The Topology of Full and Weak Belief

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Abstract

We introduce a new topological semantics for belief logics in which the belief modality is interpreted as the interior of the closure of the interior operator. We show that the system **wKD45**, a weakened version of **KD45**, is sound and complete w.r.t. the class of all topological spaces. Moreover, we point out a problem regarding updates on extremally disconnected spaces that appears in the setting of [1] and show that our proposal for topological belief semantics on all topological spaces constitutes a solution for it. While generalizing the topological belief semantics proposed in [1] to all spaces, we model conditional beliefs and updates and give complete axiomatizations of the corresponding logics.

Keywords: Topological models, epistemic and doxastic logic, updates, conditional beliefs, (hereditarily) extremally disconnected spaces.

1 Introduction

Understanding the relation between knowledge and belief is an issue of central importance in formal epistemology. Especially after the birth of the knowledge-first epistemology in [34], the question of what exactly distinguishes an item of belief from an item of knowledge and how one can be defined in terms of the other has become even more pertinent. There are basically two main approaches to analyse the knowledge-belief relation: on the one hand, one can start with the weakest notion of justified true belief (JTB) and enhance it by adding new conditions X that render the enhanced analysis JTB+X immune to Gettier-style counterexamples [16]. On the other hand, one can take a preferred notion of knowledge as primitive and weaken it to obtain a "good" (e.g. consistent, strong, introspective, possibly false) notion of belief. Most of the proposals found in the literature responding to this issue fall under the first approach. Among this category, we can mention the conception of knowledge as *correctly justified belief*: *not only the content of belief has to be true, but its justification has to be correct*. One possible implementation of this approach is via topologies under the *interior-based semantics*. According to the interior semantics, a proposition (set of possible worlds) P is known if there exists some "true evidence" (i.e. an open set U containing the real world x) that entails P (i.e., $x \in U \subseteq P$). Other responses to the Gettier challenge, falling under the first category, include the *defeasibility analysis of knowledge* [20, 19], the *sensitivity account* [24], the *contextualist account* [12] and the *safety account* [28][1].

The second approach, fits in line with Williamson's knowledge-first epistemology which challenges the 'conceptual priortiy of belief over knowledge' [34] and reverts the relation by given priority to knowledge. When knowledge has priorty, other attitudes (e.g. beliefs) should be explainable or definable in terms of it.

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[1] For an overview of responses to the Gettier challenge and a detailed discussion, we refer the reader to [18, 26].

One of the few philosphers who has worked out a formal system that ties in with this second approach is R. Stalnaker. In [29], Stalnaker uses a relational semantics for knowledge based on *reflexive, transitive and directed* Kripke models. In his work, he analyzes the relation between knowledge and belief and builds a combined modal system for these notions with the axioms extracted from his analysis. He intends to capture a strong notion of belief based on the conception of "subjective certainty"

$$B\varphi \to BK\varphi$$

meaning that *believing implies believing that one knows* [29, p. 179]. Stalnaker refers to this concept as "strong belief", but following our previous work in [1] we prefer to call it *full belief*[2]. In fact, the above axiom holds biconditionally in his system and belief therefore becomes *subjectively indistinguishable from knowledge: an agent (fully) believes φ iff she (fully) believes that she knows φ* [1]. Moreover, Stalnaker argues that the 'true' logic of knowledge is **S4.2** and that (full) belief can be defined as the *epistemic possibility of knowledge*. More precisely,

$$B\varphi = \neg K \neg K\varphi$$

meaning that *an agent believes φ iff she does't know that she does't know φ.*

In [1] we generalized Stalnaker's semantics from a relational setting to a topological setting. In particular we gave a topological semantics for full belief by extending the interior semantics for knowledge with a semantic clause for the belief modality via the *closure of the interior* operator and showed that our proposed semantics on extremally disconnected spaces constitutes the canonical (most general) semantics for Stalnaker's axiom. In this way, we did generalize Stalnaker's formalization by making it independent from its relational semantics. We moreover focused on the unimodal cases for knowledge and belief and proved that while the knowledge logic of extremally disconnected spaces under the interior-based semantics is indeed **S4.2**, its belief logic under our proposed topological semantics is **KD45**. In this paper, we provide a brief presentation of the work done in [1] in Section 3, we refer to [1, 25] for a more detailed discussion and proofs. This setting, however, comes with a problem when extended to a dynamic setting by adding update modalities in order to capture the action of learning (conditioning with) new "hard" (true) information P. In general, conditioning with new "hard" (true) information P is modeled by simply deleting the "non-P" worlds from the initial model. Its natural topological analogue, as recognized in [4, 5, 35] among others, is a topological update operator, using the restriction of the original topology to (the subspace corresponding to) the set P. This interpretation, however, cannot be implemented smoothly on extremally disconnected spaces due to their non-hereditary nature: we cannot guarantee that the subspace induced by any arbitrary true proposition P is extremally disconnected since extremally disconnectedness is not a hereditary property and thus the structural properties, in particular extremally disconnectedness, of our topological models might not be preserved. We can solve this problem by modelling updates on the topological spaces whose every subspace is extremally disconnected, i.e., by modelling updates on hereditarily extremally disconnected spaces. However, the class of hereditarily extremally disconnected spaces is quite restricted.

In this paper, we propose another solution for this problem via arbitrary topological spaces. More precisely, we do it so by introducing a topological semantics for belief based on *all* topological spaces in terms of the *interior of the closure of the interior operator*. It is important that this semantics coincides with the topological belief semantics introduced in [1] on extremally disconnected space, thus, we here generalize the semantics proposed in [1] to all topological spaces. Further, we show that while the complete logic of knowledge is actually **S4** (due to McKinsey and Tarksi [22]), the complete logic of belief is a weaker system than **KD45**, namely the logic **wKD45**. We also formalize a notion of conditional belief $B^{\varphi}\psi$ by *relativizing* the semantic clause for simple belief modality to the extension of the learnt formula φ and updates $\langle !\varphi\rangle\psi$ again as a topological update operator using the *restriction* of the initial topology to its subspace induced by the new information φ and show that we no longer encounter the problem about updates arised in the case of extremally disconnected spaces: *updates on all topological spaces behave 'nicely'.*

[2] We adopt this terminology mainly to avoid a clash with the very different notion of strong belief (due to Battigalli and Siniscalchi [6]) that is standard in epistemic game theory. At the same time we emphasize the similarity between the intuitions behind Stalnaker's notion and ones behind Van Fraassen's probabilistic concept of full belief [15].

2 Background

2.1 Topological Preliminaries

We start by introducing the basic topological concepts that will be used throughout this paper. For more detailed discussion we refer the reader to [13, 14].

A ***topological space*** is a pair (X, τ), where X is a non-empty set and τ is a family of subsets of X containing X and $\emptyset$ and is closed under finite intersections and arbitrary unions. The set X is called ***space***. The subsets of X belonging to τ are called ***open sets*** (or ***opens***) in the space; the family τ of open subsets of X is called a ***topology*** on X. Complements of opens are called ***closed sets***. An open set containing $x \in X$ is called an ***open neighbourhood*** of x. The ***interior*** $\mathrm{Int}(A)$ of a set $A \subseteq X$ is the largest open set contained in A whereas the ***closure*** $\mathrm{Cl}(A)$ of A is the least closed set containing A. It is easy to see that Cl is the De Morgan dual of Int (and vice versa) and can be written as $\mathrm{Cl}(A) = X \setminus \mathrm{Int}(X \setminus A)$.

2.2 The Interior Semantics for Modal (Epistemic) Logic

In this section, we provide the formal background for the aforementioned interior-based topological semantics for modal (epistemic) logic that originated in the work of McKinsey and Tarski [22]. While presenting some important completeness results (concerning logics of knowledge) of previous works, we also explain the connection between the interior semantics and standard Kripke semantics and focus on the topological (evidence-based) interpretation of knowledge.

Syntax. We consider the standard unimodal language $\mathcal{L}_K$ with a countable set of propositional letters Prop, Boolean operators $\neg$ and $\wedge$ and a modal operator K. Formulas of $\mathcal{L}_K$ are defined as usual by the following grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi$$

where $p \in$ Prop. Abbreviations for the connectives $\vee$, $\rightarrow$, $\leftrightarrow$ are standard. Moreover, the existential modal operator $\langle K \rangle$ and $\bot$ are defined as $\langle K \rangle\varphi := \neg K \neg\varphi$ and $\bot := p \wedge \neg p$,

Semantics. Given a topological space (X, τ), we define a ***topological model*** (or simply a ***topo-model***) as $\mathcal{M} = (X, \tau, \nu)$ where X and τ as before and $\nu : \mathrm{Prop} \rightarrow \mathcal{P}(X)$ is a valuation function.

Definition 1. *Given a topo-model $\mathcal{M} = (X, \tau, \nu)$, we define the* ***interior semantics*** *for the language $\mathcal{L}_K$ recursively as:*

$$\begin{array}{lll}
\mathcal{M}, x \models p & \textit{iff} & x \in \nu(p) \\
\mathcal{M}, x \models \neg\varphi & \textit{iff} & \textit{not } \mathcal{M}, x \models \varphi \\
\mathcal{M}, x \models \varphi \wedge \psi & \textit{iff} & \mathcal{M}, x \models \varphi \textit{ and } \mathcal{M}, x \models \psi \\
\mathcal{M}, x \models K\varphi & \textit{iff} & (\exists U \in \tau)(x \in U \wedge \forall y \in U,\ \mathcal{M}, y \models \varphi)
\end{array}$$

where $p \in$ Prop [3].

