Duality and universal models for the meet-implication fragment of IPC

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Abstract. In this paper we investigate the fragment of intuitionistic logic which only uses conjunction (meet) and implication, using finite duality for distributive lattices and universal models. We give a description of the finitely generated universal models of this fragment and give a complete characterization of the up-sets of Kripke models of intuitionistic logic which can be defined by meet-implication-formulas. We use these results to derive a new version of subframe formulas for intuitionistic logic and to show that the uniform interpolants of meet-implication-formulas are not necessarily uniform interpolants in the full intuitionistic logic.

Keywords: Duality, universal models, intuitionistic logic, Heyting algebras, free algebras, implicative semilattices, definability, interpolation

1 Introduction

Heyting algebras are the algebraic models of intuitionistic propositional logic, IPC. In this paper we will be concerned with the syntactic fragment of IPC consisting of the formulas which only use the connectives of conjunction ($\wedge$) and implication ($\to$), but no disjunction ($\vee$) or falsum ($\bot$). The algebraic structures corresponding to this fragment are called implicative semilattices[4]. A result due to Diego [1] says that the variety of implicative semilattices is locally finite, i.e., finitely generated algebras are finite, or equivalently, the finitely generated free algebras are finite. In logic terms, this theorem can be expressed as saying that there are only finitely many equivalence classes of $(\wedge, \to)$-formulas in IPC.
One of the key results in this paper is a dual characterization of a $(\wedge, \to)$-subalgebra of a given Heyting algebra generated by a fixed set of elements (Theorem 21). This theorem leads to a proof of Diego's theorem and a characterization of the n-universal models of the $(\wedge, \to)$-fragment of IPC (Theorem 22) as submodels of the universal model, in the same spirit as the proof by de Jongh

[4] In less recent literature, these are also called Brouwerian semilattices.

et al. in [2]. The first characterization of this model was obtained by Köhler [3] using his duality for finite implicative meet-semilattices. Our slightly different approach in this paper also enables us to obtain new results about the $(\wedge, \to)$-fragment of IPC. In particular, in Theorem 24, we give a full characterization of the up-sets of a Kripke model which can be defined by $(\wedge, \to)$-formulas. Since our characterization in particular applies to the n-universal model of IPC, this may be considered as a first step towards solving the complicated problem of characterizing the up-sets of the n-universal models which are definable by intuitionistic formulas (also see our more detailed remarks in Section 5). Building on this result, we use the de Jongh formulas for IPC to construct formulas that play an analogous role in the $(\wedge, \to)$-fragment. Finally, we use the characterization of $(\wedge, \to)$-definable subsets of the n-universal models of IPC to show that a uniform interpolant of a $(\wedge, \to)$-formula in intuitionistic logic may not be equivalent to a $(\wedge, \to)$-formula.

A word on methodology. The two essential ingredients to our proofs are, on the one hand, Birkhoff duality for finite distributive lattices and, on the other hand, the theory of n-universal models for IPC ([4], [5]). Our methods in this paper are directly inspired by the theory of duality for $(\wedge, \to)$-homomorphisms as developed in [3], [6], [7], [8], and also by the observations about the relation between the n-universal models and duality for Heyting algebras in [9]. However, we made an effort to write this paper in such a way to be as self-contained as possible, and in particular we do not require the reader to be familiar with any of these results. In particular, we give a brief introduction to duality for finite distributive lattices and its connection to Kripke semantics for IPC in Section 2, and we do not need to go into the intricacies of duality for implicative meet-semilattices, instead opting to give direct proofs of the duality-theoretic facts that we need.

The paper is organized as follows: in Section 2 we present the necessary preliminaries about IPC and Heyting algebras in the context of duality for distributive lattices; in Section 3 we study the meet-implication fragment of IPC and prove our main theorems mentioned above; in Section 4 we apply these results to define $(\wedge, \to)$-de Jongh formulas and analyze semantically the uniform interpolation in the $(\wedge, \to)$-fragment of IPC. In Section 5 we summarize our results and give suggestions on where to go from here.

2 Algebra, semantics and duality

We briefly outline the contents of this section. In Subsection 2.1, we recall the definitions and basic facts about adjunctions between partially ordered sets, Heyting algebras, and implicative meet-semilattices. Subsection 2.2 contains the preliminaries about duality theory that we will need in this paper. In Subsection 2.3 we show how to define the usual Kripke semantics for IPC via duality, and in Subsection 2.4 we recall how the universal and canonical models for IPC are related to free finitely generated Heyting algebras via duality.

2.1 Adjunction, Heyting algebras, implicative meet-semilattices

Since the notion of adjunction is crucial to logic in general, and in particular to intuitionistic logic, we recall some basic facts about it right away. An adjunction can be understood as an invertible rule that ties two logical connectives or terms. The typical example in intuitionistic logic is the adjunction between $\wedge$ and $\to$, which can be expressed by saying that the following (invertible) rule is derivable in IPC.

$$\frac{p \wedge q \vdash r}{p \vdash q \to r} \tag{1}$$

Recall that an *adjunction* between partially ordered sets A and B is a pair of functions $f : A \leftrightarrows B : g$ such that, for all $a \in A$ and $b \in B$, $f(a) \leq b$ if, and only if, $a \leq g(b)$; notation: $f \dashv g$. In this case, we say that f is *lower adjoint* to g and g is *upper adjoint* to f. Note that the derivability of rule (1) in IPC says exactly that, for any ψ, the function $\varphi \mapsto \varphi \wedge \psi$ on the Lindenbaum algebra for IPC (cf. Example 3(b) below) is lower adjoint to the function $\chi \mapsto \psi \to \chi$. The following general facts about adjunctions are well-known and will be used repeatedly in this paper.

Proposition 1. *Let A and B be partially ordered sets and let $f : A \leftrightarrows B : g$ be an adjunction. The following properties hold:*

1. *If f is surjective, then $fg = \mathrm{id}_B$, and therefore g is injective and the image of g is $\{a \in A \mid gf(a) \leq a\}$;*
2. *The function f preserves any joins (suprema) which exist in A and the function g preserves any meets (infima) which exist in B;*
3. *For any $b \in B$, $g(b)$ is the maximum of $\{a \in A \mid f(a) \leq b\}$. In particular, the fact that g is upper adjoint to f uniquely determines g.*

Moreover, if C and D are complete lattices and $f : C \to D$ is a function which preserves arbitrary joins, then f has an upper adjoint.

