MODAL LOGIC FOR BELIEF AND PREFERENCE CHANGE

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Modal logic for belief and preference change

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Approved for the University Committee on Graduate Studies.

Abstract

In my thesis, I show that *Order Logic* interpreted over preorders provides a unifying framework for individuals and groups to analyze believe and preference change. Order Logic is a modal logic with three modalities complete for the class of transitive and reflexive frames whose fragments and extensions yield various formalisms to analyze the dynamics of beliefs and preferences. The analysis proceeds in two steps: 1) I give static logics for belief and preference and 2) I introduce dynamic modalities to analyze actions over models. I investigate four kinds of doxastic and preference logics: *Relational Doxastic Logic*, *Binary Preference Logic*, *Ceteris Paribus Logic* and *Group Order Logic*. The actions I consider are of two kinds. In a first time, I integrate three well-known dynamic actions. The first one is public announcement, the second lexicographic upgrade and the last preference upgrade, exemplifying state elimination, state reordering and link cutting respectively. In a second time, I introduce new kinds of actions: *agenda expansion* and *agent promotion*. All actions are incorporated into static logics via compositional analysis, appealing to reduction axioms. This uniform completeness strategy consists in giving axioms that transform formulas with action modalities to equivalent formulas in the static language, reducing completeness of the dynamic logic to that of the static one.

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Contents

Chapter 1

Introduction

1.1 Case example

I used to believe that my good friend Robert had never been to Europe. I have known him for a long time and he had told me on various occasions that he had never set foot in Europe. One day, while walking in Paris, I saw a silhouette that looked strangely familiar. Since I did not know anybody in Paris, my first reaction was to infer that this person was just a member of my species looking like other members I had seen before. Getting closer to the person, however, I started thinking that his hair and jacket looked strangely like Robert's. My belief that Robert had never been to Europe was getting more seriously challenged. Yet, I was not ready to change it, since Robert had never told me about plans to go to Europe; he is quite sedentary. Knowing him well, I had serious doubts that he could have planned to go without telling me. I thus formed the belief that there was a man in Paris who looked surprisingly like my dear friend.

But when I heard the stranger saying: "Patrick, mon ami", I suddenly realized that Robert was in Paris. I then had a conflict in my beliefs. On the one hand, I believed that Robert had never been to Europe. On the other hand, I believed that Robert was in Paris. Furthermore, since I believe that Paris is in Europe and that if Robert is in Paris, then he is in Europe, I formed the belief that Robert was in Europe, following my belief in the rule of Modus Ponens. I then had a contradiction in my beliefs, namely that Robert was and was not in Europe. This contradiction called for a revision and several alternatives were available to me. I could have stopped believing that Paris was in Europe, but that would have shaken the very core of my beliefs about the world in which I live. Had I stopped believing in Modus Ponens, my revision would have affected my entire knowledge. No, all I had to do was to drop my belief that Robert had never been to Europe - as well as some accompanying beliefs. Once he had told me the story of his presence in Paris, I have formed new beliefs about Robert and added them to my stock of existing beliefs. I have also retracted other beliefs that did not cohere with the new information, for example that Robert had never crossed the Atlantic Ocean or seen the Eiffel tower.

Living in a dynamic world with incomplete information about it, changing beliefs facing new information is something we do on a daily basis. Being rational animals, however, our belief changes are not arbitrary; I did not start believing in vampires because I saw Robert in Paris and I will believe that $2 + 2 = 4$ until I die. This non-arbitrariness in belief change is a sign of rationality in action; changeability is a common feature of rationality. But change does not only occur with beliefs. It is also typical, perhaps even more importantly, with preferences. Let us pursue the story to illustrate this.

Once I had given a greeting hug to Robert, I invited him for a drink. Robert said that he would prefer wine over beer and beer over coffee. Paris is not a good place to have beer, so I decided to take Robert for wine. But how did I know that Robert would actually prefer wine over coffee? He did not explicitly tell me so. I made the inference because I expected his preferences to be transitive, for I know Robert to be a rational man; if he prefers wine over beer and beer over coffee, then he must prefer wine over coffee. Off we were to get wine. Of course, as we were having wine, I ordered some cheese. In the platter, there was Camembert and goat cheese. Robert was only eating the latter, claiming to prefer it over Camembert. I told him that Camembert is not the same in Paris as he was used to and convinced him to try it. He did and indeed liked it better than the goat cheese. This triggered an update in his preferences. Various reactions might have been expected from him. It might have been that the experience of tasting the Camembert was so strong as to reverse his preference altogether, or it might have been that Robert took it as an exception: "yes I do prefer goat cheese over Camembert, in general, but I must confess that this Camembert is much better."

This again illustrates rational attitudes facing new information, this time about preferences. Just as with beliefs, it is reasonable to expect principles underlying preference change and Logic to be an appropriate tool to formalize them. There is still another aspect of preference and belief change that is important, namely group belief and preference change. Rationality is not something that applies only to individuals.

Consider the following continuation of my encounter with Robert in Paris.

After our wine and cheese snack, Robert and I decided to share a desert. Among the choices offered to us, we both preferred chocolate and strawberry deserts, in that order, but Robert told me that he was allergic to nuts. This automatically ruled out the chocolate cake, as the waiter informed us. In that case, by my commitment to participate in a group decision about the desert, I had to comply and accept that my first choice would not be satisfied. The rational choice for the group consisting of Robert and I was to order the strawberry desert.

Analysis

The story of my encounter with Robert in Paris reflects the kind of reasoning investigated in my thesis: belief and preference change, for individuals and groups. When I have updated my belief that Robert had never been to Europe, I have performed what is called *belief revision*. When I have inferred that Robert prefers wine over coffee, I have applied a principle of Preference Logic: transitivity. If the experience of tasting Camembert had led Robert to change his preferences so that Camembert became better than Goat Cheese for him, then he would have upgraded his preference for Camembert. If, instead, he had taken the experience as an exception to a rule, then he would have included it under a *ceteris paribus* clause - everything else being equal. Finally, when Robert and I have decided to opt for the strawberry desert instead of the chocolate cake, we have aggregated our preferences so as to maximize satisfaction - and minimize death toll.

This simple story exemplifies cases of casual belief and preference change that we take for granted, but for which entire forests have been transformed into research papers and books. For instance, my reluctance to conclude that Robert was in Paris, appealing to appearance patterns in the human kind, reflected an important problem in belief revision: to find an economical and effective revision policy. I was not expecting to meet Robert in Paris and I tried to make minimal changes to explain the strange encounter. But when the incoming information was too strong, I was forced to revise them, and I did it in a way that fundamental beliefs, such as the geography of the world or logical inferences, were untouched. Minimal change is usually linked to a hierarchy among beliefs, often called *entrenchment* of beliefs, and a great deal of research has been conducted to provide conditions of entrenchment that guide minimal revision (see for instance [22]).

The story is also an illustration of notions to be conceptualized and formalized, not a list of problems to be solved with mathematical tools; this is intentional. Contrast it with old problems such as the barber who only shaves those who do not shave themselves, or recent problems such as *Sleeping Beauty*, both appealing to Mathematics for solutions. I do not intend to use Logic along these lines. There are similar problems that pertain to belief and preference, think of various paradoxes coming out of solution concepts in game theory, for example backward induction, but my investigation is not oriented towards them. I am drawing more into the logical analysis than mere problem solving.

My standpoint is thus primarily analytical, geared towards conceptualization. I do not claim, however, that all there is to belief and preference is definable and derivable from Logic; that would be foolish. I think, rather, that beliefs and preferences are better understood in a dynamic environment and that taking a stance on dynamics of beliefs and preferences shows something that is left out in more traditional cumulative and static stories. We change our mind all the time, but we do it rationally and Logic is a good tool to investigate this. I am thus conceptually committed to the claim that rationality is not only a static state of mind prescribing beliefs and preferences to be held over others, but also a disposition to behave in certain ways facing incoming information and changing environment. My logical standpoint, focusing on dynamics, is thus innovative from a philosophical point of view and enriches existing analytical treatments.

Of course, another important outcome of my approach is to provide advances in Logic itself. Indeed, the systems developed in this thesis come with a plethora of formal and mathematical results. This other aspect of my research is prominent in the various chapters, which contains both conceptual analysis of concepts relevant in Philosophy and technical results important in Mathematics. Logic is thus a nice intermediate setting between the two disciplines.

Hence, my thesis is another step in the quest for understanding concepts and their usage in daily life, a step taken with the help of Logic, more specifically Modal Logic. This choice deserves a discussion which occupies most of the Introduction. I am of the opinion that Logic is an important tool in Philosophy that has become underestimated and even neglected in the last decades. I hope to dispel curses against it by defending its usage and the outcomes expected from it. Doors have been shut between Philosophy and Logic which should be re-opened before they become sealed for good.

1.2 Modeling and Modal Logic

To formalize dynamics, a great deal of work needs to be done to find appropriate static notions; this is is made clear in the various results obtained throughout the thesis. For the static models of belief and preference, I follow a tradition initiated by von Wright in [75] and championed, to name landmarks, in the seminal work of Hintikka in Epistemic Logic [31] and Prior in Temporal Logic [51]. This tradition is to use Modal Logic as a tool for the conceptual analysis of notions like permission (von Wright), knowledge and belief (Hintikka) and time (Prior). Likewise, I take a standard Modal Logic, but with an interpretation as *Order Logic*, which I use to formalize various notions of belief and preference.

This choice needs justification, since the Modal Logic I use is expressively quite limited. Firstly, I only use Propositional Modal Logic and thus do not resort to the expressive power of first-order quantifiers. Due to its expressive limitations, Modal Logic can only make general distinctions; it cannot give a fine-grained analysis of its concepts. Secondly, Modal Logic is inherently qualitative in that it can talk about being more or less plausible, or more preferred - as we will see repeatedly - but cannot express quantitative notions, for example of having a belief of degree 0.95 in x. Modal Logic is thus only a fragment of First-Order Logic, which is itself expressively limited compared to probabilistic models that rely on the full power of Mathematics. Why then should one restrains her inquiries in a system which is thus limited, when stronger ones are available?

My answer is that this relative poverty of Modal Logic is precisely what makes it advantageous. Firstly, it forces one to make concepts explicit and be clear-minded about what is claimed. If one is not clear about the notions to be formalized, the logic obtained will yield unwanted principles. Vice versa, the principles assumed by the logic provide clear grounds for philosophical discussions. This has been the case, for instance, with the *positive introspection* epistemic axiom $K\varphi \to KK\varphi$: if I know φ, then I know that I know it. This axiom has initiated a big debate about the notion of knowledge. Secondly, Modal Logic is also closer to realistic applications of its concepts in computer science. It is for one decidable, unlike First-Order Logic, but shares with it nice formal properties such as compactness, and is of course complete. There is a balance between conceptual expressive power and practical application, and Modal Logic finds a nice equilibrium between the two. In the same line of thoughts, my thesis can be seen as lying in between conceptual work in philosophy and foundational work in computer science and artificial intelligence. I hope that this situation will actually help building bridges between them. Finally, constraints given by the expressive power of Modal Logic force creativity, both in finding general principles that would still hold in more expressive systems, but also in being able to draw conclusions that may seem hidden in the generality of the approach. If one can formalize notions of belief and preference in Modal Logic, then one shows a firm understanding of the concepts.

Furthermore, the worries about Modal Logic could be turned around against proponents of richer languages, namely that one should not use a system stronger than what is to be explained. There is no reason to have a formalization of belief change depending ultimately on the continuum hypothesis! Modal Logic is not sufficient to encompass all there is to say about belief and preference, granted, but then Mathematics with no restrains would be saying too much. Going bottom-up, starting with poor languages and enriching them as we go along is just as valuable as a top-down approach, giving oneself unlimited expressive powers to begin with and then looking for weaker systems with better control. My hope is that the two approaches will meet in the middle, but for that we need workers from both ends of the tunnel, and I choose the Modal Logic one.

Following this discussion, I need to be honest about the terminology used in this thesis. When I talk about beliefs and preferences, I mean some primitive and encompassing notions, the kind that can be formalized and put (theoretically) inside a machine. These notions are thus only a part of the full notions of belief and preferences that humans and other animals possess. For instance, although I talk about plausibility order and hierarchies of beliefs, I do not differentiate between basic beliefs, such as that the sky is blue, or more abstract ones, such as a belief in God; anything that can be expressed in a proposition is subject to be a belief in this thesis, likewise with preferences. Beliefs are thus not ordered according to kinds, origins or formations, but solely with respect to plausibility. It is about this hierarchy in terms of plausibility that Modal Logic can reason efficiently. The problem of formalizing this kind of simplified notion is already difficult and is a good starting point. If we can get that straight, then we may undertake more complex analysis that might lead to a richer understanding of belief and preference change.

My contribution to the conceptual analysis of the concepts of belief and preference is more important with respect to their dynamic aspect. In this regard, formal tools are more decisive, as they can clearly display rules that govern belief and preference change. In a modal logic setting, the dynamics can be firmly grasped and the relative poverty of statics yields a perspicuous analysis of dynamics.

Lakatos and positive heuristic of a programme

In developing logics for beliefs and preferences, one is prone to stay confined to introspective analysis, isolated in her arm-chair investigation. By moving to dynamics, however, one opens horizons to other disciplines, such as Computer Science, embedding her research in a growing inter-disciplinary scientific paradigm. The dynamic logical approach situates modeling in a scientific endeavor, bringing out links to other disciplines that may have been unforeseen otherwise. I make a little digression here and argue that the kind of modeling I use does exactly that by echoing Lakatos' account of the development of physics in Philosophy of Science, in particular what he calls the *positive heuristic of a research programme* in [35]:

> "The positive heuristic sets out a programme which lists a chain of ever more complicated *models* simulating reality: the scientist's attention is riveted on building his models following instructions which are laid down in the positive part of his programme. He ignores the *actual* counterexamples, the available '*data*'." (Op. cit., p.50)

To support this claim, Lakatos takes the example of Newton and the development of his programme for a planetary system. Newton's first studies were with a two-body system consisting of a fixed point-like sun and a moving point-like planet. Once he could manage this simple system, he moved to a two-body system revolving around a common center of gravity. This change, claims Lakatos, was not motivated by available data, since no anomaly was yet present in the model itself. Eventually, Newton created a system in which the sun and the planets were not point-like objects anymore, motivated by the fact that there cannot be infinite density. An so on and so forth until he could publish the Principia. The point is that Newton started with a simple unrealistic model and transformed his results into a research programme which gradually handled complicated planetary systems. Each step were motivated by obvious shortcomings of previous models which he had endorsed to get his system going.

> "Most, if not all, Newtonian 'puzzles', leading to a series of new variants superseding each other, were foreseeable at the time of Newton's first

> naive model and no doubt Newton and his colleagues *did* foresee them: Newton must have been fully aware of the blatant falsity of his first variants. Nothing shows the existence of a positive heuristic of a research programme clearer that this fact: this is why one speaks of 'models' in research programmes." (Op. cit. p.51)

To this claim, one could add that the development of Dynamic Logic and in a broader scope of artificial intelligence is just as good an example of the existence a positive heuristic of a research programme, one which we are witnessing in progress and to which I am contributing in the present thesis.[1] The notions of belief and preference outlined in my thesis are defeasible empirically, but the principles they sustain might be at the core of more complex philosophical investigations which would be defeated themselves if they ignored them.

1.3 Preorders, statics and dynamics

A general result coming out of my thesis - although not contained in a single theorem - is that a great deal of belief and preference change can be understood by analyzing comparative structures, interpreted as plausibility for belief or betterness for preference. To accomplish this, two things needs to analyzed: 1) statics and 2) dynamics.

The class of static comparative structure over which I base all the research in this thesis is the class of *preorders*. Preorders are reflexive and transitive relations that provide qualitative hierarchy between states. They can be seen as graphs whose nodes are sets of equivalent states with respect to the order, as in Figure 1.1. In the figure, states to the right are *higher* (or *better* or *more plausible*) in the order, for instance states u and v are higher (or better or more plausible) than state t. Two states are put in the same cluster if they are of equal rank with respect to the order, for example states u and v. Finally, two states are incomparable if neither is higher than the other, for instance states u and s.

In this thesis, the class of preorders provides a uniform setting to investigate notions of belief and preference, depending on which interpretation is given to the order. In the case of belief, I use preorders to say that φ is believed if it is true in the

[1] I do not wish to claim that I am in a similar situation as Newton, one of the greatest minds of all time... oh I will: it's just like Newton!

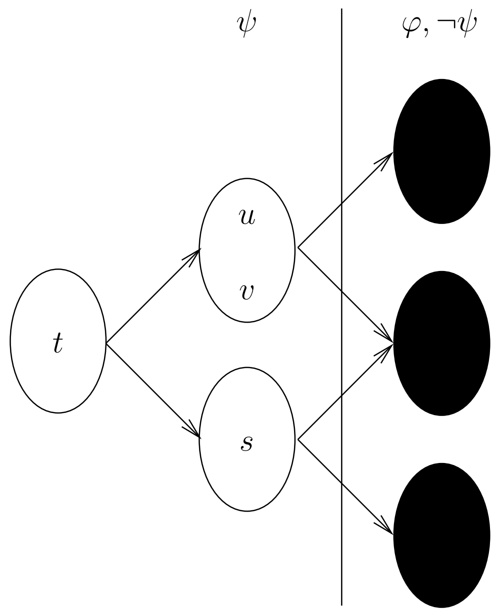

Figure 1.1: Graphical representation of a preorder, where states are equivalent with respect to the order if they are in the same cluster, and states to the right are ranked *higher* than those to the right.

most plausible worlds according to a *plausibility* relation. For example, φ is believed in the structure represented in Figure 1.1, if the figure is interpreted as providing a plausibility order, since φ is true in all states that are most plausible - to the right. In the case of preference, various notions of preference can be defined over preorders, one of them stating that φ is preferred to ψ if every φ-state is better than every ψ-state in a preorder interpreted as a *betterness* relation. This is the case in Figure 1.1.

Figure 1.1 represents, in a simplistic way, one of the main point of my thesis, namely that from a logical point of view, beliefs and preferences can be understood as comparative reasoning and a logic for preorders provides the core of this kind of reasoning.

For the dynamics part, my investigation falls under the growing paradigm of *Dynamic Logic* (cf. [72]). Dynamic Logic is the study of model change, either over states or accessibility relations. Three kinds of actions can be performed over static models: 1) adding or eliminating states, 2) reordering states, or 3) adding or eliminating accessibility links. A simple case of world elimination is illustrated in Figure 1.2. Formally, a dynamic structure is superposed over a static one: one starts with a static model and performs an action over it to end-up in a different static model. This might best be understood via completeness results, as is apparent in the various chapters. Completeness results for dynamification of static logics use the technique

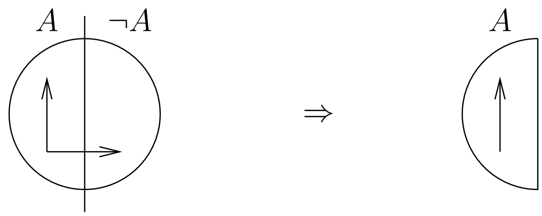

Figure 1.2: Illustration of the dynamic action of removing $\neg A$-states from a static structure.

of *compositional analysis via reduction axioms*, which consists in giving principles that analyze the effect of actions from the point of view of the original model in which they are performed. In other words, actions are encoded in static models and compositional analysis shows how to decode them.

Dynamic Logic can be seen as giving a *constructive* notion of dynamics, in the sense that it formalizes explicitly how models are transformed, as opposed to a postulational approach - especially when talking about belief change - which can be seen as providing desiderata of specific actions without describing how they work on models. Both approaches are valuable, but my thesis shows how the constructive one is better suited for the unifying treatment of belief and preference change in Order Logic.

Chapter 2 is devoted to a logic defined over the class of preorders, called *Order Logic*, and introduces three exemplars of dynamic actions which recur throughout the thesis. The rest of the chapters build on Order Logic, by isolating belief (Chapter 3) and preference (Chapter 4) fragments, or by considering extensions, the *Ceteris Paribus* Logic (Chapter 5) and Aggregation Logic (Chapter 6).[2] Order Logic is thus a good balance between its preference and belief fragments and its ceteris paribus and aggregation extensions. In the next section, I give a more detailed overview of the thesis.

1.4 Overview of the thesis

The main thrust of my thesis is a logical study of preorders and various interpretations that yield formalisms for belief and preference change. Every chapter is constructed in the same way. Each chapter first presents static models and give their complete axiomatization. Completeness results occupy an important place in my thesis. One reason is that they are intrinsically important: with them, one can see the logical

[2]The content of Chapters 2 to 5 is taken from two published papers, [23] for Chapter 3 and [68] for the three others. Chapter 6 presents unpublished material so far.

inferences, or patterns of reasoning, that various semantics sustain. They also secure semantics, for they show that one has control over it. Another reason for spending efforts on completeness results is their applicability in the second division of each chapter: dynamics. Indeed, once the static part is settled, each chapter proceeds to the *dynamification* of the static models. This is made via the introduction of known actions, such as public announcement (Chapters 3 and 4), or of new actions, guided by special features of the static models (Chapters 5 and 6). Again, completeness results for the dynamification play an important role. Here, however, I use the technique of *compositional analysis* (cf. [33, 70, 72]). Compositional analysis reduces completeness of a dynamic system to the completeness of the static one via *reduction axioms*, hence the importance of completeness results for the static parts.

Chapter 2 sets the stage for the next chapters and is devoted to an uninterpreted logic defined over the class of preorders. This logic, called *Order Logic*, is a simple multi-modal logic with one diamond $\Diamond^{\leq}$ defined over a *weak* relation $\preceq$ and a second diamond $\Diamond^{<}$ defined over the strict subrelation $\prec$ of $\preceq$. In addition, I introduce in the basic language the existential modality $E\varphi$, which is true at a state if φ is true somewhere in the model. The existential modality is of tremendous help in the next chapters, as it can distinguish minimal states, and so express doxastic statements, or talk about global features of models, thus expressing preferences of the kind "every φ-state is better than every ψ-state".

Order Logic is in itself a standard modal logic that can be found, perhaps cryptically, in most introductory books. A similar language was studied by Boutilier in [8], although Boutilier takes as primitive inverse modalities and defines the existential modality with it; this is formally equivalent. The language I use originated more recently in [71] and [73], and was applied in [41, 55]. In all of these, the logic is referred to as *Preference Logic*, but I decided to call it *Order Logic*, reserving the terminology of preference for the binary preference logic studied in Chapter 4. I prefer this nomenclature for the general logic, and the more specific terms 'Doxastic' and 'Preference Logic' presented in the other chapters. The completeness result for Order Logic is taken from [68] and is based on the technique of *bulldozing* introduced by Segerberg in [58].

For the dynamic parts of this chapter, I introduce three well-known actions: 1) public announcement ([72]), 2) lexicographic upgrade ([67]) and 3) preference upgrade ([69]). I choose these three actions because they exemplify important kinds of actions: 1) world-elimination, 2) reordering and 3) link-cutting. Furthermore, the second and third action have been introduced with the intent of dynamifying beliefs and preferences respectively. It is thus natural to see how they apply in my setting. The last section of Chapter 2 shows how the technique of compositional analysis is applied throughout the thesis, thus facilitating its recurrences in future chapters.

Chapter 3 shows that the framework of the previous chapter is adequate for what it is meant to do with respect to Dynamic Logic and beliefs: formalizing belief change. This chapter is embedded in a tradition initiated in the seminal belief revision paper [1] (see also [22]) and formalized in Modal Logic by Segerberg in various papers (see [40] for the most recent presentation). Segerberg's logic, called *Dynamic Doxastic Logic* (DDL), is modeled using what he called *onions*, which are like Lewis' systems of spheres (cf. Lewis73), but centered around a set of world instead of a single world. To see how this approach to belief revision can be treated inside Order Logic, I investigate a generalization of DDL over non-connected systems of spheres. One contribution on the static part is the generalization of Segerberg's models to include non-linear systems of spheres, thus allowing to deal with relational belief revision. This in itself provides a nice extra to the linear, or functional, analysis of belief change. To introduce dynamics, I first show how Relational Doxastic Logic can be seen as a fragment of Order Logic. To achieve this, however, one obvious obstacles has to be overcome, namely that Segerberg uses neighborhood models, whereas Order Logic is set in a standard Kripkean framework. One important theorem here is that an important fragment of Relational Doxastic Logic is equivalent to a conditional logic, investigated independently by Veltman ([74]) and Burgess ([10]), called the *Minimal Conditional Logic*. I finally show how van Benthem's lexicographic action can be introduced in Relational Doxastic Logic by the standard method of compositional analysis alluded to above.

In Chapter 4, I investigate a different fragment of Order Logic, the fragment of binary preferences. Binary preferences are statements of the form $\varphi P \psi$, comparing two sentences and saying that one is preferred to the other. This was von Wright's approach in his seminal [76], a work to which I appeal on various occasions, especially as being the investigator of the notion of *ceteris paribus* preferences, the main topic of Chapter 5. Here, the existential modality of Order Logic is used to capture the global feature of binary preference statements. For instance, one may say that φ is preferred to ψ if every φ-state is better than every ψ-state, or if one φ-state is better than every ψ-state, and so on. Many binary preference statements can be defined in this fashion and I present eight definitions, written in the form $\varphi \leq_{\forall\forall} \psi$ or $\varphi \leq_{\forall\exists} \psi$, to take two examples. Again, these fragments have been studied in [71, 73, 41, 55] and [68]. Once I have shown how each of these binary preference definitions lead to fragments of Order Logic, I focus on a specific fragment, the $\leq_{\forall\forall}$ fragment, and axiomatize it, following [68]. Similar results for the $\leq_{\forall\exists}$ fragment can be found in [26]. Finally, I introduce dynamics, this time focusing on preference upgrade; this is straightforward, and follows results of [69].

Chapter 5 is the best instantiation of the interplay between Logic and Philosophy described in the beginning of this introduction. I consider therein the notion of *ceteris paribus* - translated as *all other things being equal* - and give a thorough logical analysis. '*Ceteris paribus*' belongs to the folklore of many disciplines, ranging from Economics (cf., [49]) to Philosophy of Science (cf., [12]), but is never precisely defined nor used in consistent ways. The formalization I provide originated in the work of von Wright [76] and was further analyzed in [19]. So-called *ceteris paribus* clauses are typically used to account for *defeaters* of laws ([20]), so that laws can still be stated even though they may fail on occasions ruled out by the *ceteris paribus* clauses. I differentiate between two main general meaning ascribed to these clauses, which I call the *equality* and *normality* reading of *ceteris paribus* and I then focus my attention on the former one. The equality reading of *ceteris paribus* is naturally analyzed in a logical setting, and the reasoning it sustains is displayed explicitly. This is of great conceptual value, but the logic also raises interesting mathematical questions, since *ceteris paribus* variants of logics are situated in between basic and Infinitary Modal Logic, a situation which was monopolized by Propositional Dynamic Logic and the μ-calculus so far. Again, this raises interesting formal questions between the two approaches.

For the dynamics, I show how one can introduce public announcement and preference upgrade; this is easy. The more interesting part is in the new kind of actions suggested by *Ceteris Paribus* Logic, interpreted in terms of *research agenda* and the addition of formulas to the agenda. The subject of a research agenda, however, seems to be better situated in a multi-agent setting and this calls for a logic of aggregation, provided in the next and final chapter.

Finally, Chapter 6 presents another extension of the language of Order Logic, this time with so-called *nominals*, to get a system of aggregation. I call the The resulting logic *Group Order Logic*. The results of this chapter are based on those of [2]. The interest of this latter paper is in providing a mechanism for the aggregation of preference relation, called *lexicographic reordering*, which satisfies nice aggregation properties, such as independence of irrelevant alternatives, without being dictatorial. [2] thus presents a *possibility* result, to be contrasted with the famous Arrow's impossibility result ([6]) in social choice theory or more recent ones in the field of judgment aggregation (see for example [18]). My contribution in this Chapter is to modalize the algebraic approach of [2] and to show that the logic consists in the simple extension of Order Logic with nominals. I also investigate group binary preferences and show how to introduce the action of public announcement and preference upgrade in the logic; again, this is straightforward. Finally, I introduce yet another kind of action, this time over so-called priority graphs providing hierarchies among agents. The new action is that of *promotion* of an agent inside a (sub)group.

Beliefs and preferences hang together in various areas. They both play, for instance, an important role in game theory, but their interaction is so complicated and rich that there is still a lot to be understood from a logical point of view. The unifying system presented in this thesis might shed some lights on their interplay.

Chapter 2

Setting the stage: Order Logic

For every logic presented in the thesis, I work in two stages. I first present the *static* logic and then introduce dynamics as transformations on models, either over the states or the relations. In most cases, dynamification is performed by introducing well-known actions, but I also discuss new kinds of actions in Chapters 5 and 6, building on *ceteris paribus* and *group preference* logics. In this preparatory chapter, I give the basic framework whose fragments and extensions are the subject of the remaining chapters.

The static logic advocated in this chapter is a basic modal logic with three diamonds, one defined over the accessibility relation $\preceq$, the other over its strict subrelation $\prec$ and the last one, the existential modality. The class of models targeted is isolated by the accessibility relation $\preceq$, which is restrained to *preorders*, i.e., reflexive and transitive models. Various notions of preferences and beliefs are defined over this class, but to start with, I take a more general standpoint and talk about *Order Logic*. Order Logic has been at the core of different systems under various guises. The version I use is based on the work of a few authors in selected papers, in particular Boutilier [8] and van Benthem, van Otterloo and Roy [71].[1]

My choice for preorders over connected orders is guided by the following considerations. With connected orders, three comparisons obtain: 1) x is better than y, 2) y is better than x and 3) indifference between x and y. With preorders, a fourth kind of comparison can be made, namely 4) x and y are incomparable. The difference between incomparability and indifference is an important one, both for beliefs and preferences, and this is my reason to start my investigations with preorders. This

[1] The results presented in this chapter have been obtained in collaboration with van Benthem and Roy, soon to be published in [68].

choice becomes important in Chapter 3, since most research in belief revision has been conducted over connected orders. One reason might be that the conditional aspect of belief revision makes a treatment over preorders almost intractable. An advantage of the constructive dynamic approach used in Chapter 3 is to dissolve these problems, by formalizing belief revision inside a standard Kripkean framework.

The background for the basic dynamic logic tools presented in section 2.2 can be found in several places. In this chapter, I focus on three basic actions: 1) public announcement [72], 2) lexicographic upgrade [67] and 3) preference upgrade [69]. These three actions provide a nice sample of typical actions over models, respectively 1) state elimination, 2) relation change and 3) link cutting. More actions could be analyzed in a similar fashion, but I prefer a standpoint closer to the intended interpretation of the basic language.

2.1 Order Logic

Let PROP be a set of propositional letters. The starting language, denoted $\mathcal{L}_{\mathcal{O}}$, is inductively defined by the following rules:

$$\mathcal{L}_{\mathcal{O}} := p \mid \varphi \vee \psi \mid \neg\varphi \mid \Diamond^{\leq}\varphi \mid \Diamond^{<}\varphi \mid E\varphi$$

The class of formulas of $\mathcal{L}_{\mathcal{O}}$ is denoted 'FORM'. The intended reading of $\Diamond^{\leq}\varphi$ is "φ is true in a state that is considered to be at least as good as the current state", whereas that of $\Diamond^{<}\varphi$ is "φ is true in a state that is considered to be strictly better than the current state". $E\varphi$ is interpreted as "there is a state where φ is true".[2]

I write $\Box^{\leq}\varphi$ to abbreviate $\neg\Diamond^{\leq}\neg\varphi$, and use $\Box^{<}\varphi$ and $U\varphi$ for the duals of $\Diamond^{<}\varphi$ and $E\varphi$ respectively.

Order models

Definition 2.1.1 [Models] An *order model* $\mathfrak{M}$ is a triple $\mathfrak{M} = \langle W, \preceq, V\rangle$ where W is a set of states, $\preceq$ is a reflexive and transitive relation (a *preorder*) and V is a standard propositional valuation. The strict subrelation $\prec$ is defined in terms of $\preceq$: $u \prec v := u \preceq v\ \&$ not $v \preceq u$. Finally, a *pointed order model* is a pair $\mathfrak{M}, u$ where

[2] I could let the language have multi-agents by indexing the modalities with members of a set of agents. I omit this in the present chapter for ease of notation and readability. When the need for multi-agent arises, I will make it explicit.

$u \in W$. $\lhd$

Interpretation

Definition 2.1.2 [Truth definition] Formulas of $\mathcal{L}_{\mathcal{O}}$ are interpreted in pointed order models.

$$
\begin{array}{lll}
\mathfrak{M}, u \models p & \text{iff} & u \in V(p) \\
\mathfrak{M}, u \models \neg\varphi & \text{iff} & \mathfrak{M}, u \not\models \varphi \\
\mathfrak{M}, u \models \varphi \vee \psi & \text{iff} & \mathfrak{M}, u \models \varphi \text{ or } \mathfrak{M}, u \models \psi \\
\mathfrak{M}, u \models \Diamond^{\leq}\varphi & \text{iff} & \exists v \text{ s.t. } u \preceq v \,\&\, \mathfrak{M}, v \models \varphi \\
\mathfrak{M}, u \models \Diamond^{<}\varphi & \text{iff} & \exists v \text{ s.t. } u \prec v \,\&\, \mathfrak{M}, v \models \varphi \\
\mathfrak{M}, u \models E\varphi & \text{iff} & \exists v \text{ s.t. } \mathfrak{M}, v \models \varphi
\end{array}
$$

$\lhd$

Definition 2.1.3 A formula φ is said to be *satisfiable* in a model $\mathfrak{M}$ if there is a state u such that $\mathfrak{M}, u \models \varphi$ and *valid* if it is true at every state in every model. $\lhd$

Expressive power

From time to time, I appeal to the notions of *modal equivalence* and *bisimulation* to investigate the expressive power of the various logics presented in the thesis. These notions are by now well-understood (see for instance [7]) and I content myself with listing the definitions and proposition required latter on.

Definition 2.1.4 [Modal equivalence] Two pointed models $\mathfrak{M}, u$ and $\mathfrak{M}', v$ are *modally equivalent*, noted $\mathfrak{M}, u \leftrightsquigarrow \mathfrak{M}', v$, iff they satisfy exactly the same formulas of $\mathcal{L}_{\mathcal{O}}$, i.e. $\forall \varphi \in \textsc{form}, \mathfrak{M}, u \models \varphi$ iff $\mathfrak{M}', v \models \varphi$. $\lhd$

Definition 2.1.5 [Bisimulation] Two order pointed models $\mathfrak{M}, u$ and $\mathfrak{M}', v$ are *bisimilar* (written $\mathfrak{M}, u \leftrightarrows \mathfrak{M}', v$) if there is a relation $R \subseteq \mathfrak{M} \times \mathfrak{M}'$ such that:

1. If sRt then for all $p \in \textsc{prop}$, $s \in V(p)$ iff $t \in V'(p)$,

2. (Forth) if sRt and $s \preceq s'$ ($s \prec s'$) then there is a $t' \in W'$ such that $t \preceq' t'$ ($t \prec' t'$ respectively) and $s'Rt'$,

3. (Back) if sRt and $t \preceq' t'$ ($t \prec' t'$) then there is a $s' \in W$ such that $s \preceq s'$ ($s \prec s'$ respectively) and $s'Rt'$,

4. For all $s \in W$, there is a $t \in W'$ such that sRt, and

5. For all $t \in W'$, there is a $s \in W$ such that sRt.

$\lhd$

Definition 2.1.5 defines a notion of what is often called a *total* bisimulation, due to clauses 4 and 5, which are included to take care of the existential modality in Proposition 2.1.6.

Proposition 2.1.6 *For every $u \in \mathfrak{M}$ and $\varphi \in \mathfrak{M}'$, if $\mathfrak{M}, u \leftrightarrows \mathfrak{M}', v$, then $\mathfrak{M}, u \leftrightsquigarrow \mathfrak{M}', v$.*

Proposition 2.1.6 can be used, for instance, to show that the modality $\Diamond^{<}\varphi$ is not definable in terms of $\Diamond^{\leq}\varphi$ - even though the strict relation $\prec$ is defined in terms of $\preceq$. I prove this in the following Fact.

Fact 2.1.7 *The modality $\Diamond^{<}$ is not definable with $\Diamond^{\leq}$.*

Proof. A simple bisimulation argument establishes this latter claim. Let there be two models $\mathfrak{M}_1 = \{u\}$ with $\preceq_1 = \{(u, u)\}$, $V_1(p) = \{u\}$ and $\mathfrak{M}_2 = \{s, t\}$ with $\preceq_2 = \{(s, t), (t, t)\}$, $V_2(p) = \{s, t\}$. Then, p is strictly better at s, since there is a state t such that $s \preceq_1 t \,\&\, t \not\preceq s$ and $\mathfrak{M}, t \models p$, but p is not strictly better at u. But modal formulas of $\mathcal{L}_{\mathcal{O}}$ are invariant under bisimulation, thus u and s satisfy the same formulas. Therefore, $\mathcal{L}_{\mathcal{O}}$ cannot define the strict subrelation $\prec$ of $\preceq$. QED

Axiomatization

Let us call $\Lambda^{\mathcal{L}_{\mathcal{O}}}$ the logic of order models. This logic has two well-known fragments, namely $S4$ for $\Diamond^{\leq}$ and $S5$ for E. For $\Diamond^{<}$, the logic contains K. In addition, there are *interaction* axioms relating the three modalities. $\Lambda^{\mathcal{L}_{\mathcal{O}}}$ is thus axiomatized by:

$$\Diamond^{\leq}(\varphi \to \psi) \to (\Diamond^{\leq}\varphi \to \Diamond^{\leq}\psi) \tag{2.1}$$

$$\Diamond^{<}(\varphi \to \psi) \to (\Diamond^{<}\varphi \to \Diamond^{<}\psi) \tag{2.2}$$

$$E(\varphi \to \psi) \to (E\varphi \to E\psi) \tag{2.3}$$

$$\varphi \to \Diamond^{\leq}\varphi \tag{2.4}$$

$$\varphi \to E\varphi \tag{2.5}$$

$$\Diamond^{\leq}\Diamond^{\leq}\varphi \to \Diamond^{\leq}\varphi \tag{2.6}$$

$$EE\varphi \to E\varphi \tag{2.7}$$

$$\varphi \to UE\varphi \tag{2.8}$$

$$\Diamond^{\leq}\varphi \to E\varphi \tag{2.9}$$

$$\vdash \Diamond^{<}\varphi \to \Diamond^{\leq}\varphi \tag{2.10}$$

$$\vdash \Diamond^{\leq}\Diamond^{<}\varphi \to \Diamond^{<}\varphi \tag{2.11}$$

$$\vdash \Diamond^{<}\Diamond^{\leq}\varphi \to \Diamond^{<}\varphi \tag{2.12}$$

$$\varphi \wedge \Diamond^{\leq}\psi \to (\Diamond^{<}\psi \vee \Diamond^{\leq}(\psi \wedge \Diamond^{\leq}\varphi)) \tag{2.13}$$

$\Lambda_{\mathcal{L}_\mathcal{O}}$ has the usual rules of Modus Ponens (if φ and $\varphi \to \psi$ are provable, then ψ is provable) and Necessitation (if ψ is provable, then $\Box\psi$ is provable, where $\Box$ stands for any of the box-modalities). I did not include a transitivity axiom for $\Diamond^{<}$, as it is derivable:

Fact 2.1.8 *Transitivity of $\Diamond^{<}$ is derivable, i.e., $\vdash \Diamond^{<}\Diamond^{<}\varphi \to \Diamond^{<}\varphi$.*

Proof. Assume that $\vdash \Diamond^{<}\Diamond^{<}\varphi$, then Axiom 2.10 implies that $\vdash \Diamond^{\leq}\Diamond^{<}\varphi$. By Axiom 2.11, $\vdash \Diamond^{<}\varphi$. QED

Fact 2.1.8 reflects that, in order models, transitivity of $\prec$ is derived from transitivity of $\preceq$. Similarly, Axioms 2.9 and 2.10 together imply that $\vdash \Diamond^{<}\varphi \to E\varphi$.

Completeness of $\mathcal{L}_\mathcal{O}$

It is not trivial to show completeness with respect to the class of models where $\prec$ is *irreflexive*, for this property is not expressible in ordinary modal logic, as I have already shown. Known techniques to cope with this difficulty include the introduction of the "Gabbay Irreflexivity Rule" [21], "bulldozing" the canonical model [58] or extending the language with hybrid modalities. I resort to the bulldozing technique below.

Order models present a further challenge, namely that $\prec$ in our modal is a specific strict subrelation of $\preceq$, as I have stressed numerous times now. If $\prec$ is the intended strict subrelation of $\preceq$, then I say that $\prec$ is *adequate* with respect to $\preceq$. The following definition makes this precise.

Definition 2.1.9 A relation $\prec$ is called *adequate with respect to* $\preceq$ if the following are equivalent:

1. $w \prec v$
2. (a) $w \preceq v$ and

   (b) $v \not\preceq w$.

If only the direction from (2) to (1) holds, then the relation $\prec$ is said to be *quasi-adequate with respect to* $\preceq$. $\lhd$

It should be clear that Axiom 2.10 takes care of the implication from (1) to (2.a), and I show below how to adapt the bulldozing technique to ensure that (2.b) also holds. Quasi-$\prec$-adequacy is taken care of by Axiom 2.13, as the following correspondence argument shows.

Fact 2.1.10 *1. If a model $\mathfrak{M}$ is based on a quasi-$\prec$-adequate frame, then $\mathfrak{M}, u \models \varphi \wedge \Diamond^{\leq}\psi \rightarrow (\Diamond^{<}\psi \vee \Diamond^{\leq}(\psi \wedge \Diamond^{\leq}\varphi))$ for every state u.*

2. For every frame $\mathfrak{F}$, if $\mathfrak{F} \models \varphi \wedge \Diamond^{\leq}\psi \rightarrow (\Diamond^{<}\psi \vee \Diamond^{\leq}(\psi \wedge \Diamond^{\leq}\varphi))$, then $\mathfrak{F}$ is quasi-$\prec$-adequate.

Proof of Fact 2.1.10

1. Take any model based on a quasi-$\prec$-adequate frame, and a state $u \in W$ such that $\mathfrak{M}, u \models \varphi \wedge \Diamond^{\leq}\psi$. This means that there is a v such that $u \preceq v$ and $\mathfrak{M}, v \models \psi$. Now, either $v \preceq u$ or not. In the first case, $\mathfrak{M}, v \models \psi \wedge \Diamond^{\leq}\varphi$, and thus $\mathfrak{M}, u \models \Diamond^{\leq}(\psi \wedge \Diamond^{\leq}\varphi)$. In the second case, because $\mathfrak{M}$ is based on a quasi-$\prec$-adequate frame, $u \prec v$. Therefore, $\mathfrak{M}, u \models \Diamond^{<}\psi$.

2. Suppose that $u \leq v$ and $v \not\leq u$. Take a model $\mathfrak{M}$ with a valuation V on $\mathfrak{F}$ such that $V(p) = \{u\}$ and $V(q) = \{v\}$. Thus, $\mathfrak{M}, u \models p \wedge \Diamond^{\leq}q$. By Axiom 2.13, $\mathfrak{M}, u \models \Diamond^{<}q \vee \Diamond^{\leq}(q \wedge \Diamond^{\leq}p)$. Thus, for some w, either $u < w \,\&\, w = v$ (i.e., $u < v$) - and we are done - or $u \leq v \leq u$.

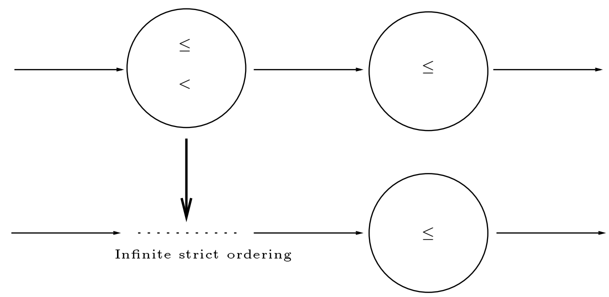

Figure 2.1: The canonical model $\mathfrak{M}^c$ and its bulldozed counterpart B, where the $\prec$-clusters are replaced with infinite strict orderings, indicated with the dotted line in the picture. The bulldozing technique I use describes just how to get appropriate strict orderings.

◀

Theorem 2.1.11 *The logic $\Lambda^{\mathcal{LO}}$ is sound and complete with respect to the class of order models.*

Proof.

Over order models, it is a routine argument to show soundness for K, $S4$ and $S5$, as well as for the inclusion Axioms 2.11 and 2.12. Soundness of Axiom 2.13 was shown in Fact 2.1.10 and I have shown in Fact 2.1.8 that transitivity of $\Diamond^<$ is derivable.

For completeness, I show that every $\Lambda^{\mathcal{LO}}$-consistent set Φ of formula has a model. I appeal to the standard definition of the canonical model $\mathfrak{M}^c = \langle W, \preceq, V\rangle$ for $\Lambda^{\mathcal{LO}}$ (cf. [7]). I also use the fact that I can extend Φ to a maximally consistent set (MCS) Γ that contains every formula $E\varphi$ or its negation. I call the set $\{\varphi : E\varphi \in \Gamma \text{ or } U\varphi \in \Gamma\}$ the E-theory of Γ, and I call the restriction of $\mathfrak{M}^c$ to the set of MCS Δ that have the same E-theory as Γ its E-submodel. In the E-submodel, E is a genuine global modality and, by Axiom 2.10, this submodel contains the submodel generated by Γ. From now on, when referring to $\mathfrak{M}^c$, I mean one of its E-submodels. I also use u, v to refer to MCS in W.

It is a standard result of modal logic that every consistent set Φ is satisfiable in $\mathfrak{M}^c$, but this model is *not* an order model in the intended sense. To see this, I introduce some terminology. Given an order model $\mathfrak{M}$, a subset C of W is called a $\preceq$*-cluster* iff $u \preceq v$ for all $u, v \in C$; $\prec$-clusters are defined in the same way. Clearly, if a model contains $\prec$-clusters, it is not $\prec$-adequate, thus not an order model. The difficulty in showing completeness for the class of order models is to guarantee the absence of $\prec$-clusters in $\mathfrak{M}^c$. This is exactly what the "bulldozing" technique achieves (cf. [7, p.221-222]). The crux of this transformation is to substitute infinite strict orderings for $\prec$-clusters, as shown in Figure 2.1. The following lemma is required in the main proof.

