between bisimulations and potential isomorphisms. On the one hand, they can be viewed as parametrized bisimulations, while on the other hand, they are families of finite partial isomorphisms satisfying some further conditions.

Not every modally definable frame class is closed under images of bisimulation systems. A typical example is the confluence property, defined by the first-order formula $\forall xyz(xRy \land xRz \to \exists u.(yRu \land zRu))$, and also by the modal formula $\Diamond\Box p \to \Box\Diamond p$. It was shown by Gargov and Goranko [46] that confluence is not definable by means of pure $\mathcal{H}(\mathsf{E})$-formulas. Their proof can be modified to show that the class of confluent frames is not closed under images of bisimulation systems.

4.2.11. PROPOSITION. *The class of confluent frames is not closed under images of bisimulation systems.*

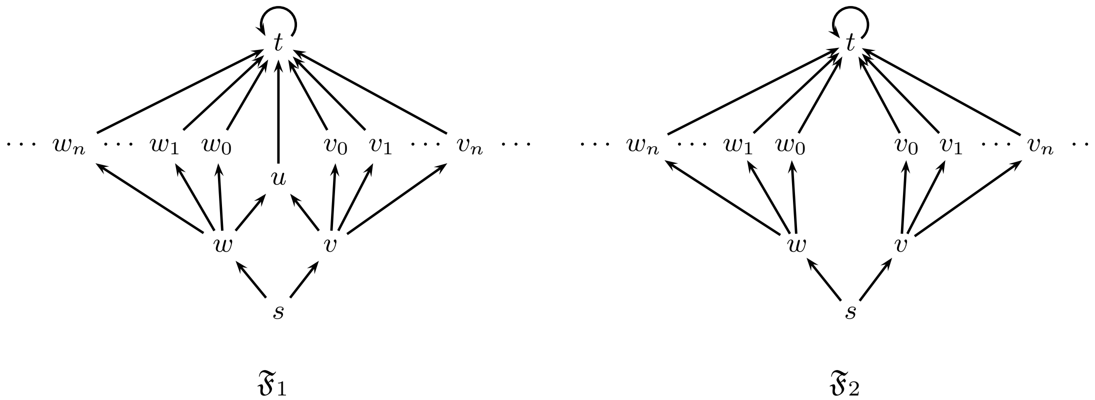

Figure 4.1: Confluence is not definable by pure formulas

Proof: Consider the two frames depicted in Figure 4.1. Notice that $\mathfrak{F}_1$ is identical to $\mathfrak{F}_2$, except for the additional point u (and its incoming and outgoing arrows). For any finite set $X \subseteq^{fin} \mathfrak{F}_2$, let $\mathcal{Z}(X) = Id_{\mathfrak{F}_2} \cup \{(u, w_k), (u, v_l)\}$, for $w_k, v_l \notin X$ (note that such w_k and v_l exist). As is not hard to see, $\mathcal{Z}$ is a bisimulation system from $\mathfrak{F}_1$ to $\mathfrak{F}_2$. However, $\mathfrak{F}_1$ is confluent, whereas $\mathfrak{F}_2$ does not.

Incidentally, the frame $\mathfrak{F}_2$ used above was first introduced by Venema [98], and the same frame was used by Gargov and Goranko [46] to show that confluence is not definable by means of pure $\mathcal{H}(\mathsf{E})$-formulas. □

As it happens, confluence can be defined by a pure $\mathcal{H}$-formula using backward looking modalities, namely by the formula $\Diamond i \rightarrow \Box\Diamond\Diamond^{-1} i$. This raises the question whether there are modally definable frame conditions that are not preserved under bisimulation systems even in the presence of backward looking modalities. Indeed there are such. Call a relation R *atomic* if it satisfies $\forall x \exists y (Rxy \wedge \forall z (Ryz \rightarrow y = z))$. Let K_{at} be the (elementary) class of bimodal frames $\mathfrak{F} = (W, R_1, R_2)$, in which R_1 is transitive and atomic and in which R_2 is the converse of R_1. Then K_{at} is defined by the conjunction of the transitivity axiom $\Diamond_1\Diamond_1 p \rightarrow \Diamond_1 p$, the McKinsey axiom $\Box_1\Diamond_1 p \rightarrow \Diamond_1\Box p$ and the axioms $p \rightarrow \Box_1\Diamond_2 p$ and $p \rightarrow \Box_2\Diamond_1 p$.

4.2.12. Proposition. *The class K_{at} is not closed under images of bisimulation systems.*

Proof: We will construct a bisimulation system from $(\mathbb{N}, \geq, \leq)$ to $(\mathbb{Z}, \geq, \leq)$. Clearly, the former is atomic while the latter is not, and hence the result follows. For any finite $X \subseteq \mathbb{Z}$, let $\mathcal{Z}(X) = \{(0, m) \mid m \leq \min X\} \cup \{(n, n - 1 + \min X) \mid n > 0\}$. It is not hard to see that for all finite $X \subseteq \mathbb{Z}$, $\mathcal{Z}(X)$ is a bisimulation between $(\mathbb{N}, \geq, \leq)$ and $(\mathbb{Z}, \geq, \leq)$ respecting X, and hence $\mathcal{Z}$ is a bisimulation system. □

It follows that K_{at} is not definable by pure $\mathcal{H}(\mathsf{E})$-formulas, as was shown already by [17].

Some lemmas

The following lemmas are of a more technical nature. They will be put to good use in the next section.

4.2.13. Lemma. *If $\mathfrak{G}_i$ is an ultrafilter morphic image of $\mathfrak{F}_i$, for $i = 1, 2$, then $\mathfrak{G}_1 \uplus \mathfrak{G}_2$ is an ultrafilter morphic image of $\mathfrak{F}_1 \uplus \mathfrak{F}_2$.*

Proof: Let $f_i : \mathfrak{F}_i \to \mathfrak{ue}\mathfrak{G}_i$ be surjective ultrafilter morphisms ($i = 1, 2$), and define $f : (\mathfrak{F}_1 \uplus \mathfrak{F}_2) \to (\mathfrak{ue}\mathfrak{G}_1 \uplus \mathfrak{ue}\mathfrak{G}_2)$ such that $f(w) = f_i(w)$ for $w \in \mathfrak{F}_i$. Then f is easily seen to be a surjective bounded morphism. Moreover, f is injective on principal ultrafilters, in the sense that $|f^{-1}(u)| = 1$ for all principal $u \in (\mathfrak{ue}\mathfrak{G}_1 \uplus \mathfrak{ue}\mathfrak{G}_2)$. Next, observe that $\mathfrak{ue}\mathfrak{G}_1 \uplus \mathfrak{ue}\mathfrak{G}_2$ is isomorphic to $\mathfrak{ue}(\mathfrak{G}_1 \uplus \mathfrak{G}_2)$. Moreover, the natural isomorphism $g : (\mathfrak{ue}\mathfrak{G}_1 \uplus \mathfrak{ue}\mathfrak{G}_2) \cong \mathfrak{ue}(\mathfrak{G}_1 \uplus \mathfrak{G}_2)$, which maps every ultrafilter $u \in \mathfrak{ue}\mathfrak{G}_i$ to the ultrafilter $\{X \uplus Y \mid X \in u \text{ and } Y \subseteq \mathfrak{G}_{3-i}\}$, preserves principality of ultrafilters. Hence the concatenation $f \cdot g$ is a surjective ultrafilter morphism from $\mathfrak{F}_1 \uplus \mathfrak{F}_2$ to $\mathfrak{ue}(\mathfrak{G}_1 \uplus \mathfrak{G}_2)$. □

4.2.14. Lemma. *If $\mathfrak{F}_1$ and $\mathfrak{G}_1$ are elementarily equivalent and $\mathfrak{F}_2$ and $\mathfrak{G}_2$ are elementarily equivalent then $\mathfrak{G}_1 \uplus \mathfrak{G}_2$ and $\mathfrak{F}_1 \uplus \mathfrak{F}_2$ are elementarily equivalent.*

Proof: A simple Ehrenfeucht-Fraïssé game argument establishes the result: by elementary equivalence, Duplicator has a winning strategy in every finite round Ehrenfeucht-Fraïssé game on $(\mathfrak{F}_1, \mathfrak{G}_1)$ and on $(\mathfrak{F}_2, \mathfrak{G}_2)$. These strategies naturally combine into a winning strategy for any finite round game on $(\mathfrak{F}_1 \uplus \mathfrak{F}_2, \mathfrak{G}_1 \uplus \mathfrak{G}_2)$: whenever Spoiler picks an element of some $\mathfrak{F}_i$ or $\mathfrak{G}_i$ ($1 \leq i \leq 2$), Duplicator responds with an element of $\mathfrak{G}_i$ respectively $\mathfrak{F}_i$, using his winning strategy for the game on $(\mathfrak{F}_i, \mathfrak{G}_i)$. In this way, Duplicator clearly maintains partial isomorphisms between $\mathfrak{F}_1$ and $\mathfrak{G}_1$ and between $\mathfrak{F}_2$ and $\mathfrak{G}_2$, and therefore also between $\mathfrak{F}_1 \uplus \mathfrak{F}_2$ and $\mathfrak{G}_1 \uplus \mathfrak{G}_2$. □

4.2.15. Lemma. *If there are bisimulation systems from $\mathfrak{F}_1$ to $\mathfrak{G}_1$ and from $\mathfrak{F}_2$ to $\mathfrak{G}_2$, then there is a bisimulation system from $\mathfrak{F}_1 \uplus \mathfrak{F}_2$ to $\mathfrak{G}_1 \uplus \mathfrak{G}_2$.*

Proof: Let f_1 and f_2 be the given bisimulations, and for all finite $X \subseteq \mathfrak{G}_1 \uplus \mathfrak{G}_2$, let $f(X) = f_1(X \cap \mathfrak{G}_1) \cup f_2(X \cap \mathfrak{G}_2)$. It is not hard to see that f is a bisimulation system from $\mathfrak{F}_1 \uplus \mathfrak{F}_2$ to $\mathfrak{G}_1 \uplus \mathfrak{G}_2$. □

4.3 Frame definability

In this section, we answer the question which elementary frame classes are definable by a set of formulas of hybrid logic. The results will be stated in terms of the operations on frames discussed in the previous section. The proofs in this section are inspired by Van Benthem's model theoretic proof of the Goldblatt-Thomason theorem [12].

As a point of notation, recall that for a frame $\mathfrak{F} = (W, (R_\Diamond)_{\Diamond \in \text{MOD}})$ and a set $X \subseteq W$, we use $m_\Diamond(X)$ to denote the set $\{w \in W \mid \exists v \in X.(wR_\Diamond v)\}$.

4.3.1. THEOREM. *An elementary frame class* K *is definable by a set of* $\mathcal{H}(@)$*-formulas iff* K *is closed under ultrafilter morphic images and generated subframes.*

Proof: The left-to-right direction was proved already in the previous section. For the right-to-left-direction, we proceed as follows. Let $Th(\mathsf{K})$ be the set of $\mathcal{H}(@)$-formulas valid on K, and suppose $\mathfrak{F} \models Th(\mathsf{K})$. It is our task to show that $\mathfrak{F} \in \mathsf{K}$. For each subset $A \subseteq W$, where W is the domain of $\mathfrak{F}$, introduce a proposition letter p_A. For every $w \in W$, introduce a nominal i_w. Let Δ be the set consisting of the following formulas, for all $A \subseteq W$, $v \in W$ and $\Diamond \in$ MOD.

$$\begin{aligned}
&p_{-A} \leftrightarrow \neg p_A \\
&p_{A \cap B} \leftrightarrow p_A \wedge p_B \\
&p_{m_\Diamond(A)} \leftrightarrow \Diamond p_A \\
&i_v \leftrightarrow p_{\{v\}}
\end{aligned}$$

Let $\Delta_{\mathfrak{F}} = \{@_{i_v} \Box_1 \cdots \Box_n \delta \mid v \in W, \delta \in \Delta, \text{ and } \Diamond_1, \ldots, \Diamond_n \in \text{MOD} \text{ with } n \in \omega\}$. Intuitively, $\Delta_{\mathfrak{F}}$ provides a full description of the frame $\mathfrak{F}$. Clearly, $\Delta_{\mathfrak{F}}$ is satisfiable on $\mathfrak{F}$, namely at any point, under the natural valuation that sends p_A to A and i_w to $\{w\}$.

Claim 1: $\Delta_{\mathfrak{F}}$ is satisfiable on K.

Proof of claim: By compactness (recall that K is elementary), it suffices to show that every finite conjunction δ of elements of $\Delta_{\mathfrak{F}}$ is satisfiable on K. But this follows immediately: δ is satisfiable on $\mathfrak{F}$ and $\mathfrak{F} \models Th(\mathsf{K})$, hence $\neg\delta \notin Th(\mathsf{K})$, i.e., δ is satisfiable on K. ⊣

Let $(\mathfrak{G}, V) \models \Delta_{\mathfrak{F}}$ with $\mathfrak{G} \in \mathsf{K}$. Since K is closed under generated subframes, we may assume that $\mathfrak{G}$ is generated by the set of points that are named by a nominal. It then follows that the model $(\mathfrak{G}, V)$ globally satisfies Δ. Let $(\mathfrak{G}^*, V^*)$ be an ω-saturated elementary extension of $(\mathfrak{G}, V)$. By elementarity, $\mathfrak{G}^* \in \mathsf{K}$ and $(\mathfrak{G}^*, V^*)$ globally satisfies Δ.

Claim 2: $\mathfrak{ue}\mathfrak{F}$ is an ultrafilter morphic image of $\mathfrak{G}^*$.

Proof of claim: For any $v \in \mathfrak{G}^*$, let $f(v) = \{A \subseteq W \mid (\mathfrak{G}^*, V^*), v \models p_A\}$. We claim that f is a bounded morphism from $\mathfrak{G}^*$ onto $\mathfrak{ue}\mathfrak{F}$, and $|f^{-1}(u)| = 1$ for all principal ultrafilters $u \in \mathfrak{ue}\mathfrak{F}$.

▷ *$f(v)$ is an ultrafilter on $\mathfrak{F}$.*

Follows immediately from the the fact that Δ is globally satisfied in the model $(\mathfrak{G}^*, V^*)$.

▷ *f is surjective*

Take any $u \in \mathfrak{ue}\mathfrak{F}$. To prove surjectiveness, we will show that that the set $\{p_A \mid A \in u\}$ is satisfiable in $(\mathfrak{G}^*, V^*)$. By ω-saturatedness, it suffices to show finitely satisfiability. Take $A_1, \ldots, A_n \in u$. Then $\bigcap_k A_k \in u$ and hence, $\bigcap_k A_k \neq \emptyset$. Let $v \in \bigcap_k A_k$. Then $\Delta_{\mathfrak{F}} \models @_{i_v} p_{\bigcap_k A_k}$, and hence $(\mathfrak{G}^*, V^*) \models @_{i_v} p_{\bigcap_k A_k}$.

▷ *Forth-condition: If $vR^*_\Diamond v'$ then $f(v) R^{\mathfrak{ue}}_\Diamond f(v')$*

By the definition of $R^{\mathfrak{ue}}_\Diamond$, it suffices to show that whenever $A \in f(v')$, $m_\Diamond A \in f(v)$. Suppose $A \in f(v')$. Then $(\mathfrak{G}^*, V^*), v' \models p_A$, hence by the global truth of Δ, $(\mathfrak{G}^*, V^*), v \models p_{m_\Diamond A}$, and therefore $m_\Diamond A \in f(v)$.

▷ *Back condition: If $f(v) R^{\mathfrak{ue}}_\Diamond u$, there is a $v' \in \mathfrak{G}^*$ s.t. $f(v') = u$ and $vR^*_\Diamond v'$*

We have to find a $\Diamond$-successor of v that satisfies $\{p_A \mid A \in u\}$. By ω-saturatedness, it suffices to show that this theory is finitely satisfiable in the set of $\Diamond$-successors of v. Take any $A_1 \ldots A_n \in u$. Then $\bigcap_i A_i \in u$ and hence, $m_\Diamond(\bigcap_i A_i) \in f(v)$. So, $(\mathfrak{G}^*, V^*), v \models p_{m_\Diamond(\bigcap_i A_i)}$, and hence, by global truth of Δ, v has a successor satisfying $p_{A_1}, \ldots, p_{A_n}$.

▷ *$|f^{-1}(u)| = 1$ for all principal ultrafilters $u \in \mathfrak{ue}\mathfrak{F}$*

Suppose $f(x) = f(y) = \pi_w$ for some $x, y \in \mathfrak{G}^*$ and $w \in \mathfrak{F}$. Then by definition, x and y satisfy the proposition letter $p_{\{w\}}$. By global truth of Δ, x and y are both named by the nominal i_w. Hence, $x = y$. ⊣

Since K is closed under ultrafilter morphic images, we conclude that $\mathfrak{F} \in \mathsf{K}$. □

4.3.2. Corollary. *An elementary frame class is definable by a set of $\mathcal{H}(\mathsf{E})$-formulas iff it is closed under ultrafilter morphic images.*

Proof: The global modality is definable by an $\mathcal{H}(@)$-formula, namely the formula $\mathsf{E}i$ (more precisely, this formula expresses that the accessibility relation of the modality E is the total relation). It follows that a frame class K is $\mathcal{H}(\mathsf{E})$-definable iff the class $\mathsf{K}' = \{(W, (R_\Diamond)_{\Diamond \in \text{MOD}}, R_\mathsf{E}) \mid (W, (R_\Diamond)_{\Diamond \in \text{MOD}}) \in \mathsf{K} \text{ and } R_\mathsf{E} = W^2\}$ is $\mathcal{H}(@)$-definable. Clearly, K' is closed under generated subframes. Furthermore, one can easily see that K' is closed under ultrafilter morphic images iff K is. The result follows. □

Gargov and Goranko [46] gave a similar characterization of the $\mathcal{H}(\mathsf{E})$-definable elementary frame classes, cf. Section 11.1 for a comparison of the two.

Next, we investigate frame definability in $\mathcal{H}$. This case for is a little more complicated. We need the following lemma. Recall that a $\mathcal{H}$-formula is nominal bounded if it is a conjunction of formulas of the form

$$\Big(\bigwedge_{k=1\dots m}(\Diamond_{k,1}\cdots\Diamond_{k,n_k}i_k)\Big) \quad\to\quad \varphi$$

where $\Diamond_{1,1},\dots,\Diamond_{m,n_m}\in$ MOD and φ contains no nominals besides $i_1,\dots,i_m$.

4.3.3. LEMMA. *Let K be a class of frames, and let $Th_{\mathcal{H}}(\mathsf{K})$ and $Th_{\mathcal{H}(@)}(\mathsf{K})$ be the set of $\mathcal{H}$-formulas and $\mathcal{H}(@)$-formulas, respectively, valid on K, and let $Th^{nb}_{\mathcal{H}}(\mathsf{K})$ be the set of nominal bounded $\mathcal{H}$-formulas valid on K. For all point-generated frames $\mathfrak{F}_w$, $\mathfrak{F}_w\models Th_{\mathcal{H}(@)}(\mathsf{K})$ iff $\mathfrak{F}_w\models Th_{\mathcal{H}}(\mathsf{K})$ iff $\mathfrak{F}_w\models Th^{nb}_{\mathcal{H}}(\mathsf{K})$.*

Proof: The left-to-right-directions are immediate. Now, suppose $\mathfrak{F}_w\models Th^{nb}_{\mathcal{H}}(\mathsf{K})$. Consider any $\varphi\in Th_{\mathcal{H}(@)}(\mathsf{K})$. By Theorem 3.3.2 (and by prefixing φ by $@_i$ for some new nominal i, if necessary), we may assume that φ is of the form $\bigwedge_m\bigvee_n @_{i_{m,n}}\varphi_{m,n}$, where each $\varphi_{m,n}$ is an $\mathcal{H}$-formula. For $k\in\omega$, let ψ_k be the following nominal bounded $\mathcal{H}$-formula:

$$\Big(\bigwedge_{m,n}\Diamond^{\leq k}(i_{m,n})\Big)\to\bigwedge_{m}\bigvee_{n}\Box^{\leq k}(i_{m,n}\to\varphi_{m,n})$$

Clearly, φ implies ψ_k, and therefore, $\psi_k\in Th^{nb}_{\mathcal{H}}(\mathsf{K})$, for each $k\in\omega$. It follows that $\mathfrak{F}_w\models\psi_k$ for all $k\in\omega$. But then $\mathfrak{F}_w\models\varphi$. For, suppose not. Then there is a valuation V and a world v such that $(\mathfrak{F}_w,V),v\not\models\varphi$. Since φ is a Boolean combination of @-prefixed formulas, its truth is not dependent on the world of evaluation, and hence $(\mathfrak{F}_w,V),w\not\models\varphi$. Now, let k be the maximal distance from the root w to a world named by one of the (finitely many) nominals occurring in the formula. Then, clearly, $(\mathfrak{F}_w,V),w\not\models\psi_k$. But this contradicts the fact that $\mathfrak{F}_w\models\psi_k$. □

4.3.4. THEOREM. *An elementary frame class K is definable by a set of $\mathcal{H}$-formulas iff the following closure conditions hold.*

1. *K is closed under ultrafilter morphic images.*
2. *K is closed under generated subframes.*
3. *For any frame $\mathfrak{F}$, if every point generated subframe of $\mathfrak{F}$ is a proper generated subframe of a frame in K, then $\mathfrak{F}\in\mathsf{K}$.*

Proof: The left-to-right direction was proved already in the previous section. For the right-to-left-direction, we proceed as follows. Let $Th(\mathsf{K})$ be the set of $\mathcal{H}$-formulas valid on K, and suppose $\mathfrak{F} \models Th(\mathsf{K})$. It is our task to show that $\mathfrak{F} \in \mathsf{K}$. If $\mathfrak{F}$ is point-generated, then by Lemma 4.3.3, in combination with Theorem 4.3.1, $\mathfrak{F} \in \mathsf{K}$, and we are done. In the remainder of this proof, we will assume that $\mathfrak{F}$ is *not* point-generated. Take any point-generated subframe $\mathfrak{F}_w = (W, (R_\diamond)_{\diamond \in \text{MOD}})$ of $\mathfrak{F}$. In what follows, we will show that $(\mathfrak{F}_w \uplus \mathfrak{F}_w) \in \mathsf{K}$. It then follows by the third closure condition that $\mathfrak{F} \in \mathsf{K}$.

For each subset $A \subseteq W$, introduce a proposition letter p_A, and for every $w \in W$, introduce a nominal i_w. Furthermore, let $i_\emptyset$ a distinct nominal. Let Δ be the set consisting of the following formulas, for all $A \subseteq W$, $v \in W$ and $\diamond \in \text{MOD}$.

$$\begin{array}{l} p_{-A} \leftrightarrow \neg p_A \\ p_{A \cap B} \leftrightarrow p_A \wedge p_B \\ p_{m_\diamond(A)} \leftrightarrow \diamond p_A \\ i_v \leftrightarrow p_{\{v\}} \end{array}$$

Let $\Delta_{\mathfrak{F}_w} = \{i_w\} \cup \{\Box_1 \cdots \Box_n \delta \mid \delta \in \Delta \text{ and } \diamond_1, \ldots, \diamond_n \in \text{MOD with } n \in \omega\} \cup \{\Box_1 \cdots \Box_n \neg i_\emptyset \mid \diamond_1, \ldots, \diamond_n \in \text{MOD with } n \in \omega\}$. Intuitively, $\Delta_{\mathfrak{F}_w}$ provides a full description of the frame $\mathfrak{F}$, from the perspective of w. Clearly, $\Delta_{\mathfrak{F}_w}$ is satisfiable on $\mathfrak{F}$, namely at w, under any valuation that sends each p_A to A, each i_w to $\{w\}$ and $i_\emptyset$ to some point not reachable from w in finitely many steps.

Claim 1: $\Delta_{\mathfrak{F}_w}$ is satisfiable on K.

Proof of claim: By compactness (recall that K is elementary), it suffices to show that every finite conjunction δ of elements of $\Delta_{\mathfrak{F}_w}$ is satisfiable on K. But this follows immediately: δ is satisfiable on $\mathfrak{F}$ and $\mathfrak{F} \models Th(\mathsf{K})$, hence $\neg\delta \notin Th(\mathsf{K})$, i.e., δ is satisfiable on K. $\dashv$

Let $(\mathfrak{G}, V), v \models \Delta_{\mathfrak{F}_w}$, with $\mathfrak{G} \in \mathsf{K}$. Let $\mathfrak{G}_v$ be the subframe of $\mathfrak{G}$ generated by v. By construction, $\mathfrak{G}_v$ is a proper generated subframe of $\mathfrak{G}$. Hence, by the third closure condition, $(\mathfrak{G}_v \uplus \mathfrak{G}_v) \in \mathsf{K}$.

By construction, all nominals except $i_\emptyset$ denote a point in $(\mathfrak{G}, V)$ that is reachable from v. Hence we can think of V as a valuation for the frame $\mathfrak{G}_v$ by removing $i_\emptyset$ from our vocabulary. In this way, we obtain a point-generated model $(\mathfrak{G}_v, V)$ that globally satisfies Δ, and such that $(\mathfrak{G}_v, V), v \models p_A$ for all $A \subseteq W$ with $w \in A$.

Let $(\mathfrak{G}^*_w, V^*)$ be an ω-saturated elementary extension of $(\mathfrak{G}_w, V)$. By elementarity, $\mathfrak{G}^*_w \in \mathsf{K}$, $(\mathfrak{G}^*_w . V^*)$ globally satisfies Δ and $(\mathfrak{G}^*_w, V^*), w \models p_A$ for all $A \subseteq W$ with $w \in A$.

Claim 2: $\mathfrak{ue}\mathfrak{F}_w$ is an ultrafilter morphic image of $\mathfrak{G}^*_w$.

Proof of claim: For any $u \in \mathfrak{G}^*_w$, let $f(u) = \{A \subseteq W \mid (\mathfrak{G}^*_w, V^*), u \models p_A\}$. One can show that f is an ultrafilter morphism from $\mathfrak{G}^*_w$ onto $\mathfrak{ue}\mathfrak{F}_w$, using similar arguments as for Claim 2 in the proof of Theorem 4.3.1. We will only show surjectiveness, since this part of the proof deviates slightly from the case for $\mathcal{H}(@)$.

Take any $u \in \mathfrak{ue}\mathfrak{F}_w$. To prove surjectiveness, we will show that the set $\{p_A \mid A \in u\}$ is satisfiable in $(\mathfrak{G}^*_w, V^*)$. By ω-saturatedness, it suffices to show finitely satisfiability. Take $A_1, \ldots, A_n \in u$. Then $\bigcap_i A_i \in u$ and hence, $\bigcap_i A_i \neq \emptyset$. Let $s \in \bigcap_i A_i$. By point-generatedness, s is reachable from w in a finite number of steps. But then there are $\diamond_1, \ldots, \diamond_n$ such that $\mathfrak{F}_w, w \models p_{m_{\diamond_1} \cdots m_{\diamond_n}(\bigcap_i A_i)}$, and hence $\mathfrak{G}^*_v, V^*, v \models p_{m_{\diamond_1} \cdots m_{\diamond_n}(\bigcap_i A_i)}$. Hence, by global truth of Δ, there is a point in $(\mathfrak{G}^*_w, V^*)$, that satisfies $p_{(\bigcap_i A_i)}$, and hence satisfies $p_{A_1}, \ldots, p_{A_n}$. ⊣

We have shown that $\mathfrak{F}_w$ is an ultrafilter morphic image of $\mathfrak{G}^*_w$. It follows by Lemma 4.2.13 that $(\mathfrak{F}_w \uplus \mathfrak{F}_w)$ is an ultrafilter morphic image of $(\mathfrak{G}^*_w \uplus \mathfrak{G}^*_w)$. By Lemma 4.2.14, $(\mathfrak{G}^*_w \uplus \mathfrak{G}^*_w)$ is elementarily equivalent to $(\mathfrak{G}_w \uplus \mathfrak{G}_w)$, which, as we saw earlier, is in K. We conclude that $(\mathfrak{F}_w \uplus \mathfrak{F}_w) \in \mathsf{K}$. □

Theorem 4.3.4 can be simplified for the case of nominal bounded $\mathcal{H}$-formulas. By Proposition 4.2.3, frame classes defined by nominal bounded $\mathcal{H}$-formulas reflect point-generated subframes.

4.3.5. THEOREM. *An elementary frame class* K *is definable by a set of nominal bounded* $\mathcal{H}$*-formulas iff* K *is closed under ultrafilter morphic images and generated subframes and* K *reflects point-generated subframes.*

Proof: The left-to-right direction was proved already in the previous section. For the right-to-left-direction, we proceed as follows. Let $Th(\mathsf{K})$ be the set of nominal bounded $\mathcal{H}$-formulas valid on K, and suppose $\mathfrak{F} \models Th(\mathsf{K})$. By preservation under generated subframes, $\mathfrak{F}_w \models Th(\mathsf{K})$ for all point-generated subframes $\mathfrak{F}_w$ of $\mathfrak{F}$. It follows from Lemma 4.3.3 and Theorem 4.3.1 that $\mathfrak{F}_w \in \mathsf{K}$ for all point-generated subframes $\mathfrak{F}_w$ of $\mathfrak{F}$. Since K reflects point-generated subframes, we conclude that $\mathfrak{F} \in \mathsf{K}$. □

4.4 Frame definability by pure formulas

In this section, we will characterize the elementary frame classes that are defined by pure formulas. Recall that a pure formula is one that contains no proposition letters (but nominals are allowed). Every frame class defined by pure formulas is elementary, as can be seen from the standard translation. It follows that non-elementary modal frame classes such as defined by $\Box\Diamond p \to \Diamond\Box p$ cannot be defined by means of pure formulas. Furthermore, we saw in Section 4.2 that the class of confluent frames, which is elementary and defined by the modal formula $\Diamond\Box p \to \Box\Diamond p$, is not definable by means of pure formulas either.

Pure formulas are interesting, since, as we will see in Section 5.4, there is a general completeness result for extensions of the basic hybrid logic with pure axioms, much like the case of Sahlqvist axioms in the basic modal language. The question for a model theoretic characterization of the frame classes definable by pure hybrid formulas has been asked first by [46].

Incidentally, another characterization of the pure formulas is given by Theorem 5.2.10, in terms of persistence under the passage from a particular type of general frame to the underlying Kripke frame.

4.4.1. Theorem. *A frame class* K *is definable by means of a pure* $\mathcal{H}(@)$ *formula iff* K *is elementary, closed under generated subframes and closed under images of bisimulation systems.*

Proof: Let $PTh(\mathsf{K})$ be the set of pure $\mathcal{H}(@)$ formulas valid on K. By compactness, it suffices to show that for all frames $\mathfrak{F}$, if $\mathfrak{F} \models PTh(\mathsf{K})$ then $\mathfrak{F} \in \mathsf{K}$.

Suppose $\mathfrak{F} \models PTh(\mathsf{K})$. For every point $w \in W$, where W is the domain of $\mathfrak{F}$, introduce a nominal i_w, and let V be the natural valuation with $V(i_w) = \{w\}$. Let $\Delta_{\mathfrak{F}}$ consist of all pure formulas of the form $@_{i_w}\varphi$ true in the model $(\mathfrak{F}, V)$. Intuitively, $\Delta_{\mathfrak{F}}$ provides a full description of the frame $\mathfrak{F}$. Clearly, $\Delta_{\mathfrak{F}}$ is satisfiable on $\mathfrak{F}$, namely under the valuation V.

Claim 1: $\Delta_{\mathfrak{F}}$ is satisfiable on K.

Proof of claim: By compactness (recall that K is elementary), it suffices to show that every finite conjunction δ of elements of $\Delta_{\mathfrak{F}}$ is satisfiable on K. But this follows immediately: δ is satisfiable on $\mathfrak{F}$ and $\mathfrak{F} \models PTh(\mathsf{K})$, hence $\neg\delta \notin PTh(\mathsf{K})$, i.e., δ is satisfiable on K. $\dashv$

Let $(\mathfrak{G}, U) \models \Delta_{\mathfrak{F}}$, with $\mathfrak{G} \in \mathsf{K}$. Since K is closed under generated subframes, we may assume that $\mathfrak{G}$ is generated by the set of points that are named by a nominal.

Claim 2: For all pure $\mathcal{H}(@)$-formulas φ, $(\mathfrak{F}, V) \models \varphi$ iff $(\mathfrak{G}, U) \models \varphi$. Equivalently, φ is satisfied at a point in $(\mathfrak{F}, V)$ iff φ is satisfied at a point in $(\mathfrak{G}, U)$.

Proof of claim: Suppose $(\mathfrak{F}, V), w \models \varphi$. Then $(\mathfrak{F}, V) \models @_{i_w}\varphi$. It follows that $@_{i_w}\varphi \in \Delta_{\mathfrak{F}}$, and hence $(\mathfrak{G}, U) \models @_{i_w}\varphi$.

Conversely, suppose $(\mathfrak{G}, U), v \models \varphi$. Since $(\mathfrak{G}, U)$ is generated by points named by nominals, there is a nominal i and modalities $\Diamond_1, \ldots, \Diamond_n \in \text{MOD}$ $(n \in \omega)$ such that $(\mathfrak{G}, U) \models @_i\Diamond_1 \cdots \Diamond_n\varphi$. It follows that $(\mathfrak{F}, V) \models @_i\Diamond_1 \cdots \Diamond_n\varphi$ (for if not, then $@_i\Box_1 \cdots \Box_n\neg\varphi \in \Delta_{\mathfrak{F}}$). $\dashv$

Let $(\mathfrak{F}^*, V^*)$ and $(\mathfrak{G}^*, U^*)$ be ω-saturated elementary extensions. By elementarity, $\mathfrak{G}^* \in \mathsf{K}$. In what follows, we will construct a bisimulation system from $\mathfrak{G}^*$ to $\mathfrak{F}^*$. Fix any $w_1, \ldots, w_n \in \mathfrak{F}^*$, and pick corresponding new nominals $j_1, \ldots, j_n$. We will write $(\mathfrak{F}^*, V^*, w_1, \ldots, w_n)$ for the expansion of $(\mathfrak{F}^*, V^*)$ in which $j_1, \ldots, j_n$ denote $w_1, \ldots, w_n$, respectively.

Claim 3: There are $v_1, \ldots, v_n \in \mathfrak{G}^*$ such that the models $(\mathfrak{F}^*, V^*, w_1, \ldots, w_n)$ and $(\mathfrak{G}^*, U^*, v_1, \ldots, v_n)$ globally satisfy exactly the same pure $\mathcal{H}(@)$-formulas of the extended language (i.e., including nominals $j_1, \ldots, j_n$).

Proof of claim: Let Γ be the following set of first-order formulas:

$$\begin{array}{l}\{\ \ \forall x.ST_x(\varphi) \mid (\mathfrak{F}^*, V^*, w_1, \ldots, w_n) \models \varphi\} \cup \\ \{\neg\forall x.ST_x(\varphi) \mid (\mathfrak{F}^*, V^*, w_1, \ldots, w_n) \not\models \varphi\}\end{array}$$

It is our task to show that Γ is satisfied in some expansion of $(\mathfrak{G}^*, U^*)$. Since $(\mathfrak{G}^*, U^*)$ is ω-saturated, it suffices to show that Γ is finitely realizable, in the sense that for all $\varphi_1, \ldots, \varphi_m \in \Gamma$, there are $v_1, \ldots, v_n$ such that $(\mathfrak{G}^*, U^*, v_1, \ldots, v_n)$ satisfies $\varphi_1, \ldots, \varphi_m$.

Take any $\varphi_1, \ldots, \varphi_m \in \Gamma$. By definition, $(\mathfrak{F}^*, V^*, w_1, \ldots, w_n)$ satisfies $\varphi_1, \ldots, \varphi_m$. Since $(\mathfrak{F}, V)$ is an elementary submodel of $(\mathfrak{F}^*, V^*)$, there are $w'_1, \ldots, w'_n$ such that $(\mathfrak{F}, V, w'_1, \ldots, w'_n)$ satisfies $\varphi_1, \ldots, \varphi_m$. Recall that $w'_1, \ldots, w'_n$ are named by the nominals $i_{w'_1}, \ldots, i_{w'_n}$. Hence, for all formulas φ, we have that $(\mathfrak{F}, V, w'_1, \ldots, w'_n) \models \varphi$ iff $(\mathfrak{F}, V) \models \varphi\ [j_1/i_{w'_1}, \ldots, j_n/i_{w'_n}]$. Let $v_1, \ldots, v_n$ be the denotation of the nominals $i_{w'_1}, \ldots, i_{w'_n}$ in the model $(\mathfrak{G}, U)$. By claim 2, $(\mathfrak{F}, V) \models \varphi\ [j_1/i_{w'_1}, \ldots, j_n/i_{w'_n}]$ iff $(\mathfrak{G}, U) \models \varphi\ [j_1/i_{w'_1}, \ldots, j_n/i_{w'_n}]$. It follows that $(\mathfrak{F}, V, w_1, \ldots, w_n) \models \varphi_i$ iff $(\mathfrak{G}, U, v_1, \ldots, v_n) \models \varphi_i$, for all $1 \leq i \leq m$. Hence, $(\mathfrak{G}, V, v_1, \ldots, v_n)$ satisfies $\varphi_1, \ldots, \varphi_m$. ⊣

Define the binary relation Z between the domains of $\mathfrak{G}^*$ and $\mathfrak{F}^*$ such that sZt iff $(\mathfrak{G}^*, U^*, v_1, \ldots, v_n), s$ and $(\mathfrak{F}^*, V^*, w_1, \ldots, w_n), t$ agree on all pure $\mathcal{H}(@)$-formulas of the extended language.

Claim 4: Z is a total bisimulation between $\mathfrak{G}^*$ and $\mathfrak{F}^*$ respecting $w_1, \ldots, w_n$.

Proof of claim: By Theorem 4.1.2, Z is a bisimulation between $\mathfrak{G}^*$ and $\mathfrak{F}^*$. To see that Z is a *total* bisimulation, take any $s \in \mathfrak{G}^*$, and let $\Gamma = \{ST_x(\varphi) \mid (\mathfrak{G}^*, U^*, v_1, \ldots, v_n), s \models \varphi\}$. It follows from Claim 3 that every finite subset of Γ is realized in $(\mathfrak{F}^*, V^*, w_1, \ldots, w_n)$. Hence, by ω-saturatedness, there is a point t such that $(\mathfrak{F}^*, V^*, w_1, \ldots, w_n), t \models \Gamma$, and therefore sZt. The other direction (i.e., $\forall s \in \mathfrak{F}^*\ \exists t \in \mathfrak{G}^*$ s.t. tZs) is proved symmetrically. Finally, that Z respects $w_1, \ldots, w_n$ is immediate from the construction. ⊣

We have constructed a bisimulation system from $\mathfrak{G}^*$ to $\mathfrak{F}^*$. By closure under images of bisimulation systems, $\mathfrak{F}^* \in \mathsf{K}$ and hence, by elementarity, $\mathfrak{F} \in \mathsf{K}$. □

4.4.2. Corollary. *An frame class is definable by a pure $\mathcal{H}(\mathsf{E})$ formula iff it is elementary and closed under bisimulation systems.*

Proof: The global modality is definable by a pure $\mathcal{H}(@)$-formula, namely the formula $\mathsf{E}i$. Hence, a frame class K is definable by a pure $\mathcal{H}(\mathsf{E})$-formula iff the class $\mathsf{K}' = \{(W, (R_\diamond)_{\diamond \in \text{MOD}}, R_\mathsf{E}) \mid (W, (R_\diamond)_{\diamond \in \text{MOD}}) \in \mathsf{K} \text{ and } R_\mathsf{E} = W^2\}$ is definable by a pure $\mathcal{H}(@)$-formula. K' is clearly closed under generated subframes. Furthermore, one can easily see that K' is closed under images of bisimulation systems iff K is. The result follows. □

As before, the case of $\mathcal{H}$ is slightly more complicated.

4.4.3. Lemma. *Let K be a class of frames, and let $PTh_{\mathcal{H}}(\mathsf{K})$ and $PTh_{\mathcal{H}(@)}(\mathsf{K})$ be the set of pure $\mathcal{H}$-formulas and pure $\mathcal{H}(@)$-formulas, respectively, valid on K, and let $PTh^{nb}_{\mathcal{H}}(\mathsf{K})$ be the set of nominal-bounded pure $\mathcal{H}$-formulas valid on K. For all point-generated frames $\mathfrak{F}_w$, $\mathfrak{F}_w \models PTh_{\mathcal{H}(@)}(\mathsf{K})$ iff $\mathfrak{F}_w \models PTh_{\mathcal{H}}(\mathsf{K})$ iff $\mathfrak{F}_w \models PTh^{nb}_{\mathcal{H}}(\mathsf{K})$.*

Proof: Analogous to the proof of Lemma 4.3.3. □

4.4.4. Theorem. *An frame class K is definable by a pure $\mathcal{H}$-formula iff K is elementary and the following closure conditions hold.*

1. *K is closed under images of bisimulation systems.*
2. *K is closed under generated subframes.*
3. *For any frame $\mathfrak{F}$, if every point generated subframe of $\mathfrak{F}$ is a proper generated subframe of a frame in K, then $\mathfrak{F} \in \mathsf{K}$.*

Proof: Let $PTh(\mathsf{K})$ be the set of pure $\mathcal{H}$-formulas valid on K. . By compactness, it suffices to show that for all frames $\mathfrak{F}$, if $\mathfrak{F} \models PTh(\mathsf{K})$ then $\mathfrak{F} \in \mathsf{K}$.

Suppose $\mathfrak{F} \models PTh(\mathsf{K})$. If $\mathfrak{F}$ is point-generated, then by Lemma 4.4.3, in combination with Theorem 4.4.1, $\mathfrak{F} \in \mathsf{K}$, and we are done. In the remainder of this proof, we will assume that $\mathfrak{F}$ is *not* point-generated. Take any point-generated subframe $\mathfrak{F}_w = (W, (R_\diamond)_{\diamond \in \text{MOD}})$ of $\mathfrak{F}$. In what follows, we will show that $(\mathfrak{F}_w \uplus \mathfrak{F}_w) \in \mathsf{K}$. It then follows by the third closure condition that $\mathfrak{F} \in \mathsf{K}$.

For every point $w \in W$, introduce a nominal i_w. Furthermore, introduce a distinct nominal $i_\emptyset$. Let V be any valuation for $\mathfrak{F}$ such that $V(i_w) = \{w\}$ and $V(i_\emptyset) = \{v\}$ for some v not reachable from w in any finite number of steps. Let $\Delta_{\mathfrak{F}_w}$ consist of all pure $\mathcal{H}$-formulas true at $(\mathfrak{F}, V), w$.

Claim 1: $\Delta_{\mathfrak{F}_w}$ is satisfiable on K.

Proof of claim: By compactness (recall that K is elementary), it suffices to show that every finite conjunction δ of elements of $\Delta_{\mathfrak{F}_w}$ is satisfiable on K. But this follows immediately: δ is satisfiable on $\mathfrak{F}$ and $\mathfrak{F} \models PTh(\mathsf{K})$, hence $\neg\delta \notin PTh(\mathsf{K})$, i.e., δ is satisfiable on K. $\dashv$

Let $(\mathfrak{G}, U), v \models \Delta_{\mathfrak{F}_w}$, with $\mathfrak{G} \in \mathsf{K}$. Let $\mathfrak{G}_v$ be the subframe of $\mathfrak{G}$ generated by v. By construction, $\mathfrak{G}_v$ is a proper generated subframe of $\mathfrak{G}$. Hence, by the third closure condition, $(\mathfrak{G}_v \uplus \mathfrak{G}_v) \in \mathsf{K}$.

By construction, all nominals except $i_\emptyset$ denote a point in $(\mathfrak{G}, U)$ that is reachable from v. Hence we can think of U as a valuation for the frame $\mathfrak{G}_v$ by removing $i_\emptyset$ from our vocabulary. Similarly, we can conceive of V as a valuation for the frame $\mathfrak{F}_w$ by removing $i_\emptyset$ from our vocabulary. In this way, we obtain point-generated model $(\mathfrak{G}_v, U)$ and $(\mathfrak{F}_w, V)$ such that for every pure $\mathcal{H}$-formula φ not containing the nominal $i_\emptyset$, $(\mathfrak{F}_w, V), w \models \varphi$ iff $(\mathfrak{G}_v, U), v \models \varphi$.

Claim 2: For all $\mathcal{H}$-formulas φ, $(\mathfrak{F}_w, V) \models \varphi$ iff $(\mathfrak{G}_v, U) \models \varphi$. Equivalently, φ is satisfied at a point in $(\mathfrak{F}_w, V)$ iff φ is satisfied at a point in $(\mathfrak{G}_v, U)$.

Proof of claim: Suppose $(\mathfrak{F}_w, V), u \models \varphi$. Since $\mathfrak{F}_w$ is generated by w, there are $\Diamond_1, \ldots, \Diamond_n \in \textsc{mod}$ such that $(\mathfrak{F}_w, V), w \models \Diamond_1 \cdots \Diamond_n \varphi$. It follows that $(\mathfrak{G}_w, V), w \models \Diamond_1 \cdots \Diamond_n \varphi$, and hence there is a point in $(\mathfrak{G}_v, V)$ satisfying φ. The converse direction is proved similarly. $\dashv$

Let $(\mathfrak{F}^*_w, V^*)$ and $(\mathfrak{G}^*_v, U^*)$ be ω-saturated elementary extensions. In what follows, we will construct a bisimulation system from $\mathfrak{G}^*_v$ to $\mathfrak{F}^*_w$. Fix any $w_1, \ldots, w_n \in \mathfrak{F}^*_w$, and pick corresponding new nominals $j_1, \ldots, j_n$. We will write $(\mathfrak{F}^*_w, V^*, w_1, \ldots, w_n)$ for the expansion of $(\mathfrak{F}^*_w, V^*)$ in which $j_1, \ldots, j_n$ denote $w_1, \ldots, w_n$, respectively.

Claim 3: There are $v_1, \ldots, v_n \in \mathfrak{G}^*_w$ such that the models $(\mathfrak{F}^*_w, V^*, w_1, \ldots, w_n)$ and $(\mathfrak{G}^*_w, U^*, v_1, \ldots, v_n)$ globally satisfy exactly the same pure $\mathcal{H}$-formulas of the extended language (i.e., including nominals $j_1, \ldots, j_n$ but not $i_\emptyset$).

Proof of claim: Analogous to Claim 3 in the proof of Theorem 4.4.1. $\dashv$

Define the binary relation Z between the domains of $\mathfrak{G}^*_v$ and $\mathfrak{F}^*_w$ such that sZt iff $(\mathfrak{G}^*_v, U^*, v_1, \ldots, v_n), s$ and $(\mathfrak{F}^*_w, V^*, w_1, \ldots, w_n), t$ agree on all pure $\mathcal{H}$-formulas of the extended language. Then Z is a total bisimulation between $\mathfrak{G}^*_w$ and $\mathfrak{F}^*_w$ respecting $w_1, \ldots, w_n$ (the proof is analogous to that of Claim 4 in the proof of Theorem 4.4.1).

Hence, we have constructed a bisimulation system from $\mathfrak{G}^*_w$ to $\mathfrak{F}^*_w$. It follows by Lemma 4.2.15 that there is a bisimulation system from $(\mathfrak{G}^*_w \uplus \mathfrak{G}^*_w)$ to $(\mathfrak{F}^*_w \uplus \mathfrak{F}^*_w)$. By Lemma 4.2.14, $(\mathfrak{G}^*_w \uplus \mathfrak{G}^*_w)$ is elementarily equivalent to $(\mathfrak{G}_w \uplus \mathfrak{G}_w)$, which, as we saw earlier, is in K. We conclude that $(\mathfrak{F}_w \uplus \mathfrak{F}_w) \in \mathsf{K}$. $\Box$

For nominal bounded $\mathcal{H}$-formulas, we again obtain a simpler result.

4.4.5. THEOREM. *An elementary frame class* K *is definable by a pure nominal bounded* $\mathcal{H}$*-formula iff* K *is closed under images of bisimulation systems and generated subframes and* K *reflects point-generated subframes.*

Proof: Let $PTh(\mathsf{K})$ be the set of nominal bounded $\mathcal{H}$-formulas valid on K, and suppose $\mathfrak{F} \models PTh(\mathsf{K})$. By preservation under generated subframes, $\mathfrak{F}_w \models PTh(\mathsf{K})$ for all point-generated subframes $\mathfrak{F}_w$ of $\mathfrak{F}$. It follows from Lemma 4.4.3 and Theorem 4.4.1 that $\mathfrak{F}_w \in \mathsf{K}$ for all point-generated subframes $\mathfrak{F}_w$ of $\mathfrak{F}$. Since K reflects point-generated subframes, we conclude that $\mathfrak{F} \in \mathsf{K}$.

Hence, $PTh(\mathsf{K})$ defines K. By compactness and the fact that every conjunction of nominal bounded $\mathcal{H}$-formulas is nominal bounded, it follows that K is defined by a single nominal bounded $\mathcal{H}$-formula. □

We end this section with an open question.[2]

4.4.6. QUESTION. *How do ultrafilter morphisms and bisimulation systems relate?* It follows from the above results that whenever an elementary frame class is closed under images of bisimulation systems, it is also closed under ultrafilter morphic images. Is there a more direct proof of this fact, and does it hold also for non-elementary frame classes?

[2] Ian Hodkinson (p.c.) has found an interesting partial answer to this question. For frames $\mathfrak{F}, \mathfrak{G}$, let us say that $\mathfrak{G}$ is a *pseudo-bisimulation image* of $\mathfrak{F}$ if there exist elementary extensions $\mathfrak{F}^+$ and $\mathfrak{G}^+$, and a total bisimulation Z between $\mathfrak{F}^+$ and $\mathfrak{G}^+$ that respects the domain of $\mathfrak{G}$.

By compactness or ultrapowers, every bisimulation system image is also a pseudo-bisimulation image. Furthermore, that all pure $\mathcal{H}(\mathsf{E})$-formulas are preserved under pseudo-bisimulation images: let $\mathfrak{G}$ be a pseudo-bisimulation image of $\mathfrak{F}$ and assume for contradiction that $\mathfrak{F} \models \varphi$ and $\mathfrak{G} \not\models \varphi$, for some pure $\mathcal{H}(\mathsf{E})$-formula φ. V is also a valuation for $\mathfrak{G}^+$, and, since $\mathfrak{G} \preceq \mathfrak{G}^+$, we have that $\mathfrak{G}^+, V, w \not\models \varphi$. Let $\mathfrak{G}, V, w \not\models \varphi$, and let Z be a bisimulation between elementary extensions $\mathfrak{F}^+$ and $\mathfrak{G}^+$ respecting the domain of $\mathfrak{G}$. We can "pull back" V and w along Z to obtain a valuation U for $\mathfrak{F}^+$ and a world w', such that $\mathfrak{F}^+, U, w' \not\models \varphi$. Hence, $\mathfrak{F}^+ \not\models \varphi$, and it follows by elementarity that $\mathfrak{F} \not\models \varphi$, a contradiction.

It follows that all results in the present section could have be phrased in terms of pseudo-bisimulation images.

Next, it can be shown that every ultrafilter morphic image is also a pseudo-bisimulation image. For suppose $f : \mathfrak{F} \to \mathfrak{ue}\mathfrak{G}$ is a surjective bounded morphism such that $|f^{-1}(u)| = 1$ for all principal u. By [21, Theorem 3.17], $\mathfrak{ue}\mathfrak{G}$ is a bounded morphic image of some elementary extension $\mathfrak{G}^+$ of $\mathfrak{G}$. Let $g : \mathfrak{G}^+ \to \mathfrak{ue}\mathfrak{G}$ be the relevant surjective bounded morphism. It is easily checked that for each principal ultrafilter $g^{-1}(\pi_w) = \{w\}$ for each principal ultrafilter π_w. Finally, define a binary relation Z between the domains of $\mathfrak{F}$ and $\mathfrak{G}^+$ by letting xZy iff $f(x) = g(y)$. Then Z is a total bisimulation respecting the domain of $\mathfrak{G}$. Hence (taking $\mathfrak{F} = \mathfrak{F}^+$), $\mathfrak{G}$ is a pseudo-bisimulation image of $\mathfrak{F}$.

4.5 Which classes definable in hybrid logic are elementary?

In the previous sections, we characterized the elementary frame classes that are definable in hybrid logic. In the present section, we ask the converse question: *which frame classes definable in hybrid logic are elementary?*

First, let us take the model theoretic perspective. Recall from Section 2.3 that a modally definable frame class is elementary iff it is closed under elementary equivalence iff it is closed under ultrapowers. One might ask whether this also holds for frame classes definable in our hybrid languages. The answer is *No*.

4.5.1. PROPOSITION. *There is a frame class* K *definable in* $\mathcal{H}$ *such that* K *is closed under elementary equivalence (and hence under ultrapowers) while* K *is not* Δ*-elementary (i.e., defined by a set of first-order formulas).*

Proof: Consider the class K of bi-modal frames consisting the finite strict total orderings, with $<$ and $>$ relations. This class is defined by the following $\mathcal{H}$-formulas.

$p \to (GPp \wedge HFp)$	"$<$ and $>$ are each others converse"
$G(Gp \to p) \to Gp$	"$<$ is transitive and conversely well-founded"
$H(Hp \to p) \to Hp$	"$>$ is transitive and conversely well-founded"
$i \vee Fi \vee Pi$	"$<$ satisfies trichotomy"

Since K consists only of finite frames, it is clearly closed under elementary equivalence. Nevertheless, K is not Δ-elementary, as a simple compactness argument establishes. □

Incidentally, the standard proof of Theorem 2.3.6 still applies to frame classes definably by bounded $\mathcal{H}$ formulas, since these classes are closed under taking disjoint unions and generated subframes. Also, since the hybrid languages $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$ are all fragments of universal second order logic (on Kripke frames), we still have that whenever a definable frame class is Δ-elementary (i.e., defined by a set of first-order sentences), it is elementary.

Next, let us consider the syntactic approach to characterizing the elementary classes. Clearly, we cannot expect a complete syntactic characterization of the elementary frame conditions. However, there are natural ways to extend the Sahlqvist-Van Benthem correspondence theorem to hybrid languages. Here, we will give a version for the language $\mathcal{H}$.

Call an $\mathcal{H}$-formula φ *positive* (*negative*) if every occurrence of a proposition letter in φ occurs positively (negatively). Note that no requirements are made on the nominals. Let a *boxed atom* be a proposition letter prefixed by any number of boxes. A hybrid Sahlqvist antecedent is a formula built up from $\top$, $\bot$, boxed atoms and negative formulas using $\wedge$, $\vee$ and diamonds. A hybrid Sahlqvist implication is an implication $\varphi \to \psi$ in which ψ is positive and φ is a hybrid Sahlqvist antecedent. A hybrid Sahlqvist formula is a formula that is built up from Sahlqvist implications by freely applying boxes and conjunctions and by applying disjunctions only between formulas that do not share any proposition letters.

In fact, the only difference between modal and hybrid Sahlqvist axioms is that in the latter, nominals are allowed throughout the formula.

4.5.2. Theorem. *Every hybrid Sahlqvist formula defines an elementary class of frames.*

Proof: The proof for modal logic given in [21] generalizes straightforwardly to the hybrid case. However, since the proof is rather tedious, we will give a separate argument.

Let φ be any hybrid Sahlqvist formula. Define φ' to be the result of replacing in φ every nominal i by a new modal constant (i.e., nullary modality) δ_i, which is temporarily added to the language. Then it is easily seen that φ' is a modal Sahlqvist formula of the extended language, and hence corresponds to a first-order frame condition, say χ. Replace in χ all subformulas of the form $R_{\delta_i}x$ by $x = y_i$, where y_i is a new first-order variable picked for the nominal i, and let χ' be the universal closure of the resulting first-order formula. Then χ' is easily seen to define the same class of frames as the original formula φ. □

In particular, every pure formula, being positive in all proposition letters, is equivalent to a hybrid Sahlqvist formula $\top \to \varphi$, hence elementary. Further generalizations of the class of hybrid Sahlqvist formulas are possible, and have been described by Goranko and Sahlqvist [59].[3]

In contrast to the above, the *completeness* theorem for Sahlqvist formulas, Corollary 2.4.6, does *not* generalize to hybrid Sahlqvist formulas as defined above, as we will see in the next chapter (cf. Theorem 5.4.3).

Incidentally, observe how, in the above proof, we reduced hybrid formulas to modal formulas by replacing nominals by modal constants. Similar reductions will be used in the next chapter to derive hybrid completeness results from modal completeness resuls.

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[3]One relatively simple extension the class of hybrid Sahlqvist formulas, pointed out to me by Goranko (p.c.), is obtained by generalizing the notion of a *boxed atom* to formulas of the form

$$\Box_1(\varphi_1 \to \Box_2(\varphi_2 \to \ldots \Box_n(\varphi_n \to p)\ldots))$$

where each φ_i is negative in all proposition letters, and further requiring that no head of such a "generalized boxed atom" of the Sahlqvist formula (the head being the proposition letter p in the above formula) occurs in the body of a generalized box-formula.

Chapter 5

Axiomatizations and completeness

One of the most important results in modal logic is the Sahlqvist completeness theorem. From the model theoretic perspective that we take in this dissertation, this result is best summarized as follows.

> If a frame class K is definable by a set of modal Sahlqvist formulas, then the modal logic of K (i.e., the set of modal formulas valid on K) is completely axiomatized by adding these Sahlqvist formulas as axioms to the basic modal logic $\mathbf{K}_{\mathcal{M}}$.

While this result covers many interesting frame classes, there are natural properties such as irreflexivity cannot be defined by modal formulas. One of the reasons why hybrid logics have become popular is that there is a general completeness result for hybrid logics that applies to many frame classes not definable by modal Sahlqvist formulas. Recall that a hybrid formula is *pure* if it contains no proposition letters (but possibly contains nominals). The following analogue of the Sahlqvist completeness theorem can be obtained for hybrid logics.[1]

> If a frame class K is definable by a set of pure hybrid formulas, then the hybrid logic of K is completely axiomatized by adding the relevant formulas as axioms to the basic hybrid logic.

For the hybrid language $\mathcal{H}$, this fact, viz. the completeness of logics axiomatized by pure formulas, was already observed in the 1980s by Gargov et al. [47].

Besides this, we still have that all hybrid logics axiomatized by modal Sahlqvist formulas are complete:

> If a frame class K is definable by a set of modal Sahlqvist formulas, then the hybrid logic of K is completely axiomatized by adding the relevant formulas as axioms to the basic hybrid logic.