We let $[\![\varphi]\!]^{\mathcal{M}} = \{x \in X \mid \mathcal{M}, x \models \varphi\}$ denote the ***extension*** of a modal formula φ in a topo-model $\mathcal{M}$, i.e., the *extension* of a formula φ in a topo-model $\mathcal{M}$ is defined as the set of points in $\mathcal{M}$ satisfying φ. We skip the index when it is clear in which model we are working. It is now easy to see that $[\![K\varphi]\!] = \mathrm{Int}([\![\varphi]\!])$ and $[\![\langle K \rangle\varphi]\!] = \mathrm{Cl}([\![\varphi]\!])$. We use this extensional notation throughout the paper as it makes clear the fact that the modalities, K and $\langle K \rangle$, are interpreted in terms of specific and *natural* topological operators. More precisely, K and $\langle K \rangle$ are modelled as the *interior* and the *closure* operators, respectively.

We say that φ is true in a topo-model $\mathcal{M} = (X, \tau, \nu)$ if $[\![\varphi]\!]^{\mathcal{M}} = X$, and that φ is valid in (X, τ) if $[\![\varphi]\!]^{\mathcal{M}} = X$ for all topo-models $\mathcal{M}$ based on (X, τ), and finally we say that φ is valid in a class of topological spaces if φ is valid in every member of the class [31]. Soundness and completeness w.r.t. the interior semantics are defined as usual.

Theorem 1 (McKinsey and Tarski, 1944)**. S4** *is sound and complete w.r.t. the class of all topological spaces under the interior semantics.*

[3]Originally, McKinsey and Tarski [22] introduce the interior semantics for the basic modal language. Since we talk about this semantics in the context of *knowledge*, we use the basic *epistemic* language.

2.2.1 Topological interpretation of knowledge: open sets as pieces of evidences

One of the reasons as to why the interior operator is interpreted as knowledge is that the Kuratowski properties (see, e.g., [13, 14]) of the interior operator amount to **S4** axioms written in topological terms. This implies that (as we can also read from Theorem 1), *topologically*, knowledge is *Truthful*

$$K\varphi \to \varphi,$$

Positively Introspective

$$K\varphi \to KK\varphi,$$

but *not necessarily* Negatively Introspective

$$\neg K\varphi \to K\neg K\varphi.$$

From a philosophical point of view, the principle of Negative Introspection is arguably the most controversial axiom regarding the characterization of knowledge. It leads to some undesirable consequences, such as Voorbraak's paradox (see e.g., [33, 1]), and rejected by some prominent people in the field such as Hintikka [17], Lenzen [21], Stalnaker [29] (among others).

Another argument in favour of *knowledge as the interior operator* conception is of a more 'semantic' nature: the interior semantics provides a deeper insight into the evidence-based interpretation of knowledge. We can interpret opens in a topological model as 'pieces of evidence' and, in particular, open neighborhoods of a state x as the pieces of *true* (*sound, correct*) evidence that are observable by the agent at state x. If an open set U is included in the extension of a proposition φ in a topo-model $\mathcal{M}$, i.e. if $U \subseteq [\![\varphi]\!]^{\mathcal{M}}$, we say that *the piece of evidence U entails (supports, justifies) the proposition* φ. Recall that, for any topo-model $\mathcal{M} = (X, \tau, \nu)$, any $x \in X$ and any $\varphi \in \mathcal{L}_K$, we have

$$x \in [\![K\varphi]\!]^{\mathcal{M}} \text{ iff } (\exists U \in \tau)(x \in U \wedge U \subseteq [\![\varphi]\!]^{\mathcal{M}}).$$

Thus, taking open sets as pieces of evidence and in fact open neighborhoods of a point x as *true* pieces of evidence (that the agent can observe at x), we obtain the following evidence-based interpretation for knowledge: *the agent knows* φ *iff she has a true piece of evidence U that justifies* φ. In other words, *knowing* φ is the same as *having a correct justification for* φ. The necessary and sufficient conditions for one's belief to qualify as knowledge consist in it being not only truthful, but also in having a correct (evidential) justification. Therefore, the interior semantics implements the widespread intuitive response to Gettier's challenge: knowledge is *correctly* justified belief (rather than being simply true justified belief) [1].

2.2.2 Connection between Kripke frames and topological spaces.

The interior semantics is closely related to the standard Kripke semantics of **S4** (and of its normal extensions): every reflexive and transitive Kripke frame corresponds to a special kind of (namely, Alexandroff) topological spaces. We now briefly explain this connection since it will be use in later sections in our completeness proofs.

Let us now fix some notation and terminology. We denote a ***Kripke frame*** by $\mathcal{F} = (X, R)$, a ***Kripke model*** by $M = (X, R, \nu)$ and $||\varphi||^M$ denotes the *extension* of a formula φ in a Kripke model $M = (X, R, \nu)$ [4]. A topological space (X, τ) is called ***Alexandroff*** if τ is closed under arbitrary intersections, i.e., $\bigcap \mathcal{A} \in \tau$ for any $\mathcal{A} \subseteq \tau$. Equivalently, a topological space (X, τ) is Alexandroff iff every point in X has a least neighborhood. As mentioned, there is a one-to-one correspondence between reflexive and transitive Kripke frames and Alexandroff spaces. More precisely, given a reflexive and transitive Kripke frame $\mathcal{F} = (X, R)$, we can construct a topological space, indeed an Alexandroff space, $\mathcal{X} = (X, \tau_R)$ by defining τ_R to be the set of all upsets[5] of $\mathcal{F}$. Moreover, the evaluation of modal formulas in a reflexive and transitive Kripke model coincides with their evaluation in the corresponding (Alexandroff) topological space (see e.g., [23, p. 306]).

[4] The reader who is not familiar with the standard Kripke semantics is referred to [7, 11] for an extensive introduction on the topic.

[5] A set $A \subseteq X$ is called an *upset* of (X, R) if for each $x, y \in X$, xRy and $x \in A$ imply $y \in A$.

2.2.3 Normal extensions of S4: the logics S4.2 and S4.3

There are two other knowledge systems, namely **S4.2** and **S4.3** , that are of particular interest in this work. Both **S4.2** and **S4.3** are strengthenings of **S4** which are defined as

$$\begin{array}{lcl} \mathbf{S4.2} & := & \mathbf{S4} + \langle K\rangle K\varphi \to K\langle K\rangle\varphi, \text{ and} \\ \mathbf{S4.3} & := & \mathbf{S4} + K(K\varphi \to \psi) \vee K(K\psi \to \varphi) \end{array}$$

where $\mathbf{L} + \varphi$ denotes the smallest logic containing $\mathbf{L}$ and φ.

We recall that a topological space (X, τ) is ***extremally disconnected*** if the closure of every open subset of X is open and it is ***hereditarily extremally disconnected*** if every subspace of (X, τ) is extremally disconnected. We here would like to remind that extremally disconnectedness is, in general, not a hereditary property[6].

Theorem 2 (Folklore)**.** **S4.2** *is sound and complete w.r.t. the class of extremally disconnected spaces under the interior semantics.*

Theorem 3 ([2, 31])**.** **S4.3** *is sound and complete w.r.t. the class of hereditarily extremally disconnected spaces under the interior semantics.*

We give a few examples of extremally disconnected and hereditarily extremally disconnected spaces. Alexandroff spaces corresponding to reflexive, transitive and directed Kripke frames are extremally disconnected but not necessarily hereditarily extremally disconnected. Another classical example of an (non-hereditarily) extremally disconnected space is the Stone-Čech compactification $\beta(\mathbb{N})$ of the set of natural numbers with a discrete topology [27]. For hereditarily extremally disconnected spaces, we can think of Alexandroff spaces corresponding to total preoreders, in particular, corresponding to linear Kripke frames. Another interesting and non-Alexandroff example of an hereditarily extremally disconnected space is the topological space $(\mathbb{N}, \tau)$ where $\mathbb{N}$ is the set of natural numbers and $\tau = \{\emptyset, \text{ all cofinite subsets of } \mathbb{N}\}$. In this space, the set of all finite subsets of $\mathbb{N}$ together with $\emptyset$ and X completely describes the set of closed subsets w.r.t. $(\mathbb{N}, \tau)$. It is not hard to see that for any $U \in \tau$, $\mathrm{Cl}(U) = \mathbb{N}$ and $\mathrm{Int}(F) = \emptyset$ for any closed F with $F \neq X$. For more examples of hereditarily extremally disconnected spaces, we refer to [8].

3 The Topology of Full Belief: Overview of [1]

3.1 Stalnaker's Combined Logic of Knowledge and Belief

In his paper [29], Stalnaker focuses on the properties of knowledge and belief and the relation between the two and approaches the problem of understanding the concrete relation between knowledge and belief from an unusual perspective. Unlike most research in the formal epistemology literature, he starts with a chosen notion of knowledge and weakens it to obtain belief. He bases his analysis on a conception of belief as "subjective certainty": *from the point of the agent in question, her belief is subjectively indistinguishable from her knowledge* [1]. In this section, we briefly introduce Stalnaker's proposal of the 'true' logic of knowledge and belief and point out some aspects of his work which are fundamentally important to ours. In this paper, following [1, 25], we will refer to Stalnaker's notion as "full belief".