Lemma 2.1.12 *For any $\preceq$-cluster C in $\mathfrak{M}^c$, if any two states $u, v \in C$ are such that $u \prec v$ then for all $s, t \in C$, $s \prec t$.*

Proof. Assume that, within a $\preceq$-cluster C, there are two states $u, v \in C$ such that $u \prec v$. I show that for any s, t in C, $s \prec t$. This amounts to showing that $\Diamond^{<}\varphi \in s$ for any $\varphi \in t$. Consider an arbitrary $\varphi \in t$. Since C is a $\preceq$-cluster, $\Diamond^{\leq}\varphi \in v$, and $u \prec v$ implies that $\Diamond^{<}\Diamond^{\leq}\varphi \in u$, from which it follows that $\Diamond^{<}\varphi \in u$ by Axiom 2.13. But since C is a $\preceq$-cluster, $\Diamond^{\leq}\Diamond^{<}\varphi \in s$, and Axiom 2.11 implies that $\Diamond^{<}\varphi \in s$, as required. QED

Bulldozing is now applied to those clusters containing $\prec$-links. This is done by the following steps:

1. Index the $\preceq$-clusters that contain $\prec$ links with an index set I.

2. Choose an arbitrary strict ordering $\prec^i$ on each C_i. Observe that, by Lemma 2.1.12, any $\prec^i$ so chosen is a subrelation of $\prec$ on C_i.

3. For each cluster C_i, define C_i^β as $C_i \times \mathbb{Z}$.

4. Build the bulldozed model $Bull(\mathfrak{M}^c) = \langle B, \preceq', \prec', V \rangle$ as follows.

   - Call W^- the set of MCS that are not $\prec$-clusters ($W - \bigcup_{i \in I} C_i$), and let $B = W^- \cup \bigcup_{i \in I} C_i^\beta$. I use $x, y, z...$ to range over elements of B. Note that if $x \notin W^-$, then x is a pair (u, n) for $u \in W$ and $n \in \mathbb{Z}$.
   - Define the map $\beta : B \to W$ by $\beta(x) = x$ if $x \in W^-$ and $\beta(x) = u$ otherwise, i.e., if x is a pair (u, n) for some u and n.
   - Now, the key step of the construction: defining, in a truth-preserving way, an adequate version of $\prec$. There are four cases to consider:

     Case 1: x or y is in W^-. In this case the original relation $\prec$ was adequate (cf. Definition 2.1.9), and is thus directly copied into $Bull(\mathfrak{M}^c)$: $x \prec' y$ iff $\beta(x) \prec \beta(y)$.

     Case 2: $\beta(x) \in C_i$, $\beta(y) \in C_j$ and $i \neq j$. Here, $\beta(x)$ and $\beta(y)$ are in different clusters and the original $\prec$ link between them is adequate. Put again $x \prec' y$ iff $\beta(x) \prec \beta(y)$.

Case 3: $\beta(x), \beta(y) \in C_i$ for some i. In this case, $x = (u, m)$ and $y = (v, n)$ for some m, n. There are two sub-cases to be considered:

Case 3.1: If $m \neq n$, use the natural strict ordering on $\mathbb{Z}$: $(u, m) \prec' (v, n)$ iff $m < n$.

Case 3.2: Otherwise, if $m = n$, use the adequate (i.e. strict) sub-relation $\prec^i$ chosen above: $(u, m) \prec' (v, m)$ iff $u \prec^i v$.

- To define the relation $\preceq'$, there are again two cases to be considered, in order to make $\prec'$ adequate:

  Case 1: If $x \in W^-$ or $y \in W^-$, take the original relation $\preceq$: $x \preceq' y$ iff $\beta(x) \preceq \beta(y)$

  Case 2: Otherwise (x and y are not in W^-), take the reflexive closure of $\prec'$: $x \preceq' y$ iff $x \prec' y$ or $x = y$.

- The valuation on $Bull(\mathfrak{M}^c)$ is based on the valuation on $\mathfrak{M}^c$: $x \in V'(p)$ iff $\beta(x) \in V(p)$.

$Bull(\mathfrak{M}^c)$ is, as indented, an adequate model:

Observation 2.1.13 *$Bull(\mathfrak{M}^c)$ is $\prec'$-adequate.*

Proof of Observation In $\mathfrak{M}^c$, given that Axiom 2.12 is a Sahlqvist formula, if $u \preceq v$ and $v \not\preceq u$, then $u \prec v$. This property is transferred to $Bull(\mathfrak{M}^c)$ if u and v are in different $\preceq$-clusters, or if they are not in the same cluster and then $u \prec' v$ by definition. If u and v are in the same $\prec$-cluster, then $\prec'$ is constructed so as to be adequate by taking $\preceq'$ to be the reflexive closure of $\prec'$. This implication would not hold in $\mathfrak{M}^c$ only in $\prec$-clusters. ◀

All that remains to be shown is that $Bull(\mathfrak{M}^c)$ and the canonical model satisfy the same formulas. This is done by showing that $Bis = \{(x, u), (u, x) : u = \beta(x)\}$ is a total bisimulation.

Claim 1 *Bis* is a total bisimulation.

Proof of Claim 1 Observe first that β is a surjective map, which establishes totality. The definition of V' yields the condition on proposition letters automatically. It remains to show that the back and forth condition hold for $\preceq'$ and $\prec'$.

($\preceq'$) Forth condition: assume that $x \preceq' y$. Given that Bis is total, all I have to show is that there is a $u \in W$ such that $\beta(x) \preceq u = \beta(y)$. If either x or $y \in W^-$, the result follows directly from case 1 of the definition of $\preceq'$. Otherwise, if $x = y$, then axiom T imply that $\beta(x) = \beta(y)$. Finally, if $x \neq y$, then case 2 of the definition of $\preceq'$ implies that $x \prec' y$. But then cases 2, 3.1 and 3.2 of $\prec'$ imply that $\beta(x) \prec \beta(y)$, and so $\beta(x) \preceq \beta(y)$, since $\prec$ is included in $\preceq$ by Axiom 2.10.

Back condition: assume that $\beta(x) \preceq u$. I have to find a $y \in B$ such that $\beta(y) = u$ and $x \preceq' y$. The only tricky case is when $\beta(x)$ and u are in the same $\prec$-cluster. This means that $x = (v, m)$ for some m. Take any y such that $y = (u, n)$ and $m < n$. By the definition of $\prec'$, $x \prec' y$ and so $x \preceq' y$ by case 2 of the definition of $\preceq'$.

($\prec'$) The argument for $\prec$ follows the same steps as for $\preceq$. I indicate the key observations. It should be clear that for all $x, y \in B$, if $x \prec' y$ then $\beta(x) \prec \beta(y)$. I show that if $\beta(x) \prec u$ then there is a $y \in B$ such that $x \prec' y$ and $\beta(y) = u$.

1. If u is in W^-, then $\beta^{-1}(u)$ is unique and $x \prec' \beta^{-1}(u)$.
2. If $u \in C_i$ for some i, $\beta^{-}(u)$ is the set $\{(u, n) : n \in \mathbb{Z}\}$. If $\beta(x) \in W^-$ or $\beta(x) \in C_j$ with $i \neq j$, let $y = (u, n)$ for an arbitrary element of this set.
3. Finally, if $\beta(x)$ and u are in the same cluster. Then $x = (v, m)$ for some $m \in \mathbb{Z}$. Take any n such that $m < n$, then the pair $y = (u, n)$ has the required properties.

◀

This conclues our proof of the completeness theorem for $\mathcal{L}_{\mathcal{O}}$. QED

2.2 Dynamics

In the remainder of this chapter, I introduce the three main actions of 1) *public announcement*, 2) *lexicographic upgrade* and 3) *preference upgrade*. I then discuss a general technique for completeness results, known as *compositional analysis* via *reduction axioms*, which I use numerous times in the thesis. Compositional analysis is a way of getting completeness for extended dynamic languages by reducing the analysis of the action modalities to the static language, thus reducing completeness of the dynamic language to that of the static one.

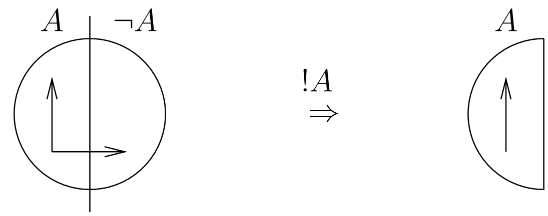

Figure 2.2: The effect of publicly announcing A

Public announcement

A public announcement of some information A is simply the truthful announcement of A. If A is true at a state u and is announced, then all $\neg A$-state are deleted from the model along with accessibility relations from A to $\neg A$-states. Public announcements are represented by modalities of the form $\langle !A\rangle\varphi$ for every A and the modalities are interpreted by:

$$\mathfrak{M}, u \models \langle !A\rangle\varphi \quad \text{iff} \quad \mathfrak{M}, u \models A \,\&\, \mathfrak{M}|_A, u \models \varphi \tag{2.14}$$

where $\mathfrak{M}|_A$ is the submodel whose domain is given by the set of states that satisfy A ($W|_A$) with a corresponding restriction of the accessibility relation to $W|_A$. The effect of announcing A is depicted in Figure 2.2. The left model is divided into two zones, the A and the $\neg A$-zones. The right model is the result of publicly announcing A, thus eliminating all $\neg A$-states as well as the relations to or from $\neg A$-states.

Lexicographic upgrade

Lexicographic upgrade, denoted '$\Uparrow A$', was first analyzed in the dynamic logic approach to belief revision by van Benthem in [67]. His goal was to provide a framework for belief revision in a dynamic setting, rather than a conditional one as, for instance, in *DDL* (cf. [40]). His approach is advantageous over traditional alternatives found in the literature in various ways. Firstly, it is all worked-out in a modal logic setting, rather than with conditionals. This is technically advantageous because the language is much simpler and comes with many technical results that apply to it directly. Secondly, it is not restricted to revisions with *factual* information, as is the case in *DDL*. Finally, it can deal with iteration in a straightforward way, something that has been a major problem in belief revision.

Lexicographic upgrade, unlike public announcement, acts on links between states

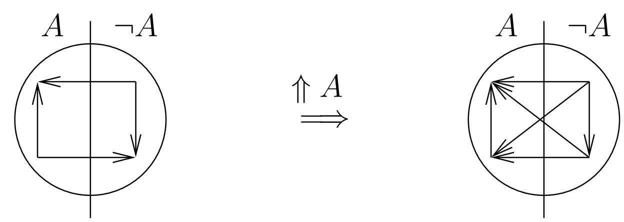

Figure 2.3: Illustration of lexicographic upgrade

rather than on states themselves. It can be seen as an adjustment on the relation so as to make incoming information of most importance. Van Benthem ([67], p.141) describes it in the following way:

> "$\Uparrow P$ is an instruction for replacing the current ordering relation $\leq$ between worlds by the following: *all P-worlds become better than all $\neg P$-worlds, and withing those two zones, the old ordering remains.*"

In the notation of propositional dynamic logic (PDL, cf., [50])[3], the updated relation $\preceq^{\Uparrow A}$ is defined by:

$$\preceq^{\Uparrow A} = (?A\,;\,\preceq\,;\,?A) \cup (?\neg A\,;\,\preceq\,;\,?\neg A) \cup (?\neg A\,;\,\top\,;\,?A) \tag{2.15}$$

A graphical representation of lexicographic upgrade is provided in Figure 2.3. The model on the left is again divided into two zones and links are seen to go across the zones in two directions. After A has been upgraded, the links within each zones are preserved, links from A to $\neg A$ are reversed and a link is added from every $\neg A$-state to every A-state.

The language of Order Logic with lexicographic upgrade is $\mathcal{L}_{\mathcal{O}}$ augmented with a lexicographic upgrade modality $\langle \Uparrow A\rangle\varphi$, whose semantics is given by:

$$\mathfrak{M}, u \models \langle \Uparrow A\rangle\varphi \quad \text{iff} \quad \mathfrak{M}_{\Uparrow A}, u \models \varphi \tag{2.16}$$

where $\mathfrak{M}_{\Uparrow} = \langle W, \preceq^{\Uparrow A}, V\rangle$.

Preference upgrade

In [69], van Benthem and Liu showed that a preference *upgrade* can be seen as a relation change. The relation change they describe is that of a public suggestion

[3]For a good introduction to PDL, see [7].

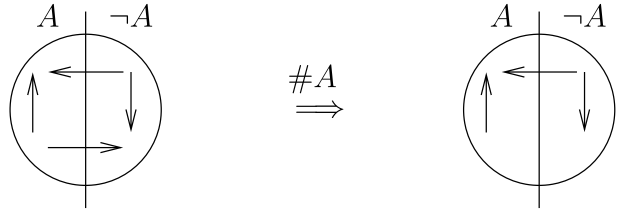

Figure 2.4: Illustration of preference upgrade

to make A better than $\neg A$, i.e., the change in the model is such that every links from $\neg A$-state to A-state is deleted while keeping the relation unchanged in the two respective zones. Preference upgrade is denoted by $\#A$ and its action on models can defined by the following:

Definition 2.2.1 Given a model $\mathfrak{M} = \langle W, \preceq, V\rangle$, the *upgraded model* by A is given by $\mathfrak{M}_{\#A} = \langle W, \preceq^{\#A}, V\rangle$, where

$$\preceq^{\#A} = \preceq - \{(u, v) : \mathfrak{M}, u \models A \,\&\, \mathfrak{M}, v \models \neg A\} \tag{2.17}$$

$\lhd$

In the notation of PDL, the updated relation $\preceq^{\#A}$ is defined by:

$$\preceq^{\#A} = \preceq - (?\neg A\,;\, \preceq;\, ?A) \tag{2.18}$$

Preference upgrade is depicted in Figure 2.4. As above, to get an Order Logic with preference upgrade, one augments $\mathcal{L}_{\mathcal{O}}$ with a modality $\langle \#A\rangle\varphi$ with semantics given by:

$$\mathfrak{M}, u \models \langle \#A\rangle\varphi \quad \text{iff} \quad \mathfrak{M}_{\#A}, u \models \varphi \tag{2.19}$$

Axiomatization and completeness

A great tool that came about with the rise of dynamic logic is the so-called *compositional analysis via reduction axioms*. Reduction axioms analyze the effect of actions in the base language, thus reducing the completeness of the extended logic to that of the basic one. Reduction axioms have a twofold advantage: 1) they provide an explicit analysis of actions on models and 2) they provide completeness for free. For instance, a typical principle analyzing epistemic effect of public announcement is the following reduction axiom:

$$\langle !A\rangle\Diamond\varphi \quad\leftrightarrow\quad A \wedge \Diamond\langle !A\rangle\varphi \tag{2.20}$$

Axiom 2.20 can be read as stating that a φ-state is accessible after the public announcement of A if and only if A is true in the current world, thus can be announced, and there is an accessible state which becomes a φ-state after the announcement of A. Formally, the validity of Axiom 2.20 can be seen by the following argument:

$$\begin{array}{llll}
\mathfrak{M}, u \models \langle !A\rangle\Diamond\varphi & \text{iff} & \mathfrak{M}, u \models A \,\&\, \mathfrak{M}|_A, u \models \Diamond\varphi & (2.14)\\
& \text{iff} & \mathfrak{M}, u \models A \,\&\, \exists v : u \preceq |_A v \,\&\, \mathfrak{M}|_A, v \models \varphi & \text{(Truth-Definition)}\\
& \text{iff} & \mathfrak{M}, u \models A \,\&\, \exists v : u \preceq v \,\&\, \mathfrak{M}, v \models \langle !A\rangle\varphi & (2.14!)\\
& \text{iff} & \mathfrak{M}, u \models A \,\&\, \mathfrak{M}, u \models \Diamond\langle !A\rangle\varphi & \text{(Truth-Definition)}\\
& \text{iff} & \mathfrak{M}, u \models A \wedge \Diamond\langle !A\rangle\varphi & \text{(Truth-Definition)}
\end{array}$$

Notice the important step from line 2 to line 3. In the first direction, since $u \preceq |_A v$, also $\mathfrak{M}|_A, v \models A$. Furthermore, $u \preceq |_A v \Rightarrow u \preceq v$, by definition. In the other direction, $\mathfrak{M}, v \models \langle !A\rangle\varphi \Rightarrow \mathfrak{M}, v \models A$. Hence $u \preceq v, \mathfrak{M}, u \models A$ and $\mathfrak{M}, v \models A$ implies that $u \preceq |_A v$.

One striking feature of axiom 2.20 is that, on the left-hand side, the action modality $\langle !A\rangle$ is outside the scope of $\Diamond$, whereas on the right-hand side, it is inside it. Since there are reduction axioms for every component of the basis language, one can push the action modalities all the way to propositional letters, where they do not act any further and can be fully eliminated.

Theorem 2.2.2 *The Order Logic with public announcement, lexicographic and preference upgrade is axiomatized by 1) $\Lambda^{\mathcal{L}_O}$ and 2) the following reduction axioms for each of the action modalities:*

$$\langle !A\rangle p \quad\leftrightarrow\quad A \wedge p \tag{2.21}$$

$$\langle !A\rangle\neg\varphi \quad\leftrightarrow\quad A \wedge \neg\langle !A\rangle\varphi \tag{2.22}$$

$$\langle !A\rangle(\varphi \vee \psi) \quad\leftrightarrow\quad \langle !A\rangle\varphi \vee \langle !A\rangle\psi \tag{2.23}$$

$$\langle !A\rangle\Diamond^{\leq}\varphi \quad\leftrightarrow\quad A \wedge \Diamond^{\leq}\langle !A\rangle\varphi \tag{2.24}$$

$$\langle !A\rangle\Diamond^{<}\varphi \quad\leftrightarrow\quad A \wedge \Diamond^{<}\langle !A\rangle\varphi \tag{2.25}$$

$$\langle !A\rangle E\varphi \quad\leftrightarrow\quad A \wedge E\langle !A\rangle\varphi \tag{2.26}$$

$$\begin{aligned}
\langle\Uparrow A\rangle p &\leftrightarrow p && (2.27)\\
\langle\Uparrow A\rangle\neg\varphi &\leftrightarrow \neg\langle\Uparrow A\rangle\varphi && (2.28)\\
\langle\Uparrow A\rangle(\varphi\vee\psi) &\leftrightarrow \langle\Uparrow A\rangle\varphi\vee\langle\Uparrow A\rangle\psi && (2.29)\\
\langle\Uparrow A\rangle\Diamond^{\leq}\varphi &\leftrightarrow A\wedge\Diamond^{\leq}(A\wedge\langle\Uparrow A\rangle\varphi) &&\\
&\vee\ \neg A\wedge\Diamond^{\leq}(\neg A\wedge\langle\Uparrow A\rangle\varphi) &&\\
&\vee\ \neg A\wedge E(A\wedge\langle\Uparrow A\rangle\varphi) && (2.30)\\
\langle\Uparrow A\rangle\Diamond^{<}\varphi &\leftrightarrow A\wedge\Diamond^{<}(A\wedge\langle\Uparrow A\rangle\varphi) &&\\
&\vee\ \neg A\wedge\Diamond^{<}(\neg A\wedge\langle\Uparrow A\rangle\varphi) &&\\
&\vee\ \neg A\wedge E(A\wedge\langle\Uparrow A\rangle\varphi) && (2.31)\\
\langle\Uparrow A\rangle E\varphi &\leftrightarrow E\langle\Uparrow A\rangle\varphi && (2.32)
\end{aligned}$$

$$\begin{aligned}
\langle\#A\rangle p &\leftrightarrow p && (2.33)\\
\langle\#A\rangle\neg\varphi &\leftrightarrow \neg\langle\#A\rangle\varphi && (2.34)\\
\langle\#A\rangle(\varphi\vee\psi) &\leftrightarrow \langle\#A\rangle\varphi\vee\langle\#A\rangle\psi && (2.35)\\
\langle\#A\rangle\Diamond^{\leq}\varphi &\leftrightarrow A\wedge\Diamond^{\leq}(A\wedge\langle\#A\rangle\varphi) &&\\
&\vee\ \neg A\wedge\Diamond^{\leq}\langle\#A\rangle\varphi && (2.36)\\
\langle\#A\rangle\Diamond^{<}\varphi &\leftrightarrow A\wedge\Diamond^{<}(A\wedge\langle\#A\rangle\varphi) &&\\
&\vee\ \neg A\wedge\Diamond^{<}\langle\#A\rangle\varphi && (2.37)\\
\langle\#A\rangle E\varphi &\leftrightarrow E\langle\#A\rangle\varphi && (2.38)
\end{aligned}$$

Proof. Notice first that no special work has to be done for the completeness part, since the axioms reduce the analysis of an arbitrary formula of the extended language to that of $\mathcal{L}_{\mathcal{O}}$ and the corresponding complete logic $\Lambda^{\mathcal{L}_{\mathcal{O}}}$. To see this, consider an arbitrary formula φ. Working inside-out, consider (one of) the innermost occurrence of an action modality. By applying successively the relevant axioms listed above until only propositional letters are in the scope of that modality, its occurrence can be eliminated using the relevant axiom among 2.21, 2.33 or 2.27. This procedure can be iterated until φ is transformed into and equivalent formula φ' containing no action modalities. The completeness of the extended logic is therefore reduced to that of

$\Lambda^{\mathcal{L}_{\mathcal{O}}}$.

Thus, unlike in most cases of completeness proofs, the interesting part for dynamic logic is the soundness of the axioms! The soundness of axiom 2.24 has already been proved in the discussion preceding the statement of the theorem. I show that Axioms 2.36 and 2.30 are also sound. First Axiom 2.36:

$$\begin{aligned}
\mathfrak{M}, u \models \langle \# A\rangle \Diamond^{\leq}\varphi \quad &\text{iff} \quad \mathfrak{M}_{\# A}, u \models \Diamond^{\leq}\varphi \\
&\text{iff} \quad \exists v : u \preceq^{\# A} v \,\&\, \mathfrak{M}_{\# A}, v \models \varphi \\
&\text{iff} \quad \exists v : u \preceq^{\# A} v \,\&\, \mathfrak{M}, v \models \langle \# A\rangle\varphi \quad (*)
\end{aligned}$$

Now, either $\mathfrak{M}, u \models A$ or $\mathfrak{M}, u \models \neg A$. In the first case, since $u \preceq^{\# A} v$, it must be that $u \preceq v$ and $\mathfrak{M}, v \models A$. Thus, $(*)$ iff $\exists v : u \preceq v \,\&\, \mathfrak{M}, v \models A \wedge \langle \# A\rangle\varphi$ iff $\mathfrak{M}, u \models A \wedge \Diamond^{\leq}(A \wedge \langle \# A\rangle\varphi)$. In the second case, $u \preceq^{\# A} v$ iff $u \preceq v$, thus $(*)$ iff $\exists v : u \preceq v \,\&\, \mathfrak{M}, v \models \langle \# A\rangle\varphi$ iff $\mathfrak{M}, u \models \neg A \wedge \Diamond^{\leq}\langle \# A\rangle\varphi$.

Second, Axiom 2.30:

$$\begin{aligned}
\mathfrak{M}, u \models \langle \Uparrow A\rangle \Diamond^{\leq}\varphi \quad &\text{iff} \quad \mathfrak{M}_{\Uparrow A}, u \models \Diamond^{\leq}\varphi \\
&\text{iff} \quad \exists v : u \preceq^{\Uparrow A} v \,\&\, \mathfrak{M}_{\Uparrow A}, v \models \varphi \\
&\text{iff} \quad \exists v : u \preceq^{\Uparrow A} v \,\&\, \mathfrak{M}, v \models \langle \Uparrow A\rangle\varphi \quad (**)
\end{aligned}$$

Now, many cases need to be considered: 1) $\mathfrak{M}, u \models A$, 2) $\mathfrak{M}, u \models \neg A$, 3) $\mathfrak{M}, v \models A$ and 4) $\mathfrak{M}, v \models \neg A$. Given that $u \preceq^{\Uparrow A} v$, the first case implies that $\mathfrak{M}, v \not\models \neg A$. Thus, $(**)$ iff $\exists u : u \preceq v \,\&\, \mathfrak{M}, v \models A \wedge \langle \Uparrow A\rangle\varphi$ iff $\mathfrak{M}, u \models A \wedge \Diamond^{\leq}(A \wedge \langle \Uparrow A\rangle\varphi)$. Now, assume that $\mathfrak{M}, u \models \neg A$. If $\mathfrak{M}, v \models \neg A$, then $(**)$ iff $\exists v : u \preceq v \,\&\, \mathfrak{M}, v \models \neg A \wedge \langle \Uparrow A\rangle\varphi$ iff $\mathfrak{M}, u \models \neg A \wedge \Diamond^{\leq}(\neg A \wedge \langle \Uparrow A\rangle\varphi)$. The remaining case is when $\mathfrak{M}, u \models \neg A$ and $\mathfrak{M}, v \models A$. In this case, regardless of whether $u \preceq v$ or not, it must be that $u \preceq^{\Uparrow A} v$ and $(**)$ iff $\exists v : \mathfrak{M}, v \models A \wedge \langle \Uparrow A\rangle\varphi$ iff $\mathfrak{M}, u \models \neg A \wedge E(A \wedge \langle \Uparrow A\rangle\varphi)$. This completes the proof. QED

Summary

This concludes the exposition of Order Logic. In this chapter, the main contribution is the completeness Theorem 2.1.11, whose proof applies Segerberg's bulldozing technique. The technique has been used on various occasions, but not in the present setting of Order Logic interpreted over preorders. In the remainder of the thesis, I show how Order Logic fulfills its telos in providing a general setting to formalize belief and preference change for individuals and groups. In the next two chapters, I look at two important fragments of $\mathcal{L}_{\mathcal{O}}$: the relational belief and universal binary fragments. As was argued in the introduction, these two logics can easily be embedded into Preference Logic via definitions making essential use of the existential modality. In the remaining chapters, I look at extensions, one to incorporate *ceteris paribus* clauses and the other to aggregation of orders into group ones.

Chapter 3

Relational Belief Revision

When I have revised my belief that Robert had never been to Europe, I had to incorporate a belief that Robert had to use some means of transportation to cross the ocean. Given our times, it was more likely that Robert had taken a plane to Europe rather than a ship, so I have revised my beliefs by incorporating the belief that Robert had taken a plane to Europe, but I had no clue what company he had been flying with. Was it an American or a European airline? Northwest Airlines, Air France, Lufthansa...? I did not know and it was over my capacities to use such a fine grained plausibility order and return a unique revised belief set. It may have seem natural for me to endorse a relational revision attitude instead, and get a multitude of new belief sets, each having the new belief that Robert had taken a plane to Europe, but in each one with a different airline. Which one of these new belief sets is accurate would have to be decided by extra-logical means: asking Robert.

The initial motivation for a formalization of relational beliefs was to get a generalization of the functional approach to belief revision known as *AGM* (cf. [1]). *AGM* is functional in the sense that an *AGM* revision operator, given a belief set Γ and a sentence φ, returns the unique revised belief set Γ' minimal with respect to some ordering. The problem was first studied by Rabinowicz and Lindström in [39] and by Cantwell in [11].

I consider that the best logical analysis of the *AGM* paradigm so far is found in Segerberg's work on *Dynamic Doxastic Logic* (*DDL*, cf. [40, 59, 60, 61]). I thus take *DDL* has the paradigm of belief revision and the starting point of my investigation in this chapter. My goal is to show that a generalization of *DDL* to a relational doxastic system is best treated as a fragment of Order Logic. To achieve this, however, some preparatory work has to be done on the static models, since *DDL* is framed in a conditional logic setting and its semantics uses different kinds of models, closer to neighborhood semantics. Hence, a great deal of work in this chapter is to bring this kind of semantics closer to that of Order Logic. I first generalize DDL to a relational doxastic logic, called *Broccoli Logic* (BL). I then show how Broccoli Logic can be dynamified by introducing the dynamic action of lexicographic upgrade. Let me elaborate.

Broccoli Logic is based on two basic modalities, $[\varphi]\psi$ and $[\varphi\rangle\psi$. The first modality is a standard conditional modality whose semantics is very close to that of *Minimal Conditional Logic* (MCL, cf. [10]). In fact, and this is one important result in this chapter (Theorem 3.3.14), the $[\varphi]\psi$ fragment of Broccoli Logic is the same as MCL, but in a different guise. The second modality, however, presents some difficulty and I have not succeeded in identifying the exact fragment of Order Logic that corresponds to full BL. This is a question pertaining to conditional logic that I leave open, although I present preliminary steps for a solution in the appendix. Nevertheless, Broccoli Logic *is* a fragment of Order Logic, as I show in Theorem 3.3.3. On the basis of this standpoint, I show how Broccoli Logic can be dynamified by incorporating lexicographic upgrade.

Introducing dynamics into BL in this fashion is where I am parting from DDL and this deserves a justification. Since the inaugural work [1], most of the research in belief revision has been conducted in what I call the *postulational* paradigm - as opposed to the *constructive* paradigm discussed below. In this approach to belief revision, one provides a set of principles that *any* revision policy should satisfy. For instance, given a belief set T (a *theory*) and a formula φ, the revision of T by φ is written as $T * \varphi$ and a typical postulate of belief revision is that $\varphi \in T * \varphi$, stating that φ is part the belief set $T * \varphi$ obtained by revising T with φ. A postulational approach provides a set of postulates in that spirit and a typical theorem about revision is along the following lines: an operator $*$ is a belief revision operator iff it satisfies every postulate. A set of revision postulates can thus be understood as a set of desiderata that a revision operator - any one (cf. [54]) - should satisfy, but they do not identify a single operator nor do they describe what actions on a model a revision operator performs. DDL follows this tradition by providing a direct translation of the AGM postulates in the object language, using two translation keys (cf. [40]): 1) from $\varphi \in T$ in AGM to $B\varphi$ in DDL and 2) from $\psi \in T * \varphi$ in AGM to $[*\varphi]B\psi$ in

DDL. With this translation manual, for instance, the AGM postulate:

$$T * \varphi \subseteq Cn(T \cup \{\varphi\})$$

becomes the DDL axiom:

$$[*\varphi]B\psi \to B(\varphi \to \psi)$$

Under the *constructive* paradigm, as we have seen in chapter 2, the emphasis is on the actions. Here, one chooses her favorite revision policy, explicitly describes its effects on static models, and then shows how to analyze it via compositional analysis. This paradigm is the one adopted in this thesis and I show in the present Chapter that relational belief revision can be understood in this fashion. Hence, I bring BL under the scope of Order Logic in two steps: 1) I show that its static part is a sublogic of Order Logic and 2) I show that its dynamification can be treated inside the *constructive* paradigm.

Before we proceed, let me say a final word about the result presented in Appendix A. As I have stated above, I have not succeeded in providing a complete system for BL with its two modalities $[\varphi]\psi$ and $[\varphi\rangle\psi$ - one source of difficulty is given in Section 3.2. I have, however, succeeded in proving a completeness result for the *Minimal Relational Logic*. I call this logic minimal in the same way that K is a called a minimal modal logic with respect to $S4$ and $S5$. I have thus succeeded in axiomatizing the minimal logic containing the modalities $[\varphi]\psi$ and $[\varphi\rangle\psi$. This is an interesting result in itself, but more relevant to the field of neighborhood logic, which is tangential to the main thrust of this thesis.

3.1 Doxastic Logic

In this section, I make an excursion in the general doxastic logic defined over preorders. In the next section, I move to a conditional approach, *dynamic doxastic logic*. In subsequent sections, I show that the two approaches are fundamentally similar, via the *representation* Theorem 3.3.14.

It is typical in logic to define beliefs in terms of a plausibility order and say that φ is believed, written '$B\varphi$', if it is true in every *most plausible* state in the order. In Order Logic, one can represent this by interpreting $\preceq$ as a *plausibility* order and say that u is *most plausible* if there is no v such that $u \prec v$. A semantics for belief operators is thus usually given as:

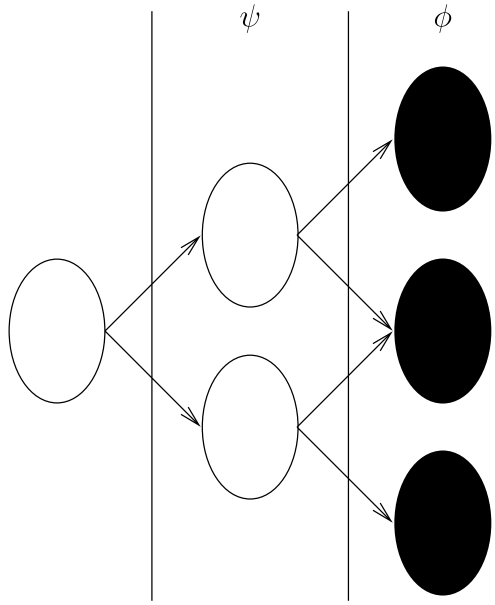

Figure 3.1: Single figure representing both the absolute belief in φ and the conditional belief $[\varphi]\psi$. Most plausible states, those to the right, are all φ and ψ-states.

Definition 3.1.1

$$\mathfrak{M}, u \models B\varphi \quad \text{iff} \quad \mathfrak{M}, v \models \varphi \text{ for all most plausible states } v. \tag{3.1}$$

$\lhd$

Definition 3.1.1 provides a notion of *absolute* belief and is represented in Figure 3.1. The notion of 'absolute belief' is definable in $\mathcal{L}_{\mathcal{O}}$, as the following fact establishes.

Fact 3.1.2 *$B\varphi$ is definable in $\mathcal{L}_{\mathcal{O}}$. That is:*

$$\mathfrak{M}, u \models B\varphi \quad \textit{iff} \quad \mathfrak{M}, u \models U(\Box^{<}\bot \rightarrow \varphi) \tag{3.2}$$

Proof. In the first direction, Assume that $\mathfrak{M}, u \models B\varphi$, then $\mathfrak{M}, v \models \varphi$ for every most plausible v-state. Let w be arbitrary such that $\Box^{<}\bot$, then there is no w' such that $w \prec w'$, i.e., w is a most plausible state. Thus $\mathfrak{M}, w \models \Box^{<}\bot \rightarrow \varphi$. Therefore, $\mathfrak{M}, u \models U(\Box^{<}\bot \rightarrow \varphi)$. In the other direction, Assume that $\mathfrak{M}, u \not\models B\varphi$, then there is a most plausible v-state such that $\mathfrak{M}, v \models \neg\varphi$. But since v is most plausible, $\mathfrak{M}, v \models \Box^{<}\bot$, thus $\mathfrak{M}, v \models \Box^{<}\bot \wedge \neg\varphi$, i.e., $\mathfrak{M}, v \models \neg(\Box^{<}\bot \rightarrow \varphi)$. Therefore, $\mathfrak{M}, u \not\models U(\Box^{<}\bot \rightarrow \varphi)$. QED

More generally though, beliefs are often formalized as conditional statements: ψ

is true in the most plausible φ-states. This is sometimes written in Lewis's notation as '$\varphi \,\Box\!\!\rightarrow \psi$' or as '$B(\psi|\varphi)$', but in this thesis, I use Chellas' notation ([14]) and write '$[\varphi]\psi$' instead. The semantics for the more general conditional belief, again depicted in Figure 3.1, is given by:

Definition 3.1.3

$$\mathfrak{M}, u \models [\varphi]\psi \quad \text{iff} \quad \mathfrak{M}, v \models \psi \text{ for all } \text{ most plausible } \varphi - \text{states } v. \tag{3.3}$$

$\triangleleft$

As was the case with absolute beliefs, conditional beliefs can be defined in $\mathcal{L}_{\mathcal{O}}$:

Fact 3.1.4 *$[\varphi]\psi$ is definable in $\mathcal{L}_{\mathcal{O}}$. That is:*

$$\mathfrak{M}, u \models [\varphi]\psi \quad \textit{iff} \quad \mathfrak{M}, u \models U(\varphi \wedge \neg\Diamond^{<}\varphi \rightarrow \psi) \tag{3.4}$$

Proof. The proof is similar to the proof of Fact 3.1.2. In the first direction, assuming that ψ is true in every most plausible φ-state, it is clear that ψ is true whenever $\varphi \wedge \neg\Diamond^{<}\varphi$ is. In the other direction, if $\mathfrak{M}, u \not\models [\varphi]\psi$, then there is a most plausible φ-state v such that $\mathfrak{M}, v \models \neg\psi$. Thus, $\mathfrak{M}, v \models \varphi \wedge \neg\Diamond^{<}\varphi \wedge \neg\psi$, which implies that $\mathfrak{M}, u \not\models U(\varphi \wedge \neg\Diamond^{<}\varphi \rightarrow \psi)$, as required. QED

In the next section, I consider the conditional approach to doxastic logic for belief revision.

Belief revision and *DDL*

Belief revision is the study of theory change in which a set of formulas is ascribed to an agent as a belief set revisable in the face of new information (cf., [22, 53]). A dominant view in belief revision is the so-called *AGM* paradigm, which describes a functional notion of revision (cf. [1]). A natural semantics in terms of sphere systems (cf. [37]) was given by Grove in [24] and a logical axiomatization was extensively studied by Segerberg (cf. [40]). The resulting logic is called "dynamic doxastic logic" (*DDL*). In this section, I present the outline of the static core of *DDL*. The fragment of *DDL* that I consider here is a simple propositional language augmented with a conditional modality $[\varphi]\psi$.

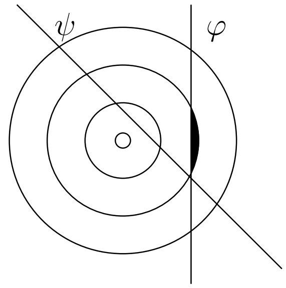

Figure 3.2: Illustration of the semantics for the conditional belief $[\varphi]\psi$.

DDL models are based on what Segerberg calls *onions*. An onion is simply a linearly ordered sphere system that satisfy the limit condition:

Definition 3.1.5 [Onions] Let W be a nonempty set. An *onion* $\mathcal{O} \subseteq \mathcal{P}(W)$ is a linearly ordered set of subsets of W satisfying the following condition (the limit condition): for all $X \subseteq U$:

$$\bigcup \mathcal{O} \cap X \neq \emptyset \Rightarrow \exists Z \in \mathcal{O} \text{ s.t. } \forall Y \in \mathcal{O}(Y \cap X \neq \emptyset \text{ iff } Z \subseteq Y)$$

◁

The limit condition states that every set intersecting an onion intersects a smallest element. Let W be a set of sets, and let $W \bullet X = \{Y \in W : Y \cap X \neq \emptyset\}$. Segerberg uses the more succinct notation '$Z\mu(W \bullet X)$' to express that Z is minimal in W, in the sense that there is no Y in W properly contained in Z. In the case of onions, due to linearity, it is natural to write $Z\mu(\mathcal{O} \bullet X)$. The limit condition can then be written as:

$$\bigcup \mathcal{O} \cap X \neq \emptyset \Rightarrow \exists Z\mu(\mathcal{O} \bullet X).$$

The semantics for conditionals $[\varphi]\psi$, depicted in Figure 3.2, is given by:

Definition 3.1.6

$$\mathfrak{M}, u \models [\varphi]\psi \quad \text{iff} \quad \forall Z\mu(\mathcal{O} \bullet |\varphi|)(Z \cap |\varphi| \subseteq |\psi|) \tag{3.5}$$

◁

An alternative presentation of conditional logic, available since the beginnings of research in this field, is with selection functions ([62]). In my setting, selection functions are illuminating for the axiomatization of the static core of DDL, and for discussion about the generalization sought for later.

Definition 3.1.7 [Selection functions] A function $f : \mathcal{P}(W) \to \mathcal{P}(W)$ is a *selection function* if it satisfies the following conditions, where $X, Y \subseteq W$:

$$\begin{array}{ll} f(X) \subseteq X & \text{(INC)} \\ X \subseteq Y \Rightarrow (f(X) \neq \emptyset \Rightarrow f(Y) \neq \emptyset) & \text{(MON)} \\ X \subseteq Y \Rightarrow (X \cap f(Y) \neq \emptyset \Rightarrow f(X) = X \cap f(Y)) & \text{(ARR)} \end{array}$$

$\lhd$

The third condition is called the *Arrow condition*. The Arrow condition is a source of difficulty in generalizing this setting to the non-linear case.
Let W be a finite set and let F be a selection function on W. Let

$$\begin{array}{rcl} S_0 & = & F(W) \\ S_{n+1} & = & S_n \cup F(W - S_n) \end{array}$$

Since W is finite, there is a smallest m such that $S_{m+1} = S_m$. I leave to the reader to verify that the set $\mathcal{O}_F = \{S_n : n < m\}$ is an onion and that $\mathcal{O}_F$ and F agree.[1] Hence, models for onions may be given in terms of selection functions.

Definition 3.1.8 [Onion selection models] Let W be a set, F a selection function on W and V a valuation on a given set of propositional variables, then the triple $\mathfrak{M} = (W, F, V)$ is an *onion selection model*. $\lhd$

The truth-definition for the modality $[\varphi]\psi$ in onion selection models is given by:

$$\mathfrak{M}, u \models [\varphi]\psi \text{ iff } F(|\varphi|) \subseteq |\psi|. \tag{3.6}$$

The axiomatization of the static core of DDL, or onion logic, builds on the three conditions for selection functions given in Definition 3.1.7.

[1] $\mathcal{O}_F$ and F agree iff

1. $\mathcal{O}_F \cap X \neq \emptyset \Rightarrow FX = X \cap S_k$ for some k.
2. $\mathcal{O}_F \cap X = \emptyset \Rightarrow FX = \emptyset$.

Theorem 3.1.9 *The complete logic for onions consists of the following set of axioms:*

$$\langle\varphi\rangle\psi \equiv \neg[\varphi]\neg\psi \tag{3.7}$$

$$[\varphi](\psi \to \theta) \to ([\varphi]\psi \to [\varphi]\theta) \tag{3.8}$$

$$[\varphi]\varphi \tag{3.9}$$

$$\langle\varphi\rangle\psi \to \langle\psi\rangle\top \tag{3.10}$$

$$\langle\varphi\rangle\psi \to ([\varphi \wedge \psi]\theta \equiv [\varphi](\psi \to \theta)) \tag{3.11}$$

Axioms 3.9, 3.10 and 3.11 are obvious analogues of conditions $(INC), (MON)$ and (ARR) of definition 3.1.7. The total resulting system is Lewis' famous conditional logic VC without an assumption of centrality, provided that we add an assumption of centrality (cf. [37, 44]).

Relational doxastic logic

To get a proper modal logic for relational beliefs, I introduce a further generalization of the conditional presented in Definition 3.1.3. Instead of defining a conditional belief $[\varphi]\psi$ in terms of the (unique) set of minimal φ-states, I define it in terms of multiple sets of minimal φ-states whose members are mutually incomparable. I use two kinds of conditional beliefs, respectively written as '$[\varphi]\psi$' and '$[\varphi\rangle\psi$'. I call the resulting logic 'relational conditional belief' logic. Its language is defined by the following rules:

$$\mathcal{L}_{\mathcal{B}} := p \mid \varphi \wedge \psi \mid \neg\varphi \mid [\varphi]\psi \mid [\varphi\rangle\psi$$

The intended reading of the modalities is: "ψ is true at every state in every set of minimal φ-states" and "ψ is true in at least one state in each sets of minimal φ-states", respectively. In terms of revision by φ, one can think of the modalities as standing for "ψ is believed in every revision by φ" and "ψ is consistent with every revision by φ", respectively.

Models for this logic are based on a generalization of onions, called *broccoli flowers* in [23]. In a relational setting, the limit condition can be generalized in various ways and I consider two options below.

Definition 3.1.10 [Broccoli flowers] Let W be a nonempty set. A *broccoli flower* $\mathcal{B} \subseteq \mathcal{P}(W)$ is a set of subsets satisfying some *generalized limit condition* - to be specified below. $\lhd$

There are (at least) two ways to specify the generalized limit condition of Definition 3.1.10. I present two obvious candidates. Let $\mathcal{B}|X = \{Y \cap X : Y \in \mathcal{B}\}$. For all $X \subseteq W$, if $\bigcup \mathcal{B} \cap X \neq \emptyset$, either:

$$\exists S \subseteq \mathcal{B}, \forall Y \in \mathcal{B}(Y \cap X \neq \emptyset \Rightarrow \exists Z \in S(Z\mu(\mathcal{B} \bullet X) \wedge Z \subseteq Y)) \tag{3.12}$$
$$\exists S \subseteq \mathcal{B}, \forall Y \in \mathcal{B}(Y \cap X \neq \emptyset \Rightarrow \exists Z \in S((Z \cap X)\mu((\mathcal{B}|X) \bullet X) \wedge Z \subseteq Y)). \tag{3.13}$$

Intuitively, a generalized limit condition states that every set intersecting a broccoli flower intersects every members of a set S of smallest elements of the flower. In the first case, the members of S are minimal sets of the broccoli flower that have a non-empty intersection with X. In the second case, the members of S have a minimal intersection with X. In the remainder of this chapter, I work with 3.12.

Definition 3.1.11 [Broccoli models] $\mathfrak{M} = (W, \{\mathcal{B}_u\}_{u \in W}, V)$ is a *broccoli model* if W is a set of worlds, $\{\mathcal{B}_u\}_{u \in W}$ is a family of broccoli flowers for each world $u \in W$ satisfying 3.12, and V is a valuation assigning sets of worlds to propositions. ◁

In what follows, I suppress the index u.

Definition 3.1.12 [Broccoli semantics] I say that φ is true at world u in a broccoli model $\mathfrak{M}$, written $\mathfrak{M}, u \vDash \varphi$ iff (taking standard truth definitions for the propositional and the Boolean cases):

$$\mathfrak{M}, u \vDash [\varphi\rangle\!]\psi \quad \text{iff} \quad \forall Z\mu(\mathcal{B} \bullet |\varphi|)(Z \cap |\varphi| \subseteq |\psi|) \tag{3.14}$$
$$\mathfrak{M}, u \vDash [\varphi\rangle\psi \quad \text{iff} \quad \forall Z\mu(\mathcal{B} \bullet |\varphi|)(Z \cap |\varphi| \cap |\psi| \neq \emptyset) \tag{3.15}$$

Here, as usual, $|\varphi| = \{u : \mathfrak{M}, u \vDash \varphi\}$, the *associated proposition* to φ.

