Logic, Algebra and Topology

Investigations into canonical extensions, duality theory and point-free topology

Jacob Vosmaer

Logic, Algebra and Topology

Investigations into canonical extensions, duality theory and point-free topology

ILLC Dissertation Series DS-2010-10

INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

For further information about ILLC-publications, please contact

Institute for Logic, Language and Computation
Universiteit van Amsterdam
Science Park 904
1098 XH Amsterdam
phone: +31-20-525 6051
fax: +31-20-525 5206
e-mail: illc@uva.nl
homepage: http://www.illc.uva.nl/

Logic, Algebra and Topology

Investigations into canonical extensions, duality theory and point-free topology

Academisch Proefschrift

ter verkrijging van de graad van doctor aan de
Universiteit van Amsterdam
op gezag van de Rector Magnificus
prof. dr. D.C. van den Boom
ten overstaan van een door het college voor
promoties ingestelde commissie, in het openbaar
te verdedigen in de Agnietenkapel
op dinsdag 14 december 2010, te 14.00 uur

door

Jacob Vosmaer

geboren te Amsterdam.

Promotiecommissie

Promotores: Prof. dr. M. Gehrke
Prof. dr. Y. Venema

Overige leden: Prof. dr. J.F.A.K. van Benthem
Dr. G. Bezhanishvili
Prof. dr. D.H.J. de Jongh
Prof. dr. A. Jung
Dr. A. Palmigiano
Prof. dr. H.A. Priestley

Faculteit der Natuurwetenschappen, Wiskunde en Informatica
Universiteit van Amsterdam
Science Park 904
1098 XH Amsterdam

This research was supported by the Netherlands Organization for Scientific Research (NWO) VICI grant 639.073.501.

Printed and bound by Ipskamp Drukkers.

ISBN: 978–90–5776–214–7

voor Liesbeth en Therus

Contents

Acknowledgments **xi**

1 Introduction **1**
1.1 Logic, algebra and topology 1
1.2 Mathematical surroundings 1
1.3 Survey of Contents 5

2 Canonical extensions: a domain theoretic approach **9**
2.1 Canonical extension via filters, ideals and topology 10
2.1.1 The canonical extension of a lattice 11
2.1.2 Topologies on posets and completions 15
2.1.3 Characterizing the canonical extension via the δ-topology . 18
2.1.4 Basic properties of the canonical extension 24
2.1.5 Conclusions and further work 27
2.2 Canonical extensions of maps I: order-preserving maps 27
2.2.1 The lower and upper extensions of an order-preserving map 28
2.2.2 Operators and join-preserving maps 33
2.2.3 Canonical extension as a functor I: lattices only 42
2.2.4 Conclusions and further work 44
2.3 Canonical extensions via dcpo presentations 45
2.3.1 Dcpo presentations 47
2.3.2 A dcpo presentation of the canonical extension 48
2.3.3 Extending maps via dcpo presentations 50
2.3.4 Conclusions and further work 51

3 Canonical extensions and topological algebra **53**
3.1 Topological algebra 54
3.1.1 Compact Hausdorff algebras 55
3.1.2 Profinite algebras and profinite completions 56

3.1.3 Topological lattices . . . 63
3.2 Canonical extensions of maps II: maps into profinite lattices . . . 68
3.2.1 Extending maps via lim inf and lim sup . . . 69
3.2.2 Maps into profinite lattices . . . 74
3.2.3 Canonical extension and function composition . . . 78
3.2.4 Conclusions and further work . . . 82
3.3 Canonical extension as a functor II: lattice-based algebras . . . 83
3.3.1 Order types and canonical extension types . . . 84
3.3.2 Preservation of homomorphisms . . . 87
3.3.3 Canonicity . . . 89
3.3.4 Conclusions and further work . . . 93
3.4 Profinite completion and canonical extension . . . 93
3.4.1 Universal properties of canonical extension . . . 94
3.4.2 Finitely generated varieties . . . 99
3.4.3 Canonical extension and monotone topological algebras . . . 105
3.4.4 Conclusions and further work . . . 109

4 Duality, profiniteness and completions **111**
4.1 Dualities for distributive lattices with operators . . . 112
4.1.1 Semi-topological DLO's . . . 113
4.1.2 Colimits of ordered Kripke frames . . . 116
4.1.3 Duality for profinite DLO's . . . 120
4.1.4 Profinite completion via duality . . . 122
4.1.5 Conclusions and further work . . . 127
4.2 A brief survey of subcategories of **DLO** . . . 127
4.2.1 Distributive lattices . . . 128
4.2.2 Boolean algebras . . . 129
4.2.3 Heyting algebras . . . 130
4.3 Duality for topological Boolean algebras with operators . . . 132
4.3.1 Duality for Boolean topological BAO's . . . 133
4.3.2 Ultrafilter extensions of image-finite Kripke frames . . . 137
4.3.3 Conclusions and further work . . . 139

5 Coalgebraic modal logic in point-free topology **141**
5.1 Introduction . . . 141
5.2 Preliminaries . . . 144
5.2.1 Basic mathematics . . . 144
5.2.2 Category theory . . . 145
5.2.3 Relation lifting . . . 147
5.2.4 Frames and their presentations . . . 152
5.2.5 Powerlocales via $\Box$ and $\Diamond$ . . . 154
5.3 The T-powerlocale construction . . . 157
5.3.1 Introducing the T-powerlocale . . . 157

5.3.2 Basic properties of the T-powerlocale 161
5.3.3 Two examples of the T-powerlocale construction 165
5.3.4 Categorical properties of the T-powerlocale 171
5.3.5 T-powerlocales via flat sites 179
5.4 Preservation results 185
5.4.1 Regularity and zero-dimensionality 185
5.4.2 Compactness 191
5.5 Further work 196

A Preliminaries **199**
A.1 Set theory 199
A.2 Category theory 200
A.2.1 Categories and functors 200
A.2.2 Adjunctions of categories 202
A.3 Order theory and domain theory 202
A.3.1 Pre-orders and partial orders 202
A.3.2 Adjunctions of partially ordered sets 204
A.3.3 Dcpo's 204
A.3.4 Order and topology 205
A.4 Lattice theory 205
A.4.1 Semilattices and suplattices 205
A.4.2 Lattices and complete lattices 206
A.4.3 Distributive lattices, Heyting algebras and Boolean algebras 206
A.5 Completions 207
A.5.1 The ideal (and filter) completion of a pre-order 207
A.5.2 The MacNeille completion of a pre-order 209
A.6 Universal algebra 210
A.6.1 Ω-algebras 210
A.6.2 Homomorphic images, subalgebras and products 210
A.6.3 Varieties 212
A.6.4 Terms and equations 214
A.7 General topology 214
A.7.1 Topological spaces 214
A.7.2 Continuity 216
A.7.3 Separation and compactness 216
A.8 Duality for ordered Kripke frames 217

Bibliography **219**

Index **227**

Samenvatting **231**

Abstract **233**

Acknowledgments

In this section I would like to acknowledge the difference that certain people made for me while I was doing my PhD research.

Scientific contacts For starters, I would like to thank my main supervisors (*promotores)* Mai Gehrke and Yde Venema, and the Netherlands Organisation for Scientific Research (NWO). The research reported on in this dissertation was carried out at the Institute for Logic, Language and Computation of the University of Amsterdam with NWO funding, as part of the VICI 'Algebra and Coalgebra' project of which Yde Venema is the principal investigator. For four years, starting in November 2006, I have been able to benefit from his supervision and mentorship. In my first year Alessandra Palmigiano served as my secondary supervisor, for which I am also very grateful. During my second year, Yde attracted Mai Gehrke as my second main supervisor. Since Mai is based at the Radboud University in Nijmegen, my interactions with her have been slightly less frequent than those with Yde, but on every occasion it was both a great pleasure and a great help to see her. Time and again, Mai and Yde have shown great concern not only for my success as a young academic, but also for my general wellbeing. This has meant a great deal to me.

I would also like to thank my co-authors for selected papers, namely Guram Bezanishvili, Mai Gehrke, Yde Venema and Steve Vickers. My first publication in a peer-reviewed journal [16] was a joint publication with Guram Bezanishvili. I learned a lot from working with him and I am grateful for the high standards he demanded from both of us. The two papers I co-authored with Mai Gehrke were written in a breeze; I enjoyed working on them very much. Finally, I am pleased to have been collaborating with Yde Venema and Steve Vickers on an as yet unpublished paper, on which Chapter 5 is based.

While I was working on my PhD during the last four years, my immediate research environment consisted of Yde's Algebra & Coalgebra group and its 'extended family', which is grounded in the University of Amsterdam but extends far beyond it. Many members of this family are mentioned elsewhere in these acknowledgments; of those who are not, I would like to mention Nick Bezhanishvili, Vincenzo Ciancia, Dion Coumans, Sam van Gool, Helle Hansen, Christian Kissig, Clemens Kupke, Loes Olde Loohuis and Alexandra Silva.

My work also benefited greatly from several visits to other universities I made during the last four years. In early 2008, I visited Hilary Priestley at the University of Oxford. I enjoyed my discussions with her a great deal. In late 2009, I visited Steve Vickers and Achim Jung at the University of Birmingham. This was another very nice opportunity for mathematical discussions, and it led to the collaboration with Steve and Yde on which Chapter 5 is based. Finally I would like to mention my visit to Mai Gehrke and Paul-André Melliès at Université Paris 7 – Denis Diderot. I enjoyed having the opportunity to give a very technical and interactive talk in Paul-André's Semantics Seminar.

At this stage, I have already mentioned most members of my PhD committee, which, besides of Mai and Yde, consists of Johan van Benthem, Guram Bezhanishvili, Dick de Jongh, Achim Jung, Alessandra Palmigiano and Hilary Priestley. They took upon them the task to critically judge my work, for which I am most grateful. Their interest and efforts mean a lot to me. I feel fortunate knowing that this dissertation has been subjected to their scrutiny. I am also grateful to Guram, Hilary, Mai and Yde for pointing out many typos in the manuscript; I assume full responsibility for all that remain.

Concluding this brief discussion of scientific contacts, I would like to acknowledge the discussions I had with people I met at OAL 2007 in Nashville, TANCL 2007 in Oxford, BLAST 2008 in Denver, AiML 2008 in Nancy, MLNL 2009 in Nijmegen, TACL 2009 in Amsterdam, the Coalgebraic Logics workshop in 2009 at Dagstuhl, ALCOP 2010 in London and MLNL 2010 in Utrecht. I will not try to list all the people I have enjoyed speaking with at these various occasions; I would like to single out Rob Goldblatt and Alexander Kurz because of the insightful discussions I have had with them.

University contacts While I conducted my PhD research I was based at the Institute for Logic, Language and Computation (ILLC) at the University of Amsterdam (UvA). The ILLC is run by managing director Ingrid van Loon and a secretarial staff which, during my time at the institute, has consisted of Karin Gigengack, Marjan Veldhuisen, Peter van Ormondt and Tanja Kassenaar. These people have made numerous positive contributions to my work environment at the UvA, for which I am most grateful. At this point I should also mention René Goedman, the receptionist at the Euclides building, who was really a part of our institute during the time it was based at said building.

Naturally, the scientific staff, post-docs and PhD students of the ILLC were also an important part of my work environment. Out of all these people, I would like to explicitly mention my office mates Fernando Velázquez Quesada, Gaëlle

Fontaine, Levan Uridia and Raul Andres Leal, who were always open to having discussions.

Concluding this very brief overview of university contacts, I would like to thank the secretarial staff of the ILLC for organizing the weekly Monday morning coffee break, and scientific director Leen Torenvliet, who organized the institute's weekly Friday afternoon drinks together with Peter van Ormondt.

Other contacts To conclude these acknowledgements, I would like to mention some people who provided healthy distractions from my research, thus contributing to it. The following people distracted me at the university: Gaëlle, Olivia, Reut and Teju. Other distractions were provided by Bert-Jan, Jael, Lodewijk, Nynke and Rembert. In addition, Nila provided a writer's retreat during the final stages of writing my dissertation. I am grateful to all these friends.

For a long time, my rowing endeavours at my rowing club A.S.R. Nereus succesfully diverted my attention away from research. Of all the people there who made rowing such a wonderful thing to spend my time on, I would like to single out my coach Roel Luijnenburg, who has been a source of invaluable advice.

Finally, I would like to thank my siblings Sara and Arend, my aunt Martine Vosmaer and my parents Liesbeth and Therus Vosmaer-de Bruin for supporting me.

Amsterdam, November 2010

Chapter 1

Introduction

1.1 Logic, algebra and topology

The connection between logic, algebra and topology is intricate and well-known. The archetypical examples of this connection are provided by the works of G. Boole and M.H. Stone. In the 1840s, Boole, one of the founding fathers of modern logic, made his 'laws of thought' the subject of mathematical investigation by treating logic as if it were algebra [21]. Boole's algebraic analysis of logic led to the study of *Boolean algebras*. In the 1930s, Stone showed that Boolean algebras can be faithfully represented through using topology as (the duals of) *Stone spaces*, which was a revolutionary contribution to both algebra and topology [81]. Out of these two ideas, the algebraization of logic and duality between algebra and topology, many research fields have sprung. In this dissertation, we present mathematical investigations which contribute to some of the research areas of logic, mathematics and computer science which developed out of the combined work of Boole and Stone: canonical extensions, extended Stone duality and point-free topology.

1.2 Mathematical surroundings

We will begin by sketching some of the mathematical context in which our work may be viewed.

Stone duality

Boolean algebras are the mathematical models Boole used to study 'the laws of thought', or classical propositional logic as we would call it nowadays. Stone's famous duality theorem for Boolean algebras is a two-edged sword. One edge amounts to the fact that any Boolean algebra $\mathbb{B}$ can be represented as the algebra of closed-and-open (clopen) subsets of a topological space X: $\mathbb{B} \simeq X^*$, where $(\cdot)^*$ is the function that assigns to a topological space X its algebra of clopen sets, with union, intersection and complementation as its algebra operations. More specifically, what Stone showed is that given $\mathbb{B}$, one can define a topological space $\mathbb{B}_*$ such that $\mathbb{B} \simeq (\mathbb{B}_*)^*$. In addition, he showed that such a space $\mathbb{B}_*$ is always compact, Hausdorff and zero-dimensional (recall that a space X is zero-dimensional if the clopen subsets of X form a basis for its topology). We call such spaces *Boolean spaces* or *Stone spaces*; in this dissertation we will use the former term. The other edge of the sword is now that if X is a Boolean space, then $X \simeq (X^*)_*$.

The constructions $(\cdot)_*$ and $(\cdot)^*$ are not merely defined on Boolean algebras and Boolean spaces: they are also defined on functions. Specifically, if $f\colon \mathbb{A} \to \mathbb{B}$ is a Boolean algebra homomorphism, then $f_*\colon \mathbb{B}_* \to \mathbb{A}_*$ is a continuous function, and conversely, if $g\colon X \to Y$ is a continuous function between Boolean spaces then $g^*\colon Y^* \to X^*$ is a Boolean algebra homomorphism. These assignments are *contravariant* functors, meaning the directions of the arrows are reversed. For this reason, the equivalence between Boolean algebras and Boolean spaces is called a *dual* equivalence or a *duality* of categories. Stone duality has proven to be a very influential mathematical discovery, see e.g. [54, Introduction]. Two research areas which are based on Stone duality play a central role in this dissertation: *relational semantics for modal logic* and *point-free topology*.

Relational semantics for modal logic

For our purposes, modal logic consists of propositional logic enriched with *modal* connectives, often denoted $\Box$, $\Diamond$, meant to express modalities such as 'possibly φ', 'agent a knows that φ', 'at some time in the future, φ', 'φ holds with probability p', etc. A wide class of modal logics, known as *normal modal logics*, admit relational semantics involving *Kripke frames* [19]. Kripke frames are structures consisting of a set X enriched with finitary relations $R_\Box \subseteq X^{n+1}$, for each modality $\Box$ involved.

Extended Stone duality

Stone duality for Boolean algebras can be extended to a representation theory for *modal algebras*, the algebraic counterparts of normal modal logics. This extended duality, known as Jónsson-Tarski duality [58], shows how one can give any modal algebra a representation as a topological relational structure. Many properties of modal logics can be understood in terms of Jónsson-Tarksi duality; however, if we want to also involve Kripke frames, i.e. discrete topological structures, then we run into a second duality, usually referred to as *discrete duality*. Discrete duality for modal algebras is based on the well-known duality between complete atomic Boolean algebras and sets [83], which in its turn is based on the fact that a complete atomic algebra can be uniquely described in terms of its set of atoms. The duality between complete atomic Boolean algebras and sets can be extended into a duality between complete, atomic, completely additive modal algebras (*perfect modal algebras*) and Kripke frames.

The connections between these two dualities are sketched in Figure 1.1. The horizontal connections are the dualities: the functors $(\cdot)_*$ and $(\cdot)^*$ for Jónsson-Tarski duality, and $(\cdot)_+$ and $(\cdot)^+$ for the discrete duality. The vertical connections in the middle are in themselves rather innocuous: on the left, we have the inclusion of the category of perfect modal algebras into the category of all modal algebras; this inclusion exists because any perfect modal algebra is a fortiori a modal algebra. On the right, we have the forgetful functor U, which takes a topological relational structure and strips it of its topology, yielding a bare, discrete relational structure.

$$
\begin{tikzcd}
\text{logical calculi} \arrow[d, leftrightsquigarrow] & \\
\text{modal algebras} \arrow[r, shift left, "(\cdot)_*"] \arrow[d, phantom, "\rotatebox[origin=c]{90}{$\subseteq$}"] & \text{descriptive general frames} \arrow[l, shift left, "(\cdot)^*"] \arrow[d, "U"] \\
\text{perfect modal algebras} \arrow[r, shift left, "(\cdot)_+"] & \text{Kripke frames} \arrow[l, shift left, "(\cdot)^+"] \arrow[d, leftrightsquigarrow] \\
& \text{relational semantics}
\end{tikzcd}
$$

Figure 1.1: The double duality diagram for modal algebras and Kripke frames

In the upper left corner of Figure 1.1, we have logical calculi consisting of Hilbert systems, Gentzen calculi, natural deduction, etc. for modal logics. The squigly arrow connecting the logical calculi with modal algebras is covered by *algebraic logic* [20], which tells us how to capture logical calculi in systems of algebras. The squigly arrow in the bottom right corner, connecting Kripke frames and relational semantics, is covered by the model theory of modal logic, which tells us how to interpret modal formulas using Kripke frames [19]. The study of the logical properties of normal modal logics can now be understood as a study of the connections through the square between the logical calculi in the upper left and the relational semantics in the lower right.

Canonical extensions

The vertical connection on the left side of Figure 1.1 is the subject of study in the theory of *canonical extensions* [58]. Canonical extensions are a construction that allows one to pass from modal algebras to perfect modal algebras, thus adding a new arrow to the diagram.

If one takes a modal algebra $\mathbb{A}$ from the upper left corner and 'pulls it around clockwise', one obtains a perfect modal algebra $(U\mathbb{A}_*)^+$, into which $\mathbb{A}$ can be embedded in a natural way. It turns out that the embedding of $\mathbb{A}$ into $(U\mathbb{A}_*)^+$ can be described in purely abstract terms as an embedding $e\colon \mathbb{A} \to \mathbb{A}^\delta$ of $\mathbb{A}$ into a perfect modal algebra $\mathbb{A}^\delta$, the canonical extension of $\mathbb{A}$, without even assuming the existence of the two dualities in Figure 1.1. This latter point can be significant because Jónsson-Tarski duality is dependent on the Axiom of Choice, a feature it inherits from Stone duality. Certain properties of modal logics can now be understood as properties of canonical extensions of modal algebras. An added advantage of this stems from the fact that canonical extensions can be defined for algebras associated with many other logics besides modal logic. This makes canonical extensions into a tool for a *generalized, uniform investigation* of logical properties of a wide range of logics.

Coalgebraic modal logic

In coalgebraic modal logic, a somewhat different view on the double duality diagram of Figure 1.1 is adopted. One separates modal logic into a Boolean base and a modal 'add-on', and similarly, Kripke frames are separated into their underlying set of states and a *transition type*. In the case of Kripke frames, the transition type corresponds to the relations on the frame which are used to interpret modal connectives. Technically, this is made precise by describing abstract transition systems as *coalgebras* and specifying two functors. One on Boolean algebras, describing the supra-Boolean logical structure, and one on sets describing the transition type of the coalgebras involved, resulting in a diagram as in Figure 1.2. In the case of basic modal logic, T is the powerset functor and L is the free $\wedge$-semilattice functor. In coalgebraic logic, one now studies modal

$$\begin{tikzcd} \mathbf{BA} \arrow[loop left, "L"] \arrow[r, bend right, "U\circ(\cdot)_*"'] & \mathbf{Set} \arrow[l, bend right, "(\cdot)^+"'] \arrow[loop right, "T"] \end{tikzcd}$$

Figure 1.2: The duality diagram of coalgebraic logic

logics (specified by the functor L) in relation to coalgebras (specified by T), seen as abstract transition systems.

Another coalgebraic interpretation of duality for modal algebras can be found in the fact that if one considers the free $\wedge$-semilattice functor M, which allows one to see modal algebras as M-algebras over Boolean algebras, then one can show that M dually corresponds to the *Vietoris hyperspace functor* V on Boolean spaces. Descriptive general frames can then be seen as V-coalgebras in the category of Boolean spaces.

Point-free topology

Although Stone called his duality theorem a representation theorem for Boolean algebras, one might as well see it as a representation theorem for Boolean spaces. When dealing with a problem involving Boolean spaces, one can translate it using Stone duality into a problem involving Boolean algebras and then use algebraic rather than topological methods to analyze the problem. Moreover, at any stage of the analysis, one can switch back to looking at Boolean spaces. In *point-free topology* [54], this duality-based view is taken as the primary way to look at topology. This approach consists of two conceptual ingredients. The first is to generalize Stone duality so that it encompasses all topological spaces, at the price of weakening the dual equivalence of Stone duality for Boolean spaces to a dual adjunction between topological spaces and *frames*, the algebras that arise when one views Stone duality at this generalized level. Frames are complete lattices in which infinite joins distribute over finite meets; a frame homomorphism is a map which preserves finite meets and infinite joins. Just as Boolean algebras can be seen as models for classical propositional logic, frames can be seen as models for *geometric propositional logic*, which is a logic with finite conjunctions and infinite disjunctions. The second conceptual ingredient in the framework of point-free topology is to *view* frames as 'point-free spaces', which we call *locales*, rather than as algebras. In this dissertation we will take a predominantly algebraic approach to point-free topology however, meaning that we will mostly deal with frames.

1.3 Survey of Contents

We will now give a broad overview of the contents of this dissertation.

Canonical extensions

The framework of canonical extensions as we present it consists of two components: a construction on lattices, which is a lattice completion, and a construction for extending maps between lattices. Canonical extensions of maps are used both to define the canonical extensions of a given lattice-based algebra $\mathbb{A}$, since the operations on $\mathbb{A}$ are maps between powers of $\mathbb{A}$, and to define extensions of homomorphisms between lattice-based algebras $f\colon \mathbb{A} \to \mathbb{B}$. In our discussion of canonical extensions in Chapters 2 and 3, our focus is on three subjects: topological and categorical properties of canonical extensions, and the view on canonical extensions as dcpo algebras.

Regarding the *topological properties* of canonical extensions, we would like to point out the topological characterization of the canonical extension of a lattice in §2.1.3, our broad discussion of topological properties of canonical extensions of maps, beyond what was previously known, in §2.2 and §3.2, and our investiagations into universal properties of canonical extensions with respect to topological lattice-based algebras in §3.4. This brings us to the *categorical properties* of canonical extensions. One of the crucial facts behind these universal properties is the fact, established in §3.3, that canonical extensions preserve surjective algebra homomorphisms, a fact which was previously only known to hold under assumption of distributivity.

In our discussion of canonical extensions of order-preserving maps in §2.2, the ideal and filter completion functors play an important role. They reprise this role when we look at *canonical extensions as dcpo algebras* in §2.3 and §3.3.3. Here we show how canonicity results for distributive lattices with operators (DLO's) can be understood using more general results about dcpo algebras.

Duality for topological lattice-based algebras

When studying distributive lattices with operators (or DLO's for short), a class of algebras which arises naturally in algebraic logic, one has two categorical dualities at one's disposal. The first one is the extended Priestley duality between 'plain' DLO's and topological ordered Kripke frames, known as relational Priestley spaces. The second one is the discrete duality between so-called perfect DLO's and ordered Kripke frames *sans* topology. In our discussion of duality for DLO's in Chapter 4, we propose that the proper perspective on perfect DLO's is to regard them as *semi-topological* DLOs. We will use discrete duality for semi-topological DLO's to provide a number of dual characterization results. In §4.1, we characterize profinite DLO's as the duals of hereditarily finite ordered Kripke frames, and accordingly we dually characterize the profinite completion of a DLO $\mathbb{A}$ relative to the prime filter frame of $\mathbb{A}$. In §4.2, we briefly discuss how these results specialize to distributive lattices, Boolean algebras and Heyting algebras. Finally, in §4.3, we characterize compact Hausdorff Boolean algebras with operators as the duals of image-finite Kripke frames, and we use this duality to study ultrafilter extensions of Kripke frames using duality.

Powerlocales and coalgebraic logic

In Chapter 5 we use techniques from *coalgebraic logic* to describe and generalize the *Vietoris powerlocale* construction from point-free topology. The idea is the following: we take an axiomatization of coalgebraic logics for T-coalgebras, where T is an arbitrary weak pullback-preserving standard set functor, known as the *Carioca axiomatization*, and we then use this axiomatization to algebraically describe a new construction on locales, the *T-powerlocale* construction in §5.3. The usual Vietoris powerlocale is now the $\mathcal{P}_\omega$-powerlocale, where $\mathcal{P}_\omega$ is the finite powerset functor. We then proceed to prove a number of properties of T-powerlocales. For instance, we show in §5.3.5 that T-powerlocales admit a *flat site presentation*, which is a technical property that allows us to disentangle the roles of conjunctions and disjunctions. Moreover, in §5.4 we show that the T-powerlocale construction *preserves (point-free) topological properties* such as regularity and the combination of compactness and zero-dimensionality.

Acknowledgments

Chapters 2 and 3 contain results from joint work with M. Gehrke [42, 43]. Chapter 5 is based on joint work with S. Vickers and Y. Venema.

Chapter 2

Canonical extensions: a domain theoretic approach

In this chapter, we provide a perspective on the toolkit of canonical extensions of lattices and order-preserving maps with an eye to connections with domain theory. This leads to new results and new insights into the reasons why and when canonical extensions work.

Canonical extensions were introduced by Jónsson and Tarski in 1951 [58, 59], as part of the representation theory for Boolean algebras with operators (BAOs), which play an important role in algebraic logic. Examples of BAOs considered by Jónsson & Tarski were e.g. relation algebras, cylindric algebras and closure algebras. Another example, which was not immediately recognized, is the class of modal algebras, which are used to study modal logic via algebra. The connection between BAOs and modal logic, or rather the connection between BAOs and Kripke semantics for modal logic via Stone duality became more widely recognized in the 1970s via the work of Thomason [87] and Goldblatt [46]; see [48] for historical context. The algebraic approach to BAO representation theory via the purely algebraic theory of canonical extensions was revived in the 1990s and 2000s in the work of Jónsson, Gehrke and Harding [57, 38, 34, 39] and De Rijke and Venema [31], and this approach has remained active since. Moreover canonical extensions have been generalized from a representation theory for Boolean algebras with operators to a more general representation theory *toolkit* for lattice-based algebras. We will wait with defining canonical extensions of lattice-based algebras until Chapter 3. In this chapter, we will introduce the reader to two important parts of the canonical extensions toolkit: canonical extensions of lattices and canonical extensions of order-preserving maps between lattices. Moreover, while doing so we will demonstrate the utility of methods from *domain theory* in relation to canonical extensions.

Domain theory was pioneered by D.S. Scott, with the aim to "give a mathematical semantics for high-level computer languages" [80]. It has been used to study subjects as diverse as recursive equations, semantics for untyped λ-calculus, computability and partial information (see [1]). In this chapter we will borrow several techniques from the study of domain theory and continuous lattices when studying canonical extensions:

- Order topologies. Order topologies are used heavily in domain theory, and they occur very naturally when studying canonical extensions [39]. We offer the most extensive treatment of the topological properties of canonical extensions to date, and we present two results in §2.1.3 which make clear that the topologies on canonical extensions are both central, even defining, and natural.

- Filter and ideal completions. Many results about canonical extensions can be understood by looking at an intermediate level of filters and ideals, rather than only at the canonical extensions themselves [44]. In fact, the filter and ideal *completion functors* play a central role in the theory of canonical extensions of order-preserving maps – a fact which was foreshadowed in [41]. We revisit and expand upon the results of [34], showing how the filter completion and the ideal completion play an important role 'under the hood'.

- Dcpo presentations. Directed complete partial orders are central in domain theory. We already indicated that the canonical extension is intimately connected with the filter completion and the ideal completion. In fact, we can present the canonical extension of a lattice $\mathbb{L}$ as a dcpo generated by the filter completion $\mathcal{F}\mathbb{L}$.

This chapter is organized as follows. In §2.1, we introduce canonical extensions of lattices, both classically and using a new topological characterization in §2.1.3. In §2.2, we develop the theory of canonical extensions of order-preserving maps with an emphasis on the role of the filter and ideal completion. Finally, in §2.3 we present an alternative characterization of canonical extensions using dcpo presentations. Furthermore, at the end of each section we provide a discussion of the contributions in that section and suggestions for further work.

2.1 Canonical extension via filters, ideals and topology

In this section we want to introduce the canonical extension of a bounded lattice, which is a well-studied lattice completion, together with an improved topological perspective on this completion. The key to this topological perspective lies in understanding the role that the ideal and the filter completion play with respect to the canonical extension.

In §2.1.1, we introduce the classical definition of canonical extensions of lattices. In §2.1.2, we will introduce several important topologies on partial orders and lattice completions. In §2.1.3, we will present the two main new results of this section, which characterize canonical extensions of lattices topologically, and which identify the characterizing topologies of canonical extensions. Finally, in §2.1.4, we present assorted additional properties of canonical extensions of lattices that will be of use later on.

2.1.1 The canonical extension of a lattice

In this subsection, we will introduce the canonical extension $\mathbb{L}^\delta$ of a lattice $\mathbb{L}$. We will define a concrete construction on a given lattice, using filters and ideals, and we will then state a uniqueness result which tells us that any completion of $\mathbb{L}$ satisfying certain abstract order-theoretic properties is in fact isomorphic to $\mathbb{L}^\delta$. From that point on, we will no longer concern ourselves with the actual concrete construction of $\mathbb{L}^\delta$; rather, we will show how one can understand $\mathbb{L}^\delta$ through its abstract characterization.

Overlapping sets, filters and ideals

Before we go ahead and introduce the canonical extension, we would like to introduce a very elegant concept from constructive mathematics. We will often want to talk about sets U, V which have a non-empty intersection, i.e. $U \cap V \neq \emptyset$. We would like to think of this relation on sets as a positive property.

2.1.1. Definition. Given two sets $U, V \subseteq X$, we write $U \between V$ (U and V *overlap*) if $U \cap V \neq \emptyset$.

What makes the overlapping relation interesting is that it interacts nicely with several operations and relations on sets.

2.1.2. Lemma. *Let X be a set and let $U, V \subseteq X$ such that $U \between V$.*

1. *If $U', V' \subseteq X$ such that $U \subseteq U'$ and $V \subseteq V'$, then also $U' \between V'$.*
2. *If $f\colon X \to Y$ is a function to another set Y, then also $f[U] \between f[V]$.*

Consequently, if $f\colon \mathbb{P} \to \mathbb{Q}$ is an order-preserving map and if $F \in \mathcal{F}\,\mathbb{P}$, $I \in \mathcal{I}\,\mathbb{P}$ such that $F \between I$, then also $\mathcal{F}\,f(F) \between \mathcal{I}\,f(I)$.

Proof Parts (1) and (2) are elementary. For the last part it suffices to recall that $\mathcal{F}\,f(F) := \uparrow f[F]$ and $\mathcal{I}\,f[F] := \downarrow f[F]$. ∎

The following lemma provides a further indication that the overlapping relation also interacts in a nice way with filters and ideals. Recall from §A.5.1 that given a poset $\mathbb{P}$, $\mathcal{I}\,\mathbb{P} := \langle \mathrm{Idl}\,\mathbb{P}, \subseteq \rangle$ is the *ideal completion* of $\mathbb{P}$, and dually that $\mathcal{F}\,\mathbb{P} := \langle \mathrm{Filt}\,\mathbb{P}, \supseteq \rangle$ is the *filter completion* of $\mathbb{P}$.

2.1.3. Lemma. *Let $\mathbb{P}$ be a poset. If $F \in \mathcal{F}\,\mathbb{P}$ and $I \in \mathcal{I}\,\mathbb{P}$ such that $F \between I$, then $\downarrow(F \cap I) = I$ and $\uparrow(F \cap I) = F$.*

Proof We only show the first case, since the other follows by order duality. Take $y \in I$, we will show that there exists $z \in F \cap I$ such that $y \leq z$. Since $F \between I$, there exists $x \in F \cap I$. Because $x, y \in I$, there exists $z \in I$ such that $x \leq z$ and $y \leq z$. Because $x \in F$ and F is an upper set, $z \in F$. But then $y \leq z \in F \cap I$, so that $I \subseteq \downarrow(F \cap I)$. The other inclusion follows immediately from the fact that $F \cap I \subseteq I$ and I is a lower set. ∎

Existence and uniqueness of the canonical extension

We will now present the canonical extension first as a concrete construction on lattices, and later as a *completion* of lattices which is unique up to isomorphism. The particular concrete construction we have chosen, which seems to go back to [44], is not the only possible one. In light of the uniqueness theorem however, the concrete construction of the canonical extension we choose now is not particularly important. Our construction will be a two-stage construction on a given lattice $\mathbb{L}$. The first stage in the construction consists of creating a pre-order. The order relation we use goes back to [44].

2.1.4. Definition. Let $\mathbb{L}$ be a lattice. We define a structure $\operatorname{Int}\mathbb{L} := \langle \mathcal{F}\,\mathbb{L} \uplus \mathcal{I}\,\mathbb{L}, \sqsubseteq \rangle$, where for all $F, F' \in \mathcal{F}\,\mathbb{L}$ and $I, I' \in \mathcal{I}\,\mathbb{L}$,

$$
\begin{array}{rl}
F \sqsubseteq I & \text{if } F \between I; \\
F \sqsubseteq F' & \text{if } F \supseteq F'; \\
I \sqsubseteq I' & \text{if } I \subseteq I'; \\
I \sqsubseteq F & \text{if } I \times F \subseteq {\leq_{\mathbb{L}}},
\end{array}
$$

i.e. $I \sqsubseteq F$ iff for all $a \in I$ and for all $b \in F$, we have $a \leq b$.

The following fact is well-known, cf. [44, p. 11].

2.1.5. Lemma. *Let $\mathbb{L}$ be a lattice. Then $\operatorname{Int}\mathbb{L}$ is a pre-order.*

Proof It suffices to show that $\sqsubseteq$ is transitive. For this, we need to make a case distinction. Let $F, F' \in \mathcal{F}\,\mathbb{L}$ and $I, I' \in \mathcal{I}\,\mathbb{L}$.

$$
\begin{array}{ll}
F \sqsubseteq F' \sqsubseteq I \Rightarrow F \sqsubseteq I & \text{by Lemma 2.1.2;} \\
F \sqsubseteq I \sqsubseteq I' \Rightarrow F \sqsubseteq I' & \text{idem;} \\
I \sqsubseteq I' \sqsubseteq F \Rightarrow I \sqsubseteq F & \text{easy,}
\end{array}
$$

since if $I \subseteq I'$ and $I' \times F \subseteq \leq_{\mathbb{L}}$ then $I \times F \subseteq I' \times F \subseteq \leq_{\mathbb{L}}$;

$$\begin{array}{ll} I \sqsubseteq F \sqsubseteq F' \Rightarrow I \sqsubseteq F' & \text{idem;} \\ I \sqsubseteq F \sqsubseteq I' \Rightarrow I \sqsubseteq I' & \text{see below;} \\ F \sqsubseteq I \sqsubseteq F' \Rightarrow F \sqsubseteq F' & \text{see below.} \end{array}$$

The last two cases have essentially the same proof; we only discuss the latter. If $F \sqsubseteq I \sqsubseteq F'$, i.e. if $F \between I$ and $I \times F' \subseteq \leq_{\mathbb{L}}$, then we need to show that also $F \sqsubseteq F'$, i.e. that $F \supseteq F'$. Let $a \in F'$. Since $F \between I$, there exists $b \in F \cap I$. Since $I \times F' \subseteq \leq_{\mathbb{L}}$, we have $b \leq a$. But then $a \in F$; it follows that $F \supseteq F'$. ∎

Note that in other places where $\mathrm{Int}\,\mathbb{L}$ is introduced [44, 32, 41], one also takes a quotient of the pre-order to make it into a partial order. We do not bother with this because the pre-order is flattened into a partial order in the second stage of constructing the canonical extension anyway. This second stage consists of taking the MacNeille completion (see §A.5.2) of $\mathrm{Int}\,\mathbb{L}$.

2.1.6. Definition. We define the *canonical extension* of $\mathbb{L}$ to be $\mathbb{L}^\delta := \overline{\mathrm{Int}\,\mathbb{L}}$, with $i\colon \mathrm{Int}\,\mathbb{L} \to \mathbb{L}^\delta$ the embedding of the MacNeille completion. We map $\mathbb{L}$ to $\mathbb{L}^\delta$ by setting $e_{\mathbb{L}}\colon a \mapsto i(\downarrow a)$.

We now have a way to constuct a complete lattice $\mathbb{L}^\delta$, given a lattice $\mathbb{L}$, and a function $e_{\mathbb{L}}\colon \mathbb{L} \to \mathbb{L}^\delta$. We see below that $e_{\mathbb{L}}$ is in fact a lattice completion, and that we can characterize it up to isomorphism of completions.

2.1.7. Definition. Given a (bounded) lattice $\mathbb{L}$ and a complete lattice $\mathbb{C}$, we call a lattice embedding $e\colon \mathbb{L} \to \mathbb{C}$ a *completion* of $\mathbb{L}$. We say two completions of $e\colon \mathbb{L} \to \mathbb{C}$ and $e'\colon \mathbb{L} \to \mathbb{C}'$ are isomorphic if there exists an isomorphism $h\colon \mathbb{C} \to \mathbb{C}'$ such that $he = e'$.

$$\begin{tikzcd} \mathbb{C} \arrow[r, "h"] & \mathbb{C}' \\ \mathbb{L} \arrow[u, "e"] \arrow[ur, "e'"'] & \end{tikzcd}$$

Before we state the basic uniqueness result concerning canonical extensions, we would like to point out that any completion $e\colon \mathbb{L} \to \mathbb{C}$ induces two auxiliary maps $e^{\mathcal{F}}\colon \mathcal{F}\,\mathbb{L} \to \mathbb{C}$ and $e^{\mathcal{I}}\colon \mathcal{I}\,\mathbb{L} \to \mathbb{C}$, if we exploit the fact that a complete lattice $\mathbb{C}$ is simultaneously a dcpo and a co-dcpo.

2.1.8. Definition. Given a lattice completion $e\colon \mathbb{L} \hookrightarrow \mathbb{C}$, we define $e^{\mathcal{I}}\colon \mathcal{I}\,\mathbb{L} \to \mathbb{C}$ by $I \mapsto \bigvee^{\uparrow} e[I]$. It is easy to see that $e^{\mathcal{I}}\colon \mathcal{I}\,\mathbb{L} \to \mathbb{C}$ is the unique Scott-continous extension of $e\colon \mathbb{L} \to \mathbb{C}$, the existence of which is stipulated by Fact A.5.3.

$$\begin{tikzcd} & \mathcal{I}\,\mathbb{L} \arrow[d, dotted, "e^{\mathcal{I}}"] \\ \mathbb{L} \arrow[ur, "\downarrow"] \arrow[r, "e"'] & \mathbb{C} \end{tikzcd}$$

Dually, we define $e^{\mathcal{F}}\colon \mathcal{F}\,\mathbb{L} \to \mathbb{C}$ by $F \mapsto \bigwedge\!\downarrow e[F]$. We emphasize that $e^{\mathcal{I}}(\downarrow x) = e^{\mathcal{F}}(\uparrow x) = e(x)$ for all $x \in \mathbb{L}$. We refer to $\{e^{\mathcal{F}}(F) \mid F \in \mathcal{I}\,\mathbb{L}\}$ and $\{e^{\mathcal{I}}(I) \mid I \in \mathcal{I}\,\mathbb{L}\}$ as the *filter* and *ideal elements of* $\mathbb{C}$, respectively.[1]

Filter and ideal elements play a crucial role in defining and understanding the canonical extension.

2.1.9. FACT ([41], PROPOSITION 3.6). *Let $\mathbb{L}$ be a lattice, then the map $e_{\mathbb{L}}\colon \mathbb{L} \to \mathbb{L}^\delta$ defined above is a lattice embedding. Moreover, if $e\colon \mathbb{L} \to \mathbb{C}$ is a completion of $\mathbb{L}$ such that*

1. *for all $x \in \mathbb{C}$,*

$$x = \bigvee \{e^{\mathcal{F}}(F) \mid e^{\mathcal{F}}(F) \le x,\ F \in \mathcal{F}\,\mathbb{L}\} = \bigwedge \{e^{\mathcal{I}}(I) \mid e^{\mathcal{I}}(I) \ge x,\ I \in \mathcal{I}\,\mathbb{L}\};$$

2. *for all $F \in \mathcal{F}\,\mathbb{L}$, $I \in \mathcal{I}\,\mathbb{L}$, if $e^{\mathcal{F}}(F) \le e^{\mathcal{I}}(I)$ then $F \between I$;*

then there exists a (unique) isomorphism of completions $h\colon \mathbb{C} \to \mathbb{L}^\delta$, i.e. such that $he = e_{\mathbb{L}}$.

We will refer to a lattice completion $e\colon \mathbb{L} \to \mathbb{C}$ satisfying the conditions above as *a* canonical extension of $\mathbb{L}$. The first condition of Fact 2.1.9, which is traditionally called *density*, states that given a canonical extension $e\colon \mathbb{L} \to \mathbb{C}$,

- the filter elements of $\mathbb{C}$ are join-dense, and
- the ideal alements of $\mathbb{C}$ are meet-dense.

The second condition (traditionally known as *compactness*) could also have been stated as:

- for all $F \in \mathcal{F}\,\mathbb{L}$, $I \in \mathcal{I}\,\mathbb{L}$, $e^{\mathcal{F}}(F) \le e^{\mathcal{I}}(I)$ iff $F \between I$,

since $F \between I$ implies that there is $a \in F \cap I$, so that $\bigwedge e[F] \le e(a) \le \bigvee e[I]$. From here on, we will simply refer to the density and compactness properties of the canonical extension instead of explicitly referring to Fact 2.1.9.

We conclude this subsection with a class of examples of canonical extensions which is both important and trivial. Recall that a poset $\mathbb{P}$ is said to satisfy the *ascending chain condition* (ACC) if for every countable chain $x_0 \le x_1 \le x_2 \le \cdots \le x_i \le \cdots$, there exists a $n \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $x_k \le x_n$.

2.1.10. FACT. *Let $\mathbb{L}$ be a (bounded) lattice. The identity embedding $id_{\mathbb{L}}\colon \mathbb{L} \to \mathbb{L}$ is a canonical extension of $\mathbb{L}$ iff $\mathbb{L}$ satisfies both the ascending chain condition (ACC) and the descending chain condition (DCC).*

[1] This is an abuse of language, since strictly speaking they are the filter and ideal elements of $e\colon \mathbb{L} \to \mathbb{C}$.

2.1.11. Example. Examples of lattices satisfying ACC and DCC include finite lattices, and lattices such as $\mathbb{M}_\infty$, where

$$\mathbb{M}_\infty := \langle \{0,1\} \cup \{a_n \mid n \in \mathbb{N}\}, \leq \rangle,$$

and $x \leq y$ iff $x = 0$ or $y = 1$, see Figure 2.1.

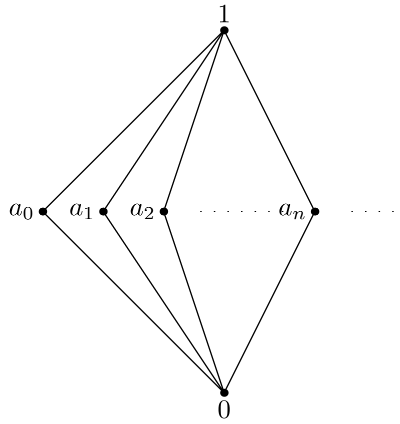

Figure 2.1: The lattice $\mathbb{M}_\infty$ is a fixed point of the canonical extension

2.1.12. Convention. If $\mathbb{L}$ is a finite lattice, then we define $\mathbb{L}^\delta := \mathbb{L}$ in light of Fact 2.1.10.

2.1.2 Topologies on posets and completions

We will now introduce two families of topologies which are defined on posets, and one which is defined on completions. The Scott topologies and the interval topologies are defined on any poset. The δ topologies are defined on any lattice completion. All three families come in three kinds: one where every open set is an upper set, one where every open set is a lower set, and the join of these upper and lower topologies, where every *basic* open is an intersection of an open upper set and an open lower set.

2.1.13. Definition. Let $\mathbb{P} = \langle P, \leq \rangle$ be a poset.

- By $\iota^\uparrow(\mathbb{P}) := \langle \{P \setminus \downarrow x \mid x \in P\} \rangle$ we denote the *upper interval topology* of $\mathbb{P}$, and $\iota^\downarrow(\mathbb{P}) := \iota^\uparrow(\mathbb{P}^{op})$.
- By $\sigma^\uparrow(\mathbb{P})$ we denote the *Scott topology* on $\mathbb{P}$: $U \subseteq P$ is $\sigma^\uparrow$-open if U is an upper set which is inaccessible by directed joins, or equivalently if $P \setminus U$ is a lower set closed under all existing directed joins. By $\sigma^\downarrow(\mathbb{P})$ we denote $\sigma^\uparrow(\mathbb{P}^{op})$.

Let $e\colon \mathbb{L} \to \mathbb{C}$ be a lattice completion. We define two topologies on $\mathbb{C}$:

- $\delta^{\uparrow}(\mathbb{C}) := \big\langle \{\uparrow e^{\mathcal{F}}(F) \mid F \in \mathcal{F}\,\mathbb{L}\} \big\rangle$ and $\delta^{\downarrow}(\mathbb{C}) := \big\langle \{\downarrow e^{\mathcal{I}}(I) \mid I \in \mathcal{I}\,\mathbb{L}\} \big\rangle$.

Recall from §A.7 that if τ and τ' are topologies on a set X, then $\tau \vee \tau' := \big\langle \{U \cap V \mid U \in \tau, V \in \tau'\} \big\rangle$ is the least topology on X containing τ and τ'. We define $\sigma(\mathbb{P}) := \sigma^{\uparrow}(\mathbb{P}) \vee \sigma^{\downarrow}(\mathbb{P})$ (the *bi-Scott topology*) and $\delta(\mathbb{C}) := \delta^{\uparrow}(\mathbb{C}) \vee \delta^{\downarrow}(\mathbb{C})$.

Below, if e.g. $f\colon \mathbb{C} \to \mathbb{M}$ is some map, we will say that f is $(\delta^{\uparrow}, \sigma^{\uparrow})$*-continuous* if $f\colon \langle C, \delta^{\uparrow}(\mathbb{C})\rangle \to \langle M, \sigma^{\uparrow}(\mathbb{M})\rangle$ is a continuous function.

2.1.14. Remark. Observe that given a completion $e\colon \mathbb{L} \hookrightarrow \mathbb{C}$, the filter and ideal elements of $\mathbb{C}$ are often referred to as the 'closed' and 'open' elements of $\mathbb{C}$ in the canonical extension literature (cf. [34, Lemma 3.3]). This makes our definitions of the $\delta^{\uparrow}$ and $\delta^{\downarrow}$ topologies equivalent with those in [39], where they are called $\sigma^{\uparrow}$ and $\sigma^{\downarrow}$ respectively.

Although we will not use it, it is worth noting that $\iota := \iota^{\uparrow} \vee \iota^{\downarrow}$ is the usual interval topology. For example, $\iota(\mathbb{R})$ is the topology generated by

$$\{\{z \in \mathbb{R} \mid x < z < y\} \mid x, y \in \mathbb{R}\},$$

i.e. the usual topology on the real line.

2.1.15. Lemma. *Let $e\colon \mathbb{L} \to \mathbb{C}$ be a lattice completion.*

1. *The following set is a base for $\delta^{\uparrow}(\mathbb{C})$:*

$$\{\uparrow \textstyle\bigvee_{F \in S} e^{\mathcal{F}}(F) \mid S \subseteq \mathcal{F}\,\mathbb{L} \text{ finite}\}.$$

2. *The following set is a base for $\delta^{\downarrow}(\mathbb{C})$:*

$$\{\downarrow \textstyle\bigwedge_{I \in T} e^{\mathcal{I}}(I) \mid T \subseteq \mathcal{I}\,\mathbb{L} \text{ finite}\}.$$

3. *The following set is a base for $\delta(\mathbb{C})$:*

$$\{\uparrow \textstyle\bigvee_{F \in S} e^{\mathcal{F}}(F) \cap \downarrow \bigwedge_{I \in T} e^{\mathcal{I}}(I) \mid S \subseteq \mathcal{F}\,\mathbb{L}, T \subseteq \mathcal{I}\,\mathbb{L} \text{ finite}\}.$$

Proof It suffices to show that (1) holds. First, observe that if $S \subseteq \mathcal{F}\,\mathbb{L}$, then since $\mathbb{C}$ is complete, it follows by order theory that

$$\textstyle\bigcap_{F \in S} \uparrow e^{\mathcal{F}}(F) = \uparrow \bigvee_{F \in S} e^{\mathcal{F}}(F). \tag{2.1}$$

Next, observe that since $\{\uparrow e^{\mathcal{F}}(F) \mid F \in \mathcal{F}\,\mathbb{L}\}$ is a subbase for $\delta^{\uparrow}(\mathbb{C})$ by definition, it follows by general topology that

$$\{\textstyle\bigcap_{F \in S} \uparrow e^{\mathcal{F}}(F) \mid S \subseteq \mathcal{F}\,\mathbb{L} \text{ finite}\}$$

is a base for $\delta^{\uparrow}(\mathbb{C})$. It now follows by (2.1) that (1) holds. ∎

We are considering topologies defined on ordered sets, and the order plays an important role in defining these topologies. Below we will establish some elementary facts about the interaction between our topologies and the order.

2.1.16. LEMMA. *Let $\mathbb{P}$ be a poset. If $O \subseteq P$ is σ-open, then $P \setminus O$ is closed under all existing directed joins and codirected meets.*

Proof We only show the case of directed joins. Towards a contradiction, let $S \subseteq P \setminus O$ be a directed set such that $\bigvee S \in O$. Then by definition of σ, there must exist a $\sigma^{\uparrow}$-open $U \subseteq \mathbb{P}$ and a $\sigma^{\downarrow}$-open $V \subseteq \mathbb{P}$ such that $\bigvee S \in U \cap V \subseteq O$. Since U is Scott-open and $\bigvee S \in U$, there is some $x \in S \cap U$. Since V is a lower set, we also get $x \in V$, which is a contradiction. ∎

2.1.17. LEMMA. *Let $\mathbb{P}$ be a poset and let $e\colon \mathbb{L} \to \mathbb{C}$ be a lattice completion.*

1. *$\iota^{\uparrow}(\mathbb{P}) \subseteq \sigma^{\uparrow}(\mathbb{P})$ and $\iota^{\downarrow}(\mathbb{P}) \subseteq \sigma^{\downarrow}(\mathbb{P})$;*
2. *$\{U \in \sigma(\mathbb{P}) \mid U$ is an upper set$\} = \sigma^{\uparrow}(\mathbb{P})$;*
3. *$\{U \in \delta(\mathbb{C}) \mid U$ is an upper set$\} = \delta^{\uparrow}(\mathbb{C})$.*

Consequently, an order-preserving map $f\colon \mathbb{P} \to \mathbb{Q}$ is (σ, σ)-continuous iff it is both $(\sigma^{\uparrow}, \sigma^{\uparrow})$-continuous and $(\sigma^{\downarrow}, \sigma^{\downarrow})$-continuous.

Proof (1). This is easy to see: any subbasic $\iota^{\uparrow}$-open $\downarrow x$ is also $\sigma^{\uparrow}$-open, since obviously $\downarrow x$ is a lower set closed under directed joins.

(2). Suppose that U is a σ-open set such that U is an upper set. Then all we have to do to show that U is $\sigma^{\uparrow}$-open, is to show that $P \setminus U$ is closed under directed joins. But this follows immediately from Lemma 2.1.16.

(3). Suppose that $U \subseteq \mathbb{C}$ is a δ-open upper set. To show that U is $\delta^{\uparrow}$-open, it suffices to show that for every $x \in U$, there exists a $\delta^{\uparrow}$-open $U' \subseteq U$ such that $x \in U'$. Take $x \in U$. Since U is δ-open, by Lemma 2.1.15(3), there exist finite sets $S \subseteq \mathcal{F}\,\mathbb{L}$ and $T \subseteq \mathcal{I}\,\mathbb{L}$ such that

$$x \in \uparrow \bigvee_{F \in S} e^{\mathcal{F}}(F) \cap \downarrow \bigwedge_{I \in T} e^{\mathcal{I}}(I) \subseteq U.$$

Then $\bigvee_{F \in S} e^{\mathcal{F}}(F) \in U$, so since U is an upper set,

$$U' := \uparrow \bigvee_{F \in S} e^{\mathcal{F}}(F) \subseteq U.$$

By Lemma 2.1.15(1), U' is $\delta^{\uparrow}$-open; since $x \in U' \subseteq U$ was arbitrary it follows that U is δ-open.

For the last claim of the lemma, suppose that $f\colon \mathbb{P} \to \mathbb{Q}$ is order preserving. If f is both $(\sigma^{\uparrow}, \sigma^{\uparrow})$ and $(\sigma^{\downarrow}, \sigma^{\downarrow})$-continuous, then it follows by general topology (Lemma A.7.3) that f is $(\sigma^{\uparrow} \vee \sigma^{\downarrow}, \sigma^{\uparrow} \vee \sigma^{\downarrow})$-continuous, so since $\sigma := \sigma^{\uparrow} \vee \sigma^{\downarrow}$, f is (σ, σ)-continuous. Conversely, if f is (σ, σ)-continuous and $U \subseteq \mathbb{Q}$ is an upper set, then $f^{-1}(U)$ is also an upper set since f is order-preserving. Now by part (2) above, $f^{-1}(U)$ is $\sigma^{\uparrow}$-open; since $U \subseteq \mathbb{Q}$ was arbitrary, it follows that f is $(\sigma^{\uparrow}, \sigma^{\uparrow})$-continuous. The argument for $(\sigma^{\downarrow}, \sigma^{\downarrow})$-continuity is analogous. ∎

When forming, say, the topology $\sigma = \sigma^\uparrow \vee \sigma^\downarrow$ on a poset $\mathbb{P}$, we are creating many new open sets. A priori, it is possible that σ contains new open upper sets which are not $\sigma^\uparrow$-open. The lemma above tells us that in the case of the σ and δ topologies, this does not happen.

Recall that the σ and ι-topologies were defined using order duality: $\iota^\downarrow(\mathbb{P}) := \iota^\uparrow(\mathbb{P}^{op})$. The following lemma states that we could have done the same with the δ topologies.

2.1.18. Lemma. *Let $e\colon \mathbb{L} \to \mathbb{C}$ be a lattice completion. Then the following topologies on $\mathbb{C}$ coincide: $\delta^\uparrow(\mathbb{L}) = \delta^\downarrow(\mathbb{L}^{op})$. Consequently, $\delta(\mathbb{L}) = \delta(\mathbb{L}^{op})$.*

Proof This follows easily from Fact A.5.2. ∎

Of course, the above lemma also holds, by definition, for the Scott topology. At this point we come to a property which the Scott topology notoriously lacks.

2.1.19. Lemma. *Let $e_1\colon \mathbb{L}_1 \to \mathbb{C}_1$ and $e_2\colon \mathbb{L}_2 \to \mathbb{C}_2$ be two lattice completions. Then the following topologies on $\mathbb{C}_1 \times \mathbb{C}_2$ coincide: $\delta^\uparrow(\mathbb{C}_1) \times \delta^\uparrow(\mathbb{C}_2) = \delta^\uparrow(\mathbb{C}_1 \times \mathbb{C}_2)$ and $\delta^\downarrow(\mathbb{C}_1) \times \delta^\downarrow(\mathbb{C}_2) = \delta^\downarrow(\mathbb{C}_1 \times \mathbb{C}_2)$.*

Proof We will show that $\delta^\uparrow(\mathbb{C}_1) \times \delta^\uparrow(\mathbb{C}_2) = \delta^\uparrow(\mathbb{C}_1 \times \mathbb{C}_2)$. Recall that

$$\begin{aligned}\delta^\uparrow(\mathbb{C}_1 \times \mathbb{C}_2) &= \langle\{{\uparrow_{\mathbb{C}_1\times\mathbb{C}_2}}(e_1 \times e_2)^{\mathcal{F}}(F) \mid F \in \mathcal{F}(\mathbb{L}_1 \times \mathbb{L}_2)\}\rangle \\ &= \langle\{{\uparrow_{\mathbb{C}_1\times\mathbb{C}_2}}(e_1^{\mathcal{F}}(F_1), e_2^{\mathcal{F}}(F_2)) \mid (F_1, F_2) \in \mathcal{F}\mathbb{L}_1 \times \mathcal{F}\mathbb{L}_2\}\rangle,\end{aligned}$$

where the last equality follows from fact that $\mathcal{F}$ commutes with products (Fact A.5.4). Moreover, if $(F_1, F_2) \in \mathcal{F}\mathbb{L}_1 \times \mathcal{F}\mathbb{L}_2$ so that $(e_1^{\mathcal{F}}(F_1), e_2^{\mathcal{F}}(F_2)) \in \mathbb{C}_1 \times \mathbb{C}_2$, then again by Fact A.5.4 (applied to $\mathbb{C}_1 \times \mathbb{C}_2$), we see that

$${\uparrow_{\mathbb{C}_1\times\mathbb{C}_2}}(e_1^{\mathcal{F}}(F_1), e_2^{\mathcal{F}}(F_2)) = {\uparrow_{\mathbb{C}_1}}\left(e_1^{\mathcal{F}}(F_1)\right) \times {\uparrow_{\mathbb{C}_2}}\left(e_2^{\mathcal{F}}(F_2)\right).$$

But we know from general topology that

$$\delta^\uparrow(\mathbb{C}_1) \times \delta^\uparrow(\mathbb{C}_2) = \langle\{{\uparrow_{\mathbb{C}_1}}\left(e_1^{\mathcal{F}}(F_1)\right) \times {\uparrow_{\mathbb{C}_2}}\left(e_2^{\mathcal{F}}(F_2)\right) \mid F_1 \in \mathcal{F}\mathbb{L}_1, F_2 \in \mathcal{F}\mathbb{L}_2\}\rangle,$$

so it follows that $\delta^\uparrow(\mathbb{C}_1) \times \delta^\uparrow(\mathbb{C}_2) = \delta^\uparrow(\mathbb{C}_1 \times \mathbb{C}_2)$. ∎

The above lemma is not true of the Scott topology: there exist lattices $\mathbb{L}$, $\mathbb{M}$ such that $\sigma^\uparrow(\mathbb{L} \times \mathbb{M}) \neq \sigma^\uparrow(\mathbb{L}) \times \sigma^\uparrow(\mathbb{M})$ [45, Thm. II-4.11].

2.1.3 Characterizing the canonical extension via the δ-topology

In this subsectoin, we present the two main new results of this section. Firstly, we prove a new, *topological* characterization theorem for the canonical extension. Secondly, we show how the δ-toplogies can be given a natural description as *subspace* topologies with respect to the Scott and co-Scott topology on superstructures of $\mathbb{L}^\delta$.

The interaction between the δ-topologies and the maps $e^{\mathcal{F}}$ and $e^{\mathcal{I}}$

We start with a lemma which tells us that the compactness property of canonical extensions is in fact a topological property.

2.1.20. LEMMA. *Let $e\colon \mathbb{L} \hookrightarrow \mathbb{C}$ be a completion of $\mathbb{L}$. Then the following are equivalent:*

1. *for all $F \in \mathcal{F}\,\mathbb{L}$ and $I \in \mathcal{I}\,\mathbb{L}$, $e^{\mathcal{F}}(F) \leq e^{\mathcal{I}}(I)$ iff $F \between I$.*
2. *$e^{\mathcal{I}}\colon \mathcal{I}\,\mathbb{L} \to \mathbb{C}$ is $(\sigma^{\uparrow}, \delta^{\uparrow})$-continuous and $e^{\mathcal{F}}\colon \mathcal{F}\,\mathbb{L} \to \mathbb{C}$ is $(\sigma^{\downarrow}, \delta^{\downarrow})$-continuous.*

Proof Assume (1) holds. Let $\uparrow e^{\mathcal{F}}(F)$ be a subbasic open set of $\delta^{\uparrow}$ (for some $F \in \mathcal{F}\,\mathbb{L}$). We will show that $U := (e^{\mathcal{I}})^{-1}\big(\uparrow e^{\mathcal{F}}(F)\big)$ is Scott-open in $\mathcal{I}\,\mathbb{L}$. Since $e^{\mathcal{I}}$ is order-preserving, we see that U must be an upper set. Now let $S \subseteq \mathcal{I}\,\mathbb{L}$ be directed; then $\bigvee S = \bigcup S$. If $\bigvee S \in U$, then

$$e^{\mathcal{I}}(\bigvee S) = e^{\mathcal{I}}(\bigcup S) \in \uparrow e^{\mathcal{F}}(F),$$

i.e. $e^{\mathcal{I}}(\bigcup S) \geq e^{\mathcal{F}}(F)$. It follows by (1) that there is $a \in F \cap \bigcup S$, i.e. there is an $I \in S$ such that $a \in F \cap I$. But then $F \between I$, so by (1), it follows that $e^{\mathcal{F}}(F) \leq e^{\mathcal{I}}(I)$, so that $I \in (e^{\mathcal{I}})^{-1}\big(\uparrow e^{\mathcal{F}}(F)\big) = U$. Since S was arbitrary it follows that U is Scott-open. The proof of the statement for $e^{\mathcal{F}}$ is the order dual of the above; it follows that (2) holds.

Conversely, assume that (2) holds and let $F \in \mathcal{F}\,\mathbb{L}$, $I \in \mathcal{I}\,\mathbb{L}$. If $F \between I$ then there is $a \in F \cap I$, so we see that

$$e^{\mathcal{F}}(F) = \bigwedge e[F] \leq e(a) \leq \bigvee e[I] = e^{\mathcal{I}}(I).$$

Now suppose that $e^{\mathcal{F}}(F) \leq e^{\mathcal{I}}(I)$, so that $I \in U := (e^{\mathcal{I}})^{-1}(\uparrow e^{\mathcal{F}}(F))$. By (2), U is Scott-open. Now since $I = \bigvee_{a \in I} \downarrow a$ is a directed join, it follows that there is some $a \in I$ such that $\downarrow a \in U$, i.e. $e^{\mathcal{I}}(\downarrow a) \in \uparrow e^{\mathcal{F}}(F)$. Equivalently, $e^{\mathcal{F}}(F) \in \downarrow e^{\mathcal{I}}(\downarrow a)$, so by an argument analogous to that above, there is some $b \in F$ such that $e^{\mathcal{F}}(\uparrow b) \leq e^{\mathcal{I}}(\downarrow a)$. But now

$$e(b) = e^{\mathcal{F}}(\uparrow b) \leq e^{\mathcal{I}}(\downarrow a) = e(a),$$

so that $b \leq a$. Since $a \in I$ and I is a lower set, we also get $b \in I$, so that $F \between I$. It follows that (1) holds. ∎

The above lemma effectively translates one of the defining properties (compactness) of the canonical extension into a topological property of lattice completions. The following lemmas establish some topological properties of $e^{\mathcal{F}}$ and $e^{\mathcal{I}}$ which will be of use later. First, we show that $e^{\mathcal{I}}\colon \mathcal{I}\,\mathbb{L} \to \mathbb{C}$ is a Scott-continuous lattice homomorphism under certain topological assumptions.

2.1.21. Lemma. *Let $\mathbb{L}$ be a bounded lattice and let $e\colon \mathbb{L} \hookrightarrow \mathbb{C}$ be a completion of $\mathbb{L}$.*

1. *$e^{\mathcal{I}}\colon \mathcal{I}\,\mathbb{L} \to \mathbb{C}$ preserves all joins (including 0), and the top element 1.*
2. *If $e^{\mathcal{I}}\colon \mathcal{I}\,\mathbb{L} \to \mathbb{C}$ is $(\sigma^{\uparrow}, \delta^{\uparrow})$-continuous and $\delta^{\uparrow}$ is T_0, then $e^{\mathcal{I}}\colon \mathcal{I}\,\mathbb{L} \to \mathbb{C}$ preserves finite meets.*

Proof (1). Preservation of 0 and 1 follows from the fact that $e\colon \mathbb{L} \to \mathbb{C}$ is a bounded lattice embedding; if we look at e.g. the top element of $\mathcal{I}\,\mathbb{L}$, i.e. $\downarrow 1$, then $e^{\mathcal{I}}(\downarrow 1) = \bigvee e[\downarrow 1] = e(1) = 1$. The argument for 0 is identical. Now for binary joins, recall that the join of two ideals $I_1, I_2 \in \mathcal{I}\,\mathbb{L}$ is $I_1 \vee I_2 := \downarrow\{a_1 \vee a_2 \mid a_1 \in I_1, a_2 \in I_2\}$. Now

$$e^{\mathcal{I}}(I_1 \vee I_2) = \bigvee e[I_1 \vee I_2] = \bigvee\{e(a_1 \vee a_2) \mid a_1 \in I_1, a_2 \in I_2\} =$$
$$\bigvee\{e(a_1) \vee e(a_2) \mid a_1 \in I_1, a_2 \in I_2\} = \bigvee e[I_1] \vee \bigvee e[I_2] = e^{\mathcal{I}}(I_1) \vee e^{\mathcal{I}}(I_2),$$

where the penultimate equality follows from the fact that I_1 and I_2 are directed. Since $e^{\mathcal{I}}$ is Scott-continuous by Fact A.5.3, it follows that $e^{\mathcal{I}}$ preserves all joins.

(2). We will only consider binary meets. Assume that $\delta^{\uparrow}$ is T_0 and let $I_1, I_2 \in \mathcal{I}\,\mathbb{L}$. Since $e^{\mathcal{I}}$ is order-preserving, we only need to show that $e^{\mathcal{I}}(I_1) \wedge e^{\mathcal{I}}(I_2) \leq e^{\mathcal{I}}(I_1 \cap I_2)$. Suppose not, then since $\delta^{\uparrow}$ is T_0 there must be some $\delta^{\uparrow}$-open U such that $e^{\mathcal{I}}(I_1) \wedge e^{\mathcal{I}}(I_2) \in U$ and $e^{\mathcal{I}}(I_1 \cap I_2) \notin U$. We see that also $e^{\mathcal{I}}(I_1) \in U$ and $e^{\mathcal{I}}(I_2) \in U$; moreover, without loss of generality we may assume $U = \bigcap_{1 \leq i \leq n} \uparrow e^{\mathcal{F}}(F_n)$ for some finite set $\{F_1, \ldots, F_n\} \subseteq \mathcal{F}\,\mathbb{L}$. Since $e^{\mathcal{I}}\colon \mathcal{I}\,\mathbb{L} \to \mathbb{C}$ is $(\sigma^{\uparrow}, \delta^{\uparrow})$-continuous, $(e^{\mathcal{I}})^{-1}(U)$ is Scott-open. Now since $I_1 = \bigvee_{a \in I_1} \downarrow a \in (e^{\mathcal{I}})^{-1}(U)$ is a directed join, there must be some $a_1 \in I_1$ such that $e^{\mathcal{I}}(\downarrow a_1) = e(a_1) \in U$, and similarly there must be some $a_2 \in I_2$ such that $e(a_2) \in U$. Since $U = \bigcap_{1 \leq i \leq n} \uparrow e^{\mathcal{F}}(F_n)$, it follows that for all $1 \leq i \leq n$, we have that $e(a_1), e(a_2) \in \uparrow e^{\mathcal{F}}(F_i)$, so also $e(a_1) \wedge e(a_2) = e(a_1 \wedge a_2) \in \uparrow e^{\mathcal{F}}(F_i)$. Since i was arbitrary, it follows that $e(a_1 \wedge a_2) \in U$. Since we also have that $a_1 \wedge a_2 \in I_1 \cap I_2$, it follows that

$$e^{\mathcal{I}}(I_1 \cap I_2) = \bigvee e[I_1 \cap I_2] \geq e(a_1 \wedge a_2) \in U,$$

which is a contradiction since we assumed U is an upper set not containing $e^{\mathcal{I}}(I_1 \cap I_2)$. It follows that $e^{\mathcal{I}}(I_1 \cap I_2) = e^{\mathcal{I}}(I_1) \wedge e^{\mathcal{I}}(I_2)$. ∎

Next, we show that under the assumptions of Lemma 2.1.21, the $\delta^{\uparrow}$-topology on a completion $e\colon \mathbb{L} \to \mathbb{C}$ has a base of principal lower sets.

2.1.22. Corollary. *Under the assumptions of Lemma 2.1.21, $\{\downarrow e^{\mathcal{I}}(I) \mid I \in \mathcal{I}\,\mathbb{L}\}$ is not only a subbase for $\delta^{\downarrow}$, but in fact a base.*

Proof We will show that $\{\downarrow e^{\mathcal{I}}(I) \mid I \in \mathcal{I}\,\mathbb{L}\}$ is closed under finite intersections. Let $I_1, I_2 \in \mathcal{I}\,\mathbb{L}$, then

$$\downarrow e^{\mathcal{I}}(I_1) \cap \downarrow e^{\mathcal{I}}(I_2) = \downarrow(e^{\mathcal{I}}(I_1) \wedge e^{\mathcal{I}}(I_2)) = \downarrow e^{\mathcal{I}}(I_1 \cap I_2). \quad ∎$$

The δ-topology determines the canonical extension

Now that we have proved the required technical results about the δ-topologies, we can present the first main result of this section: we show how the δ-topologies, which arise naturally on canonical extensions, in fact *characterize* it.

2.1.23. THEOREM. *Let $\mathbb{L}$ be a bounded lattice and let $e\colon \mathbb{L} \hookrightarrow \mathbb{C}$ be a completion of $\mathbb{L}$. Then $e\colon \mathbb{L} \hookrightarrow \mathbb{C}$ is a canonical extension of $\mathbb{L}$ iff*

1. *$e^{\mathcal{I}}\colon \mathcal{I}\,\mathbb{L} \to \mathbb{C}$ is $(\sigma^{\uparrow}, \delta^{\uparrow})$-continuous and $e^{\mathcal{F}}\colon \mathcal{F}\,\mathbb{L} \to \mathbb{C}$ is $(\sigma^{\downarrow}, \delta^{\downarrow})$-continuous,*
2. *$\delta^{\uparrow}$ and $\delta^{\downarrow}$ are both T_0.*

Proof First assume that $e\colon \mathbb{L} \to \mathbb{C}$ is a canonical extension of $\mathbb{L}$. It follows from Lemma 2.1.20 that (1) holds. Moreover, if $x, y \in \mathbb{C}$ and $x \not\leq y$, then there must be some $F \in \mathcal{F}\,\mathbb{L}$ and $I \in \mathcal{I}\,\mathbb{L}$ such that $e^{\mathcal{F}}(F) \leq x$ and $e^{\mathcal{F}}(F) \not\leq y$. It follows that $x \in {\uparrow}\, e^{\mathcal{F}}(F)$ and $y \notin {\uparrow}\, e^{\mathcal{F}}(F)$, so $\delta^{\uparrow}$ is T_0. The proof for $\delta^{\downarrow}$ is analogous; it follows that (2) holds.

Conversely, assume that (1) and (2) hold. It follows from Lemma 2.1.20 that condition (2) of Fact 2.1.9 holds. Moreover, if $x, y \in \mathbb{C}$ and $x \not\leq y$, then since $\delta^{\uparrow}$ is T_0, there must exist some finite $S \subseteq \mathcal{F}\,\mathbb{L}$ such that

$$x \in \textstyle\bigcup_{F \in S} e^{\mathcal{F}}(F) \not\ni y.$$

It follows that there must be some $F \in S$ such that $y \notin {\uparrow}\, e^{\mathcal{F}}(F)$; now we see that $e^{\mathcal{F}}(F) \leq x$ and $e^{\mathcal{F}}(F) \not\leq y$. An analogous argument shows that since $\delta^{\downarrow}$ is T_0, there must be some $I \in \mathcal{I}\,\mathbb{L}$ such that $y \leq e^{\mathcal{I}}(I)$ and $x \not\leq e^{\mathcal{I}}(I)$. Since $x, y\mathbb{C}$ were arbitrary, it follows that (1) of Fact 2.1.9 holds, so $e\colon \mathbb{L} \to \mathbb{C}$ is a canonical extension of $\mathbb{L}$. ∎

Explaining the δ-topology

We now present the second main result of this section, which sheds a different light on the definition of the δ-topology. In particular, we will focus on the $\delta^{\uparrow}$-topology. We will show that there is a natural way to embed $\mathbb{L}^{\delta}$ in $\mathcal{I}\,\mathcal{F}\,\mathbb{L}$, and that the $\delta^{\uparrow}$-topology on $\mathbb{L}^{\delta}$ is simply the Scott topology on $\mathcal{I}\,\mathcal{F}\,\mathbb{L}$, restricted to $\mathbb{L}^{\delta}$.

So how do we embed $\mathbb{L}^{\delta}$ in $\mathcal{I}\,\mathcal{F}\,\mathbb{L}$? The key insight is that since $e^{\mathcal{F}}\colon \mathcal{F}\,\mathbb{L} \to \mathbb{L}^{\delta}$ preserves all finite joins (by Lemma 2.1.24), the map $\mathcal{I}\, e^{\mathcal{F}}\colon \mathcal{I}\,\mathcal{F}\,\mathbb{L} \to \mathcal{I}\,\mathbb{L}^{\delta}$ has a left adjoint, namely $(e^{\mathcal{F}})^{-1}\colon \mathcal{I}\,\mathbb{L}^{\delta} \to \mathcal{I}\,\mathcal{F}\,\mathbb{L}$.

2.1.24. LEMMA. *Let $\mathbb{L}, \mathbb{M}$ be lattices and let $f\colon \mathbb{L} \to \mathbb{M}$ be a map preserving $\vee$ and 0.*

1. *The inverse image function f^{-1} maps ideals of $\mathbb{M}$ to ideals of $\mathbb{L}$;*

2. $\mathcal{I} f \dashv f^{-1}$, *i.e. for all* $I \in \mathcal{I}\,\mathbb{L}$, $J \in \mathcal{I}\,\mathbb{M}$, *we have*

$$\mathcal{I} f(I) \subseteq J \text{ iff } I \subseteq f^{-1}(J);$$

Proof (1). If $J \in \mathcal{I}\,\mathbb{M}$, then $f^{-1}(J)$ is a lower set since f is order-preserving. Moreover, if $a, b \in f^{-1}(J)$, then $f(a), f(b) \in J$, so $f(a) \vee f(b) = f(a \vee b) \in J$, so that $a \vee b \in f^{-1}(J)$. Finally, since $0 = f(0) \in J$, it follows that $0 \in f^{-1}(J)$, so that $f^{-1}(J)$ is non-empty.

(2). If $\mathcal{I} f(I) = {\downarrow} f[I] \subseteq J$, then for every $a \in I$, $f(a) \in J$, i.e. $a \in f^{-1}(J)$, so $\mathcal{I} f(I) \subseteq J$ implies $I \subseteq f^{-1}(J)$. Conversely, if $I \subseteq f^{-1}(J)$, then for every $a \in I$, $f(a) \in J$, i.e. $f[I] \subseteq J$. Since J is a lower set, we also get $\mathcal{I} f(I) = {\downarrow} f[I] \subseteq J$. ∎

We now define $g \colon \mathbb{L}^\delta \to \mathcal{I}\,\mathcal{F}\,\mathbb{L}$ as $g := (e^{\mathcal{F}})^{-1} \circ {\downarrow_{\mathbb{L}^\delta}}$. This map g will be the embedding that shows that $\mathbb{L}^\delta$ is isomorphic to a subposet of $\mathcal{I}\,\mathcal{F}\,\mathbb{L}$.

2.1.25. Theorem. *Let* $e \colon \mathbb{L} \to \mathbb{L}^\delta$ *be a canonical extension.*

1. *The following diagram commutes:*

$$\begin{tikzcd}
\mathcal{I}\,\mathcal{F}\,\mathbb{L} \arrow[r, shift left, "\mathcal{I}\, e^{\mathcal{F}}"] & \mathcal{I}\,\mathbb{L}^\delta \arrow[l, shift left, "(e^{\mathcal{F}})^{-1}"] \\
\mathcal{F}\,\mathbb{L} \arrow[u, "\downarrow_{\mathcal{F}\mathbb{L}}"] \arrow[r, "e^{\mathcal{F}}"'] & \mathbb{L}^\delta \arrow[u, "\downarrow_{\mathbb{L}^\delta}"']
\end{tikzcd}$$

2. *The composite* $g := (e^{\mathcal{F}})^{-1} \circ {\downarrow_{\mathbb{L}^\delta}}$ *is a* $(\delta^\uparrow, \sigma^\uparrow)$*-continuous homeomorphic embedding.*

Proof (1). We need to show that

$$\mathcal{I}\, e^{\mathcal{F}} \circ {\downarrow_{\mathcal{F}\mathbb{L}}} = {\downarrow_{\mathbb{L}^\delta}} \circ e^{\mathcal{F}} \tag{2.2}$$

and that

$$(e^{\mathcal{F}})^{-1} \circ {\downarrow_{\mathbb{L}^\delta}} \circ e^{\mathcal{F}} = {\downarrow_{\mathcal{F}\mathbb{L}}}. \tag{2.3}$$

The validity of (2.2) follows from the fact that $\downarrow$ is a natural transformation by Fact A.5.3(1). To see why (2.3) holds, first observe that since $e^{\mathcal{F}} \colon \mathcal{F}\,\mathbb{L} \to \mathbb{L}^\delta$ preserves all finite joins by Lemma 2.1.21, it follows by Lemma 2.1.24(2) that $(e^{\mathcal{F}})^{-1} \colon \mathcal{I}\,\mathbb{L}^\delta \to \mathcal{I}\,\mathcal{F}\,\mathbb{L}$ is right adjoint to $\mathcal{I}\, e^{\mathcal{F}}$. Since $\mathcal{I}\, e^{\mathcal{F}}$ is an order-embedding by Fact A.5.3(5), it follows by Fact A.3.3(2) that

$$(e^{\mathcal{F}})^{-1} \circ \mathcal{I}\, e^{\mathcal{F}} = \mathrm{id}_{\mathcal{I}\mathcal{F}\mathbb{L}}\,. \tag{2.4}$$

Now we see that

$$\begin{aligned}
(e^{\mathcal{F}})^{-1} \circ {\downarrow_{\mathbb{L}^\delta}} \circ e^{\mathcal{F}} &= (e^{\mathcal{F}})^{-1} \circ \mathcal{I}\, e^{\mathcal{F}} \circ {\downarrow_{\mathcal{F}\mathbb{L}}} && \text{by (2.2)},\\
&= {\downarrow_{\mathcal{F}\mathbb{L}}} && \text{by (2.4)}.
\end{aligned}$$

(2). We now define $g := (e^{\mathcal{F}})^{-1} \circ \downarrow_{\mathbb{L}^\delta}$; observe that

$$g(x) = (e^{\mathcal{F}})^{-1}(\downarrow_{\mathbb{L}^\delta} x) = \{F \in \mathcal{F}\,\mathbb{L} \mid e^{\mathcal{F}}(F) \leq x\}, \tag{2.5}$$

which is an ideal of $\mathcal{F}\,\mathbb{L}$. This map will be the embedding of $\mathbb{L}^\delta$ into $\mathcal{I}\,\mathcal{F}\,\mathbb{L}$. Conversely, there is a natural map from $\mathcal{I}\,\mathcal{F}\,\mathbb{L}$ to $\mathbb{L}^\delta$: since $e^{\mathcal{F}}\colon \mathcal{F}\,\mathbb{L} \to \mathbb{L}^\delta$ is an order preserving map from $\mathcal{F}\,\mathbb{L}$ to a dcpo $\mathbb{L}^\delta$, the universal property of ideal completion tells us that there exists a unique Scott-continuous map $(e^{\mathcal{F}})^{\mathcal{I}}\colon \mathcal{I}\,\mathcal{F}\,\mathbb{L} \to \mathbb{L}^\delta$ such that $(e^{\mathcal{F}})^{\mathcal{I}} \circ \downarrow_{\mathcal{F}\,\mathbb{L}} = e^{\mathcal{F}}$; namely

$$(e^{\mathcal{F}})^{\mathcal{I}}\colon I \mapsto \bigvee e^{\mathcal{F}}[I] = \bigvee \{e^{\mathcal{F}}(F) \mid F \in I\}, \tag{2.6}$$

where $I \in \mathcal{I}\,\mathcal{F}\,\mathbb{L}$.

$$\begin{tikzcd}
 & \mathcal{I}\,\mathcal{F}\,\mathbb{L} \arrow[d, dotted, "(e^{\mathcal{F}})^{\mathcal{I}}"] \\
\mathcal{F}\,\mathbb{L} \arrow[ur, "\downarrow_{\mathcal{F}\,\mathbb{L}}"] \arrow[r, "e^{\mathcal{F}}"'] & \mathbb{L}^\delta
\end{tikzcd}$$

Now observe that $(e^{\mathcal{F}})^{\mathcal{I}} \circ g = \mathrm{id}_{\mathbb{L}^\delta}$: take $x \in \mathbb{L}^\delta$, then

$$\begin{aligned}
&(e^{\mathcal{F}})^{\mathcal{I}} \circ g(x) && \\
&= \bigvee \{e^{\mathcal{F}}(F) \mid F \in g(x)\} && \text{by (2.6),} \\
&= \bigvee \{e^{\mathcal{F}}(F) \mid e^{\mathcal{F}}(F) \leq x\} && \text{by (2.5),} \\
&= x && \text{by join-density of filter elements.}
\end{aligned}$$

Now it is easy to see that g is an order embedding:

$$\begin{aligned}
&g(x) \leq g(y) && \\
&\Rightarrow (e^{\mathcal{F}})^{\mathcal{I}} \circ g(x) \leq (e^{\mathcal{F}})^{\mathcal{I}} \circ g(y) && \text{since } (e^{\mathcal{F}})^{\mathcal{I}} \text{ is order-preserving,} \\
&\Rightarrow x \leq y && \text{since } (e^{\mathcal{F}})^{\mathcal{I}} \circ g = \mathrm{id}_{\mathbb{L}^\delta}.
\end{aligned}$$

We will now show that $g\colon \mathbb{L}^\delta \to \mathcal{I}\,\mathcal{F}\,\mathbb{L}$ is a $(\delta^\uparrow, \sigma^\uparrow)$-continuous homeomorphic embedding, meaning that for every $\delta^\uparrow$-open $U \subseteq \mathbb{L}^\delta$ there exists a $\sigma^\uparrow$-open $U' \subseteq \mathcal{I}\,\mathcal{F}\,\mathbb{L}$ such that $U = g^{-1}(U')$. It suffices to show this for the case that U is a basic open, i.e. for the case that $U = \uparrow_{\mathbb{L}^\delta} e^{\mathcal{F}}(F)$ for an arbitrary $F \in \mathcal{F}\,\mathbb{L}$. Since $F \in \mathcal{F}\,\mathbb{L}$, we know that $\downarrow_{\mathcal{F}\,\mathbb{L}} F$ is a compact element of $\mathcal{I}\,\mathcal{F}\,\mathbb{L}$, so that $\uparrow_{\mathcal{I}\,\mathcal{F}\,\mathbb{L}}(\downarrow_{\mathcal{F}\,\mathbb{L}} F)$ is $\sigma^\uparrow$-open. Now

$$\begin{aligned}
&g^{-1}\left(\uparrow_{\mathcal{I}\,\mathcal{F}\,\mathbb{L}} \downarrow_{\mathcal{F}\,\mathbb{L}} F\right) && \\
&= g^{-1}\left(\uparrow_{\mathcal{I}\,\mathcal{F}\,\mathbb{L}} g \circ e^{\mathcal{F}}(F)\right) && \text{since } g \circ e^{\mathcal{F}} = \downarrow_{\mathcal{F}\,\mathbb{L}}, \\
&= \{y \in \mathbb{L}^\delta \mid g \circ e^{\mathcal{F}}(x) \leq g(y)\} && \text{by def. of } g^{-1} \text{ and } \uparrow, \\
&= \{y \in \mathbb{L}^\delta \mid e^{\mathcal{F}}(F) \leq y\} && \text{because } g \text{ is an ord. emb.,} \\
&= \uparrow_{\mathbb{L}^\delta} e^{\mathcal{F}}(F).
\end{aligned}$$

It follows that g is a homeomorphic embedding. ∎

The fact that $g\colon \mathbb{L}^\delta \to \mathcal{I}\,\mathcal{F}\,\mathbb{L}$ is a $(\delta^\uparrow, \sigma^\uparrow)$-homeomorphic embedding tells us two things. Firstly, that $\mathbb{L}^\delta$ is isomorphic to a subposet of $\mathcal{I}\,\mathcal{F}\,\mathbb{L}$. Secondly, this tells us that the $\delta^\uparrow$-topology is precisely the topology that $\mathbb{L}^\delta$ inherits as a subspace of $\mathcal{I}\,\mathcal{F}\,\mathbb{L}$, where the latter is endowed with the $\sigma^\uparrow$-topology. Naturally, this result dualizes if we view $\mathbb{L}^\delta$ as a subposet of $\mathcal{F}\,\mathcal{I}\,\mathbb{L}$.