Proof. Straightforward; cf., e.g., [10, 7.23–7.34]. ☐

Recall that a tuple $(A, \wedge, \vee, \to, 0, 1)$ is a *Heyting algebra* if $(A, \wedge, \vee, 0, 1)$ is a bounded lattice, and the operation $\to$ is upper adjoint to $\wedge$, i.e., for any $a, b, c \in A$,

$$a \wedge b \leq c \iff a \leq b \to c. \tag{2}$$

The equation (2) says that $b \to c$ is the maximum of $\{a \in A \mid a \wedge b \leq c\}$; therefore, a lattice admits at most one "Heyting implication", i.e., an operation $\to$ such that it becomes a Heyting algebra. The lattices underlying Heyting algebras are always distributive (in fact, for any $a \in A$, the function $b \mapsto a \wedge b$ preserves any join that exists in A, since it is a lower adjoint). All finite distributive lattices admit a Heyting implication. A *Heyting homomorphism* is a map between Heyting algebras that preserves each of the operations. An implicative meet-semilattice is a "Heyting algebra without disjunction". More precisely, an *implicative meet-semilattice* is a tuple $(A, \wedge, \rightarrow)$ such that $(A, \wedge)$ is a semilattice, and equation (2) holds. We will write $(\wedge, \rightarrow)$-homomorphism to abbreviate "homomorphism of implicative meet-semilattices". Note that any implicative meet-semilattice has a largest element, 1, which is preserved by any $(\wedge, \rightarrow)$-homomorphism. Also note that finite implicative meet-semilattices are distributive lattices, but $(\wedge, \rightarrow)$-homomorphisms do not necessarily preserve joins. However, *surjective* $(\wedge, \rightarrow)$-homomorphisms do preserve join (cf. [3, Lemma 2.4 and the remark thereafter]):

Lemma 2. *If $f : A \to B$ is a surjective $(\wedge, \rightarrow)$-homomorphism between Heyting algebras, then f is join-preserving.*

Proof. First of all, we have $0_B = f(a)$ for some $a \in A$, and $0_A \leq a$, so that $f(0_A) = 0_B$. Now let $a, a' \in A$. Pick $c \in A$ such that $f(c) = f(a) \vee f(a')$. Now

$$\begin{aligned} f(a \vee a') \to (f(a) \vee f(a')) &= f(a \vee a') \to f(c) \\ &= f((a \vee a') \to c) \\ &= f((a \to c) \wedge (a' \to c)) \\ &= (f(a) \to f(c)) \wedge (f(a') \to f(c)) \\ &= (f(a) \vee f(a')) \to f(c) = 1, \end{aligned}$$

so $f(a \vee a') \leq f(a) \vee f(a')$. The other inequality holds because f is order-preserving. □

Example 3. (a) An important example of a Heyting algebra is the collection of upward closed sets ('up-sets') of a partially ordered set $(X, \leq)$, ordered by inclusion; we denote this Heyting algebra by $\mathcal{U}(X)$. The Heyting implication of two up-sets U and V is given by the formula

$$U \to V = ({\downarrow}(U \cap V^c))^c, \tag{3}$$

that is, a point x is in $U \to V$ if, and only if, for all $y \geq x$, $y \in U$ implies $y \in V$. The reader who is familiar with models for IPC will recognize the similarity between this condition and the interpretation of a formula $\varphi \to \psi$ in a model; we will recall the precise connection between the two in 2.3 below.

(b) Another example of a Heyting algebra, of a more logical nature, is that of the Lindenbaum algebra for IPC; we briefly recall the definition. Fix a set of propositional variables P and consider the collection $F(P)$ of all propositional formulas whose variables are in P. Define a pre-order $\preceq$ on $F(P)$ by saying, for $\varphi, \psi \in F(P)$, that $\varphi \preceq \psi$ if, and only if, ψ is provable from φ in IPC. The Lindenbaum algebra is defined as the quotient of $F(P)$ by the equivalence relation $\approx := (\preceq) \cap (\preceq)^{-1}$. The Lindenbaum algebra is the free Heyting algebra over the set P, i.e., any function from P to a Heyting algebra H lifts uniquely to a Heyting homomorphism from the Lindenbaum algebra over P to H. We will denote the free Heyting algebra over P by $F_{HA}(P)$. Note that the same construction can be applied to the $(\wedge, \vee)$- and $(\wedge, \rightarrow)$-fragments of IPC to yield the free distributive lattice $F_{DL}(P)$ and the free implicative meet-semilattice $F_{\wedge,\rightarrow}(P)$, respectively. □

2.2 Duality

We briefly recall the facts about duality that we will need. Let D be a distributive lattice. We recall the definition of the *dual poset*, D_*, of D. The points of D_* are the prime filters of D, i.e., up-sets $F \subseteq D$ which contain finite meets of their subsets and have the property that if $a \vee b \in F$, then $a \in F$ or $b \in F$. The partial order on D_* is the inclusion of prime filters. The map $\eta : D \to \mathcal{U}(D_*)$ which sends $d \in D$ to $\{F \in D_* \mid d \in F\}$ is (assuming the axiom of choice) an embedding of distributive lattices, which is called the *canonical extension* of D. If D is finite, then η is an isomorphism, so that any finite distributive lattice is isomorphic to the lattice of up-sets of its dual poset. The assignments $X \mapsto \mathcal{U}(X)$ and $D \mapsto D_*$ between finite posets and finite distributive lattices extend to a dual equivalence, or duality, of categories: homomorphisms from a distributive lattice D to a distributive lattice E are in a natural bijective correspondence with order-preserving maps from E_* to D_*. A homomorphism $h : D \to E$ is sent to the map $h_* : E_* \to D_*$ which sends $F \in E_*$ to $h^{-1}(F)$, and an order-preserving map $f : X \to Y$ is sent to the homomorphism $f^* : \mathcal{U}(Y) \to \mathcal{U}(X)$ which sends an up-set U of Y to $f^{-1}(U)$.

If X and Y are posets, it is natural to ask which order-preserving maps $f : X \to Y$ are such that their dual $f^{-1} : \mathcal{U}(Y) \to \mathcal{U}(X)$ is a Heyting homomorphism. It turns out that these are the *p-morphisms*, i.e., the order-preserving maps which in addition satisfy the condition: for any $x \in X$, $y \in Y$, if $f(x) \leq y$, then there exists $x' \geq x$ such that $f(x') = y$.

To end this subsection, we recall how duality yields a straight-forward description of the free distributive lattice $F_{DL}(P)$. In any category of algebras, the free algebra over a set P is the P-fold coproduct of the one-generated free algebra. Therefore, since duality transforms coproducts into products, the dual space $F_{DL}(P)_*$ is the P-fold power of the poset $F_{DL}(\{p\})_*$, the dual of the one-generated free algebra. Note that $F_{DL}(\{p\})$ is the three-element chain $\{0 \leq p \leq 1\}$, so its dual is the two-element poset $2 = \{0, 1\}$. Since products in posets are simply given by equipping the Cartesian product with the pointwise order, it follows that $F_{DL}(P)_* = 2^P$. Therefore, the free distributive lattice over P is the lattice of up-sets of 2^P; in a formula, $F_{DL}(P) = \mathcal{U}(2^P)$.

