Thirty nine years of stratified trees

(Invited Paper)

Peter van Emde Boas
ILLC, FNWI, University of Amsterdam,
and
Bronstee.com Software & Services B.V., Heemstede
the Netherlands
Email: peter@bronstee.com

***Abstract*—The stratified tree, also called van Emde Boas tree, is a data structure implementing the full repertoire of instructions manipulating a single subset A of a finite ordered Universe $U = [0...u-1]$. Instructions include $member$, $insert$, $delete$, min, max, $predecessor$ and $successor$, as well as composite ones like $extract-min$. The processing time per instruction is $O(loglog(u))$. Hence it improves upon the traditional comparison based tree structures for dense subsets A; if A is sparse, meaning that the size $n = \#A = O(log(u))$ the improvement vanishes.**

Examples exist where this improvement helps to speed-up algorithmic solutions of real problems; such applications can be found for example in graph algorithms, computational geometry and forwarding of packets on the internet.

The structure was invented during a short postdoc residence at Cornell University in the fall of 1974. In the sequel of this paper I will use the original name Stratified Trees which was used in my own papers on this data structure.

There are two strategies for understanding how this $O(loglog(u))$ improvement can be obtained. Today a direct recursive approach is used where the universe is divided into a cluster of $\sqrt{u}$ galaxies each of size $\sqrt{u}$; the set manipulation instructions decompose accordingly in a instruction at the cluster and galaxy level, but one of these two instructions is always of a special trivial type. The processing complexity thus satisfies a recurrence $T(u) = T(\sqrt{u}) + O(1)$. Consequently $T(u) = O(loglog(u))$.

However, this recursive approach requires address calculations on the arguments which use multiplicative arithmetical instructions. These instructions are not allowed in the Random Access Machine model (RAM) which was the standard model in the developing research area of design and analysis of algorithms in 1974. Therefore the early implementations of the stratified trees are based on a different approach which best is described as a binary-search-on-levels strategy. In this approach the address calculations are not required, and the structure can be implemented using pointers. The downside of this approach is that it leads to rather complex algorithms, which are still hard to present correctly even today. Another bad consequence was the super linear space consumption of the data structure, which was only eliminated three years later.

In this paper I want to describe the historical backgrounds against which the stratified trees were discovered and implemented. I do not give complete code fragments implementing the data structure and the operations; they can be found in the various textbooks and papers mentioned, including a Wikipedia page. Code fragments appearing in this paper are copied verbatim from the original sources; the same holds for the figures[1].

I. Introduction

The *Stratified Trees* which in the literature frequently are named *van Emde Boas Trees* were discovered in the fall of 1974 during a short postdoc residence at Cornell University. They were designed in order to improve the time complexity of manipulating a subset A of some finite ordered set U, called the *Universe*. The size of the Universe U is denoted u, and without loss of generality we may assume that $U = [0...u-1]$. The size of the set A is denoted n. The case where U should be considered an infinite set is realised in practice by letting u be much larger than n.

Manipulation of such a dynamic set A is a core algorithmic task in areas like graph algorithms, computational geometry and databases, but also in applications like message passing on the Internet; applications which still had to be invented in 1974.

The instructions which we consider are:

1) $insert(x)$: adds element x to A
2) $delete(x)$: removes x from A
3) $member(x)$: tests whether $x \in A$
4) $card$: yields the size n of A
5) min: yields the smallest element in A
6) max: yields the largest element in A
7) $successor(x)$: yields the smallest element in A strictly larger than x
8) $predecessor(x)$: yields the largest element in A strictly smaller than x

.

There are also composite instructions like:

1) $extract-min$: yields the smallest element in A and removes it from A
2) $allmin(x)$: removes all elements $\leq x$ from A

.

Finally there is an instruction which only can be understood in terms of the binary representation of the numbers in U:

To appear in Proc. ISCIM 2013, Sep 26–28, Tirana, Albania. Work supported by grant CR 62-50 Netehrlands Organization for the Advancement of Pure Research (Z.W.O., currently called N.W.O).

[1]Due to te IEEE two column format used by ISCIM 2013 the reader will have to use a magnifying glass when reading a printed version; however the proceedings are digital anyhow

$neighbour(x)$ which yields either the predecessor or the successor of x in A, selecting out of these two possible answers the result which has the most significant bits in common with x. The purpose of this instruction will become clear when we consider the elements in U to be the leaves of a binary tree. Then the neighbour of x is the element in A which is closest in distance when traversing the edges of this tree.

These instructions have preconditions in order to be properly defined: inserting an element already present in A shouldn't be allowed. Similarly removing an element which is not in A is illegal as well. If A is empty, min and max are not defined, and neither is the successor of an element $x \geq max$. It is left to the programmer to ensure that these errors either don't occur or are resolved by an appropriate action.

When implementing the above instructions there is a conflict between the efficiency requirement of the instructions best supported by direct access like $member$, $insert$ and $delete$, and the instructions involving the ordering which are best supported by a list structure like min, max, and $predecessor$ or $successor$ for elements in the set. When implemented as an array or a bit string, $member$, $insert$ and $delete$ take constant time, while the order based instructions require linear time in the worst case; on linked lists the situation is reversed.

In order to reduce the complexity of the complete set of instructions several ingenious data structures where invented. Examples exisiting in 1974 are AVL-trees, 2-3 trees, heaps, etc. [2] yielding time complexity $O(log(n))$ for these instructions, or subsets thereoff. A well known example of such a subset is a *priority queue*, a data structure supporting $insert$ and $extractmin$ with processing time $O(log(n))$ when implemented on a binary heap.

The data structures mentioned above manipulate elements based on binary comparisons; their behaviour is independent of the size of the Universe u and therefore they work also for infinite universes, like the set of real numbers. As a consequence the time complexity $O(log(n))$ is perfectly reasonable, given the fact that with the above instructions (in fact just the two priority queue instructions are sufficient) we can sort a set of real numbers, and it is well known that for sorting numbers an $\Omega(nlog(n))$-lower bound holds for comparison based algorithms.

However, in 1974 is was known that this lower bound on sorting numbers didn't apply to non-comparison based algorithms, as illustrated by the linear time complexity of sorting methods like radix sort and bucket sort [22]. Therefore the question whether the processing time $O(log(n))$ for our instruction repertoire could be improved by going beyond the comparison based model, was still open. The stratified trees provided a positive answer to this question, substituting the processing complexity $O(log(n))$ by $O(loglog(u))$ where u is the size of the Universe rather than the size of the set being manipulated. Consequently, the improvement vanishes when the set A is *sparse* in the sense that $n = O(log(u))$. But in areas like graph algorithms where the set A consists of the nodes or edges in the graph, the set is non-sparse so an improvement indeed is achieved.

The rest of this paper is structures as follows: section II presents the two approaches towards obtaining the $O(loglog(u))$ improvement, the recursive cluster galaxy decomposition and the binary search on levels. Section III describes the state of machine based complexity theory around 1974, and explains the prohibition of multiplicative instructions in the RAM model. In section IV the binary search on levels approach is explained, but for full details the reader is referred to the original source [48]. Section V describes the developments during the first five years when I still was working on stratified trees myself; how the original design was improved and how the problem of super-linear space consumption was solved. It also tells how the recursive decomposition approach returned to the scene. Code fragments describing this approach are presented; this approach has become the dominant one today. Section VI gives a (necessarily incomplete) survey of work on stratified trees by other authors, involving applications, improvements and lower bounds. Section VII gives a conclusion.

II. Two approaches

There are two roads towards obtaining the processing time improvement of $O(loglog(u))$. It is important to understand these approaches and the difference between them.

A. The recursive cluster galaxy decomposition approach

Today the structure of the stratified trees is best understood in terms of a recursive decomposition of the Universe: the Universe consists of a *cluster* of *galaxies* both of size $\sqrt{u}$. Each Galaxy covers the elements of some interval within the universe, and the cluster represents the ordered sequence of these intervals. Both cluster and galaxies can be decomposed recursively until the universe has become sufficiently small. In theoretical descriptions this means that the recursion terminates at universes of size 2 ; more practice oriented approaches let the recursion terminate at universe size equal the word size of typical computers like 16, 32 or 64. Furthermore for each (sub)structure we maintain the number of inserted elements and the largest and smallest element in a few local variables.

The idea behind this recursive approach is simple. For small universe size (bottom case of the recursion) all instructions take constant time. Similarly one can simply store all the elements in the set A in a few local variables as long as n is small (say $n < 2$). Consider now the case that we want to insert an element x in A at a stage where A contains already some elements. We find the galaxy where x should be inserted, and two possible cases arise: either this galaxy is already populated and it suffices to insert x in this galaxy by a recursive call, or this galaxy is still uninhabited and we must insert this galaxy in the cluster first, but then the action of inserting the element in this newly inserted galaxy will be trivial since it will be its only inhabitant.

A similar analysis holds for the delete instruction; if we remove the last element from some galaxy this removal itself is trivial but then the galaxy itself should be removed from the cluster; otherwise the element only has to be removed from its galaxy. For the successor instruction the decomposition considers two cases as well: either the element is smaller than the maximal element in its galaxy and then the successor is to be found inside this galaxy, or the successor is the smallest element in the successor galaxy in the cluster.