2.1.4 Basic properties of the canonical extension

In this section, we will introduce assorted basic properties of canonical extensions which will be of use later on. The first observations concern the interaction with products and the operation of taking the order dual of a lattice. After that, we will prove certain topological and order-theoretical properties of the maps $e^{\mathcal{I}}$ and $e^{\mathcal{F}}$ and of the δ-topologies. In particular, we will see that canonical extensions satisfy two distributive laws with respect to joins of filter elements and meets of ideal elements. We conclude the subsection with a result about the internal structure of the lattices that arise as canonical extensions.

We begin by showing that canonical extensions commute with finite products and order duals of lattices.

2.1.26. LEMMA. *Let $e_1\colon \mathbb{L}_1 \to \mathbb{L}_1^\delta$ and $e_2\colon \mathbb{L}_2 \to \mathbb{L}_2^\delta$ be canonical extensions.*

1. *$e_1 \times e_2\colon \mathbb{L}_1 \times \mathbb{L}_2 \to \mathbb{L}_1^\delta \times \mathbb{L}_2^\delta$ is a canonical extension of $\mathbb{L}_1 \times \mathbb{L}_2$.*
2. *$e_1^{op}\colon \mathbb{L}_1^{op} \to (\mathbb{L}_1^\delta)^{op}$ is a canonical extension of $\mathbb{L}_1^{op}$.*

Proof (1). We will verify that $e_1 \times e_2\colon \mathbb{L}_1 \times \mathbb{L}_2 \to \mathbb{L}_1^\delta \times \mathbb{L}_2^\delta$ satisfies the topological conditions of Theorem 2.1.23. Since the product of two T_0 spaces is again T_0, it follows from Lemma 2.1.19 that both $\delta^\uparrow(\mathbb{L}_1^\delta \times \mathbb{L}_2^\delta)$ and $\delta^\downarrow(\mathbb{L}_1^\delta \times \mathbb{L}_2^\delta)$ are T_0.

Now since $e_i^{\mathcal{I}}\colon \mathcal{I}\,\mathbb{L}_i \to \mathbb{L}_i^\delta$ is $(\sigma^\uparrow, \delta^\uparrow)$-continuous for $i = 1, 2$, it follows from Fact A.5.4, Lemma 2.1.19 and general topology that

$$e_1^{\mathcal{I}} \times e_2^{\mathcal{I}}\colon\ \mathcal{I}\,\mathbb{L}_1 \times \mathcal{I}\,\mathbb{L}_2 \to \mathbb{L}_1^\delta \times \mathbb{L}_2^\delta \text{ is } (\sigma^\uparrow, \delta^\uparrow)\text{-continuous.}$$

Let $h\colon \mathcal{I}(\mathbb{L}_1 \times \mathbb{L}_2) \to \mathcal{I}\,\mathbb{L}_1 \times \mathcal{I}\,\mathbb{L}_2$ be the order-isormorphism witnessing that $\mathcal{I}(\mathbb{L}_1 \times \mathbb{L}_2) \simeq \mathcal{I}\,\mathbb{L}_1 \times \mathcal{I}\,\mathbb{L}_2$; observe that h is $(\sigma^\uparrow, \sigma^\uparrow)$-continuous. Now consider the diagram in Figure 2.2. The upper left triangle commutes by Fact A.5.4, the upper right triangle commutes by the universal property of $\downarrow_{\mathbb{L}_1 \times \mathbb{L}_2}\colon \mathbb{L}_1 \times \mathbb{L}_2 \to \mathcal{I}(\mathbb{L}_1 \times \mathbb{L}_2)$ (Fact A.5.3), and the lower triangle commutes by definition of $(e_1 \times e_2)^{\mathcal{I}}$. Now since $(e_1^{\mathcal{I}} \times e_2^{\mathcal{I}}) \circ h$ is $(\sigma^\uparrow, \delta^\uparrow)$-continuous, so is $(e_1 \times e_2)^{\mathcal{I}}$, which is what we needed to show. The argument showing that $(e_1 \times e_2)^{\mathcal{F}}\colon\ \mathcal{F}(\mathbb{L}_1 \times \mathbb{L}_2) \to \mathbb{L}_1^\delta \times \mathbb{L}_2^\delta$ is $(\sigma^\downarrow, \delta^\downarrow)$-continuous is identical; it follows by Theorem 2.1.23 that $e_1 \times e_2\colon \mathbb{L}_1 \times \mathbb{L}_2 \to \mathbb{L}_1^\delta \times \mathbb{L}_2^\delta$ is a canonical extension of $\mathbb{L}_1 \times \mathbb{L}_2$.

(2). This follows readily from the order duality between filters and ideals and Lemma 2.1.18. ∎

Figure 2.2: Finite products of canonical extensions

$$\begin{tikzcd}
 & \mathcal{I}\,\mathbb{L}_1 \times \mathcal{I}\,\mathbb{L}_2 \arrow[ddr, "e_1^{\mathcal{I}} \times e_2^{\mathcal{I}}"] & \\
 & \mathcal{I}(\mathbb{L}_1 \times \mathbb{L}_2) \arrow[u, "h"'] \arrow[dr, "(e_1 \times e_2)^{\mathcal{I}}"'] & \\
\mathbb{L}_1 \times \mathbb{L}_2 \arrow[uur, "\downarrow_{\mathbb{L}_1} \times \downarrow_{\mathbb{L}_2}"] \arrow[ur, "\downarrow_{\mathbb{L}_1 \times \mathbb{L}_2}"'] \arrow[rr, "e_1 \times e_2"'] & & \mathbb{L}_1^\delta \times \mathbb{L}_2^\delta
\end{tikzcd}$$

The fact that canonical extension commutes with taking finite products is in fact an instance of the more powerful result that canonical extensions commute with Boolean products (see §A.6). An alternative reference for the following result is [43, Theorem 5.1].

2.1.27. Fact ([34]). *Let $\mathbb{L}$ be a lattice and let $(p_x \colon \mathbb{L} \to \mathbb{M}_x)_{x \in X}$ be a Boolean product decomposition of $\mathbb{L}$. Then $\mathbb{L}^\delta \cong \prod_X \mathbb{M}_x^\delta$.*

Parts (1) and (2) of the following lemma can be found in [41]. Part (3) however, which we will use very frequently, is new. In light of Theorem 2.1.25, part (3) below is perhaps not that surprising.

2.1.28. Lemma. *Let $e \colon \mathbb{L} \to \mathbb{L}^\delta$ be the canonical extension of a bounded lattice $\mathbb{L}$.*

1. *(a) $e^{\mathcal{F}} \colon \mathcal{F}\,\mathbb{L} \to \mathbb{L}^\delta$ is an $\vee, \bigwedge$-embedding;*
   (b) $e^{\mathcal{I}} \colon \mathcal{I}\,\mathbb{L} \to \mathbb{L}$ is an $\wedge, \bigvee$-embedding;
2. *for all $x \in \mathbb{L}^\delta$,*
   (a) $\{e^{\mathcal{F}}(F) \mid e^{\mathcal{F}}(F) \leq x\}$ is directed;
   (b) $\{e^{\mathcal{I}}(I) \mid x \leq e^{\mathcal{I}}(I)\}$ is co-directed;
3. *(a) $\sigma^{\uparrow}(\mathbb{L}^\delta) \subseteq \delta^{\uparrow}(\mathbb{L}^\delta)$;*
   (b) $\sigma^{\downarrow}(\mathbb{L}^\delta) \subseteq \delta^{\downarrow}(\mathbb{L}^\delta)$;
   (c) $\sigma(\mathbb{L}^\delta) \subseteq \delta(\mathbb{L}^\delta)$;
4. *$e[L]$ is dense in $\langle L^\delta, \delta(\mathbb{L}^\delta)\rangle$ and in $\langle L^\delta, \sigma(\mathbb{L}^\delta)\rangle$.*

Proof (1). We will only prove part (a). It follows from (the order dual of) Lemma 2.1.21 that $e^{\mathcal{F}}$ is a $\vee, \bigwedge$-homomorphism. To show that it is an embedding, we will show that $e^{\mathcal{F}}$ is order-reflecting. Assume that $F \not\supseteq F'$; we will show that $e^{\mathcal{F}}(F) \not\leq e^{\mathcal{F}}(F')$. By assumption, there exists $a \in F' \setminus F$, so that $F \cap {\downarrow} a = \emptyset$

and $F' \between {\downarrow} a$. It follows by the compactness property of canonical extension that $e^{\mathcal{F}}(F) \nleq e^{\mathcal{I}}({\downarrow} a)$ and $e^{\mathcal{F}}(F') \leq e^{\mathcal{I}}({\downarrow} a)$. Now by the meet-density of ideal elements, it follows that $e^{\mathcal{F}}(F) \nleq e^{\mathcal{F}}(F')$.

(2). We only show part (a). This follows from (1), since ${\downarrow} x$ is an ideal. $\{e^{\mathcal{F}}(F) \mid e^{\mathcal{F}}(F) \leq x\}$

(3). Suppose $U \subseteq \mathbb{L}^{\delta}$ is Scott-open and that $x \in U$. Then $x = \bigvee\{e^{\mathcal{F}}(F) \mid e^{\mathcal{F}}(F) \leq x\}$ is a directed join by (1), so there must be some F such that $e^{\mathcal{F}}(F) \leq x$ and $e^{\mathcal{F}}(F) \in U$. It follows that $x \in {\uparrow} e^{\mathcal{F}}(F) \subseteq U$; hence U is $\delta^{\uparrow}$-open.

(4). Consider a non-empty basic open U of $\langle L^{\delta}, \delta(\mathbb{L}^{\delta})\rangle$, i.e. there are $F \in \mathcal{F}\,\mathbb{L}$, $I \in \mathcal{I}\,\mathbb{L}$ such that $U = \{x \in L^{\delta} \mid e^{\mathcal{F}}(F) \leq x \leq e^{\mathcal{I}}(I)\}$. Since $U \neq \emptyset$, it must be that $e^{\mathcal{F}}(F) \leq e^{\mathcal{I}}(I)$, so that $F \between I$. Take $a \in F \cap I$, then $\bigwedge e[F] \leq e(a) \leq \bigvee e[I]$ so that $e(a) \in U$. It follows that $e[L]$ is dense in $\langle L^{\delta}, \delta(\mathbb{L}^{\delta})\rangle$. Since $\sigma \subseteq \delta$ by (3), it is also the case that $e[L]$ is dense in $\langle L^{\delta}, \sigma(\mathbb{L}^{\delta})\rangle$. ∎

The distributive law below, which is similar to Lemma 3.2 of [34], is a very powerful result. We will in fact use it to characterize the canonical extension as a dcpo in §2.3.

2.1.29. Lemma. *Let $e\colon \mathbb{L} \to \mathbb{L}^{\delta}$ be the canonical extension of a lattice $\mathbb{L}$. Then for all $S \subseteq \mathcal{F}\,\mathbb{L}$ and $S' \subseteq \mathcal{I}\,\mathbb{L}$,*

$$\bigvee\{e^{\mathcal{F}}(F) \mid F \in S\} = \bigwedge\{e^{\mathcal{I}}(I) \mid \forall F \in S,\ F \between I\}$$

and

$$\bigwedge\{e^{\mathcal{I}}(I) \mid I \in S'\} = \bigvee\{e^{\mathcal{F}}(F) \mid \forall I \in S',\ F \between I\}.$$

Proof We only prove the first statement. Since the ideal elements of $\mathbb{L}^{\delta}$ are meet-dense, we see that

$$\bigvee\{e^{\mathcal{F}}(F) \mid F \in S\} = \bigwedge\{e^{\mathcal{I}}(I) \mid \textstyle\bigvee_{F \in S} e^{\mathcal{F}}(F) \leq e^{\mathcal{I}}(I)\}.$$

Now observe that $\bigvee_{F \in S} e^{\mathcal{F}}(F) \leq e^{\mathcal{I}}(I)$ iff for all $F \in S$, $e^{\mathcal{F}}(F) \leq e^{\mathcal{I}}(I)$ iff for all $F \in S$, $F \between I$. The (first) statement of the lemma follows. ∎

We conclude this subsection with a well-known result from the canonical extension literature, which plays an important role in the duality theory for canonical extensions. Let $\mathbb{L}$ be a complete lattice. Recall that an element $p \in \mathbb{L}$ is called *completely join-irreducible* if for all $S \subseteq \mathbb{L}$ such that $p = \bigvee S$, there exists $a \in S$ such that $p = a$. We denote the set of completely join-irreducible elements of $\mathbb{L}$ by $\mathrm{J}^{\infty}(\mathbb{L})$. Completely meet-irreducible elements are defined dually; we denote them by $\mathrm{M}^{\infty}(\mathbb{L})$. The following fact, which is related to Stone duality, requires the Axiom of Choice.

2.1.30. Fact ([34], Lemma 3.4). *Let $\mathbb{L}$ be a lattice. Then $\mathbb{L}^{\delta}$ is join-generated by $\mathrm{J}^{\infty}(\mathbb{L}^{\delta})$ and meet-generated by $\mathrm{M}^{\infty}(\mathbb{L}^{\delta})$.*

2.1.5 Conclusions and further work

This section served two purposes: to make the reader familiar with the classical definition and well-known properties of the canonical extension, and to add a number of fundamental *topological* results on canonical extensions. The work of Gehrke and Jónsson [39] has shown that the topological perspective on canonical extensions is worthwile, however the basic topological properties of canonical extensions in the general setting of (not necessarily distributive) lattices have not previously been studied. One fundamental difference between the distributive and the non-distributive settings is that the Scott topology and the topology generated by principal up-sets of completely join irreducibles no longer coincide.

All the results in §2.1.1 are known from the work of Gehrke and Harding [34], save perhaps the small technical results concerning the overlap relation. It should be noted however that the emphasis on the auxiliary maps $e^{\mathcal{F}}\colon \mathcal{F}\mathbb{L} \to \mathbb{L}^\delta$ and $e^{\mathcal{I}}\colon \mathcal{I}\mathbb{L} \to \mathbb{L}^\delta$ is a departure from the view on canonical extensions furthered in [34, 39].

In §2.1.3 we presented results which were first reported at TACL 2009 in Amsterdam. Both the topological characterization theorem (Theorem 2.1.23) and the result which casts the δ-topologies as subspace topologies (Theorem 2.1.25) were previously unknown. The topological characterization theorem was inspired by the work of Theunissen and Venema [86] on MacNeille completions of lattice-based algebras.

Further work

- It would be very interesting to see if Fact 2.1.27, which deals with canonical extensions of Boolean products of lattices, can be given a new proof reducing it to a statement about the ideal completion and filter completion of lattices.

- It would also be interesting to see if there is a version of the topological characterization theorem (Theorem 2.1.23) which does not rely on $e^{\mathcal{F}}$ and $e^{\mathcal{I}}$, while still giving a topological characterization of $\mathbb{L}^\delta$.

- An alternative approach to canonical extensions using filters and ideals has been developed by Gehrke, Jansana & Palmigiano [37], using *logical filters* rather than order filters. It would be interesting to see if their approach also admits a topological characterization.

2.2 Canonical extensions of maps I: order-preserving maps

In the previous section, we have studied the canonical extension as a construction on lattices to some length. We will now turn to the subject of canonical extensions of maps $f\colon \mathbb{L} \to \mathbb{M}$ between lattices, which is probably an even richer subject. In this dissertation, we will consider two approaches to obtaining a canonical extension of a map $f\colon \mathbb{L} \to \mathbb{M}$ between lattices.

- The first approach is to assume that f is order-preserving, so that we may first extend $f\colon \mathbb{L} \to \mathcal{F}\,\mathbb{M}$ to maps $\mathcal{F}\,f\colon \mathcal{F}\,\mathbb{L} \to \mathbb{M}$ and $\mathcal{I}\,f\colon \mathcal{I}\,\mathbb{L} \to \mathcal{I}\,\mathbb{M}$ acting on filters and ideals, respectively, and then work from there. This is the approach we will take in the current section.

- The second approach is to assume that $\mathbb{M}^\delta$ has nice topological properties, in which case we can develop a substantial part of the basic theory of extensions of maps without making any assumptions about the map $f\colon \mathbb{L} \to \mathbb{M}$. We will pursue this approach in §3.2.

It is an interesting open question whether the two approaches above form a *dichotomy* of some sort. We will return to this question in Remark 3.2.22.

In §2.2.1, we will first give the basic definition of the lower ($f^\triangledown$) and the upper ($f^\vartriangle$) canonical extension of an order-preserving map $f\colon \mathbb{L} \to \mathbb{M}$, and we will characterize these extensions as a largest and smallest continuous extension of f to a map $\mathbb{L}^\delta \to \mathbb{M}^\delta$, respectively (Theorem 2.2.4). We will then show in §2.2.2 that if we assume that f preserves joins (or dually, meets), even in only one coordinate, then this vastly improves the behaviour of $f^\triangledown$ and $f^\vartriangle$ (Theorem 2.2.18). Finally in §2.2.3 we will put this good behaviour to use by showing that canonical extensions of lattice homomorphisms are particularly well-behaved, so well in fact that canonical extension is a functor on the category of lattices and lattice homomorphisms (Theorem 2.2.24). All along, we will see that almost every result we prove about $f^\triangledown$ and $f^\vartriangle$ reduces to statements about filters and ideals.

2.2.1 The lower and upper extensions of an order-preserving map

In this subsection we will discuss the two canonical [34] ways to extend an order-preserving map $f\colon \mathbb{L} \to \mathbb{M}$ to a map $f'\colon \mathbb{L}^\delta \to \mathbb{M}^\delta$, namely the lower and the upper canonical extension. We will then prove some of the basic facts that hold true for any order-preserving map. We conclude the subsection with a topological characterization theorem, which tells us that $f^\triangledown$ and $f^\vartriangle$ can be seen as the largest and smallest continuous extension of f, respectively.

Consider the following diagram, where $e_{\mathbb{L}}\colon \mathbb{L} \to \mathbb{L}^\delta$ and $e_{\mathbb{M}}\colon \mathbb{M} \to \mathbb{M}^\delta$ are the canonical extensions of $\mathbb{L}$ and $\mathbb{M}$, respectively:

$$\begin{tikzcd}
\mathbb{L} \arrow[r, "\uparrow_{\mathbb{L}}"] \arrow[d, "f"'] & \mathcal{F}\,\mathbb{L} \arrow[r, "e^{\mathcal{F}}_{\mathbb{L}}"] \arrow[d, "\mathcal{F}\,f"] & \mathbb{L}^\delta \arrow[d, dashed, "?"] \\
\mathbb{M} \arrow[r, "\uparrow_{\mathbb{M}}"'] & \mathcal{F}\,\mathbb{M} \arrow[r, "e^{\mathcal{F}}_{\mathbb{M}}"'] & \mathbb{M}^\delta
\end{tikzcd}$$

(Observe that we factor $e_{\mathbb{L}}\colon \mathbb{L} \to \mathbb{L}^\delta$ as $e(a) = e^{\mathcal{F}}_{\mathbb{L}}(\uparrow a)$.) We would like to find a function that can take the place of the '?' in the diagram. To do this, we use the fact that any $x \in \mathbb{L}^\delta$ is approximated from below by $S_x := \{F \in \mathcal{F}\,\mathbb{L} \mid e^{\mathcal{F}}_{\mathbb{L}}(F) \leq x\}$. Now each $F \in S_x$ can be mapped into $\mathbb{L}^\delta$ via the assignment $F \mapsto e^{\mathcal{F}}_{\mathbb{M}} \circ \mathcal{F}\, f(F)$.

2.2.1. Definition. Let $f\colon \mathbb{L} \to \mathbb{M}$ be an order-preserving map between lattices. Then we define $f^\triangledown\colon \mathbb{L}^\delta \to \mathbb{M}^\delta$, the *lower extension of* f, as follows:

$$f^\triangledown\colon x \mapsto \bigvee \big\{e^{\mathcal{F}}_{\mathbb{M}} \circ \mathcal{F}\, f(F) \mid e^{\mathcal{F}}_{\mathbb{L}}(F) \leq x\big\}.$$

Dually, we define $f^\vartriangle\colon \mathbb{L}^\delta \to \mathbb{M}^\delta$, the *upper extension of* f, as follows:

$$f^\vartriangle\colon x \mapsto \bigwedge \big\{e^{\mathcal{I}}_{\mathbb{M}} \circ \mathcal{I}\, f(I) \mid x \leq e^{\mathcal{I}}_{\mathbb{L}}(I)\big\}.$$

2.2.2. Remark. There is a slight discrepancy between our definition of $f^\triangledown$ and $f^\vartriangle$ when we compare it with the working definition found in e.g. [34, Lemma 4.3]. Using our notation, the working definition of [34] amounts to

$$f^\triangledown\colon x \mapsto \bigvee \big\{\textstyle\bigwedge e_{\mathbb{M}} \circ f[F] \mid e^{\mathcal{F}}_{\mathbb{L}}(F) \leq x\big\}.$$

The difference is that we use $\mathcal{F}\, f(F)$ rather than $f[F]$; however this difference is inconsequential. If $F \in \mathcal{F}\,\mathbb{L}$, then $\mathcal{F}\, f(F) = {\uparrow} f[F] \supseteq f[F]$, so

$$\begin{aligned} e^{\mathcal{F}}_{\mathbb{M}}(\mathcal{F}\, f(F)) &= \textstyle\bigwedge e_{\mathbb{M}}\big[\,\mathcal{F}\, f(F)\big] && \text{by def. of } e^{\mathcal{F}}_{\mathbb{M}}, \\ &\leq \textstyle\bigwedge e_{\mathbb{M}} \circ f[F] && \text{since } \mathcal{F}\, f(F) \supseteq f[F]. \end{aligned}$$

Conversely, since $e_{\mathbb{M}}$ is order-preserving, we know by Fact A.3.1 that for all $U \subseteq \mathbb{M}$, ${\uparrow} e_{\mathbb{M}}[U] \supseteq e_{\mathbb{M}}[{\uparrow} U]$. Now we see that

$$\begin{aligned} \textstyle\bigwedge e_{\mathbb{M}} \circ f[F] &= \textstyle\bigwedge {\uparrow} e_{\mathbb{M}} \circ f[F] && \text{by order theory,} \\ &\leq \textstyle\bigwedge e_{\mathbb{M}}\big[{\uparrow} f[F]\big] && \text{since } {\uparrow} e_{\mathbb{M}} \circ f[F] \supseteq e_{\mathbb{M}}\big[{\uparrow} f[F]\big]. \end{aligned}$$

In the following lemma we see that $f^\triangledown$ and $f^\vartriangle$ mingle well with the auxilliary maps induced by $e_{\mathbb{L}}\colon \mathbb{L} \to \mathbb{L}^\delta$ and $e_{\mathbb{M}}\colon \mathbb{M} \to \mathbb{M}^\delta$, or alternatively, that $f^\triangledown$ and $f^\vartriangle$ behave well on filter and ideal elements.

2.2.3. Lemma ([34]). *Let $f\colon \mathbb{L} \to \mathbb{M}$ be order-preserving. Then $f^\triangledown\colon \mathbb{L}^\delta \to \mathbb{M}^\delta$ and $f^\vartriangle\colon \mathbb{L}^\delta \to \mathbb{M}^\delta$ are order-preserving maps, which satisfy the following additional properties:*

1. $f^\triangledown \circ e^{\mathcal{F}}_{\mathbb{L}} = e^{\mathcal{F}}_{\mathbb{M}} \circ \mathcal{F}\, f$;
2. $f^\vartriangle \circ e^{\mathcal{I}}_{\mathbb{L}} = e^{\mathcal{I}}_{\mathbb{M}} \circ \mathcal{I}\, f$;

3. $f^\triangledown \leq f^\vartriangle$*;*

4. $f^\triangledown \circ e^{\mathcal{I}}_{\mathbb{L}} = f^\vartriangle \circ e^{\mathcal{I}}_{\mathbb{L}}$*;*

5. $f^\vartriangle \circ e^{\mathcal{F}}_{\mathbb{L}} = f^\triangledown \circ e^{\mathcal{F}}_{\mathbb{L}}$*.*

Proof It is easy to see why $f^\triangledown$ and $f^\vartriangle$ are order-preserving. Take $x, y \in \mathbb{L}^\delta$ such that $x \leq y$ and consider $f^\triangledown$. Then

$$\{F \in \mathcal{F}\,\mathbb{L} \mid e^{\mathcal{F}}_{\mathbb{L}}(F) \leq x\} \subseteq \{F \in \mathcal{F}\,\mathbb{L} \mid e^{\mathcal{F}}_{\mathbb{L}}(F) \leq y\},$$

so also

$$\bigvee\{e^{\mathcal{F}}_{\mathbb{M}} \circ \mathcal{F}\,f(F) \in \mathcal{F}\,\mathbb{L} \mid e^{\mathcal{F}}_{\mathbb{L}}(F) \leq x\} \leq \bigvee\{e^{\mathcal{F}}_{\mathbb{M}} \circ \mathcal{F}\,f(F) \in \mathcal{F}\,\mathbb{L} \mid e^{\mathcal{F}}_{\mathbb{L}}(F) \leq y\},$$

i.e. $f^\triangledown(x) \leq f^\triangledown(y)$. Below, we will only prove statements (1), (3) and (4), since (2) and (5) are order duals of (1) and (4).

(1). Let $F \in \mathcal{F}\,\mathbb{L}$; the set $\{F' \in \mathcal{F}\,\mathbb{L} \mid e^{\mathcal{F}}_{\mathbb{L}}(F') \leq e^{\mathcal{F}}_{\mathbb{L}}(F)\}$ has a maximal element, viz. F. It follows that

$$\begin{aligned} & f^\triangledown\big(e^{\mathcal{F}}_{\mathbb{L}}(F)\big) \\ &= \bigvee\{e^{\mathcal{F}}_{\mathbb{M}} \circ \mathcal{F}\,f(F') \mid e^{\mathcal{F}}_{\mathbb{L}}(F') \leq e^{\mathcal{F}}_{\mathbb{L}}(F)\} && \text{by definition of } f^\triangledown, \\ &= e^{\mathcal{F}}_{\mathbb{M}} \circ \mathcal{F}\,f(F) && \text{since } e^{\mathcal{F}}_{\mathbb{M}} \circ \mathcal{F}\,f \text{ is order-preserving.} \end{aligned}$$

(3). Let $x \in \mathbb{L}^\delta$. Recall that $f^\triangledown(x) = \bigvee\{e^{\mathcal{F}}_{\mathbb{M}} \circ \mathcal{F}\,f(F) \mid e^{\mathcal{F}}_{\mathbb{L}}(F) \leq x\}$ and $f^\vartriangle(x) = \bigwedge\{e^{\mathcal{I}}_{\mathbb{M}} \circ \mathcal{I}\,f(I) \mid x \leq e^{\mathcal{I}}_{\mathbb{L}}(I)\}$; we now have to show that

$$\bigvee\{e^{\mathcal{F}}_{\mathbb{M}} \circ \mathcal{F}\,f(F) \mid e^{\mathcal{F}}_{\mathbb{L}}(F) \leq x\} \leq \bigwedge\{e^{\mathcal{I}}_{\mathbb{M}} \circ \mathcal{I}\,f(I) \mid x \leq e^{\mathcal{I}}_{\mathbb{L}}(I)\}. \tag{2.7}$$

Take $F \in \mathcal{F}\,\mathbb{L}$, $I \in \mathcal{I}\,\mathbb{L}$ such that $e^{\mathcal{F}}_{\mathbb{L}}(F) \leq x \leq e^{\mathcal{I}}(I)$, then it follows by compactness that $F \between I$. Now by Lemma 2.1.2, we get that $\mathcal{F}\,f(F) \between \mathcal{I}\,f(I)$, so that also $e^{\mathcal{F}}_{\mathbb{M}} \circ \mathcal{F}\,f(F) \leq e^{\mathcal{I}}_{\mathbb{M}} \circ \mathcal{I}\,f(I)$. It follows that (2.7) holds.

(4). Let $I \in \mathcal{I}\,\mathbb{L}$. We claim that

$$\forall J \in \mathcal{I}\,\mathbb{M}, \big[\forall F \in \mathcal{F}\,\mathbb{L},\, F \between I \Rightarrow \mathcal{F}\,f(F) \between J\big] \text{ iff } \mathcal{I}\,f(I) \subseteq J. \tag{2.8}$$

Take $J \in \mathcal{I}\,\mathbb{M}$ and suppose that the left-hand side of (2.8) holds. Take $a \in I$, then $\uparrow a \between I$, so by our assumption regarding J, $\uparrow f(a) \between J$, i.e. $f(a) \in J$. It follows that $f[I] \subseteq J$ and thus $\mathcal{I}\,f(I) = \downarrow f[I] \subseteq J$. Conversely, suppose that $\mathcal{I}\,f(I) \subseteq J$ and that $F \in \mathcal{F}\,\mathbb{L}$ such that $F \between I$. Then there is some $a \in F \cap I$. Since $a \in F$, we also get $f(a) \in f[F] \subseteq \uparrow f[F] = \mathcal{F}\,f(F)$. Since $a \in I$, we get that $f(a) \in f[I] \subseteq \downarrow f[I] = \mathcal{I}\,f(I)$. Since we assumed that $\mathcal{I}\,f(I) \subseteq J$, it follows that $f(a) \in J$. But then $\mathcal{F}\,f(F) \between J$. It follows that (2.8) holds.

We can now see that

$$\begin{aligned}
f^{\triangledown}\big(e^{\mathcal{I}}_{\mathbb{L}}(I)\big) &= \bigvee\{e^{\mathcal{F}}_{\mathbb{M}} \circ \mathcal{F}\,f(F) \mid e^{\mathcal{F}}_{\mathbb{L}}(F) \leq e^{\mathcal{I}}_{\mathbb{L}}(I)\} && \text{by def. of } f^{\triangledown}, \\
&= \bigvee\{e^{\mathcal{F}}_{\mathbb{M}} \circ \mathcal{F}\,f(F) \mid F \between I\} && \text{by compactness,} \\
&= \bigwedge\{e^{\mathcal{I}}_{\mathbb{M}}(J) \mid \forall F\big[F \between I \Rightarrow \mathcal{F}\,f(F) \between J\big]\} && \text{by Lemma 2.1.29,} \\
&= \bigwedge\{e^{\mathcal{I}}_{\mathbb{M}}(J) \mid \mathcal{I}\,f(I) \subseteq J\} && \text{by (2.8),} \\
&= e^{\mathcal{I}}_{\mathbb{M}} \circ \mathcal{I}\,f(I) && \text{by order theory,} \\
&= f^{\vartriangle}\big(e^{\mathcal{I}}_{\mathbb{L}}(I)\big) && \text{by (2).} \quad \blacksquare
\end{aligned}$$

It is known that the lower extension of an arbitrary map $f\colon \mathbb{L} \to \mathbb{M}$ between *distributive* lattices $\mathbb{L}$, $\mathbb{M}$ can be characterized as the largest continuous extension of f [39, Theorem 2.15]. The result below, which is new, tells us that we can say the same about maps between non-distributive lattices, *if* we assume that f is order-preserving. We will return to this issue in §3.2.2.

2.2.4. Theorem. *Let $f\colon \mathbb{L} \to \mathbb{M}$ be an order-preserving map between lattices.*

1. *(a) The map $f^{\triangledown}\colon \mathbb{L}^{\delta} \to \mathbb{M}^{\delta}$ is $(\delta^{\uparrow}, \sigma^{\uparrow})$-continuous and $f^{\triangledown} \circ e_{\mathbb{L}} = e_{\mathbb{M}} \circ f$.*
   (b) The map $f^{\vartriangle}\colon \mathbb{L}^{\delta} \to \mathbb{M}^{\delta}$ is $(\delta^{\downarrow}, \sigma^{\downarrow})$-continuous and $f^{\vartriangle} \circ e_{\mathbb{L}} = e_{\mathbb{M}} \circ f$.
2. *Let $f'\colon \mathbb{L}^{\delta} \to \mathbb{M}^{\delta}$ be an order-preserving extension of $f\colon \mathbb{L} \to \mathbb{M}$, i.e. assume that $f' \circ e_{\mathbb{L}} = e_{\mathbb{M}} \circ f$.*
   (a) If f' is $(\delta^{\uparrow}, \iota^{\uparrow})$-continuous, then $f' \leq f^{\triangledown}$.
   (b) If f' is $(\delta^{\downarrow}, \iota^{\downarrow})$-continuous, then $f^{\vartriangle} \leq f'$.

Proof We will only prove the statements concerning $f^{\triangledown}$, since the proofs for those concerning $f^{\vartriangle}$ are identical modulo order duality.

(1). Let $x \in \mathbb{L}^{\delta}$; we will first show that $f^{\triangledown}$ is locally $(\delta^{\uparrow}, \sigma^{\uparrow})$-continuous at x. Suppose that $f^{\triangledown}(x) \in U$, where $U \subseteq \mathbb{M}^{\delta}$ is a $\sigma^{\uparrow}$-open set. Recall that

$$f^{\triangledown}(x) = \bigvee\{e^{\mathcal{F}}_{\mathbb{M}} \circ \mathcal{F}\,f(F) \mid e^{\mathcal{F}}_{\mathbb{L}}(F) \leq x\},$$

and observe that this join is directed by Lemma 2.1.28(1). Since $f^{\triangledown}(x) \in U$, it follows by definition of the $\sigma^{\uparrow}$-topology that some element of the join above lies in U, i.e. that there must be some $F \in \mathcal{F}\,\mathbb{L}$ such that $e^{\mathcal{F}}_{\mathbb{L}}(F) \leq x$ and $e^{\mathcal{F}}_{\mathbb{M}} \circ \mathcal{F}\,f(F) \in U$. Now $\uparrow e^{\mathcal{F}}_{\mathbb{L}}(F)$ is a $\delta^{\uparrow}$-open neighborhood of x; it remains to be shown that $f^{\triangledown}[\uparrow e^{\mathcal{F}}_{\mathbb{L}}(F)] \subseteq U$. But this is easy to see: since $f^{\triangledown}$ is order-preserving, it follows by Fact A.3.1 that $f^{\triangledown}[\uparrow e^{\mathcal{F}}_{\mathbb{L}}(F)] \subseteq \uparrow f^{\triangledown}(e^{\mathcal{F}}_{\mathbb{L}}(F))$. Now by Lemma 2.2.3(1), $f^{\triangledown}(e^{\mathcal{F}}_{\mathbb{L}}(F)) = e^{\mathcal{F}}_{\mathbb{M}}(F) \circ \mathcal{F}\,f(F)$; since we assumed that $e^{\mathcal{F}}_{\mathbb{M}}(F) \circ \mathcal{F}\,f(F) \in U$ and we know that U is an upper set, it follows that $\uparrow f^{\triangledown}(e^{\mathcal{F}}_{\mathbb{L}}(F)) \subseteq U$. Since U was arbitrary, it follows that $f^{\triangledown}$ is locally $(\delta^{\uparrow}, \sigma^{\uparrow})$-continuous at x; since $x \in \mathbb{L}^{\delta}$ was arbitrary, it follows that $f^{\triangledown}$ is $(\delta^{\uparrow}, \sigma^{\uparrow})$-continuous.

To see that $f^\triangledown \circ e_{\mathbb{L}} = e_{\mathbb{M}} \circ f$, observe that the right square below commutes by Lemma 2.2.3(2) and the left square commutes by the universal property of $\mathcal{F}$.

$$\begin{tikzcd}
\mathbb{L} \arrow[r, "\uparrow_{\mathbb{L}}"] \arrow[d, "f"'] & \mathcal{F}\,\mathbb{L} \arrow[r, "e^{\mathcal{F}}_{\mathbb{L}}"] \arrow[d, "\mathcal{F}\,f"] & \mathbb{L}^\delta \arrow[d, "f^\triangledown"] \\
\mathbb{M} \arrow[r, "\uparrow_{\mathbb{M}}"'] & \mathcal{F}\,\mathbb{M} \arrow[r, "e^{\mathcal{F}}_{\mathbb{M}}"'] & \mathbb{M}^\delta
\end{tikzcd}$$

(2). Let $f' : \mathbb{L}^\delta \to \mathbb{M}^\delta$ be an order-preserving extension of $f : \mathbb{L} \to \mathbb{M}$. We will show something stronger than statement (2)(a): we will show that for all $x \in \mathbb{L}^\delta$, if f' is locally $(\delta^\uparrow, \iota^\uparrow)$-continuous at x, then $f'(x) \leq f^\triangledown(x)$. Suppose towards a contradiction that f' is locally continuous but that we have $x \in \mathbb{L}^\delta$ such that $f'(x) \not\leq f^\triangledown(x)$. Now since $\downarrow f^\triangledown(x)$ is $\iota^\uparrow$-closed, by local continuity there must be some $\delta^\uparrow$-open set $U \subseteq \mathbb{L}^\delta$ such that $x \in U$ and $f'[U] \subseteq \mathbb{M}^\delta \setminus \downarrow f^\triangledown(x)$. We may assume that U is a basic open set, i.e. that $U = \uparrow e^{\mathcal{F}}_{\mathbb{L}}(F)$ for some $F \in \mathcal{F}\,\mathbb{L}$, so it must be the case that $e^{\mathcal{F}}_{\mathbb{L}}(F) \leq x$ and $f'(e^{\mathcal{F}}_{\mathbb{L}}(F)) \notin \downarrow f^\triangledown(x)$. But now we can use the fact that f' is order-preserving to see that

$$\begin{aligned}
& f'\left(e^{\mathcal{F}}_{\mathbb{L}}(F)\right) && \\
&= f'(\textstyle\bigwedge e_{\mathbb{L}}[F]) && \text{by definition of } e^{\mathcal{F}}_{\mathbb{L}}, \\
&\leq \textstyle\bigwedge f' \circ e_{\mathbb{L}}[F] && \text{since } f' \text{ is order-preserving}, \\
&= \textstyle\bigwedge e_{\mathbb{M}} \circ f[F] && \text{because } f' \circ e_{\mathbb{L}} = e_{\mathbb{M}} \circ f, \\
&= e^{\mathcal{F}}_{\mathbb{M}} \circ \mathcal{F}\,f(F) && \text{by Remark 2.2.2}, \\
&\leq f^\triangledown(x) && \text{by definition of } f^\triangledown.
\end{aligned}$$

This is a contradiction because we also assumed that $f'(e^{\mathcal{F}}_{\mathbb{L}}(F)) \notin \downarrow f^\triangledown(x)$; it follows that indeed, local continuity of f' at x implies that $f'(x) \leq f^\triangledown(x)$. Thus, if f' is continuous at every $x \in \mathbb{L}^\delta$, it follows that $f' \leq f^\triangledown$. ∎

Even though $f^\triangledown$ is not Scott-continuous in general, it is when we restrict it to ideal elements. The following result is known from the study of duality theory for Heyting algebras and modal algebras, but it holds in fact at the level of generality of lattices and order-preserving maps.

2.2.5. Corollary (Esakia's Lemma). *Let $f : \mathbb{L} \to \mathbb{M}$ be an order-preserving map between lattices $\mathbb{L}, \mathbb{M}$. Then $f^\triangledown \circ e^{\mathcal{I}}_{\mathbb{L}} = f^\vartriangle \circ e^{\mathcal{I}}_{\mathbb{L}}$ is Scott-continuous and $f^\vartriangle \circ e^{\mathcal{F}}_{\mathbb{L}} = f^\triangledown \circ e^{\mathcal{F}}_{\mathbb{L}}$ is co-Scott continuous.*

Proof We only consider the first statement. The equality follows by Lemma 2.2.3; the continuity follows since $e^{\mathcal{I}}_{\mathbb{L}}$ is $(\sigma^\uparrow, \delta^\uparrow)$-continuous by Lemma 2.1.20 and $f^\triangledown$ is $(\delta^\uparrow, \sigma^\uparrow)$-continuous by Theorem 2.2.4. ∎

As another application of Lemma 2.2.3, we will show that the canonical extension of an order embedding is again an order embedding.

2.2.6. Lemma. *Let $\mathbb{L}$, $\mathbb{M}$ be lattices. If $f\colon \mathbb{L} \to \mathbb{M}$ is an order embedding, then so are $f^{\triangledown}$ and $f^{\vartriangle}$.*

Proof Let $f\colon \mathbb{L} \to \mathbb{M}$ be an order embedding. We will only prove part (1); part (2) follows by order duality. Let $x, y \in \mathbb{L}^\delta$ and suppose that $f^{\triangledown}(x) \leq f^{\triangledown}(y)$. We will show that

$$\forall F \in \mathcal{F}\,\mathbb{L}, \forall I \in \mathcal{I}\,\mathbb{L}, \text{ if } e^{\mathcal{F}}_{\mathbb{L}}(F) \leq x \text{ and } y \leq e^{\mathcal{I}}_{\mathbb{L}}(I), \text{ then } e^{\mathcal{F}}_{\mathbb{L}}(F) \leq e^{\mathcal{I}}_{\mathbb{L}}(I). \quad (2.9)$$

This is not hard to see. Take $F \in \mathcal{F}\,\mathbb{L}$ and $I \in \mathcal{I}\,\mathbb{L}$ such that $e^{\mathcal{F}}_{\mathbb{L}}(F) \leq x$ and $y \leq e^{\mathcal{I}}_{\mathbb{L}}(I)$. Then

$$\begin{aligned}
e^{\mathcal{F}}_{\mathbb{M}} \circ \mathcal{F}\,f(F) &= f^{\triangledown}\left(e^{\mathcal{F}}_{\mathbb{L}}(F)\right) && \text{by Lemma 2.2.3 (1),}\\
&\leq f^{\triangledown}(x) && \text{since } e^{\mathcal{F}}_{\mathbb{L}}(F) \leq x,\\
&\leq f^{\triangledown}(y) && \text{by assumption,}\\
&\leq f^{\triangledown}\left(e^{\mathcal{I}}_{\mathbb{L}}(I)\right) && \text{since } y \leq e^{\mathcal{I}}_{\mathbb{L}}(I),\\
&= e^{\mathcal{I}}_{\mathbb{M}} \circ \mathcal{I}\,f(I) && \text{by Lemma 2.2.3 (2).}
\end{aligned}$$

Consequently, $\mathcal{F}\,f(F) \between \mathcal{I}\,f(I)$, i.e. $\uparrow f[F] \between \downarrow f[I]$. It follows that there must exist $a \in F$ and $b \in I$ such that $f(a) \leq f(b)$. Since f is an order embedding, $a \leq b$, so $F \between I$. It follows that $e^{\mathcal{F}}_{\mathbb{L}}(F) \leq e^{\mathcal{I}}_{\mathbb{L}}(I)$, concluding our proof of (2.9). It now follows from the density of filter and ideal elements that $x \leq y$, concluding our proof. ∎

2.2.2 Operators and join-preserving maps

So far we have seen how to take an order-preserving map $f\colon \mathbb{L} \to \mathbb{M}$ and extend it covariantly to maps $\mathcal{F}\,f\colon \mathcal{F}\,\mathbb{L} \to \mathcal{F}\,\mathbb{M}$ and $\mathcal{I}\,f\colon \mathcal{I}\,\mathbb{L} \to \mathcal{I}\,\mathbb{M}$ at the intermediate level of filters and ideals, and from there to maps $f^{\triangledown}\colon \mathbb{L}^\delta \to \mathbb{M}^\delta$ and $f^{\vartriangle}\colon \mathbb{L}^\delta \to \mathbb{M}^\delta$. In this section we will see that if $f\colon \mathbb{L} \to \mathbb{M}$ preserves binary joins, then the set-theoretic inverse function $f^{-1}\colon \mathcal{P}(\mathbb{L}) \to \mathcal{P}(\mathbb{M})$ becomes a *partial* function from $\mathcal{I}\,\mathbb{M}$ to $\mathcal{I}\,\mathbb{L}$. This contravariant partial extension of f at the intermediate level will then tell us a lot about the topological properties of $f^{\triangledown}\colon \mathbb{L}^\delta \to \mathbb{M}^\delta$.

Applying join-preserving maps to filters and ideals

We begin by making the observation that when an order-preserving map $g\colon \mathbb{L} \to \mathbb{M}$ preserves binary joins, we not only get a map $\mathcal{I}\,g\colon \mathcal{I}\,\mathbb{L} \to \mathcal{I}\,\mathbb{M}$, defined as $\mathcal{I}\,g(I) := \downarrow g[I]$, but also a well-behaved *partial* map $g^{-1}\colon \mathcal{I}\,\mathbb{M} \to \mathcal{I}\,\mathbb{L}$.

2.2.7. Lemma. *Let $\mathbb{L}$, $\mathbb{M}$ be lattices and let $g\colon \mathbb{L} \to \mathbb{M}$ be a map preserving binary joins. Then*

1. *$\forall J \in \mathcal{I}\,\mathbb{M}$, $g^{-1}(J) \in \mathcal{I}\,\mathbb{L}$ iff $g^{-1}(J) \neq \emptyset$;*

2. $\forall F \in \mathcal{F}\,\mathbb{L}$, $\forall J \in \mathcal{I}\,\mathbb{M}$, $\mathcal{F}\,g(F) \between J$ *iff* $g^{-1}(J) \in \mathcal{I}\,\mathbb{L}$ *and* $F \between g^{-1}(J)$.

Proof (1). Let $J \in \mathcal{I}\,\mathbb{M}$. Since ideals are non-empty by definition, the left-to-right implication is immediate. For the converse, first observe that $g^{-1}(J)$ is non-empty by assumption. Moreover, g^{-1} is order-preserving, so by Fact A.3.1(3), $g^{-1}(J)$ is a lower set. Moreover, if $a, b \in g^{-1}(J)$, then $g(a), g(b) \in J$, so since J is an ideal we get $g(a) \vee g(b) \in J$. Since g preserves binary joins, we see that $g(a \vee b) = g(a) \vee g(b) \in J$, so that $a \vee b \in g^{-1}(J)$. It follows that $g^{-1}(J)$ is an ideal of $\mathbb{L}$.

(2). Let $F \in \mathcal{F}\,\mathbb{L}$ and $J \in \mathcal{I}\,\mathbb{M}$. If $\mathcal{F}\,g(F) \between J$, then $\uparrow g[F] \between J$, i.e. there must be some $b \in (\uparrow g[F]) \cap J$. Since $b \in \uparrow g[F]$, there is some $a \in F$ such that $g(a) \leq b$. Since $b \in J$ and J is a lower set, it follows that $g(a) \in J$. But then $a \in F \cap g^{-1}(J)$, so that $F \between g^{-1}(J)$. Moreover, since $g^{-1}(J) \neq \emptyset$, it follows by (1) that $g^{-1}(J) \in \mathcal{I}\,\mathbb{L}$. Conversely, if $F \between g^{-1}(J)$ then there must exist $a \in F \cap g^{-1}(J)$, i.e. $a \in F$ and $g(a) \in J$. Since $g(a) \in g[F] \subseteq \uparrow g[F]$, it follows that $\uparrow g[F] \between J$. ∎

Up to this point, we have only considered canonical extensions of unary maps $f \colon \mathbb{L} \to \mathbb{M}$. We would now like to state several more detailed results, concerning canonical extensions of n-ary maps. This is non-problematic since canonical extensions commute with finite products; however, we would like to ignore the technical difference between, say, $(\mathbb{L}_1 \times \mathbb{L}_2)^\delta$ and $\mathbb{L}_1^\delta \times \mathbb{L}_2^\delta$ whenever possible.

2.2.8. Convention. If $f \colon \mathbb{L}_1 \times \cdots \times \mathbb{L}_n \to \mathbb{M}$ is an n-ary order-preserving map, then we regard $\mathcal{F}\,f$ as a map $\mathcal{F}\,\mathbb{L}_1 \times \cdots \times \mathcal{F}\,\mathbb{L}_n \to \mathcal{F}\,\mathbb{M}$ and $f^\triangledown$ as a map $\mathbb{L}_1^\delta \times \cdots \times \mathbb{L}_n^\delta \to \mathbb{M}^\delta$, rather than as maps $\mathcal{F}(\mathbb{L}_1 \times \cdots \times \mathbb{L}_n) \to \mathcal{F}\,\mathbb{M}$ and $(\mathbb{L}_1 \times \cdots \times \mathbb{L}_n)^\delta \to \mathbb{M}^\delta$. This is justified by Fact A.5.4 and Lemma 2.1.26. Using this convention, $f^\triangledown$ is calculated as follows for $(x_1, \ldots, x_n) \in \mathbb{L}_1^\delta \times \cdots \times \mathbb{L}_n^\delta$:

$$f^\triangledown(x_1, \ldots, x_n) = \bigvee\{e^{\mathcal{F}}_{\mathbb{M}} \circ \mathcal{F}\,f(F_1, \ldots, F_n) \mid e^{\mathcal{F}}_{\mathbb{L}_1}(F_1) \leq x_1, \ldots, e^{\mathcal{F}}_{\mathbb{L}_n}(F_n) \leq x_n\},$$

and $f^\vartriangle(x_1, \ldots, x_n)$ is calculated similarly.

We will now work towards the main technical lemma of this section, which concerns canonical extensions of binary maps $f \colon \mathbb{L}_1 \times \mathbb{L}_2 \to \mathbb{M}$ which preserve joins in only one coordinate. We will first prove a result about the extension $\mathcal{F}\,f \colon \mathcal{F}\,\mathbb{L}_1 \times \mathcal{F}\,\mathbb{L}_2 \to \mathcal{F}\,\mathbb{M}$ of such a map f, which will also be of use to us in §2.3.3.

2.2.9. Lemma. *Let $f \colon \mathbb{L}_1 \times \mathbb{L}_2 \to \mathbb{M}$ be an order-preserving map between lattices $\mathbb{L}_1$, $\mathbb{L}_2$ and $\mathbb{M}$ which preserves binary joins in its first coordinate. Let $S \cup \{F'\} \subseteq \mathcal{F}\,\mathbb{L}_1$ and $G \in \mathcal{F}\,\mathbb{L}_2$. If $S \neq \emptyset$ and*

$$\forall I \in \mathcal{I}\,\mathbb{L}_1,\ [\forall F \in S,\ F \between I] \Rightarrow F' \between I, \tag{2.10}$$

then also

$$\forall J \in \mathcal{I}\,\mathbb{M},\ [\forall F \in S,\ \mathcal{F}\,f(F, G) \between J] \Rightarrow \mathcal{F}\,f(F', G) \between J. \tag{2.11}$$

Proof For each $b \in \mathbb{L}_2$, we define $f_b \colon \mathbb{L}_1 \to \mathbb{M}$ as $f_b \colon a \mapsto f(a, b)$. It follows from our assumptions above that for each $b \in \mathbb{L}_2$, $f_b \colon \mathbb{L}_1 \to \mathbb{M}$ is a map preserving binary joins. Now we claim that

$$\forall b \in G,\ \forall F \in \mathcal{F}\,\mathbb{L}_1,\ \mathcal{F}\,f(F, G) \supseteq \mathcal{F}\,f_b(F). \tag{2.12}$$

After all,

$$\begin{aligned} \mathcal{F}\,f(F, G) &= \uparrow f[F, G] && \text{by definition of } \mathcal{F}\,f, \\ &\supseteq \uparrow f[F, \{b\}] && \text{since } G \supseteq \{b\}, \\ &= \uparrow f_b[F] && \text{by definition of } f_b, \\ &= \mathcal{F}\,f_b(F). && \text{by definition of } \mathcal{F}\,f_b. \end{aligned}$$

Now suppose that (2.10) holds and that $J \in \mathcal{I}\,\mathbb{M}$ such that $\forall F \in S$, $\mathcal{F}\,f(F, G) \between J$. We need to show that $\mathcal{F}\,f(F', G) \between J$. We define $I_1 := \bigcup_{b \in G} f_b^{-1}(J)$. We claim that

$$I_1 \in \mathcal{I}\,\mathbb{L}_1, \text{ i.e. } I_1 \text{ is an ideal of } \mathbb{L}_1. \tag{2.13}$$

To establish this we need to show three things: we want that I_1 is a lower set, that I_1 is directed and that I_1 is non-empty. Since f_b is order-preserving for each $b \in G$, it follows by Fact A.3.1(3) that I_1 is a union of lower sets and hence, itself a lower set. To see that I_1 is directed, first observe that

$$\text{for all } b, b' \in G, \text{ if } b \geq b' \text{ then } f_b^{-1}(J) \subseteq f_{b'}^{-1}(J). \tag{2.14}$$

After all, if $a \in f_b^{-1}(J)$, then $f(a, b) \in J$. Since f is order-preserving and $b' \leq b$, we see that $f(a, b') \leq f(a, b) \in J$. Since J is a lower set, it follows that $f(a, b') \in J$, so that $a \in f_{b'}^{-1}$; since $a \in f_b^{-1}$ was arbitrary, it follows that (2.14) holds. Now since G is a filter, it is co-directed; consequently, $I_1 := \bigcup_{b \in G} f_b^{-1}(J)$ is a directed union. To see why I_1 is a directed subset of $\mathbb{L}_1$, consider $a, a' \in I_1$. Because I_1 is a directed union, there must exist some $b \in G$ such that $a, a' \in f_b^{-1}(J)$. Now $f_b^{-1}(J)$ is non-empty, so it is an ideal by Lemma 2.2.7(1); consequently, there must exist some $c \in f_b^{-1}(J)$ such that $a, a' \leq c$. Since $f_b^{-1}(J) \subseteq I_1$, it follows that I_1 is directed. Finally, to see that I_1 is non-empty, observe that since S is non-empty, there is some $F \in S$ such that $\mathcal{F}\,f(F, G) \between J$. Since $\mathcal{F}\,f(F, G) := \uparrow f[F, G]$, this means that there is some $c \in (\uparrow f[F, G]) \cap J$. Since $c \in \uparrow f[F, G]$, there must be $a \in F$ and $b \in G$ such that $f(a, b) \leq c$. Since J is a lower set and $c \in J$, it follows that $f(a, b) \in J$. But then also $f_b(a) \in J$, so that $a \in f_b^{-1}(J)$; since $f_b^{-1}(J) \subseteq I_1$, it follows that $I_1 \neq \emptyset$. We conclude that (2.13) holds. Next, we observe that

$$\text{If } \forall F \in S,\ \mathcal{F}\,f(F, G) \between J, \text{ then } \forall F \in S, \exists b \in G,\ F \between f_b^{-1}(J). \tag{2.15}$$

Suppose that the left-hand side of (2.15) holds and take $F \in S$. Then $\mathcal{F} f(F, G) \between J$, so as we have seen above, there must exist $a \in F$ and $b \in G$ such that $a \in f_b^{-1}(J)$. It follows that $a \in F \cap f_b^{-1}(J)$, so that $F \between f_b^{-1}(J)$. Since $F \in S$ was arbitrary, it follows that (2.15) holds. Recall that we assumed that $J \in \mathcal{I}\,\mathbb{M}$ such that $\forall F \in S$, $\mathcal{F} f(F, G) \between J$; we now see that

$$
\begin{aligned}
&\forall F \in S,\ \mathcal{F} f(F, G) \between J && \text{by assumption,}\\
&\Rightarrow \forall F \in S, \exists b \in G,\ F \between f_b^{-1}(J) && \text{by (2.15),}\\
&\Rightarrow \forall F \in S,\ F \between I_1 && \text{since } I_1 = \textstyle\bigcup_G f_b^{-1}(J),\\
&\Rightarrow F' \between I_1 && \text{by (2.13) and (2.10),}\\
&\Rightarrow \exists b \in G,\ F' \between f_b^{-1}(J) && \text{by def. of } I_1,\\
&\Rightarrow \exists b \in G,\ \mathcal{F} f_b(F') \between J && \text{by Lemma 2.2.7(2),}\\
&\Rightarrow \mathcal{F} f(F, G) \between J && \text{by (2.12) and L. 2.1.2(1).}
\end{aligned}
$$

Since $J \in \mathcal{I}\,\mathbb{M}$ was arbitrary, it follows that (2.11) holds. ∎

The following well-known lemma is perhaps the most powerful technical result in the theory of canonical extensions of maps between lattices.

2.2.10. Lemma ([34]). *Let $e_1 \colon \mathbb{L}_1 \to \mathbb{L}_1^\delta, e_2 \colon \mathbb{L}_2 \to \mathbb{L}_2^\delta$ and $e_\mathbb{M} \colon \mathbb{M} \to \mathbb{M}^\delta$ be canonical extensions of lattices, and let $f \colon \mathbb{L}_1 \times \mathbb{L}_2 \to \mathbb{M}$ be a order-preserving map which preseves binary joins in the first coordinate. Then $f^\triangledown \colon \mathbb{L}_1^\delta \times \mathbb{L}_2^\delta \to \mathbb{M}^\delta$ preseves all non-empty joins in the first coordinate.*

Proof Let $T \subseteq \mathbb{L}_1^\delta$ be a non-empty set and let $y \in \mathbb{L}_2^\delta$. To show that $f^\triangledown(\bigvee T, y) = \bigvee_{x \in T} f^\triangledown(x, y)$, it suffices to show that

$$f^\triangledown(\textstyle\bigvee T, y) \leq \bigvee_{x \in T} f^\triangledown(x, y), \tag{2.16}$$

since $f^\triangledown$ is order-preserving. Using the definition of $f^\triangledown$, one can show that

$$f^\triangledown(\textstyle\bigvee T, y) = \bigvee \{f^\triangledown(\bigvee T, e_2^\mathcal{F}(G)) \mid e_2^\mathcal{F}(G) \leq y\},$$

and that

$$\textstyle\bigvee_{x \in T} f^\triangledown(x, y) = \bigvee \{\bigvee_{x \in T} f^\triangledown(x, e_2^\mathcal{F}(G)) \mid e_2^\mathcal{F}(G) \leq y\}.$$

Thus we see that it suffices to show that for arbitrary $G \in \mathcal{F}\,\mathbb{L}_2$,

$$f^\triangledown(\textstyle\bigvee T, e_2^\mathcal{F}(G)) \leq \bigvee_{x \in T} f^\triangledown(x, e_2^\mathcal{F}(G)). \tag{2.17}$$

Fix $G \in \mathcal{F}\,\mathbb{L}_2$ and define $S := \{F \in \mathcal{F}\,\mathbb{L}_1 \mid \exists x \in T,\ e_1^\mathcal{F}(F) \leq x\}$; we see that $\bigvee_{F \in S} e_1^\mathcal{F}(F) = \bigvee T$. Now if we look at the left-hand side of (2.17) then we see that

$$
\begin{aligned}
&f^\triangledown(\textstyle\bigvee T, e_2^\mathcal{F}(G))\\
&= f^\triangledown(\textstyle\bigvee_{F \in S} e_1^\mathcal{F}(F), e_2^\mathcal{F}(G))\\
&= \textstyle\bigvee \{e_\mathbb{M}^\mathcal{F} \circ \mathcal{F} f(F', G) \mid e_1^\mathcal{F}(F') \leq \bigvee_{F \in S} e_1^\mathcal{F}(F)\} && \text{by def. of } f^\triangledown.
\end{aligned}
$$

The right-hand side of (2.17) reduces as follows:

$$\begin{aligned}
&\bigvee_{x\in T} f^{\triangledown}(x, e_2^{\mathcal{F}}(G)) \\
&= \bigvee_{x\in T}\bigvee\{e_{\mathbb{M}}^{\mathcal{F}} \circ \mathcal{F}f(F,G) \mid e_1^{\mathcal{F}}(F) \le x\} && \text{by def. of } f^{\triangledown}, \\
&= \bigvee\{e_{\mathbb{M}}^{\mathcal{F}} \circ \mathcal{F}f(F,G) \mid F \in S\} && \text{by def. of } S, \\
&= \bigwedge\{e_{\mathbb{M}}^{\mathcal{I}}(J) \mid \forall F \in S,\ \mathcal{F}f(F,G) \between J\} && \text{by L. 2.1.29.}
\end{aligned}$$

Thus we see that to show that (2.17) holds, it suffices to show that for all $F' \in \mathcal{F}\mathbb{L}_1$ such that $e_1^{\mathcal{F}}(F') \le \bigvee_{F\in S} e_1^{\mathcal{F}}(F)$ and for all $J \in \mathcal{I}\mathbb{M}$ such that $\forall F \in S$, $\mathcal{F}f(F,G) \between J$, we have that

$$e_{\mathbb{M}}^{\mathcal{F}} \circ \mathcal{F}f(F',G) \le e_{\mathbb{M}}^{\mathcal{I}}(J). \tag{2.18}$$

Now given the fact that $e_1^{\mathcal{F}}(F') \le \bigvee_{F\in S} e_1^{\mathcal{F}}(F)$, we know from Lemma 2.1.29 that

$$e_1^{\mathcal{F}}(F') \le \bigwedge\{e_1^{\mathcal{I}}(I) \mid \forall F \in S,\ F \between I\},$$

i.e. for all $I \in \mathcal{I}\mathbb{L}_1$, if $\forall F \in S$, $F \between I$, then $F' \between I$. But now it follows from Lemma 2.2.9 and our assumption about J that $\mathcal{F}f(F',G) \between J$, so that (2.18) holds. Since F' and J were arbitrary, it follows that (2.17) holds, which concludes our proof. ∎

Operators and dual operators

Obviously, we can use Lemma 2.2.10 to make claims about join-preserving maps, but that is not all. Operators form another example of maps which satisfy the conditions of Lemma 2.2.10.

2.2.11. Definition. Let $f\colon \mathbb{L}_1 \times \cdots \times \mathbb{L}_n \to \mathbb{M}$ be an n-ary order-preserving map between lattices. We call f an *operator* if f preserves binary joins in each coordinate, i.e. if for all $i \le n$, for all $a_1, \ldots, a_n \in \mathbb{L}_1 \times \cdots \times \mathbb{L}_n$ and all $b \in \mathbb{L}_i$, we have

$$f(a_1, \ldots, a_i \vee b, \ldots, a_n) = f(a_1, \ldots, a_i, \ldots, a_n) \vee f(a_1, \ldots, b, \ldots, a_n).$$

If all lattices involved are complete and if f preserves *all* non-empty joins in each coordinate, then we call f a *complete operator*.

We call f a *normal operator* if f is an operator and for all $i \le n$, for all $a_1, \ldots, a_n \in \mathbb{L}_1 \times \cdots \times \mathbb{L}_n$,

$$a_i = 0 \Rightarrow f(a_1, \ldots, a_i, \ldots, a_n) = 0.$$

In other words, f is a normal operator if it also preserves the empty join in each coordinate.

A *dual operator* (complete dual operator, etc.) is an n-ary map which preserves binary meets (all non-empty meets, etc.) in each coordinate.

Operators arise, for instance, in the algebraic semantics for modal logics [19, Ch. 5], where they correspond to existential modalities (usually denoted by '$\Diamond$').

2.2.12. Example. The property of being a (normal) operator is weaker than that of being a join-homomorphism. Consider the map $f\colon \{0,1\} \times \{0,1\} \to \{0,1\}$, with the usual order on $\{0,1\}$, defined as

$$f\colon (a,b) \mapsto a \wedge b.$$

Then f is an operator, in fact a normal operator, because $\{0,1\}$ is a distributive lattice. However, f is not a join-homomorphism:

$$f((0,1) \vee (1,0)) = f(1,1) = 1 \wedge 1 = 1,$$

but

$$f(0,1) \vee f(1,0) = (0 \wedge 1) \vee (1 \wedge 0) = 0 \vee 0 = 0.$$

This is quite different from the situation for directed joins; if $g\colon \mathbb{D}_1 \times \cdots \times \mathbb{D}_n \to \mathbb{E}$ is a map between dcpo's which preserves directed joins in each coordinate, then by Fact A.3.4, g preserves directed joins in $\mathbb{D}_1 \times \cdots \times \mathbb{D}_n$.

In Lemma 2.2.10, we only considered binary joins, i.e. non-empty finite joins. Canonical extensions also behave well with respect to maps which preserve the empty join.

2.2.13. Lemma. *Let $e_1\colon \mathbb{L}_1 \to \mathbb{L}_1^\delta, e_2\colon \mathbb{L}_2 \to \mathbb{L}_2^\delta$ and $e_\mathbb{M}\colon \mathbb{M} \to \mathbb{M}^\delta$ be canonical extensions of lattices, and let $f\colon \mathbb{L}_1 \times \mathbb{L}_2 \to \mathbb{M}$ be a order-preserving map.*

1. *If $\forall b \in \mathbb{L}_2$, $f(0,b) = 0$, then also $\forall y \in \mathbb{L}_2^\delta$, $f^\triangledown(0,y) = 0$;*
2. *If $\forall b \in \mathbb{L}_2$, $f(1,b) = 1$, then also $\forall y \in \mathbb{L}_2^\delta$, $f^\triangledown(1,y) = 1$;*

Proof We will only prove (1), since (2) is just the order dual of (1). First, observe that

$$\forall F \in \mathcal{F}\,\mathbb{L}_1, \forall G \in \mathcal{F}\,\mathbb{L}_2,\ 0 \in F \Rightarrow 0 \in \mathcal{F}\,f(F,G). \tag{2.19}$$

Take $F \in \mathcal{F}\,\mathbb{L}_1$ and $G \in \mathcal{F}\,\mathbb{L}_2$ such that $0 \in F$, then since G must be non-empty, there is some $b \in G$. Now

$$f(0,b) \in f[F,G] \subseteq {\uparrow} f[F,G] = \mathcal{F}\,f(F,G),$$

so since $f(0,b) = 0$ by assumption, we see that $0 \in \mathcal{F}\,f(F,G)$. Next, observe that it is a basic fact about canonical extensions that for any lattice $\mathbb{L}$,

$$\forall a \in \mathbb{L}, \forall F \in \mathcal{F}\,\mathbb{L},\ a \in F \text{ iff } e_\mathbb{L}^\mathcal{F}(F) \leq e_\mathbb{L}(a). \tag{2.20}$$

After all, $a \in F$ iff $F \between \downarrow a$ iff $e^{\mathcal{F}}_{\mathbb{L}}(F) \leq e^{\mathcal{I}}_{\mathbb{L}}(\downarrow a) = e_{\mathbb{L}}(a)$. Now we can see that for any $y \in \mathbb{L}^{\delta}_2$,

$$
\begin{aligned}
& f^{\triangledown}(0, y) & \\
& = f^{\triangledown}(e(0), y) & \text{since } e(0) = 0, \\
& = \bigvee\{e_{\mathbb{M}} \circ \mathcal{F} f(F, G) \mid e^{\mathcal{F}}_1(F) \leq e_1(0), e^{\mathcal{F}}_2(G) \leq y\} & \text{by def. of } f^{\triangledown}, \\
& = \bigvee\{e_{\mathbb{M}} \circ \mathcal{F} f(F, G) \mid 0 \in F, e^{\mathcal{F}}_2(G) \leq y\} & \text{by (2.20)}, \\
& \leq \bigvee\{e_{\mathbb{M}}(F') \mid 0 \in F'\} & \text{by (2.19)}, \\
& = \bigvee\{e_{\mathbb{M}}(F') \mid e_{\mathbb{M}}(F') \leq e(0)\} & \text{by (2.20)}, \\
& = 0, &
\end{aligned}
$$

which is what we wanted to show. ∎

We can now state an immediate corollary of Lemmas 2.2.10 and 2.2.13:

2.2.14. Corollary ([34]). *Let $f\colon \mathbb{L}_1 \times \cdots \times \mathbb{L}_n \to \mathbb{M}$ be an n-ary order-preserving map between lattices.*

1. *If f is a (normal) operator, then $f^{\triangledown}$ is a complete (normal) operator.*
2. *If f is a dual (normal) operator, then $f^{\vartriangle}$ is a complete dual (normal) operator.*

Topological properties of operators and join-preserving maps

It is one of the characteristic features of canonical extension that maps between lattices ($f\colon \mathbb{L} \to \mathbb{M}$) have both a lower ($f^{\triangledown}\colon \mathbb{L}^{\delta} \to \mathbb{M}^{\delta}$) and an upper extension ($f^{\vartriangle}\colon \mathbb{L}^{\delta} \to \mathbb{M}^{\delta}$). These two extensions need not necessarily be different.

2.2.15. Definition. We say that an order-preserving map $f\colon \mathbb{L} \to \mathbb{M}$ is *smooth* if $f^{\triangledown} = f^{\vartriangle}$. If we want to emphasize that f is smooth, we will refer to the canonical extension of f as f^{δ} rather than $f^{\triangledown}$ or $f^{\vartriangle}$.

In light of the recurring topological themes in this chapter, it may not come as a surprise that smoothness of a map $f\colon \mathbb{L} \to \mathbb{M}$ is a *topological* property.

2.2.16. Lemma. *an order-preserving map $f\colon \mathbb{L} \to \mathbb{M}$ between lattices is smooth iff $f^{\triangledown}\colon \mathbb{L}^{\delta} \to \mathbb{M}^{\delta}$ is (δ, σ)-continuous.*

Proof If $f^{\triangledown} = f^{\vartriangle}$, then $f^{\triangledown}$ is both $(\delta^{\uparrow}, \sigma^{\uparrow})$-continuous and $(\delta^{\downarrow}, \sigma^{\downarrow})$-continuous; hence $f^{\triangledown}$ is also (δ, σ)-continuous. Conversely, if $f^{\triangledown}$ is (δ, σ)-continuous, then it follows from Lemma 2.1.17 and the fact that $f^{\triangledown}$ is order-preserving that $f^{\triangledown}$ is $(\delta^{\downarrow}, \sigma^{\downarrow})$-continuous. It follows by the order dual of Theorem 2.2.4 that $f^{\vartriangle} \leq f^{\triangledown}$. Since $f^{\triangledown} \leq f^{\vartriangle}$ by Lemma 2.2.3, we find that $f^{\triangledown} = f^{\vartriangle}$. ∎

Before we proceed to the main result about topological properties of operators and join-preserving maps, we prove another technical lemma which says, intuitively, that if $f\colon \mathbb{L} \to \mathbb{M}$ preserves binary joins, then $f^{\triangledown}\colon \mathbb{L}^{\delta} \to \mathbb{M}^{\delta}$ has a kind of weak, partial right adjoint.

2.2.17. Lemma. *Let $f\colon \mathbb{L} \to \mathbb{M}$ be an order-preserving map preserving binary joins.*

$$\forall x \in \mathbb{L}^{\delta}, \forall J \in \mathcal{I}\,\mathbb{M},\ f^{\triangledown}(x) \leq e^{\mathcal{I}}_{\mathbb{M}}(J) \text{ iff } f^{-1}(J) \in \mathcal{I}\,\mathbb{L} \text{ and } x \leq e^{\mathcal{I}}_{\mathbb{L}} \circ f^{-1}(J).$$

Proof Let $x \in \mathbb{L}^{\delta}$ and $J \in \mathcal{I}\,\mathbb{M}$. We define $S := \{F \in \mathcal{F}\,\mathbb{L} \mid e^{\mathcal{F}}_{\mathbb{L}}(F) \leq x\}$; observe that S is always non-empty since at least $\uparrow 0 \in S$. Now if $f^{\triangledown}(x) \leq e^{\mathcal{I}}_{\mathbb{M}}(J)$, then since $f^{\triangledown}(x) = \bigvee_{F \in S} e^{\mathcal{F}}_{\mathbb{M}} \circ \mathcal{F}\, f(F)$, we see that $\forall F \in S, e^{\mathcal{F}}_{\mathbb{M}} \circ \mathcal{F}\, f(F) \leq e^{\mathcal{I}}_{\mathbb{M}}(J)$. By basic properties of canonical extension it follows that $\forall F \in S, \mathcal{F}\, f(F) \between J$. Since $S \neq \emptyset$, there is at least one $F \in \mathcal{F}\,\mathbb{L}$ such that $\mathcal{F}\, f(F) \between J$, so by Lemma 2.2.7(2), $f^{-1}(J) \in \mathcal{I}\,\mathbb{L}$. It also follows by Lemma 2.2.7(2) that $\forall F \in S, F \between f^{-1}(J)$, so that $\forall F \in S, e^{\mathcal{F}}_{\mathbb{L}}(F) \leq e^{\mathcal{I}}_{\mathbb{L}} \circ f^{-1}(J)$. Since $x = \bigvee_{F \in S} e^{\mathcal{F}}_{\mathbb{L}}(F)$, it follows that $x \leq e^{\mathcal{I}}_{\mathbb{L}} \circ f^{-1}(J)$. The proof of the converse implication is analogous. ∎

We now arrive at the main theorem about topological properties of operators and join-preserving maps. Parts (1) and (3) were already known from [34].

2.2.18. Theorem. *Let $f\colon \mathbb{L} \to \mathbb{M}$ be an order-preserving map between lattices.*

1. *(a) If f is an operator, then $f^{\triangledown}$ is $(\sigma^{\uparrow}, \sigma^{\uparrow})$-continuous.*
   (b) If f is a dual operator, then $f^{\triangledown}$ is $(\sigma^{\downarrow}, \sigma^{\downarrow})$-continuous.
2. *(a) If f preserves binary joins, then $f^{\triangledown}\colon \mathbb{L}^{\delta} \to \mathbb{M}^{\delta}$ is $(\delta^{\downarrow}, \delta^{\downarrow})$-continuous.*
   (b) If f preserves binary meets, then $f^{\vartriangle}\colon \mathbb{L}^{\delta} \to \mathbb{M}^{\delta}$ is $(\delta^{\uparrow}, \delta^{\uparrow})$-continuous.
3. *If f preserves binary joins* or *binary meets, then f is smooth.*
4. *If f preserves binary joins* and *binary meets, then f is smooth and $f^{\delta}\colon \mathbb{L}^{\delta} \to \mathbb{M}^{\delta}$ is (δ, δ)-continuous.*

Proof We will only show the proofs for the statements about $f^{\triangledown}$, since the proofs for the statements about $f^{\vartriangle}$ are order dual.

(1). By Lemma 2.2.10, $f^{\triangledown}$ preserves all non-empty joins in each coordinate, so a fortiori $f^{\triangledown}$ preserves directed joins in each coordinate. It follows from Fact A.3.4 that $f^{\triangledown}\colon \mathbb{L}^{\delta}_1 \times \cdots \times \mathbb{L}^{\delta}_n \to \mathbb{M}^{\delta}$ preserves directed joins.

(2). We will show that $(f^{\triangledown})^{-1}$ maps basic $\delta^{\downarrow}$-open sets to $\delta^{\downarrow}$-open sets. Let $J \in \mathcal{I}\,\mathbb{M}$; we need to show that $(f^{\triangledown})^{-1}\big(\downarrow e^{\mathcal{I}}_{\mathbb{M}}(J)\big)$ is $\delta^{\downarrow}$-open. If $(f^{\triangledown})^{-1}\big(\downarrow e^{\mathcal{I}}_{\mathbb{M}}(J)\big)$ is empty then we are done. If not, then it follows from Lemma 2.2.17 that

$$(f^{\triangledown})^{-1}\big(\downarrow e^{\mathcal{I}}_{\mathbb{M}}(J)\big) = \downarrow e^{\mathcal{I}}_{\mathbb{L}}\big(f^{-1}(J)\big), \tag{2.21}$$

so we see that $(f^\triangledown)^{-1}\big(\downarrow e^{\mathcal{I}}_{\mathbb{M}}(J)\big)$ is in fact a basic $\delta^\downarrow$-open set.

(3). Suppose that f preserves binary joins; then by (2), $f^\triangledown\colon \mathbb{L}^\delta \to \mathbb{M}^\delta$ is $(\delta^\downarrow, \delta^\downarrow)$-continuous. Since $\sigma^\downarrow \subseteq \delta^\downarrow$ (Lemma 2.1.28(3), it follows that $f^\triangledown$ is $(\delta^\downarrow, \sigma^\downarrow)$-continuous. On the other hand, by Theorem 2.2.4(1) we know that $f^\triangledown$ is $(\delta^\uparrow, \sigma^\uparrow)$-continuous. It now follows from Lemma 2.2.16 that f is smooth.

(4). It follows from (3) that f is smooth. Now by (2), $f^\triangledown$ is $(\delta^\downarrow, \delta^\downarrow)$-continuous and $f^\vartriangle$ is $(\delta^\uparrow, \delta^\uparrow)$-continuous. Since f is smooth, i.e. since $f^\triangledown = f^\vartriangle$, it now follows from general topology that $f^\delta := f^\triangledown$ is (δ, δ)-continuous. ∎

We now turn to an interesting question which we have neglected so far. We took it as part of our definition that $e_{\mathbb{L}}\colon \mathbb{L} \to \mathbb{L}^\delta$ is a lattice embedding, which means that the meet and join of $\mathbb{L}^\delta$ a priori 'play nice' with those of $\mathbb{L}$. If we look at meet and join as maps $\vee_{\mathbb{L}}\colon \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ and $\wedge_{\mathbb{L}}\colon \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ however, we can also ask ourselves what are the canonical extensions of $\vee_{\mathbb{L}}$ and $\wedge_{\mathbb{L}}$. Are these indeed the join and meet of $\mathbb{L}^\delta$? Fortunately, the answer is yes.

2.2.19. Lemma ([34]). *Let $e\colon \mathbb{L} \to \mathbb{L}^\delta$ be a canonical extension. Then $(\vee^{\mathbb{L}})^\triangledown = (\vee^{\mathbb{L}})^\vartriangle = \vee^{\mathbb{L}^\delta}$ and $(\wedge^{\mathbb{L}})^\triangledown = (\wedge^{\mathbb{L}})^\vartriangle = \wedge^{\mathbb{L}^\delta}$.*

Proof We will only consider $\vee$, since the other case follows by order duality. Since $\vee^{\mathbb{L}}\colon \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ is associative, it is a join-preserving map, so by Theorem 2.2.18 $(\vee^{\mathbb{L}})^\triangledown = (\vee^{\mathbb{L}})^\vartriangle$. It follows from order theory that for all $x, y \in \mathbb{L}^\delta$,

$$x \vee^{\mathbb{L}^\delta} y = \bigwedge\{z \in \mathbb{L}^\delta \mid x, y \leq z\}.$$

By meet-density of ideal elements, this reduces to

$$x \vee^{\mathbb{L}^\delta} y = \textstyle\bigwedge_{I \in S} e^{\mathcal{I}}(I),$$

where $S := \{I \in \mathcal{I}\,\mathbb{L} \mid x, y \leq e^{\mathcal{I}}(I)\}$. On the other hand,

$$x(\vee^{\mathbb{L}})^\vartriangle y = \textstyle\bigwedge_{J \in S'} e^{\mathcal{I}}(J),$$

where $S' := \{\mathcal{I}\vee^{\mathbb{L}}(J_1, J_2) \mid x \leq e^{\mathcal{I}}(J_1), y \leq e^{\mathcal{I}}(J_2)\}$. Since $\mathcal{I}\vee^{\mathbb{L}} = \vee^{\mathcal{I}\,\mathbb{L}}$, we see that if $I \in S$ then $\mathcal{I}\vee^{\mathbb{L}}(I, I) = I \in S'$, so $S' \subseteq S$. Conversely, if $\mathcal{I}\vee^{\mathbb{L}}(J_1, J_2) \in S'$ and $x \leq e^{\mathcal{I}}(J_1), y \leq e^{\mathcal{I}}(J_2)$, then also $x, y \leq e^{\mathcal{I}}(\mathcal{I}\vee^{\mathbb{L}}(J_1, J_2))$, so $\mathcal{I}\vee^{\mathbb{L}}(J_1, J_2) \in S$ and hence $S' \subseteq S$. It follows that $(\vee^{\mathbb{L}})^\vartriangle = \vee^{\mathbb{L}^\delta}$. ∎

Now if $\mathbb{L}$ is a distributive lattice, then we know that $\wedge_{\mathbb{L}}\colon \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ is an operator. Consequently, by Corollary 2.2.14, $(\wedge_{\mathbb{L}})^\triangledown = \wedge^\delta_{\mathbb{L}}$ is a complete operator, so as a bonus we get the following well-known corollary:

2.2.20. Corollary. *If $\mathbb{L}$ is a distributive lattice then so is $\mathbb{L}^\delta$.*

2.2.21. Remark. Canonical extensions of distributive lattices have much stronger properties than just being distributive. We will return to this subject in §4.1.

2.2.3 Canonical extension as a functor I: lattices only

So far in this chapter we have seen that canonical extension is a construction on lattices and maps between lattices. This raises the very natural question whether canonical extension is a *functor*, and if so between which *categories*. In this subsection we will see that canonical extension is a functor from the category of bounded lattices and lattice homomorphisms to the category of complete lattices and complete homomorphisms.

In order to establish this result, we will first prove several facts about the interaction between canonical extensions and compositions of order-preserving maps. The most basic such result is the well-known fact [34] that if we have two order-preserving maps

$$\mathbb{L}_1 \xrightarrow{f} \mathbb{L}_2 \xrightarrow{g} \mathbb{L}_3 ,$$

then it is always the case that $(gf)^\triangledown \leq g^\triangledown f^\triangledown$ and dually, $g^\vartriangle f^\vartriangle \leq (gf)^\vartriangle$. This result is supplemented by the observation that if we make certain continuity assumptions about $f^\triangledown$ or $g^\triangledown$, we can prove the reverse inequality. This fact was already known in the case of order-preserving maps between distributive lattices [39], however the result is new for the non-distributive case. Armed with these results we can then prove Theorem 2.2.24, which says that canonical extension is a functor from the category of lattices to the category of complete lattices. Theorem 2.2.24 extends a known result from [34] with a new observation about the continuity properties of canonical extensions of lattice homomorphisms. We will revisit the subject of compositions of canonical extensions of maps in §3.2.3, and the subject of functorial behaviour of canonical extension in §3.3.

We will now first state several results about canonical extensions of compositions of order-preserving maps. We begin with a well-known result.

2.2.22. Lemma ([34]). *Let $e_i\colon \mathbb{L}_i \to \mathbb{L}_i^\delta$ be canonical extensions of lattices $\mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3$ and let $f\colon \mathbb{L}_1 \to \mathbb{L}_2$ and $g\colon \mathbb{L}_2 \to \mathbb{L}_3$ be order-preserving maps. Then the following inequalities hold:*

$$(gf)^\triangledown \leq g^\triangledown f^\triangledown \leq \left\{ \begin{array}{c} g^\triangledown f^\vartriangle \\ g^\vartriangle f^\triangledown \end{array} \right\} \leq g^\vartriangle f^\vartriangle \leq (gf)^\vartriangle.$$

Proof The inequalities $g^\triangledown f^\triangledown \leq g^\triangledown f^\vartriangle$ and $g^\triangledown f^\triangledown \leq g^\vartriangle f^\triangledown$ follow from Lemma 2.2.3. For the first inequality, observe that

$$(gf)^\triangledown(x) = \bigvee\{e_3^{\mathcal{F}} \circ \mathcal{F}\, gf(F) \mid e_1^{\mathcal{F}}(F) \leq x\} =$$
$$\bigvee\{e_3^{\mathcal{F}} \circ \mathcal{F}\, g \circ \mathcal{F}\, f(F) \mid e_1^{\mathcal{F}}(F) \leq x\} \leq \bigvee\{e_3^{\mathcal{F}} \circ \mathcal{F}\, g(F') \mid e_2^{\mathcal{F}}(F') \leq f^\triangledown(x)\},$$

where the inequality directly above follows from the fact that if $e_1^{\mathcal{F}}(F) \leq x$, then also

$$e_2^{\mathcal{F}} \circ \mathcal{F}\, f(F) = f^\triangledown\big(e_1^{\mathcal{F}}(F)\big) \leq f^\triangledown(x).$$

The other inequalities in the statement of the lemma follow by order duality. ∎

Next, we present a handful of corollaries of the above lemma.