2.3 Semantics via duality

Notation. Throughout the rest of this paper, we fix a finite set of propositional variables $P = \{p_1, \ldots, p_n\}$. We denote the free algebras over P by $F_{HA}(n)$, $F_{DL}(n)$, etc.

In this paper, a *frame* is a poset $(M, \leq)$. A *model* is a triple $(M, \leq, c)$, where $(M, \leq)$ is a poset and c, the *coloring*, is an order-preserving function from M to 2^n. The coloring c yields, via duality, a distributive lattice homomorphism $c^* : \mathcal{U}(2^n) \to \mathcal{U}(M)$. As noted at the end of 2.2, $\mathcal{U}(2^n)$ is the free distributive lattice over the set of generators n. By the universal property of the free Heyting algebra, the lattice homomorphism c^* has a unique extension to a Heyting homomorphism, v, from the free n-generated Heyting algebra to the Heyting algebra $\mathcal{U}(M)$, as in diagram (4).

$$\begin{array}{ccc} & F_{HA}(n) & \\ & \uparrow & \searrow^{v} \\ (2^n)^* = & F_{DL}(n) & \xrightarrow{c^*} \mathcal{U}(M) \end{array} \tag{4}$$

A point x in a model M is said to *satisfy* a formula φ if, and only if, $x \in v(\varphi)$; we employ the usual notation: $M, x \models \varphi$. Note that, as an alternative to the above algebraic description, one may equivalently define the satisfaction relation for models by induction on the complexity of formulas; see e.g. [5, Def. 2.1.8]. A model is said to satisfy φ if every point of the model satisfies φ. A *p-morphism* f from a model M to a model N is a p-morphism between the underlying frames of M and N which in addition satisfies, for any $x \in M$, $c_N(f(x)) = c_M(x)$. From the above definitions, it is clear that p-morphisms preserve truth, i.e., $M, x \models \varphi$ if, and only if, $N, f(x) \models \varphi$, for any formula φ. A *generated submodel* of M is a submodel M' such that the inclusion $f : M' \hookrightarrow M$ is a p-morphism, or equivalently, such that M' is an up-set. We say M' is a *p-morphic image* of M if there exists a surjective p-morphism $f : M \twoheadrightarrow M'$.
Recall that a *general frame* is a tuple $(M, \leq, A)$, where $(M, \leq)$ is a poset and A is a subalgebra of the Heyting algebra of up-sets of M. The elements of the algebra A are called the *admissible sets* of the general frame. An important subclass consists of those general frames $(M, \leq, A)$ for which $(M, \leq)$ is the dual poset of the Heyting algebra A; these are precisely the *descriptive* general frames.[5]
An *admissible coloring* is a coloring $c : M \to 2^n$ with the additional property that, for each $1 \leq i \leq n$, the set $\{x \in M \mid c(x)_i = 1\}$ is admissible. By the latter description and duality, admissible colorings c on a descriptive frame $(M, \leq, A)$ correspond to homomorphisms $c^* : F_{DL}(n) \to A$. A diagram analogous to (4), replacing $\mathcal{U}(M)$ by A, is now used to define the semantics of a general model $(M, \leq, A, c)$.

2.4 Canonical and universal models

The dual poset of the free n-generated Heyting algebra, $F_{HA}(n)$, is called the *canonical frame* and is denoted by $C(n)$. In logic terms, points in the canonical frame are so-called "theories with the disjunction property". The canonical frame carries a natural coloring c, which is the dual of the inclusion $F_{DL}(n) \hookrightarrow F_{HA}(n)$. Concretely, $c(x)_i = 1$ if, and only if, the variable p_i is an element of x. The model thus defined is called the *canonical model*, and is also denoted by $C(n)$.[6]

[5] For an equivalent characterization of descriptive general frames as the 'compact refined' general frames, cf. e.g. [5, Def. 2.3.2, Thm. 2.4.2].

[6] The canonical frame and model are also known as the Henkin frame and model.

Note that, by the embedding $\eta : F_{HA}(n) \hookrightarrow \mathcal{U}(C(n))$, any element φ of $F_{HA}(n)$ defines an up-set $\eta(\varphi) = \{x \in C(n) \mid \varphi \in x\}$ of $C(n)$. Since η is in particular a Heyting homomorphism that extends c^*, it is equal to the semantics map v for $C(n)$ defined in (4). Concretely, this means that, for any $x \in C(n)$ and $\varphi \in F_{HA}(n)$, we have $C(n), x \models \varphi$ if, and only if, $\varphi \in x$; this fact is often referred to as the *truth lemma*.
Let $\widehat{F_{HA}(n)}$ be the profinite completion of $F_{HA}(n)$; recall from [11, Thm 4.7] that $\widehat{F_{HA}(n)}$ is the Heyting algebra of up-sets of $C(n)_{\text{fin}} := \{x \in C(n) \mid {\uparrow}x \text{ is finite}\}$, the image-finite part of $C(n)$. The generated submodel $C(n)_{\text{fin}}$ of $C(n)$ is known as the *universal model* and denoted by $U(n)$.

Lemma 4. *The map $v : F_{HA}(n) \to \mathcal{U}(U(n))$ is injective.*

Proof. Cf., e.g., [5, Thm 3.2.20]. □

Importantly, the universal model can be described by an inductive top-down construction, as follows.

Theorem 5. *The universal model $U(n)$ is the unique image-finite model satsfying all of the following conditions:*

1. *there are 2^n maximal points with mutually distinct colors in $U(n)$;*
2. *for any $x \in U(n)$ and $c' < c(x)$, there is a unique point $x' \in U(n)$ with $c(x') = c'$ and ${\uparrow}x' = \{x'\} \cup {\uparrow}x$;*
3. *for any finite antichain $S \subseteq U(n)$ and $c' \leq \min\{c(x) \mid x \in S\}$, there is a unique point $x' \in U(n)$ with $c(x') = c'$ and ${\uparrow}x' = \{x'\} \cup \bigcup_{x \in S} {\uparrow}x$.*

Proof. Cf. [5, Sec. 3.2]. □

The following important fact states a 'universal property' for the universal model. Following the terminology of [5, Sec 3.1], we say that a frame M has *finite depth* $\leq m$ if every chain in M has size at most m.