◁

Figure 3.3 illustrates the semantics of both operators. In the left figure, all minimal φ-sets are contained in $|\psi|$, and $|\psi|$ intersects each minimal φ-set in the right figure.

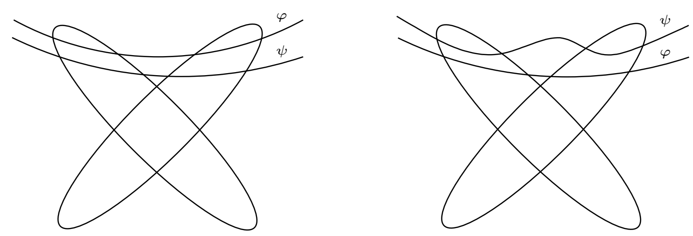

Figure 3.3: Broccoli semantics of the operators $[\varphi]\psi$ and $[\varphi\rangle\psi$.

3.2 Generalized selection functions

As we have seen above, the semantics for Onions can be given in two different ways, in a sphere system representation or with selection function. In this section, I outline difficulties in providing a semantics for BL with a generalized notion of selection functions. I show what properties generalized selection function should satisfy in broccoli models, and I point to a difficulty of the generalization, namely to find an appropriate Arrow condition for broccoli models.

Consider the issue of generalizing the format of selection functions for onions to a non-linear setting. (INC) and (MON) are easily generalized in BL to the following conditions, for all $X, Y \subseteq U$:

$$Y \in F(X) \Rightarrow Y \subseteq X \qquad \text{(INC}^*\text{)}$$
$$Y \subseteq X \text{ and } \exists Z \in F(Y) \text{ s.t. } Z \neq \emptyset \Rightarrow \exists Z \in F(X) \text{ s.t. } Z \neq \emptyset) \qquad \text{(MON}^*\text{)}$$

with the identical corresponding axioms 3.9 and 3.10. On the one hand, if $\neg\langle\varphi]\top \in u$ for some world $u \in U$ (i.e. if there is no revision by φ) then $[\varphi]\varphi \in u$ by Axiom A.7. But if there is no revision by φ, then $F(X)$ is empty, and (INC^*) holds vacuously. On the other hand, if there is a revision by φ, then 3.9 and (INC^*) express the same thing, namely that members of $F(|\varphi|)$ are contained in $|\varphi|$. Similar considerations will convince the reader that 3.10 and (MON^*) go together.

A difficulty arises when attempting to generalize condition (ARR) in a similar fashion, as the condition seems to require linearity.[2] One way to see this is by looking at the failure of axiom 3.11 in broccoli models. Only one half of 3.11 can be kept in BL, viz. $\langle\varphi\rangle\psi \rightarrow ([\varphi \wedge \psi]\theta \rightarrow [\varphi](\psi \rightarrow \theta))$. The other half makes a crucial appeal to linearity, as may be seen from the counter-model of figure 3.4. It is an open question to find an appropriate generalization of (ARR) that yields a generalized

[2]The exact relationship between the Arrow condition and linearity is still an open question.

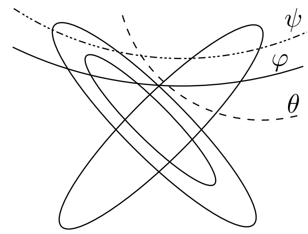

Figure 3.4: Counter-model to $\langle\varphi\rangle\psi \rightarrow ([\varphi](\psi \rightarrow \theta) \rightarrow [\varphi \wedge \psi]\theta)$

selection function for BL. This promises to be a difficult task. But instead of pursuing this enterprise further, I show in the remainder of this chapter that Broccoli Logic can be treated inside Order Logic, thus facilitating the quest for a dynamic system appropriate for relational doxastic logic.

3.3 Broccoli logic and Order Logic

In this section, I show that static BL is a fragment of Order Logic. Firstly, assuming models to be finite[3], I show that the broccoli operators $[\varphi]\psi$ and $[\varphi\rangle\psi$ can be translated in $\mathcal{L}_{\mathcal{O}}$. Secondly, I show that the $[\varphi]\psi$ fragment of BL, which I call BL^- is identical to *Minimal Conditional Logic* (MCL). This latter result show exactly which fragment of Order Logic BL^- is.

In the remainder of this chapter, I go back and forth between Order Models and Broccoli Models. For this, I appeal to the following definitions. Once again, assuming models to be finite.

Definition 3.3.1 Let $\mathcal{B}$ be a broccoli model. An *induced order model* $\mathfrak{M}^{\mathcal{B}}$ is given by $\mathfrak{M}^{\mathcal{B}} = \langle W, \leq, \preceq\rangle$, where $\leq$ is such that 1) $\forall x \in W, x \preceq x$ and 2) $\forall x, y \in W$ and $X, Y \in \mathcal{B}$, $X \subset Y, x \in X$ and $y \in Y - X$ implies that $y \preceq x$. $\lhd$

An induced an order model from a broccoli model is pictured in Figure 3.5.

Definition 3.3.2 Let $\mathfrak{M}$ be an order model. Let $C(x) = \{y \in W : x \preceq y\}$, then $BROC(\mathfrak{M}) = \{C(x) : x \in W\}$ is the *Induced Broccoli Model*. $\lhd$

Figure 3.6 shows how to get a broccoli model from an order model.

[3] I make this assumption to avoid complications with the limit condition. Furthermore, one may argue that the intuitions for BL are better understood in the finite case. For a good discussion of finite models vs infinite models with the limit assumption, see [38]. Notice also that this issue does not arise in Order Logic, giving yet another motivation to work with it.

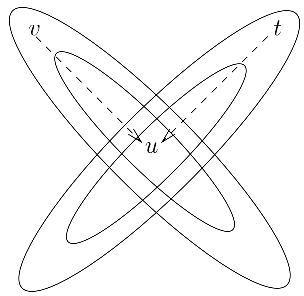

Figure 3.5: Induced order model (dotted lines) from a finite broccoli model (arrows).

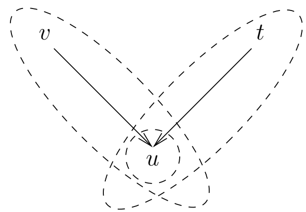

Figure 3.6: Induced broccoli model (dotted lines) from a finite order model (arrows).

Since models are finite, a generalized limit condition for $BROC(\mathfrak{M})$ obtains for free. Therefore, the semantics for the broccoli operator makes sense in this context:

$$BROC(\mathfrak{M}), u \vDash [\varphi]\psi \quad \text{iff} \quad \forall Z\mu(BROC(u) \bullet |\varphi|)(Z \cap |\varphi| \subseteq |\psi|) \tag{3.16}$$

$$BROC(\mathfrak{M}), u \vDash [\varphi\rangle\psi \quad \text{iff} \quad \forall Z\mu(BROC(u) \bullet |\varphi|)(Z \cap |\varphi| \cap |\psi| \neq \emptyset) \tag{3.17}$$

Looking at Figures 3.5 and 3.6 make it clear that the classes of finite broccoli and order models are the same. Thus, it makes sense to compare different languages over them. In the next theorem, I show that the modalities $[\varphi]\psi$ and $[\varphi\rangle\psi$ can be expressed in $\mathcal{L}_{\mathcal{O}}$.

Theorem 3.3.3 *Let $\mathcal{B}$ be a Broccoli model, then*

$$\mathcal{B}, u \models [\varphi]\psi \quad \Leftrightarrow \quad \mathfrak{M}^{\mathcal{B}}, u \models U(\varphi \to \Diamond^{\leq}((\varphi \wedge \Box^{<}\neg\varphi) \wedge \Box^{\preceq}(\varphi \to \psi))) \tag{3.18}$$

$$\mathcal{B}, u \models [\varphi\rangle\psi \quad \Leftrightarrow \quad \mathfrak{M}^{\mathcal{B}}, u \models U(\varphi \to \Diamond^{\leq}(\varphi \wedge \psi \wedge \Box^{<}\neg\varphi)) \tag{3.19}$$

Proof. First, equation 3.18. In the first direction, assume that $\mathfrak{M}, u \models [\varphi]\psi$, then $\forall Z\mu(\mathcal{B} \bullet |\varphi|)(Z \cap |\varphi| \subseteq |\psi|)$. Let v be a state such that $\mathfrak{M}, v \models \varphi$. Then, $v \in Y$ for some $Y \in \mathcal{B}$. There are two cases to consider: 1) Y is minimal in $\mathcal{B}$ 2) y is not minimal in $\mathcal{B}$. In the first case, $\mathfrak{M}^{\mathcal{B}}, v \models \varphi \wedge \Box^{<}\neg\varphi$, and $Z \cap |\varphi| \subseteq |\psi|$

implies that $\mathfrak{M}^{\mathcal{B}}, v \models \Box^{\preceq}(\varphi \rightarrow \psi)$. In the second case, by the generalized limit condition, $\exists Z \in S(Z\mu(\mathcal{B} \bullet |\varphi|) \wedge Z \subset Y))$. By the induced relation $\preceq$ (Definition 3.3.1), $\forall w \in Z \cap |\varphi|, v \preceq w$. Take any state w in $Z \cap |\varphi|$, then $\mathfrak{M}^{\mathcal{B}}, w \models \varphi \wedge \Box^{<}\neg\varphi$ and $\forall w' \preceq w \preceq w' \in Z \cap |\varphi|, \mathfrak{M}, w' \models \varphi \wedge \psi$. Therefore, $\mathfrak{M}, u \models U(\varphi \rightarrow \Diamond^{\preceq}(\varphi \wedge \Box^{\preceq}(\varphi \rightarrow \psi)))$.

In the second direction, assume that $\mathfrak{M}^{\mathcal{B}}, u \models U(\varphi \rightarrow \Diamond^{\preceq}(\varphi \wedge \Box^{\preceq}(\varphi \rightarrow \psi)))$, then $\forall v : \mathfrak{M}^{\mathcal{B}}, v \models \varphi \Rightarrow \exists v \preceq w : \mathfrak{M}^{\mathcal{B}}, w \models \varphi \wedge \Box^{<}\neg\varphi \,\&\, \mathfrak{M}^{\mathcal{B}}, w \models \Box^{\preceq}(\varphi \rightarrow \psi)$. Consider such a w and let $Z \in \mathcal{B}$ be such that $\{v : v \preceq w \preceq v \;\&\, \mathfrak{M}^{\mathcal{B}}, v \models \varphi\} \cap Z \neq \emptyset$. By the generalized limit condition, there is a $Z' \subseteq Z$ minimal in $\mathcal{B}$ such that $\{v : v \preceq w \preceq v \;\&\, \mathfrak{M}^{\mathcal{B}}, v \models \varphi\} \cap Z' \neq \emptyset$. Then $Z' \cap |\varphi| \subseteq |\psi|$, since $\mathfrak{M}^{\mathcal{B}}, w \models \Box^{\preceq}(\varphi \rightarrow \psi)$. Therefore, by the broccoli truth-definition, $\mathcal{B}, u \models [\varphi]\psi$, as needed.

A similar argument establishes the second equation, as can be seen by realizing that the right-hand-side of 3.19 states that for every φ-state, there is a minimal φ-state that is also a ψ-state. The detail of this proof are left to the reader. QED

Theorem 3.3.3 shows that BL can be treated inside Order Logic. There are many further mathematical question that could be treated here, especially with respect to the limit condition over infinite models, but I do not treat them here. Instead, I show that more can be achieved with respect to the $[\varphi]\psi$ fragment of BL, by showing that it is identical to MCL.

Minimal Conditional Logic

Minimal conditional logic (MCL) was studied by Stalnaker, Pollock, Burgess and Veltman to capture the idea that a conditional $\varphi \Rightarrow \psi$ is true if an only if the conjunction $\varphi \wedge \neg\psi$ is less possible than the conjunction $\varphi \wedge \psi$, and no more. Their modeling comes with a reflexive and transitive $\preceq$-order for each world x and no spheres need occur. In a sphere system, two worlds lying on the same sphere agree on which worlds are farther away and which are closer. This assumption is dropped in MCL: if two worlds x and y are equally far away in the underlying order from world u and if some world z is farther away than y, then no conclusions may be drawn as to whether z is farther from u than x - or vice versa. Instead of changing the onion picture by allowing non-linearly ordered sphere system as in BL, MCL ignores spheres altogether. In this section, I show that the logic of $[\varphi]\psi$ under the minimal conditional or the broccoli interpretation is the same, i.e., that the $[\varphi]\psi$ fragment of $\mathcal{L}_{\mathcal{B}}$ is the same as minimal conditional logic.

Minimal Conditional Logic

Definition 3.3.4 A *Minimal conditional logic model* is a triple $\langle U, \{\leq_x\}_{x \in V}\}, V\rangle$, where U and V are as above, and $\leq_x$ is a preorder for each $x \in U$. $\lhd$

The relation $y \leq_x z$ may be read as "according to world x, world y is no farther away than world z". Let $W_u = \{y : \exists z, y \preceq_u z\}$ be the *zone of entertainability* for world $u \in U$. Intuitively, worlds outside the zone of entertainability for u are worlds so far away that their distance from any given world is not evaluable. The *minimal conditional logic language* contains a set of propositional variables, together with negation $\neg$, disjunction $\vee$ and a counterfactual modality $[\varphi]$ for every formula φ.

Definition 3.3.5 $[MCL$ semantics$]$ A formula $[\varphi]\psi$ is true at world u in a model $\mathfrak{M}$, written $\mathfrak{M}, u \vDash [\varphi]\psi$, iff:

$$\forall y, \mathfrak{M}, y \models \varphi \Rightarrow (\exists z \preceq_u y : \mathfrak{M}, z \models \varphi \,\&\, \forall w \leq_u z, (\mathfrak{M}, w \models \varphi \Rightarrow \mathfrak{M}, w \models \psi)) \quad (3.20)$$

$\lhd$

From inspection of the truth-condition, the following fact is immediate, giving a first hint at the main result, Theorem 3.3.14, of this section:

Fact 3.3.6 *The modality $[\varphi]\psi$ of MCL is definable in $\mathcal{L}_{\mathcal{O}}$ by:*

$$[\varphi]\psi \Leftrightarrow U(\varphi \rightarrow \Diamond^{\leq}(\varphi \wedge \Box^{\leq}(\varphi \rightarrow \psi))) \quad (3.21)$$

Notice that the semantic definition of $[\varphi]\psi$ does not contain a minimality condition. If models are finite, however, then there is a minimal set of worlds $z \in U$ such that $z \in (V(\varphi) \cap V(\psi))$. In this case, Definition 3.3.5 becomes:

Definition 3.3.7 $[MCL$ semantics$]$ A formula $[\varphi]\psi$ is true at world u in the model $\mathfrak{M}$, written $\mathfrak{M}, u \vDash [\varphi]\psi$, iff:

$$\forall y, \mathfrak{M}, u \models \varphi \Rightarrow (\exists X : \forall z \in X, z \leq_u y \,\&\, \mathfrak{M}, z \models \varphi \wedge \psi \,\&\, \forall w <_u z(\mathfrak{M}, w \not\models \varphi))(3.22)$$

$\lhd$

Figure 3.7 depicts a simple model satisfying $[\varphi]\psi$. There are two minimal φ-worlds, z and z', and ψ is true at both worlds. Hence, ψ is true at every minimal φ-world.

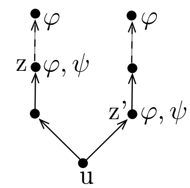

Figure 3.7: Simple model such that $[\varphi]\psi$ is true at world u. The dotted arrows stand for sequences of $\leq$-related worlds.

Axiomatization

Theorem 3.3.8 (Burgess [10]) *The following set of axioms, with the same set of rules as for minimal relational logic presented in section A.1, is complete for MCL:*

$$[\varphi]\varphi \tag{3.23}$$

$$[\varphi]\psi \wedge [\varphi]\theta \to [\varphi](\psi \wedge \theta) \tag{3.24}$$

$$[\varphi](\psi \wedge \theta) \to [\varphi]\psi \tag{3.25}$$

$$[\varphi]\psi \wedge [\varphi]\theta \to [\varphi \wedge \psi]\theta \tag{3.26}$$

$$[\varphi]\psi \wedge [\theta]\psi \to [\varphi \vee \theta]\psi \tag{3.27}$$

Here are some examples of derivable theses.

Example 3.3.9 $MCL \vdash [\varphi\psi \wedge [\varphi \wedge \psi]\theta \to [\varphi]\theta$

Proof. Assume 1) $\vdash [\varphi]\psi$ and 2) $\vdash [\varphi \wedge \psi]\theta$. By Axiom 3.23 $\vdash [\varphi \wedge \neg\psi](\varphi \wedge \neg\psi)$ and by Axiom 3.25, $\vdash [\varphi \wedge \neg\psi]\neg\psi$. Hence, by monotonicity in the consequent (3.25 again), $\vdash [\varphi \wedge \neg\psi](\neg\psi \vee \theta)$. Now, from assumption 2) and Axiom 3.25, $\vdash [\varphi \wedge \psi](\neg\psi \vee \theta)$. Combining the latter two results, $\vdash [\varphi](\neg\psi \vee \theta)$. But since $\vdash [\varphi]\psi$ by assumption (1), $\vdash [\varphi]\theta$, as desired. QED

Example 3.3.10 $MCL \vdash \langle\varphi\rangle\psi \to \langle\psi\rangle T$

Proof. I prove the contrapositive. Assume that $\vdash [\psi]\bot$. Then both $\vdash [\psi]\neg\psi$ and $\vdash [\psi]\varphi$. Hence, by Axiom 3.26, $\vdash [\psi \wedge \varphi]\neg\psi$. But $\vdash [\neg\psi \wedge \varphi](\neg\psi \wedge \varphi)$ is an instance of Axiom 3.23 and by Axiom 3.25, $\vdash [\neg\psi \wedge \varphi]\neg\psi$. Therefore, $\vdash [\varphi]\neg\psi$. QED

Example 3.3.11 $MCL \vdash [\varphi \wedge \psi]\theta \to [\varphi](\psi \to \theta)$.

Proof. Assume $\vdash [\varphi \wedge \psi]\theta$. By monotonicity, $\vdash [\varphi \wedge \psi](\neg\psi \vee \theta)$. But $\vdash [\varphi \wedge \neg\psi](\neg\psi \vee \theta)$. Therefore, $\vdash [\varphi](\neg\psi \vee \theta)$, i.e., $\vdash [\varphi](\psi \to \theta)$. QED

As can be seen from Axiom 3.23 and examples 3.3.10 and 3.3.11, 3.9, 3.10 and one direction of 3.11 of section 3.2 are derivable in MCL. Thus, MCL has the properties sought for in BL, and I show that it has *all* the properties of BL. The general reason behind these considerations becomes clear in the next subsection.

$BL^- = MCL$

A finite MCL model $\mathfrak{M} = \langle U, \leq, V \rangle$ can be transformed into a broccoli model by constructing a broccoli flower at each world of $\mathfrak{M}$. This is made precise in the following definition, a generalization of Definition 3.3.2.

Definition 3.3.12 Let $\mathfrak{M}$ be an minimal conditional model. Let $C_x(y) = \{z \in W : y \preceq_x z\}$, then $BROC(x) = \{C(y) : y \in W_x\}$ is the *Induced Broccoli flower* at x. Finally, an *induced Broccoli Model* $BROC(\mathfrak{M})$ is given by:

$$BROC(\mathfrak{M}) = \{BROC(x) : x \in W\} \tag{3.28}$$

$\triangleleft$

The main result of this section now follows from Lemma 3.3.13.

Lemma 3.3.13 $\mathfrak{M}, x \vDash [\varphi]\psi$ *iff* $BROC(\mathfrak{M}), x \vDash [\varphi]\psi$.

Proof. In the one direction, assume that $\mathfrak{M}, x \vDash [\varphi]\psi$. To simplify notation, I write C_w instead of $C_x(w)$. Let $C_w\mu(BROC(x) \bullet |\varphi|)$, and let $v \in C_w \cap |\varphi|$. By the truth definition for $[\varphi]\psi$, $\exists z \leq_x v$ such that $\mathfrak{M}, z \vDash \varphi$ and $\forall y \leq_x z, \mathfrak{M}, y \models \varphi \Rightarrow \mathfrak{M}, y \models \psi$. Now, if $v \not\leq_x z$, then $z < v$, which implies that $C_z \subset C_v \subseteq C_w$ (the latter inclusion uses the transitivity of $\leq_x$), contradicting the minimality of C_w. Thus, $v \in |\psi|$, which implies that $C_w \cap |\varphi| \subseteq |\psi|$. Therefore, since v was chosen arbitrarily, $BROC(\mathfrak{M}), x \vDash [\varphi]\psi$.

In the other direction, assume that $BROC(\mathfrak{M}), x \vDash [\varphi]\psi$ and suppose that $\mathfrak{M}, y \vDash \varphi$ for some $y \in U$. Then $C_y \cap |\varphi| \neq \emptyset$. Hence, $\exists C_w \subseteq C_y$ such that $C_w\mu(BROC(x) \bullet |\varphi|)$, since $\mathfrak{M}$ is finite, and $C_w \cap |\varphi| \subseteq |\psi|$. But since $C_w \subseteq C_y, w \leq_x y$. Assume that w is not a minimal world satisfying $\varphi \wedge \psi$ with respect to $\leq_x$, then $\exists w' <_x w$ such that $\mathfrak{M}, w' \vDash \varphi \wedge \psi$. This implies that $C'_w \subset C_w$ and $C'_w \cap |\varphi| \cap |\psi| \neq \emptyset$, contradicting the minimality of C_w. Therefore, w is a minimal world satisfying $\varphi \wedge \psi$ and since $w \leq_x y$, we get that $\mathfrak{M}, x \models [\varphi]\psi$. QED

Now for the main theorem:

Theorem 3.3.14 $BL^- = MCL$.

Proof. To show that MCL is BL, I show 1) that all axioms of Section 3.3 are valid in BL, whose semantics were given in section 3.1 and 2) that if a principle is not derivable in MCL, then there is a broccoli countermodel.

Showing that the MCL axioms are valid in the BL-models of Section 3.1 is straightforward. I show that Axiom 3.26 is valid and leave the others to the reader. Let $\mathfrak{M}$ be an arbitrary broccoli model and let $u \in U$ be arbitrary. If $\neg\langle\varphi]\top \notin u$, i.e., if there is no revision by φ, then the thesis is vacuously true. Hence, assume that there is a revision by φ. Assume furthermore that $\mathfrak{M}, u \models [\varphi]\psi \wedge [\varphi]\theta$. Since $\mathfrak{M}, u \models [\varphi]\psi, |\varphi| \cap |\psi| \neq \emptyset$. Let $Z\mu(\mathcal{B} \bullet |\varphi \wedge \psi|)$ be a minimal set of $\mathcal{B}$ intersecting $|\varphi \wedge \psi|$. Then for every $z \in Z, x \in |\varphi| \cap |\psi|$ implies that $z \in |\varphi| \subseteq |\theta|$. Hence, $\mathfrak{M}, u \models [\varphi \wedge \psi]\theta$.

To show that if a principle is not provable in MCL, then there is a broccoli countermodel to φ, I use the completeness result of Burgess. If $MCL \nvdash \varphi$ for some φ, then there is a finite model $\mathfrak{M} = (U, \leq, V)$ and a world $u \in U$ such that $\mathfrak{M}, u \not\models \varphi$. [4] By Lemma 3.3.13, $BROC(\mathfrak{M}), u \not\models \varphi$. Therefore, $BROC(\mathfrak{M})$ is a broccoli countermodel to φ. This completes the proof of Theorem 3.3.14. QED

Corollary 3.3.15 *BL is decidable.*

As was noticed above, Theorem 3.3.14 shows that the logic of $[\varphi]\psi$ under the minimal conditional or the broccoli interpretation is the same, and so are their axiomatizations. Thus, Theorem 3.3.14 yields a completeness theorem for the $[\varphi]\psi$ fragment of broccoli logic. The axiomatization of the full language $\mathcal{L}_{\mathcal{B}}$ over broccoli models, or over MCL models, is still an open question. Nevertheless, Theorem 3.3.3 shows that the full logic is still a fragment of Order Logic and I rely on this fact to introduce dynamics in the next section.

[4] Burgess proves that MCL has the finite model property.

3.4 Dynamics

In this section, I show how to incorporate lexicographic upgrade in the full *static* broccoli language $\mathcal{L}_{\mathcal{B}}$.[5] To make this work properly, $\mathcal{L}_{\mathcal{B}}$ has to be expanded with the existential modality $E\varphi$, and of course with a lexicographic upgrade modality $\langle \Uparrow A\rangle\varphi$. The resulting language, called $\mathcal{L}_{\mathcal{B}+}$, is defined by:

$$\mathcal{L}_{\mathcal{B}+} := p \mid \varphi \wedge \psi \mid \neg\varphi \mid [\varphi]\psi \mid [\varphi\rangle\psi \mid E\varphi \mid \langle \Uparrow A\rangle\varphi$$

Theorem 3.4.1 *The complete logic of relational doxastic logic is axiomatized by 1) some complete static relational doxastic logic*[6] *and 2) the following reduction axioms:*

$$\begin{aligned}
\langle \Uparrow A\rangle[\varphi]\psi \quad &\leftrightarrow \quad E(A \wedge \langle \Uparrow A\rangle\varphi) \wedge [A \wedge \langle \Uparrow A\rangle\varphi]\langle \Uparrow A\rangle\psi \\
&\vee \quad \neg E(A \wedge \langle \Uparrow A\rangle\varphi) \wedge [\langle \Uparrow A\rangle\varphi]\langle \Uparrow A\rangle\psi \qquad (3.29)\\
\langle \Uparrow A\rangle[\varphi\rangle\psi \quad &\leftrightarrow \quad E(A \wedge \langle \Uparrow A\rangle\varphi) \wedge [A \wedge \langle \Uparrow A\rangle\varphi\rangle\langle \Uparrow A\rangle\psi \\
&\vee \quad \neg E(A \wedge \langle \Uparrow A\rangle\varphi) \wedge [\langle \Uparrow A\rangle\varphi\rangle\langle \Uparrow A\rangle\psi \qquad (3.30)
\end{aligned}$$

Proof. It is enough to show the soundness of 3.29 and 3.30.

For the first direction of 3.29, assume that $\mathfrak{M}, u \models \langle \Uparrow A\rangle[\varphi]\psi$, then $\mathfrak{M}_{\Uparrow A}, u \models [\varphi]\psi$. By the truth-definition 3.1.12, $\forall Z\mu(\mathcal{B}_{\Uparrow A} \bullet |\varphi|)(Z \cap |\varphi| \subseteq |\psi|)$. Now, either $\exists w' \in \mathfrak{M}_{\Uparrow A}$ such that $\mathfrak{M}_{\Uparrow A}, w' \models A \wedge \varphi$ or not. In the first case, when $\mathfrak{M}, u \models E(A \wedge \langle \Uparrow A\rangle\varphi)$, because of the lexicographic upgrade, it must be that every $Z\mu(\mathcal{B}_{\Uparrow A} \bullet |\varphi|)$ is such that $(Z \cap |\varphi| \subseteq |A \wedge \psi|)$, so $\forall Z\mu(\mathcal{B} \bullet |A \wedge \langle \Uparrow A\rangle\varphi|)(Z \cap |A \wedge \langle \Uparrow A\rangle\varphi| \subseteq |A \wedge \langle \Uparrow A\rangle\psi|)$. Therefore, $\mathfrak{M}, u \models [A \wedge \langle \Uparrow A\rangle\varphi]\langle \Uparrow A\rangle\psi$. In the second case, when $\mathfrak{M}, u \models \neg E(A \wedge \langle \Uparrow A\rangle\varphi)$, the minimal z-states satisfying φ in $\mathfrak{M}_{\Uparrow A}$ are the same as in $\mathfrak{M}$, and they satisfy ψ after the upgrade, so $\mathfrak{M}, u \models [\langle \Uparrow A\rangle\varphi]\langle \Uparrow A\rangle\psi$, as needed.

In the other direction, consider first the case where $\mathfrak{M}, u \models E(A \wedge \langle \Uparrow A\rangle\varphi)$. This says that there exists a state v that becomes a φ-state after the upgrade by A. Now, $\mathfrak{M}, u \models [A \wedge \langle \Uparrow A\rangle\varphi]\langle \Uparrow A\rangle\psi$ implies that $\forall Z\mu(\mathcal{B} \bullet |A \wedge \langle \Uparrow A\rangle\varphi|)(Z \cap |A \wedge \langle \Uparrow A\rangle\varphi| \subseteq |\langle \Uparrow A\rangle\psi|)$. Hence, $\forall Z\mu(\mathcal{B}_{\Uparrow A} \bullet |A \wedge \varphi|)(Z \cap |A \wedge \varphi| \subseteq |\psi|)$. Now, because of the upgrade of A-states in $\mathcal{B}$, the sets Z minimal in $\mathcal{B}_{\Uparrow A} \bullet |A \wedge \varphi|$ are the same as those minimal

[5] Notice that [67] takes a more general standpoint on dynamics for belief revision, of which lexicographic upgrade is but one instance. I choose to work with the lexicographic upgrade as it is simple in character and makes a clear case for the constructive approach.

[6] This is still an open question, although a complete axiomatization for the $[\varphi]\psi$ fragment has been obtained in Theorem 3.3.14.

in $\mathcal{B}_{\Uparrow A} \bullet |\varphi|$. Thus, $\forall Z\mu(\mathcal{B}_{\Uparrow A} \bullet |\varphi|)(Z \cap |\varphi| \subseteq |\psi|)$, which implies that $\mathfrak{M}_{\Uparrow A}, u \models [\varphi]\psi$. Therefore, $\mathfrak{M}, u \models \langle\Uparrow A\rangle[\varphi]\psi$. Finally, assume that $\mathfrak{M}, u \models \neg E(A \wedge \langle\Uparrow A\rangle\varphi)$. This says that after upgrading A, there is no state v such that $\mathfrak{M}, v \models A \wedge \varphi$. Hence, the sets Z minimal in $\mathcal{B}_{\Uparrow A} \bullet |\varphi|$ are the same as those in $\mathcal{B} \bullet |\langle\Uparrow A\rangle\varphi|$. Now, $\mathfrak{M}, u \models [\langle\Uparrow A\rangle\varphi]\langle\Uparrow A\rangle\psi$ implies that $\forall Z\mu(\mathcal{B} \bullet |\langle\Uparrow A\rangle\varphi|)(Z \cap |\langle\Uparrow A\rangle\varphi| \subseteq |\langle\Uparrow A\rangle\psi|)$. Hence, $\forall Z\mu(\mathcal{B}_{\Uparrow A} \bullet |\varphi|)(Z \cap |\varphi| \subseteq |\psi|)$, which implies that $\mathfrak{M}_{\Uparrow A}, u \models [\varphi]\psi$. Therefore, $\mathfrak{M}, u \models \langle\Uparrow A\rangle[\varphi]\psi$, and this completes the proof.

Now, for the first direction of 3.30, assume that $\mathfrak{M}, u \models \langle\Uparrow A\rangle[\varphi\rangle\psi$, then $\mathfrak{M}_{\Uparrow A}, u \models [\varphi\rangle\psi$. By the truth-definition 3.1.12, $\forall Z\mu(\mathcal{B}_{\Uparrow A} \bullet |\varphi|)(Z \cap |\varphi| \cap |\psi| \neq \emptyset)$. Now, either $\exists w' \in \mathfrak{M}_{\Uparrow A}$ such that $\mathfrak{M}_{\Uparrow A}, w' \models A \wedge \varphi$ or not. In the first case, when $\mathfrak{M}, u \models E(A \wedge \langle\Uparrow A\rangle\varphi)$, because of the lexicographic upgrade, it must be that every $Z\mu(\mathcal{B}_{\Uparrow A} \bullet |\varphi|)$ is such that $(Z \cap |\varphi| \cap |A \wedge \psi| \neq \emptyset)$, so $\forall Z\mu(\mathcal{B} \bullet |A \wedge \langle\Uparrow A\rangle\varphi|)(Z \cap |A \wedge \langle\Uparrow A\rangle\varphi| \cap |A \wedge \langle\Uparrow A\rangle\psi| \neq \emptyset)$. Therefore, $\mathfrak{M}, u \models [A \wedge \langle\Uparrow A\rangle\varphi\rangle\langle\Uparrow A\rangle\psi$. In the second case, when $\mathfrak{M}, u \models \neg E(A \wedge \langle\Uparrow A\rangle\varphi)$, the minimal z-states satisfying φ in $\mathfrak{M}_{\Uparrow A}$ are the same as in $\mathfrak{M}$, and for every set of minimal states among them, at least one of them satisfy ψ after the upgrade, so $\mathfrak{M}, u \models [\langle\Uparrow A\rangle\varphi\rangle\langle\Uparrow A\rangle\psi$, as needed.

In the other direction, consider first the case where $\mathfrak{M}, u \models E(A \wedge \langle\Uparrow A\rangle\varphi)$. This says that there exists a state v that becomes a φ-state after the upgrade by A. Now, $\mathfrak{M}, u \models [A \wedge \langle\Uparrow A\rangle\varphi\rangle\langle\Uparrow A\rangle\psi$ implies that $\forall Z\mu(\mathcal{B} \bullet |A \wedge \langle\Uparrow A\rangle\varphi|)(Z \cap |A \wedge \langle\Uparrow A\rangle\varphi| \cap |\langle\Uparrow A\rangle\psi| \neq \emptyset)$. Hence, $\forall Z\mu(\mathcal{B}_{\Uparrow A} \bullet |A \wedge \varphi|)(Z \cap |A \wedge \varphi| \cap |\psi| \neq \emptyset)$. Now, because of the upgrade of A-states in $\mathcal{B}$, the sets Z minimal in $\mathcal{B}_{\Uparrow A} \bullet |A \wedge \varphi|$ are the same as those minimal in $\mathcal{B}_{\Uparrow A} \bullet |\varphi|$. Thus, $\forall Z\mu(\mathcal{B}_{\Uparrow A} \bullet |\varphi|)(Z \cap |\varphi| \cap |\psi| \neq \emptyset)$, which implies that $\mathfrak{M}_{\Uparrow A}, u \models [\varphi\rangle\psi$. Therefore, $\mathfrak{M}, u \models \langle\Uparrow A\rangle[\varphi\rangle\psi$. Finally, assume that $\mathfrak{M}, u \models \neg E(A \wedge \langle\Uparrow A\rangle\varphi)$. This says that after upgrading A, there is no state v such that $\mathfrak{M}, v \models A \wedge \varphi$. Hence, the sets Z minimal in $\mathcal{B}_{\Uparrow A} \bullet |\varphi|$ are the same as those in $\mathcal{B} \bullet |\langle\Uparrow A\rangle\varphi|$. Now, $\mathfrak{M}, u \models [\langle\Uparrow A\rangle\varphi]\langle\Uparrow A\rangle\psi$ implies that $\forall Z\mu(\mathcal{B} \bullet |\langle\Uparrow A\rangle\varphi|)(Z \cap |\langle\Uparrow A\rangle\varphi| \cap |\langle\Uparrow A\rangle\psi| \neq \emptyset)$. Hence, $\forall Z\mu(\mathcal{B}_{\Uparrow A} \bullet |\varphi|)(Z \cap |\varphi| \cap |\psi| \neq \emptyset)$, which implies that $\mathfrak{M}_{\Uparrow A}, u \models [\varphi\rangle\psi$. Therefore, $\mathfrak{M}, u \models \langle\Uparrow A\rangle[\varphi\rangle\psi$, and this completes the proof. QED

Alternative approach via translation in $\mathcal{L}_{\mathcal{O}}$

There is yet another way of getting compositional analysis for lexicographic upgrade by translating everything into Order Logic and performing reduction there. I show in the next Chapter that this method is better suited for binary preference statements, since once the reduction is performed inside Order Logic, one can translate back in the binary preference fragment, thus obtaining reduction axioms for free. In the case of relational doxastic logic, I have not succeeded in translating back in the Broccoli language, but the investigation is illuminating enough to be worthwhile.

To simplify the proof of the next theorem, the following lemmas, proved in Appendix A, are needed.

Lemma 3.4.2 *Let $*$ stand for either $\leq$ or $<$, then:*

$$\begin{aligned} \langle\Uparrow A\rangle\Box^*\varphi &\Leftrightarrow A \to \Box^*(A \to \langle\Uparrow A\rangle\varphi) \\ &\wedge \neg A \to \Box^*(\neg A \to \langle\Uparrow A\rangle\varphi) \\ &\wedge \neg A \to U(A \to \langle\Uparrow A\rangle\varphi) \end{aligned} \tag{3.31}$$

Lemma 3.4.3

$$A \wedge \langle\Uparrow A\rangle(\varphi \wedge \Box^*(\varphi \to \psi)) \Leftrightarrow A \wedge \langle\Uparrow A\rangle\varphi \wedge \Box^*(A \wedge \langle\Uparrow A\rangle\varphi \to \langle\Uparrow A\rangle\psi) \tag{3.32}$$

Lemma 3.4.4

$$\begin{aligned} \neg A \wedge \langle\Uparrow A\rangle(\varphi \wedge \Box^*(\varphi \to \psi)) &\Leftrightarrow \neg A \wedge \langle\Uparrow A\rangle\varphi \\ &\wedge \Box^*(\neg A \wedge \langle\Uparrow A\rangle\varphi \to \langle\Uparrow A\rangle\psi) \\ &\wedge U(A \wedge \langle\Uparrow A\rangle\varphi \to \langle\Uparrow A\rangle\psi) \end{aligned} \tag{3.33}$$

Theorem 3.4.5 *Given the following abbreviations:*

$$\begin{aligned} \beta &= A \wedge \langle\Uparrow A\rangle\varphi \wedge \Box^{\leq}(A \wedge \langle\Uparrow A\rangle \to \langle\Uparrow A\rangle\psi) \\ \gamma &= \neg A \wedge \langle\Uparrow A\rangle\varphi \wedge \Box^{\leq}(\neg A \wedge \langle\Uparrow A\rangle\varphi \to \langle\Uparrow A\rangle\psi) \wedge U(A \wedge \langle\Uparrow A\rangle\varphi \to \langle\Uparrow A\rangle\psi) \\ \beta' &= A \wedge \langle\Uparrow A\rangle\varphi \wedge \langle\Uparrow A\rangle\psi \wedge \Box^{<}(A \to \langle\Uparrow A\rangle\neg\varphi) \\ \gamma' &= \neg A \wedge \langle\Uparrow A\rangle\varphi \wedge \langle\Uparrow A\rangle\psi \wedge \Box^{<}(\neg A \to \langle\Uparrow A\rangle\neg\varphi) \wedge U(A \to \langle\Uparrow A\rangle\neg\varphi) \end{aligned}$$

the reduction axioms for $\langle\Uparrow A\rangle[\varphi]\psi$ and $\langle\Uparrow A\rangle[\varphi\rangle\psi$ are given by:

$$\langle\Uparrow A\rangle[\varphi]\psi \Leftrightarrow [A \wedge \langle\Uparrow A\rangle\varphi]\langle\Uparrow A\rangle\psi \wedge U((\langle\Uparrow A\rangle\varphi \wedge \neg A) \to (\Diamond^{\leq}\gamma \vee E\beta)) \tag{3.34}$$

$$\langle\Uparrow A\rangle[\varphi\rangle\psi \Leftrightarrow U[\langle\Uparrow A\rangle\varphi \to (A \wedge \Diamond^{\leq}\beta') \vee (\neg A \wedge \Diamond^{\leq}\gamma') \vee (\neg A \wedge E\beta')] \tag{3.35}$$

Let us first look at the proof and then discuss what the principles state.

Proof. I use the following abbreviations:

$$\begin{aligned}\alpha &= \varphi \wedge \Box^{\leq}(\varphi \to \psi) \\ \delta &= \varphi \wedge \psi \wedge \Box^{<}\neg\varphi\end{aligned}$$

$$\begin{aligned}
\langle \Uparrow A\rangle[\varphi]\psi &\Leftrightarrow \langle \Uparrow A\rangle U(\varphi \to \Diamond^{\leq}\alpha) && \text{(Fact 3.3.3)}\\
&\Leftrightarrow U\langle \Uparrow A\rangle(\varphi \to \Diamond^{\leq}\alpha) && \text{(Thm 2.2.2)}\\
&\Leftrightarrow U(\langle \Uparrow A\rangle\varphi \to \langle \Uparrow A\rangle\Diamond^{\leq}\alpha) && \text{(Thm 2.2.2)}\\
&\Leftrightarrow U[\langle \Uparrow A\rangle\varphi \to A \wedge \Diamond^{\leq}(A \wedge \langle \Uparrow A\rangle\alpha)\\
&\vee \quad \neg A \wedge \Diamond^{\leq}(\neg A \wedge \langle \Uparrow A\rangle\alpha)\\
&\vee \quad \neg A \wedge E(A \wedge \langle \Uparrow A\rangle\alpha)] && \text{(Thm 2.2.2)}\\
&\Leftrightarrow U[\langle \Uparrow A\rangle\varphi \to (A \wedge \Diamond^{\leq}\beta) \vee (\neg A \wedge \Diamond^{\leq}\gamma) \vee (\neg A \wedge E\beta)]\\
& && \text{(3.4.3 and 3.4.4)}\\
&\Leftrightarrow U[(\langle \Uparrow A\rangle\varphi \wedge A \to \Diamond^{\leq}\beta) \wedge ((\langle \Uparrow A\rangle\varphi \wedge \neg A) \to (\Diamond^{\leq}\gamma \vee E\beta)] && \text{(Logic)}\\
&\Leftrightarrow U(\langle \Uparrow A\rangle\varphi \wedge A \to \Diamond^{\leq}\beta) \wedge U(((\langle \Uparrow A\rangle\varphi \wedge \neg A) \to (\Diamond^{\leq}\gamma \vee E\beta)) && \text{(Modal Logic)}\\
&\Leftrightarrow [A \wedge \langle \Uparrow A\rangle\varphi]\langle \Uparrow A\rangle\psi \wedge U(((\langle \Uparrow A\rangle\varphi \wedge \neg A) \to (\Diamond^{\leq}\gamma \vee E\beta)) && \text{(Fact 3.3.3)}
\end{aligned}$$

$$\begin{aligned}
\langle \Uparrow A\rangle[\varphi\rangle\psi &\Leftrightarrow \langle \Uparrow A\rangle U(\varphi \to \Diamond^{\leq}\delta) && \text{(Fact 3.3.3)}\\
&\Leftrightarrow U\langle \Uparrow A\rangle(\varphi \to \Diamond^{\leq}\delta) && \text{(Thm 2.2.2)}\\
&\Leftrightarrow U(\langle \Uparrow A\rangle\varphi \to \langle \Uparrow A\rangle\Diamond^{\leq}\delta) && \text{(Thm 2.2.2)}\\
&\Leftrightarrow U[\langle \Uparrow A\rangle\varphi \to A \wedge \Diamond^{\leq}(A \wedge \langle \Uparrow A\rangle\delta)\\
&\vee \quad \neg A \wedge \Diamond^{\leq}(\neg A \wedge \langle \Uparrow A\rangle\delta)\\
&\vee \quad \neg A \wedge E(A \wedge \langle \Uparrow A\rangle\delta)] && \text{(Thm 2.2.2)}\\
&\Leftrightarrow U[\langle \Uparrow A\rangle\varphi \to (A \wedge \Diamond^{\leq}\beta') \vee (\neg A \wedge \Diamond^{\leq}\gamma') \vee (\neg A \wedge E\beta')]\\
& && \text{(3.4.3 and 3.4.4)}
\end{aligned}$$

QED

The technique of the proof is quite clear. The first step is to translate a formula of the form $\langle \Uparrow A\rangle[\varphi]\psi$ into $\mathcal{L}_{\mathcal{O}}$ and then using facts about Order Logic to find the reduction principles for the original formula. Ideally, as is the case for binary preference statement in the next chapter, one can translate back into the smaller language, thus obtaining reduction principles for free, by mechanical manipulations.

Let us consider what Axiom 3.34 state. The right-hand-side, as was the case previously, distinguishes various cases, depending on whether there is a state v such that $\mathfrak{M}_{\Uparrow A}, v \models A \wedge \varphi$. If this is the case, then we recover $[A \wedge \langle \Uparrow A\rangle\varphi]\langle \Uparrow A\rangle\psi$, as in

Theorem 3.4.1. Otherwise, the second conjunct explains what happens if the formula is evaluated in a $\neg A$-state. Either there are accessible states in the model before upgrade where $\langle \Uparrow A \rangle \varphi$ holds, which can be either A or $\neg A$ states, analyzed by γ. If not, the appeal to the existential modality becomes crucial. Here, it must be that the most plausible φ-states after upgrade were in the A-region, but that that region was not accessible in $\mathfrak{M}$, hence the appeal to $E\beta$.

Summary

Order Logic has passed its first test of providing a nice setting to investigate relational doxastic logic and its dynamics. Problems left open in the direct investigation starting with minimal relational logic are easily solved by seeing it as a fragment of Order Logic. Dynamics can be applied directly to $\mathcal{L}_{\mathcal{B}}$, provided that a complete system for the full broccoli logic can be obtained - an open question. By using translation inside $\mathcal{L}_{\mathcal{O}}$, however, we get reduction axioms for free, by mechanical manipulations using Theorem 2.2.2. The axioms are complex, admittedly, but this is to be expected, given the complexity inherent in Broccoli Logic. Seeing relational doxastic logic as Broccoli Logic with two operators $[\varphi]\psi$ and $[\varphi\rangle\psi$ is a first contribution of this chapter. A second important result is the representation Theorem 3.3.14 giving rigid boundaries to the fragment BL^{-} or Order Logic. Finally, the Minimal Relational Logic presented in Appendix A is a new system, which hopefully proves to be fruitful in future research in relational belief revision. In the next chapter, I investigate another important fragment of Order Logic: the binary preference fragment.