2.2.23. Corollary. *Let $f\colon \mathbb{L}_1 \to \mathbb{L}_2$ and $g\colon \mathbb{L}_2 \to \mathbb{L}_3$ be order-preserving maps between lattices.*

1. *If $g^\triangledown f^\triangledown$ is $(\delta^\uparrow, \sigma^\uparrow)$-continuous then $g^\triangledown f^\triangledown = (gf)^\triangledown$;*

2. *If $g^\triangledown$ is $(\sigma^\uparrow, \sigma^\uparrow)$-continuous then $g^\triangledown f^\triangledown = (gf)^\triangledown$;*

3. *If $f^\triangledown$ is $(\delta^\uparrow, \delta^\uparrow)$-continuous then $g^\triangledown f^\triangledown = (gf)^\triangledown$.*

Proof (1). If $g^\triangledown f^\triangledown$ is $(\delta^\uparrow, \sigma^\uparrow)$-continuous then by Theorem 2.2.4, $g^\triangledown f^\triangledown \leq (gf)^\triangledown$. By Lemma 2.2.22, $(gf)^\triangledown \leq g^\triangledown f^\triangledown$. Statements (2) and (3) are instances of (1). ∎

Recall from Definition 2.2.15 that we call a map $f\colon \mathbb{L} \to \mathbb{M}$ smooth if $f^\triangledown = f^\vartriangle$, and that we refer to the canonical extension of f as $f^\delta\colon \mathbb{L}^\delta \to \mathbb{M}^\delta$ in that case. Additionally, recall from §A.4 that we denote the category of bounded lattices and lattice homomorphisms by **Lat**, and the category of complete lattices and complete lattice homomorphisms by **CLat**. We can now state a fundamental theorem about canonical extensions of lattices. Most of this theorem was already known, see e.g. [34]; part (1) is a new observation however.

2.2.24. Theorem. *Let $\mathbb{L}$, $\mathbb{M}$ be lattices and let $f\colon \mathbb{L} \to \mathbb{M}$ be a lattice homomorphism. Then f is smooth and*

1. *$f^\delta\colon \mathbb{L}^\delta \to \mathbb{M}^\delta$ is a complete lattice homomorphism which is both (δ, δ)-continuous and (σ, σ)-continuous;*

2. *If f is injective, then so is f^δ;*

3. *If f is surjective, then so is f^δ.*

*In fact, canonical extension defines a functor from **Lat** to **CLat** and $e_\mathbb{L}\colon \mathbb{L} \to \mathbb{L}^\delta$ is a natural transformation.*

Proof (1). Since f preserves binary joins and binary meets, it follows by Lemma 2.2.10 that f^δ preserves all non-empty joins and meets. It follows from Lemma 2.2.13 that f^δ preserves 0 and 1. Now since complete homomorphisms preserve directed joins and co-directed meets a fortiori, it follows that f^δ is (σ, σ)-continuous. Finally it follows from Theorem 2.2.18(2) that f^δ is (δ, δ)-continuous.

(2). This follows from Lemma 2.2.6, since f preserves binary joins.

(3). Assume f is surjective; then by (1), f^δ is a complete homomorphism, and we already know that $\mathcal{F}\,f$ is surjective (see §A.5.1). Let $x \in \mathbb{M}^\delta$; we now make a straightforward computation:

$$\begin{aligned}
& f^\delta\left(\bigvee\{e^{\mathcal{F}}_{\mathbb{L}}(F) \mid e^{\mathcal{F}}_{\mathbb{M}} \circ \mathcal{F}\,f(F) \leq x\}\right) & \\
&= \bigvee\{f^\delta \circ e^{\mathcal{F}}_{\mathbb{L}}(F) \mid e^{\mathcal{F}}_{\mathbb{M}} \circ \mathcal{F}\,f(F) \leq x\} & \text{because } f \text{ is a complete hom.,} \\
&= \bigvee\{e^{\mathcal{F}}_{\mathbb{M}} \circ \mathcal{F}\,f(F) \mid e^{\mathcal{F}}_{\mathbb{M}} \circ \mathcal{F}\,f(F) \leq x\} & \text{by Lemma 2.2.3,} \\
&= \bigvee\{e^{\mathcal{F}}_{\mathbb{M}}(F') \mid e^{\mathcal{F}}_{\mathbb{M}}(F') \leq x\} & \text{because } \mathcal{F}\,f \text{ is surjective,} \\
&= x & \text{by join-densite of filter elements.}
\end{aligned}$$

Since $x \in \mathbb{M}^\delta$ was arbitrary it follows that f^δ is surjective.

To see that canonical extension is a functor, we will first look at compositions of homomorphisms: consider lattice homomorphisms $f\colon \mathbb{L}_1 \to \mathbb{L}_2$ and $g\colon \mathbb{L}_2 \to \mathbb{L}3$. Since g^δ is Scott-continuous by (1), it follows by Corollary 2.2.23 that $g^\delta f^\delta = (gf)^\delta$. To see that canonical extension preserves the identity function, we use that fact that $\mathcal{F}$ is a functor. Let $x \in \mathbb{L}^\delta$ be arbitrary, then

$$\begin{aligned}
(id_{\mathbb{L}})^\delta(x) &= \bigvee\{e^{\mathcal{F}}_{\mathbb{L}} \circ \mathcal{F}\,id_{\mathbb{L}}(F) \mid e^{\mathcal{F}}_{\mathbb{L}}(F) \leq x\} & \text{by definition,} \\
&= \bigvee\{id_{\mathcal{F}\mathbb{L}} \circ e^{\mathcal{F}}_{\mathbb{L}}(F) \mid e^{\mathcal{F}}_{\mathbb{L}}(F) \leq x\} & \text{because } \mathcal{F} \text{ is a functor,} \\
&= \bigvee\{e^{\mathcal{F}}_{\mathbb{L}}(F) \mid e^{\mathcal{F}}_{\mathbb{L}}(F) \leq x\} & \text{by definition of } id_{\mathcal{F}\mathbb{L}}, \\
&= x & \text{by join-density of filter elements,} \\
&= id_{\mathbb{L}^\delta}(x) &
\end{aligned}$$

This proves that canonical extension is a functor on lattices and lattice homomorphisms. The fact that $e_{\mathbb{L}}\colon \mathbb{L} \to \mathbb{L}^\delta$ is a natural transformation follows from the fact that f^δ is an extension of f. ∎

We conclude this section with a minor observation about canonical extensions of sublattices which will be useful later.

2.2.25. Corollary. *If $\mathbb{L}$ is a sublattice of $\mathbb{M}$, then $\mathbb{L}^\delta$ is isomorphic to a complete sublattice of $\mathbb{M}^\delta$.*

2.2.4 Conclusions and further work

The main contribution of this section lies in the technical results concerning filters and ideals, and the results about topological properties of $f^\triangledown$ and $f^\vartriangle$. Most of the topological results in this section can also be found in our paper with M. Gehrke [43]. Many of these where inspired by what was known about distributive lattices from the work of Gehrke and Jónsson [39]. It was not known however that distributivity of the lattices involved, which is a central assumption in [39], is in no way essential when one wants to discuss topological properties of canonical extensions of order-preserving maps. In the work of Ghilardi and Meloni [44], the action of join-preserving maps on filters and ideals receives the attention it deserves, although the authors nowhere refer explicitly to canonical extensions. The idea to define canonical extensions of *order-preserving* maps, rather than join-preserving maps or operators, via the filter and ideal completion seems to have been introduced by Gehrke and Priestley in [41], but in that paper most attention was directed at lattice homomorphisms.

The definition of $f^\triangledown$ and $f^\vartriangle$ for an order-preserving map in §2.2.1 is well known from the work of Gehrke and Harding [34]. Theorem 2.2.4, which describes $f^\triangledown$ and $f^\vartriangle$ as the largest and smallest continuous extensions of an order-preserving map f, respectively, was previously only known to hold for distributive lattices. It raises questions which we will return to in Remark 3.2.22. The result concerning preservation of order embeddings (Lemma 2.2.6) is also an improvement over what was previously known (viz. that the canonical extension of an injective lattice homomorphism is again injective).

The technical results about order-preserving maps applied to filters and ideals in §2.2.2 are new, although similar results can be found in [44]. It was already known from [34] that canonical extensions of join-preserving maps and operators are very well-behaved (Theorem 2.2.18); the topological results we presented are new however.

Further work

- In this section, we have made much use of the filter completion and the ideal completion, which we borrowed from domain theory. It would be interesting to see if there are more domain-theoretic tools available that we can use to better understand and develop the theory of canonical extensions. We will see an example of this in the next section, where we use dcpo presentations to describe canonical extensions.

- Another interesting question is to see whether the domain theory tools and topological methods of this section can also be applied in the setting of monotone partially ordered algebras, as studied by Dunn et al. in [32].

- A more specific question which needs to be resolved is whether the canonical extension of *any* order embedding is again an order embedding, cf. Lemma 2.2.6.

2.3 Canonical extensions via dcpo presentations

Thus far, we have explored canonical extensions of lattices and order-preserving maps via topological methods. We will conclude this chapter with a section, based on results from [42], in which we show that canonical extensions can also be understood via *dcpo presentations*, a technique from domain theory [60]. Apart from being interesting in its own right, the perspective on canonical extensions using dcpo presentations sheds a different light on the issue of extending maps between lattices to maps between their canonical extensions, and ultimately on the question whether inequations valid on a distributive lattice with operators $\mathbb{A}$ are also valid on its canonical extension $\mathbb{A}^\delta$, i.e. the question when inequations are canonical.

The idea of dcpo presentations is to give a unique description of a dcpo $\mathbb{D}$, that is, of a directed complete partial order, by specifiying a partially ordered or pre-ordered set of generators P such that each element of $\mathbb{D}$ is the directed join of the generators below it. If one imposes no relations on the generators other than the order, that is, if one freely adds all directed joins to P, then this is equivalent to taking the ideal completion of P. Thus we can give presentations of algebraic dcpos. If, on the other hand, one imposes relations of the shape $a \leq \bigvee U$, for $\{a\} \cup U \subseteq P$, then we may get a presentation for any dcpo, see [60].

To see how we may use this to get a dcpo presentation of the canonical extension of a lattice $\mathbb{L}$, recall from Theorem 2.1.25 that there exists an order embedding $g\colon \mathbb{L}^\delta \to \mathcal{I}\mathcal{F}\mathbb{L}$ such that $g \circ e^{\mathcal{F}}_{\mathbb{L}} = {\downarrow_{\mathcal{F}\mathbb{L}}}$.

$$\begin{tikzcd}
 & & \mathbb{L}^\delta \arrow[dr, "g"] & \\
\mathbb{L} \arrow[urr, "e_{\mathbb{L}}"] \arrow[r, "\uparrow_{\mathbb{L}}"'] & \mathcal{F}\mathbb{L} \arrow[ur, "e^{\mathcal{F}}_{\mathbb{L}}"'] \arrow[rr, "\downarrow_{\mathcal{F}\mathbb{L}}"'] & & \mathcal{I}\mathcal{F}\mathbb{L}
\end{tikzcd}$$

where $g = \left(e^{\mathcal{F}}_{\mathbb{L}}\right)^{-1} \circ {\downarrow_{\mathbb{L}^\delta}}$ and $e^{\mathcal{F}}_{\mathbb{L}}$ are embeddings. What this diagram tells us is that the canonical extension of $\mathbb{L}$ 'sits between' $\mathcal{F}\mathbb{L}$ and $\mathcal{I}\mathcal{F}\mathbb{L}$, the structure one obtains by freely adding *all* directed joins to $\mathcal{F}\mathbb{L}$. The idea of the dcpo presentation of canonical extension is to be selective and only add *some* directed joins to $\mathcal{F}\mathbb{L}$, so that we obtain $\mathbb{L}^\delta$. Because of the way that filters, ideals and canonical extensions dualize with respect to order (see Lemma 2.1.26), we can just as well regard $\mathbb{L}^\delta$ as an object sitting in between $\mathcal{I}\mathbb{L}$ and $\mathcal{F}\mathcal{I}\mathbb{L}$; this would lead to a description of $\mathbb{L}^\delta$ via a *co-dcpo* presentation over $\mathcal{I}\mathbb{L}$. We choose not to engage in this exercise of dualization.

Since dcpo presentations are characterized externally, that is, by conditions on their behaviour with respect to maps, it is quite natural to expect that we can use them to describe canonical extensions of maps between lattices. In this section, we will restrict our attention to maps of the type $f\colon \mathbb{L}^n \to \mathbb{L}$, with n a natural number. This choice is dictated by economy rather than necessity: we will develop just enough of the technique of extending maps between lattices to canonical extensions via dcpo presentations to allow us to prove a canonicity result in §3.3.3. The key observation is that under the right assumptions, the canonical extension of a map can be seen as an instance of an extension via dcpo presentations, so that one can apply results about dcpo presentations to canonical extensions.

2.3.1 Dcpo presentations

In this subsection, we introduce dcpo presentations, which are a technical tool for uniquely specifying a dcpo without spelling out its entire structure. This is an instance of a very general algebra technique, namely that of specifications by generators and relations.

2.3.1. Definition. A *dcpo presentation* [60] is a triple $\langle P, \sqsubseteq, \lhd \rangle$ where

- $\langle P, \sqsubseteq \rangle$ is a pre-order;
- $\lhd \subseteq P \times \mathcal{P}(P)$ is a binary relation such that $a \lhd U$ only if $U \subseteq P$ is non-empty and directed.

An order-preserving map $f\colon P \to \mathbb{D}$ to a dcpo $\mathbb{D}$ is *cover-stable* if for all $a \lhd U$, $f(a) \leq \bigvee f[U]$.

In other words, a dcpo presentation consists of a pre-ordered set of generators $\langle P, \sqsubseteq \rangle$ together with set of relations of the form $a \leq \bigvee U$. But what does it mean for a dcpo presentation to uniquely describe, i.e. to present a dcpo?

2.3.2. Definition. A dcpo presentation $\langle P, \sqsubseteq, \lhd \rangle$ *presents* a dcpo $\mathbb{D}$ if there exists a cover-stable order-preserving map $\eta\colon P \to \mathbb{D}$ such that for all dcpos $\mathbb{E}$, if $f\colon P \to \mathbb{E}$ is a cover-stable order-preserving map then there exists a unique Scott-continuous $f'\colon \mathbb{D} \to \mathbb{E}$ such that $f' \circ \eta = f$. If this is the case, we say that $\langle P, \sqsubseteq, \lhd \rangle$ presents $\mathbb{D}$ via η.

$$\begin{tikzcd} & \mathbb{D} \arrow[d, dotted, "f'"] \\ P \arrow[ur, "\eta"] \arrow[r, "f"'] & \mathbb{E} \end{tikzcd}$$

We may ask ourselves if every dcpo presentation uniquely describes, i.e. presents a dcpo. This is indeed the case; one can show this by constructing a dcpo from a given presentation using so-called C-ideals. For more information we refer the reader to [60].

2.3.3. Fact. *Every dcpo presentation presents a dcpo.*

We conclude this subsection with two trivial examples of dcpo presentations.

2.3.4. Example. If $\mathbb{P} = \langle P, \leq \rangle$ is a poset, then $\langle P, \leq, \emptyset \rangle$ presents $\mathcal{I}\,\mathbb{P}$ via $\downarrow\colon \mathbb{P} \to \mathcal{I}\,\mathbb{P}$. This follows from the universal property of the ideal completion. What makes this example trivial is the fact that there are *no* relations imposed on the generators.

If $\mathbb{D}$ is a dcpo, then $\langle D, \leq, \lhd_{\mathbb{D}} \rangle$, where $a \lhd_{\mathbb{D}} U$ iff $a \leq \bigvee U$, presents $\mathbb{D}$ itself via $id_{\mathbb{D}}\colon \mathbb{D} \to \mathbb{D}$. This is a trivial example because there are as many generators as there are elements in the dcpo being presented and the order is already fully specified: the main reason for considering presentations of objects via generators and relations, rather than the objects themselves, is that the presentations can be simpler to describe than the objects they are presenting. In the current example, this is not the case.

2.3.2 A dcpo presentation of the canonical extension

We now define a dcpo presentation given a lattice $\mathbb{L}$, with the aim of showing that this dcpo presentation presents $\mathbb{L}^\delta$. This is a two-stage process: first we take $\mathbb{L}$ and we define a presentation $\Delta(\mathbb{L})$, using $\mathcal{F}\,\mathbb{L}$ as the set of generators. Then, we show that $\Delta(\mathbb{L})$ presents $\mathbb{L}^\delta$.

2.3.5. Definition. Given a lattice $\mathbb{L}$, we define a dcpo presentation $\Delta(\mathbb{L}) := \langle \mathcal{F}\,\mathbb{L}, \supseteq, \lhd_{\mathbb{L}} \rangle$, where for all $F \in \mathcal{F}\,\mathbb{L}$ and $S \subseteq \mathcal{F}\,\mathbb{L}$ directed,

$$F \lhd_{\mathbb{L}} S \text{ iff } \forall I \in \mathcal{I}\,\mathbb{L},\ \big[\forall F' \in S,\ F' \between I\big] \Rightarrow F \between I.$$

We want to emphasize that this definition is not being pulled out of a hat. Firstly, since the filter elements of $\mathbb{L}^\delta$ are join-dense in $\mathbb{L}^\delta$, it is not a strange idea to take the filters of $\mathbb{L}$ as generators. Secondly, we know by Lemma 2.1.29 that if $e\colon \mathbb{L} \to \mathbb{L}^\delta$ is the canonical extension of $\mathbb{L}$, then

$$\bigvee\{e^{\mathcal{F}}(F) \mid F \in S\} = \bigwedge\{e^{\mathcal{I}}(I) \mid \forall F \in S,\ F \between I\},$$

and we will see in the proof of the theorem below that this equation is essentially equivalent to the relations of the shape $F \lhd_{\mathbb{L}} S$ we are imposing on $\Delta(\mathbb{L})$.

2.3.6. Theorem. *Let $\mathbb{L}$ be a lattice and let $e\colon \mathbb{L} \to \mathbb{L}^\delta$ be its canonical extension. Then $\Delta(\mathbb{L})$ presents $\mathbb{L}^\delta$ via $e^{\mathcal{F}}\colon\ \mathcal{F}\,\mathbb{L} \to \mathbb{L}^\delta$.*

Proof Observe that $e^{\mathcal{F}}\colon\ \mathcal{F}\,\mathbb{L} \to \mathbb{L}^\delta$ is order-preserving by Lemma 2.1.28(1); we now first need to show that $e^{\mathcal{F}}$ is cover-stable. We will show something stronger: for all $F \in \mathcal{F}\,\mathbb{L}$ and $S \subseteq \mathcal{F}\,\mathbb{L}$ directed,

$$F \lhd_{\mathbb{L}} S \text{ iff } e^{\mathcal{F}}(F) \leq \bigvee_{F' \in S} e^{\mathcal{F}}(F'). \tag{2.22}$$

The key to this observation is Lemma 2.1.29, which states that $\bigvee_{F' \in S} e^{\mathcal{F}}(F') = \bigwedge\{e^{\mathcal{I}}(I) \mid \forall F' \in S, F' \between I\}$. Now

$e^{\mathcal{F}}(F) \leq \bigvee_{F' \in S} e^{\mathcal{F}}(F')$	
iff $e^{\mathcal{F}}(F) \leq \bigwedge\{e^{\mathcal{I}}(I) \mid \forall F' \in S, F' \between I\}$	by Lemma 2.1.29,
iff $\forall I \in \mathcal{I}\,\mathbb{L}$, $[\forall F' \in S, F' \between I] \Rightarrow e^{\mathcal{F}}(F) \leq e^{\mathcal{I}}(I)$	by order theory,
iff $\forall I \in \mathcal{I}\,\mathbb{L}$, $[\forall F' \in S, F' \between I] \Rightarrow F \between I$	(†),
iff $F \lhd_{\mathbb{L}} S$	by def. of $\lhd_{\mathbb{L}}$,

where (†) follows from the basic fact about canonical extensions that $e^{\mathcal{F}}(F) \leq e^{\mathcal{I}}(I)$ iff $F \between I$. It follows that (2.22) holds.

Next, suppose that $f\colon \mathcal{F}\mathbb{L} \to \mathbb{D}$ is an order-preserving cover-stable map to a dcpo $\mathbb{D}$. We define $f'\colon \mathbb{L}^\delta \to \mathbb{D}$ by setting

$$f'\colon x \mapsto \bigvee\{f(F) \mid e^{\mathcal{F}}(F) \leq x\}.$$

We need to show (1) that f' is well-defined and Scott-continuous, (2) that $f' \circ e^{\mathcal{F}} = f$ and (3) that f' is unique with respect to properties (1) and (2).

(1). For any $x \in \mathbb{L}^\delta$, the set $\downarrow x$ is an ideal. Since $e^{\mathcal{F}}$ is a $\vee$-homomorphism (Lemma 2.1.28(1)), it follows that $(e^{\mathcal{F}})^{-1}(\downarrow x) = \{F \mid e^{\mathcal{F}}(F) \leq x\}$ is also an ideal and hence, directed, Since f is order-preserving it follows that $f'(x)$ is a directed join, so that indeed f' is well-defined. To see that f' is Scott-continous, take $S \subseteq \mathbb{L}^\delta$ directed. Observe that

$$
\begin{aligned}
&\bigvee S \\
&= \bigvee_{x\in S} \bigvee\{e^{\mathcal{F}}(F) \mid e^{\mathcal{F}}(F) \leq x\} && \text{by join-density of filter elements,} \\
&= \bigvee \textstyle\bigcup_{x\in S}\{e^{\mathcal{F}}(F) \mid e^{\mathcal{F}}(F) \leq x\} && \text{by associativity of } \textstyle\bigvee, \\
&= \bigvee e^{\mathcal{F}}\left[\textstyle\bigcup_{x\in S}\{F \mid e^{\mathcal{F}}(F) \leq x\}\right] && \text{by elementary set theory.}
\end{aligned}
$$

It is not hard to see that $\bigcup_{x\in S}\{F \mid e^{\mathcal{F}}(F) \leq x\}$ is a directed union of directed sets; consequently, we will simply assume that S is a directed set of filter elements; say $S = \{e^{\mathcal{F}}(F) \mid F \in S'\}$ where $S' \subseteq \mathcal{F}\mathbb{L}$ is directed. Now observe that

$$f'\big(\textstyle\bigvee_{F\in S'} e^{\mathcal{F}}(F)\big) = \bigvee\left\{f(F') \mid e^{\mathcal{F}}(F') \leq \textstyle\bigvee_{F\in S'} e^{\mathcal{F}}(F)\right\} = \bigvee\{f(F') \mid F' \lhd_{\mathbb{L}} S'\},$$

where the last equality follows by (2.22). Since f is cover-stable, $F' \lhd_{\mathbb{L}} S'$ implies $f(F') \leq \bigvee f[S']$. We see that

$$f'\big(\textstyle\bigvee_{F\in S'} e^{\mathcal{F}}(F)\big) = \bigvee\{f(F') \mid F' \lhd_{\mathbb{L}} S'\} \leq \displaystyle\bigvee_{F\in S'} f(F) = \bigvee_{F\in S'} f'\big(e^{\mathcal{F}}(F)\big),$$

so that it follows that f' is Scott-continuous.

(2). To see that $f' \circ e^{\mathcal{F}} = f$, observe that

$$f' \circ e^{\mathcal{F}}(F) = \bigvee\{f(F') \mid e^{\mathcal{F}}(F') \leq e^{\mathcal{F}}(F)\}.$$

Because $e^{\mathcal{F}}$ is an order embedding, the join in the RHS above has a maximal element, viz. $f(F)$. It follows that $f' \circ e^{\mathcal{F}} = f$.

(3). Suppose that $f''\colon \mathbb{L}^\delta \to \mathbb{D}$ is Scott-continuous and that $f'' \circ e^{\mathcal{F}} = f$. Take $x \in \mathbb{L}^\delta$, then since $x = \bigvee\{e^{\mathcal{F}}(F) \mid e^{\mathcal{F}}(F) \leq x\}$ is a directed join, we see that

$$
\begin{aligned}
f''(x) = f''\big(\textstyle\bigvee\{e^{\mathcal{F}}(F) \mid e^{\mathcal{F}}(F) \leq x\}\big) &= \textstyle\bigvee\{f'' \circ e^{\mathcal{F}}(F) \mid e^{\mathcal{F}}(F) \leq x\} \\
&= \textstyle\bigvee\{f(F) \mid e^{\mathcal{F}}(F) \leq x\} = f'(x).
\end{aligned}
$$

It follows that $f' = f''$ so that f' is unique. ∎

What have we learned now? Firstly, we have discovered a new characterization of the canonical extension of a lattice $\mathbb{L}$, namely as a certain dcpo generated by the filters of $\mathbb{L}$. Secondly, we have paved the way for applying general results about dcpo algebras to canonical extensions of lattice-based algebras, as we will see later.

2.3.3 Extending maps via dcpo presentations

We will now briefly look at extensions of maps via dcpo presentations. The goal is to be able to use results about dcpo presentations to prove facts about canonical extensions. Specifically, we would like to be able to lift a map $f\colon \mathbb{L}^n \to \mathbb{L}$ through our two-stage construction, first to a map defined on $\mathcal{F}\,\mathbb{L}$ and then via the dcpo presentation to a map on $\mathbb{L}^\delta$. We first state the facts we need for the second stage of extending f.

Let $\langle P, \sqsubseteq, \lhd\rangle$ be a dcpo presentation and let $f\colon P^n \to P$ be an order-preserving map. We say f is cover-stable if f preserves covers in each coordinate, i.e. for all $1 \leq i \leq n$, for all $a_1, \ldots, a_n \in P$, for all $U \subseteq P$,

$$a_i \lhd U \Rightarrow f(a_1, \ldots, a_n) \lhd \{f(a_1 \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n) \mid b \in U\}.$$

2.3.7. Fact ([60]). *Let $\langle P, \sqsubseteq, \lhd\rangle$ be a dcpo presentation which presents a dcpo $\mathbb{D}$ via $\eta\colon P \to D$. If $f\colon P^n \to P$ is a cover-stable order-preserving map, then there exists a unique Scott-continuous map $\overline{f}\colon \mathbb{D}^n \to \mathbb{D}$ which extends f, i.e. such that $\overline{f} \circ \eta^n = \eta \circ f$.*

Suppose we are given a map $f\colon \mathbb{L}^n \to \mathbb{L}$, and recall that $\mathcal{F}\,f(F_1, \ldots, F_n) := {\uparrow} f[F_1, \ldots, F_n]$. How do we know if $\mathcal{F}\,f\colon (\mathcal{F}\,\mathbb{L})^n \to \mathcal{F}\,\mathbb{L}$ is cover-stable? The best condition we know of is the rather strong requirement that f is an operator, i.e. if f preserves binary joins in each coordinate.

2.3.8. Lemma. *Let $\mathbb{L}$ be a lattice. If $f\colon \mathbb{L}^n \to \mathbb{L}$ is an operator, then $\mathcal{F}\,f\colon (\mathcal{F}\,\mathbb{L})^n \to \mathcal{F}\,\mathbb{L}$ is cover-stable with respect to $\Delta(\mathbb{L})$. Consequently, $\mathcal{F}\,f$ extends to a map $\overline{\mathcal{F}\,f}\colon (\mathbb{L}^\delta)^n \to \mathbb{L}^\delta$; moreover, $\overline{\mathcal{F}\,f} = f^\triangledown$.*

Proof Let $e\colon \mathbb{L} \to \mathbb{L}^\delta$ be the canonical extension of $\mathbb{L}$ and let $f\colon \mathbb{L}^n \to \mathbb{L}$ be an operator; to lighten the notation, we assume that $n = 2$.

First, we need to show that $\mathcal{F}\,f$ is cover-stable with respect to $\Delta(\mathbb{L})$. It suffices to show that $\mathcal{F}\,f$ is cover-stable in its first coordinate. So suppose that $\{F, G\} \cup S \subseteq \mathcal{F}\,\mathbb{L}$ such that $F \lhd_{\mathbb{L}} S$; we want to show that

$$\mathcal{F}\,f(F, G) \lhd_{\mathbb{L}} \{\mathcal{F}\,f(F', G) \mid F' \in S\}.$$

By definition of $\lhd_{\mathbb{L}}$, this amounts to showing that if

$$\forall I \in \mathcal{I}\,\mathbb{L}_1,\ [\forall F' \in S,\ F \between I] \Rightarrow F \between I,$$

then also

$$\forall J \in \mathcal{I}\,\mathbb{M},\ [\forall F' \in S,\ \mathcal{F}\,f(F', G) \between J] \Rightarrow \mathcal{F}\,f(F, G) \between J.$$

But that is exactly the statement of Lemma 2.2.9, so it follows immediately that $\mathcal{F}\,f$ is cover-stable.

Now, observe that by Fact 2.3.7, we know that $\mathcal{F}\,f$ has a Scott-continuous extension $\overline{\mathcal{F}\,f}$ such that $\overline{\mathcal{F}\,f} \circ (e^{\mathcal{F}} \times e^{\mathcal{F}}) = e^{\mathcal{F}} \circ \mathcal{F}\,f$. We will show that $\overline{\mathcal{F}\,f} = f^{\triangledown}$. Take $x, y \in \mathbb{L}^{\delta}$, then

$$\overline{\mathcal{F}\,f}(x, y) = \overline{\mathcal{F}\,f}\left(\bigvee\{e^{\mathcal{F}}(F) \mid e^{\mathcal{F}}(F) \leq x\}, \bigvee\{e^{\mathcal{F}}(G) \mid e^{\mathcal{F}}(G) \leq y\}\right),$$

by join-density of filter elements. Now since each of the joins above is directed (Lemma 2.1.28) and $\overline{\mathcal{F}\,f}$ is Scott-continuous, we see that

$$\begin{aligned}
\overline{\mathcal{F}\,f}(x, y) &= \bigvee \left\{\overline{\mathcal{F}\,f}(e^{\mathcal{F}}(F), e^{\mathcal{F}}(G)) \mid e^{\mathcal{F}}(F) \leq x,\ e^{\mathcal{F}}(G) \leq y\right\} \\
&= \bigvee \left\{e^{\mathcal{F}} \circ \mathcal{F}\,f(F, G) \mid e^{\mathcal{F}}(F) \leq x,\ e^{\mathcal{F}}(G) \leq y\right\} && \text{by F. 2.3.7,} \\
&= f^{\triangledown}(x, y),
\end{aligned}$$

where the last equality follows by definition of $f^{\triangledown}$. Since $x, y \in \mathbb{L}^{\delta}$ were arbitrary, it follows that $\overline{\mathcal{F}\,f} = f^{\triangledown}$. ∎

Thus, we see that canonical extensions of operators can be described using dcpo presentation techniques. This concludes our discussion of dcpo presentations for now. We will use what we have learned here later on, in §3.3.3.

2.3.4 Conclusions and further work

This section is based on a paper of M. Gehrke and the author [42], which was written to demonstrate how a canonicity result for distributive lattices with operators from [38] can be seen as a special case of a result concerning dcpo algebras from [60], see §3.3.3.

What we have seen here (and what we will see in §3.3.3) is part of the intersection of the structures and results which can be described both using dcpo algebras and canonical extensions. It would be interesting to further explore the overlapping area of the two fields.

Chapter 3

Canonical extensions: topological algebra and categorical properties

In Chapter 2, we have presented a toolkit of results which allow us to work with canonical extensions of bounded lattices and order-preserving maps, using techniques from domain theory. In this chapter, we will start looking at canonical extensions of *lattice-based algebras*, which is the main application area of canonical extensions, and we will see that canonical extensions are finely intertwined with *topological algebra*.

As we discussed in Chapter 1, the main application of canonical extensions is to provide representation theorems for lattice-based algebras, which arise in algebraic logic. Via such representation theorems (e.g. the Jónsson-Tarski theorem [58]), questions about logics can be reduced to questions about canonical extensions, and this is where the canonical extensions toolkit can be very useful. (We do point out however that in this chapter we are concerned with the mathematical properties of canonical extensions, rather than their applications in logic.)

Topological algebra, the study of algebras with *continuous* operations, has its origins in classical mathematics, perhaps most notably in Galois theory: Galois groups of field extensions are profinite groups [77], and profinite algebras are an important example of topological algebras (see §3.1.2). We will see that profinite lattices arise naturally as canonical extensions of lattices in finitely generated varieties of lattices.

The connections between canonical extensions and topological algebra are many and intricate. The core connection, however, is that *canonical extensions have universal properties with respect to topological algebras.* Meaning, that if we have a lattice-based algebra $\mathbb{A}$ and an algebra homomorphism $f\colon \mathbb{A} \to \mathbb{B}$, where $\mathbb{B}$ is a topological algebra, then there exists a unique continuous homomorphism

$f' \colon \mathbb{A}^\delta \to \mathbb{B}$, extending f.

$$\begin{tikzcd}
 & \mathbb{A}^\delta \arrow[d, dotted, "f'"] \\
\mathbb{A} \arrow[ur, "e_{\mathbb{A}}"] \arrow[r, "f"'] & \mathbb{B}
\end{tikzcd}$$

This is subject to certain assumptions on the algebras $\mathbb{A}$ and $\mathbb{B}$, which we will review in §3.4.

We begin this chapter with a review of preliminary facts about topological algebras in §3.1. After that, we will first expand the canonical extension toolkit with results that exploit topological lattice properties of canonical extensions of *lattices* in §3.2. We will then define lattice-based algebras and canonical extenions of lattice-based algebras in §3.3. Finally in §3.4, we discuss the connection between topological lattice-based *algebras* and canonical extensions.

3.1 Topological algebra

In this section we will review certain useful facts about two important classes of topological algebras, namely compact Hausdorff algebras and profinite algebras, which form a subclass of the compact Hausdorff algebras. After that we will take a closer look at topological lattices, which will lead to a characterization theorem for Boolean topological lattices. The idea behind topological algebra is very simple. Given an algebra signature Ω, an Ω-algebra is a structure $\mathbb{A} = \langle A, (\omega_{\mathbb{A}})_{\omega\in\Omega}\rangle$ consisting of a set A and functions $\omega_{\mathbb{A}} \colon A^{\mathrm{ar}(\omega)} \to A$ for all $\omega \in \Omega$. A topological Ω-algebra is a structure $\langle A, (\omega_{\mathbb{A}})_{\omega\in\Omega}, \tau_{\mathbb{A}}\rangle$ such that $\langle A, \tau_{\mathbb{A}}\rangle$ is a topological space and each $\omega_{\mathbb{A}} \colon A^{\mathrm{ar}(\omega)} \to A$ is a $(\tau^{\mathrm{ar}(\omega)}, \tau)$-continuous function. This makes a topological algebra $\mathbb{A} = \langle A, (\omega_{\mathbb{A}})_{\omega\in\Omega}\tau_{\mathbb{A}}\rangle$ into an object which has one foot in the world of algebra and one foot in the world of topology.

3.1.1. Example. If $\mathbb{A} = \langle A, (\omega_{\mathbb{A}})_{\omega\in\Omega}\rangle$ is an Ω-algebra, then $\langle A, (\omega_{\mathbb{A}})_{\omega\in\Omega}, \mathcal{P}(A)\rangle$ is a topological algebra, where $\mathcal{P}(A)$ is the discrete topology on A. Although this example is rather trivial, we shall see when defining profinite algebras that it is very important.

In §3.1.1, we will first briefly look at compact Hausdorff topological algebras, or compact Hausdorff algebras for short, which form the most general class of topological algebras we will consider. We will look at the way the subalgebra construction behaves when it comes to compact Hausdorff algebras, and we will consider the compactification functor for compact Hausdorff algebras. After that, we will review some facts concerning profinite algebras in §3.1.2. Profinite algebras will play an important role later on in this chapter, so we will go into some detail to make the reader familiar with their definition and construction. Finally, in §3.1.3 we will look at a particular class of topological algebras, namely topological lattices.

We will review some of the key facts about compact Hausdorff lattices, most notably the fact (due to J.D. Lawson) that the topology on a compact Hausdorff lattice is unique. We will then specialize further by looking at Boolean topological lattices, i.e. lattices with a compact Hausdorff zero-dimensional topology and continuous meet and join. What is nice about Boolean topological lattices is the fact that one can characterize them order-theoretically, a result due to H.A. Priestley.

3.1.1 Compact Hausdorff algebras

In this subsection we will state several facts about topological algebras for which the topology is compact Hausdorff. One of the things which make compact Hausdorff algebras particularly interesting is the fact every Ω-algebra $\mathbb{A}$ has a unique compactification $\eta_{\mathbb{A}}\colon \mathbb{A} \to \beta\,\mathbb{A}$ with the universal property that whenever we have a compact Hausdorff algebra $\langle \mathbb{B}, \tau\rangle$ and an algebra homomorphism $f\colon \mathbb{A} \to \mathbb{B}$, then there exists a unique continuous $f'\colon\ \beta\,\mathbb{A} \to \mathbb{B}$.

$$\begin{tikzcd} & \beta\,\mathbb{A} \arrow[d, dotted, "f'"] \\ \mathbb{A} \arrow[ur, "\eta_{\mathbb{A}}"] \arrow[r, "f"'] & \mathbb{B} \end{tikzcd}$$

We like to think of $\beta\,\mathbb{A}$ as an extension of $\mathbb{A}$, and of f' as a continuous extension of f, however there is one important caveat to this perspective: in general, the natural map $\eta_{\mathbb{A}}\colon \mathbb{A} \to \beta\,\mathbb{A}$ is *not* an embedding. This is quite contrary to the notion of compactification from general topology, where a compactification of a space X is a compact space Y of which X is a dense *subspace*.

3.1.2. FACT ([84]). *For any Ω-algebra $\mathbb{A}$ there exists a compact Hausdorff Ω-algebra $\langle \beta\,\mathbb{A}; \tau_{\mathbb{A}}\rangle$ and a homomorphism $\eta_{\mathbb{A}}\colon \mathbb{A} \to \beta\,\mathbb{A}$ such that*

1. *$\eta_{\mathbb{A}}[\mathbb{A}]$ is a dense subalgebra of $\langle \beta\,\mathbb{A}, \tau_{\beta\,\mathbb{A}}\rangle$,*
2. *for every compact Hausdorff algebra $\mathbb{B}$ and every $f\colon \mathbb{A} \to \mathbb{B}$, there exists a unique continuous homomorphism $f'\colon\ \beta\,\mathbb{A} \to \mathbb{B}$ such that $f' \circ \eta_{\mathbb{A}} = f$.*

$$\begin{tikzcd} & \beta\,\mathbb{A} \arrow[d, dotted, "f'"] \\ \mathbb{A} \arrow[ur, "\eta_{\mathbb{A}}"] \arrow[r, "f"'] & \mathbb{B} \end{tikzcd}$$

3.1.3. REMARK. Recall that a variety $\mathcal{V}$ is called *residually small* if there exist a bound on the cardinality of the subdirectly irreducible algebras in $\mathcal{V}$. It has been conjectured [12, p. 519] that a sufficient condition for $\eta_{\mathbb{A}}\colon \mathbb{A} \to \beta\,\mathbb{A}$ being injective is that $\mathbb{A} \in \mathcal{V}$ for some residually small variety $\mathcal{V}$. However, at this time only the converse of this conjecture is known to hold.

3.1.4. Fact (Corollary of Th. 1.2 of [85]). *Let $\mathcal{V}$ be a variety. If $\eta_{\mathbb{A}}\colon \mathbb{A} \to U\beta\mathbb{A}$ is an embedding for every $\mathbb{A}$ in $\mathcal{V}$, then $\mathcal{V}$ is residually small.*

3.1.5. Fact (Th. 2.1.9 of [33]). *Let $f, g\colon X \to Y$ be continuous maps from a topological space X into a Hausdorff space Y. If there is a dense subset $Z \subseteq X$ such that $f \restriction Z = g \restriction Z$, then $f = g$.*

The following Lemma is an elementary fact from topological algebra. Observe the distinction between the topological algebra $\langle \mathbb{B}, \tau \rangle$ and the discrete algebras $\mathbb{A}$, $\mathbb{B}$.

3.1.6. Lemma. *Let $\langle \mathbb{B}, \tau \rangle$ be a Hausdorff topological algebra and let $\mathbb{A}$ be a subalgebra of $\mathbb{B}$ such that $\mathbb{A}$ is dense in $\langle \mathbb{B}, \tau \rangle$. Then $\mathbb{B} \in \mathrm{HSP}(\mathbb{A})$.*

Proof We will show that if $s \approx t$ is an equation such that $\mathbb{A} \models s \approx t$, then also $\mathbb{B} \models s \approx t$. Suppose s and t use n variables. Consider the term functions $s_{\mathbb{A}}\colon \mathbb{A}^n \to \mathbb{A}$ and $t_{\mathbb{A}}\colon \mathbb{A}^n \to \mathbb{A}$. Because $\mathbb{A} \models s \approx t$ and $\mathbb{A}$ is a subalgebra of $\mathbb{B}$, we know that $s_{\mathbb{B}}$ and $t_{\mathbb{B}}$ agree on A^n. Now because $\mathbb{B}$ is a topological algebra, the term functions $s_{\mathbb{B}}\colon \mathbb{B}^n \to \mathbb{B}$ and $t_{\mathbb{B}}\colon \mathbb{B}^n \to \mathbb{B}$ are continuous. But A^n is dense in the Hausdorff space $\langle B^n, \tau^n \rangle$, so by Fact 3.1.5, $s_{\mathbb{B}} = t_{\mathbb{B}}$. We conclude that $\mathbb{B} \models s \approx t$. ∎

3.1.7. Corollary. *If $\mathbb{A}$ is an Ω-algebra, then $\beta\,\mathbb{A} \in \mathrm{HSP}(\mathbb{A})$.*

3.1.2 Profinite algebras and profinite completions

In this subsection, we will introduce two notions which are central to this chapter, namely profinite algebras and profinite completions. Profinite algebras are a very natural and well-behaved example of topological algebras, which have been studied extensively in the setting of groups [77] because of the connections between Galois theoy and profinite groups.

To be able to say what profinite algebras and profinite completions are, we must first introduce the notion of a limit of a diagram of algebras. The way one defines a limit is from the bottom up, through a universal property with respect to a diagram of objects. Roughly, this means that every mapping going into the diagram of algebras has to factor uniquely through the limit of the diagram. In this sense, a limit is a single object which represents a collection of objects, viz. the diagram of which it is a limit. At the same time, however, we can view a limit from the top down as a single object which can be broken down into a representation in the form of the diagram of which it is a limit. This perspective is particularly salient in the case of *profinite algebras*, which are those algebras which arise as limits of diagrams of *finite* algebras. It is precisely the fact that a profinite algebra $\mathbb{A} \simeq \varprojlim_I \mathbb{A}_i$ can be broken down into a diagram of finite algebras $\mathbb{A}_i$ that allows us to define a topology τ on $\mathbb{A}$ making $\langle \mathbb{A}, \tau \rangle$ a topological algebra.

A second important consequence of the fact that every profinite algebra $\mathbb{B}$ has a categorical representation as a diagram, is that it allows us to associate with each algebra $\mathbb{A}$ a unique profinite algebra $\hat{\mathbb{A}}$, the *profinite completion* of $\mathbb{A}$ and a natural map $\mu_{\mathbb{A}}\colon \mathbb{A} \to \hat{\mathbb{A}}$, such that whenever we have a homomorphism $f\colon \mathbb{A} \to \mathbb{B}$ from $\mathbb{A}$ to a profinite algebra $\mathbb{B}$, then f will factor through the natural map $\mu_{\mathbb{A}}\colon \mathbb{A} \to \hat{\mathbb{A}}$:

$$\begin{tikzcd} & \hat{\mathbb{A}} \arrow[d, dotted, "f'"] \\ \mathbb{A} \arrow[ur, "\mu_{\mathbb{A}}"] \arrow[r, "f"'] & \mathbb{B} \end{tikzcd}$$

We will make frequent use of the universal property of $\mu_{\mathbb{A}}\colon \mathbb{A} \to \hat{\mathbb{A}}$ in §3.4.

Poset-indexed limits of Ω-algebras

We will now show how to define limits of diagrams of algebras, where we restrict ourselves to the case in which the diagrams are indexed by a poset (see Remark 3.1.11). This will provide us with the technical means to define profinite algebras and profinite completions later on. Fix an algebra signature Ω and let $\mathbf{Alg}_\Omega$ denote the category of Ω-algebras and Ω-algebra homomorphisms. Let $\langle I, \leq \rangle$ be a poset. An *I-indexed diagram in* $\mathbf{Alg}_\Omega$ is a functor from I to $\mathbf{Alg}_\Omega$. In other words, an I-indexed diagram is an assigmnent of an Ω-algebra $\mathbb{A}_i$ to each $i \in I$, and a homomorphism $f_{ij}\colon \mathbb{A}_i \to \mathbb{A}_j$ to each $(i,j) \in I \times I$ such that $i \leq j$, with the additional restrictions that

- for all $i \in I$, $f_{ii} = \mathrm{id}_{\mathbb{A}_i}\colon \mathbb{A}_i \to \mathbb{A}_i$;
- if $i \leq j \leq k$, then $f_{jk} \circ f_{ij} = f_{ik}$.

We denote this diagram by $\langle \mathbb{A}_i, f_{ij} \rangle_I$. If the index poset I of a diagram $\langle \mathbb{A}_i, f_{ij} \rangle_I$ is e.g. co-directed, we say that $\langle \mathbb{A}_i, f_{ij} \rangle_I$ is a co-directed diagram of algebras. (Recall that a poset I is co-directed if for all $i, j \in I$ there exists $k \in I$ such that $k \leq i, j$.)

A *cone* to a diagram of algebras $\langle \mathbb{A}_i, f_{ij} \rangle_I$ in $\mathbf{Alg}_\Omega$ is an I-indexed collection of maps $(g_i\colon \mathbb{A} \to \mathbb{A}_i)_I$ with a common domain $\mathbb{A}$ such that for all $i, j \in I$ with $i \leq j$, we have $f_{ij} \circ g_i = g_j$.

$$\begin{tikzcd} \mathbb{A} \arrow[d, "g_i"'] \arrow[dr, "g_j"] & \\ \mathbb{A}_i \arrow[r, "f_{ij}"'] & \mathbb{A}_j \end{tikzcd}$$

Let $(h_i\colon \mathbb{B} \to \mathbb{A}_i)_I$ be another cone to $\langle \mathbb{A}_i, f_{ij} \rangle_I$ and let $e\colon \mathbb{B} \to \mathbb{A}$ be an algebra homomorphism. We say e is a *map of cones* if for all $i \in I$, $g_i \circ e = h_i$.

$$\begin{tikzcd} \mathbb{B} \arrow[r, "e"] \arrow[d, "h_i"'] & \mathbb{A} \arrow[dl, "g_i"] \\ \mathbb{A}_i & \end{tikzcd}$$

We call $(g_i \colon \mathbb{A} \to \mathbb{A}_i)_I$ a *limiting cone* if for all $(h_i \colon \mathbb{B} \to \mathbb{A}_i)_I$, there exists a *unique* map of cones $e \colon \mathbb{B} \to \mathbb{A}$. We then call $\mathbb{A}$ the *limit* of $\langle \mathbb{A}_i, f_{ij} \rangle_I$, writing $\mathbb{A} \simeq \varprojlim_I \mathbb{A}_i$.

3.1.8. Example. If I is a discrete poset, meaning that $i \leq j$ iff $i = j$, then the limit of an I-indexed diagram $\langle \mathbb{A}_i, f_{ij} \rangle_I$ is simply the product: $\varprojlim_I \mathbb{A}_i \simeq \prod_I \mathbb{A}_i$. Indeed, if we have a collection of maps $(\mathbb{B} \xrightarrow{h_i} \mathbb{A}_i)_I$, then these define a unique map $e \colon \mathbb{B} \to \prod_I \mathbb{A}_i$, viz.

$$e \colon a \mapsto (h_i(a))_{i \in I} \,.$$

Moreover, e is a map of cones: if $a \in \mathbb{A}$ and $i \in I$, then $\pi_i \circ e(a) = h_i(a)$.

Now that we have abstractly defined what limits are, we may ask ourselves whether every diagram of algebras actually has a limit.

3.1.9. Fact. *Every poset-indexed diagram $\langle \mathbb{A}_i, f_{ij} \rangle_I$ in $\mathbf{Alg}_\Omega$ has a limiting cone, which may be computed as follows:*

$$\varprojlim_I \mathbb{A}_i = \{\alpha \in \textstyle\prod_I \mathbb{A}_i \mid \forall i, j \in I,\, i \leq j \Rightarrow f_{ij}(\alpha(i)) = \alpha(j)\} \,.$$

Moreover, $\varprojlim_I \mathbb{A}_i$ is a subalgebra of $\prod_I \mathbb{A}_i$.

Because varieties of algebras (§A.6) are closed under taking products and subalgebras, we get as a corollary that varieties are closed under taking limits.

3.1.10. Corollary. *If $\mathcal{V}$ is a variety of Ω-algebras, and if $\langle \mathbb{A}_i, f_{ij} \rangle_I$ is a diagram of Ω-algebras such that $\mathbb{A}_i \in \mathcal{V}$ for each $i \in I$, then also $\varprojlim_I \mathbb{A}_i \in \mathcal{V}$.*

We now know enough about limits of diagrams of algebras to define profinite algebras and profinite completions.

3.1.11. Remark. What we have described here is really only a special case of the categorical notion of limit. The special case of poset-indexed diagrams will allow us to define what profinite algebras and profinite completions are. For a discussion of the category theory at work here, see [69, §III.4].

Profinite Ω-algebras

We will now define profinite algebras and we will sketch why profinite algebras are *topological* algebras. Let us start by simply giving the definition.

3.1.12. Definition. Let $\mathbb{A}$ be an Ω-algebra. We say that $\mathbb{A}$ is a *profinite Ω-algebra* if there exists a poset-indexed diagram $\langle \mathbb{A}_i, f_{ij} \rangle_I$ such that

- $\mathbb{A} \simeq \varprojlim_I \mathbb{A}_i$;

- for each $i \in I$, $\mathbb{A}_i$ is finite;
- the index poset I is co-directed, i.e. for all $i, j \in I$, there exists $k \in I$ such that $k \leq i, j$.

Let $\langle \mathbb{A}_i, f_{ij} \rangle_I$ be a co-directed diagram of finite Ω-algebras and let $(\pi_i \colon \mathbb{A} \to \mathbb{A}_i)_I$ be a limiting cone for this diagram, i.e. assume that $\mathbb{A} \simeq \varprojlim_I \mathbb{A}_i$. We will now sketch how we can use the limiting cone $(\pi_i \colon \mathbb{A} \to \mathbb{A}_i)_I$ to define a topology on $\mathbb{A}$, such that for each $\omega \in \Omega$, $\omega_{\mathbb{A}}$ is continuous. First observe that the following collection of sets forms a base for a topology on $\mathbb{A}$:

$$\left\{ \pi_i^{-1}(U) \mid U \subseteq \mathbb{A}_i,\, i \in I \right\}.$$

To see why, consider two basic opens $\pi_i^{-1}(U)$ and $\pi_j^{-1}(V)$, where $U \subseteq \mathbb{A}_i$ and $V \subseteq \mathbb{A}_j$. We will show that there exists a $k \in I$ and $W \subseteq \mathbb{A}_k$ such that

$$\pi_i^{-1}(U) \cap \pi_j^{-1}(V) = \pi_k^{-1}(W).$$

The key observation is the fact that by co-directedness of the index poset I, there must exist $k \in I$ such that $k \leq i, j$, so there exist maps $f_{ki} \colon \mathbb{A}_k \to \mathbb{A}_i$ and $f_{kj} \colon \mathbb{A}_k \to \mathbb{A}_j$. Because $(\pi_i \colon \mathbb{A} \to \mathbb{A}_i)_I$ is a cone, it must be the case that

$$\pi_i = f_{ki} \circ \pi_k \text{ and } \pi_j = f_{kj} \circ \pi_k. \tag{3.1}$$

We now see that

$$\begin{aligned}
&\pi_i^{-1}(U) \cap \pi_j^{-1}(V) \\
&= (f_{ki} \circ \pi_k)^{-1}(U) \cap (f_{kj} \circ \pi_k)^{-1}(V) && \text{by (3.1),} \\
&= \pi_k^{-1} \circ f_{ki}^{-1}(U) \cap \pi_k^{-1} \circ f_{kj}^{-1}(V) && \text{by basic set theory,} \\
&= \pi_k^{-1}\left(f_{ki}^{-1}(U) \cap f_{kj}^{-1}(V)\right) && \text{idem.}
\end{aligned}$$

Thus we see that if we take $W := f_{ki}^{-1}(U) \cap f_{kj}^{-1}(V)$, then indeed $\pi_i^{-1}(U) \cap \pi_j^{-1}(V) = \pi_k^{-1}(W)$. We call the topology generated by $\{\pi_i^{-1}(U) \mid U \subseteq \mathbb{A}_i,\, i \in I\}$ the *profinite topology* on $\mathbb{A}$. Note that the profinite topology on $\mathbb{A}$ is determined by the limiting cone $(\pi_i \colon \mathbb{A} \to \mathbb{A}_i)_I$; it is the coarsest topology which makes all the projection maps $\pi_i \colon \mathbb{A} \to \mathbb{A}_i$ continuous with respect to the discrete topology on $\mathbb{A}_i$. We will now sketch why $\mathbb{A}$ is a topological algebra with respect to the profinite topology. Consider an operation $\omega \in \Omega$ and suppose for the sake of simplicity that ω is unary. We want to show that $\omega_{\mathbb{A}}$ is continuous. It suffices to show that for a basic open set $\pi_i^{-1}(U)$, where $U \subseteq \mathbb{A}_i$, it is also the case that $\omega_{\mathbb{A}}^{-1}(\pi_i^{-1}(U))$ is open. The key observation is the fact that $\pi_i \colon \mathbb{A} \to \mathbb{A}_i$ is an Ω-algebra homomorphism; this allows us to see that

$$\begin{aligned}
&\omega_{\mathbb{A}}^{-1}\left(\pi_i^{-1}(U)\right) \\
&= (\pi_i \circ \omega_{\mathbb{A}})^{-1}(U) && \text{by basic set theory,} \\
&= (\omega_{\mathbb{A}_i} \circ \pi_i)^{-1}(U) && \text{because } \pi_i \text{ is an } \Omega\text{-hom.,} \\
&= \pi_i^{-1}\left(\omega_{\mathbb{A}_i}^{-1}(U)\right) && \text{by basic set theory,}
\end{aligned}$$

so that $\omega_{\mathbb{A}}^{-1}(\pi_i^{-1}(U))$ is in fact a basic open.

Thus, we see that if we are given a profinite algebra $\mathbb{A} \simeq \varprojlim_I \mathbb{A}_i$, then $\mathbb{A}$ is a topological algebra in its profinite topology, which we defined in a natural way using the limiting cone $(\pi_i \colon \mathbb{A} \to \mathbb{A}_i)_I$. We will now record the fundamental fact that every profinite topology is Boolean, i.e. every profinite topology is compact, Hausdorff and zero-dimensional.

3.1.13. Fact ([8]). *If $\mathbb{A} \simeq \varprojlim \mathbb{A}_i$ is a profinite algebra, then $\mathbb{A}$ is a Boolean topological algebra in its profinite topology.*

3.1.14. Remark. We have chosen to first describe profinite algebras using universal algebra, viz. as the limit of a diagram of finite algebras $\langle \mathbb{A}_i, f_{ij} \rangle_I$, and then to indicate that one can define a topology on $\varprojlim_I \mathbb{A}_i$, making $\varprojlim_I \mathbb{A}_i$ a topological algebra. Alternatively, one can start by endowing each algebra $\mathbb{A}_i$ with the discrete topology, and then show that one can also take limits of *topological* algebras, so that it follows immediately that $\varprojlim_I \mathbb{A}_i$ is a topological algebra. One can then show that $\varprojlim_I \mathbb{A}_i$ is a closed subalgebra of $\prod_I \mathbb{A}_i$, so that it follows from general topology that the profinite topology on $\varprojlim_I \mathbb{A}_i$ is a Boolean topology.

Next, we would like to know how we can construct new profinite algebras from one or more given profinite algebras. The most straightforward construction we can use to create new profinite algebras is to take products.

3.1.15. Fact ([8]). *If $\{\mathbb{A}_i \mid i \in I\}$ is a set of profinite algebras, then $\prod_I \mathbb{A}_i$ is also profinite.*

(An alternative reference for the above fact is [77, Proposition 2.2.1].) Now what about subalgebras? That is, when is a subalgebra of a profinite algebra again profinite? It turns out that here the profinite topology is very useful.

3.1.16. Fact ([8]). *Let $\mathbb{A}$ be a profinite algebra. If $\mathbb{B}$ is subalgebra of $\mathbb{A}$ which is closed in the profinite topology, then $\mathbb{B}$ is also profinite.*

(An alternative reference for the above fact is [77, Corollary 1.1.8].) For other constructions, such as homomorphisms, things are more complicated. We will return to this matter in the special case of canonical extensions of profinite lattices with Lemma 3.2.11.

Before we move on to profinite completions, we mention a result regarding an important question in the study of profinite algebras. We have seen above in Fact 3.1.13 that every profinite Ω-algebra is a Boolean topological algebra in its profinite topology. Conversely, it well known that every Boolean *space* is profinite. This raises the following question: given a Boolean topological algebra $\langle \mathbb{A}, \tau \rangle$, can we find a co-directed diagram of finite algebras $\langle \mathbb{A}_i, f_{ij} \rangle_I$ such that $\mathbb{A} \simeq \varprojlim_I \mathbb{A}_i$ and τ is the profinite topology, i.e. the topology induced by $(\pi_i \colon \mathbb{A} \to \mathbb{A}_i \rangle_I$?

3.1.17. Fact ([24, 28]). *Let $\langle \mathbb{A}, \tau \rangle$ be a Boolean topological algebra and suppose that $\mathbb{A} \in \mathcal{V}$ for some variety $\mathcal{V}$. If*

1. *$\mathcal{V}$ is finitely generated and congruence distributive, or*
2. *$\mathcal{V}$ has equationally definable principle congruences,*

then there exists a co-directed diagram of finite algebras $\langle \mathbb{A}_i, f_{ij} \rangle_I$ in $\mathcal{V}$ such that $\mathbb{A} \simeq \varprojlim_I \mathbb{A}_i$ and τ is the profinite topology on $\mathbb{A}$.

We will return to this question in §4.2.

Profinite completion of Ω-algebras

We now arrive at a very important property of profinite algebras, namely that every algebra $\mathbb{A}$ has a profinite completion, denoted $\hat{\mathbb{A}}$, and a natural map $\mu_{\mathbb{A}} \colon \mathbb{A} \to \hat{\mathbb{A}}$ which has the universal property that for every algebra homomorphism $f \colon \mathbb{A} \to \mathbb{B}$ to a profinite algebra $\mathbb{B}$, there exists a unique continuous homomorphism $f' \colon \hat{\mathbb{A}} \to \mathbb{B}$ such that $f' \circ \mu_{\mathbb{A}} = f$.

$$\begin{tikzcd} & \hat{\mathbb{A}} \arrow[d, dotted, "f'"] \\ \mathbb{A} \arrow[ur, "\mu_{\mathbb{A}}"] \arrow[r, "f"'] & \mathbb{B} \end{tikzcd}$$

In our discussion of the profinite completion we will focus on the construction of $\hat{\mathbb{A}}$ and the natural map $\mu_{\mathbb{A}} \colon \mathbb{A} \to \hat{\mathbb{A}}$, since we will use these later in this chapter when we describe the fundamental connection between canonical extensions and profinite completions in §3.4.1.

Fix an Ω-algebra $\mathbb{A}$; we will now describe how to construct $\hat{\mathbb{A}}$. We will do this by specifying the diagram of which $\hat{\mathbb{A}}$ is the limit. We define the following index poset:

$$\Phi_{\mathbb{A}} = \{\theta \in \operatorname{Con} \mathbb{A} \mid \mathbb{A}/\theta \text{ is finite}\},$$

where the order is the inclusion relation. Now if $\theta, \psi \in \Phi_{\mathbb{A}}$ such that $\theta \subseteq \psi$, then the Isomorphism Theorems of universal algebra [23, §II.6] tell us that the map

$$f_{\theta\psi} \colon a/\theta \mapsto a/\psi$$

is a well-defined, surjective Ω-algebra homomorphism. It is now easy to see that $\langle \mathbb{A}/\theta, f_{\theta\psi} \rangle_{\Phi_{\mathbb{A}}}$ forms a diagram in $\mathbf{Alg}_{\Omega}$; we define

$$\hat{\mathbb{A}} := \varprojlim_{\Phi_{\mathbb{A}}} \mathbb{A}/\theta.$$

But while we are at it, we can immediately also define the natural map $\mu_{\mathbb{A}} \colon \mathbb{A} \to \hat{\mathbb{A}}$. The quotient maps $(\mu_\theta \colon \mathbb{A} \to \mathbb{A}/\theta)_{\Phi_{\mathbb{A}}}$, defined as

$$\mu_\theta \colon a \mapsto a/\theta,$$

form a cone to the diagram $\langle \mathbb{A}/\theta, f_{\theta\psi}\rangle_{\Phi_\mathbb{A}}$. Thus we can define $\mu_\mathbb{A} \colon \mathbb{A} \to \hat{\mathbb{A}}$ to be the unique map of cones from $\mathbb{A}$ to $\hat{\mathbb{A}}$ induced by $(\mu_\theta \colon \mathbb{A} \to \mathbb{A}/\theta)_{\Phi_\mathbb{A}}$. We summarize these two definitions below.

3.1.18. Definition. Let $\mathbb{A}$ be an Ω-algebra. We define $\hat{\mathbb{A}}$, the *profinite completion* of $\mathbb{A}$, to be the limit of the diagram of finite quotients of $\mathbb{A}$.

$$\hat{\mathbb{A}} := \varprojlim_{\Phi_\mathbb{A}} \mathbb{A}/\theta$$

Additionally, we define $\mu_\mathbb{A} \colon \mathbb{A} \to \hat{\mathbb{A}}$ to be the map of cones induced by $(\mu_\theta \colon \mathbb{A} \to \mathbb{A}/\theta)_{\Phi_\mathbb{A}}$.

Now that we have defined the profinite completion, we record three important facts about it. The first is the universal property that we referred to before. For general categorical reasons, this property *defines* the profinite completion up to isomorphism.

3.1.19. Fact ([8]). *Let $f \colon \mathbb{A} \to \mathbb{B}$ be an Ω-algebra homomorphism between Ω-algebras $\mathbb{A}$ and $\mathbb{B}$. If $\mathbb{B}$ is profinite then there exists a unique continuous homomorphism $f' \colon \hat{\mathbb{A}} \to \mathbb{B}$ such that $f' \circ \mu_\mathbb{A} = f$.*

$$\begin{tikzcd} & \hat{\mathbb{A}} \arrow[d, dotted, "f'"] \\ \mathbb{A} \arrow[ur, "\mu_\mathbb{A}"] \arrow[r, "f"'] & \mathbb{B} \end{tikzcd}$$

The second important fact is that strictly speaking, the name 'profinite *completion*' can be seen as a misnomer. One would expect a completion of an algebra $\mathbb{A}$ to be an extension of $\mathbb{A}$; however, the natural map $\mu_\mathbb{A} \colon \mathbb{A} \to \hat{\mathbb{A}}$ is not always an embedding. Recall that an Ω-algebra $\mathbb{A}$ is *residually finite* if for all $a, b \in \mathbb{A}$ with $a \neq b$, there exists a finite Ω-algebra $\mathbb{B}$ and an Ω-homomorphism $f \colon \mathbb{A} \to \mathbb{B}$ such that $f(a) \neq f(b)$.

3.1.20. Fact. *An algebra $\mathbb{A}$ is residually finite iff the natural map $\mu_\mathbb{A} \colon \mathbb{A} \to \hat{\mathbb{A}}$ is injective. A variety $\mathcal{V}$ is residually finite iff for every $\mathbb{A} \in \mathcal{V}$, $\mu_\mathbb{A} \colon \mathbb{A} \to \hat{\mathbb{A}}$ is injective.*

In light of the above fact, we must make a distinction between a given algebra $\mathbb{A}$ and its image $\mu_\mathbb{A}[\mathbb{A}]$ in the statement of the fact below.

3.1.21. Fact. *The subalgebra $\mu_\mathbb{A}[\mathbb{A}]$ is dense in $\hat{\mathbb{A}}$.*

3.1.3 Topological lattices

We conclude this section with several observations about compact Hausdorf lattices and the characterization theorem for Boolean topological lattices.

Compact Hausdorff lattices and lattice-based algebras are an interesting class of topological algebras because a compact Hausdorff topology on a compact Hausdorff lattice is always unique and *intrinsic*, meaning that the topology is uniquely determined by the algebra structure. This means that admitting a compact Hausdorff topology is *a property of lattices*. Moreover, by extension, admitting a compact Hausdorff topology is a property of lattice-based algebras. In this subsection we will present both known and new results which help us to understand topological properties of compact Hausdorff, Boolean topological and profinite lattices via order theory and lattice theory. In particular, we will present necessary and sufficient conditions for a lattice to admit a unique Boolean topology (Theorem 3.1.26), and we will look at sufficient conditions for establishing that sublattices of profinite lattices are again profinite.

Compact Hausdorff lattices

Let us start by recording the fundamental fact that the topology on a compact Hausdorff lattice is intrinsic and unique.

3.1.22. Fact ([68]). *Let $\mathbb{L}$ be a lattice. There exists at most one topology τ on $\mathbb{L}$ such $\langle \mathbb{L}, \tau \rangle$ is a compact Hausdorff topological lattice.*

This fact has many important consequences, the first of which is that we may now refer to a compact Hausdorff lattice $\langle \mathbb{L}, \tau \rangle$ simply by referring to $\mathbb{L}$, since the topology τ is intrinsic. Recall that **Lat** is the category of bounded lattices and lattice homomorphisms, and that **CLat** is the category of complete lattices and complete lattice homomorphisms. By **KHausLat** we denote the category of compact Hausdorff topological lattices and continuous lattice homomorphisms; additionally, we are interested in **BoolLat**, which has Boolean topological lattices as its objects and continuous lattice homomorphisms as its morphisms, and in Pro-$\mathbf{Lat}_f$, which is the category of profinite lattices and continuous lattice homomorphisms.

3.1.23. Fact. *Let $\mathbb{L}$ be a compact Hausdorff lattice. Then*

1. *the unique topology making $\mathbb{L}$ a compact Hausdorff topological lattice is $\sigma(\mathbb{L})$, the bi-Scott topology;*
2. *$\mathbb{L}$ is complete;*
3. *a lattice homomorphism $f\colon \mathbb{L} \to \mathbb{M}$ between compact Hausdorff lattices is continuous iff it is complete.*

Consequently,

4. **KHausLat**, **BoolLat** *and* Pro- $\mathbf{Lat}_f$ *are full subcategories of* **CLat**.

For proofs of the statements in the above fact we refer the reader to [54, §VII-1.7] and [45, §VII-2]. We now shift our attention from compact Hausdorff lattices to Boolean topological lattices and profinite lattices.

Boolean topological lattices

We will now consider Boolean topological lattices, which form a subclass of compact Hausdorff lattices. What is particularly nice about Boolean topological lattices is that one can characterize them order-theoretically, a result which was first published by H.A. Priestley [76]. Below, we will discuss this result and its proof in some detail.

Since Boolean topological lattices are ordered Boolean topological spaces, we may ask ourselves if they are perhaps Priestley spaces, that is if they satisfy the Priestley separation axiom. This is indeed the case.

3.1.24. Lemma. *If $\mathbb{L}$ is a Boolean topological lattice, then $\langle \mathbb{L}, \sigma\rangle$ is a Priestley space, i.e. for all $x, y \in \mathbb{L}$ such that $x \not\leq y$ there exists a clopen upper set U such that $x \in U \not\ni y$.*

Proof Let $x, y \in \mathbb{L}$ with $x \not\leq y$. Then by [54, Lemma VII-1.5] and [54, Corollary VII-1.2], there exists an σ-open upper set $U' \subseteq \mathbb{L}$ such that $x \in U' \not\ni y$. Now because we assumed that the σ-topology on $\mathbb{L}$ is zero-dimensional, there exists some clopen $U \subseteq U'$ such that $x \in U$. Observe that $y \notin {\uparrow} U$, since ${\uparrow} U \subseteq U'$. Now it follows by [5, Lemmas 2 and 3] that ${\uparrow} U$ is clopen. ∎

So now that we know that there exists an abundance of clopen upper sets in a Boolean topological lattice, we may ask ourselves what these sets look like. Recall that given a lattice $\mathbb{L}$, we denote the set of its compact elements by $\mathrm{K}\,\mathbb{L}$; $p \in \mathbb{L}$ is compact if for all directed $S \subseteq \mathbb{L}$ such that $\bigvee S$ exists, we have that $p \leq \bigvee S$ only if ${\uparrow} p \between S$.

3.1.25. Lemma. *Let $\mathbb{L}$ be a compact Hausdorff lattice and let $U, V \subseteq \mathbb{L}$ be an upper and a lower set, respectively. Then*

1. *U is clopen iff there exists a finite $Z \subseteq \mathrm{K}\,\mathbb{L}$ such that $U = {\uparrow} Z$;*

2. *V is clopen iff there exists a finite $W \subseteq \mathrm{K}(\mathbb{L}^{op})$ such that $V = {\downarrow} W$.*

Proof We will only prove (1), since (2) is just its order dual. Suppose that $U \subseteq \mathbb{L}$ is a clopen upper set. We denote the minimal elements of U by $\min U$:

$$\min U := \{p \in U \mid \forall x \in \mathbb{L},\, x < p \Rightarrow x \notin U\}.$$

Now we claim that

$$U = \uparrow \min U. \tag{3.2}$$

To see why, suppose that $x \in U$. Then $\downarrow x$ is $\sigma^{\uparrow}$-closed and hence, σ-closed; consequently $U \cap \downarrow x$ is σ-closed. Now by Lemma 2.1.16, $U \cap \downarrow x$ is closed under co-directed meets, so a fortiori it is closed under taking meets of chains. In other words, $U \cap \downarrow x$ is a partial order which is closed under meets of chains; it follows by (the order dual of) Zorn's Lemma that there exists a minimal element $p \in U \cap \downarrow x$. It is easy to see that p is also minimal in U; it follows that (3.2) holds. Next we claim that

$$\min U \subseteq \mathrm{K}\,\mathbb{L}. \tag{3.3}$$

To see why, consider $p \in \min U$ and suppose that $p \leq \bigvee S$ for some directed $S \subseteq \mathbb{L}$; we must show that there is an $x \in S$ such that $p \leq x$. Because $\mathbb{L}$ is a compact Hausdorff lattice, we know that $\wedge\colon \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ is (σ, σ)-continuous; by Lemma 2.1.17, it follows that $\wedge$ is $(\sigma^{\uparrow}, \sigma^{\uparrow})$-continuous, i.e. that $\wedge$ preserves directed joins. So since $p \leq \bigvee S$, we see see that

$$p = p \wedge \bigvee S = \bigvee p \wedge S.$$

Now suppose towards a contradiction that $p \wedge x < p$ for all $x \in S$; then since p is minimal in U, it follows that $p \wedge S \subseteq \mathbb{L} \setminus U$. Because U is open, it follows by Lemma 2.1.16 that $\bigvee p \wedge S \in \mathbb{L} \setminus U$. But this contradicts the fact that

$$\bigvee p \wedge S = p \in U.$$

We conclude that there must be some $x \in S$ such that $p \leq x$. It follows that $p \in \mathrm{K}\,\mathbb{L}$. Now recall that by (3.2),

$$U = \uparrow \min U = \textstyle\bigcup_{p \in U} \uparrow p.$$

Since U is σ-closed and each $\uparrow p$ is σ-open by (3.3) and Fact A.5.5(2), it follows by compactness that there must exist some finite $Z \subseteq \min U \subseteq \mathrm{K}\,\mathbb{L}$ such that $U = \uparrow Z$.

Conversely, suppose that $U = \uparrow Z$ for some finite $Z \subseteq \mathrm{K}\,\mathbb{L}$. Since

$$U = \uparrow Z = \textstyle\bigcup_{p \in Z} \uparrow p,$$

we see firstly that U is open by Fact A.5.5(2). Secondly, we see that U is closed because it is a finite union of $\sigma^{\downarrow}$-closed sets, and $\sigma^{\downarrow} \subseteq \sigma$. It follows that U is clopen. ∎

A lattice $\mathbb{L}$ is called *bi-algebraic* if both $\mathbb{L}$ and $\mathbb{L}^{op}$ are algebraic. By $\lambda(\mathbb{L}) := \sigma^{\uparrow}(\mathbb{L}) \vee \iota^{\downarrow}$ we denote the *Lawson topology* of $\mathbb{L}$ [45].

We are now ready to prove the main result of this section, which is the Characterization Theorem for Boolean topological lattices, due to H.A. Priestley. Observe however that condition (2) below is new.

3.1.26. THEOREM ([76]). *Let $\mathbb{L}$ be a lattice. The following are equivalent:*

1. *$\mathbb{L}$ is a Boolean topological lattice;*
2. *$\mathbb{L}$ is complete and there exist $P, Q \subseteq \mathbb{L}$ such that*
   (a) *P is join-dense in $\mathbb{L}$ and Q is meet-dense in $\mathbb{L}$;*
   (b) *for every $p \in P$, there is a finite $Z \subseteq \mathbb{L}$ such that $\mathbb{L} \setminus {\uparrow} p = {\downarrow} Z$;*
   (c) *for every $q \in Q$, there is a finite $Z \subseteq \mathbb{L}$ such that $\mathbb{L} \setminus {\downarrow} p = {\uparrow} Z$;*
3. *$\mathbb{L}$ is complete and bi-algebraic, $\sigma^{\uparrow} = \iota^{\uparrow}$ and $\sigma^{\downarrow} = \iota^{\downarrow}$.*

Proof (1) $\Rightarrow$ (2). We will first argue that $\mathbb{L}$ is bi-algebraic. To see that $\mathbb{L}$ is algebraic, it suffices to show that if $x, y \in \mathbb{L}$ and $x \not\leq y$, then there exists a $p \in \mathrm{K}\,\mathbb{L}$ such that $p \leq x$ and $p \not\leq y$. But this is easy to see: by Lemma 3.1.24, there exists a clopen upper set $U \subseteq \mathbb{L}$ such that $x \in U \not\ni y$. By Lemma 3.1.25, $U = {\uparrow} Z$ for some finite $Z \subseteq \mathrm{K}\,\mathbb{L}$. But then it follows that there must be some $p \in Z$ such that $p \leq x$; moreover, since $y \notin {\uparrow} Z$, it follows that $p \not\leq y$. The argument for showing that $\mathbb{L}$ is co-algebraic is identical. If we now define $P := \mathrm{K}\,\mathbb{L}$ and $Q := \mathrm{K}\,\mathbb{L}^{op}$ then it follows that (2)(a) holds. To see that (2)(b) holds, take any $p \in P = \mathrm{K}\,\mathbb{L}$. It follows by Lemma 3.1.25 that ${\uparrow} p$ is a clopen upper set, so $\mathbb{L} \setminus {\uparrow} p$ must be a clopen lower set. Applying Lemma 3.1.25 again, we see that there must be some $W \subseteq \mathrm{K}\,\mathbb{L}^{op}$ such that $\mathbb{L} \setminus {\uparrow} p = {\downarrow} W$. The proof for (2)(c) is order dual.

(2) $\Rightarrow$ (3). We will first show that

$$\sigma^{\uparrow} = \iota^{\uparrow} \text{ and } \sigma^{\downarrow} = \iota^{\downarrow}. \tag{3.4}$$

We will only show the first part of (3.4), since the other follows by order duality. Recall from Lemma 2.1.17(1) that $\iota^{\uparrow} \subseteq \sigma^{\uparrow}$, so it suffices to show that $\sigma^{\uparrow} \subseteq \iota^{\uparrow}$. Now observe that by assumption (2)(b), we know that for every $p \in P$, we have

$$\mathbb{L} \setminus {\uparrow} p = {\downarrow} Z = \textstyle\bigcup_{q \in Z} {\downarrow} q,$$

for some finite set Z, so that

$${\uparrow} p = \mathbb{L} \setminus (\mathbb{L} \setminus {\uparrow} p) = \mathbb{L} \setminus \left(\textstyle\bigcup_{q \in Z} {\downarrow} q\right) = \textstyle\bigcap_{q \in Z} (\mathbb{L} \setminus {\downarrow} q).$$

We see that ${\uparrow} p$ is a finite intersection of $\iota^{\uparrow}$-open sets; since $p \in P$ was arbitrary, we see that

$$\text{for all } p \in P, {\uparrow} p \text{ is } \iota^{\uparrow}\text{-open.} \tag{3.5}$$

Now let U be a $\sigma^{\uparrow}$-open set and let $x \in U$. Because P is join-dense in $\mathbb{L}$, we know that $x = \bigvee({\downarrow} x \cap P)$. Since U is $\sigma^{\uparrow}$-open, it follows that there must be some finite $Z \subseteq {\downarrow} x \cap P$ such that $\bigvee Z \in U$. Now observe that

$$\begin{aligned} x &\in {\uparrow} \textstyle\bigvee Z && \text{since } Z \subseteq {\downarrow} x \cap P, \\ &= \textstyle\bigcap_{p \in Z} {\uparrow} z && \text{by order theory,} \\ &\subseteq U && \text{since } \textstyle\bigvee Z \in U. \end{aligned}$$

Since $\bigcap_{p\in Z} \uparrow z$ is $\iota^{\uparrow}$-open by (3.5) and since $x \in U$ was arbitrary, it follows that U is $\iota^{\uparrow}$-open. Since U was an arbitrary $\sigma^{\uparrow}$-open set it follows that $\sigma^{\uparrow} \subseteq \iota^{\uparrow}$ and consequently, (3.4) holds. Now we will show that

$$\mathbb{L} \text{ is bi-algebraic.} \tag{3.6}$$

We first show that $P \subseteq \mathrm{K}\,\mathbb{L}$. If $p \in P$, then by (3.5) and (3.4), $\uparrow p$ is $\sigma^{\uparrow}$-open. But then it follows immediately that $p \in \mathrm{K}\,\mathbb{L}$: if $p \leq \bigvee S$ for some directed $S \subseteq \mathbb{L}$, then $\bigvee S \in \uparrow p$, so there exists some $y \in S$ such that $p \leq y$. Now we see that for any $x \in \mathbb{L}$,

$$\begin{aligned} x &= \textstyle\bigvee(\downarrow x \cap P) && \text{because } P \text{ is join-dense in } \mathbb{L}, \\ &= \textstyle\bigvee(\downarrow x \cap \mathrm{K}\,\mathbb{L}) && \text{by order theory because } P \subseteq \mathrm{K}\,\mathbb{L}, \end{aligned}$$

so that we see that $\mathbb{L}$ is algebraic. By an order dual argument it follows that $\mathbb{L}$ is also co-algebraic, so that (3.6) holds.

(3) $\Rightarrow$ (1). We will first show that the σ-topology on $\mathbb{L}$ is a Boolean topology, i.e. it is compact, Hausdorff and zero-dimensional. Recall that the Lawson topology on $\mathbb{L}$ is defined as $\lambda = \sigma^{\uparrow} \vee \iota^{\downarrow}$. Since $\sigma := \sigma^{\uparrow} \vee \sigma^{\downarrow}$, it follows by our assumption that in our case, $\lambda = \sigma$. Since $\mathbb{L}$ is algebraic by assumption, it follows by [45, Theorem III.1.10] that the Lawson topology on $\mathbb{L}$ is compact and Hausdorff; consequently, so is the σ-topology. Now if $x, y \in \mathbb{L}$ such that $x \not\leq y$, then since $\mathbb{L}$ is algebraic, there exists a $p \in \mathrm{K}\,\mathbb{L}$ such that $p \leq x$ and $p \not\leq y$, i.e. $x \in \uparrow p \not\ni y$. Because $p \in \mathrm{K}\,\mathbb{L}$, $\uparrow p$ is $\sigma^{\uparrow}$-open. Because $\uparrow p$ is a principal upper set, it is $\iota^{\downarrow}$-closed. By our assumption that $\sigma^{\uparrow} = \iota^{\uparrow}$ and $\sigma^{\downarrow} = \iota^{\downarrow}$, it follows that $\uparrow p$ is σ-clopen, so since $x, y \in \mathbb{L}$ were arbitrary, it follows that the σ-topology is totally disconnected; consequently, the σ-topology is a Boolean topology.

Finally, we will show that $\mathbb{L}$ is a topological lattice in its σ-topology. We know by associativity that $\wedge \colon \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ preserves co-directed meets, so that $\wedge$ is $(\sigma^{\downarrow}, \sigma^{\downarrow})$-continuous. Because $\mathbb{L}$ is algebraic, we know by [45, Proposition I-1.14] that $\wedge$ preserves directed joins, i.e. that $\wedge$ is $(\sigma^{\uparrow}, \sigma^{\uparrow})$-continuous. It follows that $\wedge$ is (σ, σ)-continuous; the argument for $\vee \colon \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ is order-dual. We conclude that $\mathbb{L}$ is a Boolean topological lattice. ∎

3.1.27. Corollary. *If $\mathbb{L}$ is a Boolean topological lattice, then $\sigma = \lambda$, i.e. its intrinsic Boolean topology is the Lawson topology.*

We conclude this section with an application of the Characterization Theorem above. Recall that we saw above that a closed sublattice of a profinite lattice is again profinite. Using Theorem 3.1.26 in conjunction with a technical result from domain theory, we can prove the following:

3.1.28. Lemma. *Let $\mathbb{L}$ be a profinite lattice. If $\mathbb{L}'$ is a complete subalgebra of $\mathbb{L}$ then $\mathbb{L}'$ is closed and hence, profinite.*

Proof Because $\mathbb{L}$ is algebraic by Theorem 3.1.26, it is a continuous lattice in the sense of [45]. Consequently, we may apply [45, Theorem III.1.11] to conclude that $\mathbb{L}'$ must be closed in the Lawson topology of $\mathbb{L}$. Now by Corollary 3.1.27, we know that $\sigma = \lambda$, so $\mathbb{L}'$ is a σ-closed subalgebra of $\mathbb{L}$. Since the profinite topology on $\mathbb{L}$ must be the σ-topology (as there can be only one compact Hausdorff topology on $\mathbb{L}$ which is compatible with $\wedge$ and $\vee$), we see that $\mathbb{L}'$ is closed in the profinite topology on $\mathbb{L}$. It now follows from Lemma 3.1.16 that $\mathbb{L}'$ is itself profinite. ∎

This concludes our preliminaries for this chapter on topological algebra and topological lattices.

3.2 Canonical extensions of maps II: maps into profinite lattices

In §2.2, we defined the canonical extension of an order-preserving map $f\colon \mathbb{L} \to \mathbb{M}$ by first extending f to a map $\mathcal{F} f\colon \mathcal{F}\mathbb{L} \to \mathcal{F}\mathbb{M}$, mapping a filter F to $\uparrow f[F]$, and subsequently extending $\mathcal{F} f$ to a continuous extension $f^{\triangledown}\colon \mathbb{L}^{\delta} \to \mathbb{M}^{\delta}$, and dually via $\mathcal{I}\,\mathbb{L}$ when constructing $f^{\vartriangle}$. This approach fails, however, if we drop the assumption that f is order-preserving, because $\mathcal{F} f$ is then no longer defined. In this section we will introduce a different way of continuously extending $f\colon \mathbb{L} \to \mathbb{M}$ to a map $f^{\triangledown}\colon \mathbb{L}^{\delta} \to \mathbb{M}^{\delta}$, which does not depend on any properties of f. We can still characterize $f^{\triangledown}$ as a largest continuous extension of f if we make additional assumptions about $\mathbb{M}$, the codomain of f.

Some of the definitions and results in this section are generalizations from the case of distributive lattices studied by Gehrke and Jónsson [39]. As we indicated in our paper with M. Gehrke [43] however, distributivity of the underlying lattice is not an essential property for ensuring good behaviour of the canonical extension. Rather, what matters is that the underlying lattice lies in a finitely generated variety. We will make use of the technical lemmas from this section in the rest of this chapter, when we look at canonical extensions of lattice-based algebras rather than plain lattices.

The section is organized as follows. First, we show that there is a natural way to define a lower and upper extension of an arbitrary map $f\colon \mathbb{L} \to \mathbb{C}$, from a lattice into a complete lattice, to a continuous map $f'\colon \mathbb{L}^{\delta} \to \mathbb{C}$, viz. the $\liminf$ and $\limsup$ extension of f. We then use $\liminf$ and $\limsup$ to extend an abitrary function $f\colon \mathbb{L} \to \mathbb{M}$ to a continuous function $f^{\triangledown}\colon \mathbb{L}^{\delta} \to \mathbb{M}^{\delta}$ in §3.2.1. We will then show in §3.2.2 how we can view $f^{\triangledown}\colon \mathbb{L}^{\delta} \to \mathbb{M}^{\delta}$ as a *maximal* continuous extension of $f\colon \mathbb{L} \to \mathbb{M}$ if we assume that $\mathrm{HSP}\,\mathbb{M}$ is finitely generated. Finally, in §3.2.3 we investigate properties of extension of compositions of arbitrary maps. We conclude this section with an overview of the contributions and further work in §3.2.4.