Proposition 6. *If M is a model on n variables of finite depth $\leq m$, then there exists a unique p-morphism $f : M \to U(n)$. Moreover, the image of f has depth $\leq m$.*

Proof. [7] We prove the statement by induction on m. First let M be a model of depth 0. In this case, there is clearly a unique p-morphism from M to $U(n)$, namely the one which sends each point in M to the unique maximal point in $U(n)$ of the same colour. Now let M be a model of depth $m + 1$, for $m \geq 0$. Let $x \in M$ be arbitrary; we will define $f(x) \in U(n)$. Note that, for every $y > x$, the submodel $M_y := {\uparrow}y$ generated by y has depth $\leq m$. Thus, for each $y > x$, let $f_y : M_y \to U(n)$ be the unique p-morphism; the image of f_y has depth $\leq m$ by the induction hypothesis. Therefore, the set $S := \bigcup_{y>x} \text{im}(f_y)$ has depth $\leq m$

[7] This fact is well-known, cf. e.g. [6, p. 428]. We briefly recall the proof here. Also cf., e.g., [12, Thm. 3.2.3], for more details. Note, however, that we do not assume here that M is finite, only that M has finite depth.

in $U(n)$. If S is empty, then x is maximal, and we define $f(x)$ to be the unique maximal point of $U(n)$ that has the same colour as x. Otherwise, S has finitely many minimal points, $s_0, \ldots, s_k$, say. Pick points $y_0, \ldots, y_k$ in M such that $s_i \in \mathrm{im}(f_{y_i})$. If $k = 0$ and $c(y_0) = c(x)$, then we define $f(x) := s_0$. Otherwise, by Theorem 5, there is a unique point s in $U(n)$ whose immediate successors are $s_0, \ldots, s_k$ such that $c(s) = c(x)$; we define $f(x) := s$. It is straightforward to check that f defined in this manner is the unique p-morphism from M to $U(n)$, and clearly the image of f has depth $\leq m + 1$. □

Remark 7. Two points x and x' in a model M of finite depth are bisimilar if, and only if, the unique p-morphism f in Proposition 6 sends them to the same point of $U(n)$.

Theorem 8. *For each $x \in U(n)$ there exist formula φ_w and ψ_w such that*

1. $v(\varphi_w) = {\uparrow}x$,
2. $v(\psi_w) = U(n) \setminus {\downarrow}x$.

Formulas φ_w and ψ_w are called de Jongh formulas.

Proof. Cf. [5, Thm. 3.3.2]. □

Note that the de Jongh formula ψ_w has the following property: a frame G refutes ψ_w iff there is a generated subframe of G p-morphically mapped onto the frame generated by w. This way de Jongh formulas correspond to the so-called Jankov or splitting formulas, see [5, Sec. 3.3] for the details.

3 Separated points and the meet-implication fragment

In this section we use a duality for Heyting algebras and $(\wedge, \to)$-homomorphisms for characterizing n-universal models of the $(\wedge, \to)$-fragment of IPC (Theorem 22) and for characterizing $(\wedge, \to)$-definable up-sets of n-universal models of IPC (Theorem 24). The main technical contribution is the characterization of the dual model of the $(\wedge, \to)$-subalgebra of a Heyting algebra A generated by a finite set of generators (Theorem 21). Our proofs rely on a discrete duality and do not use topology. They can be extended to Priestley [13] and Esakia [14] dualities by adding topology, but we will not use this (explicitly) in this paper. In the study of the meet-implication fragment, the following notion of 'separated point' in a model will be crucial.[8]

Definition 9. *Let $M = (M, \leq, c)$ be a model. A point $x \in M$ is separated if, and only if, there is a variable p such that $M, x \not\models p$ and, for all $y > x$, $M, y \models p$.*

Remark 10. Note that a point x is separated precisely when there exists a variable p such that x is maximal in the down-set $v(p)^c$, the complement of $v(p)$.

[8] This notion has it roots in [2]. Our 'separated' points are precisely those points which are 'not inductive and not full' in the terminology of [2, Def. 5].

The following alternative characterization of separated points relates them to the $(\wedge, \rightarrow)$-fragment.

Lemma 11. *Let x be a point in a model M. The following are equivalent:*

1. *The point x is separated;*
2. *There exists a $(\wedge, \rightarrow)$-formula φ such that x is maximal in the complement of $v(\varphi)$.*

Proof. It is clear that (1) implies (2) by Remark 10. For (2) implies (1), we prove the contrapositive. Suppose that x is not separated. We prove the negation of (2), i.e., x is not maximal in $(v(\varphi))^c$ for any $(\wedge, \rightarrow)$-formula φ, by induction on complexity of φ. For $\varphi = p$, this is clear by Remark 10. For $\varphi = \psi \wedge \chi$, note that $v(\varphi)^c = v(\psi)^c \cup v(\chi)^c$. From this equality, it follows that if x were maximal in $v(\varphi)^c$, it would already be maximal in either $v(\psi)^c$ or $v(\chi)^c$, which contradicts the induction hypothesis. For $\varphi = \psi \rightarrow \chi$, suppose that x is maximal in $v(\varphi)^c$. We will prove that x is also maximal in $v(\chi)^c$, which again contradicts the induction hypothesis. By maximality of x, all $y > x$ satisfy $\psi \rightarrow \chi$. However, x does not satisfy $\psi \rightarrow \chi$, so we must have that $x \in v(\psi) \cap v(\chi)^c$. Since $v(\psi)$ is an up-set, we conclude that, for all $y > x$, $y \in v(\psi)$, and therefore $y \in v(\chi)$. Hence, x is maximal in $v(\chi)^c$, as required. □

For a model M, we denote by M^s the submodel consisting of the separated points of M. That is, the order and coloring on M^s are the restrictions of the corresponding structures on M. (Note that the model M^s is a submodel, but almost never a *generated* submodel, i.e. is up-set, of M!)

Lemma 12. *Let M be a model on n variables. The submodel $(M^s, \leq|_{M^s}, c|_{M^s})$ has finite depth $\leq n$.*

Proof. Let C be a chain in M^s. For any $x, y \in M^s$, if $x < y$, then $c(x) < c(y)$, since x is separated and $y > x$ in M. Therefore, $\{c(x) \mid x \in C\}$ is a chain in the poset $(2^n, \leq)$, so that it must have size $\leq n$. Hence, C has size at most n. □

Definition 13 (The model $M_{\wedge,\rightarrow}$). *Let M be a model and M^s its submodel of separated points. Let $f : M^s \rightarrow U(n)$ be the unique p-morphism which exists by Proposition 6 and Lemma 12. Define $M_{\wedge,\rightarrow} := f(M^s)$ to be the generated submodel of $U(n)$ consisting of those points in the image of f, as in the following diagram.*

$$\begin{array}{ccc} & & U(n) \\ & \overset{f}{\nearrow} & \uparrow \\ M \supseteq M^s & \twoheadrightarrow & M_{\wedge,\rightarrow} \end{array} \tag{5}$$

Remark 14. Note that $M_{\wedge,\rightarrow}$ is a generated submodel of $U(n)$ of depth $\leq n$ and in particular, that $M_{\wedge,\rightarrow}$ is a finite model. Moreover, note that $M_{\wedge,\rightarrow} \subseteq U(n)^s$: let $y = f(x) \in M_{\wedge,\rightarrow}$ be arbitrary. We prove that y is separated. Since x is separated in M, we may pick a propositional variable p such that $M, x \models p$ and $M, x' \not\models p$ for all $x' > x$. Since f preserves the colouring, $U(n), y \models p$. Also, for any $y' > y = f(x)$, there exists $x' > x$ such that $f(x') = y'$, because f is a p-morphism. By assumption, $M, x' \not\models p$, and therefore, since f preserves the colouring, $U(n), y' \not\models p$. Thus, y is separated in $U(n)$. (The same argument shows that any p-morphism preserves separated points.) □

In Theorem 21 below, we will show that, for any descriptive model $(M, \leq, A, c)$, the model $M_{\wedge,\to}$ is dual to the $(\wedge, \to)$-subalgebra of A that is generated by the admissible up-sets $v(p_1), \dots, v(p_n)$. We need two lemmas.