Inspection of these decompositions shows that each instruction is implemented by similar instructions on the cluster and galaxy level (one of each) which would result in an estimate of the processing time $T(u) \leq T(\sqrt{u}) + T(\sqrt{u}) + O(1)$ and this recurrence solves to $T(u) = O(log(u))$ which means that there is no improvement over the traditional comparison based structures. However our analysis shows that one of the two recursive calls is always of a special type which can be performed in constant time. Hence the time estimate becomes $T(u) \leq T(\sqrt{u}) + O(1)$ which resolves to $T(u) = O(loglog(u))$.

B. The binary search on levels approach

Regardless the simplicity of the strategy as sketched above, history shows that the stratified trees originally were designed using a completely different approach. This second approach is best described as a *binary search on levels* strategy.

In this approach the elements of the Universe U are represented by the leaves of a complete binary tree of height $h = log(u)$. At each element of the tree (both at the leaves and the internal nodes) we have a mark-bit. A set A is represented by marking its elements together with all internal nodes on a the path from a marked leaf to the root of the tree as illustrated in figure 1. The marked nodes will be called *present* in the sequel.

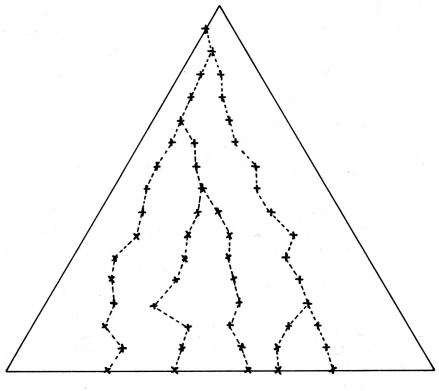


Fig. 1. Example of a five-element set represented using mark-bits

Additionally, we build a doubly linked list connecting the leaves corresponding to the elements of A. This data structure evidently supports the instructions $member$, $card$, min and max in constant time.

$insert(x)$ is implemented by traversing the path from the leaf representing x towards the root until a marked internal node v is encountered. This node v will become the lowest *branch point* on the path of the newly inserted element x. All nodes traversed during this walk are marked, Subsequently starting at this already marked node a downward traversal along marked nodes is performed, always selecting the rightmost (leftmost) marked son, depending on whether leaf x is a right (left) descendant of node v. The leaf thus found is the *neighbour* of x in the set A. It is either the predecessor or the successor of x depending on whether x is a right or left descendant of v. Having found this neighbour, the leaf x now can be inserted in the doubly linked list in constant time.

In order to delete an element $x \in A$, a similar traversal starting at leaf x is performed until a branch point (an internal node with two marked sons) is encountered. Nodes encountered during this traversal are unmarked. Next, leaf x is removed from the doubly linked list in constant time.

In order to compute the predecessor and successor of x, a similar upward traversal is performed, which yields the lowest branch point. A downward traversal subsequently takes us to the neighbour, from which the successor or predecessor can be found in constant time in case we happen to arrive at the wrong side of x in the sequence of leaves.

The time consumed by these traversals is bounded by the height of the tree h. Therefore the resulting processing time is $O(h) = O(log(u))$. But the actual time used is determined by the distance of this lowest branch point from the leaves.

The simple observation that in the above approach time saved by not traversing all the way up to the root could be used for spending more time at the lowest branch point encountered, allowed me to improve the worst case time bound. If the lowest branch point is encountered at level k above the leaves (counting from the leaves at level 0) we can spend the remaining $h - k$ time steps by allowing $h - k + 1$ sons for all nodes at level k. The consequence is that in the resulting tree of height h we are not representing a Universe of size 2^k but one of size $h!$. When $u = h!$ we have $h = O(log(u)/loglog(u))$ and this yields an improvement over the $O(log(u))$ bound.

Discussing this improvement to John Hopcroft it immediately became clear that further improvements could be expected. The strategy as explained above uses a trivial linear search strategy at this lowest branch point. If we organise the sons of this branch point in a binary tree, we can allow for branching degrees growing even faster: branching degree 2^{h-k+1} at level k which would result in a time complexity $O(\sqrt{log(u)})$. A true application of the *Divide and Conquer* heuristic propagated by the early researchers in the field of algorithm design suggests the use of the very structure I was designing, recursively, at the internal nodes.

There was yet another way of looking at the problem. In this approach the key step is the location of the lowest branch point on the path from a present leaf to the root, or the lowest present ancestor in case the leaf is not present. If these points could be located using binary search on the path connecting the leaf with the root the time complexity could be reduced from $O(log(u))$ to $O(loglog(u))$. Such a method of binary search on levels had been used previously with success by Aho, Hopcroft and Ullman for finding lowest common ancestors in trees [1].

By the end of 1974 a solution following these suggestions had been obtained [45]. Conceptually it is far more complicated than the recursive approach. Moreover I had to struggle with yet another complication, having to do with the required address calculations used by our algorithms. They use multiplicative instructions, and these were not allowed on the machine model I was using. This complication was, at this stage of the history, fatal for the further exploration of the recursive decomposition approach.

In order to understand why these multiplicative instructions were problematic we first must take a look at the machine

model used for Analysis of Algorithms when this research area was being created: the Random Access Machine.

III. Machine based complexity theory around 1974

Computability theory as we know it today was developed in the years 1920-1940 with computational models described by Post, Church, Gödel, Kleene and above all the simple and widely accepted model invented by Turing. The idea of investigating also the time and space complexity of these models of computation originates from the 1960-ies. The concept of a "*feasible computation*" was introduced by Edmonds [13]. Hartmanis and Stearns [18] started the research on the complexity of the Turing machine model. Manuel Blum [8] created the extension of Recursion Theory known as *Abstract Complexity Theory*. By 1972 the fundamental Complexity Classes P, NP, PSPACE, EXPTIME, ... had been defined and the P = NP ? problem became a part of our life.

None of these approaches provides definite answers for the real life programmer who wants to solve real problems on a real computer. The structural approach which focusses on the relation between the fundamental complexity classes, compares models based on mutual simulations with a polynomial time bounded overhead and a constant factor overhead in space [52]. However, for practical design and analysis of algorithms a quadratic improvement is highly relevant. In the abstract theory the situation is even worse: all models are basically equivalent, including the pathological ones.

When Knuth started his book series on the art of computer programming [21] he followed a different approach. He introduces in this book a simplified model of a real von Neumann computer called MIX. Algorithms are analysed by a precise estimate of the number of machine instructions on this MIX model. This immediately raises the question to which extent the results obtained by Knuth depend on the details of the MIX model.

When Aho, Hopcroft and Ullman wrote their textbook [2], they based their work on a compromise between these two extremes. They use the Random Access Machine (RAM) as described by Cook and Reckhow [9] as a model. It differs from the MIX by having a reduced set of instructions, but it is also a theoretical model since it has unbounded memory and word size. Furthermore they consider a stored program version of the model called RASP, and prove that the two models can simulate each other in real time.

They present their algorithms in a higher programming language called *Pidgin Algol* which makes the basic control features of programming like conditionals, loop constructs procedures, functions and recursion directly available. It was safe to do so since by 1974 it was well known how to compile such features in machine instructions. There were no hidden time overheads between the perceived instructions in the Pidgin Algol program and the resulting Machine code program on the RAM. What one looses is information on the precise constant factors involved, but that was felt to be less important; the asymptotic growth of the complexity was the target of the research program, as illustrated by the abundant invocation of the Landau O(f(n)) expressions.

The RAM model is basically a single register machine having an infinite memory of infinite length storage cells called words. The memory contents are non-negative integers. The instruction code includes loading constants, and the contents of memory locations using both direct and indirect addressing (where the index of the memory cell being loaded is itself the content of another memory location). For storing data in memory the machine can use both direct and indirect addressing. As flow of control the machine has unconditional and conditional jumps (testing for $= 0$ or positiveness of the accumulator contents). All arithmetic is performed on the single accumulator register. The arithmetic instructions allowed are addition, subtraction, multiplication and (integer) division. The programmer should beware for typical errors like the production of negative results or division by zero.

An earlier model of this type is the register machine introduced already in 1963 by Shepherson and Sturgis [39]. This model in its basic form only has increment and decrement as arithmetic instructions, and it lacks indirect addressing. It is equivalent to the Minsky multi-counter machine [30]. Still this model is Turing Universal, even if only two registers are being used.

For the time complexity Aho and his colleagues use two measures. In the uniform measure one just counts the number of RAM instructions performed. In the logarithmic time measure each instruction is charged for the number of bits in the arguments of the instructions, regardless whether these arguments are addresses or data. Similarly one can for a space measure either count the number of memory locations used, or weigh every location by the size of the values being stored there.

They prove that with respect to the logarithmic time measure the RAM and RASP model are equivalent to the multi tape Turing machine modulo a quadratic time overhead. But if the uniform time measure is used, there is a problem with the multiplication and division instructions, since they can generate exponentially large results in a linear number of steps. Therefore one should exclude multiplicative arithmetic when using the uniform time measure; if only additive arithmetic the uniform time measure and the logarithmic time measure are equivalent up to a quadratic overhead.