[1] This result applies to any of the languages $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$. A precise formulation will be given later on, after the basic axioms and rules for the hybrid languages have been introduced.

This was observed for $\mathcal{H}(\mathsf{E})$ by Gargov and Goranko [46]. As we will show in this chapter, it also holds for $\mathcal{H}$ and $\mathcal{H}(@)$.

In this chapter, we will prove the following new results. We will show in this chapter that there is a pure formula φ and a modal Sahlqvist formula ψ such that the logic obtained by adding φ and ψ as axioms to the basic hybrid logic is incomplete. Secondly, we will show that every axiomatization of the basic hybrid logic of which all extensions with pure formulas are complete must contain either inference rules with syntactic side conditions, or infinitely many infinite rules.

A final contribution of this chapter is in the development of a theory of general frames for hybrid logics. Two-sorted general frame are introduced and studied, and it is shown how the existing completeness proofs for hybrid logics can be recast in terms of completeness and persistence arguments with respect to classes of two-sorted general frames.

Some of the results reported in this chapter are taken from [19, 30].

5.1 The axiomatizations

For each of the hybrid languages $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$, we will now give two basic axiomatizations. The difference between these two axiomatizations lies each time in the addition of two inference rules.

5.1.1. Definition. *For any set of $\mathcal{H}$-formulas Σ, $\mathbf{K}_{\mathcal{H}}\Sigma$ is the smallest set containing all axioms in Table 5.1 and Σ closed under the rules in Table 5.1, except for the (Name) and (Paste) rule. $\mathbf{K}^{+}_{\mathcal{H}}\Sigma$ is defined similarly, closing in addition under the (Name) and (Paste) rules.*

5.1.2. Definition. *For any set of $\mathcal{H}(@)$-formulas Σ, $\mathbf{K}_{\mathcal{H}(@)}\Sigma$ is the smallest set containing all axioms in Table 5.2 and Σ closed under the rules in Table 5.2, except for the ($Name_{@}$) and (BG) rule. $\mathbf{K}^{+}_{\mathcal{H}(@)}\Sigma$ is defined similarly, closing in addition under the (Name) and (BG) rule.*

5.1.3. Definition. *For any set of $\mathcal{H}(\mathsf{E})$-formulas Σ, $\mathbf{K}_{\mathcal{H}(\mathsf{E})}\Sigma$ is the smallest set containing all axioms in Table 5.3 and Σ closed under the rules in Table 5.3, except for the (Name) and (BG_{E}) rules. $\mathbf{K}^{+}_{\mathcal{H}(\mathsf{E})}\Sigma$ is defined similarly, closing in addition under the (Name) and (BG_{E}) rules.*

It should be clear to the reader that all basic axioms are sound, and that all inference rules preserve validity with respect to any class of frames.

One note is in order concerning the (*NameLite*) rule. This rule is peculiar, in that it is admissible in every consistent logic $\mathbf{K}_{\mathcal{H}}\Sigma$. The only role of (*NameLite*) is to render logics that derive $\neg i$, for some nominal i, inconsistent, reflecting the fact that $\neg i$ is not valid on any frame. As is not hard to see, without the

Table 5.1: Axioms and inference rules of $\mathbf{K}_{\mathcal{H}}$

Axioms and inference rules of $\mathbf{K}_{\mathcal{H}}$	
(CT)	$\vdash \varphi$, for all classical tautologies φ
$(Dual)$	$\vdash \Diamond p \leftrightarrow \neg\Box\neg p$, for $\Box \in$ MOD
(K)	$\vdash \Box(p \to q) \to \Box p \to \Box q$, for $\Box \in$ MOD
(Nom)	$\vdash \Diamond_1 \cdots \Diamond_n(i \wedge p) \to \Box_{n+1} \cdots \Box_{n+m}(i \to p)$, for $\Box_1, \ldots, \Box_{n+m} \in$ MOD $(n, m \geq 0)$
(MP)	If $\vdash \varphi \to \psi$ and $\vdash \varphi$ then $\vdash \psi$
(Nec)	If $\vdash \varphi$ then $\vdash \Box\varphi$, for $\Box \in$ MOD
$(Subst)$	If $\vdash \varphi$ then $\vdash \varphi\sigma$, where σ is a substitution that uniformly replaces proposition letters by formulas and nominals by nominals.
$(NameLite)$	If $\vdash \neg i$ then $\vdash \bot$
Additional inference rules of $\mathbf{K}^{+}_{\mathcal{H}}$	
$(Name)$	If $\vdash i \to \varphi$ then $\vdash \varphi$, for i not occurring in φ
$(Paste)$	If $\vdash \Diamond_1 \ldots \Diamond_n(i \wedge \Diamond_{n+1}(j \wedge \varphi)) \to \psi$ then $\vdash \Diamond_1 \cdots \Diamond_n(i \wedge \Diamond_{n+1}\varphi) \to \psi$, for $\Diamond_1, \ldots, \Diamond_{n+1} \in$ MOD $(n \geq 0)$, $j \neq i$ and j not occurring in φ, ψ

Table 5.2: Axioms and rules of $\mathbf{K}_{\mathcal{H}(@)}$

Axioms and inference rules of $\mathbf{K}_{\mathcal{H}(@)}$	
(CT)	$\vdash \varphi$, for all classical tautologies φ
$(Dual)$	$\vdash \Diamond p \leftrightarrow \neg\Box\neg p$, for $\Box \in$ MOD
(K)	$\vdash \Box(p \to q) \to \Box p \to \Box q$, for $\Box \in$ MOD
$(K_@)$	$\vdash @_i(p \to q) \to @_i p \to @_i q$ for $i \in$ NOM
$(Selfdual)$	$\vdash \neg @_i p \leftrightarrow @_i \neg p$
(Ref)	$\vdash @_i i$
$(Intro)$	$\vdash i \wedge p \to @_i p$
$(Back)$	$\vdash \Diamond @_i p \to @_i p$, for $\Box \in$ MOD
$(Agree)$	$\vdash @_i @_j p \to @_j p$
(MP)	If $\vdash \varphi \to \psi$ and $\vdash \varphi$ then $\vdash \psi$
(Nec)	If $\vdash \varphi$ then $\vdash \Box\varphi$, for $\Box \in$ MOD
$(Nec_@)$	If $\vdash \varphi$ then $\vdash @_i\varphi$, for $i \in$ NOM
$(Subst)$	If $\vdash \varphi$ then $\vdash \varphi\sigma$, where σ is a substitution that uniformly replaces proposition letters by formulas and nominals by nominals.
Additional inference rules of $\mathbf{K}^{+}_{\mathcal{H}(@)}$	
$(Name_@)$	If $\vdash @_i\varphi$ then $\vdash \varphi$, for i not occurring in φ.
(BG)	If $\vdash @_i\Diamond j \to @_j\varphi$ then $\vdash @_i\Box\varphi$, for $i \neq j$ and j not occurring in φ.

Table 5.3: Axioms and rules of $\mathbf{K}_{\mathcal{H}(\mathsf{E})}$

Axioms and inference rules of $\mathbf{K}_{\mathcal{H}(\mathsf{E})}$	
(CT)	$\vdash \varphi$, for all classical tautologies φ
$(Dual)$	$\vdash \Diamond p \leftrightarrow \neg\Box\neg p$, for $\Box \in \textsc{mod}$
(K)	$\vdash \Box(p \to q) \to \Box p \to \Box q$, for $\Box \in \textsc{mod}$
$(Dual_{\mathsf{A}})$	$\vdash \mathsf{E}p \leftrightarrow \neg\mathsf{A}\neg p$
(K_{A})	$\vdash \mathsf{A}(p \to q) \to \mathsf{A}p \to \mathsf{A}q$
(Ref_{E})	$\vdash p \to \mathsf{E}p$
$(Trans_{\mathsf{E}})$	$\vdash \mathsf{EE}p \to \mathsf{E}p$
(Sym_{E})	$\vdash p \to \mathsf{AE}p$
$(Incl_{\Diamond})$	$\vdash \Diamond p \to \mathsf{E}p$, for $\Diamond \in \textsc{mod}$
$(Incl_i)$	$\vdash \mathsf{E}i$
(Nom_{E})	$\vdash \mathsf{E}(i \wedge p) \to \mathsf{A}(i \to p)$
(MP)	If $\vdash \varphi \to \psi$ and $\vdash \varphi$ then $\vdash \psi$
(Nec)	If $\vdash \varphi$ then $\vdash \Box\varphi$, for $\Box \in \textsc{mod}$
(Nec_{A})	If $\vdash \varphi$ then $\vdash \mathsf{A}\varphi$
$(Subst)$	If $\vdash \varphi$ then $\vdash \varphi\sigma$, where σ is a substitution that uniformly replaces proposition letters by formulas and nominals by nominals.
Additional inference rules of $\mathbf{K}^{+}_{\mathcal{H}(\mathsf{E})}$	
$(Name)$	If $\vdash i \to \varphi$ then $\vdash \varphi$, for i not occurring in φ.
(BG_{E})	If $\vdash \mathsf{E}(i \wedge \Diamond j) \to \mathsf{E}(j \wedge \varphi)$ then $\vdash \mathsf{E}(i \wedge \Box\varphi)$, for $i \neq j$ and j not occurring in φ.

(*NameLite*) rule, $\mathbf{K}_{\mathcal{H}}\{\neg i\}$ would be a consistent logic. Incidentally, (*NameLite*) is a special case of the (*Name*) rule.

Another, perhaps more elegant axiomatization for $\mathcal{H}$ was given by [47], based on the notion of necessity forms and possibility forms [51]. For a fixed symbol \$, necessity forms are defined as follows.

1. \$ is a necessity form.
2. If φ is a necessity form and ψ is an $\mathcal{H}$-formula, then $\psi \to \varphi$ is a necessity form.
3. If φ is a necessity form and $\Box \in \textsc{mod}$ then $\Box\varphi$ is a necessity form.

Possibility forms are defined similarly, replacing implications by conjunctions and boxes by diamonds. Given a possibility form M and a formula ψ, $M(\psi)$ will denote the result of replacing the unique occurence of \$ in M by ψ. Likewise for necessity forms. Now, the (*Nom*) axiom scheme and the (*Paste*) rule may be replaced by the following:

(*Nom′*) $\vdash M(i \wedge \varphi) \to L(i \to \varphi)$
where $M(\$)$ is a possibility form and $L(\$)$ is a necessity form

(*Cov*) If $\vdash L(\neg i)$ then $\vdash L(\bot)$,
where $L(\$)$ a necessity form not containing the nominal i

It is not hard to see that (*Nom*) and (*Nom′*) are interderivable, as well as the rules (*Paste*) and (*Cov*). Moreover, (*NameLite*) can be seen as the simplest possible instance of (*Cov*).

In what follows we will stick to the axiomatization given in Table 5.1.

5.2 General frames for hybrid logic

Recall the definition of general frames in Section 2.4. In the setting of hybrid logic, it seems most natural to consider general frames with two sorts of admissible sets, one for arbitrary formulas and one for nominals. The second is naturally included in the first. This is reflected in the following definition.

5.2.1. Definition. *A two-sorted general frame is a structure* $\mathfrak{F} = (W, (R_\Diamond)_{\Diamond\in\textsc{mod}}, \mathbb{A}, \mathbb{B})$, *where* $(W, (R_\Diamond)_{\Diamond\in\textsc{mod}}, \mathbb{A})$ *is a general frame,* $\mathbb{B} \subseteq W$ *is non-empty and for all* $w \in \mathbb{B}$, $\{w\} \in \mathbb{A}$.

Admissible valuations and validity are defined in the expected way: proposition letters denote sets in $\mathbb{A}$ and nominals denote points in $\mathbb{B}$. Since the set $\mathbb{B}$ is only used for the interpretation of the nominals, Definition 5.2.1 collapses to the more traditional one for modal logic, except for one small but important difference. The non-emptiness condition on $\mathbb{B}$ implies that $\mathbb{A}$ contains at least one singleton.

There are general frames that do not contain any singleton admissible set. We might call such general frames *atomless*.[2] Atomless general frames trivialize the notion of validity for hybrid logic, since they admit no hybrid valuations. In particular, the hybrid formula $\bot$ is valid on atomless frames, since, trivially, it holds under every hybrid valuation. Surprisingly, there exist consistent normal modal logics that have *only* atomless general frames [99]. This has some consequences for hybrid logic, as we will see later in Corollary 5.3.4.

Descriptive two-sorted general frames

Recall from Section 2.4 that every modal logic is strongly sound and complete with respect to a class of descriptive general frames [21]. In order to obtain a similar result to hybrid logics, we generalize the notion of descriptiveness to two-sorted general frames.

5.2.2. Definition. *A two-sorted general frame* $(W, (R_\Diamond)_{\Diamond\in\text{MOD}}, \mathbb{A}, \mathbb{B})$ *is descriptive if* $(W, (R_\Diamond)_{\Diamond\in\text{MOD}}, \mathbb{A})$ *is descriptive in the traditional sense.*

Call a formula *d2-persistent* if its validity is preserved under the passage from a descriptive two-sorted general frame to the underlying Kripke frame. One would like to know which formulas are d2-persistent. Let us first consider modal formulas. Clearly, every d-persistent modal formula (i.e., modal formula that is persistent with respect to descriptive general frames, as defined in Section 2.4) is d2-persistent. The converse does not hold: [99] shows the existence of a consistent modal formula that only has atomless general frames. It follows that this formula is not d-persistent (every Kripke frame has atoms) but that it is d2-persistent (it has no two-sorted general frames).

Next, let us consider hybrid formulas. Nominals enhance the expressive power of the language not only on the level of Kripke frame but also on the level of descriptive two-sorted general frames. The simplest example is the formula i, which defines the class of two-sorted general frames that have exactly one world. This formula is clearly d2-persistent. For another example, consider the conjunction φ of $p \to \Diamond(i \wedge \Diamond p)$ and $\Diamond\Diamond q \to \Diamond q$. Both with respect to descriptive two-sorted general frames and with respect to Kripke frames, φ expresses that the accessibility relation $R_\Diamond$ is the universal relation on the domain. Hence, φ is d2-persistent. Since validity of φ is not preserved under taking disjoint unions, φ is not equivalent (on Kripke frames or on descriptive two-sorted general frames) to a modal formula.

[2]Note that this terminology is a bit misleading: even if a general frame is atomless, the corresponding Boolean algebra with operators might still contain atoms (in the usual algebraic sense), and might even be atomic.

Strongly descriptive general frames

While we saw some d2-persistent hybrid formulas these cases are rather exceptional. In general, very few formulas involving nominals are d2-persistent. This suggests that we look for another, more restricted type of general frames. There is another reason to restrict the class of general frames under consideration: the additional inference rules of $\mathbf{K}^+_{\mathcal{H}}\Sigma$, $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma$ and $\mathbf{K}^+_{\mathcal{H}(\mathsf{E})}\Sigma$ do not preserve validity with respect to descriptive two-sorted general frames, in general.

5.2.3. Definition. *A two-sorted general frame* $(W, (R_\Diamond)_{\Diamond\in\text{MOD}}, \mathbb{A}, \mathbb{B})$ *is* strongly descriptive *if it is descriptive and it satisfies the following further conditions:*

(i) For all $X \in \mathbb{A}$*, if* $X \neq \emptyset$ *then* $X \cap \mathbb{B} \neq \emptyset$*.*

(ii) For all $X \in \mathbb{A}$ *and* $w \in \mathbb{B}$*, if* $\{v \in X \mid wRv\} \neq \emptyset$ *then* $\{v \in X \mid wRv\} \cap \mathbb{B} \neq \emptyset$*.*

Note that in strongly descriptive two-sorted general frame, we have that $w \in \mathbb{B}$ iff $\{w\} \in \mathbb{A}$. For this reason, when talking about strongly descriptive two-sorted general frames, we may leave out the qualification 'two-sorted'. The second sort $\mathbb{B}$ is already implicitly given by the underlying general frame.

5.2.4. Remark. From an algebraic perspective (cf. [21, Chapter 5]), strongly descriptive general frames correspond to Boolean algebras with operators that satisfy the following additional requirements:

1. For every element $a \neq \bot$ of the algebra, there is an atom i such that $i \leq a$.
2. For every element a of the algebra and for every atom i, if $i \leq \Diamond a$, then there is an atom j such that $j \leq a$ and $i \leq \Diamond j$.

The first condition is known as *atomicity*, and the second condition is equivalent to *complete additivity*, provided that the Boolean algebra is atomic. It is an easy exercise to show that whenever $\mathfrak{F}$ is strongly descriptive, then the corresponding algebra $\mathfrak{F}^*$ satisfies these two conditions, and conversely, whenever a Boolean algebra with operators $\mathfrak{A}$ satisfies these two conditions, the general ultrafilter frame $\mathfrak{A}_*$ is strongly descriptive.

Call a formula *sd-persistent* if its validity is preserved under the passage from strongly descriptive general frames to the underlying Kripke frame. Clearly, every d2-persistent formula is sd-persistent. However, many hybrid formulas that are not d2-persistent are sd-persistent. Consider for instance the $\mathcal{H}$-formula $i \to \Diamond i$. This formula is easily seen not to be d2-persistent. Nevertheless it is sd-persistent: Suppose a strongly descriptive general frame is not reflexive. Then, by d-persistence, $p \to \Diamond p$ can be falsified on it, i.e., there is a valuation V such that $p \wedge \neg\Diamond p$ is satisfiable under V. By strong descriptiveness, $V(p)$ contains an element of $\mathbb{B}$, say w. It follows that $i \wedge \neg\Diamond i$ is satisfiable under any valuation that sends i to $\{w\}$.

Discrete general frames

The last class of two-sorted general frames that we will consider is the class of *discrete* two-sorted general frames.

5.2.5. DEFINITION. *A two-sorted general frame* $(W, (R_\diamond)_{\diamond\in\text{MOD}}, \mathbb{A}, \mathbb{B})$ *is* discrete *if* $\mathbb{B} = W$.

As was the case with strongly descriptive two-sorted general frames, discrete two-sorted general frames are not really two-sorted: since $\mathbb{B} = W$, the admissible valuations for the nominals are already implicit in the underlying general frame. Hence, we will simply refer to these structures as *discrete general frames*.

An important source of discrete general frames is the following.

5.2.6. DEFINITION. *Given a strongly descriptive two-sorted general frame* $\mathfrak{F} = (W, (R_\diamond)_{\diamond\in\text{MOD}}, \mathbb{A}, \mathbb{B})$*, let* $\mathsf{dsf}\mathfrak{F} = (\mathbb{B}, (R_\diamond \cap (\mathbb{B}\times\mathbb{B}))_{\diamond\in\text{MOD}}, \{X\cap\mathbb{B} \mid X \in \mathbb{A}\}, \mathbb{B})$.

The notation dsf stands for *discrete subframe*, a name that is justified by the following proposition.

5.2.7. PROPOSITION. *For all strongly descriptive two-sorted general frames* $\mathfrak{F}$*,* $\mathsf{dsf}\mathfrak{F}$ *is a discrete two-sorted general frame.*

Proof: Let $\mathfrak{F} = (W, (R_\diamond)_{\diamond\in\text{MOD}}, \mathbb{A}, \mathbb{B})$ be any strongly descriptive two-sorted general frame. It is clear from the definition that $\mathsf{dsf}\mathfrak{F}$ is discrete. It remains to be shown that the set of admissible sets is closed under the Boolean operations and under the operations corresponding to the modalities.

▷ *Complement*
Suppose $Y \in \{X\cap\mathbb{B} \mid X \in \mathbb{A}\}$. Let $X \in \mathbb{A}$ be such that $Y = X\cap\mathbb{B}$. Then $\mathbb{B}\backslash Y = (W\backslash X)\cap\mathbb{B}$, and hence, since $W\backslash X \in \mathbb{A}$, it follows that $\mathbb{B}\backslash Y \in \{X\cap\mathbb{B} \mid X \in \mathbb{A}\}$.

▷ *Intersection*
Suppose $Y_1, Y_2 \in \{X\cap\mathbb{B} \mid X \in \mathbb{A}\}$. Let $X_1, X_2 \in \mathbb{A}$ be such that $Y_1 = X_1\cap\mathbb{B}$ and $Y_2 = X_2\cap\mathbb{B}$. Then $Y_1\cap Y_2 = X_1\cap X_2\cap\mathbb{B}$, and hence, since $X_1\cap X_2 \in \mathbb{A}$, it follows that $Y_1\cap Y_2 \in \{X\cap\mathbb{B} \mid X \in \mathbb{A}\}$.

▷ *Modalities*
Suppose $Y \in \{X\cap\mathbb{B} \mid X \in \mathbb{A}\}$, and let $\diamond Y = \{w \in \mathbb{B} \mid \exists v \in Y \text{ such that } wR_\diamond v\}$. Let $X \in \mathbb{A}$ be such that $X\cap\mathbb{B} = Y$, and let $\diamond X = \{w \in W \mid \exists v \in X \text{ such that } wR_\diamond v\}$. We claim that $\diamond Y = \diamond X\cap\mathbb{B}$, and hence $\diamond Y \in \{X\cap\mathbb{B} \mid X \in \mathbb{A}\}$.

[⊆] Suppose $w \in \diamond Y$. Since $Y \subseteq X$, it follows by monotonicity that $w \in \diamond X$. Furthermore, since $\diamond Y \subseteq \mathbb{B}$, we have that $w \in \diamond X\cap\mathbb{B}$.

[⊇] Suppose $w \in \diamond X\cap\mathbb{B}$. Then there is a $v \in X$ such that $wR_\diamond v$. It follows by the strong descriptiveness of $\mathfrak{F}$ that there is a $v \in X\cap\mathbb{B} = Y$ such that $wR_\diamond v$. Hence, $w \in \diamond Y$. □

5.2.8. Proposition. *For all strongly descriptive two-sorted general frames $\mathfrak{F}$ and $\mathcal{H}(\mathsf{E})$-formulas φ, $\mathfrak{F} \models \varphi$ iff $\mathsf{dsf}\mathfrak{F} \models \varphi$.*

⇒: We proceed by contraposition. Let $\mathfrak{F} = (W, (R_\diamond)_{\diamond \in \text{MOD}}, \mathbb{A}, \mathbb{B})$ be a strongly descriptive two-sorted general frame, and suppose $(\mathsf{dsf}\mathfrak{F}, V), v \not\models \varphi$ for some admissible valuation V and world $v \in \mathbb{B}$. Let V' be any admissible valuation for $\mathfrak{F}$ such that $V(p) = V'(p) \cap \mathbb{B}$ for $p \in \text{PROP}$ and $V(i) = V'(i)$ for $i \in \text{NOM}$. It is clear from the definition of $\mathsf{dsf}\mathfrak{F}$ that such valuations exist. A straightforward inductive argument establishes that for all $\mathcal{H}(\mathsf{E})$ formulas ψ, $(\mathfrak{F}, V'), v \models \psi$ iff $(\mathsf{dsf}\mathfrak{F}, V), v \models \psi$ (the only non-trivial step in the induction argument concerns formulas of the form $\diamond\varphi$, and here we use the fact the $\mathfrak{F}$ is strongly descriptive). It follows that $(\mathfrak{F}, V'), v \not\models \varphi$, and hence $\mathfrak{F} \not\models \varphi$.

[⇐] Again, we proceed by contraposition. Let $\mathfrak{F} = (W, (R_\diamond)_{\diamond \in \text{MOD}}, \mathbb{A}, \mathbb{B})$ be a strongly descriptive two-sorted general frame, and suppose $(\mathfrak{F}, V), w \not\models \varphi$ for some admissible valuation V and world $w \in W$. It follows from the first clause of Definition 5.2.3 that $(\mathfrak{F}, V), v \not\models \varphi$ for some $v \in \mathbb{B}$. Let V' be the valuation for $\mathsf{dsf}\mathfrak{F}$ given by $V'(p) = V(p) \cap \mathbb{B}$ for $p \in \text{PROP}$ and $V'(i) = V(i)$ for $i \in \text{NOM}$. It is clear from the definition that V' is an admissible valuation for $\mathsf{dsf}\mathfrak{F}$. Furthermore, a straightforward induction argument shows that for all $\mathcal{H}(\mathsf{E})$-formulas ψ and for all worlds $u \in \mathbb{B}$, $(\mathfrak{F}, V), u \models \varphi$ iff $(\mathsf{dsf}\mathfrak{F}, V'), u \models \varphi$ (the only non-trivial step in the induction argument concerns formulas of the form $\diamond\varphi$, and here we use the fact that $\mathfrak{F}$ is strongly descriptive). It follows that $(\mathsf{dsf}\mathfrak{F}, V'), v \not\models \varphi$, and hence $\mathsf{dsf}\mathfrak{F} \not\models \varphi$. □

5.2.9. Remark. As pointed out by T. Litak (p.c.), it is also possible to turn a discrete two-sorted general frame into a strongly descriptive one. It suffices to observe that if $\mathfrak{F}$ is a discrete two-sorted general frame, then the corresponding algebra $\mathfrak{F}^*$ is atomic and completely additive, hence the general ultrafilter frame $(\mathfrak{F}^*)_*$ is strongly descriptive, cf. Remark 5.2.4. It can even be shown that for strongly descriptive $\mathfrak{F}$, $((\mathsf{dsf}\mathfrak{F})^*)_* = \mathfrak{F}$, and for discrete $\mathfrak{F}$, $\mathsf{dsf}((\mathfrak{F}^*)_*) = \mathfrak{F}$. This shows that discrete two-sorted general frames and strongly descriptive two-sorted general frames are atomic and completely additive BAOs in two Gestalts. This duality can be pursued further, but we will not do so here.

Call a formula *di-persistent* if its validity is preserved under the passage from a discrete general frame to the underlying Kripke frame. From Section 2.4 we already know that every very simple modal Sahlqvist formula is di-persistent, as well as every shallow modal formula.

The most important class of di-persistent formulas is formed by the pure formulas, i.e., formulas that do not contain proposition letters, only nominals. All pure formulas are di-persistent. Moreover, every di-persistent formula defines the same class of discrete general frames as a pure formula.

5.2.10. Theorem. *Every pure $\mathcal{H}$-formula is di-persistent. Conversely, every di-persistent $\mathcal{H}$-formula defines the same class of discrete general frames as a pure $\mathcal{H}$-formula. The same holds for the languages $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$.*

Proof: We will only prove the case for the language $\mathcal{H}$. The first part of the result is obvious. Next, suppose φ is a di-persistent $\mathcal{H}$-formula, and let Σ be the set consisting of all pure instantiations of φ, i.e., $\Sigma = \{\varphi^\sigma \mid \sigma$ is a substitution that maps every proposition letter to a pure formula$\}$. We will show that Σ defines the same class of discrete general frames as φ. It then follows by compactness that φ is equivalent on discrete general frames to a finite conjunction of elements of Σ (note that compactness may be applied since the discrete general frames form an elementary class).

Let $\mathfrak{F}$ be any discrete two-sorted general frame. If $\mathfrak{F} \models \varphi$, then clearly, $\mathfrak{F} \models \Sigma$. Conversely, suppose $\mathfrak{F} \models \Sigma$. Let $\mathfrak{G}$ be the smallest discrete frame based on the underlying Kripke frame of $\mathfrak{F}$. More precisely, let V be any valuation for $\mathfrak{F}$ under which every point in $\mathfrak{F}$ is named by a nominal, and let $\mathfrak{G}$ be the discrete general frame in which the admissible subsets are precisely those definable under V by means of pure $\mathcal{H}$ formulas. Clearly, $\mathfrak{G} \models \varphi$. By di-persistence, we obtain that φ is valid on the underlying Kripke frame of $\mathfrak{G}$ (which is also the underlying Kripke frame of $\mathfrak{F}$), and hence, $\mathfrak{F} \models \varphi$. $\square$

In particular, it follows that every very simple Sahlqvist formula defines the same class of Kripke frames as a pure $\mathcal{H}$-formula.

5.3 Completeness with respect to general frames

We will now prove completeness of the axiomatizations of $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$ with respect to the types of general frames introduced in the previous section. The general pattern will be as follows: the axiomatizations without the extra inference rules are complete with respect to descriptive two-sorted general frames, whereas the axiomatizations with the extra inference rules are complete with respect to strongly descriptive two-sorted general frames and discrete general frames.

Recall that an axiomatization is *sound* for a class of semantic structures if every derivable formula is semantically valid, *complete* if every semantically valid formula is derivable, and *strongly complete* if whenever a set of formulas Γ is semantically unsatisfiable, there is a finite conjunction γ of elements of Γ such that $\neg\gamma$ is derivable.

Finally, we say that a formula φ defines a class K of general frames of some type (e.g., descriptive) if for all general frames $\mathfrak{F}$ of the relevant type, $\mathfrak{F} \in \mathsf{K}$ iff $\mathfrak{F} \models \varphi$.

Descriptive two-sorted general frames

First, let us consider the axiomatization $\mathbf{K}_{\mathcal{H}}$ and its extensions.

5.3.1. THEOREM. *Let Σ be a set of $\mathcal{H}$-formulas. $\mathbf{K}_{\mathcal{H}}\Sigma$ is sound and strongly complete for the class of descriptive two-sorted general frames defined by Σ.*

Proof: We will only prove completeness. For the purpose of this proof, we will temporarily adopt an alternative, purely modal semantics of the language $\mathcal{H}$, by treating nominals as modal constants (i.e, nullary modalities). Let a *non-standard frame* be a structure $\mathfrak{F} = (W, (R_\Diamond)_{\Diamond\in\text{MOD}}, (S_i)_{i\in\text{NOM}})$, where each $R_\Diamond$ is a binary relation on W and each S_i is a subset of W, interpreting the nominal i. Non-standard general frames and non-standard models are defined similarly.

Now, suppose Γ is a $\mathbf{K}_{\mathcal{H}}\Sigma$-consistent set of $\mathcal{H}$-formulas. Then by Theorem 2.4.3, Γ is satisfiable on a descriptive non-standard general frame $\mathfrak{F} = (W, (R_\Diamond)_{\Diamond\in\text{MOD}}, (S_i)_{i\in\text{NOM}}, \mathbb{A})$ such that $\mathfrak{F} \models \mathbf{K}_{\mathcal{H}}\Sigma$.[3] Without loss of generality, we may assume that $\mathfrak{F}$ is point-generated.

Now recall that $\mathbf{K}_{\mathcal{H}}\Sigma$ contains the following axiom scheme.

$$(\mathit{Nom}) \qquad \vdash \Diamond_1 \cdots \Diamond_n(i \wedge p) \to \Box_{n+1} \cdots \Box_{n+m}(i \to p)$$

Each instance of (Nom) is a Sahlqvist formula, and therefore d-persistent.[4] Hence, each instance, being valid on $\mathfrak{F}$, is valid on its underlying (non-standard) Kripke frame. Using these facts, and considering the first-order correspondents of the formulas involved, it is easily seen that each $|S_i| \leq 1$ for all $i \in \text{NOM}$ (for, if $|S_i| \geq 2$ for some $i \in \text{MOD}$, then some instance of (Nom) could be falsified at the root of $\mathfrak{F}$).

We can now distinguish three cases:

1. $|S_i| = 1$ for all $i \in \text{NOM}$. This is the simplest case. Let $\mathfrak{H}$ be the (standard) two-sorted general frame $(W, (R_\Diamond)_{\Diamond\in\text{MOD}}, \mathbb{A}, \mathbb{B})$, where $\mathbb{B} = \bigcup_{i\in\text{NOM}} S_i$. Clearly, Γ is satisfiable on $\mathfrak{H}$. It is also an easy exercise to show (using closure under substitution) that $\mathfrak{H} \models \Sigma$. Finally, since descriptiveness is preserved under taking reducts of general frames, $\mathfrak{H}$ is a descriptive two-sorted general frame.

2. $|S_i| = 0$ for some $i \in \text{NOM}$, but not for all.

   Let j be a nominal such that $S_j \neq \emptyset$. For $i \in \text{NOM}$, let $S'_i = S_j$ if $S_i = \emptyset$ and $S'_i = \emptyset$ otherwise. Let $\mathfrak{F}' = (W, (R_\Diamond)_{\Diamond\in\text{MOD}}, (S'_i)_{i\in\text{NOM}}, \mathbb{A})$, and let $\mathfrak{G}$ be the disjoint union of $\mathfrak{F}$ and $\mathfrak{F}'$.[5] By construction, $\mathfrak{G}$ is a descriptive general non-standard $\mathcal{H}$-frame. Furthermore, it is easily seen that $\mathfrak{G} \models \mathbf{K}_{\mathcal{H}}\Sigma$, and that Γ is satisfiable on $\mathfrak{G}$. Hence, we can proceed as in the first case.

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[3] Actually, we use here a slightly more general version of Theorem 2.4.3, that applies to languages with modal constants (see for instance [21])

[4] Here, by an instance, we mean a particular choce of modalities $\Diamond_1, \ldots, \Diamond_{n+m} \in \text{MOD}$ $(n, m \in \omega)$. Furthermore, we use here the general definition of Sahlqvist formulas given in [21], which applies to multi-modal languages with modalities that are not necessarily unary.

[5] Disjoint unions of Kripke frames were defined on page 10. The disjoint union of two general frames, $(\mathfrak{F}, \mathbb{A}) \uplus (\mathfrak{G}, \mathbb{A}')$, is defined as $(\mathfrak{F} \uplus \mathfrak{G}, \mathbb{A}'')$, where $\mathbb{A}'' = \{X \uplus Y \mid X \in \mathbb{A} \text{ and } Y \in \mathbb{A}'\}$).

3. $|S_i| = 0$ for all $i \in$ NOM. By the rule (*NameLite*), and the fact that $\mathbf{K}_{\mathcal{H}}\Sigma \nvdash \bot$, the formula i is consistent, and hence satisfiable on a point-generated descriptive non-standard $\mathcal{H}$-frame $\mathfrak{G}$ with $\mathfrak{G} \models \mathbf{K}_{\mathcal{H}}\Sigma$. By closure under disjoint union, $\mathfrak{F} \uplus \mathfrak{G} \models \mathbf{K}_{\mathcal{H}}\Sigma$, and by bisimulation invariance, Γ is satisfiable on $\mathfrak{F} \uplus \mathfrak{G}$. Hence, we can proceed as in the second case. □

It is important for this result that the logic includes the inference rule (*NameLite*). In fact, there are modal formulas φ such that $\mathbf{K}_{\mathcal{H}}\{\varphi\}$ without this rule is not complete for *any* class of descriptive frames. This follows from a more general result. For Σ a set of $\mathcal{H}$-formulas, define $\mathbf{K}^{-}_{\mathcal{H}}\Sigma$ to be the axiomatization $\mathbf{K}_{\mathcal{H}}\Sigma$ minus the (*NameLite*) rule. Then the following conservativity result holds.

5.3.2. Proposition. *For every set of modal formulas Σ and modal formula φ, $\mathbf{K}^{-}_{\mathcal{H}}\Sigma \models \varphi$ iff $\mathbf{K}_{\mathcal{M}}\Sigma \models \varphi$.*

Proof: We will only prove the left-to-right direction, since the other direction follows immediately from the fact that $\mathbf{K}^{-}_{\mathcal{H}}\Sigma$ extends $\mathbf{K}_{\mathcal{M}}\Sigma$. The proof will proceed by contraposition, and we will make use of the non-standard semantics of $\mathcal{H}$ introduced in the proof of Theorem 5.3.1.

Suppose $\mathbf{K}_{\mathcal{M}}\Sigma \not\models \varphi$. Then there is a descriptive general frame $\mathfrak{F} = (W, (R_\diamond)_{\diamond \in \text{MOD}}, \mathbb{A})$ with an admissible valuation V and a world $w \in W$ such that $\mathfrak{F} \models \Sigma$ and $\mathfrak{F}, V, w \not\models \varphi$. Let $\mathfrak{F}'$ be the non-standard general $\mathcal{H}$-frame $(W, (R_\diamond)_{\diamond \in \text{MOD}}, (S_i)_{i \in \text{NOM}}, \mathbb{A})$ where $S_i = \emptyset$ for all $i \in$ NOM. It is easily seen that $\mathfrak{F}' \models \mathbf{K}^{-}_{\mathcal{H}}\Sigma$ and $\mathfrak{F}' \not\models \varphi$. It follows that $\mathbf{K}^{-}_{\mathcal{H}}\Sigma \not\models \varphi$. □

5.3.3. Proposition. *There is a modal formula φ such that $\mathbf{K}^{-}_{\mathcal{H}}\{\varphi\}$ is not complete for any class of two-sorted general frames.*

Proof: From [99], we know that there is a modal formula φ such that the modal logic $\mathbf{K}_{\mathcal{M}}\{\varphi\}$ is consistent, and such that every general frame on which φ is valid is atomless (i.e., has no singleton admissibles). It follows that there is no two-sorted general frame on which φ is valid (every two-sorted general frame contains an admissible singleton set). Hence, if $\mathbf{K}^{-}_{\mathcal{H}}\{\varphi\}$ would be complete for any class of two-sorted general frames, it would have to be inconsistent. However, it follows from Proposition 5.3.2 that $\mathbf{K}^{-}_{\mathcal{H}}\{\varphi\}$ is consistent. □

5.3.4. Corollary. *For Σ a set of modal formulas, $\mathbf{K}_{\mathcal{H}}\Sigma$ is in general not conservative over $\mathbf{K}^{-}_{\mathcal{H}}\Sigma$ or $\mathbf{K}_{\mathcal{M}}\Sigma$.*

Next, let us consider the languages $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$.

5.3.5. Theorem. *Let Σ be a set of $\mathcal{H}(@)$-formulas. $\mathbf{K}_{\mathcal{H}(@)}\Sigma$ is sound and strongly complete for the class of descriptive two-sorted general frames defined by Σ.*

Proof: We will only prove completeness. For the purpose of this proof, we will temporarily adopt an alternative, purely modal semantics of the language $\mathcal{H}(@)$, by treating nominals as modal constants and satisfaction operators as unary modalities. Let a non-standard frame be a structure $\mathfrak{F} = (W, (R_\diamond)_{\diamond\in\text{MOD}}, (R_i)_{i\in\text{NOM}}, (S_i)_{i\in\text{NOM}})$, where each $R_\diamond$ is a binary relation on W, each R_i is a binary relation on W interpreting the the satisfaction operator $@_i$, and $S_i \subseteq W$ interprets the nominal i, taken as a modal constant. Non-standard general frame and non-standard models are defined similarly.

Now, suppose Γ is a $\mathbf{K}_{\mathcal{H}(@)}\Sigma$-consistent set of $\mathcal{H}(@)$-formulas. Then by Theorem 2.4.3, Γ is satisfiable on a descriptive non-standard general frame $\mathfrak{F} = (W, (R_\diamond)_{\diamond\in\text{MOD}}, (R_i)_{i\in\text{NOM}}, (S_i)_{i\in\text{NOM}}, \mathbb{A})$ such that $\mathfrak{F} \models \mathbf{K}_{\mathcal{H}(@)}\Sigma$. Without loss of generality, we may assume that $\mathfrak{F}$ is point-generated.

Recall that $\mathbf{K}_{\mathcal{H}(@)}\Sigma$ contains the distribution axiom for satisfaction operators ($K_@$), the necessitation rule for satisfaction operators, and the following axiom schemes.

$@_j @_i p \to @_i p$	$\forall xyz(R_j xy \wedge R_i yz \to R_i xz)$
$\diamond @_i p \to @_i p$	$\forall xyz(R_\diamond xy \wedge R_i yz \to R_i xz)$
$i \wedge p \to @_i p$	$\forall x(S_i x \to R_i xx)$
$@_i i$	$\forall x \exists y(R_i xy \wedge S_i y)$
$@_i p \leftrightarrow \neg @_i \neg p$	$\forall xyz(R_i xy \wedge R_i xz \to y = z)$

Each of the axioms is in Sahlqvist form (taken as a modal formula). Their first-order correspondents are indicated as well.[6] By d-persistence, each of these formulas is valid on the underlying (non-standard) Kripke frame of $\mathfrak{F}$. Together with the fact that $\mathfrak{F}$ is point-generated, this implies that $|S_i| = 1$ and $R_i = W \times S_i$ for each $i \in \text{NOM}$.

Let $\mathfrak{F}' = (W, (R_\diamond)_{\diamond\in\text{MOD}}, \mathbb{A}, \mathbb{B})$ with $\mathbb{B} = \bigcup_{i\in\text{NOM}} S_i$. It is an easy exercise to show (using closure under substitution) that $\mathfrak{F}' \models \Sigma$ and that Γ is satisfiable on $\mathfrak{F}'$. Finally, $\mathfrak{F}'$ is a descriptive two-sorted general frame. □

5.3.6. THEOREM. *Let Σ be a set of $\mathcal{H}(\mathsf{E})$-formulas. $\mathbf{K}_{\mathcal{H}(\mathsf{E})}\Sigma$ is sound and strongly complete for the class of descriptive two-sorted general frames defined by Σ.*

Proof: We will only prove completeness. For the purpose of this proof, we will temporarily adopt an alternative, purely modal semantics of the language $\mathcal{H}(\mathsf{E})$, by treating nominals as modal constants and interpreting the global modality as an ordinary unary modalities. Let a non-standard frame be a structure $\mathfrak{F} = (W, (R_\diamond)_{\diamond\in\text{MOD}}, R_\mathsf{E}, (S_i)_{i\in\text{NOM}})$, where each $R_\diamond$ is a binary relation on W, R_E is a binary relation on W interpreting the modality E, and $S_i \subseteq W$ interprets the nominal i, taken as a modal constant. Non-standard general frame and non-standard models can be defined similarly.

[6] Here, we exploit the fact that in the presence of the (*Selfdual*) axiom $@_i p \leftrightarrow \neg @_i \neg p$, the satisfaction operators may be interpreted not only as boxes but also as diamonds.

Now, suppose Γ is a $\mathbf{K}_{\mathcal{H}(@)}\Sigma$-consistent set of $\mathcal{H}(\mathsf{E})$-formulas. Then by Theorem 2.4.3, Γ is satisfiable on a descriptive non-standard general frame $\mathfrak{F} = (W, (R_\Diamond)_{\Diamond\in\text{MOD}}, R_\mathsf{E}, (S_i)_{i\in\text{NOM}}, \mathbb{A})$ such that $\mathfrak{F} \models \mathbf{K}_{\mathcal{H}(@)}\Sigma$. Without loss of generality, we may assume that $\mathfrak{F}$ is point-generated.

Recall that $\mathbf{K}_{\mathcal{H}(\mathsf{E})}\Sigma$ contains the distribution axiom and necessitation rule for E, as well as the following axiom schemes.

$p \to \mathsf{E}p$	$\forall x.R_\mathsf{E}xx$
$\mathsf{EE}p \to \mathsf{E}p$	$\forall xyz.(R_\mathsf{E}xy \wedge R_\mathsf{E}yz \to R_\mathsf{E}xz)$
$p \to \mathsf{AE}p$	$\forall xy.(R_\mathsf{E}xy \to R_\mathsf{E}yx)$
$\Diamond p \to \mathsf{E}p$	$\forall xy.(R_\Diamond xy \to R_\mathsf{E}xy)$
$\mathsf{E}i$	$\forall x\exists y.(R_\mathsf{E}xy \wedge S_iy)$
$\mathsf{E}(i \wedge p) \to \mathsf{A}(i \to p)$	$\forall xyz.(R_\mathsf{E}xy \wedge R_\mathsf{E}xz \wedge S_iy \wedge S_iz \to y = z)$

Each of the axioms is in Sahlqvist form (taken as a modal formula). Their first-order correspondents are indicated as well. By d-persistence, each of these formulas is valid on the underlying (non-standard) Kripke frame of $\mathfrak{F}$. Together with the fact that $\mathfrak{F}$ is point-generated, this implies that $R_\mathsf{E} = W \times W$ and $|S_i| = 1$ and for each $i \in \text{NOM}$.

Let $\mathfrak{F}' = (W, (R_\Diamond)_{\Diamond\in\text{MOD}}, \mathbb{A}, \mathbb{B})$ with $\mathbb{B} = \bigcup_{i\in\text{NOM}} S_i$. It is an easy exercise to show (using closure under substitution) that $\mathfrak{F}' \models \Sigma$ and that Γ is satisfiable on $\mathfrak{F}'$. Finally, $\mathfrak{F}'$ is a descriptive two-sorted general frame. □

Strongly descriptive two-sorted general frames

Descriptive two-sorted general frames do not provide an adequate semantics for $\mathbf{K}^+_{\mathcal{H}}$, $\mathbf{K}^+_{\mathcal{H}(@)}$ and $\mathbf{K}^+_{\mathcal{H}(\mathsf{E})}$, since the additional inference rules of these logics do not preserve validity on such frames. Strongly descriptive two-sorted general frames do provide an adequate semantics.

5.3.7. Proposition. *All inference rules of $\mathbf{K}^+_{\mathcal{H}}$, $\mathbf{K}^+_{\mathcal{H}(@)}$ and $\mathbf{K}^+_{\mathcal{H}(\mathsf{E})}$ preserve validity on strongly descriptive general frames.*

Proof: By way of example, we discuss the (*Name*) rule of $\mathbf{K}^+_{\mathcal{H}}\Sigma$. Let K be a class of strongly descriptive frames. We proceed by contraposition: suppose $\mathsf{K} \not\models \varphi$ and suppose that the nominal i does not occur in φ. Then $\neg\varphi$ is satisfiable on a (strongly descriptive) $\mathfrak{F} \in \mathsf{K}$ under some valuation V. Let $[\![\neg\varphi]\!]$ be the set of points in K satisfying $\neg\varphi$ under the valuation V. Note that $[\![\neg\varphi]\!] \neq \emptyset$, and hence by strong descriptiveness, $[\![\neg\varphi]\!] \cap \mathbb{B} \neq \emptyset$. Then by extending the valuation V such that i denotes a point in $[\![\neg\varphi]\!] \cap \mathbb{B}$, we can satisfy $i \wedge \neg\varphi$, and hence $\mathsf{K} \not\models i \to \varphi$. □

5.3.8. Corollary. *For any set Σ of $\mathcal{H}$-formulas, $\mathbf{K}^+_{\mathcal{H}}\Sigma$ is sound for the class of strongly descriptive frames defined by Σ. Similarly for $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma$ and $\mathbf{K}^+_{\mathcal{H}(\mathsf{E})}\Sigma$.*

As we will now show, these logics are not only sound but also strongly complete with respect to the relevant class of strongly descriptive frames. First, let us consider the language $\mathcal{H}$.

5.3.9. LEMMA. *Let Σ be any set of $\mathcal{H}$-formulas. Every $\mathbf{K}^+_{\mathcal{H}}\Sigma$-consistent set Γ can be extended to a maximal $\mathbf{K}^+_{\mathcal{H}}\Sigma$-consistent set Γ^+ such that*

1. *One of the elements of Γ^+ is a nominal*
2. *For all $\Diamond_1 \cdots \Diamond_n(i \wedge \Diamond_{n+1}\varphi) \in \Gamma^+$ there is a nominal j such that $\Diamond_1 \cdots \Diamond_n(i \wedge \Diamond_{n+1}(j \wedge \varphi)) \in \Gamma^+$*

Proof: By expanding the language with new nominals, we can ensure that a countably infinite number of nominals do not occur in Γ, while preserving consistency. Let $(i_n)_{n\in\mathbb{N}}$ be an enumeration of a countably infinite set of nominals not occurring in Γ, and let $(\varphi_n)_{n\in\mathbb{N}}$ be an enumeration all $\mathcal{H}$-formulas of the extended language.

Let Γ^0 denote $\Gamma\cup\{i_0\}$. The (*Name*) rule guarantees that Γ_0 is $\mathbf{K}^+_{\mathcal{H}}\Sigma$-consistent, for suppose not. Then there are $\varphi_1, \ldots, \varphi_n \in \Gamma$ such that $\vdash_{\mathbf{K}^+_{\mathcal{H}}\Sigma} i_0 \to \neg(\varphi_1 \wedge \cdots \wedge \varphi_n)$. Since i_0 does not occur in $\varphi_1, \ldots, \varphi_n$, by the (*Name*) rule, $\vdash_{\mathbf{K}^+_{\mathcal{H}}\Sigma} \neg(\varphi_1 \wedge \cdots \wedge \varphi_n)$, and hence Γ is already $\mathbf{K}^+_{\mathcal{H}}\Sigma$-inconsistent.

For $k \in \mathbb{N}$, define Γ^{k+1} as follows. If $\Gamma^k \cup \{\varphi_k\}$ is $\mathbf{K}^+_{\mathcal{H}}\Sigma$-inconsistent, then $\Gamma^{k+1} = \Gamma^k$. Otherwise:

1. $\Gamma^{k+1} = \Gamma^k \cup \{\varphi_k\}$ if φ_k is not of the form $\Diamond_1 \cdots \Diamond_n(i \wedge \Diamond_{n+1}\varphi)$.
2. $\Gamma^{k+1} = \Gamma^k \cup \{\varphi_k,\ \Diamond_1 \cdots \Diamond_n(i \wedge \Diamond_{n+1}(i_m \wedge \varphi))\}$ if φ_k is of the form $\Diamond_1 \cdots \Diamond_n(i \wedge \Diamond_{n+1}\varphi)$, where i_m is the first new nominal that does not occur in Γ^k or φ_k.

Each step preserves consistency: if Γ^k is $\mathbf{K}^+_{\mathcal{H}}\Sigma$-consistent, then so is Γ^{k+1}. The only non-trivial case concerns the second clause, and we will prove also in this case, consistency is preserved.

Let $\Gamma^k \cup \{\varphi_k\}$ be $\mathbf{K}^+_{\mathcal{H}}\Sigma$-consistent, let φ_k be of the form $\Diamond_1 \cdots \Diamond_n(i \wedge \Diamond_{n+1}\varphi)$, and suppose for the sake of contradiction that $\Gamma^{k+1} = \Gamma^k \cup \{\varphi_k, \Diamond_1 \cdots \Diamond_n(i \wedge \Diamond_{n+1}(i_m \wedge \varphi))\}$ is $\mathbf{K}^+_{\mathcal{H}}\Sigma$-inconsistent. Then there are $\varphi_1, \ldots, \varphi_l \in \Gamma^k$ such that

$$\vdash_{\mathbf{K}^+_{\mathcal{H}}\Sigma} \Big(\varphi_k \wedge \Diamond_1 \cdots \Diamond_n \Diamond_{n+1}(i_m \wedge \varphi)\Big) \to \neg(\varphi_1 \wedge \cdots \wedge \varphi_l)$$

It follows by the rule (*Paste*) that $\vdash_{\mathbf{K}^+_{\mathcal{H}}\Sigma} \varphi_k \to \neg(\varphi_1 \wedge \cdots \wedge \varphi_l)$. But this contradicts the fact that $\Gamma^k \cup \{\varphi_k\}$ is $\mathbf{K}^+_{\mathcal{H}}\Sigma$-consistent. We conclude that Γ^{k+1} is consistent.

Since $\mathbf{K}^+_{\mathcal{H}}\Sigma$-consistency is preserved at each stage, it follows that $\Gamma^+ = \bigcup_{n<\omega} \Gamma^n$ is consistent. It is easy to see that Γ^+ also satisfies the other requirements. □

5.3.10. THEOREM. *Let Σ be a set of $\mathcal{H}$-formulas. $\mathbf{K}^+_{\mathcal{H}}\Sigma$ is strongly sound and complete for the class of strongly descriptive two-sorted general frames defined by Σ.*

Proof: Let Γ be any $\mathbf{K}^+_{\mathcal{H}}\Sigma$-consistent set of formulas. Let Γ^+ be the maximal consistent set extending Γ obtained from Lemma 5.3.9. Applying Theorem 5.3.1, we obtain a descriptive two-sorted general frame $\mathfrak{F}$ such that $\mathfrak{F} \models \Sigma$ and Γ^+ is satisfiable on $\mathfrak{F}$. It follows from the properties of Γ^+ and the construction of $\mathfrak{F}$ that $\mathfrak{F}$ is in fact strongly descriptive.[7] $\Box$

Next, let us consider the language $\mathcal{H}(@)$.

5.3.11. LEMMA. *The following rule is derivable in $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma$:*

> *If $\vdash @_i\Diamond j \wedge @_j\varphi \rightarrow \psi$ then $\vdash @_i\Diamond\varphi \rightarrow \psi$, provided $i \neq j$ and j does not occur in φ or ψ.*

Proof: Suppose $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma \vdash @_i\Diamond j \wedge @_j\varphi \rightarrow \psi$. Let k be a new nominal. Then by the Necessitation rule for the satisfaction operators, $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma \vdash @_k(@_i\Diamond j \wedge @_j\varphi \rightarrow \psi)$. Then latter formula is semantically equivalent to $@_i\Diamond j \rightarrow @_j(\varphi \rightarrow @_k\psi)$. By Theorem 5.3.5, this equivalence is provable in $\mathbf{K}_{\mathcal{H}(@)}$ and hence in $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma$. It follows that $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma \vdash @_i\Diamond j \rightarrow @_j(\varphi \rightarrow @_k\psi)$. By the rule (BG), $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma \vdash @_i\Box(\varphi \rightarrow @_k\psi)$. The latter formula is semantically equivalent to $@_k(@_i\Diamond\varphi \rightarrow \psi)$. By Theorem 5.3.5, this equivalence is provable in $\mathbf{K}_{\mathcal{H}(@)}$ and hence in $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma$. It follows that $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma \vdash @_k(@_i\Diamond\varphi \rightarrow \psi)$. By the name rule, $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma \vdash @_i\Diamond\varphi \rightarrow \psi$. $\Box$

5.3.12. LEMMA. *Every $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma$-consistent set Γ can be extended to a maximal $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma$-consistent set Γ^+ such that*

1. *One of the elements of Γ^+ is a nominal*
2. *For all $@_i\Diamond\varphi \in \Gamma$ there is a nominal j such that $@_i\Diamond j \in \Gamma$ and $@_j\varphi \in \Gamma$.*

Proof: By expanding the language with new nominals, we can ensure that a countably infinite number of nominals do not occur in Γ, while preserving consistency. Let $(i_n)_{n\in\mathbb{N}}$ be an enumeration of a countably infinite set of nominals not occurring in Γ, and let $(\varphi_n)_{n\in\mathbb{N}}$ be an enumeration all $\mathcal{H}(@)$-formulas of the extended language.

[7] Here, we assume without loss of generality that the non-standard general frame $\mathfrak{F}$ used in the proof of Theorem 5.3.1 is a point-generated subframes of the canonical (non-standard) general frame. Furthermore, we use the fact that strong descriptiveness is preserved under taking disjoint unions of finitely many general frames.

Let Γ^0 denote $\Gamma \cup \{i_0\}$. The rule ($Name_@$) guarantees that Γ_0 is consistent, for suppose not. Then there are $\varphi_1, \ldots, \varphi_n$ such that $\vdash_{\mathbf{K}^+_{\mathcal{H}(@)}\Sigma} i_0 \to \neg(\varphi_1 \wedge \cdots \wedge \varphi_n)$. By the Necessitation rule and the K axiom for the satisfaction operators, it follows that $\vdash_{\mathbf{K}^+_{\mathcal{H}(@)}\Sigma} @_{i_0} i_0 \to @_{i_0}\neg(\varphi_1 \wedge \cdots \wedge \varphi_n)$. Since $\vdash_{\mathbf{K}^+_{\mathcal{H}(@)}\Sigma} @_{i_0} i_0$, it follows that $\vdash_{\mathbf{K}^+_{\mathcal{H}(@)}\Sigma} @_{i_0}\neg(\varphi_1 \wedge \cdots \wedge \varphi_n)$, and hence, by the ($Name_@$) rule, $\vdash_{\mathbf{K}^+_{\mathcal{H}(@)}\Sigma} \neg(\varphi_1 \wedge \cdots \wedge \varphi_n)$. But this contradicts the fact that Γ is consistent.

For $k \in \mathbb{N}$, define Γ^{k+1} as follows. If $\Gamma^k \cup \{\varphi_k\}$ is $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma$-inconsistent, then $\Gamma^{k+1} = \Gamma^k$. Otherwise:

1. $\Gamma^{k+1} = \Gamma^k \cup \{\varphi_k\}$ if φ_k is not of the form $@_i\Diamond\psi$.

2. $\Gamma^{k+1} = \Gamma^k \cup \{\varphi_k, @_i\Diamond i_m, @_{i_m}\psi\}$ if φ_k is of the form $@_i\Diamond\psi$,
 where i_m is the first new nominal that does not occur in Γ^k or φ_k.

Each step preserves consistency: if Γ^k is $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma$-consistent, then so is Γ^{k+1}. The only non-trivial case concerns the second clause, and we will prove that also in this case, consistency is preserved.

Let $\Gamma^k \cup \{\varphi_k\}$ be $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma$-consistent, let φ_k be of the form $@_i\Diamond\psi$, and suppose for the sake of contradiction that $\Gamma^{k+1} = \Gamma^k \cup \{\varphi_k, @_i\Diamond i_m \wedge @_{i_m}\psi\}$ is not $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma$-consistent. Then there are $\varphi_1, \ldots, \varphi_n \in \Gamma^k$ such that $\vdash_{\mathbf{K}^+_{\mathcal{H}(@)}\Sigma} (\varphi_k \wedge @_i\Diamond i_m, @_{i_m}\psi) \to \neg(\varphi_1 \wedge \cdots \wedge \varphi_n)$. It follows by Lemma 5.3.11 that $\vdash_{\mathbf{K}^+_{\mathcal{H}(@)}\Sigma} \varphi_k \to \neg(\varphi_1 \wedge \cdots \wedge \varphi_n)$. But this contradicts the fact that $\Gamma^k \cup \{\varphi_k\}$ is $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma$-consistent. We conclude that Γ^{k+1} is consistent.

Since $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma$-consistency is preserved at each stage, it follows that $\Gamma^+ = \bigcup_{n<\omega}\Gamma^n$ is $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma$-consistent. It is easy to see that Γ^+ also satisfies the other requirements. □

5.3.13. Theorem. *Let Σ be a set of $\mathcal{H}(@)$-formulas. $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma$ is strongly sound and complete for the class of strongly descriptive two-sorted general frames defined by Σ.*

Proof: Let Γ be any $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma$-consistent set of formulas. Let Γ^+ be the maximal consistent set extending Γ obtained from Lemma 5.3.12. Applying Theorem 5.3.5, we obtain a descriptive two-sorted general frame $\mathfrak{F}$ such that $\mathfrak{F} \models \Sigma$ and Γ^+ is satisfiable on $\mathfrak{F}$. It follows from the properties of Γ^+ and the construction of $\mathfrak{F}$ that $\mathfrak{F}$ is in fact strongly descriptive.[8] □

Finally, let us consider the language $\mathcal{H}(\mathsf{E})$.