The *bimodal language* $\mathcal{L}_{KB}$ of *knowledge and (full) belief* is given by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid B\varphi$$

where $p \in \mathsf{Prop}$. Abbreviations for the connectives $\vee$, $\to$ and $\leftrightarrow$ are standard. The existential modalities $\langle K\rangle$ and $\langle B\rangle$ are defined as $\neg K \neg$ and $\neg B \neg$ respectively. We call Stalnaker's system, given in the following table, **KB**:

[6] A topological property is said to be *hereditary* if for any topological space (X, τ) that has the property, every subspace of (X, τ) also has it [14, p. 68].

We refer to [1, 25] for a discussion on the axioms of **KB** and continue with some conclusions of philosophical importance derived by Stalnaker in [29] and stated in the following proposition:

Proposition 1 (Stalnaker)**.** *The following equivalence is provable in the system* **KB***:*

$$B\varphi \leftrightarrow \langle K\rangle K\varphi. \tag{1}$$

Moreover, the axioms

(K) $B(\varphi \to \psi) \to (B\varphi \to B\psi)$

(D) $B\varphi \to \langle B\rangle\varphi$

(4) $B\varphi \to BB\varphi$

(5) $\neg B\varphi \to B\neg B\varphi$

of the system **KD45** *and the* (.2)*-axiom* $\langle K\rangle K\varphi \to K\langle K\rangle\varphi$ *of the system* **S4.2** *are provable in* **KB**.

Proposition 1 thus shows that full belief is definable in terms of knowledge as "epistemic possibility of knowledge" via equation (1), the 'true' logic of belief is **KD45** and the 'true' logic of knowledge is **S4.2**.

3.2 The Topological Semantics of Full Belief

In [1, 25], we propose a topological semantics for full belief and knowledge by extending the interior semantics for knowledge with a semantic clause for belief. We interpret the belief modality B as the *closure of the interior operator* on *extremally disconnected spaces* and prove several topological soundness and completeness results for both bimodal and unimodal cases, in particular for **KB** and **KD45**, w.r.t. their proposed semantics. We now briefly overview the topological semantics for full belief introduced in [1, 25] and state the completeness results. The proofs can be found in [25].

Definition 2 (Topological Semantics for Full Belief and Knowledge)**.** *Given a topo-model* $\mathcal{M} = (X, \tau, \nu)$*, the semantics for the formulas in* $\mathcal{L}_{KB}$ *is defined for Boolean cases and* $K\varphi$ *the same way as in the interior semantics. The semantics for* $B\varphi$ *is defined as*

$$[\![B\varphi]\!]^{\mathcal{M}} = \mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]^{\mathcal{M}})).$$

Truth and validity of a formula, soundness and completeness are defined the same way as in the interior semantics

	Stalnaker's Axioms	
(K)	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$	Knowledge is additive
(T)	$K\varphi \to \varphi$	Knowledge implies truth
(KK)	$K\varphi \to KK\varphi$	Positive introspection for K
(CB)	$B\varphi \to \neg B\neg\varphi$	Consistency of belief
(PI)	$B\varphi \to KB\varphi$	(Strong) positive introspection of B
(NI)	$\neg B\varphi \to K\neg B\varphi$	(Strong) negative introspection of B
(KB)	$K\varphi \to B\varphi$	Knowledge implies Belief
(FB)	$B\varphi \to BK\varphi$	Full Belief
	Inference Rules	
(MP)	From φ and $\varphi \to \psi$ infer ψ.	Modus Ponens
(K-Nec)	From φ infer $K\varphi$.	Necessitation

Table 1: Stalnaker's System **KB**

Proposition 2. *A topological space validates all the axioms and rules of Stalnaker's system* **KB** *(under the semantics given above) iff it is extremally disconnected.*

Theorem 4. *The sound and complete logic of knowledge and belief on extremally disconnected spaces is given by Stalnaker's system* **KB**.

Besides, as far as full belief is concerned, the above topological semantics constitutes the *most general extensional semantics* for Stalnaker's system **KB** [1, 25].

This work [1, 25] further proceeds focusing on the two unimodal cases $\mathcal{L}_K$ having only K and $\mathcal{L}_B$ having only B as their modalities, respectively. As mentioned, Stalnaker's combined logic of knowledge and belief yields the system **S4.2** as the logic of knowledge and **KD45** as the logic of belief (see Proposition 1). It has been already proven that **S4.2** is complete w.r.t. the class of extremally disconnected spaces under the interior semantics. This begs the question of topological soundness and completeness for **KD45** under the proposed semantics for belief in terms of the *closure and the interior operator*.

3.2.1 Unimodal case for belief: KD45

In this section, we consider the unimodal language $\mathcal{L}_B$ having B as its only modality and focus on the new topological semantics for this language in which the *closure of the interior* operator is taken to be the only primitive operator. We refer to this semantics capturing only belief as *the topological belief semantics* as in [25].

Recall that the language $\mathcal{L}_B$ is given by

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid B\varphi$$

and again we denote $\neg B \neg$ with $\langle B \rangle$. Given a topo-model $\mathcal{M} = (X, \tau, \nu)$, the semantics for the formulas in $\mathcal{L}_B$ is defined for Boolean cases the same way as in the interior semantics. The semantic clause for $B\varphi$ reads

$$[\![B\varphi]\!]^{\mathcal{M}} = \mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]^{\mathcal{M}})).$$

Theorem 5 ([1, 25])**.** **KD45** *is sound and complete w.r.t. the class of extremally disconnected spaces under the topological belief semantics.*

Theorem 5 therefore shows that the belief logic of extremally disconnected spaces is **KD45** when B is interpreted as the closure of the interior operator.

Summing up, a new topological semantics for belief on *extremally disconnected* spaces is proposed in [1, 25] and it has been proven, in this setting, that the complete logic of knowledge and belief is Stalnaker's system **KB**, the complete logic of knowledge is **S4.2** and the complete logic of belief is **KD45** in this setting. These nice results on extremally disconnected spaces, however, encounter problems when extended to a dynamic setting by adding update modalities formalized as model restriction by means of subspaces.

3.2.2 Topological semantics for update modalities

We now consider the language $\mathcal{L}_{!KB}$ obtained by adding to the language $\mathcal{L}_{KB}$ (existential) dynamic update modalities $\langle !\varphi \rangle \psi$ meaning that φ *is true and after the agent learns the new information* φ, ψ *becomes true*. As also observed in [4, 5, 35], the topological analogue of updates corresponds to taking the restriction of a topology τ on X to a subset $P \subseteq X$, i.e., it corresponds to the restriction of the original topology to its subspace induced by the new, true information P.

Given a topological space (X, τ) and a non-empty set $P \subseteq X$, a space (P, τ_P) is called a ***subspace*** of (X, τ) where $\tau_P = \{U \cap P : U \in \tau\}$. It is well-known that the closure and interior operators in the restricted semantics (P, τ_P), denoted by Cl_{τ_P} and Int_{τ_P} respectively, satisfy the following equations for every $A \subseteq P$:

$\mathrm{Cl}_{\tau_P}(A) = \mathrm{Cl}(A) \cap P,$

$\mathrm{Int}_{\tau_P}(A) = \mathrm{Int}((X \setminus P) \cup A) \cap P.$

Now given a topo-model (X, τ, ν) and $\varphi \in \mathcal{L}_{KB!}$, we denote by $\mathcal{M}_\varphi$ the ***restricted model*** $\mathcal{M}_\varphi = ([\![\varphi]\!], \tau_{[\![\varphi]\!]}, \nu_{[\![\varphi]\!]})$ where $[\![\varphi]\!] = [\![\varphi]\!]^{\mathcal{M}}$ and $\nu_{[\![\varphi]\!]}(p) = \nu(p) \cap [\![\varphi]\!]$ for any $p \in \text{Prop}$. Then, the semantics for the dynamic language $\mathcal{L}_{!KB}$ is obtained by extending the semantics for $\mathcal{L}_{KB}$ with:

$$[\![\langle !\varphi\rangle\psi]\!]^{\mathcal{M}} = [\![\psi]\!]^{\mathcal{M}_\varphi}.$$

To explain the problem: Given that the underlying *static* logic of knowledge and belief is the logic of extremally disconnected spaces (see e.g., Theorem 2, 4 and 5) and extremally disconnectedness is not inherited by arbitrary subspaces, we cannot guarantee that the restricted model induced by an arbitrary formula φ remains extremally disconnected. Under the topological belief semantics, both the (K)-axiom (also known as the axiom of *Normality*)

$$B(\varphi \wedge \psi) \to (B\varphi \wedge B\psi)$$

and the (D)-axiom (also named as the *Consistency of Belief*)

$$B\varphi \to \langle B\rangle\varphi$$

characterize extremally disconnected spaces [25, 2]. Therefore, if the restricted model is not extremally disconnected, the agent comes to have inconsistent beliefs after an update with true information and thus comes to believe everything: *she goes crazy!* To be more precise, we illustrate this problem with the following example:

> Consider the topo-model $\mathcal{M} = (X, \tau, \nu)$ where $X = \{x_1, x_2, x_3, x_4\}$, $\tau = \{X, \emptyset, \{x_4\}, \{x_2, x_4\}, \{x_3, x_4\}, \{x_2, x_3, x_4\}\}$ and $\nu(p) = \{x_4\}$ and $\nu(q) = \{x_2, x_4\}$ for some $p, q \in \text{Prop}$. It is easy to check that (X, τ) is an extremally disconnected space and $Bq \to \langle B\rangle q$ is valid on $\mathcal{M}$. We stipulate that x_1 is the actual world and the agent receives the information $\neg p$ from an infallible, truthful source. The updated (i.e., restricted) model is then $\mathcal{M}_{\neg p} = ([\![\neg p]\!]^{\mathcal{M}}, \tau_{\neg p}, \nu_{\neg p})$ where $[\![\neg p]\!]^{\mathcal{M}} = \{x_1, x_2, x_3\}$, $\tau_{\neg p} = \{[\![\neg p]\!]^{\mathcal{M}}, \emptyset, \{x_2\}, \{x_3\}, \{x_2, x_3\}\}$, $\nu_{\neg p}(p) = \emptyset$ and $\nu_{\neg p}(q) = \{x_2\}$. Here, $([\![\neg p]\!]^{\mathcal{M}}, \tau_{\neg p})$ is not an extremally disconnected space since $\{x_3\}$ is an open subset of $([\![\neg p]\!]^{\mathcal{M}}, \tau_{\neg p})$ but $\text{Cl}_{\tau_{\neg p}}(\{x_3\}) = \{x_1, x_3\}$ is not open in $([\![\neg p]\!]^{\mathcal{M}}, \tau_{\neg p})$. Moreover, as $x_1 \in [\![Bq]\!]^{\mathcal{M}_{\neg p}} = \text{Cl}_{\tau_{\neg p}}(\text{Int}_{\tau_{\neg p}}(\{x_2\})) = \{x_1, x_2\}$ and $x_1 \in [\![B\neg q]\!]^{\mathcal{M}_{\neg p}} = \text{Cl}_{\tau_{\neg p}}(\text{Int}_{\tau_{\neg p}}(\{x_1, x_3\})) = \{x_1, x_3\}$, the agent comes to believe both q and $\neg q$, and thus she believes $q \wedge \neg q = \bot$.

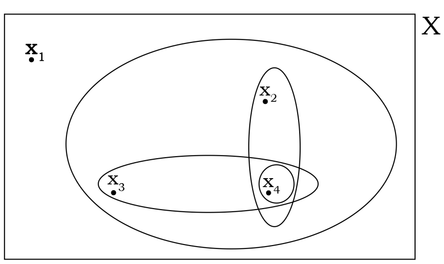

Figure 1:
(X, τ)

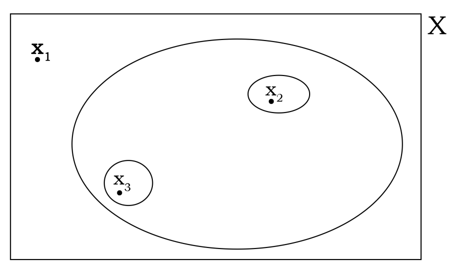

Figure 2:
$([\![\neg p]\!]^{\mathcal{M}}, \tau_{\neg p})$

One possible solution for this problem is a further limitation on the class of spaces we work with: we can restrict our attention to *hereditarily extremally disconnected spaces*, thereby, we guarantee that no model restriction leads to inconsistent beliefs. As the logic of hereditarily extremally disconnected spaces under the interior semantics is **S4.3**, the underlying static logic, in this case, would consist in **S4.3** as the logic of knowledge but again **KD45** as the logic of belief. In on-going work [2], we examine this solution and plan to present it in a future journal publication. In this paper, we present another solution which approaches the issue from the opposite direction: we propose to work with all topological spaces instead working with a restricted class. This solution, unsurprisingly, leads to a weakening of the underlying static logic of knowledge and belief. It is very well-known that the knowledge logic of all topological spaces is **S4** and here we will explore the (weak) belief logic of all topological spaces under the topological belief semantics.

Recall that given an extremally disconnected space (X, τ), we have

$$\mathrm{Cl}(\mathrm{Int}(A)) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$$

for any $A \subseteq X$. Hence, given a topo-model $\mathcal{M} = (X, \tau, \nu)$, the semantic clause for the belief modality can be written in the following equivalent forms

$$[\![B\varphi]\!]^{\mathcal{M}} \stackrel{(1)}{=} \mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]^{\mathcal{M}})) \stackrel{(2)}{=} \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]^{\mathcal{M}})))$$

if (X, τ) is an extremally disconnected space. However, $\mathrm{Cl}(\mathrm{Int}(A)) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$ is not always the case for all topological spaces and all $A \subseteq X$; the equation demands the restriction to extremally disconnected spaces. Besides, if we evaluate B as the closure of the interior operator on *all* topological spaces, we obtain that neither the (K)-axiom nor the (D)-axiom is sound. We thus concentrate on the latter equation: we interpret B as the *interior of the closure of the interior* operator on all topological spaces. This semantics boils down to the topological belief semantics introduced in Section 3.2 on *extremally disconnected spaces* and differs from it in general. Moreover, given that K is interpreted as the interior operator on topological spaces, equation (1) makes

$$B\varphi \leftrightarrow \langle K\rangle K\varphi$$

and equation (2) makes

$$B\varphi \leftrightarrow K\langle K\rangle K\varphi$$

valid on all topological spaces. However, on the syntactic side, while $\mathbf{S4.2} \vdash \langle K\rangle K\varphi \leftrightarrow K\langle K\rangle K\varphi$, we have $\mathbf{S4} \nvdash \langle K\rangle K\varphi \leftrightarrow K\langle K\rangle K\varphi$ and B as $K\langle K\rangle K$ is the only alternative holding the property of being equivalent to $\langle K\rangle K$ in **S4.2** and being not equivalent to $\langle K\rangle K$ in **S4** (see [11]). This constitutes another justification as to why we consider $B\varphi$ as $K\langle K\rangle K\varphi$ and interpret it as the interior of the closure of the interior operator.

3.3 The Topological Semantics of Weak Belief: wKD45

We define the logic **wKD45** as

$$\mathbf{wKD45} = \mathbf{K} + (B\varphi \to \langle B\rangle\varphi) + (B\varphi \to BB\varphi) + (B\langle B\rangle B\varphi \to B\varphi).$$

and call it *weak* **KD45**. This logic is weaker than **KD45** since it is obtained by replacing the 5-axiom with the axiom $B\langle B\rangle B\varphi \to B\varphi$ and while $B\langle B\rangle B\varphi \to B\varphi$ is a theorem of **KD45**, the 5-axiom is not a theorem of **wKD45**. More precisely, $\mathbf{KD45} \vdash B\langle B\rangle B\varphi \to B\varphi$ but $\mathbf{wKD45} \nvdash \langle B\rangle\varphi \to B\langle B\rangle\varphi$. We find it hard to give a direct and clear interpretation for this axiom as is given for the axiom of Negative Introspection, since it is too complex in the sense that it includes three consecutive modalities. However, we can interpret it on the basis of the axioms that we have already given an interpretation, in particular, based on the interpretation of Negative Introspection. It is easier to see the correspondence if we state the weak axiom in the following equivalent form:

$$\neg B\varphi \to \langle B\rangle B\neg B\varphi.$$

Recall that the principle of Negative Introspection says that *if an agent does not believe φ, then she believes that she does not believe φ*. On the other hand, taking the reading of Negative Introspection as the reference point, one possible doxastic reading for this axiom can be given as *if the agent does not believe φ, then it is doxastically possible to her that she believes that she does not believe φ.*

Semantics. Let $\mathcal{M} = (X, \tau, \nu)$ be a topo-model. The semantic clauses for the propositional variables and the Boolean connectives are the same as in the interior semantics. For the modal operator B, we put

$$[\![B\varphi]\!]^{\mathcal{M}} \quad = \quad \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]^{\mathcal{M}})))$$

and the semantic clause for $\langle B\rangle$ is easily obtained as

$$[\![\langle B\rangle\varphi]\!]^{\mathcal{M}} \;=\; \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}([\![\varphi]\!]^{\mathcal{M}}))).$$

Validity of a formula is defined as usual. We call this semantics **w**-*topological belief semantics* referring to the system **wKD45** for which we will prove soundness and completeness. This way we distinguish it from the *topological belief semantics* presented in Section 3.2 w.r.t. to which we proved the soundness and completeness of the system **KD45**. Throughout this section, we use the notation $[\varphi]^{\mathcal{M}}$ for the extension of a formula $\varphi \in \mathcal{L}_K$ w.r.t. the *interior semantics* in order to make clear in which semantics we work. We reserve the notation $[\![\varphi]\!]^{\mathcal{M}}$ for the extensions of the formulas $\varphi \in \mathcal{L}_B$ w.r.t. the **w**-*topological belief semantics*. We skip the index when confusion is unlikely to occur.