That there is a real problem here follows from results obtained by Pratt and Stockmeyer [35], Hartmanis and Simon [17] and Bertoni e.a. [7] or Schönhage [37]: the RAM model with multiplicative arithmetic and uniform time measure is equivalent to models of parallel computation, meaning that it can perform PSPACE complete computations in polynomial time. It is a member of the second machine class [52].

The Turing Universality of the two counter Minsky machine shows that just counting memory cells is not a good idea. Therefore the uniform space measure is never used, unless there is an explicit limit on the size of the values being stored. For the logarithmic space measure there is an issue on how to deal with storage cells which are unused during a computation while cells further down in the program are used. Also the size of the addresses of the memory cells should contribute to the logarithmic cost of such a cell. For details see [40].

The question remains which of these measures is suitable for analysing the complexity of the stratified trees. There

are several reasons for not using the logarithmic time measure. In the first place the main competitive algorithms use comparison based manipulation of lists and trees which are build using pointers. Frequently one just counts the number of coomparisons. Address calculations hardly play a role and arithmetic is hardly used at all other than for counting the size of (sub)structures.

Another argument is that use of the logarithmic measure will entail an operation complexity $\Omega(log(u))$ for just looking at the value being manipulated, so improvement to sub-logarithmic cost is impossible anyhow.

The space measure was a less interesting topic; for the tree based structures space is traditionally measured by the number of nodes and edges in the structure which amounts to the use of the uniform space measure.

However, using the combination of both uniform time and space measures comes with a price: it is no longer appropriate to use multiplicative instructions. These instructions were perceived to be more expensive than the additive ones, both in practice and in theory; the linear time multiplication algorithm of Schönhage [38] still had to be invented.

It is not difficult to see that multiplicative instructions are used in our recursive decomposition for address calculations. Assume that $u = v.w$ and that we divide the universe $[0..u-1]$ in a cluster of v galaxies each consisting of w elements each. Then element x will be represented by entry $x \bmod w$ in galaxy $x \div w$; conversely entry a in galaxy b represents element $w.b + a$ in the universe. This shows that the relation between the arguments of our operations and those involved in the recursive calls are computed using the multiplicative operations.

Another construction we can't use any more is a two-dimensional array; its indexing formula uses a multiplication.

However, in our algorithms we will in general only multiply and divide by powers of two. This makes the restriction even more unreasonable, since these arithmetical operations are realized by bit-shifts which are available on many real life computers. So we are facing a situation where theory seems to be more restrictive than practice.

Fact is that most authors discussing stratified trees, in particular when discussing the recursive approach, simply are disregarding this prohibition of multiplicative instructions; as long as the arguments don't increase above the initial values they consider them to be allowed.

IV. Implementing binary search on levels

The binary search on levels approach uses a complicated decomposition structure on the binary tree which was used in the silly method in section II. I will explain how this tree is decomposed in so-called *canonical sub-trees*, what sort of static information we need in order to navigate this tree, and how the dynamic set A is represented in this structure by assigning appropriate values to fields located at the internal nodes of the tree. By way of example the code fragment for the $neighbour$ instruction is presented and explained. Complete details can be found in the source reference [48].

A. Canonical sub-trees

For reasons of simplicity we assume in this section that $u = 2^{2^l}$; the height of the tree structure we will describe equals $h = 2^l$, and the number l will be called the *rank* of the tree. The cluster of galaxies decomposition of the universe corresponds to dividing the tree in a top-tree of height $h/2 = 2^{l-1}$ and rank $l-1$ representing the cluster whereas the bottom trees of rank $l-1$ whose roots are actually the leaves of the top tree represent the galaxies. Each of these trees can be decomposed again in a top and bottom trees of rank $l-2$. The trees of rank 0 are trivial: a root with two sons. The trees obtained in this recursive decomposition are called the *canonical sub-trees* in the papers where the stratified trees were first introduced.

The concept of rank is also ascribed to the levels in the tree and therefore also to the nodes in the tree. The leaves of a tree are located at level 0 and the root is at level $h = 2^l$. The rank of level $j > 0$ is the largest number d such that $2^d | j$ and $2^{d+1} \not| j$. By default $rank(0) = l+1$. Note that if $rank(j) = d$ we have $rank(j + 2^d) > d$ and $rank(j - 2^d) > d$; moreover the ranks of these two numbers are different. All numbers i with $j - 2^d < i < j + 2^d$ and $i \neq j$ have $rank(i) < d$.[2] The rank of some node v is the rank of its level in the tree.

A canonical sub-tree of rank d is a sub-tree of height 2^d with a root at some level j with $rank(j) \geq d$ and leaves at the closest level with rank $\geq d$. It follows that either the root or the leaves have rank equal d. If the root has rank d the canonical sub-tree is called a *bottom tree*, and it is called a *top tree* otherwise.

Canonical sub-trees are furthermore divided in a left part and a right part, the left (right) part consisting of the root and its left (right) sons and their descendants within the sub-tree. This division adds another element of complexity to our structure which has no counterpart in the recursive cluster and galaxies approach; we shall see in the sequel of this section why this further division had to be added.

A final concept is the *reach* of some node v; if $rank(v) = d$ node v is leaf of an uniquely determined top tree of rank d and the root of a bottom tree of rank d. The root of this top tree and the leaves of this bottom tree have rank $> d$. Now $reach(v)$ consists of this bottom tree together with the left or right part of the top tree containing v. See figure 2. This figure introduces also names for the relevant sub-trees: $C(v)$ is the rank $d+1$ canonical sub-tree containing v, $UC(v)$ is the top-tree of $C(v)$ and $LC(v)$ is the bottom-tree of $C(v)$ having v as its root. The left and right parts of $LC(v)$ are named $LLC(v)$ and $RLC(v)$.

B. Static information

The next topic to consider is how we can navigate in this tree and its canonical sub-trees. A useful way of indexing nodes in a binary tree is to assign index 1 to the root, and assigning indices $2.m$ and $2.m + 1$ to the left and right son of the node with index m. Moving j levels up from the node with index m yields node $m \div 2^j$. The 2^j descendants of node m have indices $2^j.m, \ldots 2^j.m + 2^j - 1$. The leaves of our

[2] The idea of such a rank function is well illustrated by the lengths of the lines on the British side of a ruler; an abstract definition which will work for general heights h is sketched in [48].

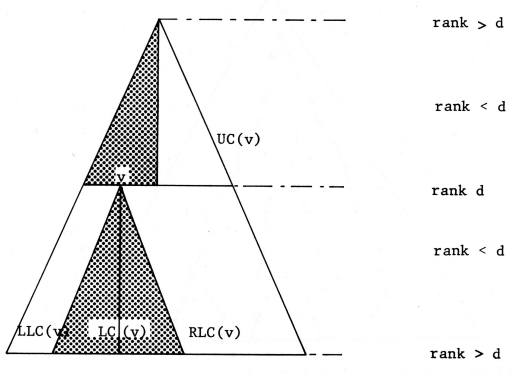


Fig. 2. Canonical sub-trees and the reach of some node

complete tree which will represent the elements of the universe have indices $2^h, \ldots 2^h + u - 1$.

The problem is that these expressions which allow us to freely navigate in the tree use multiplicative operations; once again - these operations look perfectly innocent since they are easily evaluated by bit-shifts - but still they were considered illegal in the RAM model with the uniform time measure in 1974. Therefore another solution was invented: pre-compute sufficiently many of these navigation steps in advance, so that you can use table look-up when you need to perform this step.

It turns out that the required navigation steps for our algorithms to work are the ancestors at levels $1, 2, \ldots 2^d$ of some node v of rank d; these are precisely the roots of those canonical sub-trees of rank $0, 1, \ldots d$ having v as a leaf. In order to have a format which doesn't depend on the rank of some node it was decided to store at node v pointers toward the roots of all canonical sub-trees containing v. There are $loglog(u)$ pointers of this type at every node yielding a space consumption of $\Omega(u.loglog(u))$ for storing just this static information which will never change during the manipulation of some subset A. The fact that you don't need pointers to ancestors at level 2^l for $l > rank(v)$ doesn't help us since the leaves at the bottom of the tree all have maximal rank and represent a majority of the nodes in the tree. Using a rank dependent format would save at best a constant factor in space.

This solution to our navigation problems also explains why the original implementation of the stratified tree used pointers rather than addresses; the stratified tree is a solution of our set manipulation problem in the pointer based model of computation, and this is relevant in relation to lower bound results which we will encounter later in this survey.

C. Dynamic information

In our silly solution as illustrated in figure 1 we represent some set A by marking all leaves representing the elements of A and all ancestors of some marked node. Marked nodes are called to be *present*. The various operations were implemented by locating either the lowest branch-point (a present node with two present sons) above some present node or the lowest marked ancestor above some non-present node (for the Insert and Neighbour instructions); this node will be called a *focal-point* in our further discussion. The key challenge now is that we don't want to spend time $\Omega(log(u))$ by a linear traversal of these ancestor paths connecting leaves with the root; instead we want to use binary search on levels.