[8]Here, we assume without loss of generality that the non-standard general frame $\mathfrak{F}$ used in the proof of Theorem 5.3.5 is a point-generated subframes of the canonical (non-standard) general frame.

5.3.14. Lemma. *Every* $\mathbf{K}^+_{\mathcal{H}(\mathsf{E})}\Sigma$*-consistent set* Γ *can be extended to a maximal* $\mathbf{K}^+_{\mathcal{H}(\mathsf{E})}\Sigma$*-consistent set* Γ^+ *such that*

1. *One of the elements of* Γ^+ *is a nominal*
2. *For all* $\mathsf{E}(i \wedge \Diamond\varphi) \in \Gamma$ *there is a nominal* j *such that* $\mathsf{E}(i \wedge \Diamond j) \in \Gamma$ *and* $\mathsf{E}(j \wedge \varphi) \in \Gamma$.

Proof: Analogous to the proof of Lemma 5.3.12. □

5.3.15. Theorem. *Let* Σ *be a set of* $\mathcal{H}(\mathsf{E})$*-formulas.* $\mathbf{K}^+_{\mathcal{H}(\mathsf{E})}\Sigma$ *is strongly sound and complete for the class of strongly descriptive two-sorted general frames defined by* Σ.

Proof: Analogous to the proof of Theorem 5.3.13, using Lemma 5.3.14. □

Discrete two-sorted general frames

We will now show that, besides strongly descriptive frames, discrete frames also offer an suitable semantics for $\mathbf{K}^+_{\mathcal{H}}$, $\mathbf{K}^+_{\mathcal{H}(@)}$ and $\mathbf{K}^+_{\mathcal{H}(\mathsf{E})}$, in the sense that for all sets Σ of formulas of the relevant language, $\mathbf{K}^+_{\mathcal{H}}\Sigma$, $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma$ and $\mathbf{K}^+_{\mathcal{H}(\mathsf{E})}\Sigma$ are sound and strongly complete with respect to the class of discrete frames defined by Σ.

There are two routes for constructing discrete frames from consistent sets of formulas: either directly by a Henkin-style construction, or using our earlier results by transforming a strongly descriptive frame into a discrete one. We have opted for the latter.

5.3.16. Theorem. $\mathbf{K}^+_{\mathcal{H}}\Sigma$, $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma$ *and* $\mathbf{K}^+_{\mathcal{H}(\mathsf{E})}\Sigma$ *are strongly sound and complete for the class of discrete two-sorted general frames defined by* Σ*, where* Σ *is any set of* $\mathcal{H}$*-,* $\mathcal{H}(@)$ *or* $\mathcal{H}(\mathsf{E})$*-formulas, respectively.*

Proof: We will prove the case for $\mathcal{H}$, since the other cases are similar. Let Γ be any $\mathbf{K}^+_{\mathcal{H}}\Sigma$-consistent set of formulas. Pick a new nominal i. By the (*Name*) rule, $\Gamma \cup \{i\}$ is also $\mathbf{K}^+_{\mathcal{H}}\Sigma$-consistent. Hence, by Theorem 5.3.10, $\Gamma \cup \{i\}$ is satisfiable on a strongly descriptive two-sorted general frame $\mathfrak{F} = (W, (R_\Diamond)_{\Diamond\in\textsc{mod}}, \mathbb{A}, \mathbb{B})$ with $\mathfrak{F} \models \Sigma$. Let V be an admissible valuation for $\mathfrak{F}$ and let w be a world such that $(\mathfrak{F}, V), w \models \Gamma \cup \{i\}$. Note that $w \in \mathbb{B}$. Let V' be the valuation for $\mathsf{dsf}\mathfrak{F}$ given by $V'(p) = V(p) \cap \mathbb{B}$ for $p \in \textsc{prop}$ and $V'(i) = V(i)$ for $i \in \textsc{nom}$. It is clear from the definition of $\mathsf{dsf}\mathfrak{F}$ that V' is admissible.

A straightforward induction argument shows that for all $\mathcal{H}(\mathsf{E})$-formulas φ and for all worlds $v \in \mathbb{B}$, $(\mathfrak{F}, V), v \models \varphi$ iff $(\mathsf{dsf}\mathfrak{F}, V'), v \models \varphi$. The only non-trivial step in the induction argument concerns formulas of the form $\Diamond\varphi$, and here we use the fact the $\mathfrak{F}$ is strongly descriptive.

It follows that $(\mathsf{dsf}\mathfrak{F}, V'), w \models \Gamma$. By Proposition 5.2.8, $\mathsf{dsf}\mathfrak{F} \models \Sigma$. Hence, Γ is satisfiable on the class of discrete two-sorted general frames defined by Σ. □

5.4 Completeness with respect to Kripke frames

As corollaries of the results of the previous section, we obtain a number of results on completeness with respect to Kripke frames. In this section, we will again call Kripke frames simply frames.

Firstly, recall from Section 5.2 that pure formulas, very simple modal Sahlqvist formulas and shallow modal formulas are di-persistent. By Theorem 5.3.16, we obtain the following.

5.4.1. COROLLARY. *Let Σ be any set of pure $\mathcal{H}(@)$-formulas, very simple modal Sahlqvist formulas and/or shallow modal formulas. Then $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma$ is strongly complete for the class of frames defined by Σ. Similar for $\mathbf{K}^+_{\mathcal{H}}\Sigma$ and $\mathbf{K}^+_{\mathcal{H}(\mathsf{E})}\Sigma$.*

Completeness results for hybrid logics axiomatized by pure formulas have been around for a long time, cf. [25, 46].

Next, recall that modal Sahlqvist formulas and shallow modal formulas are d2-persistent. By Theorem 5.3.1, 5.3.5 and 5.3.6, we obtain the following.

5.4.2. COROLLARY. *Let Σ be a set of modal Sahlqvist formulas and/or shallow modal formulas. Then $\mathbf{K}_{\mathcal{H}}\Sigma$, $\mathbf{K}_{\mathcal{H}(@)}\Sigma$ and $\mathbf{K}_{\mathcal{H}(\mathsf{E})}\Sigma$ are strongly complete for the class of frames defined by Σ.*

As an immediate corollary, we obtain completeness for $\mathbf{K}^+_{\mathcal{H}}\Sigma$, $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma$ and $\mathbf{K}^+_{\mathcal{H}(\mathsf{E})}\Sigma$ in the case where Σ is a set of modal Sahlqvist formulas. In [46], this result was already obtained for $\mathcal{H}(\mathsf{E})$.

Corollary 5.4.2 may still be generalized. Recall from Section 5.2 that the hybrid formulas i and $(p \to \Diamond(i \wedge \Diamond p)) \wedge (\Diamond\Diamond q \to \Diamond q)$ are also d2-persistent, and that they define the class of frames with one element and the class of frames with the universal relation, respectively. Corollary 5.4.2 holds also for axiomatizations that include besides modal Sahlqvist formulas and shallow modal formulas also these formulas.

It is natural to ask whether Corollary 5.4.1 and 5.4.2 can be combined. The following result states that this is not possible.

5.4.3. THEOREM. *There is a pure $\mathcal{H}$-formula φ and a modal Sahlqvist formula ψ such that the hybrid logics $\mathbf{K}^+_{\mathcal{H}}\{\varphi,\psi\}$, $\mathbf{K}^+_{\mathcal{H}(@)}\{\varphi,\psi\}$ and $\mathbf{K}^+_{\mathcal{H}(\mathsf{E})}\{\varphi,\psi\}$ are not complete for any class of frames.*

Proof: Consider the following axioms. The first-order frame conditions they define are given as well.

(*Confluence*)	$\Diamond\Box p \to \Box\Diamond p$	$\forall xyz(Rxy \wedge Rxz \to \exists u(Ryu \wedge Rzu))$
(*NoGrid*)	$\Diamond(i \wedge \Diamond j) \to \Box(\Diamond j \to i)$	$\forall xyzu(Rxy \wedge Rxz \wedge Ryu \wedge Rzu \to y = z)$
(*Func*)	$\Diamond p \to \Box p$	$\forall xyz(Rxy \wedge Rxz \to y = z)$

(*Confluence*) is a Sahlqvist formula and (*NoGrid*) is pure. As can be easily seen from the first-order correspondents, every frame validating (*Confluence*) and (*NoGrid*) validates (*Func*). However, (*Func*) is not derivable from the axioms (*Confluence*) and (*NoGrid*). To see this, consider the countably branching tree of infinite depth. Let $\mathfrak{F}$ be the discrete two-sorted general frame based on this structure in which the admissible sets are exactly the finite and co-finite sets [21]. Then $\mathfrak{F} \models$ (*Confluence*). For suppose $\mathfrak{F}, V, w \Vdash \Diamond\Box p$. Since $V(p)$ admissible, it must be either finite or co-finite. Since w satisfies $\Diamond\Box p$, there must be a point with only successors satisfying p. Since every point has infinitely many successors, it follows that $V(p)$ must be infinite, hence co-finite. It follows that every world has a successor satisfying p, and therefore, $\mathfrak{F}, V, w \models \Box\Diamond p$.

Finally, observe that $\mathfrak{F} \models$ (*NoGrid*) and $\mathfrak{F} \not\models$ (*Func*). □

It was shown in [98] that if attention is restricted to *versatile* frames (i.e., frames containing for each modality also its converse), all modal Sahlqvist formulas are di-persistent.[9] It follows that Corollary 5.4.2 and 5.4.1 *can* be combined in the case of tense logics. In connection to this, it is also worth mentioning Goranko and Vakarelov [59], who proved, in the context of reversive hybrid polyadic modal logic, that every Sahlqvist formula is provably frame equivalent to a pure formula.

Confluence seems to be the most natural frame condition that is definable by a Sahlqvist formula but not by a pure formula (cf. Section 4.2). One might therefore ask if there is still a systematic way to obtain complete axiomatizations for frame classes definable by a set of pure formulas together with the confluence formula. One possibility is to replace the confluence axiom by the following inference rule.

> *If* $\vdash @_i\Diamond j \wedge @_i\Diamond k \to @_j\Diamond l \wedge @_k\Diamond l \to \psi$ *then* $\vdash \psi$,
> *provided* i, j, k, l *are distinct and* l *does not occur in* ψ.

Read from bottom to top, this rule says that in order to prove a formula φ, one may introduce a new nominal l, and assume that $@_i\Diamond j \wedge @_i\Diamond k \to @_j\Diamond l \wedge @_k\Diamond l$. It was proved in [19] that for all sets Σ of pure $\mathcal{H}(@)$-formulas, the axiomatization $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma$ extended with the above rule is complete for the class of confluent frames defined by Σ. In fact, the authors show that this strategy for obtaining complete axiomatizations can be applied not only to the confluence property, but to a wider class of properties not definable by pure formulas. Goranko and Vakarelov [58] provide similar results for $\mathcal{M}(\mathsf{D})$, the extension of the basic modal language with the difference operator.

The completeness results mentioned so far only apply to elementary, or at least canonical logics. There are a number of non-elementary complete modal logics. Examples include **GL**, **Grz** and **PDL**. One might wonder whether the

[9] It is more common to speak about versatile languages than versatile languages. However, in order to prevent any further proliferation of hybrid languages, and since all languages we treat are already multi-modal in general, we have chosen to define versatility in terms of frames.

corresponding hybrid logics are also complete. In Chapter 8, we will show that this is indeed the case. In fact, we will show for a reasonable class of modal formulas φ that completeness of $\mathbf{K}_{\mathcal{M}}\{\varphi\}$ implies completeness of $\mathbf{K}_{\mathcal{H}}\{\varphi\}$ and $\mathbf{K}_{\mathcal{H}(@)}\{\varphi\}$. In connection to this, it is worth noting that general completeness results for non-elementary hybrid logics have been proved [83, 69], but that these results crucially involve the use of ω-rules, i.e., inference rules with infinitely many antecedents.

5.5 On the status of the non-orthodox rules

Corollary 5.4.1 crucially depends on the additional inference rules of $\mathbf{K}^+_{\mathcal{H}}$, $\mathbf{K}^+_{\mathcal{H}(@)}$ and $\mathbf{K}^+_{\mathcal{H}(\mathsf{E})}$. These rules are non-orthodox, in the sense that they involve syntactic side conditions. Such kinds of rules, sometimes called Gabbay-Burgess-style rules, were first introduced by Burgess [26] and Gabbay [44] around 1980, in the context of temporal logic. It is natural to ask if a result along the lines of Corollary 5.4.1 could be obtained without the use of such rules. A number of things can be said in this respect.

Recall that a frame class K is called *versatile* if for each modality $\Diamond$ there is a modality $\Diamond^-$ such that the accessibility relation of $\Diamond^-$ is the converse of the accessibility relation of $\Diamond$ for all frames in K. A typical example of a versatile frame class is the class of symmetric frames, as the converse of a symmetric relation is the same relation. In can be shown that, on versatile frame classes, the rules (*Paste*), (*BG*) and (BG_{E}) are derivable.[10]

In the remainder of this section, we will define the notion of an *orthodox inference rule*, and we will show that every axiomatization for $\mathcal{H}(@)$ that complete-for-pure-extensions in the sense of Corollary 5.4.2 contains either non-orthodox

[10] Here, we will give a derivation of the (*BG*) rule as an example. The other rules can be derived in a similar way.

1. $\vdash @_i\Diamond j \to @_j\varphi$ (Assumption)
2. $\vdash @_j\Box\Diamond^{-1}j$ (Tense axiom)
3. $\vdash @_j\Diamond^{-1}i \to @_j\Diamond^{-1}(i \wedge \Diamond k)$ (From 2, by (K), ($K_@$), (*Nec*) and ($Nec_@$))
4. $\vdash i \wedge \Diamond j \to @_i\Diamond j$ ((*Intro*))
5. $\vdash @_j\Diamond^{-1}(i \wedge \Diamond j) \to @_j\Diamond^{-1}@_i\Diamond j$ (From 4, by (K), ($K_@$), (*Nec*) and ($Nec_@$))
6. $\vdash @_j\Diamond^{-1}@_i\Diamond j \to @_i\Diamond j$ (By (*Back*) and (*Agree*))
7. $\vdash @_j\Diamond^{-1}i \to @_i\Diamond j$ (From 3, 5 and 6 by (*MP*))
8. $\vdash @_j\Diamond^{-1}i \to @_j\varphi$ (From 7 and 1 by (*MP*))
9. $\vdash @_j(\Diamond^{-1}i \to \varphi)$ (From 8 by ($K_@$) and (*Selfdual*))
10. $\vdash \Diamond^{-1}i \to \varphi$ (From 9 by (*Name*))
11. $\vdash @_i\Box\Diamond^{-1}i \to @_i\Box\varphi$ (From 10, by (K), ($K_@$), (*Nec*) and ($Nec_@$))
12. $\vdash @_i\Box\Diamond^{-1}i$ (From 2 by (*Subst*))
13. $\vdash @_i\Box\varphi$ (From 11 and 12, by (*MP*))

Cf. Goranko [54] for a more general discussion of the derivability of such rules in versatile languages.

rules or infinitely many rules.

By an orthodox inference rule we mean a rule of the form

$$\frac{\vdash \varphi_1(\alpha_1,\dots,\alpha_n) \quad \& \quad \cdots \quad \& \quad \vdash \varphi_k(\alpha_1,\dots,\alpha_n)}{\vdash \psi(\alpha_1,\dots,\alpha_n)}$$

Here, $\alpha_1,\dots,\alpha_n$ are variables ranging over arbitrary formulas, and are implicitly universally quantified. As usual, the formulas above the line indicate the premises of the rule, and the formula below the line indicates the conclusion. In the presence of a modus ponens rule (together with enough propositional axioms), we can assume without loss of generality that there is only a single antecedent (a big conjunction), hence all orthodox rules can be assumed to be of the form

$$\frac{\vdash \varphi(\alpha_1,\dots,\alpha_n)}{\vdash \psi(\alpha_1,\dots,\alpha_n)}$$

In fact, we may assume that φ and ψ do not contain any proposition letters (any proposition letter p occurring in φ or ψ may be safely replaced by a variable α_{n+1}). In other words, we may assume that φ and ψ are built up from $\alpha_1,\dots,\alpha_n$ and nominals, using the Boolean connectives, modal operators and satisfaction operators. The rank of such a rule will be n. For example, the rank of the Nec rule is 1. A rule *preserves validity* on a class of frames F, if for all formulas $\alpha_1,\dots,\alpha_n$, $\mathsf{F} \models \varphi(\alpha_1,\dots,\alpha_n)$ implies $\mathsf{F} \models \psi(\alpha_1,\dots,\alpha_n)$. We can now prove the desired result: no finite collection of orthodox rules can be complete for all pure extensions, even if we take as axioms all validities of $\mathcal{H}(@)$.

5.5.1. THEOREM. *Let Λ be any axiomatic system that contains as axioms all $\mathcal{H}(@)$-formulas that are valid on every frame, and that contains a finite number of orthodox inference rules, plus modus ponens and substitution rule. Then there is a set of $\mathcal{H}(@)$-formulas Σ such that Λ extended with the formulas in Σ as axioms is not sound and complete with respect to the class of frames defined by Σ.*

Proof: Let n be the maximal rank of the orthodox rules of Λ — this information is all we need to construct a pure extension that is incomplete with respect to the frame class it defines. Define Σ to be the set consisting of the S5 axioms, together with the following pure formula:

$$\bigwedge_{1\leq l\leq 2^n+2} \Diamond i_l \quad \to \bigvee_{1\leq k<l\leq 2^n+2} \Diamond(i_k \wedge i_l).$$

Let $\Lambda + \Sigma$ be the axiomatic system Λ enriched by the axioms in Σ (closed under modus ponens, substitution and the other rules of Λ). Let F be the class of frames defined by Σ, i.e., the class of all S5 frames in which each world has at most 2^n+1 successors. Either the rules of Λ preserve validity on F or they do not. If they do not, soundness is lost and there is nothing to prove, so assume that the rules of Λ do preserve validity on F. We shall now show that $\Lambda + \Sigma$ is not complete for F.

Let M be the class of models based on frames in F. Let $\mathcal{F} = (W, R)$ be the frame with $W = \{1, \ldots, 2^n + 2\}$ and $R = W^2$. Clearly, $\mathcal{F} \notin \mathsf{F}$. Finally, let $\mathsf{M}' = \mathsf{M} \cup \{(\mathcal{F}, V) \mid V$ is a valuation for $\mathcal{F}$ such that $V(i) = V(j)$ for all nominals $i, j\}$. We shall show that $\Lambda + \Sigma$ is sound for the class of models M'.

Claim 1: All axioms of $\Lambda + \Sigma$ are valid on M'. Moreover, validity on M' is closed under modus ponens and under uniform substitution of formulas for proposition letters and nominals for nominals.

Proof of claim: The proof of Claim 1 is straightforward and is left to the reader. $\dashv$

Claim 2: All formulas valid on F with at most n proposition letters are valid on M'.

Proof of claim: Let φ be a formula with at most n proposition letters, and suppose for the sake of contradiction that $\mathsf{F} \models \varphi$ and $\mathsf{M}' \not\models \varphi$. Then there is a valuation V and a world w such that $\mathcal{F}, V, w \Vdash \neg\varphi$, and such that V assigns the same world to each nominal. Consider the bisimulation contraction of $(\mathfrak{F}, V)$ with respect to the proposition letters and nominals occurring in φ, i.e., the quotient of $(\mathfrak{F}, V)$ with respect to the largest auto-bisimulation, also called *strongly extensional quotient* [1]. Since only n proposition letters occur in φ, and all nominals are true at the same world, the bisimulation contraction of $(\mathfrak{F}, V)$ (over this restricted vocabulary) has at most $2^n + 1$ worlds; hence, its underlying frame is in F. It follows that $\mathsf{F} \not\models \varphi$, which contradicts our initial assumption. $\dashv$

Claim 3: All inference rules of Λ preserve validity on M'.

Proof of claim: Consider any rule ρ of Λ of the form

$$\frac{\vdash \varphi(\alpha_1, \ldots, \alpha_m)}{\vdash \psi(\alpha_1, \ldots, \alpha_m)}$$

with $m \leq n$, and suppose that $\mathsf{M}' \models \varphi(\alpha_1, \ldots, \alpha_m)$ for particular formulas $\alpha_1, \ldots, \alpha_m$. Uniformly substitute $\top$ for each of the proposition letters occurring in $\alpha_1, \ldots, \alpha_m$. We then obtain pure formulas $\beta_1, \ldots, \beta_m$, and by Claim 1 it follows that $\mathsf{M}' \models \varphi(\beta_1, \ldots, \beta_m)$. Let $p_1, \ldots, p_m$ be new, distinct proposition letters. Then it follows that

$$\mathsf{M}' \models \varphi((p_1 \lhd \Box\varphi(p_1, \ldots, p_m) \rhd \beta_1), \ldots, (p_m \lhd \Box\varphi(p_1, \ldots, p_m) \rhd \beta_m))$$

where $(\varphi \lhd \psi \rhd \chi)$ is shorthand for $(\psi \wedge \varphi) \vee (\neg\psi \wedge \chi)$. Hence

$$\mathsf{F} \models \varphi((p_1 \lhd \Box\varphi(p_1, \ldots, p_m) \rhd \beta_1), \ldots, (p_m \lhd \Box\varphi(p_1, \ldots, p_m) \rhd \beta_m))$$

Since ρ preserves validity on F,

$$\mathsf{F} \models \psi((p_1 \lhd \Box\varphi(p_1, \ldots, p_m) \rhd \beta_1), \ldots, (p_m \lhd \Box\varphi(p_1, \ldots, p_m) \rhd \beta_m))$$

Since this formula contains at most n proposition letters, it follows by Claim 2 that

$$\mathsf{M}' \models \psi((p_1 \lhd \Box\varphi(p_1, \ldots, p_m) \rhd \beta_1), \ldots, (p_m \lhd \Box\varphi(p_1, \ldots, p_m) \rhd \beta_m))$$

By closure under uniform substitution (Claim 1), it follows that

$$\mathsf{M}' \models \psi((\alpha_1 \lhd \Box\varphi(\alpha_1, \ldots, \alpha_m) \rhd \beta_1), \ldots, (\alpha_m \lhd \Box\varphi(\alpha_1, \ldots, \alpha_m) \rhd \beta_m))$$

Recall that $\mathsf{M}' \models \varphi(\alpha_1, \ldots, \alpha_m)$. It follows that $\mathsf{M}' \models (\alpha_i \lhd \Box\varphi(\alpha_1, \ldots, \alpha_m) \rhd \beta_i) \leftrightarrow \alpha_i$. Hence, $\mathsf{M}' \models \psi(\alpha_1, \ldots, \alpha_m)$. ⊣

It follows that $\Lambda + \Sigma$ is sound with respect to M'. But now consider the following formula

$$\eta = \bigwedge_{1 \le i \le 2^n+2} \Diamond p_i \quad \rightarrow \bigvee_{1 \le i < j \le 2^n+2} \Diamond(p_i \wedge p_j)$$

Notice that $\mathsf{M}' \not\models \eta$. By Claim 1–3, it follows that $\Lambda + \Sigma \nvdash \eta$. However $\mathsf{F} \models \eta$. It follows that $\Lambda + \Sigma$ is not complete for F. □

Chapter 6

Interpolation and Beth definability

In this chapter, we study interpolation and Beth definability properties of hybrid logics. Recall the interpolation property from Section 2.5. In the setting of hybrid logic, there is a choice to be made concerning the definition of the interpolation property. The first, and more conservative option is to require that the interpolant of a valid implication must contain only proposition letters occurring both in the antecedent and in the consequent. No restriction is made on the occurrence of nominals in the interpolant. The more daring option would be to require that both the proposition letters and the nominals occurring in the interpolant occur both in the antecedent and the consequent. We will refer to these options as *interpolation over proposition letters* and *interpolation over proposition letters and nominals*. Note that we only consider local interpolation.

Areces, Blackburn and Marx [5] were the first to consider interpolation in the context of hybrid languages. They proved that $\mathcal{H}(@)$ does not have interpolation over proposition letters and nominals, with respect to the class of all frames, but that interpolation may be regained by extending the language of $\mathcal{H}(@)$ with state variables and a $\downarrow$-binder. The language obtained in this way, $\mathcal{H}(@,\downarrow)$, will be discussed in detail in Chapter 9 of this thesis.

Subsequent results were proved by Conradie in his Masters thesis [34]. He showed that $\mathcal{H}(@)$ lacks interpolation over proposition letters and nominals even with respect to the class of S5 frames, but that it has the Beth property with respect to this class of frames.

This chapter presents the following new results. First, we will show that the languages $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$ have interpolation over proposition letters with respect to many frame classes, including the class of all frames. As a corollary, we will obtain the Beth property for $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$. On the other hand, we will see that the Beth property fails for $\mathcal{H}$.

Next, we will show that $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$ lack interpolation over nominals in a strong sense. In fact, we will show that the least expressive extension of $\mathcal{H}(@)$ with interpolation over proposition letters and nominals is $\mathcal{H}(@,\downarrow)$, and that the least expressive extension of $\mathcal{H}(\mathsf{E})$ with interpolation over proposition letters and nominals is the first order correspondence language $\mathcal{L}^1$.

The results presented in this chapter are based on [30] and [28].

6.1 Motivations for studying interpolation

Before we plunge into technical details, let us briefly discuss different types of interpolation, and motivations for studying them.

The first question to be addressed is probably why interpolation is important. One answer to this question is that interpolation is important as a modularity principal. Suppose there are two system specifications, knowledge bases, or in general, sets of formulas, Σ and Γ. Now, suppose Σ and Γ contradict each other. Then the interpolation property (in combination with compactness) tells us that there is a sentence φ in the common language, on which Σ and Γ disagree. In other words, there are no unexpected interactions.

Other reasons why interpolation is important include the fact that it can be used as a lemma for proving the Beth property and other preservation theorems (cf. Craig's original article [35]), and that interpolation has been considered an indicator for the existence of nice, cut-free sequent calculi for the logic in question, cf. for instance [8, page 17].

A more detailed discussion of interpolation and motivations for studying it can be found in Hoogland's dissertation [65].

Apart from the general motivation for studying interpolation, there is the following issue.

> What type of interpolation should a good hybrid logic have? Interpolation over nominals, or only over proposition letters? And, what about modalities?

It is hard to give a general answer, but a few things can be said. In order to be able to derive the Beth property, it is enough to have interpolation over proposition letters. On the other hand, when interpolation is used as a modularity principle, interpolation over nominals is desirable as well. In tense logics, where there are two modal operators, there is no obvious need for interpolation over modalities. On the other hand, from the viewpoint of *description logics*, where modalities are considered non-logical operators, just like proposition letters, interpolation over modalities is desirable.

Finally, it should be mentioned that, besides the type of interpolation studied in this thesis, which is sometimes called *local interpolation* or *arrow interpolation*, there is another type of interpolation called *global interpolation* or *turnstile interpolation*. More information about the latter type of interpolation and its relation to the local interpolation property can be found in [65].

6.2 Interpolation over proposition letters and the Beth property

We saw in Section 2.5 that the basic modal language has interpolation relative to any elementary class of frames closed under bisimulation products and generated subframes. As we will now show, this result generalizes to hybrid logic, in the sense that the languages $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$ have interpolation over proposition letters relative to such frame classes.

For any formula φ, let $\textsc{prop}(\varphi)$ denote the set of proposition letters occurring in φ. We say that a hybrid $\mathcal{L}$ has *interpolation over proposition letters* relative to a frame class K if the following holds: for all $\mathcal{L}$-formulas φ, ψ, if $\mathsf{K} \models \varphi \to \psi$ then there is a $\mathcal{L}$-formula ϑ such that $\mathsf{K} \models \varphi \to \vartheta$, $\mathsf{K} \models \vartheta \to \psi$ and $\textsc{prop}(\vartheta) \subseteq \textsc{prop}(\varphi) \cap \textsc{prop}(\psi)$.

6.2.1. Theorem. *Let* K *be any elementary frame class closed under generated subframes and bisimulation products. Then* $\mathcal{H}(@)$ *has interpolation over proposition letters relative to* K*.*

Proof: Let K be any elementary frame class closed under generated subframes and bisimulation products, let $\mathsf{K} \models \varphi \to \psi$, and suppose for the sake of contradiction that there is no interpolant for this implication. Let $Cons(\varphi)$ be the set of $\mathcal{H}(@)$-formulas χ such that $\mathsf{K} \models \varphi \to \chi$ and $\textsc{prop}(\chi) \subseteq \textsc{prop}(\varphi) \cap \textsc{prop}(\psi)$. By the same argument used in the proof of Theorem 2.5.3, we can construct models $\mathfrak{M}, \mathfrak{N}$ based on frames in K, with corresponding worlds w, v, such that

(1.) $\mathfrak{M}, w \models Cons(\varphi) \cup \{\neg\psi\}$

(2.) $\mathfrak{N}, v \models Cons(\varphi) \cup \{\varphi\}$,

(3.) For all $\mathcal{H}(@)$-formulas ϑ with $\textsc{prop}(\vartheta) \subseteq \textsc{prop}(\varphi) \cap \textsc{prop}(\psi)$, $\mathfrak{M}, w \models \vartheta \Leftrightarrow \mathfrak{N}, v \models \vartheta$.

Since K is closed under generated subframes, we may assume that $\mathfrak{M}$ and $\mathfrak{N}$ are generated by w respectively v, together with all points named by nominals.

Let $\mathfrak{M}^+$ and $\mathfrak{N}^+$ be ω-saturated elementary extensions of $\mathfrak{M}$ and $\mathfrak{N}$. Since K is elementary, the underlying frames of $\mathfrak{M}^+$ and $\mathfrak{N}^+$ are in K. Define the binary relation Z between the domains of $\mathfrak{M}^+$ and $\mathfrak{N}^+$ by letting dZe if for all $\mathcal{H}(@)$-formulas χ with $\textsc{prop}(\chi) \subseteq \textsc{prop}(\varphi) \cap \textsc{prop}(\psi)$ then $\mathfrak{M}^+, d \models \chi \Leftrightarrow \mathfrak{N}^+, e \models \chi$. In other words, dZe if d and e cannot be distinguished by a $\mathcal{H}(@)$-formula in the common vocabulary of φ and ψ. With the *common vocabulary of* φ *and* ψ we mean the vocabulary that contains all nominals, but that contains only the proposition letters that occur both in φ and in ψ. Note that, by construction, wZv.

Claim 1: Z is a total $\mathcal{H}(@)$-bisimulation between $\mathfrak{M}^+$ and $\mathfrak{N}^+$, with respect to the common vocabulary of φ and ψ.

Proof of claim: It follows from Theorem 4.1.2 that Z is an $\mathcal{H}(@)$-bisimulation between $\mathfrak{M}^+$ and $\mathfrak{N}^+$, with respect to the common vocabulary of φ and ψ. It only remains to show that Z is a *total* $\mathcal{H}(@)$-bisimulation. Let d be any point of $\mathfrak{M}^+$, and let $\Gamma = \{ST_x(\varphi) \mid \mathfrak{M}^+, d \models \varphi\}$. We will show that Γ is realized by some point e of $\mathfrak{N}^+$, and hence dZe. By ω-saturatedness of $\mathfrak{N}^+$, it suffices to show that every finite subset of Γ is realized in $\mathfrak{N}^+$.

Let $ST_x(\gamma_1), \ldots, ST_x(\gamma_n) \in \Gamma$. Since $\mathfrak{M}^+$ is an elementary extension of $\mathfrak{M}$ and $\mathfrak{M}^+ \models \exists x.(ST_x(\gamma_1) \wedge \cdots \wedge ST_x(\gamma_n))$, we have that $\mathfrak{M} \models \exists x.(ST_x(\gamma_1) \wedge \cdots \wedge ST_x(\gamma_n))$. Since $\mathfrak{M}$ is generated by w together with all points named by constants, either $\mathfrak{M}, w \models \Diamond_1 \cdots \Diamond_n(\gamma_1 \wedge \cdots \wedge \gamma_n)$ or $\mathfrak{M}, w \models @_i\Diamond_1 \cdots \Diamond_n(\gamma_1 \wedge \cdots \wedge \gamma_n)$ for some nominal i and sequence of modalities $\Diamond_1 \cdots \Diamond_n$. In either case, it follows by (3.) that $\gamma_1 \wedge \cdots \wedge \gamma_n$ is true at some point in $\mathfrak{N}$, hence is true at that point in $\mathfrak{N}^+$.

A symmetric argument shows that for every point e of $\mathfrak{N}^+$ there is a point d of $\mathfrak{M}^+$ such that dZe. $\dashv$

Let $\mathfrak{F}$ and $\mathfrak{G}$ be the underlying frames of $\mathfrak{M}^+$ and $\mathfrak{N}^+$. Then, in particular, Z is a total frame bisimulation between $\mathfrak{F}$ and $\mathfrak{G}$. Hence, by Proposition 2.5.2, there is a bisimulation product $\mathfrak{H} \in \mathsf{K}$ of $\mathfrak{F}$ and $\mathfrak{G}$ of which the domain is Z. By the definition of bisimulation products, the natural projections $f : \mathfrak{H} \to \mathfrak{F}$ and $g : \mathfrak{H} \to \mathfrak{G}$ are surjective bounded morphisms. For any proposition letter $p \in \textsc{prop}(\varphi)$, let $V(p) = \{u \mid \mathfrak{M}^+, f(u) \models p\}$, and for any proposition letter $p \in \textsc{prop}(\psi)$, let $V(p) = \{u \mid \mathfrak{N}^+, g(u) \models p\}$. The properties of Z guarantee that this V is well-defined for $p \in \textsc{prop}(\varphi) \cap \textsc{prop}(\psi)$. For any nominal i, let $V(i) = \{u \mid \mathfrak{M}^+, f(u) \models i\} = \{u \mid \mathfrak{N}^+, g(u) \models i\}$. Again, the properties of Z guarantee that $V(i)$ is well-defined, and that it is a singleton set for each nominal i.

Finally, by a standard argument, the graph of f is a $\mathcal{H}(@)$-bisimulation between $(\mathfrak{H}, V)$ and $\mathfrak{M}^+$ with respect to the proposition letters and nominals occuring in φ, and the graph of g is a bisimulation between $(\mathfrak{H}, V)$ and $\mathfrak{N}^+$ with respect to the proposition letters and nominals occuring in ψ. It follows that $(\mathfrak{H}, V), \langle w, v\rangle \models \varphi \wedge \neg\psi$. This contradicts our initial assumption that $\mathsf{K} \models \varphi \to \psi$. $\Box$

6.2.2. Corollary. *Let* K *be any elementary frame class closed under bisimulation products. Then* $\mathcal{H}(\mathsf{E})$ *has interpolation over proposition letters relative to* K.

Proof: Given a frame class K, let K' be the class $\{(W, (R_\Diamond)_{\Diamond \in \textsc{mod}}, R_\mathsf{E}) \mid (W, (R_\Diamond)_{\Diamond \in \textsc{mod}}) \in \mathsf{K}$ and $R_\mathsf{E} = W^2\}$. Clearly, every $\mathcal{H}(\mathsf{E})$-formula, when interpreted on K, can be seen as an $\mathcal{H}$-formula interpreted on K'. This, together with the fact that @-operators are definable in terms of E, implies that $\mathcal{H}(\mathsf{E})$ has interpolation on K if $\mathcal{H}(@)$ has interpolation on K'. Now, K is trivially closed under generated subframes, and it is not hard to see that K' is closed under bisimulation products iff K is. Finally, K' is elementary iff K is. Combining these observations, the result follows. □

The case for $\mathcal{H}$ turns out to be more complicated.

6.2.3. THEOREM. *Let* K *be any elementary frame class satisfying the following conditions.*

1. K *is closed under generated subframes and bisimulation products*

2. *For any frame* $\mathfrak{F}$*, if every point-generated subframe of* $\mathfrak{F}$ *is a proper generated subframe of a frame in* K*, then* $\mathfrak{F} \in \mathsf{K}$*.*

Then $\mathcal{H}$ *has interpolation over proposition letters relative to* K*.*

Proof: Let K be any elementary frame class satisfying the given conditions, let $\mathsf{K} \models \varphi \to \psi$, and suppose for the sake of contradiction that there is no interpolant for this implication. Let $Cons(\varphi)$ be the set of $\mathcal{H}$-formulas χ such that $\mathsf{K} \models \varphi \to \chi$ and $\textsc{prop}(\chi) \subseteq \textsc{prop}(\varphi) \cap \textsc{prop}(\psi)$. By the same argument used in the proof of Theorem 2.5.3, we can construct models $\mathfrak{M}, \mathfrak{N}$ based on frames in K, with corresponding worlds w, v, such that

(1.) $\mathfrak{M}, w \models Cons(\varphi) \cup \{\neg\psi\}$

(2.) $\mathfrak{N}, v \models Cons(\varphi) \cup \{\varphi\}$,

(3.) For all $\mathcal{H}$-formulas ϑ with $\textsc{prop}(\vartheta) \subseteq \textsc{prop}(\varphi) \cap \textsc{prop}(\psi)$, $\mathfrak{M}, w \models \vartheta \Leftrightarrow \mathfrak{N}, v \models \vartheta$.

We can distinguish two cases.

(a) Suppose every point of $\mathfrak{M}$ named by a nominal is reachable from w. It follows from (3.) that also every point of $\mathfrak{N}$ named by a nominal is reachable from v. Let $\mathfrak{M}_w$ and $\mathfrak{N}_v$ be the submodels of $\mathfrak{M}$ and $\mathfrak{N}$ generated by w and v respectively, and let $\mathfrak{M}_w^+$ and $\mathfrak{N}_v^+$ be ω-saturated elementary extensions of these. Note that, since K is elementary and closed under generated subframes, the underlying frames of $\mathfrak{M}_w^+$ and $\mathfrak{N}_v^+$ are in K. Define the binary relation Z between the domains of $\mathfrak{M}_w^*$ and $\mathfrak{N}_v^*$ by letting dZe if d and e cannot be distinguished by a $\mathcal{H}$-formula in the common vocabulary of φ and ψ. With the *common vocabulary of* φ *and* ψ we mean the vocabulary that consists of all nominals, plus those proposition letters that occur both in φ and in ψ. By construction, wZv. A similar argument as for Claim 1 in the proof of Theorem 6.2.1 that Z is a total $\mathcal{H}(@)$-bisimulation between $\mathfrak{M}_w^+$ and $\mathfrak{N}_v^+$, with respect to the common vocabulary of φ and ψ. We may now proceed as in the proof of Theorem 6.2.1 to show that $\varphi \land \neg\psi$ is satisfiable on a frame in K, which contradicts out initial assumption that $\mathsf{K} \models \varphi \to \psi$.

(b) Suppose not every point of $\mathfrak{M}$ named by a nominal is reachable from w. It follows from (3.) that also not every point of $\mathfrak{N}$ named by a nominal is reachable from v. Let $\mathfrak{M} = (\mathfrak{F}, V)$ and $\mathfrak{N} = (\mathfrak{G}, V')$, and let $\mathfrak{F}_w$ and $\mathfrak{G}_v$ be the subframes of $\mathfrak{F}$ and $\mathfrak{G}$ generated by w and v, respectively. Let $\mathfrak{F}'_w$ be a disjoint isomorphic copy of $\mathfrak{F}_w$, and consider the frames $\mathfrak{F}_w \uplus \mathfrak{F}'_w$ and $\mathfrak{G}_v \uplus \mathfrak{F}'_w$. It follows from the closure conditions of K that these frames are in K.

Define respective valuations V_1 and V_2 for $\mathfrak{F}_w \uplus \mathfrak{F}'_w$ and $\mathfrak{G}_v \uplus \mathfrak{F}'_w$, as follows, where x is a fixed element of $\mathfrak{F}'_w$.

$$\begin{aligned}
V_1(p) &= V(p) \cap \mathfrak{F}_w \\
V_1(i) &= \begin{cases} \{u\} & \text{if } V(i) = \{u\} \text{ with } u \in \mathfrak{F}_w \\ \{x\} & \text{otherwise} \end{cases} \\
V_2(p) &= V'(p) \cap \mathfrak{G}_v \\
V_2(i) &= \begin{cases} \{u\} & \text{if } V'(i) = \{u\} \text{ with } u \in \mathfrak{G}_w \\ \{x\} & \text{otherwise} \end{cases}
\end{aligned}$$

A simple argument using $\mathcal{H}$-bisimulations shows that $(\mathfrak{F}_w \uplus \mathfrak{F}'_w, V_1), w$ and $(\mathfrak{G}_v \uplus \mathfrak{F}'_w, V_2), v$ still agree on all $\mathcal{H}$-formulas in the common vocabulary of φ and ψ, and that it is still the case that $(\mathfrak{F}_w \uplus \mathfrak{F}'_w, V_1), w \models \varphi$ and $(\mathfrak{G}_v \uplus \mathfrak{F}'_w, V_2), v \models \neg\psi$. Finally, we proceed as in (a) using ω-saturated elementary extensions of $(\mathfrak{F}_w \uplus \mathfrak{F}'_w, V_1)$ and $(\mathfrak{G}_v \uplus \mathfrak{F}'_w, V_2)$. □

As a corollary of these interpolation results, we obtain the Beth property for hybrid logics of elementary frame classes closed under bisimulation products. Let us briefly recall the definition of the Beth property. We will use $\models^{glo}_{\mathsf{K}}$ to refer to the global entailment relation, relative to the frame class K, i.e., $\Sigma \models^{glo}_{\mathsf{K}} \varphi$ means that for all models $\mathfrak{M}$ based on a frame in K, if $\mathfrak{M}$ globally satisfies all formulas in Σ then $\mathfrak{M}$ globally satisfies φ. For a set of formulas $\Sigma(p)$ containing the proposition letter p (and possibly other proposition letters and nominals), we say that $\Sigma(p)$ *implicitly defines* p, relative to a frame class K, if $\Sigma(p) \cup \Sigma(p') \models^{glo}_{\mathsf{K}} p \leftrightarrow p'$. Here, p' is a proposition letter not occurring in Σ, and $\Sigma(p')$ is the result of replacing all occurrences of p by p' in $\Sigma(p)$. A language $\mathcal{L}$ is said to have the Beth property relative to a frame class K if whenever a set of $\mathcal{L}$-formulas $\Sigma(p)$ implicitly defines a proposition letter p, relative to K, then there is a formula ϑ in which p does not occur, such that $\Sigma \models^{glo}_{\mathsf{K}} p \leftrightarrow \vartheta$. The relevant formula φ is called an *explicit definition* of p, relative to Σ and K.

6.2.4. THEOREM. *If* K *is a elementary frame class closed under generated subframes and bisimulation products, then* $\mathcal{H}(@)$ *has the Beth property relative to* K. *If* K *is a elementary frame class closed under bisimulation products, then* $\mathcal{H}(\mathsf{E})$ *has the Beth property relative to* K.

Proof: The basic argument is the same as in the proof of Theorem 2.5.4. We will only prove the result for $\mathcal{H}(@)$, since the argument for $\mathcal{H}(\mathsf{E})$ is similar. Furthermore, for ease of presentation we restrict attention to the uni-modal case. The proof generalizes easily to languages containing more modalities.

Let $\Sigma(p)$ be any set of $\mathcal{H}(@)$-sentences containing the proposition letter p (and possibly other proposition letters and nominals), and suppose Σ implicitly defines the proposition letter p, relative to K. Let p' be a new proposition letter, and let $\Sigma(p')$ be the result of replacing all occurrences of p in Σ by p'. Then, by the definition of implicit definability, $\Sigma(p) \cup \Sigma(p') \models^{glo}_{\mathsf{K}} p \leftrightarrow p'$. Let $\Gamma(p) = \{\Box^n\varphi, @_i\Box^n\varphi \mid \varphi \in \Sigma(p), n \in \omega, i \in \textsc{nom}\}$, and define $\Gamma(p')$ similarly.

Claim 1: $\Gamma(p) \cup \Gamma(p') \models_{\mathsf{K}} p \leftrightarrow p'$.

Proof of claim: Suppose $\mathfrak{M}, w \models \Gamma(p) \cup \Gamma(p')$ for some model $\mathfrak{M}$ based on a frame in K. Let $\mathfrak{M}_w$ be the submodel of $\mathfrak{M}$ generated by w. By closure under generated subframes, the underlying frame of $\mathfrak{M}'$ is also in K. By construction of Γ, $\mathfrak{M}_w$ globally satisfies $\Sigma(p)$ and $\Sigma(p')$. It follows that $\mathfrak{M}, w$ globally satisfies $p \leftrightarrow p'$, and hence, $\mathfrak{M}, w \models p \leftrightarrow p'$. ⊣

By compactness, there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0(p) \cup \Gamma_0(p') \models_{\mathsf{K}} p \leftrightarrow p'$. It follows that $\models_{\mathsf{K}} (p \wedge \bigwedge\Gamma_0(p)) \to (\bigwedge\Gamma_0(p') \to p')$. Let ϑ be an interpolant for this implication. Then the following facts hold.

1. The proposition letters p and p' do not occur in ϑ.
2. $\models_{\mathsf{K}} (p \wedge \bigwedge\Gamma_0(p)) \to \vartheta$.
3. $\models_{\mathsf{K}} \vartheta \to (\bigwedge\Gamma_0(p') \to p')$, and hence, by uniform substitution, $\models_{\mathsf{K}} \vartheta \to (\bigwedge\Gamma_0(p) \to p)$.

We conclude that $\Gamma_0(p) \models_{\mathsf{K}} p \leftrightarrow \vartheta$, and hence $\Sigma(p) \models^{glo}_{\mathsf{K}} p \leftrightarrow \vartheta$. □

Surprisingly, the same does *not* hold for $\mathcal{H}$. Call a frame $\mathfrak{F}$ n-cyclic ($n \in \omega$) if $\mathfrak{F} \models p \to \Box\Diamond^{\leq n}p$, i.e., if every transition (w, v) in in $\mathfrak{F}$ is part of a directed cycle of length at most $n+1$. Call a frame cyclic if it is n-cyclic for some $n \in \omega$. Cyclicity is a rather strong condition. For instance, reflexive transitive frames are in general not cyclic, although symmetric frames are.

6.2.5. Proposition. *Let* K *be any frame class that contains a non-cyclic frame. Then* $\mathcal{H}$ *lacks the Beth property relative to* K.

Proof: Let $\Sigma = \{p \to i, j \wedge q \to \Diamond(i \wedge p), j \wedge \neg q \to \Diamond(i \wedge \neg p)\}$. Then Σ implicitly defines p, since in any model that globally satisfies Σ, p holds nowhere besides possibly at the point named i, and it holds there iff q holds at the point named j. Now, assume for the sake of contradiction that there is an explicit definition of p, i.e., a $\mathcal{H}$-formula φ not containing p such that $\Sigma \models^{glo}_{\mathsf{K}} p \leftrightarrow \varphi$. Let n be the modal depth of φ.

Since K contains a non-cyclic frame, we can find a frame $\mathfrak{F} \in \mathsf{K}$ with worlds w, v such that wRv and w is not reachable from v in n or less steps. Let V_1 be the valuation for $\mathfrak{F}$ that sends i to v, j to w, p to $\{v\}$ and q to $\{w\}$. Let V_2 be the valuation that sends i to v, j to w and that sends p and q to $\emptyset$. Note that $(\mathfrak{F}, V_1)$ and $(\mathfrak{F}, V_2)$ both globally satisfy Σ. A straightforward argument shows that $(\mathfrak{F}, V_1), v$ and $(\mathfrak{F}, V_2), v$ cannot be distinguished by any $\mathcal{H}$-formula of modal depth n. It follows that φ cannot distinguish these points. This contradicts the fact that $(\mathfrak{F}, V_1), v \models \varphi$ and $(\mathfrak{F}, V_2), v \models \neg\varphi$. □

6.3 Interpolation over nominals

We will now consider interpolation over nominals. What follows now can be seen as a warming up for Section 6.4, where a strong negative interpolation result will be given that generalizes the results of this section.

Recall that for a formula φ, $\textsc{prop}(\varphi)$ denotes the set of proposition letters occurring in φ. Likewise, let $\textsc{nom}(\varphi)$ denote the set of nominals occurring in φ. For $\mathcal{L}$ one of the languages $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$, and for K a class of frames, we say that $\mathcal{L}$ has *interpolation over nominals* relative to K if the following holds: for all $\mathcal{L}$-formulas φ, ψ, if $\mathsf{K} \models \varphi \to \psi$ then there is a $\mathcal{L}$-formula ϑ such that $\mathsf{K} \models \varphi \to \vartheta$, $\mathsf{K} \models \vartheta \to \psi$, and $\textsc{nom}(\vartheta) \subseteq \textsc{nom}(\varphi) \cap \textsc{nom}(\psi)$.

It is quite easy to see that this version of interpolation fails for $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$, relative to the class of all frames. Consider for instance the valid implication $i \wedge \Diamond i \to (j \to \Diamond j)$. An interpolant for this implication has to express that the current world is related to itself, without using any nominals. An easy bisimulation argument shows that this is not possible, not even in the language $\mathcal{H}(\mathsf{E})$.

Two strategies can be used in order to repair this failure of interpolation: one can either restrict attention to a specific class of frames, or extend the expressivity of the language so that the relevant interpolants can be expressed. In the remainder of this section, we follows the first strategy, and in the next section, we follow the second one. Our results will be formulated in terms of the hybrid language $\mathcal{H}(@, \downarrow)$, that will be introduced in detail in Chapter 9, where its syntax and semantics are given, and also interpolation for this language is studied. To appreciate the following theorem, it is worth noting that $\mathcal{H}(@, \downarrow)$ is a very expressive, undecidable language, and that it has interpolation over nominals and proposition letters (relative to many frame classes).

6.3.1. Theorem. *Let K be any frame class. If $\mathcal{H}(@)$ has interpolation over nominals on K then it is as expressive as $\mathcal{H}(@, \downarrow)$ on K.*

Proof: Suppose that $\mathcal{H}(@)$ has interpolation over nominals on K. We will show that every $\mathcal{H}(@, \downarrow)$ sentence φ is equivalent (on K) to an $\mathcal{H}(@)$ formula. We proceed by induction on the length of φ. The only interesting case is where φ is of the form $\downarrow x.\psi(x)$. Let i and j be nominals not occurring in $\downarrow x.\psi(x)$. By induction, we know that $\psi(i)$ and $\psi(j)$ are equivalent to $\mathcal{H}(@)$ formulas $\psi'(i)$ and $\psi'(j)$ respectively. Now, the following implication is valid:

$$\mathsf{K} \models i \land \psi'(i) \to (j \to \psi'(j))$$

Let ϑ be any interpolant for this valid implication. We will show that ϑ is equivalent to $\downarrow x.\psi(x)$.

Consider any model $\mathfrak{M}$ and world w such that $\mathfrak{M}, w \models \downarrow x.\psi(x)$. Let $\mathfrak{M}[i/w]$ be the model that differs from $\mathfrak{M}$ only in the fact that i denotes w. Since i does not occur in $\downarrow x.\psi(x)$, we have that $\mathfrak{M}[i/w], w \models \downarrow x.\psi(x)$, hence $\mathfrak{M}[i/w], w \models i \land \psi(i)$. It follows that $\mathfrak{M}[i/w], w \models \vartheta$. Since i does not occur in ϑ, it follows that $\mathfrak{M}, w \models \vartheta$. Conversely, suppose $\mathfrak{M}, w \models \vartheta$. Let $\mathfrak{M}[j/w]$ be the model that differs from $\mathfrak{M}$ only in the fact that j denotes w. Since j does not occur in ϑ, we have that $\mathfrak{M}[j/w], w \models \vartheta$. It follows that $\mathfrak{M}[j/w], w \models j \to \psi(j)$, and hence $\mathfrak{M}[j/w], w \models \downarrow x.\psi(x)$. Since j does not occur in $\downarrow x.\psi(x)$, it follows that $\mathfrak{M}, w \models \downarrow x.\psi(x)$. □

6.3.2. THEOREM. *Let* K *be any frame class. If* $\mathcal{H}(E)$ *has interpolation over nominals on* K *then it is expressively complete for* $\mathcal{L}^1$ *on* K.

Proof: The proof is similar to that for Theorem 6.3.1, using the fact that $\mathcal{H}(\mathsf{E}, \downarrow)$ is expressively equivalent to the first-order correspondence language $\mathcal{L}^1$. □

These results can be interpreted as very strong negative interpolation results. For instance, as a corollary of Theorem 6.3.2, we obtain the following.

6.3.3. COROLLARY. $\mathcal{H}(\mathsf{E})$ *lacks interpolation over nominals on any non-empty modally definable frame class.*

Proof: Let K be any non-empty modally definable frame class, and let $\mathfrak{F} \in \mathsf{K}$. Let $\mathfrak{G}$ be the disjoint union of three isomorphic copies of $\mathfrak{F}$. By closure under disjoint unions, $\mathfrak{G} \in \mathsf{K}$. Let $w \in \mathfrak{F}$, and let w_1, w_2, w_3 denote the disjoint copies of w in $\mathfrak{G}$. Let V and V' be valuations for $\mathfrak{G}$ such that $V(p) = \{w_1\}$ and $V'(p) = \{w_1, w_2\}$. One can easily see that the models $(\mathfrak{G}, V)$ and $(\mathfrak{G}, V')$ are $\mathcal{H}(\mathsf{E})$-bisimilar. It follows that the $\mathcal{L}^1$-formula $\exists xy.(x \neq y \land Px \land Py)$ is not expressible in $\mathcal{H}(\mathsf{E})$ on K. □

We leave it as an open question whether there is an analogue of Theorem 6.3.1 and 6.3.2 for $\mathcal{H}$. At any rate, it is clear that interpolation over nominals fails also for $\mathcal{H}$ on many frame classes.

6.4 Repairing interpolation

In this section we are again concerned with interpolation over nominals. As we mentioned in the previous section, one way to repair the failure of interpolation for $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$, is to increase the expressivity of the language, such that the required interpolants can be expressed. In this section, we show that $\mathcal{H}(@,\downarrow)$ is the least expressive extension of $\mathcal{H}(@)$ with interpolation, and that the first-order correspondence language $\mathcal{L}^1$ is the least expressive extension of $\mathcal{H}(\mathsf{E})$ with interpolation. With *interpolation*, we will mean interpolation over proposition letters and nominals.

In order to state these results precisely, we need to give an abstract definition of what counts as a language. We will now give such a definition. We will assume a fixed set of (unary) modalities $\textsc{mod}$. A signature is a pair $\sigma = (\textsc{prop}_\sigma, \textsc{nom}_\sigma)$ of disjoint sets containing proposition letters and nominals respectively. We will often be sloppy by using σ to denote the union $\textsc{prop}_\sigma \cup \textsc{nom}_\sigma$. For instance, we will write $\sigma \subseteq \tau$ instead of $\textsc{prop}_\sigma \subseteq \textsc{prop}_\tau$ & $\textsc{nom}_\sigma \subseteq \textsc{nom}_\tau$.

Given a signature σ, a (pointed, but not necessarily point-generated) *σ-model* is a structure $\mathfrak{M} = (\mathfrak{F}, V, w)$ where $\mathfrak{F} = (W, R_\diamond)_{\diamond \in \textsc{mod}}$ is a frame, $V : \textsc{prop}_\sigma \cup \textsc{nom}_\sigma \to \wp(W)$ a valuation and $w \in W$ a world. As usual, we require that $|V(i)| = 1$ for all $i \in \textsc{nom}_\sigma$. The class of all σ-models is denoted by $\mathsf{Str}[\sigma]$. Furthermore, for any class of frames F, $\mathsf{Str}_\mathsf{F}[\sigma]$ will denote the class of σ-models of which the underlying frame belongs to F.

Two operations on models will be useful later on. Firstly, a renaming $\rho : \sigma \to \tau$ is a mapping from σ to τ that respects the sorting: it maps elements of $\textsc{prop}_\sigma$ to elements of $\textsc{prop}_\tau$ and elements of $\textsc{nom}_\sigma$ to elements of $\textsc{nom}_\tau$. For any model $\mathfrak{M} = (\mathfrak{F}, V, w) \in \mathsf{Str}[\tau]$ and renaming $\rho : \sigma \to \tau$, let $\mathfrak{M}^\rho$ be the σ-model $(\mathfrak{F}, \rho\cdot V, w)$. Secondly, if $\mathfrak{M} \in \mathsf{Str}[\tau]$ and $\sigma \subseteq \tau$, then $\mathfrak{M} \upharpoonright \sigma$ denotes the σ-reduct of $\mathfrak{M}$, i.e., the σ-model that is obtained from $\mathfrak{M}$ by "forgetting" the interpretation of $\tau \backslash \sigma$. We write $\mathsf{K} \upharpoonright \sigma$ for $\{\mathfrak{M} \upharpoonright \sigma \mid \mathfrak{M} \in \mathsf{K}\}$.

6.4.1. Definition (Hybrid languages). *A* hybrid language *is a pair $(\mathcal{L}, \models_\mathcal{L})$, where $\mathcal{L}$ is a map from signatures to sets of formulas, and $\models_\mathcal{L}$ is a relation between formulas and models satisfying the following conditions.*

1. **Expansion Property.** *If $\sigma \subseteq \tau$ then $\mathcal{L}[\sigma] \subseteq \mathcal{L}[\tau]$. Furthermore, for all $\varphi \in \mathcal{L}[\sigma]$ and $\mathfrak{M} \in \mathsf{Str}[\tau]$, $\mathfrak{M} \models_\mathcal{L} \varphi$ iff $\mathfrak{M} \upharpoonright \sigma \models_\mathcal{L} \varphi$. For $\mathfrak{M} \in \mathsf{Str}[\sigma]$, the statement $\mathfrak{M} \models \varphi$ is defined (i.e., true or false) if and only if $\varphi \in \mathcal{L}[\sigma]$. Otherwise, it is undefined.*

2. **Renaming Property** *For all $\varphi \in \mathcal{L}[\sigma]$ and renamings $\rho : \sigma \to \tau$, there is a $\psi \in \mathcal{L}[\tau]$ such that for all $\mathfrak{M} \in \mathsf{Str}[\tau]$, $\mathfrak{M} \models \psi$ iff $\mathfrak{M}^\rho \models \varphi$.*

Definition 6.4.1 is inspired by similar ones occurring in the literature on abstract model theory [8]. Since the definition is rather general, one might ask what is still *modal*, or *hybrid*, about these languages. The two main distinctively modal features in Definition 6.4.1 are (1) the fact that that the structures we work with are pointed, reflecting the fact that modal formulas are always evaluated locally, and (2) the strict distinction between modalities on the one hand and proposition letters and nominals on the other hand. The importance of this distinction will become clear later on, when we'll consider specific classes of frames.