Definition 3 (Translation $(.)^{\circledast} : \mathcal{L}_B \to \mathcal{L}_K$)**.** *For any $\varphi \in \mathcal{L}_B$, the translation $(\varphi)^{\circledast}$ of φ into $\mathcal{L}_K$ is defined recursively as follows:*

1. $(\bot)^{\circledast} = \bot$
2. $(p)^{\circledast} = p$, *where* $p \in \mathrm{Prop}$
3. $(\neg\varphi)^{\circledast} = \neg\varphi^{\circledast}$
4. $(\varphi \wedge \psi)^{\circledast} = \varphi^{\circledast} \wedge \psi^{\circledast}$
5. $(B\varphi)^{\circledast} = K\langle K\rangle K\varphi^{\circledast}$
6. $(\langle B\rangle\varphi)^{\circledast} = \langle K\rangle K\langle K\rangle\varphi^{\circledast}$

Proposition 3. *For any topo-model $\mathcal{M} = (X, \tau, \nu)$ and for any formula $\varphi \in \mathcal{L}_B$ we have*

$$[\![\varphi]\!]^{\mathcal{M}} = [\varphi^{\circledast}]^{\mathcal{M}}.$$

Proof. We prove the lemma by induction on the complexity of φ. The cases for

1. $\varphi = \bot$,
2. $\varphi = p$,
3. $\varphi = \neg\psi$, and
4. $\varphi = \psi \wedge \chi$

are straightforward. Now let $\varphi = B\psi$, then

$$\begin{array}{lcll} [\![\varphi]\!]^{\mathcal{M}} & = & [\![B\psi]\!]^{\mathcal{M}} & \\ & = & \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}([\![\psi]\!]^{\mathcal{M}}))) & \text{(by the } \mathbf{w}\text{-topological belief semantics for } \mathcal{L}_B) \\ & = & \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}([\psi^{\circledast}]^{\mathcal{M}}))) & \text{(by I.H.)} \\ & = & [K\langle K\rangle K\psi^{\circledast}]^{\mathcal{M}} & \text{(by the interior semantics for } \mathcal{L}_K.) \\ & = & [(B\psi)^{\circledast}]^{\mathcal{M}} & \text{(by the translation } ^{\circledast}.) \\ & = & [\varphi^{\circledast}]^{\mathcal{M}}. & \end{array}$$

$\square$

3.4 Soundness of wKD45

The proof of soundness follows as usual.

Proposition 4. *For any topo-model $\mathcal{M} = (X, \tau, v)$ and any $\varphi \in \mathcal{L}_B$ we have*

1. $[\![B\varphi \to \langle B\rangle\varphi]\!] = X$
2. $[\![B\varphi \to BB\varphi]\!] = X,$
3. $[\![B\langle B\rangle B\varphi \to B\varphi]\!] = X.$

Proof. Let $\mathcal{M} = (X, \tau, v)$ be a topo-model and $\varphi \in \mathcal{L}_B$. Note that for any $\varphi, \psi \in \mathcal{L}_B$ we have

$$[\![\varphi \to \psi]\!] = X \text{ iff } [\![\varphi]\!] \subseteq [\![\psi]\!]. \tag{2}$$

1. By equation 2, it suffices to show that $[\![B\varphi]\!] \subseteq [\![\langle B\rangle B\varphi]\!]$ and the proof follows:

$$\begin{array}{rcll} [\![B\varphi]\!] & = & \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]))) & \\ & \subseteq & \mathrm{Int}(\mathrm{Cl}(([\![\varphi]\!]))) & \text{(by (I2) and (C2))} \\ & \subseteq & \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}([\![\varphi]\!]))) & \text{(by (C2))} \\ & = & [\![\langle B\rangle\varphi]\!]. & \end{array}$$

2. Similar to part-(1), it suffices to show that $[\![B\varphi]\!] \subseteq [\![BB\varphi]\!]$. As known, the interior of a closed set is an open domain[7] [14, p. 20]. We then have

$$[\![B\varphi]\!] = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]))) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!])))))$$

as $\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]))$ is closed in (X, τ). Then, we have

$$\begin{array}{rcll} \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]))))) & = & \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!])))))) & \text{(by (I4))} \\ & = & [\![BB\varphi]\!] & \end{array}$$

Therefore, we obtain $[\![B\varphi]\!] = [\![BB\varphi]\!]$ which implies $[\![B\varphi \to BB\varphi]\!] = X$.

3. The proof proceeds in a similar way as in above cases:

$$\begin{array}{rcll} [\![B\langle B\rangle B\varphi]\!] & = & \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(([\![\varphi]\!]))))))))) & \\ & \subseteq & \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]))) & \text{(by the argument on open domains in (2))} \\ & = & [\![B\varphi]\!]. & \end{array}$$

Therefore, by equation 2, we have $[\![B\langle B\rangle B\varphi \to B\varphi]\!] = X$.

$\square$

Lemma 1. *For any $\varphi \in \mathcal{L}_K$,* **S4** $\vdash K\langle K\rangle K\varphi \wedge K\langle K\rangle K\psi \to K\langle K\rangle K(\varphi \wedge \psi)$.

Proof. We know that **S4** is complete w.r.t. the class of reflexive and transitive Kripke frames. So, it suffices to show that $K\langle K\rangle K\varphi \wedge K\langle K\rangle K\psi \to K\langle K\rangle K(\varphi \wedge \psi)$ is valid on all reflexive and transitive Kripke frames.
Let $\mathcal{F} = (X, R)$ be a reflexive and transitive Kripke frame, $\mathcal{M} = (X, R, v)$ a model on $\mathcal{F}$ and $x \in X$. Suppose $x \in \|K\langle K\rangle K\varphi \wedge K\langle K\rangle K\psi\|$. Hence, $x \in \|K\langle K\rangle K\varphi\|$ and $x \in \|K\langle K\rangle K\psi\|$. This implies

$$R(x) \subseteq \|\langle K\rangle K\varphi\| \text{ and } R(x) \subseteq \|\langle K\rangle K\psi\| \tag{3}$$

Note that, since R is reflexive, $R(x) \neq \emptyset$. Now let $y \in R(x)$. Then, by 3, $y \in \|\langle K\rangle K\varphi\|$ and $y \in \|\langle K\rangle K\psi\|$. Hence, there exists a $z_1 \in X$ with yRz_1 such that $z_1 \in \|K\varphi\|$. Since R is transitive and $xRyRz_1$, we have xRz_1. Then, by 3, $z_1 \in \|\langle K\rangle K\psi\|$. Thus, there exists $z_2 \in X$ with z_1Rz_2 such that $z_2 \in \|K\psi\|$. Since R is transitive and z_1Rz_2, $z_1 \in \|K\varphi\|$ implies that $z_2 \in \|K\varphi\|$ as well. Hence, $z_2 \in \|K\varphi \wedge K\psi\|$ implying $z_2 \in \|K(\varphi \wedge \psi)\|$. Since R is transitive and yRz_1Rz_2, $y \in \|\langle K\rangle K(\varphi \wedge \psi)\|$. Since y has been chosen arbitrarily from $R(x)$, it holds for all $y \in R(x)$. Therefore, $x \in \|K\langle K\rangle K(\varphi \wedge \psi)\|$. Therefore, by the completeness of **S4**, we have **S4** $\vdash K\langle K\rangle K\varphi \wedge K\langle K\rangle K\psi \to K\langle K\rangle K(\varphi \wedge \psi)$. $\square$

[7] A subset A of a topological space is called an *open domain* if $A = \mathrm{Int}(\mathrm{Cl}(A))$ [14, p. 20]. In the literature, an open domain is also called *regular open*.

Proposition 5. *The* K*-axiom $B\varphi \wedge B\psi \to B(\varphi \wedge \psi)$ is valid in all topological spaces w.r.t. the* **w***-topological belief semantics.*

Proof. Let (X, τ) be a topological space and $\mathcal{M} = (X, \tau, \nu)$ be a topo-model on it. Also suppose $\varphi, \psi \in \mathcal{L}_B$. Then,

$$\begin{aligned}
[\![B\varphi \wedge B\psi \to B(\varphi \wedge \psi)]\!]^{\mathcal{M}} &= [(B\varphi \wedge B\psi \to B(\varphi \wedge \psi))^{\circledast}]^{\mathcal{M}} \quad \text{(by Proposition 3)}\\
&= [K\langle K\rangle K\varphi^{\circledast} \wedge K\langle K\rangle K\psi^{\circledast} \to K\langle K\rangle K(\varphi^{\circledast} \wedge \psi^{\circledast})]^{\mathcal{M}}\\
&= X \quad \text{(by Lemma 1)}
\end{aligned}$$

Hence, (X, τ) validates $B\varphi \wedge B\psi \to B(\varphi \wedge \psi)$. □

Theorem 6. *The logic* **wKD45** *is sound w.r.t. the class all topological spaces in* **w***-topological belief semantics.*

Proof. Follows from Propositions 4 and 5. □

3.4.1 Completeness of wKD45

We prove the completeness of **wKD45** by using the translation $^{\circledast}$ from the language $\mathcal{L}_B$ into the language $\mathcal{L}_K$ and the topological completeness of **S4**.

For the topological completeness proof of **wKD45** we also make use of the completeness of **wKD45** and **S4** in the standard Kripke semantics. We first recall some frame conditions concerning the relational completeness of the respective systems.

We denote the set of final clusters of a transitive Kripke frame (X, R) by $\mathfrak{C}_R$. A transitive Kripke frame (X, R) having at least one final cluster is called ***weak cofinal*** if for each $x \in X$ there is a $C \in \mathfrak{C}_R$ such that for all $y \in C$ we have xRy. In fact, every finite reflexive and transitive frame is weak cofinal. We call a weak cofinal frame a ***weak brush*** if $X \setminus \bigcup \mathfrak{C}_R$ is an irreflexive anti-chain, i.e., for each $x, y \in X \setminus \bigcup \mathfrak{C}_R$ we have $\neg(xRy)$. A weak brush with a singleton $X \setminus \bigcup \mathfrak{C}_R$ is called a ***weak pin***.[8] By definition, every weak brush and every weak pin is transitive and also serial. Finally, we say that a transitive frame (X, R) is of depth n if there is a chain of points $x_1 R x_2 R \ldots R x_n$ such that $\neg(x_{i+1} R x_i)$ for any $i \leq n$ and there is no chain of greater length satisfying this condition. It is hard to draw a generic picture of a weak brush, but the following figures illustrate weak pins and how a weak brush could look like (where top squares correspond to final clusters).