Consider some canonical sub-tree T of rank d whose root v is present. There are two cases to consider: either there is just a single present leaf in this canonical sub-tree, and one might as well store at v some information which will allow us to locate this leaf; otherwise there are at least two present leaves in T which means that somewhere in T a branch-point exists (it may be the root v itself). Consider now the leaves of the canonical sub-tree T' of rank $d-1$ with root v (the top-tree in the recursive decomposition of T). If there are at least two present leaves in T' there will be a branch-point inside T'; otherwise if w is the unique present leaf in T' there will be a branch-point inside the canonical sub-tree of rank $d-1$ with root w (one of the bottom-trees in the recursive decomposition of T). In order to determine which of the two cases arises we should store some information at the present nodes on this half-way level where w is located.

This leads to the concept of an *active* node. An internal node is active if information is stored there which is required for locating some focal-point or branch-point during some possible operation. A leaf is active if it is present, and by definition the root is active when A is non-empty (in our original implementation this was achieved by stipulating that $u - 1 \in A$ permanently, but this restriction turned out to be superfluous).

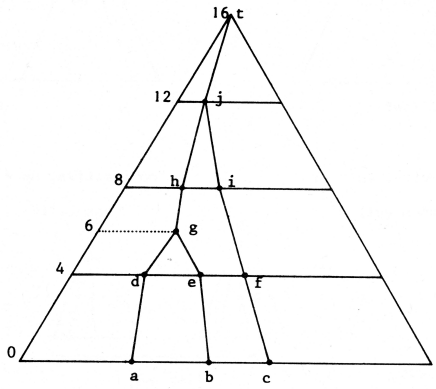
Whether some internal node should be active or not is determined by an invariant called the *properness condition* [48]: *An internal node v is active iff there exists a branch-point in the interior of the reach of v*; i.e. there exists a branch-point $u \in R(v)$ which is neither the top nor a leaf of $C(v)$.

At an inactive node no information is stored; its fields will have default values.

The information stored at active leaves consists of two pointers towards their present predecessor and successor in the set A as far as they exist, together with a mark-bit indicating that the leaf is present. For an active internal node v the data consists of four pointers and a mark-bit [3]; the pointers $lmin$ and $lmax$ store the least and largest present leaf in $LLC(v)$ if such leaves exist, and $rmin$ and $rmax$ do so for $RLC(v)$. Note that these are leaves in $LC(v)$ which means that they are not necessarily leaves of the entire tree. The mark-bit ub indicates whether there exists a branch-point in between v and the top of $UC(v)$. Note that an active internal node is present and hence it must have at least one present descendant leaf in $LC(v)$, so at least two of the four pointers $lmin$, $lmax$, $rmin$ and $rmax$ are defined. It is possible to have $lmin(v) = lmax(v)$ which indicates that there is no branch-point in between v and $lmin(v)$; if in this situation also $rmin(v)$ is defined the node v itself is a branch-point. Figure 3 illustrates the assignment of values to these pointers at internal nodes for a three-element set A.

The representation of a set A now is determined as follows: the leaves representing the elements of A are declared to be

[3] Our implementation used different record formats for leaves and internal nodes in the tree; the case-dependent record format in PASCAL made it easy to implement this difference.



	LEVEL	RANK	LMIN	LMAX	RMIN	RMAX	UB
t	16	4	a	c	∿	∿	–
j	12	2	h	h	i	i	–
h	8	3	∿	∿	a	b	+
i	8	3	∿	∿	c	c	+
g	6	1	d	d	e	e	–
d	4	2	a	a	∿	∿	+
e	4	2	∿	∿	b	b	+
f	4	2	∿	∿	∿	∿	∿

Fig. 3. Example of the internal pointers for a three-element set

present and active and the pointers in the doubly linked list are given their appropriate values. If $A \neq \emptyset$ the root is declared to be present and its four pointers are assigned their proper values. Next for internal nodes, processing them by rank decreasing from $h-1$ to 0 it is checked whether they should be made active according the properness condition, and if so their fields obtain the required values. [4]

D. Operations

Given the complexity of the representation of the dynamic set A presented above it should come as no surprise that the actual implementations of the operations are complex as well. Certainly if compared with the code which results from implementing the recursive cluster-galaxy decomposition scheme which can be found in the textbook presentations of my structure [10], [16], [27], [60]. The key operations which do the work are insert, delete and neighbour. A full explanation of these operations would require me to copy the entire section 5 from our original publication [48] which would fill many pages. Therefore I rather refer the reader to the original sources. I include and discuss the code for the neighbour instruction, directly copied from [48].

The programming language used is PASCAL [56]. This code was produced by my students Kaas and Zijlstra [25] who on behalf of this contribution earned co-authorship on the journal publication of my FOCS 1975 paper [47].

The instruction neighbour has only a single argument when called from the outside; however, in its recursive structure it requires additional parameters for navigational purposes. Hence the overall structure of this function consists of the declaration of an internal function $neighb$ which actually performs the work, which is subsequently invoked on the appropriate arguments.

[4] evidently this is not the algorithm which was used for maintaining the representation of a dynamic set A in our implementation.

```
function neighbour(j: integer): integer;
  var pt: ptr;
  function neighb(leaf, top, pmin, pmax: ptr; order: integer): ptr;
  var y, z, nb, hl: ptr; pos: 1 . . n;
  begin pos:= leaf↑.position;
    if (pmin = nil) or ((pmin = pmax) and (pmin = leaf)) then
      if pos <= top↑.position then neighb:= yourmin(leaf, top)
      else neighb:= yourmax(leaf, top)
    else if pmin↑.position > pos then neighb:= pmin
    else if pmax↑.position < pos then neighb:= pmax
    else
    begin hl:= leaf↑.fathers[order−1]; y:= minof(hl); z:= maxof(hl);
      if ((y = z) and (y = leaf)) or (hl↑.ub = undef) then
      begin nb:= neighb(hl, top, pmin↑.fathers[order−1],
          pmax↑.fathers[order−1], order−1);
        if hl↑.position < nb↑.position then neighb:= minof(nb)
        else neighb:= maxof(nb)
      end
      else neighb:= neighb(leaf, hl, mymin(leaf, hl),
          mymax(leaf, hl), order−1)
    end
  end;
begin pt:= elements[j];
  pt:= neighb(pt, root, mymin(pt, root), mymax(pt, root), h);
  if pt = nil then neighbour:= 0 else neighbour:= pt↑.position
end;
```

Fig. 4. The code for the neighbour instruction

The arguments of this internal function are: $leaf$ (the ancestor of the original argument at the bottom level of the canonical sub-tree under consideration), top (the top of this sub-tree), $pmin$ and $pmax$ (the leftmost and rightmost present leaves in the half-tree containing leaf) and $order$ (the rank of the sub-tree). Of the four pointer fields at the node top we first have to inspect those which are located on the same side as $leaf$; in order to simplify the code this choice has been hidden into the auxiliary functions $mymin$, $mymax$, $yourmin$ and $yourmax$. The auxiliary functions $minof$ and $maxof$ yield the least and largest leaves in the entire sub-tree. The left-right comparison between two nodes is performed by comparing the $position$ field of the nodes which is part of the static information and filled with the in-order number assigned to the node when the tree is initialized in an in-order traversal.

There are several cases where the neighbour can be computed without a recursive call: when the half-tree containing leaf doesn't contain a present leaf ($pmin = nil$), or when leaf is actually its unique active leaf ($pmin = pmax = leaf$); a third case is when there are several active leaves but leaf is outside the range between $pmin$ and $pmax$.

Otherwise we must inspect the ancestor hl of $leaf$ on the level half way in between $leaf$ and top. In case hl has no (other) present leaf (indicated by hl being inactive or $minof(hl) = maxof(hl) = leaf$) a recursive call is performed on the top-tree with root top yielding the neighbour at that level nb. The neighbour of $leaf$ now is $minof(nb)$ or $maxof(nb)$ depending on the position of hl compared to nb. If hl has another present leaf a recursive call is performed on the bottom tree with root hl which yields the neighbour at the right level directly.

As promised: this procedure either terminates in constant time or performs a single recursive call with $order$ decreased by 1, and this suffices for obtaining time complexity $O(loglog(u))$.

The reader should note however that this procedure in fact behaves differently from what one would expect if one follows the recursive cluster-galaxy decomposition approach.

It has to do with the conditions allowing the $neighb$ function to terminate without a recursive call. The cases where there is no present leaf or when leaf is the unique present leaf in the recursive approach will look at the entire sub-tree and not just the half-tree to which $leaf$ belongs; the very concepts of left- and right sub-trees are simply not defined in the recursive approach.

The operations insert and delete are structured in a similar fashion, but they are even more intricate since they have side effects on the structure.

Finally some remarks on the initialization routine. It performs a mixture of a pre-order and an in-order traversal of the tree being constructed and builds what is to become a proper representation of the empty set A. When creating a node it must assign proper values to the father pointers and to the position fields. What are the proper values depends on the rank and the level of a node. Multiplicative instructions for evaluating the rank for a given level are prevented by precomputing these ranks and storing the results in a table. Details can be found in [48] but the code is difficult to understand due to various hidden side effects.

V. The development of the original, and the revival of the recursive decomposition approach

The code in the journal paper discussed in the previous section represents the final stage of a sequence of presentations, starting with the 1974 Cornell report. I describe in this section how the perspective on what operations are essential gradually changed during this development. Next we look at Knuth's classroom note from 1977 and its importance for reviving the recursive decomposition approach.