Chapter 4

Binary Preference Logic

When Robert said that he preferred wine over beer, he meant something more general than the mere comparison of two objects (or states, as in Order Logic). His statement was about wine and beer in general. In a similar fashion, preferring blue over red is a general standpoint, not a specific comparison. Nevertheless, there are cases of preferences between particulars, for instance of a book over another one, or of this person over that one. When Robert, after having tasted the cheeses presented to him, acknowledged that he preferred the Camembert over the goat cheese, he was then comparing the two token cheeses in the platter. Preferences thus operate at two levels: 1) over simple objects and 2) over sets of objects. The first level is the one covered by Order Logic, whereas the second one is the subject of the present chapter, in which I show how to lift preferences over objects to preferences over sets of objects.

Corresponding to this stratification of preferences over objects and sets of objects, there are two approaches investigating the relation between the two levels. The first one is top-down, from general preferences to specific comparisons, the second one bottom-up, from basic preferences over objects to general preferences. The difference may be illustrated as follows. I may say, in the top-down approach, that I prefer this car over that one because the former is blue and the latter red, and I prefer blue over red. Going bottom-up, I would say instead that I prefer blue over red because I prefer every blue object to every red object, including this blue car over that red one. The top-down approach is investigated in a logical setting by Liu and de Jongh in [42]. Their strategy is to take as primitive a constraint sequence and derive preferences over objects from it. For instance, if I want to buy a house and my constraints are such that I preferred living in a peaceful neighborhood at an affordable price and within reasonable distance from my work (in this order), then I would choose my house accordingly. The preference of the house settled as my final choice is for the one that maximizes satisfaction of my constraints.

Both approaches are advantageous in their own right and I see them as being complementary rather than in opposition. I do not, however, pursue this issue further here. I refer to Liu's Ph.D. thesis [41] for a more thorough discussion of the top-down vs bottom-up approaches to preferences. In this chapter, I opt for the bottom-up approach, taking basic preferences over states (Order Logic) and deriving global preferences over sets of objects, or propositions.

An important work in the field of preference logic, to which I pay special attention in the present chapter and the next one, is the seminal work of von Wright [76]. This work itself takes roots in Halldén [25] on the logic of *betterness*. There is indeed a very close kinship between the two books, but I take von Wright as being more fundamental to my investigations.[1] Von Wright gives us enough material for the purpose of this thesis, and historical considerations of his work is the subject of Section 4.1.

A side remark, before we proceed, is that Order Logic, as presented in Chapter 2, has been called 'Preference Logic' in other presentations (notably [68, 8, 71]). The reasons for that should become clear in the present chapter: due to the existential modality, order logic can express numerous preferential statement. But to read a formula of the form $\Diamond^{\leq}\varphi$ as a preference of φ is indeed abusive; preferential statements are typically comparative: I prefer x over y. To say that I prefer x without comparing x to anything else would, but for exceptional cases, be meaningless as a preferential statement. For this reason, I choose to take Order Logic as a general logic of comparison in Chapter 2, which can be instantiated as plausibility, in the case of beliefs in Chapter 3, or as betterness, in the case of preferences in the present Chapter.

The chapter is divided as follows. I first provide historical consideration on preference logic seen from a logical point of view, focusing on von Wright's seminal work [76] in Section 4.1. In Section 4.2, I show how the Order Logic of Chapter 2 can be used to express a plethora of binary preferential statements between propositions. In Section 4.3, I focus on one of these fragments, the $\forall\forall$ fragment. Finally, in Section 4.4, I introduce dynamics in the latter fragment with the preference upgrade action already discussed in Chapter 2. One important feature that has been left out in the

[1] One reason for choosing von Wright is his insistence on the notion of *ceteris paribus* preferences, the main subject of Chapter 5, which turned out to yield very interesting logical results. Von Wright's insight on *ceteris paribus* yet again proved to be fruitful and leading to interesting logical systems. More about this later.

discussion so far is the notion of *ceteris paribus* preferences; I reserve this for Chapter 5.

4.1 Von Wright's preference logic: Historical considerations

Adequately understanding von Wright's conception of *ceteris paribus* preferences is a difficult task, given the lack of semantic considerations in his work. Leaving this scholarly task aside, I appeal to what appears to be his fundamental intuitions and use them as landmarks to situate my proposal.

Von Wright uses a propositional language whose propositional variables range over states of affairs, augmented with a binary preference relation P such that "pPq" expresses that the states of affairs p are preferred to the states of affairs q. There is a restriction in the inductive definition of the language, namely that in '$\varphi P\psi$', 'φ' and 'ψ' can only be 'factual' propositional formulas, i.e., formulas without preference operators. Von Wright's formalism, as is commonly the case in the early development of modal logic, is almost purely syntactical. Essentially, given a preference statement, one manipulates it syntactically until it is in what von Wright calls *normal form*. If the resulting sentence is *consistent*, then so is the original sentence. This whole procedure of sentence manipulation can be seen as giving the meaning of von Wright's notion of preference. Indeed, his whole discussion can be summarized in the following syntactic principles:

1. $\varphi P\psi \rightarrow \neg(\psi P\varphi)$
2. $\varphi P\psi \wedge \psi P\xi \rightarrow \varphi P\xi$
3. $\varphi P\psi \equiv (\varphi \wedge \neg\psi)P(\neg\varphi \wedge \psi)$
4. (a) $\varphi P(\psi \vee \xi) \equiv \varphi P\psi \wedge \varphi P\xi$
   (b) $(\varphi \vee \psi)P\xi \equiv \varphi P\xi \wedge \psi P\xi$
5. $\varphi P\psi \equiv [(\varphi \wedge r)P(\psi \wedge r)] \wedge [(\varphi \wedge \neg r)P(\psi \wedge \neg r)]$, where r is any propositional variable not occurring in either φ or ψ.

The first two principles express *asymmetry* and *transitivity* of preference respectively, and are typical assumptions about preferential relations. The asymmetry of the relation is obvious with a notion of *strict* preference; if one strictly prefers p to q, then it is not the case that one also strictly prefers q to p.

Transitivity has a strong intuitive appeal, although it has often been questioned (see, for a good discussion, [27]). I leave the discussion of paradoxes involving transitive preferences aside. From a logical point of view, transitivity of preferences is natural. Even though it may fail at psychologically, assuming the logical core of preferences to be transitive is reasonable.

The third principle is known as *conjunctive expansion*: given two states of affairs p and q, to say that p is preferred to q is to say that a state of affairs with $p \wedge \neg q$ is preferred to a state of affairs with $\neg p \wedge q$. Conjunctive expansion predates von Wright, and was introduced in the field of deontic logic by Halldén in [25].[2]

The fourth principle analyzes disjunctions in terms of conjunctions in preference expressions. For instance, if I prefer flying to taking either a bus or a train, then I prefer flying to taking a bus, and I prefer flying to taking a train. This requirement seems natural, and I show below (cf. Section 5.3) that it follows from Order Logic.

The final principle, which is the leitmotiv of the next chapter, is what makes preferences *unconditional*, in von Wright's terminology. It says that a change in the world might influence the preference order between two states of affairs, but if all conditions stay constant in the world, then so does the preference order. 'Ceteris paribus' is the terminology commonly used to express this feature. Here is a formal expression, given by von Wright. Let φ be a formula. Denote by $PL(\varphi)$ the set of propositional letters that occur in φ, and which von Wright calls the *universe of discourse*. Suppose $r \notin PL(\varphi P \psi)$, then replace every formula $\varphi P \psi$ by the conjunction

$$(\varphi \wedge r) P (\psi \wedge r) \wedge (\varphi \wedge \neg r) P (\psi \wedge \neg r).$$

Von Wright calls this principle *amplification*. Amplification is applied for every r in the complement of $PL(\varphi P \psi)$ with respect to the set of propositional letters. Amplification guarantees that every r in the universe of discourse of a formula that is not directly relevant to the evaluation of a preference subformula is kept constant. This

[2] A similar principle is also found in the literature on verisimilitude [65]. For a philosophical criticism, see [32]. For an interesting (but short) discussion of Halldén's principle, see [13], in which Castañeda, after showing a counterexample to the principle, still provides the following in its defense:

> "When St. Paul said "better to marry than to burn" he meant "it is better to marry and not burn than not to marry and burn"."

would not be the case, for example, if we could have a resulting sentence of the form $r \wedge \neg b P u \wedge b$, which expresses something of the form "I prefer having my umbrella and my boots over having my raincoat but no boots." The loss of my boots in this example would reverse my preference for my raincoat over my umbrella. Amplification guarantees that only the universe of discourse of a preference statement is relevant in the evaluation of the comparison.

It would be hard to get a better understanding of von Wright's notion of *ceteris paribus* by further consideration of the postulates and I end this discussion here. The main purpose of the next chapter is to provide a precise semantics for *ceteris paribus*. I have the advantage of more than thirty years of development in modal logic and tools are now available that make a semantic treatment of *ceteris paribus* feasible. For now, I focus on my own framework for preference logic, implementing one key idea alluded to above: global (or general) preferences. Von Wright's notion of preference is further investigated in Section 5.3.

4.2 Binary preferences

In the introduction to the present chapter, I have claimed that a general preference of say, blue over red, may be derived from the preference of every blue objects over every red objects. But it might be required instead that every blue objects be preferred to some red objects, or that some blue ones be preferred to some red ones, and so on. In this section, I investigate various binary notion of preferences. As was the case with relational belief logic in Chapter 3, logics for binary preferences may be seen as fragments of Order Logic. The various fragments that can be isolated, however, depend on an assumption of totality of $\preceq$. I first present eight binary preference relations with their intended meaning. I then show that four of them can be defined in $\mathcal{L}_{\mathcal{O}}$ with no special assumptions, but that the four others cannot be so defined without assuming $\preceq$ to be total.

Definition 4.2.1 [Binary preference statements]

$$\mathfrak{M}, u \models \varphi \leq_{\exists\exists} \psi \quad \text{iff} \quad \exists s, \exists t : \mathfrak{M}, s \models \varphi \,\&\, \mathfrak{M}, t \models \psi \,\&\, s \preceq t \tag{4.1}$$
$$\mathfrak{M}, u \models \varphi \leq_{\forall\exists} \psi \quad \text{iff} \quad \forall s, \exists t : \mathfrak{M}, s \models \varphi \Rightarrow \mathfrak{M}, t \models \psi \,\&\, s \preceq t \tag{4.2}$$
$$\mathfrak{M}, u \models \varphi <_{\exists\exists} \psi \quad \text{iff} \quad \exists s, \exists t : \mathfrak{M}, s \models \varphi \,\&\, \mathfrak{M}, t \models \psi \,\&\, s \prec t \tag{4.3}$$
$$\mathfrak{M}, u \models \varphi <_{\forall\exists} \psi \quad \text{iff} \quad \forall s, \exists t : \mathfrak{M}, s \models \varphi \Rightarrow \mathfrak{M}, t \models \psi \,\&\, s \prec t \tag{4.4}$$
$$\mathfrak{M}, u \models \varphi <_{\forall\forall} \psi \quad \text{iff} \quad \forall s, \forall t : \mathfrak{M}, s \models \varphi \,\&\, \mathfrak{M}, t \models \psi \Rightarrow s \prec t \tag{4.5}$$
$$\mathfrak{M}, u \models \varphi >_{\exists\forall} \psi \quad \text{iff} \quad \exists s, \forall t : \mathfrak{M}, s \models \varphi \,\&\, \mathfrak{M}, t \models \psi \Rightarrow t \prec s \tag{4.6}$$
$$\mathfrak{M}, u \models \varphi \leq_{\forall\forall} \psi \quad \text{iff} \quad \forall s, \forall t : \mathfrak{M}, s \models \varphi \,\&\, \mathfrak{M}, t \models \psi \Rightarrow s \preceq t \tag{4.7}$$
$$\mathfrak{M}, u \models \varphi \geq_{\exists\forall} \psi \quad \text{iff} \quad \exists s, \forall t : \mathfrak{M}, s \models \varphi \,\&\, \mathfrak{M}, t \models \psi \Rightarrow t \preceq s \tag{4.8}$$

$\lhd$

The formulas $\varphi \leq_{\exists\exists} \psi$ and $\varphi <_{\exists\exists} \psi$ may be read as "there is a ψ-state that is at least as good as a φ-state", and "there is a ψ-state that is strictly better than a φ-state" respectively. The other comparative statements, $\varphi \leq_{\forall\exists} \psi$ and $\varphi <_{\forall\exists} \psi$, can be read as "for every φ-state, there is a ψ-state that is at least as good" and as "for every φ-state, there is a strictly better ψ-state" respectively. The other connectives receive similar intuitive readings.

Fact 4.2.2 *The first four preference operators of Definition 4.2.1 can be defined in $\mathcal{L}_{\mathcal{O}}$.*

$$\varphi \leq_{\exists\exists} \psi \quad := \quad E(\varphi \wedge \Diamond^{\leq}\psi) \tag{4.9}$$
$$\varphi \leq_{\forall\exists} \psi \quad := \quad U(\varphi \to \Diamond^{\leq}\psi) \tag{4.10}$$
$$\varphi <_{\exists\exists} \psi \quad := \quad E(\varphi \wedge \Diamond^{<}\psi) \tag{4.11}$$
$$\varphi <_{\forall\exists} \psi \quad := \quad U(\varphi \to \Diamond^{<}\psi) \tag{4.12}$$

Proof. The first and third definitions are obvious, and the second and last are proved in a similar fashion. I thus show 4.10:

$$\begin{array}{rll}\mathfrak{M}, u \models \varphi \leq_{\forall\exists} \psi & \text{iff} & \forall s, \exists t : \mathfrak{M}, s \models \varphi \Rightarrow \mathfrak{M}, t \models \psi \,\&\, s \preceq t \\ & \text{iff} & \forall s : \mathfrak{M}, s \models \varphi \Rightarrow \exists t : s \preceq t \,\&\, \mathfrak{M}, t \models \psi \\ & \text{iff} & \forall s : \mathfrak{M}, s \models \varphi \Rightarrow \mathfrak{M}, s \models \Diamond^{\leq}\psi \\ & \text{iff} & \forall s : \mathfrak{M}, s \models \varphi \rightarrow \Diamond^{\leq}\psi \\ & \text{iff} & \mathfrak{M}, u \models U(\varphi \rightarrow \Diamond^{\leq}\psi)\end{array}$$

QED

If totality is assumed, the four other operators of Definition 4.2.1 can also be defined in $\mathcal{L}_{\mathcal{O}}$. I give the translations before showing that the assumption of totality is crucial.

Fact 4.2.3 *The remaining four preference operators of Definition 4.2.1 can be defined in $\mathcal{L}_{\mathcal{O}}$, assuming totality.*

$$\varphi <_{\forall\forall} \psi \;:=\; U(\psi \rightarrow \Box^{\leq}\neg\varphi) \tag{4.13}$$

$$\varphi >_{\exists\forall} \psi \;:=\; E(\varphi \wedge \Box^{\leq}\neg\psi) \tag{4.14}$$

$$\varphi \leq_{\forall\forall} \psi \;:=\; U(\psi \rightarrow \Box^{<}\neg\varphi) \tag{4.15}$$

$$\varphi \geq_{\exists\forall} \psi \;:=\; E(\varphi \wedge \Box^{<}\neg\psi) \tag{4.16}$$

Proof. I give the proof for the first two cases, the two others being similar.

$$\begin{array}{rlll}\mathfrak{M}, u \models \varphi <_{\forall\forall} \psi & \text{iff} & \forall s, t : \mathfrak{M}, s \models \varphi \,\&\, \mathfrak{M}, t \models \psi \Rightarrow s \prec t & \\ & \text{iff} & \forall s, t : \mathfrak{M}, s \models \varphi \,\&\, \mathfrak{M}, t \models \psi \Rightarrow t \not\preceq s & \text{Totality!} \\ & \text{iff} & \forall s, t : \mathfrak{M}, t \models \psi \Rightarrow (t \preceq s \Rightarrow \mathfrak{M}, s \models \neg\varphi) & \text{Logic} \\ & \text{iff} & \forall t : \mathfrak{M}, t \models \psi \Rightarrow \mathfrak{M}, t \models \Box^{\leq}\neg\varphi & \\ & \text{iff} & \mathfrak{M}, u \models U(\psi \rightarrow \Box^{\leq}\neg\varphi) & \end{array}$$

$$\begin{array}{rlll}\mathfrak{M}, u \models \varphi >_{\exists\forall} \psi & \text{iff} & \exists s, \forall t : \mathfrak{M}, s \models \varphi \,\&\, \mathfrak{M}, t \models \psi \Rightarrow t \prec s & \\ & \text{iff} & \exists s, \forall t : \mathfrak{M}, s \models \varphi \,\&\, t \not\prec s \Rightarrow \mathfrak{M}, t \models \neg\psi & \\ & \text{iff} & \exists s, \forall t : \mathfrak{M}, s \models \varphi \,\&\, s \preceq t \Rightarrow \mathfrak{M}, t \models \neg\psi & \text{Totality!} \\ & \text{iff} & \exists s : \mathfrak{M}, s \models \varphi \,\&\, \mathfrak{M}, s \models \Box^{\leq}\neg\psi & \text{Logic} \\ & \text{iff} & \mathfrak{M}, u \models E(\varphi \wedge \Box^{\leq}\neg\psi) & \end{array}$$

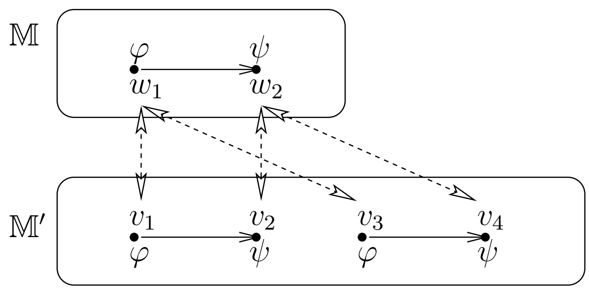

Figure 4.1: $\varphi <_{\forall\forall} \psi$ is not definable on totally ordered models.

QED

Notice that the four last connectives are the duals of the others, assuming totality. For example, Equation 4.13 can be obtained from the dual of 4.1 by:

$$\varphi <_{\forall\forall} \psi \quad \Leftrightarrow \quad \neg(\psi \leq_{\exists\exists} \varphi) \quad \Leftrightarrow \quad \neg E(\psi \wedge \Diamond^{\leq}\varphi) \quad \Leftrightarrow \quad U(\psi \rightarrow \Box^{\leq}\neg\varphi)$$

Fact 4.2.4 *The connectives $\varphi <_{\forall\forall} \psi$, $\varphi >_{\exists\forall} \psi$, $\varphi \leq_{\forall\forall} \psi$ and $\varphi \geq_{\exists\forall} \psi$ are not definable in their intended meaning in terms of $\mathcal{L}_{\mathcal{O}}$ on non-totally ordered models.*

Proof of Fact Consider the models in Figure 4.1. The $\preceq$ relations are given by the black arrows, while the bisimulation is indicated by the dashed lines. The same model may be used to analyze all four cases, but I only prove the $\varphi <_{\forall\forall} \psi$ case. First, since w_1 is the only φ-state in $\mathfrak{M}$, and the only world that it can see is a ψ-state, $\mathfrak{M}, w_1 \models \varphi <_{\forall\forall} \psi$. But $\mathfrak{M}', v_1 \not\models \varphi \leq_{\forall\forall} \psi$, since v_4 is a ψ-state that is not preferred to v_1. Since the states w_1 and v_1 are bisimilar, they are modally equivalent with respect to $\mathcal{L}_{\mathcal{O}}$, hence no formula in $\mathcal{L}_{\mathcal{O}}$ defines $\varphi <_{\forall\forall} \psi$, since w_1 and v_1 disagree on its truth-value.

◀

4.3 The $\forall\forall$ fragment

The binary preference formulas constitute only a small part of $\mathcal{L}_{\mathcal{O}}$. To show how to handle such notions of preference directly, I focus in this section on a fragment, denoted $\mathcal{L}_{\mathcal{O}}{}^{<_{\forall\forall}}$, based on the binary preference modalities $\varphi \leq_{\forall\forall} \psi$ and $\varphi <_{\forall\forall} \psi$. I investigate its expressive power and axiomatize it completely with respect to totally ordered preference models. I make this assumption about totality of models following

Fact 4.2.4 because I want $\leq_{\exists\exists}$ and $<_{\exists\exists}$ to be the duals of $\leq_{\forall\forall}$ and $<_{\forall\forall}$ respectively. Hence, the fragment $\mathcal{L_O}^{<_{\forall\forall}}$ that I investigate is generated by the following rule:

$$\mathcal{L_O}^{<_{\forall\forall}} := p \mid \varphi \wedge \psi \mid \neg\varphi \mid \varphi \leq_{\forall\forall} \varphi \mid \varphi <_{\forall\forall} \psi$$

Interpretation

The truth definition for the propositional letters and Booleans is standard. The interpretation of $\varphi \leq_{\forall\forall} \psi$, $\varphi <_{\forall\forall} \psi$ and their duals $\varphi \leq_{\exists\exists} \psi$ and $\varphi <_{\exists\exists} \psi$ is given in Definition 4.2.1.

Expressivity of $\mathcal{L_O}^{<_{\forall\forall}}$

As I have stressed many times already, the modalities of $\mathcal{L_O}^{<_{\forall\forall}}$ act globally. A formula $\varphi \leq_{\forall\forall} \psi$ is true in a model if certain conditions are met everywhere in the model. It should be expected that the global modality E to definable in this fragment. Indeed:

Fact 4.3.1 *The existential modality $E\varphi$ of $\mathcal{L_O}$ is expressible in $\mathcal{L_O}^{<_{\forall\forall}}$ by $\varphi \leq_{\exists\exists} \varphi$.*

Proof. I show that $\mathfrak{M}, u \models \varphi \leq_{\exists\exists} \varphi$ iff there exists a $v \in W$ such that $\mathfrak{M}, v \models \varphi$. But $\mathfrak{M}, u \models \varphi \leq_{\exists\exists} \varphi$ iff there exists $s, t \in W$ such that $\mathfrak{M}, s \models \varphi$, $\mathfrak{M}, t \models \varphi$ and $s \preceq t$ from the truth-definition. Hence, there exists $v \in W$ such that $\mathfrak{M}, v \models \varphi$. The other direction follows from the reflexivity of $\preceq$, since $v \preceq v$ and $\mathfrak{M}, v \models \varphi$ imply that $\mathfrak{M}, u \models \varphi \leq_{\exists\exists} \varphi$. QED

As a consequence of Lemma 4.3.1, The universal modality $U\varphi$ is also expressible in $\mathcal{L_O}^{<_{\forall\forall}}$ by $\neg\varphi <_{\forall\forall} \neg\varphi$.

To further investigate the expressivity of $\mathcal{L_O}^{<_{\forall\forall}}$, a notion slightly weaker than bisimulation, called *double-simulation*, is sufficient (cf. [73]).

Definition 4.3.2 (Double-simulation) *A relation $\rightleftharpoons$ is a* double-simulation *between two preference models $\mathfrak{M}, w$ and $\mathfrak{M}', v$, noted $\mathfrak{M}, w \rightleftharpoons \mathfrak{M}', v$, iff*

1. *For all $p \in \textsc{prop}$, $s \rightleftharpoons t \Rightarrow s \in V(p)$ iff $t \in V'(p)$.*
2. *For all $s, t \in W$ with $s \preceq t\,(s \prec t) : \exists s', t' \in W' : s \rightleftharpoons s', t \rightleftharpoons t' \,\&\, s' \preceq' t'\,(s' \prec t')$.*
3. *For all $s', t' \in W'$ with $s' \preceq' t'\,(s' \prec t') : \exists s, t \in W : s' \rightleftharpoons s, t' \rightleftharpoons t \,\&\, s \preceq t\,(s \prec t)$.*

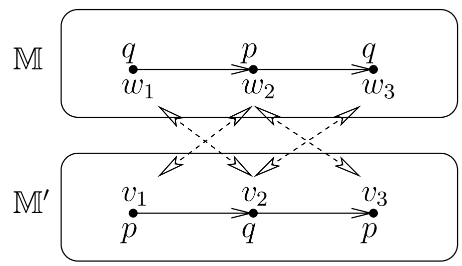

Figure 4.2: Double-similar, but not bisimilar models.

The following proposition shows that bisimulation and double-simulation indeed differ.

Proposition 4.3.3 *For any preference models $\mathfrak{M}$ and $\mathfrak{M}'$, if $\mathfrak{M}, w \leftrightarroweq \mathfrak{M}', v$ then $\mathfrak{M}, w \rightleftharpoons \mathfrak{M}', v$, but there are some preference models for which $\mathfrak{M}, w \rightleftharpoons \mathfrak{M}', v$ and $\mathfrak{M}, w \not\leftrightarroweq \mathfrak{M}', v$.*

Proof. If $\mathfrak{M}, w \leftrightarroweq \mathfrak{M}', v$, then the relation which establishes a bisimulation between $\mathfrak{M}, w$ and $\mathfrak{M}', v$ also establishes a double-simulation. This establishes the first claim.

For the second claim, consider the model in Figure 4.2 (reflexive arrows omitted). The pointed models $\mathfrak{M}, v_1$ and $\mathfrak{M}', w_2$ are double-similar, but not modally equivalent, since $\mathfrak{M}, v_1 \models \Diamond(p \wedge \Diamond q)$ but $\mathfrak{M}', w_2 \not\models \Diamond(p \wedge \Diamond q)$. Hence, the two models are not bisimilar. QED

A usual argument by induction on formulas using the duals of $\leq_{\forall\forall}$ and $<_{\forall\forall}$ establishes the following proposition, which I state without proving it:

Proposition 4.3.4 *Let $\mathfrak{M}, w$ and $\mathfrak{M}', w'$ be two pointed preference models. Then $\mathfrak{M}, w \rightleftharpoons \mathfrak{M}', v$ implies that $\mathfrak{M}, w \leftrightsquigarrow \mathfrak{M}', v$.*

Proposition 4.3.4 can be applied to show that ${\mathcal{L}_{\mathcal{O}}}^{<_{\forall\forall}}$ is less expressive than $\mathcal{L}_{\mathcal{O}}$. I show a number of expressive limitations of ${\mathcal{L}_{\mathcal{O}}}^{<_{\forall\forall}}$.

Fact 4.3.5 *The following connectives and frame properties are not definable in ${\mathcal{L}_{\mathcal{O}}}^{<_{\forall\forall}}$:*

1. *The modal diamonds $\Diamond^{\leq}$ and $\Diamond^{<}$,*

2. *$\leq_{\forall\exists}$, as defined in $\mathcal{L}_{\mathcal{O}}$,*

3. *Reflexivity and transitivity of $\preceq$,*

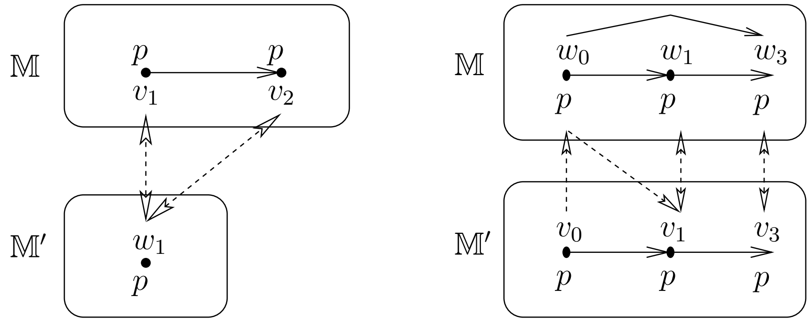

Figure 4.3: Double-similar models

4. Quasi-adequacy, as introduced in Section 2.1.

Proof of Fact 4.3.5

1. Consider the pair of models in Figure 4.2. The pointed models $\mathfrak{M}, w_3$ and $\mathfrak{M}', v_2$ are double-similar but not modally equivalent, since $\mathfrak{M}, w_3 \not\models *p$ and $\mathfrak{M}, v_2 \models *p$, where $*$ stands for either $\Diamond^{\leq}$ or $\Diamond^{<}$.

2. Consider the double-similar pointed models $\mathfrak{M}, v_1$ and $\mathfrak{M}', w_2$ in Figure 4.2. $M, w_2 \models p \leq_{\forall\exists} q$ but $M', v_1 \not\models p \leq_{\forall\exists} q$.

3. Figure 4.3 displays two pairs of double-similar models. In the left figure, $\mathfrak{M}, v_1 \leftrightsquigarrow \mathfrak{M}', w_1$, from Proposition 4.3.4, but reflexivity is not preserved and thus not definable in $\mathcal{L_O}^{<_{\forall\forall}}$. The right-hand figure shows that transitivity is not definable either.

4. Consider the pair of double-similar models $\mathfrak{M}, w_3$ and $\mathfrak{M}', v_1$ from Figure 4.4. The dashed arrows indicate that $w_3 \preceq w_4$, but neither $w_4 \preceq w_3$ nor $w_3 \prec w_4$. Nevertheless, $\mathfrak{M}, w_3 \leftrightsquigarrow \mathfrak{M}', v_1$, from 4.3.4, and therefore quasi-adequacy is not definable in $\mathcal{L_O}^{<_{\forall\forall}}$.

◀

Axiomatization

To simplify notation, I use the abbreviation $E\varphi$ for $\varphi \leq_{\exists\exists} \varphi$ and $U\varphi$ for $\neg\varphi <_{\forall\forall} \neg\varphi$ in the next definition. In the axioms listed below, $*$ stands for either $\leq_{\forall\forall}$ or $<_{\forall\forall}$.

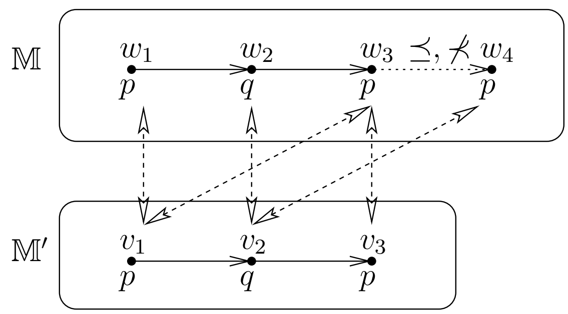

Figure 4.4: No quasi-adequacy

Definition 4.3.6 The complete logic $\Lambda^{\mathcal{L}\mathcal{O}^-}$ is the following set of formulas, along with all propositional tautologies, and closed under the inference rules of necessitation for U and substitution of logical equivalents.

$$\varphi \leq_{\exists\exists} \psi \leftrightarrow \neg(\psi <_{\forall\forall} \varphi) \tag{4.17}$$

$$\varphi <_{\exists\exists} \psi \leftrightarrow \neg(\psi \leq_{\forall\forall} \varphi) \tag{4.18}$$

$$\varphi * \psi \wedge U(\xi \rightarrow \psi) \rightarrow (\varphi * \xi) \tag{4.19}$$

$$\varphi * \psi \wedge U(\xi \rightarrow \varphi) \rightarrow \xi * \psi \tag{4.20}$$

$$\varphi \leq_{\forall\forall} \psi \wedge E\varphi \wedge E\psi \rightarrow \varphi \leq_{\exists\exists} \psi \tag{4.21}$$

$$\varphi <_{\forall\forall} \psi \wedge E\varphi \wedge E\psi \rightarrow \varphi <_{\exists\exists} \psi \tag{4.22}$$

$$\varphi * \xi \wedge \xi * \psi \ \wedge E\xi \rightarrow \varphi * \psi \tag{4.23}$$

$$U\varphi \rightarrow \varphi \tag{4.24}$$

$$U\neg\varphi \vee U\neg\psi \rightarrow \varphi * \psi \tag{4.25}$$

$$\varphi * \psi \rightarrow U(\varphi * \psi) \tag{4.26}$$

$$\varphi <_{\forall\forall} \psi \rightarrow \varphi \leq_{\forall\forall} \psi \tag{4.27}$$

$\lhd$

Completeness

Theorem 4.3.7 *The logic $\Lambda^{\mathcal{L}\mathcal{O}^-}$ is sound and complete with respect to the class of totally ordered preference models.*

Soundness does not present special difficulties and I focus on completeness. As above, I show that every $\Lambda^{\mathcal{L}_{\mathcal{O}}^-}$-consistent set Φ of formulas has a model. I use the definition of the canonical model $\mathfrak{M}^{\mathcal{L}_{\mathcal{O}}^-} = \langle W^{\mathcal{L}_{\mathcal{O}}^-}, \preceq^{\mathcal{L}_{\mathcal{O}}^-}, V^{\mathcal{L}_{\mathcal{O}}^-}\rangle$ for language of arbitrary similarity types as given in [7], Definition 4.24, where the relation is defined by:

$$u \preceq^{\mathcal{L}_{\mathcal{O}}} v \text{ iff for all formulas } \varphi \text{ and } \psi,\ \varphi \in u \text{ and } \psi \in v \text{ implies that } \varphi \leq_{\exists\exists} \psi \in u.$$

The strict sub-relation of $\preceq^{\mathcal{L}_{\mathcal{O}}}$ is then defined by $w \prec^{\mathcal{L}_{\mathcal{O}}} v$ iff $w \preceq^{\mathcal{L}_{\mathcal{O}}} v$ and not $v \preceq^{\mathcal{L}_{\mathcal{O}}} sw$. For the remainder of the proof, I omit the superscript $\mathcal{L}_{\mathcal{O}}^-$. With this definition in hand, I can readily use the Existence Lemma 4.26 and the Truth Lemma 4.2.4 of [7]. I state them without proofs.

Lemma 4.3.8 *Existence Lemma. If $\varphi \leq_{\exists\exists} \psi \in w$, then there are $u, v \in W$ such that $\varphi \in u$, $\psi \in v$ and $u \preceq v$.*

Lemma 4.3.9 *Truth-Lemma. For any formula φ, $\mathfrak{M}, w \models \varphi$ iff $\varphi \in w$.*

From these two lemmas, it follows that $\Lambda^{\mathcal{L}_{\mathcal{O}}^-}$ is complete with respect to the class of all models. What remains to be shown is that it is complete with respect to the class of totally ordered preference models. This result follows from the following lemma:

Lemma 4.3.10 *The relation $\preceq$ defined above is (1) reflexive, (2) total and (3) transitive*

Proof.

1. I show that for all u, v, $v \preceq v$, i.e., that $\forall \varphi, \psi(\varphi \in u \,\&\, \psi \in v \Rightarrow \varphi \leq_{\exists\exists} \psi \in u)$. But $\varphi \in u \,\&\, \psi \in v$ implies that $E(\varphi \wedge \psi) := (\varphi \wedge \psi \leq_{\exists\exists} \varphi \wedge \psi) \in u$, which implies that $\varphi \leq_{\exists\exists} \psi \in u$ by the monotonicity Axioms 4.19 and 4.20.

2. I show that for all u, v, $u \preceq v$ or $v \preceq u$. Assume that $\neg u \preceq v$. I show that $v \preceq u$, i.e., $\forall \varphi, \psi(\varphi \in v \& \psi \in u \Rightarrow \varphi \leq_{\exists\exists} \psi \in v)$. Let $\varphi \in v$ and $\psi \in u$ be arbitrary. I show that $\varphi \leq_{\exists\exists} \psi \in v$.

   From the assumption that $\neg u \preceq v$ and the definition of the relation $\preceq$, it follows that $\exists \xi, \sigma : \xi \in u$ and $\sigma \in v$ and $\xi \leq_{\exists\exists} \sigma \notin u$. Hence (1) $\xi \wedge \psi \in u$, (2) $\sigma \wedge \varphi \in v$ and (3) $\sigma <_{\forall\forall} \xi \in u$, using the duality Axiom 4.17. (3) together with axioms

4.19 and 4.20 imply that $\sigma \wedge \varphi <_{\forall\forall} \xi \wedge \psi \in u$. Let $\alpha := \xi \wedge \psi$ and $\beta := \sigma \wedge \varphi$. From the duality axiom, (4) $\neg(\alpha \leq_{\exists\exists} \beta) \in u$.

Now suppose that $\varphi \leq_{\exists\exists} \psi \notin v$, then $\psi <_{\forall\forall} \varphi \in u$, using the duality axiom and Axiom 4.26 successively, which implies that $\alpha <_{\forall\forall} \beta \in u$ from axioms 4.19 and 4.20. But (1) and (2) imply that $E\alpha \in u$ and $E\beta \in u$ and thus $\alpha <_{\exists\exists} \beta \in u$ from axiom 4.22. Finally, axiom 4.27 gives that $\alpha \leq_{\exists\exists} \beta \in u$, contradicting (4). Therefore, $\varphi \leq_{\exists\exists} \psi \in u$ and hence $v \preceq u$, as required.

3. I need to show that $u \preceq v \,\&\, v \preceq s \Rightarrow u \preceq s$. Using Logic and totality of $\preceq$ as proved above, it is enough to show that $(v \not\prec u \,\&\, s \not\prec v) \Rightarrow s \not\prec u$. Hence, I need to show that there is a $\varphi \in s$ and $\psi \in u$ such that $\varphi \leq_{\exists\exists} \psi \notin s$, i.e., $\psi <_{\forall\forall} \varphi \in s$. But $v \not\preceq u$ and totality imply that there is a $\varphi' \in v$ and $\psi' \in u$ such that $\psi' <_{\forall\forall} \varphi' \in v$ and $s \not\preceq v$ implies that there is $\varphi'' \in s$ and $\psi'' \in v$ such that $\psi'' <_{\forall\forall} \varphi'' \in s$. By the monotonicity Axioms 4.19 and 4.20 and Axiom 4.26, it follows that $\psi' <_{\forall\forall} (\varphi' \wedge \psi'') \in s$ and $(\varphi' \wedge \psi'') <_{\forall\forall} \varphi'' \in s$. Furthermore, $\varphi' \wedge \psi'' \in v$ implies that $E(\varphi' \wedge \psi'') \in s$. Therefore, $\psi' <_{\forall\forall} \varphi'' \in s$ by the transitivity Axiom 4.22, which was required to show.

QED

4.4 Dynamics

To introduce dynamics in $\mathcal{L}_{\mathcal{O}}^{<_{\forall\forall}}$, I apply the technique seen in the previous chapter of translating everything in $\mathcal{L}_{\mathcal{O}}$ and performing the reduction there, using Theorem 2.2.2. Here, however, the technique is more successful, as one can translate the result back into $\mathcal{L}_{\mathcal{O}}^{<_{\forall\forall}}$.

The axioms listed in the next theorem are stated in terms of the existential binary preference statement $\leq_{\exists\exists}$ in order to allow direct appeal to Theorem 2.2.2.

Theorem 4.4.1 *The complete logic of universal binary preference logic with preference upgrade is given by 1) the axiomatization given in Definition 4.3.6 and 2) the* *following list of reduction axioms:*

$$\begin{aligned}
\langle\#A\rangle p &\leftrightarrow p && (4.28)\\
\langle\#A\rangle\neg\varphi &\leftrightarrow \neg\langle\#A\rangle\varphi && (4.29)\\
\langle\#A\rangle(\varphi\vee\psi) &\leftrightarrow \langle\#A\rangle\varphi\vee\langle\#A\rangle\psi && (4.30)\\
\langle\#A\rangle(\varphi\leq_{\exists\exists}\psi) &\leftrightarrow (A\wedge\langle\#A\rangle\varphi)\leq_{\exists\exists}(A\wedge\langle\#A\rangle\psi) &&\\
&\vee\ (\neg A\wedge\langle\#A\rangle\varphi)\leq_{\exists\exists}\langle\#A\rangle\psi && (4.31)\\
\langle\#A\rangle(\varphi<_{\exists\exists}\psi) &\leftrightarrow (A\wedge\langle\#A\rangle\varphi)<_{\exists\exists}(A\wedge\langle\#A\rangle\psi) &&\\
&\vee\ (\neg A\wedge\langle\#A\rangle\varphi)<_{\exists\exists}\langle\#A\rangle\psi && (4.32)\\
\langle\#A\rangle E\varphi &\leftrightarrow E\langle\#A\rangle\varphi && (4.33)
\end{aligned}$$

Proof. As we saw in Chapter 2, all that I need to show is that the axioms are sound. The only distinguishing cases are Axioms 4.31 and 4.32. I show soundness of the former, appealing to Fact 4.2.2 and Theorem 2.2.2. The strategy consists in translating $\langle\#A\rangle\varphi\leq_{\exists\exists}\psi$ into $\mathcal{L}_{\mathcal{O}}$ and performing compositional analysis there, before translating back to the binary preference language.

$$\begin{aligned}
\langle\#A\rangle(\varphi\leq_{\exists\exists}\psi) &\leftrightarrow \langle\#A\rangle E(\varphi\wedge\Diamond^{\leq}\psi) && \text{Fact 4.2.2}\\
&\leftrightarrow E\langle\#A\rangle(\psi\wedge\Diamond^{\leq}\psi) && \text{Theorem 2.2.2}\\
&\leftrightarrow E(\langle\#A\rangle\psi\wedge\langle\#A\rangle\Diamond^{\leq}\psi) && \text{Theorem 2.2.2}\\
&\leftrightarrow E(\langle\#A\rangle\psi\wedge(A\wedge\Diamond^{\leq}(A\wedge\langle\#A\rangle\psi) &&\\
&\quad\vee\neg A\wedge\Diamond^{\leq}\langle\#A\rangle\psi)) && \text{Theorem 2.2.2}\\
&\leftrightarrow E(A\wedge\langle\#A\rangle\varphi\wedge\Diamond^{\leq}(A\wedge\langle\#A\rangle\psi) &&\\
&\quad\vee(\neg A\wedge\langle\#A\rangle\varphi\wedge\Diamond^{\leq}\langle\#A\rangle\psi)) && \text{Logic}\\
&\leftrightarrow E(A\wedge\langle\#A\rangle\varphi\wedge\Diamond^{\leq}(A\wedge\langle\#A\rangle\psi)) &&\\
&\quad\vee E(\neg A\wedge\langle\#A\rangle\varphi\wedge\Diamond^{\leq}\langle\#A\rangle\psi)) && \text{Modal Logic}\\
&\leftrightarrow (A\wedge\langle\#A\rangle\varphi)\leq_{\exists\exists}(A\wedge\langle\#A\rangle\psi) &&\\
&\quad\vee(\neg A\wedge\langle\#A\rangle\varphi)\leq_{\exists\exists}\langle\#A\rangle\psi && \text{Fact 4.2.2}
\end{aligned}$$

QED

Once again, Order Logic shows to be a good setting to investigate preferences. It is indeed well suited to analyze dynamics of binary preferences, as the last theorem demonstrates.

Summary

In this chapter, I have investigated the binary preference fragment of Order Logic. I have first looked back at the rise of Preference Logic in the seminal work of von Wright. I have then suggested eight binary preference definitions that lead to various fragments of Order Logic. I have also shown how to axiomatize a specific fragment, the $\forall\forall$ one, and provided a completeness result. Introducing the dynamic action of preference upgrade was straightforward and has shown how Order Logic can be used to get reduction axioms for its fragments, by translating everything in it, performing the reduction there, and then translating back. The main contributions presented in this chapter are twofold. The first one is the completeness result for the $\forall\forall$ fragment contained in Theorem 4.3.7. The second one, completed in the next Chapter with *ceteris paribus* logic, is a thorough investigation of von Wright's preference logic, both semantically and axiomatically.

Chapter 5

Ceteris Paribus Logic

Even though Robert has acknowledged to prefer the Camembert over the goat cheese served to us, he might not have changed his mind about his former preference. After all, Paris's cheese is quite different from what he could find back home. It would still have been rational for him to withhold his preference for goat cheese over Camembert and take his experience in Paris as an exception; preferences can be defeated.

Typically, preferences allowing for defeaters are expressed with *ceteris paribus* clauses, most commonly translated as "all other things being equal". In this chapter, I present a modal logic for defeasible preferences based on a strict reading of "all other things being equal" given by mathematical equivalence classes. I call it *Ceteris Paribus Logic* and denote it 'CPL'. CPL is a new kind of modal logic interesting both for its conceptual and mathematical implications. Conceptually, it analyzes perspicuously defeasible preferences and more generally the meaning of '*ceteris paribus*'. Mathematically, it yields extensions of Modal Logic with an infinitary side comparable to that of Propositional Dynamic Logic, yet independent from it. Furthermore, its dynamification provides a new kind of action over models, which I express in terms of 'research agenda'. The techniques I develop in this chapter could be applied to any basic modal logic, but I work out the initial presentation over the Order Logic of Chapter 2.[1]

The chapter is divided as follows. In Section 5.1, I present two senses typically given to *ceteris paribus*, which I call the *normality* and the *equality* reading. In Section 5.2, I formalize the latter reading, first by giving its semantics, then axiomatizing

[1]The results contained in the chapter have been obtained in collaboration with Johan van Benthem and Olivier Roy in [68]. In this paper, we have taken the liberal nomenclature of 'Preference Logic' instead of the 'Order Logic' used here.

it along with a completeness result for the *finite* fragment. In Section 5.3, I come back to von Wright's preference logic and the set of postulates discussed in Section 4.1. In Section 5.4, I abstract away from Preference and Order Logics and investigate mathematical questions pertaining to *ceteris paribus* logic. Dynamification of *ceteris paribus* logic is investigated in Section 5.5, where I show how public announcement and preference upgrade can be introduce, but also discuss two new actions over models, called *agenda expansion* and *contraction*. In Appendix B, I discuss an application of the equality reading of *ceteris paribus* to Game Theory, by showing that the Nash Equilibrium solution concept can be represented inside *ceteris paribus* logic.

5.1 Different senses of *ceteris paribus*

In the present section, I distinguish two senses '*ceteris paribus*': 1) "all other things being *normal*" and 2) "all other things being *equal*", which I call the *normality* and *equality* readings of *ceteris paribus* respectively. I first discuss the normality reading and discuss to what extent it is already analyzable in Order Logic. I then consider the equality reading and compare it to the first one.