Lemma 15. *Let $f : H \to K$ be a function between Heyting algebras with an upper adjoint $g : K \to H$. Then f preserves binary meets if, and only if, for all $a \in H$, $b \in K$, the equality $a \to g(b) = g(f(a) \to b)$ holds.*

Proof. Let $a \in H$ be arbitrary. Consider the following two diagrams.

$$\begin{array}{ccc} H & \xrightarrow{\;f\;} & K \\ {\scriptstyle a \wedge -}\downarrow & & \downarrow{\scriptstyle f(a) \wedge -} \\ H & \xrightarrow{\;f\;} & K \end{array} \qquad\qquad \begin{array}{ccc} H & \xleftarrow{\;g\;} & K \\ {\scriptstyle a \to -}\uparrow & & \uparrow{\scriptstyle f(a) \to -} \\ H & \xleftarrow{\;g\;} & K \end{array} \tag{6}$$

A way to express the assertion that f preserves binary meets is that, for all $a \in H$, the left diagram in (6) commutes. By uniqueness of adjoints, the left diagram in (6) commutes if, and only if, the right diagram in (6) commutes. □

Lemma 15 and its proof are very similar to, and were in fact directly inspired by, the *Frobenius condition* in [15, Def. p. 157] and the remark following it; we leave further exploration of the precise connection to future research.

Definition 16. *We say that a model M is a* model with borders *or that M has* borders[9] *if, for any variable p, $v(p)^c = {\downarrow}\max(v(p)^c)$.*

Proposition 17.

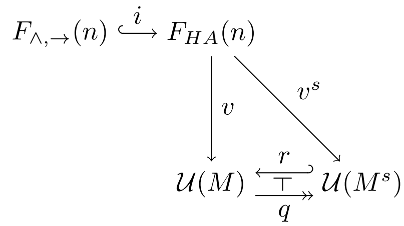
1. *Every image-finite model has borders.*
2. *Every descriptive model has borders.*

Proof. Item (1) is straightforward. For the proof of (2) see e.g., [5, Thm. 2.3.24].

The following lemma now provides the key connection between the construction of M^s and the $(\wedge, \to)$-fragment.

[9] Points in $\max(v(p)^c)$ are often called p-border points.

Lemma 18. *Let M be a model with borders. Consider the following diagram:*



(7)

where i is the natural inclusion, v and v^s are the valuation maps of M and M^s, respectively, q is the lattice homomorphism dual to the inclusion $M^s \hookrightarrow M$, and r is its upper adjoint. Then

$$r \circ v^s \circ i = v \circ i. \tag{8}$$

Remark 19. Note that, by Proposition 1, the function r sends an up-set V of M^s to the up-set $\{x \in M \mid \forall y \geq x, y \in M^s : y \in V\}$ of M. Therefore, the equality (8) says precisely that, for all $\varphi \in (\wedge, \to)$, $x \in M$, we have

$$M, x \models \varphi \iff \forall y \geq x, y \in M^s : M^s, y \models \varphi. \tag{9}$$

In this sense, Lemma 18 is an algebraic rendering of the crucial ingredient to [2, Proof of Thm. 1]. The proof we give here is different in spirit.

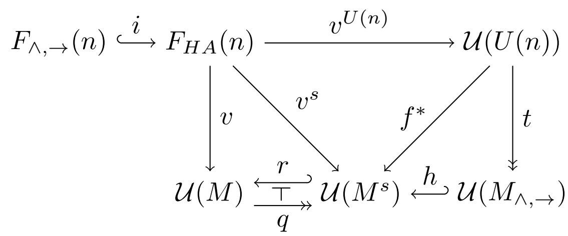
Proof (of Lemma 18). By Proposition 1, $qr = \mathrm{id}$ since q is surjective. We prove the equality (8) by induction on the complexity of $\varphi \in F_{\wedge,\to}(n)$, and suppress the function i in our notation. If φ is a variable p, note that $v^s(p) = qv(p)$ by definition, so we need to prove $v(p) = rqv(p)$. The inequality $v(p) \leq rqv(p)$ holds because r is upper adjoint to q. For the other inequality, suppose that $x \notin v(p)$. As M is a model with borders we can pick $y \in \max(v(p)^c)$ such that $y \geq x$. Then $y \in M^s$, so that $x \notin rqv(p)$, as required. The step $\varphi = \psi \wedge \chi$ is clear from the fact that both v and rv^s preserve binary meets (r is meet-preserving since it is an upper adjoint). Finally, suppose that (8) holds for formulas ψ and χ; we prove (8) for $\psi \to \chi$. Note in particular that $qv(\psi) = qrv^s(\psi) = v^s(\psi)$, where we use that $qr = \mathrm{id}$. We now have $rv^s(\psi \to \chi) = r(v^s(\psi) \to v^s(\chi)) = r(qv(\psi) \to v^s(\chi)) = v(\psi) \to rv^s(\chi)$, where the last equality holds by Lemma 15, since q preserves binary intersections. By the induction hypothesis, $rv^s(\chi) = v(\chi)$. Hence, $rv^s(\psi \to \chi) = v(\psi) \to v(\chi) = v(\psi \to \chi)$, as required. □

Corollary 20. *Let M be a model with borders and let M^s and $f : M^s \to M_{\wedge,\to}$ be as in Definition 13. Then for each $w \in M^s$ and a $(\wedge, \to)$-formula φ, we have*

$$M, w \models \varphi \iff M_{\wedge,\to}, f(w) \models \varphi. \tag{10}$$

Proof. By (9) we have that $M, w \models \varphi \iff \forall y \geq w, y \in M^s : M^s, y \models \varphi$. So, if $w \in M^s$, then $M^s, w \models \varphi$, and as f is a p-morphism we also have $M_{\wedge,\to}, f(w) \models \varphi$. Conversely, if $M_{\wedge,\to}, f(w) \models \varphi$, then $M^s, w \models \varphi$. Therefore, $M^s, y \models \varphi$ for each $y \in M^s$ with $y \geq w$. By (9) again this implies that $M, w \models \varphi$. □