A. The improvements of the original presentation

The first papers I published on the stratified trees all are based on the binary search on levels approach sketched in the previous section. The oldest paper is the Cornell technical report [45]. The representation used in this original paper is almost identical to the one used in the eventual journal paper [48], but the perspective on the operations is quite different. The primitive operations are the location of the lowest branch (present) point above a given (non) present node (called the focal point previously), and the deletion (insertion) of the path segment between the argument leaf and the focal point. The instruction neighbour is mentioned but its importance remains unrecognised. And to my own dissatisfaction I was unable to design a recursive version of the code for the delete instruction.

The second paper was written for a seminar talk at IRIA in the spring of 1975 [46]. The same representation is used but the operations are different and the neighbour instruction has become more important. Hardly any code fragments are given.

Around this time I had assigned the task of programming a real implementation of my structure as a possible assignment for a term project to a group of students in an advanced programming course which I was teaching with some colleagues during the spring of 1975. Two students, Kaas and Zijlstra, accepted the challenge and eventually produced the PASCAL code contained in a report of the department of Mathematics in my institute [25]. Aside from repairing some hideous errors in my original code and contributing some valuable ideas they found a recursive version of the delete instruction. Having this code available I submitted the result for the FOCS 1975 conference in Berkeley, where it was accepted and published [47]. The binary search on levels approach still is used, but in the expository introduction the idea of the recursive cluster galaxy approach is mentioned. Nothing is done with this idea. The FOCS paper includes code fragments from the Kaas and Zijlstra program.

The journal paper [48] actually merges the two previous reports and Kaas and Zijlstra became co-authors. A preprint is dated December 1975.

In August 1976 a symposium was organised on interfaces between computer science and operations research. For this symposium I gave a presentation on developments in data structures. The text of this presentation was eventually published in 1978 [50], but since this is two years later than the presentation it is uncertain to what extent its contents correspond with the talk given. Fact is that in this paper the cluster galaxy decomposition idea is used as a starting point and the stratified trees are presented as being the result of unwinding the recursion in the recursive data structure implementing this decomposition.

In 1977 I found a solution [49] to another nasty feature of my original structure: its super-linear space consumption. It is based on the cluster galaxy decomposition idea used in a non-recursive way. The structure of the cluster and the galaxies are different: the cluster representing m galaxies of size k is implemented by the original structure, but the galaxies are implemented by some structure using linear space with a worse time complexity. A binary tree based structure like my silly structure from figure 1 is OK, but even a linear list will work here. When one realises that every instruction is decomposed in an instruction operating on the cluster level and one on the galaxy level, and uses the fact that the storage being used is the sum of the space for the cluster and that for the galaxies, one obtains the following estimates:

Time: $O(loglog(m)) + O(k)$

Space: $O(m.loglog(m)) + m.O(k)$

Taking $m = u/loglog(u)$ and $k = loglog(u)$ solves the problem.

The code fragments from this paper illustrate the simplicity of this approach. See figure 5. One of my worst mistakes is that I, having arrived at this idea of a hybrid mixture of data structures, didn't use this for other purposes; it became an important tool for designing dynamic data structures in areas like computational geometry and was used extensively by my colleagues from Utrecht Jan van Leeuwen and Mark Overmars [32].

B. Knuth's classroom note and its impact on reviving the recursive decomposition approach

I had exchanged papers and reprints with Don Knuth. In reply of the preprint of [49] he send me a copy of his classroom note [23]. To my knowledge this is the first written version of the fully recursive cluster and galaxy decomposition approach.

```
proc insertun(x);
      begin y:=galaxy(x);
      if emptygal(y)  then begin insertgal(x,y);insertcl(y) end
                      else insertgal(x,y)
      end;

proc deleteun(x);
      begin y:=galaxy(x); deletegal(x,y);
      if  emptygal(y)  then  deletecl(y)
      end;

fun  memberun(x);
      begin y:=galaxy(x);
      if  membercl(y)  then memberun := membergal(x,y)
                       else memberun := false
      end;

 fun  successorun(x);
       begin  y:=galaxy(x); t:=successorgal(x,y);
       if t = nil  # no successor in  Sy #
       then begin  z := successorcl(y); successorun := mingal(z) end
       else  successorun := t
       end;
```

Fig. 5. Code fragments for the non recursive cluster galaxy decomposition

Multiplicative instructions are used without hesitation. Both the asymptotic complexity and practical implementations for small universes are considered (when $u = 2^{20}$ it consumes 242937 bytes, sufficiently small to fit in core on the IBM 360). For this case a three level decomposition with explicit addresses and bit-shift operations is presented; see figure 6 for the relevant code fragments. The programming language used is inspired by Simula. Note also the final sentence in this page: this is the source of the famous quote from Knuth on program correctness.

```
procedure insert0 (integer x);
if size[0] = 0 then (size[0] ← 1; least[0] ← greatest[0] ← x)
else begin if size[0] = 1 then (insert1(least[0] ↱ 8,1);
           insert 1(least[0], 18 x (least[0] ↱ 8) + 19));
   if size[18 x (x ↱ 8) + 19] = 0 then insert1(x ↱ 8,1);
   insert1(x, 18 x (x ↱ 8) + 19);
   size[0] ← size[0] + 1;
   if x < least[0] then least[0] ← x
   else if x > greatest[0] then greatest[0] ← x;
   end;
procedure insert1 (integer x, i);
if size[i] = 0 then (size[i] ← 1; least[i] ← greatest[i] ← x)
else begin if size[i] = 1 then (insert2(least[i] ↱ 4, i+1);
           insert2(least[i], (least[i] ↱ 4) mod 16 + i+2));
   if size[(x ↱ 4) mod 16 + i+2] = 0 then insert2(x ↱ 4, i+1);
   insert2(x, (x ↱ 4) mod 16 + i+2);
   size[i] ← size[i]+1;
   if x < least[i] then least[i] ← x
   else if x > greatest[i] then greatest[i] ← x;
   end;
procedure insert2 (integer x, i)
begin B[i] ← B[i] ∨ (2 ↑ (x mod 16));
   size[i] ← size[i]+1;
   if x < least[i] then least[i] ← x
   else if x > greatest[i] then greatest[i] ← x;
   end;

The implementation of deletion would be similar.  It is safe to use  0
and  2^16-1  for  -∞  and  +∞ .
     Beware of bugs in the above code; I have only proved it correct, not
tried it.
```

Fig. 6. the final page of the classroom note by Knuth

In the letter which was included with the classroom note Knuth observes that his recursive top-down approach eliminates the need for the concepts of rank, canonical sub-trees and the branch-points. Proving correctness would be much easier, so Knuth claimed part of the $ 10.– prize for a correctness proof for the procedures which I had offered in my presentation at FOCS 1975 (I must have done so in my talk - the text in the proceedings doesn't mention this prize). In my reply I conceded on the issue of the unneeded complexity of my original structure, pointing towards the problem with the multiplicative instructions which I was told not to use. I also observe that due to the absence of the left-right half sub-tree distinction the original structure can't be obtained by unwinding the recursive version, contrary to what I suggested in [50]. The prize was intended for a correctness proof of the original non-recursive version, so I rejected his claim but by separate mail I have send him one of my traditional gifts (a Dutch windmill tile).

Knuth also informed me that he intends to include my structure in volume 4 of his series of textbooks. As we all know, the production of this volume was delayed by some 30 years, and now that the first part of volume 4 has appeared, I found that my structure is destined to be included in the second part of volume 4.

The final paper on the stratified trees I wrote myself is another contribution to a colloquium at the Mathematical Centre in Amsterdam [51]. This paper is written in Dutch. Three sections discuss the stratified trees. I first describe the $O(log(u))$ recursive cluster galaxy decomposition approach following the algorithms as presented in figure 5. The next section sketches the tree model as presented in [48], [49]. In the final section I present my own version of the fully recursive $O(loglog(u))$ implementation suggested by Knuth with a full reference to his classroom note. I show some code fragments, this time written in some version of Algol 68 [55], in figure 7 and figure 8.

In figure 7 is shown how I dealt with the multiplicative instructions. Thinking in terms of bit-strings they are called $head$, $tail$ and $conc$ respectively, but since the implied galaxy size q depends on the recursive level there is another hidden argument called $order$. Therefore $head$ and $tail$ become binary operators, and $conc$ becomes a procedure since Algol 68 doesn't allow to write ternary operators as operators. Actually these operations were implemented by table look-up. Note that we don't use a two-dimensional array since its indexing formula would reintroduce the multiplicative instructions we want to eliminate. Instead we use an array of arrays. The final code fragment shows how these tables are precomputed by counting; another code fragment using side effects on s.

Figure 8 shows the insert operation in the recursive decomposition scheme. The data structure itself consists of three integers, a pointer towards the top structure representing the cluster, and a pointer towards an array of bottom structures representing the galaxies. It is not stated explicitly that these pointers should be nil at the bottom of the recursion when $order = 0$.