Ceteris paribus as normality

Ceteris paribus as "all other things being normal" is taken to mean that, under *normal conditions*, something ought to be the case. This is the sense that plays a role, for instance, in the philosophical debate between Schiffer and Fodor over psychological laws, in which Fodor argued that ceteris paribus laws are necessary to provide special sciences with scientific explanation [20, 57]. A typical example given to illustrate this reading is the preference of red over white wine, unless when eating fish. Having fish with wine is taken as an atypical situation that defeats the original preference; it is a *defeater* of the general rule taken into account by the *ceteris paribus* clause.

To some extent, the basic preference language is sufficient to express the "all other things being normal" reading. Consider the preference alluded above of red over white wine. I assume that when saying "I prefer red over white wine, unless I'm having fish", one expresses that under normal conditions (having meat, cheese, pasta, salad, etc.), one prefers red wine to white wine. To simplify the exposition, I assume that the normal conditions for comparing red and white wine are all those where fish is not served. This is illustrated in Figure 5.1, where f stands for 'fish', m for 'meat',

Figure 5.1: Model of a preference for red wine over white wine under *normal* conditions. w stands for white wine, r for red, f for fish and m for meat.

r for 'red wine' and w for 'white wine'. Building on Definition 4.2.1, to express that red wine is preferred to white wine in normal conditions, I write:

$$(\neg f \wedge w) \leq_{\forall\forall} (\neg f \wedge r).$$

More generally, if the normal conditions are given by a set of formulas[2], then I can express that ψ is preferred to φ in normal conditions.

Fact 5.1.1 *Given a set of normal conditions $C = \{\varphi_1, ..., \varphi_n\}$, $\varphi P \psi$ in normal conditions translates as:*

$$\varphi \wedge \bigwedge C \quad \leq_{\forall\forall} \quad \psi \wedge \bigwedge C$$

Thus, I *can* express preferences *ceteris paribus* as "all other things being normal" in the base language, given a full description of a particular situation. But the logic itself does not provide the set of normal conditions, nor does it guide the choice of conditions - this is relegated to the modeler. Indeed, the weak reading of *ceteris paribus* only says that certain patterns of preferences hold in a restricted set of controlled conditions, a set that varies quite arbitrarily. In other words, given a set of normal conditions C in the language, then I can specify the preferences conjoined with C, leaving the $not - C$ case open.

But often (the usual situation) one cannot define the relevant normal conditions and then needs to incorporates the normal conditions in the formalism with some extra plausibility structure for each world. That is, it might be that a preference is defined with respect to a set of normal conditions without this set being fully describable, because, for instance, not all normal conditions are known. One may still want to apply logical reasoning in such cases and one way to do it is by taking an abstract view on normality and introducing a normality order between worlds [36]. This is a typical strategy in non-monotonic logic. The most plausible worlds in that structure provide the normal conditions for the evaluation of the preference relation.

[2]A set of normal conditions is called a *completer* in [20]

The normality sense of *ceteris paribus* thus links up with a well-established tradition in non-monotonic logic, which I do not pursue further here.

Ceteris paribus as equality

The equality reading of *ceteris paribus* is less frequent in the literature. In the field of preference logic, as I already noted multiple times, von Wright is the main proponent of this reading. Rather than providing a set of normal conditions, the equality reading identifies facts to be kept *constant* in preferential relations. This receives a natural mathematical interpretation in terms of equivalence classes, as is formally explained in [19], namely to divide a space of possibilities into equivalence classes and ignore comparison links that go across them.

The idea behind the equality reading is that reasoning may be conducted with a certain body of knowledge kept constant. The example given in Section 4.1 when talking about amplification is the example given by Von Wright himself. It expressed a preference of a raincoat over an umbrella when the consideration of having boots is kept constant. That is, if I have my boots, then I prefer my raincoat over my umbrella and similarly if I do not have my boots, I still prefer my raincoat over my umbrella. But I do prefer an umbrella and boots over a raincoat and no boots. In this case, I say that the preference of my raincoat over my umbrella is *ceteris paribus* with respect to having my boots. This is illustrated in figure 5.2. In short, the equality reading specifies, for some definable partition of the domain, that the same preferences must hold in every zone. In the remainder of this chapter, I formalize this equality reading of *ceteris paribus*.

5.2 Equality-based *ceteris paribus* Order Logic

In this section, I generalize the Order Language $\mathcal{L}_{\mathcal{O}}$ by relativizing the modalities with respect to sets of formulas representing the conditions to be kept "equal". I call the resulting language $\mathcal{L}_{\mathcal{CP}}$.

General setting

Definition 5.2.1 [Language] Let PROP be a set of propositions, and let Γ be a set of formulas of the base language (to be specified below). The language $\mathcal{L}_{\mathcal{CP}}$ is defined

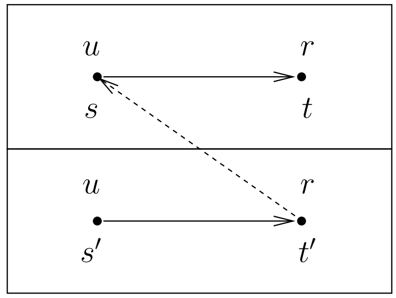

Figure 5.2: A simple illustration of a *ceteris paribus* preference of a raincoat (r) over an umbrella (u). Arrows point to preferred states. The model is divided into two equivalence classes, in each of which every r-state is preferred to every u-state. Only the dotted arrow indicates a preference for u over r, but the arrow goes across the equivalence classes, which I count as a violation of "all other things being equal".

by the inductive rules:

$$\mathcal{L}_{\mathcal{CP}} := p \mid \neg\varphi \mid \varphi \vee \psi \mid \langle\Gamma\rangle^{\leq}\varphi \mid \langle\Gamma\rangle^{<}\varphi \mid \langle\Gamma\rangle\varphi$$

The set Γ is restricted to formulas of the base language, i.e., members of PROP, Boolean combinations of them, or modalities of the form $\langle\emptyset\rangle\varphi$, $\langle\emptyset\rangle^{\leq}\varphi$ or $\langle\emptyset\rangle^{<}\varphi$. $\lhd$

To simplify the exposition in the rest of the paper, I introduce a new piece of notation. Given a set of formulas Γ, if w and v are two states such that for all $\varphi \in \Gamma$, $\mathfrak{M}, w \models \varphi$ iff $\mathfrak{M}, v \models \varphi$, then I say that w and v are *equivalent with respect to* Γ, and I write $w \equiv_\Gamma v$.

Definition 5.2.2 [*Ceteris paribus* models] A *ceteris paribus preference model* is a quadruple $\mathfrak{M} = \langle W, \preceq, \trianglelefteq_\Gamma, V\rangle$, where:

- $\langle W, \preceq, V\rangle$ is a standard order model as in Definition 2.1.1 and
- $\trianglelefteq_\Gamma$ is a binary relation such that $w \trianglelefteq_\Gamma v$ iff a) $w \preceq v$, and b) $w \equiv_\Gamma v$.

$\lhd$

The strict subrelation $\lhd_\Gamma$ is defined by a) $w \prec v$ and b) $w \equiv_\Gamma v$. As above, a *pointed preference model* is a pair $\mathfrak{M}, w$ where $w \in W$. The notation $w \equiv_\Gamma v$ makes it explicit that the *ceteris paribus* preferential relation is the intersection of two relations: the basic preference relation and the equivalence relation with respect to the truth-valuation of the formulas in Γ.

Definition 5.2.3 [Truth definition] Formulas of $\mathcal{L}_{\mathcal{CP}}$ are interpreted in pointed *ceteris paribus* preference models. The truth conditions for the proposition letters and the Booleans are standard. Here are the three crucial clauses:

$$\begin{array}{lll} \mathfrak{M}, w \models \langle\Gamma\rangle^{\leq}\varphi & \text{iff} & \exists v \text{ such that } w \trianglelefteq_{\Gamma} v \,\&\, \mathfrak{M}, v \models \varphi \\ \mathfrak{M}, w \models \langle\Gamma\rangle^{<}\varphi & \text{iff} & \exists v \text{ such that } w \triangleleft_{\Gamma} v \,\&\, \mathfrak{M}, v \models \varphi \\ \mathfrak{M}, w \models \langle\Gamma\rangle\varphi & \text{iff} & \exists v \text{ such that } w \equiv_{\Gamma} v \,\&\, \mathfrak{M}, v \models \varphi \end{array}$$

$\lhd$

$\mathcal{L}_{\mathcal{CP}}$ vs $\mathcal{L}_{\mathcal{O}}$

The following facts show that the *ceteris paribus* variant of Order Logic is a proper extension of it, since it can recover its modalities, but not, in general, the other way around.

Fact 5.2.4 *The modalities $\Diamond^{\leq}\varphi, \Diamond^{<}\varphi$ and the existential modality $E\varphi$ of $\mathcal{L}_{\mathcal{O}}$ are expressible in $\mathcal{L}_{\mathcal{CP}}$.*

Proof. The following equivalences hold:

$$\begin{array}{llll} 1. & \mathfrak{M}, w \models_{\mathcal{L}_{\mathcal{O}}} \Diamond^{\leq}\varphi & \text{iff} & \mathfrak{M}, w \models_{\mathcal{L}_{\mathcal{CP}}} \langle\emptyset\rangle^{\leq}\varphi \\ 2. & \mathfrak{M}, w \models_{\mathcal{L}_{\mathcal{O}}} \Diamond^{<}\varphi & \text{iff} & \mathfrak{M}, w \models_{\mathcal{L}_{\mathcal{CP}}} \langle\emptyset\rangle^{<}\varphi \\ 3. & \mathfrak{M}, w \models_{\mathcal{L}_{\mathcal{O}}} E\varphi & \text{iff} & \mathfrak{M}, w \models_{\mathcal{L}_{\mathcal{CP}}} \langle\emptyset\rangle\varphi. \end{array}$$

I show how to prove the first claim, and leave the two other cases to the reader.

In the one direction, assume that $\mathfrak{M}, w \models \Diamond^{\leq}\varphi$. This implies that $\exists v(w \preceq v \,\&\, \mathfrak{M}, v \models \varphi)$. But, $w \equiv_{\emptyset} v$ is vacuously true. Hence, $\exists v(w \trianglelefteq_{\emptyset} v \,\&\, \mathfrak{M}, v \models \varphi)$. Therefore, by the semantic definition, $\mathfrak{M}, w \models \langle\emptyset\rangle^{\leq}\varphi$.

In the other direction, assume that $\mathfrak{M}, w \models \langle\emptyset\rangle^{\leq}\varphi$. Then $\exists v(w \trianglelefteq_{\emptyset} v \& \mathfrak{M}, v \models \varphi)$ by the semantic definition. This implies that $\exists v(w \preceq v \,\&\, w \equiv_{\emptyset} v \,\&\, \mathfrak{M}, v \models \varphi)$, by Definition 5.2.2. Hence, $\exists v(w \preceq v \,\&\, \mathfrak{M}, v \models \varphi)$. Therefore, $\mathfrak{M}, w \models \Diamond^{\leq}\varphi$. QED

In the other direction, I show that $\mathcal{L}_{\mathcal{CP}}$ reduces to $\mathcal{L}_{\mathcal{O}}$ for finite sets of "equality conditions".

Fact 5.2.5 *If Γ is a finite set of formulas, then the modalities $\langle\Gamma\rangle^{\leq}\varphi, \langle\Gamma\rangle^{<}\varphi$ and $\langle\Gamma\rangle\varphi$ are expressible in the order language $\mathcal{L}_{\mathcal{O}}$.*

Proof. Let $\Gamma = \{\varphi_1, ...\varphi_n\}$. Consider the set Δ of all possible conjunctions of formulas and negated formulas taken from Γ, i.e., the set of all formulas α of the form $\alpha := \bigwedge_{\varphi_i \in \Gamma} \pm\varphi_i (1 \leq i \leq n)$, where $+\varphi_i = \varphi_i$ and $-\varphi_i = \neg\varphi_i$. Then,

1. $\mathfrak{M}, w \models_{\mathcal{L_{CP}}} \langle\Gamma\rangle^{\leq}\varphi$ iff $\mathfrak{M}, w \models_{\mathcal{L_O}} \bigvee_{\alpha\in\Delta}(\alpha \wedge \Diamond^{\leq}(\alpha \wedge \varphi))$
2. $\mathfrak{M}, w \models_{\mathcal{L_{CP}}} \langle\Gamma\rangle^{<}\varphi$ iff $\mathfrak{M}, w \models_{\mathcal{L_O}} \bigvee_{\alpha\in\Delta}(\alpha \wedge \Diamond^{<}(\alpha \wedge \varphi))$
3. $\mathfrak{M}, w \models_{\mathcal{L_{CP}}} \langle\Gamma\rangle\varphi$ iff $\mathfrak{M}, w \models_{\mathcal{L_O}} \bigvee_{\alpha\in\Delta}(\alpha \wedge E(\alpha \wedge \varphi))$

I prove the first case.

In the first direction, assume that $\mathfrak{M}, w \models \langle\Gamma\rangle^{\leq}\varphi$, then $\exists v(w \trianglelefteq^{\Gamma} v \,\&\, \mathfrak{M}, v \models \varphi)$. But one and only one $\alpha \in \Delta$ is satisfied in w (since Δ is an exhaustive list of the possible valuations of formulas in Δ, and since the α's are mutually inconsistent), which implies that $\mathfrak{M}, w \models \pm\varphi_i, 1 \leq i \leq n$, where $\pm\varphi_i = \varphi_i$ if $\mathfrak{M}, w \models \varphi_i$ and $\pm\varphi_i = \neg\varphi_i$ if $\mathfrak{M}, w \not\models \varphi_i$. But $w \trianglelefteq_\Gamma v$ implies that $w \equiv_\Gamma v$, hence $\mathfrak{M}, v \models \pm\varphi_i, 1 \leq i \leq n$. Thus, $\mathfrak{M}, v \models \alpha$, and $\mathfrak{M}, v \models \alpha \wedge \varphi$. Since $w \trianglelefteq_\Gamma v$, also $w \preceq v$, which implies that $\mathfrak{M}, w \models \Diamond^{\leq}(\alpha \wedge \varphi)$ by the semantic definition. But $\mathfrak{M}, w \models \alpha$, therefore, $\mathfrak{M}, w \models \alpha \wedge \Diamond^{\leq}(\alpha \wedge \varphi)$ and finally $\mathfrak{M}, w \models \bigvee_{\alpha\in\Delta}(\alpha \wedge \Diamond^{\leq}(\alpha \wedge \varphi))$.

In the other direction, assume that $\mathfrak{M}, w \models \bigvee_{\alpha\in\Delta}(\alpha \wedge \Diamond^{\leq}(\alpha \wedge \varphi))$. For the same reason as above, it must that there is an $\alpha \in \Delta$ such that $\mathfrak{M}, w \models \alpha \wedge \Diamond^{\leq}(\alpha \wedge \varphi)$. Hence, there exists a $v \in W$ such that $w \preceq v, \mathfrak{M}, v \models \alpha$ and $\mathfrak{M}, v \models \varphi$. Thus, there exists a v such that $\mathfrak{M}, v \models \pm\varphi_i\,(1 \leq i \leq n)$, where $\pm\varphi_i = \varphi_i$ if $\mathfrak{M}, w \models \varphi_i$ and $\pm\varphi_i = \neg\varphi_i$ if $\mathfrak{M}, w \not\models \varphi_i$. Hence, $w \equiv_\Gamma v$. By Definition 5.2.2, $w \trianglelefteq_\Gamma v$ and $\mathfrak{M}, v \models \varphi$. Therefore, by the semantic definition, $\mathfrak{M}, w \models \langle\Gamma\rangle^{\leq}\varphi$. QED

Of course, if Γ is infinite, this simple translation no longer works. I discuss the infinite case in Section 5.4. But even in the finite case, $\mathcal{L_{CP}}$ gives control over the reasoning involving 'equal conditions' - hence it is worthwhile to determine its logic explicitly.

Axiomatization

The expressivity of CPL varies with assumption imposed on the structures of the sets Γ in $\langle\Gamma\rangle\varphi$. In this section, I prove completeness when Γ is assumed to be finite. The infinite case is still unsettled and I leave it as an open question. Notice that the completeness of the finite fragment has already been obtained via translation in basic Order Logic provided in Fact 5.2.5, in a way analogous to compositional analysis. Nevertheless, a direct completeness proof for this logic is instructive, as it reveals the reasoning that *ceteris paribus* modalities sustains. Furthermore, the axiomatization suggests a dynamic reading of the CP-modalities, investigated in Section 5.5.

I call $\Lambda^{\mathcal{L}_{\mathcal{CP}}}$ the logic of *ceteris paribus* preference models. As above, $\Lambda^{\mathcal{L}_{\mathcal{CP}}}$ has several well-known fragments: $S4$ for $\langle\Gamma\rangle^{\leq}\varphi$, K for $\langle\Gamma\rangle^{<}\varphi$ (transitivity being derivable from the inclusion axioms given below), and $S5$ for $\langle\Gamma\rangle\varphi$. To these are added the following interaction axioms:

- Inclusion axioms:

$$\langle\Gamma\rangle^{<}\varphi \to \langle\Gamma\rangle^{\leq}\varphi \tag{5.1}$$
$$\langle\Gamma\rangle^{\leq}\varphi \to \langle\Gamma\rangle\varphi \tag{5.2}$$

- Mixed axioms for $\langle\Gamma\rangle^{\leq}$ and $\langle\Gamma\rangle^{<}$:

$$\langle\Gamma\rangle^{\leq}\langle\Gamma\rangle^{<}\varphi \to \langle\Gamma\rangle^{<}\varphi \tag{5.3}$$
$$\langle\Gamma\rangle^{<}\langle\Gamma\rangle^{\leq}\varphi \to \langle\Gamma\rangle^{<}\varphi \tag{5.4}$$
$$(\psi \wedge \langle\Gamma\rangle^{\leq}\varphi) \to (\langle\Gamma\rangle^{<}\varphi \vee \langle\Gamma\rangle^{\leq}(\varphi \wedge \langle\Gamma\rangle^{\leq}\psi)) \tag{5.5}$$

- *Ceteris paribus* reflexivity, when $\varphi \in \Gamma$:

$$\langle\Gamma\rangle\varphi \to \varphi \tag{5.6}$$
$$\langle\Gamma\rangle\neg\varphi \to \neg\varphi \tag{5.7}$$

- Mixed axioms for Γ:
  - $\Gamma \subseteq \Gamma'$:

$$\langle\Gamma'\rangle\varphi \to \langle\Gamma\rangle\varphi \tag{5.8}$$
$$\langle\Gamma'\rangle^{\leq}\varphi \to \langle\Gamma\rangle^{\leq}\varphi \tag{5.9}$$
$$\langle\Gamma'\rangle^{<}\varphi \to \langle\Gamma\rangle^{<}\varphi \tag{5.10}$$

- I also have some axioms reminiscent of cautious monotonicity for our 3 modalities:

$$\pm\varphi \wedge \langle\Gamma\rangle(\alpha \wedge \pm\varphi) \to \langle\Gamma \cup \{\varphi\}\rangle\alpha \tag{5.11}$$

$$\pm\varphi \wedge \langle\Gamma\rangle^{\leq}(\alpha \wedge \pm\varphi) \to \langle\Gamma \cup \{\varphi\}\rangle^{\leq}\alpha \tag{5.12}$$

$$\pm\varphi \wedge \langle\Gamma\rangle^{<}(\alpha \wedge \pm\varphi) \to \langle\Gamma \cup \{\varphi\}\rangle^{<}\alpha \tag{5.13}$$

I show the soundness of Axioms 5.6 and 5.11.

Proof. For Axiom 5.6, assume that $\mathfrak{M}, w \models \langle\Gamma\rangle\varphi$ and that $\varphi \in \Gamma$. Then there exists a state v such that $w \equiv_\Gamma v$ and $\mathfrak{M}, v \models \varphi$. Hence, $\mathfrak{M}, w \models \varphi$, since $\varphi \in \Gamma$.

I give the argument for $\pm\varphi = \varphi$ for axiom 5.11. Assume that (1) $\mathfrak{M}, w \models \varphi$ and (2) $\mathfrak{M}, w \models \langle\Gamma\rangle(\alpha \wedge \varphi)$. From (2), $\exists v(w \equiv_\Gamma v \;\&\; \mathfrak{M}, v \models \alpha \wedge \varphi)$, which implies that $\mathfrak{M}, v \models \varphi$. But from (1), $\mathfrak{M}, w \models \varphi$. Hence, $w \equiv_{\Gamma\cup\{\varphi\}} v$. Therefore, by the truth definition, $\mathfrak{M}, w \models \langle\Gamma \cup \{\varphi\}\rangle\alpha$. QED

By way of illustration, I derive another principle of the logic.

Example 5.2.6 $\vdash [\Gamma]^{\leq}\varphi \wedge \langle\Gamma\rangle^{\leq}\alpha \to \langle\Gamma \cup \{\varphi\}\rangle^{\leq}\alpha$.

Proof of example 5.2.6.

i.	$\vdash [\Gamma]^{\leq}\varphi \wedge \langle\Gamma\rangle^{\leq}\alpha \to \langle\Gamma\rangle^{\leq}(\alpha \wedge \varphi)$	modal logic
ii.	$\vdash [\Gamma]^{\leq}\varphi \to \varphi$	Axiom T
iii.	$\vdash [\Gamma]^{\leq}\varphi \wedge \langle\Gamma\rangle^{\leq}\alpha \to \langle\Gamma \cup \{\varphi\}\rangle^{\leq}\alpha$	$i) - ii)$, Axiom 5.12

QED

Completeness

Theorem 5.2.7 (Completeness) *The logic $\Lambda^{\mathcal{L}_{\mathcal{CP}}}$ is sound and complete with respect to the class of* ceteris paribus *frames.*

I already proved the soundness of two axioms above and the others do not present special difficulties. For completeness, I use the following canonical model.

Definition 5.2.8 The canonical model $\mathfrak{M}^{\Lambda^{\mathcal{L}_{\mathcal{CP}}}} = \langle W^{\Lambda^{\mathcal{L}_{\mathcal{CP}}}}, \trianglelefteq_\Gamma^{\Lambda^{\mathcal{L}_{\mathcal{CP}}}}, \equiv_\Gamma^{\Lambda^{\mathcal{L}_{\mathcal{CP}}}}, V^{\Lambda^{\mathcal{L}_{\mathcal{CP}}}}\rangle$, with

$$\begin{array}{ccc}
\mathfrak{M}^c & \xrightarrow{\leftrightarrows} & Bull(\mathfrak{M}^c) \\
\downarrow & & \downarrow \\
\mathfrak{M}^{\Lambda^{\mathcal{L}_{CP}}} & \xrightarrow{\leftrightarrows} & Bull(\mathfrak{M}^{\Lambda^{\mathcal{L}_{CP}}})
\end{array}$$

Figure 5.3: Commuting diagram showing that applying bulldozing to $\preceq$ in $\mathfrak{M}^c$ or to $\trianglelefteq$ in $\mathfrak{M}^{\Lambda^{\mathcal{L}_{CP}}}$ amounts to the same operation.

- $W^{\Lambda^{\mathcal{L}_{CP}}}$ the set of all maximal consistent sets of $\Lambda^{\mathcal{L}_{CP}}$,
- $w \equiv_\Gamma^{\Lambda^{\mathcal{L}_{CP}}} v$ iff for all $\psi \in \Gamma$, $\psi \in w$ iff $\psi \in v$,
- $w \trianglelefteq_\Gamma^{\Lambda^{\mathcal{L}_{CP}}} v$ iff a) for all $\varphi \in v$, $\langle \Gamma \rangle^{\leq} \varphi \in w$.

$\lhd$

I define $\preceq^{\Lambda^{\mathcal{L}_{CP}}}$ as $\trianglelefteq_\emptyset^{\Lambda^{\mathcal{L}_{CP}}}$. I omit the superscript $\Lambda^{\mathcal{L}_{CP}}$ for the rest of the completeness proof. I further assume that the bulldozing technique of Theorem 2.1.11 has been carried through on the relation $\trianglelefteq_\emptyset$. This is expressed in the following lemma:

Lemma 5.2.9 *The diagram of Figure 5.3 commutes, i.e., taking the bulldoze M^c and then its* ceteris paribus *variant is the same as bulldozing the* ceteris paribus *variant of $\mathfrak{M}^c$.*

Proof. The result follows from the realization that the horizontal lines in Diagram 5.3 are bisimulations. For a detailed proof, consult [68]. QED

What remains to be shown is an Existence Lemma for the new modalities and that the relation $\trianglelefteq_\Gamma$ is the intended comparison relation, i.e., the intersection of the relations $\preceq$ and $\equiv_\Gamma$.

Lemma 5.2.10 (Existence Lemma) *For any state $w \in W$, if $\langle \Gamma \rangle^{\leq} \varphi \in w$, then there exists a state $v \in W$ such that $w \trianglelefteq_\Gamma v$ and $\varphi \in v$.*

Proof. Suppose that $\langle \Gamma \rangle^{\leq} \varphi \in w$. For every $\psi_i \in \Gamma$, let $\pm\psi_i = \psi_i$ if $\psi \in w$, and $\pm\psi_i = \neg\psi_i$ if $\psi_i \notin w$. Let $v^- = \{\varphi\} \cup \{\xi : [\Gamma]^{\leq}\xi \in w\} \cup \{\pm\psi : \psi \in \Gamma\}$. I claim that v^- is consistent. Indeed, on the assumption that it is not, a standard argument shows that $\vdash [\Gamma]^{\leq}\xi_1 \wedge ... \wedge [\Gamma]^{\leq}\xi_m \wedge [\Gamma]^{\leq} \pm \psi_1 \wedge ... \wedge [\Gamma]^{\leq} \pm \psi_n \to [\Gamma]^{\leq}\neg\varphi$, for some m, n. Now, $[\Gamma]^{\leq}\xi_i \in w, 1 \leq i \leq m$ by definition of v^-. Furthermore, $\pm\psi_i \in w$ implies that $[\Gamma] \pm \psi_i \in w$, using Axiom 5.6 and 5.7, which in turns implies that $[\Gamma]^{\leq}\psi_i \in w$

by Axiom 5.2. Hence, $[\Gamma]^{\leq}\xi_1 \wedge ... \wedge [\Gamma]^{\leq}\xi_m \wedge [\Gamma]^{\leq} \pm \psi_1 \wedge ... \wedge [\Gamma]^{\leq} \pm \psi_n \in w$, and thus $[\Gamma]^{\leq}\neg\varphi \in w$ by Modus Ponens. But this contradicts the initial assumption that $\langle\Gamma\rangle^{\leq}\varphi \in w$. Hence, v^- is consistent. By Lindenbaum's Lemma, there exists a maximal consistent extension v of v^-, and v is such that $[\Gamma]^{\leq}\psi \in w$, implies that $\psi \in v$ for all ψ. Thus $w \preceq v$ from the definition of the $\preceq$-relation in the canonical model. Furthermore, $w \equiv_\Gamma v$ by the construction of v. Therefore, $w \trianglelefteq_\Gamma v$ and $\varphi \in v$.
QED

Corollary 5.2.11 (to the proof of Lemma 5.2.10) *For any state $w \in W$, if $\langle\Gamma\rangle\varphi \in w$, then there exists a state $v \in W$ such that $w \equiv_\Gamma v$ and $\varphi \in v$.*

Proof. Consider $v^- = \{\varphi\} \cup \{\pm\psi : \psi \in \Gamma\}$, and proceed as above. QED

Lemma 5.2.12 $\trianglelefteq_\Gamma = \preceq \cap \equiv_\Gamma$.

Proof.

The first direction follows from the definition of $\trianglelefteq^\Gamma$ in the canonical model.

In the other direction, assume that $w \preceq v$ and that $w \equiv_\Gamma v$. In the first case, let $\varphi \in v$ and consider $\psi \in \Gamma$ such that, without loss of generality, $\psi \in v$. Then $\varphi \wedge \psi \in v$, which implies that $\langle\emptyset\rangle^{\leq}(\varphi \wedge \psi) \in w$, since $w \preceq v$. But $w \equiv_\Gamma v$ implies that $\psi \in w$. Hence, $\psi \wedge \langle\emptyset\rangle^{\leq}(\varphi \wedge \psi)$, which implies that $\langle\{\psi\}\rangle^{\leq}\varphi \in w$, using Axiom 5.12. Since Γ is finite, the same procedure can be repeated for every $\psi \in \Gamma$. Therefore, $\langle\Gamma\rangle^{\leq}\varphi \in w$, as required.

QED

This completes the proof-theoretical analysis of CPL. I leave the question of axiomatizing infinitary CPL open. The remainder of this chapter explores various applications of CPL.

5.3 Coming back to von Wright; *Ceteris paribus* counterparts of binary preference statements

In this section, I show how to define *ceteris paribus* counterparts of the binary preference statements as given in Definition 4.2.1. By the *ceteris paribus* counterparts, I mean preference statements that compare states with respect to relevant information and all other information is kept 'equal'. This type of comparison is more restrictive than the preferences I have been considering so far. The definition I give is consonant with von Wright's and a good way of testing this is by analyzing von Wright's postulates from Section 4.1. The only binary relation that I consider here is $<_{\forall\forall}$, since this is the one I attribute to von Wright, and also to avoid replicating the same pattern too many times. I first introduce more notation, give the definition of preferences *ceteris paribus* and then investigate resulting properties by comparing them with von Wright's notion.

Let $PL(\varphi) = \{p \in \textsc{prop} : p \text{ occurs in } \varphi\}$, let Γ be a set, and let $cp(\Gamma) = \textsc{prop} - \bigcup\{PL(\varphi) : \varphi \in \Gamma\}$. Then $\langle cp(\Gamma)\rangle^{\leq}\varphi$ expresses that there exists a φ-state at least as good as the current state in which the propositional information outside of Γ is equal. Assuming models to be total (cf., Fact 4.2.3), an equality-based notion of *ceteris paribus* preference close to von Wright's is defined as:

$$\varphi P \psi := [\emptyset](\psi \to [cp(\{\psi, \varphi\})]^{\leq}\neg\varphi) \tag{5.14}$$

This definition captures the essence of von Wright's definition. First, it is a strict preference of the $<_{\forall\forall}$-type. Second, the modality $[\emptyset]$ provides the global reach of preferences. The evaluation of a preference statement at a state depends on every state in the model. Finally, the *ceteris paribus* clause is with respect to the propositional information not mentioned in either φ or ψ.

To test my definition against von Wright's, I show which postulates from Section 4.1 are preserved under my translation. For those which are not, I provide a justification for their rejection. For the sake of simplicity, I assume the *ceteris paribus* clause to be with respect to the same set Γ.

First principle: Asymmetry of strict preferences

The first postulate holds in CPL if neither φ nor ψ equals $\perp$, and if the model contains at least one φ-state and one ψ-state. This is because $[\emptyset](\psi \to [\Gamma]^{\leq}\neg\varphi)$ is vacuously true in both of the two first cases, and models with a single state that has only φ or ψ provides a counterexample in the latter cases. Hence, this postulates only hold for genuine strict preferences. However, these failures of the first principle are not alarming. It is not clear what a preference amounts to when a contradiction is involved. Likewise, if something does not possibly exists, then a preference comparison involving it is meaningless. Furthermore, these are simple consequences of universal preferences in the lines of "all φ-states are preferred to all ψ-states"; such preferences may hold vacuously. But I am not bound to the universal preference relations, and I could have chosen another formulation that behave differently with this principle.

Second principle: Transitivity of preferences

Transitivity is not valid either under my translation. A model in which there is one state w with ξ and φ true at w provides a counterexample. Since, $\mathfrak{M}, w \models \neg\psi$, $\mathfrak{M}, w \models \psi \rightarrow [\Gamma]^{\leq}\neg\varphi$ ($\varphi P \psi$) and $\mathfrak{M}, w \models [\Gamma]^{\leq}\neg\psi$ implies that $\mathfrak{M}, w \models \xi \rightarrow [\Gamma]^{\leq}\neg\psi$ ($\psi P \xi$). But $\mathfrak{M}, w \models \varphi$ implies that $\mathfrak{M}, w \not\models [\Gamma]^{\leq}\neg\varphi$, which in turns implies that $\mathfrak{M}, w \not\models \xi \rightarrow [\Gamma]^{\leq}\neg\varphi$ (not $\varphi P \xi$). It might seem strange that transitivity is not preserved here, but it should be expected. Indeed, it is not the case in general that relations between states should be preserved when lifted to sets of states. For preferences, this was noted in [19], Theorem 3. Nevertheless, this counterexample may be seen as a degenerate case of preference evaluation. It still holds that in any model with worlds w, v and t such that , $\mathfrak{M}, v \models \psi$, $w \preceq v$ and $v \preceq t$, then also $w \preceq t$. This is reflected for instance in $\Lambda^{\mathcal{LO}^-}$ by Axiom 6, where the transitivity between φ and ψ is guaranteed by the existence of a ξ-state.

Von Wright's principles of asymmetry and transitivity are not preserved in general under my translation; they fail in degenerate cases. Indeed, the underlying strict preferential relation is asymmetric and transitive and those properties are transferred to preferences among formulas in most cases. My formalism reveals that the validity of those principles depends on the satisfiability of the formulas occurring in the scope of preference modalities.

Third principle: Conjunctive expansion

Under translation 5.14, the third postulate (conjunctive expansion) becomes:

$$[\emptyset](\psi \rightarrow [\Gamma]^{\leq}\neg\varphi) \equiv [\emptyset]((\neg\varphi \wedge \psi) \rightarrow [\Gamma]^{\leq}(\neg\varphi \vee \psi)). \tag{5.15}$$

The principle holds only from left to right. Indeed, assume that $\mathfrak{M}, w \models \neg\varphi \wedge \psi$ for some w arbitrary, suppose there is a v such that $w \trianglelefteq_\Gamma v$. But $\mathfrak{M}, w \models \psi$ implies that $\mathfrak{M}, w \models [\Gamma]^{\leq}\neg\varphi$, and thus $\mathfrak{M}, v \models \neg\varphi \vee \psi$. As v was chosen arbitrarily, $\mathfrak{M}, w \models [\Gamma]^{\leq}(\neg\varphi \vee \psi)$, which implies that $\mathfrak{M}, w \models (\neg\varphi \wedge \psi) \rightarrow [\Gamma]^{\leq}(\neg\varphi \vee \psi)$. As w was chosen arbitrarily, we get that $[\emptyset]((\neg\varphi \wedge \psi) \rightarrow [\Gamma]^{\leq}(\neg\varphi \vee \psi)$ is valid. The other direction is not valid in general. A model with a single state w with $\mathfrak{M}, w \models \varphi \wedge \psi$

provides a counterexample. Here, $(\neg\varphi\wedge\psi) \to [\Gamma]^{\leq}(\neg\varphi\vee\psi)$ is vacuously true, whereas $\psi \to [\Gamma]^{\leq}\neg\varphi$ does not hold. Once again, my modeling helps to understand exactly where conjunctive expansion can be falsified.

Fourth principle: Distribution

The fourth principle is entirely preserved under translation 5.14. The resulting thesis is:

$$[\emptyset](\psi \vee \xi \to [\Gamma]^{\leq}\neg\varphi) \equiv [\emptyset](\psi \to [\Gamma]^{\leq}\neg\varphi) \wedge [\emptyset](\xi \to [\Gamma]^{\leq}\neg\varphi) \tag{5.16}$$

I show the soundness of the left to right direction. If $\mathfrak{M}, w \models \psi$, then $\mathfrak{M}, w \models \psi \vee \xi$, which implies that $\mathfrak{M}, w \models [\Gamma]^{\leq}\neg\varphi$ by assumption. Similarly if $\mathfrak{M}, w \models \xi$, then $\mathfrak{M}, w \models [\Gamma]^{\leq}\neg\xi$. Here, I am in complete agreement with von Wright, and the principle comes out as a theorem of my logic.

Fifth principle: *Ceteris paribus*

The *ceteris paribus* clause of von Wright's notion of preference is probably the major test for my definition. Fortunately, it comes out as a theorem of CPL, supporting my equality reading of von Wright's *ceteris paribus* preferences.

In the next equation, I assume that r does not occur in either φ or ψ and thus that $r \in \Gamma$. The principle is then translated as:

$$[\emptyset](\psi \to [\Gamma]^{\leq}\neg\varphi) \equiv [\emptyset]((\psi\wedge r) \to [\Gamma]^{\leq}(\neg\varphi\vee\neg r))\wedge[\emptyset]((\psi\wedge\neg r) \to [\Gamma]^{\leq}(\neg\varphi\vee r)) \tag{5.17}$$

The first direction does not present special difficulties, nor does it use the *ceteris paribus* clause in a crucial way. I prove the right to left direction. Let w be arbitrary and assume that $\mathfrak{M}, w \models \psi$. I distinguish two cases: 1) $\mathfrak{M}, w \models r$, and 2) $\mathfrak{M}, w \not\models r$. Under the first assumption, we get that $\mathfrak{M}, w \models \psi \wedge r$, and thus that $\mathfrak{M}, w \models [\Gamma]^{\leq}(\neg\varphi \vee \neg r)$. Let v be arbitrary such that $w \trianglelefteq_{\Gamma} v$, then $\mathfrak{M}, v \models \neg\varphi \vee \neg r$. But since $r \in \Gamma$ and $w \equiv_{\Gamma} v$, it follows that $\mathfrak{M}, v \models r$ and hence that $\mathfrak{M}, v \models \neg\varphi$. A similar argument shows that if $\mathfrak{M}, w \not\models r$, then $\mathfrak{M}, v \models \neg\varphi$, using the second conjunct. In either case, we get that $\mathfrak{M}, w \models [\Gamma]^{\leq}\neg\varphi$, which completes the proof.

Final remarks on preferences

Comparing my formalism against von Wright's is instructive in many ways. It shows that his postulates would only be complete for specific classes of models, something which was lacking altogether in [76]. Moreover, it produces a workable calculus for explicit reasoning about *ceteris paribus* preferences. I conclude here the discussion of von Wright's preference logic, both in the base case and its *ceteris paribus* variation.

5.4 Mathematical perspective

To formalize *ceteris paribus* I based my analysis on the basic Order Logic, but I could have taken any modal language and relativize modalities with respect to sets of sentences. It is a natural question to inquire what mathematical properties this adaptation has in general. In this section, I adopt this general outlook on the *ceteris paribus* variant of modal logic and show that its full infinitary version lies in between basic and infinitary modal logics. This adds technical interest to the formalism, in addition to its conceptual motivations.

Given a modal logic whose diamonds are defined over a family of relations $\{R\}$, *ceteris paribus* diamonds can always be defined over the intersection of the members of $\{R\}$ with $\equiv_\Gamma$. Hence, given a modal language $\mathcal{L}$, and a normal modal logic Λ in $\mathcal{L}$, I investigate the language $\mathcal{L}_\Gamma$ whose modalities are the modalities of $\mathcal{L}$ relativized to sets of sentences. The logic defined over $\mathcal{L}_\Gamma$ is denoted Λ^Γ. Without loss of generality, I assume that $\mathcal{L}$ contains only one diamond $\Diamond$, and some logic Λ defined over this language. Accordingly, in the remainder of this section, I consider a *ceteris paribus* logic Λ^Γ containing only one diamond $\langle\Gamma\rangle$. As in the case of the basic preference language, the semantics for this diamond is given by the intersection of the relation R of the logic Λ with the modal equivalence $w \equiv_\Gamma v$, which I write R_Γ. Furthermore, I no longer restrict the sets of formulas in the *ceteris paribus* diamonds to the base language. I only require them to be sets. The next proposition shows that I am justified in viewing Λ^Γ as a modal logic.

Proposition 5.4.1 *If Λ is bisimulation-invariant, then so is the corresponding* ceteris paribus *logic Λ^Γ.*

Proof. I proceed by induction on the complexity of formulas, where every member of Γ in $\varphi = \langle\Gamma\rangle\psi$ is of lower complexity than φ by the definition of well-formed formulas.

Let $\mathfrak{M}$ and $\mathfrak{M}'$ be two models such that $\mathfrak{M}, u \leftrightarrows \mathfrak{M}', v$ and assume that $\mathfrak{M}, u \models \langle\Gamma\rangle\varphi$. Then, there is a u' such that both uRu' and $u \equiv_\Gamma u'$ and $\mathfrak{M}, u' \models \varphi$. But since $\mathfrak{M}, u \leftrightarrows \mathfrak{M}'v$, there is a corresponding v' such that $vR'v'$ and $\mathfrak{M}, u' \leftrightarrows \mathfrak{M}', v'$. By the inductive hypothesis, $\mathfrak{M}', v' \models \varphi$. I claim that $v \equiv_\Gamma v'$. To prove the claim, let $\gamma \in \Gamma$ be such that $\mathfrak{M}, v \models \gamma$. By the inductive hypothesis, and since γ is of lower complexity than φ, $\mathfrak{M}', u \models \gamma$. Since $u \equiv_\Gamma u'$, we also have that $\mathfrak{M}, u' \models \gamma$. But by the inductive hypothesis again, since $\mathfrak{M}, u' \leftrightarrows \mathfrak{M}', v'$, we also get that $\mathfrak{M}', v' \models \gamma$. Similarly, for every $\gamma \in \Gamma$ such that $\mathfrak{M}, v \models \neg\gamma$, $\mathfrak{M}', v' \models \neg\gamma$. Therefore, $v \equiv_\Gamma v'$, which implies by truth-definition that $\mathfrak{M}', v \models \langle\Gamma\rangle\varphi$, as required. QED

Expressivity of Λ^Γ

I now investigate the additional expressive power imbued to a modal logic by taking its *ceteris paribus* variation. By way of illustration, I show that the resulting logic can express that a point in a model sees a finite chain of successor of any length. One consequence of this fact for the *ceteris paribus* Order Logic is that it does not have the finite model property. I take these results in turn.

Proposition 5.4.2 *Let $\Gamma = \{\langle\emptyset\rangle_n\top : n \in \mathbb{N}\}$ and let $\varphi = \langle\Gamma\rangle\top$. Then $\mathfrak{M}, s \models \varphi$ iff there is a state $t \in W$ such that sRt and t has finite chains of (not necessarily distinct) successors of any length.*

Proof. If there is a state $t \in W$ such that $sR_\Gamma t$ and t has finite chains of successors of any length, then $\mathfrak{M}, t \models \langle\emptyset\rangle_n\top$ for every $n \in \mathbb{N}$. But $s \equiv_\Gamma t$ implies that $\mathfrak{M}, s \models \langle\emptyset\rangle_n\top$ for every $n \in \mathbb{N}$. Therefore, $\mathfrak{M}, s \models \varphi$ by the truth-definition.

In the other direction, assume that $\mathfrak{M}, s \models \langle\Gamma\rangle\top$. By the truth definition, there is a state t such that $sR_\Gamma t$ and $\mathfrak{M}, t \models \top$. I show by induction that t has a chain of n successors of any length, i.e., that $\mathfrak{M}, t \models \langle\emptyset\rangle_n\top$ for every $n \in \mathbb{N}$. The base case is trivial, since $\langle\emptyset\rangle_n\top$ reduces to $\top$ and both s and t satisfy $\top$. Assume that t has a chain of n successors (not necessarily distinct), then $\mathfrak{M}, t \models \langle\emptyset\rangle_n\top$. Since $sR_\Gamma t$, $\mathfrak{M}, s \models \langle\emptyset\rangle\langle\emptyset\rangle_n\top = \langle\emptyset\rangle_{n+1}\top$. Since $\langle\emptyset\rangle_{n+1}\top \in \Gamma$ and $s \equiv_\Gamma t$, I get that $\mathfrak{M}, t \models \langle\emptyset\rangle_{n+1}\top$. This completes the proof. QED

Corollary 5.4.3 Ceteris paribus *(strict) modal logic lacks the finite model property.*

Proof. Let $\Gamma' = \{\langle\emptyset\rangle^<_n\top : n \in \mathbb{N}\}$, let $\varphi = \langle\Gamma'\rangle^<\top$ and assume that $\mathfrak{M}, s \models \varphi$. From Proposition 5.4.2, there exists a t such that $s \lhd t$ and t sees a finite chain of successors of any length. But since every modality in Γ' is strict, t must see a finite chain of n different successors for every $n \in \mathbb{N}$. Therefore, t must be at the root of a tree with infinitely many states. QED

I have not been able to prove this result for a modal logic without a strict interpretation of accessibility - and must leave this as an open question.

CPL vs $ML_{\infty,\omega}$

We saw in Fact 5.2.4 that the $\mathcal{L}_{\mathcal{CP}}$ modalities are expressible in $\mathcal{L}_{\mathcal{O}}$ if Γ is a finite set. I now show that the unrestricted *ceteris paribus* modality $\langle\Gamma\rangle\varphi$ of the present section is expressible in $ML_{\infty,\omega}$, the modal logic which allows infinite conjunctions and disjunctions, but only finite nesting of modalities. The definition I provide is actually the same as in Lemma 5.2.4, but this time using infinite conjunctions and disjunctions.

Proposition 5.4.4 *The modalities $\langle\Gamma\rangle\varphi$ are expressible in $\mathcal{L}_{\infty,\omega}$.*

Proof. Let $\Gamma = \{\varphi_i : i \in I\}$ be an arbitrary set of formulas. Let Δ contain all possible (infinite) conjunctions of formulas and negated formulas taken from Γ, i.e., all formulas α of the form $\alpha := \bigwedge_{i \in I} \pm\varphi_i(1 \leq i \leq n)$, where $+\varphi_i = \varphi_i$ and $-\varphi_i = \neg\varphi_i$. Then,

$$\mathfrak{M}, w \models_{\mathcal{L}_{\mathcal{CP}}} \langle\Gamma\rangle\varphi \quad \text{iff} \quad \mathfrak{M}, w \models_{\mathcal{L}_{\infty,\omega}} \bigvee_{\alpha\in\Delta}(\alpha \wedge \Diamond^{\leq}(\alpha \wedge \varphi))$$

The argument now proceeds in the same way as in the proof of Lemma 5.2.4. QED

Combining the results of the last sections, *ceteris paribus* logic is a modal logic that lies in between basic and infinitary modal logics. Its syntax and expressivity are infinitary in character, by the construction of diamonds with infinite sets Γ. Still, it does not use a full-blown syntax as in $ML_{\infty,\omega}$ with its infinite conjunctions and disjunctions.