3.2.1 Extending maps via lim inf and lim sup

Our first goal in this subsection is to extend an arbitrary map $f\colon \mathbb{L} \to \mathbb{C}$ from a lattice $\mathbb{L}$ to a complete lattice $\mathbb{C}$ to a continuous map $f'\colon \mathbb{L}^\delta \to \mathbb{C}$ in such a way that $f' \circ e = f$, where $e\colon \mathbb{L} \to \mathbb{L}^\delta$ is the canonical extension of $\mathbb{L}$.

$$\begin{tikzcd} \mathbb{L}^\delta \arrow[r, "f'"] & \mathbb{C} \\ \mathbb{L} \arrow[u, "e"] \arrow[ur, "f"'] & \end{tikzcd}$$

We exploit the fact that $e[L]$ is dense in $\langle L, \delta(\mathbb{L})\rangle$ (by Lemma 2.1.28), meaning that for 'a lot' of points in $\mathbb{L}^\delta$, our function $f'\colon \mathbb{L}^\delta \to \mathbb{C}$ is already defined. For an arbitrary $x \in \mathbb{L}^\delta$ and a δ-open neighborhood U of x, we get a set of values $f[e^{-1}(U)]$ in $\mathbb{C}$ approximating $f'(x)$. We can now take the infimum (meet) or supremum (join) of this set of approximating values. Thus, we get an inf- or sup-approximant of $f'(x)$ for every open neighborhood of x. Intuitively, we can now define $f'(x)$ to be the 'limit' over all open neighborhoods of x of these approximants.

Recall that the δ-topology on $\mathbb{L}^\delta$ has as its base the collection of sets

$$\{\uparrow e^{\mathcal{F}}(F) \cap \downarrow e^{\mathcal{I}}(I) \mid F \in \mathcal{F}\,\mathbb{L},\ I \in \mathcal{I}\,\mathbb{L}\}.$$

We now arrive at the following definition:

3.2.1. Definition. Given a function $f\colon \mathbb{L} \to \mathbb{C}$, where $\mathbb{C}$ is a complete lattice, we define $\liminf f\colon \mathbb{L}^\delta \to \mathbb{C}$, where

$$\liminf f(x) = \bigvee \left\{ \bigwedge f[F \cap I] \mid e^{\mathcal{F}}(F) \le x \le e^{\mathcal{I}}(I) \right\}.$$

Dually, we define $\limsup f\colon \mathbb{L}^\delta \to \mathbb{C}$ as

$$\limsup f(x) = \bigwedge \left\{ \bigvee f[F \cap I] \mid e^{\mathcal{F}}(F) \le x \le e^{\mathcal{I}}(I) \right\}.$$

First, observe that these definitions follow the pattern we sketched above. The reader may notice however that the expressions $f[F \cap I]$ are a lot simpler than the $f[e^{-1}(U)]$ we arrived at before. The reason lies in the following lemma:

3.2.2. Lemma. *Let $e\colon \mathbb{L} \to \mathbb{L}^\delta$ be a canonical extension and let $F \in \mathcal{F}\,\mathbb{L}$, $I \in \mathcal{I}\,\mathbb{L}$. Then $e^{\mathcal{F}}(F) \le e(a) \le e^{\mathcal{I}}(I)$ iff $a \in F \cap I$.*

Proof We show that $e^{\mathcal{F}}(F) \le e(a)$ iff $a \in F$; the statement then follows by order duality. Now, observe that $e^{\mathcal{F}}(F) \le e(a) = e^{\mathcal{F}}(\uparrow a)$ iff $F \supseteq \uparrow a$, since $e^{\mathcal{F}}(F)$ is an embedding (by Lemma 2.1.28). But $F \supseteq \uparrow a$ iff $a \in F$. ∎

The following lemma shows that $\liminf f$ indeed gives us a continuous extension of f.

3.2.3. LEMMA. *Let $f\colon \mathbb{L} \to \mathbb{C}$ be a function from a lattice $\mathbb{L}$ to a complete lattice $\mathbb{C}$ and let $e\colon \mathbb{L} \to \mathbb{L}^\delta$ be the canonical extension of $\mathbb{L}$. Then*

1. $\liminf f$ *is $(\delta, \sigma^\uparrow)$-continuous;*
2. $\limsup f$ *is $(\delta, \sigma^\downarrow)$-continuous;*
3. $\liminf f \leq \limsup f$;
4. $\liminf f \circ e = \limsup f \circ e = f$.

$$\begin{tikzcd} \mathbb{L}^\delta \arrow[r, "\liminf f"] & \mathbb{C} \\ \mathbb{L} \arrow[u, "e"] \arrow[ur, "f"'] & \end{tikzcd}$$

Proof (1). First, observe that given $x \in \mathbb{L}^\delta$,

$$\big\{\bigwedge f[F \cap I] \mid e^{\mathcal{F}}(F) \leq x \leq e^{\mathcal{I}}(I)\big\}$$

is a directed set: take $F_1, F_2 \in \mathcal{F}\,\mathbb{L}$ and $I_1, I_2 \in \mathcal{I}\,\mathbb{L}$ such that $e^{\mathcal{F}}(F_i) \leq x \leq e^{\mathcal{I}}(I_i)$ for $i = 1, 2$. Then

$$\begin{aligned} e^{\mathcal{F}}(F_1 \cap F_2) &= e^{\mathcal{F}}(F_1) \vee e^{\mathcal{F}}(F_2) && \text{since } e^{\mathcal{F}} \text{ is a homomorphism,} \\ &\leq x && \text{by assumption,} \\ &\leq e^{\mathcal{I}}(I_1) \wedge e^{\mathcal{I}}(I_2) && \text{idem,} \\ &= e^{\mathcal{I}}(I_1 \cap I_2) && \text{because } e^{\mathcal{I}} \text{ is a homomorphism.} \end{aligned}$$

Moreover, since $F_1 \cap F_2 \cap I_1 \cap I_2 \subseteq F_i \cap I_i$ for $i = 1, 2$, we get

$$\bigwedge f[F_1 \cap F_2 \cap I_1 \cap I_2] \geq \bigwedge f[F_i \cap I_i] \text{ for } i = 1, 2;$$

it follows that $\liminf f(x)$ is a directed join. We will use this fact to show that $\liminf f$ is locally continuous at x; since x is arbitrary this suffices to show that $\liminf f$ is continuous. Suppose that $U \subseteq C$ is Scott-open and $\liminf f(x) \in U$, then since $\liminf f(x)$ is a directed join, there must be some $F' \in \mathcal{F}\,\mathbb{L}$, $'I \in \mathcal{I}\,\mathbb{L}$ such that $e^{\mathcal{F}}(F') \leq x \leq e^{\mathcal{I}}(I')$ and $\bigwedge f[F' \cap I'] \in U$. But then for all $y \in \mathbb{L}^\delta$ such that $e^{\mathcal{F}}(F') \leq y \leq e^{\mathcal{I}}(I')$, we have

$$\begin{aligned} \liminf f(y) &= \bigvee \big\{\bigwedge f[F \cap I] \mid e^{\mathcal{F}}(F) \leq y \leq e^{\mathcal{I}}(I)\big\} && \text{by definition of } \liminf, \\ &\geq \bigwedge f[F' \cap I'] && \text{since } e^{\mathcal{F}}(F') \leq y \leq e^{\mathcal{I}}(I'), \\ &\in U, \end{aligned}$$

so that it follows that $\liminf f$ is locally $(\delta, \sigma^{\uparrow})$-continuous. Part (2) is just the order dual of (1).

(3). Let $x \in \mathbb{L}^{\delta}$; we want to show that

$$\bigvee \left\{ \bigwedge f[F \cap I] \mid e^{\mathcal{F}}(F) \leq x \leq e^{\mathcal{I}}(I) \right\} \leq \bigwedge \left\{ \bigvee f[F \cap I] \mid e^{\mathcal{F}}(F) \leq x \leq e^{\mathcal{I}}(I) \right\}.$$

It suffices to show that for all $F, F' \in \mathcal{F}\,\mathbb{L}$ and $I, I' \in \mathcal{I}\,\mathbb{L}$ such that $e^{\mathcal{F}}(F) \leq x \leq e^{\mathcal{I}}(I)$ and $e^{\mathcal{F}}(F') \leq x \leq e^{\mathcal{I}}(I')$, we have that $\bigwedge f[F \cap I] \leq \bigvee f[F' \cap I']$. First, observe that we have that

$$e^{\mathcal{F}}(F \cap F') = e^{\mathcal{F}}(F) \vee e^{\mathcal{F}}(F') \leq x \leq e^{\mathcal{I}}(I) \wedge e^{\mathcal{I}}(I') = e^{\mathcal{I}}(I \cap I'),$$

as before. It follows by the compactness property of canonical extensions that $(F \cap F') \between (I \cap I')$, i.e. $F \cap F' \cap I \cap I' \neq \emptyset$. But then

$$\begin{aligned}
\bigwedge f[F \cap I] &\leq \bigwedge f[F \cap F' \cap I \cap I'] && \text{since } (F \cap I) \supseteq (F \cap F' \cap I \cap I'), \\
&\leq \bigvee f[F \cap F' \cap I \cap I'] && \text{since } F \cap F' \cap I \cap I' \neq \emptyset, \\
&\leq \bigvee f[F' \cap I'] && \text{since } (F \cap F' \cap I \cap I') \subseteq (F' \cap I').
\end{aligned}$$

Since $F, F' \in \mathcal{F}\,\mathbb{L}$ and $I, I' \in \mathcal{I}\,\mathbb{L}$ were arbitrary it now follows that $\liminf f(x) \leq \limsup f(x)$.

(4). We only show that $\liminf f \circ e = f$; the other equality follows by order duality. Let $a \in \mathbb{L}$, then

$$\begin{aligned}
\liminf f \circ e(a) &= \bigvee \left\{ \bigwedge f[F \cap I] \mid e^{\mathcal{F}}(F) \leq e(a) \leq e^{\mathcal{I}}(I) \right\} && \text{by definition,} \\
&= \bigvee \left\{ \bigwedge f[F \cap I] \mid a \in F \cap I \right\} && \text{by Lemma 3.2.2,} \\
&\leq f(a) && \text{by order theory.}
\end{aligned}$$

Conversely, since $e^{\mathcal{F}}(\uparrow a) \leq e(a) \leq e^{\mathcal{I}}(\downarrow a)$, we see that

$$f(a) = \bigwedge f[\uparrow a \cap \downarrow a] \leq \liminf f \circ e(a). \qquad \blacksquare$$

3.2.4. Lemma. *Let $f\colon \mathbb{L}_1 \to \mathbb{C}_1$ and $g\colon \mathbb{L}_2 \to \mathbb{C}_2$ be functions from lattices $\mathbb{L}_1$, $\mathbb{L}_2$ to complete lattices $\mathbb{C}_1$, $\mathbb{C}_2$. Then*

$$\liminf(f \times g) = (\liminf f) \times (\liminf g)$$
$$\limsup(f \times g) = (\limsup f) \times (\limsup g)$$

Proof We will only prove the first statement, since the proof of the second statement is identical modulo order duality. Although we will have to do some bookkeeping and there is a lot of notation, the proof of this lemma is essentially very easy. We first make a number of observations. For starters, if $(x_1, x_2) \in \mathbb{L}_1^{\delta} \times \mathbb{L}_2^{\delta}$, then

$$\begin{aligned}
&\left\{ (F, I) \mid e^{\mathcal{F}}_{\mathbb{L}_1 \times \mathbb{L}_2}(F) \leq (x_1, x_2) \leq e^{\mathcal{I}}_{\mathbb{L}_1 \times \mathbb{L}_2}(I) \right\} \\
&= \left\{ (F_1 \times F_2, I_1 \times I_2) \mid e^{\mathcal{F}}_{\mathbb{L}_i}(F_i) \leq x_i \leq e^{\mathcal{I}}_{\mathbb{L}_i}(I_i),\ i = 1, 2 \right\},
\end{aligned} \tag{3.7}$$

where F and I are filters and ideals of $\mathbb{L}_1 \times \mathbb{L}_2$, and F_i and I_i are filters and ideals of $\mathbb{L}_i$ for $i = 1, 2$. This follows from fact that every filter of $\mathbb{L}_1 \times \mathbb{L}_2$ is of the form $F_1 \times F_2$ for some $F_1 \in \mathcal{F}\mathbb{L}_1$, $F_2 \in \mathcal{F}\mathbb{L}_2$. Next, we claim that for all filters and ideals $F_1, I_1 \subseteq \mathbb{L}_1$ and $F_2, I_2 \subseteq \mathbb{L}_2$,

$$(F_1 \times F_2) \cap (I_1 \times I_2) = (F_1 \cap I_1) \times (F_2 \cap I_2). \tag{3.8}$$

This follows from basic set theory. Finally, we claim that for all filters and ideals $F_1, I_1 \subseteq \mathbb{L}_1$ and $F_2, I_2 \subseteq \mathbb{L}_2$,

$$f \times g\left[(F_1 \times F_2) \cap (I_1 \times I_2)\right] = f\left[(F_1 \cap I_1)\right] \times g\left[(F_2 \cap I_2)\right]. \tag{3.9}$$

This follows from (3.8) and basic set theory, since $f \times g(a, b) = (f(a), g(b))$ for $(a, b) \in \mathbb{L}_1 \times \mathbb{L}_2$. Let $(x_1, x_2) \in \mathbb{L}_1^\delta \times \mathbb{L}_2^\delta$; we will show that $\liminf(f \times g)(x_1, x_2) = (\liminf f(x_1), \liminf g(x_2))$.

$$\begin{aligned}
&\liminf f \times g(x_1, y_1) \\
&= \bigvee \left\{ \bigwedge f \times g[F \cap I] \mid e^{\mathcal{F}}_{\mathbb{L}_1 \times \mathbb{L}_2}(F) \leq (x, y) \leq e^{\mathcal{I}}_{\mathbb{L}_1 \times \mathbb{L}_2}(I) \right\} \\
&\qquad\qquad\qquad\qquad \text{by definition of } \liminf, \\
&= \bigvee \left\{ \bigwedge f \times g[F_1 \times F_2 \cap I_1 \times I_2] \mid e^{\mathcal{F}}_{\mathbb{L}_i}(F_i) \leq x_i \leq e^{\mathcal{I}}_{\mathbb{L}_i}(I_i),\ i = 1, 2 \right\} \\
&\qquad\qquad\qquad\qquad \text{by (3.7)}, \\
&= \bigvee \left\{ \bigwedge f\left[(F_1 \cap I_1)\right] \times g\left[(F_2 \cap I_2)\right] \mid e^{\mathcal{F}}_{\mathbb{L}_i}(F_i) \leq x_i \leq e^{\mathcal{I}}_{\mathbb{L}_i}(I_i),\ i = 1, 2 \right\} \\
&\qquad\qquad\qquad\qquad \text{by (3.9)}, \\
&= \bigvee \left\{ \left(\bigwedge f\left[(F_1 \cap I_1)\right], \bigwedge g\left[(F_2 \cap I_2)\right]\right) \mid e^{\mathcal{F}}_{\mathbb{L}_i}(F_i) \leq x_i \leq e^{\mathcal{I}}_{\mathbb{L}_i}(I_i),\ i = 1, 2 \right\} \\
&\qquad\qquad\qquad\qquad \text{because } \bigwedge \text{ is computed component-wise}, \\
&= \Big(\bigvee \left\{ \bigwedge f\left[(F_1 \cap I_1)\right] \mid e^{\mathcal{F}}_{\mathbb{L}_1}(F_1) \leq x_1 \leq e^{\mathcal{I}}_{\mathbb{L}_1}(I_1) \right\}, \\
&\qquad \bigvee \left\{ \bigwedge g\left[(F_2 \cap I_2)\right] \mid e^{\mathcal{F}}_{\mathbb{L}_2}(F_2) \leq x_2 \leq e^{\mathcal{I}}_{\mathbb{L}_2}(I_2) \right\} \Big) \\
&\qquad\qquad\qquad\qquad \text{because } \bigvee \text{ is computed component-wise}, \\
&= (\liminf f(x_1), \liminf g(x_2)) \\
&\qquad\qquad\qquad\qquad \text{by definition of } \liminf.
\end{aligned}$$

Because $(x_1, x_2) \in \mathbb{L}_1^\delta \times \mathbb{L}_2^\delta$ was arbitrary, it follows that $\liminf(f \times g) = (\liminf f) \times (\liminf g)$. ∎

3.2.5. Remark. The technical results about lim inf may be seen even more as topological results rather than canonical extension results. This viewpoint is discussed further in [39, §2.3].

Thus we see that the lim inf-construction we have defined above has the desirable property that it gives us a continuous extension of an arbitrary map into a complete lattice. We now have a good candidate definition for extending an arbitrary map $f\colon \mathbb{L} \to \mathbb{M}$ to a map from $\mathbb{L}^\delta$ to $\mathbb{M}^\delta$: simply take the map $\liminf(e_{\mathbb{M}} \circ f)\colon \mathbb{L}^\delta \to \mathbb{M}^\delta$.

$$\begin{tikzcd}
\mathbb{L}^\delta \arrow[r, "\liminf(e_{\mathbb{M}}\circ f)"] & \mathbb{M}^\delta \\
\mathbb{L} \arrow[u, "e_{\mathbb{L}}"] \arrow[r, "f"'] & \mathbb{M} \arrow[u, "e_{\mathbb{M}}"']
\end{tikzcd}$$

As the above diagram illustrates, $\liminf(e_{\mathbb{M}} \circ f)$ is indeed an extension of $f\colon \mathbb{L} \to \mathbb{M}$ to a map from $\mathbb{L}^\delta$ to $\mathbb{M}^\delta$. But before we take this as the definition, we must ask ourselves an important question: is this proposed extension compatible with the extensions of order-preserving maps we studied in §2.2?

3.2.6. LEMMA. *If $f\colon \mathbb{L} \to \mathbb{M}$ is order-preserving, then $f^\triangledown = \liminf(e_{\mathbb{M}} \circ f)$ and $f^\vartriangle = \limsup(e_{\mathbb{M}} \circ f)$.*

Proof Let $x \in \mathbb{L}^\delta$. Recall that

$$f^\triangledown(x) = \bigvee \left\{ e^{\mathcal{F}}_{\mathbb{M}} \mathcal{F} f(F) \mid e^{\mathcal{F}}_{\mathbb{L}}(F) \leq x \right\},$$

and that

$$\liminf(e_{\mathbb{M}} \circ f)(x) = \bigvee \left\{ \bigwedge e_{\mathbb{M}} \circ f[F \cap I] \mid e^{\mathcal{F}}_{\mathbb{L}}(F) \leq x \leq e^{\mathcal{I}}_{\mathbb{L}}(I) \right\}.$$

We claim that

$$e^{\mathcal{F}}_{\mathbb{M}} \mathcal{F} f(F) = \bigwedge e_{\mathbb{M}} \circ f[F \cap I] \text{ whenever } e^{\mathcal{F}}_{\mathbb{M}}(F) \leq e^{\mathcal{I}}_{\mathbb{M}}(I). \tag{3.10}$$

First, observe that

$$\begin{aligned}
e^{\mathcal{F}}_{\mathbb{M}} \mathcal{F} f(F) &= \bigwedge e_{\mathbb{M}} [\uparrow f[F]] && \text{by definition,} \\
&= \bigwedge e_{\mathbb{M}} \circ f[F] && \text{since } e_{\mathbb{M}} \text{ is order-preserving.}
\end{aligned}$$

Moreover, we see that

$$\begin{aligned}
\bigwedge e_{\mathbb{M}} \circ f[F \cap I] &= \bigwedge e_{\mathbb{M}} \circ f\,[\uparrow(F \cap I)] && \text{since } f \text{ is order-preserving,} \\
&= \bigwedge e_{\mathbb{M}} \circ f[F] && \text{by Lemma 2.1.3,}
\end{aligned}$$

where we need the fact that $e^{\mathcal{F}}_{\mathbb{M}}(F) \leq e^{\mathcal{I}}_{\mathbb{M}}(I)$ in order to apply Lemma 2.1.3. It follows that (3.10) holds; it is now easy to see that $f^\triangledown(x) = \liminf(e_{\mathbb{M}} \circ f)(x)$. ∎

In light of Lemma 3.2.6, we can now safely state the following definition, which subsumes Definition 2.2.1.

3.2.7. Definition. If $f\colon \mathbb{L} \to \mathbb{M}$ is an abitrary function between lattices, then we define $f^\triangledown\colon \mathbb{L}^\delta \to \mathbb{M}^\delta$ as follows:

$$f^\triangledown := \liminf(e_\mathbb{M} \circ f).$$

Dually, we define $f^\vartriangle\colon \mathbb{L}^\delta \to \mathbb{M}^\delta$:

$$f^\vartriangle := \limsup(e_\mathbb{M} \circ f).$$

As a corollary of Lemma 3.2.3, we can now list the following properties of our newly defined extensions $f^\triangledown$ and $f^\vartriangle$.

3.2.8. Corollary. *Let $f\colon \mathbb{L} \to \mathbb{M}$ be an arbitrary function between lattices $\mathbb{L}$, $\mathbb{M}$. Then*

1. *$f^\triangledown\colon \mathbb{L}^\delta \to \mathbb{M}^\delta$ is $(\delta, \sigma^\uparrow)$-continuous;*
2. *$f^\vartriangle\colon \mathbb{L}^\delta \to \mathbb{M}^\delta$ is $(\delta, \sigma^\downarrow)$-continuous;*
3. *$f^\triangledown \leq f^\vartriangle$.*

Proof This follows immediately from Lemma 3.2.3 and Definition 3.2.7. ∎

3.2.2 Maps into profinite lattices

Recall from §2.2 that if $f\colon \mathbb{L} \to \mathbb{M}$ is an order-preserving map, then $f^\triangledown\colon \mathbb{L}^\delta \to \mathbb{M}^\delta$ is the largest $(\delta^\uparrow, \iota^\uparrow)$-continuous extension of f, where $\iota^\uparrow$ is the upper interval topology. We would like to prove a similar result in the case that f is not necessarily order-preserving. We begin with a maximality result concerning the $\liminf$-construction for maps $f\colon \mathbb{L} \to \mathbb{C}$, where we use the additional assumption that $\mathbb{C}$ is profinite, which is a very strong property. It is an interesting and open question whether the assumption that $\mathbb{C}$ is profinite is essential; see Remark 3.2.22.

3.2.9. Theorem. *Let $f\colon \mathbb{L} \to \mathbb{C}$ be a function from a lattice $\mathbb{L}$ to a profinite lattice $\mathbb{C}$, and let $f'\colon \mathbb{L}^\delta \to \mathbb{C}$ be an extension of f, i.e. assume that $f' \circ e_\mathbb{L} = f$.*

$$\begin{tikzcd}
 & \mathbb{L}^\delta \arrow[d, "f'"] \\
\mathbb{L} \arrow[ur, "e_\mathbb{L}"] \arrow[r, "f"'] & \mathbb{C}
\end{tikzcd}$$

1. *If $f'\colon \mathbb{L}^\delta \to \mathbb{C}$ is $(\delta, \iota^\uparrow)$-continuous, then $f' \leq \liminf f$.*
2. *If $f'\colon \mathbb{L}^\delta \to \mathbb{C}$ is $(\delta, \iota^\downarrow)$-continuous, then $\limsup f \leq f'$.*

Proof By order duality it suffices to prove part (1). We will prove something stronger, in fact: we will show that for all $x \in \mathbb{L}^\delta$, if $f' \colon \mathbb{L}^\delta \to \mathbb{C}$ is *locally* $(\delta, \iota^\uparrow)$-continuous at x, then $f'(x) \leq \liminf f(x)$. Towards a contradiction, suppose that $f'(x) \not\leq \liminf f(x)$. By Theorem 3.1.26, $\mathbb{C}$ is algebraic, so there must exist a compact element $p \in \mathrm{K}\,\mathbb{C}$ such that $p \leq f'(x)$ and $p \not\leq \liminf f(x)$. Now $\uparrow p$ is $\sigma^\uparrow$-open (Fact A.5.5), so again by Theorem 3.1.26, it follows that $\uparrow p$ is $\iota^\uparrow$-open. By local continuity of f' at x, there must exist $F \in \mathcal{F}\,\mathbb{L}$ and $I \in \mathcal{I}\,\mathbb{L}$ such that $e^{\mathcal{F}}_{\mathbb{L}}(F) \leq x \leq e^{\mathcal{I}}_{\mathbb{L}}(I)$ and

$$f'\left[\{y \in \mathbb{L}^\delta \mid e^{\mathcal{F}}_{\mathbb{L}}(F) \leq y \leq e^{\mathcal{I}}_{\mathbb{L}}(I)\}\right] \subseteq \uparrow p. \tag{3.11}$$

Now for every $a \in F \cap I$, by Lemma 3.2.2 we have $e^{\mathcal{F}}_{\mathbb{L}}(F) \leq e_{\mathbb{L}}(a) \leq e^{\mathcal{I}}_{\mathbb{L}}(I)$, so by (3.11), $f' \circ e_{\mathbb{L}}(a) = f(a) \in \uparrow p$. Since $a \in F \cap I$ was arbitrary, it follows that $f[F \cap I] \subseteq \uparrow p$, hence $\bigwedge f[F \cap I] \geq p$. Since $e^{\mathcal{F}}_{\mathbb{L}}(F) \leq x \leq e^{\mathcal{I}}_{\mathbb{L}}(I)$, it follows by definition of lim inf that

$$\liminf f(x) \geq \bigwedge f[F \cap I] \geq p.$$

But this contradicts our assumption that $p \not\leq \liminf f(x)$. It follows that indeed, $f'(x) \liminf f(x)$. ∎

3.2.10. Corollary. *Let $f \colon \mathbb{L} \to \mathbb{C}$ be a function from a lattice $\mathbb{L}$ to a profinite lattice $\mathbb{C}$. If $f' \colon \mathbb{L}^\delta \to \mathbb{C}$ is a (σ, σ)-continuous function such that $f' \circ e_{\mathbb{L}} = f$, we have $f' = \liminf f = \limsup f$.*

Proof If f' is (σ, σ)-continuous, then by Lemma 2.1.28(3), f' is also (δ, σ)-continuous. This has two immediate consequences. Firstly, since $\sigma^\uparrow \subseteq \sigma$ by definition, we see that f' is $(\delta, \sigma^\uparrow)$-continuous, so by Theorem 3.2.9, $f' \leq \liminf f$. Secondly, since $\sigma^\downarrow \subseteq \sigma$, we get that f' is $(\delta, \sigma^\downarrow)$-continuous, so it follows again by Theorem 3.2.9 that $\limsup f \leq f'$. We conclude that

$$\limsup f \leq f' \leq \liminf f.$$

Since $\liminf f \leq \limsup f$ by Lemma 3.2.3(3), it follows that $\limsup f = f' = \liminf f$. ∎

With the help of the above theorem, we can make sure that $f^\triangledown \colon \mathbb{L}^\delta \to \mathbb{M}^\delta$ is the largest continuous extension of $f \colon \mathbb{L} \to \mathbb{M}$ *if* $\mathbb{M}^\delta$ happens to be profinite. Fortunately, there is a condition on $\mathbb{M}$ that guarantees that this will be the case.

3.2.11. Lemma. *Let $\mathbb{L}$ be a profinite lattice and let $\mathbb{M}$ be an arbitrary lattice. If there exists a complete surjective homomorphism $h \colon \mathbb{L} \to \mathbb{M}^\delta$ then $\mathbb{M}^\delta$ is a Boolean topological lattice.*

Proof We know from Fact 2.1.30 that $\mathbb{M}^\delta$ is join-generated by its completely join-irreducibles $\mathrm{J}^\infty(\mathbb{M}^\delta)$ and meet-generated by its completely meet-irreducibles $\mathrm{M}^\infty(\mathbb{M}^\delta)$. We will show that

$$\begin{aligned} &\forall p \in \mathrm{J}^\infty(\mathbb{M}^\delta), \exists Z \subseteq \mathbb{M}^\delta \text{ finite, such that } \mathbb{M}^\delta \setminus {\uparrow} p = {\downarrow} Z, \text{ and} \\ &\forall p \in \mathrm{M}^\infty(\mathbb{M}^\delta), \exists Z \subseteq \mathbb{M}^\delta \text{ finite, such that } \mathbb{M}^\delta \setminus {\downarrow} p = {\uparrow} Z. \end{aligned} \tag{3.12}$$

Since $h\colon \mathbb{L} \to \mathbb{M}^\delta$ is a complete homomorphism, it has a left adjoint $h^\flat\colon \mathbb{M}^\delta \to \mathbb{L}$. Since $h^\flat \dashv h$ and h is surjective, we know from Fact A.3.3 that $h \circ h^\flat = \mathrm{id}_{\mathbb{M}^\delta}$. We claim that

$$h^\flat \text{ maps elements of } \mathrm{J}^\infty(\mathbb{M}^\delta) \text{ to elements of } \mathrm{K}\,\mathbb{L}. \tag{3.13}$$

Let $p \in \mathrm{J}^\infty(\mathbb{M}^\delta)$; we will show that $h^\flat(p) \in \mathrm{K}\,\mathbb{L}$. Let $S \subseteq \mathbb{L}$ be a directed set such that $h^\flat(p) \leq \bigvee S$, then

$$\begin{aligned} h^\flat(p) &= h^\flat(p) \wedge \bigvee S && \text{by order theory,} \\ &= \bigvee \big(h^\flat(p) \wedge S\big) && \text{by } \sigma\text{-continuity of } \wedge. \end{aligned}$$

Now we see that

$$\begin{aligned} p &= h \circ h^\flat(p) && \text{because } h \circ h^\flat = \mathrm{id}_{\mathbb{M}^\delta}, \\ &= h\left(\bigvee(h^\flat(p) \wedge S)\right) && \text{because } h^\flat(p) = \bigvee(h^\flat(p) \wedge S), \\ &= \bigvee\left(h \circ h^\flat(p) \wedge h[S]\right) && \text{because } h \text{ is a complete lattice hom.,} \\ &= \bigvee\left(p \wedge h[S]\right) && \text{because } h \circ h^\flat = \mathrm{id}_{\mathbb{M}^\delta}. \end{aligned}$$

Since p is completely join-irreducible, it follows that there must exist $x \in S$ such that $p = p \wedge h(x)$. But then $p \leq h(x)$, so since $h^\flat \dashv h$, we see that $h^\flat(p) \leq x$; it follows that $p \in \mathrm{K}\,\mathbb{L}$ and we may conclude that (3.13) holds.

Now if $p \in \mathrm{J}^\infty(\mathbb{M}^\delta)$, so that $h^\flat(p) \in \mathrm{K}\,\mathbb{L}$, we know by Theorem 3.1.26 that there exists a finite $Z \subseteq \mathbb{L}$ such that $\mathbb{L} \setminus {\uparrow} h^\flat(p) = {\downarrow} Z$. We will show that

$$\mathbb{M}^\delta \setminus {\uparrow} p = {\downarrow} h[Z]. \tag{3.14}$$

Recall from Fact A.3.3 that $h^\flat$ is an order embedding because h is surjective; we now see that

$$\begin{aligned} &x \in {\downarrow} h[Z] \\ \text{iff } &\exists q \in Z,\, x \leq h(q) && \text{by def. of } {\downarrow}\cdot, \\ \text{iff } &\exists q \in Z,\, h^\flat(x) \leq q && \text{because } h^\flat \dashv h, \\ \text{iff } &h^\flat(x) \in {\downarrow} Z && \text{by def. of } {\downarrow}\cdot, \\ \text{iff } &h^\flat(p) \not\leq h^\flat(x) && \text{because } \mathbb{L} \setminus {\uparrow} h^\flat(p) = {\downarrow} Z, \\ \text{iff } &p \not\leq x && \text{because } h^\flat \text{ is an order embedding.} \end{aligned}$$

It follows that (3.14) holds. Now because Z is finite, so is $h[Z]$. It follows that (3.12) holds (the case for $q \in \mathrm{M}^\infty(\mathbb{M}^\delta)$ follows by order duality). Now we may apply Theorem 3.1.26 to conclude that $\mathbb{M}^\delta$ is a Boolean topological lattice. ∎

3.2.12. Theorem. *Let* $\mathbb{L}$ *be a lattice such that* $\mathrm{HSP}(\mathbb{L})$ *is finitely generated. Then* $\mathbb{L}^\delta$ *is profinite and* $\mathbb{L}^\delta \in \mathrm{HSP}(\mathbb{L})$

Proof Since $\mathrm{HSP}(\mathbb{L})$ is finitely generated, there must exist some finite lattice $\mathbb{M}$ such that $\mathrm{HSP}(\mathbb{L}) = \mathrm{HSP}(\mathbb{M})$. Since $\mathbb{M}$ is finite, by Fact A.6.3 we have that $\mathrm{HSP}(\mathbb{M}) = \mathrm{HSP_B}(\mathbb{M})$. Since $\mathbb{M}$ clearly is profinite and has the property that $\mathbb{M} \in \mathrm{HSP}(\mathbb{M}) = \mathrm{HSP}(\mathbb{L})$, it suffices to show that this property is preserved as we apply $\mathrm{P_B}$, S and H.

If $\mathbb{L} \in \mathrm{P_B}(\mathbb{M})$, then there exists a Boolean decomposition $(p_x \colon \mathbb{L} \to \mathbb{M})_{x \in X}$ for some Boolean space X. By Fact 2.1.27, we have that $\mathbb{L}^\delta \simeq \mathbb{M}^X$. It follows by Fact 3.1.15 that $\mathbb{L}^\delta$ is profinite; it is also immediate that $\mathbb{L}^\delta \in \mathrm{HSP}(\mathbb{L})$.

If $\mathbb{L} \in \mathrm{SP_B}(\mathbb{M})$, then there exists some $\mathbb{L}' \in \mathrm{P_B}(\mathbb{M})$ such that $\mathbb{L}$ is a subalgebra of $\mathbb{L}'$. It follows from Theorem 2.2.24 that $\mathbb{L}^\delta$ is (isomorphic to) a complete subalgebra of $\mathbb{L}'^\delta$. By the above, $\mathbb{L}'^\delta$ is profinite and $\mathbb{L}'^\delta \in \mathrm{HSP}(\mathbb{L})$, so it immediately follows that $\mathbb{L}^\delta \in \mathrm{HSP}(\mathbb{L})$. Moreover, it follows by Lemma 3.1.28 that $\mathbb{L}^\delta$ is profinite.

Finally, if $\mathbb{L} \in \mathrm{HSP_B}(\mathbb{M})$, then there exists $\mathbb{L}' \in \mathrm{SP_B}(\mathbb{M})$ and a surjective homomorphism $h \colon \mathbb{L}' \to \mathbb{L}$. By Theorem 2.2.24, $h^\delta \colon \mathbb{L}'^\delta \to \mathbb{L}^\delta$ is a complete surjective homomorphism; it follows immediately that $\mathbb{L}^\delta \in \mathrm{HSP}(\mathbb{M}) = \mathrm{HSP}(\mathbb{L})$. Since $\mathbb{L}'^\delta$ is profinite by the above, it follows from Lemma 3.2.11 that $\mathbb{L}^\delta$ is a Boolean topological lattice. Since $\mathbb{L}^\delta \in \mathrm{HSP}(\mathbb{M})$, which is a finitely generated congruence distributive variety, it follows from Fact 3.1.17 that $\mathbb{L}^\delta$ is profinite. ∎

3.2.13. Remark. In fact, we will later see in §3.4.2 the statement that $\mathbb{L}^\delta$ is profinite is equivalent to saying that $\mathbb{L}^\delta$ is the profinite completion of $\mathbb{L}$. In this light, the above theorem is a consequence of the main result in [50].

3.2.14. Remark. In our proof of Theorem 3.2.12, we invoke Fact 3.1.17 to show that under the assumptions of the theorem, a Boolean topological quotient of a profinite lattice must again be profinite. In a recent paper, Gehrke et al. show that a Boolean topological quotient of a profinite algebra is always profinite, using a duality argument [35].

We can now state a powerful result about canonical extensions of arbitrary maps, which echoes Theorem 2.2.4.

3.2.15. Corollary. *Let* $f \colon \mathbb{L} \to \mathbb{M}$ *be an arbitrary function between lattices* $\mathbb{L}$ *and* $\mathbb{M}$*; furthermore assume that* $\mathrm{HSP}(\mathbb{M})$ *is finitely generated. Let* $f' \colon \mathbb{L}^\delta \to \mathbb{M}^\delta$ *be an extension of* f*, i.e. assume that* $f' \circ e_{\mathbb{L}} = e_{\mathbb{M}} \circ f$*. Then*

1. *if* $f'\colon \mathbb{L}^\delta \to \mathbb{M}^\delta$ *is* $(\delta, \iota^\uparrow)$*-continuous, then* $f' \leq f^\triangledown$*;*
2. *if* $f'\colon \mathbb{L}^\delta \to \mathbb{M}^\delta$ *is* $(\delta, \iota^\downarrow)$*-continuous, then* $f^\vartriangle \leq f'$*;*

Proof We only prove (1). If HSP($\mathbb{M}$) is finitely generated, then by Theorem 3.2.12, $\mathbb{M}^\delta$ is profinite and consequently $e_\mathbb{M} \circ f\colon \mathbb{L} \to \mathbb{M}^\delta$ is a map into a profinite lattice. It now follows by Theorem 3.2.9 that $f' \leq \liminf(e_\mathbb{M} \circ f) = f^\triangledown$. ∎

3.2.16. Corollary. *Let* $f\colon \mathbb{L} \to \mathbb{M}$ *be an arbitrary function between lattices* $\mathbb{L}$ *and* $\mathbb{M}$*; furthermore assume that* HSP($\mathbb{M}$) *is finitely generated. If* $f'\colon \mathbb{L}^\delta \to \mathbb{M}^\delta$ *is a* (σ, σ)*-continuous function such that* $f' \circ e_\mathbb{L} = e_\mathbb{M} \circ f$ *then* $f' = f^\triangledown = f^\vartriangle$*, i.e.* f *is smooth.*

3.2.3 Canonical extension and function composition

We conclude this section with a series of technical lemmas about the interaction between canonical extension and function composition, and composition with lattice homomorphisms in particular.

If we look at a function composition with a lattice homomorphism on the left, we need no other assumptions to show that canonical extension commutes with function composition. Recall that if $h\colon \mathbb{L} \to \mathbb{M}$ is a lattice homomorphism then h is smooth (by Theorem 2.2.18), so we write h^δ instead of $h^\triangledown$ or $h^\vartriangle$.

3.2.17. Lemma. *Let* $e_i\colon \mathbb{L}_i \to \mathbb{L}_i^\delta$ *be canonical extensions of lattices* $\mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3$*; let* $f\colon \mathbb{L}_1 \to \mathbb{L}_2$ *be an arbitrary map and let* $h\colon \mathbb{L}_2 \to \mathbb{L}_3$ *be a lattice homomorphism. Then* $h^\delta f^\triangledown = (hf)^\triangledown$*.*

Proof Recall from Theorem 2.2.24 that $h^\delta\colon \mathbb{L}_2^\delta \to \mathbb{L}_3^\delta$ is a complete homomorphism.

$$\begin{tikzcd}
\mathbb{L}_1^\delta \arrow[r, "f^\triangledown"] & \mathbb{L}_2^\delta \arrow[r, "h^\delta"] & \mathbb{L}_3^\delta \\
\mathbb{L}_1 \arrow[u, "e_1"] \arrow[r, "f"] & \mathbb{L}_2 \arrow[u, "e_2"] \arrow[r, "h"] & \mathbb{L}_3 \arrow[u, "e_3"']
\end{tikzcd}$$

Let $x \in \mathbb{L}_1^\delta$; it takes an easy computation to see that

$$\begin{aligned}
h^\delta f^\triangledown(x) &= h^\delta \circ \liminf(e_2 \circ f)(x) && \text{by definition of } f^\triangledown, \\
&= h^\delta \left(\bigvee \left\{ \bigwedge e_2 \circ f[F \cap I] \mid e_1^\mathcal{F}(F) \leq x \leq e_1^\mathcal{I}(I) \right\} \right) && \text{by definition of } \liminf \\
&= \bigvee \left\{ \bigwedge h^\delta \circ e_2 \circ f[F \cap I] \mid e_1^\mathcal{F}(F) \leq x \leq e_1^\mathcal{I}(I) \right\} && \text{since } h^\delta \text{ preserves } \bigvee, \bigwedge, \\
&= \bigvee \left\{ \bigwedge e_3 \circ h \circ f[F \cap I] \mid e_1^\mathcal{F}(F) \leq x \leq e_1^\mathcal{I}(I) \right\} && \text{since } h^\delta \circ e_2 = e_3 \circ h, \\
&= \liminf(e_3 \circ h \circ f)(x) && \text{by definition of } \liminf, \\
&= (hf)^\triangledown(x) && \text{by definition of } (hf)^\triangledown.
\end{aligned}$$

Since $x \in \mathbb{L}_1^\delta$ was arbitrary it follows that $h^\delta f^\triangledown = (hf)^\triangledown$. ∎

If we look at lattice homomorphisms on the right, the situation is more complicated. Consider the following picture, where h is a lattice homomorphism and g is an arbitrary map:

$$\mathbb{L}_1 \xrightarrow{h} \mathbb{L}_2 \xrightarrow{g} \mathbb{L}_3$$

When considering the question whether $(gh)^\triangledown = g^\triangledown h^\delta$, it turns out that it matters if h is surjective or not. We first record the following observation about the action of surjective lattice homomorphisms on lattice filters and ideals.

3.2.18. LEMMA. *Let $h\colon \mathbb{L} \to \mathbb{M}$ be a surjective lattice homomorphism. Then for all $F \in \mathcal{F}\,\mathbb{L}$, $I \in \mathcal{I}\,\mathbb{L}$,*

$$F \between I \Rightarrow h[F \cap I] = \mathcal{F}\,h(F) \cap \mathcal{I}\,h(I).$$

Proof Let $F \in \mathcal{F}\,\mathbb{L}$ and $I \in \mathcal{I}\,\mathbb{L}$ and suppose that $F \between I$. Since $F \cap I \subseteq F$, we see that

$$h[F \cap I] \subseteq h[F] \subseteq {\uparrow}\,h[F] = \mathcal{F}\,h(F),$$

and similarly $h[F \cap I] \subseteq \mathcal{I}\,h(I)$, so that

$$h[F \cap I] \subseteq \mathcal{F}\,h(F) \cap \mathcal{I}\,h(I).$$

For the converse, assume that $c \in \mathcal{F}\,h(F) \cap \mathcal{I}\,h(I) = {\uparrow}\,h[F] \cap {\downarrow}\,h[I]$, so there exist $a \in F$ and $b \in I$ such that

$$h(a) \leq c \leq h(b).$$

Since $F \between I$, by Lemma 2.1.3 we know that $F = {\uparrow}(F \cap I)$ and $I = {\downarrow}(F \cap I)$. Consequently, we may assume without loss of generality that $a, b \in F \cap I$. Since $h\colon \mathbb{L} \to \mathbb{M}$ is surjective, there must exist some $c' \in \mathbb{L}_1$ such that $h(c') = c$. Define $c'' := (c' \vee a) \wedge b$. We will show that $h(c'') = c$ and that $c'' \in F \cap I$, so that $c \in h[F \cap I]$. For the first claim, observe that

$$\begin{aligned}
h(c'') &= h\left((c' \vee a) \wedge b\right) && \text{by definition,}\\
&= (h(c') \vee h(a)) \wedge h(b) && \text{because } h \text{ is a homomorphism,}\\
&= (c \vee h(a)) \wedge h(b) && \text{because } h(c') = c,\\
&= c && \text{because } h(a) \leq c \leq h(b).
\end{aligned}$$

For the second claim, observe that since $a \leq c' \vee a$, we also get

$$a \wedge b \leq (c' \vee a) \wedge b = c''.$$

Since $a, b \in F$, we also have $a \wedge b \in F$, so that $c'' \in F$. Since $b \in I$ and

$$c'' = (c' \vee a) \wedge b \leq b,$$

we also see that $c'' \in I$, so that $c'' \in F \cap I$. It follows that $\mathcal{F}\,h(F) \cap \mathcal{I}\,h(I) \subseteq h[F \cap I]$. ∎

3.2.19. LEMMA. *Let $e_i\colon \mathbb{L}_i \to \mathbb{L}_i^\delta$ be canonical extensions of lattices $\mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3$; let $h\colon \mathbb{L}_1 \to \mathbb{L}_2$ be a* surjective *lattice homomorphism and let $g\colon \mathbb{L}_2 \to \mathbb{L}_3$ be an arbitrary map. Then $(gh)^\triangledown = g^\triangledown h^\delta$.*

Proof Recall that

$$\begin{aligned}(gh)^\triangledown(x) &= \liminf(e_3 \circ g \circ h)(x)\\ &= \bigvee \left\{ \textstyle\bigwedge e_3 \circ g \circ h[F \cap I] \mid e_1^{\mathcal{F}}(F) \leq x \leq e_1^{\mathcal{I}}(I) \right\}\end{aligned}$$

and

$$g^\triangledown h^\delta(x) = \bigvee \left\{ \textstyle\bigwedge e_3 \circ g[F' \cap I'] \mid e_2^{\mathcal{F}}(F') \leq h^\delta(x) \leq e_2^{\mathcal{I}}(I) \right\}.$$

We will show that

$$\left\{ g \circ h[F \cap I] \mid e_1^{\mathcal{F}}(F) \leq x \leq e_1^{\mathcal{I}}(I) \right\} = \left\{ g[F' \cap I'] \mid e_2^{\mathcal{F}}(F') \leq h^\delta(x) \leq e_2^{\mathcal{I}}(I') \right\}, \tag{3.15}$$

which is sufficient to show that $(gh)^\triangledown(x) = g^\triangledown h^\delta(x)$. Take an element of the left-hand side of (3.15), i.e. take $F \in \mathcal{F}\,\mathbb{L}_1$, $I \in \mathcal{I}\,\mathbb{L}_1$ such that $e_1^{\mathcal{F}}(F) \leq x \leq e_1^{\mathcal{I}}(I)$. We will show that $g \circ h[F \cap I]$ is an element of the right-hand side of (3.15), because

$$g \circ h[F \cap I] = g\left[\mathcal{F}\,h(F) \cap \mathcal{I}\,h(I)\right],$$

and that $e_2^{\mathcal{F}} \circ \mathcal{F}\,f(F) \leq x \leq e_2^{\mathcal{I}} \circ \mathcal{I}\,h(I)$. The former follows immediately from Lemma 3.2.18; we see that the latter holds since

$$\begin{aligned} e_2^{\mathcal{F}} \circ \mathcal{F}\,h(F) &= h^\delta \circ e_1^{\mathcal{F}}(F) && \text{by Lemma 2.2.3(1),}\\ &\leq h^\delta(x) && \text{since } e_1^{\mathcal{F}}(F) \leq x \leq e_1^{\mathcal{I}}(I),\\ &\leq h^\delta \circ e_1^{\mathcal{I}}(I) && \text{idem,}\\ &= e_2^{\mathcal{I}} \circ \mathcal{I}\,h(I) && \text{by Lemma 2.2.3(2).}\end{aligned}$$

Thus we have shown that the left-hand side of (3.15) is contained in the right-hand side.

Conversely, consider an element of the right-hand side of (3.15), i.e. take $F' \in \mathcal{F}\,\mathbb{L}_2$ and $I' \in \mathcal{I}\,\mathbb{L}_2$ such that $e_2^{\mathcal{F}}(F') \leq h^\delta(x) \leq e_2^{\mathcal{I}}(I')$. We will show that $g[F' \cap I']$ is an element of the left-hand side of (3.15), because

$$g[F' \cap I'] = g \circ h\left[h^{-1}(F) \cap h^{-1}(I)\right],$$

and that $e_1^{\mathcal{F}} \circ h^{-1}(F') \leq x \leq e_1^{\mathcal{I}} \circ h^{-1}(I')$. The first claim follows from the fact that h is surjective, so that $\mathcal{F}\,h$ and $\mathcal{I}\,h$ are also surjective:

$$\begin{aligned} & g \circ h\left[h^{-1}(F') \cap h^{-1}(I')\right]\\ &= g\left[\mathcal{F}\,h \circ h^{-1}(F') \cap \mathcal{I}\,h \circ h^{-1}(I')\right] && \text{by Lemma 3.2.18,}\\ &= g[F' \cap I'] && \text{by surj. of } \mathcal{F}\,h \text{ and } \mathcal{I}\,h.\end{aligned}$$

For the second claim, note that since $e_2^{\mathcal{F}}(F') \leq h^\delta(x) \leq e_2^{\mathcal{I}}(I')$, we have that $h^\delta(x) \in {\uparrow}\, e_2^{\mathcal{F}}(F') \cap {\downarrow}\, e_2^{\mathcal{I}}(I')$, so that

$$\begin{aligned} x &\in (h^\delta)^{-1}\left({\uparrow}\, e_2^{\mathcal{F}}(F') \cap {\downarrow}\, e_2^{\mathcal{I}}(I')\right) \\ &= (h^\delta)^{-1}\left({\uparrow}\, e_2^{\mathcal{F}}(F')\right) \cap (h^\delta)^{-1}{\downarrow}\left(e_2^{\mathcal{I}}(I')\right) \quad \text{because } (h^\delta)^{-1} \text{ commutes with } \cap, \\ &= {\uparrow}\, e_1^{\mathcal{F}} \circ h^{-1}(F') \cap {\downarrow}\, e_1^{\mathcal{I}} \circ h^{-1}(I'), \end{aligned}$$

where the last equality follows from claim (2.21) from the proof of Theorem 2.2.18(2) since h preserves both binary joins and meets. It follows that the right-hand side of (3.15) is contained in the left-hand side. ∎

Recall the picture we had before, where h is a lattice homomorphism and g is an arbitrary map:

$$\mathbb{L}_1 \xrightarrow{h} \mathbb{L}_2 \xrightarrow{g} \mathbb{L}_3$$

It turns out that if h is not surjective, we need to make several strong assumptions if we want to prove that $(gh)^\triangledown = g^\triangledown \circ h^\delta$. For starters, we assume that $\mathrm{HSP}(\mathbb{L}_3)$ is finitely generated.

3.2.20. Lemma. *Let $e_i\colon \mathbb{L}_i \to \mathbb{L}_i^\delta$ be canonical extensions of lattices $\mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3$ and let $f\colon \mathbb{L}_1 \to \mathbb{L}_2$ and $g\colon \mathbb{L}_2 \to \mathbb{L}_3$ be arbitrary maps. Furthermore, assume that* $\mathrm{HSP}(\mathbb{L}_3)$ *is finitely generated.*

1. *If $g^\triangledown f^\triangledown$ is $(\delta, \sigma^\uparrow)$-continuous, then $g^\triangledown f^\triangledown \leq (gf)^\triangledown$;*
2. *If $g^\triangledown f^\triangledown$ is $(\delta, \sigma^\downarrow)$-continuous, then $g^\triangledown f^\triangledown \geq (gf)^\triangledown$;*
3. *If $g^\triangledown f^\triangledown$ is (δ, σ)-continuous, then $g^\triangledown f^\triangledown = (gf)^\triangledown$.*

Proof (1). It is easy to see that $g^\triangledown f^\triangledown$ extends $gf\colon \mathbb{L}_1 \to \mathbb{L}_3$, since both squares below commute:

$$\begin{tikzcd} \mathbb{L}_1^\delta \arrow[r, "f^\triangledown"] & \mathbb{L}_2^\delta \arrow[r, "g^\triangledown"] & \mathbb{L}_3^\delta \\ \mathbb{L}_1 \arrow[u, "e_1"] \arrow[r, "f"] & \mathbb{L}_2 \arrow[u, "e_2"] \arrow[r, "g"] & \mathbb{L}_3 \arrow[u, "e_3"'] \end{tikzcd}$$

Now since we assumed that $g^\triangledown f^\triangledown$ is $(\delta, \sigma^\uparrow)$-continuous, and since $\iota^\uparrow \subseteq \sigma^\uparrow$, it follows that $g^\triangledown f^\triangledown \leq (gf)^\triangledown$, since $(gf)^\triangledown$ is the largest $(\delta, \iota^\uparrow)$-continuous extension of gf by Corollary 3.2.15.

(2). If $g^\triangledown f^\triangledown$ is $(\delta, \sigma^\downarrow)$-continuous, then

$$\begin{aligned} g^\triangledown f^\triangledown &\geq (gf)^\vartriangle && \text{by Corollary 3.2.15,} \\ &\geq (gf)^\triangledown && \text{by Lemma 3.2.8(3).} \end{aligned}$$

(3). If $g^\triangledown f^\triangledown$ is (δ, σ)-continuous, then a forteriori $g^\triangledown f^\triangledown$ is $(\delta, \sigma^\uparrow)$-continuous and $(\delta, \sigma^\downarrow)$-continuous, since $\sigma^\uparrow, \sigma^\downarrow \subseteq \sigma$. The statement now follows by (1) and (2). ∎

We can now straightforwardly apply Lemma 3.2.20 to the case where we have a lattice homomorphism on the right.

3.2.21. Lemma. *Let $e_i\colon \mathbb{L}_i \to \mathbb{L}_i^\delta$ be canonical extensions of lattices $\mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3$; let $h\colon \mathbb{L}_1 \to \mathbb{L}_2$ be a lattice homomorphism and let $g\colon \mathbb{L}_2 \to \mathbb{L}_3$ be an arbitrary map. Furthermore, assume that* HSP$(\mathbb{L}_3)$ *is finitely generated. Then $g^\triangledown h^\delta \leq (gh)^\triangledown$.*

If we additionally assume that $g^\triangledown : \mathbb{L}_2^\delta \to \mathbb{L}_3^\delta$ is (σ,σ)-continuous, then $g^\triangledown h^\delta = (gh)^\triangledown$.

Proof By Theorem 2.2.24, h^δ is (δ,δ)-continuous. Since $g^\triangledown$ is $(\delta,\sigma^\uparrow)$-continuous by Corollary 3.2.8, it follows by general topology that $g^\triangledown h^\delta$ is $(\delta,\sigma^\uparrow)$-continuous. Since $h^\delta = h^\triangledown$, it follows by Lemma 3.2.20(1) that $g^\triangledown h^\delta \leq (gh)^\triangledown$.

If $g^\triangledown$ is (σ,σ)-continuous, then by Lemma 2.1.28(3), $g^\triangledown$ is also (δ,σ)-continuous. By Theorem 2.2.24, h^δ is (σ,σ)-continuous, so we see that $g^\triangledown \circ h^\delta$ is (δ,σ)-continuous. It follows by Lemma 3.2.20(3) that $g^\triangledown h^\delta = (gh)^\triangledown$. ∎

3.2.4 Conclusions and further work

In this section we investigated canonical extensions of arbitrary maps between lattices, rather than extensions of order-preserving maps (which we studied in §2.2). Canonical extensions of arbitrary maps between *distributive* lattices have been studied extensively by Gehrke and Jònsson [39]. This section is based on our paper with M. Gehrke [43]; our contribution lies primarily in two observations:

- The results in [39] hold not only for maps between distributive lattices, but more generally for maps between lattices which lie in finitely generated varieties.
- When considering the canonical extensions $f^\triangledown : \mathbb{L}^\delta \to \mathbb{M}^\delta$ and $f^\vartriangle : \mathbb{L}^\delta \to \mathbb{M}^\delta$ of a map $f\colon \mathbb{L} \to \mathbb{M}$, the natural topology one should be using on the codomain $\mathbb{M}^\delta$ is the $\sigma^\uparrow$-topology, respectively the $\sigma^\downarrow$-topology. This is a departure from [39], where one would have considered the $\iota^\uparrow$-topology and the $\iota^\downarrow$-topology on $\mathbb{M}^\delta$. Our choice for the σ topologies was inspired by Y. Venema's treatment of canonicity for BAOs [89] and the work on MacNeille completions by Theunissen and Venema [86].

Most of the results from §3.2.1 were only known to hold for distributive lattices from [39]. What was not known before however, is that one does not need to make any assumptions about the lattices or the maps involved to prove the results in §3.2.1. In §3.2.2 we introduced topological algebra into the picture of canonical extensions. Theorem 3.2.12, which says that $\mathbb{L}^\delta$ is profinite if HSP$(\mathbb{L})$ is finitely generated, can be regarded as a corollary of a result of J. Harding [50]. It is interesting to note that many of the proofs in §3.2.3 are direct adaptations of the proofs for distributive lattices from [39], which further supports our claim above that all results in that paper may be generalized from distributive lattices to lattices lying in a finitely generated variety.

3.2.22. Remark. We now have two sets of circumstances under which $f^\triangledown : \mathbb{L}^\delta \to \mathbb{M}^\delta$ is the largest continuous extension of $f : \mathbb{L} \to \mathbb{M}$ (and dually, we have conditions under which $f^\vartriangle$ is the smallest continuous extension of f).

- If f is order-preserving, then $f^\triangledown$ is the largest $(\delta^\uparrow, \sigma^\uparrow)$-continuous extension of f (Theorem 2.2.4).
- If HSP($\mathbb{M}$) is finitely generated, then $f^\triangledown$ is the largest $(\delta, \sigma^\uparrow)$-continuous extension of f (Corollary 3.2.15).

It would be interesting to see if there is a unifying explanation for these continuity properties.

3.3 Canonical extension as a functor II: lattice-based algebras

In this section, we want to change our perspective on canonical extension from a construction on lattices to a construction on lattice-based algebras. Previously, canonical extensions of lattice-based algebras have only been considered under certain additional assumptions, such as monotonicity of all algebra operations [34], or distributivity of the underlying lattice [39]. In contrast, we will define canonical extensions for lattice-based algebras without making any further assumptions about the algebra operations or the shape of the lattice (other than boundedness).

Once we have defined canonical extensions of lattice-based algebras, it does not follow straightforwardly that the canonical extension construction applied to lattice based-algebras is well-defined on algebra homomorphisms, i.e. whether canonical extension is a *functor*. In fact, it is already known from [39] that in general this is not the case, unless we make certain assumptions about either the algebras or the homomorphisms involved. We will discuss two ways to improve the behaviour of canonical extensions of homomorphisms. Firstly, if $h : \mathbb{A} \to \mathbb{B}$ is a homomorphism between lattice-based algebras and every algebra operation is monotone, i.e. order-preserving or order-reversing in each coordinate, then $h^\delta : \mathbb{A}^\delta \to \mathbb{B}^\delta$ is also an algebra homomorphism. This is already known from [34]. If we do not know whether all algebra operations on $\mathbb{A}$ and $\mathbb{B}$ are monotone, but we do know that h is surjective, then $h^\delta : \mathbb{A}^\delta \to \mathbb{B}^\delta$ is also an algebra homomorphism. If the assumption of surjectivity of h is dropped, we can no longer guarantee that h^δ will be a homomorphism. This preservation of surjective algebra homomorphisms was already known for the distributive case from [39]. We will see however that distributivity is not needed for this result.

A second question we may ask ourselves is:

> For which equations is validity on a lattice-based algebra $\mathbb{A}$ preserved when moving to its canonical extension $\mathbb{A}^\delta$?

This topic is known as *canonicity*. One could argue that the study of canonicity is actually the raison d'être of the body of work on canonical extensions that this chapter and the previous are contributing to. The approaches to proving canonicity occupy a spectrum ranging from the imposing of very restrictive abstract conditions on $\mathbb{A}$ (e.g. demanding that HSP($\mathbb{A}$) is finitely generated) which guarantee preservation of validity of *all* equations, to sophisticated toolkits that allow one to decide whether validity of *one* given equation is preserved by looking at the syntactic shape of the given equation. The results on canonical extensions in this dissertation make a technical contribution to the methods of proving canonicity at both ends of the above-mentioned spectrum, but remain foundational. We will not go into any concrete applications of canonicity.

This section is organized as follows. First, we set up the technicalities of defining canonical extensions of lattice-based algebras. In particular we need to take some care when extending order-reversing maps, and for every algebra operation $\omega_{\mathbb{A}} : A^n \to A$, we need to choose whether we want its canonical extension to be $\omega_{\mathbb{A}}^{\triangledown}$ or $\omega_{\mathbb{A}}^{\vartriangle}$. After that we look at preservation of homomorphisms; we conclude this section with a discussion of the relation of this work to the field of canonicity. We discuss the contribution of this section and possible further work on p. 93.

3.3.1 Order types and canonical extension types

Our goal in this subsection is to define canonical extensions for any algebra with a lattice reduct. So let us first make this notion of lattice-based algebra a little more precise.

3.3.1. DEFINITION. Let Ω be an algebraic signature and let $\mathrm{ar}\colon \Omega \to \mathbb{N}$ be its associated arity function (see §A.6). We say that Ω is a lattice-based similarity type if $\{\wedge, \vee, 0, 1\} \subseteq \Omega$; in the remainder of this section we will always assume that we are dealing with lattice-based signatures. Given a lattice-based similarity type Ω, a *lattice-based Ω-algebra* is an Ω-algebra $\mathbb{A} = \langle A, (\omega_{\mathbb{A}})_{\omega\in\Omega}\rangle$ such that $\langle A, \wedge_{\mathbb{A}}, \vee_{\mathbb{A}}, 0_{\mathbb{A}}, 1_{\mathbb{A}}\rangle$ is a lattice. We denote the lattice reduct of $\mathbb{A}$ by $\mathbb{A}^l$. We will usually suppress the subscripts on algebra operations, writing $\wedge$ instead of $\wedge_{\mathbb{A}}$, etc.

By $\mathbf{LatAlg}_\Omega$ we denote the category of lattice-based Ω-algebras and Ω-algebra homomorphisms. If it is clear from the context what Ω is, we will simply speak of lattice-based algebras and algebra homomorphisms.

Now, we would like to define canonical extensions of lattice-based algebras in such a way that we can profit maximally from the results in §2.2. In particular, we want to see order-reversing maps as a variation of order-preserving maps. The reason this is possible is that a map $f\colon \mathbb{L} \to \mathbb{M}$ is order-reversing if and only if $f\colon \mathbb{L}^{op} \to \mathbb{M}$ is order-preserving. So if e.g. $g\colon \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ is a function that is order-reversing in its first coordinate and order-preserving in its second coordinate, then equivalently, $g\colon \mathbb{L}^{op} \times \mathbb{L} \to \mathbb{L}$ is order-preserving. The following definition will provide us with notation for dealing with such maps in a uniform way.

3.3.2. Definition. An *order type* of arity n is an element $o \in \{1, op\}^n$. If $o = (o_1, \dots, o_n)$ is an order type and $\mathbb{L}$ is a lattice, then we define

$$\mathbb{L}^o := \mathbb{L}^{o_1} \times \cdots \times \mathbb{L}^{o_n},$$

where $\mathbb{L}^1 := \mathbb{L}$ and $\mathbb{L}^{op}$ is the usual order dual of $\mathbb{L}$. If $f\colon \mathbb{L} \to \mathbb{M}$ is a function, then we define $f^o\colon \mathbb{L}^o \to \mathbb{M}^o$ as follows:

$$f^o\colon (x_1, \dots, x_n) \to (f(x_1), \dots, f(x_n)) .$$

It is easy to see that f^o is again a lattice homomorphism.

Observe that the usual product construction of lattices is a special case of the above construction: if we take $o_i = 1$ for $i = 1, \dots, n$, then $\mathbb{L}^o = \mathbb{L}^n$ and likewise for lattice homomorphisms.

We can now define monotone lattice-based algebras, i.e. lattice-based algebras with operations that are either order-preserving or order-reversing in each coordinate.

3.3.3. Definition. Given an algebraic signature Ω, an *order signature* is a function $\mathrm{ord}\colon \Omega \to \{1, op\}^*$ such that for all $\omega \in \Omega$, $\mathrm{ord}(\omega) \in \{1, op\}^{\mathrm{ar}(\omega)}$. An ord-monotone lattice based Ω-algebra $\mathbb{A}$ is a lattice-based Ω-algebra such that for all $\omega \in \Omega$, $\omega_{\mathbb{A}}\colon \mathbb{A}^{\mathrm{ord}(\omega)} \to \mathbb{A}$ is order-preserving.

By $\mathbf{MLatAlg}_{(\Omega,\mathrm{ord})}$ we denote the category of ord-monotone lattice based Ω-algebras and Ω-algebra homomorphisms. If it is clear from the context what Ω and ord are, we will simply speak of monotone lattice-based algebras.

3.3.4. Example. Heyting algebras are an example of monotone lattice-based algebras. Their signature is $\Omega = \{\to, \wedge, \vee, 0, 1\}$, where $\mathrm{ar}(\to) = 2$, and their order types are as follows: $\mathrm{ord}(\to) = (op, 1)$ and $\mathrm{ord}(\wedge) = \mathrm{ord}(\vee) = (1, 1)$. Indeed,

$$\begin{aligned} \wedge_{\mathbb{A}}&\colon \mathbb{A} \times \mathbb{A} \to \mathbb{A}, \\ \vee_{\mathbb{A}}&\colon \mathbb{A} \times \mathbb{A} \to \mathbb{A}, \\ \to_{\mathbb{A}}&\colon \mathbb{A}^{op} \times \mathbb{A} \to \mathbb{A}, \end{aligned}$$

are all order-preserving.

We know that canonical extensions commute with products and taking order duals, modulo isomorphism. In practice it is rather cumbersome to explicitly deal with said isomorphisms all the time however.

3.3.5. Convention. Let $\mathbb{L}$ be a lattice and let o be an order type. It follows by Lemma 2.1.26 that $(\mathbb{L}^\delta)^o$ is a canonical extension of $\mathbb{L}^o$. In light of this fact, we will identify $(\mathbb{L}^\delta)^o$ and $(\mathbb{L}^o)^\delta$. Consequently, if $h\colon \mathbb{L} \to \mathbb{M}$ is a lattice homomorphism, then we also identify $(h^\delta)^o\colon (\mathbb{L}^\delta)^o \to (\mathbb{M}^\delta)^o$ and $(h^o)^\delta\colon (\mathbb{L}^o)^\delta \to (\mathbb{M}^o)^\delta$.

We can now finally define canonical extensions of lattice-based algebras.

3.3.6. Definition. Let Ω be a lattice-based similarity type. A *canonical extension type* for Ω is a function $\beta\colon \Omega \to \{\nabla, \Delta\}$. Given a lattice-based Ω-algebra $\mathbb{A} = \langle A, (\omega_{\mathbb{A}})_{\omega\in\Omega}\rangle$, the *$\beta$-canonical extension of* $\mathbb{A}$ is the algebra $\mathbb{A}^\delta := \langle A^\delta, (\omega_{\mathbb{A}^\delta})_{\omega\in\Omega}\rangle$. For each $\omega \in \Omega$, the map $\omega_{\mathbb{A}^\delta}\colon (\mathbb{A}^\delta)^n \to \mathbb{A}^\delta$, where $n = \mathrm{ar}(\omega)$, is

$$(\omega_{\mathbb{A}})^{\beta(\omega)}\colon \left(\mathbb{A}^\delta\right)^n \to \mathbb{A}^\delta$$

If it is clear from the context what β is, we will simply speak of the canonical extension of $\mathbb{A}$.

If $\mathbb{A}$ is a monotone lattice-based algebra with respect to some order signature ord, then we define $\omega_{\mathbb{A}^\delta}$ as

$$(\omega_{\mathbb{A}})^{\beta(\omega)}\colon \left(\mathbb{A}^\delta\right)^{\mathrm{ord}(\omega)} \to \mathbb{A}^\delta.$$

Observe that we are really using Convention 3.3.5 above: strictly speaking, $(\omega_{\mathbb{A}})^{\beta(\omega)}$ is a map $(\mathbb{A}^o)^\delta \to \mathbb{A}^\delta$.

We can now finally speak of canonical extensions of algebras. We should keep in mind however that a priori, a lattice-based algebra has many different canonical extensions: for every algebra operation we may choose either the upper or the lower canonical extension. Sometimes, however, these choices do not arise. Recall that a map $f\colon \mathbb{L} \to \mathbb{M}$ between lattices $\mathbb{L}$ and $\mathbb{M}$ is called smooth if $f^\nabla = f^\Delta$. Analogously, we say a lattice-based Ω-algebra $\mathbb{A}$ is smooth if for every $\omega \in \Omega$, $(\omega_{\mathbb{A}})^\nabla = (\omega_{\mathbb{A}})^\Delta$. Observe that in case $\mathbb{A}$ is smooth, $\mathbb{A}$ has a unique canonical extension, since it does not matter whether $\beta(\omega) = \nabla$ or $\beta(\omega) =\Delta$ for any $\omega \in \Omega$.

3.3.7. Lemma. *Fix a lattice-based signature Ω and a canonical extension type β. Let $\mathbb{A}$ be a lattice-based Ω-algebra such that* $\mathrm{HSP}(\mathbb{A}^l)$ *is finitely generated. If $\mathbb{A}^\delta$ is a Boolean topological algebra then $\mathbb{A}$ is smooth.*

Proof Suppose that $\mathbb{A}^\delta$ is a Boolean topological algebra; we need to show that for every $\omega \in \Omega$, $\omega_{\mathbb{A}}$ is smooth. Take $\omega \in \Omega$ and without loss of generality, assume that $\beta(\omega) = \nabla$. Because $\mathbb{A}^\delta$ is a Boolean topological algebra, we know that $(\omega_{\mathbb{A}})^\nabla$ must be (σ, σ)-continuous, since the topology on $\mathbb{A}^\delta$ is the σ-topology (Fact 3.1.23(1)). It follows by Corollary 3.2.16 that $\omega_{\mathbb{A}}$ is smooth; this is where we use the fact that $\mathrm{HSP}(\mathbb{A}^l)$ is finitely generated. Since $\omega \in \Omega$ was arbitrary, it follows that $\mathbb{A}$ is smooth. ∎

Finite lattice-based algebras in particular satisfy the conditions of the above lemma, although it is also easy to see directly that if $\mathbb{A}$ is a finite lattice-based algebra, then $\mathbb{A} \simeq \mathbb{A}^\delta$. Just as we did with lattices in Convention 2.1.12, we will therefore just define $\mathbb{A}^\delta := \mathbb{A}$ in case $\mathbb{A}$ is finite.

3.3.8. Convention. If $\mathbb{A}$ is a finite lattice-based algebra, then we define $\mathbb{A}^\delta := \mathbb{A}$.

We conclude this subsection with some observations about $(\cdot)^l$, the forgetful functor from the category of lattice-based Ω-algebras to the category of lattices.

3.3.9. Fact. *Let $\mathbb{A}$ be a lattice-based Ω-algebra. Then*

1. $(\mathbb{A}^\delta)^l = (\mathbb{A}^l)^\delta$*;*
2. *If* $\mathrm{HSP}(\mathbb{A})$ *is a finitely generated variety of Ω-algebras, then* $\mathrm{HSP}(\mathbb{A}^l)$ *is a finitely generated variety of lattices;*
3. *If $\mathbb{A}$ is a profinite Ω-algebra, then $\mathbb{A}^l$ is a profinite lattice.*

3.3.2 Preservation of homomorphisms

In this subsection we will prove two main results on preservation of algebra homomorphisms by canonical extensions. These results are very important in light of one of the main subjects of this chapter: the relation between canonical extensions and profinite completions. The latter is characterized externally, in terms of algebra homomorphisms. Consequently, homomorphisms form a very natural element of our discourse. Canonical extensions do not behave perfectly on homomorphisms; interestingly, they do behave well enough.

3.3.10. Theorem ([34]). *Fix a lattice-based similarity type Ω, an order type* ord *and a canonical extension type β. If $\mathbb{A}$ and $\mathbb{B}$ are monotone lattice-based algebras and if $h\colon \mathbb{A} \to \mathbb{B}$ is an algebra homomorphism, then $h^\delta\colon \mathbb{A}^\delta \to \mathbb{B}^\delta$ is also an algebra homomorphism.*

Proof We know by Theorem 2.2.24(1) that $h^\delta\colon \mathbb{A}^\delta \to \mathbb{B}^\delta$ is a lattice homomorphism which is both (σ,σ)-continuous and (δ,δ)-continuous. To show that it is an algebra homomorphism, consider an arbitrary $\omega \in \Omega$. Let $\mathrm{ord}(\omega) = o$, so that $\omega_\mathbb{A}\colon A^o \to A$ is an order-preserving map. Finally, without loss of generality, assume that $\beta(\omega) = \triangledown$, so that $\omega_{\mathbb{A}^\delta} = (\omega_\mathbb{A})^\triangledown$. By Convention 3.3.5, it suffices to show that the following diagram commutes:

$$\begin{tikzcd}
(A^o)^\delta \arrow[r, "(\omega_\mathbb{A})^\triangledown"] \arrow[d, "(h^o)^\delta"'] & A \arrow[d, "h^\delta"] \\
(B^o)^\delta \arrow[r, "(\omega_\mathbb{B})^\triangledown"] & B
\end{tikzcd}$$

It follows from Lemma 3.2.17 that

$$h^\delta \circ (\omega_{\mathbb{A}})^\triangledown = (h \circ \omega_{\mathbb{A}})^\triangledown. \tag{3.16}$$

Secondly, since h^o is a fortiori a $\wedge$-homomorphism, it follows by Theorem 2.2.18(2) that $(h^o)^\delta$ is $(\delta^\uparrow, \delta^\uparrow)$-continuous, so by Corollary 2.2.23(3), we see that

$$(\omega_{\mathbb{B}})^\triangledown \circ (h^o)^\delta = (\omega_{\mathbb{B}} \circ h^o)^\triangledown. \tag{3.17}$$

Now because $h\colon \mathbb{A} \to \mathbb{B}$ is an Ω-algebra homomorphism, we see that $h \circ \omega_{\mathbb{A}} = \omega_{\mathbb{B}} \circ h^o$, so that

$$\begin{aligned} h^\delta \circ (\omega_{\mathbb{A}})^\triangledown &= (h \circ \omega_{\mathbb{A}})^\triangledown && \text{by (3.16)}, \\ &= (\omega_{\mathbb{B}} \circ h^o)^\triangledown && \text{since } h \circ \omega_{\mathbb{A}} = \omega_{\mathbb{B}} \circ h^o, \\ &= (\omega_{\mathbb{B}})^\triangledown \circ (h^o)^\delta && \text{by (3.17)}. \end{aligned}$$

Since $\omega \in \Omega$ was arbitrary, we conclude that h^δ is an Ω-algebra homomorphism. ∎

Thus we see that canonical extension maps algebra homomorphisms to algebra homomorphisms, provided we are looking at *monotone* lattice-based algebras. Since all other things we ask of functors (commuting with function composition, preserving the identity function) already follow from Theorem 2.2.24, we get the following result.

3.3.11. Corollary ([34]). *Fix a lattice-based similarity type Ω, an order type* ord *and a canonical extension type β. Then β-canonical extension forms a functor from* $\mathbf{MLatAlg}_{(\Omega,\mathrm{ord})}$ *to* $\mathbf{MLatAlg}_{(\Omega,\mathrm{ord})}$.

Now we turn to the more general situation where we do not assume that every operation of our lattice-based algebras is monotone. The following theorem was already known for distributive lattice-based algebras [39, Theorem 3.7]; interestingly, however, distributivity is not a necessary condition.

3.3.12. Theorem. *Fix a lattice-based similarity type Ω and a canonical extension type β. Let $\mathbb{A}$ and $\mathbb{B}$ be lattice-based algebras. If $h\colon \mathbb{A} \to \mathbb{B}$ is a surjective algebra homomorphism, then so is $h^\delta\colon \mathbb{A}^\delta \to \mathbb{B}^\delta$.*

Proof As in the proof of Theorem 3.3.10, we must show that for arbitrary $\omega \in \Omega$, we have that $h^\delta \circ \omega_{\mathbb{A}^\delta} = \omega_{\mathbb{B}^\delta} \circ (h^\delta)^n$, where $n = \mathrm{ar}(\omega)$. Without loss of generality, we again assume that $\beta(\omega) = \triangledown$; now as before, showing that

$$h^\delta \circ (\omega_{\mathbb{A}})^\triangledown = (\omega_{\mathbb{B}})^\triangledown \circ (h^\delta)^n$$

boils down to showing that the following diagram commutes; the difference is that we do not have to bother with order types.

$$\begin{tikzcd} (A^n)^\delta \arrow[r, "(\omega_{\mathbb{A}})^\triangledown"] \arrow[d, "(h^n)^\delta"'] & A \arrow[d, "h^\delta"] \\ (B^n)^\delta \arrow[r, "(\omega_{\mathbb{B}})^\triangledown"] & B \end{tikzcd}$$

It follows from Lemma 3.2.17 that

$$h^\delta \circ (\omega_{\mathbb{A}})^\triangledown = (h \circ \omega_{\mathbb{A}})^\triangledown.$$

For the other direction, observe that since $h\colon A \to B$ is surjective, so is $h^n\colon A^n \to B^n$, so by Lemma 3.2.19 we have that

$$(\omega_{\mathbb{B}})^\triangledown \circ (h^n)^\delta = (\omega_{\mathbb{B}} \circ h^n)^\triangledown.$$

Just like in the proof of Theorem 3.3.10, we see that the diagram above commutes because h is an Ω-algebra homomorphism, i.e. because $h \circ \omega_{\mathbb{A}} = \omega_{\mathbb{B}} \circ h^n$. Since $\omega \in \Omega$ was arbitrary we see that $h^\delta\colon \mathbb{A}^\delta \to \mathbb{B}^\delta$ is an Ω-algebra homomorphism. ∎

Unfortunately, it is not possible to improve on the above theorem, for [39, Example 3.8] provides an example of a distributive lattice-based algebra $\mathbb{B}$ with a subalgebra $\mathbb{A}$ such that $\mathbb{A}^\delta$ is *not* a subalgebra of $\mathbb{B}^\delta$. However, we will see that for the main result of §3.4.1, our Theorem 3.3.12 gives us just enough to work with.

3.3.3 Canonicity

To conclude this section we will make a few remarks about the preservation of (equations and) inequations by canonical extensions of lattice-based algebras, i.e. canonicity of inequations. This very rich subject is probably the single most important application of canonical extensions in logic. This has to do with the relation between canonical extensions and Stone duality, a subject we will look at in Chapter 4. The exact details of the applications in logic aside, the question is whether we can show that if a given inequation $s \preccurlyeq t$ is valid on a lattice-based algebra $\mathbb{A}$ (see §A.6), then $s \preccurlyeq t$ is also valid on $\mathbb{A}^\delta$. We will very briefly discuss three approaches to this question.

Finitely generated varieties

The first approach is one of brute force. If we assume that HSP($\mathbb{A}$) is finitely generated, then *every* inequation valid on $\mathbb{A}$ is also valid on $\mathbb{A}^\delta$. The reason for this is that under these assumptions, canonical extensions coincide with profinite completions; see §3.4.2. It holds for any algebra $\mathbb{A}$ that $\hat{\mathbb{A}}$, the profinite completion of $\mathbb{A}$, lies in the variety generated by $\mathbb{A}$ (see §3.1.2). Therefore, an equation $s \approx t$ is valid on $\hat{\mathbb{A}}$ iff it is valid on $\mathbb{A}$; since every inequation $s \preccurlyeq t$ can be encoded as an equation $s \vee t \approx t$, the same holds for inequations. Profinite completions are always well-behaved in this sense, and if $\mathrm{HSP}(\mathbb{A})$ is finitely generated, this good behaviour of $\hat{\mathbb{A}}$ is also exhibited by $\mathbb{A}^\delta$ since the two are isomorphic. This good behaviour comes at a price, however. There are many interesting varieties of lattice-based algebras which are not finitely generated. To name two of the most illustrious, we mention the variety of Heyting algebras and the variety of modal algebras.

Our contribution to this first approach of using finitely generated varieties is that we show in §3.4.2 that this works for arbitrary lattice-based algebras, rather than only for monotone lattice-based algebras [34, Corollary 6.9] or distributive lattice-based algebras [39, Corollary 4.6].

Sahlqvist-style theorems

The second approach to proving canonicity focusses on one inequation $s \preccurlyeq t$ at a time, relying on syntactic criteria on s and t to prove a result. As an example, we give a proof for a result that goes back to [38]; the form in which we state it is closer to [34] though. The particular proof we employ (via dcpo algebras) was first presented in [42].

In the proof of the theorem that follows, we want to exploit properties of dcpo algebras and dcpo presentations. First of all, we have to explain what a dcpo algebra is. An Ω-dcpo algebra is simply an ordered Ω-algebra $\mathbb{A}$ such that $\langle A, \leq\rangle$ is a dcpo and such that each $\omega_{\mathbb{A}} \colon A^{\mathrm{ar}(\omega)} \to A$ is Scott-continuous. In §2.3, we have considered dcpo presentations, which allowed us to describe dcpos in an economical fashion. This technique can be extended so that it is also applicable to dcpo algebras.

3.3.13. Definition. An Ω-*dcpo algebra presentation* consists of a structure $\langle P, \sqsubseteq, \lhd, (\omega_P)_{\omega\in\Omega}\rangle$ such that $\langle P, \sqsubseteq, \lhd\rangle$ is a dcpo presentation, $\langle P, (\omega_P)_{\omega\in\Omega}\rangle$ is an Ω-algebra and each $\omega_P \colon P^{\mathrm{ar}(\omega)} \to P$ is cover-stable.

As usual, given a definition of a particular kind of algebra presentation, one needs to show that each such presentation actually presents an object. This is exactly what the following fact tells us.

3.3.14. Fact ([60]). *Let $\langle P, \sqsubseteq, \lhd, (\omega_P)_{\omega\in\Omega}\rangle$ be a dcpo algebra presentation. Suppose that $\langle P, \sqsubseteq, \lhd\rangle$ presents a dcpo $\mathbb{D}$ via $\eta \colon P \to \mathbb{D}$. Then*

1. *There exist unique Scott-continuous algebra operations $\omega_{\mathbb{D}}$ on $\mathbb{D}$ such that $\eta \colon P \to \mathbb{D}$ is an Ω-algebra homomorphism;*

2. *For all inequations $s \preccurlyeq t$, if $\langle P, (\omega_P)_{\omega\in\Omega}\rangle \models s \preccurlyeq t$ then also $\langle \mathbb{D}, (\omega_{\mathbb{D}})_{\omega\in\Omega}\rangle \models s \preccurlyeq t$.*

Now that we have prepared our magic wand, we can go ahead and present our theorem. Observe that we make a distinction between the signature of the algebra $\mathbb{A}$ and the signature used in the inequation $s \preccurlyeq t$ that we want to preserve. The reason for this is that demanding that each operation in the algebra signature Ω is an operator would restrict the applicability of the theorem to distributive lattices, since it would require that $\wedge \colon \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ is an operator.

3.3.15. Theorem ([34]). *Let $\mathbb{A}$ be a lattice-based Ω-algebra and let $s \preccurlyeq t$ be an inequation. If for each ω occurring in s or t, $\omega_{\mathbb{A}}$ is an operator and $\omega_{\mathbb{A}^\delta} = (\omega_{\mathbb{A}})^{\triangledown}$, then $\mathbb{A} \models s \preccurlyeq t$ implies $\mathbb{A}^\delta \models s \preccurlyeq t$.*

Proof We present the proof of the theorem as found in [42]. Let $s(x_1, \ldots, x_n)$ and $t(x_1, \ldots, x_n)$ be terms and let Ω' denote the set of function symbols occurring in s or t. We assume that each $\omega \in \Omega'$ is an operator. Fix any canonical extension type $\beta \colon \Omega \to \{\triangledown, \vartriangle\}$ such that $\beta(\omega) = \triangledown$ for all $\omega \in \Omega'$. Now, suppose that $\mathbb{A} \models s \preccurlyeq t$; we need to show that $\mathbb{A}^\delta \models s \preccurlyeq t$

Since for each $\omega \in \Omega'$, we assumed $\omega_{\mathbb{A}}$ is an operator, we see by Lemma 2.3.8 that $\mathcal{F}\,\omega_{\mathbb{A}}$ is cover-stable for each $\omega \in \Omega'$. It follows that $\langle \mathcal{F}\,\mathbb{A}, \supseteq, (\mathcal{F}\,\omega_{\mathbb{A}})_{\omega \in \Omega'} \rangle$ is a dcpo algebra presentation. Moreover, the dcpo algebra it presents is the Ω'-reduct of $\mathbb{A}^\delta$, since $\overline{\mathcal{F}\,\omega_{\mathbb{A}}} = (\omega_{\mathbb{A}})^{\triangledown}$ for each $\omega \in \Omega'$ by Lemma 2.3.8. Now by Fact 3.3.14, it follows that if we can show that $\langle \mathcal{F}\,\mathbb{A}, \supseteq, (\mathcal{F}\,\omega_{\mathbb{A}})_{\omega \in \Omega'} \rangle \models s \preccurlyeq t$, then we automatically get that $\mathbb{A}^\delta \models s \preccurlyeq t$. But the former is easy to see: because each operation in s and t is order-preserving, it follows from the fact that $\mathcal{F}$ is a functor from the category of partially ordered sets and order-preserving maps to the category of co-dcpos that

$$\mathcal{F}\,s_{\mathbb{A}} = s_{\mathcal{F}\,\mathbb{A}}, \tag{3.18}$$

and likewise for t. This can be shown by an easy induction on the complexity of s. (See §A.6 for a reminder about term functions and universal algebra.)