Theorem 21. *Let M be a model with borders. Consider the following diagram:*



(11)

where the left part of the diagram is defined as in (7), $v^{U(n)}$ is the valuation on $U(n)$, and the triangle $f^* = h \circ t$ is the dual of the triangle in (5). The following properties hold:

1. *The image of the composite rh is equal to the $(\wedge, \to)$-subalgebra of $\mathcal{U}(M)$ that is generated by $v(p_1), \dots, v(p_n)$;*
2. *The implicative meet-semilattice $\mathcal{U}(M_{\wedge,\to})$ is isomorphic to the $(\wedge, \to)$-subalgebra of $\mathcal{U}(M)$ that is generated by $v(p_1), \dots, v(p_n)$.*

Proof. It clearly suffices to prove (1), as (2) then follows immediately. Let B denote the $(\wedge, \to)$-subalgebra of $\mathcal{U}(M)$ that is generated by $v(p_1), \dots, v(p_n)$. Note that $B = \mathrm{im}(vi)$. We show that $\mathrm{im}(rh)$ is equal to $B = \mathrm{im}(vi)$. To this end, note first that, chasing the diagram (11), we have:

$$vi = rv^s i = rf^* v^{U(n)} i = rhtv^{U(n)} i, \tag{12}$$

where we use Lemma 18 and the fact that $v^s = f^* v^{U(n)}$, since f is a p-morphism of models. In particular, B contains $\mathrm{im}(r \circ h)$ and $\mathrm{im}(r \circ h)$ contains $v(p_1), \dots, v(p_n)$. It thus remains to show that $\mathrm{im}(r \circ h)$ is a $(\wedge, \to)$-subalgebra of $\mathcal{U}(M)$, or equivalently, that $r \circ h$ preserves $\wedge$ and $\to$. Since r is an upper adjoint and h is a Heyting homomorphism, $r \circ h$ preserves $\wedge$. Moreover, using Lemma 15 and the fact that $qr = \mathrm{id}$, we have, for any $U, V \in \mathcal{U}(M_{\wedge,\to})$, that

$$rh(U) \to rh(V) = r(qrh(U) \to h(V)) = r(h(U) \to h(V)) = rh(U \to V),$$

where the last step uses that h is a Heyting homomorphism. $\square$

We now use this theorem to prove three facts about the $(\wedge, \to)$-fragment of IPC. The first is Diego's theorem.

Theorem 22. *For any n, we have $F_{\wedge,\to}(n) \cong \mathcal{U}(U(n)_{\wedge,\to})$. In particular, the variety of implicative meet-semilattices is locally finite.*

Proof. Apply Theorem 21 to the general model $(U(n), \leq, \mathcal{U}(U(n)), c)$. Since the valuation $v : F_{HA}(n) \to \mathcal{U}(U(n))$ is injective (Lemma 4), vi is injective. By (12) in the proof of Theorem 21, $vi = rhtv^{U(n)}i$. Since rh is also injective, it now follows easily that $tv^{U(n)}i : F_{\wedge,\to}(n) \to \mathcal{U}(U(n)_{\wedge,\to})$ is an isomorphism. The 'in particular'-part now follows becuase $U(n)_{\wedge,\to}$ is finite by Lemma 12. $\square$

Theorem 23. *For any $\varphi \in F_{HA}(n)$, there exists $s(\varphi) \in F_{\wedge,\to}(n)$ such that, for any model M and $x \in M^s$, we have:*

$$M^s, x \models \varphi \iff M^s, x \models s(\varphi).$$

Proof. Since $F_{\wedge,\to}(n)$ is finite by Theorem 22, it is a Heyting algebra, in which the supremum is given by

$$\varphi \veebar \psi = \bigwedge \{\chi \in F_{\wedge,\to} \mid \varphi \leq \chi \text{ and } \psi \leq \chi\}.$$

Hence, there exists a unique Heyting homomorphism $s : F_{HA}(n) \to F_{\wedge,\to}(n)$ such that $s(p) = p$ for all propositional variables p. Note that si is the identity on $F_{\wedge,\to}(n)$, so s is surjective. Also note that $tv^{U(n)}i$ is surjective: if $U \in \mathcal{U}(M_{\wedge,\to})$, then $rh(U)$ is in the image of vi, by Theorem 21, so pick $\psi \in F_{\wedge,\to}(n)$ such that $rh(U) = vi(\psi)$. Now, by (12), $rh(U) = rhtv^{U(n)}i(\psi)$, so $U = tv^{U(n)}i(\psi)$, since rh is injective. We conclude that $tv^{U(n)}is$ is a surjective $(\wedge,\to)$-preserving map, and therefore it is a Heyting homomorphism by Lemma 2. Now, $htv^{U(n)}is$ is also a Heyting homomorphism and $htv^{U(n)}is(p) = htv^{U(n)}(p) = v^s(p)$. By uniqueness of the map v^s, we conclude that $htv^{U(n)}is = v^s$, as required. $\square$

We now give a characterization of the $(\wedge,\to)$-definable subsets in any model.

Theorem 24. *Let M be a model with borders. Let $U \subseteq M$ be an up-set. The following are equivalent:*

1. *There exists a $(\wedge,\to)$-formula φ such that $v(\varphi) = U$;*
2. *For all $x \in M$, if, for all $z \in M^s$ such that $z \geq x$, there exists $y \in U \cap M^s$ bisimilar to z in M^s, then $x \in U$;*
3. *For all $x \in M$,*
   (*a*) *if all separated points above x are in U, then $x \in U$, and*
   (*b*) *if $x \in M^s$ and there exists $x' \in U \cap M^s$ which is bisimilar to x in M^s, then $x \in U$.*

Proof. By Theorem 21(1), the up-sets which are definable by a $(\wedge,\to)$-formula are precisely the up-sets in the image of rh. Let $h^\flat$ denote the lower adjoint of h, which is given explicitly by sending $S \in \mathcal{U}(M^s)$ to $f(S) \in \mathcal{U}(M_{\wedge,\to})$. By Proposition 1(1), applied to the adjunction $h^\flat q \dashv rh$, an up-set U is in $\mathrm{im}(rh)$ if, and only if, $rhh^\flat q(U) \subseteq U$. Writing out the definitions of r, h, $h^\flat$ and q, we see that this condition is equivalent to:

$$\forall x \in M, \text{ if } \big(\forall z \in M^s \text{ if } z \geq x \text{ then } z \in f^{-1}(f(U \cap M^s))\big) \text{ then } x \in U.$$

This condition is in turn equivalent to (2), using Remark 7. If (2) holds, then (3a) is clear. For (3b), suppose x is separated and there exists $x' \in U \cap M^s$ which is bisimilar to x in M^s. By bisimilarity, for any $z \in M^s$ with $z \geq x$, there exists $y \in M^s$ with $y \geq x'$ and y bisimilar to z in M^s. Moreover, since U is an up-set containing x', we have $y \in U$. Using (2), we conclude that $x \in U$. Now assume (3) and let $x \in M$ be a point such that for all $z \in M^s$ with $z \geq x$, there exists $y \in U \cap M^s$ bisimilar to z in M^s. If z is any separated point above x, then it follows from applying (3b) to z that $z \in U$. Therefore, by (3a), $x \in U$. $\square$

4 Subframe formulas and uniform interpolation

In this section we will apply the results obtained in the previous section to define $(\wedge, \to)$-versions of de Jongh formulas. We will show in Theorem 28 that these formulas correspond to subframe formulas in just the same way as de Jongh formulas correspond to Jankov formulas. We will also use the characterization of $(\wedge, \to)$-definable up-sets of $U(n)$ to prove that uniform interpolants in the $(\wedge, \to)$-fragment of IPC are not always given by the IPC-uniform interpolants (Example 31).