CPL vs PDL

Another system between the basic modal logic and $ML_{\infty,\omega}$ is the well-known propositional dynamic logic (PDL). PDL has a finite syntax with only implicit infinitary expressive power via the Kleene-star operator. To better situate *ceteris paribus* modal logic (CPL) in the landscape of modal logics, I compare it with PDL and show that they are expressively independent.

Consider a simple version of PDL with one primitive program φ and with diamonds $\langle\pi\rangle\varphi$ and $\langle\pi^*\rangle\varphi$. The intended reading of those diamonds is "there is an execution of the program π that leads to a state where φ is true" and "after finitely many execution of the program π, there is a state where φ is true." Notice that since I only work with one program, the choice and composition diamonds $\langle\pi\cup\pi\rangle\psi$ and $\langle\pi;\pi\rangle\varphi$ reduce to $\langle\pi\rangle\psi$ and $\langle\pi\rangle\langle\pi\rangle\psi$ respectively. Accordingly, I only treat the $\langle\pi\rangle\varphi$ and $\langle\pi^*\rangle\varphi$ cases in the proofs below.

Proposition 5.4.5 *The* ceteris paribus *modality $\langle\Gamma\rangle\varphi$ is not definable in PDL.*

Proof. Let x and y be two states such that xRy. Let $\mathcal{T} = \{t_i : t_i \text{ is a finite tree }\}$ be the set of all finite trees. For every $t_i \in \mathcal{T}$ with root w_i, let xRw_i, and similarly for y. Then x and y can access the root of every finite tree in one step. I further assume that the propositional valuation is empty. This is illustrated in Figure 5.4. I show 1) that states x and y are equivalent in PDL, but that 2) there is a formula $\varphi \in \mathcal{L}_{\mathcal{CP}}$ such that $x \models \varphi$ but $y \not\models \varphi$.

The first claim is proved by induction on the inductive definition of well-formed-formulas of $\mathcal{L}_{\mathcal{PDL}}$. I show that for every $\varphi \in \mathcal{L}_{\mathcal{PDL}}$, $x \models \varphi$ iff $y \models \varphi$. That $y \models \varphi \Rightarrow x \models \varphi$ is obvious, since x and y see the same submodel, i.e., every root of a finite tree model. I show that $x \models \varphi \Rightarrow y \models \varphi$.

The basis and the Boolean cases are obvious. The interesting cases are $\varphi = \langle\pi\rangle\psi$ and $\varphi = \langle\pi^*\rangle\psi$. In either case, the only problematic situation is if $\mathfrak{M}, x \models \langle\pi\rangle\psi$ or $\mathfrak{M}, w \models \langle\pi^*\rangle\psi$ and $\mathfrak{M}, y \models \psi$. It is sufficient to show that if $\mathfrak{M}, y \models \psi$ then $\mathfrak{M}, y \models \langle\pi\rangle\psi$. Thus, suppose that $\mathfrak{M}, y \models \psi$. I use the *pruning* lemma of [34] for the μ-calculus, which states that if $\mathfrak{M}, w \models \varphi$, then there is a tree-like model $\mathfrak{M}'$ whose branching is bounded by the size $|\varphi|$ of φ and such that φ is satisfiable at the root of this tree. Furthermore, I can assume that the depth of $\mathfrak{M}'$ is bounded by the modal depth of ψ and thus that $\mathfrak{M}'$ is a finite tree. But since every finite tree is in T, $\mathfrak{M}' = t_i$ for some $t_i \in T$. This means that there is a successor z of y that is the root of the tree t_i and such that $\mathfrak{M}, z \models \psi$. therefore, by the truth-definition, $\mathfrak{M}, y \models \langle\pi\rangle\psi$.

To prove the second claim, let $\Gamma = \{\langle\emptyset\rangle^i[\emptyset]\bot : i \geq 1, i \in \mathfrak{N}\}$, and let $\varphi = \langle\Gamma\rangle\top$. I show that $x \models \varphi$, but that $y \not\models \varphi$. Since x and y are the roots of every finite tree, each sees a finite branch of any length greater or equal to 1. Hence, for every $n \in \mathfrak{N}$, $x \models \langle\emptyset\rangle^i[\emptyset]\bot$, and $y \models \langle\emptyset\rangle^i[\emptyset]\bot$. Hence, for every $\xi \in \Gamma$, $x \models \xi$ iff $y \models \xi$.

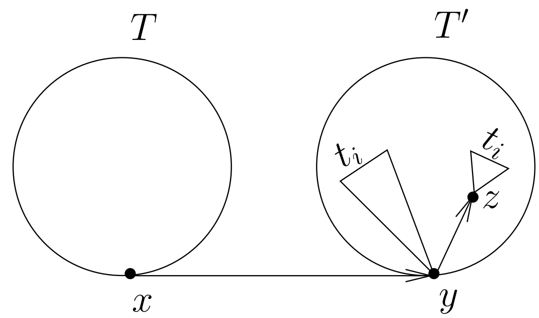

Figure 5.4: T and T' are the collection of all finite trees seen by x and y in one step.

Therefore, $x \models \langle\Gamma\rangle\top$. Now, no successors of y is such that it sees a finite branch of any length, as this would only be the case if it was the root of an infinite tree, contrary to our assumption. Hence, there is no state accessible from y which agrees on the truth-valuation of every member of Γ. Therefore, $y \not\models \langle\Gamma\rangle\top$. This completes the proof. QED

To get the previous results, I have used a lemma about the μ-calculus that bounds the branching of trees for the satisfiability of formulas. This does not hold with the *ceteris paribus* logic, and one should expect that the above argument also establishes that the μ-calculus cannot express the *ceteris paribus* modality either (Yde Venema, p.c.).

The following Proposition was first proved game-theoretically by Shivaram Lingamneni, but I present a different proof.

Proposition 5.4.6 *The PDL modality $\langle\pi^*\rangle\varphi$ is not definable in CPL.*

Proof. It is enough to show that for every n, there are two models such that no formulas of depth n can distinguish between them, whereas PDL can. Fix n and consider the models depicted in Figures 5.5. Clearly, in PDL, $\langle\pi^*\rangle\neg p$ is true at u_o but false at v_0. Now, for every $i \leq n$, it is also clear that $[\emptyset]^{n-i}p$ is true at u_i - notice that it is never the case that $[\emptyset]\bot$, because of the reflexive loops, so that we get a genuine match between states in the two models. Thus, using Axioms 5.8-5.10, $[\Gamma]^{n-i}p$ for every Γ. Hence, there is no CPL formula of depth $i \leq n$ that can see u_{n+1} from u_i. Therefore, no formula of depth n can distinguish u_o and v_o. Since n was chosen arbitrarily, this holds for every n.

QED

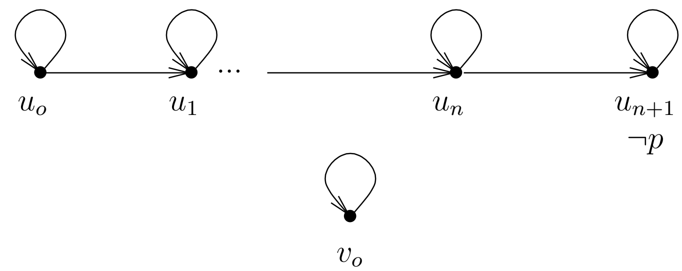

Figure 5.5: The models of Proposition 5.4.6, with every propositional letter true at every world, except p false at u_{n+1}.

5.5 Dynamics

Introducing the dynamic actions of public announcement and preference upgrade in the CPL version of Order Logic is straightforward, as I show in the present section. In the next section, I show that CPL can do more, by looking at new kinds of actions suggested by CPL modalities.

Public announcement and preference upgrade

To find a reduction principle for public announcement with *ceteris paribus* modalities, one needs to pay special attention to $\equiv_\Gamma$ in the original model $\mathfrak{M}$ and in its submodel $\mathfrak{M}|_A$ after announcement of A. Given a set of sentences Γ, let $\Gamma_{!A} := \{\langle !A\rangle\gamma : \gamma \in \Gamma\}$. The reduction axiom for public announcement is given in the following fact:

Fact 5.5.1 *The reduction axiom for CPL with public announcement is:*

$$\langle !A\rangle\langle\Gamma\rangle\varphi \quad\leftrightarrow\quad A \wedge \langle\Gamma_{!A}\rangle\langle !A\rangle\varphi \tag{5.18}$$

Proof of Fact The result follows from the observation that $u \equiv_{\Gamma_{!A}} v$ in $\mathfrak{M}$ iff $u \equiv_\Gamma v$ in $\mathfrak{M}|_A$. Indeed, let $\gamma \in \Gamma$ and consider two states u and v such that $\mathfrak{M}|_A, u \models \gamma$ and $\mathfrak{M}|_A, v \models \gamma$. Then u and v both satisfy γ after the announcement of A and they thus agree on $\langle !A\rangle\gamma$ in $\mathfrak{M}$. ◀

CPL can thus function at once in presence of information action updates. Similarly, preference upgrade can be introduce in *ceteris paribus* logic, this time taking $\Gamma_{\#A} := \{\langle \#A\rangle\gamma : \gamma \in \Gamma\}$:

Fact 5.5.2

$$\begin{aligned} \langle \# A\rangle\langle\Gamma\rangle\varphi & \leftrightarrow & A \wedge \langle \Gamma_{\# A}\rangle (A \wedge \langle \# A\rangle \varphi) \\ & \vee & \neg A \wedge \langle \Gamma_{\# A}\rangle\langle \# A\rangle\varphi \end{aligned} \tag{5.19}$$

Proof. Once again, the result follows from the observation that $u \equiv_{\Gamma_{\# A}} v$ in $\mathfrak{M}$ iff $u \equiv_{\Gamma} v$ in $\mathfrak{M}_{\# A}$. QED

Introducing public announcement and preference upgrade in CPL is thus fairly simple. But CPL can do more. Indeed, Axioms 5.11-5.13 in Section 5.2 suggest that it can reason with addition of formulas to the set Γ, bringing out the dynamic intuitions behind CPL. Indeed, one can see a formula occurring in the set Γ as splitting a model in two, one zone where it is true and the other where it is false. The next section elaborates on this intuition.

5.6 Agenda expansion: a new kind of dynamics

Consider the following CPL validity:

$$\begin{aligned} \langle \Gamma \cup A\rangle\varphi & \leftrightarrow & A \wedge \langle \Gamma\rangle (A \wedge \varphi) \\ & \vee & \neg A \wedge \langle \Gamma\rangle(\neg A \wedge \varphi) \end{aligned} \tag{5.20}$$

The right to left direction is an axiom of CPL. For the other direction, assume that $\mathfrak{M}, u \models \langle \Gamma \cup A\rangle\varphi$ and $\mathfrak{M}, u \models A$. Then there exists a v such that $u \preceq_{\Gamma\cup A} v$ and $\mathfrak{M}, v \models \varphi$. But $A \in \Gamma \cup A$ and $u \preceq_{\Gamma\cup A} v$ implies that $\mathfrak{M}, v \models A$ and $u \preceq_{\Gamma} v$ respectively. Hence, $\mathfrak{M}, u \models A \wedge \langle \Gamma \cup A\rangle\varphi$. The same argument applies in case $\mathfrak{M}, u \models \neg A$, which completes the proof.

The interest of (5.20) lies in having the form of a reduction axiom analyzing the *addition* of a sentence A to a set Γ in terms of Γ itself. On the basis of this standpoint, one may argue that CPL deals, implicitly, with dynamics of sets of distinguished formulas, called a *research agenda*. I use the term 'agenda' in a liberal way and only to refer to a set of distinguished formulas in a dynamic setting. I do not use 'agenda' in the sense of a schedule, where things have to be done in a specific order, say by importance. My use of the term is closer to instances such as a committee having to make a decision depending on agreed or prescribed items. For instance, a hiring committee might have to evaluate candidates with respect to given skills expected

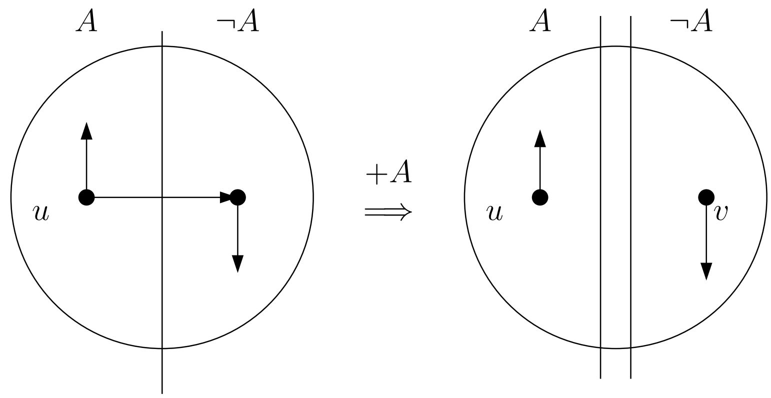

Figure 5.6: Simple representation of an agenda expansion. The double-line in the right model divides the model into an A-zone and a $\neg A$-zone. After the expansion, state v is no longer accessible from state u and no links are affected in either the A or the $\neg A$-zone.

from them, say being good in teaching as well as in research, and the candidates who fulfills best both criteria would be hired, irrespective of other properties they may have, such as sex or age. I discuss in what ways a *ceteris paribus agenda* distinguishes sentences after I have presented the formalism.

Equation 5.20 suggests introducing a primitive action of "agenda change" as well as a modality $\langle +A\rangle\varphi$ corresponding to agenda *expansion*. I achieve this by modifying *ceteris paribus* modalities $\langle\Gamma\rangle\varphi$ to *ceteris paribus* actions $\langle +A\rangle\varphi$ of adding a formula A to the agenda. This action can be analyzed compositionally with reduction axioms, just as the other dynamic modalities studied above.

The agenda expansion language, denoted $\mathcal{L}_{\mathcal{CPA}^+}$, is inductively defined by the following rules:

$$\mathcal{L}_{\mathcal{CPA}^+} := p \mid \varphi \vee \psi \mid \neg\varphi \mid \Diamond\varphi \mid \langle +A\rangle\varphi.$$

Models

Models have an additional component $\mathcal{A}$ consisting of a set of sentences.

Definition 5.6.1 [Models] An *agenda model* $\mathfrak{M}$ is a tuple $\mathfrak{M} = \langle W, \mathcal{A}, \preceq, V\rangle$ where $\langle W, \preceq, V\rangle$ is a standard modal model and $\mathcal{A}$ is a set of formula, called the *agenda*. $\lhd$

I use the notation '$\mathfrak{M} + A$' to denote the expansion of an agenda model $\mathfrak{M}$ given by $\mathfrak{M} + A = \langle W, \mathcal{A} \cup \{A\}, \preceq, V\rangle$ and I write $\mathcal{A} \cup A$ instead of $\mathcal{A} \cup \{A\}$.

Interpretation

Definition 5.6.2 [Truth definition] Let $\preceq_{\mathcal{A}} = \preceq \cap \equiv_{\mathcal{A}}$. The truth conditions for the propositions and the Booleans are standard.

$$\begin{array}{lll} \mathfrak{M}, u \models \Diamond\varphi & \text{iff} & \exists v \text{ such that } u \preceq_{\mathcal{A}} v \text{ and } \mathfrak{M}, v \models \varphi \\ \mathfrak{M}, u \models \langle +A \rangle\varphi & \text{iff} & \mathfrak{M} + A, u \models \varphi \end{array}$$

Satisfaction and validity over classes of models are defined as usual. ◁

Notice that the relation $\preceq$ is always in the background, but only a subsets of its links is available for $\Diamond$, depending on the agenda. Adding a formula to the agenda has the effect of reducing the number of available links from $\preceq_{\mathcal{A}}$, as with preference upgrade, but unlike public announcement, it does not eliminate worlds. In *Agenda Logic*, I have removed the explicit information about the ceteris paribus set Γ in the earlier operators $\langle \Gamma \rangle$ and relegated it to an implicit agenda given by the model, making the modality $\Diamond$ essentially context-dependent. The effect of agenda expansion is illustrated in figure 5.6.

Fact 5.6.3 *From (5.20), a reduction axiom for the modality $\langle +A \rangle\varphi$ in the base language is already available, providing a completeness proof for agenda expansion logic, denoted '$\Lambda^{\mathcal{L}_{CP\mathcal{A}^+}}$. In this logic, 5.20 becomes:*

$$\begin{array}{rcl} \langle +A \rangle \Diamond\varphi & \leftrightarrow & A \wedge \Diamond(A \wedge \langle +A \rangle\varphi) \\ & \vee & \neg A \wedge \Diamond(\neg A \wedge \langle +A \rangle\varphi) \end{array} \tag{5.21}$$

Proof of Fact A simple observation establishes the fact, namely that the action of adding A to the agenda eliminates links between A and $\neg A$-states, splitting the model into two disjoint components. Hence, if A is true and it is possible to go to an A-state such that φ is true after removing links to $\neg A$-states, then φ is possible after removing links to $\neg A$-states; and similarly for $\neg A$. ◀

Putting this analysis together with that of the preceding section, arbitrary dynamic formulas of public announcement, preference upgrade and agenda expansion can be reduced to equivalent ones in the basic language of CPL. Therefore,

Theorem 5.6.4 *The complete* ceteris paribus *logic of public announcement, preference upgrade and agenda expansion is axiomatized by (a) the complete system for*

CPL, (b) the reduction principle (5.18) given in Fact 5.5.1, (c) the reduction principle (5.19) given in Fact 5.5.2, and (d) the reduction principle (5.20).

Some interesting questions regarding the combined logic of public announcement and agenda expansion are not fully answered by the previous completeness result. For instance, there is the interesting general issues, which might be displayed with formulas, whether we have valid schematic laws for the following complexes:

$\langle !A\rangle\langle +B\rangle\varphi$: agenda addition after an announcement
$\langle +A\rangle\langle \# B\rangle\varphi$: preference upgrade after an agenda change

Furthermore, unlike for the case of public announcement, $\langle +A\rangle\langle +B\rangle\varphi$ is not equivalent to a formula with only one action of the form $\langle + *(A,B)\rangle\varphi$, where $*(A,B)$ is some formulas in terms of A and B. In other words, even though two successive public announcements are always equivalent to a single announcement, successive expansions are not in general equivalent to a single expansion. A modality that would be equivalent to $\langle +A\rangle\langle +B\rangle\varphi$ would be a 4-event action that divides the model in four equivalence classes.

As was the case with the other dynamic actions investigated in this thesis, agenda expansion can be expressed in the language of PDL.

Fact 5.6.5

$$\preceq_{\mathcal{A}+A} = \preceq_{\mathcal{A}} -((?A\,;\,\preceq_{\mathcal{A}}\,;\,?\neg A)\cup(?\neg A,;\,\preceq_{\mathcal{A}}\,;\,?A)) \tag{5.22}$$

Fact 5.6.5 shows that expansion can be defined in PDL in a way similar to preference upgrade, as can be seen from the definition of the latter in 2.18. Indeed, the following fact shows that expansion is a special case of upgrade.

Fact 5.6.6 *Agenda expansion is a special case of preference upgrade.*

Proof.

$$\begin{aligned}\preceq_{\mathcal{A}+A} &= \preceq_{\mathcal{A}} -((?A\,;\,\preceq_{\mathcal{A}}\,;\,?\neg A)\cup(?\neg A,;\,\preceq_{\mathcal{A}}\,;\,?A))\\ &= \preceq^{\# A}\cap\preceq^{\#\neg A}\end{aligned} \tag{5.23}$$

QED

Corollary 5.6.7 *The modality $\langle +A\rangle\varphi$ is expressible in terms of $\langle \# A\rangle\varphi$.*

Proof. Follows from Fact 5.6.6:

$$\langle +A\rangle\varphi \Leftrightarrow A \wedge \langle \# A\rangle\varphi \vee \neg A \wedge \langle \# \neg A\rangle\varphi$$

QED

$\Lambda^{\mathcal{L}_{\mathcal{CPA}}{}^{+}}$ thus analyzes the action of adding a formula A to a set of distinguished sentences $\mathcal{A}$ in a *ceteris paribus* setting. An obvious follow-up to this result is to investigate the retraction of a formula from the agenda. The remainder of this chapter explores this.

5.7 A challenge: agenda contraction

Analogously to $\mathcal{L}_{\mathcal{CPA}}{}^{+}$, the *agenda contraction* language $\mathcal{L}_{\mathcal{CPA}}{}^{-}$ is given by:

$$\mathcal{L}_{\mathcal{CPA}^{-}} := p \mid \varphi \vee \psi \mid \neg\varphi \mid \Diamond\varphi \mid \langle -A\rangle\varphi.$$

An obvious way to define agenda contraction is to take the converse operation of expansion on $\mathfrak{M}$: set-theoretical subtraction. Models are thus as in Definition 5.6.1 with $\mathfrak{M} - A$ defined by $\langle W, \mathcal{A} - \{A\}, \preceq, V\rangle$. Satisfaction of $\langle -A\rangle\varphi$ is then defined by:

$$\mathfrak{M}, u \models \langle -A\rangle\varphi \text{ iff } \mathfrak{M} - A, u \models \varphi \tag{5.24}$$

It is clear that any action of contraction of $\mathfrak{M}$ by A would be pictured as some reversal of Figure 5.6, going from a universe split into independent A and $\neg A$ zones, to a universe where links are reintroduced between them. A first challenge is to axiomatize this version of contraction, but it is not clear how one should proceed. Unfortunately, the technique of compositional analysis does not seem to be applicable, even in the presence of the existential modality. To see this, consider how compositional analysis is applied throughout the thesis. Reduction axioms always have the same form: the left-hand side of the axiom states that a certain formula holds after a model change and the right-hand side analyzes the conditions of the original model for this action to yield the said formula. In other words, the right-hand-side of a reduction axiom states when the action can be performed and predicts which formulas hold after the action has taken place. Hence the term ‘reduction’: the axiom reduces the analysis of a dynamic action to conditions of a static model. In the case of contraction, however, even though the relation $\preceq$ is always in the background of $\mathfrak{M}$, only a subset of its links are available in the model before contraction. Some information in the original model is thus lacking for analyzing the effect of contraction. I leave the question of axiomatizing agenda contraction open.[3]

A second problem, conceptually motivated, is what I call the problem of *successful* contraction. By a *successful* contraction, I mean that the re-introduction of links actually obtains. Consider the following simple example.

Example 5.7.1 Let there be a model with two worlds u and v such that both A and B are true at u and false at v and where $u \preceq v$. Let $\mathcal{A} = \{A, B\}$, then $\mathcal{A} - A = \{B\}$ is not enough to introduce a link between u and v, as B is not equal in both worlds. In this case, B also need to be retracted from the agenda for a link to be introduced between the A and the $\neg A$ zones. In this model, a successful contraction by A would have to be accompanied by the removal of B from $\mathcal{A}$, so that $\mathcal{A} - A = \emptyset$.

This problem is conceptually similar to the problem of successful contraction in belief revision, where the mere subtraction of A from a belief set is not sufficient to guarantee its retraction, for instance if B and $B \to A$ are in the belief set. But what counts as a successful contraction? How many and which links should be reintroduced? The introduction of one link between the two zones would be a minimal contraction and the introduction of a link from each A-state to every $\neg A$-state would be a maximal one. Intermediate strategies would be to introduce (at least) one link from each A-state to a $\neg A$ state - and vice-versa. But again, this can obtain in many ways.

To overcome this problem, care as to be taken that the contraction be *successful* and this may require that additional sentences from the agenda be retracted along with A. I use the notation '$\mathcal{A}^*$' to stand for the successful contraction of $\mathcal{A}$ by A.

Building on the previous example, I can make three further observations about successful contraction: 1) contraction is sensitive to states, 2) a contraction followed by an expansion does not always return the original agenda and 3) contraction by A may yield new links in each of the A and the $\neg A$ zone.

Observation 5.7.2 *Successful contraction $\mathcal{A}^*$ is state-dependent.*

[3] Notice that the converse action to preference upgrade would be just as problematic. This suggests that, in general, compositional analysis fails to deal with link (re)-introduction. This is a limitation worth further investigation, but not pursued here.

Proof of Observation Let $\mathcal{A} = \{A, B\}$ and consider a model with three worlds $\{u_1, u_2, v_1\}$ with the relation $\preceq$ being the universal relation and where $V(A) = \{u_1, u_2\}$ and $V(B) = \{u_2\}$. Notice that $\preceq_{\mathcal{A}} = \emptyset$, since no two worlds satisfy the same formulas from $\mathcal{A}$. Now, for contraction by A to be successful, links have to be re-introduced between the A and the $\neg A$ zones. To introduce a link between u_1 and v_1, retracting A from $\mathcal{A}$ is enough, since B is false in both worlds. Thus, $\mathcal{A}^*_{u_1} = \{B\}$. At u_2, however, the sole retraction of A is not sufficient, as B is not preserved between u_2 and v_1. In this case, $\mathcal{A}^*_{u_2} = \emptyset$. ◀

Observation 5.7.3 $(\mathcal{A} - A) + A \subseteq \mathcal{A}$

Proof of Observation Consider the same model as in Observation 5.7.2. Then $(\mathcal{A}_{u_2} - A) + A = \emptyset + A = \{A\} \subset \mathcal{A}$. ◀

Observation 5.7.4 *Contraction by A may yield additional accessible states in each of the A and the $\neg A$-zones.*

Proof of Observation Take again the model from example 5.7.2. Remember that $\preceq_{\mathcal{A}} = \emptyset$, even thought $\preceq$ is the universal relation. Hence, it is not the case that $u_2 \preceq_{\mathcal{A}} u_1$. But after contraction by A considered at u_2, since $\mathcal{A}^*_{u_2} = \emptyset$, it is the case that $u_2 \preceq_{\emptyset} u_1$. Therefore, not only links have been added between u_2 and v_1, but also inside the A-zone. ◀

Figure 5.7 represents a simple case of successful agenda contraction that summarizes the above considerations.

Summary

I have now completed the applications of Order Logic to belief and preference change for individuals, as well as the formalization of the equality reading of *ceteris paribus* logic. The latter was the main contribution of the present chapter. To achieve this, I have relativized the modalities of Order Logic with respect to sets of sentences Γ standing for the *other things* to be kept equal. I have provided a completeness results for the finite fragment of CPL, namely when the sets Γ are taken to be finite. This has shown explicitly the specific reasoning principles of the equality reading of *ceteris paribus*. This new kind of modal logic, called CPL, has raised interesting mathematical questions, particularly with respect to basic and infinitary

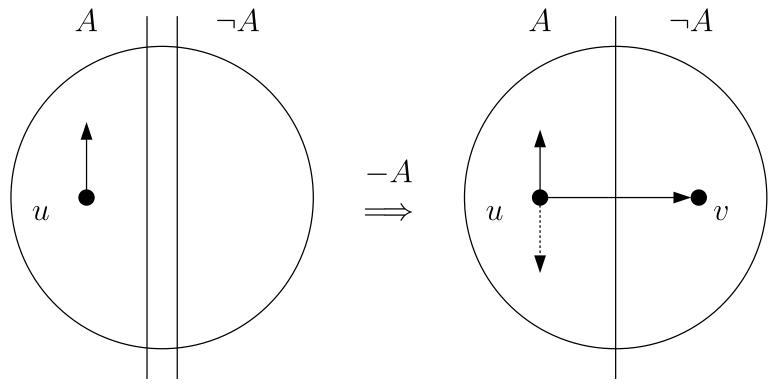

Figure 5.7: Simple representation of a successful agenda contraction. The double-line in the left model divides the model into an A-zone and a $\neg A$-zone. After the contraction, state u can access a state v in the $\neg A$-zone as well as additional states in the A-zone.

modal logics as well as logics lying in between them. After having introduced the actions of public announcement and preference upgrade, I have investigated new kinds of actions suggested by the form of CPL axioms dealing with the addition of formulas to Γ. The interpretation I have given to these new action was in terms of a *research agenda*.

Now, the notion of a research agenda seems to pertain to groups rather than individuals. Recall the example of a committee having to make decisions following specific items on the agenda. This suggests that agenda expansion would be an action more appropriate to a group logic. For this, however, a static Order Logic for groups is needed an the next and final chapter takes exactly this as a starting point.

Chapter 6

Group Order Logic

When Robert and I have decided to share a dessert, we have implicitly agreed to aggregate our preferences so as to find the best dessert for *us*. This meant that we had to choose a dessert that would satisfy both of us and we agreed that chocolate and strawberry desserts were the two best ones - in this order. I would have preferred a cheesecake to either one, but Robert hates it, in which case it would have been unfair to him if, as a group, we had one. On the other hand, his allergy to nuts has ruled out the chocolate cake, just as a meat dish is ruled-out when a vegetarian is part of a couple who agrees to order only one dish to be shared; this is just matters of fairness. In a way, one can see such cases as being a mixture of democratic aggregation of preferences with compensating options in case of disagreement. The democratic part in Robert and I's case was in opting for either chocolate or strawberry desserts, whereas the compensating part was choosing the latter because of Robert's allergy.

In this chapter, I formalize aggregation of individual preferences using lexicographic reordering, following [2]. In this setting, groups of agents are taken to be ordered and aggregation proceeds in a compensating way: if every member of the group opt for option x, then x is the group preference, otherwise the group endorses the preferences of the most influential agents. To achieve this, it is enough to expand $\mathcal{L}_{\mathcal{O}}$ with so-called nominals [3]. I call the resulting logic *Group Order Logic* and denote it '*GOL*'.

The chapter is divided as follows. In Section 6.1, I collect the results from [2] over which I base the rest of the chapter. In Section 6.2, I define *GOL* and provide its complete axiomatization. In Section 6.3, I investigate various applications of *GOL*. I show how the complete relational algebra of [2] can be derived and how to lift comparisons over states to preferences over sets of states. Finally, in Section 6.4, I introduce dynamics in the logic. This comes in two stages: 1) public announcement and preference upgrade, and 2) and *agent promotion*. Agent promotion is the action of changing the hierarchy of the group by putting one agent on top. My approach is thus advantageous in three ways: 1) it provides a simple modal logic for preference aggregation, 2) it lifts group preferences from objects to sets of objects and 3) it yields a straightforward dynamification of group preferences with a new kind of action over the hierarchy of the group.

6.1 Lexicographic reordering

In this section, I collect the results from [2] needed for GOL. The fundamental tools used in this paper are *priority graphs* and *priority operators*. A priority graph imposes a hierarchy among basic relations and a priority operator maps these relations to a single relation lexicographically, following the hierarchy provided by the graph. Although the results hold for the aggregation of arbitrary orders, I confine myself to the special case of preorders. No generality is lost by this choice. I thus take a priority graph as a hierarchy imposed on a group of agents and a priority operator as the aggregation of their preferences.[1]

Let W be a set containing at least two elements, standing for the set of objects to be compared and over which agents give their preference orders $\preceq_i$. For the remainder of this chapter, I identify agents with the order they give on W, but I keep the notation $i, j, ...$ to refer to agents instead of $\preceq_i, \preceq_j, ...$, for the sake of readability. For two objects $u, v \in W$, I say that agent i prefers v over u if $u \preceq_i v$; this is the order given by i between u and v.

To aggregate preferences, a partial strict order $<$ is imposed over the agents, which can occur multiple times in the order. To help building the intuition, one can think of this order as providing credibility or reliability criteria. Suppose you have a committee of scientists (geologists, physicists, chemists etc.) investigating the effect of human societies on global climate change. It is natural to expect the scientists to have more weights on the interpretation of the results in their respective fields. Physicists would be given more weight in answering questions pertaining to physics, geologists to geological questions, and so on. One can think of the multiple occurrences of

[1] To keep notation consistent in the thesis, I write $i < j$ to express that j is strictly better than i, the opposite notation of that used in [2]. I also draw pictures for priority graphs by putting best agents on top of the graph where [2] puts them at the bottom. The reader is asked to keep that in mind if reading this chapter and [2] in parallel.

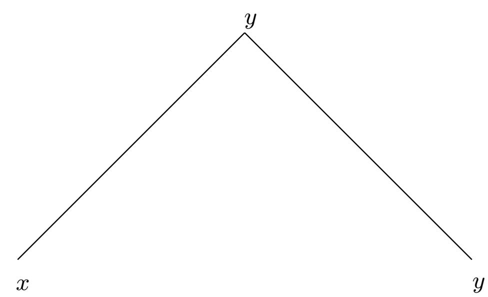

Figure 6.1: Graphical representation of the priority graph g defined in example 6.1.2. In the figure, a variable occurring above another variable is prioritized.

agents in the order as representing exactly this: expertise. The hierarchy of agents is represented in a *priority graph*.

Definition 6.1.1 [Priority graph] Let X be a set of variables. A *priority graph* is a tuple $P = \langle N, <, V \rangle$ where N is a set, $<$ is a strict partial order on N and V is function from N to X. $\triangleleft$

Something has to be said about the use of variables in priority graphs. First, as I have noted above, agents are identified with the order they give on W. Two agents giving the same order can thus be identified in priority graphs, thus eliminating useless replication of the same information in graphs. Second, priority graphs allow variable to occur several times, so that assigning agents to variables allows a representation of expertise in graphs; in one occurrence an agent is the expert and in another, it is dominated. Finally, the repetition of variables in priority graphs increases the expressivity, although I do not prove this; the reader can consult [2] for a proof.

Example 6.1.2 Let $N = \{i, j, k\}$, $V(i) = V(j) = y$, $V(k) = x$ and let $j < i, k < i$. The priority graph $g = \langle N, <, V \rangle$ is represented graphically in figure 6.1. In the figure, as well as in the remainder of this chapter, a variable x occurs above a variable y if $y < x$. Notice that this priority graph is equivalent to a simpler one using only two nodes, x and y, with x below y.

Next, I define the notion of a priority operator. A priority operator aggregates individual preferences lexicographically according to the priority graph. A lexicographic order $\leq$ between two elements a and b is an order such that:

$$(a, b) \leq (a', b') \text{ iff } a < a' \text{ or } (a = a' \text{ and } b \leq b')$$

This definition generalizes to n-tuples in the following way:

$$(a_1, a_2, ...a_n) \leq (b_1, b_2, ..., b_n) \text{ iff } \exists m > 0, \forall i < m (a_i = b_i \text{ and } a_m <_m b_m)$$

A familiar example of a lexicographic ordering is the alphabetic order used in a dictionary, where the priority is given to letters on the left. For instance, the word 'animal' comes before the word 'apple', since there is a position (the second position) such that the letters occurring in the first position are identical and the letters occurring in the second position are such that $n \leq p$.

In the case of priority graphs with single occurrences of variables, a priority operator o orders the relations of the graph lexicographically when:

$$a \preceq b \text{ iff } \forall x \in V(a \preceq_x b \text{ or } \exists y \in V(x < y \,\&\, a \prec_y b))$$

Since priority graphs allow variables to occur multiple times, a further generalization of the lexicographic rule is needed:

Definition 6.1.3 [Priority operator] A priority graph g denotes a *priority operator* o if:

$$ao((\preceq_x)_{x \in X} b \Leftrightarrow \forall i \in N(a \preceq_{V(i)} b \vee \exists j \in N(i < j \wedge a \prec_{V(j)} b)) \tag{6.1}$$

$\lhd$

Example 6.1.4 Consider the priority graph given in Example 6.1.2. Let $a, b \in M$, then according to Definition 6.1.3:

$$\begin{aligned} ao(\preceq_x, \preceq_y)b \quad \text{iff} \quad & (a \preceq_{V(i)} b \wedge a \preceq_{V(j)} b \wedge a \preceq_{V(k)} b) \vee a \prec_{V(k)} b \\ \text{iff} \quad & (a \preceq_x b \wedge a \preceq_y b) \vee a \prec_y b \end{aligned}$$

Therefore, the group consisting of $\{i, j, k\}$ considers b better than a if they reach a consensus or if both i and j strictly prefer b to a.

The next two theorems, for whose proof the reader should consult [2], are crucial in modalizing Group Order Logic. Theorem 6.1.6 states that every priority operator is equivalent to one build from two fundamental operators given in Definition 6.1.5 and Theorem 6.1.7 provides a complete relational algebra in terms of these operators.

Definition 6.1.5 The two operators $x \parallel y$ and x/y are called the *but* and *on the*

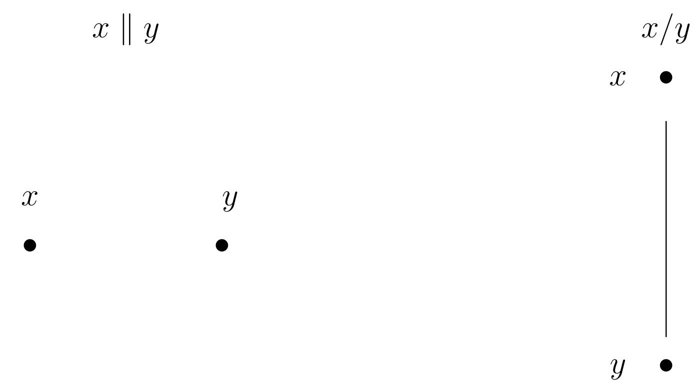

Figure 6.2: The priority graph of the *but* and *on the other hand* operators.

other hand operators respectively and are defined by:

$$\begin{aligned} x \parallel y &= x \cap y \\ x/y &= (x \cap y) \cup x^{<} \end{aligned}$$

$\triangleleft$

The two operators are depicted in Figure 6.2. Here are the two crucial theorems:

Theorem 6.1.6 *Any finitary priority operator is denoted by a term build from $/$, $\parallel$ and the variables occurring in the priority graph for the operator.*

Theorem 6.1.7 *An equation is true in all preferential algebras iff it is derivable from the following axioms:*

$$x \parallel x = x \tag{6.2}$$

$$x \parallel (y \parallel z) = (x \parallel y) \parallel z \tag{6.3}$$

$$x \parallel y = y \parallel x \tag{6.4}$$

$$(x/x) = x \tag{6.5}$$

$$(x/y)/z = x/(y/z) \tag{6.6}$$

$$x/(y \parallel z) = (x/y) \parallel (x/z) \tag{6.7}$$

$$(x/y) \parallel y = x \parallel y \tag{6.8}$$

6.2 Modal logic for order aggregation

As I mentioned above, the language for Group Order Logic is obtained by expanding $\mathcal{L}_{\mathcal{O}}$ with so-called nominals. In contrast to propositional variables, which are typically evaluated at sets of states, nominal are evaluated at single states. In this respect, states in models can be interpreted as simple objects and nominals as names for these objects. I introduce nominals in the logic to lift individual orders to group orders, using lexicographic reordering. with the existential modality, I then lift group orders among objects to group preferences over sets of objects, just as I did in Section 4.2 in the individual case. There are thus two orthogonal lifts in the present chapters, one from individual to group orders over objects and the other from group orders over objects to group preferences over sets of objects - or propositions.

Language and semantics

Definition 6.2.1 (Language) *Let* PROP *be a set of propositional variables with* $p \in$ PROP *and* NOM *a set of nominals with* $s \in$ NOM*. I use the letters* i, j, k *for single agents, but in the spirit of Andréka et all, these variables range over whole preference relations - agents are identified with their orders. The language* $\mathcal{L}_{\mathcal{GP}}$ *is defined by the following recursive rules:*

$$
\begin{array}{rcl}
\varphi & := & s \mid p \mid \neg\varphi \mid \varphi \vee \psi \mid \langle X\rangle^{\leq}\varphi \mid \langle X\rangle^{<}\varphi \mid E\varphi \\
X & := & i \mid X/Y \mid X \parallel Y
\end{array}
$$

The intended reading of the modalities is as follows. $\langle i\rangle^{\leq}\varphi$ stands for 'agent i thinks that an accessible state where φ holds is at least as good as the current state' and $\langle i\rangle^{<}\varphi$ stands for 'according to agent i, there is an accessible state that is strictly better where φ holds'. Complex modalities with capital variables X, Y, Z and their combinations with the operators / and $\parallel$ stand for group orders. Giving an intuitive and succinct reading of modalities for groups of more than two agents is not straightforward, although the intention is quite clear. I illustrate the meaning of these modalities with a simple group of two agents i and j. In the case where there is a hierarchy, say of a master i and a student j, the modality $\langle i/j\rangle\varphi$ is read as 'the group consisting of a master i and a student j considers an accessible state where φ holds to be at least as good as the current state.' If the two agents i and j have the same weight in the decision making, the case where $i \parallel j$, the modality $\langle i \parallel j\rangle^{\leq}\varphi$ is read as 'the group consisting of two agents of incomparable rank considers an accessible state where φ holds to be at least as good as the current state'. The strict modalities receive the obvious analogous readings.

Definition 6.2.2 (Models) *Models are tuples* $\mathfrak{M} = \langle W, G, \{\preceq_X\}_{X \in G}, V \rangle$*, where* W *is a set of states,* G *is a set of graphs,* $\{\preceq_X\}_{X \in G}$ *is a family of relations induced by priority graphs and* $V : \textsc{prop} \cup \textsc{nom} \to \mathcal{P}(W)$ *is valuation that assigns a singleton set for members of* NOM.

Definition 6.2.3 (Semantics)

$$\begin{array}{lll}
\mathfrak{M}, u \models p & \textit{iff} & u \in V(p) \\
\mathfrak{M}, u \models s & \textit{iff} & \{u\} = V(s) \\
\mathfrak{M}, u \models \neg\varphi & \textit{iff} & \mathfrak{M}, u \not\models \varphi \\
\mathfrak{M}, u \models \varphi \vee \psi & \textit{iff} & \mathfrak{M}, u \models \varphi \textit{ or } \mathfrak{M}, u \models \psi \\
\mathfrak{M}, u \models \langle X \rangle^{\leq}\varphi & \textit{iff} & \exists v \in W \textit{ s.t. } u \preceq_X v \,\&\, \mathfrak{M}, v \models \varphi \\
\mathfrak{M}, u \models \langle X \rangle^{<}\varphi & \textit{iff} & \exists v \in W \textit{ s.t. } u \prec_X v \,\&\, \mathfrak{M}, v \models \varphi \\
\mathfrak{M}, u \models E\varphi & \textit{iff} & \exists v \textit{ s.t. } \mathfrak{M}, v \models \varphi
\end{array}$$

In case $X = \{i\}$*,* $\preceq_i$ *is the betterness relation for a single agent. Complex relations* $\preceq_X$ *are recursively reduced to individual preference relations using* $\preceq_{X/Y} = (\preceq_X \cap \preceq_Y) \cup \prec_Y$ *and* $\preceq_{X \| Y} = \preceq_X \cap \preceq_Y$*. The strict subrelations* $\prec_X$ *are defined in the standard way, i.e.,* $u \prec_X v$ *iff* $u \preceq_X v \,\&\, \neg(v \preceq_X u)$.

Remark 6.2.4 The hybrid binder $@_i\varphi$ is definable with the existential modality and nominals by $E(i \wedge \varphi)$. Therefore, Group Order Logic is a superset of the hybrid logic $\mathcal{H}(@)$ (cf. [3]).

Axiomatization

Theorem 6.2.5 *Let* $\mathrm{MOD} = \{\langle X \rangle^{\leq}\varphi, \langle X \rangle^{<}\varphi, E\varphi, [X]^{\leq}\varphi, [X]^{<}\varphi, U\varphi\}$*. The following set of axioms is complete for Group Order Logic. I call the logic* Λ_{GP}.

1. *Classical tautologies.*

2. *Normality axioms, with* $\Diamond, \Box \in \mathrm{MOD}$*:*

$$\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi \tag{6.9}$$

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \tag{6.10}$$

$$\tag{6.11}$$

3. Axioms for the existential modality:

$$\varphi \to E\varphi \tag{6.12}$$

$$EE\varphi \to E\varphi \tag{6.13}$$

$$\varphi \to UE\varphi \tag{6.14}$$

$$Ei \tag{6.15}$$

$$E(i \wedge \varphi) \to U(i \to \varphi) \tag{6.16}$$

4. Axioms defining properties of $\preceq$ and $\prec$:

$$s \to \langle X\rangle^{\leq} s \quad \textit{Reflexivity of } \leq_X \tag{6.17}$$

$$\langle X\rangle^{\leq}\langle X\rangle^{\leq} s \to \langle X\rangle^{\leq} s \quad \textit{Transitivity of} \leq_X \tag{6.18}$$

$$s \to \neg\langle X\rangle^{<} s \quad \textit{Irreflexivity of } <_X \tag{6.19}$$

$$s \to \neg\langle X\rangle^{<}\langle X\rangle^{<} s \quad \textit{Assymetry of } <_X \tag{6.20}$$

$$\langle X\rangle^{<} s \to \langle X\rangle^{\leq} s \quad \textit{Inclusion} \tag{6.21}$$

$$\langle X\rangle^{\leq}\langle X\rangle^{<} s \to \langle X\rangle^{<} s \quad \textit{Mix 1} \tag{6.22}$$

$$\langle X\rangle^{<}\langle X\rangle^{\leq} s \to \langle X\rangle^{<} s \quad \textit{Mix 2} \tag{6.23}$$

$$s \wedge \langle X\rangle^{\leq} t \to (\langle X\rangle^{<} t \vee \langle X\rangle^{\leq}(t \wedge \langle X\rangle^{\leq} s) \quad \textit{Mix 3} \tag{6.24}$$

5. Mixed axioms:

$$\langle X\rangle^{\leq}\varphi \to E\varphi \tag{6.25}$$

$$\langle X\rangle^{<}\varphi \to E\varphi \tag{6.26}$$

6. *Group axioms:*

$$\langle X \parallel Y\rangle^{\leq} s \leftrightarrow \langle X\rangle^{\leq} s \wedge \langle Y\rangle^{\leq} s \tag{6.27}$$

$$\langle X/Y\rangle^{\leq} s \leftrightarrow (\langle X\rangle^{\leq} s \wedge \langle Y\rangle^{\leq} s) \vee \langle X\rangle^{<} s \tag{6.28}$$

$$\langle X \parallel Y\rangle^{<} s \leftrightarrow (\langle X\rangle^{<} s \wedge \langle Y\rangle^{\leq} s) \vee (\langle X\rangle^{\leq} s \wedge \langle Y\rangle^{<} s) \tag{6.29}$$

$$\langle X/Y\rangle^{<} s \leftrightarrow (\langle X\rangle^{\leq} s \wedge \langle Y\rangle^{<} s) \vee \langle X\rangle^{<} s \tag{6.30}$$

In addition, Λ_{GP} has the rules of Modus Ponens, Necessitation and the hybrid logic rules NAME *and* PASTE*.*

Remark 6.2.6 1. Transitivity of $\langle X\rangle^{<}\varphi$ is derivable, as it was in Order Logic.