- Suppose that $s = x_i$. Then $s_{\mathbb{A}} \colon \mathbb{A}^n \to \mathbb{A}$ is simply the i-th projection function $\pi_i \colon \mathbb{A}^n \to \mathbb{A}$, and by Fact A.5.4, $\mathcal{F}\,\pi_i \colon (\mathcal{F}\,\mathbb{A})^n \to \mathcal{F}\,\mathbb{A}$ is again the i-th projection function.

- Now suppose that $s = \omega(t_1, \ldots, t_m)$, where $m = \mathrm{ar}(\omega)$ and by induction hypothesis $\mathcal{F}(t_i)_{\mathbb{A}} = (t_i)_{\mathcal{F}\,\mathbb{A}}$. Then

$$\begin{aligned}
&\mathcal{F}\,s_{\mathbb{A}} \\
&= \mathcal{F}\,(\omega_{\mathbb{A}} \circ ((t_1)_{\mathbb{A}} \times \cdots \times (t_m)_{\mathbb{A}})) && \text{by definition of } s_{\mathbb{A}}, \\
&= \mathcal{F}\,\omega_{\mathbb{A}} \circ \mathcal{F}\,((t_1)_{\mathbb{A}} \times \cdots \times (t_m)_{\mathbb{A}}) && \text{since } \mathcal{F} \text{ is a functor,} \\
&= \mathcal{F}\,\omega_{\mathbb{A}} \circ (\mathcal{F}(t_1)_{\mathbb{A}} \times \cdots \times \mathcal{F}(t_m)_{\mathbb{A}}) && \text{since } \mathcal{F} \text{ preserves finite products,} \\
&= \mathcal{F}\,\omega_{\mathbb{A}} \circ ((t_1)_{\mathcal{F}\,\mathbb{A}} \times \cdots \times (t_m)_{\mathcal{F}\,\mathbb{A}}) && \text{by induction hypothesis,} \\
&= s_{\mathcal{F}\,\mathbb{A}} && \text{by definition of } s_{\mathcal{F}\,\mathbb{A}}.
\end{aligned}$$

We see that

$$\begin{aligned} s_{\mathcal{F}\,\mathbb{A}} &= \mathcal{F}\, s_{\mathbb{A}} && \text{by (3.18),} \\ &\leq \mathcal{F}\, t_{\mathbb{A}} && \text{since } s_{\mathbb{A}} \leq t_{\mathbb{A}} \text{ by assumption,} \\ &= t_{\mathcal{F}\,\mathbb{A}} && \text{by (3.18).} \end{aligned}$$

Thus, we see that the validity of $s \preccurlyeq t$ lifts from $\mathbb{A}$ to $\langle \mathcal{F}\,\mathbb{A}, \supseteq, (\mathcal{F}\,\omega_{\mathbb{A}})_{\omega\in\Omega'}\rangle$ by functorial properties of $\mathcal{F}$, and from there to $\mathbb{A}^\delta$ by Fact 3.3.14. ∎

Results of this kind form an example of Sahlqvist canonicity [79], that is canonicity of inequations based on their syntactic shape; for examples see [34, 40, 39]. It should be noted that what we are discussing here is only half of what Sahlqvist theory is about: Sahlqvist correspondence (see e.g. [19, Ch. 3]) is as important as the canonicity we have just discussed.

Ad hoc analysis

The third approach is one of ad hoc analysis. In this case, the idea is to prove only what is needed to show that validity on $\mathbb{A}$ of *one* given inequation $s \preccurlyeq t$ is preserved by canonical extensions. First, we need to think of the term functions induced by $s(x_1, \ldots, x_n)$ and $t(x_1, \ldots, x_n)$, i.e. $s_{\mathbb{A}} \colon \mathbb{A}^n \to \mathbb{A}$ and $t_{\mathbb{A}} \colon \mathbb{A}^n \to \mathbb{A}$. Validity of the inequation then becomes equivalent to the statement that $s_{\mathbb{A}} \leq t_{\mathbb{A}}$, and the goal becomes to prove that $s_{\mathbb{A}^\delta} \leq t_{\mathbb{A}^\delta}$. For the moment, let us assume that for every $\omega \in \Omega$, we have that $\omega_{\mathbb{A}^\delta} = (\omega_{\mathbb{A}})^\triangledown$. If canonical extensions commuted with composition of arbitrary functions it would now be a breeze to show that validity of $s \preccurlyeq t$ is preserved, because then we would see that

$$s_{\mathbb{A}^\delta} = (s_{\mathbb{A}})^\triangledown \leq (t_{\mathbb{A}})^\triangledown = t_{\mathbb{A}^\delta}.$$

In practice however, we do not know a priori if the equalities $s_{\mathbb{A}^\delta} = (s_{\mathbb{A}})^\triangledown$ and $(t_{\mathbb{A}})^\triangledown = t_{\mathbb{A}^\delta}$ hold, or even the inequalities $s_{\mathbb{A}^\delta} \leq (s_{\mathbb{A}})^\triangledown$ and $(t_{\mathbb{A}})^\triangledown \leq t_{\mathbb{A}^\delta}$, which would already be sufficient. The ad hoc analysis approach now consists of using specific knowledge about the algebra operations $\omega_{\mathbb{A}}$ occuring in $s_{\mathbb{A}}$ and $t_{\mathbb{A}}$, in conjunction with the results about the interaction of function composition and canonical extensions from §2.2.3 and §3.2.3, to show that $s_{\mathbb{A}^\delta} \leq (s_{\mathbb{A}})^\triangledown$ and $(t_{\mathbb{A}})^\triangledown \leq t_{\mathbb{A}^\delta}$. This analysis would also have to take into account whether $\omega_{\mathbb{A}^\delta} = (\omega_{\mathbb{A}})^\triangledown$ or $\omega_{\mathbb{A}^\delta} = (\omega_{\mathbb{A}})^\vartriangle$ for every ω in s and t.

One can see the Sahlqvist approach to canonicity as an organized version of the ad-hoc analysis we sketched above. It should also be noted that there is a limit to the complexity of the inequations for which we can prove canonicity results, since it is undecidable in general whether a given inequation is canonical [64, Thm. 9.6.1].

3.3.4 Conclusions and further work

Most of the results in §3.3 are well-known from the work of Gehrke and Harding [34]; the purpose of this section is simply to set straight the definitions of canonical extensions of lattice-based algebras, expanding upon canonical extensions of lattices. In our discussion of canonicity using our dcpo approach, we present a new proof of a known canonicity theorem (Theorem 3.3.15). The new proof we present was first published by M. Gehrke and the author in [42].

Further work

It may be interesting to see if Theorem 3.3.12, which states that surjective homomorphisms are preserved by canonical extensions, can be stretched even further, e.g. to ordered algebras with monotone [32] or non-monotone operations.

3.4 Profinite completion and canonical extension

In this section we will explore the connections between canonical extensions and profinite completions of lattice-based algebras. These connection are very rich and in some cases also very intricate.

The most basic connection between canonical extension and profinite completion is Theorem 3.4.1, which states that the profinite completion $\hat{\mathbb{A}}$ of any lattice-based algebra $\mathbb{A}$ can be seen as a complete quotient of $\mathbb{A}^\delta$, the canonical extension of $\mathbb{A}$. As a consequence, without making any assumptions about $\mathbb{A}$, we can prove that $\mathbb{A}^\delta$ has a universal property with respect to profinite algebras (Corollary 3.4.2). This result is completely general and it shows that there is really a fundamental connection between the canonical extension and the profinite completion of a lattice-based algebra. In general, however, $\mathbb{A}^\delta$ itself is not profinite, unless we make additional assumptions about $\mathbb{A}$. The most extreme assumption we can make is that $\mathrm{HSP}(\mathbb{A})$ is finitely generated; in this case, $\mathbb{A}^\delta$ *is* the profinite completion of $\mathbb{A}$ (Theorem 3.4.12). A consequence of this is that $\mathbb{A}^\delta \in \mathrm{HSP}(\mathbb{A})$ if $\mathrm{HSP}(\mathbb{A})$ is finitely generated, which is a well-known canonicity result.

We conclude the section with two results which sit between the basic connection (profinite completion as a quotient of canonical extension) and the strongest connection (canonical extensions coinciding with profinite completions in finitely generated varieties). It turns out that if we restrict our attention to monotone lattice-based algebras $\mathbb{A}$, then we can prove a universal property of canonical extensions with respect to Boolean topological monotone lattice-based algebras with profinite lattice reducts (Theorem 3.4.14), and a theorem characterizing certain retracts of canonical extensions (Theorem 3.4.16). The prime example of such monotone lattice-based algebras is the class of distributive lattices with operators, which we will revisit in Chapter 4. We discuss the contribution of this section and possible further work in §3.4.4.

3.4.1 Universal properties of canonical extension

In this subsection we will discuss an interesting property of the canonical extension, namely that for any lattice-based algebra $\mathbb{A}$ and any homomorphism $f\colon \mathbb{A} \to \mathbb{B}$ to a profinite lattice-based algebra $\mathbb{B}$, there exists a unique complete homomorphism $f'\colon \mathbb{A}^\delta \to \mathbb{B}$ such that $f' \circ e_\mathbb{A} = f$ (Corollary 3.4.2).

$$\begin{tikzcd} & \mathbb{A}^\delta \arrow[d, dotted, "f'"] \\ \mathbb{A} \arrow[ur, "e_\mathbb{A}"] \arrow[r, "f"'] & \mathbb{B} \end{tikzcd}$$

We will arrive at this result by first showing that the profinite completion $\hat{\mathbb{A}}$ is a quotient of $\mathbb{A}^\delta$; the canonical extension then inherits the universal property of $\mu_\mathbb{A}\colon \mathbb{A} \to \hat{\mathbb{A}}$.

Let $\mathbb{A}$ be a lattice-based algebra. Recall that $\hat{\mathbb{A}}$, the profinite completion of $\mathbb{A}$, is the limit of the finite quotients of $\mathbb{A}$. These finite quotients are arranged in a diagram $\langle \mathbb{A}/\theta, f_{\theta\psi}\rangle_{\theta,\psi\in\Phi_\mathbb{A}}$, where

$$\Phi_\mathbb{A} = \{\theta \in \mathrm{Con}\,\mathbb{A} \mid \mathbb{A}/\theta \text{ finite}\},$$

and $f_{\theta\psi}\colon \mathbb{A}/\theta \to \mathbb{A}/\psi$ is defined if $\theta \subseteq \psi$, as

$$f_{\theta\psi}\colon a/\theta \mapsto a/\psi.$$

The profinite completion of $\mathbb{A}$, denoted $\hat{\mathbb{A}}$, is the limiting cone over this diagram (see §3.1.2) and it is characterized by the property that for every cone $(f_\theta\colon \mathbb{B} \to \mathbb{A}/\theta)_{\Phi_\mathbb{A}}$, there exists a unique map of cones $f\colon \mathbb{B} \to \hat{\mathbb{A}}$ over $\langle \mathbb{A}/\theta, f_{\theta\psi}\rangle_{\Phi_\mathbb{A}}$. This is how we defined the natural map $\mu_\mathbb{A}\colon \mathbb{A} \to \hat{\mathbb{A}}$, namely, using the fact that $(\mu_\theta\colon \mathbb{A} \to \mathbb{A}/\theta)_{\Phi_\mathbb{A}}$, where $\mu_\theta\colon a \mapsto a/\theta$, is a cone over the diagram $\langle \mathbb{A}/\theta, f_{\theta\psi}\rangle_{\Phi_\mathbb{A}}$.

3.4.1. Theorem. *Let Ω be a lattice-based similarity type and let $\mathbb{A}$ be a lattice-based Ω-algebra. Then there exists an Ω-algebra homomorphism $\nu_\mathbb{A}\colon \mathbb{A}^\delta \to \hat{\mathbb{A}}$ such that*

$$\begin{tikzcd} & \mathbb{A}^\delta \arrow[d, "\nu_\mathbb{A}"] \\ \mathbb{A} \arrow[ur, "e_\mathbb{A}"] \arrow[r, "\mu_\mathbb{A}"'] & \hat{\mathbb{A}} \end{tikzcd}$$

1. *$\nu_\mathbb{A} \circ e_\mathbb{A} = \mu_\mathbb{A}$;*

2. *$\nu_\mathbb{A}$ is (σ,σ)-continuous, i.e. a complete homomorphism;*

3. *$\nu_\mathbb{A}$ is surjective;*

4. *$\nu_\mathbb{A} = \liminf \mu_\mathbb{A} = \limsup \mu_\mathbb{A}$;*

Proof Let Ω be the similarity type of $\mathbb{A}$, and fix a canonical extension type β. We claim that

$$(\mu_\theta^\delta\colon \mathbb{A}^\delta \to \mathbb{A}/\theta)_{\Phi_\mathbb{A}} \text{ is a cone over the diagram } \langle \mathbb{A}/\theta, f_{\theta\psi}\rangle_{\Phi_\mathbb{A}}. \tag{3.19}$$

Observe that $\mu_\theta^\delta\colon \mathbb{A}^\delta \to \mathbb{A}/\theta$ is well-defined: since $\mu_\theta\colon \mathbb{A} \to \mathbb{A}/\theta$ is a surjective Ω-algebra homomorphism, it follows by Theorem 3.3.12 that $\mu_\theta^\delta\colon \mathbb{A}^\delta \to (\mathbb{A}/\theta)^\delta$ is also an Ω-algebra homomorphism; since $(\mathbb{A}/\theta)^\delta = \mathbb{A}/\theta$ by Convention 3.3.8, we see that indeed $\mu_\theta^\delta\colon \mathbb{A}^\delta \to \mathbb{A}/\theta$ is an Ω-algebra homomorphism. To show that $(\mu_\theta^\delta\colon \mathbb{A}^\delta \to \mathbb{A}/\theta)_{\Phi_\mathbb{A}}$ is a cone over $\langle \mathbb{A}/\theta, f_{\theta\psi}\rangle_{\Phi_\mathbb{A}}$, take $\theta, \psi \in \Phi_\mathbb{A}$ such that $\theta \subseteq \psi$. Since $(\mu_\theta\colon \mathbb{A} \to \mathbb{A}/\theta)_{\theta\in\Phi_\mathbb{A}}$ is a cone, we know that $f_{\theta\psi} \circ \mu_\theta = \mu_\psi$; we want to show that it is also the case that $f_{\theta\psi} \circ \mu_\theta^\delta = \mu_\psi^\delta$, that is, we want to show that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{A}^\delta \arrow[d, "\mu_\theta^\delta"'] \arrow[dr, "\mu_\psi^\delta"] & \\
\mathbb{A}/\theta \arrow[r, "f_{\theta\psi}"'] & \mathbb{A}/\psi
\end{tikzcd}$$

But this is easy to see:

$$\begin{aligned}
f_{\theta\psi} \circ \mu_\theta^\delta &= f_{\theta\psi}^\delta \circ \mu_\theta^\delta && \text{since } (\mathbb{A}/\theta)^\delta = \mathbb{A}/\theta \text{ and } (\mathbb{A}/\psi)^\delta = \mathbb{A}/\psi,\\
&= (f_{\theta\psi} \circ \mu_\theta)^\delta && \text{by Th. 2.2.24, since } f_{\theta\psi} \text{ and } \mu_\theta \text{ are latt. hom.'s},\\
&= \mu_\psi^\delta && \text{because } f_{\theta\psi} \circ \mu_\theta = \mu_\psi.
\end{aligned}$$

Now that we know that (3.19) holds, it follows from the fact that $\hat{\mathbb{A}}$ is the limit of $\langle \mathbb{A}/\theta, f_{\theta\psi}\rangle_{\Phi_\mathbb{A}}$ that there exists a unique map of cones $\nu_\mathbb{A}\colon \mathbb{A}^\delta \to \hat{\mathbb{A}}$ over $\langle \mathbb{A}/\theta, f_{\theta\psi}\rangle_{\Phi_\mathbb{A}}$.

(1). We will show that $e_\mathbb{A}\colon \mathbb{A} \to \mathbb{A}^\delta$ is a map of cones over $\langle \mathbb{A}/\theta, f_{\theta\psi}\rangle_{\Phi_\mathbb{A}}$. That means that we must show that for all $\theta \in \Phi_\mathbb{A}$, $\mu_\theta^\delta \circ e_\mathbb{A} = \mu_\theta$, i.e. that the following diagram commutes.

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "e_\mathbb{A}"] \arrow[d, "\mu_\theta"'] & \mathbb{A}^\delta \arrow[dl, "\mu_\theta^\delta"] \\
\mathbb{A}/\theta &
\end{tikzcd}$$

It is a basic property of canonical extensions of maps that $\mu_\theta^\delta \circ e_\mathbb{A} = e_{\mathbb{A}/\theta} \circ \mu_\theta$. Since $\mathbb{A}/\theta$ is finite, we know that $(\mathbb{A}/\theta)^\delta = \mathbb{A}/\theta$. But that means that $e_{\mathbb{A}/\theta} = \mathrm{id}_{\mathbb{A}/\theta}$, so that indeed $\mu_\theta^\delta \circ e_\mathbb{A} = \mu_\theta$. Since $\theta \in \Phi_\mathbb{A}$ was arbitrary, it follows that $e_\mathbb{A}\colon \mathbb{A} \to \mathbb{A}^\delta$ is a map of cones. Now since $\nu_\mathbb{A}\colon \mathbb{A}^\delta \to \hat{\mathbb{A}}$ is also a map of cones, we obtain a map of cones $\nu_\mathbb{A} \circ e_\mathbb{A}\colon \mathbb{A} \to \hat{\mathbb{A}}$ over $\langle \mathbb{A}/\theta, f_{\theta\psi}\rangle_{\Phi_\mathbb{A}}$. Since $\hat{\mathbb{A}}$ is the limit of $\langle \mathbb{A}/\theta, f_{\theta\psi}\rangle_{\Phi_\mathbb{A}}$ and both $\mu_\mathbb{A}\colon \mathbb{A} \to \hat{\mathbb{A}}$ and $\nu_\mathbb{A} \circ e_\mathbb{A}\colon \mathbb{A} \to \hat{\mathbb{A}}$ are maps of cones over $\langle \mathbb{A}/\theta, f_{\theta\psi}\rangle_{\Phi_\mathbb{A}}$ from $\mathbb{A}$ to $\hat{\mathbb{A}}$, it follows that $\nu_\mathbb{A} \circ e_\mathbb{A} = \mu_\mathbb{A}$.

(2). Since for each $\theta \in \Phi_{\mathbb{A}}$, $\mu_\theta \colon \mathbb{A} \to \mathbb{A}/\theta$ is a lattice homomorphism, it follows by Theorem 2.2.24 that $\mu_\theta^\delta \colon \mathbb{A}^\delta \to \mathbb{A}/\theta$ is (σ,σ)-continuous. By Fact 3.1.13, the σ-topology on $\hat{\mathbb{A}}$ is generated by the base

$$\{\pi_\theta^{-1}(U) \mid \theta \in \Phi_{\mathbb{A}},\, U \subseteq \mathbb{A}/\theta\}.$$

Now if we take a basic open set $\pi_\theta^{-1}(U)$, then we see that

$$\begin{aligned} (\nu_{\mathbb{A}})^{-1}\left(\pi_\theta^{-1}(U)\right) &= (\pi_\theta \circ \nu_{\mathbb{A}})^{-1}(U) && \text{by properties of } (\cdot)^{-1}, \\ &= (\mu_\theta^\delta)^{-1}(U) && \text{because } \nu_{\mathbb{A}} \text{ is a map of cones,} \end{aligned}$$

which is a σ-open subset of $\mathbb{A}^\delta$ since $\mu_\theta^\delta \colon \mathbb{A}^\delta \to \mathbb{A}/\theta$ is (σ,σ)-continuous. We conclude that $\nu_{\mathbb{A}}$ is (σ,σ)-continuous.

(3). Given $p \in \mathrm{K}\,\hat{\mathbb{A}}$, we define $F_p := \{a \in \mathbb{A} \mid p \le \mu_{\mathbb{A}}(a)\}$. We claim that

$$\forall p \in \mathrm{K}\,\hat{\mathbb{A}},\ p = \bigwedge \mu_{\mathbb{A}}[F_p]. \tag{3.20}$$

It is easy to see that $p \le \bigwedge \mu_{\mathbb{A}}[F_p]$; towards a contradiction suppose that the converse is not the case, i.e. suppose that $\bigwedge \mu_{\mathbb{A}}[F_p] \not\le p$. Then because $\hat{\mathbb{A}}$ is (bi-) algebraic, there must exist $q \in \mathrm{K}\,\hat{\mathbb{A}}$ such that $q \le \bigwedge \mu_{\mathbb{A}}[F_p]$ and $q \not\le p$. The former tells us

$$\forall a \in F_p,\ q \le \mu_{\mathbb{A}}(a).$$

The latter tells us that $\uparrow p \setminus \uparrow q \neq \emptyset$. Now $\uparrow p$ and $\uparrow q$ are σ-clopen (Lemma 3.1.25), so it follows that $\uparrow p \setminus \uparrow q$ is σ-clopen. Now since $\mu_{\mathbb{A}}[\mathbb{A}]$ is dense in $\hat{\mathbb{A}}$ (Fact 3.1.21), it follows that there must be some $a \in \mathbb{A}$ such that $\mu_{\mathbb{A}}(a) \in \uparrow p \setminus \uparrow q$. But then $p \le \mu_{\mathbb{A}}(a)$, so that $a \in F_p$, and $q \not\le \mu_{\mathbb{A}}(a)$, which is a contradiction. It follows that $\bigwedge \mu_{\mathbb{A}}[F_p] \le p$ so that (3.20) holds. Now take $x \in \hat{\mathbb{A}}$, then

$$\begin{aligned} x &= \bigvee \left\{ p \in \mathrm{K}\,\hat{\mathbb{A}} \mid p \le x \right\} && \text{because } \hat{\mathbb{A}} \text{ is algebraic,} \\ &= \bigvee \{ \bigwedge \mu_{\mathbb{A}}[F_p] \mid p \le x \} && \text{by (3.20),} \\ &= \bigvee \{ \bigwedge \nu_{\mathbb{A}} \circ e_{\mathbb{A}}[F_p] \mid p \le x \} && \text{by (1),} \\ &= \nu_{\mathbb{A}} \left(\bigvee \{ \bigwedge e_{\mathbb{A}}[F_p] \mid p \le x \} \right) && \text{since } \nu_{\mathbb{A}} \text{ is complete by (2).} \end{aligned}$$

Since $x \in \hat{\mathbb{A}}$ was arbitrary, it follows that $\nu_{\mathbb{A}}$ is surjective.

(4). By Fact 3.3.9(3), $(\hat{\mathbb{A}})^l$ is a profinite lattice. Now by (1) and (2) above, we can apply Corollary 3.2.10 to see that $\nu_{\mathbb{A}} = \liminf \mu_{\mathbb{A}} = \limsup \mu_{\mathbb{A}}$. ∎

Now that we have a unique complete homomorphism $\nu_{\mathbb{A}} \colon \mathbb{A}^\delta \to \hat{\mathbb{A}}$ from the canonical extension to the profinite completion, it is not very difficult to show that $e_{\mathbb{A}} \colon \mathbb{A} \to \mathbb{A}^\delta$ has a universal property with respect to profinite algebras.

3.4.2. Corollary. *Let $f\colon \mathbb{A} \to \mathbb{B}$ be a homomorphism from a lattice-based algebra $\mathbb{A}$ to a profinite lattice-based algebra $\mathbb{B}$. Then there exists a unique complete homomorphism $f'\colon \mathbb{A}^\delta \to \mathbb{B}$ such that $f' \circ e_{\mathbb{A}} = f$. In fact, $f' = \liminf f = \limsup f$.*

$$\begin{tikzcd} & \mathbb{A}^\delta \arrow[d, dotted, "f'"] \\ \mathbb{A} \arrow[ur, "e_{\mathbb{A}}"] \arrow[r, "f"'] & \mathbb{B} \end{tikzcd}$$

Proof Assume that we have a homomorphism $f\colon \mathbb{A} \to \mathbb{B}$ to a profinite algebra $\mathbb{B}$. By the universal property of $\mu_{\mathbb{A}}\colon \mathbb{A} \to \hat{\mathbb{A}}$, the profinite completion of $\mathbb{A}$, there exists a unique continuous, i.e. complete, homomorphism $\tilde{f}\colon \hat{\mathbb{A}} \to \mathbb{B}$ such that $\tilde{f} \circ \mu_{\mathbb{A}} = f$. Recall from Theorem 3.4.1 that there exists a complete homomorphism $\nu_{\mathbb{A}}\colon \mathbb{A}^\delta \to \hat{\mathbb{A}}$ such that $\nu_{\mathbb{A}} \circ e_{\mathbb{A}} = \mu_{\mathbb{A}}$. We now define $f' := \tilde{f} \circ \nu_{\mathbb{A}}$. It follows immediately that f' is a complete homomorphism; moreover

$$\begin{aligned} f' \circ e_{\mathbb{A}} &= \tilde{f} \circ \nu_{\mathbb{A}} \circ e_{\mathbb{A}} && \text{by definition of } f', \\ &= \tilde{f} \circ \mu_{\mathbb{A}} && \text{because } \nu_{\mathbb{A}} \circ e_{\mathbb{A}} = \mu_{\mathbb{A}}, \\ &= f && \text{because } \tilde{f} \circ \mu_{\mathbb{A}} = f. \end{aligned}$$

Now since $f'\colon \mathbb{A}^\delta \to \mathbb{B}$ is a complete, i.e. (σ, σ)-continuous homomorphism and $\mathbb{B}$ is profinite, it follows from Corollary 3.2.10 that $f' = \liminf f = \limsup f$, so that f' is unique. ∎

At this point we would like to remind the reader that even though any homomorphism $f\colon \mathbb{A} \to \mathbb{B}$ from a lattice-based algebra $\mathbb{A}$ to a profinite lattice-based algebra $\mathbb{B}$ can be extended to a continuous $f'\colon \mathbb{A}^\delta \to \mathbb{B}$, $\mathbb{A}^\delta$ itself need not be a profinite algebra. In fact, $\mathbb{A}^\delta$ need not even be a topological lattice, as we will see in the example below.

3.4.3. Example. We will present an example of a lattice $\mathbb{L}$ such that its canonical extension $\mathbb{L}^\delta$ is not meet-continuous. Consider the lattice $\mathbb{L} = \langle L, \wedge, \vee, 0, 1\rangle$ where

$$L = \{0, 1\} \cup \{a_{ij} \mid i, j \in \mathbb{N}\},$$

0 is the bottom, 1 is the top, and

$$a_{ij} \geq a_{kl} \quad \iff \quad (i + j \leq k + l \text{ and } i \geq k).$$

It is not hard to show that the poset $\langle L, \leq\rangle$ is a lattice. This lattice, see Figure 3.1, is non-distributive by [23, Theorem 3.6], since

$$\{1, a_{20}, a_{11}, a_{00}, a_{02}\}$$

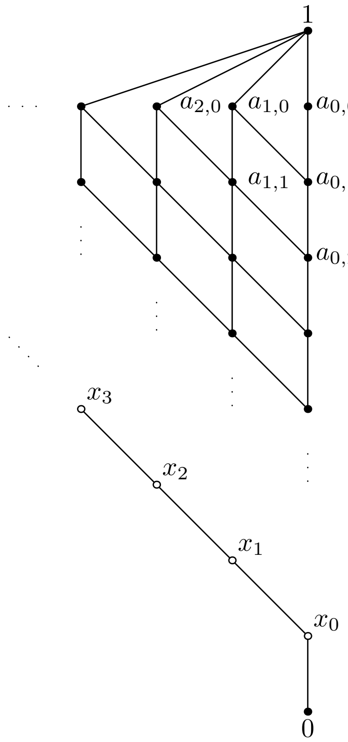

Figure 3.1: The lattice $\mathbb{M}$, with its sublattice $\mathbb{L}$ denoted by the solid dots.

is a (non-bounded) sublattice of $\mathbb{L}$ which is isomorphic to $\mathbb{N}_5$ (see [23]). We now add the elements $x_i, i \in \mathbb{N}$ to $\mathbb{L}$, with $x_i \leq a_{ij}$ for all $i, j \in \mathbb{N}$, and $x_i \leq x_j$ for all $i \leq j \in \mathbb{N}$, to obtain a new lattice $\mathbb{M}$. We claim that $\mathbb{M}$, depicted in Figure 3.1, is the canonical extension of its sublattice $\mathbb{L}$. Rather than proving this in detail, we provide the reader with the following hints:

- $\mathbb{M}$ is complete;
- every element of $\mathbb{M}$ is a filter element, i.e. for all $b \in \mathbb{M}$, there exists a filter $F \in \mathcal{F}\mathbb{L}$ such that $b = \bigwedge F$;
- every element of $\mathbb{L}$ is an ideal element of $\mathbb{M}$, i.e. only the elements x_i for $i \in \mathbb{N}$ are *not* ideal elements of $\mathbb{M}$.

Armed with these hints it is not hard to prove that $\mathbb{M}$ is the canonical extension of $\mathbb{L}$. To see that $\mathbb{L}^\delta$ is not meet-continuous note that

$$a_{00} \wedge (\bigvee_{i=0}^{\infty} x_i) = a_{00} \wedge 1 = a_{00}$$

while

$$\bigvee_{i=0}^{\infty}(a_{00} \wedge x_i) = \bigvee_{i=0}^{\infty} x_0 = x_0.$$

Now if $\mathbb{L}^\delta$ were a topological lattice in its σ-topology, then a fortiori $\wedge\colon \mathbb{L}^\delta \times \mathbb{L}^\delta \to \mathbb{L}^\delta$ would have to be $(\sigma^\uparrow, \sigma^\uparrow)$-continuous, i.e. meet-continuous; however we have just demonstrated that this is not the case. It follows that $\mathbb{L}^\delta$ is not a topological lattice in its σ-topology.

We conclude this subsection with a result stating that $\nu_{\mathbb{A}}\colon \mathbb{A}^\delta \to \hat{\mathbb{A}}$ *also* has a universal property, namely with respect to complete homomorphisms $f\colon \mathbb{A}^\delta \to \mathbb{B}$ from $\mathbb{A}^\delta$ to a profinite algebra $\mathbb{B}$.

3.4.4. Corollary. *Fix a canonical extension type β and consider the canonical extension $e\colon \mathbb{A} \to \mathbb{A}^\delta$ of a lattice-based algebra $\mathbb{A}$. If $f\colon \mathbb{A}^\delta \to \mathbb{B}$ is a complete homomorphism to a profinite lattice-based algebra $\mathbb{B}$, then there exists a unique complete homomorphism $f'\colon \hat{\mathbb{A}} \to \mathbb{B}$ such that $f' \circ \nu_{\mathbb{A}} = f$.*

$$\begin{tikzcd}
 & \hat{\mathbb{A}} \arrow[d, dotted, "f'"] \\
\mathbb{A}^\delta \arrow[ur, "\nu_{\mathbb{A}}"] \arrow[r, "f"'] & \mathbb{B}
\end{tikzcd}$$

Proof Suppose that $f\colon \mathbb{A}^\delta \to \mathbb{B}$ is a complete homomorphism to a profinite lattice-based algebra $\mathbb{B}$. Then $f \circ e_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{B}$ is a homomorphism from $\mathbb{A}$ to a profinite algebra $\mathbb{B}$, so by the universal property of $\mu_{\mathbb{A}}\colon \mathbb{A} \to \hat{\mathbb{A}}$, there exists a complete homomorphism $f'\colon \hat{\mathbb{A}} \to \mathbb{B}$ such that $f' \circ \mu_{\mathbb{A}} = f \circ e_{\mathbb{A}}$.

$$\begin{tikzcd}
 & & \hat{\mathbb{A}} \arrow[d, dotted, "f'"] \\
\mathbb{A} \arrow[urr, "\mu_{\mathbb{A}}"] \arrow[r, "e_{\mathbb{A}}"'] & \mathbb{A}^\delta \arrow[r, "f"'] & \mathbb{B}
\end{tikzcd}$$

We want to show that $f' \circ \nu_{\mathbb{A}} = f$. By Theorem 3.4.1(1), we see that

$$f \circ e_{\mathbb{A}} = f' \circ \mu_{\mathbb{A}} = f' \circ \nu_{\mathbb{A}} \circ e_{\mathbb{A}}.$$

Since both $f\colon \mathbb{A}^\delta \to \mathbb{B}$ and $f' \circ \nu_{\mathbb{A}}\colon \mathbb{A}^\delta \to \mathbb{B}$ are complete homomorphisms agreeing on $e_{\mathbb{A}}[\mathbb{A}]$, it follows by Corollary 3.4.2 that $f = f' \circ \nu_{\mathbb{A}}$, which is what we wanted to show. ∎

3.4.2 Finitely generated varieties

In §3.4.1, we saw that every homomorphism $f\colon \mathbb{A} \to \mathbb{B}$ from a lattice-based algebra $\mathbb{A}$ to a profinite lattice-based algebra $\mathbb{B}$ factors through the canonical extension $e_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{A}^\delta$, but that we cannot expect $\mathbb{A}^\delta$ itself to be profinite. In this subsection we will study sufficient and sometimes necessary conditions for $\mathbb{A}^\delta$ being profinite, namely the condition that HSP($\mathbb{A}$), the variety generated by $\mathbb{A}$, is finitely generated. A different approach to canonical extensions in this setting, using *natural extension* techniques and focussing on prevarieties rather than varieties, can be found in [26].

Recall that a lattice-based Ω-algebra $\mathbb{A}$ is smooth if for every $\omega \in \Omega$, $(\omega_{\mathbb{A}})^{\triangledown} = (\omega_{\mathbb{A}})^{\vartriangle}$, and that in case $\mathbb{A}$ is smooth, $\mathbb{A}$ has a unique canonical extension, since it does not matter whether $\beta(\omega) = \triangledown$ or $\beta(\omega) = \vartriangle$ for any $\omega \in \Omega$.

We will now prove a series of lemmas which will later help us to prove Theorem 3.4.10, which says that if $\mathrm{HSP}(\mathbb{A})$ is finitely generated, then $\mathbb{A}^{\delta}$ is profinite. The proof of Theorem 3.4.10 uses the observation, due to Jònsson, that if $\mathbb{B}$ is a finite algebra and $\mathrm{HSP}(\mathbb{B})$ is congruence distributive, then $\mathrm{HSP}(\mathbb{B}) = \mathrm{HSP}_B(\mathbb{B})$, where P_B stands for taking Boolean products (see §A.6). To exploit this fact, we will need the following technical lemmas about Boolean products, subalgebras and homomorphic images of lattice-based algebras.

3.4.5. Lemma. *Let $\mathbb{A}$ be a lattice-based Ω-algebra and let $(p_x \colon \mathbb{A} \to \mathbb{B})_{x \in X}$ be a Boolean power decomposition of $\mathbb{A}$. If $\mathbb{B}$ is finite then $\mathbb{A}^{\delta}$ is profinite.*

Proof Suppose that $\mathbb{A}$ has a Boolean decomposition $(p_x \colon \mathbb{A} \to \mathbb{B})_{x \in X}$ where $\mathbb{B}$ is a finite algebra. Since $\mathbb{B}$ is finite, it follows by Fact 2.1.10 that $\mathbb{B} \simeq \mathbb{B}^{\delta}$. We now know by Fact 2.1.27 that on the lattice level, $\mathbb{B}^X$ is a canonical extension of $\mathbb{A}$, where $e \colon \mathbb{A} \to \mathbb{B}^X$ is defined as

$$e \colon a \mapsto (p_x(a))_{x \in X}.$$

We will now show that

$$\text{for all } \omega \in \Omega,\ \omega_{\mathbb{B}^X} = (\omega_{\mathbb{A}})^{\triangledown} = (\omega_{\mathbb{A}})^{\vartriangle}. \tag{3.21}$$

This observation has two consequences. Firstly, it shows that $\mathbb{B}^X$ is the (unique) canonical extension of $\mathbb{A}$ and that $\mathbb{A}$ is smooth. Secondly, since a product of profinite algebras is profinite, it follows that $\mathbb{A}^{\delta} \simeq \mathbb{B}^X$ is a profinite algebra.

So let us show that (3.21) holds. Pick $\omega \in \Omega$ and let n be the arity of ω. We will first show that $\omega_{\mathbb{B}^X}$ is a (σ, σ)-continuous extension of $\omega_{\mathbb{A}}$. First, observe that it follows from universal algebra that since each $p_x \colon \mathbb{A} \to \mathbb{B}$ is an Ω-algebra homomorphism, so is $e \colon \mathbb{A} \to \mathbb{B}^X$, i.e. the following diagram commutes and $\omega_{\mathbb{B}^X}$ is indeed an extension of $\omega_{\mathbb{A}}$.

$$\begin{tikzcd}
(\mathbb{B}^X)^n \arrow[r, "\omega_{\mathbb{B}^X}"] & \mathbb{B}^X \\
\mathbb{A}^n \arrow[u, "e^n"] \arrow[r, "\omega_{\mathbb{A}}"'] & \mathbb{A} \arrow[u, "e"']
\end{tikzcd}$$

Now since $\mathbb{B}^X$ is profinite (Fact 3.1.15) and $\mathbb{B}^X$ is a lattice-based algebra, we know that $\omega_{\mathbb{B}^X}$ is (σ, σ)-continuous. We would now like to apply Corollary 3.2.16 to conclude that (3.21) holds, but in order to do that we need to know that $\mathrm{HSP}(\mathbb{A}^l)$ is finitely generated. Since we assumed that $\mathbb{A}$ is a Boolean power of $\mathbb{B}$, we know that $\mathbb{A} \in \mathrm{HSP}(\mathbb{B})$, so by Fact A.6.2, $\mathrm{HSP}(\mathbb{A})$ is finitely generated. It follows by Fact 3.3.9 that $\mathrm{HSP}(\mathbb{A}^l)$ is finitely generated; we may now apply Corollary 3.2.16 to conclude that (3.21) holds. It follows that $\mathbb{A}^\delta \simeq \mathbb{B}^X$ so that $\mathbb{A}^\delta$ is profinite. ∎

The next lemma shows that the property of having a profinite canonical extension is inherited by subalgebras of lattice-based algebras.

3.4.6. Lemma. *Let $\mathbb{A}$, $\mathbb{B}$ be lattice-based algebras and assume that $\mathrm{HSP}(\mathbb{B}^l)$ is finitely generated. If $\mathbb{A}$ is a subalgebra of $\mathbb{B}$ and if $\mathbb{B}^\delta$ is profinite, then $\mathbb{A}^\delta$ is (isomorphic to) a closed subalgebra of $\mathbb{B}^\delta$ and, consequently, profinite.*

Proof Fix a canonical extension type β. Let us denote the embedding of $\mathbb{A}$ into $\mathbb{B}$ by $h\colon \mathbb{A} \to \mathbb{B}$. Below, we will show that $h^\delta\colon \mathbb{A}^\delta \to \mathbb{B}^\delta$ is an Ω-algebra homomorphism, so that $\mathbb{A}^\delta \simeq h[\mathbb{A}^\delta]$ is isomorphic to a subalgebra of $\mathbb{B}^\delta$. It then follows from Corollary 2.2.25 and Lemma 3.1.28 that $h[\mathbb{A}^\delta]$ is closed and hence, profinite by Fact 3.1.16.

So let us show that $h^\delta\colon \mathbb{A}^\delta \to \mathbb{B}^\delta$ is an Ω-algebra homomorphism. Take $\omega \in \Omega$ and let $n := \mathrm{ar}(\omega)$. Without loss of generality, assume that $\beta(\omega) = \triangledown$. We will show that the following diagram commutes:

$$\begin{tikzcd}
(\mathbb{B}^\delta)^n \arrow[r, "(\omega_\mathbb{B})^\triangledown"] & \mathbb{B}^\delta \\
(\mathbb{A}^\delta)^n \arrow[u, "(h^\delta)^n"] \arrow[r, "(\omega_\mathbb{A})^\triangledown"'] & \mathbb{A}^\delta \arrow[u, "h^\delta"']
\end{tikzcd}$$

Since we assumed that $\mathbb{B}^\delta$ is a profinite algebra, we know that

$$(\omega_\mathbb{B})^\triangledown\colon (\mathbb{B}^\delta)^n \to \mathbb{B}^\delta \text{ is } (\sigma,\sigma)\text{-continuous.} \tag{3.22}$$

Now we see that

$$\begin{aligned}
(\omega_\mathbb{B})^\triangledown \circ (h^\delta)^n &= (\omega_\mathbb{B})^\triangledown \circ (h^n)^\delta && \text{by Convention 3.3.5,}\\
&= (\omega_\mathbb{B} \circ h^n)^\triangledown && \text{by Lemma 3.2.21 and (3.22),}\\
&= (h \circ \omega_\mathbb{A})^\triangledown && \text{because } h \text{ is an } \Omega\text{-homomorphism,}\\
&= h^\delta \circ (\omega_\mathbb{A})^\triangledown && \text{by Lemma 3.2.17.}
\end{aligned}$$

Since $\omega \in \Omega$ was arbitrary, it follows that $h^\delta\colon \mathbb{A}^\delta \to \mathbb{B}^\delta$ is an Ω-algebra homomorphism. ∎

3.4.7. REMARK. In Lemmas 3.3.7, 3.4.5 and 3.4.6 above, we are assuming that the lattice-based algebras involved have a lattice reduct lying in a finitely generated variety. The reason for this is that we want to be able to apply results that are consequences of Corollary 3.2.15. If we wanted, however, we could replace the assumptions concerning finitely generated varieties by restricting our results to *monotone* lattice-based algebras. We would then have the results that follow from Theorem 2.2.4 at our disposal, in particular Corollary 2.2.23.

Now that we have studied profiniteness of canonical extensions in relation with Boolean products and subalgebras, the only thing left to consider is homomorphic images. Here, we need to do a little more work, starting with a technical lemma about continuity properties of complete surjective lattice homomorphisms. Observe that the property that the following lemma ascribes to complete surjective lattice homomorphisms $h\colon \mathbb{L} \to \mathbb{M}$ is *weaker* than saying that h is an open map: h only preserves forward images of open sets of the shape $h^{-1}(U)$, for $U \subseteq \mathbb{M}$.

3.4.8. LEMMA. *Let $h\colon \mathbb{L} \to \mathbb{M}$ be a complete surjective lattice homomorphism between complete lattices $\mathbb{L}$ and $\mathbb{M}$. Let $U \subseteq \mathbb{M}$. If $h^{-1}(U)$ is $\sigma^{\uparrow}$-open ($\sigma^{\downarrow}$-open, σ-open) in $\mathbb{L}$, then U is $\sigma^{\uparrow}$-open ($\sigma^{\downarrow}$-open, σ-open) in $\mathbb{M}$.*

Proof We will only treat the case for the $\sigma^{\uparrow}$-topology; the other cases follow by order duality. Observe that since $h\colon \mathbb{L} \to \mathbb{M}$ is a complete homomorphism, it has a left adjoint $h^{\flat}\colon \mathbb{M} \to \mathbb{L}$. By Fact A.3.3(1), $h^{\flat}$ preserves all joins. Moreover, since h is surjective, we know that $h \circ h^{\flat} = \mathrm{id}_{\mathbb{M}}$ (Fact A.3.3(2)). Moreover,

$$\forall x \in U,\ h^{\flat}(x) \in h^{-1}(U). \tag{3.23}$$

After all, if $x \in U$, then because $h \circ h^{\flat} = \mathrm{id}_{\mathbb{M}}$, we see that $h \circ h^{\flat}(x) = x \in U$, so that $h^{\flat}(x) \in h^{-1}(U)$. Now suppose that $U \subseteq \mathbb{M}$ such that $h^{-1}(U)$ is $\sigma^{\uparrow}$-open. We will show that U itself is then also $\sigma^{\uparrow}$-open. First, we show that U is an upper set. If $x \in U$ and $x \leq y$, then since $h^{\flat}$ is order-preserving, we see that $h^{\flat}(x) \leq h^{\flat}(y)$. By (3.23), we see that $h^{\flat}(x) \in h^{-1}(U)$. Since we assumed that $h^{-1}(U)$ is an upper set, it follows that $h^{\flat}(y) \in h^{-1}(U)$. Now

$$\begin{aligned} y &= h \circ h^{\flat}(y) && \text{since } h \circ h^{\flat} = \mathrm{id}_{\mathbb{M}}, \\ &\in h\left[h^{-1}(U)\right] && \text{since } h^{\flat}(y) \in h^{-1}(U), \\ &= U && \text{since } h \text{ is surjective.} \end{aligned}$$

It follows that U is an upper set. Next, suppose that $S \subseteq \mathbb{M}$ is directed and that $\bigvee S \in U$. Then we see that

$$\begin{aligned} \bigvee h^{\flat}[S] &= h^{\flat}(\bigvee S) && \text{because } h^{\flat} \text{ preserves all joins,} \\ &\in h^{-1}(U) && \text{by (3.23) since } \bigvee S \in U. \end{aligned}$$

Now because $h^{-1}(U)$ is $\sigma^{\uparrow}$-open, there must exist $x \in S$ such that $h^{\flat}(x) \in h^{-1}(U)$. Now

$$\begin{aligned} x &= h \circ h^{\flat}(x) && \text{since } h \circ h^{\flat} = \mathrm{id}_{\mathbb{M}}, \\ &\in h\left[h^{-1}(U)\right] && \text{since } h^{\flat}(x) \in h^{-1}(U), \\ &= U && \text{since } h \text{ is surjective.} \end{aligned}$$

It follows that U is $\sigma^{\uparrow}$-open. ∎

We are now ready to state the result about homomorphic images of algebras which have a profinite canonical extension. Note that this result is decidedely weaker than the corresponding results about Boolean products and subalgebras above, since we only get a Boolean topological algebra rather than a profinite algebra.

3.4.9. Lemma. *Let $\mathbb{A}$, $\mathbb{B}$ be lattice-based algebras. If $h\colon \mathbb{B} \to \mathbb{A}$ is a surjective Ω-algebra homomorphism and if $\mathbb{B}^{\delta}$ is profinite, then $\mathbb{A}^{\delta}$ is a Boolean topological algebra.*

Proof Fix a canonical extension type β. It follows from Lemma 3.2.11 that $(\mathbb{A}^{\delta})^{l}$ is a Boolean topological lattice, so we already know that the σ-topology on $\mathbb{A}^{\delta}$ is a Boolean topology. What remains to be shown is that $\mathbb{A}^{\delta}$ is a topological algebra, i.e. that for each $\omega \in \Omega$, $(\omega_{\mathbb{A}})^{\beta(\omega)}\colon (\mathbb{A}^{\delta})^{n} \to \mathbb{A}^{\delta}$ is (σ,σ)-continuous, where $n = \mathrm{ar}(\omega)$. Without loss of generality, suppose that $\beta(\omega) = \triangledown$. Consider the following diagram:

$$\begin{array}{ccc} (\mathbb{A}^{\delta})^{n} & \xrightarrow{(\omega_{\mathbb{A}})^{\triangledown}} & \mathbb{A}^{\delta} \\ {\scriptstyle (h^{\delta})^{n}}\uparrow & & \uparrow{\scriptstyle h^{\delta}} \\ (\mathbb{B}^{\delta})^{n} & \xrightarrow[(\omega_{\mathbb{B}})^{\triangledown}]{} & \mathbb{B}^{\delta} \end{array}$$

We know that this diagram commutes because $h\colon \mathbb{B} \to \mathbb{A}$ is a surjective Ω-algebra homomorphism, so Theorem 3.3.12 applies. Let $U \subseteq \mathbb{A}^{\delta}$ be σ-open; we want to show that $\left((\omega_{\mathbb{A}})^{\triangledown}\right)^{-1}(U)$ is σ-open. First, observe that

$$\begin{aligned} \left((h^{n})^{\delta}\right)^{-1} \circ \left((\omega_{\mathbb{A}})^{\triangledown}\right)^{-1}(U) &= \left((\omega_{\mathbb{A}})^{\triangledown} \circ (h^{n})^{\delta}\right)^{-1}(U) && \text{by properties of } (\cdot)^{-1}, \\ &= \left(h^{\delta} \circ (\omega_{\mathbb{B}})^{\triangledown}\right)^{-1}(U) && \text{because } h^{\delta} \text{ is an } \Omega\text{-hom.} \end{aligned}$$

Now $(\omega_{\mathbb{B}})^{\triangledown}$ is (σ,σ)-continuous because $\mathbb{B}^{\delta}$ is profinite (Fact 3.1.23(1)), and h^{δ} is (σ,σ)-continuous because h is a lattice homomorphism (Theorem 2.2.24); consequently, $h^{\delta} \circ (\omega_{\mathbb{B}})^{\triangledown}$ is (σ,σ)-continuous. It follows that $\left(h^{\delta} \circ (\omega_{\mathbb{B}})^{\triangledown}\right)^{-1}(U)$ is σ-open. Now since $(h^{n})^{\delta}$ is a complete homomorphism, it follows by Lemma 3.4.8 that $\left((\omega_{\mathbb{A}})^{\triangledown}\right)^{-1}(U)$ is σ-open. Since $U \subseteq \mathbb{A}^{\delta}$ was arbitrary, it follows that $(\omega_{\mathbb{A}})^{\triangledown}$ is (σ,σ)-continuous. Since $\omega \in \Omega$ was arbitrary, it follows that $\mathbb{A}^{\delta}$ is a Boolean topological algebra. ∎

Now that we have our technical lemmas sorted out, we are ready to prove the first main theorem of this subsection.

3.4.10. THEOREM. *If $\mathbb{A}$ is a lattice-based algebra such that* HSP($\mathbb{A}$) *is finitely generated, then $\mathbb{A}^\delta$ is profinite and hence, $\mathbb{A}$ is smooth.*

Proof Suppose that $\mathbb{B}$ is a finite lattice-based algebra such that $\mathbb{A} \in \mathrm{HSP}(\mathbb{B})$. Because HSP($\mathbb{B}$) is congruence distributive, we may conclude by Fact A.6.3 that $\mathrm{HSP}(\mathbb{B}) = \mathrm{HSP}_B(\mathbb{B})$. If $\mathbb{A} \in \mathrm{SP}_B(\mathbb{B})$, then by Lemmas 3.4.5 and 3.4.6 it follows that $\mathbb{A}$ is profinite. If $\mathbb{A} \in \mathrm{HSP}_B(\mathbb{B})$ however, then Lemma 3.4.9 only tells us that $\mathbb{A}$ is a Boolean topological algebra. But, since $\mathbb{A} \in \mathrm{HSP}(\mathbb{B})$ and HSP($\mathbb{B}$) is congruence distributive and finitely generated, we may use Fact 3.1.17 to conclude that $\mathbb{A}$ is profinite. It follows by Lemma 3.3.7 that $\mathbb{A}$ is smooth. ∎

3.4.11. REMARK. Observe that the above proof uses Fact 3.1.17, which is a powerful result from [24]. In a recent paper, Gehrke et al. [35] have showed that any Boolean topological quotient of a profinite algebra is again profinite, employing an argument based on Stone duality.

The following result is a generalization of the main result of [50]. There, the theorem was stated for monotone lattice-based algebras. It was communicated to us at the time of writing of this chapter (June/July 2010) that Theorem 3.4.12 has been independently discovered by M.J. Gouveia [49]; see §3.4.4 for further discussion.

3.4.12. THEOREM. *Fix a lattice-based similarity type Ω and let $\mathcal{V}$ be a variety of lattice-based Ω-algebras.*

1. *If $\mathcal{V}$ is finitely generated then for every $\mathbb{A} \in \mathcal{V}$, $\mathbb{A} \xrightarrow{e_\mathbb{A}} \mathbb{A}^\delta$ is the profinite completion of $\mathbb{A}$;*

2. *If Ω is finite and for every $\mathbb{A} \in \mathcal{V}$, $\mathbb{A} \xrightarrow{e_\mathbb{A}} \mathbb{A}^\delta$ is the profinite completion of $\mathbb{A}$, then $\mathcal{V}$ is finitely generated.*

Proof (1). Let $\mathbb{A} \in \mathcal{V}$; it follows from Theorem 3.4.10 that $\mathbb{A}^\delta$ is profinite. To show that $e_\mathbb{A} \colon \mathbb{A} \to \mathbb{A}^\delta$ is the profinite completion of $\mathbb{A}$, we only have to show that $e_\mathbb{A} \colon \mathbb{A} \to \mathbb{A}^\delta$ has the universal property that characterizes the profinite completion, namely, that if $\mathbb{B}$ is a profinite algebra and $f \colon \mathbb{A} \to \mathbb{B}$ is a homomorphism, then there exists a unique continuous homomorphism $f' \colon \mathbb{A}^\delta \to \mathbb{B}$ such that $f' \circ e_\mathbb{A} = f$. But this follows immediately from Corollary 3.4.2.

(2). The proof for the second statement is identical to that in [50]. The proof uses more background knowledge than we assume for the rest of our discourse; we include it for cognoscenti. If for every $\mathbb{A} \in \mathcal{V}$, $\mathbb{A} \xrightarrow{e_\mathbb{A}} \mathbb{A}^\delta$ is the profinite completion of $\mathbb{A}$, then the natural map from $\mathbb{A}$ to $\hat{\mathbb{A}}$ is injective for every $\mathbb{A} \in \mathcal{V}$, since

$e_{\mathbb{A}} \colon \mathbb{A} \to \mathbb{A}^\delta$ is always injective. It follows by Fact 3.1.20 that $\mathcal{V}$ is residually finite. Now since we assumed that Ω is finite, it follows by [61, Theorem 4.1] that there exists a finite bound on the size of subdirect irreducibles in $\mathcal{V}$; consequently, $\mathcal{V}$ is finitely generated. ∎

We conclude this subsection with a canonicity result that is well-known in other, less general forms (e.g. [34, Corollary 6.9]).

3.4.13. COROLLARY. *Let $\mathbb{A}$ be a lattice-based algebra. If* $\mathrm{HSP}(\mathbb{A})$ *is finitely generated, then* $\mathbb{A}^\delta \in \mathrm{HSP}(\mathbb{A})$*; consequently, every equation which is valid on $\mathbb{A}$ is also valid on $\mathbb{A}^\delta$.*

Proof If $\mathrm{HSP}(\mathbb{A})$ is finitely generated, then by Theorem 3.4.12(1), $\mathbb{A}^\delta \simeq \hat{\mathbb{A}}$. Since $\hat{\mathbb{A}}$ is a subalgebra of a product of quotients of $\mathbb{A}$, we know that $\hat{\mathbb{A}} \in \mathrm{HSP}(\mathbb{A})$. The statement now follows. ∎

3.4.3 Canonical extension and monotone topological algebras

We conclude this section with a universal property of canonical extensions with respect to Boolean topological lattice-based algebras and a corresponding retraction theorem. Our motivation for studying this universal property comes from modal algebras. We will see in §4.1 that Boolean topological modal algebras correspond, via Stone duality, to image-finite Kripke frames. In light of this motivating example, we will make two assumptions about the Boolean topological lattice-based algebras $\mathbb{B}$ we are dealing with in this subsection.

- We only consider *monotone* lattice-based algebras $\mathbb{B}$;
- we only consider Boolean topological lattice-based algebras $\mathbb{B}$ such that $\mathbb{B}^l$, the lattice reduct of $\mathbb{B}$, is a profinite lattice.

We will see in §4.1 that modal algebras, and more generally distributive lattices with operators, indeed satisfy the conditions above. Moreover, modal algebras will provide us with a clear example why the conditions above are strictly weaker than assuming that $\mathbb{B}$ is profinite.

3.4.14. THEOREM. *Fix a lattice-based similarity type Ω and a canonical extension type β. Let $\mathbb{A}$ be a monotone lattice-based Ω-algebra. If $f \colon \mathbb{A} \to \mathbb{B}$ is an Ω-homomorphism to a Boolean topological monotone lattice-based Ω-algebra $\mathbb{B}$ and if $\mathbb{B}^l$ is profinite, then there exists a unique complete Ω-algebra homomorphism $f' \colon \mathbb{A}^\delta \to \mathbb{B}$ such that $f' \circ e_{\mathbb{A}} = f$.*

$$\begin{tikzcd} & \mathbb{A}^\delta \arrow[d, dotted, "f'"] \\ \mathbb{A} \arrow[ur, "e_{\mathbb{A}}"] \arrow[r, "f"'] & \mathbb{B} \end{tikzcd}$$

Proof Suppose that we have an Ω-homomorphism from $f\colon \mathbb{A} \to \mathbb{B}$ where $\mathbb{A}$ and $\mathbb{B}$ are monotone lattice-based algebras and $\mathbb{B}$ is a Boolean topological algebra, and suppose that $\mathbb{B}^l$ is profinite. Under these assumptions Corollary 3.4.2 tells us that there exists a unique continuous *lattice* homomorphism $f'\colon (\mathbb{A}^\delta)^l \to \mathbb{B}^l$ such that $f' \circ e_\mathbb{A} = f$. Moreover, by Corollary 3.2.10 we know that

$$f' = \liminf f = \limsup f. \tag{3.24}$$

We now have to show that $f'\colon \mathbb{A}^\delta \to \mathbb{B}$ is in fact an Ω-algebra homomorphism.

Let $\omega \in \Omega$ and let $n := \mathrm{ar}(\omega)$. Without loss of generality, assume that $\beta(\omega) = \triangledown$. We now want to show that the following diagram commutes:

$$\begin{tikzcd}
(\mathbb{A}^\delta)^n \arrow[r, "(\omega_\mathbb{A})^\triangledown"] \arrow[d, "(f')^n"'] & \mathbb{A}^\delta \arrow[d, "f'"] \\
\mathbb{B}^n \arrow[r, "\omega_\mathbb{B}"'] & \mathbb{B}
\end{tikzcd}$$

In order to show that $\omega_\mathbb{B} \circ (f')^n = f' \circ (\omega_\mathbb{A})^\triangledown$, we need to make a few observations. Firstly, observe that

$$\omega_\mathbb{B}\colon \mathbb{B}^n \to \mathbb{B} \text{ is both Scott- and co-Scott-continuous,} \tag{3.25}$$

so that $\omega_\mathbb{B}$ preserves both directed joins and co-directed meets. Since $\mathbb{B}$ is a Boolean topological algebra, we know that $\omega_\mathbb{B}$ is (σ, σ)-continuous. Since $\omega_\mathbb{B}$ is monotone, it follows by Lemma 2.1.17 that (3.25) holds.

Our second observation is that

$$\omega_\mathbb{B} \circ f^n = f' \circ e_\mathbb{A} \circ \omega_\mathbb{A}. \tag{3.26}$$

Since $f\colon \mathbb{A} \to \mathbb{B}$ is an Ω-homomorphism, we know that $\omega_\mathbb{B} \circ f^n = f \circ \omega_\mathbb{A}$. Now we may use the fact that $f = f' \circ e_\mathbb{A}$ to conclude that (3.26) holds. Now let $x \in (\mathbb{A}^\delta)^n = (\mathbb{A}^n)^\delta$; we see that

$$\begin{aligned}
&\omega_\mathbb{B} \circ (f')^n(x) \\
&= \omega_\mathbb{B} \circ (\liminf f)^n\,(x) && \text{by (3.24),} \\
&= \omega_\mathbb{B} \circ \liminf(f^n)(x) && \text{by Lemma 3.2.4,} \\
&= \omega_\mathbb{B}\left(\bigvee \left\{\bigwedge f^n[F \cap I] \mid e^\mathcal{F}_{\mathbb{A}^n}(F) \leq x \leq e^\mathcal{I}_{\mathbb{A}^n}(I)\right\}\right) && \text{by def. of } \liminf, \\
&= \bigvee \left\{\bigwedge \omega_\mathbb{B} \circ f^n[F \cap I] \mid e^\mathcal{F}_{\mathbb{A}^n}(F) \leq x \leq e^\mathcal{I}_{\mathbb{A}^n}(I)\right\} && \text{by (3.25) } (\dagger), \\
&= \bigvee \left\{\bigwedge f' \circ e_\mathbb{A} \circ \omega_\mathbb{A}[F \cap I] \mid e^\mathcal{F}_{\mathbb{A}^n}(F) \leq x \leq e^\mathcal{I}_{\mathbb{A}^n}(I)\right\} && \text{by (3.26),} \\
&= f'\left(\bigvee \left\{\bigwedge e_\mathbb{A} \circ \omega_\mathbb{A}[F \cap I] \mid e^\mathcal{F}_{\mathbb{A}^n}(F) \leq x \leq e^\mathcal{I}_{\mathbb{A}^n}(I)\right\}\right) && \text{since } f' \text{ is complete,} \\
&= f' \circ \liminf(e_\mathbb{A} \circ \omega_\mathbb{A})(x) && \text{by def. of } \liminf \\
&= f' \circ (\omega_\mathbb{A})^\triangledown(x) && \text{by definition of } (\omega_\mathbb{A})^\triangledown,
\end{aligned}$$

where (†) makes use of the fact that $F \cap I$ is both directed and co-directed for any filter F and ideal I, and the fact that $f^n \colon \mathbb{A}^n \to \mathbb{B}^n$ is order-preserving, so that $f^n[F \cap I]$ is again both directed and co-directed.

Since $x \in (\mathbb{A}^\delta)^n$ was arbitrary we see that $\omega_{\mathbb{B}} \circ (f')^n = f' \circ (\omega_{\mathbb{A}})^\triangledown$. (In case $\beta(\omega) = \triangle$, we would have used the fact that $f' = \limsup f$ in the derivation above.) Since $\omega \in \Omega$ was arbitrary, it follows that $f' \colon \mathbb{A}^\delta \to \mathbb{B}$ is an Ω-homomorphism. This concludes our proof. ∎

To conclude this subsection, we will prove a theorem that tells us when a monotone lattice-based algebra $\mathbb{A}$ with a profinite lattice reduct, i.e. a structure which has a reduct which is already a topological algebra, is in fact a topological algebra in its entire signature. Before we come to this theorem, we first need a technical lemma.

3.4.15. Lemma. *Let $\mathbb{L}$ and $\mathbb{M}$ be complete lattices such that there exist complete homomorphisms $g \colon \mathbb{L}^\delta \to \mathbb{L}$ and $h \colon \mathbb{M}^\delta \to \mathbb{M}$ such that $g \circ e_{\mathbb{L}} = \mathrm{id}_{\mathbb{L}}$. Then if $f \colon \mathbb{L} \to \mathbb{M}$ is an order-preserving map so that either $h \circ f^\triangledown = f \circ g$* or *$h \circ f^\triangle = f \circ g$, then f is (σ, σ)-continuous.*

$$\begin{tikzcd} & \mathbb{L}^\delta \arrow[r, "f^\triangledown \text{ or } f^\triangle"] \arrow[d, "g"] & \mathbb{M}^\delta \arrow[d, "h"] \\ \mathbb{L} \arrow[ur, "e_{\mathbb{L}}"] \arrow[r, "\mathrm{id}_{\mathbb{L}}"'] & \mathbb{L} \arrow[r, "f"'] & \mathbb{M} \end{tikzcd}$$

Proof Throughout the proof, we will suppose that $h \circ f^\triangledown = f \circ g$; the other case is order dual. We will first show that $f \colon \mathbb{L} \to \mathbb{M}$ preserves directed joins. Recall that by the universal property of $\downarrow_{\mathbb{L}} \colon \mathbb{L} \to \mathcal{I}\,\mathbb{L}$, we can factor $e_{\mathbb{L}} \colon \mathbb{L} \to \mathbb{L}^\delta$ as $e_{\mathbb{L}} = e_{\mathbb{L}}^{\mathcal{I}} \circ \downarrow_{\mathbb{L}}$ through $\mathcal{I}\,\mathbb{L}$, the ideal completion of $\mathbb{L}$. We may now draw the big diagram in Figure 3.2, about which we make two observations. Firstly, we claim

$$\begin{tikzcd} & & \mathbb{L}^\delta \arrow[r, "f^\triangledown"] \arrow[dd, "g"] & \mathbb{M}^\delta \arrow[dd, "h"] \\ & \mathcal{I}\,\mathbb{L} \arrow[ur, "e_{\mathbb{L}}^{\mathcal{I}}"] & & \\ \mathbb{L} \arrow[ur, "\downarrow_{\mathbb{L}}"] \arrow[rr, "\mathrm{id}_{\mathbb{L}}"'] & & \mathbb{L} \arrow[r, "f"'] & \mathbb{M} \end{tikzcd}$$

Figure 3.2: A closer look at $f^\triangledown$ via $\mathcal{I}\,\mathbb{L}$.

that

$$g \circ e_{\mathbb{L}}^{\mathcal{I}} \text{ is } (\sigma^\uparrow, \sigma^\uparrow)\text{-continuous.} \tag{3.27}$$

$$\begin{tikzcd}
 & & \mathbb{L}^\delta \arrow[r, "f^\triangledown"] \arrow[dd, "g"'] & \mathbb{M}^\delta \arrow[dd, "h"] \\
 & \mathcal{F}\,\mathbb{L} \arrow[ur, "e^{\mathcal{F}}_{\mathbb{L}}"] & & \\
\mathbb{L} \arrow[ur, "\uparrow_{\mathbb{L}}"] \arrow[rr, "\mathrm{id}_{\mathbb{L}}"'] & & \mathbb{L} \arrow[r, "f"'] & \mathbb{M}
\end{tikzcd}$$

Figure 3.3: A closer look at $f^\triangledown$ via $\mathcal{F}\,\mathbb{L}$.

To see why, first observe that since g is a complete homomorphism, it also preserves all directed joins, so that g is $(\sigma^\uparrow, \sigma^\uparrow)$-continuous. Since $\sigma^\uparrow \subseteq \delta^\uparrow$ by Lemma 2.1.28(3), it follows that $g\colon \mathbb{L}^\delta \to \mathbb{L}$ is $(\delta^\uparrow, \sigma^\uparrow)$-continuous. Now since $e^{\mathcal{I}}\colon \mathcal{I}\,\mathbb{L} \to \mathbb{L}$ is $(\sigma^\uparrow, \delta^\uparrow)$-continuous by Theorem 2.1.23, it follows that (3.27) holds. Our second observation is that

$$h \circ f^\triangledown \circ e^{\mathcal{I}}_{\mathbb{L}} \text{ is } (\sigma^\uparrow, \sigma^\uparrow)\text{-continuous.} \tag{3.28}$$

Similarly to g, as we have seen above, it is the case that $h\colon \mathbb{M}^\delta \to \mathbb{M}$ is $(\sigma^\uparrow, \sigma^\uparrow)$-continuous. Now since $f^\triangledown \circ e^{\mathcal{I}}_{\mathbb{L}}$ is $(\sigma^\uparrow, \sigma^\uparrow)$-continuous by Corollary 2.2.5, it follows that (3.28) holds.

We can now show that $f\colon \mathbb{L} \to \mathbb{M}$ is $(\sigma^\uparrow, \sigma^\uparrow)$-continuous, i.e. that f preserves directed joins. Let $S \subseteq \mathbb{L}$ be directed, then

$$\begin{aligned}
f(\bigvee S) &= f\left(\bigvee g e^{\mathcal{I}}_{\mathbb{L}} \downarrow_{\mathbb{L}}[S]\right) && \text{since } g \circ e^{\mathcal{I}}_{\mathbb{L}} \circ \downarrow_{\mathbb{L}} = \mathrm{id}_{\mathbb{L}}, \\
&= f g e^{\mathcal{I}}_{\mathbb{L}} (\bigvee \downarrow_{\mathbb{L}}[S]) && \text{by (3.27)}, \\
&= h f^\triangledown e^{\mathcal{I}}_{\mathbb{L}} (\bigvee \downarrow_{\mathbb{L}}[S]) && \text{since } h \circ f^\triangledown = f \circ g, \\
&= \bigvee h f^\triangledown e^{\mathcal{I}}_{\mathbb{L}} \downarrow_{\mathbb{L}}[S] && \text{by (3.28)}, \\
&= \bigvee f g e^{\mathcal{I}}_{\mathbb{L}} \downarrow_{\mathbb{L}}[S] && \text{since } h \circ f^\triangledown = f \circ g, \\
&= \bigvee f[S] && \text{since } g \circ e^{\mathcal{I}}_{\mathbb{L}} \circ \downarrow_{\mathbb{L}} = \mathrm{id}_{\mathbb{L}}.
\end{aligned}$$

The proof that f preserves co-directed meets is similar. We need to consider the diagram in Figure 3.3, about which we make the two following familiar observations. Firstly, for reasons analogous to those for (3.27), it follows that

$$g \circ e^{\mathcal{F}}_{\mathbb{L}} \text{ is } (\sigma^\downarrow, \sigma^\downarrow)\text{-continuous.} \tag{3.29}$$

Secondly, as before with (3.28) we see that

$$h \circ f^\triangledown \circ e^{\mathcal{F}}_{\mathbb{L}} \text{ is } (\sigma^\downarrow, \sigma^\downarrow)\text{-continuous.} \tag{3.30}$$

Now if $S \subseteq \mathbb{L}$ is co-directed, then we can make the same step-by-step argument as above to show that $f(\bigwedge S) = \bigwedge f[S]$. Having established that f is both $(\sigma^\uparrow, \sigma^\uparrow)$-continuous and $(\sigma^\downarrow, \sigma^\downarrow)$-continuous, we see that it follows by general topology (Lemma A.7.3) that $f\colon \mathbb{L} \to \mathbb{M}$ is (σ, σ)-continuous. ∎

In its most general form, we say an algebra $\mathbb{A}$ is a retract of an algebra $\mathbb{B}$ if there exist homomorphisms $f\colon \mathbb{A} \to \mathbb{B}$ and $g\colon \mathbb{B} \to \mathbb{A}$ such that $gf = \mathrm{id}_{\mathbb{A}}$. So the question when a lattice-based algebra is a retract of its canonical extension is the question under what circumstances there exists a homomorphism $g\colon \mathbb{A}^\delta \to \mathbb{A}$ such that $g \circ e_{\mathbb{A}} = \mathrm{id}_{\mathbb{A}}$. We will answer this question below under two additional requirements. Firstly, we will only consider complete homomorphisms $g\colon \mathbb{A}^\delta \to \mathbb{A}$. Secondly, we will assume that $\mathbb{A}$ has a profinite lattice reduct.

3.4.16. Theorem. *Fix a lattice-based signature Ω and let $\mathbb{A}$ be a monotone lattice-based Ω-algebra such that $\mathbb{A}^l$ is profinite. Then there exists a unique complete* lattice *homomorphism $g\colon (\mathbb{A}^\delta)^l \to \mathbb{A}^l$ such that $g \circ e_{\mathbb{A}} = \mathrm{id}_{\mathbb{A}}$. Moreover, the following are equivalent:*

1. *$g\colon \mathbb{A}^\delta \to \mathbb{A}$ is an Ω-algebra homomorphism;*
2. *$\mathbb{A}$ is a Boolean topological algebra.*

Proof Since we assumed that $\mathbb{A}^l$ is profinite, the existence of a unique complete lattice homomorphism $g\colon (\mathbb{A}^\delta)^l \to \mathbb{A}^l$ extending $\mathrm{id}_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{A}$ follows from Corollary 3.4.2. To show that (2) implies (1), we can apply Theorem 3.4.14 to $\mathrm{id}_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{A}$.

We will now show that (1) implies (2). Since $\mathbb{A}^l$ is already a Boolean topological lattice, we only need to show that for each $\omega \in \Omega$, $\omega_{\mathbb{A}}\colon \mathbb{A}^o \to \mathbb{A}$ is continuous, where o is the order type of ω, and the topology we are considering is $\sigma(\mathbb{A})$, the bi-Scott topology. Since we assumed that $g\colon \mathbb{A}^\delta \to \mathbb{A}$ is an Ω-algebra homomorphism, we know that the following diagram commutes:

$$\begin{tikzcd}
 & (\mathbb{A}^\delta)^o \arrow[r, "\omega_{\mathbb{A}^\delta}"] \arrow[d, "g^o"] & \mathbb{A}^\delta \arrow[d, "g"] \\
\mathbb{A}^o \arrow[ur, "(e_{\mathbb{A}})^o"] \arrow[r, "\mathrm{id}_{\mathbb{A}^o}"'] & \mathbb{A}^o \arrow[r, "\omega_{\mathbb{A}}"'] & \mathbb{A}
\end{tikzcd}$$

Now regardless of whether $\omega_{\mathbb{A}^\delta} = (\omega_{\mathbb{A}})^\triangledown$ or $\omega_{\mathbb{A}^\delta} = (\omega_{\mathbb{A}})^\vartriangle$, Lemma 3.4.15 tells us that $\omega_{\mathbb{A}}$ is (σ, σ)-continuous, since $\omega_{\mathbb{A}}$ is order-preserving and g is a complete homomorphism. Since $\omega \in \Omega$ was arbitrary, it follows that $\mathbb{A}$ is a topological algebra. ∎

3.4.4 Conclusions and further work

At this point in the narrative of this dissertation, §3.4 is a high point that is best enjoyed standing on a lot of ground work developed in Chapters 2 and 3. To a large extent, Chapters 2 and 3 were shaped to support the individual results in §3.4. Since this section contributes to an active field, we will now put our results in some perspective.

The results in §3.4.1 are an improvement of results from [96]. In [96], we relied on duality theory to establish the fundamental connection between the canonical extension and the profinite completion (Theorem 3.4.1), so we could only state the result for distributive lattices with operators. The connection between canonical extensions and profinite completions via duality has been studied more extensively in the cases of Heyting algebras [15, 14, 16] and distributive lattices and Boolean algebras [16]; also see §4.1.4. It is only now that we have extended (the topological) part of the canonical extension theory (in Chapter 2 and §3.2, and with Theorem 3.3.12), that we are able to state and prove Theorem 3.4.1 in full generality, without using duality theory. The central reasons for the main point of Theorem 3.4.1, viz. the fact that the profinite completion $\hat{\mathbb{A}}$ of a lattice-based algebra $\mathbb{A}$ is always a complete quotient of the canonical extension $\mathbb{A}^\delta$, are the fact that canonical extensions preserve surjective algebra homomorphisms and the fact that finite algebras are a fixed point of canonical extensions.

In §3.4.2, we set out to improve on J. Harding's result [50], which says that profinite completions and canonical extensions of monotone lattice-based algebras coincide in finitely generated varieties, using a proof strategy centered around the fact that HSP = HSP_B in finitely generated congruence-distributive varieties. At the time of writing of this chapter, we learned that M.J. Gouveia has a paper in press [49] which contains the same result as our Theorem 3.4.12. Unfortunately we cannot presently compare our approach in §3.4.2 with that of [49], because we have not seen this paper yet. Our results in §3.4.3, concerning universal properties of canonical extensions with respect to Boolean topological lattice-based algebras, are similar to unpublished work of Gehrke & Harding, who investigated the question when canonical extensions are a *reflector* when applied to a lattice-based algebras.

Further work

We believe that there is a lot more work to be done on the subject of canonical extensions and topological algebra. We will name a few possible more concrete questions.

- Even though we know that the canonical extension of a lattice is not always a topological lattice in its σ-topology (Example 3.4.3), this does not rule out that there is a meaningful way to see the canonical extension of a lattice $\mathbb{L}$ as a topological lattice. One possibility would be to consider 'mixed' topologies on $\mathbb{L}^\delta$, such as $\sigma^\uparrow \vee \delta^\downarrow$ or $\sigma^\downarrow \vee \delta^\uparrow$, much like the Lawson topology $\sigma^\uparrow \vee \iota^\downarrow$ [45].
- Given the fact that Boolean topological lattices can be characterized order-theoretically (Theorem 3.1.26), it would be interesting to see if we can find further universal properties of canonical extensions with respect to Boolean topological algebras.

Chapter 4

Duality, profiniteness and completions

In §3.4, we saw that there are several strong connections between canonical extensions, profinite completions and topological lattice-based algebras. In this chapter, we will see that some of these connections can also be interpreted through the discrete duality for distributive lattices with operators (DLO's).

Discrete duality is one of the ingredients of the 'double duality diagram' for modal logic (Fig. 1.1, p. 3) from Chapter 1. For DLO's, the diagram looks as follows.

$$\begin{tikzcd}
\text{DLO's} \arrow[r, shift left, "\text{extended}"] & \text{relational Priestley spaces} \arrow[l, shift left, "\text{Priestley duality}"] \arrow[d, "\text{forget topology}"] \\
\text{semi-topological DLO's} \arrow[u, "\text{are included in}"] \arrow[r, shift left, "\text{discrete}"] & \text{ordered Kripke frames} \arrow[l, shift left, "\text{duality}"]
\end{tikzcd}$$

We point out two reasons why the discrete duality results we present are interesting. Firstly, these results provide *algebraic* insight into interesting classes of (ordered) Kripke frames. Secondly, these results provide *spacial* insight into interesting classes of lattice-based algebras. This latter perspective will be the primary one in this chapter.

This chapter is organized as follows. In §4.1, we discuss a general duality for profinite DLO's, and a way to present profinite completions of DLO's using duality. We will see that profinite DLO's correspond to hereditarily finite Kripke frames. In §4.2, we provide a brief survey of how the results from §4.1 specialize to known results concerning distributive lattices, Boolean algebras and Heyting algebras. Finally, in §4.3, we will restrict our attention to Boolean algebras with operators, and we will see that in that case, we can also characterize Boolean topological algebras via duality: Boolean topological Boolean algebras with operators are dual to image-finite Kripke frames.

4.1 Dualities for distributive lattices with operators

In this section, we will describe discrete duality for profinite distributive lattices with operators (DLO's), and we will discuss the relation between canonical extensions, profinite completions and duality. Recall that distributive lattices correspond to Priestley spaces via Priestley duality, and that complete, bi-algebraic distributive lattices correspond to partially ordered sets via *discrete duality.* Goldblatt [47] showed that discrete duality for distributive lattices can be extended to a duality relating $\mathbf{DLO}^+$, the category of *semi-topological* DLO's, to $\mathbf{OKFr}$, the category of *ordered Kripke frames.* Moreover, there is a strong and non-coincidental connection between canonical extensions of DLO's and discrete duality: given a DLO $\mathbb{A}$, one can naturally construct an ordered Kripke frame $\mathbb{A}_\bullet$ via extended Priestley duality, which is then the discrete dual of $\mathbb{A}^\delta$, the canonical extension of $\mathbb{A}$.

The two main points of this section are the following. Firstly, we show that Pro-$\mathbf{DLO}_f$, the category of profinite DLO's, forms a subcategory of $\mathbf{DLO}^+$, the category of semi-topological DLO's, and that the discrete duality for semi-topological DLO's restricts to a duality between profinite DLO's and *hereditarily finite* ordered Kripke frames.

$$
\begin{array}{ccc}
\mathbf{DLO}^+ & \simeq & \mathbf{OKFr}^{op} \\
\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\
\text{Pro-}\,\mathbf{DLO}_f & \simeq & \mathbf{H}_\omega\mathbf{OKFr}^{op}
\end{array}
$$

Secondly, we show that $\hat{\mathbb{A}}$, the profinite completion of a DLO $\mathbb{A}$, corresponds to a *generated subframe* of $\mathbb{A}_\bullet$, the dual of $\mathbb{A}^\delta$.

$$
\begin{tikzcd}
\mathbb{A} \arrow[dr, "e_{\mathbb{A}}"] \arrow[ddr, bend right, "\mu_{\mathbb{A}}"'] \arrow[drr, dotted, no head, bend left] & & \\
& \mathbb{A}^{\delta} \arrow[r, dotted, no head] \arrow[d, "\nu_{\mathbb{A}}"] & \mathbb{A}_\bullet \arrow[d, phantom, "\rotatebox{90}{$\sqsubseteq$}"] \\
& \hat{\mathbb{A}} \arrow[r, dotted, no head] & H_\omega \mathbb{A}_\bullet
\end{tikzcd}
$$

This section is organized as follows. First, in §4.1.1, we introduce semi-topological DLO's, and we show how semi-topological DLO's are a generalization of Boolean topological DLO's and profinite DLO's. Next, in §4.1.2, we take a closer look at the category of ordered Kripke frames; specifically, we show how one can construct direted colimits of ordered Kripke frames. In §4.1.3, we use a categorical argument to show that the category of profinite DLO's is dually equivalent to the category of hereditarily finite ordered Kripke frames. Finally, in §4.1.4, we show how the relation between canonical extensions and profinite completions can be understood using duality. In §4.1.5 we provide conclusions and suggestions for further work.

4.1.1 Semi-topological DLO's

In this subsection we will present several facts and results on topological DLO's which we will need further on, and which additionally serve to motivate the phrase 'semi-topological DLO'. Let us start by restricting the notion of DLO we will consider in this section.

4.1.1. Convention. In full generality, a DLO is a monotone distributive lattice-based algebra (see §3.3.1) $\mathbb{A}$ for a signature $\Omega = \{0, 1, \wedge, \vee\} \uplus \Omega_j \uplus \Omega_m$ such that for each $\omega \in \Omega_j$, $\omega_{\mathbb{A}} \colon \mathbb{A}^{\mathrm{ord}(\omega)} \to \mathbb{A}$ is a normal operator, and $\omega \in \Omega_m$, $\omega_{\mathbb{A}} \colon \mathbb{A}^{\mathrm{ord}(\omega)} \to \mathbb{A}$ is a dual normal operator. To simplify our presentation, we fix two natural numbers n, m and we will only consider DLO's with a single normal operator $\Diamond \colon \mathbb{A}^n \to \mathbb{A}$ and dual normal operator $\Box \colon \mathbb{A}^m \to \mathbb{A}$.

4.1.2. Remark. We choose not to explicitly deal with order-reversing operators in this chapter, to simplify the presentation. However, all results we present generalize to operators which are order-preserving in some coordinates and order-reversing in others.

4.1.3. Example. Examples of DLO's are:

- distributive lattices (with *no* operators);
- Heyting algebras $\mathbb{A} = \langle A, \wedge, \vee, \to, 0, 1\rangle$, as the Heyting implication $\to \colon \mathbb{A}^{op} \times \mathbb{A} \to \mathbb{A}$ is a dual normal operator;
- Boolean algebras $\mathbb{A} = \langle A, \wedge, \vee, \neg, 0, 1\rangle$, as Boolean negation $\neg \colon \mathbb{A}^{op} \to \mathbb{A}$ is both a normal operator and a dual normal operator by De Morgan's laws;
- modal algebras $\mathbb{A} = \langle A, \wedge, \vee, \neg, 0, 1, \Box\rangle$, since these satisfy $\Box(x \wedge y) = \Box x \wedge \Box y$ and $\Box 1 = 1$.

We may now define semi-topological DLO's, which are the main objects of study in this section, together with ordered Kripke frames (which we will introduce in §4.1.2).

4.1.4. Definition. Let $\mathbb{A} = \langle A, \wedge, \vee, 0, 1, \Diamond, \Box \rangle$ be a DLO. We say $\mathbb{A}$ is a *semi-topological DLO* if

- $\mathbb{A}$ is complete,
- $\mathbb{A}$ is bi-algebraic,
- $\Diamond\colon \mathbb{A}^n \to \mathbb{A}$ is a complete operator (preserves all joins in each coordinate),
- $\Box\colon \mathbb{A}^m \to \mathbb{A}$ is a complete dual operator (preserves all meets in each coordinate).

By $\mathbf{DLO}^+$ we denote the category of semi-topological DLO's and complete homomorphisms.

Semi-topological DLO's are usually called perfect DLO's in the literature. We hope that the facts and results in the remainder of this subsection help to convince the reader that there are good reasons for the name we have chosen. The first reason for using the phrase 'semi-topological' lies in the fact that if we were to consider DLO's without any operators, i.e. 'naked' distributive lattices, then 'semi-topological' distributive lattices are topological lattices.

4.1.5. Fact. *Let $\mathbb{L}$ be a distributive lattice. Then the following are equivalent:*

1. *$\mathbb{L}$ is semi-topological (i.e. complete and bi-algebraic);*
2. *$\mathbb{L}$ is a profinite lattice.*

There are many equivalent characterizations of profinite lattices such as the fact above, see e.g. [27, Th. 2.5].

We see now that semi-topological DLO's come equiped with a natural, Boolean topology, and that $\vee$ and $\wedge$ are always continuous with respect to this topology. The reason a semi-topological DLO $\mathbb{A}$ may fail to be finite lies in the (lack of) continuity properties of $\Diamond$ and $\Box$. We show below that we can expect *some* continuity from $\Diamond$ and $\Box$, however.