From Definition 13 and the remark following it, we know that $U(n)_{\wedge,\to}$ is a finite up-set of $U(n)$. So every up-set of $U(n)_{\wedge,\to}$ is also a finite up-set of $U(n)$, and hence is definable by an intuitionistic formula (cf. Theorem 8). The next theorem shows that these up-sets are also definable by the $(\wedge, \to)$-formulas that are the s-images of these intuitionistic formulas.

Theorem 25. *Let $U \subseteq U(n)_{\wedge,\to} \subseteq U(n)$ be defined in $U(n)$ by an intuitionistic formula φ. Then U is defined in $U(n)_{\wedge,\to}$ by the $(\wedge, \to)$-formula $s(\varphi)$.*

Proof. First note that $U(n)_{\wedge,\to} \subseteq U(n)^s$, by applying Remark 14 to $M = U(n)$. Now let $U \subseteq U(n)_{\wedge,\to}$ be defined by an intuitionistic formula φ. By Theorem 23, $U(n)^s, x \models \varphi$ if, and only if, $U(n)^s, x \models s(\varphi)$. Therefore, U is defined by $s(\varphi)$ in $U(n)^s$. Finally, as f is a p-morphism of models, we obtain that U is defined by $s(\varphi)$ in $U(n)_{\wedge,\to}$. □

For each $w \in U(n)$ we let φ_w denote its de Jongh formula (see Theorem 8). We let I_w denote the set of immediate succesors of w and $\varphi_{I_w} := \bigvee\{\varphi_v : v \in I_w\}$, $\xi_w := s(\varphi_w) \to s(\varphi_{I_w})$. We have the following useful corollary of Theorems 8 and 25.

Corollary 26. *Let $w \in U(n)_{\wedge,\to}$. Then*

1. $\uparrow w = v(s(\varphi_w))$,
2. $\uparrow I_w = v(s(\varphi_{I_w}))$,
3. $U(n)_{\wedge,\to} \setminus \uparrow w = v(\xi_w)$.

Proof. Items (1) and (2) follow directly from Theorems 8 and 25. Item (3) follows from (1) and (2). □

We need one more auxiliary lemma before proving the main theorem of this section.

Lemma 27. *For each finite rooted frame F, there exist $n \in \omega$ and a colouring $c : F \to 2^n$ such that $M = (F, c)$ is isomorphic to a generated submodel of $U(n)_{\wedge,\to}$.*

Proof. Let $n := |F|$ and enumerate the points of F as $x_1, \dots, x_n$. Define $c(x_i)_j$, the j^{th} coordinate of the colour of the point x_i, to be 1 if $x_i \geq x_j$, and 0 otherwise. All points in $M = (F, c)$ have distinct colors, and are in particular separated, so $M = M^s$. Let f be the unique p-morphism from $M = M^s$ to $U(n)$ from Proposition 6, its image is $M_{\wedge,\to}$. Recall from Remark 14 that $M_{\wedge,\to}$ is a submodel of $U(n)^s$. Let g be the unique p-morphism from $U(n)^s$ onto $U(n)_{\wedge,\to}$. Since the composite $gf : M \to U(n)_{\wedge,\to}$ preserves colours, it is injective, and it is therefore an isomorphism onto a generated submodel of $U(n)_{\wedge,\to}$. □

Theorem 28. *Let F be a finite rooted frame and let $M = (F, c)$ be the model on F defined in the proof of Lemma 27. There exists a $(\wedge, \to)$-formula $\beta(F)$ such that for each descriptive model N we have*

$$N \not\models \beta(F) \iff M \text{ is a p-morphic image of } N^s.$$

Proof. By Lemma 27, M is isomorphic to a generated submodel of $U(n)_{\wedge,\to}$. Without loss of generality, we will assume in the rest of this proof that M actually *is* a generated submodel of $U(n)_{\wedge,\to}$. Since the model M is rooted, there exists $w \in U(n)_{\wedge,\to}$ such that $M = {\uparrow}w$. We define $\beta(F) := \xi_w = s(\varphi_w) \to s(\varphi_{I_w})$ and prove that $\beta(F)$ satisfies the required property.
First note that, as follows from Corollary 26, $s(\varphi_w)$ defines the up-set of $U(n)_{\wedge,\to}$ generated by w and $s(\varphi_{I_w})$ defines the up-set of $U(n)_{\wedge,\to}$ generated by the set of proper successors of w. Therefore, w is the only point of $U(n)_{\wedge,\to}$ that satisfies $s(\varphi_w)$ and refutes $s(\varphi_{I_w})$.
Suppose that $N \not\models \beta(F)$. Since N is descriptive, we can find a successor u of v such that $N, u \models s(\varphi_w)$, $N, u \not\models s(\varphi_{I_w})$ and every proper successor of u satisfies $s(\varphi_{I_w})$ (see e.g., [5, Thm. 2.3.24]). By Lemma 11, this implies that $u \in N^s$. Let $f : N^s \to U(n)_{\wedge,\to}$ be the unique p-morphism as in Proposition 6. Because $u \in N^s$, $s(\varphi_w)$ is a $(\wedge, \to)$-formula, and $N, u \models s(\varphi_w)$, Corollary 20 entails that $U(n)_{\wedge,\to}, f(u) \models s(\varphi_w)$. By the same argument we also have that $U(n)_{\wedge,\to}, f(u) \not\models s(\varphi_{I_w})$. Thus, we obtain that $U(n)_{\wedge,\to}, f(u) \models s(\varphi_w)$ and $U(n)_{\wedge,\to}, f(u) \not\models s(\varphi_{I_w})$. We have shown in the previous paragraph that this implies $f(u) = w$. Therefore, as f is a p-morphism, we obtain that F is a p-morphic image of N^s.
For the other direction, let $f : N^s \to M$ be a surjective p-morphism. Since f is surjective, pick $u \in N^s$ such that $f(u) = w$. As w satisfies $s(\varphi_w)$ and refutes $s(\varphi_{I_w})$, and both are $(\wedge, \to)$-formulas, the same argument as in the proof of Corollary 20 gives that $N, u \models s(\varphi_w)$ and $N, u \not\models s(\varphi_{I_w})$. Hence, $N \not\models \beta(F)$. □

The formula $\beta(F)$ defined in Theorem 28 is called the *subframe formula of F*.
Recall that, for any formula φ in n variables, we say a Heyting algebra A *satisfies the equation* $\varphi \approx 1$, notation $A \models \varphi \approx 1$, if $\overline{v}(\varphi) = 1$ under each assignment $v : \{p_1, \dots, p_n\} \to A$. If there is an assignment v under which $\overline{v}(\varphi) \neq 1$, we say that A *refutes the equation* $\varphi \approx 1$.