Some shorthand notation will be convenient. Firstly, by a slight abuse of notation, we will use $\mathcal{L}$ also to refer to the pair $(\mathcal{L}, \models_{\mathcal{L}})$. Secondly, given a model $\mathfrak{M} = (\mathfrak{F}, V, w)$ and an element v of the domain of $\mathfrak{F}$, we will use $(\mathfrak{M}, v)$ to denote the model $(\mathfrak{F}, V, v)$. Thus, with $\mathfrak{M}, v \models \varphi$ we mean $(\mathfrak{F}, V, v) \models \varphi$. Next, for $\varphi \in \mathcal{L}[\sigma]$, let $\mathsf{Mod}^{\sigma}_{\mathcal{L}}(\varphi) = \{\mathfrak{M} \in \mathsf{Str}[\sigma] \mid \mathfrak{M} \models_{\mathcal{L}} \varphi\}$. For $\mathfrak{M} \in \mathsf{Str}[\sigma]$ and $\varphi \in \mathcal{L}[\sigma]$, let $[\![\varphi]\!]^{\mathfrak{M}}_{\mathcal{L}} = \{v \mid \mathfrak{M}, v \models \varphi\}$, i.e., the subset of the domain of $\mathfrak{M}$ defined by φ. Finally, the symbol $\models$ will be used not only to refer to the satisfaction relation, but also to the *local consequence* relation: for $\Phi \cup \{\psi\} \subseteq \mathcal{L}[\sigma]$, we say that $\Phi \models_{\mathcal{L}} \varphi$ iff for all $\mathfrak{M} \in \mathsf{Str}[\sigma]$, it holds that if $\mathfrak{M} \models_{\mathcal{L}} \varphi$ for $\varphi \in \Phi$ then $\mathfrak{M} \models_{\mathcal{L}} \psi$.

Often, we will restrict attention to a specific frame class F. In these cases, we will write $\mathsf{Mod}^{\sigma}_{\mathcal{L},\mathsf{F}}(\varphi)$ for $\{\mathfrak{M} \in \mathsf{Str}_{\mathsf{F}}[\sigma] \mid \mathfrak{M} \models_{\mathcal{L}} \varphi\}$. Likewise, for $\Phi \cup \{\psi\} \subseteq \mathcal{L}[\sigma]$, we say that $\Phi \models_{\mathcal{L},\mathsf{F}} \varphi$ iff $\bigcap_{\varphi\in\Phi} \mathsf{Mod}^{\sigma}_{\mathcal{L},\mathsf{F}}(\varphi) \subseteq \mathsf{Mod}^{\sigma}_{\mathcal{L},\mathsf{F}}(\psi)$.

6.4.2. Definition (Extensions of hybrid languages). *Let $\mathcal{L}, \mathcal{L}'$ be hybrid languages. Then $\mathcal{L}'$ extends $\mathcal{L}$ relative to a frame class F (notation: $\mathcal{L} \subseteq_{\mathsf{F}} \mathcal{L}'$) if the following holds for all signatures σ and proposition letters $p_1, \ldots, p_n$ ($n \geq 0$).*

- *For each $\varphi \in \mathcal{L}[\sigma \cup \{p_1, \ldots, p_n\}]$ and $\psi_1, \ldots, \psi_n \in \mathcal{L}'[\sigma]$, there is a formula of $\mathcal{L}'[\sigma]$, which we will denote by $\varphi^{[\vec{p}/\vec{\psi}]}$, such that for all $\mathfrak{M} \in \mathsf{Str}_{\mathsf{F}}[\sigma]$, $\mathfrak{M} \models_{\mathcal{L}'} \varphi^{[\vec{p}/\vec{\psi}]}$ iff $\mathfrak{M}^{[p_1 \mapsto [\![\psi_1]\!]^{\mathfrak{M}}_{\mathcal{L}'}, \ldots, p_n \mapsto [\![\psi_n]\!]^{\mathfrak{M}}_{\mathcal{L}'}]} \models_{\mathcal{L}} \varphi$.*

Note that Definition 6.4.2 concerns *expressive* extensions rather than *axiomatic* extensions. As a special case (take $n = 0$), we have that whenever $\mathcal{L} \subseteq_{\mathsf{F}} \mathcal{L}'$ and $\varphi \in \mathcal{L}[\sigma]$, there is a $\psi \in \mathcal{L}'[\sigma]$ such that $\mathsf{Mod}^{\sigma}_{\mathcal{L},\mathsf{F}}(\varphi) = \mathsf{Mod}^{\sigma}_{\mathcal{L}',\mathsf{F}}(\psi)$. However, Definition 6.4.2 provides more information: it ensures that $\mathcal{L}'$ is closed under the basic operations of $\mathcal{L}$, such as negation. For, suppose $\mathcal{L} \subseteq_{\mathsf{F}} \mathcal{L}'$ and $\mathcal{L}$ has negation. Then for any $\varphi \in \mathcal{L}'$, $(\neg p)^{[p/\varphi]}$ expresses the negation of φ. Definitions like Definition 6.4.2 are quite common in the literature on abstract model theory. Incidentally, such definitions makes sense only for languages $\mathcal{L}$ that are closed under substitution of formulas for proposition letters, since otherwise it might happen that $\mathcal{L} \not\subseteq \mathcal{L}$. All languages that we will be concerned with are closed under substitution.

The languages $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$ are hybrid languages in the sense of Definition 6.4.1. Similarly, $\mathcal{H}(@, \downarrow)$ is a hybrid language if we consider only sentences, not formulas with free variables. Finally, the first-order correspondence language $\mathcal{L}^1$ constitutes a hybrid language, if we consider only formulas with at most one free variable.

Finally, let us define interpolation, by which we will mean interpolation over proposition letters and nominals. Using the terminology of this section, interpolation can be defined as follows.

6.4.3. Definition (Interpolation). *A hybrid language $\mathcal{L}$ has* interpolation *on a frame class* F *if for all* $\varphi \in \mathcal{L}[\sigma]$ *and* $\psi \in \mathcal{L}[\tau]$ *such that* $\varphi \models_{\mathcal{L},\mathsf{F}} \psi$, *there is a* $\vartheta \in \mathcal{L}[\sigma \cap \tau]$ *such that* $\varphi \models_{\mathcal{L},\mathsf{F}} \vartheta$, *and* $\vartheta \models_{\mathcal{L},\mathsf{F}} \psi$.

The reader should keep in mind that $\models_{\mathcal{L},\mathsf{F}}$ denotes the *local* entailment relation.

Now for the main result of this section.

6.4.4. Theorem. *Then the following hold for any frame class* F.

(i) For all hybrid languages $\mathcal{L}$, *if* $\mathcal{H}(@) \subseteq_{\mathsf{F}} \mathcal{L}$ *and* $\mathcal{L}$ *has interpolation on* F *then* $\mathcal{H}(@, \downarrow) \subseteq_{\mathsf{F}} \mathcal{L}$

(ii) For all hybrid languages $\mathcal{L}$, *if* $\mathcal{H}(\mathsf{E}) \subseteq_{\mathsf{F}} \mathcal{L}$ *and* $\mathcal{L}$ *has interpolation on* F *then* $\mathcal{L}^1 \subseteq_{\mathsf{F}} \mathcal{L}$

These results can be interpreted as general negative interpolation results, or, from another perspective, as characterizations. For instance, since $\mathcal{H}(@, \downarrow)$ has interpolation (as will be shown in Chapter 9), Theorem 6.4.4(i) characterizes $\mathcal{H}(@, \downarrow)$ as the smallest extension of $\mathcal{H}(@)$ that has interpolation. Similarly, when combined with Lindström's characterization of first-order logic [77], Theorem 6.4.4(ii) singles out first-order logic as the unique extension of $\mathcal{H}(\mathsf{E})$ with interpolation, compactness and the Löwenheim-Skolem property.

Note that Theorem 6.3.1 and 6.3.2 are special cases of Theorem 6.4.4.

The remainder of this section is devoted to the proof of Theorem 6.4.4. First, we prove an adapted version of well-known lemma relating interpolation with *projective classes* [8].

6.4.5. Definition (Projective classes). *Let* σ *be a signature, and let* $\mathsf{K} \subseteq \mathsf{Str}_{\mathsf{F}}[\sigma]$. *Then* K *is a* projective class *of a hybrid language* $\mathcal{L}$ *relative to a frame class* F *if there is a* $\varphi \in \mathcal{L}[\tau]$ *with* $\tau \supseteq \sigma$, *such that* $\mathsf{K} = \mathsf{Mod}^{\tau}_{\mathcal{L},\mathsf{F}}(\varphi) \restriction \sigma$.

6.4.6. Definition (Negation). *A hybrid language* $\mathcal{L}$ has negation *on* F *if for each* $\varphi \in \mathcal{L}[\sigma]$ *there is an formula of* $\mathcal{L}[\sigma]$, *which we will denote by* $\neg\varphi$, *such that* $\mathsf{Mod}_{\mathcal{L},\mathsf{F}}(\psi) = \mathsf{Str}_{\mathsf{F}}[\sigma] \backslash \mathsf{Mod}_{\mathcal{L},\mathsf{F}}(\varphi)$.

6.4.7. Lemma. *Let* $\mathcal{L}$ *be a hybrid language with negation that has interpolation on a frame class* F, *and let* $\mathsf{K} \subseteq \mathsf{Str}_{\mathsf{F}}[\sigma]$, *for some signature* σ. *If both* K *and* $\mathsf{Str}_{\mathsf{F}}[\sigma] \backslash \mathsf{K}$ *are projective classes of* $\mathcal{L}$ *relative to* F, *then there is a* $\varphi \in \mathcal{L}[\sigma]$ *such that* $\mathsf{K} = \mathsf{Mod}_{\mathcal{L},\mathsf{F}}(\varphi)$.

Proof: Since K is a projective class, there is a formula $\varphi \in \mathcal{L}[\tau]$, with $\sigma \subseteq \tau$, such that $\mathsf{K} = \mathsf{Mod}_{\mathcal{L},\mathsf{F}}(\varphi) \restriction \sigma$. Likewise, since $\mathsf{Str}_{\mathsf{F}}[\sigma]\backslash\mathsf{K}$ is a projective class, there is a formula $\psi \in \mathcal{L}[\tau']$, with $\sigma \subseteq \tau'$, such that $\mathsf{Str}_{\mathsf{F}}[\sigma]\backslash\mathsf{K} = \mathsf{Mod}_{\mathcal{L},\mathsf{F}}(\psi) \restriction \sigma$. Without loss of generality, we may assume that $\tau \cap \tau' = \sigma$ (by the Renaming property of $\mathcal{L}$). It follows that $\varphi \models_{\mathcal{L},\mathsf{F}} \neg\psi$. Since $\mathcal{L}$ has interpolation, there must be a $\vartheta \in \mathcal{L}[\sigma]$ such that $\varphi \models_{\mathcal{L},\mathsf{F}} \vartheta$ and $\vartheta \models_{\mathcal{L},\mathsf{F}} \neg\psi$. As a last step, we will show that $\mathsf{Mod}_{\mathcal{L},\mathsf{F}}(\vartheta) = \mathsf{K}$.

Suppose $\mathfrak{M} \in \mathsf{K}$. Then $\mathfrak{M} = \mathfrak{N} \restriction \sigma$ for some $\mathfrak{N} \in \mathsf{Mod}_{\mathcal{L},\mathsf{F}}(\varphi)$. Since $\varphi \models_{\mathcal{L},\mathsf{F}} \vartheta$, it follows that $\mathfrak{N} \models \vartheta$. By the Expansion property, $\mathfrak{M} \models \vartheta$. Conversely, suppose $\mathfrak{M} \notin \mathsf{K}$. Then $\mathfrak{M} = \mathfrak{N} \restriction \sigma$ for some $\mathfrak{N} \in \mathsf{Mod}_{\mathcal{L},\mathsf{F}}(\psi)$. Since $\vartheta \models_{\mathcal{L},\mathsf{F}} \neg\psi$, it follows that $\mathfrak{N} \not\models \vartheta$. By the Expansion property, $\mathfrak{M} \not\models \vartheta$. □

The property expressed in Lemma 6.4.7 may be called Δ-*interpolation*, by analogy to the notion of Δ-interpolation in [8]. It is a slightly weaker condition than interpolation, and arguably more natural from a model theoretic perspective. Incidentally, it should be mentioned that Theorem 6.4.4 may be strengthened by replacing the condition of interpolation by that of Δ-interpolation.

Using Lemma 6.4.7, we can show that if the $\downarrow$-binder is added to a hybrid language with interpolation extending $\mathcal{H}(@)$, then the expressivity of the language in question does not increase. This is expressed in the following lemma.

6.4.8. Lemma. *Let $\mathcal{L}$ be a hybrid language with interpolation on a frame class F, such that $\mathcal{H}(@) \subseteq_{\mathsf{F}} \mathcal{L}$. Then for all $\varphi \in \mathcal{L}[\sigma]$ and $i \in \mathrm{NOM}_\sigma$, there is a formula of $\mathcal{L}[\sigma\backslash\{i\}]$, which we will denote by $\downarrow i.\varphi$, such that $\mathsf{Mod}_{\mathcal{L},\mathsf{F}}(\downarrow i.\varphi) = \{(\mathfrak{F}, V, w) \in \mathsf{Str}_{\mathsf{F}}[\sigma\backslash\{i\}] \mid (\mathfrak{F}, V^{[i\mapsto\{w\}]}, w) \models \varphi\}$.*

Proof: Let $\mathsf{K}_{\downarrow i.\varphi} = \{(\mathfrak{F}, V, w) \in \mathsf{Str}_{\mathsf{F}}[\sigma\backslash\{i\}] \mid (\mathfrak{F}, V^{[i\mapsto\{w\}]}, w) \models \varphi\}$. $\mathsf{K}_{\downarrow i.\varphi}$ is projectively defined by $i \wedge \varphi$ and its complement is projectively defined by $i \wedge \neg\varphi$. Since $\mathcal{L}$ has negation and has interpolation on F, by Lemma 6.4.7 $\mathsf{K}_{\downarrow i.\varphi} = \mathsf{Mod}_{\mathcal{L},\mathsf{F}}(\psi)$ for some $\psi \in \mathcal{L}[\sigma\backslash\{i\}]$. □

We are now ready to prove Theorems 6.4.4(i) and 6.4.4(ii).

Proof of Theorem 6.4.4(i): Let $\mathcal{L}$ be any hybrid language with interpolation on a frame class F, such that $\mathcal{H}(@) \subseteq_{\mathsf{F}} \mathcal{L}$. Let $\varphi \in \mathcal{H}(@,\downarrow)[\sigma \cup \{p_1,\ldots,p_n\}]$ and $\psi_1,\ldots,\psi_n \in \mathcal{L}[\sigma]$. We will show that there is a formula $\chi \in \mathcal{L}[\sigma]$ that is $[\vec{p}/\vec{\psi}]$-equivalent to φ on F, meaning that

$$\text{for all } \mathfrak{M} \in \mathsf{Str}_{\mathsf{F}}[\sigma],\ \mathfrak{M} \models_{\mathcal{L}} \chi \text{ iff } \mathfrak{M}^{[p_1\mapsto[\![\psi_1]\!]^{\mathfrak{M}}_{\mathcal{L}},\ldots,p_n\mapsto[\![\psi_n]\!]^{\mathfrak{M}}_{\mathcal{L}}]} \models_{\mathcal{H}(@)} \varphi$$

The proof proceeds by induction on the length of φ. The base case (where φ is a proposition letter or nominal from σ, or φ is $\top$ or φ is p_i for some $i \leq n$) follows immediately from the fact that $\mathcal{H}(@) \subseteq_{\mathsf{F}} \mathcal{L}$. For the inductive step, we will only prove the cases for negation and for the $\downarrow$-binder, since the other cases are similar to the one for negation.

Let φ be of the form $\neg\psi$. By induction hypothesis, ψ is $[\vec{p}/\vec{\psi}]$-equivalent on F to some $\chi \in \mathcal{L}[\sigma]$. Let q be any proposition letter not in σ and distinct from $p_1, \dots, p_n$. Since $\mathcal{H}(@) \subseteq_{\mathsf{F}} \mathcal{L}$ and $(\neg q) \in \mathcal{H}(@)[\sigma \cup \{q\}]$, Definition 6.4.2 guarantees the existence of a formula $(\neg p)^{[p/\chi]} \in \mathcal{L}[\sigma]$ that expresses the negation of ψ on F. It follows that $(\neg p)^{[p/\chi]}$ is $[\vec{p}/\vec{\psi}]$-equivalent on F to φ.

Let φ be of the form $\downarrow x.\psi$. Let i be any nominal not in σ. By the induction hypothesis, we know that there is some $\chi \in \mathcal{L}[\sigma \cup \{i\}]$ that is $[\vec{p}/\vec{\psi}]$-equivalent on F to $\psi[x/i]$. By Lemma 6.4.8 it follows that $\downarrow x.\psi$ is $[\vec{p}/\vec{\psi}]$-equivalent on F to $\downarrow i.\chi \in \mathcal{L}[\sigma]$. □

Proof of Theorem 6.4.4(ii): Similar to the proof of Theorem 6.4.4(i). We will only discuss the inductive step for formulas of the form $\exists y.\psi$.

Let $\varphi \in \mathcal{L}^1[\sigma]$ be of the form $\exists y.\psi$. By the definition of $\mathcal{L}^1$, φ contains at most one free variable, say x (in case φ contains no free variables, let x be any variable distinct from y). Let i, j be distinct nominals (constants) not in σ. By induction hypothesis, $\varphi[x/i, y/j] \in \mathcal{L}^1[\sigma \cup \{i, j\}]$ is $[\vec{p}/\vec{\psi}]$-equivalent on F to some $\chi \in \mathcal{L}[\sigma \cup \{i, j\}]$. By Lemma 6.4.8 and by the fact that $\mathcal{H}(\mathsf{E}) \subseteq_{\mathsf{F}} \mathcal{L}$, we obtain a formula $\downarrow i.\mathsf{E}\downarrow j.\chi \in \mathcal{L}[\sigma]$ that is easily shown to be $[\vec{p}/\vec{\psi}]$-equivalent to φ on F □

6.4.9. Remark. It should be noted that, while the results in this section have been formulated for languages with unary modalities only, the proof can easily be adapted to the general case where modal operators can have any arity.

Secondly, while we have chosen to formulate the results in this section in terms of interpolation over proposition letters and nominals, inspection of the proofs shows that the results hold even if we would replace this notion of interpolation by the weaker *interpolation over nominals*.

Chapter 7

Translations from hybrid to modal logics

In this chapter, we will show that for certain frame classes K, there is a translation from hybrid formulas to modal formulas that preserves satisfiability with respect to K. There are at least two reasons to be interested in such translation. One reason is that they allow us to apply theorem provers developed for modal logics to hybrid logics. The second reason is that such translations make it possible to derive results on hybrid logics from results on modal logics. The translations that will be provided in this chapter allow us to do both.

One of the results we will prove is the following.

> If a frame class K admits polynomial filtration (cf. Section 2.6), then there is a polynomial translation from $\mathcal{H}(\mathsf{E})$ to $\mathcal{M}(\mathsf{E})$ preserving satisfiability with respect to K.

Similar results are proved for $\mathcal{H}$ and $\mathcal{H}(@)$. The translations are modular enough to give rise to transfer results concerning complexity, (uniform) interpolation and axiomatizations. These transfer results will be presented in the next chapter.

All proofs in this chapter make use of filtrations. It has been observed by several authors that if the basic modal language $\mathcal{M}$ admits filtration with respect to a frame class K, then $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$ also admit filtration with respect to K (see for instance [17, 46]). It follows that if decidability of a modal logic is proved using filtrations, the corresponding hybrid logic is also decidable. The translations presented in this chapter, however, allow for a much more fine-grained analysis. As will be shown in the next chapter, the translations give rise to transfer of complexity bounds, as well as other properties such as interpolation.

The results reported in this chapter are partly taken from [15].

7.1 From $\mathcal{H}(\mathsf{E})$ to $\mathcal{M}(\mathsf{E})$

The first case that we will consider is the simplest case: we will translate formulas of $\mathcal{H}(\mathsf{E})$ to $\mathcal{M}(\mathsf{E})$, which is the extension of the basic modal language with the global modality. Recall the definition of filtration in Section 2.6. For an $\mathcal{H}(\mathsf{E})$-formula $\varphi(i_1, \dots, i_n)$ let $\varphi[\vec{i}/\vec{p_i}]$ denote the $\mathcal{M}(\mathsf{E})$-formula obtained by uniformly replacing each nominal i_k by distinct new proposition letter p_{i_k}.

7.1.1. THEOREM. *Let K be a frame class that admits filtration. Let $\varphi(i_1, \dots, i_n)$ be any $\mathcal{H}(\mathsf{E})$-formula. Then φ is satisfiable on K iff the $\mathcal{M}(\mathsf{E})$-formula*

$$\varphi^* = \varphi[\vec{i}/\vec{p_i}] \;\wedge \bigwedge_{1 \le k \le n} \mathsf{E} p_{i_k} \wedge \bigwedge_{\substack{1 \le k \le n \\ \psi \in \Sigma_{\varphi[\vec{i}/\vec{p_i}]}}} \mathsf{E}(p_{i_k} \wedge \psi) \to \mathsf{A}(p_{i_k} \to \psi)$$

is satisfiable on K, where $\Sigma_{\varphi[\vec{i}/\vec{p_i}]}$ is the filtration set of $\varphi[\vec{i}/\vec{p_i}]$.

Proof:

[$\Rightarrow$] Suppose $(\mathfrak{F}, V), w \models \varphi$ with $\mathfrak{F} \in \mathsf{K}$. Let V' be any valuation that agrees with V on all proposition letters occurring in φ, and such that $V'(p_{i_k}) = V(i_k)$ for each nominal i_k. Clearly, $(\mathfrak{F}, V'), w \models \varphi[\vec{i}/\vec{p_i}]$. The truth of the remaining conjuncts of φ^* at w under V' follows directly from the fact that $V'(p_{i_k})$ is a singleton set for each $k = 1, \dots, n$.

[$\Leftarrow$] Suppose $(\mathfrak{F}, V), w \models \varphi^*$ with $\mathfrak{F} = (W, R) \in \mathsf{K}$. Our task is to construct a hybrid model satisfying φ.

We will filtrate $(\mathfrak{F}, V)$. Let $\Sigma = \Sigma_{\varphi[\vec{i}/\vec{p_i}]}$. Since K admits filtration, there exists a model $\mathfrak{M} = (W/_{\sim_\Sigma}, R_\Sigma, V_\Sigma)$ such that $(W/_{\sim_\Sigma}, R_\Sigma) \in \mathsf{K}$ and such that for all $v \in W$ and $\psi \in \Sigma$, $\mathfrak{M}, [v] \models \psi$ iff $(\mathfrak{F}, V), v \models \psi$. In particular, $\mathfrak{M}, [w] \models \varphi[\vec{i}/\vec{p_i}]$.

Claim 1: $V_\Sigma(p_{i_k})$ contains exactly one point (for $k = 1, \dots, n$).

Proof of claim: $V_\Sigma(p_{i_k})$ is easily seen to be non-empty: by the second conjunct of φ^*, $\mathfrak{M}, v \models p_{i_k}$ for some v. By the definition of filtration, $[v] \in V_\Sigma(p_{i_k})$.

Next, suppose $[v], [v'] \in V_\Sigma(p_{i_k})$. Then $v, v' \in V(p_{i_k})$, by the definition of V_Σ. Since $(\mathfrak{F}, V), w \models \mathsf{E}(p_{i_k} \wedge \psi) \to \mathsf{A}(p_{i_k} \to \psi)$ for all $\psi \in \Sigma$, it follows that v, v' agree on formulas in Σ. Indeed, if $v \models \psi$ then $w \models \mathsf{E}(p_{i_k} \wedge \psi)$, so $w \models \mathsf{A}(p_{i_k} \to \psi)$ and therefore $v' \models \psi$. Thus, $v \sim_\Sigma v'$ and so $[v] = [v']$. ⊣

Replacing each p_{i_k} by the corresponding i_k, we therefore obtain a hybrid model again, which furthermore satisfies φ at $[w]$. We conclude that φ is satisfiable on K. □

7.1.2. COROLLARY. *Let K be a frame class that admits filtration. Let $\varphi(i_1, \dots, i_n)$ be any $\mathcal{H}(\mathsf{E})$-formula. Then φ is valid on K iff the modal formula*

$$\Big(\bigwedge_{1 \le k \le n} \mathsf{E} p_{i_k} \wedge \bigwedge_{\substack{1 \le k \le n \\ \psi \in \Sigma_{\neg\varphi}}} \mathsf{E}(p_{i_k} \wedge \psi) \to \mathsf{A}(p_{i_k} \to \psi) \Big) \to \varphi[\vec{i}/\vec{p_i}]$$

is valid on K, where $\Sigma_{\neg\varphi[\vec{i}/\vec{p_i}]}$ is the filtration set of $\neg\varphi[\vec{i}/\vec{p_i}]$.

In the rest of this chapter, we will give similar results for the hybrid languages $\mathcal{H}$ and $\mathcal{H}(@)$. However, the situation for these languages is substantially more complicated, for the following reason. By Proposition 8.1.2, there can be no polynomial reduction from $\mathcal{H}$ or $\mathcal{H}(@)$ to modal logic that preserves satisfiability on symmetric frames. Nevertheless, the class of symmetric frames admits filtration [31]. Hence, Theorem 7.1.1 cannot be adapted to $\mathcal{H}$ or $\mathcal{H}(@)$ without further restrictions.

We will consider two classes of modal logics, namely logics that admit filtration and have a master modality, and logics that are axiomatized by modal formulas in which every occurrence of a proposition letter is in the scope of at most one modal operator. Note that the logic of symmetric frames does not fall in either class.

7.2 From $\mathcal{H}$ to $\mathcal{M}$ in case of a master modality

We say that a frame class K *has a master modality*, if there is a modal formula $\varphi(p)$ containing no proposition letter besides p, such that for all models $\mathfrak{M}$ based on a frame in K, and worlds w, $\mathfrak{M}, w \models \varphi(p)$ iff p holds somewhere in the submodel of $\mathfrak{M}$ generated by w. It follows that, if $\varphi(\psi)$ is obtained by uniformly replacing p by ψ in φ, $\mathfrak{M}, w \models \varphi(\psi)$ iff ψ holds somewhere in the submodel of $\mathfrak{M}$ generated by w. We will use $\diamondplus$ to denote the master modality.

For a $\mathcal{H}$-formula $\varphi(i_1, \ldots, i_n)$ let $\varphi[\vec{i}/\vec{p_i}]$ denote the modal formula obtained from φ by uniformly replacing each nominal i_k by distinct new proposition letter p_{i_k}.

7.2.1. Theorem. *Let K be any frame class closed under generated subframes, disjoint unions and isomorphic copies that admits filtration and that has a master modality. Let $\varphi(i_1, \ldots, i_n)$ be any $\mathcal{H}$-formula. Then φ is satisfiable on K iff the modal formula*

$$\varphi^* = \varphi[\vec{i}/\vec{p_i}] \;\wedge \bigwedge_{\substack{1 \le k \le n \\ \psi \in \Sigma_{\varphi[\vec{i}/\vec{p_i}]}}} \diamondplus(p_{i_k} \wedge \psi) \to \boxplus(p_{i_k} \to \psi)$$

is satisfiable on K, where $\diamondplus$ is the master modality of K and $\Sigma_{\varphi[\vec{i}/\vec{p_i}]}$ is the filtration set of $\varphi[\vec{i}/\vec{p_i}]$.

Proof: For simplicity, we only prove the case for uni-modal logics. The proof generalizes straightforwardly to the general case.

[$\Rightarrow$] Suppose $(\mathfrak{F}, V), w \models \varphi$ with $\mathfrak{F} \in \mathsf{K}$. Let V' be any valuation that agrees with V on all proposition letters occurring in φ, and such that $V'(p_{i_k}) = V(i_k)$ for each nominal i_k. Clearly, $(\mathfrak{F}, V'), w \models \varphi[\vec{i}/\vec{p_i}]$. The truth of the second conjunct of φ^* at w under V' follows directly from the fact that $V'(p_{i_k})$ is a singleton set for each $k = 1, \ldots, n$.

[$\Leftarrow$] Suppose $(\mathfrak{F}, V), w \models \varphi^*$ with $\mathfrak{F} = (W, R) \in \mathsf{K}$. Without loss of generality, we can assume that $\mathfrak{F}$ is generated by w (note that φ^* is a purely modal formula). Our task is to construct a hybrid model satisfying φ.

First, we will filtrate $(\mathfrak{F}, V)$. Let $\Sigma = \Sigma_{\varphi[\vec{i}/\vec{p_i}]}$. Since K admits filtration, there exists a model $\mathfrak{M} = (W/_{\sim_\Sigma}, R_\Sigma, V_\Sigma)$ such that $(W/_{\sim_\Sigma}, R_\Sigma) \in \mathsf{K}$ and such that for all $v \in W$ and $\psi \in \Sigma$, $\mathfrak{M}, [v] \models \psi$ iff $(\mathfrak{F}, V), v \models \psi$. In particular, $\mathfrak{M}, [w] \models \varphi[\vec{i}/\vec{p_i}]$.

Claim 1: $V_\Sigma(p_{i_k})$ contains at most one point (for $k = 1, \ldots, n$).

Proof of claim: Suppose $[v], [v'] \in V_\Sigma(p_{i_k})$. Then $v, v' \in V(p_{i_k})$, by the definition of V_Σ. Since $(\mathfrak{F}, V), w \models \oplus(p_{i_k} \wedge \psi) \to \boxplus(p_{i_k} \to \psi)$ for all $\psi \in \Sigma$, it follows that v, v' agree on formulas in Σ. Indeed, if $v \models \psi$ then $w \models \oplus(p_{i_k} \wedge \psi)$, so $w \models \boxplus(p_{i_k} \to \psi)$ and therefore $v' \models \psi$. Thus, $v \sim_\Sigma v'$ and so $[v] = [v']$. $\dashv$

If every p_{i_k} is true at *exactly* one point, then the proof is finished, since we can consider $(W/_{\sim_\Sigma}, R_\Sigma)$ to be a hybrid model for φ. In general, however, this need not be the case: p_{i_k} could be true nowhere. So, we need to ensure that for every p_{i_k} there is indeed a point where p_{i_k} is true. Let $\mathfrak{G}$ be the disjoint union of two isomorphic copies of $(W/_{\sim_\Sigma}, R_\Sigma)$. For convenience, we will use $[v]_1$ and $[v]_2$ to refer to the two distinct copies of a world $[v] \in W/_{\sim_\Sigma}$. Since K is closed under disjoint unions, $\mathfrak{G} \in \mathsf{K}$. Define the valuation V' for $(W/_{\sim_\Sigma}, R_\Sigma)$ by putting $V'(p) = \{v_1 \mid v \in V_\Sigma(p)\}$ for each proposition letter p occurring in φ, and for each nominal $k = 1, \ldots, n$,

$$V'(p_{i_k}) = \begin{cases} \{[v]_1\} & \text{if } V_\Sigma(p_{i_k}) = \{[v]\} \\ \{[w]_2\} & \text{if } V_\Sigma(p_{i_k}) = \emptyset \end{cases}$$

Intuitively speaking, the only role of the second disjoint copy of $(W/_{\sim_\Sigma}, R_\Sigma)$ is to provide enough points so that we can make each p_{i_k} true somewhere, without affecting the truth of φ at $[w]$. Indeed, a simple bisimulation argument shows that $(\mathfrak{G}, V'), [w] \models \varphi[\vec{i}/\vec{p_i}]$.

By construction, V' assigns to each p_{i_k} a singleton set. Replacing each p_{i_k} by the corresponding i_k, we therefore obtain a hybrid model again, which furthermore satisfies φ at $[w]_1$. We conclude that φ is satisfiable on K. □

7.2.2. COROLLARY. *Let* K *be any frame class closed under generated subframes, disjoint unions and isomorphic copies that admits filtration and that has a master modality. Let* $\varphi(i_1, \ldots, i_n)$ *be any* $\mathcal{H}$*-formula. Then* φ *is valid on* K *iff the modal formula*

$$\Big(\bigwedge_{\substack{1 \le k \le n \\ \psi \in \Sigma_{\neg\varphi[\vec{i}/\vec{p_i}]}}} \oplus(p_{i_k} \wedge \psi) \to \boxplus(p_{i_k} \to \psi) \Big) \to \varphi[\vec{i}/\vec{p_i}]$$

is valid on K*, where* $\oplus$ *is the master modality of* K *and* $\Sigma_{\neg\varphi[\vec{i}/\vec{p_i}]}$ *is the filtration set of* $\neg\varphi[\vec{i}/\vec{p_i}]$.

7.3 From $\mathcal{H}(@)$ to $\mathcal{M}$ in case of a master modality

In order to translate $\mathcal{H}(@)$-formulas into modal formulas, we will need to make use of an extra modality. Let $\diamondsuit$ be a new modality. For every frame class K, let $\mathsf{Exp}_1(\mathsf{K})$ be the result of expanding the frames in K with an extra binary relation. More precisely, let $\mathsf{Exp}_1(\mathsf{K}) = \{(W, (R_\diamond)_{\diamond \in \text{MOD}}, R_{\diamondsuit}) \mid (W, (R_\diamond)_{\diamond \in \text{MOD}}) \in \mathsf{K}$ and $R_{\diamondsuit} \subseteq W \times W\}$.

For an $\mathcal{H}(@)$-formula $\varphi(i_1, \ldots, i_n)$, let $\varphi[\vec{i}/\vec{p_i}, @_i/\diamondsuit(p_i \wedge \cdot)]$ denote the formula obtained from φ by uniformly replacing each nominal i_k by distinct new proposition letter p_{i_k}, and replacing each satisfaction operator $@_{i_k}(\cdot)$ by $\diamondsuit(p_{i_k} \wedge \cdot)$. We will use $\diamondsuit^{\leq 1}\psi$ as a shorthand for $\psi \vee \diamondsuit\psi$ and $\boxdot^{\leq 1}\psi$ as a shorthand for $\psi \wedge \boxdot\psi$.

7.3.1. Theorem. *Let* K *be a frame class closed under generated subframes that admits filtration and that has a master modality. Let* φ *be any* $\mathcal{H}(@)$*-formula in* $@$*-normal form (cf. Definition 3.3.1). Then* φ *is satisfiable on* K *iff the modal formula*

$$\varphi^* = \varphi[\vec{i}/\vec{p_i}, @_i/\diamondsuit(p_i \wedge \cdot)] \;\wedge\; \bigwedge_{1 \leq k \leq n} \diamondsuit p_{i_k} \;\wedge$$
$$\bigwedge_{\substack{1 \leq k \leq n \\ \psi \in \Sigma}} \Big(\diamondsuit^{\leq 1} \oplus(p_{i_k} \wedge \psi) \;\rightarrow\; \boxdot^{\leq 1} \boxplus(p_{i_k} \rightarrow \psi)\Big)$$

is satisfiable on $\mathsf{Exp}_1(\mathsf{K})$*, where* $\oplus$ *is the master modality of* K *and* Σ *is the filtration set of the formula* $\bigwedge\{\chi[\vec{i}/\vec{p_i}] \mid \chi \in Sub(\varphi)$ *and* χ *contains no* $@$*-operators*$\}$.

Proof: For simplicity, we only prove the case for uni-modal logics. The proof generalizes straightforwardly to the general case.

[$\Rightarrow$] Suppose $(W, R_\diamond, V), w \models \varphi$ with $(W, R_\diamond) \in \mathsf{K}$. Let $R_{\diamondsuit}$ be the total relation on the domain of $\mathfrak{F}$, and let V' be any valuation that agrees with V on all proposition letters occurring in φ, and such that $V'(p_{i_k}) = V(i_k)$ for each nominal i_k. Then, clearly, $(W, R_\diamond, R_{\diamondsuit}, V'), w \models \varphi[\vec{i}/\vec{p_i}, @_i/\diamondsuit(p_{i_k} \wedge \cdot)]$. The truth of the remainder of φ^* follows directly from the construction of the model.

[$\Leftarrow$] Suppose $\mathfrak{M}, w \models \varphi^*$ with $\mathfrak{M} = (\mathfrak{F}, V)$ and $\mathfrak{F} = (W, R, R') \in \mathsf{Exp}_1(\mathsf{K})$. Let $\mathfrak{F}' = (W, R)$ and let $\mathfrak{M}' = (\mathfrak{F}', V)$. Note that $\mathfrak{F}' \in \mathsf{K}$. Next, let $\mathfrak{N}$ be submodel of $\mathfrak{M}'$ generated by $\{w\} \cup \{v \in W \mid wR'v\}$. By the truth of the second conjunct of φ^* at $(\mathfrak{M}, w)$, every p_{i_k} is true somewhere in $\mathfrak{N}$. By the truth of the third conjunct of φ^* at $(\mathfrak{M}, w)$, any two points in $\mathfrak{N}$ that satisfy the same p_{i_k} agree on all formulas in Σ (recall that no formula in this set contains any $\diamondsuit$-modality). Since $\mathfrak{N}$ is a generated submodel of $\mathfrak{M}'$, we also have that $\mathfrak{N}$ is based on a frame in K.

Since K admits filtration, there exists a model $\mathfrak{N}_\Sigma = (W_\Sigma, R_\Sigma, V_\Sigma)$ based on a frame in K, such that for all points v in $\mathfrak{N}$ and formulas $\psi \in \Sigma$, $\mathfrak{N}_\Sigma, [v] \models \psi$ iff $\mathfrak{N}, v \models \psi$.

Claim 1: p_{i_k} is true at precisely one world in $\mathfrak{N}_\Sigma$ (for $k = 1, \ldots, n$).

Proof of claim: As we observed above, p_{i_k} is true at some world v of $\mathfrak{N}$. Since $p_{i_k} \in \Sigma$, it follows that $\mathfrak{N}_\Sigma, [v] \models p_{i_k}$. As for uniqueness, suppose $\mathfrak{N}_\Sigma, [v] \models p_{i_k}$ and $\mathfrak{N}_\Sigma, [v'] \models p_{i_k}$. Then $\mathfrak{N}, v \models p_{i_k}$ and $\mathfrak{N}, v' \models p_{i_k}$. As we noted above, this implies that $(\mathfrak{N}, v)$ and $(\mathfrak{N}, v')$ agree on all formulas in Σ. Thus, by definition, $[v] = [v']$. ⊣

By the above claim, we can consider $\mathfrak{N}_\Sigma$ to be a hybrid model. We extend the valuation of $\mathfrak{N}_\Sigma$ to the nominals $i_1, \ldots, i_n$, by letting $\mathfrak{N}_\Sigma, [v] \models i_k$ iff $\mathfrak{N}_\Sigma, [v] \models p_{i_k}$.

Claim 2: The following holds for all $\chi \in Sub(\varphi)$ not containing any @-operators, and for $k = 1 \ldots n$.

1. $\mathfrak{N}_\Sigma, [w] \models \chi$ iff $\mathfrak{M}, w \models \chi[\vec{i}/\vec{p_i}]$
2. $\mathfrak{N}_\Sigma, [w] \models @_{i_k}\chi$ iff $\mathfrak{M}, w \models \diamondsuit(p_{i_k} \wedge \chi[\vec{i}/\vec{p_i}])$

Proof of claim: 1. By construction, $\mathfrak{N}_\Sigma, [w] \models \chi$ iff $\mathfrak{N}_\Sigma, [w] \models \chi[\vec{i}/\vec{p_i}]$. By the definition of filtration, $\mathfrak{N}_\Sigma, [w] \models \chi[\vec{i}/\vec{p_i}]$ iff $\mathfrak{N}, w \models \chi[\vec{i}/\vec{p_i}]$. By invariance under generated submodels, $\mathfrak{N}, w \models \chi[\vec{i}/\vec{p_i}]$ iff $\mathfrak{M}, w \models \chi[\vec{i}/\vec{p_i}]$.

2. First, suppose $\mathfrak{N}_\Sigma, [w] \models @_{i_k}\chi$. Then there is a point $[v]$ such that $\mathfrak{N}_\Sigma, [v] \models i_k \wedge \chi$, hence $\mathfrak{N}_\Sigma, [v] \models p_{i_k} \wedge \chi[\vec{i}/\vec{p_i}]$. Since $p_{i_k} \in \Sigma$ and $\chi[\vec{i}/\vec{p_i}] \in \Sigma$, it follows that $\mathfrak{N}, v \models p_{i_k} \wedge \chi[\vec{i}/\vec{p_i}]$. By invariance under generated submodels, $\mathfrak{M}, v \models p_{i_k} \wedge \chi[\vec{i}/\vec{p_i}]$. By the truth of the second conjunct of φ^* at $(\mathfrak{M}, w)$, there is an R'-successor u of w such that $\mathfrak{M}, u \models p_{i_k}$. By the truth of the third conjunct of φ^* at $(\mathfrak{M}, w)$, and the fact that v and u are both elements of the generated submodel $\mathfrak{N}$, we have that $\mathfrak{M}, u \models \chi[\vec{i}/\vec{p_i}]$. Hence, $\mathfrak{M}, w \models \diamondsuit(p_{i_k} \wedge \chi[\vec{i}/\vec{p_i}])$.
   Conversely, suppose $\mathfrak{M}, w \models \diamondsuit(p_{i_k} \wedge \chi[\vec{i}/\vec{p_i}])$. Then there is an R'-successor u of w such that $\mathfrak{M}, u \models p_{i_k} \wedge \chi[\vec{i}/\vec{p_i}]$. By invariance under generated submodels, it follows that $\mathfrak{N}, u \models p_{i_k} \wedge \chi[\vec{i}/\vec{p_i}]$. Hence, by the definition of filtration, $\mathfrak{N}_\Sigma, [u] \models p_{i_k} \wedge \chi[\vec{i}/\vec{p_i}]$. Hence, $\mathfrak{N}_\Sigma, [w] \models @_{i_k}\chi$. ⊣

Since φ is in @-normal form, it is a Boolean combination of formulas of the form χ or $@_i\chi$, where χ is a subformula of φ not containing any satisfaction operators. Hence, Claim 2 together with a simple induction argument yield that $\mathfrak{N}_\Sigma, [w] \models \varphi$ iff $\mathfrak{M}, w \models \varphi[\vec{i}/\vec{p_i}, @_i/\diamondsuit(p_i \wedge \cdot)]$. Hence, $\mathfrak{N}_\Sigma, [w] \models \varphi$. □

7.3.2. Corollary. *Let* K *be a frame class closed under generated subframes that admits filtration and that has a master modality. Let φ be any hybrid $\mathcal{H}(@)$-formula in @-normal form. Then φ is valid on* K *iff the modal formula*

$$\Big(\bigwedge_{1 \le k \le n} \diamondsuit p_{i_k} \wedge \bigwedge_{\substack{1 \le k \le n \\ \psi \in \Sigma}} \big(\diamondsuit^{\le 1} \diamondplus(p_{i_k} \wedge \psi) \to \boxdot^{\le 1} \boxplus(p_{i_k} \to \psi)\big)\Big) \\ \to \varphi[\vec{i}/\vec{p_i}, @_i/\diamondsuit(p_i \wedge \cdot)]$$

is valid on $\mathsf{Exp}_1(\mathsf{K})$*, where* $\oplus$ *is the master modality of* K *and* Σ *is the filtration set of the formula* $\neg \bigwedge\{\chi[\vec{i}/\vec{p_i}] \mid \chi \in Sub(\varphi)$ *and* χ *contains no* @*-operators*$\}$*.*

7.4 From $\mathcal{H}$ to $\mathcal{M}$ in case of shallow axioms

Not many frame classes have a master modality. In particular, the class of all frames does not have the master modality. In this section, we will provide a translation that works for frame classes defined by shallow modal formulas. Recall that a modal formula is shallow if every occurrence of a proposition letter is in the scope of at most one modal operator.

Before we give the proof in full generality, we will first consider the most simple case, namely the class of all frames. In what follows, we will assume that MOD is finite, and we will use $\langle\cup\rangle\psi$ as a shorthand for $\bigvee_{\Diamond\in\text{MOD}}\Diamond\psi$ and we will use $[\cup]\psi$ as a shorthand for $\bigwedge_{\Diamond\in\text{MOD}}\Box\psi$. Furthermore, we will use $\langle\cup\rangle^{\leq n}\psi$ as a shorthand for $\psi \vee \langle\cup\rangle\psi \vee \langle\cup\rangle\langle\cup\rangle\psi \vee \cdots \vee \langle\cup\rangle^n\psi$, and we will use $[\cup]^{\leq n}\psi$ as a shorthand for $\psi \wedge [\cup]\psi \wedge [\cup][\cup]\psi \wedge \cdots \wedge [\cup]^n\psi$.

7.4.1. THEOREM. *An* $\mathcal{H}$*-formula* $\varphi(i_1, \ldots, i_n)$ *is satisfiable iff the modal formula*

$$\varphi^* \quad = \quad \varphi[\vec{i}/\vec{p_i}] \;\wedge \bigwedge_{\substack{1\leq k\leq n \\ \psi\in Sub(\varphi[\vec{i}/\vec{p_i}])}} \Big(\langle\cup\rangle^{\leq md(\varphi)}(p_{i_k} \wedge \psi) \to [\cup]^{\leq md(\varphi)}(p_{i_k} \to \psi)\Big)$$

is satisfiable.

Proof: The left to right implication is easy to prove. Now suppose that φ^* is satisfiable. Let $\mathfrak{M}, w \models \varphi^*$, with $\mathfrak{M} = (\mathfrak{F}, V)$ and $\mathfrak{F} = (W, (R_\Diamond)_{\Diamond\in\text{MOD}})$. Without loss of generality, we can assume that $\mathfrak{F}$ is generated by w. For every point $v \in W$, let $d_{\mathfrak{F}}(v)$ be the minimal number of $\big(\bigcup_{\Diamond\in\text{MOD}} R_\Diamond\big)$-steps in which v is reachable from the root w. Consider the equivalence relation $\sim_{Sub(\varphi[\vec{i}/\vec{p_i}])}$. Two worlds stand in this equivalence relation if they satisfy the same subformulas of $\varphi[\vec{i}/\vec{p_i}]$. For any $\sim_{Sub(\varphi[\vec{i}/\vec{p_i}])}$-equivalence class $[v]$, choose a representative $f[v] \in [v]$ such that for any $v' \in [v]$ we have $d_{\mathfrak{F}}(f[v]) \leq d_{\mathfrak{F}}(v')$. Note that while $f[w] = w$, these representatives are in general not unique. Also note that for every $v \in W$ and $\psi \in Sub(\varphi[\vec{i}/\vec{p_i}])$, $\mathfrak{M}, v \models \psi$ iff $\mathfrak{M}, f[v] \models \psi$.

Let $W' = \{f[v] \mid v \in W\}$. Define the relation $R_{\Diamond}$ ($\Diamond \in$ MOD) on W' by putting $f[u]R_{\Diamond}f[v]$ iff there is a $v' \in [v]$ with $f[u]R_\Diamond v'$. Define a valuation V' on W' by letting $f[w] \in V'(p)$ iff $w \in V(p)$ for all $p \in Sub(\varphi[\vec{i}/\vec{p_i}])$. Let $\mathfrak{F}' = (W', R')$ and $\mathfrak{M}' = (\mathfrak{F}', V')$.

> **Claim 1:** For any $\psi \in Sub(\varphi[\vec{i}/\vec{p_i}])$ and a point $v \in W$, $\mathfrak{M}, f[v] \models \psi$ iff $\mathfrak{M}', f[v] \models \psi$
>
> **Proof of claim:** By the induction on the complexity of ψ. If ψ is a proposition letter, then the claim holds by the definition of V'. The Boolean cases are obvious. Finally, let $\psi = \Diamond\chi$.

[⇒] Suppose that $\mathfrak{M}, f[v] \models \Diamond\chi$. Then there is a point $u \in W$ such that $f[v]R_\Diamond u$ and $\mathfrak{M}, u \models \chi$. Since $\chi \in Sub(\varphi[\vec{i}/\vec{p_i}])$ and $u \sim_{Sub(\varphi[\vec{i}/\vec{p_i}])} f[u]$, we have that $\mathfrak{M}, f[u] \models \chi$. By the induction hypothesis, it follows that $\mathfrak{M}', f[u] \models \chi$. Finally, we have that $f[v]R_{\diamond} f[u]$, by the definition of $R_{\diamond}$. Hence, $\mathfrak{M}', f[v] \models' \Diamond\chi$.

[⇐] Suppose that $\mathfrak{M}', f[v] \models \Diamond\chi$. Then there is an $f[u] \in W'$ such that $f[v]R_{\diamond} f[u]$ and $\mathfrak{M}', f[u] \models \chi$. By the induction hypothesis, $\mathfrak{M}, f[u] \models \chi$. Also, by the definition of $R_{\diamond}$, there must be a $u' \in [u]$ such that $f[v]R_\Diamond u'$. Since $\chi \in Sub(\varphi[\vec{i}/\vec{p_i}])$ and $u' \sim_{Sub(\varphi[\vec{i}/\vec{p_i}])} f[u]$, it follows that $\mathfrak{M}, u' \models \chi$. We conclude that $\mathfrak{M}, f[v] \models \Diamond\chi$. ⊣

Let us define $d_{\mathfrak{F}'}$ similar to $d_{\mathfrak{F}}$. Note that $\mathfrak{F}'$ need not be point-generated anymore. For worlds $f[v] \in W'$ that are not reachable from $f[w] = w$, let $d_{\mathfrak{F}'}(f[v]) = \infty$.

Claim 2: $d_{\mathfrak{F}}(f[v]) \leq d_{\mathfrak{F}'}(f[v])$, for all $v \in W$

Proof of claim: If $d_{\mathfrak{F}'}(f[v]) = \infty$, the claim obviously holds. Otherwise, the proof proceeds by induction on $d_{\mathfrak{F}'}(f[v])$. The base case, with $d_{\mathfrak{F}'}(f[v]) = 0$, only applies if $f[v] = w$, in which case the claim clearly holds. Next, suppose $d_{\mathfrak{F}'}(f[v]) = n + 1$. By definition, there must be a path of the form

$$f[w] = w \xrightarrow{R'_{\diamond_1}} \cdots \xrightarrow{R'_{\diamond_n}} f[u] \xrightarrow{R'_{\diamond_{n+1}}} f[v]$$

It follows that $d_{\mathfrak{F}'}(f[u]) \leq n$, and hence by the induction hypothesis, $d_{\mathfrak{F}}(f[u]) \leq d_{\mathfrak{F}'}(f[u]) \leq n$. Since $f[u]R'_{\diamond_{n+1}} f[v]$, by the definition of $R'_{\diamond_{n+1}}$ we have that there is a $v' \in [v]$ such that $f[u]R_{\diamond_{n+1}} v'$. This implies that $d_{\mathfrak{F}}(v') \leq n + 1$. By the definition of f, we know that $d_{\mathfrak{F}}(f[v]) \leq d_{\mathfrak{F}}(v')$, because $v' \in [v]$. Therefore, $d_{\mathfrak{F}}(f[v]) \leq n + 1$. ⊣

Claim 3: For all $k = 1 \ldots n$, there is at most one world $f[v] \in W'$ such that $d_{\mathfrak{F}'}(f[v]) \leq md(\varphi)$ and $\mathfrak{M}', f[v] \models p_{i_k}$.

Proof of claim: Suppose $\mathfrak{M}', f[v] \models p_{i_k}$ and $\mathfrak{M}', f[u] \models p_{i_k}$, with $d_{\mathfrak{F}'}(f[v]), d_{\mathfrak{F}'}(f[u]) \leq md(\varphi)$. By Claim 2, $d_{\mathfrak{F}}(f[v]), d_{\mathfrak{F}}(f[u]) \leq md(\varphi)$. Furthermore, $\mathfrak{M}, f[v] \models p_{i_k}$ and $\mathfrak{M}, f[u] \models p_{i_k}$. By our initial assumption, $\mathfrak{M}, w \models \varphi^*$, hence $f[v] \sim_{Sub(\varphi)} f[u]$, which implies that $f[v] = f[u]$. ⊣

From Claim 1, we immediately deduce that $\mathfrak{M}', w \models \varphi[\vec{i}/\vec{p_i}]$. The valuation of $p_{i_1} \ldots p_{i_n}$ can be restricted to the worlds with depth $\leq md(\varphi)$ without affecting the truth of $\varphi[\vec{i}/\vec{p_i}]$ at w. In this way, by Claim 3, we make sure that every p_{i_k} is true at at most one world. Finally, applying the same argument as in the proof of Theorem 7.2.1, we conclude that the original hybrid formula φ is satisfiable. □

We will now proceed to the general case, for frame classes that are defined by finitely many shallow formulas, or, equivalently by a single shallow formula. Recall that a modal formula is closed if it contains no proposition letters.

7.4.2. THEOREM. *Let* K *be a frame class defined by a shallow modal formula* ψ_{K}*. Then an* $\mathcal{H}$*-formula* $\varphi(i_1, \ldots, i_n)$ *is satisfiable on* K *iff the modal formula*

$$\varphi^* \quad = \quad \varphi[\vec{i}/\vec{p_i}] \ \wedge \bigwedge_{\substack{1 \le k \le n \\ \psi \in \Sigma}} \Big(\langle \cup \rangle^{\le md(\varphi)} (p_{i_k} \wedge \psi) \to [\cup]^{\le md(\varphi)} (p_{i_k} \to \psi) \Big)$$

is satisfiable on K*, where* Σ *consists of all subformulas of* $\varphi[\vec{i}/\vec{p_i}]$ *plus all closed subformulas of* ψ_{K}*.*

Proof: We use the same construction as in the proof of Theorem 7.4.1, but now we use a richer filtration set, that includes also all closed subformulas of ψ_{K}. It suffices to show that the constructed frame $\mathfrak{F}'$ is in K. Let V be a valuation for $\mathfrak{F}'$, and let $x \in W'$ such that $(\mathfrak{F}', V), x \models \varphi$. Define V' such that $v \in V'(p)$ iff $f[v] \in V(p)$. We claim that for all shallow axioms χ of L and for all $v \in W$, $(\mathfrak{F}, V'), f[v] \models \chi$ iff $(\mathfrak{F}', V), f[v] \models \chi$.

This, we prove by induction on χ. Note that χ is shallow, and hence we may assume that χ is generated by the following recursive definition:

> $\chi ::= \top \mid p \mid \neg\chi \mid \chi_1 \wedge \chi_2 \mid \Diamond\psi$, where ψ is any Boolean combination of proposition letters and closed formulas (i.e., formulas containing no proposition letters).

The only non-trivial case in the induction is when χ is of the form $\Diamond\psi$ where ψ is a Boolean combination of proposition letters and closed formulas. In this case, we reason as follows.

[⇒] Suppose $(\mathfrak{F}, V'), f[v] \models \Diamond\psi$. Then there is a $u \in W$ such that $f[v]R_\Diamond u$ and $(\mathfrak{F}, V'), u \models \psi$. By the definition of V' and the fact that all closed subformulas of ψ are in the filtration set, it follows that $(\mathfrak{F}', V), f[u] \models \psi$. By definition of $R_\Diamond$, $f[v]R_\Diamond f[u]$. Hence, $(\mathfrak{F}', V), f[v] \models \Diamond\psi$.

[⇐] Suppose $(\mathfrak{F}', V), f[v] \models \Diamond\psi$. Then there is an $f[u] \in W'$ such that $(\mathfrak{F}', V), f[u] \models \psi$ and $f[v]R_\Diamond f[u]$. By definition of $R_\Diamond$, there is a $u' \in [u]$ such that $f[v]R_\Diamond u'$. By the definition of V' and the fact that all closed subformulas of ψ are in the filtration set, it follows that $(\mathfrak{F}, V'), u' \models \psi$. Hence, $(\mathfrak{F}, V'), f[v] \models \Diamond\psi$. □

Note that the length of φ^* is in general exponential in the length of φ, but polynomial in case of uni-modal languages.

7.4.3. COROLLARY. *Let* K *be a frame class defined by a shallow modal formulas* ψ_{K}. *Then a* $\mathcal{H}$*-formula* $\varphi(i_1,\ldots,i_n)$ *is valid on* K *iff*

$$\varphi^* \quad = \quad \bigwedge_{\substack{1\leq k\leq n\\ \psi\in\Sigma}} \Big(\langle\cup\rangle^{\leq md(\varphi)}(p_{i_k}\wedge\psi)\rightarrow[\cup]^{\leq md(\varphi)}(p_{i_k}\rightarrow\psi)\Big)\rightarrow\varphi[\vec{i}/\vec{p_i}]$$

is valid on K*, where* Σ *consists of all subformulas of* $\neg\varphi[\vec{i}/\vec{p_i}]$ *plus all closed subformulas of* ψ_{K}.

7.5 From $\mathcal{H}(@)$ to $\mathcal{M}$ in case of shallow axioms

In order to translate $\mathcal{H}(@)$-formulas to modal formulas, we again need to make use of an extra modality. We follow the same notation conventions as in the previous two sections.

7.5.1. THEOREM. *Let* K *be a frame class defined by a shallow modal formula* ψ_{K}. *Let* φ *be any* $\mathcal{H}(@)$*-formula in @-normal form. Then* φ *is satisfiable on* K *iff the modal formula*

$$\varphi^* = \varphi[\vec{i}/\vec{p_i}, @_i/\diamondsuit(p_i\wedge\cdot)] \;\wedge\; \bigwedge_{1\leq k\leq n}\diamondsuit p_{i_k} \;\wedge$$
$$\bigwedge_{\substack{1\leq k\leq n\\ \psi\in\Sigma}}\Big(\diamondsuit^{\leq 1}\langle\cup\rangle^{\leq md(\varphi)}(p_{i_k}\wedge\psi) \;\rightarrow\; \boxdot^{\leq 1}[\cup]^{\leq md(\varphi)}(p_{i_k}\rightarrow\psi)\Big)$$

is satisfiable on $\mathsf{Exp}_1(\mathsf{K})$*, where* Σ *consists of the subformulas of* $\varphi[\vec{i}/\vec{p_i}]$ *containing no satisfaction operators plus the closed subformulas of* ψ_{K}.

Proof: For simplicity, we only prove the case for uni-modal logics. The proof generalizes straightforwardly to the general case.