4.1.6. Lemma. *Let $\mathbb{A}$ be a DLO. Then $\mathbb{A}$ is a semi-topological DLO if and only if*

1. *$\mathbb{A}^l$ is a profinite lattice;*
2. *$\Diamond\colon \mathbb{A}^n \to \mathbb{A}$ is $(\sigma^\uparrow, \sigma^\uparrow)$-continuous;*
3. *$\Box\colon \mathbb{A}^m \to \mathbb{A}$ is $(\sigma^\downarrow, \sigma^\downarrow)$-continuous.*

Proof Suppose that $\mathbb{A}$ is a semi-topological DLO. Firstly, we see that $\mathbb{A}^l$, the lattice reduct of $\mathbb{A}$, is complete and bi-algebraic, so by Fact 4.1.5, $\mathbb{A}^l$ is a profinite lattice. Secondly, since $\Diamond\colon \mathbb{A}^n \to \mathbb{A}$ preserves all joins in each coordinate, a fortiori it also preserves directed joins in each coordinate. It follows from Fact A.3.4 that $\Diamond$ is $(\sigma^\uparrow, \sigma^\uparrow)$-continuous. The argument for $\Box$ is analogous.

Conversely, suppose that $\mathbb{A}$ is a DLO satisfying (1), (2) and (3). It follows by Fact 4.1.5 that $\mathbb{A}$ is complete and bi-algebraic. Moreover, by Fact A.3.4, $\Diamond\colon \mathbb{A}^n \to \mathbb{A}$ preserves directed joins in each coordinate. But then $\Diamond$ preserves both finite joins (because it is an operator) and directed joins in each coordinate, it follows by Fact A.4.1 that f preserves all joins in each coordinate. An order dual argument shows that $\Box\colon \mathbb{A}^m \to \mathbb{A}$ preserves all meets in each coordinate; consequently we may conclude that $\mathbb{A}$ is a semi-topological DLO. ∎

We will now see that two important classes of topological DLO's, namely Boolean topological DLO's and profinite DLO's, are semi-topological. This is interesting for two reasons. Firstly, it tells us that we can apply the duality for semi-topological DLO's to Boolean topological DLO's and profinite DLO's, so that we may better understand them. Secondly, the fact that Boolean topological DLO's are semi-topological is a further motivation for the use of the phrase 'semi-topological'.

4.1.7. Definition. By **BoolDLO** we denote the category of Boolean topological DLO's and continuous homomorphisms. By Pro-$\mathbf{DLO}_f$ we denote the category of profinite DLO's and continuous homomorphism.

We know for general reasons that Pro-$\mathbf{DLO}_f$ is a subcategory of **BoolDLO** (Fact 3.1.13). If we restrict our attention to distributive lattices rather than DLO's, this inclusion can also be reversed, as was first demonstrated by K. Numakura [72].

4.1.8. Fact. Pro-$\mathbf{DL}_f \cong$ **BoolDL**.

We will now show how semi-topological DLO's can be seen as a generalization of the profinite DLO's and Boolean topological DLO's.

4.1.9. Corollary. **BoolDLO** *is a full subcategory of* $\mathbf{DLO}^+$*. Consequently, so is* Pro-$\mathbf{DLO}_f$*.*

Proof We will show that every Boolean topological DLO is a semi-topological DLO. Since all continuous DLO homomorphisms are complete homomorphisms by Fact 3.1.23(3), it then follows that **BoolDLO** is a full subcategory of $\mathbf{DLO}^+$. Moreover, since by Fact 3.1.13 every profinite DLO is also a Boolean topological DLO, it follows that Pro-$\mathbf{DLO}_f$ is also a full subcategory of $\mathbf{DLO}^+$.

Suppose $\mathbb{A}$ is a Boolean topological DLO. Then $\mathbb{A}^l$, the lattice reduct of $\mathbb{A}$, is profinite by Fact 4.1.8. Also, the operation $\Diamond\colon \mathbb{A}^n \to \mathbb{A}$ is continuous with respect to the (unique!) topology on $\mathbb{A}$, i.e. with respect to the bi-Scott topology (Fact 3.1.23(1)). Since $\Diamond$ is order-preserving, it is also $(\sigma^{\uparrow}, \sigma^{\uparrow})$-continuous (by Lemma 2.1.17). By the same argument if follows that $\Box$ is $(\sigma^{\downarrow}, \sigma^{\downarrow})$-continuous, so we may conclude by Corollary 4.1.6 that $\mathbb{A}$ is a semi-topological DLO. This concludes our proof. ∎

4.1.2 Colimits of ordered Kripke frames

In this subsection we will introduce ordered Kripke frames, which correspond to semi-topological DLO's via discrete duality, and we will show how to construct directed colimits of ordered Kripke frames. Moreover, we will see that directed unions of Kripke frames are an example of directed colimits.

Ordered Kripke frames

We begin defining **OKFr**, the category of ordered Kripke frames and bounded morphisms, which is dually equivalent to $\mathbf{DLO}^{+}$, the category of semitopological DLO's and continuous homomorphisms.

4.1.10. Definition. An *ordered Kripke frame* [47] consists of a tuple $\mathfrak{F} = \langle X_{\mathfrak{F}}, \leq_{\mathfrak{F}}, R_{\mathfrak{F}}, Q_{\mathfrak{F}} \rangle$ where

1. $\langle X_{\mathfrak{F}}, \leq_{\mathfrak{F}} \rangle$ is a partial order;
2. $R_{\mathfrak{F}} \subseteq X \times X^n$ is an $(n+1)$-ary relation such that
   (a) $\forall x \in X$, $R_{\mathfrak{F}}[x] := \{\bar{y} \in X^n \mid x \; R_{\mathfrak{F}} \; \bar{y}\}$ is a lower set;
   (b) $\forall \bar{y} \in X^n$, $\{x \in X \mid x \; R_{\mathfrak{F}} \; \bar{y}\}$ is an upper set;
3. $Q_{\mathfrak{F}} \subseteq X \times X^m$ is an $(m+1)$-ary relation such that
   (a) $\forall x \in X$, $Q_{\mathfrak{F}}[x]$ is a upper set;
   (b) $\forall \bar{y} \in X^n$, $\{x \in X \mid x \; Q_{\mathfrak{F}} \; \bar{y}\}$ is a lower set.

Observe that the conditions on $R_{\mathfrak{F}}$ and $Q_{\mathfrak{F}}$ can be summarized as follows:

$$\geq_{\mathfrak{F}} ; R_{\mathfrak{F}} ; (\geq_{\mathfrak{F}})^n = R_{\mathfrak{F}} \text{ and } \leq_{\mathfrak{F}} ; Q_{\mathfrak{F}} ; (\leq_{\mathfrak{F}})^m = Q_{\mathfrak{F}}.$$

A *bounded morphism* between two ordered Kripke frames $\mathfrak{F} = \langle X_{\mathfrak{F}}, \leq_{\mathfrak{F}}, R_{\mathfrak{F}}, Q_{\mathfrak{F}} \rangle$ and $\mathfrak{G} = \langle X_{\mathfrak{G}}, \leq_{\mathfrak{G}}, R_{\mathfrak{G}}, Q_{\mathfrak{G}} \rangle$ is a function $f \colon X_{\mathfrak{F}} \to X_{\mathfrak{G}}$ such that

1. $\forall x, y \in \mathfrak{F}$, if $x \leq_{\mathfrak{F}} y$ then $f(x) \leq_{\mathfrak{G}} f(y)$ (f is order-preserving);
2. $\forall x, y_1, \ldots, y_n \in \mathfrak{F}$, if $x \; R_{\mathfrak{F}} \; (y_1, \ldots, y_n)$ then $f(x) \; R_{\mathfrak{G}} \; (f(y_1), \ldots, f(y_n))$ and similarly for Q (f has the *forth property*);

3. $\forall x \in \mathfrak{F}$, $\forall y_1', \ldots, y_n' \in \mathfrak{G}$, if $f(x)\ R_{\mathfrak{G}}\ (y_1', \ldots, y_n')$ then $\exists y_1, \ldots, y_n \in \mathfrak{F}$ such that $x\ R_{\mathfrak{F}}\ (y_1, \ldots, y_n)$ and $f(y_1) = y_1'$, $\ldots$, $f(y_n) = y_n'$ (f has the *back property*).

By **OKFr** we denote the category of ordered Kripke frames and bounded morphisms. We will sometimes omit the subscripts when specifying a frame, simply writing $\mathfrak{F} = \langle X, \leq, R, Q\rangle$.

For details about the following fact, see e.g. [40, §2.3]. Of the two functors that witness this duality, we will only use $(\cdot)^+ \colon \mathbf{OKFr}^{op} \to \mathbf{DLO}^+$, which we will describe in more detail in Definition 4.1.21.

4.1.11. FACT. *The categories* $\mathbf{DLO}^+$ *and* **OKfr** *are dually equivalent.*

Directed colimits of ordered Kripke frames

We will now describe directed colimits of ordered Kripke frames. First, we recall what colimits are. A *cocone* to a poset-indexed diagram of ordered Kripke frames $\langle \mathfrak{F}_i, f_{ij}\rangle_I$ in **OKFr** is an I-indexed collection of maps $(g_i \colon \mathfrak{F}_i \to \mathfrak{F})_I$ with a common codomain $\mathfrak{F}$ such that for all $i, j \in I$ with $i \leq j$, we have $g_j \circ f_{ij} = g_i$.

$$\begin{tikzcd} \mathfrak{F} & \\ \mathfrak{F}_i \arrow[u, "g_i"] \arrow[r, "f_{ij}"'] & \mathfrak{F}_j \arrow[ul, "g_j"'] \end{tikzcd}$$

Let $(h_i \colon \mathfrak{F}_i \to \mathfrak{G})_I$ be another cocone to $\langle \mathfrak{F}_i, f_{ij}\rangle_I$ and let $e \colon \mathfrak{F} \to \mathfrak{G}$ be a bounded morphism. We say e is a *map of cocones* if for all $i \in I$, $e \circ g_i = h_i$.

$$\begin{tikzcd} \mathfrak{G} & \mathfrak{F} \arrow[l, "e"'] \\ \mathfrak{F}_i \arrow[u, "h_i"] \arrow[ur, "g_i"'] & \end{tikzcd}$$

We call $(g_i \colon \mathfrak{F}_i \to \mathfrak{F})_I$ a *limiting cocone* if for all $(h_i \colon \mathfrak{F}_i \to \mathfrak{G})_I$, there exists a *unique* map of cocones $e \colon \mathfrak{F} \to \mathfrak{G}$. We then call $\mathfrak{F}$ the *colimit* of $\langle \mathfrak{F}_i, f_{ij}\rangle_I$, writing $\mathfrak{F} \simeq \varinjlim_I \mathfrak{F}_i$. In this chapter, we will only consider *directed* diagrams of ordered Kripke frames, meaning that we will always assume that for all $i, j \in I$, there exists a $k \in I$ such that $i, j \leq k$. What is nice about directed colimits of ordered Kripke frames is that we can construct them in an elegant way.

Suppose that $\langle \mathfrak{F}_i, f_{ij}\rangle_{i,j\in I}$ is a directed diagram of ordered Kripke frames $\mathfrak{F}_i = \langle X_i, \leq_i, R_i, Q_i\rangle$. We define a relation $\sim_I$ on the disjoint union $\biguplus_I X_i$: for all $x \in X_i$ and $y \in X_j$,

$$x \sim_I y :\Leftrightarrow \exists k \geq i, j,\ f_{ik}(x) = f_{jk}(y).$$

It is easy to see that $\sim_I$ is reflexive and symmetric; moreover, since I is directed, one can also easily show that $\sim_I$ is transitive and hence, an equivalence relation. What we have just described is how to construct directed colimits in the category of sets. We will now define an ordered Kripke frame structure on the set of $\sim_I$-equivalence classes, i.e. on $\biguplus_I X_i/{\sim_I} = \{[x] \mid \exists i \in I,\, x \in X_i\}$. We define the relations as follows:

$$[x] \leq_I [y] \quad \text{if } \exists i \in I \text{ and } x', y' \in \mathfrak{F}_i \text{ such that } x \sim_I x',\ y \sim_I y' \text{ and } x' \leq_i y',$$

and similarly for R and Q. Now we define

$$\biguplus_I \mathfrak{F}_i/{\sim_I} := \langle \biguplus_I X_i/{\sim_I}, \leq_I, R_I, Q_I \rangle.$$

4.1.12. Lemma. *Let $\langle \mathfrak{F}_i, f_{ij} \rangle_{i,j\in I}$ be a directed diagram of ordered Kripke frames. Then $\biguplus_I \mathfrak{F}_i/{\sim_I}$ is the colimit of $\langle \mathfrak{F}_i, f_{ij} \rangle_{i,j\in I}$.*

Proof Suppose that we have a cocone for $\langle \mathfrak{F}_i, f_{ij} \rangle_{i,j\in I}$, i.e. an ordered Kripke frame $\mathfrak{G} = \langle Y, \leq, R', Q' \rangle$ and bounded morphisms $g_i \colon \mathfrak{F}_i \to \mathfrak{G}$ such that for all $i \leq j$, $g_j \circ f_{ij} = g_i$. We want to find a unique bounded morphism $g\colon \biguplus_I \mathfrak{F}_i/{\sim_I} \to \mathfrak{G}$ extending all the g_i. We know from basic category theory that the function

$$\begin{aligned} g\colon \biguplus X_i/{\sim_I} &\to Y \\ [x] &\mapsto g_i(x) \qquad\qquad \text{for } x \in \mathfrak{F}_i, \end{aligned}$$

is the unique *function* extending all the g_i. What remains to be shown is that g is a bounded morphism. To see why e.g. the order $\leq_I$ is preserved by g, suppose that $[x] \leq_I [y]$. Then there must exist some $i \in I$ and $x', y' \in \mathfrak{F}_i$ such that $x \sim_I x'$, $y \sim_I y'$ and $x' \leq_i y'$. Now we see that

$$\begin{aligned} g([x]) &= g([x']) && \text{since } x \sim_I x', \\ &= g_i(x') && \text{since } x' \in \mathfrak{F}_i, \\ &\leq g_i(y') && \text{since } g_i \text{ is order-preserving,} \\ &= g([y']) && \text{since } y' \in \mathfrak{F}_i, \\ &= g([y]) && \text{since } y \sim_I y'. \end{aligned}$$

Similar arguments show that the relations R_I and Q_I are preserved. Finally, suppose that $g([x])R'(y'_1, \ldots, y'_n)$. We have to find $[y_1], \ldots, [y_n] \in \biguplus_I \mathfrak{F}_i/{\sim_I}$ such that

$$[x]R_I([y_1], \ldots, [y_n]) \text{ and } g([y_1]) = y'_1, \ldots, g([y_n]) = y'_n.$$

Let $i \in I$ such that $x \in \mathfrak{F}_i$; then since $g([x]) = g_i(x)$ we see that $g_i(x)R'(y'_1, \ldots, y'_n)$. Since g_i is a bounded morphism, it follows that there exist $y_1, \ldots, y_n \in X_i$ such that $xR_i(y_1, \ldots, y_n)$ and $g_i(y_1) = y'_1$, etc. It follows that $[x]R_I([y_1], \ldots, [y_n])$, and since $g([y_1]) = g_i(y_1)$, etc, we see that g is indeed a bounded morphism. It follows that $\biguplus_I \mathfrak{F}_i/{\sim_I}$ is the colimit of $\langle \mathfrak{F}_i, f_{ij} \rangle_{i,j\in I}$. ∎

Colimits via generated subframes

We will now see that diagrams of ordered Kripke frames can occur as collections of generated subframes of a given frame $\mathfrak{F}$, and that directed colimits of such diagrams can be computed by simply taking a union. First, we will define generated subframes of ordered Kripke frames.

4.1.13. Definition. Given ordered Kripke frames $\mathfrak{F}' = \langle X', \leq', R', Q' \rangle$ and $\mathfrak{F} = \langle X, \leq, R, Q \rangle$, we say $\mathfrak{F}'$ is a *substructure* of $\mathfrak{F}$ if

- $X' \subseteq X$;
- $\leq' = \leq \restriction (X' \times X')$ and similarly for R' and Q'

Observe that if $\mathfrak{F}'$ is a substructure of $\mathfrak{F}$, then the relations on $\mathfrak{F}'$ are all induced by the underlying set X'. We may therefore refer to $\mathfrak{F}'$ as the substructure of $\mathfrak{F}$ induced by X'. We say $\mathfrak{F}'$ is a *generated subframe* of $\mathfrak{F}$ (notation: $\mathfrak{F}' \sqsubseteq \mathfrak{F}$) if additionally, we have that

- $\forall x \in X'$, $\forall y_1, \ldots, y_n \in X$, if $xR(y_1, \ldots, y_n)$, then $\{y_1, \ldots, y_n\} \subseteq X'$, and similarly for Q ($\mathfrak{F}'$ is closed under R- and Q-successors).

We present a well-known fact which will be of use in §4.1.4. Let $f\colon \mathfrak{F}' \to \mathfrak{F}$ be a bounded morphism between ordered Kripke frames $\mathfrak{F} = \langle X, \leq, R, Q \rangle$ and $\mathfrak{F}' = \langle X', \leq', R', Q' \rangle$. We define $f[\mathfrak{F}']$ to be the substructure of $\mathfrak{F}$ induced by $f[X']$.

4.1.14. Lemma. *Let $f\colon \mathfrak{F}' \to \mathfrak{F}$ be a bounded morphism between ordered Kripke frames. Then $f[\mathfrak{F}']$ is a generated subframe of $\mathfrak{F}$.*

Proof We need to show that $f[\mathfrak{F}']$ is a substructure of $\mathfrak{F}$, and that $f[\mathfrak{F}']$ is closed under R- and Q-successors. The former follows by the definition of $f[\mathfrak{F}']$ above; the latter from the fact that f, as a bounded morphism, has the back property. ∎

It is not hard to see that the generated subframe relation $\sqsubseteq$ on **OKFr** is a partial order. Consequently, we can view a collection of generated subframes as a *diagram* in the category of ordered Kripke frames and bounded morphisms. Let $\{\mathfrak{F}_i \mid i \in I\}$ be a collection of generated subframes of $\mathfrak{F}$. We may impose an order on I by defining $i \leq j :\Leftrightarrow \mathfrak{F}_i \sqsubseteq \mathfrak{F}_j$. If we denote the embedding from $\mathfrak{F}_i$ to $\mathfrak{F}_j$ by $f_{ij}\colon \mathfrak{F}_i \to \mathfrak{F}_j$, then $\langle \mathfrak{F}_i, f_{ij} \rangle_{i,j \in I}$ becomes a diagram of ordered Kripke frames: if $\mathfrak{F}_i \sqsubseteq \mathfrak{F}_j \sqsubseteq \mathfrak{F}_k$, then $\mathfrak{F}_i \sqsubseteq \mathfrak{F}_k$, so that the following diagrams commutes.

$$\begin{tikzcd}
\mathfrak{F}_j \arrow[r, "f_{jk}"] & \mathfrak{F}_k \\
\mathfrak{F}_i \arrow[u, "f_{ij}"] \arrow[ur, "f_{ik}"'] &
\end{tikzcd}$$

We call $\langle \mathfrak{F}_i, f_{ij}\rangle_{i,j\in I}$ the *diagram associated with* $\{\mathfrak{F}_i \mid i \in I\}$; whenever possible we will not refer explicitly to the embedding maps f_{ij}. One of the pleasant properties of *directed* diagrams of generated subframes is that we can easily compute their colimit. If $\{\mathfrak{F}_i \mid i \in I\}$ is a set of generated subframes of $\mathfrak{F}$, we denote by $\bigcup\{\mathfrak{F}_i \mid i \in I\}$ the substructure of $\mathfrak{F}$ induced by $\bigcup\{X_i \mid i \in I\}$.

4.1.15. LEMMA. *Let $\mathfrak{F} = \langle X, \leq, R, Q\rangle$ be an ordered Kripke frame and let $\{\mathfrak{F}_i \mid i \in I\}$ be a collection of generated subframes of $\mathfrak{F}$. Then $\bigcup_I \mathfrak{F}_i$ is also a generated subframe of $\mathfrak{F}$.*

Moreover, if we assume that $\{\mathfrak{F}_i \mid i \in I\}$ is directed, then $\bigcup_I \mathfrak{F}_i$ is the colimit of the diagram associated with $\{\mathfrak{F}_i \mid i \in I\}$.

Proof Since $\bigcup_I \mathfrak{F}_i$ is a substructure of $\mathfrak{F}$ by definition, in order to show that $\bigcup_I \mathfrak{F}_i \sqsubseteq \mathfrak{F}$ it suffices to show that $\bigcup_I \mathfrak{F}_i$ is closed under R-successors and Q-successors. So suppose that $x \in \bigcup_I \mathfrak{F}_i$ and that $xR(y_1, \ldots, y_n)$ for some $y_1, \ldots, y_n \in \mathfrak{F}$. Then there must be some $i \in I$ such that $x \in \mathfrak{F}_i$. Now since $\mathfrak{F}_i \sqsubseteq \mathfrak{F}$ by assumption, we see that $y_1, \ldots, y_n \in \mathfrak{F}_i$. It follows that $\bigcup_I \mathfrak{F}_i$ is closed under R-successors; the argument for Q-successors is identical.

To see that $\bigcup_I \mathfrak{F}_i$ is the colimit of the diagram associated with $\{\mathfrak{F}_i \mid i \in I\}$, suppose we have a co-cone of bounded morphisms $(g_i\colon \mathfrak{F}_i \to \mathfrak{G})_{i\in I}$ to some frame $\mathfrak{G}$. Saying that $(g_i\colon \mathfrak{F}_i \to \mathfrak{G})_{i\in I}$ is a cocone means that whenever $x \in \mathfrak{F}_i \sqsubseteq \mathfrak{F}_j$, then $g_j(x) = g_i(x)$.

$$\begin{tikzcd}
\mathfrak{F}_j \arrow[r, "g_j"] & \mathfrak{G} \\
\rotatebox{90}{$\sqsubseteq$} & \\
\mathfrak{F}_i \arrow[uur, "g_i"'] &
\end{tikzcd}$$

But this is precisely sufficient for allowing us to uniquely define a map $g\colon \bigcup_I \mathfrak{F}_i \to \mathfrak{G}$ which commutes with all of the g_i: given $x \in \bigcup_I \mathfrak{F}_i$, pick any $i \in I$ such that $x \in \mathfrak{F}_i$ and map x to $g_i(x)$. To see that $g\colon \bigcup_I \mathfrak{F}_i \to \mathfrak{G}$ is order-preserving and has the forth-property, i.e. that it preserves each of the relations $\leq$, R and Q, suppose that e.g. $x, y \in \bigcup_I \mathfrak{F}_i$ and $x \leq y$. Then there must exist $i, j \in I$ such that $x \in \mathfrak{F}_i$ and $y \in \mathfrak{F}_j$. Since we assumed that $\{\mathfrak{F}_i \mid i \in I\}$ is directed, there must exist some $k \in I$ such that $\mathfrak{F}_i \sqsubseteq \mathfrak{F}_k$ and $\mathfrak{F}_j \sqsubseteq \mathfrak{F}_k$. But now it follows from the fact that $g_k\colon \mathfrak{F}_k \to \mathfrak{G}$ is a bounded morphism that $g(x) = g_k(x) \leq g_k(y) = g(y)$. The argument to show that $g\colon \bigcup_I \mathfrak{F}_i \to \mathfrak{G}$ has the back-property is identical to that from the proof of Lemma 4.1.12. It follows that $\bigcup_I \mathfrak{F}_i$ is the colimit of $\{\mathfrak{F}_i \mid i \in I\}$. ∎

4.1.3 Duality for profinite DLO's

In this subsection, we will show that the category of profinite distributive lattices with operators and the category of hereditarily finite ordered Kripke frames are dually equivalent. Moreover, this duality is the restriction of the duality between semi-topological DLO's and ordered Kripke frames.

$$\begin{array}{ccc} \mathbf{DLO}^+ & \simeq & \mathbf{OKFr}^{op} \\ \cup\!\mid & & \cup\!\mid \\ \text{Pro-}\,\mathbf{DLO}_f & \simeq & \mathbf{H}_\omega\mathbf{OKFr}^{op} \end{array}$$

We start by defining the category in the lower right hand corner of the above diagram.

4.1.16. Definition. An ordered Kripke frame $\mathfrak{F}$ is *hereditarily finite* if for all $x \in \mathfrak{F}$, there exists a finite $\mathfrak{F}' \sqsubseteq \mathfrak{F}$ such that $x \in \mathfrak{F}'$. By $\mathbf{H}_\omega\mathbf{OKFr}$ we denote the full subcategory of $\mathbf{OKFr}$ whose objects are all hereditarily finite ordered Kripke frames.

One can see the property of being hereditarily finite as a *local* finiteness property. We will see below that we can express this property categorically.

4.1.17. Lemma. *Let $\mathfrak{F}$ be an ordered Kripke frame. The following are equivalent:*

1. *$\mathfrak{F}$ is hereditarily finite;*
2. *$\mathfrak{F} \simeq \varinjlim_I \mathfrak{F}_i$ for some directed diagram of finite frames.*

Proof (1) $\Rightarrow$ (2). If $\mathfrak{F}$ is hereditarily finite then for every $x \in \mathfrak{F}$, there is a finite $\mathfrak{F}' \sqsubseteq \mathfrak{F}$ such that $x \in \mathfrak{F}'$. It follows that $\mathfrak{F} = \bigcup\{\mathfrak{F}' \sqsubseteq \mathfrak{F} \mid \mathfrak{F}' \text{ finite}\}$. It follows by the first part of Lemma 4.1.15 that the diagram associated with $\{\mathfrak{F}' \sqsubseteq \mathfrak{F} \mid \mathfrak{F}' \text{ finite}\}$ is directed: if $\mathfrak{F}_0 \sqsubseteq \mathfrak{F}$ and $\mathfrak{F}_1 \sqsubseteq \mathfrak{F}$ are finite generated subframes of $\mathfrak{F}$, then $\mathfrak{F}_0 \cup \mathfrak{F}_1$ is also a generated subframe of $\mathfrak{F}$, which obviously is still finite. It now follows from the second part of Lemma 4.1.15 that $\mathfrak{F}$ is a directed colimit of finite frames.

(2) $\Rightarrow$ (1). Suppose that $\langle \mathfrak{F}_i, f_{ij}\rangle_{i,j\in I}$ is a directed diagram of finite ordered Kripke frames; we will show that $\biguplus \mathfrak{F}_i/{\sim_I}$ is hereditarily finite. This is sufficient since by Lemma 4.1.12, $\varinjlim_I \mathfrak{F}_i \simeq \biguplus \mathfrak{F}_i/{\sim_I}$. Recall that each $\mathfrak{F}_i$ can be mapped to $\bigcup \mathfrak{F}_i/{\sim_I}$ by sending $x \in \mathfrak{F}_i$ to $[x]$, the $\sim_I$-equivalence class of x. Let us denote these maps by $h_i \colon \mathfrak{F}_i \to \biguplus \mathfrak{F}_i/{\sim_I}$. Let $[x] \in \biguplus \mathfrak{F}_i/{\sim_I}$; we want to show that there is a finite $\mathfrak{F}' \sqsubseteq \biguplus \mathfrak{F}_i/{\sim_I}$ such that $x \in \mathfrak{F}'$. Since $[x] \in \biguplus \mathfrak{F}_i/{\sim_I}$, we know that $x \in \biguplus \mathfrak{F}_i$, so there is some $i \in I$ such that $x \in \mathfrak{F}_i$. It follows that $h_i(x) = [x]$, so that $[x] \in h_i[\mathfrak{F}_i]$. Now we know from Fact 4.1.14 that $h_i[\mathfrak{F}_i] \sqsubseteq \biguplus \mathfrak{F}_i/{\sim_I}$, and since $\mathfrak{F}_i$ is finite, so is $h_i[\mathfrak{F}_i]$. Since $[x] \in \biguplus \mathfrak{F}_i/{\sim_I}$ was arbitrary, it follows that $\biguplus \mathfrak{F}_i/{\sim_I}$ is hereditarily finite. ∎

Now that we have characterized the category of hereditarily finite ordered Kripke frames categorically, it is easy to prove the duality result of this subsection.

4.1.18. THEOREM. Pro-$\mathbf{DLO}_f \cong (\mathbf{H}_\omega\mathbf{OKFr})^{op}$.

Proof We already know that $\mathbf{DLO}^+ \simeq \mathbf{OKFr}^{op}$ (Fact A.8.1). Since Pro-$\mathbf{DLO}_f$ is a full subcategory of $\mathbf{DLO}^+$ and $\mathbf{H}_\omega\mathbf{OKFr}$ is a full subcategory of $\mathbf{OKFr}$, it suffices to show that the functor $(\cdot)_+ \colon \mathbf{DLO}^+ \to \mathbf{OKFr}^{op}$ maps profinite DLO's to hereditarily finite ordered Kripke frames and vice versa for $(\cdot)^+ \colon \mathbf{OKFr}^{op} \to \mathbf{DLO}^+$; we need not concern ourselves with morphisms.

Suppose that $\mathbb{A}$ is a profinite DLO, i.e. that $\mathbb{A} \simeq \varprojlim_I \mathbb{A}_i$ for some co-directed diagram $\langle \mathbb{A}_i, f_{ij}\rangle_{i,j\in I}$. We will show that $\mathbb{A}_+$ is a hereditarily finite ordered Kripke frame. Since being a limit is a categorical property, it is stable under categorical equivalenc. Since $(\cdot)_+ \colon \mathbf{DLO}^+ \to \mathbf{OKFr}^{op}$ is a *contravariant* functor however, we see that $\mathbb{A}_+$ must be a *colimit* for the diagram $\langle(\mathbb{A}_i)_+, (f_{ij})_+\rangle_{i,j\in I}$, and since the arrows are reversed, this diagram is directed rather than co-directed. Since each $\mathbb{A}_i$ is finite, we see by Fact A.8.1 that each $(\mathbb{A}_i)_+$ is finite as well. We have now established that

$$\mathbb{A}_+ \simeq \varinjlim_I (\mathbb{A}_i)_+,$$

for a directed diagram of finite frames $\langle(\mathbb{A}_i)_+, (f_{ij})_+\rangle_{i,j\in I}$. It follows by Lemma 4.1.17 that $\mathbb{A}_+$ is hereditarily finite.

Conversely, if $\mathfrak{F}$ is hereditarily finite, then by Lemma 4.1.17 we may assume that $\mathfrak{F} \simeq \varinjlim_I \mathfrak{F}_i$ for some directed diagram of finite frames $\langle \mathfrak{F}_i, f_{ij}\rangle_{i,j\in I}$. It now follows by an argument analogous to the one above that $\langle \mathfrak{F}_i^+, f_{ij}^+\rangle_{i,j\in I}$ is a co-directed diagram of finite DLO's and that $\mathfrak{F}^+ \simeq \varprojlim_I \mathfrak{F}_i^+$; consequently, $\mathfrak{F}^+$ is profinite. This concludes our proof. ∎

4.1.4 Profinite completion via duality

In this subsection, we will show that $\hat{\mathbb{A}}$, the profinite completion of a DLO $\mathbb{A}$, corresponds to the largest hereditarily finite generated subframe of $\mathbb{A}_\bullet$, the prime filter frame of $\mathbb{A}$. We illustrate this with the following picture:

$$\begin{tikzcd}
\mathbb{A} \arrow[dr, "e_{\mathbb{A}}"] \arrow[ddr, bend right, "\mu_{\mathbb{A}}"'] \arrow[drr, dotted, no head, bend left] & & \\
& \mathbb{A}^\delta \arrow[r, dotted, no head] \arrow[d, "\nu_{\mathbb{A}}"] & \mathbb{A}_\bullet \arrow[d, phantom, "\sqcup\!|"] \\
& \hat{\mathbb{A}} \arrow[r, dotted, no head] & H_\omega\mathbb{A}_\bullet
\end{tikzcd}$$

The left side of the picture above follows from the commutative diagram of Theorem 3.4.1, which states that the profinite completion $\mu_{\mathbb{A}} \colon \mathbb{A} \to \hat{\mathbb{A}}$ of a lattice-based algebra $\mathbb{A}$ can be factored through $e_{\mathbb{A}} \colon \mathbb{A} \to \mathbb{A}^\delta$, the canonical extension of $\mathbb{A}$, via a surjective complete homomorphism $\nu_{\mathbb{A}} \colon \mathbb{A}^\delta \to \hat{\mathbb{A}}$. The right-hand side of the picture above shows two dual structures which are derived from our DLO $\mathbb{A}$. The prime filter frame of $\mathbb{A}$, denoted $\mathbb{A}_\bullet$, is known to be the discrete dual of $\mathbb{A}^\delta$. What we will see below is that $\hat{\mathbb{A}}$, the profinite completion of $\mathbb{A}$, is dual to $H_\omega\mathbb{A}_\bullet$, which is the largest hereditarily finite generated subframe of $\mathbb{A}_\bullet$.

Canonical extensions via duality

Recall that given a DLO $\mathbb{A} = \langle A, \wedge, \vee, 0, 1, \Box_\mathbb{A}, \Diamond_\mathbb{A}\rangle$, we can think of four different canonical extensions of $\mathbb{A}$, depending on how we choose to extend $\Box_\mathbb{A}$ and $\Diamond_\mathbb{A}$: by taking $(\Box_\mathbb{A})^\triangledown$ and $(\Diamond_\mathbb{A})^\triangledown$, $(\Box_\mathbb{A})^\vartriangle$ and $(\Diamond_\mathbb{A})^\vartriangle$, $(\Box_\mathbb{A})^\triangledown$ and $(\Diamond_\mathbb{A})^\vartriangle$, or $(\Box_\mathbb{A})^\vartriangle$ and $(\Diamond_\mathbb{A})^\triangledown$. Out of these four choices, the last one has the best continuity properties if we go by Theorem 2.2.18 (1).

4.1.19. Convention. If $\mathbb{A} = \langle A, \wedge, \vee, 0, 1, \Box_\mathbb{A}, \Diamond_\mathbb{A}\rangle$ is a DLO, then we define $\mathbb{A}^\delta = \langle A^\delta, \wedge, \vee, 0, 1, (\Box_\mathbb{A})^\vartriangle, (\Diamond_\mathbb{A})^\triangledown\rangle$, meaning that we take the lower extension of the operator(s) of $\mathbb{A}$ and the upper extension of the dual operator(s).

We will now discuss how the canonical extension of a DLO $\mathbb{A}$ can be constructed via $\mathbb{A}_*$, the *extended Priestley dual* of $\mathbb{A}$. First, one takes $\mathbb{A}_*$ and one strips this structure of its topology, resulting in a discrete ordered Kripke frame we denote by $\mathbb{A}_\bullet$, the *prime filter frame* of $\mathbb{A}$. This construction is in fact a functor $(\cdot)_\bullet\colon \mathbf{DLO} \to \mathbf{OKFr}^{op}$.

4.1.20. Definition. The prime filter frame of a DLO $\mathbb{A} = \langle A, \wedge, \vee, 0, 1, \Diamond_\mathbb{A}, \Box_\mathbb{A}\rangle$ is defined as follows:

$$\mathbb{A}_\bullet := \langle X_{\mathbb{A}_\bullet}, \leq_{\mathbb{A}_\bullet}, R_{\mathbb{A}_\bullet}, Q_{\mathbb{A}_\bullet}\rangle,$$

where

- $\langle X_{\mathbb{A}_\bullet}, \leq_{\mathbb{A}_\bullet}\rangle$ is the set of prime filters of $\mathbb{A}$, ordered by inclusion;
- $F\ R_{\mathbb{A}_\bullet}\ (G_1, \ldots, G_n)$ iff $\Diamond_\mathbb{A}[G_1, \ldots, G_n] \subseteq F$;
- $F\ Q_{\mathbb{A}_\bullet}\ (G_1, \ldots, G_m)$ iff $\Box_\mathbb{A}^{-1}(F) \subseteq \bigcup_{i=1}^m b_i(A, G_i)$, where

$$b_i(U, V) := \underbrace{U \times \cdots \times U}_{i-1} \times V \times \underbrace{U \times \cdots \times U}_{m-i}.$$

If $f\colon \mathbb{A} \to \mathbb{B}$ is a DLO homomorphism, then $f_\bullet\colon \mathbb{B}_\bullet \to \mathbb{B}_\bullet$ is defined simply as

$$f_\bullet\colon F \mapsto f^{-1}(F).$$

The next step in constructing $\mathbb{A}^\delta$ from $\mathbb{A}$ is to take the *complex algebra* of $\mathbb{A}_\bullet$. The complex algebra functor $(\cdot)^+\colon \mathbf{OKfr}^{op} \to \mathbf{DLO}^+$ is part of the discrete duality between semi-topological DLO's and ordered Kripke frames (Fact 4.1.11).

4.1.21. Definition. If $\mathfrak{F} = \langle X, \leq, R, Q\rangle$ is an ordered Kripke frame, then we define

$$\mathfrak{F}^+ := \langle \mathrm{Up}(X), \cap, \cup, \emptyset, X, \langle R\rangle, [Q]\rangle,$$

the complex algebra of $\mathfrak{F}$, where

- $\langle \mathrm{Down}(X), \cap, \cup, \emptyset, X\rangle$ is the lattice of lower sets of $\langle X, \leq\rangle$;
- $\langle R\rangle \colon (U_1, \ldots, U_n) \mapsto \{x \in X \mid R[x] \between U_1 \times \cdots \times U_n\}$;
- $[Q] \colon (U_1, \ldots, U_m) \mapsto \{x \in X \mid Q[x] \subseteq \bigcup_{i=1}^{m} b_i(X, U_i)\}$.

If $f \colon \mathfrak{F} \to \mathfrak{G}$ is a bounded morphism, then we define $f^+ \colon \mathfrak{G}^+ \to \mathfrak{F}^+$ as

$$f^+ \colon U \mapsto f^{-1}(U).$$

Having defined the functors $(\cdot)_\bullet \colon \mathbf{DLO} \to \mathbf{OKFr}^{op}$ and $(\cdot)^+ \colon \mathbf{OKFr}^{op} \to \mathbf{DLO}^+$, we can now describe the canonical extension of a DLO $\mathbb{A}$ using duality.

4.1.22. Fact ([38]). *Let $\mathbb{A}$ be a DLO. Then $e \colon \mathbb{A} \to (\mathbb{A}_\bullet)^+$, where*

$$e \colon a \mapsto \{F \in \mathbb{A}_\bullet \mid a \in F\},$$

is the canonical extension of $\mathbb{A}$.

The above fact is no coincidence; canonical extensions were developed by Jónsson and Tarski [58] for *Boolean algebras* with operators precisely to study properties of duality-based representations such as that in Fact 4.1.22. Note that rather than defining $\mathbb{A}^\delta$ first on the lattice reduct of $\mathbb{A}$, and then adding extensions of $\Box$ and $\Diamond$ as we did in §3.3, we can define the canonical extension of a DLO directly. This is because we restricted our notion of canonical extension in Convention 4.1.19.

4.1.23. Convention. For convenience, we redefine $e_\mathbb{A} \colon \mathbb{A} \to \mathbb{A}^\delta$, the canonical extension of a DLO $\mathbb{A}$, as $e_\mathbb{A} \colon \mathbb{A} \to (\mathbb{A}_\bullet)^+$ (see Fact 4.1.22 above) for the remainder of this chapter.

4.1.24. Remark. Observe that the functor $(\cdot)^+ \colon \mathbf{OKFr}^{op} \to \mathbf{DLO}^+$ maps an ordered Kripke frame to its collection of *lower* sets. Some authors define Priestley duality differently, using upper sets rather than lower sets. This is possible only if one also adapts the definition of $(\cdot)_+ \colon \mathbf{DLO}^+ \to \mathbf{OKFr}^{op}$. Both approaches are mathematically equivalent, as long as the readers are aware which approach is being used.

The dual of the profinite completion

In the introduction to this subsection, we mentioned that the profinite completion of $\mathbb{A}$ correspondes to 'the largest hereditarily finite generated subframe' of $\mathbb{A}_\bullet$. Below, we will make this notion precise.

4.1.25. LEMMA. $\mathbf{H}_\omega\mathbf{OKFr}$ *is a co-reflective subcategory of* $\mathbf{OKFr}$.

Proof The statement of the lemma boils down to the following. For each frame $\mathfrak{F}$ we need to find a hereditarily finite frame $\mathfrak{F}'$ and a bounded morphism $h\colon \mathfrak{F}' \to \mathfrak{F}$, such that whenever $f\colon \mathfrak{G} \to \mathfrak{F}$ is a bounded morphism from a hereditarily finite frame $\mathfrak{G}$ to $\mathfrak{F}$, then there exists a unique $f'\colon \mathfrak{G} \to \mathfrak{F}'$ such that $h \circ f' = f$.

$$\begin{tikzcd}
 & \mathfrak{F} \\
\mathfrak{G} \arrow[ur, "f"] \arrow[r, dotted, "f'"'] & \mathfrak{F}' \arrow[u, "h"']
\end{tikzcd}$$

Concretely, we will define $\mathfrak{F}'$ to be the following generated subframe of $\mathfrak{F}$:

$$H_\omega\mathfrak{F} = \bigcup\{\mathfrak{G} \mid \mathfrak{G} \sqsubseteq_\omega \mathfrak{F}\}.$$

Observe $H_\omega\mathfrak{F}$ is indeed a generated subframe of $\mathfrak{F}$ by Lemma 4.1.15, and by the same lemma, we see that $H_\omega\mathfrak{F}$ is a colimit of a directed diagram of finite frames. We will take $h\colon H_\omega\mathfrak{F} \to \mathfrak{F}$ to simply be the embedding map. Now we must show that $H_\omega\mathfrak{F}$ has the desired universal property. What we will show is that if $f\colon \mathfrak{G} \to \mathfrak{F}$ is a bounded morphism from a hereditarily finite frame $\mathfrak{G}$ to $\mathfrak{F}$, then $f[\mathfrak{G}] \sqsubseteq H_\omega\mathfrak{F}$. But this is easy to see: if $x \in \mathfrak{G}$, then since $\mathfrak{G}$ is hereditarily finite, there exists a finite $\mathfrak{G}' \sqsubseteq \mathfrak{G}$ such that $x \in \mathfrak{G}'$. It follows that $f(x) \in f[\mathfrak{G}']$. Since $\mathfrak{G}'$ is finite, so is $f[\mathfrak{G}']$. Now since also $f[\mathfrak{G}'] \sqsubseteq \mathfrak{F}$ by Fact 4.1.14, we see that

$$f(x) \in f[\mathfrak{G}'] \sqsubseteq H_\omega\mathfrak{F}.$$

Since $x \in \mathfrak{G}$ was arbitrary, it follows that $f[\mathfrak{G}] \sqsubseteq H_\omega\mathfrak{F}$. We may therefore take $f'\colon \mathfrak{G} \to H_\omega\mathfrak{F}$ to be f with its codomain restricted to $H_\omega\mathfrak{F}$. It follows from basic set theory that f' is the unique map such that $h \circ f' = f$. ∎

By basic category theory, $H_\omega\colon \mathbf{OKFr} \to \mathbf{H}_\omega\mathbf{OKFr}$ is now a functor [69, §IV.3]; if $f\colon \mathfrak{F} \to \mathfrak{G}$ is a bounded morphism then $H_\omega f\colon H_\omega\mathfrak{F} \to H_\omega\mathfrak{G}$ is simply the restriction of f to $H_\omega\mathfrak{F}$.

4.1.26. REMARK. Observe that our definition of $H_\omega\mathfrak{F}$ above leaves open the possibility that $H_\omega\mathfrak{F}$ is the *empty frame*. This will happen if every point in $\mathfrak{F}$ generates an infinite generated subframe. Not all textbooks on modal logic universally agree with us that the empty frame is actually a frame, cf. [19, Definition 1.19].

We can now show how to construct the profinite completion $\hat{\mathbb{A}}$ of a DLO $\mathbb{A}$ via duality.

4.1.27. THEOREM. *Let $\mathbb{A}$ be a DLO. Then $m\colon \mathbb{A} \to (H_\omega\mathbb{A}_\bullet)^+$, where*

$$m\colon a \mapsto \{F \in H_\omega\mathbb{A}_\bullet \mid a \in F\},$$

is the profinite completion of $\mathbb{A}$.

Proof Since the profinite completion of $\mathbb{A}$ can be characterized via a universal property (Fact 3.1.19), it suffices to show the following three things:

1. $(H_\omega\mathbb{A}_\bullet)^+$ is profinite;
2. $m\colon \mathbb{A} \to (H_\omega\mathbb{A}_\bullet)^+$ is a DLO algebra homomorphism;
3. for every DLO homomorphism $f\colon \mathbb{A} \to \mathbb{B}$, where $\mathbb{B}$ is a profinite DLO, there exists a unique continuous DLO homomorphism $f'\colon (H_\omega\mathbb{A}_\bullet)^+ \to \mathbb{B}$ such that $f' \circ m = f$.

(1). This follows immediately from the fact that $H_\omega\colon \mathbf{OKFr} \to \mathbf{H}_\omega\mathbf{OKFr}$ is a functor and Theorem 4.1.18.

(2). Let us denote the embedding $H_\omega\mathbb{A}_\bullet \sqsubseteq \mathbb{A}_\bullet$ by $h\colon H_\omega\mathbb{A}_\bullet \to \mathbb{A}_\bullet$. Then

$$h^+\colon (\mathbb{A}_\bullet)^+ \to (H_\omega\mathbb{A}_\bullet)^+$$

is a complete DLO homomorphism. Now it is an easy calculation to show that

$$m = h^+ \circ e_{\mathbb{A}}; \tag{4.1}$$

consequently, $m\colon \mathbb{A} \to (H_\omega\mathbb{A}_\bullet)^+$ is a DLO algebra homomorphism.

(3). Suppose that $f\colon \mathbb{A} \to \mathbb{B}$ and that $\mathbb{B}$ is a profinite DLO. Then by Corollary 3.4.2, there exists a unique continuous homomorphism $f''\colon (\mathbb{A}_\bullet)^+ \to \mathbb{B}$ such that

$$f'' \circ e_{\mathbb{A}} = f. \tag{4.2}$$

Now since $(\cdot)_+$ and $(\cdot)^+$ form a duality, there exists a bijection

$$\varphi\colon \operatorname{Hom}((\mathbb{A}_\bullet)^+, \mathbb{B}) \rightleftarrows \operatorname{Hom}(\mathbb{B}_+, \mathbb{A}_\bullet)\colon \varphi^{-1},$$

which is natural in $\mathbb{A}_\bullet$ and $\mathbb{B}$. This bijection gives us a bounded morphism $\varphi(f'')\colon \mathbb{B}_+ \to \mathbb{A}_\bullet$. Since $\mathbb{B}_+$ is image-finite by Theorem 4.1.18, it follows from Lemma 4.1.25 that there exists a unique $g\colon \mathbb{B}_+ \to H_\omega\mathbb{A}_\bullet$ such that $h \circ g = \varphi(f'')$.

$$\begin{tikzcd} & H_\omega\mathbb{A}_\bullet \arrow[d, "h"] \\ \mathbb{B}_+ \arrow[ur, dotted, "g"] \arrow[r, "\varphi(f'')"'] & \mathbb{A}_\bullet \end{tikzcd}$$

Observe that it is a consequence of the naturality of φ^{-1} that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{OKFr}}(\mathbb{B}_+, H_\omega\mathbb{A}_\bullet) \arrow[r, "\varphi^{-1}"] \arrow[d, "h\circ_"] & \mathrm{Hom}_{\mathbf{DLO}^+}((H_\omega\mathbb{A}_\bullet)^+, \mathbb{B}) \arrow[d, "_\circ h^+"] \\
\mathrm{Hom}_{\mathbf{OKFr}}(\mathbb{B}_+, \mathbb{A}_\bullet) \arrow[r, "\varphi^{-1}"'] & \mathrm{Hom}_{\mathbf{DLO}^+}((\mathbb{A}_\bullet)^+, \mathbb{B})
\end{tikzcd}$$

Now we see that

$$\begin{aligned}
f'' &= \varphi^{-1} \circ \varphi(f'') && \text{since } \varphi \text{ is a bijection,} \\
&= \varphi^{-1}(h \circ g) && \text{since } h \circ g = \varphi(f''), \\
&= \varphi^{-1}(g) \circ h^+ && \text{by naturality of } \varphi^{-1}.
\end{aligned}$$

Now we define $f' := \varphi^{-1}(g)$. Then we see that

$$\begin{aligned}
f' \circ m &= f' \circ h^+ \circ e_{\mathbb{A}} && \text{by (4.1),} \\
&= \varphi^{-1}(g) \circ h^+ \circ e_{\mathbb{A}} && \text{by definition of } f', \\
&= f'' \circ e_{\mathbb{A}} && \text{since } \varphi^{-1}(g) \circ h^+ = f'', \\
&= f && \text{by (4.2).}
\end{aligned}$$

Moreover, it follows from the fact that $g \colon \mathbb{B}_+ \to H_\omega\mathbb{A}_\bullet$ is unique that $f' = \varphi^{-1}(g)$ is unique. ∎

4.1.5 Conclusions and further work

The duality for profinite DLO's we presented in §4.1.3 was inspired by the general categorical approach of Johnstone [54, Ch. VI], and the results on the Heyting algebra case due to G. & N. Bezhanishvili [14]. The dual characterization of profinite completion we present in §4.1.4 was inspired by results on the Heyting algebra case due to G. Bezhanishvili et al. [15]. In [15], the dual of the profinite completion of a Heyting algebra is described using the Nachbin order compactification; this is a different approach than ours, which uses the fact that hereditarily finite ordered Kripke frames form a co-reflective subcategory of the category of ordered Kripke frames (Lemma 4.1.25).

An interesting question for further work is whether one can characterize **BoolDLO** non-trivially using duality. We will provide a partial answer to this question in §4.3; we revisit this question explicitly in §4.3.3.

4.2 A brief survey of subcategories of DLO

In this section, we will briefly sketch how the two main results of §4.1 specialize to three well-known subcategories of DLO: the categories of distributive lattices, Boolean algebras and Heyting algebras. We will now recall the main results from §4.1. Firstly, we showed that the discrete duality for semi-topological DLO's restricts to a duality for profinite DLO's and hereditarily finite ordered Kripke frames.

$$\begin{array}{ccc} \mathbf{DLO}^{+} & \simeq & \mathbf{OKFr}^{op} \\ \cup\!| & & \cup\!| \\ \text{Pro-}\,\mathbf{DLO}_f & \simeq & \mathbf{H}_{\omega}\mathbf{OKFr}^{op} \end{array}$$

Secondly, the profinite completion $\hat{\mathbb{A}}$ of a DLO $\mathbb{A}$ dually corresponds to the largest hereditarily finite generated subframe of $\mathbb{A}_{\bullet}$, the prime filter frame of $\mathbb{A}$.

$$\begin{tikzcd}
\mathbb{A} \arrow[dr, "e_{\mathbb{A}}"] \arrow[ddr, bend right, "\mu_{\mathbb{A}}"'] & & & \\
& \mathbb{A}^{\delta} \arrow[d, "\nu_{\mathbb{A}}"] & \simeq\ (\mathbb{A}_{\bullet})^{+} \arrow[d] & \mathbb{A}_{\bullet} \\
& \hat{\mathbb{A}} & \simeq\ (H_{\omega}\mathbb{A}_{\bullet})^{+} & H_{\omega}\mathbb{A}_{\bullet} \arrow[u, phantom, "\sqcup\!|"]
\end{tikzcd}$$

Below we will see that in some cases, the diagrams above collapse. Our prime examples of DLO's for which the diagrams do not collapse are Heyting algebras, which we will briefly discuss in §4.2.3, and Boolean algebras with operators. The latter will be discussed in greater detail in §4.3.

4.2.1. Remark. In our discussion of completions in this section we have not included the *MacNeille completion* (§A.5.2). Duality for MacNeille completions of Heyting algebras was discussed by Harding & Bezhanishvili [51]. The relation between canonical extensions and MacNeille completions of monotone lattice-based algebras has been studied by Gehrke, Harding & Venema [36]. Bezhanishvili & Vosmaer [16] discuss the question when canonical extensions, MacNeille completions and profinite completions of distributivel lattices, Heyting algebras and Boolean algebras are isomorphic, using duality. Finally, we would like to point out the brief discussion of the circumstances under which profinite completions and MacNeille completions of modal algebras coincide in [97].

4.2.1 Distributive lattices

In the case of distributive lattices, the discrete duality between $\mathbf{DLO}^{+}$ and **OKFr** boils down to the duality between $\mathbf{DL}^{+}$, the category of complete, bi-algebraic distributive lattices and complete lattice homomorphisms, and **Pos**, the category of partially ordered sets and order-preserving maps.

4.2.2. Fact. $\mathbf{DL}^{+} \simeq \mathbf{Pos}^{op}$.

If we look at $\mathbf{BoolDL} \subseteq \mathbf{DL}^+$, the category of Boolean topological distributive lattices and continuous homomorphisms, and Pro- $\mathbf{DL}_f$, the category of profinite distributive lattices and continuous homomorphisms, then it follows from Facts 4.1.5 and 4.1.8 that both of these categories are isomorphic to $\mathbf{DL}^+$.

4.2.3. FACT. Pro- $\mathbf{DL}_f \cong \mathbf{BoolDL} \cong \mathbf{DL}^+$.

We combine these two facts in the following picture:

$$
\begin{tikzcd}
\mathbf{DL}^+ \arrow[r, phantom, "\simeq"] \arrow[d, phantom, "\cong" sloped] & \mathbf{Pos}^{op} \\
\mathbf{BoolDL} \arrow[d, phantom, "\cong" sloped] & \\
\text{Pro-}\,\mathbf{DL}_f &
\end{tikzcd}
$$

These dualities and isomorphisms have been discussed and rediscovered many times; we refer the reader to Johnstone [54, §VI-3] and Davey et al. [27] for more details and historical discussion.

Just like the categories $\mathbf{DL}^+$ and Pro- $\mathbf{DL}_f$ collapse, we see that profinite completion and the canonical extension of a distributive lattice $\mathbb{L}$ are isomorphic.

$$
\begin{tikzcd}
\mathbb{L} \arrow[dr, "e_{\mathbb{L}}"] \arrow[ddr, bend right, "\mu_{\mathbb{L}}"'] & & & \\
& \mathbb{L}^\delta \arrow[r, phantom, "\simeq"] \arrow[d, phantom, "\wr"] & (\mathbb{L}_\bullet)^+ \arrow[d, phantom, "\|"] & \mathbb{L}_\bullet \arrow[d, phantom, "\|"] \\
& \hat{\mathbb{L}} \arrow[r, phantom, "\simeq"] & (H_\omega \mathbb{L}_\bullet)^+ & H_\omega \mathbb{L}_\bullet
\end{tikzcd}
$$

The fact that $\nu_{\mathbb{L}} \colon \mathbb{L}^\delta \to \hat{\mathbb{L}}$ is an isomorphism and that $\nu_{\mathbb{L}} \circ e_{\mathbb{L}} = \mu_{\mathbb{L}}$ is a corollary of Theorem 3.4.1, and was first published by G. Bezhanishvili et al. [15] and J. Harding [50]. The relation between $\mathbb{L}^\delta$ and $\hat{\mathbb{L}}$ via duality has been studied in greater detail in [15] and by G. Bezhanishvili and the author in [16]; also see [27].

4.2.2 Boolean algebras

For Boolean algebras, the discrete duality between $\mathbf{DLO}^+$ and **OKFr** boils down to the well-known duality between **CABA**, the category of complete, atomic Boolean algebras and complete homomorphisms, and **Set**, the category of sets and functions.

4.2.4. FACT. $\mathbf{CABA} \simeq \mathbf{Set}^{op}$.

The situation with topological Boolean algebras is similar to that for distributive lattices, but it has one important additional ingredient, which is a result due to D. Papert Strauss [74].

4.2.5. Fact. *If $\mathbb{A}$ is a compact Hausdorff Boolean algebra, then $\mathbb{A}$ is profinite.*

This surprising fact contributes to the following collapse of categories:

4.2.6. Fact. Pro- $\mathbf{BA}_f \cong \mathbf{BoolBA} \cong \mathbf{KHausBA} \cong \mathbf{CABA}$.

We combine the above facts in the following picture:

$$\begin{array}{ccc}
\mathbf{CABA} & \simeq & \mathbf{Set}^{op} \\
\cong & & \\
\mathbf{KHausBA} & & \\
\cong & & \\
\mathbf{BoolBA} & & \\
\cong & & \\
\text{Pro-}\,\mathbf{BA}_f & &
\end{array}$$

For further discussion of these dualities we refer the reader to [54, §VII-1.16].

The relation between the canonical extension and the profininte completion of a given Boolean algebra $\mathbb{A} = \langle A, \wedge, \vee, \neg, 0, 1\rangle$ is the same as in the case of distributive lattices: the map $\nu_{\mathbb{A}} : \mathbb{A}^\delta \to \hat{\mathbb{A}}$ from Theorem 3.4.1 is an isomorphism, so that $\mathbb{A}^\delta \simeq \hat{\mathbb{A}}$, and $\mathbb{A}^\delta \simeq (\mathbb{A}_\bullet)^+$, i.e. the canonical extension of $\mathbb{A}$ is isomorphic to the complex algebra of the set of ultrafilters of $\mathbb{A}$. Indeed, the embedding of $\mathbb{A}$ into $(\mathbb{A}_\bullet)^+$ is what canonical extensions were defined to describe by Jónsson and Tarski [58].

4.2.3 Heyting algebras

For Heyting algebras, the discrete duality between $\mathbf{DLO}^+$ and $\mathbf{OKFr}$ boils down to a duality between $\mathbf{HA}^+$, the category of complete bi-algebraic Heyting algebras and complete homomorphisms, and $\mathbf{IntKFr}$, the category of intuitionistic Kripke frames, which consists of partially ordered sets and bounded morphisms. A bounded morphism between intuitionistic Kripke frames $f : \mathfrak{F} \to \mathfrak{G}$, where $\mathfrak{F} = \langle X_{\mathfrak{F}}, \leq_{\mathfrak{F}}\rangle$ and $\mathfrak{G} = \langle X_{\mathfrak{G}}, \leq_{\mathfrak{G}}\rangle$, is a function $f : X_{\mathfrak{F}} \to X_{\mathfrak{G}}$ such that

$$\uparrow_{\mathfrak{G}} f(x) = f[\uparrow_{\mathfrak{F}} x],$$

for all $x \in X_{\mathfrak{F}}$. One can easily show that every bounded morphism between intuitionistic Kripke frames is order-preserving; consequently $\mathbf{IntKFr}$ is a (non-full!) subcategory of $\mathbf{Pos}$. The duality between $\mathbf{HA}^+$ and $\mathbf{IntKFr}$ is obtained by simply restricting the duality between $\mathbf{DL}^+$ and $\mathbf{Pos}$.

4.2.7. FACT. $\mathbf{HA}^+ \simeq \mathbf{IntKFr}^{op}$.

It is hard to pin down the origin of Fact 4.2.7. One early source is de Jongh & Troelstra [30]; for more categorical detail see G. Bezhanishvili [13]. As far as topological Heyting algebras are concerned, the following is known.

4.2.8. FACT. *Let $\mathbb{A}$ be a Heyting algebra. The following are equivalent:*

1. *$\mathbb{A}$ is a Boolean topological algebra;*
2. *$\mathbb{A}$ is profinite;*
3. *$\mathbb{A}$ is complete, bi-algebraic and residually finite.*

Proof The equivalence of (1) and (2) was established by Johnstone [54, Prop. VI-2.10]. The equivalence of (2) and (3) was established by G. & N. Bezhanishvili [14]. ∎

Hereditarily finite intuitionistic Kripke frames are those frames $\mathfrak{F}$ such that for all $x \in \mathfrak{F}$, $\uparrow x$ is finite. We denote the category of hereditarily finite intuitionistic Kripke frames by $\mathbf{H}_\omega\mathbf{IntKFr}$. The following result, which can be seen as a specialization of our Theorem 4.1.18, is due to G. & N. Bezhanishvili [14].

4.2.9. FACT. $\text{Pro-}\mathbf{HA}_f \simeq \mathbf{H}_\omega\mathbf{IntKFr}^{op}$.

We summarize the dualities for Heyting algebras we have discussed in the diagram below.

$$\begin{array}{ccc}
\mathbf{HA}^+ & \simeq & \mathbf{IntKFr}^{op} \\
\cup\!\mid & & \\
\mathbf{BoolHA} & & \cup\!\mid \\
\wr\!\mid\!\mid & & \\
\text{Pro-}\mathbf{HA}_f & \simeq & \mathbf{H}_\omega\mathbf{IntKFr}^{op}
\end{array}$$

4.2.10. REMARK. We have seen that for distributive lattices and Boolean algebras, there is no difference between semi-topological algebras and topological algebras. We now have our first example of a category of DLO's for which not every semi-topological DLO is a Boolean topological DLO; we will demonstrate this using duality. Observe that the inclusion $\mathbf{H}_\omega\mathbf{IntKFr}^{op} \subseteq \mathbf{IntKFr}^{op}$ is strict: consider the poset $\mathfrak{F} = \langle \mathbb{N}, \leq \rangle$ consisting of the natural numbers with their usual ordering. The poset $\mathfrak{F}$ is an intuitionistic Kripke frame, however, for all $x \in \mathfrak{F}$, $\uparrow x$ is infinite, so $\mathfrak{F}$ is a fortiori not hereditarily finite. Now since the inclusion $\mathbf{H}_\omega\mathbf{IntKFr}^{op} \subseteq \mathbf{IntKFr}^{op}$ is strict, the inclusion $\mathbf{BoolHA} \subseteq \mathbf{HA}^+$ is necessarily also strict.

Unlike the situation with distributive lattices and Boolean algebras, profinite completions of Heyting algebras do not always coincide with canonical extensions [15, Th. 5.2]; consequently, given a Heyting algebra $\mathbb{A}$, we are faced with the general picture:

$$\begin{tikzcd}
\mathbb{A} \arrow[dr, "e_{\mathbb{A}}"] \arrow[ddr, bend right, "\mu_{\mathbb{A}}"'] & & & & \\
& \mathbb{A}^{\delta} \arrow[d, "\nu_{\mathbb{A}}"] & \simeq & (\mathbb{A}_\bullet)^+ \arrow[d] & \mathbb{A}_\bullet \arrow[d, phantom, "\rotatebox{90}{$\sqsubseteq$}"] \\
& \hat{\mathbb{A}} & \simeq & (H_\omega \mathbb{A}_\bullet)^+ & H_\omega \mathbb{A}_\bullet
\end{tikzcd}$$

Here, $\mathbb{A}_\bullet$ is the prime filter frame of $\mathbb{A}$, and $H_\omega \mathbb{A}_\bullet$ is the generated subframe of $\mathbb{A}_\bullet$ induced by the set of points $x \in \mathbb{A}_\bullet$ such that $\uparrow x$ is finite. These facts, which can again be seen as a specialization of results from §4.1, were first established by G. Bezhanishvili et al. [15].

4.3 Duality for topological Boolean algebras with operators

In the previous section, we considered topological algebras in three categories of distributive lattices with operators: distributive lattices, Boolean algebras and Heyting algebras. In all three of these categories, every Boolean topological algebra is profinite. In this final section of this chapter, we will discuss the category of Boolean algebras with operators (BAO's). This category has the interesting property that *not* every Boolean topological algebra is profinite. Moreover, we can characterize Boolean topological BAO's via duality: every Boolean topological BAO is the dual of an *image-finite Kripke frame.*

Boolean algebras with operators are the original class of lattice-based algebras for which canonical extensions were defined by Jónsson and Tarksi [58]. They arise naturally as an algebraic semantics for various modal logics [19] and also more generally in algebraic logic, e.g. as relation algebras [59]. We know from §4.1 that profinite BAO's correspond to hereditarily finite Kripke frames. We will see that Boolean topological BAO's correspond to image-finite Kripke frames, i.e. frames in which each point has finitely many successors. From a coalgebraic viewpoint (see Chapter 5), image-finite Kripke frames are a very natural class of frames: they are coalgebras for the finite powerset functor.

This section is organized as follows. In §4.3.1, we establish the main result of this section, namely the duality between Boolean topological BAO's and image-finite Kripke frames, and we briefly discuss a few examples. Next, in §4.3.2, we take a closer look at the folk result that a Kripke frame $\mathfrak{F}$ can be embedded in its ultrafilter extension iff $\mathfrak{F}$ is image-finite, and we show how this result can be interpreted using duality.

4.3.1 Duality for Boolean topological BAO's

In this subsection we will show that given a BAO $\mathbb{A}$, the following three things are equivalent: (1) $\mathbb{A}$ is a Boolean topological algebra; (2) $\mathbb{A}$ is a compact Hausdorff algebra; (3) $\mathbb{A}_+$, the discrete dual of $\mathbb{A}$, is an image-finite frame. Given the facts and results we have previously stated, the proof of this result is very simple. We begin with some conventions and definitions. A distinguishing property of BAO's is that operators and dual operators are interdefinable. If $\mathbb{A} = \langle A, \wedge, \vee, \neg, 0, 1, \Diamond_{\mathbb{A}}\rangle$ is a BAO with a single normal operator $\Diamond_{\mathbb{A}} \colon \mathbb{A}^m \to \mathbb{A}$, then

$$\Box_{\mathbb{A}} \colon (a_1, \ldots, a_m) \mapsto \neg\Diamond_{\mathbb{A}}(\neg a_1, \ldots, \neg a_m)$$

is a dual normal operator. For this reason, we will afford ourselves the convenience of only considering BAO's $\mathbb{A} = \langle A, \wedge, \vee, \neg, 0, 1, \Box_{\mathbb{A}}\rangle$ where $\Box_{\mathbb{A}}$ is an m-ary dual normal operator.[1]

4.3.1. Definition. The category of semi-topological BAO's $\mathbf{BAO}^+$ consists of complete atomic Boolean algebras with operators $\mathbb{A} = \langle A, \wedge, \vee, \neg, 0, 1, \Box_{\mathbb{A}}\rangle$, where $\Box_{\mathbb{A}}$ is an m-ary complete dual normal operator, and complete BAO homomorphisms. The category of Kripke frames **KFr** consists of structures $\mathfrak{F} = \langle X, R\rangle$, where $R \subseteq X \times X^m$ is an $(m+1)$-ary relation. A morphism of Kripke frames $f \colon \mathfrak{F} \to \mathfrak{G}$ is a map satisfying the back and forth conditions of Definition 4.1.10, or equivalently, such that

$$f^m[R[x]] = R[f(x)]$$

for all $x \in \mathfrak{F}$.

Now that we have made it more precise what we mean by BAO's and Kripke frames, it is time for us to introduce image-finite Kripke frames.

4.3.2. Definition. By $\mathbf{Im}_\omega\mathbf{KFr}$ we denote the full subcategory of **KFr** whose objects are all *image-finite* Kripke frames. A frame $\mathfrak{F} = \langle X, R\rangle$ is image-finite (or finitely branching) if for all $x \in \mathfrak{F}$, $R[x]$ is finite.

4.3.3. Example. Recall that a frame $\mathfrak{F}$ is hereditarily finite if for every $x \in \mathfrak{F}$, there exists a finite generated subframe $\mathfrak{F}' \sqsubseteq \mathfrak{F}$ such that $x \in \mathfrak{F}'$. Not every image-finite frame is hereditarily finite: consider the frame $\mathfrak{F} := \langle \mathbb{N}, S\rangle$, where S is the successor relation: $x\ S\ y$ iff $y = x + 1$. The frame $\mathfrak{F}$ is image-finite, since for every $x \in \mathfrak{F}$, $S[x] = \{x + 1\}$ is a finite set. However, it is not hard to show that if $\mathfrak{F}' \sqsubseteq \mathfrak{F}$, then $\mathfrak{F}'$ must be infinite.

[1] We might as well have chosen to take an operator rather than a dual operator as primary. Our choice is a matter of taste.

We now arrive at the key lemma which will allow us to prove the duality result for Boolean topological DLO's and image-finite frames.

4.3.4. Lemma. *Let $\mathfrak{F} = \langle X, R\rangle$, with $R \subseteq X^{m+1}$, be a Kripke frame. Then $\mathfrak{F}$ is image-finite iff $[R]\colon \mathcal{P}(W)^m \to \mathcal{P}(W)$ is Scott-continuous.*

Proof For the sake of simplicity we will only treat the case where $m = 2$. Recall from §A.8 that for all $U, V \in \mathcal{P}(X)$,

$$[R](U, V) := \{x \in X \mid R[x] \subseteq U \times X \cup X \times V\}.$$

Suppose that $\langle X, R\rangle$ is image-finite; we will show that $[R]\colon \mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X)$ is Scott-continuous. By Fact A.3.4, it suffices to show that $[R]$ is Scott-continuous in each coordinate. Now suppose that $\{U_i \mid i \in I\} \subseteq \mathcal{P}(X)$ is a directed collection of sets and $V \in \mathcal{P}(X)$ is an arbitrary set, and take $x \in [R](\bigcup_I U_i, V)$. We will show that $x \in \bigcup_I [R](U_i, V)$; since $[R]$ is order-preserving this is sufficient to show that $[R]$ preserves directed joins in its first coordinate. Now since $x \in [R](\bigcup_I U_i, V)$, it follows that

$$\begin{aligned}
R[x] &\subseteq (\textstyle\bigcup_I U_i) \times X \cup X \times V && \text{by definition of } [R],\\
&= \textstyle\bigcup_I (U_i \times X) \cup X \times V && \text{by elementary set theory,}\\
&= \textstyle\bigcup_I (U_i \times X \cup X \times V) && \text{idem.}
\end{aligned}$$

Now since $\{U_i \mid i \in I\} \subseteq \mathcal{P}(X)$ is directed, so is $\{U_i \times X \cup X \times V \mid i \in I\}$. Since $R[x]$ is finite by assumption, there must be some $j \in I$ such that

$$R[x] \subseteq U_j \times X \cup X \times V.$$

Now we see that $x \in [R](U_j, V)$, so consequently $x \in \bigcup_I [R](U_i, V)$. Since x was arbitrary it follows that

$$[R](\textstyle\bigcup_I U_i, V) \subseteq \bigcup_I [R](U_i, V).$$

Since $\{U_i \mid i \in I\}$ was arbitrary, it follows that $[R]$ preserves directed joins in its first coordinate. An analogous argument shows that this also holds for the second coordinate; we conclude that $[R]$ is Scott-continuous.

For the converse, suppose that $[R]$ is Scott-continuous. Take $x \in X$; we want to show that $R[x]$ is finite. It is easy to verify that $[R](X, \emptyset) = X$. Let

$$\mathcal{S} := \{U \subseteq X \mid X \text{ finite}\},$$

then $X = \bigcup \mathcal{S}$. It follows that $x \in [R](\bigcup \mathcal{S}, \emptyset)$. Now since $[R]$ is Scott-continuous, there must be some $U_1 \in \mathcal{S}$ such that $x \in [R](U_1, \emptyset)$, so that

$$R[x] \subseteq U_1 \times X \cup X \times \emptyset = U_1 \times X.$$

An analogous argument shows that there must be a $U_2 \in \mathcal{S}$ such that $R[x] \subseteq X \times U_2$. But now we see that

$$R[x] \subseteq U_1 \times X \cap X \times U_2 = U_1 \times U_2,$$

and since both U_1 and U_2 are finite, it follows that $R[x]$ must be finite. Since $x \in X$ was arbitrary it follows that $\langle X, R\rangle$ is image-finite. ∎

The lemma above equates being image-finite with a continuity property; all that is left for us to do is to show that this continuity property precisely characterizes Boolean topological BAO's.

4.3.5. Theorem. **KHausBAO** $\cong$ **BoolBAO** $\simeq$ $\mathbf{Im}_\omega\mathbf{KFr}^{op}$.

Proof To see why **KHausBAO** $\cong$ **BoolBAO**, first observe that **BoolBAO** is a full subcategory of **KHausBAO** because every Boolean topological BAO is necessarily a compact Hausdorff BAO. Moreover, if $\mathbb{A} = \langle A, \wedge, \vee, \neg, 0, 1, \Box_\mathbb{A}\rangle$ is a compact Hausdorff BAO, then its Boolean algebra reduct $\langle A, \wedge, \vee, \neg, 0, 1\rangle$ is a compact Hausdorff Boolean algebra. By Fact 4.2.5, $\langle A, \wedge, \vee, \neg, 0, 1\rangle$ is a profinite Boolean algebra; consequently the bi-Scott topology on $\mathbb{A}$ is a Boolean topology, so that $\mathbb{A}$ is a Booleaen topological BAO. Since $\mathbb{A}$ was arbitrary it follows that each compact Hausdorff BAO is a Boolean topological BAO.

We will now show that **BoolBAO** $\simeq$ $\mathbf{Im}_\omega\mathbf{KFr}^{op}$. Since **BoolBAO** is a full subcategory of $\mathbf{BAO}^+$, and since $\mathbf{Im}_\omega\mathbf{KFr}$ is a full subcategory of **KFr**, it suffices to show that $(\cdot)_+$ maps objects of **BoolBAO** to objects of $\mathbf{Im}_\omega\mathbf{KFr}$, and conversely that $(\cdot)^+$ maps objects of $\mathbf{Im}_\omega\mathbf{KFr}$ to objects of **BoolBAO**. First, suppose that $\mathbb{A} = \langle A, \wedge, \vee, \neg, 0, 1, \Box_\mathbb{A}\rangle$ is a Boolean topological BAO. Then $\Box_\mathbb{A} \colon \mathbb{A}^m \to \mathbb{A}$ must be (σ, σ)-continuous; since $\Box_\mathbb{A}$ is order-preserving, it follows by Lemma 2.1.17 that $\Box_\mathbb{A}$ is $(\sigma^\uparrow, \sigma^\uparrow)$-continuous. Now by Lemma 4.3.4, $\mathbb{A}_+$ is image-finite.

Next, consider an image-finite Kripke frame $\mathfrak{F} = \langle X, R\rangle$. By duality, $\mathfrak{F}^+$ is a semi-topological BAO, so the Boolean reduct of $\mathfrak{F}^+$ is a profinite Boolean algebra and $[R] \colon \mathfrak{F}^+ \to \mathfrak{F}^+$ is $(\sigma^\downarrow, \sigma^\downarrow)$-continuous by Lemma 4.1.6. Now since we assumed that $\mathfrak{F}$ is image-finite, it follows by Lemma 4.3.4 that $[R]$ is $(\sigma^\uparrow, \sigma^\uparrow)$-continuous; consequently, by Lemma 2.1.17, $[R] \colon \mathfrak{F}^+ \to \mathfrak{F}^+$ is (σ, σ)-continuous, so that $\mathfrak{F}^+$ is indeed a Boolean topological algebra. This concludes our proof. ∎

We can now draw the following diagram to present the dualities for topological BAO's.

$$\begin{array}{ccc}
\mathbf{BAO}^{+} & \simeq & \mathbf{KFr}^{op} \\
\cup\mid & & \\
\mathbf{KHausBAO} & & \cup\mid \\
\wr\mid\mid & & \\
\mathbf{BoolBAO} & \simeq & \mathbf{Im}_{\omega}\mathbf{KFr}^{op} \\
\cup\mid & & \cup\mid \\
\text{Pro-}\,\mathbf{BAO}_{f} & \simeq & \mathbf{H}_{\omega}\mathbf{KFr}^{op}
\end{array}$$

4.3.6. Remark. Both inclusions $\mathbf{H}_{\omega}\mathbf{KFr}^{op} \subseteq \mathbf{Im}_{\omega}\mathbf{KFr}^{op} \subseteq \mathbf{KFr}^{op}$ are strict; consequently, the inclusions Pro- $\mathbf{BAO}_{f} \subseteq \mathbf{BoolBAO}$ and $\mathbf{KHausBAO} \subseteq \mathbf{BAO}^{+}$ are also strict.

To see why the inclusions $\mathbf{H}_{\omega}\mathbf{KFr}^{op} \subseteq \mathbf{Im}_{\omega}\mathbf{KFr}^{op} \subseteq \mathbf{KFr}^{op}$ are strict, consider the frames $\mathfrak{F} := \langle \mathbb{N}, \leq \rangle$ and $\mathfrak{G} := \langle \mathbb{N}, S \rangle$, where xSy iff $y = x + 1$. It is not hard to see that $\mathfrak{F}$ is not image-finite (cf. Example 4.3.3); it is also not hard to see that $\mathfrak{G}$ is image-finite but not hereditarily finite.

4.3.7. Remark. Recall from §3.1.1 that for general reasons, there exists a *compactification functor* $\beta\colon \mathbf{BAO} \to \mathbf{KHausBAO}$. Since $\mathbf{KHausBAO}$ forms a subcategory of $\mathbf{BAO}$, $\beta\colon \mathbf{BAO} \to \mathbf{KHausBAO}$ is a reflector from $\mathbf{BAO}$ to $\mathbf{KHausBAO}$. The behaviour of the compactification functor for BAO's is similar to that of the profinite completion. For instance, it is a corollary of Theorem 3.4.14 that for each BAO $\mathbb{A}$, there exists a unique map $h\colon \mathbb{A}^{\delta} \to \beta\mathbb{A}$ such that $h \circ e_{\mathbb{A}} = \eta_{\mathbb{A}}$, where $\eta_{\mathbb{A}}\colon \mathbb{A} \to \beta\mathbb{A}$ is the natural map from $\mathbb{A}$ to $\beta\mathbb{A}$. The compactification $\beta\mathbb{A}$ can also be described using duality, analogously to Theorem 4.1.27, using the fact that $\mathbf{Im}_{\omega}\mathbf{KFr}$ is a co-reflective subcategory of $\mathbf{KFr}$.

4.3.8. Example. Let n be a natural number and define

$$\mathrm{alt}_n := \bigvee_{0 \leq j \leq n} \Box \Big((\textstyle\bigwedge_{0 \leq i < j} p_i) \to p_j \Big) .$$

An alternative, equivalent form of this axiom is:

$$\bigwedge_{0 \leq i \leq n} \Diamond p_i \to \bigvee_{0 \leq i < j \leq n} \Diamond (p_i \wedge p_j).$$

It is known that if $\mathbb{A}$ is a BAO such that $\mathbb{A} \models \mathrm{alt}_n$, then $\mathbb{A}^{\delta} \in \mathrm{HSP}(\mathbb{A})$ [11], also see the discussion of 'logics of bounded alternativity' in [64]. We will show how this result can be understood in terms of compact Hausdorff BAO's.

Assume that $\mathbb{A} \models \mathrm{alt}_n$; then it is known (cf. [11]) that for each $x \in \mathbb{A}_{\bullet}$, $|R[x]| \leq n$. A fortiori, $\mathbb{A}_{\bullet}$ is image-finite, so $\mathbb{A}^{\delta} = (\mathbb{A}_{\bullet})^{+}$ is a compact Hausdorff BAO by

Theorem 4.3.5. We will argue that in fact, $e_{\mathbb{A}} \colon \mathbb{A} \to \mathbb{A}^\delta$ is the compactification of $\mathbb{A}$. To establish this fact, it suffices to show that for every homomorphism $f \colon \mathbb{A} \to \mathbb{B}$, where $\mathbb{B}$ is a compact Hausdorff modal algebra, there exists a unique continuous $f' \colon \mathbb{A}^\delta \to \mathbb{B}$ such that $f' \circ e_{\mathbb{A}} = f$. This is indeed the case: by Theorem 4.3.5, $\mathbb{B}$ is a Boolean topological algebra, so by Theorem 3.4.14, there exists a unique continuous $f' \colon \mathbb{A}^\delta \to \mathbb{B}$ such that $f' \circ e_{\mathbb{A}} = f$. Since $f \colon \mathbb{A} \to \mathbb{B}$ was arbitrary, we conclude that $e_{\mathbb{A}} \colon \mathbb{A} \to \mathbb{A}^\delta$ is indeed the compactification of $\mathbb{A}$. Now by Corollary 3.1.7, $\mathbb{A}^\delta \in \mathrm{HSP}(\mathbb{A})$.

4.3.2 Ultrafilter extensions of image-finite Kripke frames

The ultrafilter extension of a Kripke frame is an important construction in the model theory of modal logic [19]; for instance, it used to state the Goldblatt-Thomason theorem. In this subsection we will investigate a folklore result, namely that a Kripke frame $\mathfrak{F}$ can be embedded in its ultrafilter extension if and only if $\mathfrak{F}$ is image-finite; we will see that this result is essentially a corollary of Theorem 3.4.16.

4.3.9. Definition. Given a frame $\mathfrak{F} = \langle W, R \rangle$, we define

$$\mathfrak{ue}\,\mathfrak{F} := (\mathfrak{F}^+)_\bullet.$$

There is a natural map $i_{\mathfrak{F}} \colon \mathfrak{F} \to \mathfrak{ue}\,\mathfrak{F}$, defined by

$$i_{\mathfrak{F}} \colon x \mapsto \{W \subseteq X \mid x \in W\},$$

i.e. sending $x \in X$ to the principal ultrafilter over X generated by x.

One might ask if $i_{\mathfrak{F}} \colon \mathfrak{F} \to \mathfrak{ue}\,\mathfrak{F}$ is a bounded morphism; we will see in Theorem 4.3.11 that this is not always the case. Before we can show why, we need to explain the connection between ultrafilter extensions of Kripke frames and canonical extensions of BAO's. Observe that

$$\begin{aligned} (\mathfrak{ue}\,\mathfrak{F})^+ &= \big((\mathfrak{F}^+)_\bullet\big)^+ && \text{by definition of } \mathfrak{ue}, \\ &= \big(\mathfrak{F}^+\big)^\delta && \text{by Convention 4.1.23,} \end{aligned}$$

where the BAO $\mathfrak{F}^+$ is embedded in $(\mathfrak{ue}\,\mathfrak{F})^+$ via

$$e_{\mathfrak{F}^+} \colon W \mapsto \{F \in \mathfrak{ue}\,\mathfrak{F} \mid W \in F\}. \tag{4.3}$$

We will now show that there is a natural relation between the two maps $i_{\mathfrak{F}} \colon \mathfrak{F} \to \mathfrak{ue}\,\mathfrak{F}$ and $e_{\mathfrak{F}^+} \colon \mathfrak{F}^+ \to (\mathfrak{ue}\,\mathfrak{F})^+$. Recall that $\mathbb{A}^l$ is the lattice reduct of $\mathbb{A}$, and that if $\mathbb{A}^l$ is profinite then by Corollary 3.4.2, there exists a unique complete lattice homomorphism $g \colon (\mathbb{A}^\delta)^l \to \mathbb{A}^l$ such that $g \circ e_{\mathbb{A}} = \mathrm{id}_{\mathbb{A}}$.

4.3.10. Lemma. *Let $\mathfrak{F}$ be a Kripke frame. Then*

$$(i_{\mathfrak{F}})^+ \colon \left((\mathfrak{ue}\,\mathfrak{F})^+\right)^l \to \left(\mathfrak{F}^+\right)^l$$

is a complete Boolean algebra homomorphism and $(i_{\mathfrak{F}})^+ \circ e_{\mathfrak{F}^+} = \mathrm{id}_{\mathfrak{F}^+}$.

Proof Since the duality between semi-topological BAO's and Kripke frames is an extension of the duality between complete Boolean algebras and sets, it follows that the (set) *function* $i_{\mathfrak{F}} \colon \mathfrak{F} \to \mathfrak{ue}\,\mathfrak{F}$ corresponds to a complete Boolean algebra homomorphism

$$(i_{\mathfrak{F}})^+ \colon \left((\mathfrak{ue}\,\mathfrak{F})^+\right)^l \to \left(\mathfrak{F}^+\right)^l.$$

Moreover, an easy calculation shows that $(i_{\mathfrak{F}})^+ \circ e_{\mathfrak{F}^+} = \mathrm{id}_{\mathfrak{F}^+}$. Let us denote elements of $(\mathfrak{ue}\,\mathfrak{F})^+$ by $\mathcal{W}, \mathcal{V}$ for the duration of this proof. Then since $(i_{\mathfrak{F}})^+ := (i_{\mathfrak{F}})^{-1}$, we see that

$$(i_{\mathfrak{F}})^+ \colon \mathcal{W} \mapsto \{x \in \mathfrak{F} \mid i_{\mathfrak{F}}(x) \in \mathcal{W}\}. \tag{4.4}$$

Now let $W \in \mathfrak{F}^+$, then we see that

$$\begin{array}{ll}
(i_{\mathfrak{F}})^+ \circ e_{\mathfrak{F}^+}(W) & \\
= (i_{\mathfrak{F}})^+ \left(\{F \in \mathfrak{ue}\,\mathfrak{F} \mid W \in F\}\right) & \text{by (4.3),} \\
= \{x \in \mathfrak{F} \mid i_{\mathfrak{F}}(x) \in \{F \in \mathfrak{ue}\,\mathfrak{F} \mid W \in F\}\} & \text{by (4.4),} \\
= \{x \in \mathfrak{F} \mid W \in i_{\mathfrak{F}}(x)\} & \text{by elementary set theory,} \\
= \{x \in \mathfrak{F} \mid x \in W\} & \text{by definition of } i_{\mathfrak{F}}, \\
= W. &
\end{array}$$

It follows that $(i_{\mathfrak{F}})^+ \circ e_{\mathfrak{F}^+} = \mathrm{id}_{\mathfrak{F}^+}$. ∎

We now arrive at the main result of this subsection.