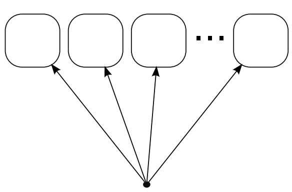

Figure 3:
Weak pin

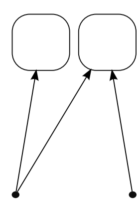

Figure 4:
An example of a weak brush

Lemma 2. *If $\mathcal{F} = (X, R)$ is a rooted* **wKD45***-frame with depth at least 2 then it is a weak pin.*

Proof. Let $\mathcal{F} = (X, R)$ be a rooted **wKD45**-frame with depth of at least 2 and x be the root. $\mathcal{F}$ is both transitive and serial since it validates the axioms D and 5. Moreover, as it is a frame of depth 2, there exists a $y_0 \in X$ such that xRy_0 and $\neg(y_0 R x)$. As $\mathcal{F}$ is serial, every maximal point of it is in a final cluster. Hence, for any $x \in X$, x is maximal point iff there is a final cluster C of $\mathcal{F}$ such that $x \in C$, i.e. the set of maximal points of $\mathcal{F}$ is $\bigcup \mathfrak{C}_R$. Recall that a weak pin is a weak cofinal frame with a singleton irreflexive $X \setminus \bigcup \mathfrak{C}_R$. We hence need to show that x is an irreflexive point and every successor of x is a maximal point. Suppose for a contradiction that xRx or there is a $t_0 \in X$ such that xRt_0 and t_0 is not a maximal point of $\mathcal{F}$.

[8] Brushes and pins are introduced in [25] and a similar terminology is used in this paper.

- *Case 1*: xRx

  Consider the valuation v on (X, R) such that $v(p) = X \setminus \{x\}$ for some $p \in$ Prop. We want to show that $x \in \|B\langle B\rangle Bp\|$ but $x \notin \|Bp\|$. Let $y \in X$ such that xRy.

  Case 1.1: $x = y$
  Since $\neg(y_0Rx)$ and xRx, we have that $y_0 \neq x$ and $x \notin R(y_0)$. Hence, $R(y_0) \subseteq (X \setminus \{x\})$. Then, as $v(p) = X \setminus \{x\}$, $R(y_0) \subseteq v(p)$ implying that $y_0 \in \|Bp\|$. Therefore, since yRy_0, $y \in \|\langle B\rangle Bp\|$.
  Case 1.2: $x \neq y$
  If yRx, then by transitivity of R we have yRy_0. Since $y_0 \in \|Bp\|$, we obtain that $y \in \|\langle B\rangle Bp\|$.
  If $\neg(yRx)$ then for all $z \in R(y)$ we have $\neg(zRx)$ by transitivity of R. Hence, for all $z \in R(y)$, $x \notin R(z)$ implying that $R(z) \subseteq (X \setminus \{x\})$ (since R is serial, $R(y) \neq \emptyset$). Therefore, $R(z) \subseteq v(p)$. Hence, as yRz, $y \in \|\langle B\rangle Bp\|$.
  Therefore $x \in \|B\langle B\rangle Bp\|$. On the other hand, as xRx and $x \notin v(p)$, $x \notin \|Bp\|$. We then have that (X, R) refutes $B\langle B\rangle Bp \to Bp$ implying that $\mathcal{F}$ cannot be a **wKD45** frame. Therefore, x is an irreflexive point.

- *Case 2*:

  There is a $t_0 \in X$ such that xRt_0 and t_0 is not a maximal point of $\mathcal{F}$. Since t_0 is not a maximal point, there exists a $z_0 \in X$ such that t_0Rz_0 but $\neg(z_0Rt_0)$. Consider the valuation v on (X, R) such that $v(p) = X \setminus \{t_0\}$ for some $p \in$ Prop. Observe that, as $t_0 \notin R(z_0)$, $R(z_0) \subseteq (X \setminus \{t_0\})$, thus, $z_0 \in \|Bp\|$. We want to show that $x \in \|B\langle B\rangle Bp\|$ but $x \notin \|Bp\|$. Let $y \in X$ such that xRy. Then, since x is an irreflexive point, $y \neq x$.

  Case 2.1: yRz_0
  Then, as $z_0 \in \|Bp\|$, we have $y \in \|\langle B\rangle Bp\|$.
  Case 2.2: $\neg(yRz_0)$
  Then, $\neg(yRt_0)$ by transitivity of R. This implies $t_0 \notin R(y)$. Therefore, $R(y) \subseteq X \setminus \{t_0\}$ meaning that $R(y) \subseteq v(p)$. Hence, $y \in \|Bp\|$. Then, by seriality and transitivity of R, we have $y \in \|\langle B\rangle Bp\|$.
  Therefore $x \in \|B\langle B\rangle Bp\|$. On the other hand, as xRt_0 and $t_0 \notin v(p)$, $x \notin \|Bp\|$. We then have that (X, R) refutes $B\langle B\rangle Bp \to Bp$ implying that every successor of x is a maximal point.

Therefore, every rooted **wKD45** frame which is of depth at least 2 is a weak pin. This implies that every rooted **wKD45** is of at most depth 2.

□

Lemma 3.

1. *Each reflexive and transitive weak cofinal frame is an* **S4**-*frame. Moreover,* **S4** *is sound and complete w.r.t. the class of finite rooted reflexive and transitive weak cofinal frames.*

2. *Each weak brush is a* **wKD45**-*frame. Moreover,* **wKD45** *is sound and complete w.r.t. the class of finite weak brushes, indeed, w.r.t. the class of finite weak pins.*

Proof. (1) is a very well-known and we refer to [7, 10]. For (2), we proved in Lemma 2 that the **wKD45**-frames are of finite depth. It is well known that every logic over **K4** that has finite depth has the finite model property (e.g., [10, Chapter 12 (tabularity)]). This implies that **wKD45** as well has the finite model property and thus it has the finite model property w.r.t. to finite rooted **wKD45**-frames. Then by Lemma 2, we have that **wKD45** is in fact complete w.r.t. to finite weak brushes and weak pins.

□

For any reflexive and transitive weak cofinal frame (X, R) we define R_B on X by

$$xR_By \text{ if } y \in \bigcup \mathfrak{C}_{R(x)}$$

for each $x, y \in X$, where $\bigcup \mathfrak{C}_{R(x)} = R(x) \cap \bigcup \mathfrak{C}_R$. In other words, $R_B(x) = \bigcup \mathfrak{C}_{R(x)}$ for each $x \in X$. Moreover, we have the following equivalence:

Lemma 4. *For any reflexive and transitive weak cofinal frame* (X, R)*,*

$$\bigcup \mathfrak{C}_{R_B} = \bigcup \mathfrak{C}_R.$$

Proof. Let (X, R) be a reflexive and transitive weak cofinal frame and $x \in X$.

($\subseteq$) Suppose $x \in \bigcup \mathfrak{C}_{R_B}$ and $x \notin \bigcup \mathfrak{C}_R$. $x \in \bigcup \mathfrak{C}_{R_B}$ means that $x \in C$ for some $C \in \mathfrak{C}_{R_B}$. As C is a final cluster, there is no $y \in X$ such that xR_By and $\neg(yR_Bx)$. On the other hand, since (X, R) is a weak cofinal frame, there is a $C' \in \mathfrak{C}_R$ such that xRz for all $z \in C'$. Hence, $C' \subseteq \bigcup \mathfrak{C}_{R(x)}$. Thus, by definition of R_B, we have $C' \subseteq R_B(x)$. However, as $x \notin \bigcup \mathfrak{C}_R$, we have that $\neg(zRx)$ and thus $\neg(zR_Bx)$ for any $z \in C'$ contradicting $x \in C$ for a final cluster C of (X, R_B).in fact, there is a unique $C \in \mathfrak{C}_{R_B}$ such that $R_B(x) = C$ since C is a final cluster.

($\supseteq$) Suppose $x \in \bigcup \mathfrak{C}_R$. Then, there is a (unique) $C \in \mathfrak{C}_R$ such that $x \in C$ and in fact $R(x) = C$. Also suppose that $x \notin \bigcup \mathfrak{C}_{R_B}$. Hence, there is a $y_0 \in X$ such that xR_By_0 and $\neg(y_0R_Bx)$. Then, $y_0 \in \bigcup \mathfrak{C}_{R(x)}$ but $x \notin \bigcup \mathfrak{C}_{R(y_0)}$ by definition of R_B. By definition of R_B, xR_By_0 implies xRy_0. Hence, as $y_0 \in R(x)$, we also have $R(y_0) = R(x) = C$. Thus, $\bigcup \mathfrak{C}_{R(y_0)} = \bigcup \mathfrak{C}_{R(x)}$. As R is reflexive, $x \in \bigcup \mathfrak{C}_{R(x)}$ and hence $x \in \bigcup \mathfrak{C}_{R(y_0)}$ contradicting $\neg(y_0R_Bx)$.

□

Lemma 5. *For any reflexive and transitive weak cofinal Kripke model* $\mathcal{M} = (X, R, v)$*, any* $\varphi \in \mathcal{L}_K$ *and any* $x \in X$*, we have*

$$\bigcup \mathfrak{C}_{R(x)} \subseteq ||\varphi||^{\mathcal{M}} \textit{ iff } x \in ||K\langle K\rangle K\varphi||^{\mathcal{M}}.$$

Proof. Let $\mathcal{M} = (X, R, v)$ be a reflexive and transitive weak cofinal model, $\varphi \in \mathcal{L}_K$ and $x \in X$.