[$\Rightarrow$] Suppose $(W,R_\diamond,V),w\models\varphi$ with $(W,R_\diamond)\in\mathsf{K}$. Let $R_{\diamondsuit}$ be the total relation on the domain of $\mathfrak{F}$, and let V' be any valuation that agrees with V on all proposition letters occurring in φ, and such that $V'(p_{i_k})=V(i_k)$ for each nominal i_k. Then, clearly, $(W,R_\diamond,R_{\diamondsuit},V'),w\models\varphi[\vec{i}/\vec{p_i},@_i/\diamondsuit(p_{i_k}\wedge\cdot)]$. The truth of the remainder of φ^* follows directly from the construction of the model.

[$\Leftarrow$] Suppose $\mathfrak{M},w\models\varphi^*$ with $\mathfrak{M}=(\mathfrak{F},V)$ and $\mathfrak{F}=(W,R,R')\in\mathsf{Exp}_1(\mathsf{K})$. Let $\mathfrak{F}'=(W,R)$ and let $\mathfrak{M}'=(\mathfrak{F}',V)$. Note that $\mathfrak{F}'\in\mathsf{K}$. Next, let $\mathfrak{N}$ be submodel of $\mathfrak{M}'$ generated by $\{w\}\cup\{v\in W\mid wR'v\}$. By the truth of the second conjunct of φ^* at $(\mathfrak{M},w)$, every p_{i_k} is true somewhere in $\mathfrak{N}$. By the truth of the third conjunct of φ^* at $(\mathfrak{M},w)$, any two points in $\mathfrak{N}$ that satisfy the same p_{i_k} agree on all formulas in Σ (recall that no formula in this set contains any $\diamondsuit$-modality). Since $\mathfrak{N}$ is a generated submodel of $\mathfrak{M}'$, we also have that $\mathfrak{N}$ is based on a frame in K. From here, we proceed as in the proof of Theorem 7.4.2. □

7.5.2. Corollary. *Let* K *be a frame class defined by a shallow modal formula* ψ_{K}*. Let* φ *be any* $\mathcal{H}(@)$*-formula in @-normal form. Then* φ *is valid on* K *iff the modal formula*

$$\Big(\bigwedge_{1\leq k\leq n} \diamondsuit p_{i_k} \ \wedge \ \bigwedge_{\substack{1\leq k\leq n\\ \psi\in\Sigma}} \big(\diamondsuit^{\leq 1}\langle\cup\rangle^{\leq md(\varphi)}(p_{i_k}\wedge\psi) \ \rightarrow \ \boxdot^{\leq 1}[\cup]^{\leq md(\varphi)}(p_{i_k}\rightarrow\psi)\big)\Big)$$

$$\rightarrow \quad \varphi[\vec{i}/\vec{p_i}, @_i/\diamondsuit(p_i\wedge\cdot)]$$

is valid on $\mathsf{Exp}_1(\mathsf{K})$*, where* Σ *consists of the subformulas of* $\neg\varphi[\vec{i}/\vec{p_i}]$ *containing no satisfaction operators plus the closed subformulas of* ψ_{K}*.*

This concludes the chapter. The translations that were introduced in this chapter will be put to use in the next chapter.

Chapter 8

Transfer

It is a natural question to ask which properties of modal logics are preserved when nominals, satisfaction operators and/or the global modality are added to the language. For example, given that the basic modal language has uniform interpolation with respect to the class of all frames, does it follow that $\mathcal{H}$ and $\mathcal{H}(@)$ have uniform interpolation with respect to the class of all frames? Given that PDL has an ExpTime-complete satisfiability problem, does it follow that PDL with nominals also has an ExpTime-complete satisfiability problem? Such questions are addressed in this chapter.

As far as the author is aware, Gargov and Goranko [46] were the first this question explicitly. They ask, for instance, whether the finite model property and decidability transfer.

Areces et al. [5] showed that when nominals are added to the basic tense logic, the complexity of the satisfiability problem increases from PSpace to ExpTime. This can be seen as a first negative transfer results. In Chapter 6 of this thesis, we saw that $\mathcal{H}$ does not have the Beth property relative to the class of all frames. Since the basic modal languages *does* have the Beth property relative to the class of all frames, and hence this gives us a second negative transfer result. In Section 8.1, we will show that, likewise, decidability, the finite model property, complexity and Kripke completeness do not transfer.

Some positive results are obtained in Section 8.2, where we show that complexity, (uniform) interpolation over proposition letters, and completeness transfer for a particular class of logics. The proofs make use of the translations provided in the previous chapter.

Some of the results reported in this chapter are taken from [15].

8.1 Negative results

Areces, Blackburn and Marx [5] show that complexity does not transfer in general (under the usual complexity-theoretic assumptions). Let K_t be the class of bi-modal frames (W, R_1, R_2) on which R_1 and R_2 are each others converse (as in the basic tense logic).

8.1.1. Proposition ([5]). *$\mathcal{H}(@)$-satisfiability for K_t is* ExpTime*-complete.*

Note that the corresponding modal problem is only PSpace-complete [21]. A uni-modal example of non-transfer of complexity is the following. Let K_B be the class of symmetric uni-modal frames.

8.1.2. Proposition. *$\mathcal{H}$-satisfiability for K_B is* ExpTime*-complete.*

Proof: For any modal formula φ, let $\varphi' = i \wedge \Box\neg i \wedge \Box\Box(\neg i \to \Diamond i) \wedge \Box\varphi^{\neg i}$, where i is any nominal and $\varphi^{\neg i}$ is obtained from φ by relativising all modalities with $\neg i$. One can easily see that φ' holds at a world w in a symmetric model $\mathfrak{M}$ iff φ holds globally in the submodel of $\mathfrak{M}$ generated by w, minus the world w itself. It follows that, on symmetric frames, φ' is satisfiable iff φ is globally satisfiable. The global satisfiability problem for modal formulas on the class of symmetric frames is ExpTime-complete [33]. Hence, the satisfiability problem for $\mathcal{H}$ on the class of symmetric frames is ExpTime-hard. That the problem is inside ExpTime follows from the fact that converse PDL with nominals is in ExpTime [36] □

Again, the corresponding modal problem is only PSpace-complete [33].

Next, we will show that decidability and the finite model property do not transfer either. Consider the bi-modal language with modalities $\Diamond_1$ and $\Diamond_2$, and let Σ consist of the following modal Sahlqvist axioms.

$\bigwedge_{1\leq k\leq 3} \Diamond_1 p_k \to \bigvee_{1\leq k<l\leq 3} \Diamond_1(p_k \wedge p_l)$	(at most 2 R_1-successors)
$\bigwedge_{1\leq k\leq 4} \Diamond_1\Diamond_1 p_k \to \bigvee_{1\leq k<l\leq 4} \Diamond_1\Diamond_1(p_k \wedge p_l)$	(at most 3 two-step R_1-successors)
$p \to \Box_2\Diamond_2 p$	(R_2 is symmetric)

8.1.3. Proposition. *$\mathbf{K}_{\mathcal{M}}\Sigma$ has the finite model property and is decidable.*

Proof: First, consider the uni-modal logic axiomatized by the first two axioms. This logic is complete for a class of frames that is closed under taking subframes, and it has the bounded width property: no point has more than two successors. It follows that this logic has the finite model property and is decidable. Second, consider the uni-modal logic given by the last axiom. This logic, which is complete for the class of symmetric frames, has the finite model property [31] and its satisfiability problem is complete for PSpace [33]. Since decidability and the finite model property are preserved under taking fusions [45], the result follows. □

8.1.4. Proposition. *$\mathbf{K}_{\mathcal{H}}\Sigma$ is undecidable and lacks the finite model property.*

Table 8.1: Axioms of Θ used to disprove transfer of completeness.

$\Diamond_1\Diamond_1 p \to \Diamond_1 p$
$\Diamond_1 p \wedge \Diamond_1 q \to \Diamond_1(p \wedge \Diamond_1 q) \vee \Diamond_1(q \wedge \Diamond_1 p) \vee \Diamond_1(p \wedge q)$
$\Diamond_1 p \to \Diamond_1(p \wedge \neg\Diamond_1 p)$
$\Diamond_2 p \to \Box_2 p$
$\Diamond_3 p \to \Box_3 p$
$p \to \Box_2\Diamond_1 p$
$\Diamond_1\Diamond_2 p \vee \Diamond_2\Diamond_1 p \to p \vee \Diamond p$
$\Diamond_3 p \to \Diamond_1 p$
$\Box_3\Box_3\bot$
$\Box_3\Box_1\Box_3\bot$
$\Diamond_1\Diamond_3 p \wedge \Diamond_3 q \to \Diamond_1\Diamond_3(p \wedge \Diamond q)$

Proof: For any uni-modal formula φ with modality $\Diamond_1$, let $\varphi^* = i \wedge \Box_2\neg i \wedge \Box_2\Box_1\Diamond_2 i \wedge \Box_2\varphi^{\neg i}$. One can easily see that φ' holds at a world w in a model $\mathfrak{M}$ iff φ holds globally in the submodel of $\mathfrak{M}$ generated by the set of R_2-successors of w along R_1, minus the world w itself. It follows that φ' is satisfiable iff φ is globally satisfiable. Global satisfiability of modal formulas on the class K_{23} is undecidable [91]. It follows that $\mathbf{K}_{\mathcal{H}}\Sigma$ is undecidable, and hence, since it is finitely axiomatizable, that it lacks the finite model property. $\Box$

Via the Thomason simulation [70, Chapter 6], this can be turned into a uni-modal example. We leave out the technical details. Incidentally, Proposition 8.1.4 also shows that the finite model property and decidability do not transfer under taking fusions of hybrid logics, since the same arguments as in the proof of Proposition 8.1.3 show that the corresponding hybrid logics have the finite model property and are decidable. Transfer of complexity under fusions of hybrid logics is actually an interesting topic by itself, and has been investigated in [48].

Finally, we will show that Kripke completeness does not transfer in general from a modal logic $\mathbf{K}_{\mathcal{M}}\Sigma$ to the hybrid logics $\mathbf{K}^+_{\mathcal{H}}\Sigma$, $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma$ and $\mathbf{K}^+_{\mathcal{H}(\mathsf{E})}\Sigma$.

8.1.5. THEOREM. *There is a set of modal formulas Σ such that $\mathbf{K}_{\mathcal{M}}\Sigma$ is Kripke complete but $\mathbf{K}^+_{\mathcal{H}}\Sigma$, $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma$ and $\mathbf{K}^+_{\mathcal{H}(\mathsf{E})}\Sigma$ are not.*

Proof: Let Θ be the set of axioms given in Table 8.1. Kracht [70, Section 9.6] proves the following, in order to establish that Kripke completeness does not transfer under addition of a global modality.

1. $\mathbf{K}_{\mathcal{M}}\Theta$ is Kripke complete.

2. $\mathsf{Fr}(\Theta) \models \mathsf{A}(\Diamond_3\top \to \Diamond_2\Diamond_3\top) \to \neg\Diamond_3\top$

3. $\mathsf{A}(\Diamond_3\top \to \Diamond_2\Diamond_3\top) \wedge \Diamond_3\top$ is satisfiable on a discrete general Θ-frame

We introduce a fourth modality. Let $\Theta' = \Theta \cup \{p \to \Box_4\Diamond_4 p\}$. Since Kripke completeness transfers under fusions, $\mathbf{K}_{\mathcal{M}}\Theta'$ is Kripke complete. Let χ be the formula

$$i \wedge \Diamond i \wedge \bigwedge_{1\leq k\leq 3} (\Box_k\Diamond_4 i) \wedge \Box_4(\Diamond_3\top \to \Diamond_2\Diamond_3\top) \to \Box_4(\neg\Diamond_3\top)$$

Let $\mathfrak{F}'$ be the expansion of $\mathfrak{F}$ with a fourth relation, viz. the total relation on the domain of $\mathfrak{F}$ (it is clear that $\mathfrak{F}'$ satisfies the requirements of a general frame with respect to the fourth relation). Also, it is easily seen that $\mathsf{Fr}(\Theta') \models \chi$, that $\neg\chi$ is satisfiable on $\mathfrak{F}'$ and that $\mathfrak{F}' \models \Theta'$. It follows by Theorem 5.3.16 that $\mathbf{K}^+_{\mathcal{H}}\Sigma \not\models \chi$, $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma \not\models \chi$ and $\mathbf{K}^+_{\mathcal{H}(\mathsf{E})}\Sigma \not\models \chi$. □

8.2 Positive results for logics admitting filtration

A note on the complexity of fusions

In Chapter 7, we showed that the satisfiability problem of $\mathcal{H}(@)$-formulas on certain frame classes K can be reduced to the satisfiability problem of modal formulas on the class $\mathsf{Exp}_1(\mathsf{K})$, which is the fusion of K with the class of all unimodal frames. While many properties of logics are preserved under taking fusions, complexity is in general not preserved. However, the translation in question uses only a very restricted class of fusion formulas. Call a modal formula of the fusion language (i.e., possibly containing the modality $\diamond$) *very simple* if no occurrence of $\diamond$ is in the scope of any other modal operator, including $\diamond$ itself (and $\boxdot$, which is shorthand for $\neg\diamond\neg$). We will show that satisfiability of very simple fusion formulas on $\mathsf{Exp}_1(\mathsf{K})$ is reducible to satisfiability of modal formulas on K. To make this precise, we use *non-deterministic polynomial time conjunctive reductions*, as defined in Appendix B.

8.2.1. LEMMA. *Let* K *be a class of frames. Then satisfiability of very simple fusion formulas on* $\mathsf{Exp}_1(\mathsf{K})$ *is non-deterministic polynomial time conjunctive reducible to satisfiability of modal formulas on* K*.*

Proof: Let a very simple formula φ of the fusion language be given. By definition, φ is generated by the following recursive definition:

$\psi ::= \chi \mid \diamond\chi \mid \neg\psi \mid \psi_1 \wedge \psi_2$,
where χ is any formula not containing the $\diamond$ modality.

Let Γ be the set of all subformulas χ of φ that contain no occurrences of $\diamond$. In order to test whether φ is satisfiable on $\mathsf{Exp}_1(\mathsf{K})$, we perform the following procedure.

1. Non-deterministically choose subsets $S_1, S_2 \subseteq \Gamma$. Intuitively, the formulas in S_1 are supposed to be the ones that are true in the actual world, whereas the formulas in S_2 are supposed to be the ones that are true in some $R_\diamond$-successor of the actual world.

2. Check in polynomial time whether φ holds under the chosen interpretation of the subformulas of φ given by S_1, S_2. This can be done using any polynomial model checking algorithm for propositional logic.

3. Check if the choices of S_1, S_2 are consistent with respect to K: (1) Check K-satisfiability of $\bigwedge_{\chi \in S_1} \chi \wedge \bigwedge_{\chi \in \Gamma \setminus S_1} \neg\chi$, and (2) for each $\chi \in S_2$, check the K-satisfiability of $\chi \wedge \bigwedge_{\chi' \in \Gamma \setminus S_2} \neg\chi'$. All in all, the number of tests is polynomial in the length of φ, and each test involves a formula of length polynomial in the length of φ.

If φ is satisfiable on $\mathsf{Exp}_1(\mathsf{K})$, then clearly, S_1 and S_2 can be picked in such a way that all tests in 2 and 3. succeed. Conversely, if these tests all succeed, then a model for φ based on a frame in $\mathsf{Exp}_1(\mathsf{K})$ is easily constructed. □

The usual complexity classes NP, PSPACE, (N)k-EXPTIME and k-EXPSPACE ($k > 0$), are closed under non-deterministic polynomial time conjunctive reductions.

Complexity

As immediate corollary of simulations introduced in the previous section (together with Lemma 8.2.1), we obtain the following.

8.2.2. THEOREM. *Let* K *be any frame class that satisfies one of the following conditions.*

1. K *admits polynomial filtration and has a master modality.*

2. K *is uni-modal and defined by a shallow modal formula.*

Then the satisfiability problem for $\mathcal{H}$ *on* K *is polynomially reducible to the satisfiability problem for modal logic on* K*, and the satisfiability problem for* $\mathcal{H}(@)$ *on* K *is non-deterministic polynomial time conjunctive reducible to the satisfiability problem for modal logic on* K*.*

8.2.3. THEOREM. *Let* K *be any frame class that admits polynomial filtration. Then the satisfiability problem for* $\mathcal{H}(\mathsf{E})$ *is polynomially reducible to the satisfiability problem for* $\mathcal{M}(\mathsf{E})$ *on* K*.*

Interpolation

Recall that a modal logic admits *simple* filtration if it admits filtration and for every formula φ we have $\Sigma_\varphi = Sub(\varphi)$. For logics admitting simple filtration, interpolation transfers.

8.2.4. THEOREM. *Let* K *be any frame class satisfying one of the following conditions:*

(a) K *has a master modality and admits simple filtration.*
(b) K *is defined by a shallow modal formula.*

If modal logic has interpolation on K*, then* $\mathcal{H}$ *and* $\mathcal{H}(@)$ *have interpolation over proposition letters on* K*.*

8.2.5. THEOREM. *Let* K *be any frame class that admits simple filtration. If* $\mathcal{M}(\mathsf{E})$ *has interpolation on* K*, then* $\mathcal{H}(\mathsf{E})$ *has interpolation over proposition letters on* K*.*

Proof: By way of example, we prove Theorem 8.2.4(a) for the language $\mathcal{H}$. All other cases are proved similarly (using the fact that interpolation transfers under fusion and replacing occurrences of $\diamondsuit$ in the obtained interpolant by $\bigvee_{i\in\textsc{nom}} @_i(\cdot)$ where necessary).

Suppose $\mathsf{K} \models \varphi \to \psi$, where $\varphi \to \psi$ is a $\mathcal{H}$-formula containing nominals $i_1, \ldots, i_n$. Let $\Sigma = Sub(\neg(\varphi \to \psi)[\vec{i}/\vec{p_i}])$ By Corollary 7.2.2,

$$\mathsf{K} \models \Big(\bigwedge_{\substack{1\le k\le n\\ \chi\in\Sigma}} \big(\oplus(p_{i_k} \wedge \chi) \to \boxplus(p_{i_k} \to \chi)\big)\Big) \to (\varphi[\vec{i}/\vec{p_i}] \to \psi[\vec{i}/\vec{p_i}])$$

The antecedent of this formula says that *for all* $1 \le k \le n$*, if two worlds* w *and* w' *in the model both satisfy* p_{i_k}*, then* w *and* w' *satisfy exactly the same formulas in* Σ. Note that every formula in Σ is a Boolean combination of subformulas of $\varphi[\vec{i}/\vec{p_i}]$ and $\psi[\vec{i}/\vec{p_i}]$. Hence, to say that w and w' satisfy the same formulas in Σ is equivalent to saying that they satisfy the same subformulas of $\varphi[\vec{i}/\vec{p_i}]$ and $\psi[\vec{i}/\vec{p_i}]$. Therefore,

$$\mathsf{K} \models \Big(\bigwedge_{\substack{1\le k\le n\\ \chi\in Sub(\varphi[\vec{i}/\vec{p_i}])\cup Sub(\psi[\vec{i}/\vec{p_i}])}} \big(\oplus(p_{i_k} \wedge \chi) \to \boxplus(p_{i_k} \to \chi)\big)\Big) \to (\varphi[\vec{i}/\vec{p_i}] \to \psi[\vec{i}/\vec{p_i}])$$

By some simple syntactic manipulations, we obtain from this that

$$\mathsf{K} \models \Big(\bigwedge_{\substack{1\le k\le n\\ \chi\in Sub(\varphi[\vec{i}/\vec{p_i}])}} \oplus(p_{i_k} \wedge \chi) \to \boxplus(p_{i_k} \to \chi)\Big) \wedge \varphi[\vec{i}/\vec{p_i}] \to$$

$$\Big(\bigwedge_{\substack{1\le k\le n\\ \chi\in Sub(\psi[\vec{i}/\vec{p_i}])}} \oplus(p_{i_k} \wedge \chi) \to \boxplus(p_{i_k} \to \chi)\Big) \to \psi[\vec{i}/\vec{p_i}]$$

Let ϑ be the modal interpolant for this implication. Note that, apart from $p_{i_1}, \dots p_{i_n}$, ϑ only contains proposition letters that occur both in φ and in ψ. By uniform substitution of formulas for proposition letters, we obtain that

$$\mathsf{K} \models \Big(\bigwedge_{\substack{1 \le k \le n \\ \chi \in Sub(\varphi)}} \oplus(i_k \wedge \chi) \to \boxplus(i_k \to \chi) \Big) \wedge \varphi \to \vartheta[\vec{p_i}/\vec{i}]$$

and

$$\mathsf{K} \models \vartheta[\vec{p_i}/\vec{i}] \to \Big(\bigwedge_{\substack{1 \le k \le n \\ \chi \in Sub(\psi)}} \oplus(i_k \wedge \chi) \to \boxplus(i_k \to \chi) \Big) \to \psi$$

Since $\oplus(i \wedge \chi) \to \boxplus(i \to \chi)$ is valid for any i and χ, it follows that $\mathsf{K} \models \varphi \to \vartheta[\vec{p_i}/\vec{i}]$ and $\mathsf{K} \models \vartheta[\vec{p_i}/\vec{i}] \to \psi$. Finally, as we mentioned above, all proposition letters occurring in $\vartheta[\vec{p_i}/\vec{i}]$ occur both in φ and in ψ. We conclude that $\vartheta[\vec{p_i}/\vec{i}]$ is an interpolant for $\varphi \to \psi$. □

Uniform interpolation

Let us define uniform interpolation for hybrid logics as follows.

8.2.6. DEFINITION. *Let $\mathcal{L}$ be one of the languages $\mathcal{H}, \mathcal{H}(@), \mathcal{H}(\mathsf{E})$, and let K be a frame class. $\mathcal{L}$ has uniform interpolation over proposition letters on K if for each formula φ and finite set of proposition letters $P \subseteq \text{PROP}(\varphi)$, there is a formula φ_P such that*

- $\text{PROP}(\varphi_P) \subseteq P$, *and*
- *For all formulas ψ, if $\text{PROP}(\psi) \cap \text{PROP}(\varphi) \subseteq P$ and $\text{NOM}(\psi) \subseteq \text{NOM}(\varphi)$, then $\models_{\mathsf{K}} \varphi \to \psi$ iff $\models_{\mathsf{K}} \varphi_P \to \psi$.*

When restricted to modal formulas, this definition becomes the usual definition of uniform interpolation for modal logics [100, 49]. Note that, in contrast to what one might expect, according to this definition the uniform interpolant φ_P does not apply in case the consequent ψ contains nominals not occurring in φ.

8.2.7. THEOREM. *Let K be any frame class satisfying one of the following conditions:*

(a) K has a master modality and admits simple filtration.

(b) K is defined by a shallow modal formula.

If modal logic has uniform interpolation on K then $\mathcal{H}$ and $\mathcal{H}(@)$ have uniform interpolation over proposition letters on K

8.2.8. THEOREM. *Let* K *be any frame class that admits simple filtration. If* $\mathcal{M}(\mathsf{E})$ *has uniform interpolation on* K *then* $\mathcal{H}(\mathsf{E})$ *has uniform interpolation over proposition letters on* K*.*

Proof: By way of example, we prove Theorem 8.2.7(a) for the language $\mathcal{H}$. All other cases are proved similarly (using the fact that uniform interpolation transfers under fusion and replacing occurrences of $\diamondplus$ in the obtained uniform interpolant by $\bigvee_{i \in \textsc{nom}} @_i(\cdot)$ where necessary).

Let φ be an $\mathcal{H}$-formula with nominals $i_1, \ldots, i_n$, and let $P \subseteq \textsc{prop}(\varphi)$. Let $P' = P \cup \{p_{i_1}, \ldots, p_{i_k}\}$. Let ϑ be a uniform interpolant over P' of the modal formula

$$\varphi^* \quad = \quad \varphi[\vec{i}/\vec{p_i}] \wedge \bigwedge_{\substack{1 \leq k \leq n \\ \chi \in Sub(\varphi[\vec{i}/\vec{p_i}])}} \big(\diamondplus(p_{i_k} \wedge \chi) \rightarrow \boxplus(p_{i_k} \rightarrow \chi)\big)$$

We claim that $\vartheta[\vec{p_i}/\vec{i}]$ is a uniform interpolant of the $\mathcal{H}$-formula φ over P. Consider any hybrid formula ψ with $\textsc{prop}(\psi) \cap \textsc{prop}(\varphi) \subseteq P$ and $\textsc{nom}(\psi) \subseteq \textsc{nom}(\varphi)$. We will show that $\mathsf{K} \models \varphi \rightarrow \psi$ iff $\mathsf{K} \models \vartheta[\vec{p_i}/\vec{i}] \rightarrow \psi$.

[$\Rightarrow$] Suppose $\mathsf{K} \models \varphi \rightarrow \psi$. Let $\Sigma = Sub(\neg(\varphi \rightarrow \psi)[\vec{i}/\vec{p_i}])$, By Corollary 7.2.2, we have that

$$\mathsf{K} \models \Big(\bigwedge_{\substack{1 \leq k \leq n \\ \chi \in \Sigma}} \big(\diamondplus(p_{i_k} \wedge \chi) \rightarrow \boxplus(p_{i_k} \rightarrow \chi)\big)\Big) \rightarrow \Big(\varphi[\vec{i}/\vec{p_i}] \rightarrow \psi[\vec{i}/\vec{p_i}]\Big)$$

The same argument as in the proof of Theorem 8.2.4 shows that

$$\mathsf{K} \models \Big(\varphi[\vec{i}/\vec{p_i}] \wedge \bigwedge_{\substack{1 \leq k \leq n \\ \chi \in Sub(\varphi[\vec{p_i}/\vec{i}])}} \diamondplus(p_{i_k} \wedge \chi) \rightarrow \boxplus(p_{i_k} \rightarrow \chi)\Big) \rightarrow$$
$$\Big(\bigwedge_{\substack{1 \leq k \leq n \\ \chi \in Sub(\psi[\vec{p_i}/\vec{i}])}} \diamondplus(p_{i_k} \wedge \chi) \rightarrow \boxplus(p_{i_k} \rightarrow \chi)\Big) \rightarrow \psi[\vec{i}/\vec{p_i}]$$

or, equivalently,

$$\mathsf{K} \models \varphi^* \rightarrow \Big(\bigwedge_{\substack{1 \leq k \leq n \\ \chi \in Sub(\psi[\vec{p_i}/\vec{i}])}} \big(\diamondplus(p_{i_k} \wedge \chi) \rightarrow \boxplus(p_{i_k} \rightarrow \chi)\big) \rightarrow \psi[\vec{i}/\vec{p_i}]\Big)$$

Since ϑ is a uniform interpolant for φ^* over P', it follows that

$$\mathsf{K} \models \vartheta \rightarrow \Big(\bigwedge_{\substack{1 \leq k \leq n \\ \chi \in Sub(\psi[\vec{p_i}/\vec{i}])}} \big(\diamondplus(p_{i_k} \wedge \chi) \rightarrow \boxplus(p_{i_k} \rightarrow \chi)\big) \rightarrow \psi[\vec{i}/\vec{p_i}]\Big)$$

By uniform substitution of formulas for proposition letters, we obtain that

$$\mathsf{K} \models \vartheta[\vec{p_i}/\vec{i}] \to \Big(\bigwedge_{\substack{1\leq k\leq n \\ \chi\in Sub(\psi)}} \big(\Diamond\!\!\!\!+ (i_k \wedge \chi) \to \boxplus(i_k \to \chi)\big) \to \psi\Big)$$

Since $\Diamond\!\!\!\!+(i \wedge \chi) \to \boxplus(i \to \chi)$ is valid for any i and χ, it follows that $\mathsf{K} \models \vartheta[\vec{p_i}/\vec{i}] \to \psi$.

[⇐] Suppose $\mathsf{K} \models \vartheta[\vec{p_i}/\vec{i}] \to \psi$. Since ϑ is a uniform interpolant for φ^*, $\mathsf{K} \models \varphi^* \to \vartheta$. It follows bu uniform substitution that

$$\mathsf{K} \models \Big(\varphi \wedge \bigwedge_{\substack{1\leq k\leq n \\ \chi\in Sub(\varphi)}} \Diamond\!\!\!\!+(i_k \wedge \chi) \to \boxplus(i_k \to \chi)\Big) \to \vartheta[\vec{p_i}/\vec{i}]$$

Since $\models \Diamond\!\!\!\!+(i \wedge \chi) \to \boxplus(i \to \chi)$ for any i and χ, it follows that $\mathsf{K} \models \varphi \to \vartheta[\vec{p_i}/\vec{i}]$, and therefore, $\mathsf{K} \models \varphi \to \psi$. □

It is known that the modal logics **K**, **GL**, **S5** and **Grz** have uniform interpolation (see [100] and [49]). From Theorem 8.2.7 and the fact that **GL** and **S5** admit simple filtration, it follows immediately that the corresponding $\mathcal{H}$-logics $\mathbf{K}_{\mathcal{H}}$, $\mathbf{S5}_{\mathcal{H}}$ and $\mathbf{GL}_{\mathcal{H}}$ have uniform interpolation over proposition letters, as well as the $\mathcal{H}(@)$-logics $\mathbf{K}_{\mathcal{H}(@)}$, $\mathbf{S5}_{\mathcal{H}(@)}$ and $\mathbf{GL}_{\mathcal{H}(@)}$ (here, with $\mathbf{S5}_{\mathcal{H}}$ we mean the $\mathcal{H}$-logic of the frame class defined by **S5**, and similar for other logics). **Grz** does not admit simple filtration. Nevertheless, we will now show that the construction used in the proof of Theorem 8.2.7 can be applied to $\mathbf{Grz}_{\mathcal{H}}$ and $\mathbf{Grz}_{\mathcal{H}(@)}$ as well.

8.2.9. THEOREM. *$\mathbf{Grz}_{\mathcal{H}}$ and $\mathbf{Grz}_{\mathcal{H}(@)}$ have uniform interpolation over proposition letters.*

Proof: **Grz** admits filtration in the following manner [23]:

> For any formula φ, let $\Sigma_\varphi = Sub(\{\varphi\} \cup \{\Diamond(\neg\psi \wedge \Diamond\psi) : \Diamond\psi \in Sub(\varphi)\})$. For any model $\mathfrak{M} = (W, R, V)$ based on a **Grz**-frame $\mathfrak{F}$, let $\mathfrak{M}_{\Sigma_\varphi} = (W/_{\sim_{\Sigma_\varphi}}, R_{\Sigma_\varphi}, V_{\Sigma_\varphi})$, where $[w]R_{\Sigma_\varphi}[v]$ if $[w] = [v]$ or the following two conditions hold:
>
> 1. for every $\Diamond\psi \in \Sigma_\varphi$, $v \models \psi \vee \Diamond\psi$ implies $w \models \Diamond\psi$, and
> 2. there exists $\Diamond\psi \in \Sigma_\varphi$ with $w \models \Diamond\psi$ and $v \not\models \Diamond\psi$.
>
> Then $\mathfrak{M}_{\Sigma_\varphi}$ is again based on a (finite) **Grz**-frame, and for all $w \in W$ and $\psi \in \Sigma_\varphi$, $\mathfrak{M}_{\Sigma_\varphi}, [w] \models \psi$ iff $\mathfrak{M}, w \models \psi$.

Now consider again the proof of Theorem 8.2.7. The crux of the proof lies in the fact that the filtration set $Sub(\neg(\varphi \to \psi))$ can be split up in two disjoint sets, such that every formula in the first set contains only symbols that occur in φ, and every formula in the second set contains only symbols that occur in ψ. As we will now show, the same holds for the filtration set of **Grz**. To see this, note that

$$\begin{array}{rcl} \Sigma_{\neg(\varphi\to\psi)} & = & Sub(\{\neg(\varphi \to \psi)\} \cup \{\Diamond(\neg\chi \wedge \Diamond\chi) \mid \Diamond\chi \in Sub(\neg(\varphi \to \psi))\}) \\ & = & Sub(\{\neg(\varphi \to \psi)\} \cup \\ & & \quad \{\Diamond(\neg\chi \wedge \Diamond\chi) \mid \Diamond\chi \in Sub(\varphi)\} \cup \{\Diamond(\neg\chi \wedge \Diamond\chi) \mid \Diamond\chi \in Sub(\psi)\}) \\ & = & \{\neg(\varphi \to \psi), \varphi \to \psi\} \cup Sub(\varphi) \cup Sub(\{\Diamond(\neg\chi \wedge \Diamond\chi) \mid \Diamond\chi \in Sub(\varphi)\}) \\ & & \qquad \cup\ Sub(\psi) \cup Sub(\{\Diamond(\neg\chi \wedge \Diamond\chi) \mid \Diamond\chi \in Sub(\psi)\}) \\ & = & \{\neg(\varphi \to \psi), \varphi \to \psi\} \cup \Sigma_\varphi \cup \Sigma_\psi \end{array}$$

Hence, every formula in $\Sigma_{\neg(\varphi\to\psi)}$ is a Boolean combination of formulas in Σ_φ and Σ_ψ. The same argument as in the proof of Theorem 8.2.7 shows that $\mathcal{H}$ and $\mathcal{H}(@)$ have uniform interpolation over proposition letters on the frame class defined by **Grz**. □

Completeness

The last topic that we will address is transfer of Kripke completeness: if $\mathbf{K}_\mathcal{M}\Sigma$ is Kripke complete, does it follow that $\mathbf{K}^+_\mathcal{H}\Sigma$, $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma$ and $\mathbf{K}^+_{\mathcal{H}(\mathsf{E})}\Sigma$ are Kripke complete? In Section 8.1, we saw already that the answer is negative. However, for the class of logics that we are considering in this section, a positive answer can be given. First, we need three lemmas.

8.2.10. LEMMA. *For every $\mathcal{H}(@)$-formula φ there is an $\mathcal{H}(@)$-formula ψ in @-normal form, such that $\mathbf{K}_{\mathcal{H}(@)} \vdash \varphi \leftrightarrow \psi$.*

Proof: Follows from Theorem 3.3.2 together with Corollary 5.4.2. □

For a given sequence of nominals $i_1, \ldots, i_n \in$ NOM, we will use $@_{\vec{i}}(\cdot)$ as a shorthand for $\bigwedge_{1\leq k\leq n} @_{i_k}\psi$. Semantically, $@_{\vec{i}}(\cdot)$ can be seen as a modality, and more precisely as a box. Indeed, as the following lemma shows, the distribution axiom and necessitation rule for this compound modality are derivable in $\mathbf{K}_{\mathcal{H}(@)}$.

8.2.11. LEMMA. *The following are derivable in $\mathbf{K}_{\mathcal{H}(@)}$, for any sequence of nominals $i_1, \ldots, i_n \in$ NOM.*

1. $\vdash @_{\vec{i}}(p \to q) \to @_{\vec{i}}p \to @_{\vec{i}}q$
2. *If* $\vdash \varphi$ *then* $\vdash @_{\vec{i}}\varphi$.

Proof: The first claim follows from Corollary 5.4.2 since $\models @_{\vec{i}}(p \to q) \to @_{\vec{i}}p \to @_{\vec{i}}q$. As for the second, if $\vdash \varphi$, then by the Nec$_@$ rule, $\vdash @_{i_k}\varphi$ for $k \leq n$. It follows that $\vdash @_{\vec{i}}\varphi$. □

8.2.12. Lemma. *If a modal logic* $\mathbf{K}_{\mathcal{M}}\Sigma$ *is complete with respect to a frame class* K *that has a master modality* $\oplus$, *then* $\mathbf{K}_{\mathcal{H}}\Sigma \vdash \oplus(i \to p) \to \boxplus(i \to p)$.

Proof: Recall from the definition of having a master modality that $\oplus\,\psi$ is shorthand for $\varphi(\psi)$, for some fixed formula $\varphi(p)$ containing no proposition letters besides p. Let n be the modal depth of φ, and let MOD be the (finite) set of modalities occurring in φ. Then the following is holds.

$$\mathsf{K} \models \quad \oplus\, p \quad \leftrightarrow \bigvee_{\substack{\Diamond_1,\ldots\Diamond_k \in \text{MOD} \\ k \leq n}} \Diamond_1 \cdots \Diamond_k p$$

Since $\mathbf{K}_{\mathcal{M}}\Sigma$ is complete for K and $\mathbf{K}_{\mathcal{H}}\Sigma$ extends $\mathbf{K}_{\mathcal{M}}\Sigma$, it follows that

$$\vdash_{\mathbf{K}_{\mathcal{H}}\Sigma} \quad \oplus\, p \quad \leftrightarrow \bigvee_{\substack{\Diamond_1,\ldots\Diamond_k \in \text{MOD} \\ k \leq md(\varphi)}} \Diamond_1 \cdots \Diamond_k p$$

By definition, $\mathbf{K}_{\mathcal{H}}\Sigma \models (Nom)$. It follows by some simple modal reasoning that $\mathbf{K}_{\mathcal{H}}\Sigma \models \oplus(i \wedge p) \to \boxplus(i \to p)$. □

We are now ready to prove our transfer result for completeness. With Kripke completeness, we will mean weak completeness: a formula is consistent in the logic iff it is satisfiable on a frame in the frame class defined by the logic.

8.2.13. Theorem. *If* $\mathbf{K}_{\mathcal{M}}\Sigma$ *is complete with respect to a frame class* K *that admits filtration and has a master modality, then* $\mathbf{K}_{\mathcal{H}}\Sigma$ *and* $\mathbf{K}_{\mathcal{H}(@)}\Sigma$ *are complete with respect to* K.

8.2.14. Theorem. *Let* Σ *be any set of modal formulas. If* $\mathbf{K}_{\mathcal{M}(\mathsf{E})}\Sigma$ *is Kripke complete and admits filtration, then* $\mathbf{K}_{\mathcal{H}(\mathsf{E})}\Sigma$ *is Kripke complete.*

Proof: We will give the proof Theorem 8.2.13 for the languages $\mathcal{H}$ and $\mathcal{H}(@)$. The proof of Theorem 8.2.14 is similar.

- First, let us prove Theorem 8.2.13 for $\mathcal{H}$. Let K be the class of frames defined by Σ, and suppose $\mathsf{K} \models \varphi$, for some $\mathcal{H}$-formula $\varphi(i_1, \ldots, i_n)$. Let $\Sigma = \Sigma_{\neg\varphi[\vec{i}/\vec{p_i}]}$. By Corollary 7.2.2,

$$\mathsf{K} \models \Big(\bigwedge_{\substack{1\leq k\leq n \\ \psi \in \Sigma}} \oplus(p_{i_k} \wedge \psi) \to \boxplus(p_{i_k} \to \psi) \Big) \to \varphi[\vec{i}/\vec{p_i}]$$

  and hence, by Kripke completeness,

$$\mathbf{K}_{\mathcal{M}}\Sigma \vdash \Big(\bigwedge_{\substack{1\leq k\leq n \\ \psi \in \Sigma}} \oplus(p_{i_k} \wedge \psi) \to \boxplus(p_{i_k} \to \psi) \Big) \to \varphi[\vec{i}/\vec{p_i}]$$

Since $\mathbf{K}_{\mathcal{H}}\Sigma$ extends $\mathbf{K}_{\mathcal{M}}\Sigma$, we have that

$$\mathbf{K}_{\mathcal{H}}\Sigma \vdash \Big(\bigwedge_{\substack{1\le k\le n\\ \psi\in\Sigma}} \diamondplus(p_{i_k} \wedge \psi) \to \boxplus(p_{i_k} \to \psi) \Big) \to \varphi[\vec{i}/\vec{p_i}]$$

By closure under substitution,

$$\mathbf{K}_{\mathcal{H}}\Sigma \vdash \Big(\bigwedge_{\substack{1\le k\le n\\ \psi\in\Sigma}} \diamondplus(i_k \wedge \psi) \to \boxplus(i_k \to \psi) \Big) \to \varphi$$

By Lemma 8.2.12 and closure under uniform substitution, $\mathbf{K}_{\mathcal{H}}\Sigma \vdash \diamondplus(i \wedge \chi) \to \boxplus(i \to \chi)$ for all i and χ, hence we conclude that $\mathbf{K}_{\mathcal{H}}\Sigma \vdash \varphi$.

- The proof of Theorem 8.2.13 for $\mathcal{H}(@)$ is more involved. Let K be the class of frames defined by Σ, and suppose $\mathsf{K} \models \varphi$, for some $\mathcal{H}(@)$-formula $\varphi(i_1, \ldots, i_n)$. By Lemma 8.2.10, we may assume that φ is in @-normal form. By Corollary 7.3.2,

$$\mathsf{Exp}_1(\mathsf{K}) \models \Big(\bigwedge_{1\le k\le n} \diamonddot p_{i_k} \;\wedge \bigwedge_{\substack{1\le k\le n\\ \psi\in\Sigma}} \big((\diamonddot)\, \diamondplus(p_{i_k} \wedge \psi) \;\to\; (\boxdot)\, \boxplus(p_{i_k} \to \psi)\big)\Big)$$
$$\to \quad \varphi[\vec{i}/\vec{p_i}, @_i/\diamonddot(p_i \wedge \cdot)]$$

Since completeness transfers under fusion and $\mathbf{K}_{\mathcal{M}}\Sigma$ is complete, the fusion logic $\mathbf{K}_{\mathcal{M}}\Sigma \oplus \mathbf{K}_{\mathcal{M}}$ is frame complete, and hence

$$\mathbf{K}_{\mathcal{M}}\Sigma \oplus \mathbf{K}_{\mathcal{M}} \vdash \Big(\bigwedge_{1\le k\le n} \diamonddot p_{i_k} \;\wedge \bigwedge_{\substack{1\le k\le n\\ \psi\in\Sigma}} \big((\diamonddot)\, \diamondplus(p_{i_k} \wedge \psi) \;\to\; (\boxdot)\, \boxplus(p_{i_k} \to \psi)\big)\Big)$$
$$\to \quad \varphi[\vec{i}/\vec{p_i}, @_i/\diamonddot(p_i \wedge \cdot)]$$

Replacing the proposition letters of the form p_{i_k} by the corresponding nominal i_k and replacing subformulas of the form $\diamonddot\psi$ by $@_{\vec{i}}\ \psi$, we obtain via Lemma 8.2.11 that

$$\mathbf{K}_{\mathcal{H}(@)}\Sigma \vdash \Big(\bigwedge_{1\le k\le n} @_{\vec{i}}\ i_k \;\wedge \bigwedge_{\substack{1\le k\le n\\ \psi\in\Sigma}} \big((@_{\vec{i}}\ )\, \diamondplus(i_k \wedge \psi) \;\to\; (\neg @_{\vec{i}}\ \neg)\, \boxplus(i_k \to \psi)\big)\Big)$$
$$\to \quad \varphi[@_i/@_{\vec{i}}\ (i \wedge \cdot)]$$

From this, it easily follows that $\mathbf{K}_{\mathcal{H}(@)}\Sigma \vdash \varphi$. □

Transfer of completeness for logics axiomatized by shallow modal formulas can be obtained in the same way, but already follows from Corollary 5.4.2.

Part II

More expressive languages

Chapter 9

The bounded fragment and $\mathcal{H}(@, \downarrow)$

The bounded fragment is a fragment of first-order logic containing formulas that use only a restricted form of quantification. More precisely, a first-order formula is bounded if it is built up from atomic formulas using the Boolean connectives and bounded quantifiation of the form $\exists x.(Rtx \wedge \varphi)$ and $\forall x.(Rtx \rightarrow \varphi)$, with t a term not containing the variable x.

Bounded formulas have been considered in the literature already for a long time. In set theory, where bounded quantifiers are of the form $\exists x.(x \in y \wedge \varphi)$ and $\forall x.(x \in y \rightarrow \varphi)$, the bounded fragment was introduced in 1965 by Levy [75], under the name Δ_0. Δ_0-formulas of set theory have the desirable property of being set-theoretically absolute, meaning that whether a Δ_0-formula $\varphi(x_1, \ldots, x_n)$ holds of sets $a_1, \ldots, a_n$ is independent of the universe of set theory in which $a_1, \ldots, a_n$ reside (cf. for instance [7]).

Bounded formulas have also been considered in the context of arithmetic, where bounded quantifiers are of the form $\exists x.(x \leq y \wedge \varphi)$ and $\forall x.(x \leq y \rightarrow \varphi)$. In fact, there is a separate field of research called *bounded arithmetic*, which is connected to complexity theory (in particular, to the polynomial hierarchy) and to propositional proof theory [27].

Around 1966, Feferman and Kreisel [40, 39] characterized the bounded fragment as the generated submodel invariant fragment of first-order logic. More precisely, they showed that a first-order formula is equivalent to a bounded formula iff it is invariant under generated submodels. Moreover, it was shown in [39] by means of a cut-free sequent calculus that the bounded fragment has interpolation.

The bounded fragment is also natural to consider from a modal logic perspective. In the preface of their book, Blackburn et al. [21] write:

> Slogan 2: Modal languages provide an internal, local perspective on relational structures.

It seems that the invariance under generated submodels is precisely what makes modal formulas local. The bounded fragment can therefore be seen as a natural generalization of the modal language. Indeed, in the late nineties hybrid logicians independently invented a language called $\mathcal{H}(@, \downarrow)$, that was subsequently proved to be a notational variant of the bounded fragment [55, 5]. Unaware of Feferman and Kreisel's early results, Areces, Blackburn and Marx [5, 20] characterized the expressivity of $\mathcal{H}(@, \downarrow)$ and proved that it has interpolation.

We already mentioned that the bounded fragment, and hence $\mathcal{H}(@, \downarrow)$, is the generated submodel invariant fragment of first-order logic. In Chapter 6, another characterization was given: $\mathcal{H}(@, \downarrow)$ is the smallest extension of $\mathcal{H}(@)$ with interpolation. A third characterization will be given in Chapter 12, where it will be shown that $\mathcal{H}(@, \downarrow)$ is precisely the intersection of first-order logic with second order propositional modal logic.

In this chapter, we will improve known results concerning frame definability, interpolation, and Beth definability for $\mathcal{H}(@, \downarrow)$. We also simplify the existing axiomatizations of $\mathcal{H}(@, \downarrow)$, thus obtaining the first axiomatization of $\mathcal{H}(@, \downarrow)$ that does not contain non-orthodox rules (i.e., rules with syntactic side conditions). Finally, we provide a number of complexity results, which show that $\mathcal{H}(@, \downarrow)$ has computational advantages over $\mathcal{L}^1$. The completeness results in Section 9.4 are taken from [19]. The complexity results in Section 9.6 are taken from [29].

9.1 Syntax and semantics

The hybrid language $\mathcal{H}(@, \downarrow)$ extends $\mathcal{H}(@)$ with state variables and the $\downarrow$-binder. Intuitively speaking, the state variables relate to first-order variables in the same way that nominal relate to first-order constants. The $\downarrow$-binder, like the first-order quantifiers, binds variables. It binds variables to the current world. For example, the formula $\downarrow x.\Diamond x$, which should read as "bind the variable x to the current world and evaluate $\Diamond x$", expresses that the current world is reflexive, i.e., $\mathfrak{M}, w \models \downarrow x.\Diamond x$ iff $(w, w) \in R_\Diamond$. Similarly, $\downarrow x.\Diamond\downarrow y.@_x\Box y$ expresses that the current world has exactly one $R_\Diamond$-successor.

Formally, let disjoint sets $\textsc{prop}, \textsc{nom}, \textsc{mod}$ be given as before, and let $\textsc{svar}$ be a countably infinite set of state variables. Then the formulas of $\mathcal{H}(@, \downarrow)$ are given by the following recursive definition.

$$\varphi ::= \top \mid p \mid t \mid \neg\varphi \mid \varphi \wedge \psi \mid \Diamond\varphi \mid @_t\varphi \mid \downarrow x.\varphi$$

where $p \in \textsc{prop}$, $t \in \textsc{nom} \cup \textsc{svar}$, $\Diamond \in \textsc{mod}$ and $x \in \textsc{svar}$. The interpretation of a state variables will is an element of the domain of the model, and the $\downarrow$-binder binds a variable to the world of evaluation. Formally, given a model $\mathfrak{M} = (W, (R_\Diamond)_{\Diamond\in\textsc{mod}}, V)$, an assignment for $\mathfrak{M}$ is a function $g : \textsc{svar} \to W$. Truth of a $\mathcal{H}(@, \downarrow)$-formula is defined relative to a model, assignment and world, as follows.

Table 9.1: Standard translation and hybrid translation

$ST_x(\top)$	$=$	$\top$
$ST_x(p)$	$=$	$P_p\,x$
$ST_x(i)$	$=$	$x = c_i$
$ST_x(y)$	$=$	$x = y$
$ST_x(\neg\varphi)$	$=$	$\neg ST_x(\varphi)$
$ST_x(\varphi \wedge \psi)$	$=$	$ST_x(\varphi) \wedge ST_x(\psi)$
$ST_x(\Diamond\varphi)$	$=$	$\exists y.(R_\Diamond xy \wedge ST_y(\varphi))$
$ST_x(@_i\varphi)$	$=$	$ST_y(\varphi)[y/i]$
$ST_x(@_y\varphi)$	$=$	$ST_y(\varphi)$
$ST_x(\downarrow y.\varphi)$	$=$	$ST_x(\varphi)[y/x]$

$HT(\top)$	$=$	$\top$
$HT(P_p t)$	$=$	$@_{t'} p$
$HT(R_\Diamond st)$	$=$	$@_{s'}\Diamond t'$
$HT(s = t)$	$=$	$@_{s'} t'$
$HT(\neg\varphi)$	$=$	$\neg HT(\varphi)$
$HT(\varphi \wedge \psi)$	$=$	$HT(\varphi) \wedge HT(\psi)$
$HT(\exists x.(Rtx \wedge \varphi))$	$=$	$@_{t'}\Diamond\downarrow x.HT(\varphi)$
$HT_x(\varphi)$	$=$	$\downarrow x.HT(\varphi)$

where t is a term of the form c_i or x, and t' denotes i or x, respectively.

$$
\begin{array}{lll}
\mathfrak{M}, g, w \models \top & & \\
\mathfrak{M}, g, w \models p & \text{iff} & w \in V(p) \\
\mathfrak{M}, g, w \models i & \text{iff} & w \in V(i) \\
\mathfrak{M}, g, w \models x & \text{iff} & w = g(x) \\
\mathfrak{M}, g, w \models \neg\varphi & \text{iff} & \mathfrak{M}, g, w \not\models \varphi \\
\mathfrak{M}, g, w \models \varphi \wedge \psi & \text{iff} & \mathfrak{M}, g, w \models \varphi \text{ and } \mathfrak{M}, g, w \models \psi \\
\mathfrak{M}, g, w \models \Diamond\varphi & \text{iff} & \text{there is a } v \in W \text{ such that } wR_\Diamond v \text{ and } \mathfrak{M}, g, v \models \varphi \\
\mathfrak{M}, g, w \models @_i\varphi & \text{iff} & \mathfrak{M}, g, v \models \varphi \text{ where } V(i) = \{v\} \\
\mathfrak{M}, g, w \models @_x\varphi & \text{iff} & \mathfrak{M}, g, g(x) \models \varphi \\
\mathfrak{M}, g, w \models \downarrow x.\varphi & \text{iff} & \mathfrak{M}, g[x := w], w \models \varphi
\end{array}
$$

If φ is a sentence of $\mathcal{H}(@,\downarrow)$ (i.e., a formula without free variables), then we will simply leave out the assignment and say $\mathfrak{M}, w \models \varphi$.

The modal depth of a $\mathcal{H}(@,\downarrow)$-formula φ, denoted by $md(\varphi)$, is defined as on page 8, not counting satisfaction operators or $\downarrow$-binders (i.e., $md(@_i\varphi) = md(\downarrow x.\varphi) = md(\varphi)$). For instance, $md(@_i\Diamond\downarrow y.@_i\Box y)$ is 2. It can be shown that, roughly speaking, a $\mathcal{H}(@,\downarrow)$-sentence of modal depth k, when evaluated at a world w, can only see the points in the model that are reachable from w or from a node named by a nominal, in at most k steps.

Recall the first-order correspondence language $\mathcal{L}^1$ defined in Section 3.2. As the standard translation ST given in Table 9.1 shows, $\mathcal{H}(@,\downarrow)$ is still a fragment of $\mathcal{L}^1$. In fact, this translation tells us a little bit more. Recall that a formula of $\mathcal{L}^1$ *bounded* if it is built up from atomic formulas using the Boolean connectives and bounded quantification of the form $\exists x.(Rtx \wedge \varphi)$ or $\forall x.(Rtx \rightarrow \varphi)$, where t is a term distinct from the variable x. Then the translation ST_x maps every $\mathcal{H}(@,\downarrow)$-sentence to a bounded formula of $\mathcal{L}^1$ that has x as its only free variable. A straightforward induction shows that for all $\mathcal{H}(@,\downarrow)$-sentence φ, models $\mathfrak{M}$ and worlds w, $\mathfrak{M}, w \models \varphi$ iff $\mathfrak{M} \models ST_x(\varphi)\ [w]$ [53, 16].

Not only does every $\mathcal{H}(@,\downarrow)$-sentence correspond to a bounded formula of $\mathcal{L}^1$, the converse holds as well. In fact, the translation HT_x given in Table 9.1 maps every bounded $\mathcal{L}^1$-formula with x as its only free variable to an $\mathcal{H}(@,\downarrow)$-sentence. Again, a simple inductive argument shows that the translation preserve truth, in the sense that $\mathfrak{M} \models \psi(x)\ [w]$ iff $\mathfrak{M}, w \models HT_x(\psi(x))$.

In other words, $\mathcal{H}(@,\downarrow)$ can be seen as a notational variant of the bounded fragment of $\mathcal{L}^1$ (given that we restrict attention to formulas with at most one free variable). In the remainder of this section, we will discuss a model theoretic characterization of this fragment.

9.2 Expressivity

Recall the notion of a *generated submodel* that was defined on page 48. A simple inductive argument shows that sentences of $\mathcal{H}(@,\downarrow)$ are invariant under generated submodels, in the following sense.

9.2.1. PROPOSITION. *Let $\mathfrak{M}$ be a generated submodel of $\mathfrak{N}$, let w be a world of $\mathfrak{M}$, and let φ be any $\mathcal{H}(@,\downarrow)$-sentence. Then $\mathfrak{M}, w \models \varphi$ iff $\mathfrak{N}, w \models \varphi$.*

This implies that properties such as $\exists x.Rxx$ or $\exists y.Ryx$, which are not invariant under generated submodels, are not expressible in $\mathcal{H}(@,\downarrow)$.

If we combine this observation with the fact that first-order formulas are invariant under potential isomorphisms (cf. Appendix A), we obtain the following result, where $\cong_p$ denotes the relation of potential isomorphism, $\equiv_{\mathcal{H}(@,\downarrow)}$ denotes the relation of indistinguishability with respect to $\mathcal{H}(@,\downarrow)$-sentences and $\mathfrak{M}_w$ denotes the submodel of $\mathfrak{M}$ generated by w.

9.2.2. PROPOSITION. *If $\mathfrak{M}_w, w \cong_p \mathfrak{N}_v, v$ then $\mathfrak{M}, w \equiv_{\mathcal{H}(@,\downarrow)} \mathfrak{N}, v$.*

This gives us a sufficient condition for $\mathcal{H}(@,\downarrow)$-indistinguishability. One might hope that it is also a necessary condition. Unfortunately, it is not the case, as the following proposition shows.

9.2.3. PROPOSITION. *There exist point-generated models $\mathfrak{M}_w$ and $\mathfrak{N}_v$ such that $\mathfrak{M}_w, w \equiv_{\mathcal{H}(@,\downarrow)} \mathfrak{N}_v, v$ and $\mathfrak{M}_w, w \not\cong_p \mathfrak{N}_v, v$.*

Proof: Consider the frames depicted in Figure 9.1. Let $\mathfrak{M}_w = (\mathfrak{F}, V)$ and $\mathfrak{N}_v = (\mathfrak{G}, U)$, where V and U are valuations that make all proposition letters false everywhere and that make all nominals true at the root. We will use w and v to refer to the roots of these frames. We will show that $\mathfrak{M}_w, w$ and $\mathfrak{N}_v, v$ satisfy the same $\mathcal{H}(@,\downarrow)$-sentences, but that they can be distinguished in first-order logic.

Let φ be any $\mathcal{H}(@,\downarrow)$-sentence, and let n be its modal depth. Let $\mathfrak{M}_w \restriction_n$ and $\mathfrak{N}_v \restriction_n$ be the submodels of $\mathfrak{M}_w$ and $\mathfrak{N}_v$ containing all points that are reachable from the root in at most n steps. Clearly, $\mathfrak{M}_w, w \models \varphi$ iff $\mathfrak{M}_w \restriction_n, w \models \varphi$ and

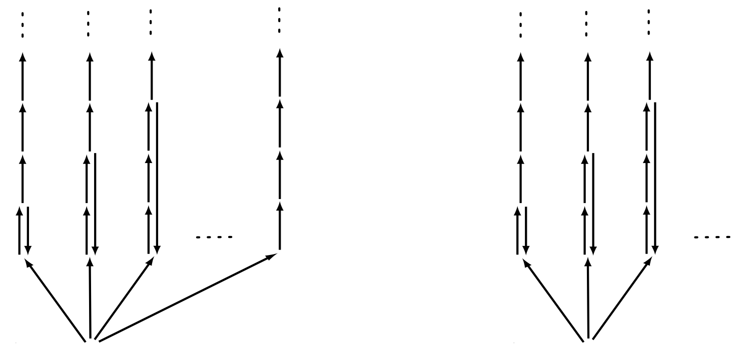


Figure 9.1: Counterexample to the converse of Proposition 9.2.2

$\mathfrak{N}_v, v \models \varphi$ iff $\mathfrak{N}_v \restriction_n, v \models \varphi$. Furthermore, it is not hard to see that $\mathfrak{M}_w \restriction_n$ and $\mathfrak{N}_v \restriction_n$ are isomorphic, and that the isomorphism connects w to v. Hence, $\mathfrak{M}_w \restriction_n, w \models \varphi$ iff $\mathfrak{N}_v \restriction_n, v \models \varphi$. We conclude that $\mathfrak{M}_w, w \models \varphi$ iff $\mathfrak{N}_v, v \models \varphi$.

Finally, note that the first-order sentence $\forall x \exists y.(Rxy \wedge \forall z.(Rzy \rightarrow z = x))$ distinguishes $\mathfrak{M}_w$ from $\mathfrak{N}_v$. It follows that $\mathfrak{M}_w, w \not\cong_p \mathfrak{N}_v, v$. □

Nevertheless, the converse of Proposition 9.2.2 holds on ω-saturated models.