The next code fragment is the first insert procedure which is invoked when inserting a first element; it only affects the three integer fields, leaving the pointers untouched. The general insert procedure uses the case construct of Algol 68 depending on the current cardinality of the structure. If that equals 0 first insert is invoked and the procedure terminates; if the cardinality equals 1, the existing element must be inserted in

```
head x = x over q,
tail x = x mod q   en
t conc u = t*q + u.
```

Multiplicative instructions

```
op tail = (int order, arg) : taila[order][arg],
proc conc = (int order, head, tail) : tail + concs[order][head] .
```

Table look-up implementation of forbidden multiplicative instructions

```
int l := 2;
  for k to max order do
    int s := -1;
      for i from 0 tot l - 1 do
       concs[k][i] := s + 1;
        for j from 0 to l - 1 do
          heada[k][s +:= 1] := i; taila[k][s] := j
       od od;
     l := s + 1
   od;
```

Precomputing the tables by counting

Fig. 7. computing addresses without multiplicative instructions in the recursive decomposition

the recursive structure first, but this generates an invocation of first insert both at the top and the bottom level and therefore it takes constant time. If the cardinality equals 2 or more this step is skipped. Now we are ready for inserting the new element. The galaxy to which it belongs is retrieved; if this galaxy is non-empty the element is inserted in this galaxy by a recursive call; otherwise the galaxy is inserted in the cluster and the element is inserted in the galaxy using first insert. Finally the min, max and $card$ fields are adjusted.

In this code fragment it is assumed that the recursive data structure has been built previously; no initialisation procedure is described in this paper.

```
mode deque = struct(int card,min,max,
                    ref deque top,
                    ref [ ] deque bottome);
```

Recursive date structure definition

```
proc first insert = (ref deque queue, int arg) void:
     (card of queue := 1; min of queue := max of queue := arg);


proc insert = (ref deque queue, int arg, order) void:
  if order = 0 then # bottom case of recursion;
                    trivial actions for small universe #
                  .....
  else case card of queue +1 in
    (first insert (queue,arg); goto done),
    (# set up internal structure for existing elt #
       int old = min of queue;
       int old tail = order tail old, old gal = order head old;
    ref deque cl = top of queue, bot = (bottome of queue) [old gal];
       first insert (cl,old gal); first insert (bot, old tail) )
  esac; # now we are ready for new insert #
   int new tail = order tail arg, new gal = order head arg;
 ref deque cl = top of queue, bot = (bottome of queue) [new gal];
   if card of bot /= 0
        then insert (bot, newtail, order-1)
        else first insert (bot, newtail);
             insert(top, new gal, order-1)
   fi; # now adjust min/max and card #
     if arg < min of queue then min of queue := arg
   elif arg > max of queue then max of queue := arg fi;
     card of queue +:= 1; done: skip fi;
```

Code of the insert instruction

Fig. 8. the insert procedure for the recursive decomposition approach

The reader will find the = symbol used at position where an assignment := would be expected; this is the Algol 68 feature of constant declarations. No code is given in this seminar talk for the delete and the neighbour instructions but they would have been structured similarly.

All together it seems that the main problems with the original presentation had been solved; the presentation had been clarified[5], the super linear space consumption had been eliminated and the forbidden address calculations had been replaced by something which was legal. So I considered the project to be completed; in fact I have not seriously looked at stratified trees since 1980 except for the fact that in my valedictory address in 2010 they are mentioned as one of 12 topics I have worked on [53].

The first textbook presentations of the stratified trees known to me are given by Mehlhorn [27] and Gonnet [16]. They both use the recursive decomposition scheme. This holds also for the modern presentation in the textbook by Cormen, Leiserson, Rivest and Stein [10]. These authors dedicate an entire chapter of their book to the stratified trees. They first present the silly structure; next they give the $O(log(u))$ recursive decomposition version and finish their chapter with the fully recursive $O(loglog(u))$ structure.

There are two differences with my own presentation in [51]. They have changed the terminology: galaxies now are called clusters, and the cluster part is called a summary. Furthermore they have chosen to store the minimal element in the structure only in the min field at the root, and not to insert it in the recursive substructures. Evidently this will save the recursive insertion of one of the two elements when a second element arrives but it also complicates the code.

The wikipedia page on the van Emde Boas trees [60] follows the same approach as Cormen et al. On the wikipedia talk page [61], discussing the presentation on the main page, the correctness of the delete procedure is under discussion. Remember Knuth's warning!

VI. Literature on stratified trees: improvements and applications

The day I am writing this section (August 26 2013) Google scholar informs me that there are 550 papers referencing [48] and 408 referencing [49]. In fact [48] used for several years to be number 1 on the result list if one queries for the name van Emde Boas in Google scholar [6].

Some of these involve applications leading to a more efficient algorithm for some problem; others involve the complexity of the set manipulation problems themselves. The variety of machine models under consideration has grown. Finally there are lower bound proofs for the operations holding for various restricted machine models, some of which indicating that under specific conditions the $\Omega(loglog(u))$ processing time for the instructions is optimal. There are other scenarios where this lower bound is invalid and where the time for the operations has been improved.

[5] in a footnote in [20] D.B. Johnson refers to [50] as a *less inaccessible simplified exposition by the original author*

[6] Only this year I lost this position to another van Emde Boas who happens to be my brother and who now occupies the first two places with papers on the taxonomy of epileptic seizures. No bad feelings; the 7340 results produced by Google scholar seem all to originate from seven authors, all close family members.

The mere fact that some paper includes a reference to our papers doesn't entail that this paper also discusses it; it may include this reference since it was referenced in some other paper which is discussed. Moreover the context could be negative (look guys, we improved this result). This literature is far to extensive to be covered in this section [7]. Therefore I have selected just some noticeable examples which I could access from home. And I won't attempt to give much of an explanation since I hardly understand the proofs myself. As told before, I have not looked seriously at stratified trees for over three decades.

A. Applications

In the four original papers there is a section dedicated to reductions between set manipulation algorithms, both in the on-line case where the instructions are to be performed upon arrival, and in the off-line case where the entire instruction sequence is given in advance and may be preprocessed. Reductions should yield at worst a constant factor time overhead and program expansion (in the off-line case). The problems considered are insert and extract min, insert and allmin, and union find. The results consist of establishing the impact of the newly invented stratified trees on the earlier results bij Aho, Hopcroft and Ullman [1], [2].

The final journal paper [48] moreover considers the problem of building a mergeable heap based on our structure. This means that many sets are being manipulated and that the instructions union and find are added. An approach based on Tritter trees with stratified trees at the nodes is proposed. Tritter trees are the trees used by the union-find algorithm by Tarjan [41]. This extends a section in the FOCS paper where the problem of manipulating multiple sets is considered.

There is an evident problem with manipulating multiple sets: the space requirement is multiplied by the number of sets being used leading to a worst case space complexity $O(u^2.loglog(u))$, which is outrageous. I suggest a solution to this problem by a proposal amounting to dynamic allocation of the structure: allocate only those parts of the stratified trees which you actually need. Analysis shows that you always will need the top tree, but you can allocate the bottom trees at the time their root becomes active. Iterating this idea the space requirement can be reduced to $O(u^{1+\epsilon})$ for every positive ϵ.

With hindsight you can observe that this idea also works for a single stratified tree, so why didn't I use this approach for solving my problem of the super linear space? Once again it has to do with the prohibition on multiplicative instructions on the RAM and the way we had coped with this problem. The space for the static structure (which in principle can be shared between multiple copies of the stratified tree) already uses space $(u.loglog(u))$ and so do the precomputed tables used for the address calculations in the case of the recursive decomposition approach as presented in [51].

One of the first applications of my structure by others is the result on the longest common subsequence problem from Hunt and Szymanski [19]. It is a dynamic programming method in string manipulation algorithms, which uses priority queue operations on positions in the strings. Note that in the paper as published, the resulting $loglog(n)$ factor is not claimed in the main theorem, but is added in a second theorem called a *theoretical result*. At the time the paper was submitted (May 1975) the authors only had access to the Cornell report.

In order to have meaningful applications one needs to consider problems where the universe is bounded and the dynamic set being manipulated is not sparse. Such situations arise for example in Graph theory, Dijkstra's shortest path algorithm being a representative example of an algorithm using a priority queue. The size of the universe evidently depends on the nature of the edge weights. Such an approach is considered in the paper by Ahuja, Mehlhorn, Orlin and Tarjan [3], but the result in this paper shows that further improvements are possible. This result was further improved by Thorup [43].

An application in computational geometry was given by de Berg, van Krevelt and Snoeyink [12]. This result was used subsequently for packet switching on the internet by Lakshman and Stiliadis [26].

B. Improvements

The first improvement known to me was proposed by Donald B. Johnson [20]. He improves the operation time from $O(loglog(u))$ to $O(loglog(d))$ where d denotes the distance in the universe between the aregument and its neighbour in the set A. The idea of the improvement uses the binary search on levels approach. Rather than finding the focal point of an operation by top-down binary search one should first inspect the ancestors of the argument at levels 1,2,4,16, ...(these are the roots of the canonical sub-trees having the argument as leaf) untill one locates an ancestor above the focal point (something which can be recognized from the values being stored there).

As observed by the author: there seems to be no way to obtain this improvement when the recursive cluster galaxy decomposition approach is used.