2. Axioms 6.27 and 6.28 analyze the (weak) modalities $\langle X \parallel Y\rangle^{\leq}$ and $\langle X/Y\rangle^{\leq}$ in terms of $\langle X\rangle^{\leq}$, $\langle X\rangle^{<}$ and $\langle Y\rangle^{<}$. Similarly, the strict modalities $\langle X||Y\rangle^{<}$ and $\langle X/Y\rangle^{<}$ can be analyzed in terms of more basic modalities by axioms 6.29 and 6.30.

Before proving completeness, some preliminary results are needed. Recall from Definition 2.1.9 that a model is called $\prec$-*adequate* if the following are equivalent:

1. $w \prec v$

2. (a) $w \preceq v$ and

   (b) $v \not\preceq w$.

In GOL, since there are multiple individual and group betterness orders, a generalization of this definition is needed.

Definition 6.2.7 A model is $\preceq_X$-*adequate* if the following are equivalent, for every priority graph X:

1. $w \prec_X v$

2. (a) $w \preceq_X v$ and

   (b) $v \not\preceq_X w$.

◁

In Order Logic, the axiomatization guarantees quasi-adequacy, namely the direction from (2) to (1). Here, resorting to nominals, better can be achieved, namely:

Fact 6.2.8 *1. If a model $\mathfrak{M}$ is based on an $\prec_X$-adequate frame, then $\mathfrak{M} \models s \wedge \langle X\rangle^{\leq}t \rightarrow (\langle X\rangle^{<}t \vee \langle X\rangle^{\leq}(t \wedge \langle X\rangle^{\leq}s))$ and $\mathfrak{M} \models s \rightarrow \neg\langle X\rangle^{<}s$.*

2. For every frame $\mathfrak{F}$, if $\mathfrak{M} \models s \wedge \langle X\rangle^{\leq}t \rightarrow (\langle X\rangle^{<}t \vee \langle X\rangle^{\leq}(t \wedge \langle X\rangle^{\leq}s)$, $\mathfrak{M} \models s \rightarrow \neg\langle X\rangle^{<}s$ and $\mathfrak{M} \models \langle X\rangle^{<}\langle X\rangle^{\leq}s \rightarrow \langle X\rangle^{<}s$, then $\mathfrak{F}$ is $\prec_X$-adequate.

Proof of Fact 6.2.8 It is easy to see that $\mathfrak{M} \models \langle X\rangle^{<}\langle X\rangle^{\leq}s \rightarrow \langle X\rangle^{<}s$. The proof of Fact 2.1.10 can be transposed here to show the result for quasi-adequacy. All that remains to be shown is that $\mathfrak{M} \models s \rightarrow \neg\langle X\rangle^{<}s$ and that $u \prec_X v \Rightarrow v \npreceq_X u$.

1. Let $\mathfrak{M}$ be based on an adequate frame, and let $u \in W$ such that $\mathfrak{M}, u \models s$. Suppose that $\mathfrak{M}, u \models \langle X\rangle^{<}s$, then there is a $v \in W$ such that $u \prec_X v$ and $\mathfrak{M}, v \models s$. But $u \prec_X v$ implies that $u \neq v$, hence that $V(s) = \{u, v\}$. This is contradiction, since s is a nominal and must be assigned a singleton set. Therefore, $\mathfrak{M}, u \models \neg\langle X\rangle^{<}s$.

2. Assume that $u \prec_X v$. Take a model $\mathfrak{M}$ with $V(s) = u$. Suppose that $v \preceq_X u$, then $\mathfrak{M}, v \models \langle X\rangle^{\leq}s$. Since $u \prec_X v$, we get that $\mathfrak{M}, u \models \langle X\rangle^{<}\langle X\rangle^{\leq}s$. Now, $\mathfrak{M}, u \models \langle X\rangle^{<}\langle X\rangle^{\leq}s \rightarrow \langle X\rangle^{<}s$, thus $\mathfrak{M}, u \models \langle X\rangle^{<}s$. Hence, $\mathfrak{M}, u \models s \wedge \langle X\rangle^{<}s$, a contradiction with $\mathfrak{M}, u \models s \rightarrow \neg\langle X\rangle^{<}s$. Therefore, $v \npreceq_X u$, as required.

◀

Proof of Theorem 6.2.5. I only show soundness of Axioms 6.29 and 6.30. I give a semantic argument here and refer the reader to the appendix for an algebraic derivation.

To show the soundness of Axioms 6.29, it is enough to show that:

$$\prec_{X\|Y} = (\prec_X \cap \preceq_Y) \cup (\preceq_X \cap \prec_Y)$$

In the first direction, assume that $u \prec_{X\|Y} v$. By definition, $u \preceq_{X\|Y} v$ and $\neg v \preceq_{X\|Y} u$. Thus, $u \preceq_X v$ and $u \preceq_Y v$. It is now enough to show that $u \prec_X v$ or $u \prec_Y v$. Suppose not, then $v \preceq_X u$ and $v \preceq_Y u$, which implies that $v \preceq_{X\|Y} u$, a contradiction.

In the other direction, assume that $(u \prec_X v \,\&\, u \preceq_Y v)$ or $(u \preceq_X v \,\&\, u \prec_Y v)$. In either case, $u \preceq_{X\|Y} v$. Now, if $\neg u \prec_{X\|Y} v$, then it must be that $v \preceq_{X\|Y} u$, i.e., $v \preceq_X u$ and $v \preceq_Y u$. Hence, $\neg u \prec_X v$ and $\neg u \prec_Y v$, a contradiction.

For Axiom 6.30, I show the following:

$$u \prec_{X/Y} v = (u \preceq_X v \,\&\, u \prec_Y v) \text{ or } u \prec_X v \tag{6.31}$$

In the first direction, assume that $u \prec_{X/Y} v$. Then $u \preceq_{X/Y} v$ and $\neg v \preceq_{X/Y} u$, which implies, by definition, that $(u \preceq_X v \,\&\, u \preceq_Y v)$ or $u \prec_X v$. In the latter case, the result follows, so I show that $(u \preceq_X v \,\&\, u \preceq_Y v) \,\&\, \neg v \preceq_{X/Y} u$ implies the right-hand side of 6.31. Since $u \preceq_X v$, it is enough to show that $u \prec_X v$ or $u \prec_Y$. Suppose not, then $v \preceq_X u$ and $v \preceq_Y u$, since $u \preceq_X v$ and $u \preceq_Y v$. Thus, $v \preceq_{X/Y} u$, a contradiction.

In the other direction, assume that the right-hand side of 6.31 holds. Since $u \prec_Y v$ implies that $uR \preceq_Y v$, we have that $uR \preceq_{X/Y} v$. Suppose that $\neg u \prec_{X/Y} v$, then we must have that $vR \preceq_{X/Y} u$, i.e., $(v \preceq_X u \,\&\, v \preceq_Y u)$ or $v \prec_X u$. The first disjunct implies that $\neg u \prec_X v$ and $\neg u \prec_Y v$, whereas the second implies that $\neg u \preceq_X v$, and we obtain a contradiction in either case.

For completeness, I use the following well-known result of hybrid logic, stated as Corollary 5.4.1 in [63]:

> Let Σ be any set of pure $\mathcal{H}(E)$-formulas[2], the $K^+_{\mathcal{H}(E)}$ is strongly complete for the class of frames defined by Σ.

From this result, every consistent set Φ is satisfiable in the canonical model (named and pasted). Furthermore, thanks to Fact 6.2.8, this model is adequate. Thus, unlike in the case of basic preference logic, there is no need for a transformation of the model to eliminate $\preceq$-clusters; nominals and Axiom 6.19 relieve us of this task. Finally, thanks to Axioms 6.27-6.30, the completeness for group modalities is reduced to that of individual modalities. QED

6.3 Applications

Group Order Logic vs equational algebra

In Chapter 3, I have shown that Broccoli Logic can be seen as a fragment of Order Logic. Two results have supported this claim: 1) Fact 3.3.3, showing that the modalities $[\varphi]\psi$ and $[\varphi\rangle\psi$ can be defined in $\mathcal{L}_{\mathcal{O}}$, and 2) Theorem 3.3.14, identifying the $[\varphi]\psi$ fragment with minimal conditional logic. Here, I do something analogical by showing that the equational algebra of Theorem 6.1.6 can be derived inside Λ_{GP}. This is yet another instance displaying the unifying power of my approach. This result is contained in the following theorem:

---

[2] A formula is pure is it has no propositional variables.

Theorem 6.3.1 *The complete relational algebra of Section is derivable in group order logic.*

Proof. I first translate the equations given in Theorem 6.1.7:

$$\langle X \parallel X\rangle^{\leq}s \leftrightarrow \langle X\rangle^{\leq}s \tag{6.32}$$
$$\langle X \parallel (Y \parallel Z)\rangle^{\leq}s \leftrightarrow \langle (X \parallel Y) \parallel Z\rangle^{\leq}s \tag{6.33}$$
$$\langle X \parallel Y\rangle^{\leq}s \leftrightarrow \langle Y \parallel X\rangle^{\leq}s \tag{6.34}$$
$$\langle (X/X)\rangle^{\leq}s \leftrightarrow \langle X\rangle^{\leq}s \tag{6.35}$$
$$\langle (X/Y)/Z)\rangle^{\leq}s \leftrightarrow \langle X/(Y/Z)\rangle^{\leq}s \tag{6.36}$$
$$\langle X/(Y \parallel Z\rangle^{\leq}s \leftrightarrow \langle (X/Y) \parallel (X/Z)\rangle^{\leq}s \tag{6.37}$$
$$\langle (X/Y) \parallel Y\rangle^{\leq}s \leftrightarrow \langle X \parallel Y\rangle^{\leq}s \tag{6.38}$$

Equations (6.32)-(6.34) are easily derivable using Axiom (6.27). I show how to derive the remaining formulas, keeping Equation 6.36 for the last, at it is the most difficult:

1. Equation (6.35):

$$\begin{array}{rcll} \langle (X/X)\rangle^{\leq}s & \leftrightarrow & (\langle X\rangle^{\leq}s \wedge \langle X\rangle^{\leq}s) \vee \langle X\rangle^{<}s & \text{(Axiom 6.28)} \\ & \leftrightarrow & \langle X\rangle^{\leq}s & \text{(Logic and Axiom 6.21)} \end{array}$$

2. Equation (6.37):

$$\begin{array}{rcll} \langle X/(Y \parallel Z)\rangle^{\leq}s & \leftrightarrow & (\langle X\rangle^{\leq}s \wedge \langle Y \parallel Z\rangle^{\leq}s) \vee \langle X\rangle^{<}s & \text{(Axiom 6.28)} \\ & \leftrightarrow & (\langle X\rangle^{\leq}s \wedge \langle Y\rangle^{\leq}s \wedge \langle Z\rangle^{\leq}s) \vee \langle X\rangle^{<}s & \text{(Axiom 6.27)} \\ & \leftrightarrow & (((\langle X\rangle^{\leq}s \wedge \langle Y\rangle^{\leq}x) \vee \langle X\rangle^{<}s) & \\ & & \wedge(((\langle X\rangle^{\leq}s \wedge \langle Z\rangle^{\leq}s) \vee \langle X\rangle^{<}s) & \text{(Logic)} \\ & \leftrightarrow & \langle (X/Y) \parallel (X/Z)\rangle^{\leq}s & \text{(Axiom 6.27, 6.28)} \end{array}$$

3. Equation (6.38). The right to left direction follows from Axiom 6.27 and Logic. I prove the left to right direction:

$$\begin{array}{rcll} \langle (X/Y) \parallel Y\rangle^{\leq}s & \rightarrow & (((\langle X\rangle^{\leq}s \wedge \langle Y\rangle^{\leq}s) \vee \langle X\rangle^{<}s) \wedge \langle Y\rangle^{\leq}s & \text{(Axiom 6.27, 6.28)} \\ & \rightarrow & (\langle X\rangle^{\leq}s \wedge \langle Y\rangle^{\leq}s) \vee (\langle X\rangle^{<}s \wedge \langle Y\rangle^{\leq}s) & \text{(Logic)} \\ & \rightarrow & \langle X\rangle^{\leq}s \wedge \langle Y\rangle^{\leq}s & \text{(Logic and Axiom 6.21)} \\ & \rightarrow & \langle X \parallel Y\rangle^{\leq}s & \text{(Axiom 6.27)} \end{array}$$

4. Equation (6.36). I first prove a preliminary lemma that is crucial in the main proof:

Lemma 6.3.2 $\vdash (\langle X\rangle^{\leq}s \wedge \langle Y\rangle^{\leq}s) \vee \langle X\rangle^{<}s \leftrightarrow \langle X\rangle^{\leq}s \wedge (\langle Y\rangle^{\leq}s \vee \langle X\rangle^{<}s)$*:*

Proof.

$$\begin{array}{rcll} (\langle X\rangle^{\leq}s \wedge \langle Y\rangle^{\leq}s) \vee \langle X\rangle^{<}s & \leftrightarrow & (\langle X\rangle^{\leq}s \vee \langle X\rangle^{<}s) \wedge (\langle Y\rangle^{\leq}s \vee \langle X\rangle^{<}s) & \text{(Logic)} \\ & \leftrightarrow & \langle X\rangle^{\leq}s \wedge (\langle Y\rangle^{\leq}s \vee \langle X\rangle^{<}s) & \text{(Axiom 6.21)} \end{array}$$

QED

I use the following abbreviations:

$$\begin{array}{rcl} \alpha & := & \langle X\rangle^{\leq}s \wedge \langle Y\rangle^{\leq}s \wedge \langle Z\rangle^{\leq}s \\ \beta & := & (\langle X\rangle^{\leq}s \wedge \langle Y\rangle^{<}s) \vee \langle X\rangle^{<}s \end{array}$$

Now for the main proof:

$$\begin{array}{rcll} \langle (X/Y)/Z)\rangle^{\leq}s & \leftrightarrow & (\langle X/Y\rangle^{\leq}s \wedge \langle Z\rangle^{\leq}s) \vee \langle X/Y\rangle^{<}s & \text{(Axiom 6.28)} \\ & \leftrightarrow & [((\langle X\rangle^{\leq}s \wedge \langle Y\rangle^{\leq}s) \vee \langle X\rangle^{<}s) \wedge \langle Z\rangle^{\leq}s] \vee \beta & \text{(Axioms6.28, 6.30)} \\ & \leftrightarrow & [\langle X\rangle^{\leq}s \wedge ((\langle Y\rangle^{\leq}s \vee \langle X\rangle^{<}s) \wedge \langle Z\rangle^{\leq}s)] \vee \beta & \text{(Lemma6.3.2)} \\ & \leftrightarrow & \alpha \vee (\langle X\rangle^{\leq}s \wedge \langle X\rangle^{<}s \wedge \langle Z\rangle^{\leq}s) \vee \beta & \text{(Logic)} \\ & \leftrightarrow & \alpha \vee \beta & \text{(Logic!)} \\ & \leftrightarrow & [((\langle Y\rangle^{\leq}s \wedge \langle Z\rangle^{\leq}s) \vee \langle Y\rangle^{<}s) \wedge \langle X\rangle^{\leq}s] \vee \langle X\rangle^{<}s & \text{(Logic)} \\ & \leftrightarrow & (\langle X\rangle^{\leq}s \wedge \langle Y/Z\rangle^{\leq}s) \vee \langle X\rangle^{<}s & \text{(Axiom 6.28)} \\ & \leftrightarrow & \langle X/(Y/Z)\rangle^{\leq}s & \text{(Axiom 6.28)} \end{array}$$

QED

Corollary 6.3.3 *The equational algebra of Theorem 6.1.7 is decidable.*

Proof. Notice that in the proof of Theorem 6.3.1, I only appealed to Axioms 6.21 and the Group Axioms 6.27-6.30. Furthermore, I did not use the existential modality. Hence, the equational algebra of Theorem 6.3.1 is derivable in the fragment of group order logic that only uses normal modalities and nominals, which is a decidable system. QED

Remark 6.3.4 The decidability of 6.3.1 is not surprising, as it is translatable into the two-variable fragment of first-order logic (p.c. Hajnal Andréka). Theorem 6.3.1 establishes that GOL subsumes the algebraic treatment in a natural way, just as regular algebra gets subsumed in a straightforward manner by PDL. It is still an open question whether the full Group Order Logic is decidable. Indeed, since the basic order relations are transitive, we are no longer in the two-variable fragment of first-order logic. Furthermore, the presence of transitivity may make a system undecidable, as in the case of the guarded fragment with transitivity (p.c. Balder Ten Cate). One might then wonder why I have introduced the existential modality in the system. The reason is that it can lift orders over objects to preferences over sets of object with it, as I show in the next section.

Binary group preferences

In chapter 4, I reduced binary preference statements among formulas to sentences using unary modalities and the existential modality. Exactly the same can be done for binary group preferences between formulas. In the next definitions, let $X \in \{i, Y \parallel Z, Y/Z\}$.

Definition 6.3.5 [Binary group preference statements]

$$\begin{aligned}
\varphi \leq^X_{\exists\exists} \psi &:= E(\varphi \wedge \langle X\rangle^{\leq}\psi) && (6.39)\\
\varphi \leq^X_{\forall\exists} \psi &:= U(\varphi \rightarrow \langle X\rangle^{\leq}\psi) && (6.40)\\
\varphi <^X_{\exists\exists} \psi &:= E(\varphi \wedge \langle X\rangle^{<}\psi) && (6.41)\\
\varphi <^X_{\forall\exists} \psi &:= U(\varphi \rightarrow \langle X\rangle^{<}\psi) && (6.42)\\
\varphi <^X_{\forall\forall} \psi &:= U(\psi \rightarrow [X]^{\leq}\neg\varphi) && (6.43)\\
\varphi >^X_{\exists\forall} \psi &:= E(\varphi \wedge [X]^{\leq}\neg\psi) && (6.44)\\
\varphi \leq^X_{\forall\forall} \psi &:= U(\psi \rightarrow [X]^{<}\neg\varphi) && (6.45)\\
\varphi \geq^X_{\exists\forall} \psi &:= E(\varphi \wedge [X]^{<}\neg\psi) && (6.46)
\end{aligned}$$

Following Fact 4.2.3, models have to be total for the last four definitions to hold. $\lhd$

Hence, as I have claimed above, GOL performs two kinds of lifts: 1) from individual orders to group orders and 2) from basic order relations between objects to binary preferences between sets of objects, or proposition.

6.4 Dynamics

Simple operations of public announcement and preference upgrade can be incorporated into Group Order Logic, given the completeness result of the previous section. As was the case in CPL, GOL suggest yet a new kind of action over priority graphs.

Public announcement and preference upgrade

Public announcement and preference upgrade are easily integrated in GOL, as can be seen in the two following theorems.

Theorem 6.4.1 *The complete logic of group order logic with public announcement is axiomatized by: 1) the logic Λ_{GP} together with 2) the reduction axioms of public announcement, notably:*

$$\langle !A\rangle\langle X\rangle^{\leq}\varphi \leftrightarrow A \wedge \langle X\rangle^{\leq}\langle !A\rangle\varphi \tag{6.47}$$

$$\langle !A\rangle\langle X\rangle^{<}\varphi \leftrightarrow A \wedge \langle X\rangle^{<}\langle !A\rangle\varphi \tag{6.48}$$

In group preference logic, the definition of individual preference upgrade can be generalized to that of group preference upgrade, which I denote '$\preceq_X^{\#A}$'. The generalized definition is given by:

Definition 6.4.2

$$\preceq_X^{\#A} = \preceq_X -(?\neg A\,;\,\preceq_X\,;\,?A) \tag{6.49}$$

◁

I say that group X has upgraded its preferences so as to make A always preferred to $\neg A$ and I expand the language with the modality $\langle \#A, X\rangle\varphi$.

Theorem 6.4.3 *The complete logic of group preference, public announcement and preference upgrade is given by: 1) the logic Λ_{GP} together with 2) the reduction axioms of public announcement logic and 3) the reduction axioms of preference upgrade, notably:*

$$\langle \#A, X\rangle\langle Y\rangle^{\leq}\varphi \leftrightarrow (\neg A \wedge \langle Y\rangle^{\leq}\langle \#A, X\rangle\varphi) \vee (\langle Y\rangle^{\leq}(\varphi \wedge \langle \#A, X\rangle\varphi) \tag{6.50}$$

$$\langle \#A, X\rangle\langle Y\rangle^{<}\varphi \leftrightarrow (\neg A \wedge \langle Y\rangle^{<}\langle \#A, X\rangle\varphi) \vee (\langle Y\rangle^{<}(\varphi \wedge \langle \#A, X\rangle\varphi) \tag{6.51}$$

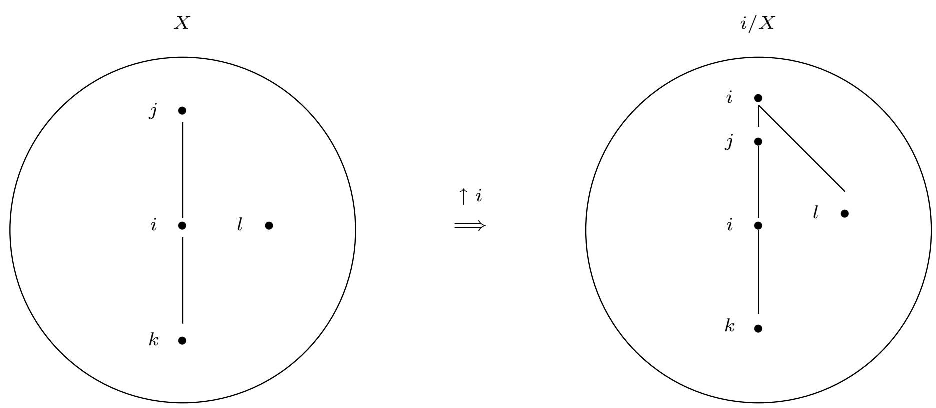

Figure 6.3: Illustration of the promotion of an agent i inside a group X.

Theorem 6.4.3 shows that GOL can be dynamified in the same as Order Logic. This is a welcome result, showing yet another time the uniform application of similar techniques in and extension of the Order Logic setting.

In the remainder of this chapter, I show that GOL, like CPL, motivates new topics in dynamic logic.

Agent promotion

Given that I base aggregation of preferences on a given hierarchy between agents, it seems natural to inquire what happens when the ranks of agents change in the hierarchy. Several reordering of group hierarchy are conceivable, but I focus my attention on an obvious first choice: putting an agent on top of the group. In the present section, I thus study a different and new kind of dynamics for group of agents, this time where the hierarchy inside the group is changed by upgrading an agent to become the master of the group. I call this action *agent promotion*, namely when an agent in a (sub)group is promoted to a higher rank.

I introduce some preliminary notations. The promotion of an an gent i in group X, simply written 'i/X', is given by the graph X' whose hierarchy is the same as in X with $j < i$ added for every $j \in X$. If X does not contain i, then $X' = X$.[3] An illustration of a promotion is provided in Figure 6.3.

The action of promoting $i \in X$ in a model $\mathfrak{M} = \langle W, G, \{\preceq_X\}_{X \in G}, V \rangle$, denoted $\mathfrak{M} \uparrow i$ is given by the model $\mathfrak{M}' = \langle W, G', \{\preceq_X\}_{X \in G'}, V \rangle$, where each graph $Y \in G$

[3]The notation just introduced is somewhat abusive, as I should write '$V(i)/X$', but I keep the original one for the sake of readability. I also illustrate graphs with nodes labeled with individual constants instead of variables.

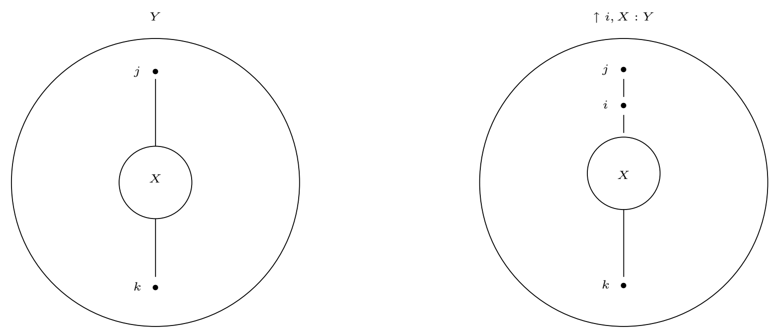

Figure 6.4: Illustration of the promotion of an agent i inside a subgroup X of Y.

that has non-empty intersection with X is replaced with the graph $Y' \in G'$ where i has been promoted in $X \cap Y$, as described above.

To talk about agent promotion in subgroups, I expand the language with a modality $\langle \uparrow i, X \rangle \varphi$, which should be read as "after promoting agent i in (sub)group X, φ is the case." The semantics of this new modality is given by:

$$\mathfrak{M}, u \models \langle \uparrow i, X \rangle \varphi \quad \text{iff} \quad \mathfrak{M} \uparrow i, u \models \varphi \tag{6.52}$$

For the axiomatization of agent promotion, I use reduction axioms viewed as syntactic relativizations. As van Benthem and Liu note in [69], "the reduction axioms for public announcement merely express the inductive facts about the modal assertion $\langle !\varphi \rangle \psi$ referring to the left-hand side, relating these on the right to relativization instructions creating $(\psi)^{\varphi}$" (p.171). On the basis of this standpoint, a reduction axiom may be seen as a syntactic relativization expressed in the principle:

$$\langle := def(R) \rangle \langle R \rangle \varphi \leftrightarrow \langle def(R) \rangle \langle R := def(R) \rangle \varphi \tag{6.53}$$

In the case of agent promotion, I denote $def(R)$ by '$\uparrow i, X : Y$', standing for the substitution of the priority graph $i/(X \cap Y)$ for every occurrence of $X \cap Y$ in Y. Notice that $\uparrow i, X : Y$ is defined over the intersection of X and Y. There are thus 4 cases that may arise: 1) $X \subseteq Y$, 2) $Y \subset X$, 3) $X \cap Y \neq \emptyset$ and $X \cap Y = \emptyset$. The first case is depicted in Figure 6.4. The second case, $Y \subset X$ implies that $\uparrow i, X : Y = \uparrow i, Y : Y = \uparrow i, Y$ is the same as in Figure 6.3. The third case is depicted in Figure 6.5. The fourth case is obvious: if $X \cap Y = \emptyset$, promoting i in X has no effect on Y. The next definition provides a recursive construction of $\uparrow i, X : Y$[4]:

[4]Thanks Alexandru Baltag for suggesting Definition 6.4.4.

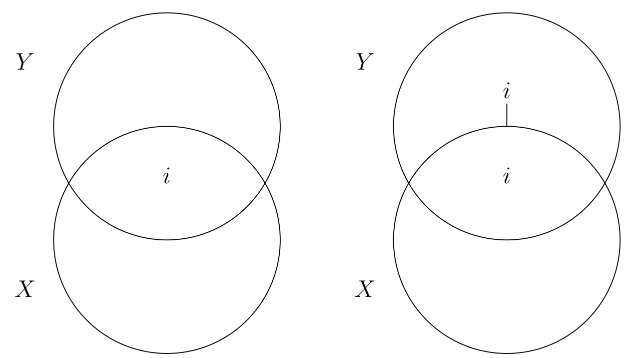

Figure 6.5: Illustration of the promotion of an agent i inside X when $X \cap Y$ but neither $X \subseteq Y$ nor $Y \subseteq X$.

Definition 6.4.4

$$\uparrow i, X : j = \begin{cases} i/j & \text{if } j \in X \\ j & \text{if } j \notin X \end{cases} \tag{6.54}$$

$$\uparrow i, X : (Y \parallel Z) = (\uparrow i, X : Y) \parallel (\uparrow i, X : Z) \tag{6.55}$$

$$\uparrow i, X : (Y/Z) = (\uparrow i, X : Y)/(\uparrow i, X : Z) \tag{6.56}$$

$\lhd$

The following theorem provides a compositional analysis of this modality in the base group preference language.

Theorem 6.4.5 *The logic of group preference with public announcement, preference upgrade and agent promotion is given by 1) Λ_{GP}, 2) the reduction axioms of public announcement, 3) the reduction axioms of preference upgrade provided in Theorem 6.4.3 and 4) the following reduction principles:*

$$\langle \uparrow i, X \rangle s \leftrightarrow s \tag{6.57}$$

$$\langle \uparrow i, X \rangle p \leftrightarrow p \tag{6.58}$$

$$\langle \uparrow i, X \rangle \neg\varphi \leftrightarrow \neg\langle \uparrow i, X \rangle \varphi \tag{6.59}$$

$$\langle \uparrow i, X \rangle (\varphi \vee \psi) \leftrightarrow \langle \uparrow i, X \rangle \varphi \vee \langle \uparrow i, X \rangle \varphi \tag{6.60}$$

$$\langle \uparrow i, X \rangle E\varphi \leftrightarrow E\langle \uparrow i, X \rangle \varphi \tag{6.61}$$

$$\langle \uparrow i, X \rangle \langle Y \rangle^{\leq} \varphi \leftrightarrow \langle \uparrow i, X : Y \rangle^{\leq} \langle \uparrow i, X \rangle \varphi \tag{6.62}$$

$$\langle \uparrow i, X \rangle \langle Y \rangle^{<} \varphi \leftrightarrow \langle \uparrow i, X : Y \rangle^{<} \langle \uparrow i, X \rangle \varphi \tag{6.63}$$

Proof. *The soundness is immediate since the definition of promotion in models and Definition 6.4.4 are in a perfect match.* QED

Summary

In this final chapter, I have shown how to extend Order Logic to Group Order Logic. For this, all that was required was to incorporate nominals into $\mathcal{L}_{\mathcal{O}}$. This addition to the language allowed two kinds of lifts: 1) from individual orders to group orders, using lexicographic upgrade and 2) from orders between states to preferences between propositions. I have also provided a complete axiomatization and shown how standard dynamic actions can be included in the logic via compositional analysis. Finally, I have investigated a new kind of action, this time acting on the group hierarchy, which I have called promotion. The innovations in this chapter were to modalize the algebraic setting of [2], thus getting a modal logic for aggregating individual orders into group orders. This modalization of the algebra has also allowed to introduce dynamics in the system as well as suggesting new kinds of dynamics, that of promotion. This puts an end to my thesis and supports once again its main point, namely that the setting of Order Logic defined over preorders is a useful and unifying setting which deals with belief and preference change, both at the individual and the group level. In the conclusion, I come back to this latter point by looking back at what as been achieved in the thesis and I discuss questions that have been left open, hoping that the reader be challenged in trying to solve them.

Chapter 7

Conclusion

In my thesis, I have shown that Order Logic ($\mathcal{L}_{\mathcal{O}}$) interpreted over preorders provides a unifying framework for individuals and groups to analyze believe and preference change. I have achieved this by producing dynamic doxastic and preference logics seen either as fragments or extensions of $\mathcal{L}_{\mathcal{O}}$. The actions I have considered were of two kinds. In a first time, I have integrated three well-known dynamic actions. The first one is public announcement, the second lexicographic upgrade and the last preference upgrade, exemplifying state elimination, state reordering and link cutting respectively. In a second time, I have introduced new kinds of actions: *agenda expansion* and *agent promotion*. All actions have been incorporated into static logics via compositional analysis, appealing to reduction axioms. This uniform completeness strategy consists in giving axioms that transform formulas with action modalities to equivalent formulas in the static language, reducing completeness of the dynamic logic to that of the static one. In this conclusion, I summarize the main results of each chapter and propose a list of open questions.

7.1 Summary

In Chapter 2, I have presented the system of Order Logic based on a language with three modalities: $\Diamond^{\leq}\varphi, \Diamond^{<}\varphi$ and $E\varphi$. The existential modality $E\varphi$ has been used throughout the thesis to isolate minimal states, in the case of belief revision, and to provide global comparative statement, in the case of preferences. Even though the strict modality $\Diamond^{<}\varphi$ cannot be defined in terms of $\Diamond^{\leq}$, I have shown via the completeness Theorem 2.1.11 that the logic is complete with respect to the classes of frames where the relation $\prec$ is adequate (Definition 2.1.9). For this last result, I have used the technique of *bulldozing*, first introduced by Segerberg in [58]. In the second part of Chapter 2, I have introduced the dynamic actions of public announcement, lexicographic upgrade and preference upgrade and shown how the method of compositional analysis can be applied to obtain completeness.

In Chapter 3, I have shown how Order Logic can treat relational belief revision. I have started my investigation with Segerberg's Dynamic Doxastic Logic (DDL) and have generalized it to a relational system which I have called *Broccoli Logic* (BL). The generalization consisted in defining two conditionals, $[\varphi]\psi$ and $[\varphi\rangle\psi$, over the class of non-linear systems of sphere. To show how this could be brought under the scope of Order Logic, I have shown that BL can be seen as one of its sublogics (Theorem 3.3.3). The main result of this chapter was to show that the $[\varphi]\psi$ fragment of Order Logic is the same as Minimal Conditional Logic. Finally, I have shown how the action of lexicographic upgrade can be incorporated in BL. This has been accomplished in two ways: 1) by direct analysis in the broccoli language and 2) by translation inside $\mathcal{L}_{\mathcal{O}}$.

Chapter 4 presented another important fragment of Order Logic, the binary preference fragment. This approach to Preference Logic originated in the work of von Wright ([76]), which I have analyzed carefully - partially in this chapter, but more importantly in Chapter 5 for *ceteris paribus* preferences. I have provided eight binary preference fragments of Order Logic in Definition 4.2.1 and I have focused my attention on the $\forall\forall$ fragment. The main result of this chapter was provided in Theorem 4.3.7, showing that the logic $\Lambda^{\mathcal{L}_{\mathcal{O}}^{-}}$ is complete with respect to the class of totally ordered preference models. Finally, I have introduced the action of preference upgrade in the latter fragment by translating everything in $\mathcal{L}_{\mathcal{O}}$, performing the reduction there, and finally translating back.

In the two remaining chapters, I have taken a different route, this time showing that Order Logic is a good system to be extended to get *ceteris paribus* preferences and group aggregation logics. Chapter 5 was focused on defeasible preferences statements, endowed with *ceteris paribus* clauses. As I have argued, there are (at least) two senses that may be given to *ceteris paribus*. The first one is "all other things being normal" (normality reading) and the second "all other things being equal" (equality reading). The normality reading is usually used to account for laws that hold in specific circumstances, for instance preferring red wine over white wine with every dish but fish. The equality reading is rather used to keep certain information constant when evaluating statements, for instance preferring raincoats over umbrellas *ceteris* *paribus* with respect to having boots. The second sense is the one I have incorporated in Order Logic, following [19] and using the mathematical notion of equivalence classes. My innovation was to modalize this notion by relativizing the modalities of Order Logic with sets of sentences Γ standing for the *other things* to be kept equal. I have provided a complete axiomatization for the variant where sets Γ are restrained to finite sets. This was accomplished by adding a list of axioms on top of $\Lambda^{\mathcal{L}_\mathcal{O}}$. I have then compared this logic to the original Preference Logic of von Wright and showed how my system could analyze his principles, showing where they hold or fail, and why. For the dynamic part, the introduction of public announcement and preference upgrade were straightforward. But new kinds of actions were considered, those of adding or subtracting formulas from a *research agenda*.

Finally, Chapter 6 considered the extension of Order Logic to Group Order Logic and order aggregation. The aggregation policy adopted was that of lexicographic reordering developed in [2]. One reason for choosing this policy was that it satisfies nice aggregation properties without being dictatorial. To modalize this system, I have introduce nominals in $\mathcal{L}_\mathcal{O}$ to lift individual preferences to group preferences. In the axiomatization, this lift was analyzed in a way analogical to compositional analysis, by reducing group statements to mixtures of individual statements. This modal logic for aggregation of orders is the main contribution of this chapter. As in the previous chapters, the introduction of public announcement and preference upgrade were straightforward and a new kind of action was studied, this time over the hierarchy imposed over the agents of the group: *agent promotion*.

It is now clear that Order Logic has fulfilled its telos and has proved to be a unifying setting to investigate preference and belief change. In the rest of this conclusion, I list open questions that could not be answered in my thesis.

7.2 Open questions

I first discuss general open questions pertaining to the methodology assumed in my thesis and then take on more specific questions relating to each chapter.

Methodology

In my thesis, I have adopted a specific methodology: the formalization of belief and preference change inside Modal Logic. This has lead to a perspicuous analysis of the dynamic aspect of beliefs and preferences, but some questions pertaining to this methodology have not been fully addressed.

One worry that may be raised with respect to completeness results via compositional analysis is that dynamification does not add expressive power to static logics. Indeed, reduction axioms show how arbitrary formulas of dynamic languages are equivalent to formulas of static languages. Nevertheless, it is advantageous to consider dynamic extensions as I do throughout my thesis, both conceptually and technically. Conceptually, the analysis of dynamics is made clear in the extended language. Even though dynamic formulas are equivalent to static ones, they isolate directly the information pertaining to actions and it is not clear how one would get the same information without resorting to the extended language. Technically, as has been shown in [43] with respect to public announcement, dynamic languages exponentially increase the succinctness of formulas without affecting the complexity. Hence, dynamification of logics extract the information from static languages to reason about actions in a succinct way without blowing up complexity. This results holds for public announcement, but has not been proved for the other actions considered in my thesis. It is reasonable to expect a similar result to hold for lexicographic and preference upgrade - and thus agenda expansion, cf., Fact 5.6.6 - but a proof is still awaiting. I leave this as an open question.

Another question that has not been fully addressed is to what extent the promised contribution to the conceptual analysis of belief and preference is enhanced from the formalism investigated. My claim is that the conceptual analysis gained from my formalisms comes from the emphasis on the dynamic aspect of preferences and beliefs. By giving clear and distinct actions pertaining to beliefs and preferences, I have shown an important aspect of these concepts that have gone mostly unnoticed in standard, more static, philosophical analyses. My conclusion is that rationality does not only apply to the formation of beliefs and preferences, but also to their management in a changing environment. I have not, however, developed a detailed philosophical account of belief and preference building on these results. This could easily form the subject of a separate thesis and I hope that my contribution will provide a good starting point for this enterprise.

A related question pertains to the multiplicity of logics used in my thesis. I have developed a plethora of systems for beliefs and preferences, but I have not taken a stance on which system is the right one. This is a methodological choice, for I think that having various systems is a good thing. For instance, depending on applications, it might be that a $\forall\forall$ definition of preference is more desirable than a $\forall\exists$, if one wants to allow for preferences of non-existent or fictional objects: I prefer vampires to zombies. In other circumstances, for example in trying to choose the preferred object among a set of objects, it could be that another binary definitions would more appropriate. One version of optimization theory with respect to preferences has been carried out in a *ceteris paribus* setting under the equality reading of Chapter 5 (cf., [9]). Multiplicity is thus valuable in practice as it provides various options depending on applications. From a philosophical standpoint, however, it might be desirable to have a more comprehensive understanding of beliefs and preferences, one that takes into account the various facets formalized in my thesis, but in a more homogeneous framework. I am of the opinion that part of analyzing the concept of preference amounts to realizing its various possible instantiations as described in my thesis, but I have not given a fuller philosophical account that does this. I leave this important question open for future research.

Meta-theoretical results regarding Order Logic and its fragment would also be desirable. It is known that Modal Logic is the bisimulation-invariant fragment of FOL and that it shares with it completeness and compactness, but is decidable, unlike FOL. Analogous questions can be asked for Order Logic and the various fragments investigated in my thesis. Does the $\forall\forall$ fragment yield meta-theoretical gain over Order Logic, say with respect to complexity? I have shown that it is indeed a fragment and a conceptual motivation is that it isolates preference reasoning restricted to this specific binary definition, but meta-theoretical results mirroring this conceptual gain should be investigated.

A feature of preferences that has been made prominent in Chapter 4 is the global aspect of binary preference statements. This can be witnessed by the heavy use of the existential modality throughout the thesis. It might then be argued that a language with first-order quantifiers would be more appropriate to define global preferences. My choice for the Modal Logic approach was guided partly by applications and partly because it yields a formalism well suited for the explicit analysis of actions. But this could all be done in FOL and it would be interesting to develop first-order versions of the various preferences defined in my thesis and see how the two formalisms compare. This could be informative in various ways, perhaps in showing that further distinctions can be captured inside FOL, or alternatively in unifying preference logics into a single system. In either case, the concept of preference would be further analyzed an better understood. The same point could be made with other formalisms, for instance a probabilistic setting could make further distinctions than either Modal Logic or FOL. A choice has to be made about the formalism appropriate to analyze concepts. Comparing various approaches would thus be illuminating in establishing which systems are better suited for specific feature of concepts to be formalized. There is thus a quest for the best formalism to study preferences and beliefs. This is something outside the scope of my thesis and I hope that future research will show where exactly my frameworks lie in the plethora of available formal tools.

Relational belief revision

An obvious problem following the work done in Chapter 3 is to find the complete conditional logic axiomatizing full BL. In Appendix A, I provide the first steps by axiomatizing Minimal Relational Logic, a logic giving the basic interaction principles between the modalities $[\varphi]\psi$ and $[\varphi\rangle\psi$. From Theorem A.1.3, it seems reasonable to conjecture that adding Axioms A.3 and A.5-A.7 might be sufficient, but I have not succeeded in proving this.

Another question that was left open in Section 3.2 was to find an appropriate selection function for Broccoli Logic. A conjecture made by Horacio Arló-Costa with respect to BL^- in private communication is to use Chellas's definition of a choice function f as a function from worlds and propositions to sets of propositions (cf. [15], p. 270). The semantics for the modality $[\varphi]\psi$ is then given by:

$$\models_w^{\mathcal{M}} [\varphi]\psi \text{ iff } |\psi|^{\mathcal{M}} \in f(w, |\varphi|^{\mathcal{M}}). \tag{7.1}$$

Notice first that 7.1 does not hold without an extra assumption of monotonicity on the choice function. Consider a model with 3 worlds w_1, w_2, w_3 such that $f(w_1, |\varphi|^{\mathcal{M}}) = \{\{w_2\}\}$, and assume that $V(\psi) = \{w_2, w_3\}$. Then ψ is true in every minimal φ-world returned by the choice function, but $|\psi|^{\mathcal{M}} \notin f(w, |\varphi|^{\mathcal{M}})$. This simple example shows that Chellas' definition of choice functions requires an extra assumption of monotonicity in order to make sure that definition 7.1 indeed provides a semantics for a conditional operator. One solution is to close the image of the choice function under supersets, and another solution is to change definition 7.1 to:

$$\models_w^{\mathcal{M}} [\varphi]\psi \text{ iff } \exists Z \in f(w, |\varphi|^{\mathcal{M}}) \text{ such that } Z \subseteq |\psi|^{\mathcal{M}}. \tag{7.2}$$

In the case of MCL, Arló-Costa's proposal is to impose the following conditions on the selection function:

1. $X \in f(w, X)$
2. $X \in f(w, Y) \wedge Y \in f(w, Z) \Rightarrow X \in f(w, Y \cup Z)$
3. $X \in f(w, Y) \wedge Y \in f(w, X) \Rightarrow Z \in f(w, X)$ iff $Z \in f(w, Y)$.
4. $Y \cap Y' \in f(w, X)$ iff $Y \in f(w, X) \wedge Y' \in f(w, X)$

A quick check shows that this is indeed a choice function for MCL models. Condition 1 corresponds to Axiom 3.23, condition 2 to Axiom 3.27, condition 4 to Axiom 3.24, and finally condition 3 is derivable using Axioms 3.25 and 3.26. The fourth condition provides an appropriate monotonicity condition.

It is still an open question what happens with the Arrow condition in broccoli logic. Arló-Costa's choice function is an appropriate selection function for MCL, and thus for BL^{-}, but the arrow condition has been lost in the process, along with linearity. It is still an open question to find an appropriate generalized limit condition for full BL.[1]

Binary preference logic

Chapter 4 is self-contained, but the literature on Preference Logic is very wide and it would be worthwhile to see exactly where my system fits. Research in Preference Logic, as I mentioned earlier, finds its roots in the work of von Wright [76], strongly influenced in Halldén's earlier manuscript [25]. An influential author in the field of preference logic is Hansson (see in particular [27], although his approach is somewhat tangential to mine, as he works at the level or preference relations not in an explicit object language for preferences. Of course, preferences have been studied and used widely in a perpetually growing body of research in economics and social choice theory. For a good survey of early literature in these fields, see [77].

My thesis shows a close link between beliefs and preferences when formalized over preorders, and this is not a coincidence. There is a long tradition of investigating beliefs as derived from preferences going back to de Finetti [17] and followed by Ramsey [52] and Savage [56]. More recent research along these lines can be found in [45]. But preferences can also be seen as being derived from beliefs, or at least influenced by them. A richer notion of preference, which takes into account both the betterness order of my Order Logic as well as a plausibility order to derive preferences,

[1]For further discussion of generalized selection functions, see Arló-Costa [4].

can be found in [36]. Finally, a good discussion of the relation between preferences and beliefs, as well as further references to the abundant literature, can be found in [41].

Another source of research on preferences can be found in game theory. Notion such as backward induction and the Nash Equilibrium have been formalized in a preference setting in [28] and [16]. Appendix B presents an analysis of the Nash Equilibrium solution concept in CPL. There are lots of conceptual and formal questions to be raised here. There is also a growing body of research in the interplay between preferences, beliefs and intentions in game theory (cf., [55]).