9.2.4. PROPOSITION. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be ω-saturated models, with worlds w and v. Then $\mathfrak{M}, w \equiv_{\mathcal{H}(@,\downarrow)} \mathfrak{N}, v$ iff $\mathfrak{M}_w, w \cong_p \mathfrak{N}_v, v$.*

Proof: We will prove the result for uni-modal language. The generalization to formulas with multiple modalities is straightforward.

Proposition 9.2.2 gives the right-to-left direction. For the other direction, we proceed as follows. Call a finite partial isomorphism f between $\mathfrak{M}_w$ and $\mathfrak{N}_v$ $\mathcal{H}(@,\downarrow)$-preserving, if for all $\mathcal{H}(@,\downarrow)$-formulas $\varphi(x_1,\ldots,x_n)$ and for all $u_1,\ldots,u_n \in dom(f)$ it holds that $\mathfrak{M}_w, w \models \varphi\ [u_1,\ldots,u_n] \Leftrightarrow \mathfrak{N}_v, v \models \varphi\ [fu_1,\ldots,fu_n]$. Define F to be the set of all $\mathcal{H}(@,\downarrow)$-preserving finite partial isomorphisms. Clearly, F is non-empty (in particular, $\{(w,v)\}$ belongs to it). Furthermore, F is a potential isomorphism. We will only prove the first of the two symmetric extension conditions, since the proof for the other is analogous.

Let $f \in F$ with $dom(f) = \{a_1,\ldots,a_n\}$, and let $b \in \mathfrak{M}_w$. Then b is reachable either from w or from some point named by a nominal i, in a finite number of steps, say l steps. Let Σ be the set of all $\mathcal{H}(@,\downarrow)$-formulas $\varphi(x_1,\ldots,x_n,y)$ such that $\mathfrak{M}_w, w \models \varphi\ [a_1,\ldots,a_n,b]$.

Claim 1: There is a $b' \in \mathfrak{N}$ such that $\mathfrak{N}, v \models \Sigma\ [f(a_1),\ldots,f(a_n),b']$.

Proof of claim: By ω-saturatedness of $\mathfrak{N}$, it suffices to prove finite satisfiability, i.e., it suffices to prove that for each conjunction σ of elements of Σ there is a $b' \in \mathfrak{N}$ such that $\mathfrak{N}, v \models \sigma\ [fa_1, \dots, fa_n, b']$.

By assumption $\mathfrak{M}_w, w \models \sigma\ [a_1, \dots, a_n, b]$. Since $b \in \mathfrak{M}_w$, there is a nominal i and an $\ell \in \omega$ such that either $\mathfrak{M}_w, w \models \downarrow z.\Diamond^\ell \downarrow y.@_z\sigma\ [a_1, \dots, a_n]$ or $\mathfrak{M}_w, w \models \downarrow z.@_i\Diamond^\ell \downarrow y.@_z\sigma\ [a_1, \dots, a_n]$, where z is a fresh variable, not occurring in σ. By invariance under generated submodels and the fact that f is $\mathcal{H}(@,\downarrow)$-preserving, we obtain that either $\mathfrak{N}, v \models \downarrow z.\Diamond^\ell \downarrow y.@_z\sigma\ [fa_1, \dots, fa_n]$ or $\mathfrak{N}, v \models \downarrow z.@_i\Diamond^\ell \downarrow y.@_z\sigma\ [fa_1, \dots, fa_n]$. Hence, there is a point b' such that $\mathfrak{N}, v \models \sigma\ [fa_1, \dots, fa_n, b']$. ⊣

Since $b \in \mathfrak{M}_w$, there is a nominal i and an $\ell \in \omega$, such that $\Diamond^\ell y \in \Sigma$ or $@_i\Diamond^\ell y \in \Sigma$, and hence, $b' \in \mathfrak{N}_v$. By invariance under generated submodels, $\mathfrak{N}_v, v \models \Sigma\ [fa_1, \dots, fa_n]$. It follows that $f \cup \{(b, b')\}$ is a partial isomorphism and that $f \cup \{(b, b')\} \in F$. □

As a corollary of this, we obtain the following characterization, which was first proved by Feferman [39] using proof theoretic techniques, and later rediscovered by [5]. Below, we include another, very short proof by compactness.

9.2.5. Theorem ([39, 5]). *Let $\varphi(x)$ be an $\mathcal{L}^1$-formula with at most one free variable. Then the following are equivalent.*

1. *$\varphi(x)$ is equivalent to the standard translation of a $\mathcal{H}(@,\downarrow)$-sentence*
2. *$\varphi(x)$ is invariant under generated submodels.*

Proof: We will prove the result for uni-modal language. The generalization to formulas with multiple modalities is straightforward.

Suppose a first-order formula $\varphi(x)$ is invariant under generated submodels. Without loss of generality, we may assume that x does not occur as a bound variable in φ. Let $\textsc{cons}$ be the set of constants occurring in φ, and let P be a new predicate. Then the following holds (by invariance under generated submodels, 2x).

$$\{\forall y.(tR^n y \to Py) \mid t \in \textsc{cons}(\varphi) \cup \{x\} \text{ and } n \in \omega\} \models \varphi \leftrightarrow \varphi^P$$

where φ^P is the result of relativising all quantifiers in φ by P. By compactness, it follows that there is an $m \in \omega$ such that

$$\bigwedge_{t \in \textsc{cons} \cup \{x\}} \forall x.(tR^{\leq m} x \to Px) \models \varphi \leftrightarrow \varphi^P$$

Let φ' be the result of relativising all quantifiers in φ by the predicate $\lambda x.\big(\bigvee_{t \in \textsc{cons} \cup \{x\}} (c_l R^{\leq m} x)\big)$. It follows that $\models \varphi \leftrightarrow \varphi^*$ Finally, modulo some simple syntactic manipulations, φ^* is a bounded sentence. Hence, it is equivalent to a sentence of $\mathcal{H}(@,\downarrow)$. □

9.3 Frame definability

Like formulas of $\mathcal{H}$ and $\mathcal{H}(@)$, $\mathcal{H}(@,\downarrow)$-sentences are preserved under taking generated subframes. This follows from Proposition 9.2.1 by the same argument used in the proof of Proposition 4.2.1. On the other hand, they are no longer preserved under taking ultrafilter morphic images. For example, consider the class K of frames in which every point has a reflexive successor. It is well known that K does not reflect ultrafilter extensions [21], and hence it is not closed under ultrafilter morphic images. Nevertheless, the $\mathcal{H}(@,\downarrow)$-sentence $\Diamond{\downarrow}x.\Diamond x$ defines K.

In order to characterize the elementary frame classes definable in $\mathcal{H}(@,\downarrow)$, we need one more motion. For $k \in \omega$, we will say that a frame class K *reflects k-point generated subframes* if the following holds for all frames $\mathfrak{F}$: if every subframe of $\mathfrak{F}$ generated by at most k points is in $\mathfrak{K}$ then $\mathfrak{F} \in \mathsf{K}$. Similarly, we say that K *reflects finitely generated subframes* if for all frames $\mathfrak{F}$, if every subframe of $\mathfrak{F}$ generated by finitely many points is in K, then $\mathfrak{F} \in \mathsf{K}$. It is not hard to see that every frame class defined by a set of $\mathcal{H}(@,\downarrow)$-sentences reflects finitely generated subframes. Likewise, every frame class defined by a set of $\mathcal{H}(@,\downarrow)$-sentences containing in total at most k nominals reflects $k+1$-point generated subframes. These observation can be strengthened into the following characterization.

9.3.1. THEOREM. *A frame class K is definable by a pure $H(@,\downarrow)$-sentence with k nominals iff K is elementary and closed under generated subframes and K reflects $k+1$-point generated subframes.*

Proof: Fix distinct nominals, $i_1,\ldots,i_k$, and let $PTh(\mathsf{K})$ be the set of pure $H(@,\downarrow)$ formulas with these nominals valid on K. Let $\mathfrak{F} \models PTh(\mathsf{K})$. We will show that $\mathfrak{F} \in \mathsf{K}$. In this way, we show that $PTh(\mathsf{K})$ defines K, and hence, by compactness, K is defined by a single pure $\mathcal{H}(@,\downarrow)$ formula with at most k nominals.

Let $\mathfrak{F}^+$ be an ω-saturated elementary extension of $\mathfrak{F}$. Since $\mathfrak{F}$ and $\mathfrak{F}^+$ are elementary equivalent, in order to show that $\mathfrak{F} \in \mathsf{K}$ it suffices to show that $\mathfrak{F}^+ \in \mathsf{K}$. In fact, by the closure properties of K, it suffices to show that every $k+1$-point generated subframe of $\mathfrak{F}^+$ is in K.

Fix worlds $w_1,\ldots,w_{k+1}$ of $\mathfrak{F}^+$, and let $\mathfrak{F}^+_{w_1,\ldots,w_{k+1}}$ be the subframe of $\mathfrak{F}^+$ generated by $w_1,\ldots,w_{k+1}$. Note that $\mathfrak{F}^+_{w_1,\ldots,w_{k+1}} \models PTh(\mathsf{K})$. Let V be the valuation that assigns the worlds $w_1,\ldots,w_k$ to the nominals $i_1,\ldots,i_k$. Note that under this valuation, w_{k+1} is not necessarily named by a nominal. Also note that $(\mathfrak{F}^+, V)$ is an ω-saturated model (expanding an ω-saturated structure with finitely many constants always results is an ω-saturated structure). Let Δ be the set of pure $\mathcal{H}(@,\downarrow)$ sentences φ (in the language with the nominals $i_1,\ldots,i_k$) such that $(\mathfrak{F}^+_{w_1,\ldots,w_{k+1}}, V), w_{k+1} \models \varphi$.

Claim 1: Δ is satisfiable on K.

Proof of claim: By compactness (recall that K is elementary), it suffices to show that every finite conjunction δ of elements of Δ is satisfiable on K. But this follows immediately: δ is satisfiable on $\mathfrak{F}$ and $\mathfrak{F} \models PTh(\mathsf{K})$, hence $\neg\delta \notin PTh(\mathsf{K})$, i.e., δ is satisfiable on K. $\dashv$

Let $(\mathfrak{G}, U), v \models \Delta$ with $\mathfrak{G} \in \mathsf{K}$. Let $(\mathfrak{G}^+, U)$ be an ω-saturated elementary extension of $(\mathfrak{G}, U)$. Then, clearly, $(\mathfrak{G}^+, U), v \models \Delta$ and $\mathfrak{G}^+ \in \mathsf{K}$. Let $v_1, \ldots, v_k$ be the worlds named by the nominals $i_1, \ldots, i_k$ under the valuation U. For convenience, we will use v_{k+1} to refer to the world v. Let $\mathfrak{G}^+_{v_1,\ldots,v_{k+1}}$ be the subframe of $\mathfrak{G}^+$ generated by $v_1, \ldots, v_{k+1}$. Clearly, $(\mathfrak{G}^+_{v_1,\ldots,v_{k+1}}, U), v_{k+1} \models \Delta$ and $\mathfrak{G}^+_{v_1,\ldots,v_{k+1}} \in \mathsf{K}$. By Proposition 9.2.4, $\mathfrak{F}^+_{w_1,\ldots,w_{k+1}}$ and $\mathfrak{G}^+_{v_1,\ldots,v_{k+1}}$ are elementarily equivalent. It follows that $\mathfrak{F}^+_{w_1,\ldots,w_{k+1}} \in \mathsf{K}$. $\Box$

As special cases of this result, we obtain the following known result.

9.3.2. Corollary ([5]). *The following are equivalent for elementary frame classes* K.

1. K *is definable by a set of nominal-free* $H(@,\downarrow)$*-sentences*
2. K *is defined by a single pure nominal-free* $H(@,\downarrow)$*-sentence*
3. K *is closed under generated subframes and reflects point-generated subframes.*

(The direction of proof is, of course, $1 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1$.) Similarly, we obtain the following, which may be also seen as a characterization of the expressive power of universal closures of bounded first-order formula (in other words, of bounded first-order formulas with parameters).

9.3.3. Corollary. *The following are equivalent for elementary frame classes* K.

1. K *is definable by a set of* $H(@,\downarrow)$*-sentences*
2. K *is defined by a single pure* $H(@,\downarrow)$*-sentence*
3. K *is closed under generated subframes and reflects finitely generated subframes.*

Proof: Follows from Theorem 9.3.1. We only need to show that if a frame class K is closed under generated subframes and reflects finitely generated subframes, then there is a $k \in \omega$ such that K reflects k-point generated subframes. This is established by a compactness argument.

Suppose K is closed under generated subframes and reflects finitely generated subframes. Let S be a new binary predicate, and for $n \in \omega$, let $\varphi^{\bigvee_{1 \leq k \leq n} S(x_k,\cdot)}$

be the result of relativising all quantifiers in φ by the given predicate. Then the following entailment is valid:

$$\{\forall xy.(R^n xy \to S(x,y)) \mid n \in \omega\} \cup \{\forall x_1,\ldots,x_n.\varphi^{\bigvee_{1\leq k\leq n} S(x_k,\cdot)} \mid n \in \omega\} \models_{\mathsf{K}} \varphi$$

For, suppose the antecedent of the entailment holds in a frame $\mathfrak{F}$. Then, whenever a world v is reachable from a world w, $S(w,v)$ holds. Hence, by preservation under generated subframes, the second part of the antecedent implies that φ holds in every finitely generated subframe of $\mathfrak{F}$. Hence, by reflection of finitely generated subframes, $\mathfrak{F} \models \varphi$.

Applying compactness, we infer that there is a $k \in \omega$ such that

$$\{\forall xy.(R^n xy \to S(x,y)) \mid n \in \omega\} \cup \{\forall x_1,\ldots,x_n.\varphi^{\bigvee_{1\leq k\leq n} S(x_k,\cdot)} \mid n \leq k\} \models_{\mathsf{K}} \varphi$$

In other words, K reflects k-point generated subframes. □

The following three results indicate, each in their own way, that the above results cannot be easily generalized.

9.3.4. Proposition. *There is an elementary frame class* K *that is closed under generated subframes, but not definable by a set of* $\mathcal{H}(@,\downarrow)$ *sentences.*

Proof: Let K be the frame class defined by the first-order condition $\forall x \exists y.(Rxy \wedge \forall z.(Rzy \to z = x))$ ("every point has a successor with in-degree 1"). This class is easily seen to be closed under generated subframes. Now consider the frames given in Figure 9.1. We will show that every pure $\mathcal{H}(@,\downarrow)$-sentence valid on $\mathfrak{F}$ is also valid on $\mathfrak{G}$. Since $\mathfrak{F} \in \mathsf{K}$ and $\mathfrak{G} \notin \mathsf{K}$, it follows that K cannot be defined by a pure $\mathcal{H}(@,\downarrow)$-sentence, and hence, by Corollary 9.3.3, K cannot be defined by a set of $\mathcal{H}(@,\downarrow)$-sentences either.

Let $\varphi(i_1,\ldots,i_n)$ be any pure $\mathcal{H}(@,\downarrow)$-sentence such that $\mathfrak{F} \models \varphi$. Let V be any valuation for $\mathfrak{G}$, and let u be any world of $\mathfrak{G}$. Viewing $\mathfrak{G}$ as a submodel of $\mathfrak{F}$, we can think of V also as a valuation for $\mathfrak{F}$ (it simply makes all proposition letters and nominals false at the extra points). Let m be the modal depth of φ and let k be the length of the longest path from the root to u or to a world named by one of the nominals $i_1,\ldots,i_n$. Let $\mathfrak{F}\restriction_{k+m}$ and $\mathfrak{G}\restriction_{k+m}$ be the subframes of $\mathfrak{F}$ and $\mathfrak{G}$ containing all points reachable from the root in at most $k+m$ steps. An inductive argument shows that $(\mathfrak{F},V),u \models \varphi$ iff $(\mathfrak{F}\restriction_{k+m},V),u \models \varphi$, and that $(\mathfrak{G},V),u \models \varphi$ iff $(\mathfrak{G}\restriction_{k+m},V),u \models \varphi$. Furthermore, $(\mathfrak{F},\restriction_{k+m},V)$ and $(\mathfrak{G}\restriction_{k+m},V)$ are easily seen to be isomorphic, and the isomorphism connects w and v. Since $\mathfrak{F} \models \varphi$, it follows by the above considerations that $(\mathfrak{G},V),u \models \varphi$. Since we made no assumptions on V or u, we conclude that $\mathfrak{G} \models \varphi$. □

9.3.5. Proposition. *Consider finite models only. There is a first-order frame condition that is is closed under generated submodels and reflects point-generated subframes (with respect to finite models), but that is not definable by a pure nominal free* $\mathcal{H}(@,\downarrow)$*-sentence (with respect to finite models).*

Proof: Let K be the class of finite frames that are disjoint unions of directed cycles. It is easily seen that K is closed under generated submodels, reflects point-generated submodels (in the finite), and is defined (in the finite) by the first-order formula $\forall x.(\exists^{=1}y.Rxy \wedge \exists^{=1}y.Ryx)$.

Suppose for the sake of contradiction that K is defined (in the finite) by a pure nominal-free $\mathcal{H}(@,\downarrow)$-formula φ with modal depth k. Consider the following two frames: $\mathfrak{F} = (\{0,\ldots,k\}, \{(n,n+1) \mid n < k\} \cup \{(k,0)\})$ and $\mathfrak{G} = (\{-1,0,\ldots,k\}, \{(n,n+1) \mid n < k\} \cup \{(k,0)\})$. A straightforward Ehrenfeucht-Fraïssé style argument shows that φ cannot distinguish between these two frames. However, $\mathfrak{F} \in \mathsf{K}$ and $\mathfrak{G} \notin \mathsf{K}$. This contradicts the fact that φ defines K.

Note that K can be defined using nominals: it is defined by $(\Diamond i \to \Box i) \wedge (@_i\Diamond k \wedge @_j\Diamond k \to @_i j)$. □

9.3.6. Proposition. *There is a monadic Π^1_1-definable frame class K that is closed under generated subframes and reflects point-generated subframes, such that K is not defined by a set of $\mathcal{H}(@,\downarrow)$-sentences.*

Proof: Let K be the class of (possibly infinite) disjoint unions of directed cycles. This class is closed under generated subframes, reflects point-generated subframes and is defined by the monadic Π^1_1-sentence $\forall x.(\exists^{=1}y.Rxy \wedge \exists^{=1}y.Ryx) \wedge \forall P.(\exists xy.(Px \wedge Rxy \wedge \neg Py) \to \exists xy.(\neg Px \wedge Rxy \wedge Py))$. Consider any set of $\mathcal{H}(@,\downarrow)$-sentence Σ such that $\mathsf{K} \models \varphi$ for all $\varphi \in \Sigma$. We will show that $(\mathbb{N}, succ) \models \varphi$ for all $\varphi \in \Sigma$. Since $(\mathbb{N}, succ) \notin \mathsf{K}$, it then follows that Σ does not define K.

Let V be any valuation for $(\mathbb{N}, succ)$, let $n \in \mathbb{N}$ and let $\varphi \in \Sigma$. Let $m = \max(\{n\} \cup \bigcup_{i \in \textsc{nom}(\varphi)} V(i))$, where $\textsc{nom}(\varphi)$ is the set of nominals occurring in φ. Let k be the modal depth of φ. Consider frame $\mathfrak{G} = (\{0,\ldots,m+k\}, \{(\ell,\ell+1) \mid \ell < m+k\} \cup \{(m+k,0)\})$, and let V' be the restriction of V to $\mathfrak{G}$. Since $\mathfrak{G} \in \mathsf{K}$, we have that $\mathfrak{G}, V', n \models \varphi$. It follows by an inductive argument that $\mathfrak{F}, V, n \models \varphi$. □

9.4 Axiomatizations and completeness

In this section, we give two axiomatizations for $\mathcal{H}(@,\downarrow)$. We show these axiomatizations, as well as extensions of them by means of pure axioms, are strongly complete for the relevant frame classes. The first axiomatization is obtained by extending $\mathbf{K}^+_{\mathcal{H}(@)}$ with a simple axiom scheme. The second axiomatization improves on the first one, since it does not contain any non-orthodox rules (besides the substitution rule).

9.4.1. Definition. *For any set of $\mathcal{H}(@,\downarrow)$-formulas Σ, $\mathbf{K}^I_{\mathcal{H}(@,\downarrow)}\Sigma$ is the smallest set containing all axioms in Table 9.2 and Σ, closed under the rules in Table 9.2.*

Table 9.2: Axioms and inference rules of $\mathbf{K}^{I}_{\mathcal{H}(@,\downarrow)}$

All axioms and inference rules of $\mathbf{K}^{+}_{\mathcal{H}(@)}$, plus	
(DA)	$\vdash @_i(\downarrow s.\varphi \leftrightarrow \varphi[s := i])$

Table 9.3: Axioms and inference rules of $\mathbf{K}^{II}_{\mathcal{H}(@,\downarrow)}$

All axioms of $\mathbf{K}^{+}_{\mathcal{H}(@)}$, plus	
(DA)	$\vdash @_i(\downarrow s.\varphi \leftrightarrow \varphi[s := i])$
$(Name_{\downarrow})$	$\vdash \downarrow s.@_s\varphi \rightarrow \varphi$, provided s does not occur in φ
$(BG_{\downarrow})$	$\vdash @_i\Box\downarrow x.@_i\Diamond x$
(MP)	If $\vdash \varphi \rightarrow \psi$ and $\vdash \varphi$ then $\vdash \psi$
$(Subst)$	If $\vdash \varphi$ then $\vdash \varphi^{\sigma}$, provided that σ is safe for φ
(Nec)	If $\vdash \varphi$ then $\vdash \Box\varphi$, for $\Box \in$ MOD
$(Nec_{@})$	If $\vdash \varphi$ then $\vdash @_i\varphi$
$(Nec_{\downarrow})$	If $\vdash \varphi$ then $\vdash \downarrow s.\varphi$

$\mathbf{K}^{II}_{\mathcal{H}(@,\downarrow)}\Sigma$ *is the smallest set containing all axioms in Table 9.3 and* Σ*, closed under the rules in Table 9.3*

Both $\mathbf{K}^{I}_{\mathcal{H}(@,\downarrow)}$ and $\mathbf{K}^{II}_{\mathcal{H}(@,\downarrow)}$ have a substitution rule, which allows replacement of terms (i.e., nominals or variables) by terms and formulas by formulas. The usual restrictions apply, to prevent free variables from becoming accidentally bound.

First, we will prove completeness of $\mathbf{K}^{I}_{\mathcal{H}(@,\downarrow)}$.

9.4.2. LEMMA. *Every* $\mathbf{K}^{I}_{\mathcal{H}(@,\downarrow)}\Sigma$*-consistent set* Γ *can be extended to a maximal* $\mathbf{K}^{I}_{\mathcal{H}(@,\downarrow)}\Sigma$*-consistent set* Γ^{+} *such that*

1. *One of the elements of* Γ^{+} *is a nominal*
2. *For all* $@_i\Diamond\varphi \in \Gamma$ *there is a nominal* j *such that* $@_i\Diamond j \in \Gamma$ *and* $@_j\varphi \in \Gamma$*.*

Proof: Analogous to the proof of Lemma 5.3.12. □

9.4.3. THEOREM. *Let* Σ *be any set of pure* $\mathcal{H}(@,\downarrow)$*-sentences.* $\mathbf{K}^{I}_{\mathcal{H}(@,\downarrow)}\Sigma$ *is sound and strongly complete for the class of frames defined by* Σ*.*

Proof: First, note that, by Corollary 5.4.2 and the fact that $\mathbf{K}^{I}_{\mathcal{H}(@,\downarrow)}\Sigma$ extends $\mathbf{K}_{\mathcal{H}(@)}$, the following validities are derivable in $\mathbf{K}^{I}_{\mathcal{H}(@,\downarrow)}$.

(a) $\vdash @_jk \rightarrow (@_j\psi \leftrightarrow @_k\psi)$

(b) $\vdash @_j(\psi_1 \wedge \psi_2) \leftrightarrow @_j\psi_1 \wedge @_j\psi_2$

(c) $\vdash @_j\neg\psi \leftrightarrow \neg @_j\psi$
(d) $\vdash @_j @_k\psi \leftrightarrow @_k\psi$
(e) $\vdash @_j\Diamond k \wedge @_k\psi \rightarrow @_j\Diamond\psi$

Now, let Γ be any $\mathbf{K}^I_{\mathcal{H}(@,\downarrow)}\Sigma$ consistent set of $\mathcal{H}(@,\downarrow)$-formulas. Let Γ^+ be a maximal $\mathbf{K}^I_{\mathcal{H}(@,\downarrow)}\Sigma$-consistent set of $\mathcal{H}(@,\downarrow)$-formulas extending Γ that satisfies the conditions of Lemma 9.4.2. For all nominals i, let $[i] = \{j \mid @_i j \in \Gamma^+\}$. Let $\mathfrak{M} = (W, (R_\Diamond)_{\Diamond\in\text{MOD}}, V)$, where

$$\begin{array}{lcl} W & = & \{[i] \mid i \text{ is a nominal occurring in } \Gamma^+\} \\ R_\Diamond & = & \{([i],[j]) \mid @_i\Diamond j \in \Gamma^+\} \\ V(p) & = & \{[i] \mid @_i p \in \Gamma^+\} \\ V(i) & = & \{[i]\} \end{array}$$

We will show that Γ is satisfied at a point in $\mathfrak{M}$ and that the underlying frame of $\mathfrak{M}$ validates Σ.

Claim 1: For all $\mathcal{H}(@,\downarrow)$-formulas φ and nominals i, $\mathfrak{M}, [i] \models \varphi$ iff $@_i\varphi \in \Gamma^+$.

Proof of claim: A straightforward induction on φ, using the properties of Γ^+ and (a) – (e). For the inductive step for formulas of the form $\downarrow x.\psi$, we use the fact that Γ^+ contains all instances of the the (*DA*) axiom scheme. $\dashv$

It follows that $\mathfrak{M}, [i] \models \Gamma^+$, for $i \in \Gamma^+$ (recall that one of the elements of Γ^+ is a nominal). Since $\mathfrak{M}$ is a named model (i.e., every point is named by a nominal) and Γ^+ contains all substitution instances of elements of Σ, all formulas in Σ are valid on the underlying frame of $\mathfrak{M}$. We conclude that Γ is satisfiable on the class of frames defined by Σ. $\Box$

Next, let us consider the second axiomatization, $\mathbf{K}^{II}_{\mathcal{H}(@,\downarrow)}$. Note that $\mathbf{K}^{II}_{\mathcal{H}(@,\downarrow)}$ differs from $\mathbf{K}^I_{\mathcal{H}(@,\downarrow)}$ only in that the ($Name_@$) and (*BG*) rules are replaced by corresponding axioms ($Name_\downarrow$) and ($BG_\downarrow$), and the rule ($Nec\downarrow$) is added.

9.4.4. Theorem. *Let Σ be any set of pure $\mathcal{H}(@,\downarrow)$-sentences. $\mathbf{K}^{II}_{\mathcal{H}(@,\downarrow)}\Sigma$ is sound and strongly complete for the class of frames defined by Σ.*

Proof: We will show that the ($Name_@$) and (*BG*) rule are derivable in $\mathbf{K}^{II}_{\mathcal{H}(@,\downarrow)}\Sigma$. It then follows that $\mathbf{K}^{II}_{\mathcal{H}(@,\downarrow)}\Sigma$ extends $\mathbf{K}^I_{\mathcal{H}(@,\downarrow)}\Sigma$, and hence is strongly complete.

First, let us consider the ($Name_@$) rule. Suppose $\mathbf{K}^{II}_{\mathcal{H}(@,\downarrow)}\Sigma \vdash @_i\varphi$, where the nominal i does not occur in φ. Let s be a variable not occurring in φ. By the rules (*Subst*) and ($Nec_\downarrow$), $\mathbf{K}^{II}_{\mathcal{H}(@,\downarrow)}\Sigma \vdash \downarrow s.@_s\varphi$. Hence, by the ($Name_\downarrow$) axiom and the rule (*MP*), $\mathbf{K}^{II}_{\mathcal{H}(@,\downarrow)}\Sigma \vdash \varphi$

Next, let us consider the (*BG*) rule. Suppose $\mathbf{K}^{II}_{\mathcal{H}(@,\downarrow)}\Sigma \vdash @_i\Diamond j \rightarrow @_j\varphi$, where j is a nominal distinct from i, and j does not occur in φ. By the (*Agree*) axiom, $\mathbf{K}^{II}_{\mathcal{H}(@,\downarrow)}\Sigma \vdash @_j @_i \Diamond j \to @_j \varphi$. By some simple modal reasoning using the (*Selfdual*) axiom, we obtain that $\mathbf{K}^{II}_{\mathcal{H}(@,\downarrow)}\Sigma \vdash @_j(@_i \Diamond j \to \varphi)$. By (*DA*) and the (*Name*$_@$) rule, which we already showed to be derivable, $\mathbf{K}^{II}_{\mathcal{H}(@,\downarrow)}\Sigma \vdash \downarrow x @_i \Diamond x \to \varphi)$. By (*Nec*) and (*K*), we obtain from this that $\mathbf{K}^{II}_{\mathcal{H}(@,\downarrow)}\Sigma \vdash \Box \downarrow x @_i \Diamond x \to \Box\varphi$. Similarly, by (*Nec*$_@$) and (*K*$_@$), $\mathbf{K}^{II}_{\mathcal{H}(@,\downarrow)}\Sigma \vdash @_i \Box \downarrow x @_i \Diamond x \to @_i \Box \varphi$. Finally, by the (*BG*$_\downarrow$) axiom, $\mathbf{K}^{II}_{\mathcal{H}(@,\downarrow)}\Sigma \vdash @_i \Box \varphi$. □

Results similar to Theorem 9.4.3 and 9.4.4 have been proved for a different axiomatization in [22]. As far as we know, however, $\mathbf{K}^{II}_{\mathcal{H}(@,\downarrow)}$ is the first complete axiomatization of $\mathcal{H}(@,\downarrow)$ without non-orthodox rules.

Interestingly, Corollary 9.3.3 has the following surprising consequence regarding finite axiomatizability. If K is an elementary frame class definable by a set of $\mathcal{H}(@,\downarrow)$-sentences, then there is a single pure $\mathcal{H}(@,\downarrow)$-sentence φ such that φ defines K, and hence, by Theorem 9.4.3 and 9.4.4, $\mathbf{K}^{I}_{\mathcal{H}(@,\downarrow)}\{\varphi\}$ and $\mathbf{K}^{II}_{\mathcal{H}(@,\downarrow)}\{\varphi\}$ are complete for K! We do not believe that similar general finite axiomatizability results can be obtained for *every* elementary class. In particular, we conjecture that the $\mathcal{H}(@,\downarrow)$-logic of the frame class defined by $\forall x \exists y (Rxy \wedge \forall z.(Rzy \to z = x)$ is not finitely axiomatizable.

In connection to the discussion in Section 5.4 about the confluence property, it may be observed that, while confluence is not definable by pure $\mathcal{H}(\mathsf{E})$ formulas, it is defined by the pure $\mathcal{H}(@,\downarrow)$-sentence $\downarrow x.\Box\downarrow y.@_x \Box\Diamond\downarrow z.@_y \Diamond z$. Hence, the completeness above results for pure extensions of $\mathbf{K}^{I}_{\mathcal{H}(@,\downarrow)}$ and $\mathbf{K}^{II}_{\mathcal{H}(@,\downarrow)}$ apply equally well to classes of confluent frames.

To conclude this section, consider again Theorem 8.2.13, which shows that under certain conditions, completeness of $\mathbf{K}_{\mathcal{M}}\Sigma$ implies completeness of $\mathbf{K}_{\mathcal{H}}\Sigma$ and $\mathbf{K}_{\mathcal{H}(@)}\Sigma$. One might ask if a similar result could be obtained for $\mathcal{H}(@,\downarrow)$. The answer is negative: consider the class K of transitive, conversely well-founded uni-modal frames. This class admits filtration and has a master modality, and its modal logic is $\mathbf{K}_{\mathcal{M}}\{\Box(\Box p \to p) \to \Box p\}$. It follows by Theorem 8.2.13 that $\mathbf{K}_{\mathcal{H}}\{\Box(\Box p \to p) \to \Box p\}$ and $\mathbf{K}_{\mathcal{H}(@)}\{\Box(\Box p \to p) \to \Box p\}$ are complete for K. Nevertheless, $\mathbf{K}^{I}_{\mathcal{H}(@,\downarrow)}\{\Box(\Box p \to p) \to \Box p\}$ and $\mathbf{K}^{II}_{\mathcal{H}(@,\downarrow)}\{\Box(\Box p \to p) \to \Box p\}$ are incomplete. This follows from the following theorem, which shows that the $\mathcal{H}(@,\downarrow)$-logic of K is not recursively axiomatizable.

9.4.5. Theorem. *The satisfiability problem for $\mathcal{H}(@,\downarrow)$-sentences on the class of transitive conversely well-founded frames is Σ^1_1-hard.*

Proof: Consider the model $(\mathbb{N}, >)$. By Theorem B.0.1, the existential second order theory of this structure is Σ^1_1-complete. We will reduce this problem to the satisfiability problem for $\mathcal{H}(@,\downarrow)$ on transitive conversely well-founded frames, thus establishing Σ^1_1-hardness of the latter problem.

Let us use $\exists_{suc} y.\varphi$ as a shorthand for $\downarrow x.\Diamond\downarrow y.@_x.\varphi$, and $\forall_{suc} y.\varphi$ as its dual. Furthermore, let us use $\Diamond_{imm}\varphi$ as a shorthand for $@_x\Diamond(\varphi \wedge \downarrow y.@_x \neg\Diamond\Diamond y)$. Let χ be the conjunction of the following formulas.

$$\begin{array}{l} \Diamond p \\ \Box(p \to \Box p) \\ \Box(p \wedge \Diamond\top \to \Diamond_{imm}\top)\ \forall_{suc}x\forall_{suc}y(@_x p \wedge @_y p \to @_x\Diamond y \vee @_y\Diamond x \vee @_x y) \\ \forall_{suc}x(@_x p \to \exists_{suc}y.@_y(p \wedge \Diamond x)) \end{array}$$

Suppose $(\mathfrak{F}, V), w \models \chi$, where $\mathfrak{F}$ is a transitive conversely well-founded frame. Then the subframe of $\mathfrak{F}$ consisting of the successors of w that satisfy p (under the valuation V) constitutes an isomorphic copy of $(\mathbb{N}, >)$.

Next, consider any Σ^1_1-formula $\exists R_1 \dots R_n.\psi$. For each $k \in \omega$, introduce a new proposition letter p_k, and for each relation R_k ($1 \leq k \leq n$), pick a new proposition letter q_{R_k}. Finally, define ψ^* inductively as follows.

$$\begin{array}{lcl} (x = y)^* & = & @_x y \\ (x > y)^* & = & @_x\Diamond y \\ R_k(x_1, \dots, x_m)^* & = & \exists_1 y_1 \dots y_m z.\big(@_z q_{R_k} \wedge \bigwedge_{\ell=1\dots n} @_{y_\ell}(p_\ell \wedge \Diamond_{imm} x_\ell \wedge \Diamond_{imm} z)\big) \\ (\neg\psi)^* & = & \neg(\psi^*) \\ (\psi_1 \wedge \psi_2)^* & = & \psi_1^* \wedge \psi_2^* \\ (\exists x.\psi)^* & = & \exists_{suc}x.(@_x p \wedge \psi^*) \end{array}$$

We will now show that $(\mathbb{N}, >) \models \exists R_1 \dots R_n.\psi$ iff $\chi \wedge \psi^*$ is satisfiable on the class of transitive conversely well-founded frames.

[⇒] Suppose $(\mathbb{N}, >, R_1, \dots, R_n) \models \psi$. Construct a new uni-modal model $\mathfrak{M} = (W, R, V)$ as follows:

$$\begin{array}{lcl} W & = & \mathbb{N} \cup \{root\} \cup \\ & & \{\langle R_k, d_1, \dots, d_m, \ell\rangle \mid R_k(d_1, \dots, d_m) \text{ and } 0 \leq \ell \leq m\} \\ R & = & \{(root, d) \mid d \in W\} \cup \\ & & \{(m, n) \in \mathbb{N}^2 \mid m > n\rangle\} \cup \\ & & \{(\langle R_k, d_1, \dots, d_m, \ell\rangle, e) \mid \ell > 0 \text{ and } e \geq d_\ell\} \cup \\ & & \{(\langle R_k, d_1, \dots, d_m, \ell\rangle, \langle R_k, d_1, \dots, d_m, 0\rangle) \mid \ell > 0\} \\ V(p) & = & \mathbb{N} \\ V(p_m) & = & \{\langle R_k, d_1, \dots, d_m, \ell\rangle \mid m = \ell\} \\ V(p_{R_m}) & = & \{\langle R_k, d_1, \dots, d_m, \ell\rangle \mid m = k \text{ and } \ell = 0\} \end{array}$$

The reader may check that the relation R is indeed transitive and conversely well-founded, and that $\mathfrak{M}, root \models \chi$. Furthermore, an inductive argument shows that $\mathfrak{M}, root \models \psi^*$.

[⇐] Suppose $\mathfrak{M}, w \models \chi \wedge \psi^*$. Then, as discussed above, the submodel of $\mathfrak{M}$ consisting of all successors of w that satisfy p is isomorphic to $(\mathbb{N}, <)$. A model $(\mathbb{N}, <, R_1, \dots, R_n)$ for ψ may now be obtained by letting $(d_1, \dots, d_m) \in R_k$ iff $\mathfrak{M}, w \models \exists_1 y_1 \dots y_m z.\big(@_z q_{R_k} \wedge \bigwedge_{\ell=1\dots n} @_{y_\ell}(p_\ell \wedge \Diamond_{imm} x_\ell \wedge \Diamond_{imm} z)\big)\ [d_1, \dots, d_m]$.

It is not hard to see that if $\mathfrak{M}, w \models \psi^*$, then ψ holds of the submodel of $\mathfrak{M}$ consisting of all successors of w that satisfy p. Conversely, if $\mathfrak{M} \models \psi$, then $\mathfrak{M}$ is easily extended to a model $\mathfrak{M}'$ such that $\mathfrak{M}', w \models \psi^*$ for some world w. It follows that $(\mathbb{N}, >) \models \exists R_1 \dots R_n.\psi$ iff $\chi \to \psi^*$ is satisfiable on the class of transitive conversely well-founded frames. □

9.5 Interpolation and Beth definability

It was proved in the 1960s by Feferman [39] that the bounded fragment satisfies the usual, first-order version of interpolation (cf. Appendix A). In other words, $\mathcal{H}(@, \downarrow)$ has interpolation, not only over proposition letters and nominals, but also over modalities. This was proved in [39] on the basis of a complete, cut-free sequent calculus for the bounded fragment. Being unaware of Feferman's article, Areces, Blackburn and Marx [5, 20], rediscovered these results.

9.5.1. Theorem ([39, 5, 20]). *$\mathcal{H}(@, \downarrow)$ has interpolation over proposition letters, nominals and modalities, with respect the class of all frames.*

Furthermore, it was shown by [20] that $\mathcal{H}(@, \downarrow)$ has interpolation over proposition letters and nominals relative to many frame classes. Here, we include a short proof of the latter result.

9.5.2. Theorem ([20]). *$\mathcal{H}(@, \downarrow)$ has interpolation over proposition letters and nominals relative to any elementary frame class defined by a set of nominal free $\mathcal{H}(@, \downarrow)$-sentences.*

Proof: Suppose K is an elementary frame class definable by means of a set of nominal free $\mathcal{H}(@, \downarrow)$-sentences. By Corollary 9.3.2, K is closed under generated subframes and reflects point-generated subframes, and K is defined by a single pure nominal free $\mathcal{H}(@, \downarrow)$-sentence χ.

Next, suppose $\models_{\mathsf{K}} \varphi \to \psi$. Let NOM and MOD be the sets of nominals and modalities, respectively, occurring in the formula $\varphi \to \psi$. Let $[\cup]\chi$ be shorthand for $\bigwedge_{\Box \in \text{MOD}} \Box\chi$. It follows from the invariance of φ and ψ under generated submodels that $\{[\cup]^n\chi, @_i[\cup]^n\chi \mid i \in \text{NOM}, n \in \omega\} \models \varphi \to \psi$. By compactness, there is an $m \in \omega$ such that $\models \big([\cup]^{\leq m}\chi \wedge \bigwedge_{i \in \text{NOM}} @_i[\cup]^{\leq m}\chi\big) \to (\varphi \to \psi)$. It follows that

$$\models \big(\varphi \wedge [\cup]^{\leq m}\chi \wedge \bigwedge_{i \in \text{NOM}(\varphi)} @_i[\cup]^{\leq m}\chi\big) \quad \to \quad ((\bigwedge_{i \in \text{NOM}(\psi)} @_i[\cup]^{\leq m}\chi) \to \psi)$$

By Theorem 9.5.1, there is an interpolant ϑ such that

1. $\models \big(\varphi \wedge [\cup]^{\leq m}\chi \wedge \bigwedge_{i \in \text{NOM}(\varphi)} @_i[\cup]^{\leq m}\chi\big) \to \vartheta$
2. $\models \vartheta \to ((\bigwedge_{i \in \text{NOM}(\psi)} @_i[\cup]^{\leq m}\chi) \to \psi)$
3. All nominals and proposition letters occurring in ϑ occur both in φ and in ψ.

Since χ is valid on K, it follows that $\models_{\mathsf{K}} \varphi \to \vartheta$ and $\models_{\mathsf{K}} \vartheta \to \psi$. $\square$

One might ask if Theorem 9.5.2 could be generalized to frame classes defined by $\mathcal{H}(@,\downarrow)$-sentences containing nominals. The answer is negative, as shown by the following result.

9.5.3. PROPOSITION. *There is an elementary frame class defined by an $\mathcal{H}(@,\downarrow)$-sentence, on which $\mathcal{H}(@,\downarrow)$ does not have interpolation over nominals.*

Proof: Let K be the class of frames satisfying $\exists x.Rxx \to \forall yz.(Ryz \to y = z)$. Then $\models_{\mathsf{K}} @_i\Diamond i \to @_j\Box j$. In fact, this formula defines K. Suppose for the sake of contradiction that this implication had an interpolant ϑ. Not that ϑ must be a formula in the empty vocabulary. Now consider the following models: $\mathfrak{M}_1 = (\{w,v\},\{(v,v)\},\{(i,v),(j,v)\})$ and $\mathfrak{M}_2 = (\{w,v\},\{(v,w)\},\{(i,v),(j,v)\})$. Clearly, $(\mathfrak{M}_1,w)$ and $(\mathfrak{M}_2,w)$ cannot be distinguished by a $\mathcal{H}(@,\downarrow)$-formula in the empty vocabulary. However, ϑ must be true in $(\mathfrak{M}_1,w)$ and false in $(\mathfrak{M}_2,w)$. $\square$

Nevertheless, Theorem 9.5.2 can be improved if one is interested only in interpolation over proposition letters.

9.5.4. THEOREM. *$\mathcal{H}(@,\downarrow)$ has interpolation over proposition letters on any $\mathcal{H}(@,\downarrow)$-definable elementary frame class.*

Proof: Suppose K is an elementary frame class definable by means of a set of $\mathcal{H}(@,\downarrow)$-sentences. By Corollary 9.3.3, K is defined by a single pure $\mathcal{H}(@,\downarrow)$-sentence $\chi(i_1,\ldots,i_n)$.

Suppose $\models_{\mathsf{K}} \varphi \to \psi$. Let NOM and MOD be the set of nominals and modalities, respectively, occurring in the formula $\varphi \to \psi$. Let $[\cup]\chi$ be shorthand for $\bigwedge_{\Box\in\text{MOD}} \Box\chi$ and let $@[\cup]^{\leq n}\chi$ be a shorthand for $\bigwedge_{t\in\text{NOM}\cup\{y\}} @_t[\cup]^{\leq n}\chi$, where y is a fresh variable. The generated submodel invariance of $\mathcal{H}(@,\downarrow)$-sentences implies that

$$\{\downarrow y.@[\cup]^{\leq k}\downarrow x_1 @[\cup]^{\leq k}\downarrow x_2 \cdots @[\cup]^{\leq k}\downarrow x_n.@[\cup]^{\leq k}\chi(x_1,\ldots,x_n) \mid n\in\omega\} \models \varphi\to\psi$$

By compactness, there is a $k\in\omega$ such that

$$\models \downarrow y.@[\cup]^{\leq k}\downarrow x_1 @[\cup]^{\leq k}\downarrow x_2 \cdots @[\cup]^{\leq k}\downarrow x_n.@[\cup]^{\leq k}\chi(x_1,\ldots,x_n) \ \to\ (\varphi\to\psi)$$

and hence

$$\models \Big((\downarrow y.@[\cup]^{\leq k}\downarrow x_1 @[\cup]^{\leq k}\downarrow x_2 \cdots @[\cup]^{\leq k}\downarrow x_n.@[\cup]^{\leq k}\chi(x_1,\ldots,x_n)) \wedge \varphi\Big) \to \psi$$

Applying Theorem 9.5.1 on this, we obtain an interpolant ϑ with the following properties.

1. $\models \Big((\downarrow y.@[\cup]^{\leq k}\downarrow x_1 @[\cup]^{\leq k}\downarrow x_2 \cdots @[\cup]^{\leq k}\downarrow x_n.@[\cup]^{\leq k}\chi(x_1,\ldots,x_n)) \wedge \varphi \Big) \to \vartheta$
2. $\models \vartheta \to \psi$
3. All proposition letters occurring in ϑ occur both in φ and in ψ.

Since χ is valid on K, it follows that $\models_{\mathsf{K}} \varphi \to \vartheta$ and $\models_{\mathsf{K}} \vartheta \to \psi$. □

9.5.5. Corollary. *$\mathcal{H}(@,\downarrow)$ has the Beth property relative to every elementary $\mathcal{H}(@,\downarrow)$-definable class of frames.*

Proof: Similar to the proof of Theorem 6.2.4. □

Here is a simple example of an elementary frame class on which $\mathcal{H}(@,\downarrow)$ lacks the Beth property. Let K be the class of frames satisfying $\exists x \forall yz.(Ryz \leftrightarrow y = x)$, and let $\Sigma = \{p \to \Box q, \neg p \to \Box \neg q\}$ Clearly, in models that are based on a frame in K and that globally satisfy Σ, q holds at a state iff p holds at the root, and hence, Σ implicitly defines q in terms of p, relative to K. In Section 2.5, we already observed that q cannot be defined explicitly in terms of p in the basic modal language, relative to Σ and K. In fact, it is not hard to see that also $\mathcal{H}(@,\downarrow)$ fails provide an explicit definition, and hence the Beth property fails also for this language, relative to K. We leave it as an open problem whether there is an elementary class closed under generated subframes, with respect to which $\mathcal{H}(@,\downarrow)$ lacks the Beth property.

9.6 Decidability and complexity

In this section, we investigate the complexity of deciding whether a formula of $\mathcal{H}(@,\downarrow)$ is satisfiable. It was shown by Areces, Blackburn and Marx [4] that this problem is undecidable, and in fact they mention that $\mathcal{H}(@,\downarrow)$ is a *conservative reduction class*. Following [24] we call a fragment of first-order logic a conservative reduction class if there is a recursive function τ mapping arbitrary first-order formulas to formulas in the fragment, such that for all formulas α, $\tau(\alpha)$ is satisfiable iff α is, and $\tau(\alpha)$ has a finite model iff α has. Clearly, every conservative reduction class has an undecidable (in fact Π^0_1-complete) satisfiability problem, as well as an undecidable (in fact Σ^0_1-complete) finite satisfiability problem [24].

9.6.1. Theorem. *$\mathcal{H}(@,\downarrow)$ is a conservative reduction class.*

Proof: It is known that the relational first-order formulas with a single, binary, relation symbol form a conservative reduction class [24]. Consider the following embedding τ from first-order logic with one binary relation to $\mathcal{H}(@,\downarrow)$, where i be a fixed nominal:

$$\begin{array}{lcl}
\tau(Rxy) & = & @_x\Diamond y \\
\tau(x=y) & = & @_x y \\
\tau(\neg\varphi) & = & \neg\tau(\varphi) \\
\tau(\varphi\wedge\psi) & = & \tau(\varphi)\wedge\tau(\psi) \\
\tau(\exists x.\varphi) & = & @_i\Diamond\downarrow x.\ \tau(\varphi)
\end{array}$$

Table 9.4: Complexity of the satisfiability problem on κ-models

	$\mathcal{H}(@,\downarrow)$	$\mathcal{L}^1$
$\kappa = 1$	NP-complete	NExpTime-complete
$\kappa = 2$	NP-complete	Decidable but not elementary
$3 \leq \kappa < \omega$	NExpTime-complete	Π^0_1-complete (co-r.e., not decidable)
$\kappa = \omega$	Σ^0_1-complete (r.e., not decidable)	Σ^1_1-complete (highly undecidable)
$\kappa > \omega$	Π^0_1-complete (co-r.e., not decidable)	Π^0_1-complete (co-r.e., not decidable)

Clearly, τ is a recursive function. We claim that for each first-order sentence φ, φ is has a (finite) model iff $\tau(\varphi)$ is has a (finite) model.

First, suppose $\mathfrak{M} \models \varphi$. Let the model $\mathfrak{M}'$ be obtained from $\mathfrak{M}$ by adding a new state w, labeled with nominal i, and by extending the relation R such that $(w, v) \in R$ for all states v of $\mathfrak{M}$. Then $\mathfrak{M}', w \models \tau(\varphi)$. Moreover, $\mathfrak{M}'$ is finite if $\mathfrak{M}$ is. Conversely, suppose $\mathfrak{M}, w \models \tau(\varphi)$. Let v be the state in $\mathfrak{M}$ labeled by the nominal s. Let $\mathfrak{M}'$ be the submodel of $\mathfrak{M}$ consisting of all successors of v. Then $\mathfrak{M}' \models \varphi$. Moreover, $\mathfrak{M}'$ is finite if $\mathfrak{M}$ is. □

In what follows, we will give a number of decidability results for more restricted classes of models. We will use our results to compare $\mathcal{H}(@,\downarrow)$ with the first-order correspondence language $\mathcal{L}^1$. For any cardinal κ, let K_κ be the class of uni-modal models in which for every node d there are strictly less than κ nodes e such that $(d, e) \in R$. In particular, K_2 is the class of models in which every points has at most one R-successor, and K_ω is the class of models in which every node has only finitely many R-successors. We will refer to elements of K_κ as κ-models for short. In what follows we will consider the satisfiability problem of $\mathcal{H}(@,\downarrow)$ and of the first-order correspondence language on κ-models, for particular κ. Our results are summarized in Table 9.4. All results generalize to to case with multiple modalities, except for the decidability of the first-order correspondence language on K_2.

9.6.2. Theorem. *The satisfiability problem of $\mathcal{H}(@,\downarrow)$ on the class of models K_κ is*

1. NP-*complete, for $\kappa = 1, 2$*
2. NExpTime-*complete, for $3 \leq \kappa < \omega$.*
3. Σ^0_1-*complete, for $\kappa = \omega$*
4. Π^0_1-*complete, for $\kappa > \omega$*

Proof: 1. The lower bound follows from the NP-hardness of propositional satisfiability. The upper bound is proved by establishing the polynomial size model property.

For $\kappa = 1, 2$, every κ-satisfiable $\mathcal{H}(@, \downarrow)$-formula is satisfiable in a κ-model with at most $O(|\varphi|^2)$ nodes. For, suppose $\mathfrak{M}, w \models \varphi$ for some κ-model $\mathfrak{M} = (W, R, V)$. Let $W' \subseteq W$ consist of all worlds that are reachable from w or from a world named by one of the nominals occurring in φ in at most $md(\varphi)$ steps, where $md(\varphi)$ is the modal depth of φ. Let $\mathfrak{M}'$ be the submodel of $\mathfrak{M}$ with domain W'. Clearly, $\mathfrak{M}'$ is a κ-model and $\mathfrak{M}'$ satisfies the cardinality requirements. Furthermore, a straightforward induction argument shows that $\mathfrak{M}', w \models \varphi$.

This leads to a non-deterministic polynomial time algorithm for testing satisfiability of an $\mathcal{H}(@, \downarrow)$-formula φ on κ-models, for $\kappa = 1, 2$. The algorithm first non-deterministically chooses a candidate model $(\mathfrak{M}, w)$ of size $O(|\varphi|^2)$, and then it tests whether $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M} \in \mathsf{K}_\kappa$. The latter tests can be performed in polynomial time using a top down model checking algorithm.

2. **[Upper bound]** For $3 \leq \kappa < \omega$, every formula satisfiable on a κ-model is satisfiable on a κ-model with at most $O(|\varphi| \cdot \kappa^{md(\varphi)})$ nodes. For, suppose $\mathfrak{M}, w \models \varphi$ for some κ-model $\mathfrak{M} = (W, R, V)$. Let $W' \subseteq W$ consist of all worlds that are reachable from w or from a world named by one of the nominals occurring in φ in at most $md(\varphi)$ steps. Let $\mathfrak{M}'$ be the submodel of $\mathfrak{M}$ with domain W'. Note that the cardinality of $\mathfrak{M}'$ is $O(|\varphi| \cdot \kappa^{|\varphi|})$, and $\mathfrak{M}'$ is still a κ-model. Furthermore, a straightforward induction argument shows that $\mathfrak{M}', w \models \varphi$.

   This leads to a non-deterministic ExpTime algorithm for testing satisfiability of an $\mathcal{H}(@, \downarrow)$-formula φ on κ-models. The algorithm first non-deterministically chooses a candidate model $(\mathfrak{M}, w)$ of size $O(|\varphi| \cdot \kappa^{|\varphi|})$, and then tests whether $\mathfrak{M}, w \models \varphi$. The latter test can be performed in time $O(|\mathfrak{M}|^{|\varphi|})$ [43], which is $O((|\varphi| \cdot \kappa^{|\varphi|})^{|\varphi|}) = O(|\varphi|^{|\varphi|} \cdot \kappa^{(|\varphi|^2)})$.

   [Lower bound] Consider monadic first-order formulas without equality, i.e., first-order formulas containing unary predicates only, without equality. Any such satisfiable formula φ of length n has a model with at most 2^n nodes, and the satisfiability problem for such formulas is NExpTime-complete [24, Section 6.2.1]. We will reduce this problem to the satisfiability problem for $\mathcal{H}(@, \downarrow)$-formulas on κ-models (for $3 \leq \kappa < \omega$), thus showing that the latter problem is NExpTime-hard.

   Fix a nominal i, and for any monadic first-order formula φ without equality, define φ^+ inductively, such that $(x = y)^+ = @_x y$, $(Px)^+ = @_x p$, $(\cdot)^+$ commutes with the Boolean connectives and $(\exists x.\psi)^+ = @_i \Diamond^{|\varphi|} \downarrow x.\psi^+$. In words, φ^+ states that φ holds in the submodel consisting of all points reachable from the point named i in exactly $|\varphi|$ many steps. In general, there can be up to $(\kappa - 1)^{|\varphi|}$ many points reachable from the point named i in exactly $|\varphi|$ many steps (in particular, this will be the case if the submodel generated by i is a $(\kappa - 1)$-ary tree). It follows that φ is satisfiable iff φ is satisfiable in a model with at most $2^{|\varphi|}$ nodes iff φ^+ is satisfiable in a κ-model, for $\kappa \geq 3$.

3. We will provide polynomial reductions between this problem and the finite satisfiability problem for first-order logic, which is Σ^0_1-complete, even in the case with only a single, binary relation [24, Section 3.2].

   Trivially, if an $\mathcal{H}(@, \downarrow)$-formula is satisfiable in a finite model, it is satisfiable in a ω-model. Conversely, if an $\mathcal{H}(@, \downarrow)$-formula is satisfiable in an ω-model then it is satisfiable in a finite model, since the modal depth of the formula provides a bound on the depth of the model. Hence, the satisfiability problem of $\mathcal{H}(@, \downarrow)$ on ω-models reduces (by the standard translation) to the satisfiability problem for first-order logic on finite models.

   Conversely, the finite satisfiability problem for first-order logic can be reduced to satisfiability of $\mathcal{H}(@, \downarrow)$ on ω-models. Fix a nominal i, and for any first-order formula φ, define φ^+ inductively, such that $(x = y)^+ = @_x y$, $(Rxy)^+ = @_x \Diamond y$, $(\cdot)^+$ commutes with the Boolean connectives and $(\exists x.\psi)^+ = @_i \Diamond \downarrow x.\psi^+$. In words, φ^+ states that φ holds in the submodel consisting of the successors of the point named i. It follows that φ is satisfiable in a finite model iff the $\mathcal{H}(@, \downarrow)$-formula φ^+ is satisfiable on an finitely branching ω-model.

4. By the Löwenheim-Skolem theorem, a first-order formula is satisfiable if and only if it is satisfiable on a finite or countably infinite model. Since $\mathcal{H}(@, \downarrow)$ is a fragment of first-order logic, the Löwenheim-Skolem theorem also applies to $\mathcal{H}(@, \downarrow)$-formulas. It follows that the satisfiability problem for $\mathcal{H}(@, \downarrow)$ on countably branching models coincides with the general satisfiability problem of $\mathcal{H}(@, \downarrow)$, which is Π^0_1-complete by Theorem 9.6.1. □

9.6.3. THEOREM. *The satisfiability problem for the first-order correspondence language $\mathcal{L}^1$ on K_κ is*

1. NEXPTIME *complete, for $\kappa = 1$*
2. *decidable but not elementary, for $\kappa = 2$*
3. Π^0_1*-complete, for $3 \leq \kappa < \omega$*
4. Σ^1_1*-complete, for $\kappa = \omega$*
5. Π^0_1*-complete, for $\kappa > \omega$*

Proof: 1. This case coincides with the satisfiability problem for monadic first-order logic (on 1-models, every formula of the form Rst is equivalent to $\bot$), which is known to be NEXPTIME complete [24].

2. Consider the satisfiability problem for first-order logic with one unary function symbol, an arbitrary number of unary relation symbols and equality ("the Rabin class"). This problem is decidable, but not elementary [24]. We will provide polynomial reductions between this problem and the satisfiability problem for first-order logic on 2-models.