($\Rightarrow$) Suppose $\bigcup \mathfrak{C}_{R(x)} \subseteq ||\varphi||^{\mathcal{M}}$. Let $y \in X$ such that xRy. As R is transitive and xRy, $R(y) \subseteq R(x)$ implying that $\bigcup \mathfrak{C}_{R(y)} \subseteq \bigcup \mathfrak{C}_{R(x)}$. Hence, by assumption, $\bigcup \mathfrak{C}_{R(y)} \subseteq ||\varphi||^{\mathcal{M}}$. Thus, there is a $C \in \mathfrak{C}_R$ such that $C \subseteq R(y)$ and $C \subseteq ||\varphi||^{\mathcal{M}}$. Since for all $z \in C$, we have $R(z) = C$ and $C \subseteq ||\varphi||^{\mathcal{M}}$, we have $C \subseteq ||K\varphi||^{\mathcal{M}}$. As $C \subseteq R(y)$, we have $y \in ||\langle K\rangle K\varphi||^{\mathcal{M}}$. Therefore, since y has been chosen arbitrarily from $R(x)$, $x \in ||K\langle K\rangle K\varphi||^{\mathcal{M}}$.

($\Leftarrow$) Suppose $\bigcup \mathfrak{C}_{R(x)} \not\subseteq ||\varphi||^{\mathcal{M}}$. This implies that there exists a $y \in \bigcup \mathfrak{C}_{R(x)}$ such that $y \notin ||\varphi||^{\mathcal{M}}$. $y \in \bigcup \mathfrak{C}_{R(x)}$ implies that there is a $C \in \mathfrak{C}_R$ such that $R(y) = C$ and $R(y) \subseteq R(x)$. As zRy for all $z \in C$ and $y \notin ||\varphi||^{\mathcal{M}}$, we have $z \notin ||K\varphi||^{\mathcal{M}}$ for all $z \in C$. Then, as $R(y) = C$, $y \notin ||\langle K\rangle K\varphi||^{\mathcal{M}}$. Then, since xRy, $x \notin ||K\langle K\rangle K\varphi||^{\mathcal{M}}$.

□

Lemma 6. *For any reflexive and transitive weak cofinal frame* (X, R)*,*

1. (X, R_B) *is a weak brush.*
2. *For any valuation* v *on* X *and for each formula* $\varphi \in \mathcal{L}_B$ *we have*

   $$||\varphi^{\circledast}||^{\mathcal{M}} = ||\varphi||^{\mathcal{M}_B}$$

   where $\mathcal{M} = (X, R, v)$ *and* $\mathcal{M}_B = (X, R_B, v)$*.*

Proof. Let (X, R) be a reflexive and transitive weak cofinal frame.

1.
   - *Transitivity:* Let $x, y, z \in X$ such that xR_By and yR_Bz. This means that $y \in \bigcup \mathfrak{C}_{R(x)}$ and $z \in \bigcup \mathfrak{C}_{R(y)}$. As R being transitive and xRy, $\bigcup \mathfrak{C}_{R(y)} \subseteq \bigcup \mathfrak{C}_{R(y)}$. Hence, $z \in \bigcup \mathfrak{C}_{R(x)}$, i.e., xR_Bz.
   - *Seriality:* Let $x \in X$. Since (X, R) is weak cofinal, there is a $y \in \bigcup \mathfrak{C}_{R(x)}$, i.e., xR_By.
   - *Irreflexive, antichain:* Suppose there is an $x \in X \setminus \bigcup \mathfrak{C}_{R_B}$ such that xR_Bx. This implies, $x \in \bigcup \mathfrak{C}_{R(x)}$, thus, $x \in \bigcup \mathfrak{C}_R$ Then, by Lemma 4, $x \in \bigcup \mathfrak{C}_{R_B}$ which contradicts our assumption. Moreover, suppose that $X \setminus \bigcup \mathfrak{C}_{R_B}$ is not an antichain, i.e., there are $x, y \in X \setminus \bigcup \mathfrak{C}_{R_B}$ such that either xR_By or yR_Bx. W.l.o.g., suppose xR_By. Hence, by definition of R_B, $y \in \bigcup \mathfrak{C}_{R(x)}$. Thus, $y \in \bigcup \mathfrak{C}_R$ and, by Lemma 4, $y \in \bigcup \mathfrak{C}_{R_B}$ contradicting $y \in X \setminus \bigcup \mathfrak{C}_{R_B}$.
2. We prove this item by induction on the complexity of φ. Let $\mathcal{M} = (X, R, v)$ be a model on (X, R). The cases for $\varphi = \bot$, $\varphi = p$, $\varphi = \neg\psi$, $\varphi = \psi \wedge \chi$ are straightforward. Let $\varphi = B\psi$.

   ($\subseteq$) Let $x \in ||(B\psi)^{\circledast}||^{\mathcal{M}} = ||K\langle K\rangle K\psi^{\circledast}||^{\mathcal{M}}$. Then, by Lemma 5, $\bigcup \mathfrak{C}_{R(x)} \subseteq ||\psi^{\circledast}||^{\mathcal{M}}$. By I.H, we obtain $\bigcup \mathfrak{C}_{R(x)} \subseteq ||\psi||^{\mathcal{M}_B}$. Since $\bigcup \mathfrak{C}_{R(x)} = R_B(x)$, we have $R_B(x) \subseteq ||\psi||^{\mathcal{M}_B}$ implying that $x \in ||B\psi||^{\mathcal{M}_B}$.

($\supseteq$) Let $x \in \|B\psi\|^{\mathcal{M}_B}$. Then, by the standard Kripke semantics, we have $R_B(x) \subseteq \|\psi\|^{\mathcal{M}_B}$. By I.H, we obtain $R_B(x) \subseteq \|\psi^{\circledast}\|^{\mathcal{M}}$. Since $\bigcup \mathfrak{C}_{R(x)} = R_B(x)$, we have $\bigcup \mathfrak{C}_{R(x)} \subseteq \|\psi^{\circledast}\|^{\mathcal{M}}$. Thus, by Lemma 5, $x \in \|K\langle K\rangle K\psi^{\circledast}\|^{\mathcal{M}} = \|(B\psi)^{\circledast}\|^{\mathcal{M}}$.

□

Lemma 7. *For any weak brush* (X, R)*,*

1. (X, R^+) *is a reflexive and transitive weak cofinal frame.*
2. *For any valuation* ν *on* X *and for each formula* $\varphi \in \mathcal{L}_B$ *we have*

$$\|\varphi\|^{\mathcal{M}} = \|\varphi^{\circledast}\|^{\mathcal{M}^+}$$

where $\mathcal{M} = (X, R, \nu)$ *and* $\mathcal{M}^+ = (X, R^+, \nu)$*.*

Proof. Let (X, R) be a serial weak brush.

1. Since R is transitive, R^+ is also transitive and it is reflexive by definition. Moreover, (X, R^+) is weak cofinal since (X, R) is a weak brush.
2. We prove (2) by induction on the complexity of φ. Let $\mathcal{M} = (X, \tau, \nu)$ be a model on (X, R). The cases for $\varphi = \bot$, $\varphi = p$, $\varphi = \neg\psi$, $\varphi = \psi \wedge \chi$ are straightforward. Let $\varphi = B\psi$.

   ($\subseteq$) Let $x \in \|B\psi\|^{\mathcal{M}}$. Then, by the standard Kripke semantics, we have $R(x) \subseteq \|\psi\|^{\mathcal{M}}$. Hence, by I.H., $R(x) \subseteq \|\psi^{\circledast}\|^{\mathcal{M}^+}$. Since (X, R) is a weak brush, $R(x) = \bigcup \mathfrak{C}_{R(x)} \subseteq \bigcup \mathfrak{C}_{R^+(x)}$. Hence, $x \in \bigcup \mathfrak{C}_{R^+(x)}$. Then, by Lemma 5, $x \in \|K\langle K\rangle K\psi^{\circledast}\|^{\mathcal{M}^+}$.

   ($\supseteq$) Let $x \in \|K\langle K\rangle K\psi^{\circledast}\|^{\mathcal{M}^+}$. Then, by Lemma 5, $\bigcup \mathfrak{C}_{R^+(x)} \|\psi^{\circledast}\|^{\mathcal{M}^+}$. Thus, by I.H., $\bigcup \mathfrak{C}_{R^+(x)} \|\psi\|^{\mathcal{M}}$. Then, by a similar argument above, $R(x) \subseteq \|\psi\|^{\mathcal{M}}$ implying that $x \in \|B\psi\|^{\mathcal{M}}$.

□

Theorem 7. *For each formula* $\varphi \in \mathcal{L}_B$*,*

$$\mathbf{S4} \vdash \varphi^{\circledast} \textit{ iff } \mathbf{wKD45} \vdash \varphi.$$

Proof. Let $\varphi \in \mathcal{L}_B$.

($\Rightarrow$) Suppose $\mathbf{wKD45} \nvdash \varphi$. By Lemma 3(2), there exists a Kripke model $\mathcal{M} = (X, R, \nu)$ where (X, R) is a finite weak pin such that $\|\varphi\|^{\mathcal{M}} \neq X$. Then, by Lemma 7, $\mathcal{M}^+$ is a model based on the finite reflexive and transitive weak cofinal frame (X, R^+) and $\|\varphi^{\circledast}\|^{\mathcal{M}^+} \neq X$. Hence, by Lemma 3(1), we have $\mathbf{S4} \nvdash \varphi^{\circledast}$.