CPL

One pressing question is to find an axiomatization for CPL with no restriction on the set Γ. Theorem 5.2.7 provides a complete system for the finite fragment, but the argument used to prove this result fails for the infinite case. Indeed, if Γ can contain infinitely many sentences from the base language, the very last argument of the completeness proof does not go through, as the same procedure would have to be repeated infinitely many times, contra the finiteness of derivations. My conjecture is that infinitary CPL can be axiomatized by introducing infinite conjunctions and disjunctions and rephrase Axioms 5.11-5.13 in terms of *colors* of Γ, denoted $Color_\Gamma$, an (infinite) conjunction of formulas and negated formulas in Γ. For instance, Axiom 5.11 would now be read as:

$$11'. \quad Color_{\Gamma'} \wedge \langle \Gamma \rangle (Color_{\Gamma'} \wedge \alpha) \rightarrow \langle \Gamma \cup \Gamma' \rangle \alpha$$

I have not succeeded in proving this conjecture and leave it for future research.

A conceptual question left open was the exact relationship between the normality and the equality readings of *ceteris paribus*. Two lessons can be drawn from the red-white wine and the raincoat-umbrella examples of Section 5.1. One is that the equality reading is stronger[2] than the normality reading. Indeed, if I prefer my raincoat over my umbrella *ceteris paribus* with respect to my boots, then I have the same preference if having my boots is taken to be among the normal conditions. As I mentioned above, given a set of normal conditions, the normality reading focuses on a set of normal states and leaves the other cases open. In the equality reading, one considers every

[2]Stronger in the sense that there are *ceteris paribus* cases in the normality reading which are not *ceteris paribus* under equality.

possible combinations of the members of Γ, which induces a partition of the space into equivalence classes, and considers the relation between states inside each class. The equivalence class where every member of C is satisfied is one among them. The second lesson is that the preference of red wine over white wine, *ceteris paribus* in the normality reading, is not *ceteris paribus* in the equality reading. Indeed, looking at figure 5.2, if having meat is kept constant, then fw is preferred to fr, although meat is not served in either case. Similarly, we get contradicting preferences if fish is to be kept constant. The equality reading is thus stronger than the normality reading. The exact relationship between the two readings is still not fully settled.

Finally, another interesting subject for further research is with the notion of Agenda used in Section 5.6. I have been using the terminology of 'agenda' to introduce *ceteris paribus* actions in a liberal way, taking it only to stand for a set of distinguished sentences. The effect of adding information to the agenda on a model is to split it into independent zones; this is formally clear. But what interpretation can this action be given more specifically?

First, agenda actions can be considered from two standpoints, which I call *local* and *global* respectively. The interpretation of the action is *local* if the effect is considered from a single state, and *global* when the effect on the whole model is studied. Let us take the two standpoints in turn.

From a local point of view, adding A to the agenda has the effect of situating the actual state in one of the A or the $\neg A$-zone. This action is studied in a different setting in [30], where PDL test actions are investigated. The usual PDL test action $?\varphi$ returns φ if the actual state is a φ-state and returns nothing otherwise. A more complex test action, called the *test whether* action and denoted $??\varphi$, returns φ is the actual state is a φ-state and $\neg\varphi$ if it is a $\neg\varphi$-state. The axiomatization given in [30] for this action is contained in a single axiom (Proposition 4, Axiom 4), which is trans-literally the reduction Axiom 5.21. Hence, from a local point of view, to put A on the agenda is to test in which of the A or the $\neg A$ zone the actual state is.

From a global point of view, agenda expansion can express the (epistemic) ability to ignore alternatives that differ with respect to independent, irrelevant or unknown features. This is especially relevant in presence of a global modality, where investigations can be made to be independent from item put on the agenda. It could, for instance, be argued that scientific knowledge should be kept independent from the existence of God exactly in this respect. If we were in a world where God did not exist, then the predictions of science should not be undermined by counterexamples involving the existence of God - God would break the laws of science, but that would not make them false. Vice versa, if we were in a world where God existed, then the predictions of science would be kept independent from divine intervention, so that counterexamples where God does not exist would not apply - a law involving divine intervention is not a scientific law. Whether or not God exists, science should be the same; this seems to be the scientific attitude of our times. But in former times, the existence of God would often be used in arguing about scientific theories, a famous instance being the correspondence between Leibniz and Clarke about Newtonian mechanics. When Leibniz offered counterexamples to the Newtonian notion of absolute space, he relied heavily on his interpretation of God, as if Newton lived in a non-God world (i.e., a non-Leibnizian-God world) and counterexamples from the God-worlds refuted his theory. It is arguable that science is now developing independently from similar considerations and that what it is becoming is kept independent from God. In other words, God has been put on the *ceteris paribus agenda* and laws of science are now independent from its existence. I do not mean to claim that God has been degraded, but rather that we have adopted an epistemic attitude of keeping our scientific inquiries independent from our beliefs in God and that *ceteris paribus* actions describe in what ways this has occurred. One could similarly argue that the debate on duality in philosophy of Mind should be kept independent from the existence of the Soul. In both cases, one builds her knowledge on reasoning involving a restrained set of alternatives and CPL actions provide a logical framework that supports this attitude and gives it clear logical boundaries.

There is a thus a notion of independence lurking in the equality reading and it would also be interesting to compare it with existing logic of dependence [64].

One more note about agenda contraction. The quasi-historical interpretation of expansion gives us some more ground to understand why contraction seems to be so difficult. Consider again the case of God being put on the *ceteris paribus agenda*. Should we come, as a community, to bring back God in scientific inquiries, how would we do so? Would we go back to the times of Leibniz and Newton, or would we impose different constraints in appealing to its existence when arguing about scientific theories? I do not see how to appeal to general principles to answer this question and this seems to echo the various difficulties in formalizing contraction encountered above. It may be that logic fails to formalize agenda contraction precisely because this is not something that happens logically, or rationally! It may be that mathematics is not the right arena to ask the question and perhaps has logic reached one of its limits.

But the question is still an important one: what is *ceteris paribus agenda contraction*? What does it mean to stop taking a piece of information to be independent in a rational debate, or in the growth of Science?

Group Order Logic

Finally, Group Order Logic also raises a lot of problems. One important conceptual question pertains to the choice of lexicographic re-ordering as an aggregation policy. My main motivation for choosing it is the *possibility* result proved in [2], namely that priority operators are the only operators satisfying nice aggregation conditions: 1) independence of irrelevant alternatives, 2) preference based, 3) unanimous with abstention, 4) transitivity preserving and 5) non-dictatorial. All these properties are justifiable for an aggregation procedure and they are interesting to compare with the conditions of Arrow's impossibility result. The fifth property, which is derivable from the 4 others, is where the divergence becomes striking; lexicographic operators are non-dictatorial. Of course, this is not a contradiction with Arrow's result, since his conditions are more general, but it provides a nice touchstone to investigate possibility results in the fields of preference and judgment aggregation, as well as to compare them with well-known and abundant impossibility results.

From [2]'s results, however, lexicographic reordering is the *only* operator satisfying the conditions listed above. This then opens questions as to the reliability of the lexicographic reordering and the quest for possibility results. Is giving more weight to certain agents in a society justifiable from a democratic point of view? What kind of voting procedures and responses to the opinions of the majority would obtain? Are we better to stay in a society where the aggregation of preferences is known to be non-uniform (because of Arrow's theorem)? But also, can we find other sets of nice conditions that are represented by operators which still yield possibility results?

Here, one should also compare this with the growing body of research in *Judgment Aggregation*, where various impossibility results of the Arrow type have been obtained (cf. [18, 48]).

It would be interesting to see how Group Order Logic and its belief and preference fragments could be applied in game theory. I have described actions such as preference upgrade, but no specific actions that could be taken by agents in games. It seems that Group Order Logic would be an appropriate logic to investigate further the connections between logic and game theory. For a starter, one should compare how the aggregation system presented in Chapter 6 relates to existing logics for game theory such as coalition logic ([46, 47]).

On a more technical side, the use of priority graphs and the axiomatization provided in Theorem 6.2.5 suggests a more general approach to the subject, namely to get a logic of graph manipulations. Firsts steps towards this are given in Appendix C, where I show in Facts C.0.22 and C.0.23 that the 'but' and 'on the other hand' priority operators can be seen as the operations on graphs of disjoint union and *sequential composition*. But many more actions on graphs could be defined, and it would be interesting to see how much can be treated in a similar fashion.

Final words

A lot of ground has been covered in this thesis, but as is now clear from the above discussion, a lot more needs to be done. I hope that the reader will be motivated in trying to answer some of the open questions.

Appendix A

Minimal Relational Logic

A.1 Minimal relational logic

In this section, I investigate the Minimal Relational Logic (MRL) behind Broccoli Logic. MRL is said to be minimal in the same way that K is the minimal normal logic for $S4$ or $S5$. I use neighborhood semantics, which seems to me to be the most convenient way to attack this problem. I also show how this minimal logic can be restrained to transitive classes of frames, by the addition of axioms similar to the $S4$ axiom $\Diamond\Diamond\varphi \to \Diamond\varphi$.

Minimal relational models are defined as follows:

Definition A.1.1 [MRL models] A *minimal relational model* is a triple (W, R, V), where W is a set of states, V is a propositional valuation and R stands for a family of relations:

$$R = \{R_\varphi : \varphi \text{ is a formula}, R_\varphi \subseteq W \times \mathcal{P}(W)\}$$

with the following (transitivity) restriction:

$$uR_{|\varphi|}X \,\&\, \forall x \in X, xR_{|\varphi|}Y_x \Rightarrow uR_{|\varphi|} \bigcup_{x \in X} Y_x \tag{A.1}$$

$\lhd$

In the special case where X is a singleton, condition A.1 reduces to the a usual transitivity condition:

$$uR_{|\varphi|}\{x\} \,\&\, xR_{|\varphi|}\{y\} \Rightarrow uR_{|\varphi|}\{y\}$$

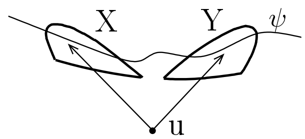

Figure A.1: Minimal relational model

Definition A.1.2 [MRL semantics] Let $\mathfrak{M}$ be a model and let $u \in W$. The truth-definition for atomic propositions, negations and disjunction is standard. The semantics for the conditionals $[\varphi]\psi$ and $[\varphi\rangle\psi$ is given by :

$$\mathfrak{M}, u \vDash [\varphi]\psi \quad \text{iff} \quad \forall X(uR_\varphi X \Rightarrow \forall v \in X, \mathfrak{M}, v \vDash \psi)$$
$$\mathfrak{M}, u \vDash [\varphi\rangle\psi \quad \text{iff} \quad \forall X(uR_\varphi X \Rightarrow \exists v \in X, \mathfrak{M}, v \vDash \psi)$$

$\lhd$

The semantics of the modalities $[\varphi]$ and $[\varphi\rangle$ contains two levels of quantification and should be read in two stages: 1) the left bracket picks out a set of φ-subsets of the universe and 2) the right bracket evaluates where ψ is true in these subsets. Notice that the semantics given by minimal relational models is a neighborhood semantics. Indeed, the relation R is a relation between worlds and subsets of the universe. The modality $[\varphi]$ is the usual neighborhood universal modality, but indexed with associated propositions $|\varphi|$. It comes with its dual modality $\langle\varphi\rangle$ with the obvious semantics. The interesting addition of our language is the modality $[\varphi\rangle$, which expresses that every set R_φ-related to u satisfies ψ in at least one point. In neighborhood terminology, this modality expresses that every φ-neighborhood contains at least one ψ-point. This latter modality also comes with its natural dual $\langle\varphi]$, which expresses that there is a minimal φ-set that is contained in $|\psi|$.[1]

Figure A.1 presents a simple minimal relational model, in which the world u is R_φ-related (illustrated with arrows) to the sets of worlds X and Y in such a way that ψ is consistent with X and Y. Hence, according to the minimal semantics of Definition A.1.2, $[\varphi\rangle\psi$ is true at u.

One way to see the link between BL and MRL is by adding restrictions on the relation $R_{|\varphi|}$ in order to get the sets X and Y of figure A.1 as sets returned under revision by φ. This is illustrated in figure A.2, where $[\varphi\rangle\psi$ is true at u, since ψ is consistent with every revision by φ. It is in this perspective that I call the relational logic

[1] The modality $\langle\varphi]\psi$ was first studied in [5]

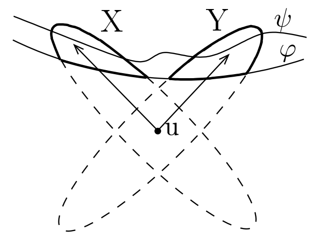

Figure A.2: Minimal relational model

of the present section *minimal*. In a full-blown broccoli logic, additional restrictions on the relation R would play the role of selecting minimal revised sets. Once these sets would be selected, the MRL would provide the logic to evaluate what holds in these sets.

Axiomatization

Theorem A.1.3 *The following set of axioms and rules, added to classical tautologies, are complete with respect to minimal relational models. I call the resulting logic* Λ_{MRL}

Axioms:

$$\langle\varphi\rangle\psi \equiv \neg[\varphi]\neg\psi \tag{A.2}$$

$$\langle\varphi]\psi \equiv \neg[\varphi\rangle\neg\psi \tag{A.3}$$

$$[\varphi](\psi \to \theta) \to ([\varphi]\psi \to [\varphi]\theta) \tag{A.4}$$

$$\langle\varphi]\psi \to \langle\varphi](\psi \vee \theta) \tag{A.5}$$

$$[\varphi]\psi \wedge \langle\varphi]\theta \to \langle\varphi](\psi \wedge \theta) \tag{A.6}$$

$$\neg\langle\varphi]\top \to [\varphi]\psi \tag{A.7}$$

$$\langle\varphi\rangle\langle\varphi\rangle\psi \to \langle\varphi\rangle\psi \tag{A.8}$$

$$\langle\varphi]\langle\varphi]\psi \to \langle\varphi]\psi \tag{A.9}$$

Rules:

1. *Modus Ponens.*

2. *Necessitation for* $[\varphi]$ *and* $[\varphi\rangle$.

3. *If φ and φ' are formulas differing only in φ having an occurrence of θ in one place where φ' has an occurrence of θ', and if $\theta \equiv \theta'$ is a theorem, then $\varphi \equiv \varphi'$ is also a theorem.*

Rule 3, *substitution of equivalents*, is applied indiscriminately inside or outside the modal operators. I count the presence of 'φ' inside $[\varphi]$ and $[\varphi\rangle$ as occurrences of φ. For example, if $\psi \equiv \theta$, then both $[\varphi]\psi \equiv [\varphi]\theta$ and $[\psi]\alpha \equiv [\theta]\alpha$ are instances of rule 3.

Axioms A.2 and A.3 provide the dual modalities of $[\varphi]$ and $[\varphi\rangle$ respectively. Axiom A.4 is a typical K axiom for the modality $[\varphi]$ and yields Modus Ponens under the scope of $[\varphi]$.[2] Axioms A.5 is a monotonicity axiom for the modality $\langle\varphi]$. Intuitively, if ψ is consistent with some revision by φ, then anything weaker than ψ is also consistent with some revision by φ. Axiom A.6 provides a minimal interaction between the modalities: If ψ is believed after every revision by φ and there is a revision by φ such that θ is believed, then there is a revision by ψ such that both ψ and θ are believed. Axiom A.7 says that if there is no revision by φ, then every $[\varphi]$-formula holds vacuously. This is akin to saying that every necessary formula holds at en end-point in an irreflexive Kripke model. Finally, Axioms A.8 and A.9 are typical transitivity axioms.

Now, Suppose that $\langle\varphi]\top \in u$ for some $u \in W$.[3] Then, for every $\psi \in u$ such that $[\varphi\rangle\psi \in u$, axiom A.6 gives that $\langle\varphi](\psi \wedge \top) \in u$. By monotonicity of $\langle\varphi]$ (axiom A.5), $\langle\varphi]\psi \in u$. Hence, if there is a revision by φ and if ψ is consistent with every revision by φ, then there is a least one revision by φ that witnesses the consistency of ψ. This is desirable for a relational belief revision logic.

Completeness

Soundness is a matter of routine. I show the soundness of Axiom A.6 and A.9 and leave the others to the reader. Assume that $\mathfrak{M}, u \vDash [\varphi]\psi \wedge \langle\varphi]\theta$. Then $\mathfrak{M}, u \vDash \langle\varphi]\theta$, i.e., $\exists X((u,X) \in R_{|\varphi|} \wedge \forall v \in X, \mathfrak{M}, v \vDash \theta)$. But $\mathfrak{M}, u \vDash [\varphi]\psi$ implies that $\forall v \in X, \mathfrak{M}, v \vDash \psi$. Hence, $\forall v \in X, \mathfrak{M}, v \vDash \psi \wedge \theta$. Therefore, $\exists X((u,X) \in R_{|\varphi|} \wedge \forall v \in X, \mathfrak{M}, v \vDash \psi \wedge \theta$, i.e., $\mathfrak{M}, u \vDash \langle\varphi](\psi \wedge \theta)$.

[2]There is no corresponding K axiom for the $[\varphi\rangle$. Consider a model M such that the set $X \subseteq W$ is the only subset of W that is φ-related to the world $u \in W$, i.e, $R_{|\varphi|} = \{(u,X)\}$. Suppose that both $|\psi| \cap X \neq \emptyset$ and $|\neg\psi| \cap X \neq \emptyset$, but that $|\theta| \cap X = \emptyset$. Then $u \vDash [\varphi\rangle(\psi \rightarrow \theta)$ (since $|\neg\psi| \cap X \neq \emptyset$) and $u \vDash [\varphi\rangle\psi$, but $u \nvDash [\varphi\rangle\theta$. Hence, $[\varphi\rangle(\psi \rightarrow \theta) \rightarrow ([\varphi\rangle\psi \rightarrow [\varphi\rangle\theta)$ is not valid.

[3]I read $\langle\varphi]\top$ as "there is a revision by φ".

Assume that $\mathfrak{M}, u \models \langle\varphi]\langle\varphi]\psi$, the there is an X with $uR_{|\varphi|}X$ and for every $x \in X, \mathfrak{M}, x \models \langle\varphi]\psi$. Hence, for every $x \in X$, there is a Y_x with $xR_{|\varphi|}Y_x$ such that for every $y \in Y_x, \mathfrak{M}, y \models \psi$. By the transitivity of $R_{|\varphi|}, uR_{|\varphi|} \bigcup_{x\in X} Y_x$ and $\mathfrak{M}, y \models \psi$ for every $y \in \bigcup_{x\in X} Y_x$. Thus, $\mathfrak{M}, u \models \langle\varphi]\psi$.

For the completeness part, let $W^{\mathcal{L}}$ consist of all maximal $\mathcal{L}$-consistent sets of formulas. For each formula φ, I define an accessibility relation $R^{\mathcal{L}}_{|\varphi|}$ between worlds and subsets of worlds of $W^{\mathcal{L}}$. For all worlds $u \in W^{\mathcal{L}}$, if $\langle\varphi]\top \notin u$, then I put $R^{\mathcal{L}}_{|\varphi|} = \emptyset$. Otherwise, for every subset $X \subseteq W^{\mathcal{L}}$ and formulas φ and ψ, I say that the ordered pair $(u, X) \in R^{\mathcal{L}}_{|\varphi|}$ iff X satisfies the following two conditions:

1. for all $x \in X$, if $[\varphi]\psi \in u$, then $\psi \in x$; and

2. for every $[\varphi\rangle\psi \in u, X$ contains at least one world v with $\psi \in v$.

Definition A.1.4 [Canonical MRL model] Let $p \in$ PROP be a proposition. Let $V^{\mathcal{L}}(p) = |p|$ and let $R^{\mathcal{L}} = \{R^{\mathcal{L}}_{|\varphi|} : \varphi$ is a formula$\}$, then the model $\mathfrak{M}^{\mathcal{L}} = (W^{\mathcal{L}}, R^{\mathcal{L}}, V^{\mathcal{L}})$ is the *canonical minimal relational model.* $\lhd$

The completeness of the proof system in section A.1 follows from a standard truth-lemma:

Lemma A.1.5 *For all $u \in W^{\mathcal{L}}$ and for all formulas θ, $\theta \in u$ iff $\mathfrak{M}, u \models \theta$.*

I give the proof of the truth-lemma once I have stated and proved the following crucial lemmas.

Lemma A.1.6 *If $\langle\varphi]\alpha \in u$, then there exists a subset $X \subseteq W^{\mathcal{L}}$ such that $R^{\mathcal{L}}_{|\varphi|}uX$, and for every world $x \in X$, $\alpha \in x$.*

Proof. Let $[\varphi\rangle\theta \in u$, and let

$$v^- = \{\beta : [\varphi]\beta \in u\} \cup \{\theta\} \cup \{\alpha\}.$$

Suppose that v^- is not consistent, then there exists $\delta_1, ..., \delta_n \in v^-$ such that $\vdash (\bigwedge \delta_i \wedge \alpha) \to \neg\theta$. For every $1 \leq i \leq n$,

$$\begin{array}{rcll}
\delta_i \in v^- & \Rightarrow & [\varphi]\delta_i \in u & \text{(Definition of } v^-\text{)} \\
& \Rightarrow & \bigwedge[\varphi]\delta_i \in u & \text{(Truth definition)} \\
& \Rightarrow & [\varphi]\bigwedge \delta_i \in u & \text{(Axiom A.4)} \\
& \Rightarrow & ([\varphi]\bigwedge \delta_i \wedge \langle\varphi]\alpha) \in u & \text{(since } \langle\varphi]\alpha \in u\text{)} \\
& \Rightarrow & \langle\varphi](\bigwedge \delta_i \wedge \alpha) \in u & \text{(axiom A.6)} \\
& \Rightarrow & \langle\varphi]\neg\theta \in u & \text{(by the monotonicity axiom A.5)} \\
& \Rightarrow & \neg[\varphi\rangle\theta \in u & \text{(axiom A.3)}
\end{array}$$

and this is a contradiction, since $[\varphi\rangle\theta \in u$ by assumption. Therefore, v^- is consistent. Let v be a maximal extension of v^-.

For every θ_i such that $[\varphi\rangle\theta_i \in u$, let w_i be obtained from the above construction, and let

$$X = \{w_i : [\varphi\rangle\theta_i \in u, \theta_i \in w_i\}.$$

Then X satisfies conditions 1) and 2) and for every $x \in X$, $\alpha \in x$. QED

Corollary A.1.7 (Corollary to the proof of lemma A.1.6) *If $[\varphi\rangle\psi \in u$, then the set $w = \{\psi\} \cup \{\theta : [\varphi]\theta \in u\}$ is consistent.*

Lemma A.1.8 *If $\langle\varphi\rangle\psi \in u$, then there exists a subset $X \subseteq W^{\mathcal{L}}$ such that $R^{\mathcal{L}}_{|\varphi|}uX$, and there exists a world $x \in X$ such that $\psi \in x$.*

Proof. Assume $\langle\varphi\rangle\psi \in u$. Then there is a maximal consistent set v such that $\psi \in v$. The proof that v exists is standard (see [7], Lemma 4.20).

By corollary A.1.7, for every formula α_i, if $[\varphi\rangle\alpha_i \in u$, then the set $w_i^- = \{\alpha_i\} \cup \{\theta : [\varphi]\theta\}$ is consistent. By Lindenbaum's lemma, there exists a maximal consistent set w_i extending w_i^- such that $\alpha_i \in w_i$. Let $W = \{w_i : [\varphi\rangle\alpha_i \in u\}$

Finally, let $X = \{v\} \cup W$. It is not difficult to check that $R^{\mathcal{L}}_{|\varphi|}uX$, and $\psi \in v$. QED

Proof.[Lemma A.1.5] Thanks to axioms A.5 and A.7, if $\langle\varphi]\top \notin u$, then $[\varphi\rangle\psi \in u$ and $[\varphi]\psi \in u$ for all ψ. Since $R^{\mathcal{L}}_{|\varphi|} = \emptyset$ when $\langle\varphi]\top \notin u$, the lemma is trivially satisfied. Thus, I assume for the remainder of the proof that $\langle\varphi]\top \in u$. The proof now proceeds by induction on the complexity of θ. The interesting cases are when $\theta = [\varphi]\psi$ or $\theta = [\varphi\rangle\psi$. The first direction ($\theta \in u \Rightarrow \mathfrak{M}, u \vDash \theta$) follows from the conditions imposed on $R^{\mathcal{L}}_{|\varphi|}$. I prove that $\mathfrak{M}, u \vDash \theta \Rightarrow \theta \in u$.

Suppose $[\varphi]\psi \notin u$. Since u is a maximal consistent set of formulas, $\neg[\varphi]\psi \in u$. By axiom A.2, this implies that $\langle\varphi\rangle\neg\psi \in u$. By lemma A.1.8, there exists a subset $X \subseteq W^{\mathcal{L}}$ such that $R^{\mathcal{L}}_{|\varphi|}uX$ and a world $x \in X$ such that $\mathfrak{M}, x \vDash \neg\psi$. Hence, by the truth-definition $\mathfrak{M}, u \vDash \langle\varphi\rangle\neg\psi$, i.e., $\mathfrak{M}, u \vDash \neg[\varphi]\psi$. Therefore, $\mathfrak{M}, u \nvDash [\varphi]\psi$.

Finally, suppose that $[\varphi\rangle\psi \notin u$, then $\neg[\varphi\rangle\psi \in u$. Hence, $\langle\varphi]\neg\psi \in u$ (axiom A.3). By lemma A.1.6, there exists a subset $X \subseteq W^{\mathcal{L}}$ such that $R^{\mathcal{L}}_{|\varphi|}uX$ and for every world $x \in X$, $\neg\psi \in x$. By the inductive hypothesis, for every $x \in X$, $\mathfrak{M}, x \vDash \neg\psi$. Therefore, by the truth-definition, $\mathfrak{M}, u \nvDash [\varphi\rangle\psi$. QED

Theorem A.1.9 *The logic Λ_{MRL} is complete with respect to the class of transitive frames, where transitivity is defined as in Definition A.1.1.*

Proof. Assume that $uR_{|\varphi|}X$ and $\forall x \in X, xR_{|\varphi|}Y_x$ and suppose that $\exists y_x \in Y_x$ such that $\psi \in y_x$. Then $\langle\varphi\rangle\psi \in x$, which implies that $\langle\varphi\rangle\langle\varphi\rangle\psi \in u$. By the transitivity Axiom A.8, $\langle\varphi\rangle\psi \in u$. Now, assume that $\psi \in y$ for every $y \in \bigcup_{x\in X} Y_x$. Thus, $\psi \in y$ for every $y \in Y_x$, which implies that $\langle\varphi]\psi \in x$ for every $x \in X$. Hence, $\langle\varphi]\langle\varphi]\psi \in u$. By the second transitivity Axiom A.9, $\langle\varphi]\psi \in u$. Therefore, since both conditions of the definition of the accessibility relation in the canonical model are met, $uR_{|\varphi|}\bigcup_{x\in X} Y_x$, as desired. QED

Proof of Lemmas needed in Section 3.4

Lemma A.1.10 *Let $*$ stand for either $\leq$ or $<$, then:*

$$\begin{aligned} \langle\Uparrow A\rangle\Box^*\varphi \;\Leftrightarrow\; & A \to \Box^*(A \to \langle\Uparrow A\rangle\varphi) \\ \wedge\; & \neg A \to \Box^*(\neg A \to \langle\Uparrow A\rangle\varphi) \\ \wedge\; & \neg A \to U(A \to \langle\Uparrow A\rangle\varphi) \end{aligned} \tag{A.10}$$

Proof. Follows with some obvious manipulations after taking the negation on both sides of Axiom 2.30. QED

Lemma A.1.11

$$A \wedge \langle\Uparrow A\rangle(\varphi \wedge \Box^*(\varphi \to \psi)) \Leftrightarrow A \wedge \langle\Uparrow A\rangle\varphi \wedge \Box^*(A \wedge \langle\Uparrow A\rangle\varphi \to \langle\Uparrow A\rangle\psi) \tag{A.11}$$

Proof.

$$\begin{array}{rcll} A \wedge \langle \Uparrow A \rangle (\varphi \wedge \Box^* (\varphi \rightarrow \psi)) & \Leftrightarrow & A \wedge \langle \Uparrow A \rangle \varphi \wedge \langle \Uparrow A \rangle \Box^* (\varphi \rightarrow \psi) & \text{(Theorem 2.2.2)} \\ & \Leftrightarrow & A \wedge \langle \Uparrow A \rangle \varphi \wedge \Box^* (A \rightarrow \langle \Uparrow A \rangle (\varphi \rightarrow \psi)) & \text{(Lemma A.1.10)} \\ & \Leftrightarrow & A \wedge \langle \Uparrow A \rangle \varphi \wedge \Box^* (A \rightarrow (\langle \Uparrow A \rangle \varphi \rightarrow \langle \Uparrow A \rangle \psi)) & \text{(Theorem 2.2.2)} \\ & \Leftrightarrow & A \wedge \langle \Uparrow A \rangle \varphi \wedge \Box^* (A \wedge \langle \Uparrow A \rangle \varphi \rightarrow \langle \Uparrow A \rangle \psi) & \text{(Logic)} \end{array}$$

QED

Lemma A.1.12

$$\begin{array}{rcl} \neg A \wedge \langle \Uparrow A \rangle (\varphi \wedge \Box^* (\varphi \rightarrow \psi)) & \Leftrightarrow & \neg A \wedge \langle \Uparrow A \rangle \varphi \\ & \wedge & \Box^* (\neg A \wedge \langle \Uparrow A \rangle \varphi \rightarrow \langle \Uparrow A \rangle \psi) \\ & \wedge & U(A \wedge \langle \Uparrow A \rangle \varphi \rightarrow \langle \Uparrow A \rangle \psi) \end{array} \tag{A.12}$$

Proof.

$$\begin{array}{rcll} \neg A \wedge \langle \Uparrow A \rangle (\varphi \wedge \Box^* (\varphi \rightarrow \psi)) & \Leftrightarrow & \neg A \wedge \langle \Uparrow A \rangle \varphi \wedge \langle \Uparrow A \rangle \Box^* (\varphi \rightarrow \psi) & \text{(Theorem 2.2.2)} \\ & \Leftrightarrow & \neg A \wedge \langle \Uparrow A \rangle \varphi & \\ & \wedge & \Box^* (\neg A \rightarrow \langle \Uparrow A \rangle (\varphi \rightarrow \psi)) & \\ & \wedge & U(A \rightarrow \langle \Uparrow A \rangle (\varphi \rightarrow \psi)) & \text{(Lemma A.1.10)} \\ & \Leftrightarrow & \neg A \wedge \langle \Uparrow A \rangle \varphi & \\ & \wedge & \Box^* (\neg A \rightarrow (\langle \Uparrow A \rangle \varphi \rightarrow \langle \Uparrow A \rangle \psi)) & \\ & \wedge & U(A \rightarrow (\langle \Uparrow A \rangle \varphi \rightarrow \langle \Uparrow A \rangle \psi)) & \text{(Theorem 2.2.2)} \\ & \Leftrightarrow & \neg A \wedge \langle \Uparrow A \rangle \varphi & \\ & \wedge & \Box^* (\neg A \wedge \langle \Uparrow A \rangle \varphi \rightarrow \langle \Uparrow A \rangle \psi) & \\ & \wedge & U(A \wedge \langle \Uparrow A \rangle \varphi \rightarrow \langle \Uparrow A \rangle \psi) & \text{(Logic)} \end{array}$$

QED

Appendix B

CPL and Nash Equilibrium

The equality reading of *ceteris paribus* can be seen to arise naturally in game theory, where concepts such as "best response" and "Nash equilibrium" implicitly use an "all other things being equal" clause. The ability of defining Nash equilibrium, furthermore, is a benchmark for modern logics of games and this problem has been solved in several ways [16, 29, 71]. This appendix offers a new solution emphasizing the *ceteris paribus* aspect of Nash equilibrium.

A Nash equilibrium is a state in which no player has incentives to unilaterally change her strategy: for every i, no alternatives are strictly better for i in which every player but i keeps the same strategies. This can be expressed in CPL by bringing out the *ceteris paribus* aspect in the Nash equilibrium solution concept. I achieve this for finite games in strategic form and show the details for a simple 2-player game with players a and b.

Consider a language with the propositional letters $a_1, ..., a_m$ and $b_1, ..., b_n$ ranging over a and b's strategies respectively and consider a $m \times n$-game matrix such as in Figure B. Identify each cell, or strategy profile, with a possible state (a_i, b_j) and take, for each player, an arbitrary total preference relation among these states. I use subscripts on the modalities for agents. For example, the notation $\langle\emptyset\rangle_a^{\leq}\varphi$ expresses that there is a better state according to a's preferences where φ holds. My goal is to express that state u is a Nash equilibrium.

In line with [29], I first express the notion of *best response*. I say that strategy a_i is a best response for a at state u if $u = (a_i, b_j)$ is at least as good as any other state, keeping b_j equal. I express this by:

$$\mathfrak{M}, u \models \neg\langle\{b_j\}\rangle_a^{<}\top$$

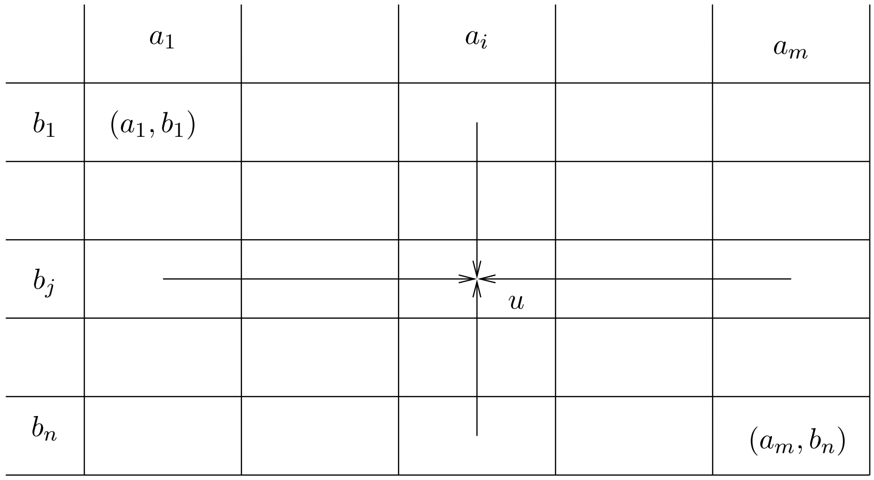

Figure B.1: Simple representation of a Nash equilibrium. The arrows indicate that $(x, b_j) \leq_i (a_i, b_j) \forall x \in \langle a_1, ..., a_m \rangle$ and $(a_i, y) \leq (a_i, b_j) \forall y \in \langle b_1, ..., b_n \rangle$.

which says that no world where b plays b_j is strictly better than u for a. Assuming totality, this is equivalent to "u is at least as good as any alternative where b plays b_j". For the Nash equilibrium, I say that every player uses its best response at u. In the two-player case, this amounts to:

$$\mathfrak{M}, u \models \neg \langle \{a_i\} \rangle_b^< \top \wedge \neg \langle \{b_j\} \rangle_a^< \top$$

This definition is *local*, since the formula defining the equilibrium depends on the current state u. A more generic global definition of best response for agent i might involve a *ceteris paribus* modality referring to the intersection $\bigcap_{i \neq j} \sim_j$ of the epistemic accessibility relations for the other agents and strict preference for i. [66] gives a solution relating this to distributed knowledge of the other players.

For the general case, let Γ be the set of all strategies of all players in the set N, and Γ_{-a} the set off all strategies minus a's.

Fact B.0.13 *A state u is a Nash equilibrium iff:*

$$\mathfrak{M}, u \models \bigwedge_{a \in N} \neg \langle \Gamma_{-a} \rangle_a^< \top.$$

This definition of the Nash equilibrium isolates its *ceteris paribus* part and shows how it may be applied in game theory. Of course, to get a more substantial definition where actions and beliefs are also involved, one would need to extend the language, build models for it and seek its logic. But we see here a glimpse of how the CPL approach might help in this research.

Appendix C

Some Algebra

I provide an algebraic derivation of the group Axioms 6.27-6.30. The result I want to show is the following:

Proposition C.0.14 *The group Axioms 6.27-6.30 are sound.*

I first introduce preliminary definitions and facts.

Definition C.0.15 Let R be a relation, then the *inverse* of R, denoted R^{-1}, is defined by:

$$R^{-1} = \{(b, a) : (a, b) \in R\}$$

$\lhd$

Fact C.0.16 *The inverse operator* -1 *distributes over* $\cap$ *and* $\cup$.

Proof. I show that $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$ and $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$.

1.
$$\begin{array}{rcl} (R \cap S)^{-1} & = & \{(b, a) : (a, b) \in R \cap S\} \\ & = & \{(b, a) : (a, b) \in R\} \,\&\, \{(b, a) : (a, b) \in S\} \\ & = & R^{-1} \cap S^{-1} \end{array}$$

2.
$$\begin{array}{rcl} (R \cup S)^{-1} & = & \{(b, a) : (a, b) \in R \cup S\} \\ & = & \{(b, a) : (a, b) \in R\} \text{ or } \{(b, a) : (a, b) \in S\} \\ & = & R^{-1} \cup S^{-1} \end{array}$$

QED

Definition C.0.17 Let R be a relation, then the *complement* or R, denoted $\neg R$, is defined by:

$$\neg R = \{(a, b) : (a, b) \notin R\}$$

$\lhd$

Fact C.0.18 $R \cap \neg(R^{-1}) = R^{<}$

Proof. Definition. QED

Fact C.0.19 $R^{<} \cap \neg(R^{-1}) = R^{<}$

Proof. I show that $R^{<} \subseteq \neg(R^{-1}$. Let $(a, b) \in R^{<}$, then $(a, b) \in R \,\&\, (a, b) \notin R^{-1}$. Hence, $(a, b) \in \neg R^{-1}$. QED

Fact C.0.20 $R \cap \neg((R^{<})^{-1}) = R$

Proof. I show that $R \subseteq \neg((R^{<})^{-1})$. Let $(a, b) \in R$ and suppose that $(a, b) \in (R^{<})^{-1}$. Then $(b, a) \in R^{<}$, i.e., $(b, a) \in R \,\&\, (b, a) \in \neg(R^{-1})$. Thus, $(b, a) \notin R^{-1}$, i.e., $(a, b) \notin R$, contradiction. Therefore, $(a, b) \in \neg(R^{<})^{-1}$. QED

Proof of Proposition C.0.14. The soundness of Axioms 6.27 and 6.28 is immediate, as they correspond to the *but* and *on the other hand* priority operators. I provide an algebraic derivation of Axioms 6.29 and 6.30. I show this by analyzing the decomposition of $(X \parallel Y)^{<}$ and $(X/Y)^{<}$ algebraically. In the next two derivations, I only use simple algebraic operations (De Morgan laws) as well as the facts just established.

The algebraic analysis of $(X \parallel Y)^{<}$, rendered in Axiom 6.29, is given by:

$$\begin{aligned}
(X \parallel Y)^{<} &= (X \parallel Y)^{\leq} \cap \neg(((X \parallel Y)^{\leq})^{-1}) \\
&= (X \cap Y) \cap \neg((X \cap Y)^{-1}) \\
&= (X \cap Y) \cap \neg(X^{-1} \cap Y^{-1}) \\
&= (X \cap Y) \cap (\neg X^{-1} \cup \neg Y^{-1}) \\
&= (X \cap Y \cap \neg X^{-1}) \cup (X \cap Y \cap \neg Y^{-1}) \\
&= (X^{<} \wedge Y) \vee (X \wedge Y^{<})
\end{aligned}$$

and the algebraic analysis of $(X/Y)^{<}$, rendered in Axiom 6.30, is given by:

$$
\begin{aligned}
(X/Y)^{<} &= (X/Y)^{\leq} \cap \neg(((X/Y)^{\leq})^{-1}) \\
&= ((X \cap Y) \cup Y^{<}) \cap \neg(((X \cap Y) \cup Y^{<})^{-1}) \\
&= ((X \cap Y) \cup Y^{<}) \cap \neg((X^{-1} \cap Y^{-1}) \cup (Y^{<})^{-1}) \\
&= ((X \cap Y) \cup Y^{<}) \cap ((\neg X^{-1} \cup \neg Y^{-1}) \cap \neg(Y^{<})^{-1}) \\
&= [((X \cap Y) \cap \neg(Y^{<})^{-1}) \cup (Y^{<} \cap \neg(Y^{<})^{-1}))] \cap (\neg X^{-1} \cup \neg Y^{-1}) \\
&= [((X \cap Y) \cup Y^{<}] \cap (\neg X^{-1} \cup \neg Y^{-1}) \\
&= [((X \cap Y) \cup Y^{<}) \cap \neg X^{-1}] \cup [((X \cap Y) \cup Y^{<}) \cap \neg Y^{-1}] \\
&= [((X \cap Y) \cup Y^{<}) \cap \neg X^{-1}] \cup [(X \cap Y^{<}) \cup Y^{<}] \\
&= (X^{<} \cap Y) \cup (Y^{<} \cap \neg X^{-1}) \cup Y^{<} \\
&= (X^{<} \cap Y) \cup Y^{<}
\end{aligned}
$$

QED

Another perspective: graph calculus

The way I have used the priority operators / and $\|$ in modalizing preference aggregation suggests a more general standpoint on the subject. From a strictly formal point of view, the modalities can be read as graph manipulations and this suggests investigating modal logics for graph calculus. Here, I content myself with restating the aggregation language in terms of graph calculus and leave for future research a full system for graph manipulations.[1]

Inspecting the structure of Axiom 6.27, it is easy to restate the operator $\|$ as an operation on graphs, namely *disjoint union* $\uplus$.

Definition C.0.21 Let $G_1 = \langle N_1, <_1, V_1\rangle$ and $G_2 = \langle N_2, <_2, V_2\rangle$ be two priority graphs. The *disjoint union* of G_1 and G_2, denoted $G_1 \uplus G_2$, is a a triple $\langle N_{G_1 \uplus G_2}, <_{G_1 \uplus G_2}, V_{G_1 \uplus G_2}\rangle$, where $N_{G_1 \uplus G_2} = N_1 \cup N_2$, $<_{G_1 \uplus G_2} = <_1 \cup <_2$ and $V_{G_1 \uplus G_2}(i) = V_1(i) \cup V_2(i)$. $\lhd$

Fact C.0.22 *Let G_1, G_2 be priority graphs, o_1, o_2 priority operators denoting G_1 and G_2 respectively and o_{1+2} denoting $G_1 \uplus G_2$, then:*

$$a o_{1+2}((\preceq_x)_{x \in N}) b \quad \textit{iff} \quad a o_1((\preceq_x)_{x \in N}) b \cap a o_2((\preceq_x)_{x \in N}) b \tag{C.1}$$

[1] Similar considerations on graph calculus can be found in [41].

Proof. The equivalence follows directly from Definition 6.1. In the one direction, since G_1 and G_2 are disjoint, it is not the case that there is a $j \in N$ with $j < k$ for any $k \in G_1, G_2$. Thus, it must be that $ao_1((\preceq_x)_{x\in N})b \cap ao_2((\preceq_x)_{x\in N})b$. In the other direction, since $ao_1((\preceq_x)_{x\in N})b \cap ao_2((\preceq_x)_{x\in N})b$, we immediately get that $ao_{1+2}((\preceq_x)_{x\in N})b$. QED

Fact C.0.22 is reflected in the following principle, which is just a restatement of Axiom 6.27, with modalities explicitly indexed with graphs and disjoint union of graphs instead of graph variables as in group preference logic:

$$\langle G_1 \uplus G_2\rangle^{\leq}s \leftrightarrow \langle G_1\rangle^{\leq}s \wedge \langle G_2\rangle^{\leq}s \tag{C.2}$$

More generally, given n graphs $G_1, ..., G_n$, the effect of taking their disjoint union is expressed in the principle:

$$\langle G_1 \uplus ... \uplus G_n\rangle^{\leq}s \leftrightarrow \langle G_1\rangle^{\leq}s \wedge ... \wedge \langle G_n\rangle^{\leq}s \tag{C.3}$$

Similarly, the priority graph operator / can be translated as an explicit operation on graph, which I call *sequential composition*, denoted $G; G'$. Sequential composition on graphs takes a set of graph with a given order, where graphs occurring on the left are given priority, and returns their lexicographic aggregation. Thus, given two graphs G_1 and G_2 such that $G_1; G_2$, sequential composition returns the order $a \preceq_{G_1;G_2} b$ if $a \prec_{G_1} b$ or finds a compensation with the common relations $a \preceq_{G_1} b$ and $a \preceq_{G_2} b$, in case it is not the case that $a \prec_{G_1} b$.

Fact C.0.23 *Let G_1, G_2 be priority graphs, o_1, o_2 priority operators denoting G_1 and G_2 respectively and $o_{1;2}$ denoting $G_1; G_2$, then:*

$$\begin{aligned} ao_{1;2}((\preceq_x)_{x\in N})b \quad &\textit{iff} \quad (ao_1((\preceq_x)_{x\in N})b \cap ao_2((\preceq_x)_{x\in N})b) \\ &\quad\;\; \textit{or } (ao_1((\prec_x)_{x\in N})b \end{aligned} \tag{C.4}$$

As above, Fact C.0.23 can be represented in the following principle, a restatement of Axiom 6.29:

$$\langle G_1; G_2\rangle^{\leq}s \quad \leftrightarrow \quad (\langle G_1\rangle^{\leq}s \wedge \langle G_2\rangle^{\leq}s) \vee \langle G_1\rangle^{<}s \tag{C.5}$$

For a general statement over n graphs, sequential decomposition can be obtained recursively, starting with the leftmost graph to obtain:

$$\langle G_1; ...; G_2\rangle^{\leq} s \quad \leftrightarrow \quad (\langle G_1\rangle^{\leq} s \wedge \langle G_2; ...; G_n\rangle^{\leq} s) \vee \langle G_1\rangle^{<} s \tag{C.6}$$

and after successive decomposition of ';', to obtain:

$$\begin{aligned} \langle G_1; ...; G_2\rangle^{\leq} s \quad \leftrightarrow \quad & (\langle G_1\rangle^{\leq} s \wedge ... \langle G_n\rangle^{\leq} s) \vee (\langle G_1\rangle^{\leq} s \wedge ... \wedge \langle G_{n-1}\rangle^{<} s) \\ & \vee ... \vee (\langle G_1\rangle^{\leq} s \wedge \langle G_2\rangle^{<} s) \vee \langle G_1\rangle^{<} s \end{aligned} \tag{C.7}$$

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