4.3.11. Theorem. *Let $\mathfrak{F} = \langle W, R\rangle$ be a Kripke frame. The natural map $i_{\mathfrak{F}} \colon \mathfrak{F} \to \mathfrak{ue}\,\mathfrak{F}$ is a bounded morphism iff $\mathfrak{F}$ is image-finite.*

Proof Recall from our discussion above that $(\mathfrak{F}^+)^\delta = (\mathfrak{ue}\,\mathfrak{F})^+$. Now if $i_{\mathfrak{F}} \colon \mathfrak{F} \to \mathfrak{ue}\,\mathfrak{F}$ is a bounded morphism, then

$$(i_{\mathfrak{F}})^+ \colon (\mathfrak{ue}\,\mathfrak{F})^+ \to \mathfrak{F}^+$$

is a BAO homomorphism, and by Lemma 4.3.10, $(i_{\mathfrak{F}})^+$ is a complete lattice algebra homomorphism such that

$$(i_{\mathfrak{F}})^+ \circ e_{\mathfrak{F}^+} = \mathrm{id}_{\mathfrak{F}^+}.$$

It now follows from Theorem 3.4.16 that $\mathfrak{F}^+$ is a Boolean topological algebra, so by Theorem 4.3.5, $\mathfrak{F}$ is image-finite.

For the converse, assume that $\mathfrak{F}$ is image-finite. By Theorem 3.4.16, there exists a unique complete lattice homomorphism

$$g\colon \left((\mathfrak{ue}\,\mathfrak{F})^{+}\right)^{l} \to \left(\mathfrak{F}^{+}\right)^{l}$$

such that $e_{\mathfrak{F}^{+}} \circ g = \mathrm{id}_{\mathfrak{F}^{+}}$. By Lemma 4.3.10 and uniqueness of g, it must be the case that $g = (i_{\mathfrak{F}})^{+}$. Now since $\mathfrak{F}^{+}$ is a Boolean topological algebra by Theorem 4.3.5, $g\colon (\mathfrak{ue}\,\mathfrak{F})^{+} \to \mathfrak{F}^{+}$ is in fact a modal algeba homomorphism. Since we have already established that $g = (i_{\mathfrak{F}})^{+}$, it follows by duality that $i_{\mathfrak{F}}$ is a bounded morphism. ∎

4.3.3 Conclusions and further work

In §4.3.1, we showed that every compact Hausdorff BAO is a Boolean topological BAO, and that these BAO's correspond to image-finite Kripke frames via discrete duality (Theorem 4.3.5). One of the key insights for this result was provided by Lemma 4.3.4, which seems to be folklore. Theorem 4.3.11, which states that a Kripke frame $\mathfrak{F}$ embeds in $\mathfrak{ue}\,\mathfrak{F}$ iff $\mathfrak{F}$ is image finite, also seems to be a folklore result. However, the proof we present, using duality for image-finite frames, and reducing the question to whether $\mathfrak{F}^{+}$ is a retract of $(\mathfrak{F}^{+})^{\delta}$, is new.

Further work

With §4.3.1 we have provided a partial answer to the question from §4.1.5 whether **BoolDLO** can be characterized in a meaningful way via duality: if we restrict our attention from DLO's to BAO's, then the answer is yes. The question whether **BoolDLO** can be characterized dually in a non-trivial way is still open however. (A trivial characterization would be to say that Boolean topological DLO's correspond to ordered Kripke frames $\mathfrak{F} = \langle X, \leq, R, Q\rangle$ such that $\langle R\rangle$ preserves co-directed intersections of lower sets, and $[Q]$ preserves directed unions of lower sets.)

Another interesting question, which is related to modal logic, is to see whether one can prove without using duality that if $\mathbb{A}$ is a modal algebra such that $\mathbb{A} \models \psi_n$, then $\mathbb{A}^{\delta}$ is a compact Hausdorff algebra, cf. Example 4.3.8.

Chapter 5

Coalgebraic modal logic in point-free topology

5.1 Introduction

In this chapter we will show how powerlocales, a construction in point-free topology, can be understood and studied via coalgebraic modal logic.

Hyperspaces and powerlocales

The *Vietoris hyperspace construction* is a topological construction on compact Hausdorff spaces, which was introduced in 1922 by L. Vietoris [95] as a generalization of the Hausdorff metric. Given a topological space X one defines a new topology τ_X on $\mathrm{K}\,X$, the set of compact subsets of X. This new topology τ_X has as its basis all sets of the form

$$\nabla\{U_1,\ldots,U_n\} := \{F \in \mathrm{K}\,X \mid F \subseteq \bigcup_{i=1}^{n} U_i \text{ and } \forall i \leq n,\ F \between U_i\},$$

where $U_1,\ldots,U_n \subseteq X$ is a finite collection of open sets and $F \between U$ is notation to indicate that $F \cap U \neq \emptyset$. Alternatively, one can use a subbasis to generate τ_X, consisting of subbasic open sets of the shape

$$\Box(U) := \{F \in \mathrm{K}\,X \mid F \subseteq U\},$$

and

$$\Diamond(U) := \{F \in \mathrm{K}\,X \mid F \between U\}.$$

To generate the basic open sets $\nabla\{U_1,\ldots,U_n\}$ from $\Box(U)$ and $\Diamond(U)$, one can use the following expression:

$$\nabla\{U_1,\ldots,U_n\} = \Box\left(\bigcup_{i=1}^{n} U_i\right) \cap \bigcap_{i=1}^{n} \Diamond(U_i).$$

In the field of *point-free topology*, a considerable amount of general topology has been recast in a way which makes it more compatible with cosntructive mathematics and topos theory. (Standard references are Johnstone [54] and Vickers [91]). The main idea is to study the lattices of open sets of topological spaces, rather than their associated sets of points. In other words, it is an approach to topology via algebra, where one studies categories of *locales* rather than of topological spaces. Locales have algebraic representations in the form of their associated *frames*. A frame is a complete lattice in which infinite joins distribute over finite meets. In accordance with the algebraic approach which is prevalent throughout this dissertation, we will only work with frames in this chapter. We will be using the word 'powerlocale' rather than 'powerframe' however, in order to remain consistent with standard terminology.

Johnstone defines a point-free, syntactic version of the Vietoris powerlocale, using $\Box(a)$ and $\Diamond(a)$. However he soon also introduces expressions of the shape

$$\Box(\bigvee A) \wedge \bigwedge_{b \in B} \Diamond(b),$$

where A and B are finite sets, which should remind the reader of the expression for $\nabla\{U_1, \ldots, U_n\}$ above. Nevertheless, the description of the Vietoris powerlocale using $\Box(a)$ and $\Diamond(a)$ is usually taken as primitive, and not without good reason: one can construct the Vietoris powerlocale by first constructing one-sided locales corresponding to the $\Box$-generators on the one hand and the $\Diamond$-generators on the other, and then joining these two one-sided powerlocales to obtain the Vietoris powerlocale [94]. The question remains however, if one can describe the Vietoris powerlocale directly in terms of its basic opens, corresponding to $\nabla\{U_1, \ldots, U_n\}$, rather than the subbasic opens $\Box(U)$ and $\Diamond(U)$. One of the main contributions of this chapter is to show that this is indeed possible.

The cover modality and coalgebraic modal logic

The reader may have noticed that the notation using $\Box$ and $\Diamond$ above is highly suggestive of modal logic; this is no coincidence. In Boolean-based modal logic, one can define a ∇-modality which is applied to finite *sets* of formulas. The ∇-modality then has the following semantics. If $\mathfrak{M} = \langle W, R, V \rangle$ is a Kripke model and α is a finite set of formulas, then for any state $w \in W$,

$$\begin{aligned} \mathfrak{M}, w \Vdash \nabla\alpha \text{ iff } & \forall a \in \alpha,\ \exists v \in R[w],\ \mathfrak{M}, v \Vdash a \text{ and} \\ & \forall v \in R[w],\ \exists a \in \alpha,\ \mathfrak{M}, v \Vdash a. \end{aligned}$$

In classical modal logic, the ∇-modality is equi-expressive with the $\Box$- and $\Diamond$-modalities, using the following translations:

$$\nabla\alpha \equiv \Box\left(\bigvee \alpha\right) \wedge \bigwedge_{a \in \alpha} \Diamond a,$$

and in the other direction, one can use

$$\Box a \equiv \nabla\{a\} \vee \nabla\emptyset, \text{ and } \Diamond a \equiv \nabla\{a, \top\}.$$

The ∇ was first introduced as a modality by Barwise & Moss [10] in the study of circularity and by Janin & Walukiewicz [53] in the study of the modal μ-calculus. It was in Moss [71] however that the ∇-modality stepped into the spotlight as a modality suitable for *coalgebraic* modal logic.

A T-coalgebra is simply a function $\sigma\colon X \to TX$, where X is the underlying set of states of the coalgebra, and a T-coalgebra morphism between coalgebras $\sigma\colon X \to TX$ and $\sigma\colon X' \to TX'$ is simply a function $f\colon X \to X'$ such that $Tf \circ \sigma = \sigma' \circ f$. Aczel [2] introduced T-coalgebras as a means to study transition systems. A natural example of such transition systems is provided by the *Kripke frames* and *Kripke models* used in the model theory of propositional modal logic: the category of Kripke frames and bounded morphisms is isomorphic to the category of P-coalgebras, where $P\colon \mathbf{Set} \to \mathbf{Set}$ is the covariant powerset functor. Universal coalgebra was later introduced by J. Rutten [78] as a theoretical framework for modelling behaviour of set-based transition systems, parametric in their transition functor $T\colon \mathbf{Set} \to \mathbf{Set}$.

Coalgebraic logics are designed and studied in order to reason formally about coalgebras; one of the main applications of this approach is the design of specification and verification languages for coalgebras, i.e. for transition systems. One influential approach to coalgebraic logic, known as *coalgebraic modal logic*, is to try and generalize properties of propositional modal logic, because the Kripke semantics of classical modal logic is coalgebra in disguise. The cover modality ∇ was introduced in coalgebraic modal logic by L. Moss [71], who made the fundamental observation that the modal semantics for ∇ can be described using the set-theoretical/categorical technique of *relation lifting*. The most recent sound and complete derivation system for coalgebraic ∇-logic, the *Carioca axiomatization*, was introduced through the collaborative efforts of Bílková, Palmigano & Venema [17] and Kupke, Kurze & Venema [66].

Contribution

The results in this chapter are all joint work with Yde Venema and Steve Vickers. We list some of the main contributions of this work:

- We introduce a *generalized* powerlocale construction, parametric in a transition functor T. The classical Vietoris powerlocale construction is an instantiation of the T-powerlocale, where we take $T = P_\omega$, the covariant finite power set functor.

- We show that the connection between the Vietoris construction and the cover modality, which was implicit in *semantic* form already from Vietoris's

1922 paper [95], can also be made explicit *syntactically* using coalgebraic modal logic. Our approach shows how to describe the Vietoris constructions syntactically using the ∇-expressions as primitives, rather than as expressions derived from $\Box$- and $\Diamond$-primitives, as it was introduced in [54]. This approach runs parallel to that of Kupke, Kurz & Venema who introduced the Boolean algebra version of the construction we apply to frames [66, 67].

- Additionally, we take first steps towards developing a *geometric* coalgebraic modal logic, i.e. a logic using finite conjunctions and infinite disjunctions. This is a step away from previous work on the Carioca axioms [73, 17, 66, 62, 67] where one only considered *finite disjunctions*. Possible future applications in coalgebra include the development of modal logics for coalgebras on compact Hausdorff spaces, rather than on discrete sets.

This chapter is organized in a more self-contained manner than the rest of this dissertation. In §5.2 we introduce preliminaries on category theory, relation lifting, frame presentations and the classical point-free presentation of the powerlocale, along with a new compactness proof. In §5.3 we introduce the T-powerlocale functor V_T. We then show that the P_ω-powerlocale is isomorphic to the classical Vietoris powerlocale and we demonstrate how one can extend natural transformations between transition functors to natural transformations between T-powerlocale functors. We conclude the section with a different presentation of T-powerlocales, which reveals that each element of a T-powerlocale has a disjunctive normal form. Finally in §5.4 we show several first preservation results: we show that V_T preserves regularity and zero-dimensionality, and the combination of zero-dimensionality and compactness.

5.2 Preliminaries

5.2.1 Basic mathematics

First we fix some mathematical notation and terminology, which will sometimes differ from that used previously in this dissertation.

Let $f : X \to X'$ be a function. Then the *graph* of f is the relation

$$Grf ::= \{(x, f(x)) \in X \times X' \mid x \in X\}$$

Given a relation $R \subseteq X \times X'$, we denote the *domain* and *range* of R by $\mathsf{dom}(R)$ and $\mathsf{rng}(R)$, respectively. Given subsets $Y \subseteq X$, $Y \subseteq X'$, the *restriction* of R to Y and Y' is given as

$$R{\restriction}_{Y \times Y'} ::= R \cap (Y \times Y').$$

The composition of two relations $R \subseteq X \times X'$ and $R' \subseteq X' \times X''$ is denoted by $R\,;R'$, whereas the composition of two functions $f : X \to X'$ and $f' : X' \to X''$ is denoted by $f' \circ f$. Thus, we have $Gr(f' \circ f) = Grf\ ;Grf'$.

We will denote by $P(X)$ and $P_\omega(X)$ the *power set* and *finite power set* of a given set X. The *diagonal on* X is the relation $\Delta_X = \{(x, x) \mid x \in X\}$. Given two sets X, Y we say that X and Y *overlap*, notation: $X \between Y$, if $X \cap Y$ is inhabited (that is, non-empty).

A *pre-order* is a pair (X, R) where R is a reflexive and transitive relation on X. Given such a pre-order we define the operations $\downarrow_{(X,R)}, \uparrow_{(X,R)} : PX \to PX$ by $\downarrow(Y) := \{x \in X \mid x\ R\ y \text{ for some } y \in Y\}$ and $\uparrow(Y) := \{x \in X \mid y\ R\ x \text{ for some } y \in Y\}$. If no confusion is likely, we will write $\downarrow_X$ or $\downarrow$ rather than $\downarrow_{(X,R)}$.

5.2.2 Category theory

We will assume familiarity with the basic notions from category theory discussed in §A.2, including those of categories, functors, natural transformations, and (co-) monads.

We let **Set** denote the category with sets as objects and functions as morphisms; endofunctors on this category will simply be called *set functors*. The most important set functor that we shall use is the covariant power set functor P, which is in fact (part) of a monad (P, μ, η), with $\eta_X : X \to P(X)$ denoting the singleton map $\eta_X : x \mapsto \{x\}$, and $\mu_X : PPX \to PX$ denoting union, $\mu_X(\mathcal{A}) := \bigcup \mathcal{A}$. The contravariant power set functor will be denoted by $\breve{P}$.

We will restrict our attention to set functors satisfying certain properties, of which the first one is crucial. In order to define it, we need to recall the notion of a (weak) pullback. Given two functions $f_0 : X_0 \to X$, $f_1 : X_1 \to X$, a *weak pullback* is a set P, together with two functions $p_i : P \to X_i$ such that $f_0 \circ p_0 = f_1 \circ p_1$, and in addition, for every triple (Q, q_0, q_1) also satisfying $f_0 \circ q_0 = f_1 \circ q_1$, there is an arrow $h : Q \to P$ such that $q_0 = h \circ p_0$ and $q_1 = h \circ p_0$, in a diagram:

$$
\begin{tikzcd}
Q \arrow[dr, dashed, "h"] \arrow[drr, bend left, "q_1"] \arrow[ddr, bend right, "q_0"'] & & \\
& P \arrow[r, "p_1"] \arrow[d, "p_0"] & X_1 \arrow[d, "f_1"] \\
& X_0 \arrow[r, "f_0"'] & X
\end{tikzcd}
$$

For (P, p_0, p_1) to be a *pullback*, we require in addition the arrow h to be unique.

A functor T *preserves weak pullbacks* if it transforms every weak pullback (P, p_0, p_1) for f_0 and f_1 into a weak pullback (TP, Tp_0, Tp_1) for Tf_0 and Tf_1. An equivalent characterization is to require T to *weakly preserve pullbacks*, that is, to turn pullbacks into weak pullbacks. In the next subsection we will see yet another, and motivating, characterization of this property.

The second property that we will impose on our set functors is that of standardness. Given two sets X and X' such that $X \subseteq X'$, let $\iota_{X,X'}$ denote the inclusion map from X into X'. A weak pullback-preserving set functor T is *standard* if it *preserves inclusions*, that is: $T\iota_{X,X'} = \iota_{TX,TX'}$ for every inclusion map $\iota_{X,X'}$.

5.2.1. REMARK. Unfortunately the definition of standardness is not uniform throughout the literature. Our definition of standardness is taken from Moss [71], while for instance Adámek & Trnková [4] have an additional condition involving so-called distinguished points. Fortunately, the two definitions are equivalent in case the functor preserves weak pullbacks, see Kupke [65, Lemma A.2.12].

The restriction to standard functors is not essential, since every set functor is 'almost standard' [4, Theorem III.4.5]: given an arbitrary set functor T, we may find a standard set functor T' such that the restriction of T and T' to all non-empty sets and non-empty functions are naturally isomorphic.

Finally, we shall require that our functors are determined by their behaviour on finite sets. Call a standard set functor T *finitary* if $TX = \bigcup\{TX' \mid X' \subseteq_\omega X\}$. Our focus on finitary functors is not so much a restriction as a convenient way to express the fact that we are interested in the *finitary version* of an arbitrary set functor, in the sense that P_ω is the finitary version of P. Generally, we may define, for a standard functor T, the functor T_ω that on objects X is defined by $T_\omega X = \bigcup\{TX' \mid X' \subseteq X\}$, while on arrows f we simply put $T_\omega f := Tf$.

5.2.2. CONVENTION. Throughout this chapter we will assume that T is a finitary, standard endofunctor on **Set** that preserves weak pullbacks.

Many set functors satisfy the conditions listed in Convention 5.2.2, guaranteeing a wide scope for the results in this chapter.

5.2.3. EXAMPLE. The identity functor *Id*, the finitary power set functor P_ω, and, for each set Q, the constant functor C_Q (given by $C_Q X = Q$ and $C_Q f = \mathrm{id}_Q$) are standard, finitary, and preserve weak pullbacks.

For a slightly more involved example, consider the finitary *multiset* functor M_ω. This functor takes a set X to the collection $M_\omega X$ of maps $\mu : X \to \mathbb{N}$ of finite support (that is, for which the set $Supp(\mu) := \{x \in X \mid \mu(x) > 0\}$ is finite), while its action on arrows is defined as follows. Given an arrow $f : X \to X'$ and a map $\mu \in M_\omega X$, we define $(M_\omega f)(\mu) : X' \to \mathbb{N}$ by putting

$$(M_\omega f)(\mu)(x') := \sum\{\mu(x) \mid f(x) = x'\}.$$

With this definition, the functor is not standard, but we may 'standardize' it by representing any map $\mu : X \to \mathbb{N}$ of finite support by its 'support graph' $\{(x, \mu x) \mid \mu x > 0\}$. As a variant of M_ω, consider the finitary probability functor D_ω, where $D_\omega X = \{\delta : X \to [0,1] \mid Supp(\delta) \text{ is finite and } \sum_{x\in X}\delta(x) = 1\}$, while the action of D_ω on arrows is just like that of M_ω.

Perhaps more importantly, the class of finitary, standard functors that preserve weak pullbacks is closed under the following operations: composition ($\circ$) , product ($\times$), co-product ($+$), and exponentiation with respect to some set D ($(\cdot)^D$). As a corollary, inductively define the following class $EKPF_\omega$ of *extended finitary Kripke polynomial functors*:

$$T ::= Id \mid P_\omega \mid C_Q \mid M_\omega \mid D_\omega \mid T_0 \circ T_1 \mid T_0 + T_1 \mid T_0 \times T_1 \mid T^D.$$

Then each extended Kripke polynomial functor falls in the scope of the work in this chapter.

As running examples in this chapter we will often take the *binary tree functor* $B = Id \times Id$, and the finitary power set functor P_ω.

An interesting result of standard functors is that they preserve finite intersections [4, Theorem III.4.6]: $T(X \cap Y) = TX \cap TY$. As a consequence, if T is finitary, for any object $\xi \in TX$ we may define

$$Base^T_X(\xi) := \bigcap \{X' \in P_\omega(X) \mid \xi \in TX'\},$$

and show that $Base^T_X(\xi)$ is the *smallest* set X such that $\xi \in TX$ [90]. In fact, the base maps provide a natural transformation $Base : T \to P_\omega$; for referencing we will mention this fact explicitly in the next section.

To facilitate the reasoning in this chapter, which will involve objects of various different types, we use a variable naming convention.

5.2.4. Convention. Let X be a set and let $T\colon \mathbf{Set} \to \mathbf{Set}$ be a functor. We use the following naming convention:

Set	Elements
X	$a, b, \ldots, x, y, \ldots$
TX	$\alpha, \beta, \ldots$
PX	$A, B, \ldots$
PTX	$\Gamma, \Delta, \ldots$
TPX	$\Phi, \Psi, \ldots$

5.2.3 Relation lifting

In §5.1, we mentioned that coalgebraic modal logic using the cover modality, as introduced by Moss, crucially uses relation lifting, both for its syntax and semantics. Relation lifting is a technique which allows one to extend a functor $T\colon \mathbf{Set} \to \mathbf{Set}$ defined on the category of sets (satisfying the conditions of Convention 5.2.2) to a functor $\overline{T}\colon \mathbf{Rel} \to \mathbf{Rel}$ on the category of sets and relations in a natural way. In this subsection we will introduce some of the basic facts and definitions about relation lifting.

Let T be a set functor. Given two sets X and X', and a binary relation R between X and X', we define the *lifted relation* $\overline{T}(R) \subseteq TX \times TX'$ as follows:

$$\overline{T}(R) := \{((T\pi)(\rho), (T\pi')(\rho)) \mid \rho \in TR\},$$

where $\pi : R \to X$ and $\pi' : R \to X'$ are the projection functions given by $\pi(x, x') = x$ and $\pi'(x, x') = x'$. In a diagram:

$$X \xleftarrow{\ \pi\ } R \xrightarrow{\ \pi'\ } X'$$

$$\begin{tikzcd}
TX & TR \arrow[l, "T\pi"'] \arrow[r, "T\pi'"] \arrow[d, two heads] \arrow[dd, bend right, "\langle T\pi, T\pi' \rangle"'] & TX' \\
 & \overline{T}R \arrow[d, hook] & \\
 & TX \times TX' \arrow[uul] \arrow[uur] &
\end{tikzcd}$$

In other words, we apply the functor T to the relation R, seen as a *span* $X \xleftarrow{\pi} R \xrightarrow{\pi'} X'$, and define $\overline{T}R$ as the image of TR under the product map $\langle T\pi, T\pi' \rangle$ obtained from the lifted projection maps $T\pi$ and $T\pi'$.

Let us first see some concrete examples.

5.2.5. Example. Fix a relation $R \subseteq X \times X'$. For the identity and constanct functors, we find, respectively:

$$\begin{aligned} \overline{Id}R &= R \\ \overline{C_Q}R &= \Delta_Q. \end{aligned}$$

The relation lifting associated with the power set functor P can be defined concretely as follows:

$$\overline{P}R = \{(A, A') \in PX \times PX' \mid \forall a \in A\, \exists a' \in A'. aRa' \text{ and } \forall a' \in A'\, \exists a \in A. aRa'\}.$$

This relation is known under many names, of which we mention that of the *Egli-Milner* lifting of R. For any standard, weak pullback preserving functor T it can be shown that the lifting of T_ω agrees with that of T, in the sense that $\overline{T_\omega}R = TR \cap (T_\omega X \times T_\omega X')$. From this it follows that

$$\text{for all } A \in T_\omega X, A' \in T_\omega X' : A\ \overline{P_\omega}R\ A' \text{ iff } A\ \overline{P}R\ A',$$

and for this reason, we shall write $\overline{P}R$ rather than $\overline{P_\omega}R$.

Relation lifting for the finitary multiset functor is slightly more involved: given two maps $\mu \in M_\omega X, \mu' \in M_\omega X'$, we put $\mu\ \overline{M_\omega}R\ \mu'$ iff

$$\begin{aligned} \text{there is some map } \rho : R \to \mathbb{N} \text{ such that } & \forall x \in X. \textstyle\sum\{\rho(x, x') \mid x' \in X'\} = 1 \\ \text{and } & \forall x' \in X'. \textstyle\sum\{\rho(x, x') \mid x \in X\} = 1. \end{aligned}$$

The definition of $\overline{D_\omega}$ is similar.

Finally, relation lifting interacts well with various operations on functors [52]. In particular, we have

$$\begin{array}{rcl}
\overline{T_0 \circ T_1}R & = & \overline{T}_0(\overline{T}_1 R) \\
\overline{T_0 + T_1}R & = & \overline{T}_0 R \cup \overline{T}_1 R \\
\overline{T_0 \times T_1}R & = & \big\{((\xi_0, \xi_1), (\xi_0', \xi_1')) \mid (\xi_i, \xi_i') \in \overline{T}_i, \text{ for } i \in \{0,1\}\big\}\,. \\
\overline{T^D}R & = & \{(\varphi, \varphi') \mid (\varphi(d), \varphi'(d) \in \overline{T}R \text{ for all } d \in D\}
\end{array}$$

5.2.6. Remark. Strictly speaking, the definition of the relation lifting of a given relation R depends on the type of the relation, i.e. given sets X, X', Y, Y' such that $R \subseteq X \times X'$ and $R \subseteq Y \times Y'$, it matters whether we look at R as a relation from X to X' or as a relation from Y to Y'. We have avoided this potential source of ambiguity by requiring the functor T to be *standard*, see also Fact 5.2.7(6).

Relation lifting has a number of properties that we will use throughout the chapter. It can be shown that relation lifting interacts well with the operation of taking the graph of a function $f : X \to X'$, and with most operations on binary relations. Most of the properties below are easy to establish — we refer to [67] for proofs.

5.2.7. Fact. *Let T be a set functor. Then the relation lifting $\overline{T}$ satisfies the following properties, for all functions $f : X \to X'$, all relations $R, S \subseteq X \times X'$, and all subsets $Y \subseteq X$, $Y' \subseteq X'$:*

1. *$\overline{T}$ extends T: $\overline{T}(Grf) = Gr(Tf)$;*
2. *$\overline{T}$ preserves the diagonal: $\overline{T}(\Delta_X) = \Delta_{TX}$;*
3. *$\overline{T}$ commutes with relation converse: $\overline{T}(R^{\smile}) = (\overline{T}R)^{\smile}$;*
4. *$\overline{T}$ is monotone: if $R \subseteq S$ then $\overline{T}(R) \subseteq \overline{T}(S)$;*
5. *$\overline{T}$ distributes over composition: $\overline{T}(R\,;S) = \overline{T}(R)\,;\overline{T}(S)$, if T preserves weak pullbacks.*
6. *$\overline{T}$ commutes with restriction: $\overline{T}(R{\restriction_{Y\times Y'}}) = \overline{T}R{\restriction_{TY\times TY'}}$, if T is standard and preserves weak pullbacks.*

Fact 5.2.7(5) plays a key role in our work. In fact, distributivity of $\overline{T}$ over relation composition is *equivalent* to T preserving weak-pullbacks; the proof of this equivalence goes back to Trnková [88].

Many proofs in this chapter will be based on Fact 5.2.7, and we will not always provide all technical details. In the lemma below we have isolated some facts that will be used a number of times; the proof may serve as a sample of an argument using properties of relation lifting.

5.2.8. Lemma. *Let X, Y be sets, let $f, g\colon X \to Y$ be two functions and let $R \subseteq X \times X$ and $S \subseteq Y \times Y$ be relations.*

1. *If (X, R) is a pre-order, then so is $(TX, \overline{T}R)$.*
2. *If $f(x)\ S\ g(x)$ for all $x \in X$, then $Tf(\alpha)\ \overline{T}S\ Tg(\alpha)$ for all $\alpha \in TX$.*
3. *If $x\ R\ y$ implies $f(x)\ S\ g(y)$ for all $x, y \in X$, then $\alpha\ \overline{T}R\ \beta$ implies $(Tf)\alpha\ \overline{T}S\ (Tg)\beta$ for all $\alpha, \beta \in TX$.*

Proof For part 1, observe that (X, R) is a pre-order iff $\Delta_X \subseteq R$ and $R\,;R \subseteq R$. Hence, if (X, R) is a pre-order, it follows from Fact 5.2.7(2,4) that $\Delta_{TX} = \overline{T}\Delta_X \subseteq \overline{T}R$, and from Fact 5.2.7(5,4) that $\overline{T}R\,;\overline{T}R = \overline{T}(R\,;R) \subseteq \overline{T}R$, implying that $(TX, \overline{T}R)$ is a pre-order as well.

For part 2, observe that the antecedent can be succinctly expressed as

$$(Grf)^{\smile}\,; Grg \subseteq S.$$

Then it follows by the properties of relation lifting that

$$\begin{aligned}
(GrTf)^{\smile}\,; GrTg &= (\overline{T}(Grf))^{\smile}\,;\overline{T}(Grg) && \text{(Fact 5.2.7(1))}\\
&= \overline{T}((Grf)^{\smile})\,;\overline{T}(Grg) && \text{(Fact 5.2.7(3))}\\
&= \overline{T}((Grf)^{\smile}\,; Grg) && \text{(Fact 5.2.7(5))}\\
&\subseteq \overline{T}S && \text{(Fact 5.2.7(4))}
\end{aligned}$$

But the inclusion $(GrTf)^{\smile}\,; GrTg \subseteq \overline{T}S$ is just another way of stating the conclusion of part 2.

For part 3, we reformulate the statement of its antecedent as

$$(Grf)^{\smile}\,; R\,; Grg \subseteq S.$$

On the basis of this we may reason, via a completely analogous argument to the one just given, that

$$(GrTf)^{\smile}\,;\overline{T}R\,; GrTg \subseteq TS,$$

which is equivalent way of phrasing the conclusion of part 3. ∎

Relation lifting interacts with the map *Base* as follows:

5.2.9. Fact ([67]). *Let T be a standard, finitary, weak pullback-preserving functor.*

1. *Base is a natural transformation $Base : T \to P_\omega$. That is, given a map $f : X \to X'$ the following diagram commutes:*

$$\begin{tikzcd}
TX \arrow[r, "Base_X"] \arrow[d, "Tf"'] & P_\omega X \arrow[d, "Pf"] \\
TX' \arrow[r, "Base_{X'}"] & P_\omega X'
\end{tikzcd}$$

2. *Given a relation $R \subseteq X \times X'$ and elements $\alpha \in TX$, $\beta \in TY$, it follows from $\alpha\, \overline{T}R\, \beta$ that $Base(\alpha)\ \overline{P}R\ Base(\beta)$.*

An interesting relation to which we shall apply relation lifting is the *membership* relation $\in$. If needed, we will denote the membership relation restricted to a given set X by the relation $\in_X \subseteq X \times PX$. Given a set X and $\Phi \in TPX$, we define

$$\lambda^T_X(\Phi) = \{\alpha \in TX \mid \alpha\, \overline{T}{\in_X}\, \Phi\}.$$

Elements of $\lambda^T(\Phi)$ will be called *lifted members of* Φ.

Properties of λ^T are intimately related to those of $\overline{T}$.

5.2.10. Fact ([67]). *The collection of maps λ^T_X forms a distributive law with respect to both the co- and the contravariant power set functor. That is, λ^T provides two natural transformations, $\lambda^T\colon TP \to PT$, and $\lambda^T : T\breve{P} \to \breve{P}T$.*

5.2.11. Remark. Rather than just a distributive law with respect to the functor P, λ^T is a distributive law over the *monad* (P, μ, η), in the sense of being also compatible with the unit η and the multiplication μ of P, as given by the following two diagrams:

$$\begin{tikzcd} TX \arrow[r, "T\eta_X"] \arrow[dr, "\eta_{TX}"'] & TPX \arrow[d, "\lambda^T_X"] \\ & PTX \end{tikzcd} \qquad \begin{tikzcd} TPPX \arrow[r, "\lambda^T_{PX}"] \arrow[d, "T\mu_X"] & PTPX \arrow[r, "P\lambda^T_X"] & PPTX \arrow[d, "\mu_X"] \\ TPX \arrow[rr, "\lambda^T_X"'] & & PTX \end{tikzcd}$$

In the terminology of [82], (T, λ^T) is a *monad opfunctor* from the monad P to itself, and there is a one-one correspondence between the monad opfunctors and the functors T equipped with extensions to endofunctors on the *Kleisli category* $\mathbf{Kl}(M)$ associated with M. (The explicit results in [82], using the 2-functor $\mathbf{Alg_C}$, are in terms of monad functors and extensions to the category of Eilenberg-Moore algebras. The results for monad opfunctors and the Kleisli category are dual.) Note that the Kleisli category of the power set monad is (isomorphic to) the category **Rel** with sets as objects, and binary relations as arrows. The correspondence mentioned then links the natural transformation λ^T to the notion of relation lifting $\overline{T}$.

5.2.12. Lemma. *Let T be a standard, finitary, weak pullback-preserving functor. Let X be some set and let $\Phi \in TPX$.*

1. *If $\varnothing \in Base(\Phi)$ then $\lambda^T(\Phi) = \varnothing$.*

2. *If $Base(\Phi)$ consists of singletons only, then $\lambda^T(\Phi)$ is a singleton.*

3. *If T maps finite sets to finite sets, then for all $\Phi \in TP_\omega X$, $|\lambda^T(\Phi)| < \omega$.*

Proof For part 1, suppose that α is a lifted member of Φ; then it follows by Fact 5.2.9 that $Base(\alpha)\ \overline{P}{\in}\ Base(\Phi)$. But from this it would follow, if $\varnothing \in Base(\Phi)$, that $Base(\alpha)$ contains a member of $\varnothing$, which is clearly impossible. Consequently, then $\lambda^T(\Phi)$ is empty.

For part 2, observe that another way of saying that $Base(\Phi)$ consists of singletons only, is that $\Phi \in TS_X$. Let $\theta_X : S_X \to X$ be the inverse of η_X, that is, θ_X is the bijection mapping a singleton $\{x\}$ to x. Clearly then, the map $T\theta_X : TX \to TS_X$ is a bijection as well. In addition, we have $(Gr\theta_X)^{\smile} = \in_X$, from which it follows by Fact 5.2.7 that $(GrT\theta_X)^{\smile} = \overline{T}{\in}$. From this it is immediate that if $\Phi \in TS_X$, then $(T\theta_X)(\Phi)$ is the unique lifted member of Φ.

Finally, we consider part 3. Since T is finitary, $\Phi \in TP_\omega X$ implies that $\Phi \in TP_\omega Y$ for some finite set Y, and from this it follows that $Base(\Phi) \subseteq P_\omega Y$. If α is a lifted member of Φ, then by Fact 5.2.9 we obtain $Base(\alpha)\ \overline{P}{\in}\ Base(\Phi)$, and so in particular we find $Base(\alpha) \subseteq \bigcup Base(\Phi) \subseteq Y$. From this it follows that $\lambda^T(\Phi) \subseteq TY$, and so by the assumption on T $\lambda^T(\Phi)$ must be finite. ∎

5.2.4 Frames and their presentations

A *frame* is a complete lattice in which finite meets distribute over arbitrary joins. The signature of frames consists of arbitrary joins and finite meets, and it will be convenient for us to include the top and bottom as well. Thus a frame will usually be given as $\mathbb{L} = \langle L, \bigvee, \wedge, 0, 1\rangle$, while we will often consider join and meet as functions $\bigvee_{\mathbb{L}} : PL \to L$ and $\bigwedge_{\mathbb{L}} : P_\omega L \to L$. This enables us for instance to define a frame homomorphism $f : \mathbb{L} \to \mathbb{M}$ as a map from L to M satisfying $f \circ \bigwedge = \bigwedge \circ (P_\omega f)$ and $f \circ \bigvee = \bigvee \circ (Pf)$. By **Fr** we denote the category of frames and frame homomorphisms. The initial frame (the lattice $\{0, 1\}$ of truth values) will be denoted as Ω, and for a given frame $\mathbb{L}$ we will let $!_{\mathbb{L}}$ denote the unique frame homomorphism from Ω to $\mathbb{L}$, omitting the subscript if $\mathbb{L}$ is clear from context.

The order relation $\leq_{\mathbb{L}}$ of a frame $\mathbb{L}$ is given by $a \leq_{\mathbb{L}} b$ if $a \wedge b = a$ (or, equivalently, $a \vee b = b$). We can adjoin an implication operation to a frame $\mathbb{L}$ by defining $a \to b := \bigvee\{c \mid a \wedge c \leq b\}$; this operation turns $\mathbb{L}$ into a Heyting algebra. As a special case of implication we can consider the *negation*: $\neg a := \bigvee\{c \mid a \wedge c = 0\}$. Neither of these two operations is preserved by frame homomorphisms. A subset S of $\mathbb{L}$ is *directed* if for every $s_0, s_1 \in S$ there is an element $s \in S$ such that $s_0, s_1 \leq s$. The join of a directed set S is often denoted as $\bigvee^{\uparrow} S$.

A *frame presentation* is a tuple $\langle G \mid R\rangle$ where G is a set of generators and $R \subseteq PP_\omega G \times PP_\omega G$ is a set of relations. A presentation $\langle G \mid R\rangle$ *presents* a frame $\mathbb{L}$ if there exists a function $f : G \to L$ which is *compatible with* R, i.e. such that

$$\text{for all } (t_1, t_2) \in R,\ \bigvee_{A \in t_1} \bigwedge (P_\omega f)A = \bigvee_{B \in t_2} \bigwedge (P_\omega f)B,$$

and for all frames $\mathbb{M}$ and functions $g : G \to M$ compatible with R, there is a unique frame homomorphism $g' \colon \mathbb{L} \to \mathbb{M}$ such that $g'f = g$. We call f the *insertion of generators* (of G in $\mathbb{L}$).

5.2.13. FACT ([91], SECTION 4.4). *Every frame presentation presents a frame.*

The details of the proof of the above fact tell us how to construct a unique frame given a presentation $\langle G \mid R\rangle$. Omitting these details of the construction, we denote this unique frame by $\mathrm{Fr}\langle G \mid R\rangle$. We will usually write $\bigvee_{i\in I}\bigwedge A_i = \bigvee_{j\in J}\bigwedge B_j$ instead of $(\{A_i \mid i \in I\}, \{B_j \mid j \in J\})$ when specifying relations. In light of the fact that $a \leq b$ iff $a \vee b = b$, we will also allow ourselves the liberty to specify inequalities of the shape $\bigvee_{i\in I}\bigwedge A_i \leq \bigvee_{j\in J}\bigwedge B_j$ as relations. It follows from the proof of Fact 5.2.13 that if $f \colon G \to \mathrm{Fr}\langle G \mid R\rangle$ is the insertion of generators, then every element of $\mathrm{Fr}\langle G \mid R\rangle$ can be written as $\bigvee_{i\in I}\bigwedge P_\omega f A$ for some $\{A_i \mid i \in I\} \in PP_\omega G$; in other words every element of $\mathrm{Fr}\langle G \mid R\rangle$ can be written as an infinite disjunction of finite conjunctions of generators.

We will now introduce flat site presentations for frames, which have as one of their main advantages that they allow us to assume that an arbitrary element of the frame being presented is an infinite disjunction of generators. A *flat site* is a triple $\langle X, \sqsubseteq, \lhd_0\rangle$, where $\langle X, \sqsubseteq\rangle$ is a pre-order and $\lhd_0 \subseteq X \times PX$ is a binary relation such that for all $b \sqsubseteq a \lhd_0 A$, there exists $B \subseteq {\downarrow}A \cap {\downarrow}b$ such that $b \lhd_0 B$. A flat site $\langle X, \sqsubseteq, \lhd_0\rangle$ presents a frame $\mathbb{L}$ if there exists a function $f \colon X \to L$ such that

- f is order-preserving,
- $1 \leq \bigvee (Pf)X$,
- for all $a, b \in X$, $f(a) \wedge f(b) \leq \bigvee (Pf)({\downarrow}a \cap {\downarrow}b)$, and
- for all $a \lhd_0 A$, $f(a) \leq \bigvee (Pf)A$

and for all frames $\mathbb{M}$ and all $g \colon X \to M$ satisfying the above two properties, there exists a unique frame homomorphism $g' \colon \mathbb{L} \to \mathbb{M}$ such that $g'f = g$. Specifically, for all $a \in \mathbb{L}$,

$$g'(a) = \bigvee\{g(x) \mid f(x) \leq a\}.$$

To put it another way, the frame presented by a flat site is

$$\begin{aligned}\mathrm{Fr}\langle X, \sqsubseteq, \lhd_0\rangle \simeq \mathrm{Fr}\langle X \mid \ & a \leq b \quad (a \sqsubseteq b),\\ & a \leq \textstyle\bigvee A \quad (a \lhd_0 A),\\ & 1 = \textstyle\bigvee X\\ & a \wedge b = \textstyle\bigvee\{c \mid c \sqsubseteq a, c \sqsubseteq b\}\rangle.\end{aligned}$$

A *suplattice* is a complete $\bigvee$-semilattice; accordingly, a suplattice homomorphism is a map which preserves $\bigvee$. A *suplattice presentation* is a triple $\langle X, \sqsubseteq, \lhd_0\rangle$ where $\langle X, \sqsubseteq\rangle$ is a preorder and $\lhd_0 \subseteq X \times PX$. A suplattice presentation $\langle X, \sqsubseteq, \lhd_0\rangle$ presents a suplattice $\mathbb{L}$ if there exists a function $f \colon X \to L$ such that

- f is order-preserving;
- for all $a \lhd_0 A$, $f(a) \leq \bigvee Pf(A)$;

and for all suplattices $\mathbb{M}$ and all functions $g \colon X \to M$ respecting the above two conditions, there exists a unique suplattice homomorphism $g' \colon \mathbb{L} \to \mathbb{M}$ such that $g' \circ f = g$. Every suplattice presentation presents a suplattice [60, Prop. 2.5]. Now observe that every flat site can also be seen as a suplattice presentation with an additional stability condition. Consequently, given a flat site $\langle X, \sqsubseteq, \lhd_0 \rangle$, we can generate two different objects with it: a frame $\mathrm{Fr}\langle X, \sqsubseteq, \lhd_0 \rangle$ and a suplattice $\mathrm{SupLat}\langle X, \sqsubseteq, \lhd_0 \rangle$. The Flat site Coverage Theorem tells us that these two objects are in fact order isomorphic.

5.2.14. FACT (TH. 5 OF [93]). *Let $\langle X, \sqsubseteq, \lhd_0 \rangle$ be a flat site. Then* $\mathrm{Fr}\langle X, \sqsubseteq, \lhd_0 \rangle \simeq \mathrm{SupLat}\langle X, \sqsubseteq, \lhd_0 \rangle$.

We record the following consequences of the above fact. Suppose that $\langle X, \sqsubseteq, \lhd_0 \rangle$ is a flat site which presents a frame $\mathbb{L}$ via $f \colon X \to L$. Then

- every element of $\mathbb{L}$ is of the shape $\bigvee Pf(A)$ for some $A \in PX$;
- we can use $\langle X, \sqsubseteq, \lhd_0 \rangle$ both to define suplattice homomorphisms and frame homomorphisms.

5.2.5 Powerlocales via $\Box$ and $\Diamond$

We will now introduce the Vietoris powerlocale. In line with our generally algebraic approach we shall define it directly as a functor on the category of frames rather than its opposite, the category of locales. In its full generality it originates (as the "Vietoris construction") in Johnstone [55], with some earlier, more restricted references in [54]. For locales it is a localic analogue of hyperspace (with Vietoris topology). The points are (in bijection with) certain sublocales of the original locale. For a full constructive description see [92].

Given a frame $\mathbb{L}$, we first define $L_\Box := \mathbb{L}$ and $L_\Diamond := \mathbb{L}$, and then

$$
\begin{array}{rll}
V\mathbb{L} := \mathrm{Fr}\langle L_\Box \oplus L_\Diamond \mid & \Box 1 = 1 & \\
& \Box(a \wedge b) = \Box a \wedge \Box b & \\
& \Box(\bigvee^{\uparrow} A) = \bigvee^{\uparrow}_{a \in A} \Box a & (A \in PL \text{ directed}) \\
& \Diamond(\bigvee A) = \bigvee_{a \in A} \Diamond a & (A \in PL) \\
& \Box a \wedge \Diamond b \leq \Diamond(a \wedge b) & \\
& \Box(a \vee b) \leq \Box a \vee \Diamond b & \\
& \rangle &
\end{array}
$$

5.2.15. REMARK. We are abusing notation when specifying the relations in the definition above. Strictly speaking, we have two maps, $\Box \colon L_\Box \to V\mathbb{L}$ for the left copy of $\mathbb{L}$ and $\Diamond \colon L_\Diamond \to V\mathbb{L}$ for the right copy of $\mathbb{L}$, so that the insertion of generators is the map $\Box \oplus \Diamond \colon L_\Box \oplus L_\Diamond \to V\mathbb{L}$.

Johnstone [55] shows that V gives a monad on the category of locales, i.e. a comonad on the category of frames. We shall not need the full strength of this here, but some of the ingredients of the comonad structure are easy to check.

- V is functorial. If $f\colon \mathbb{L} \to \mathbb{M}$ is a frame homomorphism, then the function $(\Box f) \oplus (\Diamond f)\colon L_\Box \oplus L_\Diamond \to VM$ is compatible with the relations in the presentation of $V\mathbb{L}$, so that there is a frame homomorphism $Vf\colon V\mathbb{L} \to V\mathbb{M}$ extending this map. It is also easy to show functoriality.

- The counit $i_\mathbb{L}\colon V\mathbb{L} \to \mathbb{L}$ is given by $\Box a \mapsto a$ and $\Diamond a \mapsto a$. The comultiplication $\mu_\mathbb{L}\colon V\mathbb{L} \to VV\mathbb{L}$ is given by $\Box a \mapsto \Box\Box a$ and $\Diamond a \mapsto \Diamond\Diamond a$.

There are various relations between properties of $\mathbb{L}$ and of $V\mathbb{L}$. [55] shows that $\mathbb{L}$ is regular, completely regular, zero-dimensional or compact regular iff $V\mathbb{L}$ is, and also that if $\mathbb{L}$ is locally compact then so is $V\mathbb{L}$. The same paper also mentions without proof that if $\mathbb{L}$ is compact then so is $V\mathbb{L}$, referring to a proof by transfinite induction similar to that used for the localic Tychonoff theorem in [54]. The paper leaves open the converse question, of whether $V\mathbb{L}$ being compact implies that so is $V\mathbb{L}$. We shall give here a constructive (topos-valid) proof using preframe techniques that $\mathbb{L}$ is compact iff $V\mathbb{L}$ is.

5.2.16. Definition. A frame $\mathbb{L}$ (or, more properly, its locale) is *compact* if whenever $1 \le \bigvee_i^{\uparrow} a_i$ then $1 \le a_i$ for some i.

The following constructive proof, as proposed by Steve Vickers, is a routine application of the techniques in [56].

5.2.17. Theorem. *$\mathbb{L}$ is compact iff $V\mathbb{L}$ is.*

Proof $\Rightarrow$: $\mathbb{L}$ is compact iff the function $\mathbb{L} \to \Omega$ that maps $a \in \mathbb{L}$ to the truth value of $a = 1$ is a preframe homomorphism, i.e. preserves finite meets and directed joins. This function is characterized by being right adjoint to the unique frame homomorphism $! : \Omega \to \mathbb{L}$ and so to prove compactness it suffices to define a preframe homomorphism $\mathbb{L} \to \Omega$ and show that it is right adjoint to !. If $\mathbb{L}$ is presented – as a *frame* – by generators and relations, then the "preframe coverage theorem" of [56] shows how to derive a presentation as preframe, which can then be used for defining preframe homomorphisms from $\mathbb{L}$. The strategy is to generate a $\vee$-semilattice from the generators, and add relations to ensure a $\vee$-*stability* condition analogous to the $\wedge$-stability used in Johnstone's coverage theorem [54].

Our first step is to apply the preframe coverage theorem to derive a preframe presentation of $V\mathbb{L}$. We show

$$\begin{array}{lll}
V\mathbb{L} \cong \mathrm{Fr}\langle P_\omega\mathbb{L}\times\mathbb{L} & (\text{qua } \vee\text{-semilattice}) \mid & \\
& 1 \leq (\gamma\cup\{1\}, d) & \\
& (\gamma\cup\{a\}, d)\wedge(\gamma\cup\{b\}, d) \leq (\gamma\cup\{a\wedge b\}, d) & \\
& (\gamma\cup\{\bigvee^{\uparrow} A\}, d) \leq \bigvee^{\uparrow}_{a\in A}(\gamma\cup\{a\}, d) & (A \text{ directed}) \\
& (\gamma, \bigvee^{\uparrow} A\vee d) \leq \bigvee^{\uparrow}_{a\in A}(\gamma, a\vee d) & (A \text{ directed}) \\
& (\gamma\cup\{a\}, d)\wedge(\gamma, b\vee d) \leq (\gamma, (a\wedge b)\vee d) & \\
& (\gamma\cup\{a\vee b\}, d) \leq (\gamma\cup\{a\}, b\vee d) & \\
& \rangle. &
\end{array}$$

The $\vee$-semilattice structure on $P_\omega\mathbb{L}\times\mathbb{L}$ is the product structure from $\cup$ on $P_\omega\mathbb{L}$ and $\vee$ on $\mathbb{L}$. The homomorphisms between the frame presented above and $V\mathbb{L}$ are given by

$$\Box a \mapsto (\{a\}, 0),\ \Diamond a \mapsto (\emptyset, a)$$
$$(\gamma, d) \mapsto \bigvee_{c\in\gamma} \Box c \vee \Diamond d.$$

The relations shown are $\vee$-stable, so the preframe coverage shows that

$$V\mathbb{L} \cong \mathrm{PreFr}\langle P_\omega\mathbb{L}\times\mathbb{L} \text{ (qua poset)} \mid \text{same relations as above } \rangle.$$

We can now define a preframe homomorphism $\varphi : V\mathbb{L} \to \Omega$ by

$$\varphi(\gamma, d) = \exists c\in\gamma.\, c\vee d = 1.$$

To motivate this, we want criteria for $\bigvee_{c\in\gamma}\Box c\vee\Diamond d = 1$, and intuitively this means that for every sublocale K corresponding to a point of $V\mathbb{L}$ either K is included in some $c\in\gamma$ or K and d overlap. Taking K to be the closed complement of d, we get the given condition. This is not a rigorous argument, since that closed complement is not necessarily a point of $V\mathbb{L}$. However, the rest of our argument validates the choice. The relations in the preframe presentation of $V\mathbb{L}$ are largely easy to check. We shall just mention the penultimate one. Suppose $(\gamma\cup\{a\}, d)$ and $(\gamma, b\vee d)$ are both mapped to 1. We have either some $c\in\gamma$ with $c\vee d = 1$, in which case $c\vee(a\wedge b)\vee d = 1$, or we have $a\vee d = 1$ and in addition some $c'\in\gamma$ with $c'\vee b\vee d = 1$. In this latter case $c'\vee(a\wedge b)\vee d = 1$.

Next we show that φ is right adjoint to $! : \Omega \to V\mathbb{L}$, the unique frame homomorphism defined by

$$!(p) = \bigvee\{1\mid p\} = \bigvee^{\uparrow}(\{0\}\cup\{1\mid p\})\,.$$

We must show $\varphi(!(p)) \geq p$ and $!(\varphi(\gamma, d)) \leq (\gamma, d)$.

$$\begin{aligned}
\varphi(!(p)) &= \varphi\left(\bigvee^{\uparrow}(\{0\}\cup\{1\mid p\})\right) \\
&= \varphi(\emptyset, 0)\vee\bigvee\{\varphi(\{1\}, 0)\mid p\} \geq p
\end{aligned}$$

since if p holds then the disjuncts include $\varphi(\{1\}, 0) = 1$. For the other inequality, we must show that

$$\bigvee \{1 \mid \varphi(\gamma, d)\} \leq (\gamma, d).$$

If $\varphi(\gamma, d)$ holds true then $c \vee d = 1$ for some $c \in \gamma$, so

$$1 = (\{1\}, 0) = (\{c \vee d\}, 0) \leq (\{c\}, d) \leq (\gamma, d).$$

$\Leftarrow$: Suppose in $\mathbb{L}$ we have $1 = \bigvee_i^{\uparrow} a_i$. Then in $V\mathbb{L}$ we have $1 = \Box 1 = \bigvee_i^{\uparrow} \Box a_i$ and so $1 = \Box a_i$ for some i. Applying i to both sides gives $1 = a_i$. ∎

5.3 The T-powerlocale construction

In this section we arrive at the main conceptual contribution of this chapter. Given a weak pullback-preserving, standard, finitary functor $T\colon \mathbf{Set} \to \mathbf{Set}$, we define its associated T-powerlocale functor $V_T\colon \mathbf{Fr} \to \mathbf{Fr}$ on the category of frames, using the Carioca axioms for coalgebraic modal logic. This construction truly generalizes the Vietoris powerlocale construction, because we will see that the P_ω-powerlocale is isomorphic to the Vietoris powerlocale. The other two major results in this section are the fact that one can lift a natural transformation between transition functors $\rho\colon T' \to T$ to a natural transformation $\widehat{\rho}\colon V_T \to V_{T'}$ going in the other direction, and the fact that T-powerlocales are join-generated by their generators of the shape $\nabla\alpha$. We will establish the latter fact via the stronger result by showing that $V_T\mathbb{L}$ admits a flat site presentation. The fact that $V_T\mathbb{L}$ is join-generated by its generators is not entirely surprising, since the Carioca axioms were designed with the desirability of conjunction-free disjunctive normal forms in mind [17]; however the precise mathematical formulation of this property, using flat sites and suplattices, is an improvement over what was previously known.

This section is organized as follows. In §5.3.1 we introduce the T-powerlocale construction on frames. In §5.3.2 we make technical observations about T-powerlocales. In §5.3.3, we consider two instantiations of the T-powerlocale construction, the most notable of which is the P_ω-powerlocale which is isomorphic to the classical Vietoris powerlocale. In §5.3.4 we extend the T-powerlocale construction to a functor V_T on the category of frames, and we show how one can lift natural transformations between set functors T, T' to natural transformations between powerlocale functors V_T, $V_{T'}$. We conclude this section with §5.3.5, in which we show that the T-powerlocale construction admits a flat site presentation, a corollary of which is that each element of $V_T\mathbb{L}$ has a disjunctive normal form.

5.3.1 Introducing the T-powerlocale

In this subsection, we will use the Carioca axioms for coalgebraic modal logic [17] to define the T-powerlocale $V_T\mathbb{L}$ of a given frame $\mathbb{L}$ using a frame presentation, i.e. using generators and relations. The generators of $V_T\mathbb{L}$ will be given by the set TL; in order to specify the relations we will use *relation lifting* (§5.2.3) and *slim redistributions*, which we will introduce below. In addition, we will provide an alternative presentation of $V_T\mathbb{L}$, which does not use slim redistributions. From a conceptual viewpoint, it is not immediately obvious which presentation of $V_T\mathbb{L}$ should be taken as the primary definition. Our choice to use slim redistributions in the primary definition is motivated by the existing literature [17, 66, 67].

5.3.1. Definition. Let X be a set and let $\Gamma \in P_\omega TX$. The set of all *slim redistributions* of Γ is defined as follows:

$$SRD(\Gamma) = \left\{\Psi \in TP_\omega\left(\bigcup_{\gamma\in\Gamma} Base(\gamma)\right) \mid \forall\gamma\in\Gamma,\, \gamma\, \overline{T}{\in}\, \Psi\right\}$$

Intuitively, $\Psi \in TP_\omega X$ is a slim redistribution of $\Gamma \in P_\omega TX$ if (i) Ψ is 'obtained from the material of Γ', that is:

$$\Psi \in TP_\omega\left(\bigcup_{\gamma\in\Gamma} Base(\gamma)\right),$$

and (ii) every element of Γ is a lifted member of Ψ, or equivalently, $\Gamma \subseteq \lambda^T(\Psi)$. We illustrate this with the motivating example of slim redistributions, namely slim redistribution for the finite powerset functor.

5.3.2. Example. Recall from Example 5.2.5 that if $R \subseteq X \times Y$ is a relation then $\overline{P_\omega}R \subseteq P_\omega X \times P_\omega Y$ can be characterized as follows:

$$\alpha\overline{P_\omega}R\beta \text{ iff } \forall x\in\alpha,\, \exists y\in\beta,\, xRy \text{ and } \forall y\in\beta,\, \exists x\in\alpha,\, xRy.$$

In particular, for $\in\ \subseteq X \times \mathcal{P}X$ we get $\alpha\,\overline{P_\omega}\in\Gamma$ iff $\alpha \subseteq \bigcup\Gamma$ and $\forall\gamma\in\Gamma,\, \gamma \between \alpha$. (Recall that $\gamma \between \alpha$ means that $\gamma\cap\alpha$ is inhabited.) For an order $\leq$, let us define the *upper, lower* and *convex* preorders on finite sets:

$$\begin{aligned}
&\alpha \leq_L \beta \text{ if } \alpha \subseteq {\downarrow}\beta, \text{ i.e. } \forall x\in\alpha,\, \exists y\in\beta,\, x\leq y\\
&\alpha \leq_U \beta \text{ if } {\uparrow}\alpha \supseteq \beta, \text{ i.e. } \forall y\in\beta,\, \exists x\in\alpha,\, x\leq y\\
&\alpha \leq_C \beta \text{ if } \alpha \leq_L \beta \text{ and } \alpha \leq_U \beta.
\end{aligned}$$

Thus $\overline{P_\omega}\leq$ is $\leq_C$.

Next, if $\alpha \in P_\omega S$ then

$$Base(\alpha) = \bigcap\{S' \in P_\omega(S) \mid \alpha \subseteq S'\} = \alpha.$$

From this, if $\Gamma \in P_\omega P_\omega X$ then

$$\begin{aligned}
SRD(\Gamma) &= \{\Psi \in P_\omega P_\omega\left(\bigcup\Gamma\right) \mid \forall\gamma\in\Gamma,\, (\gamma \subseteq \bigcup\Psi \text{ and } \forall\alpha\in\Psi,\, \alpha\between\gamma)\}\\
&= \{\Psi \in P_\omega P_\omega\left(X\right) \mid \bigcup\Psi = \bigcup\Gamma \text{ and } \forall\gamma\in\Gamma,\, \forall\alpha\in\Psi,\, \alpha\between\gamma\}.
\end{aligned}$$

5.3.3. Definition. Let T be a standard, finitary, weak pullback-preserving functor. Let $\mathbb{L}$ be a frame. We define the *T-powerlocale of* $\mathbb{L}$

$$V_T\mathbb{L} := \mathrm{Fr}\langle TL \mid (\nabla 1), (\nabla 2), (\nabla 3)\rangle,$$

where the relations are the *Carioca axioms* [17]:

$$\begin{array}{lll}(\nabla 1) & \nabla\alpha \leq \nabla\beta, & (\alpha\ \overline{T}{\leq}\ \beta)\\ (\nabla 2) & \bigwedge_{\alpha\in\Gamma}\nabla\alpha \leq \bigvee\{\nabla(T\bigwedge)\Psi \mid \Psi \in \mathit{SRD}(\Gamma)\}, & (\Gamma \in P_\omega TL)\\ (\nabla 3) & \nabla(T\bigvee)\Phi \leq \bigvee\{\nabla\beta \mid \beta\ \overline{T}{\in}\ \Phi\}, & (\Phi \in TPL)\end{array}$$

5.3.4. Remark. To be precise, we assume that $\nabla\colon TL \to V_T L$ is the insertion of generators, so when specifying the relations we should write e.g. $\alpha \leq \beta$ instead of $\nabla\alpha \leq \nabla\beta$. The way we have specified the relations above is more consistent with [17].

We will discuss the instantiation of these axioms for $T = P_\omega$ in some more detail in §5.3.3.

We will now present a very useful equivalent definition of $V_T\mathbb{L}$. The crucial observation behind the alternative definition of $V_T\mathbb{L}$ is the following technical lemma, which characterizes the slim redistributions of a given finite subset Γ of $\langle TL, \overline{T}{\leq}\rangle$ as the maximal lower bounds of Γ. Observe that the lemma also holds in case $\Gamma = \emptyset$.

5.3.5. Lemma. *Let $\mathbb{L}$ be a meet-semilattice (e.g., a frame) and let $\Gamma \in P_\omega TL$. Then for any $\alpha \in TL$, the following are equivalent:*

(a) $\alpha \in TL$ is a lower bound of Γ, that is, $\alpha\ \overline{T}{\leq}\ \gamma$ for all $\gamma \in \Gamma$;

(b) $\alpha\ \overline{T}{\leq}\ (T\bigwedge)\Phi$ for some $\Phi \in \mathit{SRD}(\Gamma)$.

In particular, if $\Phi \in \mathit{SRD}(\Gamma)$ then $(T\bigwedge)\Phi\ \overline{T}{\leq}\ \gamma$ for all $\gamma \in \Gamma$.

Proof Recall that

$$\mathit{SRD}(\Gamma) := \left\{\Psi \in TP\left(\textstyle\bigcup_{\gamma\in\Gamma}\mathit{Base}(\gamma)\right) \mid \Gamma \subseteq \lambda^T(\Psi)\right\}.$$

For the implication from (b) to (a), observe that for any $a \in L$ and $A \in P_\omega L$, we have that $a \in A$ implies that $\bigwedge A \leq a$. By Fact 5.2.7 it follows that for all $\gamma \in TL$ and $\Psi \in TP_\omega L$, if $\gamma\ \overline{T}{\in}\ \Psi$ then $T\bigwedge(\Psi)\ \overline{T}{\leq}\ \gamma$. Now suppose that Ψ is a slim redistribution of Γ. Then $\Gamma \subseteq \lambda^T(\Psi)$, and so $(T\bigwedge)\Psi$ is a $\overline{T}{\leq}$-lower bound of Γ. From this the implication (b) $\Rightarrow$ (a) is immediate.

For the oppositie implication, take $\alpha \in TL$ such that $\forall\gamma \in \Gamma$, $\alpha\ \overline{T}{\leq}\ \gamma$. Then by Fact 5.2.9, we obtain $\mathit{Base}(\alpha)\ \overline{P}{\leq}\ \mathit{Base}(\gamma)$ for all $\gamma \in \Gamma$. Abbreviate $C := \bigcup_{\gamma\in\Gamma}\mathit{Base}(\gamma)$, and define $f\colon \mathit{Base}(\alpha) \to PC$ as follows:

$$f\colon a \mapsto {\uparrow_L}\, a \cap C,$$

that is: $f(a) = \{c \in C \mid a \leq c\}$. Then Tf is a function

$$Tf\colon TBase(\alpha) \to TPC.$$

We claim that $\Psi := Tf(\alpha)$ is an element of $SRD(\Gamma)$ and that $\alpha \overline{T}{\leq} T\bigwedge(\Psi)$. For the first claim, since $\Phi \in TPC$, all we need to show is that $\Gamma \subseteq \lambda^T(\Psi)$, i.e. that for all $\gamma \in \Gamma$, $\gamma \overline{T}{\in} \Psi$. So suppose that $\gamma \in \Gamma$; then by assumption, $\alpha \overline{T}{\leq} \gamma$, so $Base(\alpha) \overline{P}{\leq} Base(\gamma)$. It follows from the definition of f that for all $b \in Base(\gamma)$, and all $a \in Base(\alpha)$, if $a \leq b$ then $b \in f(a)$. It follows by Fact 5.2.7 that

$$\forall \delta \in TBase(\alpha), \forall \beta \in TBase(\gamma), \delta \overline{T}{\leq} \beta \Rightarrow \beta \overline{T}{\in} Tf(\delta).$$

So in particular, since $\alpha \in TBase(\alpha)$, $\gamma \in TBase(\gamma)$ and $\alpha \overline{T}{\leq} \gamma$, we see that $\gamma \overline{T}{\in} Tf(\alpha) = \Psi$. Since $\gamma \in \Gamma$ was arbitrary, it follows that $\Gamma \subseteq \lambda^T(\Psi)$. Consequently, $\Psi \in SRD(\Gamma)$, as we wanted to show.

For the second claim, i.e. that $\alpha \overline{T}{\leq} T\bigwedge(\Psi)$, it suffices to observe that $a \leq \bigwedge f(a)$ for all $a \in Base(\alpha)$, so by Fact 5.2.7,

$$\forall \delta \in TBase(\alpha), \delta \overline{T}{\leq} T\bigwedge \circ Tf(\delta).$$

Since $\alpha \in TBase(\alpha)$ and $\Psi = Tf(\alpha)$, we get that $\alpha \overline{T}{\leq} T\bigwedge \circ Tf(\alpha) = T\bigwedge(\Psi)$. ∎

5.3.6. Corollary. *Let* $\mathbb{L}$ *be a frame. Then*

$$V_T\mathbb{L} \simeq \mathrm{Fr}\langle TL \mid (\nabla 1), (\nabla 2'), (\nabla 3)\rangle,$$

where the relations are as follows:

$$\begin{array}{lll}
(\nabla 1) & \nabla\alpha \leq \nabla\beta, & (\alpha \overline{T}{\leq} \beta) \\
(\nabla 2') & \bigwedge_{\gamma\in\Gamma}\nabla\gamma \leq \bigvee\{\nabla\alpha \mid \forall\gamma \in \Gamma, \alpha \overline{T}{\leq} \gamma\}, & (\Gamma \in P_\omega TL) \\
(\nabla 3) & \nabla(T\bigvee)\Phi \leq \bigvee\{\nabla\beta \mid \beta \overline{T}{\in} \Phi\}, & (\Phi \in TPL)
\end{array}$$

Proof Observe that the only difference between $\mathrm{Fr}\langle TL \mid (\nabla 1), (\nabla 2'), (\nabla 3)\rangle$ and the original definition of $V_T\mathbb{L}$ is that we replaced $(\nabla 2)$,

$$(\nabla 2) \quad \bigwedge_{\alpha\in\Gamma}\nabla\alpha \leq \bigvee\{\nabla(T\bigwedge)\Psi \mid \Psi \in SRD(\Gamma)\}, \quad (\Gamma \in P_\omega TL)$$

with $(\nabla 2')$. The equivalence of these two relations is an immediate corollary of Lemma 5.3.5: take any $\Gamma \in TP_\omega L$, then

$$\begin{array}{ll}
\bigvee\{\nabla T\bigwedge(\Psi) \mid \Psi \in SRD(\Gamma)\} & \\
= \bigvee\{\nabla\alpha \mid \exists\Psi \in SRD(\Gamma), \alpha \overline{T}{\leq} \nabla T\bigwedge(\Psi)\} & \text{by order theory and } (\nabla 1), \\
= \bigvee\{\nabla\alpha \mid \forall\gamma \in \Gamma, \alpha \overline{T}{\leq} \gamma\} & \text{by Lemma 5.3.5.}
\end{array}$$

It follows that $V_T\mathbb{L} \simeq \mathrm{Fr}\langle TL \mid (\nabla 1), (\nabla 2'), (\nabla 3)\rangle$. ∎

5.3.7. Remark. We will see later that both axioms $(\nabla 2)$ and $(\nabla 2')$ are equally useful. It seems that $(\nabla 2')$ has not been studied before in the literature on coalgebraic modal logic via the ∇-modality [73, 17, 62, 67].

5.3.2 Basic properties of the T-powerlocale

In this subsection we make some technical observations about slim redistributions and about the structure of the T-powerlocale. We start with two facts on slim redistributions.

5.3.8. Lemma. *$SRD(\varnothing) = T\{\varnothing\}$.*

Proof If Φ is a slim redistribution of the empty set, then by definition $\Phi \in TP_\omega(\varnothing) = T\{\varnothing\}$. Conversely, any $\Phi \in T\{\varnothing\}$ satisfies the condition that $\varnothing \subseteq \lambda^T(\Phi)$, and so $\Phi \in SRD(\varnothing)$. ∎

The following Lemma plays an essential role when defining V_T on frame homomorphisms, rather than just on frames. It is of crucial use when showing that if $f\colon \mathbb{L} \to \mathbb{M}$ is a frame homomorphism, then $V_T f\colon V_T\mathbb{L} \to V_T\mathbb{M}$ preserves conjunctions, as we will see in §5.3.4.

5.3.9. Lemma. *Let X, Y be sets and let $f\colon X \to Y$ be a function; let $\Gamma \in P_\omega TX$. Then the restriction of $TP_\omega f\colon TP_\omega X \to TP_\omega Y$ to $SRD(\Gamma)$ is a surjection onto $SRD(P_\omega Tf\Gamma)$.*

Proof Let X, Y, f and Γ be as in the statement of the Lemma, and abbreviate $\Gamma' := (P_\omega Tf)\Gamma$, $C := \bigcup_{\gamma\in\Gamma} Base(\gamma)$ and $C' := \bigcup_{\gamma'\in\Gamma'} Base(\gamma')$. Then an easy calculation shows that

$$\begin{aligned} C' &= \bigcup_{\gamma\in\Gamma} Base(Tf)(\gamma) && \text{(definition of } \Gamma'\text{)} \\ &= \bigcup_{\gamma\in\Gamma} (Pf)Base(\gamma) && (Base \text{ is natural transformation}) \\ &= (Pf)(C) && \text{(elementary set theory)} \end{aligned}$$

We will first show that $TP_\omega f$ maps slim redistributions of Γ to slim redistributions of Γ'. For that purpose, take an arbitrary element $\Phi \in SRD(\Gamma)$, and write $\Phi' := (TP_\omega f)\Phi$. We claim that $\Phi' \in SRD(\Gamma')$, and first show that

$$\Phi' \in TP_\omega C', \tag{5.1}$$

or equivalently, that $Base\Phi' \subseteq P_\omega C'$. To prove this inclusion, take an arbitrary set $A' \in Base(\Phi')$. Since by Fact 5.2.9, $Base(\Phi') = (P_\omega P_\omega f)(Base(\Phi)$, this means that A' must be of the form $(P_\omega f)(A)$ for some $A \in Base(\Phi)$. In particular, A' must be a subset of $(P_\omega f)(\bigcup Base(\Phi))$. Also, because Φ is a slim redistribution of Γ, by definition we have $Base(\Phi) \subseteq P_\omega C$, and so $\bigcup Base(\Phi) \subseteq \bigcup C$. From this it follows that $A' \subseteq (Pf)(\bigcup Base(\Phi)) \subseteq (Pf)(\bigcup C) = C'$, as required.

Second, we claim that

$$\Gamma' \subseteq \lambda^T(\Phi'). \tag{5.2}$$

To prove this, take an arbitrary element of Γ', say, $(Tf)\gamma$ for some $\gamma \in \Gamma$. We have $\gamma \overline{T}{\in} \Phi$ by the assumption that $\Phi \in SRD(\Gamma)$. But then, since $a \in A$ implies $fa \in (P_\omega f)A$ for any $a \in C$ and $A \subseteq C$, it follows by Lemma 5.2.8 that $\gamma' = (Tf)\gamma \overline{T}{\in} (TP_\omega f)(\Phi) = \Phi'$. This means that γ' is a lifted member of Φ', as required.

Clearly, the claims (5.1) and (5.2) above suffice to prove that $\Phi' \in SRD(\Gamma')$, which means that indeed, $TP_\omega f$ maps slim redistributions of Γ to slim redistributions of Γ'.

Thus it is left to prove that every slim redistribution of Γ' is of the form $(TP_\omega f)\Phi$ for some slim redistribution Φ of Γ. Take an arbitrary $\Phi' \in SRD(\Gamma')$, and recall that $\breve{P}$ denotes the contravariant power set functor. Restrict f to the map $f^- : C \to C'$, which means that $\breve{P}f^- : P_\omega C' \to P_\omega C$. It follows that $T\breve{P}f^- : TP_\omega C' \to TP_\omega C$, so that we may define $\Phi := (T\breve{P}f^-)\Phi'$, and obtain $\Phi \in TP_\omega C$. Hence, in order to prove that

$$\Phi \in SRD(\Gamma), \tag{5.3}$$

it suffices to show that $\Gamma \subseteq \lambda^T(\Phi)$. But this is an immediate consequence of the fact that λ^T is a distributive law of T over $\breve{P}$ (Fact 5.2.10), since for an arbitrary $\gamma \in \Gamma$ we may reason as follows. From $\gamma \in \Gamma$ it follows by definition of Γ' that $(Tf^-)(\gamma) = (Tf)(\gamma)$ belongs to Γ'. Since $\Gamma' \subseteq \lambda^T_Y(\Phi)$ by assumption, by definition of $\breve{P}$ we find that $\gamma \in (\breve{P}Tf)\lambda^T_Y(\Psi)$. But by $\lambda^T : T\breve{P} \to \breve{P}T$ we know that $(\breve{P}Tf)\lambda^T_Y(\Psi) = \lambda^T_X(T\breve{P}f)(\Psi) = \lambda^T_X(\Phi)$. Thus we find $\gamma \in \lambda^T(\Phi)$, as required.

Finally, observe that $f^- : C \to C'$ is surjective, so that it follows by properties of the co- and contravariant power set functors that $P_\omega f^- \circ \breve{P}f^- = \mathrm{id}_{P_\omega C'}$. From this it is immediate by functoriality of T that

$$\Phi' = (TP_\omega f^- \circ T\breve{P}f^-)\Phi' = (TP_\omega f^-)\Phi = (TP_\omega f)\Phi.$$

This finishes the proof of the Lemma. ∎

In the following lemma we gather some basic observations on the frame structure of the T-powerlocale. These facts generalize results from [67] to our geometrical setting.

5.3.10. Lemma. *Let T be a standard, finitary, weak pullback-preserving functor and let $\mathbb{L}$ be a frame.*

1. *If $\alpha \in TL$ is such that $0_{\mathbb{L}} \in Base(\alpha)$, then $\nabla\alpha = 0_{V_T\mathbb{L}}$.*
2. *If $A \subseteq L$ is such that $a \wedge b = 0_{\mathbb{L}}$ for all $a \neq b$ in A, then $\nabla\alpha \wedge \nabla\beta = 0_{V_T\mathbb{L}}$ for all $\alpha \neq \beta$ in TA.*
3. *If there is no relation R such that $\alpha \overline{T}R\, \beta$, then $\nabla\alpha \wedge \nabla\beta = 0_{V_T\mathbb{L}}$.*
4. $1_{V_T\mathbb{L}} = \bigvee\{\nabla\gamma \mid \gamma \in T\{1_{\mathbb{L}}\}\}$.

5. *For any $A \subseteq L$ such that $1_{\mathbb{L}} = \bigvee A$, we have $1_{V_T\mathbb{L}} = \bigvee\{\nabla\alpha \mid \alpha \in TA\}$.*

Proof For part 1, let $\alpha \in TL$ be such that $0_{\mathbb{L}} \in Base(\alpha)$. Consider the map $f : L \to PL$ given by

$$f(a) := \begin{cases} \emptyset & \text{if } a = 0_{\mathbb{L}}, \\ \{a\} & \text{if } a > 0_{\mathbb{L}}. \end{cases}$$

Then $\mathrm{id}_L = \bigvee \circ f$, so that $\mathrm{id}_{TL} = (T\bigvee) \circ (Tf)$ by functoriality of T. In particular, we obtain that $\alpha = (T\bigvee)(Tf)(\alpha)$, so that we may calculate

$$\begin{aligned} \nabla\alpha &= \bigvee \left\{\nabla\beta \mid \beta \,\overline{T}{\in}\, (Tf)(\alpha)\right\} && \text{(axiom } \nabla 2\text{)} \\ &\leq \bigvee \left\{\nabla\beta \mid Base(\beta) \,\overline{P}{\in}\, Base((Tf)(\alpha))\right\} && \text{(Fact 5.2.9)} \\ &= \bigvee \emptyset && (\dagger) \\ &= 0_{V_T\mathbb{L}} \end{aligned}$$

In order to justify the remaining step (†) in this calculation, observe that it follows from the naturality of *Base* (Fact 5.2.9(1)) that

$$Base((Tf)(\alpha)) = (Pf)(Base(\alpha)),$$

and so by the assumption that $0_{\mathbb{L}} \in Base(\alpha)$ we obtain $\emptyset \in Base((Tf)(\alpha))$. Now suppose for contradiction that there is some $B \subseteq L$ such that $B \,\overline{P}{\in}\, Base((Tf)(\alpha))$. Then by definition of $\overline{P}$ there is a $b \in B$ such that $b \in \emptyset$, which provides the desired contradiction. This proves (†), and finishes the proof of part 1.

For part 2, let $A \subseteq L$ be such that $a \wedge b = 0_{\mathbb{L}}$ for all $a \neq b$ in A, and take two distinct elements $\alpha, \beta \in TA$. In order to prove that $\nabla\alpha \wedge \nabla\beta = 0_{V_T\mathbb{L}}$, it suffices by axiom ($\nabla 2$) to show that

$$\nabla(T{\textstyle\bigwedge})(\Phi) = 0_{V_T\mathbb{L}}, \text{ for all } \Phi \in SRD\{\alpha, \beta\}. \tag{5.4}$$

Take an arbitrary slim redistribution Φ of $\{\alpha, \beta\}$, then by Fact 5.2.12, $Base(\Phi)$ contains a set $A_0 \subseteq_\omega A$ of size > 1. Define the map $d : Base(\Phi) \to P_\omega(A) \cup \{\{1_{\mathbb{L}}\}\}$ by putting:

$$d(B) := \begin{cases} \emptyset & \text{if } |B| > 1, \\ B & \text{if } |B| = 1, \\ \{1_{\mathbb{L}}\} & \text{if } |B| = 0. \end{cases}$$

It is straightforward to verify from the assumptions on A and the definition of d, that $\bigwedge B \leq \bigvee d(B)$, for each $B \in Base(\Phi)$. Hence it follows by Fact 5.2.7 that $(T\bigwedge)(\Phi) \,\overline{T}{\leq}\, (T\bigvee)(Td)(\Phi)$, so that by axiom ($\nabla 1$) we may conclude that

$$\nabla(T{\textstyle\bigwedge})(\Phi) \leq \nabla(T{\textstyle\bigvee})(Td)(\Phi) \tag{5.5}$$

Finally, it follows from the naturality of *Base* (Fact 5.2.9(1)) that $Base(Td)(\Phi) = (Pd)(Base(\Phi))$. Consequently, for the set $A_0 \in Base(\Phi)$ satisfying $|A_0| > 1$, we find $\emptyset = d(A_0) \in \mathit{Base}(Td)(\Phi)$, and then $0_{\mathbb{L}} = \bigvee \varnothing \in (P\bigvee)\mathit{Base}(Td)(\Phi) = \mathit{Base}(T\bigvee)(Td)(\Phi)$. Thus by part (1) of this lemma it follows that

$$\nabla(T\bigvee)(Td)(\Phi) = 0_{V_T\mathbb{L}}. \tag{5.6}$$

This finishes the proof of part 2, since (5.4) is immediate on the basis of (5.5) and (5.6).

In order to prove part 3, suppose that $\alpha, \beta \in TL$ are not linked by any lifted relation. Consider the (unique) map

$$f : L \to \{1\},$$

and define $\alpha' := (Tf)\alpha$, $\beta' := (Tf)(\beta)$. Suppose for contradiction that $\alpha' = \beta'$. Then we would find $\alpha\, \overline{T}((\mathit{Gr} f)^{\smile} ; \mathit{Gr} f)\, \beta$, contradicting the assumption on α and β. It follows that α' and β are distinct, and so by part (2) of this lemma (with $A = \{1_{\mathbb{L}}\}$), we may infer that $\nabla\alpha' \wedge \nabla\beta' = 0_{V_T\mathbb{L}}$. This means that we are done, since it follows from $\mathit{Gr} f \subseteq\, \leq$ and the definitions of α', β', that $\alpha\, \overline{T}{\leq}\, \alpha'$ and $\beta\, \overline{T}{\leq}\, \beta'$, and from this we obtain by (∇1) that

$$\nabla\alpha \wedge \nabla\beta \leq \nabla\alpha' \wedge \nabla\beta' \leq 0_{V_T\mathbb{L}}.$$

For part 4, we reason as follows:

$$\begin{aligned} 1_{V_T\mathbb{L}} &= \bigvee\{\nabla(T\bigwedge)(\Phi) \mid \Phi \in \mathit{SRD}(\emptyset)\}, && \text{(axiom } (\nabla 2) \text{ with } A = \emptyset) \\ &= \bigvee\{\nabla(T\bigwedge)(\Phi) \mid \Phi \in T\{\emptyset\}\} && \text{(Fact 5.3.8)} \\ &= \bigvee\{\nabla\gamma \mid \gamma \in T\{1_{\mathbb{L}}\}\} && (\ddagger) \end{aligned}$$

where the last step ($\ddagger$) is justified by the observation that, since the map $\bigwedge : P_\omega L \to L$ restricts to a bijection $\bigwedge : \{\emptyset\} \to \{1_{\mathbb{L}}\}$, its lifting restricts to a bijection $T\bigwedge : T\{\emptyset\} \to T\{1_{\mathbb{L}}\}$.

Finally, we turn to the proof of part 5. Let $A \subseteq L$ be such that $1_{\mathbb{L}} = \bigvee A$, and consider an arbitrary element $\Phi \in T\{A\}$. We claim that

$$\lambda^T(\Phi) \subseteq TA. \tag{5.7}$$

To see this, take an arbitrary lifted element α of Φ. It follows from $\alpha\, \overline{T}{\in}\, \Phi$ that $\mathit{Base}(\alpha)\, \overline{P}{\in}\, \mathit{Base}(\Phi)$. In particular, each $a \in \mathit{Base}(\alpha)$ must belong to some $B \in \mathit{Base}(\Phi) \subseteq \{A\}$. In other words, $\mathit{Base}(\alpha) \subseteq A$, which is equivalent to saying that $\alpha \in TA$. This proves (5.7).

By (5.7) and axiom (∇3) we obtain

$$\nabla(T\bigvee)(\Phi) \leq \bigvee\{\nabla\alpha \mid \alpha \in TA\}. \tag{5.8}$$

Now we reason as follows:

$$\begin{aligned} 1_{V_T\mathbb{L}} &= \bigvee\{\nabla\alpha \mid \alpha \in T\{1_{\mathbb{L}}\}\} && \text{(part 4)} \\ &= \bigvee\{\nabla(T\bigvee)(\Phi) \mid \Phi \in T\{A\}\} && (*) \\ &\leq \bigvee\{\nabla\alpha \mid \alpha \in TA\}, && (5.8) \end{aligned}$$

To justify the second step $(*)$, observe that if we restrict the map $\bigvee : PL \to L$ to the bijection $\bigvee : \{A\} \to \{1_{\mathbb{L}}\}$, as its lifting we obtain a bijection $T\bigvee : T\{A\} \to T\{1_{\mathbb{L}}\}$. ∎

5.3.3 Two examples of the T-powerlocale construction

In this subsection we will discuss two examples of T-powerlocales. First, we discuss the somewhat trivial example of the Id-powerlocale. After that, we will discuss the defining example of T-powerlocales, namely the P_ω-powerlocale, which is isomorphic to the classical Vietoris powerlocale.

5.3.11. Example. Let $\mathrm{Id}\colon \mathbf{Set} \to \mathbf{Set}$ be the identity functor on the category of sets. Then for all frames $\mathbb{L}$, $V_{\mathrm{Id}}\mathbb{L} \simeq \mathbb{L}$.

First recall from Example 5.2.5 that for any relation $R \subseteq X \times Y$, $\overline{\mathrm{Id}}R = R$. Moreover, if $A \in \mathrm{Id}\,P_\omega L = P_\omega L$, then it is straightforward to verify that

$$\begin{aligned} SRD(A) &= \{\Psi \in P_\omega(\textstyle\bigcup_{c\in A}\{c\}) \mid \forall c \in A, c \in \Psi\} \\ &= \{A\}. \end{aligned}$$

Consequently, the ∇-relations reduce to the following in case $T = \mathrm{Id}$:

$$\begin{array}{lll} (\nabla 1) & \nabla a \leq \nabla b, & (a \leq b) \\ (\nabla 2) & \bigwedge_{a\in A}\nabla a \leq \nabla\bigwedge A, & (A \in P_\omega L) \\ (\nabla 3) & \nabla\bigvee A \leq \bigvee\{\nabla b \mid b \in A\}. & (A \in PL) \end{array}$$

The identity $\mathrm{id}_L\colon L \to L$ obviously satisfies $(\nabla 1)$, $(\nabla 2)$ and $(\nabla 3)$. Moreover if we have a frame $\mathbb{M}$ and a function $f\colon L \to M$ which is compatible with $(\nabla 1)$, $(\nabla 2)$ and $(\nabla 3)$, then it is easy to see that f is in fact a frame homomorphism $\mathbb{L} \to \mathbb{M}$. By the universal property of frame presentations, it follows that $V_{\mathrm{Id}}\mathbb{L} \simeq \mathbb{L}$.