The key reason why space $O(u)$ is required even in situations where $n << u$ is that the direct access of an array element, given its index, requires that the storage space for the array is allocated; as is well known from a famous exercise in [2] it is not needed to also initialize this space. The solution of this problem with direct access is to use dynamic perfect hashing. Fredman, Komlós and Szemerédi [14] have shown that you can do so without loss of the $O(1)$ access time [8].

Dan Willard proposed this approach [57] in 1983; together with Fredman this evolved in the design of the Fusion trees [15]. See also the survey paper [58].

Mehlhorn and Näher [29] obtained the optimal result combining time $O(loglog(u))$ and space $O(n)$, improving on Willard who had obtained these bounds only for a static set A.

Still the improvements don't stop at this point. M. Thorup [42] and Paul Beame and Faith E. Fich [6] obtain even

[7] Frequently the search leads to the discovery that the source is hidden behind a pay-wall, and therefore only very incomplete information on the contents becomes available.

[8] In fact I have used the results of this paper in some other part of my research [40], but never had the idea to use it for improving my stratified trees

better bounds, but it all becomes very complicated. The word-length of the RAM processor becomes another relevant parameter, and general AC_0 instructions on integers are allowed. This is far far away from my original intuitions. For example, lemma 2.1 in [42] cites a result from [5] where two sorted lists of at most k keys can be merged in time $O(log(k))$; the snag being that the two lists are stored in a single RAM word, and the result should be stored in two RAM words. Evidently this will require the use of instructions way beyond what was considered to be allowed in 1974.

For improvements in completely different directions: Wang and Lin [59] obtain amortized processing time O(1) on a pipelined system with $O(loglog(u))$ layers.

Finally the question remains whether any of these ideas will work if used with real life computers on regular size data; this has been investigated by Dementiev, Kettner, Mehnert and Sanders [11].

C. Lower bounds

During one of our encounters in the 1980-ies Kurt Mehlhorn told me that his students had proven that my structure is time-optimal, at least in the pointer based model of computation used in the original presentation. I presume that he refers to the result by Mehlhorn, Näher and Alt presented in [28].

This paper considers the union-split-find problem, but this problem is equivalent to the insert-delete-predecessor problem. Just consider the points in A to be the endpoints of intervals which are split and merged by inserting or removing new endpoints, and which are to be identified by their right boundary point. The lower bound is proven using graph theoretic arguments. One investigates the directed graph formed by the pointers which can be created, destroyed or moved around, but other operations on pointers have no meaning.

The situation changes when one moves to the cell-probe model. Here the RAM computation is analysed as a big decision tree where at any step inspecting some value stored in a RAM memory location one branches depending on the possible value stored there. The branching degree at these decision nodes is exponential in the word-length of the processor. The arguments become information theoretical and involve sometimes the use of communication complexity. An example of this approach can be found in the paper by Peter Bro Miltersen [31] who improves on earlier results by Ajtai [4]. He obtains a lower bound $\Omega(\sqrt{loglog(u)}$.

The final papers I want to mention, are papers by M. Pătraşcu and M. Thorup. In their paper [33], presented at STOC 38 in 2006, they give a refined lower bound analysis in the cell probe model. The parameters involved are both the universe size u and its logarithm l, the word length w, the number of elements in the set n and the number of available memory words S. The result shows that at least on part of this parameter space the stratified trees are optimal. The results are extended to a randomized model in [34].

VII. Conclusion

It is not exceptional that results in computer science or mathematics do not obtain in their early publications the transparent treatment which will eventually surface in the literature. In most cases this is the result of the authors not yet fully understanding the problems at hand. There are however also instances where the original author is guided by scientific social constraints into a direction which makes it hard if not impossible to express his original intuition. A prime example of this process is the history of Savitch' theorem [36] stating that $PSPACE = NPSPACE$. In his original publication he was prohibited by the reviewers or the editor to use a recursive procedure, and the result is that the paper is much harder to read and to understand the simple idea behind the proof. For a direct proof following modern intuition which doesn't use any machine simulation see [54].

The van Emde Boas tree, when explained today, is always presented in the recursive cluster galaxy decomposition approach. This is not the way this structure was published in the original papers in the period 1974 to 1977, since my original papers from this period [45], [46], [47], [48] all follow the binary search on levels approach. The simpler recursive decomposition approach is worked out in details for the first time in the unpublished classroom note from Knuth [23]. This is sufficient reason for arranging to make this note finally accessible and bibliographically traceable [24].

As I have indicated the two approaches are not equivalent. The recursive approach is more abstract. There is a correspondence between the cluster galaxy substructures and the canonical sub-trees, but this correspondence breaks down when the separation between the left and right half-trees becomes part of the algorithms. The behaviour of the operations in the two models are different, as illustrated by the fact that, when the focal point happens to be located at the root of a higher ranked canonical sub-tree, in the tree-based approach recursion can terminate, while in the recursive decomposition approach the recursion will proceed to the bottom level. Another indication is the suggested impossibility to obtain Johnson's improvement [20] in the recursive decomposition approach.

My choice was primarily motivated by the prohibition of using multiplicative instructions in the RAM model; a clear instance of scientific social control by the environment. On the other hand it is evident from my writings that I was fully committed to the tree-based approach, possibly for its closeness to the silly mark-bit approach which allowed me to break the $O(log(n))$ barrier originally. As explained there is just a hint at the recursive decomposition approach in my original papers.

In the superficial survey on work during the past 33 years I have used the themes applications, improvements and lower bounds. Applications frequently obtain the label of a mere theoretical result; by the lime the $loglog(u)/log(n)$ improvement is sufficiently large to absorb the constant factor the universe size u has become large and the space consumption $O(u)$ of the improved original structure will have become prohibitive; however the improvements based on perfect hashing give new hope. I should also stress that these improvements (and also the lower bounds) consider RAM models which from the 1974 perspective are unrealistic (arbitrary AC_0 instructions and word-size $w >> log(u)$).

There is a possible version of the structure which I have

not discovered in any of the papers I have looked at. People like Gonnet [16] explicitly give the freedom to use special structures when the universe size has become sufficiently small. But the algorithmic decision to pass from storing the information at the root to starting the allocation and use of substructures always seems to occur when a second element is going to be inserted in the structure under consideration. An dually, the substructures are invalidated when this second element is removed.

As far as the recurrence relation $T(u) = T(\sqrt{u}) + O(1)$ is concerned this transition might as well occur between 2 and 3, or between k and $k+1$ for any fixed constant k. In this variation of the original idea the constant time used for setting up the substructures when the boundary is passed will increase, but one may hope that there will be far less need for going deeper into the recursion. Presumably this will not work in the worst case since an adversory may submit a chain of instructions crossing this boundary almost always, but it may yield improvements in average or amortized complexity.

ACKNOWLEDGMENT

The topic of this paper was suggested to me by Ilir Capuni suggesting that I should look at the work of the late Mihai Pătraşcu [44]. As indicated this work is way beyond my level of understanding, but it is good to read that my ideas from 39 years ago still are optimal in some parts of the relevant parameter space.

Instead this invitation has triggered me to go back into the past and adjust my own perspective on what really has happened. It has convinced me that the importance of the contribution by Don Knuth [23] is severely underestimated and it has triggered me to honour the promise I made 7 years ago to make this document accessible [24].

REFERENCES

[1] A.V. Aho, J.E. Hopcroft & J.D. Ullman, *On fincing lowest common ancestors in trees*, proc. ACM STOC 5 (1973) pp 253–265.

[2] A.V. Aho, J.E. Hopcroft & J.D. Ullman, *the design and analysis of computer algorithms*, Addison Wesley 1974.

[3] R.K. Ahuja, K. Mehlhorn, J Orlin & R.E. Tarjan, *Faster algorithms for the shortest path problem*, J. ACM 37 (1990) 213–223.

[4] M. Ajtai, *A lower bound for fiunding predecessors in Yao's cell probe model*, Combinatorica 8 (1988) 235–247.

[5] S. Albers & T. Hagerup, *Improved parallel integer sorting without concurrent writing*, Inf. and Comput. 136 (1997), 25–51; earlier presented at ACM SIAM SODA 3 (1992) 463-472.

[6] *Optimal bounds for the predecssor problem and related problems*, J. Comput. System Sci. 65 (2002) 38–72, previously presented at ACM STOC 41 (1999) 295–304.

[7] A. Bertoni, G. Mauri,& N. Sabadini, *Simulations among classes of random access machines and equivalence among numbers succinctly represented*, Ann. Discr. Math. 25 (1985) 65–90 .

[8] M. Blum, *A machine-independent theory of the complexity of recursive functions*, J. Assoc. Comput. Mach. 14 (1967) 322–336.

[9] S.A. Cook, & R.A. Reckhow, *Time bounded random access machines*, J. Comput. Syst. Sci. 7 (1973) 354–375.

[10] T.H. Cormen, C.E. Leiserson, R.L. Rivest & C. Stein, *Introduction to Algorithms*, ch. 20, the van Emde Boas tree, MIT Press, 2009, pp 531–560 .

[11] R. Dementiev, L. Kettner, J. Mehnert & P. Sanders, *Engineering a sorted list data structure for 32 bits keys*, Proc. Sixth workshop on Algorithm engineering and experiments and the first workshop on analytic algorithmics and combinatorics, New Orleans, LA, USA Jan 2004, SIAM 2004, pp. 142–151.