Corollary 29. *Let F be a finite rooted frame and A its Heyting algebra of up-sets. Then for each Heyting algebra B we have*

$$B \not\models \beta(F) \approx 1 \iff \text{there is a } (\wedge, \to)\text{-embedding } A \hookrightarrow B$$

Proof. It follows from the proof of Theorem 28 that the model (F, c) refutes $\beta(F)$. This means that, in the Heyting algebra $A = \mathcal{U}(F)$, the formula $\beta(F)$ does not evaluate to 1 under the assignment $v : p_i \mapsto c^*(p_i)$. Suppose that there is a $(\wedge, \to)$-embedding $i : A \hookrightarrow B$. Since $\beta(F)$ is a $(\wedge, \to)$-formula, under the assignment $i \circ v$, the formula $\beta(F)$ does not evaluate to 1 in B. Conversely, suppose that $B \not\models \beta(F) \approx 1$, under an assignment v. Let G be the descriptive frame with B as its algebra of admissible up-sets. The assignment v yields an admissible colouring c' on G with the property that $N = (G, c') \not\models \beta(F)$. By Theorem 28, this implies in particular that F is a p-morphic image of N^s. It now follows from Theorem 21 that A is $(\wedge, \to)$-embedded into B. □

Remark 30. Subframe formulas axiomatize a large class of logics having the finite model property [6, Ch. 11]. The frames of these logics are closed under taking subframes. Alternatively varieties of Heyting algebras corresponding to these logics are closed under $(\wedge, \to)$-subalgebras. There exist many different ways to define subframe formulas for intuionistic logic: model-theoretic [6, Ch. 11], algebraic [16], [8], and via the so-called NNIL formulas [17]. Theorem 28 gives a new way to define subframe formulas. The proof of this theorem shows that the same way de Jongh formulas for intuitionistic logic correspond to Jankov formulas [5], de Jongh formulas for the $(\wedge, \to)$-fragment of intuitionistic logic correspond to subframe formulas. This provides a different perspective on the interaction of de Jongh-type formulas and frame-based formulas such as Jankov formulas, subframe formulas etc.

We finish this section by applying the results of this paper to show that the uniform IPC-interpolant, as defined by Pitts [18], of a meet-implication formula is not necessarily equivalent to a meet-implication formula.

Example 31. As can be readily checked, the uniform interpolant of the formula $p \to (q \to p)$ in IPC with respect to the variable p is the formula $\neg\neg p$. We will use the characterization in Theorem 24 prove that $\neg\neg p$ is not equivalent to a $(\wedge, \to)$-formula. Namely, if there were a $(\wedge, \to)$-formula φ equivalent to $\neg\neg p$ is, then in particular the up-set U defined by the formula $\neg\neg p$ in the 1-universal model of IPC (see figure 1 below) would be $(\wedge, \to)$-definable. It thus suffices to show that U is not $(\wedge, \to)$-definable. To see this, note that $U(1)^s = \max v(p)^c = \{x_1, x_2\}$, and these two points are bisimilar in $U(1)^s$. Since $x_2 \in U$ but $x_1 \notin U$, U does not satisfy (3b) in Theorem 24, and is therefore not $(\wedge, \to)$-definable.

We now also prove that the least $(\wedge, \to)$-definable up-set of $U(1)$ containing U is $U(1)$ itself. Indeed, let W be a $(\wedge, \to)$-definable up-set which contains U. Then, by the above, x_1 belongs to W. It then easily follows from (3a) in Theorem 24 that every color 0 point of $U(1)$ must also belong to W. Thus, $W = U(1)$. This argument shows, via semantics, that the $(\wedge, \to)$-formula which is a uniform interpolant of $p \to (q \to p)$ is $\top$. We refer to [19] for more details on uniform interpolation in the fragments of intuitionistic logic. □

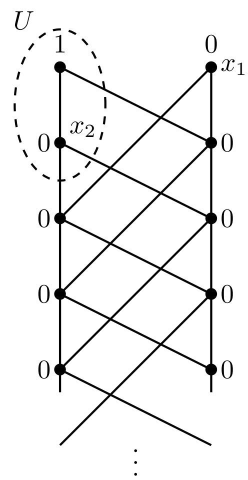


Figure 1: The 1-universal model, $U(1)$, also known as the Rieger-Nishimura ladder, with $U(1)^s = \{x_1, x_2\}$ and $U = v(\neg\neg p)$.

5 Conclusions and future work

In this paper we studied the $(\wedge, \to)$-fragment of intuitionistic logic via methods of duality theory. We gave an alternative proof of Diego's theorem and characterized $(\wedge, \to)$-definable up-sets of the n-universal model of intuitionistic logic, using duality as our main tool. Interestingly, we were able to directly use finite duality for distributive lattices and adjunction properties such as the Frobenius property (Lemma 15), without resorting to any of the existing dualities for implicative meet-semilattices. We expect that the techniques developed in Section 3 could be extended to the infinite setting in order to give a unified account of the different dualities that exist in the literature for implicative meet-semilattices, e.g., [3], [7] and [8]. We leave this as an interesting question for future work.

The characterization of $(\wedge, \to)$-definable up-sets that we gave in Theorem 24 can be considered as a first step towards solving a complicated problem of characterizing all IPC-definable up-sets of n-universal models. This problem is linked to the following interesting question. In [20] free Heyting algebras are described from free distributive lattices via step-by-step approximations of the operation $\to$. In [21], the authors explained how the construction in [20] can be understood via (finite) duality for distributive lattices. This begs the question whether one can use duality for implicative meet-semilattices to build free Heyting algebras, starting from free implicative meet-semilattices and approximating the operation of disjunction, $\vee$, step-by-step. The results of this paper can be considered as the first (or actually zeroth) step of such a step-by-step construction.

Finally, we note that [22] and [23] study n-universal models in other fragments of intuitionistic logic. We leave it to future work to investigate how the duality methods of this paper relate to the methods developed in [22] and [23].

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