   - Let φ be any first-order formula containing one unary function symbol f and any number of unary relation symbols and equality. Let R be a binary relation symbol, and let φ_R be obtained from φ by repeatedly applying the rewrite rules
     - replace atomic formulas of the form $Pf(t)$ by $\exists x.(Rtx \wedge Px)$
     - replace atomic formulas of the form $f(s) = t$ or $t = f(s)$ by $\exists x.(Rsx \wedge x = t)$

     until the function symbol f does not occur in the formula anymore (in case of nested function symbols, the above rules might need to be applied several times). It is not hard to see that φ is satisfiable iff $\varphi_R \wedge \forall x \exists y.Rxy$ is satisfiable on a 2-model.
   - Let φ be any first-order formula with one binary relation symbol R and any number of unary relation symbols. Let f be a unary function symbol and let P be a new unary relation, and let φ_f be the result of replacing all subformulas of φ of the form Rst by $Ps \wedge (t = fs)$. Intuitively, the unary predicate P represents the existence of a successor, and the unary function f encodes the successor of a node, if it exists. One can easily see that φ is satisfiable on a 2-model iff φ_f is satisfiable (simply let R denote the the graph of f, or vice versa).

   It follows that the satisfiability problem of first-order logic on 2-models is decidable but not elementary recursive.

3. It is known that the satisfiability problem for first-order sentences with a single binary relation R is Π^0_1-complete [24]. For any such first-order formula φ define φ^* as follows:

$$
\begin{array}{lcl}
(x = y)^* & = & x = y \\
(Rxy)^* & = & \exists x'y'.(\neg Rx'x' \wedge \neg Ry'y' \wedge Rx'y' \wedge Rx'x \wedge Ry'y) \\
(\neg\varphi)^* & = & \neg\varphi^* \\
(\varphi \wedge \psi)^* & = & \varphi^* \wedge \psi^* \\
(\exists x.\varphi)^* & = & \exists x(Rxx \wedge \varphi^*)
\end{array}
$$

   We claim that φ is satisfiable in a model $\mathfrak{M}$ iff φ^* is satisfiable on a 3-model $\mathfrak{M}'$. Intuitively, the reflexive nodes of $\mathfrak{M}'$ will correspond to the nodes of $\mathfrak{M}$, and the irreflexive nodes of $\mathfrak{M}'$ will be used to encode the binary relation of $\mathfrak{M}$: we think of reflexive points d, e as standing in the binary relation iff there are irreflexive points d', e' such that $(d', d) \in R$, $(d', e') \in R$ and $(e', e) \in R$. More precisely, the argument can be spelled out as follows.

[⇒] Suppose $\mathfrak{M} \models \varphi$, with $\mathfrak{M} = (D, R)$. Let D' be a set of objects obtained from D by adding by adding new objects $(d, e)_1$ and $(d, e)_2$ for all $d, e \in D$. Let $R' = \{(d, d), ((d, e)_1, d), ((d, e)_2, e) \mid d \in D\} \cup \{((d, e)_1, (d, e)_2) \mid (d, e) \in R\}$. The model (D', R') is a 3-model, and by induction on can easily show that $\mathfrak{M}' \models \varphi^*$.

[⇐] Suppose $\mathfrak{M} \models \varphi^*$ for some 3-model $\mathfrak{M} = (D, I)$. Let $D' = \{d \in D \mid (d, d) \in R\}$. Let $R' = \{(d, e) \in (D')^2 \mid (d', d') \notin R$ and $(e', e') \notin R$ and $(d', d) \in R$ and $(e', e) \in R$ and $(d', e') \in R$, for some $d', e' \in D\}$. Let $\mathfrak{M}' = (D', R')$. A straightforward induction shows that $\mathfrak{M}' \models \varphi$.

For $3 < \kappa < \omega$, it follows that a first-order formulas φ with one binary relation R is satisfiable iff $\varphi^* \wedge \forall x \exists^{\leq 2} y.Rxy$ is satisfiable on a κ-model. Hence, satisfiability of first-order formulas on κ-models is Π^0_1-hard. Finally, membership of Π^0_1 follows from the fact that the satisfiability problem for first-order formulas is in Π^0_1, since φ is satisfiable on a κ-model iff $\varphi \wedge \forall x \exists^{\leq \kappa} y.Rxy$ is satisfiable.

4. We will provide reductions between that the satisfiability problem for first-order formulas on ω-models and the problem of deciding whether an existential second order sentence holds in the model $(\mathbb{N}, <)$. This proves the result, since the latter problem is Σ^1_1-complete (cf. Theorem B.0.1).

   Let $\varphi_{(\mathbb{N},>)}$ be a first-order sentence expressing that R is a strict linear order and $\forall x \exists y.Ryx$. Then a finitely branching model satisfies $\varphi_{(\mathbb{N},>)}$ precisely if the model is isomorphic to $(\mathbb{N}, >)$. For any existential second order sentence $\varphi = \exists R_1 \ldots R_n.\psi(R_1, \ldots, R_n, >)$, let φ^* be the defined as follows, where $P_1, \ldots, P_n, N$ are new, distinct unary predicates.

$$
\begin{array}{lcl}
(x = y)^* & = & x = y \\
(x > y)^* & = & Rxy \\
(R_k x_1 \ldots x_n)^* & = & \exists y_1 \ldots y_n.\big(\bigwedge_{m=1\ldots n}(P_k y_m \wedge R y_m x_m) \wedge \\
 & & \qquad \bigwedge_{m=1\ldots n-1}(R y_m y_{m+1}) \big) \\
(\neg\varphi)^* & = & \neg\varphi^* \\
(\varphi \wedge \psi)^* & = & \varphi^* \wedge \psi^* \\
(\exists x.\varphi)^* & = & \exists x(Nx \wedge \varphi^*)
\end{array}
$$

   We claim that $(\mathbb{N}, >) \models \varphi$ iff $\varphi^* \wedge \varphi^N_{(\mathbb{N},>)}$ is satisfiable in a finitely branching model, where $\varphi^N_{(\mathbb{N},>)}$ is the result of relativising all quantifiers in $\varphi_{(\mathbb{N},>)}$ by N. The argument is similar to the one used in the proof of Theorem 9.6.3(3). The submodel consisting of the points satisfying N is the "intended model", while the elements satisfying one of the unary predicates P_k are only used to encode which tuples stand in the R_k relation. More specifically, a tuple $(d_1, \ldots, d_n)$ of points satisfying N is thought to stand in the R_k relation iff there are points $e_1, \ldots, e_n$ satisfying P_k such that $e_m R d_m$ for all $m \leq n$ and $e_m R e_{m+1}$ for all $m < n$. We will omit the details of the proof here.

Now for the other direction. First, observe that whenever a first-order formula has a finitely branching model $\mathfrak{M}$, then it has a countable such model (indeed, it suffices to take any countable elementary submodel of $\mathfrak{M}$). Now, for any first-order formula $\varphi(R, P_1, \ldots, P_n)$, let φ' be the existential second order sentence $\exists R, P_1, \ldots, P_n.(\varphi \wedge \forall x \exists y \forall z.(Rxz \rightarrow z < y))$. Observe how, on the natural numbers, the second conjunct enforces that each point has only finite many R-successors). It follows that φ is satisfiable in a countable ω-model iff φ' is true in a submodel of $(\mathbb{N}, <)$. The latter in turn holds iff $\exists Q.(\varphi')^Q$ is true in $(\mathbb{N}, <)$, where $(\varphi')^Q$ is the result of relativising all quantifiers in φ' by Q.

5. By the Löwenheim-Skolem theorem, a first-order formula is satisfiable if and only if it is satisfiable on a finite or countably infinite model. Hence, the satisfiability problem on countably branching models coincides with the general satisfiability problem, which is known to be Π^0_1-complete [24]. □

We can conclude from Table 9.4 that $\mathcal{H}(@, \downarrow)$ has computational advantages over $\mathcal{L}^1$, as least on structures with a bounded out-degree.

In [29], a fragment of $\mathcal{H}(@, \downarrow)$ is identified for which the satisfiability is decidable. The fragment consists of all $\mathcal{H}(@, \downarrow)$-sentences that are not of the form $\cdots \Box(\cdots \downarrow x.(\cdots \Box \cdots) \cdots) \cdots$. It is shown that this result is optimal, in the sense that the fragment cannot be easily extended without losing decidability.

Chapter 10

Guarded fragments

The guarded fragment is a fragment of relational first-order logic that extends the modal fragment. It was introduced by Andréka, van Benthem and Németi in the 90s [2] in order to understand why the modal language is so well behaved, computationally and model theoretically. The guarded fragment inherits many good properties from the modal language. For instance, is decidable, has the finite model property, and admits a Łos-Tarski-style preservation theorem [2]. It was shown by Hoogland and Marx [66] that, while the guarded fragment lacks interpolation, it has the Beth property. In [13], the guarded fragment was extended even further, obtaining the *loosely guarded fragment*, which is slightly more expressive than the guarded fragment, but it still satisfies the above properties.

Concrete complexity results for the satisfiability problem for guarded and loosely guarded formulas were established by Grädel [60]. To be precise, Grädel generalized the guarded and loosely guarded fragments by allowing constants to occur in the formulas (but not function symbols of positive arity), and by allowing identity statements of the form $x = x$ or $x = y$ as guards, and subsequently proved the following:

10.0.4. THEOREM (GRÄDEL [60]). *The satisfiability problem for loosely guarded formulas is 2*EXPTIME*-complete. The same problem is only* EXPTIME*-complete for loosely guarded relational formulas with a bounded number of variables, and for guarded relational formulas with a bound on the arity of the relation symbols.*

With a relational formula, we mean a formula that contains no constants (function symbols of positive arity were already excluded).

Furthermore, Grädel suggests in his paper that his results also work for (loosely) ∀-guarded formulas, i.e., formulas of which only the universal quantifiers are (loosely) guarded. However, the details are not completely spelled out.[1]

[1] Marx [79] does explicitly state and prove the *decidability* of the satisfiability problem for loosely ∀-guarded formulas.

This chapter serves three purposes. Firstly, it formulates the precise results for universally guarded formulas that may be obtained with Grädel's techniques, and it contains the details of the proofs. Secondly, and more importantly, we improve Grädel's results by showing that the qualification *'relational'* in the above theorem may be dropped. Finally, we show how guarded fragments with constants are related to hybrid logics, and we use this connection to prove a strong negative interpolation result for guarded fragments.

Concretely, we prove the following the following complexity result.

10.0.5. Theorem. *The satisfiability problem for loosely $\forall$-guarded formulas is 2*ExpTime*-complete. The same problem is only* ExpTime*-complete for loosely $\forall$-guarded formulas with a bounded number of variables and for guarded formulas with a bounded arity.*

To appreciate the additional value of Theorem 10.0.5, we must return to the original motivation behind the guarded fragment. The guarded fragment was invented in order to explain and generalize the large number of decidability and low complexity results in modal logic. The key observation is that modal operators express a guarded form of quantification, where the accessibility relations are the guards.

For explaining *decidability* results in modal logic, the first part of Theorem 10.0.4 often suffices. However, in order to explain *low complexity*, a more refined analysis is needed. Consider for instance the global consequence problem for modal formulas (*does every model that globally satisfies φ globally satisfy ψ?*). This is an ExpTime-complete problem. To understand why this problem is in ExpTime, it suffices to observe that global truth of a modal formula φ can be expressed by means of a guarded first-order formula with only two variables, namely $\forall x.(x = x \to ST_x(\varphi))$. [2] This shows the importance of bounded variable guarded fragments.

The standard translation for $\mathcal{H}(\mathsf{E})$ produces first-order formulas in the two-variable guarded fragment with an unlimited number of constants. Clearly, Theorem 10.0.4 will not allow us to prove, say, that the global consequence problem for $\mathcal{H}(@)$ is in ExpTime. Theorem 10.0.5 does, and it thereby broadens the application of guarded fragments to the field of hybrid logic. A concrete example of a complexity result from the literature that follows immediately from Theorem 10.0.5 is the ExpTime-membership of the satisfiability problem for the hybrid language $\mathcal{H}(\mathsf{E})$ [5].

The results in this chapter are taken from [94] and [28].

10.1 Normal forms for (loosely) guarded formulas

We will consider first-order languages with arbitrarily many relation symbols of any arity, constants and equality, but without function symbols of arity greater

[2] Here, we use the Vardi-style standard translation that uses only two variables.

than zero. A first-order formula φ of such a language is called *guarded* if it is built up from atomic formulas using the Boolean connectives and guarded quantifiers of the form $\exists x_1 \dots x_n.(\pi \wedge \psi)$ or $\forall x_1 \dots x_n.(\pi \to \psi)$, where π is an atomic formula and the free variables of ψ all occur in π. A formula is called $\forall$-*guarded* if it is built up from atomic formulas and negated atomic formulas using conjunction, disjunction, ordinary existential quantifiers and guarded universal quantifiers. Note that the guards π may be atomic equality statements. In particular, if a guarded formula φ has only one free variable x, then $\exists x.(x = x \wedge \varphi)$ and $\forall x.(x = x \to \varphi)$ are guarded formulas. These formulas are equivalent to $\exists x.\varphi$ and $\forall x.\varphi$, respectively.

The loosely guarded fragment is an extension of the guarded fragment. A first-order formula φ is called *loosely guarded* if it is built up from atomic formulas using the Boolean connectives and loosely guarded quantifiers of the form $\exists x_1 \dots x_n.(\pi \wedge \psi)$ or $\forall x_1 \dots x_n.(\pi \to \psi)$, where π is conjunction of atomic formulas, such that every quantified variable x_i co-occurs with every free variable $y \neq x_i$ of ψ in some conjunct of π. A formula is called *loosely* $\forall$-*guarded* if it is built up from atomic formulas and negated atomic formulas using conjunction, disjunction, ordinary existential quantifiers and loosely guarded universal quantifiers. Note that if a loosely guarded formula φ has only one free variable x, then $\exists x.(\top \wedge \varphi)$ and $\forall x.(\top \to \varphi)$ are loosely guarded.

Grädel [60] proved his main complexity results for guarded formulas using the following normal form.

10.1.1. Definition. *A (loosely) $\forall$-guarded formula is in* normal form *if it is of the form*

$$\exists \vec{x}.P\vec{x} \wedge \bigwedge_{i \in I} \forall \vec{x}.(\pi_i(\vec{x}) \to \exists \vec{y}.\varphi_i(\vec{x}, \vec{y}))$$

where, for each $i \in I$, the variables $\vec{x}, \vec{y}$ are distinct, π_i is a (loose) guard and $\varphi_i(\vec{x}, \vec{y})$ is a quantifier-free formula.

Grädel showed that every (loosely) guarded formula can be translated in polynomial time into an equisatisfiable (loosely) $\forall$-guarded formula in normal form. A slight variation of Grädel's proof works for (loosely) $\forall$-guarded sentences, thus turning it into a true normal form theorem for (loosely) $\forall$-guarded formulas. To be sure, we will spell out the proof here for the case of (loosely) $\forall$-guarded formulas.

For any formula φ, let $\textsc{width}(\varphi)$ be the maximal number of free variables of a subformula of φ, i.e., $\textsc{width}(\varphi)$ is the largest natural number n such that φ has a subformula with n free variables.

10.1.2. Proposition. *Every (loosely) $\forall$-guarded formula φ can be transformed in polynomial time into an equisatisfiable (loosely) $\forall$-guarded sentence χ in normal form. Moreover,* $\textsc{width}(\chi) \leq \textsc{width}(\varphi)$.

Proof: We first give the proof for $\forall$-guarded formulas, and then show how the proof generalizes to loosely $\forall$-guarded formulas. Let φ be $\forall$-guarded, and assume without loss of generality that no equality sign occurs inside a guard in φ. Note furthermore that, by the definition of $\forall$-guarded formulas, the negation symbol only occurs in φ at the atomic level.

If φ is quantifier-free, then we are already done. Otherwise, there are two possibilities.

1. φ contains a subformula of the form $\chi(\vec{x}) = \exists y.\psi(\vec{x}, y)$, where ψ is quantifier-free. Pick a new predicate R_χ of the appropriate arity, and let $\varphi[\chi/R_\chi]$ be the result of replacing $\chi(\vec{x})$ in φ by $R_\chi(\vec{x})$. Finally, let
$$\varphi' = \varphi[\chi/R_\chi] \wedge \forall\vec{x}.(R_\chi(\vec{x}) \rightarrow \exists y.\psi(\vec{x}, y))$$
Then φ' is equi-satisfiable to φ, and one step closer to being of the required form.

2. φ contains a subformula of the form $\chi(\vec{x}) = \forall\vec{y}.(\pi(\vec{x}, \vec{y}) \rightarrow \psi(\vec{x}, \vec{y}))$, where ψ is quantifier-free. Pick a new relation symbol R_χ with the appropriate arity, and let $\varphi[\chi/R_\chi]$ be the result of replacing $\chi(\vec{x})$ in φ by $R_\chi(\vec{x})$. Finally, let
$$\varphi' = \varphi[\chi/R_\chi] \wedge \forall\vec{x}\vec{y}.(\pi(\vec{x}, \vec{y}) \rightarrow (R_\chi(\vec{x})) \rightarrow \psi(\vec{x}, \vec{y}))$$
Then φ' is equi-satisfiable to φ, and one step closer to being of the required form.

Repeating these steps, we eventually obtain a formula of the form $\varphi''(\vec{x}) \wedge \eta$, where $\varphi''(\vec{x})$ is quantifier-free, and η is a conjunction of formulas of the form $\forall\vec{x}(\pi(\vec{x}) \rightarrow \exists y.\psi(\vec{x}, y))$. As a final step, pick a new predicate P and let $\vartheta = \exists\vec{x}.P(\vec{x}) \wedge \forall\vec{x}(P(\vec{x}) \rightarrow \varphi''(\vec{x}))) \wedge \eta$. Then ϑ is in normal form and equi-satisfiable to the original formula φ.

A slight variation of this argument works for loosely $\forall$-guarded formulas. Suppose φ is loosely $\forall$-guarded and contains a subformula of the form $\chi(\vec{x}) = \forall\vec{y}.(\pi(\vec{x}, \vec{y}) \rightarrow \psi(\vec{x}, \vec{y}))$, where ψ is quantifier-free. As before, we pick a new relation symbol R_χ with the appropriate arity, but now we also pick a new binary relation symbol Z. Also, the conjunct we add to φ is slightly different: instead of $\forall\vec{x}\vec{y}.(\pi(\vec{x}, \vec{y}) \rightarrow (R_\chi(\vec{x})) \rightarrow \psi(\vec{x}, \vec{y}))$, we add $\forall\vec{x}\vec{y}.((\pi(\vec{x}, \vec{y}) \wedge \bigwedge_{z,z' \in \{\vec{x}\}} Zzz') \rightarrow (R_\chi(\vec{x})) \rightarrow \psi(\vec{x}, \vec{y}))$. This ensures that each two variables in $\vec{x}$ co-occur in some atom of the guard, to guarantee that the universal quantifier is properly loosely guarded. Finally, to ensure that the new formula is equi-satisfiable to the original one, instead of replacing $\chi(\vec{x})$ in φ by $R_\chi(\vec{x})$, we it by $R_\chi(\vec{x}) \wedge \bigwedge_{z,z' \in \{\vec{x}\}} Zzz'$. The rest of the proof remains the same. □

In the case of loosely guarded formulas, one can furthermore ensure that the arity of the relation symbols occuring in the formula is bounded by the width. For any formula φ, let MAXARITY(φ) denote the highest arity of a relation symbol occuring in φ.

10.1.3. Proposition. *Every loosely $\forall$-guarded formula φ can be transformed in polynomial time into an equisatisfiable loosely $\forall$-guarded formula χ in normal form, such that* $\textsc{width}(\chi) \leq \max\{\textsc{width}(\varphi), 2\}$ *and* $\textsc{maxarity}(\chi) \leq \max\{\textsc{width}(\varphi), 2\}$.

Proof: The proof proceeds in two steps. First, we will reduce the arity of the relation symbols occuring in φ to two. Then, we will write the resulting formula in normal form. The latter step might increase the arity of the relation symbols again, but it will still be bounded by the width of the formula.

Let φ be any loosely $\forall$-guarded formula. For each n-ary relation symbol R occurring in φ, with $n > 2$, introduce $n + 1$ new binary relation symbols, $R_0, \ldots, R_n$. These relation symbols will be used to encode the tuples that stand in the relation R: a tuple $\langle d_1, \ldots, d_n \rangle$ will be thought to stand in the relation if each pair $\langle d_\ell, d_m \rangle$ stands in the R_0 relation ($1 \leq \ell, m \leq n$), and there exists is an element e such that $\langle e, d_\ell \rangle \in R_\ell$ for $1 \leq \ell \leq n$.

Replace each subformula of φ of the form $R(t_1, \ldots, t_n)$ that is not inside a guard by

$$\bigwedge_{1 \leq \ell, m \leq n} R_0(t_\ell, t_m) \ \wedge\ \exists u. \bigwedge_{1 \leq \ell \leq n} R_\ell(u, t_\ell)$$

If φ has a subformula of the form $\forall \vec{x}(\pi \rightarrow \psi)$, where the guard π contains a conjunct of the form $R(t_1, \ldots, t_n)$, then replace that conjunct by $\bigwedge_{1 \leq \ell, m \leq n} R_0(t_\ell, t_m)$, and replace ψ by $\exists u.(\bigwedge_{1 \leq \ell \leq n} R_\ell(u, t_\ell) \wedge \top) \rightarrow \psi$.

The resulting formula contains no relation symbols of arity greater than 2, and it is satisfiable iff the original formula φ is satisfiable. Furthermore, the width of the resulting formula is at most $\max(\textsc{width}(\varphi), 2\}$.

Finally, we apply Proposition 10.1.2 to bring the resulting formula into normal form. Inspection of the proof of Proposition 10.1.2 shows that the arity of the relation symbols added during the normal form translation is bounded by the width of the input formula. Hence, we end up with a formula with the desired properties. □

Incidentally, the constraints of bounded width and of bounded number of variables in a first-order formula are equivalent, as proved in the following theorem.

10.1.4. Proposition. *For $k \in \mathbb{N}$, every first-order formula φ of width k can be transformed in polynomial time into an equivalent formula containing k variables.*

Proof: The proof is by structural induction on the input formula φ. If φ is an atomic formula, then its width equals the number of variables occurring in it, hence the claim holds. If φ is of the form $\neg\psi$ or $\exists x.\psi$, then the claim follows immediately from the induction hypothesis (note that, in the second case, we may assume that x occurs in ψ). This leaves us with the case in which φ is a conjunction.

Let φ be of the form $\psi \wedge \chi$. By induction hypothesis, we may assume that ψ and χ each have at most k variables. We may also assume that the only variables occurring both in ψ and in χ are the ones that occur freely in ψ and in χ. It follows that the set of all variables occurring in φ can be partitioned into disjoint subsets X, Y, Z, U, V such that $free(\psi) = X \cup Y$, $free(\chi) = Y \cup Z$, $bound(\psi) \setminus free(\psi) = U$ and $bound(\chi) \setminus free(\chi) = V$. In other words,

$$\varphi(X,Y,Z) \quad = \quad \underset{\substack{\text{(additional}\\ \text{bound variables}\\ U)}}{\psi(X,Y)} \quad \wedge \quad \underset{\substack{\text{(additional}\\ \text{bound variables}\\ V)}}{\chi(Y,Z)}$$

Let W be a new set of variables, disjoint from X, Y, Z, U, V, such that $|W| = k - |X \cup Y \cup Z|$. By disjointness of the sets involved, $|W \cup Z| = k - |X \cup Y| \leq |U|$ and $|W \cup X| = k - |Y \cup Z| \leq |V|$. This means that we can safely replace the (bound) variables U in ψ by the variables $W \cup Z$, and replace the (bound) variables V in χ by the variables $X \cup W$. The resulting formula is equivalent to the original, but only contains variables in $X \cup Y \cup Z \cup W$, of which there are only k many. □

10.2 Eliminating constants

Most results on guarded formulas have been stated only for relational first-order languages, i.e., languages without constants. The results discussed in this section show how the same techniques can be applied to formulas containing constants.

Let NCONS(φ) be the number of constants occurring in φ. Grädel [60] proved the following.[3]

10.2.1. PROPOSITION. *Every (loosely) $\forall$-guarded formula φ can be transformed in polynomial time into an equisatisfiable relational (loosely) $\forall$-guarded formula χ, such that* WIDTH(χ) $\leq$ WIDTH(φ) + NCONS(φ).

For complexity reasons, we have a particular interest in formulas with a bounded width. Unfortunately, for such formulas φ, Proposition 10.2.1 does not imply a bound on the width of χ. We will now present another method to eliminate constants, that allows us to circumvent this problem.

10.2.2. PROPOSITION. *Fix a natural number $k \geq 2$. Every loosely $\forall$-guarded formula φ of width at most k can be transformed in polynomial time into an equisatisfiable relational loosely $\forall$-guarded formula χ of width at most k.*

[3] Strictly speaking, Grädel's proof for this proposition is flawed, since his translation does not correctly handle formulas containing equality. However, this problem can easily be fixed.

Proof: Consider any loosely $\forall$-guarded formula φ of width at most k. By Proposition 10.1.3, we may assume that φ is in normal form and that $\textsc{maxarity}(\varphi) \leq k$.

Let $\textsc{cons}$ be the set of constants occurring in φ. For each n-place relation symbol R occurring in φ, except for equality, and for each partial function $f : \{1, \ldots, n\} \hookrightarrow \textsc{cons}$, introduce a new relation symbol R_f with arity $n - |dom(f)|$, where $dom(f)$ is the set of all $k \in \{1, \ldots, n\}$ for which $f(k)$ is defined. For example, if R is a ternary relation symbol and $f = \{(1, c), (3, d)\}$, then R_f is a unary relation symbol, which we will also denote by $R_{c\bullet d}$. The intended interpretation of $R_{c\bullet d}(x)$ will be the same as $R(c, x, d)$. Also, for each pair of constants c, d, introduce a nullary relation symbol E_{cd}.

We will now eliminate all constants, with the help of these new relation symbols. For any sequence of variables $\vec{x}$, let $T(\vec{x})$ be the set of all partial functions from $\{\vec{x}\}$ to $\textsc{cons}$ (including the empty function). Note that there are $(\textsc{ncons} + 1)^{|\vec{x}|}$ such functions. For each $\tau \in T(\vec{x})$ and formula ψ, let ψ^τ be the result of replacing each occurrence of a variable $x \in dom(\tau)$ by $\tau(x)$. Finally, let φ^* be obtained from φ by means of the following procedure.

1. Replace each subformula of the form $\forall\vec{x}.\psi$ by $\bigwedge_{\tau \in T(\vec{x})} \forall\vec{x}.\psi^\tau$, and replace each subformula of the form $\exists\vec{y}.\psi$ by $\bigvee_{\tau \in T(\vec{y})} \exists\vec{y}.\psi^\tau$. [4]

2. Replace each atomic formula of the form $R(c_1, \ldots, c_n, x_1, \ldots, x_m)$ by $R_{c_1 \ldots c_n \bullet \ldots \bullet}(x_1, \ldots, x_m)$ (and similarly for other permutations)

3. Replace each atomic formulas of the form $c = d$ by E_{cd}, and replace each atomic formula of the form $x = c$ or $c = x$ by $\bot$.

Let χ be the conjunction of φ^* with

$$\bigwedge_{c \in \textsc{cons}} E_{cc} \wedge \bigwedge_{c,d \in \textsc{cons}} E_{cd} \rightarrow E_{dc} \wedge \bigwedge_{c,d,e \in \textsc{cons}} E_{cd} \wedge E_{de} \rightarrow E_{ce}$$

and all formulas of the form

$$\forall x_1 \ldots x_m.(R_{c_1 \ldots c_\ell \ldots c_n \bullet \ldots \bullet}(x_1, \ldots, x_m) \rightarrow (E_{c_\ell d} \rightarrow R_{c_1 \ldots d \ldots c_n \bullet \ldots \bullet}(x_1, \ldots, x_m)))$$

(including all permutations the sequence $c_1, \ldots, c_n x_1, \ldots, x_m$). [5]

Clearly, χ does not contain any constants, and is loosely $\forall$-guarded. Furthermore, the length of χ is polynomial in the length of φ, and that χ can be obtained from φ' in polynomial time.

[4] Note that this will only polynomially increase the length of the formula, due to the fact that both the width and the quantifier depth of φ is bounded (keep in mind that φ is in normal form).

[5] The number of such formulas is approximately $\textsc{nrel}(\varphi) \cdot (\textsc{ncons}(\varphi)^{\textsc{maxarity}(\varphi)})$, where $\textsc{nrel}(\varphi)$ is the number of relation symbols occurring in φ. This is polynomial in the length of φ, given that $\textsc{maxarity}(\varphi) \leq k$.

Finally, we claim that χ is satisfiable iff φ is satisfiable. One direction of this claim is easy: a model for φ is easily turned into a model for χ. As for the other direction, every model M satisfying χ can be turned into a model M' for φ in the following way: define an equivalence relation on the set CONS by putting $c \sim d$ iff $M \models E_{cd}$, extend the domain of M with one element for each equivalence class, and extend the relations to the new elements in the obvious way: $([c_1], \ldots, [c_n], e_1, \ldots, e_m) \in R$ iff $(e_1, \ldots, e_m) \in R_{c_1 \ldots c_n \bullet \ldots \bullet}$, and likewise for other permutations. It is easily seen that the resulting model M' satisfies φ. □

Note that the translation used in the above proof is polynomial only provided that the width of the input formula is bounded by a constant. Unlike Grädel's translation, it is in general exponential.

We will now proceed with the proof of Theorem 10.0.5, using the help of the above results. As we already mentioned, Grädel [60] states his main results only in terms of guarded or loosely guarded formulas. Nevertheless, the central argument on which these results are based is formulated in terms of relational loosely ∀-guarded formulas in normal form, cf. Definition 10.1.1. Specifically, Grädel shows that the satisfiability problem for such formulas is 2EXPTIME-complete, and that it becomes EXPTIME-complete if there is a bound on the width of the (normal form) formula. Together with our above results, this allows us to prove Theorem 10.0.5.

Proof of Theorem 10.0.5: The 2EXPTIME-membership of the satisfiability problem for loosely ∀-guarded formulas follows from Grädel's result by Proposition 10.2.1 and Proposition 10.1.2. The EXPTIME-membership of the satisfiability problem for loosely ∀-guarded formulas with a bounded number of variables follows from Grädel's result by Proposition 10.2.2 and Proposition 10.1.2 (if a formula φ contains at most k variables, then, trivially, $\text{WIDTH}(\varphi) \leq k$).

Finally, it is easy to see that the width of a guarded formula is bounded by the arity of the relation symbols occuring in it. Note that, in general, this does not hold for ∀-guarded formulas, nor for loosely guarded formulas. Indeed, by a similar argument as used in the proof of Proposition 10.1.3, the satisfiability problem for loosely guarded formulas with arity at most 2 is already as hard as the satisfiability problem for loosely guarded formulas in general, i.e., 2EXPTIME-complete. □

10.3 Connections with hybrid logic, and interpolation

As we already mentioned in the introduction of this chapter, guarded fragments with constants have important applications in the area of hybrid logic. Conversely, results from the hybrid logic literature may have applications to guarded fragments with constants. Here, we will discuss one such application, which concerns the interpolation property. When the guarded fragment was introduced in

[2], it was hoped that it has interpolation. Unfortunately, it was shown in [66] that this is not the case. This negative interpolation result can be strengthened for guarded fragments with constants as follows:

10.3.1. THEOREM. *Let F be any fragment of first-order logic with constants that contains all atomic formulas, is closed under the Boolean connectives and is closed under guarded quantification (i.e., if $\varphi(\vec{x}\vec{y}) \in F$ and $\alpha(\vec{x}\vec{y})$ is atomic then $\exists\vec{x}(\alpha(\vec{x}\vec{y} \wedge \varphi(\vec{x}\vec{y}) \in F$ $\forall\vec{x}(\alpha(\vec{x}\vec{y} \to \varphi(\vec{x}\vec{y}) \in F$). Furthermore suppose that F satisfies the following form of interpolation:*

> *For all formulas $\varphi(x), \psi(x) \in F$ with at most one free variable x, if $\models \varphi(x) \to \psi(x)$ then there is a formula $\vartheta(x) \in F$ such that $\models \varphi(x) \to \vartheta(x)$, $\models \vartheta(x) \to \psi(x)$, and all relation symbols and constants occurring in ϑ occur both in φ and in ψ.*

Then every first-order formula φ with at most one free variable is equivalent to a formula in F.

Proof: As was noted in Remark 6.4.9, the proof of Theorem 6.4.4(ii) does not depends on the assumption that all modalities are unary. The result also holds if hybrid languages would be defined relative to a set of modalities MOD that includes k-ary modalities with $k \neq 1$. For present purposes, we may therefore assume that modalities can have any arity.

Having noted this, consider any fragment F satisfying the requirements mentioned in the statement of the theorem. Then F constitutes a hybrid language in the following sense. For any signature $\sigma = (\text{PROP}, \text{NOM})$, let σ^* be the first-order signature that has PROP as its unary predicates, NOM as its constants, and that has a relation $R_\triangle$ of arity $n(\triangle) + 1$ for each $\triangle \in \text{MOD}$ (here we assume again a fixed, given set of modalities MOD). Fix a first-order variable x, and for all signatures σ, let $\mathcal{L}_F[\sigma]$ be the collection of first-order formulas $\varphi(x)$ in the first-order signature σ^* that are in the fragment F. Furthermore, let $\mathfrak{M}, w \models_{\mathcal{L}_F} \varphi(x)$ iff $\varphi(x)$ holds in $\mathfrak{M}$ conceived of as a first-order structure, interpreting x as w. Then $(\mathcal{L}_F, \models_{\mathcal{L}_F})$ is a hybrid language, according to Definition 6.4.1.

In fact, we will show that it follows from the requirements on F, that $\mathcal{L}_F$ extends $\mathcal{H}(\mathsf{E})$. It is easily seen that $\mathcal{L}_F$ has interpolation on the class of all frames. Consequently, Theorem 6.4.4(ii) applies and we can conclude that $\mathcal{L}^1 \subseteq \mathcal{L}_F$. In other words, every first-order formula with at most one free variable x is equivalent to a formula in the fragment F.

To see that $\mathcal{L}_F$ extends $\mathcal{H}(\mathsf{E})$, consider any $\varphi \in \mathcal{H}(\mathsf{E})[\sigma \cup \{p_1, \ldots, p_n\}]$. and $\psi_1, \ldots, \psi_n \in \mathcal{L}_F[\sigma]$. We will show that there is a formula $\varphi' \in \mathcal{L}_F[\sigma]$ that is $[\vec{p}/\vec{\psi}]$-equivalent to φ, meaning that

$$\text{for all } \mathfrak{M} \in \mathsf{Str}[\sigma],\ \mathfrak{M} \models_{\mathcal{L}_F} \varphi' \text{ iff } \mathfrak{M}^{[p_1 \mapsto [\![\psi_1]\!]^{\mathfrak{M}}_{\mathcal{L}_F}, \ldots, p_n \mapsto [\![\psi_n]\!]^{\mathfrak{M}}_{\mathcal{L}_F}]} \models_{\mathcal{H}(\mathsf{E})} \varphi$$

The proof proceeds by induction on φ. The base case (where φ is a proposition letter or nominal from σ, or φ is $\top$ or φ is p_i for some $i \leq n$) follows from the assumptions on F. The inductive inductive steps for formulas of the form $\neg\chi$, $\chi_1 \wedge \chi_2$, $\Diamond\chi$ or $@_i\chi$ also follows from the assumptions on F (cf. the Standard Translation for $\mathcal{H}(\mathsf{E})$). □

In other words, it is not possible to repair interpolation for the guarded fragment by increasing its expressivity without ending up with full first-order logic. Note the modal character of interpolation property used in Theorem 10.3.1: it applies to formulas with at most one free variable. Also note that, while this result applies to the loosely guarded fragment, it does not cover the universally guarded fragment, or other fragments that are not closed under negation.

Without proof, we state two straightforward generalization of this result. Firstly, [66] show that, while interpolation fails for the Grädel-style guarded fragment, the *purely relational* guarded fragment (i.e., without constants) does satisfy a weak version of interpolation that is strong enough to entail the Beth property. Theorem 10.3.1 can be shown to apply also to this weak version of interpolation, provided that constants are allowed again.

Secondly, in the original definition of the guarded fragment by [2], identity statements are not allowed as guards (i.e., all quantifiers must be guarded by atomic formulas of the form $Rt_1 \dots t_n$). Assuming that constants are allowed, the least expressive extension of this version of the guarded fragment with interpolation is precisely what [2] refer to as the fragment $F3$.

10.4 Discussion

We finish by discussing two open questions. The first question is the following:

> *What is the complexity of the satisfiability problem for $\forall$-guarded formulas with bounded arity?*

Note that the answer to this question does not depend on the presence of constants. Our conjecture is that this problem is ExpTime-complete.

A second interesting question would be the following question:

> *Classify, in the style of Börger et al. [24], the quantifier patterns π for which the satisfiability problem for sentences consisting of a sequence of quantifiers conform π followed by a guarded formula, is decidable.*

The satisfiability problem for $\pi = \exists^*\forall$ is still decidable, as can be seen by replacing the existentially quantified variables by constants and guarding the universal quantifier by an identity statement of the form $x = x$. On the other hand, $\pi = \forall^3$ is already a conservative reduction class, as follows from results of Grädel [60]. What about $\pi = \exists^*\forall^2$?

Chapter 11

Relation algebra and $\mathcal{M}(\mathsf{D})$

Some of the typical features of modal formulas are (1) their local nature, which shows up in the fact that they are invariant under generated submodels, (2) the decidability of the satisfiability problem, and (3) their variable free notation. For each of these properties, one may ask to what extent the basic modal language could be extended while preserving the property.

This question has been answered partly in the previous chapters. In particular, Theorem 9.2.5 tells us that the bounded fragment is the largest fragment of first-order logic that is invariant under generated submodels. Unfortunately. the bounded fragment is undecidable and does not have a variable free notation.

Likewise, the guarded fragments discussed in Chapter 10 form large, but still decidable, extensions the modal language. Unfortunately, they lack interpolation and a variable free notation.

Relation algebra, which we will discuss in this chapter, can be seen as a large fragment of first-order logic that extends the modal fragment and that (unlike the bounded fragment and the guarded fragment) preserves the variable free nature of modal formulas. Unfortunately, it lacks interpolation, and is undecidable.

Relation algebra finds its origins in the work of Augustus De Morgan, Charles Sanders Peirce and Ernst Schröder in the nineteenth century. It was further developed and systematized by Tarski and others. For a recent overview, cf. Hirsch and Hodkinson [63]. The expressions of relation algebra denote binary relations. Formally, given a countably infinite set of atomic relations symbols, $R, S, \ldots$, the terms of relation algebra are given by the following inductive definition:

$$\alpha ::= R \mid \top \mid -\alpha \mid \alpha \cap \beta \mid \alpha \circ \beta \mid \otimes\alpha \mid \delta$$

Here, $\top$ is the total relation (over the given domain), $-\alpha$ denotes the complement of the relation α, $\alpha \cap \beta$ denotes the intersection of α and β, $\alpha \circ \beta$ denotes the relational composition of α and β, $\otimes\alpha$ denotes the converse of the relation α, and δ is a constant that denotes the identity relation. Thus, each relation algebra term denotes a binary relation, and the relation denoted by a term can be computed on the basis of the denotation of the atomic relation symbols occurring in it.

Table 11.1: Translation from modal logic to relation algebra

$$
\begin{aligned}
p^* &= R_p \\
(\top)^* &= \delta \\
(\neg\varphi)^* &= -\varphi^* \cap \delta \\
(\varphi \wedge \psi)^* &= \varphi^* \cap \psi^* \\
(\Diamond\varphi)^* &= (R_\Diamond \circ \varphi^* \circ \top) \cap \delta
\end{aligned}
$$

Relation algebra is a fragment of first-order logic, in the following sense: each term of relation algebra corresponds to a first-order formula in two free variables. For instance, the term $R \circ S$ corresponds to the first-order formula $\exists z.(Rx_1z \wedge Szx_2)$. In fact, it has been shown that every term of relation algebra corresponds to a first-order formula in two free variables containing at most three variables, and vice versa [93].

The basic modal language is again a fragment of relation algebra. Since relation algebraic terms denote binary relations, it is convenient to associate with each proposition letter p a subrelation R_p of the identity relation, where $(w, w) \in R_p$ iff w satisfies p). Then, Table 11.1 provides a translation from the modal language into the language of relation algebra. It is not hard to see that a world w satisfies φ iff w stands in the relation φ^* to itself.

In this chapter, we will show that the only way to repair interpolation for relation algebra is to the extend the language such that every first-order operation on binary relations becomes definable. Roughly speaking, this means that first-order logic is the smallest extension of relation algebra with interpolation. In order to prove this, we will first consider $\mathcal{M}(\mathsf{D})$, which is the extension of the basic modal language with the difference operator. We will show that the first-order correspondence language $\mathcal{L}^1$ is the smallest extension of $\mathcal{M}(\mathsf{D})$ with interpolation. Next, we use a well known connection between $\mathcal{M}(\mathsf{D})$ and relation algebra in order to derive the above mentioned result. These results are taken from [28].

11.1 $\mathcal{M}(\mathsf{D})$ and its relation to $\mathcal{H}(\mathsf{E})$

The language of difference logic, denoted by $\mathcal{M}(\mathsf{D})$, is obtained by extending the basic modal language with a logical modality D, where $\mathsf{D}\varphi$ is interpreted as "φ holds somewhere else." More precisely, the formulas of $\mathcal{M}(\mathsf{D})$ are given by

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Diamond\varphi \mid \mathsf{D}\varphi$$

where $p \in$ PROP and $\Diamond \in$ MOD. The truth definition for the basic modal language is extended by letting $\mathfrak{M}, w \models \mathsf{D}\varphi$ iff $\mathfrak{M}, v \models \varphi$ for some world v distinct from w. In other words, the accessibility relation for D is the inequality relation.

The expressivity of $\mathcal{M}(\mathsf{D})$ with respect to models has been studied by [85]. In [46], the elementary frame properties definable in $\mathcal{M}(\mathsf{D})$ are characterized. Completeness results for $\mathcal{M}(\mathsf{D})$ can be found in [90, 98].

In this chapter, we use our results on hybrid logic to derive some further results for difference logic, and also relation algebra and first-order logic.

There is a strong connection between $\mathcal{M}(\mathsf{D})$ and the hybrid language $\mathcal{H}(\mathsf{E})$. On the one hand, nominals are definable in $\mathcal{M}(\mathsf{D})$, in the sense that $\mathsf{E}(p \wedge \neg \mathsf{D}p)$ is true in a model precisely if p has a singleton denotation. On the other hand, $\mathsf{D}\varphi$ holds at the world named by the nominal i precisely if $\mathsf{E}(\neg i \wedge \varphi)$ is true. In fact, the following has been shown.[1]

11.1.1. THEOREM ([3]). *There are polynomial translations between $\mathcal{H}(\mathsf{E})$ and $\mathcal{M}(\mathsf{D})$ that preserve validity with respect to any frame.*

It follows that for all frame classes K, K is definable in $\mathcal{H}(\mathsf{E})$ iff K is definable in $\mathfrak{M}(\mathsf{D})$, and it also follows that the satisfiability problem for $\mathcal{M}(\mathsf{D})$-formulas with respect to K has the same complexity (up to a polynomial) as that for $\mathcal{H}(\mathsf{E})$-formulas. In combination with Corollary 4.3.2, this gives us the following result.

11.1.2. COROLLARY. *An elementary frame class is definable in $\mathcal{M}(\mathsf{D})$ iff it is closed under ultrafilter morphic images.*

Gargov and Goranko give a similar characterization of the elementary frame classes definable in $\mathcal{M}(\mathsf{D})$. Their result states:

11.1.3. THEOREM ([46]). *An elementary class K is definable in $\mathcal{M}(\mathsf{D})$ iff the following closure condition holds, where we use $\neq_W$ to denote the inequality relation on the set W:*

> *If $(W, R) \in \mathsf{K}$, and $\mathfrak{ue}(W', R', \neq_{W'})$ is a bounded morphic image of $(W, R, \neq_W)$, then $(W', R') \in \mathsf{K}$.*

While the two characterizations are quite similar, we have not been able to derive our result from Gargov and Goranko's. Incidentally, the proofs are also quite different. The proof of [46] uses algebraic techniques and is not easily adapted to other hybrid languages such as $\mathcal{H}(@)$. On the other hand, our result was proved purely model theoretically and the same technique was used to characterize the frame definable power of $\mathcal{H}$ and $\mathcal{H}(@)$.

Another result on difference logic that we obtain as a corollary of our results on hybrid logics is the following:

[1] Gargov and Goranko [46] proved a similar result, but involving an exponential translation.

11.1.4. Corollary. *Let* K *be any frame class that admits polynomial filtration. Satisfiability of* $\mathcal{M}(\mathsf{D})$*-formulas on* K *is polynomially reducible to satisfiability of* $\mathcal{M}(\mathsf{E})$*-formulas on* K.

This follows immediately from Theorem 8.2.3.

11.2 Repairing interpolation for $\mathcal{M}(\mathsf{D})$

Recall the characterization of $\mathcal{H}(@, \downarrow)$ and $\mathcal{L}^1$ in terms of interpolation presented by Theorem 6.4.4. In this section, we give a similar result, using $\mathcal{M}(\mathsf{D})$ instead of $\mathcal{H}(\mathsf{E})$. More precisely, we show that the least expressive extension of $\mathcal{M}(\mathsf{D})$ with interpolation is the first-order correspondence language. The proof is similar to that of Theorem 6.4.4(ii), but a number of small modifications need to be made. In particular, the abstract notion of a hybrid language used there needs to be replaced by that of a modal language, by removing all reference to nominals. However, since the main line of the proof remains the same, in what follows, we will be slightly more concise than in Section 6.4.

We will assume a fixed set of (unary) modalities MOD. A signature σ is simply a set of proposition letters. Given a signature σ, a (pointed, but not necessarily point-generated) σ*-model* is a structure $\mathfrak{M} = (\mathfrak{F}, V, w)$ where $\mathfrak{F} = (W, R_\diamond)_{\diamond \in \textsc{mod}}$ is a frame, $V : \sigma \to \wp(W)$ a valuation and $w \in W$ a world. The class of all σ-models is denoted by $\mathsf{Str}[\sigma]$. Furthermore, for any class of frames F, $\mathsf{Str}_\mathsf{F}[\sigma]$ will denote the class of σ-models of which the underlying frame belongs to F.

For any model $\mathfrak{M} = (\mathfrak{F}, V, w) \in \mathsf{Str}[\tau]$ and function $\rho : \sigma \to \tau$, let $\mathfrak{M}^\rho$ be the σ-model $(\mathfrak{F}, \rho \cdot V, w)$. Secondly, if $\mathfrak{M} \in \mathsf{Str}[\tau]$ and $\sigma \subseteq \tau$, then $\mathfrak{M} \upharpoonright \sigma$ denotes the σ-reduct of $\mathfrak{M}$, i.e., the σ-model that is obtained from $\mathfrak{M}$ by "forgetting" the interpretation of $\tau \backslash \sigma$. We write $\mathsf{K} \upharpoonright \sigma$ for $\{\mathfrak{M} \upharpoonright \sigma \mid \mathfrak{M} \in \mathsf{K}\}$.

11.2.1. Definition (Modal languages). *A* modal language *is a pair* $(\mathcal{L}, \models_\mathcal{L}$ *), where* $\mathcal{L}$ *is a map from signatures to sets of formulas, and* $\models_\mathcal{L}$ *is a relation between formulas and models satisfying the following conditions.*

1. **Expansion Property.** *If* $\sigma \subseteq \tau$ *then* $\mathcal{L}[\sigma] \subseteq \mathcal{L}[\tau]$*. Furthermore, for all* $\varphi \in \mathcal{L}[\sigma]$ *and* $\mathfrak{M} \in \mathsf{Str}[\tau]$, $\mathfrak{M} \models_\mathcal{L} \varphi$ *iff* $\mathfrak{M} \upharpoonright \sigma \models_\mathcal{L} \varphi$*. For* $\mathfrak{M} \in \mathsf{Str}[\sigma]$*, the statement* $\mathfrak{M} \models \varphi$ *is defined (i.e., true or false) if and only if* $\varphi \in \mathcal{L}[\sigma]$*. Otherwise, it is undefined.*

2. **Renaming Property** *For all* $\varphi \in \mathcal{L}[\sigma]$ *and* $\rho : \sigma \to \tau$*, there is a* $\psi \in \mathcal{L}[\tau]$ *such that for all* $\mathfrak{M} \in \mathsf{Str}[\tau]$, $\mathfrak{M} \models \psi$ *iff* $\mathfrak{M}^\rho \models \varphi$.

We will use $\mathcal{L}$ also to refer to the pair $(\mathcal{L}, \models_\mathcal{L})$.

Given a model $\mathfrak{M} = (\mathfrak{F}, V, w)$ and an element v of the domain of $\mathfrak{F}$, we will use $(\mathfrak{M}, v)$ to denote the model $(\mathfrak{F}, V, v)$. Thus, with $\mathfrak{M}, v \models \varphi$ we mean $(\mathfrak{F}, V, v) \models \varphi$.

For $\varphi \in \mathcal{L}[\sigma]$, let $\mathsf{Mod}^{\sigma}_{\mathcal{L}}(\varphi) = \{\mathfrak{M} \in \mathsf{Str}[\sigma] \mid \mathfrak{M} \models_{\mathcal{L}} \varphi\}$. For $\mathfrak{M} \in \mathsf{Str}[\sigma]$ and $\varphi \in \mathcal{L}[\sigma]$, let $[\![\varphi]\!]^{\mathfrak{M}}_{\mathcal{L}} = \{v \mid \mathfrak{M}, v \models \varphi\}$, i.e., the subset of the domain of $\mathfrak{M}$ defined by φ.

Finally, the symbol $\models$ will be used not only to refer to the satisfaction relation, but also to the *local consequence* relation: for $\Phi \cup \{\psi\} \subseteq \mathcal{L}[\sigma]$, we say that $\Phi \models_{\mathcal{L}} \varphi$ iff for all $\mathfrak{M} \in \mathsf{Str}[\sigma]$, it holds that if $\mathfrak{M} \models_{\mathcal{L}} \varphi$ for $\varphi \in \Phi$ then $\mathfrak{M} \models_{\mathcal{L}} \psi$.

When restricting attention to a specific frame class F, we will write $\mathsf{Mod}^{\sigma}_{\mathcal{L},\mathsf{F}}(\varphi)$ for $\{\mathfrak{M} \in \mathsf{Str}_{\mathsf{F}}[\sigma] \mid \mathfrak{M} \models_{\mathcal{L}} \varphi\}$. Likewise, for $\Phi \cup \{\psi\} \subseteq \mathcal{L}[\sigma]$, we say that $\Phi \models_{\mathcal{L},\mathsf{F}} \varphi$ iff $\bigcap_{\varphi \in \Phi} \mathsf{Mod}^{\sigma}_{\mathcal{L},\mathsf{F}}(\varphi) \subseteq \mathsf{Mod}^{\sigma}_{\mathcal{L},\mathsf{F}}(\psi)$.

11.2.2. Definition (Extensions of modal languages). *Let $\mathcal{L}, \mathcal{L}'$ be modal languages. Then $\mathcal{L}'$ extends $\mathcal{L}$ relative to a frame class F (notation: $\mathcal{L} \subseteq_{\mathsf{F}} \mathcal{L}'$) if the following holds for all signatures σ and proposition letters $p_1, \ldots, p_n$ ($n \geq 0$).*

- *For each $\varphi \in \mathcal{L}[\sigma \cup \{p_1, \ldots, p_n\}]$ and $\psi_1, \ldots, \psi_n \in \mathcal{L}'[\sigma]$, there is a formula of $\mathcal{L}'[\sigma]$, which we will denote by $\varphi^{[\vec{p}/\vec{\psi}]}$, such that for all $\mathfrak{M} \in \mathsf{Str}_{\mathsf{F}}[\sigma]$, $\mathfrak{M} \models_{\mathcal{L}'} \varphi^{[\vec{p}/\vec{\psi}]}$ iff $\mathfrak{M}^{[p_1 \mapsto [\![\psi_1]\!]^{\mathfrak{M}}_{\mathcal{L}'}, \ldots, p_n \mapsto [\![\psi_n]\!]^{\mathfrak{M}}_{\mathcal{L}'}]} \models_{\mathcal{L}} \varphi$.*

The basic modal language language $\mathcal{M}$ and its extension $\mathcal{M}(\mathsf{D})$ are modal languages in the sense of Definition 11.2.1. The first-order correspondence language $\mathcal{L}^1$ also constitutes a modal language, if we consider only formulas with at most one free variable.[2]

11.2.3. Definition (Interpolation). *A modal language $\mathcal{L}$ has* interpolation *on a frame class F if for all $\varphi \in \mathcal{L}[\sigma]$ and $\psi \in \mathcal{L}[\tau]$ such that $\varphi \models_{\mathcal{L},\mathsf{F}} \psi$, there is a $\vartheta \in \mathcal{L}[\sigma \cap \tau]$ such that $\varphi \models_{\mathcal{L},\mathsf{F}} \vartheta$, and $\vartheta \models_{\mathcal{L},\mathsf{F}} \psi$.*

The reader should keep in mind that $\models_{\mathcal{L},\mathsf{F}}$ denotes the *local* entailment relation.

Now for the main result of this section.

11.2.4. Theorem. *Let $\mathcal{L}$ be any modal language, and let F be any frame class. If $\mathcal{M}(\mathsf{D}) \subseteq_{\mathsf{F}} \mathcal{L}$ and $\mathcal{L}$ has interpolation on F then $\mathcal{L}^1 \subseteq_{\mathsf{F}} \mathcal{L}$.*

The remainder of this section is devoted to the proof of Theorem 11.2.4.

11.2.5. Definition (Projective classes). *Let σ be a signature, and let $\mathsf{K} \subseteq \mathsf{Str}_{\mathsf{F}}[\sigma]$. Then K is a* projective class *of a modal language $\mathcal{L}$ relative to a frame class F if there is a $\varphi \in \mathcal{L}[\tau]$ with $\tau \supseteq \sigma$, such that $\mathsf{K} = \mathsf{Mod}^{\tau}_{\mathcal{L},\mathsf{F}}(\varphi) \upharpoonright \sigma$.*

---

[2] In this chapter, we use $\mathcal{L}^1$ to refer to the first-order correspondence language of modal logic, as opposed to the first-order correspondence language for hybrid logic (which contains in addition constants).

11.2.6. Definition (Negation). *A modal language* $\mathcal{L}$ has negation *on* F *if for each* $\varphi \in \mathcal{L}[\sigma]$ *there is an formula of* $\mathcal{L}[\sigma]$*, which we will denote by* $\neg\varphi$*, such that* $\mathsf{Mod}_{\mathcal{L},\mathsf{F}}(\psi) = \mathsf{Str}_{\mathsf{F}}[\sigma] \backslash \mathsf{Mod}_{\mathcal{L},\mathsf{F}}(\varphi)$.

11.2.7. Lemma. *Let* $\mathcal{L}$ *be a modal language with negation that has interpolation on a frame class* F*, and let* $\mathsf{K} \subseteq \mathsf{Str}_{\mathsf{F}}[\sigma]$*, for some signature* σ*. If both* K *and* $\mathsf{Str}_{\mathsf{F}}[\sigma] \backslash \mathsf{K}$ *are projective classes of* $\mathcal{L}$ *relative to* F*, then there is a* $\varphi \in \mathcal{L}[\sigma]$ *such that* $\mathsf{K} = \mathsf{Mod}_{\mathcal{L},\mathsf{F}}(\varphi)$.

Proof: Since K is a projective class, there is a formula $\varphi \in \mathcal{L}[\tau]$, with $\sigma \subseteq \tau$, such that $\mathsf{K} = \mathsf{Mod}_{\mathcal{L},\mathsf{F}}(\varphi) \restriction \sigma$. Likewise, since $\mathsf{Str}_{\mathsf{F}}[\sigma] \backslash \mathsf{K}$ is a projective class, there is a formula $\psi \in \mathcal{L}[\tau']$, with $\sigma \subseteq \tau'$, such that $\mathsf{Str}_{\mathsf{F}}[\sigma] \backslash \mathsf{K} = \mathsf{Mod}_{\mathcal{L},\mathsf{F}}(\psi) \restriction \sigma$. Without loss of generality, we may assume that $\tau \cap \tau' = \sigma$ (by the Renaming property of $\mathcal{L}$). It follows that $\varphi \models_{\mathcal{L},\mathsf{F}} \neg\psi$. Since $\mathcal{L}$ has interpolation, there must be a $\vartheta \in \mathcal{L}[\sigma]$ such that $\varphi \models_{\mathcal{L},\mathsf{F}} \vartheta$ and $\vartheta \models_{\mathcal{L},\mathsf{F}} \neg\psi$. As a last step, we will show that $\mathsf{Mod}_{\mathcal{L},\mathsf{F}}(\vartheta) = \mathsf{K}$.

Suppose $\mathfrak{M} \in \mathsf{K}$. Then $\mathfrak{M} = \mathfrak{N} \restriction \sigma$ for some $\mathfrak{N} \in \mathsf{Mod}_{\mathcal{L},\mathsf{F}}(\varphi)$. Since $\varphi \models_{\mathcal{L},\mathsf{F}} \vartheta$, it follows that $\mathfrak{N} \models \vartheta$. By the Expansion property, $\mathfrak{M} \models \vartheta$. Conversely, suppose $\mathfrak{M} \notin \mathsf{K}$. Then $\mathfrak{M} = \mathfrak{N} \restriction \sigma$ for some $\mathfrak{N} \in \mathsf{Mod}_{\mathcal{L},\mathsf{F}}(\psi)$. Since $\vartheta \models_{\mathcal{L},\mathsf{F}} \neg\psi$, it follows that $\mathfrak{N} \not\models \vartheta$. By the Expansion property, $\mathfrak{M} \not\models \vartheta$. □

11.2.8. Lemma. *Let* $\mathcal{L}$ *be a modal language with interpolation on a frame class* F*, such that* $\mathcal{M}(\mathsf{D}) \subseteq_{\mathsf{F}} \mathcal{L}$*. Then for all* $\varphi \in \mathcal{L}[\sigma]$ *and* $p \in \sigma$*, there is a formula of* $\mathcal{L}[\sigma \backslash \{p\}]$*, which we will denote by* $\downarrow p.\varphi$*, such that* $\mathsf{Mod}_{\mathcal{L},\mathsf{F}}(\downarrow p.\varphi) = \{(\mathfrak{F}, V, w) \in \mathsf{Str}_{\mathsf{F}}[\sigma \backslash \{p\}] \mid (\mathfrak{F}, V^{[p \mapsto \{w\}]}, w) \models \varphi\}$.