[12] M. de Berg, M. van Kreveld & J. Snoeyink, *Two-dimensional and three-dimensional point location in rectangular subdivisions*, J. of Algorithms 18 (1995) 256–277.

[13] J. Edmonds, *Paths,trees, and flowers*, Canad. J. Math. 17 (1965), 449–467 .

[14] M.L. Fredman, J. Komlós & E. Szemerédi, *Storing a sparse table with $O(1)$ worst case access time*, J. ACM 31 (1984) 538–544.

[15] M.L. Fredman & D.E. Willard, *Surpassing the information theoretic bound with fusion trees*, J. Comput. Syst. Sci. 47 (1993), 424-436.

[16] G.H. Gonnet, *Handbook of Algorithms and Data Structures*, Int. Comp. Sci. Series Addison Wesley, 1984, ch. 5.1.4, pp 172–174 .

[17] J. Hartmanis, & J. Simon, *On the power of multiplication in random access machines*, Proc. IEEE Switching and automata theory 15 (1974), 13–23 .

[18] J. Hartmanis, & R.E Stearns, *On the computational complexity of algorithms*, Trans. Amer. Math. Soc. 117 (1966) 285–306

[19] J.W. Hunt & T.G. Szymanski, *A fast algorithm for computing longest common subsequences*, Communications of the ACM 20 (1977) 350–353;

[20] D.B. Johnson, *A priority queue in which initialization and queue operations take $O(loglog(D))$ time*, Math. Syst. Theory 15, 1981/82, 295–309.

[21] D.E. Knuth, *The art of computer programming, volume 1: fundamental algorithms*, Addison Wesley 1968.

[22] D.E. Knuth, *The art of computer programming, volume 3: sorting and searching*, Addison Wesley 1973.

[23] D.E. Knuth, *Notes on the van Emde Boas construction of priority deques: an instructive use of recursion*, Classroom notes Stanford University, March 1977.

[24] *The correspondence between Donald E. Knuth and Peter van Emde Boas on priority deques during the spring of 1977*, facsimile edition posted at http://staff.science.uva.nl/ peter/knuthnote.pdf

[25] R. Kaas & E. Zijlstra, *A Pascal implementation of an efficient priority queue*, report dept. of Mathematics, University of Amsterdam 75-11

[26] T.V. Lakshman & D. Stiliadis, *High-speed policy-based packet forwarding using efficient multi-dimensional range matching*, proc. ACM SIGCOMM'98 (1998) 203-214.

[27] K. Mehlhorn, *Data Structures and Algorithms 1, Sorting and Searching*, EATCS Monographs on Theoretical Computer Science 1, Springer Verlag, 1984 , ch 8.2, pp 290–298.

[28] K. Mehlhorn, S. Näher & H. Alt, A lower bound for the complexity of the union-split-find problem, SIAM J. Comput. 17 (1988) 1093–1102, preprint report FB10, Informatik, Univ. des Saarlandes A 07/1986.

[29] K. Mehlhorn & S Näher, *Bounded ordered dictionaries in $O(loglog(N))$ time and $O(n)$ space*, Inf. Proc. Letters 35 (1990) 183–189.

[30] M. Minsky, *Computation, Finite and Infinite Machines*, Prentice Hall, 1972.

[31] Peter Bro Miltersen, *Lower bounds for union-split-find related problems on random access machines*, proc ACM STOC 26 (1994) 625–234.

[32] M.H. Overmars, *The design of dynamic data structures*, Springer LNCS 156, 1983.

[33] M. Pătraşcu & M Thorup *Time-Space trade-offs for predecessor seach*, Proc. ACM STOC 38 (2006) 232-240; an extended version can be found at http://arxiv.org/abs/cs/0603043.

[34] M. Pătraşcu & M Thorup *Randomization does not help searching predecessors*, Proc. ACM SIAM SODA 2007 (2007) 555-564.

[35] V.R. Pratt, & L.J. Stockmeyer, *A characterization of the power of vector machines*, J. Comput. Syst. Sci. 12 (1976) 198–221 .

[36] W.J. Savitch, *Relations between deterministic and nondeterministic tape complexities*, J. Comput. Syst. Sci. 12 (1970) 177–192.

[37] A. Schönhage, *On the power of random access machines*, Proc. ICALP 6, Springer LNCS 71 (1979) 520–529

[38] A. Schönhage, *Storage modification machines*, SIAM J. Comput. 9 (1980) 490–508.

[39] J.C. Sherpherdson, & H.E. Sturgis, *Computability of recursive functions*, J. Assoc. Comput. Mach. 10 (1963) 217–255.

[40] C. Slot,& P. van Emde Boas, *The problem of space invariance for sequential machines*, Inf. and Comp. 77 (1988) 93–122.

[41] R.E. Tarjan, *Efficinecy of a good but non linear set union agorithm*, J. Assoc Comput. Mach. 22 (1975) 215–224.

[42] M. Thorup, *On RAM priority queues*, Proc. ACM SIAM SODA 1996 (1996) 59–67.

[43] M. Thorup, *Integer priority queues with decrease key in constant time and the single source shortest path problem*, J. Comput. System Sci. 69 (2004) 330–353.

[44] M. Thorup, *Mihai Pătraşcu: Obituary and open problems*, ACM SIGACT news 44 (2013) 110–114.

[45] P. van Emde Boas, *An $O(n\log\log n)$ On-line Algorithm for the Insert-Extract Min problem*, Report Cornell University Dept. of Computer Science TR 74-221, december 1974; online accessible at https://dspace.library.cornell.edu/bitstream/1813/6060/1/74-221.pdf .

[46] P. van Emde Boas, *The on-line Insert-Extract min problem*, Séminaires IRIA, Théorie des algorithmes, des langages et de la Programmation, 1975, pp. 41-56 , preprint: report dept. of Mathematics, University of Amsterdam 75-04 .

[47] P. van Emde Boas, *Preserving order in a forest in less than logarithmic time*, Proc. IEEE FOCS 16, Berkeley, Oct 1975, pp. 75–84, preprint: Report Mathematical Centre Amsterdam, MC-ZW-55-75.

[48] P. van Emde Boas, R. Kaas & E. Zijlstra, *Design and implementation of an efficient priority queue*, Math. Syst. Theory 10 (1977) 99–128, preprint: Report Mathematical Centre Amsterdam, MC-ZW-60-75 .

[49] P. van Emde Boas, *Preserving order in a forest in less than logarithmic time and linear space*, Inf. Proc. Letters 6 (1977) 80–82, preprint: Report dept. of Mathematics, University of Amsterdam 77-05.

[50] P. van Emde Boas, *Developments in Data Structures*, in J.K. Lenstra, A.H.G. Rinnooy Kan & P. van Emde Boas eds., Interfaces between Computer Science and Operations Research, MC Tracts 99 (1978) pp. 33–61.

[51] P. van Emde Boas, *Complexiteit van verzamelingenmanipulatie*, in J.C van Vliet, ed., Colloquium Capita Datastructuren, MC Syllabi 37 (1978) pp. 43–63, in Dutch.

[52] P. van Emde Boas, *Machine models and simulations*, in J. van Leeuwen, ed., Handbook of Theoretical Computer Science, vol. A, Algorithms and Complexity Elsevier Science Publishers, Amsterdam etc. (1990), pp. 1–66, preprint: report University of Amsterdam, Institute for Logic, Language and Computation, ITLI-CT-89-02

[53] P. van Emde Boas, *Egghead betegelt badkamer*, valedictory address given march 25 2010, ed. Faculteit der Natuurwetenschappen, Wiskunde en Informatica, University of Amsterdam, in Dutch.

[54] P. van Emde Boas, *Playing Savitch and Cooking games*, in D. Damm, U.Hannemann & M. Steffen, eds., Concurrency, Compositionality and Correctness; essays in honour of Willem-Paul de Roever, Springer LNCS 5930, 2010, pp. 10–21.

[55] A. van Wijngaarden, B.J. Mailloux, J.E.L. Peck, C.H.A. Koster, M. Sintzoff, C.H. Lindsey, L.G.T. Meertens & R.G. Fisker, *Revised report on the Algorithmic Language ALGOL 68*, Acta Informatica 5, 1975, 1–236.

[56] N. Wirth, *The Programming Language PASCAL (revised report)*, in K. Jensen & N. Wirth, PASCAL User Manual and Report, Springer LNCS 18, 1974.

[57] D.E. Willard, *Log-logarithmic worst-case range queries are possible in space $\Theta(N)$*, Inf. Proc. Letters 17 (1983) 81–84.

[58] D.E. Willard, *Examining Computational Geometry, van Emde Boas trees and hashing from the persppective of the fusion tree*, SIAM J. Comput. 29 (2000) 1030–1049.

[59] Hao Wang & Bill Lin, *Pipeplined van Emde Boas tree: algorithms, analysis and applications*, IEEE INFOCOM 2007 (2007) 2471–2475.

[60] Wikipaedia page *van Emde Boas tree*, accessible at http://en.wikipedia.org/wiki/Van_Emde_Boas_tree

[61] Wikipedia page *Talk:van Emde Boas tree*, accessible at http://en.wikipedia.org/wiki/Talk:Van_Emde_Boas_tree