($\Leftarrow$) Suppose $\mathbf{S4} \nvdash \varphi^{\circledast}$. By Lemma 3(1), there exists a Kripke model $\mathcal{M} = (X, R, \nu)$ where (X, R) is a finite reflexive and transitive weak cofinal frame such that $\|\varphi^{\circledast}\|^{\mathcal{M}} \neq X$. Then, by Lemma 6, $\mathcal{M}_B$ is a model based on the (finite) weak brush (X, R_B) and $\|\varphi\|^{\mathcal{M}_B} \neq X$. Hence, by Lemma 3(2), we have $\mathbf{wKD45} \nvdash \varphi$.

□

Theorem 8. **wKD45** *is complete w.r.t. the class of all topological spaces in the* **w***-topological belief semantics.*

Proof. Let $\varphi \in \mathcal{L}_B$ such that $\mathbf{wKD45} \nvdash \varphi$. By Theorem 7, $\mathbf{S4} \nvdash \varphi^{\circledast}$. Hence, by topological completeness of **S4** w.r.t. the class of all topological spaces in the interior semantics, there exists a topo-model $\mathcal{M} = (X, \tau, \nu)$ such that $[\varphi^{\circledast}]^{\mathcal{M}} \neq X$. Then, by Proposition 3, $[\![\varphi]\!]^{\mathcal{M}} \neq X$. Thus, we found a topological space (X, τ) which refutes φ in the **w**-topological belief semantics. Hence, **wKD45** is complete w.r.t. the class of all topological spaces in the **w**-topological belief semantics. □

4 The Topology of Static and Dynamic Belief Revision

4.1 Static Belief Revision: conditional beliefs

In DEL, static belief revision captures the agent's revised beliefs about how the world was before learning new information and is implemented by conditional belief operators $B^{\varphi}\psi$. The statement $B^{\varphi}\psi$ says that *if the agent would learn φ, then she would come to believe that ψ was the case before the learning* [3, p. 12]. That means conditional beliefs are hypothetical by nature, hinting at possible future 'real' belief changes of the agent. In DEL literature, the semantics for conditional beliefs is generally given in terms of plausibility models (or equivalently, in terms of sphere models), see, e.g., [30, 3, 32].

In this section, we explore the topological analogue of static conditioning by providing topological semantics for conditional belief modalities. As conditional beliefs capture *hypothetical* belief changes of an agent in case she would learn a piece of new information φ, we can obtain the semantics for a conditional belief modality $B^{\varphi}\psi$ in a natural and standard way by relativizing the semantics for the simple belief modality to the extension of the learnt formula φ. By *relativization* we mean a local change in the sense that it only affects one occurrence of the belief modality $B\varphi$. Unlike model *restriction* in the case of updates, conditional belief semantics does not cause a change in the model, i.e. it does not lead to a global change, due to its static nature.

Syntax and Semantics. We now consider the language $\mathcal{L}_{KCB}$ obtained by adding conditional belief modalities $B^{\varphi}\psi$ to $\mathcal{L}_{KB}$ and investigate the natural topological analogue of modelling conditional beliefs.

For any subset P of a topological space (X, τ), we can generalize the belief modality B on the topo-models by relativizing the closure and the interior operators to the set P. More precisely, given a topological model $\mathcal{M} = (X, \tau, \nu)$, the additional semantic clause reads

$$[\![B^{\varphi}\psi]\!]^{\mathcal{M}} = \mathrm{Int}([\![\varphi]\!]^{\mathcal{M}} \to \mathrm{Cl}([\![\varphi]\!]^{\mathcal{M}} \cap \mathrm{Int}([\![\varphi]\!]^{\mathcal{M}} \to [\![\psi]\!]^{\mathcal{M}})))$$

where $[\![\varphi]\!]^{\mathcal{M}} \to [\![\psi]\!]^{\mathcal{M}} := (X \setminus [\![\varphi]\!]^{\mathcal{M}}) \cup [\![\psi]\!]^{\mathcal{M}}$.

One possible justification for the above semantics of conditional belief is that it validates an equivalence that generalizes the one for belief in a natural way:

Proposition 6. *The following equivalence is valid in all topological spaces wrt the refined topological semantics for conditional beliefs and knowledge*

$$B^{\varphi}\psi \;\leftrightarrow\; K(\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \psi))).$$

This shows that, just like simple beliefs, conditional beliefs can be defined in terms of knowledge and this identity corresponds to the definition of the "conditional connective ⇒" in [9]. Moreover, as a corollary of Proposition 6, we obtain that the equivalences

$$B^{\top}\psi \overset{(1)}{\leftrightarrow} K(\top \to \langle K\rangle(\top \wedge K(\top \to \psi))) \overset{(2)}{\leftrightarrow} K\langle K\rangle K\psi \overset{(3)}{\leftrightarrow} B\psi$$

valid in all topological spaces, and thus our semantics for conditional beliefs and simple beliefs (in terms of the interior of the closure of the interior operator) are perfectly compatible with each other. Last but not least, we obtain the complete logic **KCB** of knowledge and conditional beliefs w.r.t. all topological spaces in the following way:

Theorem 9. *The logic* **KCB** *of knowledge and conditional beliefs is axiomatized completely by the system* **S4** *for the knowledge modality K together with the following equivalences:*

1. $B^{\varphi}\psi \leftrightarrow K(\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \psi)))$
2. $B\varphi \leftrightarrow B^{\top}\varphi$

4.2 Dynamic Belief Revision: updates on all topological spaces

In this section, we implement updates on arbitrary topological spaces and show that the problems occurred when we work with extremally disconnected spaces do not arise here: we in fact obtain a complete dynamic logic of knowledge and conditional beliefs w.r.t. the class of all topological spaces.

We now consider the language $\mathcal{L}_{!KCB}$ obtained by adding (existential) dynamic modalities $\langle !\varphi\rangle\psi$ to $\mathcal{L}_{KCB}$ and we model $\langle !\varphi\rangle\psi$ by means of subspaces exactly the same way as formalized in Section 3.2.2 , i.e., by using the restricted model $\mathcal{M}_\varphi$ with the semantic clause

$$[\![\langle !\varphi\rangle\psi]\!]^{\mathcal{M}} = [\![\psi]\!]^{\mathcal{M}_\varphi}.$$

In this setting, however, as the underlying static logic **KCB** is the logic of all topological spaces, we implement updates on arbitrary topological spaces. Since the resulting restricted model $\mathcal{M}_\varphi$ is always based on a topological (sub)space and no additional property of the initial topology needs to be inherited by the corresponding subspace (unlike the case for extremally disconnected spaces), we do not face the problem of loosing some validities of the corresponding static system: all the axioms of **KCB** (and, in particular, of **S4** and **wKD45**) will still be valid in the restricted space. Moreover, we obtain a complete axiomatization of the dynamic logic of knowledge and conditional beliefs:

Theorem 10. *The logic obtained by adding update modalities to the language $\mathcal{L}_{KCB}$ is axiomatized completely by adding the following reduction axioms to any complete axiomatization of the logic* **KCB***:*

1. $\langle !\varphi\rangle p \leftrightarrow (\varphi \wedge p)$
2. $\langle !\varphi\rangle\neg\psi \leftrightarrow (\varphi \wedge \neg\langle !\varphi\rangle\psi)$
3. $\langle !\varphi\rangle(\psi \wedge \theta) \leftrightarrow (\langle !\varphi\rangle\psi \wedge \langle !\varphi\rangle\theta)$
4. $\langle !\varphi\rangle K\psi \leftrightarrow (\varphi \wedge K(\varphi \to \langle !\varphi\rangle\psi))$
5. $\langle !\varphi\rangle B^\theta\psi \leftrightarrow (\varphi \wedge B^{\langle !\varphi\rangle\theta}\langle !\varphi\rangle\psi)$

Proof. Proof of this theorem follows, in a standard way, by the soundness of the reduction axioms w.r.t. all topological spaces. For proof details, we refer to [25, Theorem 12, pp. 66-67]. □

5 Conclusion and Future Work

In this paper, we proposed a new topological semantics for belief in terms of the *interior of the closure of the interior* operator which coincides with the one introduced in [1, 25] on extremally disconnected spaces and diverges from it on arbitrary topological spaces. This new topological semantics for belief comes with significant advantages especially concerning static and dynamic belief revision (in particular, concerning conditional belief and update semantics) and a few disadvantages compared to the setting in [1].

In [1], we worked with the knowledge system **S4.2** and the standard belief system **KD45**, however, on a restricted class of topological spaces, namely on extremally disconnected spaces. Although the framework of [1] provides a solid ground for the static systems of knowledge and belief and the relation between the two, the topological semantics based on extremally disconnected spaces falls short of dealing with updates as shown in Section 3.2.2. In this paper, we did not only provide semantics for belief based on *all topological spaces* but we also showed that its natural extension to conditional beliefs and updates gave us 'well-behaved' semantics. In other words, while extending the class of topo-models we could work within the context of knowledge and belief, we also solved the problem about updates in the previous setting. The price we had to pay for these results, however, was a weakening of the underlying static knowledge and belief logics: we weakened the knowledge logic **S4.2** to **S4** and the belief logic **KD45** to a slightly weaker one **wKD45**.

In on-going work, we investigate a more natural axiomatizations of the logic of knowledge and conditional beliefs **KCB** and its dynamic counterpart w.r.t. arbitrary topological spaces. Moreover, we propose another solution by means of hereditarily extremally disconnected spaces for the problem of updates occurred on extremally disconnected spaces: we formalize conditional beliefs and updates on hereditarily extremally disconnected spaces and provide natural complete axiomatizations for the corresponding logics. We plan to present these results in [2].

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