Proof: Let $\mathsf{K}_{\downarrow p.\varphi} = \{(\mathfrak{F}, V, w) \in \mathsf{Str}_{\mathsf{F}}[\sigma \backslash \{p\}] \mid (\mathfrak{F}, V^{[p \mapsto \{w\}]}, w) \models \varphi\}$. $\mathsf{K}_{\downarrow p.\varphi}$ is projectively defined by $p \wedge \neg \mathsf{D} p \wedge \varphi$ and its complement is projectively defined by $p \wedge \neg \mathsf{D} p \wedge \neg\varphi$. Since $\mathcal{L}$ has negation and has interpolation on F, by Lemma 6.4.7 $\mathsf{K}_{\downarrow p.\varphi} = \mathsf{Mod}_{\mathcal{L},\mathsf{F}}(\psi)$ for some $\psi \in \mathcal{L}[\sigma \backslash \{p\}]$. □

Proof of Theorem 11.2.4: Let $\mathcal{L}$ be any modal language with interpolation over nominals on a frame class F, such that $\mathcal{M}(\mathsf{D}) \subseteq_{\mathsf{F}} \mathcal{L}$. Let $\varphi \in \mathcal{L}^1[\sigma \cup \{p_1, \ldots, p_n\}]$ and $\psi_1, \ldots, \psi_n \in \mathcal{L}[\sigma]$. We will show that there is a formula $\chi \in \mathcal{L}[\sigma]$ that is $[\vec{p}/\vec{\psi}]$-equivalent to φ on F, meaning that

$$\text{for all } \mathfrak{M} \in \mathsf{Str}_{\mathsf{F}}[\sigma],\ \mathfrak{M} \models_{\mathcal{L}} \chi \text{ iff } \mathfrak{M}^{[p_1 \mapsto [\![\psi_1]\!]^{\mathfrak{M}}_{\mathcal{L}}, \ldots, p_n \mapsto [\![\psi_n]\!]^{\mathfrak{M}}_{\mathcal{L}}]} \models_{\mathcal{H}(@)} \varphi$$

The proof proceeds by induction on the length of φ. To simplify the induction, we will temporarily extend the syntax of $\mathcal{L}^1$, by allowing unary predicates to occur as arguments of other predicates. For instance, $R(y, P)$ is allowed as an atomic formula, and it is interpreted as $\exists x.(Px \wedge Ryx)$. This change clearly does not affect the expressive power of $\mathcal{L}^1$, but it will make the inductive argument simpler.

It is not hard to see that in the base case, where φ is an atomic formula, the claim holds. Also the inductive step for formulas of the form $\neg\psi$ or $\psi_1 \wedge \psi_2$ is straightforward (cf. also the proof of Theorem 6.4.4(i)). Finally, let φ be of the form $\exists y.\psi$. By the definition of $\mathcal{L}^1$, φ contains at most one free variable, say x (in case φ contains no free variables, let x be any variable distinct from y). Let p, q be distinct proposition letters (unary predicates) not occurring in σ. By induction hypothesis, $\varphi[x/i, y/j] \in \mathcal{L}^1[\sigma \cup \{i, j\}]$ is $[\vec{p}/\vec{\psi}]$-equivalent on F to some $\chi \in \mathcal{L}[\sigma \cup \{p, q\}]$. By Lemma 11.2.8 and by the fact that $\mathcal{M}(\mathsf{D}) \subseteq_{\mathsf{F}} \mathcal{L}$, we obtain a formula $\downarrow p.\mathsf{E}\downarrow q.\chi \in \mathcal{L}[\sigma]$ that is easily shown to be $[\vec{p}/\vec{\psi}]$-equivalent to φ on F. □

11.3 An application to relation algebra

In the introduction of this chapter, we mentioned that the basic modal language can be seen as a fragment of relation algebra. As it happens, relation algebra can itself be thought of as an instance of the basic modal language, with a specific set of modalities, and interpreted on a specific class of frames.

We will consider the basic modal language over a collection of three modalities: a binary modality $\circ$, a unary modality $\otimes$, and a null-ary modality (modal constant) δ. Thus, the formulas of this language are given by

$$\varphi ::= p \mid \top \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \circ \psi \mid \otimes\varphi \mid \delta$$

The corresponding frames have three accessibility relations, one for each modality. Let SQ be the class of such frames $\mathfrak{F} = (W, R_\circ, R_\otimes, R_\delta)$ for which there is a set U such that $W = U \times U$, and

$$\begin{array}{lcl} R_\circ & = & \{((w,v),(w,u),(u,v)) \mid w,v,u \in U\} \text{ (i.e., } R_\circ \text{ denotes composition)} \\ R_\otimes & = & \{((w,v),(v,w)) \mid w,v \in U\} \text{ (i.e., } R_\otimes \text{ denotes inverse)} \\ R_\delta & = & \{(w,w) \mid w \in U\} \text{ (i.e., } R_\delta \text{ denotes the identity relation on } U) \end{array}$$

The basic modal language interpreted on the frame class SQ is known as *arrow logic*. In fact, it is relation algebra in disguise. Arrow logic is known not to have interpolation. Theorem 11.2.4 tells us what it takes to repair interpolation: it tells us that the first-order correspondence language is the smallest extension of the basic modal logic that has interpolation on SQ.

Note that while Theorem 11.2.4 was only proved for languages with unary modalities, the proof generalized to languages such as that of arrow logic, that have modalities with other arities (cf. also Remark 6.4.9).

11.3.1. THEOREM. *$\mathcal{L}^1$ is the least expressive extension of the basic modal language with interpolation on* SQ.

Proof: The difference operator is definable relative to SQ: for any formula φ, $\mathsf{D}\varphi$ is equivalent to $(\neg\delta \circ \varphi \circ \top) \vee (\top \circ \varphi \circ \neg\delta)$ [96]. Hence, in terms of Definition 11.2.2, the basic modal language $\mathcal{M}$ extends $\mathcal{M}(\mathsf{D})$ relative to SQ. It follows by Theorem 11.2.4 that every modal language extending $\mathcal{M}$ relative to SQ that has interpolation on SQ extends $\mathcal{L}^1$ relative to SQ. Finally, that $\mathcal{L}^1$ itself has interpolation relative to SQ follows immediately from the fact that SQ is an elementary frame class. □

In fact, in order to repair interpolation, an extension of the language of arrow logic was proposed, called $RL{\downarrow}$ [78]. In the same paper, it is shown that $RL{\downarrow}$ is equally expressive as $\mathcal{L}^1$ (on SQ). Hence, Theorem 11.3.1 tells us that, in some sense, the results of [78] are optimal.

We can rephrase Theorem 11.3.1 in relation algebraic terms by observing that every elementary operation on binary relations is definable in $\mathcal{L}^1$ over SQ. To make this precise, we need to introduce some terminology. Every first-order formula of the form $\varphi(R_1, \ldots, R_n, x, y)$, where $R_1, \ldots, R_n$ are binary relation symbols, defines an n-ary operation $\mathcal{O}$ on binary relations: given binary relations $R_1, \ldots, R_n$ on a set D, $\mathcal{O}(R_1, \ldots, R_n) = \{(d, e) \in D \mid (D, R_1, \ldots, R_n) \models \varphi\ [d, e]\}$. Operations on binary relations that are defined by a first-order formula in this way are called *elementary*. Examples are intersection $(R_1xy \wedge R_2xy)$, complement $(\neg Rxy)$ and composition $(\exists z.(R_1xz \wedge R_2zy))$.

11.3.2. Proposition. *Let $\mathcal{O}$ be any n-ary elementary operation on binary relations ($n \geq 0$). Then there is a formula $\chi(p_1, \ldots, p_n) \in \mathcal{L}^1[\{p_1, \ldots, p_n\}]$ (involving the modalities $\circ$, $\otimes$ and δ), such that for all models $\mathfrak{M}$ based on a frame in SQ, $[\![\chi(p_1, \ldots, p_n)]\!]^{\mathfrak{M}}_{\mathcal{L}^1} = \mathcal{O}([\![p_1]\!]^{\mathfrak{M}}_{\mathcal{L}^1}, \ldots, [\![p_n]\!]^{\mathfrak{M}}_{\mathcal{L}^1})$.*

Proof: Let $\varphi(R_1, \ldots, R_n, x, y)$ be any first-order formula defining a map from n binary relations to a single binary relation. Pick corresponding proposition letters (unary predicates) $P_1, \ldots, P_n$, and define φ^* inductively as follows

$$
\begin{array}{lcl}
(R_kxy)^* & = & \exists z.(P_k(z) \wedge R_\circ zxz \wedge R_\circ zzy) \\
(x = y)^* & = & x = y \\
\top^* & = & \top \\
(\varphi \wedge \psi)^* & = & \varphi^* \wedge \psi^* \\
(\neg\varphi)^* & = & \neg(\varphi^*) \\
(\exists x.\varphi)^* & = & \exists x.(R_\delta(x) \wedge \varphi^*)
\end{array}
$$

Finally, let $\chi(x) \in \mathcal{L}^1[\sigma]$ be the formula $\exists yz.(\varphi^*(y, z) \wedge R_\circ xyx \wedge R_\circ xxz)$. Then for all models $\mathfrak{M}$ based on a frame in SQ, $[\![\chi(p_1, \ldots, p_n)]\!]^{\mathfrak{M}}_{\mathcal{L}^1} = \mathcal{O}([\![p_1]\!]^{\mathfrak{M}}_{\mathcal{L}^1}, \ldots, [\![p_n]\!]^{\mathfrak{M}}_{\mathcal{L}^1})$. The proof of this claim is left to the reader. □

Algebraically speaking, we can conclude from this that the only way to restore interpolation for the class of representable relation algebra by expansion is to add the entire clone of elementary operations on binary relations. In particular, it does not suffice to add only finitely many elementary operations, or to add only Jónsson's Q-operators [97].

Chapter 12

Second order propositional modal logic

In this chapter, we consider the extension of the basic modal logic with propositional quantifiers introduced in 1970 by Fine [41]. The formulas of this languages are generated by the following recursive definition:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Diamond\varphi \mid \exists p.\varphi \mid \forall p.\varphi$$

The propositional quantifiers are interpreted in the expected way: $\exists p.\varphi$ is true if there is a subset X of the domain such that φ holds when the valuation is changed such that p denotes X, and similar for the universal quantifier.

In what follows, we will refer to this language as *second order propositional modal logic* (SOPML). This name that is justified by the fact that many formulas of SOPML express non-elementary properties, even on the level of models. Consider for instance the formula $\forall p.(\Box\Diamond p \to \Diamond\Box p)$. If this formula would have a first-order equivalent $\chi(x)$, then $\forall x.\chi(x)$ would define the class of frames defined by the McKinsey formula, which is known to be non-elementary. It follows that $\forall p.(\Box\Diamond p \to \Diamond\Box p)$ does not have a first-order equivalent. In other words, the standard translation cannot be extended to the full SOPML.

A formula of SOPML is *in prefix form* if it is of the form $Q_1 p_1 \cdots Q_n p_n.\varphi$, where $Q_1, \ldots, Q_n \in \{\exists, \forall\}$ and φ is a quantifier free modal formula.

It was shown by Fine [41] that second-order arithmetic can be interpreted in SOPML. This result was strengthened by Kaminski and Tiomkin [68], where it was shown that there is a satisfiability preserving translation from full second order logic to SOPML.[1] It follows immediately that the satisfiability problem for SOPML is not decidable, and in fact not analytical.

Nevertheless, not every second order formula is equivalent to a formula of SOPML. This follows from the fact that SOPML formulas are invariant under

[1] Kremer [71, 72] further strengthened this result by showing that such translation from second order logic exists already for the extension of several intuitionistic and relevance logics with propositional quantifiers.

generated subframes, as was observed by Van Benthem [11]. The precise expressive power of SOPML has not been characterized. In particular, the following questions asked by Van Benthem [11] have not been answered so far:

1. Is every SOPML formula equivalent to one in prefix form?
2. Is every bounded $\mathcal{L}^1$-formula equivalent to a formula of SOPML?

In what follows, we will answer these questions positively. Furthermore, we will show that the basic modal language is the bisimulation invariant fragment of SOPML, and we will show that $\mathcal{H}(@,\downarrow)$ has the same expressive power as the first-order definable part of SOPML.

12.0.1. Proposition. *Every formula of SOPML is equivalent to one in prefix form.*

Proof: We will prove the result for uni-modal languages. The proof generalizes straightforwardly to the multi-modal case. Let $\varphi_1(p)$ be the formula $\Diamond p \wedge \forall q(\Diamond(p \wedge q) \rightarrow \Box(p \rightarrow q))$, which holds iff there is exactly one successor satisfying p. Now, given a second order modal formula φ, one can move all quantifiers to the front of the formula using the following equivalences.

$$
\begin{array}{lcl|lcl}
\neg\exists p.\psi & = & \forall p.\neg\psi & \neg\forall p.\psi & = & \exists p.\neg\psi \\
(\exists p.\psi)\wedge\chi & = & \exists p.(\psi\wedge\chi) & (\forall p.\psi)\wedge\chi & = & \forall p.(\psi\wedge\chi) \\
\Diamond\exists p.\psi & = & \exists p.\Diamond\psi & \Diamond\forall p.\psi & = & \exists q\forall p.(\varphi_1(q)\wedge\Box(q\rightarrow\psi))
\end{array}
$$

where p does not occur in χ and q does not occur in ψ. The resulting formula might still not be in prefix form due to the newly introduced φ_1-subformulas, but it can easily be transformed in prefix form. Furthermore, it is equivalent to the original formula φ. □

The following analogue of Theorem 2.2.3 holds for second order modal logic.

12.0.2. Theorem. *Both on finite models and in general: a formula φ of SOPML is invariant under bisimulations iff φ is equivalent to a formula of the basic modal language.*

Proof: One direction follows simply from the bisimulation of modal formulas. For the other direction, we will use the notion of n-bisimulation [21]. Let $n \in \omega$, let $\mathfrak{M}, \mathfrak{N}$ be models and let w and v states of $\mathfrak{M}$ and $\mathfrak{N}$, respectively. We say that w is n-bisimilar to w' (notation: $\mathfrak{M}, w \leftrightarroweq_n \mathfrak{N}, w'$) if there exists a sequence of binary relations $Z_0, \ldots, Z_n$ satisfying the following properties:

(i) wZ_0w'

(ii) If vZ_iv' then v and v' agree on all proposition letters

(iii) If vZ_iv' for $i < n$ and $vR_\diamond u$ then there exists a u' with $v'R'_\diamond u'$ and $uZ_{i+1}u'$

(iv) If vZ_iv' for $i < n$ and $v'R'_\diamond u'$ then there exists a u with $vR_\diamond u$ and $uZ_{i+1}u'$

Consider any formula φ of SOPML that is invariant under bisimulations Let k be the modal depth of φ (i.e., the maximal nesting degree of modal operators). As a first step, we will show that φ is invariant under k-bisimulations. Suppose $(\mathfrak{M}, w) \leftrightarrows_k (\mathfrak{N}, v)$. Let $(\widehat{\mathfrak{M}}, w)$ be the tree-unraveling of $(\mathfrak{M}, w)$ and let $\widehat{\mathfrak{M}}_k$ be the submodel of $\widehat{\mathfrak{M}}$ consisting of all points reachable from w in at most k steps (along the union of all accessibility relations). By construction, $(\mathfrak{M}, w) \leftrightarrows (\widehat{\mathfrak{M}}, w)$, and $(\widehat{\mathfrak{M}}, w)$ satisfies φ iff $(\widehat{\mathfrak{M}}_k, w)$ does. Define $\widehat{\mathfrak{N}}$ and $\widehat{\mathfrak{N}}_k$ similarly. Then $(\widehat{\mathfrak{M}}_k, w) \leftrightarrows (\widehat{\mathfrak{N}}_k, v)$. Combining these observations, and using the bisimulation invariance of φ, we conclude that $(\mathfrak{M}, w)$ and $(\mathfrak{N}, v)$ agree on φ.

It is known that, if we restrict attention to the (finitely many) proposition letters occurring in φ, every model $(\mathfrak{M}, w)$ is described completely up to k-bisimulation by a single modal formula $\chi^k_{(\mathfrak{M},w)}$ of modal depth k (e.g., see Proposition 2.29 and Proposition 2.30 in [21]).

Finally, consider the set $\Sigma = \{\neg\chi^n_{(\mathfrak{M},w)} \mid (\mathfrak{M}, w) \not\models \varphi\}$. It follows from the invariance under k-bisimulations that for all $\mathfrak{M} \in \mathsf{K}$, $(\mathfrak{M}, w) \models \Sigma$ iff $(\mathfrak{M}, w) \models \varphi$. Since there are only finitely many mutually non-equivalent modal formulas of modal depth k [21, Proposition 2.29], Σ contains only finitely many formulas, modulo logical equivalence, and φ is equivalent their conjunction.

A similar argument applies to finite models, where the tree unraveling construction must be replaced by a partial unraveling, cf. [81]. □

In other words, the Van Benthem-Rosen characterization of modal logic as the bisimulation invariant fragment of first-order logic holds also if first-order logic is replaced by second order modal logic.

Note that the proof of Theorem 12.0.2 crucially depends on the use of (partial) tree unravellings, and that the result might not hold on frame classes that are not closed under this operation. In particular, consider the class of bi-modal frames (W, R_1, R_2) in which R_2 is the reflexive transitive closure of R_1 (note that this class is definable by a single modal formula). As observed by [92], results of [67] imply that, on such frames, the bisimulation invariant fragment of SOPML is the modal μ-calculus!

Second order modal logic itself can be characterized itself in terms of invariance under generated submodels. This follows from the following surprising connection between second order modal logic and $\mathcal{H}(@, \downarrow)$.

12.0.3. THEOREM. *Every nominal free $\mathcal{H}(@, \downarrow)$-sentence is equivalent to a formula of SOPML. Conversely, if a formula of SOPML has a first-order equivalent, then it is equivalent to a nominal free $\mathcal{H}(@, \downarrow)$-sentence.*

Before we will prove Theorem 12.0.3, we will list a number of its consequences. First of all, Theorem 12.0.3 shows that $\mathcal{H}(@,\downarrow)$ is, in some sense, the intersection of SOPML and first-order logic. In order to formulate this more precisely, let a *pointed model* be a pair $(\mathfrak{M}, w)$, where $\mathfrak{M}$ is a model and w is an element of $\mathfrak{M}$. Note that $\mathfrak{M}$ need not be generated by w. Modal formulas, as well as first-order formulas with one free variable, naturally define classes of pointed models. As with frame classes, we call a class of pointed models *elementary* if it is defined by a first-order formula with one free variable.

12.0.4. Corollary. *A class* K *of pointed models is definable by a nominal free* $\mathcal{H}(@,\downarrow)$*-sentence iff* K *is both elementary and definable by a formula of SOPML.*

By Theorem 9.2.5, we obtain the following.

12.0.5. Corollary. *An elementary class* K *of pointed models is defined by a formula of SOPML iff* K *it is invariant under generated submodels.*

Theorem 12.0.3 in combination with Corollary 9.3.2 also gives rise to the following analogue of the Goldblatt-Thomason theorem for second order propositional modal logic.

12.0.6. Corollary. *Let* K *be an elementary class of frames. Then the following are equivalent.*

1. K *is definable by a set of formulas of SOPML*
2. K *is defined by a single formula of SOPML*
3. K *is closed under generated subframes and reflects point-generated subframes*

In particular, if a frame class is defined by a first-order formula of the form $\forall x.\varphi(x)$, with $\varphi(x)$ bounded, then it is also defined by a formula of SOPML.

It seems not unreasonable to expect that Corollary 12.0.6 can be generalized to frame classes definable in monadic second order logic. However, inspection of the proof shows that Proposition 9.3.6 applies also to second order modal logic. In other words, there is a monadic Π^1_1-definable frame class K that is closed under generated subframes and reflects point-generated subframes, such that K is not definable in SOPML. Similarly, Proposition 9.3.5 shows that Corollary 12.0.6 does not hold on finite models.

In order to prove Theorem 12.0.3, we will extend the hybrid language $\mathcal{H}$ with a new kind of quantifiers, denoted by $\exists_k$ and $\forall_k$. Formally, for every formula φ and natural number k, we admit $\exists_k x.\varphi$ and $\forall_k x.\varphi$ as formulas, and we extend the truth definition in such a way that $\mathfrak{M}, g, w \models \exists_k x.\varphi$ iff there is a point v such that $\mathfrak{M}, g[x := v], w \models \varphi$, and v is reachable from w in at most k steps along the union of all accessibility relations (similarly for the universal quantifier). Let $\mathcal{H}(\exists_n)$ be the extension of $\mathcal{H}$ with the quantifiers $\exists_k$ and $\forall_k$ for all $k \in \omega$.

12.0.7. LEMMA. *Every $\mathcal{H}(\exists_n)$-formula is equivalent to a $\mathcal{H}(@,\downarrow)$-sentence, and conversely, every nominal free $\mathcal{H}(@,\downarrow)$-sentence is equivalent to an $\mathcal{H}(\exists_n)$-formula.*

Proof: We will prove the result for uni-modal languages. The generalization to formulas containing several modalities is straightforward.

The first part of the statement is easy to prove: let $\Diamond^{\leq n}\psi$ is shorthand for $\bigvee_{k\leq n}\Diamond^k\psi$. Then $\exists_n x.\varphi$ is equivalent to $\downarrow y.(\Diamond^{\leq n}\downarrow x.@_y\varphi)$, for y a variable distinct from x that does not occur in φ.

As for the second part, let φ be any nominal free $\mathcal{H}(@,\downarrow)$-sentence, and let m be its modal depth. Consider its standard translation $ST_x(\varphi)$, which is a bounded formula of the first-order correspondence language with no free variables besides x. For any bounded first-order formula ψ, define the $\mathcal{H}(\exists_n)$-formula ψ^* as follows.

$$
\begin{array}{lcl}
(Ryz)^* & = & \Diamond^{\leq m}(y\wedge\Diamond z)\\
(Py)^* & = & \Diamond^{\leq m}(y\wedge p)\\
(y=z)^* & = & \Diamond^{\leq m}(y\wedge z)\\
(\neg\psi)^* & = & \neg(\psi^*)\\
(\psi\wedge\chi)^* & = & \psi^*\wedge\chi^*\\
\exists z.(Ryz\wedge\psi)^* & = & \exists_m z.(\Diamond^{\leq m}(y\wedge\Diamond z)\wedge\psi^*)
\end{array}
$$

An inductive argument shows that the $\mathcal{H}(\exists_n)$-sentence $\exists_0 x.(ST_x(\varphi))^*$ is equivalent to $ST_x(\varphi)$, and hence to φ. □

Armed with Lemma 12.0.7, we can proceed with the proof of Theorem 12.0.3.

Proof of Theorem 12.0.3: Let φ be any nominal free $\mathcal{H}(@,\downarrow)$-formula. By Lemma 12.0.7, φ is equivalent to an $\mathcal{H}(\exists^{\leq n})$-formula ψ. We may assume without loss of generality that ψ is of the form $Q_1x_1\cdots Q_kx_k\chi$, where $Q_1,\ldots,Q_n\in\{\forall_n,\exists_n\mid n\in\omega\}$ and χ is quantifier free. Let ℓ be the largest natural number such that a quantifier of the form $\exists_\ell$ or $\forall_\ell$ occurs in ψ, and let $m=\ell+md(\chi)$. Let $\Diamond^{\leq m}\varphi$ be shorthand for $\bigvee_{k\leq m}\Diamond^k\varphi$, let $\Box^{\leq m}\varphi$ be shorthand for $\bigwedge_{k\leq m}\Box\varphi$, and let ϑ be the formula of SOPML obtained from ψ by replacing every subformula of the form $\exists^{\leq n}x.\psi$ by $\exists p.(\Diamond^{\leq n}p\wedge\forall q.(\Diamond^{\leq m}(p\wedge q)\rightarrow\Box^{\leq m}(p\rightarrow q)\wedge\varphi[x/p])$ and replacing every subformula of the form $\forall^{\leq n}x.\psi$ by $\forall p.(\Diamond^{\leq n}p\wedge\forall q.(\Diamond^{\leq m}(p\wedge q)\rightarrow\Box^{\leq m}(p\rightarrow q)\rightarrow\varphi[x/p])$. A simple inductive argument shows that ϑ is equivalent to the $\mathcal{H}(\exists^{\leq n})$-formula ψ, and hence to the $\mathcal{H}(@,\downarrow)$-formula φ.

For the converse direction, by Theorem 9.2.5 it suffices to observe that second order modal formulas are invariant under generated submodels. □

Chapter 13

Conclusions

Roughly speaking, this thesis contains two types of results. Results of the first type can be seen as addressing specific cells in a big table along the following two dimensions.

▷ *Extensions of the basic modal language*

The basic modal language, the hybrid languages $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$, the bounded fragment and $\mathcal{H}(@, \downarrow)$, guarded fragments, relation algebra and second order propositional modal logic

▷ *Model theoretic and computational properties*

Expressivity, frame definability, axiomatization, interpolation, the Beth property and complexity

The second type of results establish cross-connections between languages. In particular, a number of truth- or satisfiability-preserving translations between different languages are described, and certain languages are characterized in terms of others (for instance, as being a model theoretically interesting fragment, or as being the smallest extension satisfying certain properties).

Results of the first type reported in this thesis include the following.

We gave Goldblatt-Thomason-style characterizations of the elementary frame classes definable in $\mathcal{H}$, $\mathcal{H}(@)$, $\mathcal{H}(\mathsf{E})$, and $\mathcal{H}(@, \downarrow)$, both for pure formulas and for arbitrary formulas. The characterizations are based on two new operations on frames: ultrafilter morphic images and bisimulation systems.

We characterized the expressivity and frame definable power of second order propositional modal logic (SOPML). The proofs are based on the observation that the first-order definable part of SOPML coincides with the bounded fragment.

> We showed that either infinitely many rules or non-orthodox rules are needed in the axiomatizations of $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$ in order to obtain a general completeness result for pure extensions. On the other hand, we showed that there is an axiomatization for $\mathcal{H}(@,\downarrow)$ that contains only finitely many, orthodox rules, that satisfies a general completeness result for pure extensions.

Results of the second type include the following.

> We showed that $\mathcal{H}(@,\downarrow)$ is the smallest extension of $\mathcal{H}(@)$ with interpolation (i.e., interpolation over proposition letters and nominals). Likewise, we showed that $\mathcal{L}^1$ is the smallest extension of $\mathcal{H}(\mathsf{E})$ with interpolation, and that $\mathcal{L}^1$ is the smallest extension of $\mathcal{M}(\mathsf{D})$ with interpolation. The proofs are based on the simple observation that every interpolant for $(i \wedge \varphi) \to (j \to \varphi[i/j])$ (with j a nominal not occuring in φ) is equivalent to $\downarrow x.\varphi[i/x]$.
>
> We showed that, while most properties do not transfer in general from a modal logic to the corresponding $\mathcal{H}$- or $\mathcal{H}(@)$-logic, there is a large class of modal logics for which complexity, interpolation, uniform interpolation and finite axiomatization *do* transfer. The proof is based on a series of translations from $\mathcal{H}$ and $\mathcal{H}(@)$ to the basic modal language, each of which preserves satisfiability with respect to certain frame classes.

A few cells of the big table described above are still blank (in particular, which elementary frame classes are definable by guarded first-order formulas?). Also, there might still be interesting model theoretic cross-connections between fragments of first-order logic, waiting to be discovered. In fact, I hope that this thesis will contribute to the emergence of a new area of research that might be called "abstract model theory below first-order logic".

Appendix A

Basics of model theory

This section reviews a number of important results on the model theory of first order logic that are used in proofs throughout this thesis. For a more detailed treatment, cf. [64, 37]. We assume that the reader is familiar with the syntax and semantics of first-order logic. We will only consider first-order languages with constants and relation symbols but without function symbols of arity greater than zero. We will denote first-order models (or, structures) as pairs $\mathfrak{M} = (D, I)$ consisting of a domain D and an interpretation function I that assigns relations of the appropriate arity to the relation symbols and that assigns elements of D to constants. Given such a structure $\mathfrak{M}$ and a first-order formula $\varphi(x_1, \ldots, x_n)$, we will write $\mathfrak{M} \models \varphi\ [d_1, \ldots, d_n]$ if $d_1, \ldots, d_n$ are elements of the domain of $\mathfrak{M}$, such that φ holds in $\mathfrak{M}$ interpreting $x_1, \ldots, x_n$ as $d_1, \ldots, d_n$.

The first three results are easily stated.

A.0.1. Theorem (Compactness). *Let Σ be a set of first-order formulas. If every finite subset of Σ has a model, then Σ has a model.*

A.0.2. Theorem (Löwenheim-Skolem). *Let Σ be a countable set of first-order formulas. If Σ has a model then Σ has a countable model.*

A.0.3. Theorem (Craig Interpolation). *Let φ, ψ be first-order formulas, such that $\models \varphi \to \psi$. Then there is a formula ϑ such that $\models \varphi \to \vartheta$, $\models \vartheta \to \psi$ and all constants, relation symbols and function symbols occurring in ϑ occur both in φ and in ψ.*

For the remaining results we need to introduce some terminology. A model $\mathfrak{M}$ is a *submodel* of a model $\mathfrak{N}$ if the domain of $\mathfrak{M}$ is a subset of the domain of $\mathfrak{N}$ and the interpretations of every non-logical symbol in $\mathfrak{M}$ is simply the restriction of its interpretation in $\mathfrak{N}$ with respect to the domain of $\mathfrak{M}$. It follows that if an element of the domain of $\mathfrak{N}$ is named by a constant, then it is also in the domain of $\mathfrak{M}$. We say that $\mathfrak{M}$ is an *elementary submodel* of $\mathfrak{N}$ if it is a submodel, and for all first-order formulas $\varphi(x_1, \ldots, x_n)$ and elements $d_1, \ldots, d_n$ of the domain of $\mathfrak{M}$, $\mathfrak{M} \models \varphi\ [d_1, \ldots, d_n]$ iff $\mathfrak{N} \models \varphi\ [d_1, \ldots, d_n]$. In this case, we also say that $\mathfrak{N}$ is an *elementary extension* if $\mathfrak{M}$.

Given a set of models $\{\mathfrak{M}_i \mid i \in I\}$ for a relational language (i.e., without constants or function symbols), the union $\mathfrak{N} = \bigcup_{i \in I} \mathfrak{M}_i$ is defined in the natural way: the domain of $\mathfrak{N}$ is the union of the domains of $\mathfrak{M}_i$ $(i \in I)$, and the same holds for the interpretation of the relation symbols. In general, this notion can only be applied to models for relational languages. However, there are circumstances in which it can also be applied to models for languages containing constants and function symbols. An example of this is the following situation.

A.0.4. Theorem (Unions of elementary chains). *Let $(\mathfrak{M}_k)_{k \in \omega}$ be a sequence of models, such that $\mathfrak{M}_k$ is an elementary submodel of $\mathfrak{M}_{k+1}$ for all $k \in \omega$, and let $\mathfrak{M}_\omega$ be the union $\bigcup_{k \in \omega} \mathfrak{M}_k$. Then for each $k \in \omega$, $\mathfrak{M}_k$ is an elementary submodel of $\mathfrak{M}_\omega$.*

NB: $\bigcup_{i \in I} \mathfrak{M}_i$ should not be confused with the *disjoint union* of the models $\mathfrak{M}_i$ $(i \in I)$. In fact, for the above result crucially relies on the non-disjointness of the models in question.

An *ultrafilter* over a set W is a set $U \subseteq \wp(W)$ satisfying three conditions:

1. $W \in U$
2. For all $X \subseteq W$, $X \in U$ iff $(W \setminus X) \notin U$
3. For all $X \in U$ and $Y \in U$, $X \cap Y \in U$

An ultrafilter is *principal* if has a singleton element.

A.0.5. Definition (Ultraproducts). *Given a collection of models $\{\mathfrak{M}_a = (D_a, I_a) \mid a \in A\}$ and an ultrafilter U over the set A, the following defines the ultraproduct $\Pi_U \mathfrak{M}_a = (D, I)$.*

Let $\sim$ be the equivalence relation $\sim$ on the product $\Pi_{a \in A} D_a$ given by

$$f \sim g \text{ iff } \{a \in A \mid f(a) = g(a)\} \in U$$

Let D be the quotient $(\Pi_{a \in A} D_a)/\sim$. For each constant c, let

$$I(c) = [\langle I_a(c) \rangle_{a \in A}]_\sim$$

Finally, for each k-ary relation R and $[f_1], \ldots, [f_k] \in D$, let

$$([f_1], \ldots, [f_k]) \in I(R) \text{ iff } \{a \in A \mid (f_1(a), \ldots, f_k(a)) \in I_a(R)\} \in U$$

If all factor models $\mathfrak{M}_a$ are the same, then $\Pi_U \mathfrak{M}_i$ is called an *ultrapower*. Every model $\mathfrak{M}$ is isomorphic to a submodel of the ultrapower $\Pi_U \mathfrak{M}$, the isomorphism being the function that sends every element d to the equivalence class $[\langle d, d, \ldots \rangle]_\sim$.

A.0.6. Theorem (Łos). *For all models $\mathfrak{M}$, ultrafilters U and first-order sentences φ, $\Pi_U\mathfrak{M} \models \varphi$ iff $\mathfrak{M} \models \varphi$*

Related to ultraproducts are the simpler notions of products and subdirect products, which will also play a role in this thesis.

A.0.7. Definition (Products and subdirect products). *The product of a collection of models $\{\mathfrak{M}_a = (D_a, I_a) \mid a \in A\}$, (also called* cartesian product *or* direct product*, notation: $\Pi_{a\in A}\mathfrak{M}_a$) is the model (D, I), where D is the cartesian product $\Pi_{a\in A}D_a$, and for each n-ary relation R,*

$$I(R) = \{\langle d_1, \ldots, d_n\rangle \in D^n \mid \langle d_1(a), \ldots, d_n(a)\rangle \in I_a(R) \textit{ for each } a \in A\}$$

A subdirect product *of $\{\mathfrak{M}_a \mid a \in A\}$ is any submodel $\mathfrak{N}$ of the product $\Pi_{a\in A}\mathfrak{M}_a$ for which it holds that the natural projection functions from the domain of $\mathfrak{N}$ to the domains of the models $\mathfrak{M}_a$ ($a \in A$) are surjective.*

The next notion we introduce is that of *ω-saturatedness.* A *1-type* is a set of formulas in one free variable. A 1-type $\Gamma(x)$ is *realized* in a model $\mathfrak{M}$ if there is an element d of the domain of $\mathfrak{M}$ such that $\mathfrak{M} \models \Gamma\ [x : d]$. A model is said to be *1-saturated* if for all 1-types $\Gamma(x)$, if every finite subset of $\Gamma(x)$ is realized in $\mathfrak{M}$, then $\Gamma(x)$ itself is realized in $\mathfrak{M}$. One can think of 1-saturatedness as a sort of *compactness within a model.*

Given a model $\mathfrak{M}$ and a finite sequence $d_1, \ldots, d_n$ of elements of the domain of $\mathfrak{M}$, we use $(\mathfrak{M}, d_1, \ldots, d_n)$ to denote the expansion of $\mathfrak{M}$ in which the elements $d_1, \ldots d_n$ are named by additional constants $c_1, \ldots, c_n$ (each new constant c_k denotes the corresponding element d_k in the expanded model). A model $\mathfrak{M}$ is *ω-saturated* if every such expansion $(\mathfrak{M}, d_1, \ldots, d_n)$ (with $n \in \omega$) is 1-saturated. Note that we use ω and $\mathbb{N}$ interchangably to denote the set of non-negative integers.

A.0.8. Theorem (ω-Saturation). *Every model $\mathfrak{M}$ has an ω-saturated elementary extension $\mathfrak{M}^+$. In fact, $\mathfrak{M}^+$ can be constructed such that it is isomorphic to an ultrapower of $\mathfrak{M}$.*

It should be noted that this result holds regardless of the cardinality of the language (i.e., the number of non-logical symbols) [32, Theorem 6.1.4 and 6.1.8].

We say that two models, $\mathfrak{M}, \mathfrak{N}$ are *elementarily equivalent* (notation: $\mathfrak{M} \equiv_{FO} \mathfrak{N}$) if they satisfy the same first-order sentences.

One, rather trivial, sufficient condition for elementary equivalence is the existence of an *isomorphism.* An isomorphism between models $\mathfrak{M}$ and $\mathfrak{N}$ is a bijection f between the domains of $\mathfrak{M}$ and $\mathfrak{N}$ such that for all atomic formulas $\varphi(x_1, \ldots, x_n)$ and elements $d_1, \ldots, d_n$ for the domain of $\mathfrak{M}$, $\mathfrak{M} \models \varphi\ [d_1, \ldots, d_n]$

iff $\mathfrak{N} \models \varphi\ [f(d_1), \ldots, f(d_n)]$. If an isomorphism between $\mathfrak{M}$ and $\mathfrak{N}$ exists, then we say that $\mathfrak{M}$ and $\mathfrak{N}$ are isomorphic, and that $\mathfrak{N}$ is an isomorphic copy of $\mathfrak{N}$. Clearly isomorphic models satisfy the same first-order formulas. A more interesting sufficient condition for elementary equivalence is the existence of a *potential isomorphism*, a notion that will be defined next.

A *finite partial isomorphism* between models $\mathfrak{M}, \mathfrak{N}$ is a finite relation $\{(a_1, b_1), \ldots, (a_n, b_n)\}$ between the domains of $\mathfrak{M}$ and $\mathfrak{N}$ such that for all atomic formulas $\varphi(x_1, \ldots, x_n)$, $\mathfrak{M} \models \varphi\ [a_1, \ldots, a_n]$ iff $\mathfrak{N} \models \varphi\ [b_1, \ldots, b_n]$. Since equality statements are atomic formulas, every finite partial isomorphism is (the graph of) an injective partial function.

A.0.9. Definition (Potential isomorphisms). *A potential isomorphism between two models $\mathfrak{M}$ and $\mathfrak{N}$ is a non-empty collection F of finite partial isomorphisms between $\mathfrak{M}$ and $\mathfrak{N}$ that satisfies the following conditions:*

- *For all finite partial isomorphisms $Z \in F$ and for each $w \in \mathfrak{M}$, there is a $v \in \mathfrak{N}$ such that $Z \cup \{(w, v)\} \in F$.*
- *For all finite partial isomorphisms $Z \in F$ and for each $v \in \mathfrak{N}$, there is a $w \in \mathfrak{M}$ such that $Z \cup \{(w, v)\} \in F$.*

We write $\mathfrak{M}, w_1, \ldots, w_n \cong_p \mathfrak{N}, v_1, \ldots, v_n$ to indicate the existence of a potential isomorphism F between $\mathfrak{M}$ and $\mathfrak{N}$ such that $\{(w_1, v_1), \ldots, (w_n, v_n)\} \in F$.

It is well known that first-order formulas are invariant under potential isomorphisms. In other words, the existence of a potential isomorphism implies elementary equivalence. The converse does not hold in general, but it holds for ω-saturated models.

A.0.10. Theorem. *If $\mathfrak{M} \cong_p \mathfrak{N}$ then $\mathfrak{M} \equiv_{FO} \mathfrak{N}$. Conversely, if $\mathfrak{M} \equiv_{FO} \mathfrak{N}$ and $\mathfrak{M}$ and $\mathfrak{N}$ are ω-saturated, then $\mathfrak{M} \cong_p \mathfrak{N}$.*

An exact characterization of elementary equivalence can be given in terms of *Ehrenfeucht-Fraïssé games*, which can be seen as finite approximations of potential isomorphisms. The Ehrenfeucht-Fraïssé game of length n on models $\mathfrak{M}$ and $\mathfrak{N}$ (notation: $EF(\mathfrak{M}, \mathfrak{N}, n)$) is as follows. There are two players, Spoiler and Duplicator. The game has n rounds, each of which consists of a move of Spoiler followed by a move of Duplicator. Spoiler's moves consist of picking an element from one of the two models, and Duplicators response consists of picking an element of the opposite model. In this way, Spoiler and Duplicator build up a (finite) binary relation between the domains of the two models: initially, the relation is empty; each round, it is extended with another pair. The winning conditions are as follows: if at some point of the game the constructed binary relation is not a finite partial isomorphism, then Spoiler wins immediately. If after each round the relation is a finite partial isomorphism, then the game is won by Duplicator.

A.0.11. Theorem (Ehrenfeucht-Fraïssé). *Assume a first-order language with only finitely many relation symbols and function symbols.* $\mathfrak{M} \equiv_{FO} \mathfrak{N}$ *iff Duplicator has a winning strategy in the game* $EF(\mathfrak{M}, \mathfrak{N}, n)$ *for each* $n \in \omega$.

Observe that, since these games are finite zero-sum perfect information games between two-players, by Zermelo's theorem one of the two players always has a winning strategy.

In fact, Theorem A.0.11 can be strengthened: equivalence with respect to first-order formulas of quantifier depth n corresponds to Duplicator having a winning strategy in the game of n rounds. Moreover, a winning strategy for spoiler may be constructed from the distinguishing formula, and vice versa [9].

Appendix B

Basics of computability theory

We briefly review some notions from complexity theory and recursion theory that are used in this thesis. More information can be found in [24], [89] and [62].

A decision problem may be identified either with a set of strings over the alphabet $\{0, 1\}$, or with a set of natural numbers. In fact, these views can be identified by considering natural numbers as written down in binary notation. Thus, while the length of a string s is simply the number of elements of the sequence, the length of a natural number n will be the length of its binary encoding, which is approximately $\log n$. We will use $|s|$ to refer to the length of s, where s is either a bit-string or a natural number.

Given such a set L of bitstrings, or of natural numbers, the task is then to decide for a given string, or natural number, s whether $s \in L$. A problem L is called *decidable* (or, *recursive*) if there is a deterministic Turing machine that solves this problem in finite amount of time (i.e., for each input s it terminates after finitely many steps and correctly answers the question whether $s \in L$). A problem L is called *recursively enumerable* (r.e.) if there is a (not necessarily halting) deterministic Turing machine that enumerates the elements of L. A problem is *co-recursively enumerable* if its complement is recursively enumerable. Any problem that is neither recursively enumerable nor co-recursively enumerable is called *highly undecidable*.

Complexity classes

Complexity theory classifies decision problems with respect to the amount of time and space a Turing machine needs to solve them.

Consider a function $f : \mathbb{N} \to \mathbb{N}$. We say that a problem L in $\textsc{dtime}(f)$ if there is a deterministic Turing machine M and natural numbers c, d such that on any input s with $|s| > d$, M terminates after at most $c \cdot f(|s|)$ many steps and correctly answers the question whether $s \in L$. $\textsc{ntime}(f)$ is defined similarly, using non-deterministic Turing machines. A problem L in $\textsc{space}(f)$ if there is a deterministic Turing machine M and natural numbers c, d such that, on any

Table B.1: Some important complexity classes

$$
\begin{aligned}
\textsc{PTime} &= \bigcup_{k\in\mathbb{N}} \textsc{dtime}(n^k)\\
\textsc{NP} &= \bigcup_{k\in\mathbb{N}} \textsc{ntime}(n^k)\\
\textsc{PSpace} &= \bigcup_{k\in\mathbb{N}} \textsc{space}(n^k)\\
\textsc{ExpTime} &= \bigcup_{k\in\mathbb{N}} \textsc{dtime}(2^{n^k})\\
\textsc{NExpTime} &= \bigcup_{k\in\mathbb{N}} \textsc{ntime}(2^{n^k})\\
\textsc{ExpSpace} &= \bigcup_{k\in\mathbb{N}} \textsc{space}(2^{n^k})\\
\textsc{2-ExpTime} &= \bigcup_{k\in\mathbb{N}} \textsc{dtime}(2^{2^{n^k}})\\
\textsc{2-NExpTime} &= \bigcup_{k\in\mathbb{N}} \textsc{ntime}(2^{2^{n^k}})\\
\textsc{2-ExpSpace} &= \bigcup_{k\in\mathbb{N}} \textsc{space}(2^{2^{n^k}})\\
&\vdots\\
\textsc{Elementary} &= \bigcup_{k\in\mathbb{N}} k\text{-}\textsc{ExpTime}
\end{aligned}
$$

input s with $|s| > d$, M decides in finite amount of time whether $s \in L$, using at most $c \cdot f(|s|)$ many cells of the tape.

These notions can be used to define a number of important classes of decision problems that play a role in this thesis, which are listed Table B.1. Each of these classes is contained in the classes appearing below it in the list.

Reductions and completeness

A polynomial reduction from a problem L to a problem L' (more precisely, a *polynomial time many-one reduction*) is a deterministic Turing machine that, given input s, terminates after at most $f(|s|)$ many steps and produces output t such that $s \in L$ iff $t \in L'$, for some polynomial function $f : \mathbb{N} \to \mathbb{N}$. All complexity classes listed in Table B.1 are closed under polynomial reductions.

For C a class of decision problems and L a decision problem, L is said to be C-*hard* (more precisely, C*-hard under polynomial reductions*) if every problem in C can be polynomially reduced to L. A decision problem L is said to be C*-complete* if $L \in C$ and L is C-hard.

We will also make use of other types of reductions in this thesis. A *computable reduction* from a problem L to a problem L' is a deterministic Turing machine that, given input s, terminates after finitely many steps and produces output t such that $s \in L$ iff $t \in L'$. Clearly, the class of decidable decision problems is closed under computable reductions. On the other hand, the classes listed in Table B.1 are not closed under computable reductions.

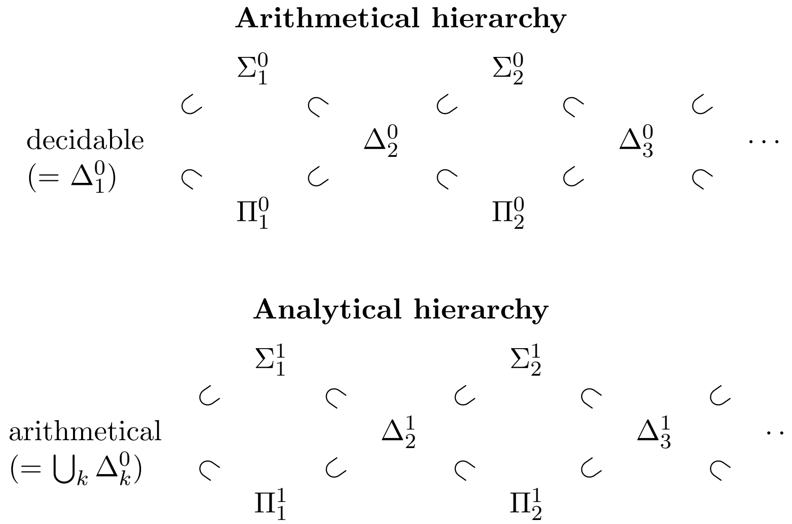
Finally, a *non-deterministic polynomial conjunctive reduction* of a problem L to a problem L' is a polynomial time non-deterministic Turing machine that, given input s (non-deterministically) produces a sequence $t_1, \ldots, t_n$ of instances of L', such that $s \in L$ iff some run of the non-deterministic Turing machine with input s produces a sequence $t_1, \ldots, t_n$ such that each t_i is in L' ($i \leq n$). Clearly, non-deterministic polynomial conjunctive reduction generalize the usual polynomial time many-one reductions. With the exception of PTime, all complexity classes listed in Table B.1 are closed under non-deterministic polynomial conjunctive reductions (the class PTime is not closed under such reductions, unless PTime = NP) [74].

Arithmetical and analytical hierarchy

While complexity theory provides the tools to classify the complexity of decidable problems, recursion theory is the proper framework for the studying and classifying undecidable problems. Recursion theory studies decision problems from the perspective of definability in first-order or second-order arithmetic.

The language of *first-order Peano arithmetic*, $\mathcal{L}^1_{PA}$, is the first-order language over the vocabulary that consists of binary relation $\leq$, function symbols $+$ and $\times$, and equality. Formulas of this language are interpreted over the natural numbers. A set L of natural numbers is called *arithmetical* if it is definable in first-order Peano arithmetic, i.e., if there is a formula $\varphi(x)$ of $\mathcal{L}^1_{PA}$ such that for all $n \in \mathbb{N}$, $n \in L$ iff $(\mathbb{N}, \leq, +, \times) \models \varphi\ [n]$. Arithmetical sets may be further classified in terms of the quantifier patterns occuring in the formulas that define them. More specifically, a set of natural numbers is said to be in Σ^0_k (with $k \geq 1$) if it is defined by a $\mathcal{L}^1_{PA}$-formula of the form $Q_1 x_1 \cdots Q_n x_n.\varphi$, with $Q_1, \ldots, Q_n \in \{\exists, \forall\}$ and φ quantifier-free, such that $Q_1 = \exists$ and the number of quantifier alternations (i.e., universal quantifiers following existential quantifiers or vice versa) in the sequence $Q_1 \ldots Q_n$ is at most $k-1$. A set of natural numbers is said to be in Π^0_k if its complement is in Σ^0_k, and in Δ^0_k if it is both in Σ^0_k and in Π^0_k. A remarkable result in recursion theory states that the decidable sets of natural numbers are precisely the ones that are in Δ^0_1, and the recursively enumerable sets are the ones in Σ^0_1.

Table B.2: Some important classes of problems in recursion theory



The language of *second-order Peano arithmetic*, $\mathcal{L}^2_{PA}$, is the second-order language over the vocabulary that consists of binary relation $\leq$, function symbols $+$ and $\times$, and equality. A set of natural numbers is called *analytical* if it is defined by a formula of $\mathcal{L}^2_{PA}$. Again, the analytical sets can be classified with respect to the quantifier patterns occuring in the defining formulas. A set of natural numbers is said to be in Σ^1_k (with $k \geq 1$) if it is defined by a $\mathcal{L}^2_{PA}$-formula of the form $Q_1 X_1 \cdots Q_n X_n.\varphi$, where $Q_1, \ldots, Q_n \in \{\exists, \forall\}$ are quantifiers over sets and φ contains only first-order quantifiers, such that $Q_1 = \exists$ and the number of quantifier alternations (i.e., universal quantifiers following existential quantifiers or vice versa) in the sequence $Q_1 \ldots Q_n$ is at most $k-1$. A set of natural numbers is said to be in Π^1_k if its complement is in Σ^1_k, and in Δ^1_k if it is both in Σ^0_k and in Π^1_k.

Table B.2 summarizes some of the above classes, and indicates their relationships. Each of the indicated inclusions is strict. Each of the classes listed in Table B.2 is closed under computable reductions. A set A of natural numbers is said to be Σ^k_ℓ*-hard* (more precisely, Σ^k_ℓ*-hard under computable reductions*) if for every set B in Σ^k_ℓ there is a computable reduction from B to A. A set of natural numbers is Σ^k_ℓ-complete if it is both in Σ^k_ℓ and Σ^k_ℓ-hard. Likewise for Π^k_ℓ and Δ^k_ℓ. When one speaks of an arbitrary decision problem as being, for instance, Σ^1_1-hard, then it is implicitly understood that the instances of the decision problem are coded into natural numbers (using a computable encoding).

The set of (codings of) true Σ^1_1 sentences of arithmetic is itself a Σ^1_1-complete set. In fact, this can be strengthened slightly, since the intended interpretation of $+$ and $\times$ in $(\mathbb{N}, \leq)$ can be defined using first-order sentences. In this way, we obtain the following.

B.0.1. THEOREM. *The existential second order theory of* $(\mathbb{N}, \leq)$ *is* Σ^1_1*-complete.*

Another example of a Σ^1_1-hard decision problem, due to Harel [62] is the recurrent tiling problem, which can be defined as follows. A *tile* is a tuple $t = \langle t_{left}, t_{right}, t_{top}, t_{bottom}\rangle$ of elements of some set C. A *tiling of* $\mathbb{N} \times \mathbb{N}$ *using a set of tiles* T is a function $f : \mathbb{N} \times \mathbb{N} \to T$ such that for all $n, m \in \mathbb{N}$, $f(n,m)_{right} = f(n+1,m)_{bottom}$ and $f(n,m)_{top} = f(n,m+1)_{bottom}$. Now, the recurrent tiling problem is the following problem:

> *given a finite set of tiles* T *and a designated tile* $t \in T$*, is there a tiling* f *of* $\mathbb{N} \times \mathbb{N}$ *using* T *such that* $f(n, 0) = t$ *for infinitely many* $n \in \mathbb{N}$*?*

B.0.2. THEOREM ([62]). *The recurrent tiling problem is* Σ^1_1*-complete.*

Here is an example of a decision problem that is not analytical.

B.0.3. THEOREM. *Satisfiability of monadic second order formulas over the signature consisting of a single binary relation is highly undecidable, and in fact not analytical.*

Proof: There is a computable satisfiability-preserving translation from arbitrary second-order formulas to monadic second order formulas in one binary relation symbol [68]. By a standard recursion theoretic argument, using the fact that the model $(\mathbb{N}, \leq, +, \times)$ is defined up to isomorphism by a second order formula, the class of satisfiable second-order formulas is not analytical (cf. [38]). The result follows. $\square$

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Samenvatting

In dit proefschrift worden verschillende uitbreidingen van de basis modale taal bestudeerd. Modeltheoretische en computationele eigenschappen van deze uitbreidingen worden onderzocht. Het proefschrift bevat grofweg twee typen resultaten. De resultaten van het eerste type behandelen specifieke cellen van een grote tabel met de volgende dimensies.

▷ *Uitbreidingen van de basis modale taal*

De basis modale taal, de hybride talen $\mathcal{H}$, $\mathcal{H}(@)$ en $\mathcal{H}(\mathsf{E})$, het bounded fragment en $\mathcal{H}(@, \downarrow)$, guarded fragmenten, relatie algebra en tweede orde propositionele modale logica.

▷ *Modeltheoretische en computationele eigenschappen*

Expressiviteit, frame definieerbaarheid, axiomatizering, interpolatie, de Beth eigenschap en complexiteit

Resultaten van het tweede type tonen kruisverbanden aan tussen talen. In het bijzonder worden verschillende waarheid- of vervulbaarheid-behoudende vertalingen tussen verschillende talen beschreven, en worden bepaalde talen gekarakteriseerd in termen van andere talen (bijvoorbeeld als zijnde een modeltheoretisch interessant fragment, of als zijnde de kleinste uitbreiding die aan bepaalde eigenschappen voldoet).

Hoofdstuk 1 geeft een algemene introductie tot het proefschrift.

Hoofdstuk 2 neemt belangrijke noties en resultaten in modale logica door vanuit een modeltheoretisch perspectief. Het bevat tevens enkele nieuwe resultaten: de niet-recursieve opsombaarheid van de eerste orde formules die behouden blijven onder ultrafilter extensies, een algemeen interpolatie-resultaat voor modale logica's, en enige resultaten betreffende modale logica's die geaxiomatizeerd zijn door ondiepe formules (i.e., formules waarin geen voorkomen van een propositieletter in het bereik is van meer dan één modale operator).

De hoofdstukken die op Hoofdstuk 2 volgen zijn onderverdeeld in twee delen. In Deel I, dat bestaat uit Hoofdstuk 3–8, worden de hybride talen $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$ in detail bestudeerd. Deze talen kunnen worden beschouwd als bescheiden uitbreidingen van de basis modale taal. Hoofdstuk 3 introduceert de talen met hun syntax en semantiek. In Hoofdstuk 4 wordt de expressiviteit bestudeerd, zowel op het niveau van modellen als op het niveau van frames. In Hoofdstuk 5 worden axiomatizeringen en volledigheid bestudeerd. Hoofdstuk 6 bevat resultaten betreffende interpolatie en de Beth eigenschap. Hoofdstuk 7 behandelt vervulbaarheid-behoudende vertalingen van $\mathcal{H}$, $\mathcal{H}(@)$ en $\mathcal{H}(\mathsf{E})$ naar de basis modale taal. Hoofdstuk 8 behandelt de algemene vraag naar de overdracht van eigenschappen van modale logica's naar corresponderende logica's in de rijkere talen $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$.

In Deel II, dat bestaat uit Hoofdstuk 9–12, worden enige meer expressieve uitbreidingen van de basis modale taal bestudeerd. Hoofdstuk 9 bestudeert het bounded fragment, en de daaraan gerelateerde hybride taal $\mathcal{H}(@, \downarrow)$. Hoofdstuk 10 bestudeert guarded fragmenten met constanten. Hoofdstuk 11 bestudeert relatie algebra. Tot slot betreft Hoofdstuk 12 tweede orde propositionele modale logica, de uitbreiding van de basis modale taal met propositionele kwantoren.

Abstract

In this thesis, several extensions of the basic modal language are studied. Model theoretic and computational properties of these extensions are investigated. Roughly speaking, the thesis contains two types of results. The first type of results can be seen as addressing specific cells in a big table along the following two dimensions.

▷ *Extensions of the basic modal language*

The basic modal language, the hybrid languages $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$, the bounded fragment and $\mathcal{H}(@, \downarrow)$, guarded fragments, relation algebra and second order propositional modal logic

▷ *Model theoretic and computational properties*

Expressivity, frame definability, axiomatization, interpolation, the Beth property and complexity

The second type of results establish cross-connections between languages. In particular, a number of truth- or satisfiability-preserving translations between different languages are described, and certain languages are characterized in terms of others (for instance, as being a model theoretically interesting fragment, or as being the smallest extension satisfying certain properties).

Chapter 1 provides a general introduction to the thesis.

Chapter 2 reviews basic notions and results of modal logic from a model theoretic perspective. It also contains several new results: the non-recursive enumerability of the first-order formulas preserved under ultrafilter extensions, a general interpolation result for modal logics, and some results concerning modal logics axiomatized by shallow formulas (i.e., formulas in which no occurence of a proposition letter is in the scope of more than one modal operator).

The chapters that follow Chapter 2 are divided in two parts. In Part I, which consists of Chapter 3–8, the hybrid languages $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$ are studied in detail. These languages can be considered modest extensions of the basic modal language. Chapter 3 introduces the languages with their syntax and semantics. In Chapter 4, their expressivity is studied, both on the level of models and on the level of frames. In Chapter 5, axiomatizations and completeness results are discussed. Chapter 6 contains results concerning interpolation and the Beth property. Chapter 7 discusses satisfiability preserving translations from $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$ to the basic modal language. Chapter 8 discusses the general question which properties transfer from modal logics to the corresponding logics in the richer languages $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$.

In Part II, consisting of Chapter 9–12, some more expressive extensions of the basic modal language are studied. Chapter 9 studies the bounded fragment, and the related hybrid language $\mathcal{H}(@, \downarrow)$. Chapter 10 studies guarded fragments with constants. Chapter 11 studies relation algebra. Finally, Chapter 12 concerns second order propositional modal logic, which is the extension of the basic modal language with propositional quantifiers.

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