We now turn to the P_ω-powerlocale. Recall from Example 5.2.3 that $P_\omega\colon \mathbf{Set} \to \mathbf{Set}$, the covariant finite power set functor, is indeed standard, weak pullback-preserving and finitary. We will now show that the P_ω-powerlocale is the Vietoris powerlocale. The equivalence of the ∇ axioms and the $\Box$, $\Diamond$ axioms on distributive lattices is already known from the work of Palmigiano & Venema [73]; what is different here is that we consider infinite joins rather than only finite joins.

We will use the presentation using ($\nabla 1$), ($\nabla 2'$) and ($\nabla 3$) as our point of departure. Recall that for all $\alpha, \beta \in P_\omega L$,

$$\begin{aligned}
&\alpha \leq_L \beta \text{ if } \alpha \subseteq {\downarrow}\beta,\\
&\alpha \leq_U \beta \text{ if } {\uparrow}\alpha \supseteq \beta,\\
&\alpha \leq_C \beta \text{ if } \alpha \leq_L \beta \text{ and } \alpha \leq_U \beta.
\end{aligned}$$

By Example 5.3.2, two of the relations presenting $V_{P_\omega}\mathbb{L}$ thus become

$$\begin{array}{ll}
(\nabla 2') & \bigwedge_{\gamma\in\Gamma} \nabla\gamma \leq \bigvee\{\nabla\alpha \mid \forall\gamma\in\Gamma,\ \alpha \leq_C \gamma\}\\
(\nabla 3) & \nabla\{\bigvee\alpha \mid \alpha\in\Phi\} \leq \bigvee\{\nabla\beta \mid \beta\in P_\omega(\bigcup\Phi) \text{ and } \forall\alpha\in\Phi,\ \alpha \between \beta\}
\end{array}$$

5.3.12. Lemma. *We consider the presentation of $V_{P_\omega}\mathbb{L}$.*

1. *In the presence of* ($\nabla 1$)*, the relation* ($\nabla 2'$) *can be replaced by*

$$\begin{array}{ll}
(\nabla 2.0) & 1 \leq \bigvee\{\nabla\beta \mid \beta \in P_\omega L\}\\
(\nabla 2.2) & \nabla\gamma_1 \wedge \nabla\gamma_2 \leq \bigvee\{\nabla\beta \mid \beta \leq_C \gamma_1, \beta \leq_C \gamma_2\}
\end{array}$$

2. *In the presence of* ($\nabla 1$) *and* ($\nabla 2$) *(or its equivalent formulations), the relation* ($\nabla 3$) *can be replaced by*

$$\begin{array}{ll}
(\nabla 3.\!\uparrow) & \nabla\left(\gamma\cup\{\bigvee^{\uparrow} S\}\right) \leq \bigvee^{\uparrow}\{\nabla(\gamma\cup\{a\}) \mid a\in S\} \quad \textit{(S directed)}\\
(\nabla 3.0) & \nabla(\gamma\cup\{0\}) \leq 0\\
(\nabla 3.2) & \nabla(\gamma\cup\{a_1\vee a_2\}) \leq \nabla(\gamma\cup\{a_1\})\vee\nabla(\gamma\cup\{a_2\})\vee\nabla(\gamma\cup\{a_1,a_2\})
\end{array}$$

Proof (1) ($\nabla 2.0$) and ($\nabla 2.2$) are special cases of ($\nabla 2'$), when Γ is empty or a doubleton. To show that they imply ($\nabla 2'$) is an induction on the number of elements needed to enumerate the finite set Γ.

(2) Each of the replacement relations is a special case of ($\nabla 3$) in which all except one of the elements of Φ are singletons. We now show that they are sufficient to imply ($\nabla 3$). First, we show for any *finite* S that

$$\nabla\left(\gamma\cup\{\bigvee S\}\right) \leq \bigvee\{\nabla(\gamma\cup\alpha) \mid \emptyset\neq\alpha\in P_\omega S\}.$$

We use induction on the length of a finite enumeration of S. The base case, S empty, is ($\nabla 3.0$). Now suppose $S = \{a\}\cup S'$. Then

$$\begin{aligned}
&\nabla\left(\gamma\cup\{\bigvee S\}\right)\\
&= \nabla\left(\gamma\cup\{a\vee\bigvee S'\}\right)\\
&\leq \nabla(\gamma\cup\{a\})\vee\nabla\left(\gamma\cup\{\bigvee S'\}\right)\vee\nabla\left((\gamma\cup\{a\})\cup\{\bigvee S'\}\right) \quad \text{(by } (\nabla 3.2))\\
&\leq \nabla(\gamma\cup\{a\})\vee\bigvee\{\nabla(\gamma\cup\alpha') \mid \emptyset\neq\alpha'\in P_\omega S'\}\\
&\qquad \vee\bigvee\{\nabla(\gamma\cup\{a\}\cup\alpha') \mid \emptyset\neq\alpha'\in P_\omega S'\} \qquad \text{(by induction)}\\
&= \bigvee\{\nabla(\gamma\cup\alpha) \mid \emptyset\neq\alpha\in P_\omega S\}.
\end{aligned}$$

Now we can use ($\nabla 3.\uparrow$) to relax the finiteness condition on S, since for an arbitrary S we have

$$\begin{aligned}\nabla\left(\gamma \cup \{\bigvee S\}\right) &= \nabla\left(\gamma \cup \left\{\bigvee^{\uparrow}\left\{\bigvee S_0 \mid S_0 \in P_\omega S\right\}\right\}\right)\\ &\leq \bigvee^{\uparrow}\left\{\nabla\left(\gamma \cup \left\{\bigvee S_0\right\}\right) \mid S_0 \in P_\omega S\right\}.\end{aligned}$$

Finally, we can use induction on the length of a finite enumeration of Φ to deduce ($\nabla 3$). More precisely, one shows by induction on n that

$$\begin{aligned}&\nabla\left(\gamma \cup \{\bigvee S_1, \ldots, \bigvee S_n\}\right)\\ &\leq \bigvee\left\{\nabla(\gamma \cup \alpha) \mid \emptyset \neq \alpha \in P_\omega\left(\bigcup_{i=1}^{n} S_i\right) \text{ and } \forall i,\, \alpha \between S_i\right\}.\end{aligned}$$

$\blacksquare$

5.3.13. Remark. Relation ($\nabla 2.0$) can be weakened even further, to

$$1 \leq \nabla\emptyset \vee \nabla\{1\}.$$

For if β is non-empty then $\beta \leq_C \{1\}$. From ($\nabla 2.2$) we can also deduce that $\nabla\emptyset \wedge \nabla\{1\} = 0$, giving that $\nabla\emptyset$ and $\nabla\{1\}$ are clopen complements.

5.3.14. Lemma. *In $V\mathbb{L}$ we have, for any $S \subseteq \mathbb{L}$,*

$$\Box\left(\bigvee S\right) = \bigvee\left\{\Box\left(\bigvee \alpha\right) \wedge \bigwedge_{a \in \alpha} \Diamond a \mid \alpha \in P_\omega S\right\}.$$

Proof $\geq$ is immediate. For $\leq$, first note that since $\bigvee S$ is a directed join $\bigvee^{\uparrow}_{\alpha \in P_\omega S} \bigvee \alpha$, we have $\Box(\bigvee S) \leq \bigvee^{\uparrow}_{\alpha \in P_\omega S} \Box(\bigvee \alpha)$ and thus we reduce to the case where S is finite. We show that for every $\alpha, \beta \in P_\omega S$ we have

$$\Box\left(\bigvee \alpha \vee \bigvee \beta\right) \wedge \bigwedge_{a \in \alpha} \Diamond a \leq \text{ RHS in statement,}$$

after which the result follows by taking $\beta = S$ and $\alpha = \emptyset$. We use P_ω-induction on β, effectively an induction on the length of an enumeration of its elements. The base case, $\beta = \emptyset$, is trivial. For the induction step, suppose $\beta = \beta' \cup \{b\}$. Then

$$
\begin{aligned}
&\Box\left(\bigvee \alpha \vee \bigvee \beta\right) \wedge \bigwedge_{a\in\alpha} \Diamond a \\
&= \Box\left(\bigvee \alpha \vee b \vee \bigvee \beta'\right) \wedge \bigwedge_{a\in\alpha} \Diamond a \\
&= \Box\left(\bigvee \alpha \vee b \vee \bigvee \beta'\right) \wedge \bigwedge_{a\in\alpha} \Diamond a \wedge \left(\Box\left(\bigvee \alpha \vee \bigvee \beta'\right) \vee \Diamond b\right) \\
&= \left(\Box\left(\bigvee \alpha \vee \bigvee \beta'\right) \wedge \bigwedge_{a\in\alpha} \Diamond a\right) \vee \left(\Box\left(\bigvee (\alpha \cup \{b\}) \vee \bigvee \beta'\right) \wedge \bigwedge_{a\in\alpha\cup\{b\}} \Diamond a\right) \\
&\leq \text{ RHS, by induction.}
\end{aligned}
$$

∎

5.3.15. THEOREM. *Let $\mathbb{L}$ be a frame. Then $V\mathbb{L} \cong V_{P_\omega}\mathbb{L}$.*

Proof First we define a frame homomorphism $\varphi : V_{P_\omega}\mathbb{L} \to V\mathbb{L}$ by $\varphi(\nabla\alpha) = \Box\left(\bigvee\alpha\right) \wedge \bigwedge_{a\in\alpha}\Diamond a$. We must check that this respects the relations. For (∇1), suppose $\alpha \leq_C \beta$. From $\alpha \leq_U \beta$ and $\alpha \leq_L \beta$ we get $\bigwedge_{a\in\alpha}\Diamond a \leq \bigwedge_{b\in\beta}\Diamond b$ and $\bigvee\alpha \leq \bigvee\beta$, giving $\varphi(\nabla\alpha) \leq \varphi(\nabla\beta)$.

For (∇2.0), we have $1 = \Box(0 \vee 1) = \Box 0 \vee (\Box 1 \wedge \Diamond 1) = \varphi\left(\nabla\emptyset\right) \vee \varphi\left(\nabla\left\{1\right\}\right)$.

For (∇2.2), $\varphi\left(\nabla\gamma_1\right) \wedge \varphi\left(\nabla\gamma_2\right)$ is

$$
\begin{aligned}
&\Box\left(\bigvee\gamma_1\right) \wedge \bigwedge_{c\in\gamma_1}\Diamond c \wedge \Box\left(\bigvee\gamma_2\right) \wedge \bigwedge_{c'\in\gamma_2}\Diamond c' \\
&= \Box\left(\bigvee\gamma_1 \wedge \bigvee\gamma_2\right) \wedge \bigwedge_{c\in\gamma_1}\Diamond c \wedge \bigwedge_{c'\in\gamma_2}\Diamond c' \\
&= \Box\left(\bigvee\gamma_1 \wedge \bigvee\gamma_2\right) \wedge \bigwedge_{c\in\gamma_1}\Diamond\left(c \wedge \bigvee\gamma_1 \wedge \bigvee\gamma_2\right) \wedge \bigwedge_{c'\in\gamma_2}\Diamond\left(c' \wedge \bigvee\gamma_1 \wedge \bigvee\gamma_2\right) \\
&= \Box\left(\bigvee\gamma_1 \wedge \bigvee\gamma_2\right) \wedge \bigwedge_{c\in\gamma_1}\Diamond\left(c \wedge \bigvee\gamma_2\right) \wedge \bigwedge_{c'\in\gamma_2}\Diamond\left(c' \wedge \bigvee\gamma_1\right) \\
&= \Box\left(\bigvee_{c\in\gamma_1}\bigvee_{c'\in\gamma_2} c \wedge c'\right) \wedge \bigwedge_{c\in\gamma_1}\bigvee_{c'\in\gamma_2}\Diamond\left(c \wedge c'\right) \wedge \bigwedge_{c'\in\gamma_2}\bigvee_{c\in\gamma_1}\Diamond\left(c \wedge c'\right)
\end{aligned}
$$

Redistributing the disjunctions of the $\Diamond$s, we find that each resulting disjunct is of the form

$$\Box\left(\bigvee_{c\in\gamma_1}\bigvee_{c'\in\gamma_2} c \wedge c'\right) \wedge \bigwedge_{cRc'}\Diamond\left(c \wedge c'\right)$$

for some $R \in P_\omega\left(\gamma_1 \times \gamma_2\right)$ such that $\gamma_1\,\overline{P_\omega}R\,\gamma_2$. Note that for any such R if we define $\beta_R = \{c \wedge c' \mid cRc'\}$ then we have $\beta_R \leq_C \gamma_i$ $(i = 1, 2)$. Now by Lemma 5.3.14 we see

$$
\begin{aligned}
&\Box\left(\bigvee_{c\in\gamma_1}\bigvee_{c'\in\gamma_2} c \wedge c'\right) \wedge \bigwedge_{cRc'}\Diamond\left(c \wedge c'\right) \\
&\leq \bigvee\left\{\Box\left(\bigvee_{cR'c'} c \wedge c'\right) \wedge \bigwedge_{c(R\cup R')c'}\Diamond\left(c \wedge c'\right) \mid R' \in P_\omega\left(\gamma_1 \times \gamma_2\right)\right\} \\
&\leq \bigvee\left\{\Box\left(\bigvee_{cR'c'} c \wedge c'\right) \wedge \bigwedge_{cR'c'}\Diamond\left(c \wedge c'\right) \mid R \subseteq R' \in P_\omega\left(\gamma_1 \times \gamma_2\right)\right\} \\
&= \bigvee\left\{\varphi\left(\nabla\beta_{R'}\right) \mid R \subseteq R' \in P_\omega\left(\gamma_1 \times \gamma_2\right)\right\}
\end{aligned}
$$

and the result follows.

For (∇3.$\uparrow$): the LHS is

$$\begin{aligned}
&\Box\left(\bigvee\gamma \vee \bigvee^{\uparrow} S\right) \wedge \bigwedge_{c\in\gamma}\Diamond c \wedge \Diamond\left(\bigvee^{\uparrow} S\right)\\
&= \bigvee^{\uparrow}\left\{\Box\left(\bigvee\gamma \vee a\right) \mid a \in S\right\} \wedge \bigvee^{\uparrow}\left\{\bigwedge_{c\in\gamma}\Diamond c \wedge \Diamond a \mid a \in S\right\}\\
&= \bigvee^{\uparrow}\left\{\Box\left(\bigvee\gamma \vee a\right) \wedge \bigwedge_{c\in\gamma}\Diamond c \wedge \Diamond a \mid a \in S\right\}
\end{aligned}$$

which is the RHS.

For (∇3.0): the LHS is

$$\Box\left(\bigvee\gamma \vee 0\right) \wedge \bigwedge_{c\in\gamma}\Diamond c \wedge \Diamond 0 = 0.$$

For (∇3.2): the LHS is

$$\begin{aligned}
&\Box\left(\bigvee\gamma \vee a_1 \vee a_2\right) \wedge \bigwedge_{c\in\gamma}\Diamond c \wedge \Diamond\left(a_1 \vee a_2\right)\\
&= \bigvee_{i=1}^{2}\bigvee\left\{\Box\left(\bigvee\beta\right) \wedge \bigwedge_{c\in\beta\cup\gamma\cup\{a_i\}}\Diamond c \mid \beta \in P_\omega\left(\gamma \cup \{a_1, a_2\}\right)\right\}\\
&\le \bigvee_{i=1}^{2}\bigvee\left\{\varphi\left(\nabla\left(\beta\cup\gamma\cup\{a_i\}\right)\right) \mid \beta \in P_\omega\left(\gamma \cup \{a_1, a_2\}\right)\right\}\\
&= \bigvee_{i=1}^{2}\bigvee\left\{\varphi\left(\nabla\beta\right) \mid \gamma\cup\{a_i\} \subseteq \beta \in P_\omega\left(\gamma \cup \{a_1, a_2\}\right)\right\}\\
&= \varphi\left(\nabla\left(\gamma\cup\{a_1\}\right)\right) \vee \varphi\left(\nabla\left(\gamma\cup\{a_2\}\right)\right) \vee \varphi\left(\nabla\left(\gamma\cup\{a_1, a_2\}\right)\right).
\end{aligned}$$

Next, we define the frame homomorphism $\psi : V\mathbb{L} \to V_{P_\omega}\mathbb{L}$ by

$$\begin{aligned}
\psi\left(\Box a\right) &= \bigvee\left\{\nabla\alpha \mid \alpha \le_L \{a\}\right\} = \nabla\emptyset \vee \nabla\{a\}\\
\psi\left(\Diamond a\right) &= \bigvee\left\{\nabla\alpha \mid \alpha \le_U \{a\}\right\} = \bigvee\left\{\nabla\left(\beta\cup\{a\}\right) \mid \beta \in P_\omega L\right\}.
\end{aligned}$$

(Observe that the expression for $\psi\left(\Diamond a\right)$ could be simplified even further to $\nabla\{1, a\}$.) We check the relations. First, it is clear that ψ respects monotonicity of $\Box$ and $\Diamond$.

$\Box$ preserves directed joins:

$$\psi\left(\Box\left(\bigvee_i^{\uparrow} a_i\right)\right) = \nabla\emptyset \vee \nabla\{\bigvee_i^{\uparrow} a_i\} = \bigvee_i^{\uparrow}\psi\left(\Box a_i\right).$$

$\Box$ preserves top immediately from (∇2.0).

$\Box$ preserves binary meets:

$$\begin{aligned}
\psi\left(\Box a_1\right) \wedge \psi\left(\Box a_2\right) &= \nabla\emptyset \vee \left(\nabla\{a_1\} \wedge \nabla\{a_2\}\right)\\
&= \nabla\emptyset \vee \bigvee\left\{\nabla\beta \mid \beta \le_C \{a_1\},\ \beta \le_C \{a_2\}\right\}\\
&= \nabla\emptyset \vee \nabla\{a_1 \wedge a_2\} = \psi\left(\Box\left(a_1 \wedge a_2\right)\right).
\end{aligned}$$

$\Diamond$ preserves joins:

$$\begin{aligned}\psi\left(\Diamond\left(\bigvee A\right)\right) &= \bigvee\left\{\nabla\left(\beta\cup\left\{\bigvee A\right\}\right)\mid\beta\in P_\omega L\right\}\\ &= \bigvee\left\{\nabla\left(\beta\cup\alpha\right)\mid\beta\in P_\omega L,\emptyset\neq\alpha\in P_\omega A\right\}\\ &= \bigvee_{a\in A}\bigvee\left\{\nabla\left(\beta\cup\{a\}\right)\mid\beta\in P_\omega L\right\}=\bigvee_{a\in A}\psi(\Diamond a).\end{aligned}$$

For the first mixed relation, and noting that $\nabla\emptyset\wedge\nabla\left(\beta\cup\{b\}\right)\leq\nabla\emptyset\wedge\nabla\{1\}=0$, we have:

$$\begin{aligned}\psi\left(\Box a\right)\wedge\psi\left(\Diamond b\right) &= \bigvee_{\beta\in P_\omega L}\left(\nabla\emptyset\vee\nabla\{a\}\right)\wedge\nabla\left(\beta\cup\{b\}\right)\\ &= \bigvee_{\beta\in P_\omega L}\nabla\{a\}\wedge\nabla\left(\beta\cup\{b\}\right)\\ &= \bigvee\left\{\nabla\gamma\mid\exists\beta,\,\gamma\leq_C\{a\},\gamma\leq_C\beta\cup\{b\}\right\}\\ &\leq \bigvee_{\beta\in P_\omega L}\nabla\left(\beta\cup\{a\wedge b\}\right)=\psi\left(\Diamond\left(a\wedge b\right)\right).\end{aligned}$$

For the second:

$$\begin{aligned}\psi\left(\Box\left(a\vee b\right)\right) &= \nabla\emptyset\vee\nabla\{a\vee b\}\\ &= \nabla\emptyset\vee\nabla\{a\}\vee\nabla\{b\}\vee\nabla\{a,b\}\\ &\leq \psi\left(\Box a\right)\vee\psi\left(\Diamond b\right)\end{aligned}$$

since $\nabla\emptyset\vee\nabla\{a\}=\psi\left(\Box a\right)$ and $\nabla\{b\}\vee\nabla\{a,b\}\leq\psi\left(\Diamond b\right)$.

It remains to show that φ and ψ are mutually inverse.

$$\varphi\left(\psi\left(\Box a\right)\right)=\varphi\left(\nabla\emptyset\vee\nabla\{a\}\right)=\Box 0\vee\left(\Box a\wedge\Diamond a\right)=\Box a$$

since $\Box 0\wedge\Diamond a\leq\Diamond\left(0\wedge a\right)=0$.

Next, to show $\varphi\left(\psi\left(\Diamond a\right)\right)=\Diamond a$, we have

$$\begin{aligned}\varphi\left(\psi\left(\Diamond a\right)\right) &= \bigvee_{\beta\in P_\omega L}\left(\Box\left(\bigvee\beta\vee a\right)\wedge\bigwedge_{b\in\beta}\Diamond b\wedge\Diamond a\right)\\ &\leq \Diamond a\\ &= \Box\left(1\vee a\right)\wedge\Diamond 1\wedge\Diamond a=\varphi\left(\nabla\{1,a\}\right)\leq\varphi\left(\psi\left(\Diamond a\right)\right).\end{aligned}$$

Finally, to show $\psi\left(\varphi\left(\nabla\alpha\right)\right)=\nabla\alpha$, we have

$$\begin{aligned}\psi\left(\varphi\left(\nabla\alpha\right)\right) &= \psi\left(\Box\left(\bigvee\alpha\right)\wedge\bigwedge_{a\in\alpha}\Diamond a\right)\\ &= \left(\nabla\emptyset\vee\nabla\left\{\bigvee\alpha\right\}\right)\wedge\bigwedge_{a\in\alpha}\bigvee_{\beta_a\in P_\omega L}\nabla\left(\beta\cup\{a\}\right).\end{aligned}$$

Now,

$$\begin{aligned}\bigwedge_{a\in\alpha}\bigvee_{\beta_a\in P_\omega L}\nabla\left(\beta\cup\{a\}\right) &= \bigvee\left\{\nabla\gamma\mid\forall a\in\alpha,\,\exists\beta_a\in P_\omega L,\,\gamma\leq_C\beta_a\cup\{a\}\right\}\\ &= \bigvee\left\{\nabla\gamma\mid\gamma\leq_U\alpha\right\}.\end{aligned}$$

Also

$$\begin{aligned}
\nabla\emptyset \wedge \bigvee\{\nabla\gamma \mid \gamma \leq_U \alpha\} &= \bigvee\{\nabla\delta \mid \delta \leq_C \emptyset, \delta \leq_U \alpha\} \\
&= \begin{cases} \nabla\alpha & \text{if } \alpha = \emptyset \\ 0 & \text{if } \alpha \neq \emptyset \end{cases} \\
\nabla\left\{\bigvee\alpha\right\} \wedge \bigvee\{\nabla\gamma \mid \gamma \leq_U \alpha\} &= \bigvee\left\{\nabla\delta \mid \delta \leq_C \left\{\bigvee\alpha\right\}, \delta \leq_U \alpha\right\} \\
&= \nabla\left(\alpha \cup \left\{\bigvee\alpha\right\}\right) \\
&= \bigvee\{\nabla(\alpha \cup \alpha') \mid \emptyset \neq \alpha' \in P_\omega\alpha\} \\
&= \begin{cases} 0 & \text{if } \alpha = \emptyset \\ \nabla\alpha & \text{if } \alpha \neq \emptyset \end{cases}
\end{aligned}$$

It follows that, whether α is empty or not, $\psi(\varphi(\nabla\alpha)) = \nabla\alpha$. ∎

5.3.4 Categorical properties of the T-powerlocale

In this section we discuss two categorical properties of the T-powerlocale construction. First we show how to extend the frame construction V_T to an endofunctor on the category **Fr** of frames. As a second topic we will see how the natural transformation $i : V_{P_\omega} \to V_{Id}$ (discussed in §5.2.5 as $i : V \to Id$) can be generalized to a natural transformation

$$\widehat{\rho} : V_T \to V_{T'},$$

for any natural transformation $\rho : T' \to T$ satisfying some mild conditions (where T and T' are two finitary, weak pullback preserving set functors).

V_T is a functor

We start with introducing a natural way to transform a frame homomorphism $f\colon \mathbb{L} \to \mathbb{M}$ into a frame homomorphism from $V_T\mathbb{L}$ to $V_T\mathbb{M}$. For that purpose we first prove the following technical lemma.

5.3.16. Lemma. *Let $\mathbb{L}, \mathbb{M}$ be frames and let $f\colon \mathbb{L} \to \mathbb{M}$ be a frame homomorphism. Then the map $\nabla \circ Tf\colon TL \to V_TM$, i.e. $\alpha \mapsto \nabla(Tf)(\alpha)$, is compatible with the relations* ($\nabla 1$)*,* ($\nabla 2$) *and* ($\nabla 3$).

Proof We abbreviate $\heartsuit := \nabla \circ Tf$, that is, for $\alpha \in TL$, we define $\heartsuit\alpha := \nabla(Tf)(\alpha)$.

In order to prove that $\heartsuit$ is compatible with ($\nabla 1$), we need to show that

$$\text{for all } \alpha, \beta \in TL: \ \alpha \overline{T}{\leq_{\mathbb{L}}} \beta \text{ implies } \heartsuit\alpha \leq_{V_T\mathbb{M}} \heartsuit\beta. \tag{5.9}$$

To see this, assume that $\alpha, \beta \in TL$ are such that $\alpha \overline{T}{\leq_{\mathbb{L}}} \beta$. From this it follows by Lemma 5.2.8 and the assumption that f is a frame homomorphism, that $(Tf)(\alpha) \overline{T}{\leq_{\mathbb{M}}} (Tf)(\beta)$. Then by $(\nabla 1)_{\mathbb{M}}$ we obtain that $\heartsuit\alpha \leq_{V_T\mathbb{M}} \heartsuit\beta$, as required.

Proving compatibility with ($\nabla 2$) boils down to showing

$$\text{for all } \Gamma \in P_\omega TL : \bigwedge_{\alpha\in\Gamma} \heartsuit\alpha \leq \bigvee\{\heartsuit(T\bigwedge)(\Psi) \mid \Psi \in SRD(\Gamma)\}. \tag{5.10}$$

For this purpose, given $\Gamma \in P_\omega TL$, let $\Gamma' \in P_\omega TM$ denote the set $(P_\omega Tf)(\Gamma) = \{(Tf)(\alpha) \mid \alpha \in \Gamma\}$. Then we may observe

$$\begin{aligned}
\bigwedge_{\alpha\in\Gamma} \heartsuit\alpha &= \bigvee\{\nabla(T\bigwedge)(\Psi) \mid \Psi \in SRD(\Gamma')\} && (\nabla 1)\\
&\leq \bigvee\{\nabla(T\bigwedge)(TP_\omega f)(\Phi) \mid \Phi \in SRD(\Gamma)\} && (\text{Lemma 5.3.9})\\
&= \bigvee\{\nabla(Tf)(T\bigwedge)(\Phi) \mid \Phi \in SRD(\Gamma)\} && (\dagger)\\
&= \bigvee\{\heartsuit(T\bigwedge)(\Phi) \mid \Phi \in SRD(\Gamma)\} && (\text{definition of } \heartsuit)
\end{aligned}$$

Here the identity marked ($\dagger$) is easily justified by f being a homomorphism: it follows from $f \circ \bigwedge = \bigwedge \circ (P_\omega f)$ and functoriality of T that $(Tf) \circ (T\bigwedge) = (T\bigwedge) \circ (TP_\omega f)$.

Finally, for compatibility with ($\nabla 3$) we need to verify that

$$\text{for all } \Phi \in TPL : \heartsuit(T\bigvee)(\Psi) \leq \bigvee\{\heartsuit\beta \mid \beta \mathrel{\overline{T}{\in}} \Phi\}. \tag{5.11}$$

To prove this, we calculate for a given $\Phi \in TPL$:

$$\begin{aligned}
\heartsuit(T\bigvee)(\Phi) &= \nabla(Tf)(T\bigvee)(\Phi) && (\text{definition of } \heartsuit)\\
&= \nabla(T\bigvee)(TPf)(\Phi) && (f \text{ a frame homomorphism})\\
&\leq \bigvee\{\nabla\beta \mid \beta \mathrel{\overline{T}{\in}} (TPf)(\Phi)\} && (\nabla 3)_{\mathbb{M}}\\
&= \bigvee\{\nabla(Tf)(\gamma) \mid \gamma \mathrel{\overline{T}{\in}} \Phi\} && (\ddagger)\\
&= \bigvee\{\heartsuit\gamma \mid \gamma \mathrel{\overline{T}{\in}} \Phi\} && (\text{definition of } \heartsuit)
\end{aligned}$$

Here the identity ($\ddagger$) follows from the observation that for all $\beta \in TM$ and $\Phi \in TPL$, we have $\beta \mathrel{\overline{T}{\in}} (TPf)(\Phi)$ iff β is of the form $\beta = (Tf)(\gamma)$ for some $\gamma \in TL$. Using Fact 5.2.7, this is easily derived from the observation that for $b \in M$ and $A \in PL$, we have $b \in (Pf)A$ iff $b = f(c)$ for some $c \in A$. ∎

Lemma 5.3.16 justifies the following definition.

5.3.17. Definition. Let $f\colon \mathbb{L} \to \mathbb{M}$ be a frame homomorphism. We define $V_T f\colon V_T\mathbb{L} \to V_T\mathbb{M}$ to be the unique frame homomorphism extending

$$\nabla \circ Tf\colon TL \to V_T M.$$

5.3.18. THEOREM. *Let T be a standard, finitary, weak pullback-preserving functor. Then the operation V_T defined above is an endofunctor on the category* **Fr**.

Proof Since for an arbitrary $f : \mathbb{L} \to \mathbb{M}$ we have ensured by definition that $V_T F$ is a frame homomorphism from $V_T\mathbb{L}$ to $V_T\mathbb{M}$, it is left to show that V_T maps the identity map of a frame to the identity map of its T-powerlocale, and distributes over function composition. We confine our attention to the second property.

Let $f : \mathbb{K} \to \mathbb{L}$ and $g : \mathbb{L} \to \mathbb{M}$ be two frame homomorphisms. In order to show that $V_T(g \circ f) = V_T g \circ V_T f$, first recall that $V_T(g \circ f)$ is by definition the unique frame homomorphism extending the map $\nabla_{\mathbb{M}} \circ T(g \circ f)\colon TK \to V_T\mathbb{M}$. Hence, it suffices to prove that the map $V_T g \circ V_T f$, which is obviously a frame homomorphism, extends $\nabla_{\mathbb{M}} \circ T(g \circ f)$. But it is easy to see why this is the case: given an arbitrary element $\alpha \in TK$, a straightforward unravelling of definitions shows that

$$(V_T g \circ V_T f)(\alpha) = V_T g(\nabla_{\mathbb{L}}(Tf)(\alpha)) = \nabla_{\mathbb{M}}(Tg)(Tf)(\alpha) = \nabla_{\mathbb{M}} T(g \circ f)(\alpha),$$

as required. ∎

Natural transformations between V_T and $V_{T'}$

Now that we have seen how each finitary, weak pulback preserving set functor T induces a functor V_T on the category of frames, we investigate the relation *between* two such functors $V_T, V_{T'}$. In fact, we have already seen an example of this: recall that in §5.2.5 we mentioned Johnstone's result [55] that the standard Vietoris functor V is in fact a *comonad* on the category of frames. In our nabla-based presentation of this functor as $V = V_{P_\omega}$, thinking of the identity functor on the category **Fr** as the Vietoris functor V_{Id}, we can see the *counit* of this comonad as a natural transformation

$$i : V_{P_\omega} \to V_{Id},$$

given by $i_{\mathbb{L}} : \nabla A \mapsto \bigwedge A$. More precisely, we can show that the map $\heartsuit : P_\omega L \to L$ given by $\heartsuit A := \bigwedge A$ is compatible with the ∇-axioms, and hence can be uniquely extended to the homomorphism $i_{\mathbb{L}}$; subsequently we can show that this i is natural in $\mathbb{L}$. Recall that in the case of a concrete topological space (X, τ), this counit corresponds on the dual side to the singleton map $\sigma_X : s \mapsto \{s\}$ which provides an embedding of a compact Hausdorff topology into its Vietoris space.

We will now see how to generalize this picture, of the natural transformation $i : V_{P_\omega} \to V_{Id}$ being induced by the singleton natural transformation $\sigma : Id \to P_\omega$, to a more general setting. First consider the following definition.

5.3.19. DEFINITION. Let T and T' be standard, finitary, weak pullback-preserving functors. A natural transformation $\rho : T' \to T$ is said to *respect relation lifting* if for any relation $R \subseteq X \times Y$ we have, for all $\alpha, \beta \in TX$

$$\alpha \, \overline{T}R \, \beta \text{ only if } \rho_X(\alpha) \, \overline{T'}R \, \rho_Y(\beta). \tag{5.12}$$

We call ρ *base-invariant* if it commutes with *Base*, that is,

$$\mathit{Base}^{T'} = \mathit{Base}^{T} \circ \rho. \tag{5.13}$$

for any set X.

5.3.20. Example. We record three examples of base-invariant natural transformations which respect relation lifting.

1. The base transformation $\mathit{Base}^T : T \to P_\omega$;
2. The singleton natural transformation $\sigma \colon \mathrm{Id} \to P_\omega$, which is in fact a special case of (1);
3. The diagonal map δ (given by $\delta_X : x \mapsto (x, x)$); it is straightforward to check that as a natural transformation, $\delta : \mathit{Id} \to \mathit{Id} \times \mathit{Id}$ also satisfies both properties of Definition 5.3.19.

As we will see now, every base-invariant natural transformation $\rho : T' \to T$ that respects relation lifting, induces a natural transformation $\widehat{\rho} \colon V_T \to V_{T'}$. In particular, the natural transformation $i : V \to \mathit{Id}$ can be seen as $i = \widehat{\sigma}$, where $\sigma : \mathit{Id} \to P_\omega$ is the singleton transformation discussed above.

5.3.21. Theorem. *Let T and T' be standard, finitary, weak pullback-preserving functors, assume that $\rho : T' \to T$ is a base-invariant natural transformation that respects relation lifting, and let $\mathbb{L}$ be a frame. Then the map from TL to $V_{T'}L$ given by*

$$\alpha \mapsto \bigvee \{\nabla\alpha' \mid \alpha' \in T'L, \rho(\alpha') \, \overline{T}{\leq} \, \alpha\}$$

specifies a frame homomorphism

$$\widehat{\rho}_{\mathbb{L}} \colon V_T\mathbb{L} \to V_{T'}\mathbb{L}$$

which is natural in $\mathbb{L}$.

Proof We let $\heartsuit : TL \to L$ denote the map given in the statement of the Theorem, that is, $\heartsuit\alpha := \bigvee \{\nabla\alpha' \mid \alpha' \in T'L, \rho(\alpha') \, \overline{T}{\leq} \, \alpha\}$. We will first prove that this map is compatible with, respectively, (∇1), (∇2) and (∇3), and then turn to the naturality of the induced frame homomorphism.

1. Claim. The map $\heartsuit$ is compatible with (∇1).

Proof of Claim To show that $\heartsuit$ is compatible with (∇1), take two elements $\alpha, \beta \in TL$ such that $\alpha \mathrel{\overline{T}{\leq}} \beta$. Then for any $\alpha' \in T'L$ such that $\rho(\alpha') \mathrel{\overline{T}{\leq}} \alpha$, by transitivity of $\overline{T}{\leq}$ (Fact 5.2.7(5)), we obtain that $\rho(\alpha') \mathrel{\overline{T}{\leq}} \beta$. From this it is immediate that $\heartsuit\alpha \leq \heartsuit\beta$, as required.

2. Claim. The map $\heartsuit$ is compatible with (∇2).

Proof of Claim For compatibility with (∇2), it suffices to show compatibility with ($\nabla 2'$). That is, for $\Gamma \in P_\omega TL$, we will verify that

$$\bigwedge\{\heartsuit\gamma \mid \gamma \in \Gamma\} \leq \bigvee\{\heartsuit\beta \mid \beta \mathrel{\overline{T}{\leq}} \gamma, \text{ for all } \gamma \in \Gamma\}. \tag{5.14}$$

We start with rewriting the left hand side of (5.14) into

$$\begin{aligned} \bigwedge\{\heartsuit\gamma \mid \gamma \in \Gamma\} &= \bigwedge\left\{\bigvee\{\nabla\gamma' \mid \rho(\gamma') \mathrel{\overline{T}{\leq}} \gamma\} \mid \gamma \in \Gamma\right\} && \text{(definition of } \heartsuit) \\ &= \bigvee\left\{\bigwedge\{\varphi_\gamma \mid \gamma \in \Gamma\} \mid \varphi \in \mathcal{C}_\Gamma\right\} && \text{(frame distributivity)} \end{aligned}$$

where we define $\mathcal{C}_\Gamma := \{\varphi : \Gamma \to T'L \mid \rho(\varphi_\gamma) \mathrel{\overline{T}{\leq}} \gamma, \text{ for all } \gamma \in \Gamma\}$.

For any map $\varphi \in \mathcal{C}_\Gamma$ we may calculate

$$\begin{aligned} &\bigwedge\{\varphi_\gamma \mid \gamma \in \Gamma\} \\ &= \bigvee\{\nabla\gamma' \mid \gamma' \mathrel{\overline{T'}{\leq}} \varphi_\gamma, \forall\gamma \in \Gamma\} && (\nabla 2') \\ &\leq \bigvee\{\nabla\gamma' \mid \rho(\gamma') \mathrel{\overline{T}{\leq}} \rho(\varphi_\gamma), \forall\gamma \in \Gamma\} && (\rho \text{ respects relation lifting}) \\ &\leq \bigvee\{\nabla\gamma' \mid \rho(\gamma') \mathrel{\overline{T}{\leq}} \gamma, \forall\gamma \in \Gamma\} && (\varphi \in \mathcal{C}_\Gamma, \text{ transitivity of } \overline{T}{\leq}) \\ &= \bigvee\left\{\bigvee\{\nabla\gamma' \mid \rho(\gamma') \mathrel{\overline{T}{\leq}} \beta\} \mid \beta \mathrel{\overline{T}{\leq}} \gamma, \forall\gamma \in \Gamma\right\} && (\text{associativity of } \textstyle\bigvee) \\ &= \bigvee\{\heartsuit\beta \mid \beta \mathrel{\overline{T}{\leq}} \gamma, \forall\gamma \in \Gamma\} && (\text{definition of } \heartsuit) \end{aligned}$$

From the above calculations, (5.14) is immediate.

3. Claim. The map $\heartsuit$ is compatible with (∇3).

Proof of Claim We need to show, for an arbitrary but fixed set $\Phi \in TPL$, that

$$\heartsuit(T\textstyle\bigvee)(\Phi) = \bigvee\{\heartsuit\alpha \mid \alpha \mathrel{\overline{T}{\in}} \Phi\}. \tag{5.15}$$

By definition, on the left hand side of (5.15) we find

$$\heartsuit(T\textstyle\bigvee)(\Phi) = \bigvee\{\nabla\beta' \mid \rho(\beta') \mathrel{\overline{T}{\leq}} (T\bigvee)(\Phi)\},$$

while on the right hand side we obtain, by definition of $\heartsuit$,

$$\begin{aligned}\bigvee\{\heartsuit\alpha \mid \alpha\ \overline{T}\!\in \Phi\} &= \bigvee\left\{\bigvee\{\nabla\alpha' \mid \rho(\alpha')\ \overline{T}\!\leq \alpha\} \mid \alpha\ \overline{T}\!\in \Phi\right\}\\ &= \bigvee\left\{\nabla\alpha' \mid \rho(\alpha')\ \overline{T}(\leq ; \in)\ \Phi\right\}\end{aligned}$$

where the latter equality is by associativity of $\bigvee$, and the compositionality of relation lifting (Fact 5.2.7(5)).

As a consequence, in order to establish the compatibility of $\heartsuit$ with (∇3), it suffices to show that

$$\nabla\beta' \leq \bigvee\left\{\nabla\alpha' \mid \rho(\alpha')\ \overline{T}(\leq ; \in)\ \Phi\right\} \text{ for any } \beta' \text{ with } \rho(\beta')\ \overline{T}\!\leq (T\bigvee)(\Phi). \quad (5.16)$$

Let β' be an arbitrary element of TL such that $\rho(\beta')\ \overline{T}\!\leq (T\bigvee)(\Phi)$. Our goal will be to find a set $\Phi' \in T'PL$ satisfying (5.20), (5.21) and (5.22) below: clearly this will satisfy to prove (5.16).

By Fact 5.2.9 we obtain that

$$Base^{T}(\rho\beta)\ \overline{P}\!\leq Base^{T}((T\bigvee)(\Phi)) = (P\bigvee)Base^{T}(\Phi),$$

and since ρ is base-invariant, we have $Base^{T'}(\beta') = Base^{T}(\rho\beta')$. Combining these facts we see that $Base^{T'}(\beta')\ \overline{P}\!\leq (P\bigvee)Base^{T}(\Phi)$. This motivates the definition of the following map $\mathcal{H} : Base^{T'}(\beta') \to P_\omega PL$:

$$\mathcal{H}(b) := \{B \in Base^{T}(\Phi) \mid b \leq \bigvee B\}.$$

From the definitions it is immediate that

$$\text{for all } b \in Base^{T'}(\beta') :\ b \leq \bigwedge\left\{\bigvee B \mid B \in \mathcal{H}(b)\right\}. \quad (5.17)$$

Also, given a set $\mathcal{B} \in P_\omega PL$, let $\mathcal{C}_B$ be the collection of choice functions on $\mathcal{B}$, that is:

$$\mathcal{C}_{\mathcal{B}} := \{f : \mathcal{B} \to L \mid f(B) \in B \text{ for all } B \in \mathcal{B}\}.$$

Then it follows by frame distributivity that

$$\bigwedge\left\{\bigvee B \mid B \in \mathcal{B}\right\} = \bigvee\left\{\bigwedge(Pf)(\mathcal{B}) \mid f \in \mathcal{C}_{\mathcal{B}}\right\}. \quad (5.18)$$

Define the map $K : P_\omega PL \to PL$ by

$$K(\mathcal{B}) := \left\{\bigwedge(Pf)(\mathcal{B}) \mid f \in \mathcal{C}_{\mathcal{B}}\right\},$$

then it follows from (5.17), (5.18) and the definitions that

$$\text{for all } b \in Base^{T'}(\beta') :\ b \leq \bigvee K(\mathcal{H}(b)). \quad (5.19)$$

As a corollary, if we define

$$\Phi' := (T'K)(T'\mathcal{H})(\beta'),$$

then it follows from (5.19), by the properties of relation lifting, that $\beta' \ \overline{T'}{\leq} \ (T'\bigvee)(\Phi')$, so that an application of (∇1) yields

$$\nabla\beta' \leq \nabla(T'\bigvee)(\Phi'). \tag{5.20}$$

Also, on the basis of an application of (∇3) we may conclude that

$$\nabla(T'\bigvee)(\Phi') \leq \bigvee\{\nabla\gamma' \mid \gamma' \ \overline{T'}{\in} \ \Phi'\}. \tag{5.21}$$

This means that we are done with the proof of (5.16) if we can show that

$$\text{for any } \gamma' \in T'L, \text{ if } \gamma' \ \overline{T'}{\in} \ \Phi' \text{ then } \rho(\gamma') \ \overline{T}(\leq ; \in) \ \Phi. \tag{5.22}$$

For a proof of (5.22), let γ' be an arbitrary T'-lifted member of Φ' and recall that $\Phi' = (TK)(T\mathcal{H})(\beta')$. Then it follows by the assumption that ρ respects relation lifting, that $\rho(\gamma') \ \overline{T}{\in} \ \rho(\Phi') = (TK)(T\mathcal{H})(\rho(\beta'))$. Given our assumption on β', this means that the relation between $\rho(\gamma')$ and Φ can be summarized as

$$\rho(\gamma') \ \overline{T}{\in} \ (TK)(T\mathcal{H})(\beta) \text{ and } \beta \ \overline{T}{\leq} \ (T\bigvee)(\Phi) \text{ for some } \beta \in TBase^{T'}(\beta'), \tag{5.23}$$

where for β we may take $\rho(\beta')$.

Returning to the ground level, observe that for any $c \in L$, $A \in Base^{T}(\Phi)$, we have

$$\text{if } c \in K\mathcal{H}(b) \text{ and } b \leq \bigvee A, \text{ for some } b \in Base^{T'}(\beta'), \text{ then } c \ (\leq ; \in) \ A. \tag{5.24}$$

To see why this is the case, assume that $c \in K\mathcal{H}(b)$ and $b \leq \bigvee A$, for some $b \in Base^{T'}(\beta')$. Then by definition of $\mathcal{H}$ we find $A \in \mathcal{H}(b)$, while $c \in K\mathcal{H}(b)$ simply means that $c = \bigwedge\{f(B) \mid B \in \mathcal{H}(b)\}$, for some $f \in \mathcal{C}_{\mathcal{H}(b)}$. But then it is immediate that $c \leq f(A)$, while $f(A) \in A$ by definition of $\mathcal{C}_{\mathcal{H}(b)}$. Thus $f(A)$ is the required element witnessing that $c \ (\leq ; \in) \ A$.

But by the properties of relation lifting, we may derive from (5.24) that

$$\begin{aligned} &\text{if } \gamma \ \overline{T}{\in} \ (TK)(T\mathcal{H})(\beta) \text{ and } \beta \ \overline{T}{\leq} \ (T\bigvee)(\Phi) \text{ for some } \beta \in TBase^{T'}(\beta'), \\ &\quad \text{then } \gamma \ \overline{T}(\leq ; \in) \ \Phi, \end{aligned} \tag{5.25}$$

so that it is immediate by (5.23) that $\rho(\gamma') \ \overline{T}(\leq ; \in) \ \Phi$. This proves (5.22).

As mentioned already, the compatibility of $\heartsuit$ with (∇3) is immediate by (5.20), (5.21) and (5.22), and so this finishes the proof of Claim 3.

As a corollary of the Claims 1–3, we may uniquely extend $\heartsuit$ to a homomorphism $\widehat{\rho}_{\mathbb{L}} : V_T\mathbb{L} \to V_{T'}\mathbb{L}$. Clearly then, in order to prove the theorem it suffices to prove the following claim.

4. CLAIM. The family of homomorphisms $\widehat{\rho}_{\mathbb{L}}$ constitutes a natural transformation $\widehat{\rho} : V_T \to V_{T'}$.

Proof of Claim Given two frames $\mathbb{L}$ and $\mathbb{M}$ and a frame homomorphism $f : \mathbb{L} \to \mathbb{M}$, we need to show that the following diagram commutes:

$$\begin{tikzcd}
V_T\mathbb{L} \arrow[r, "\widehat{\rho}_{\mathbb{L}}"] \arrow[d, "V_T f"'] & V_{T'}\mathbb{L} \arrow[d, "V_{T'} f"] \\
V_T\mathbb{M} \arrow[r, "\widehat{\rho}_{\mathbb{M}}"] & V_{T'}\mathbb{M}
\end{tikzcd}$$

To show this, take an arbitrary element $\alpha \in TL$, and consider the following calculation:

$$\begin{array}{ll}
(V_{T'} f)(\widehat{\rho}_{\mathbb{L}}(\nabla\alpha)) & \\
= (V_{T'} f)(\heartsuit\alpha) & \text{(definition of } \widehat{\rho}_{\mathbb{L}})) \\
= (V_{T'} f)\left(\bigvee\left\{\nabla\beta' \mid \rho_L(\beta')\, \overline{T}{\leq}\, \alpha\right\}\right) & \text{(definition of } \heartsuit)) \\
= \bigvee\left\{(V'_T f)(\nabla\beta') \mid \rho_L(\beta')\, \overline{T}{\leq}\, \alpha\right\} & (V_{T'} f \text{ is a frame homomorphism}) \\
= \bigvee\left\{\nabla(T' f)(\beta') \mid \rho_L(\beta')\, \overline{T}{\leq}\, \alpha\right\} & \text{(definition of } V_{T'} f)) \\
= \bigvee\left\{\nabla\delta' \mid \rho_M(\delta')\, \overline{T}{\leq}\, (Tf)(\alpha)\right\} & (\dagger) \\
= \heartsuit(Tf)(\alpha) & \text{(definition of } \heartsuit)) \\
= \widehat{\rho}_{\mathbb{M}}(\nabla(Tf)(\alpha)) & \text{(definition of } \widehat{\rho}_{\mathbb{M}})) \\
= \widehat{\rho}_{\mathbb{M}}((V_T f)(\nabla\alpha)) & \text{(definition of } V_T f))
\end{array}$$

Here the crucial step, marked (†), is proved by establishing the two respective inequalities, as follows. For the inequality $\leq$, it is straightforward to show that the set of joinands on the left hand side is included in that on the right hand side, and this follows from

$$\rho_L(\beta')\, \overline{T}{\leq}\, \alpha \text{ implies } \rho_M((T' f)(\beta'))\, \overline{T}{\leq}\, (Tf)(\alpha). \tag{5.26}$$

To prove (5.26), suppose that $\rho_L(\beta')\, \overline{T}{\leq}\, \alpha$; then it follows by the fact that f is a homomorphism, and hence, monotone, that $(Tf)(\rho_L(\beta'))\, \overline{T}{\leq}\, (Tf)(\alpha)$. But since ρ is a natural transformation, we also have $(Tf)(\rho_L(\beta')) = \rho_M(T' f)(\beta')$, and from this (5.26) is immediate.

In order to prove the opposite inequality

$$\bigvee\left\{\nabla\delta' \mid \rho_M(\delta')\, \overline{T}{\leq}\, (Tf)(\alpha)\right\} \leq \bigvee\left\{\nabla(T' f)(\beta') \mid \rho_L(\beta')\, \overline{T}{\leq}\, \alpha\right\}, \tag{5.27}$$

fix an arbitrary element $\delta' \in TL$ such that $\rho_M(\delta')\, \overline{T}{\leq}\, (Tf)(\alpha)$.

Define the map $h : Base^{T'}(\delta') \to L$ by putting

$$h(d) := \bigwedge\{a \in Base^{T}(\alpha) \mid d \leq f(a)\}.$$

Then for all $d \in Base^{T'}(\delta')$ and all $a \in Base^{T}(\alpha)$, we find that $d \leq fa$ implies $hd \leq a$; this can be expressed by the relational inclusion

$$Grf \,;\, \geq \,;\, Grh \;\subseteq\; \geq$$

so that by the properties of relation lifting we may conclude that $Gr(Tf) \,;\, \overline{T}{\geq} \,;\, Gr(Th) \;\subseteq\; \overline{T}{\geq}$, which is just another way of saying that, for all $\delta \in TBase^{T'}(\delta')$, we have

$$\delta \; \overline{T}{\leq} \; (Tf)(\alpha) \text{ only if } (Th)(\delta) \; \overline{T}{\leq} \; \alpha. \tag{5.28}$$

Now define

$$\beta' := (T'h)(\delta'),$$

then we may conclude from the fact that ρ respects relation lifting that $\rho_L(\beta') = (Th)\rho_M(\delta')$, and so by the assumption that $\rho_M(\delta') \; \overline{T}{\leq} \; (Tf)(\alpha)$, we obtain by (5.28) that

$$\rho_L(\beta') \; \overline{T}{\leq} \; \alpha. \tag{5.29}$$

Similarly, from the fact that $d \leq fhd$, for each $d \in Base^{T'}(\delta')$, we may derive that $\delta' \; \overline{T'}{\leq} \; (T'f)(\beta')$, and so by ($\nabla$1) we may conclude that

$$\nabla\delta' \leq \nabla(T'f)(\beta'). \tag{5.30}$$

Finally, (5.26) is immediate by (5.25) and (5.30).

This finishes the proof of Claim 4. ∎

5.3.22. Remark. The definition of the $\widehat{\rho}_{\mathbb{L}}\colon V_T\mathbb{L} \to V_{T'}\mathbb{L}$, using the assignment

$$\alpha \mapsto \bigvee\{\nabla\alpha' \mid \alpha' \in T'L, \rho(\alpha') \; \overline{T}{\leq} \; \alpha\},$$

is very similar to that of a right adjoint. If it were the case that $\widehat{\rho}_{\mathbb{L}}$ preserved *all* meets, then the adjoint functor theorem would allow us to define its left adjoint. However, we only have a proof that $\widehat{\rho}_{\mathbb{L}}\colon V_T\mathbb{L} \to V_{T'}\mathbb{L}$ preserves *finite* conjunctions, so it is not at all obvious at this point if there even is a left adjoint to $\widehat{\rho}_{\mathbb{L}}$. This is an interesting question for future work.

5.3.5 T-powerlocales via flat sites

In this subsection, we will show that $V_T\mathbb{L}$, the T-powerlocale of a given frame $\mathbb{L}$, has a flat site presentation as $V_T\mathbb{L} \simeq \mathrm{Fr}\langle TL, \overline{T}{\leq}, \lhd_0^{\mathbb{L}}\rangle$. It then follows by the Flat site Coverage Theorem that every element of $V_T\mathbb{L}$ has a disjunctive normal form, and that the suplattice reduct of $V_T\mathbb{L}$ has a presentation defined only in terms of the order $\overline{T}\leq$ and the lifted join function $T\bigvee\colon TPL \to TL$.

Recall that $\langle X, \sqsubseteq, \lhd_0\rangle$ is a flat site if $\langle X, \sqsubseteq\rangle$ is a pre-order and $\lhd_0$ is a basic cover relation compatible with $\sqsubseteq$. In that case, we know that $\langle X, \sqsubseteq, \lhd_0\rangle$ presents a frame $\mathrm{Fr}\langle X, \sqsubseteq, \lhd_0\rangle$, and that if we denote the insertion of generators by $\heartsuit\colon X \to \mathrm{Fr}\langle X, \sqsubseteq, \lhd_0\rangle$, then

$$\begin{aligned}\mathrm{Fr}\langle X, \sqsubseteq, \lhd_0\rangle \simeq \mathrm{Fr}\langle X \mid\ & \heartsuit a \leq \heartsuit b \quad (a \sqsubseteq b),\\ & 1 = \textstyle\bigvee\{\heartsuit a \mid a \in X\}\\ & \heartsuit a \wedge \heartsuit b = \textstyle\bigvee\{\heartsuit c \mid c \sqsubseteq a, c \sqsubseteq b\}\\ & \heartsuit a \leq \textstyle\bigvee\{\heartsuit b \mid b \in A\} \quad (a \lhd_0 A)\rangle.\end{aligned}$$

Observe that this is very similar to our presentation of $V_T\mathbb{L}$ from Corollary 5.3.6 using (∇1), ($\nabla 2'$) and (∇3), namely

$$\begin{aligned}V_T\mathbb{L} \simeq \mathrm{Fr}\langle TL \mid\ & \nabla\alpha \leq \nabla\beta \quad (\alpha\,\overline{T}\!\leq \beta),\\ & \textstyle\bigwedge_\Gamma \nabla\gamma = \bigvee\{\nabla\delta \mid \forall\gamma \in \Gamma,\, \delta\,\overline{T}\!\leq \gamma\} \quad (\Gamma \in TP_\omega L)\\ & \nabla T\textstyle\bigvee(\Phi) \leq \bigvee\{\nabla\beta \mid \beta \in \lambda^T(\Phi)\} \quad (\Phi \in TPL)\rangle.\end{aligned}$$

We will see below that if we define a cover relation $\lhd_0^{\mathbb{L}}$ which is inspired by (∇3), then we obtain a flat site $\langle TL, \overline{T}\!\leq, \lhd_0^{\mathbb{L}}\rangle$, and this flat site presents $V_T\mathbb{L}$.

So how do we go about defining a basic cover relation $\lhd_0^{\mathbb{L}} \subseteq TL \times PTL$ so we can give a presentation of $V_T\mathbb{L}$? Intuitively, we would like to take the T-lifting of the relation $\{(a, A) \in L \times PL \mid a \leq \bigvee A\} = \leq ; (Gr\bigvee)\breve{}$. However, the T-lifting of this relation is of type $TL \times TPL$, while a basic cover relation on $\langle TL, \overline{T}\!\leq\rangle$ should be of type $TL \times PTL$. We solve this by involving the natural transformation $\lambda^T : TP \to PT$, given by $\lambda^T(\Phi) := \{\beta \in TL \mid \beta\,\overline{T}\!\in \Phi\}$, assigning to each $\Phi \in TPL$ the set of its lifted members. That is, we define

$$\lhd_0^{\mathbb{L}} \;:=\; \{(\alpha, \lambda^T(\Phi)) \in L \times PTL \mid \alpha\,\overline{T}\!\leq T\textstyle\bigvee(\Phi)\}.$$

In other words: we put $\alpha \lhd_0^{\mathbb{L}} \Gamma$ iff Γ is of the form $\lambda^T(\Phi)$ for some $\Phi \in TPL$ such that $\alpha\,\overline{T}\!\leq (T\bigvee)\Phi$. We must now do two things: first, we must show that $\langle TL, \overline{T}\!\leq, \lhd_0^{\mathbb{L}}\rangle$ is a flat site, meaning that $\lhd_0$ is compatible with $\overline{T}\!\leq$. Secondly, we must show that $\langle TL, \overline{T}\!\leq, \lhd_0^{\mathbb{L}}\rangle$ presents $V_T\mathbb{L}$. The following technical observation about the relation $\alpha\,\overline{T}\!\leq T\bigvee(\Phi)$ is the main reason why $V_T\mathbb{L}$ admits a flat site presentation. The reason for introducing a $\wedge$-semilattice $\mathbb{M}$ below will become apparent in §5.4.2.

5.3.23. LEMMA. *Let $\mathbb{L}$ be a frame and let $\mathbb{M}$ be a $\wedge$-subsemilattice of $\mathbb{L}$. Then for all $\alpha \in TM$ and $\Phi \in TPM$ such that $\alpha\,\overline{T}\!\leq T\bigvee(\Phi)$, there exists $\Phi' \in TPM$ such that*

1. *$\alpha\,\overline{T}\!\leq T\bigvee(\Phi')$;*

2. $\Phi' \overline{T}{\subseteq} T \downarrow_L \circ T\eta(\alpha)$;

3. $\Phi' \overline{T}{\subseteq} T \downarrow_L(\Phi)$

Proof First, we define the following relation on $M \times PM$:

$$R := \{(a, A) \in M \times PM \mid a \leq \bigvee A\} = (\leq ; (Gr\bigvee)^{\smile}) \restriction_{M \times PM}.$$

Consider the span $M \xleftarrow{p_1} R \xrightarrow{p_2} PM$. We define the following function $f\colon R \to R$:

$$f\colon (a, A) \mapsto (a, a \wedge A),$$

where $a \wedge A := \{a \wedge b \mid b \in A\}$. To see why this function is well-defined, first observe that $a \wedge A \in PM$ because $\mathbb{M}$ is a $\wedge$-subsemilattice of $\mathbb{L}$. Moreover, by the infinite distributive law for frames, we see that if $(a, A) \in R$, i.e. if $a \leq \bigvee A$, then also $a \leq \bigvee(a \wedge A)$, so that $(a, a \wedge A) \in R$. Now observe that $f\colon R \to R$ satisfies an equation and two inequations: for all $(a, A) \in R$,

$$
\begin{aligned}
&p_1 \circ f(a, A) = a = p_1(a, A), && \text{by def. of } f,\\
&p_2 \circ f(a, A) = a \wedge A \subseteq_L \downarrow_L\{a\} = \downarrow_L \circ \eta_L \circ p_1(a, A), && \text{since } \forall b \in A,\ a \wedge b \leq a,\\
&p_2 \circ f(a, A) = a \wedge A \subseteq_L \downarrow_L A = \downarrow_L \circ p_2(a, A) && \text{since } \forall b \in A,\ a \wedge b \leq b \in A.
\end{aligned}
$$

Now consider the lifted diagram

$$TM \xleftarrow{Tp_1} TR \xrightarrow{Tp_2} TPM.$$

It follows from Lemma 5.2.8 and the equation/inequations above that for each $\delta \in TR$, we have

$$Tp_1 \circ Tf(\delta) = Tp_1(\delta), \tag{5.31}$$

$$Tp_2 \circ Tf(\delta)\ \overline{T}{\subseteq_L}\ T \downarrow_L \circ T\eta_L \circ Tp_1(\delta), \tag{5.32}$$

$$Tp_2 \circ Tf(\delta)\ \overline{T}{\subseteq_L}\ T \downarrow_L \circ Tp_2(\delta) \tag{5.33}$$

Now recall that by Fact 5.2.7,

$$\overline{T}{\leq} ; Gr(T\bigvee)^{\smile} = \overline{T}(\leq ; (Gr\bigvee)^{\smile}) = \overline{T}R,$$

so we see that $\alpha\ \overline{T} \leq T\bigvee(\Phi)$ iff $\alpha\ \overline{T}R\ \Phi$. So let $\alpha \in TM$ and $\Phi \in TPM$ such that $\alpha\ \overline{T}{\leq}\ T\bigvee(\Phi)$, i.e. such that $\alpha\ \overline{T}R\ \Phi$; we will show that there is a $\Phi' \in TPM$ satisfying properties (1)–(3). First, observe that by definition of relation lifting, there must exist some $\delta \in TR$ such that

$$Tp_1(\delta) = \alpha \text{ and } Tp_2(\delta) = \Phi.$$

We claim that $\Phi' := Tp_2 \circ Tf(\delta)$ satisfies properties (1)–(3). We know by definition of relation lifting that $(Tp_1 \circ Tf(\delta))\ \overline{T}R\ (Tp_2 \circ Tf(\delta))$. Since

$$\begin{aligned} Tp_1 \circ Tf(\delta) &= Tp_1(\delta) && \text{by (5.31)}, \\ &= \alpha && \text{by assumption}, \end{aligned}$$

it follows that $\alpha\ \overline{T}R\ \Phi'$, i.e. $\alpha\ \overline{T}{\leq}\ T\bigvee(\Phi')$; we conclude that (1) holds. Moreover, it follows immediately from (5.32) that (2) holds. Similarly, it follows immediately from (5.33) that (3) holds. ∎

In the lemma above, we use the lifted inclusion relation $T{\subseteq}$ and the lifted downset function $T{\downarrow}$. In the lemma below we record several elementary observations about the interaction between $T{\subseteq}$, $T{\downarrow}$ and the natural transformation $\lambda^T : TP \to PT$.

5.3.24. LEMMA. *Let $\langle X, \sqsubseteq\rangle$ be a pre-order, let $\alpha \in TX$ and let $\Phi, \Phi' \in TPX$. Then*

1. $\downarrow_{TX}\lambda^T(\Phi) = \lambda^T\,(T{\downarrow_X}(\Phi))$;
2. $\downarrow_{TX}\{\alpha\} = \lambda^T\,(T{\downarrow_X} \circ T\eta_X(\alpha))$;
3. *If $\Phi'\ \overline{T}\subseteq_X \Phi$, then also $\lambda^T(\Phi') \subseteq \lambda^T(\Phi)$.*

Proof (1). For all $a \in X$ and all $A \in PX$, we have $a \leq ; \in A$ iff $a \in \downarrow_X A$. Consequently,

$$\forall \alpha \in TL, \forall \Phi \in TPL,\ \alpha\ \overline{T}{\leq}\ ;\ \overline{T}{\in}\ \Phi \text{ iff } \alpha\ \overline{T}{\in}\ T\downarrow_X(\Phi).$$

Now we see that

$$\begin{aligned} \alpha \in \downarrow_{TX} \lambda^T(\Phi) &\Leftrightarrow \alpha\ \overline{T}{\leq}\ ;\ \overline{T}{\in}\ \Phi && \text{by def. of } \downarrow \text{ and } \lambda^T, \\ &\Leftrightarrow \alpha\ \overline{T}{\in}\ T\downarrow_X(\Phi) && \text{by the above}, \\ &\Leftrightarrow \alpha \in \lambda^T\,(T{\downarrow_X}(\Phi)) && \text{by def. of } \lambda^T. \end{aligned}$$

(2). For all $a, b \in X$, we have $b \leq a$ iff $b \in \downarrow_X\{a\}$. It follows by relation lifting that

$$\forall \alpha, \beta \in TX,\ \beta\ \overline{T}{\leq}\ \alpha \text{ iff } \beta\ \overline{T}{\in}\ T\downarrow_X \circ T\eta_X(\alpha).$$

It now follows by an argument analogous to that for (1) above that (2) holds.

(3). Observe that for all $A, A' \in PX$ and all $a \in X$, we have that $a \in A' \subseteq A$ implies that $a \in A$. The statement follows by relation lifting. ∎

We are now ready to prove that $\langle TL, \overline{T}{\leq}, \lhd_0^{\mathbb{L}}\rangle$ is indeed a flat site.

5.3.25. LEMMA. *If $\mathbb{L}$ is a frame then $\langle TL, \overline{T}{\leq}, \lhd_0^{\mathbb{L}}\rangle$ is a flat site.*

Proof We have already know from Lemma 5.2.8 that $\langle TL, \overline{T}{\leq}\rangle$ is a pre-order, so what remains to be shown is that the relation $\lhd_0^{\mathbb{L}}$ is compatible with the pre-order. Fix $\alpha \in TL$ and $\Phi \in TPL$ such that $\alpha \mathrel{\overline{T}{\leq}} T\bigvee(\Phi)$, so that $\alpha \lhd_0^{\mathbb{L}} \lambda^T(\Phi)$. We need to show that

$$\forall\beta \in TL,\ \text{if } \beta \mathrel{\overline{T}{\leq}} \alpha \text{ then } \exists\Gamma \in TPL \text{ s.t. } \Gamma \subseteq {\downarrow_{TL}}\{\beta\} \cap {\downarrow_{TL}}\, \lambda^T(\Phi) \text{ and } \beta \lhd_0^{\mathbb{L}} \Gamma. \tag{5.34}$$

But this is easy to see: if $\beta \mathrel{\overline{T}{\leq}} \alpha$ then since $\alpha \mathrel{\overline{T}{\leq}} T\bigvee(\Phi)$, it follows by transitivity of $\overline{T}{\leq}$ that $\beta \mathrel{\overline{T}{\leq}} T\bigvee(\Phi)$. Now by Lemma 5.3.23 there exists $\Phi' \in TPL$ such that $\alpha \mathrel{\overline{T}{\leq}} T\bigvee(\Phi')$, $\Phi' \mathrel{\overline{T}{\subseteq}} T{\downarrow_L} \circ T\eta(\beta)$ and $\Phi' \mathrel{\overline{T}{\subseteq}} T{\downarrow_L}\Phi$. Define $\Gamma := \lambda^T(\Phi')$, then it follows from the definition of $\lhd_0^{\mathbb{L}}$ that $\beta \lhd_0 \Gamma$; moreover, it now follows from Lemma 5.3.24 that $\Gamma \subseteq {\downarrow_{TL}}\{\beta\} \cap {\downarrow_{TL}}\,\lambda^T(\Phi)$. We conclude that (5.34) holds. Since $\alpha \in TL$ and $\Phi \in TPL$ were arbitrary, it follows that $\lhd_0^{\mathbb{L}}$ is compatible with the order $\overline{T}{\leq}$, so that $\langle TL, \overline{T}{\leq}, \lhd_0^{\mathbb{L}}\rangle$ is a flat site. ∎

Having established that $\langle TL, \overline{T}{\leq}, \lhd_0^{\mathbb{L}}\rangle$ is a flat site, we will now prove that it presents $V_T\mathbb{L}$, i.e. that $V_T\mathbb{L} \simeq \mathrm{Fr}\langle TL, \overline{T}{\leq}, \lhd_0^{\mathbb{L}}\rangle$.

5.3.26. Theorem. *Let $\mathbb{L}$ be a frame and let T be a standard, finitary, weak pullback-preserving functor. Then $V_T\mathbb{L}$ admits the following flat site presentation:*

$$V_T\mathbb{L} \simeq \mathrm{Fr}\langle TL, \overline{T}{\leq}, \lhd_0^{\mathbb{L}}\rangle,$$

where $\lhd_0^{\mathbb{L}} = \{(\alpha, \lambda^T(\Phi)) \in L \times PTL \mid \alpha \mathrel{\overline{T}{\leq}} T\bigvee(\Phi)\}$, and in each direction, the isomorphism is the unique frame homomorphism extending the identity map id_{TL} on the set of generators of $V_T\mathbb{L}$ and $\mathrm{Fr}\langle TL, \overline{T}{\leq}, \lhd_0^{\mathbb{L}}\rangle$, respectively.

Proof For this proof, we denote the insertion of generators from TL to $V_T\mathbb{L}$ by ∇, and from TL to $\mathrm{Fr}\langle TL, \overline{T}{\leq}, \lhd_0^{\mathbb{L}}\rangle$ by $\heartsuit$. We will show that

1. the function $\heartsuit\colon TL \to \mathrm{Fr}\langle TL, \overline{T}{\leq}, \lhd_0^{\mathbb{L}}\rangle$ is compatible with the relations (∇1), ($\nabla 2'$) and (∇3), and
2. that the function $\nabla\colon TL \to V_T\mathbb{L}$ has the following properties:
   (a) ∇ is order-preserving;
   (b) $1 = \bigvee\{\nabla\alpha \mid \alpha \in TL\}$;
   (c) for all $\alpha, \beta \in TL$, $\nabla\alpha \wedge \nabla\beta = \bigwedge\{\nabla\gamma \mid \delta \mathrel{\overline{T}{\leq}} \alpha, \beta\}$;
   (d) for all $\alpha \lhd_0^{\mathbb{L}} \Gamma$, $\nabla\alpha \leq \bigvee\{\nabla\beta \mid \beta \in \Gamma\}$.

(1). First consider (∇1). Suppose that $\alpha, \beta \in TL$ such that $\alpha \mathrel{\overline{T}{\leq}} \beta$; we have to show that $\heartsuit\alpha \leq \heartsuit\beta$. This follows immediately from the fact that $\heartsuit\colon TL \to \mathrm{Fr}\langle TL, \overline{T}{\leq}, \lhd_0^{\mathbb{L}}\rangle$ is order-preserving. Secondly, consider ($\nabla 2'$). Let $\Gamma \in P_\omega TL$, we then have to show that

$$\textstyle\bigwedge_{\gamma\in\Gamma} \heartsuit\gamma \leq \bigvee\{\heartsuit\delta \mid \forall\gamma \in \Gamma,\ \delta \mathrel{\overline{T}{\leq}} \gamma\}. \tag{5.35}$$

Recall from §5.2.4 that since $\langle TL, \overline{T}\!\leq, \lhd_0^{\mathbb{L}}\rangle$ is a flat site, we know that $1 = \bigvee\{\heartsuit\alpha \mid \alpha \in TL\}$ and that for all $\alpha, \beta \in TL$, $\heartsuit\alpha \wedge \heartsuit\beta = \bigwedge\{\heartsuit\gamma \mid \delta\ \overline{T}\!\leq \alpha, \beta\}$. It now follows by induction on the size of Γ that (5.35) holds.

Finally for (∇3), take $\Phi \in TPL$. We have to show that $\heartsuit T\bigvee(\Phi) \leq \bigvee\{\heartsuit\beta \mid \beta \in \lambda^T(\Phi)\}$. This follows immediately from the definition of $\lhd_0^{\mathbb{L}}$, since $T\bigvee(\Phi)\ \overline{T}\!\leq T\bigvee(\Phi)$. We conclude that $\heartsuit\colon TL \to \mathrm{Fr}\langle TL, \overline{T}\!\leq, \lhd_0^{\mathbb{L}}\rangle$ is compatible with the relations (∇1), (∇2) and (∇3) and thus there must be a unique frame homomorphism $f\colon V_T\mathbb{L} \to \mathrm{Fr}\langle TL, \overline{T}\!\leq, \lhd_0^{\mathbb{L}}\rangle$ which extends $\heartsuit$.

(2). We first have to show that ∇ is order-preserving, i.e. that if $\alpha\ \overline{T}\!\leq \beta$, then $\nabla\alpha \leq \nabla\beta$. This follows immediately from (∇1). Secondly, we must show that (2)(b) and (2)(c) are satisfied, but this follows immediately from ($\nabla 2'$). Finally, consider (2)(d), i.e. suppose that $\alpha \lhd_0^{\mathbb{L}} \Gamma$. By definition of $\lhd_0^{\mathbb{L}}$, there is some $\Phi \in TPL$ such that $\alpha\ \overline{T}\!\leq T\bigvee(\Phi)$ and $\lambda^T(\Phi) = \Gamma$. Now we need to show that $\nabla\alpha \leq \bigvee\{\nabla\beta \mid \beta \in \lambda^T(\Phi)\}$. This is easy to see, since

$$\begin{aligned} \nabla\alpha &\leq \nabla T\bigvee(\Phi) && \text{by } (\nabla 1), \\ &\leq \bigvee\{\nabla\beta \mid \beta \in \lambda^T(\Phi)\} && \text{by } (\nabla 3). \end{aligned}$$

It follows that (2)(d) holds; consequently, there is a unique frame homomorphism $g\colon \mathrm{Fr}\langle TL, \overline{T}\!\leq, \lhd_0^{\mathbb{L}}\rangle \to V_T\mathbb{L}$ extending ∇.

Finally, it is easy to see that

$$gf = \mathrm{id}_{V_T\mathbb{L}} \text{ and } fg = \mathrm{id}_{\langle TL, \overline{T}\leq, \lhd_0^{\mathbb{L}}\rangle},$$

so that indeed $V_T\mathbb{L} \simeq \mathrm{Fr}\langle TL, \overline{T}\!\leq, \lhd_0^{\mathbb{L}}\rangle$. ∎

In light of Theorem 5.3.26 above, we denote the insertion of generators by $\nabla\colon TL \to \mathrm{Fr}\langle TL, \overline{T}\!\leq, \lhd_0^{\mathbb{L}}\rangle$. We now arrive at the most important corollary of Theorem 5.3.26, which says that every element of $V_T\mathbb{L}$ has a disjunctive normal form.

5.3.27. Corollary. *Let $\mathbb{L}$ be a frame. Then for all $x \in V_T\mathbb{L}$, there is a $\Gamma \in PTL$ such that $x = \bigvee\{\nabla\gamma \mid \gamma \in \Gamma\}$.*

Proof By Theorem 5.3.26 we know that $V_T\mathbb{L} \simeq \mathrm{SupLat}\langle TL, \overline{T}\!\leq, \lhd_0^{\mathbb{L}}\rangle$. The statement now follows Fact 5.2.14. ∎

5.3.28. Remark. It is not hard to show that

$$\mathrm{SupLat}\langle TL, \overline{T}\!\leq, \lhd_0^{\mathbb{L}}\rangle \simeq \mathrm{SupLat}\langle TL \mid (\nabla 1), (\nabla 3)\rangle.$$

Consequently, by Theorem 5.3.26 and Fact 5.2.14, the *order* on $V_T\mathbb{L}$ is uniquely determined by the relations (∇1) and (∇3).

5.4 Preservation results

Now that we have established the T-powerlocale construction, we can set about to prove that it is well-behaved. One particular kind of good behaviour is to ask that it preserves algebraic properties. In this section, we present several initial results in this area. In particular, we show that V_T preserves regularity and zero-dimensionality of frames, and the property of being a compact zero-dimensional frame.

5.4.1 Regularity and zero-dimensionality

The purpose of this subsection is to prove that the operation V_T preserves regularity and zero-dimensionality of frames. Both of these notions are defined in terms of the well-inside relation $\lessdot$; accordingly, the main technical result of this subsection states that if $\alpha\ \overline{T}\!\lessdot\ \beta$, then also $\nabla\alpha \lessdot_{V_T\mathbb{L}} \nabla\beta$. We first recall some notions leading up to the definition of regularity.

5.4.1. Definition. Given two elements a, b of a distributive lattice $\mathbb{L}$, we say that a is *well inside* b, notation: $a \lessdot b$, if there is some c in $\mathbb{L}$ such that $a \wedge c = 0$ and $b \vee c = 1$. If $a \lessdot a$ we say a is *clopen*. We denote the clopen elements of $\mathbb{L}$ by $C_{\mathbb{L}}$.

In case $\mathbb{L}$ is a frame, in the definition of $\lessdot$, for the element c witnessing that $a \lessdot b$ we may always take the Heyting complementation $\neg a$ of a. In other words, $a \lessdot b$ iff $b \vee \neg a = 1$. Consequently, if a is clopen then $a \vee \neg a = 1$. In the sequel we will use not only this fact, but also the following properties of $\lessdot$ without warning.

5.4.2. Fact ([54], L. III.1.1). *Let $\mathbb{L}$ be a frame.*

1. $\lessdot\ \subseteq\ \leq$;
2. $\leq ; \lessdot ; \leq\ \subseteq\ \lessdot$;
3. *for $X \in PL$, if $\forall x \in X. x \lessdot y$ then $\bigvee X \lessdot y$;*
4. *for $X \in PL$, if $\forall x \in X. y \lessdot x$ then $y \lessdot \bigwedge X$;*
5. *$a \lessdot a$ iff a has a complement.*

5.4.3. Definition. A frame $\mathbb{L}$ is *regular* if every $a \in \mathbb{L}$ satisfies

$$a = \bigvee\{b \in L \mid b \lessdot a\}.$$

We say $\mathbb{L}$ is *zero-dimensional* if for all $a \in \mathbb{L}$,

$$a = \bigvee\{b \in C_{\mathbb{L}} \mid b \leq a\}.$$

We record the following useful property of $C_{\mathbb{L}}$ [54, §III-1.1]:

5.4.4. FACT. *Let $\mathbb{L}$ be a frame. Then $\langle C_{\mathbb{L}}, \wedge, \vee, 0, 1\rangle$ is a sublattice of $\mathbb{L}$.*

We define a function $\Downarrow\colon PL \to PC_{\mathbb{L}}$ which maps $A \in PL$ to $\downarrow A \cap C_{\mathbb{L}}$.

5.4.5. LEMMA. *If $\mathbb{L}$ is a zero-dimensional frame, then*

1. $\forall \alpha \in TL$, $\nabla\alpha = \bigvee\{\nabla\beta \mid \beta \in TC_{\mathbb{L}},\ \beta\,\overline{T}{\leq}\,\alpha\}$;
2. $\forall \Phi \in TPL$, $T\bigvee(\Phi) = T\bigvee \circ T\Downarrow(\Phi)$;
3. $\forall \Phi \in TPL$, $\forall \alpha \in TL$, $[\alpha \in TC_{\mathbb{L}}$ *and* $\alpha\,\overline{T}{\leq}\,;\overline{T}{\in}\,\Phi]$ *iff* $\alpha \in \lambda^T\left(T\Downarrow(\Phi)\right)$.

Similarly to (1), if $\mathbb{L}$ is regular then $\forall \alpha \in TL$, $\nabla\alpha = \bigvee\{\nabla\beta \mid \beta \in TL,\ \beta\,\overline{T}{\lessdot}\,\alpha\}$.

Proof (1). First, observe that for all $a \in L$, we have that

$$\begin{aligned} a &= \bigvee\{b \in C_{\mathbb{L}} \mid b \leq a\} && \text{by zero-dimensionality,}\\ &= \bigvee \Downarrow\{a\} && \text{by definition of } \Downarrow,\\ &= \bigvee \Downarrow \circ \eta(a) && \text{by def. of } \eta\colon \mathrm{Id}_{\mathbf{Set}} \to P. \end{aligned}$$

By relation lifting, it follows that

$$\forall \alpha \in TL,\ \alpha = T\bigvee \circ T\Downarrow \circ T\eta(\alpha). \tag{5.36}$$

Now observe that for all $a, b \in L$, we have $b \in \Downarrow\{a\}$ iff $b \in C_{\mathbb{L}}$ and $b \leq a$. By relation lifting, it follows that

$$\forall \alpha, \beta \in TL,\ \left[\beta\,\overline{T}{\in}\,T\Downarrow\circ T\eta(\alpha) \text{ iff } \beta \in TC_{\mathbb{L}} \text{ and } \beta\,\overline{T}{\leq}\,\alpha\right]. \tag{5.37}$$

Combining these two observations, we see that

$$\begin{aligned} \nabla\alpha &= \nabla\left(T\bigvee \circ T\Downarrow\circ T\eta(\alpha)\right) && \text{by (5.36),}\\ &= \bigvee\{\nabla\beta \mid \beta\,\overline{T}{\in}\,T\Downarrow\circ T\eta(\alpha)\} && \text{by } (\nabla 3),\\ &= \bigvee\{\nabla\beta \mid \beta \in TC_{\mathbb{L}},\ \beta\,\overline{T}{\leq}\,\alpha\} && \text{by (5.37).} \end{aligned}$$

(2). It follows by zero-dimensionality of $\mathbb{L}$ that for all $A \in PL$, we have $\bigvee A = \bigvee \Downarrow A$. Consequently, by relation lifting, it follows that (2) holds.

(3). Take $a \in L$ and $A \in PL$. Then

$$\begin{aligned} a \in \Downarrow A &\Leftrightarrow a \in C_{\mathbb{L}} \text{ and } \exists b \in A,\ a \leq b && \text{by definition of } \Downarrow,\\ &\Leftrightarrow a \in C_{\mathbb{L}} \text{ and } a \leq ; \in A && \text{by def. of relation composition.} \end{aligned}$$

It follows by relation lifting that

$$\forall \Phi \in TPL, \forall \alpha \in TL,\ \alpha\,\overline{T}{\in}\,T\Downarrow(\Phi) \text{ iff } \alpha \in TC_{\mathbb{L}} \text{ and } \alpha\,T{\leq}\,;T{\in}\,\Phi.$$

Now it follows by definition of $\lambda^T(\Phi)$ that (3) holds.

For the last part of the proof, first observe that if $\mathbb{L}$ is regular, then for all $a \in L$, $a = \bigvee w(a)$, where we temporarily define $w\colon L \to PL$ as

$$w\colon a \mapsto \{b \in L \mid b \leqslant a\}.$$

By relation lifting, it follows that

$$T\bigvee \circ Tw = \mathrm{id}_L\,. \tag{5.38}$$

Moreover, it follows by definition of $w\colon L \to PL$ that for all $a, b \in L$, $b \in w(a)$ iff $b \leqslant a$. Consequently,

$$\forall \alpha, \beta \in TL,\ \beta \mathrel{\overline{T}{\in}} Tw(\alpha) \text{ iff } \beta \mathrel{\overline{T}{\leqslant}} \alpha. \tag{5.39}$$

Now we see that for any $\alpha \in TL$,

$$\begin{aligned}
\nabla\alpha &= \nabla\left(T\bigvee \circ Tw(\alpha)\right) && \text{by (5.38),}\\
&= \bigvee\{\nabla\beta \mid \beta \mathrel{\overline{T}{\in}} Tw(\alpha)\} && \text{by } (\nabla 3),\\
&= \bigvee\{\nabla\beta \mid \beta \mathrel{\overline{T}{\leqslant}} \alpha\} && \text{by (5.39).} \quad \blacksquare
\end{aligned}$$

The key technical lemma of this subsection states that relation lifting preserves the $\leqslant$-relation.

5.4.6. LEMMA. *Let T be a standard, finitary, weak pullback-preserving functor and let $\mathbb{L}$ be a frame. Then*

$$\textit{for all } \alpha, \beta \in TL:\ \alpha \mathrel{\overline{T}{\leqslant}} \beta \textit{ implies } \nabla\alpha \leqslant_{V_T\mathbb{L}} \nabla\beta. \tag{5.40}$$

Proof Let $\alpha, \beta \in TL$ be such that $\alpha \mathrel{\overline{T}{\leqslant}} \beta$. Our aim will be to show that $\nabla\alpha \leqslant_{V_T\mathbb{L}} \nabla\beta$.

We may assume without loss of generality that

$$\exists f : \mathit{Base}(\alpha) \to \mathit{Base}(\beta) \text{ such that } \beta = (Tf)\alpha \text{ and } \forall a \in \mathit{Base}(\alpha),\ a \leqslant fa. \tag{5.41}$$

To justify this assumption, assume that we have a proof of (5.40) for all β satisfying (5.41). To derive (5.40) in the general case, consider arbitrary elements $\alpha, \beta' \in TL$ such that $\alpha \mathrel{\overline{T}{\leqslant}} \beta'$. In order to show that $\nabla\alpha \mathrel{\overline{T}{\leqslant}} \nabla\beta'$, consider the map $f : \mathit{Base}(\alpha) \to L$ given by $f(a) := \bigwedge\{b \in \mathit{Base}(\beta') \mid a \leqslant b\}$. On the basis of Fact 5.4.2 it is not difficult to see that $Gr(f) \subseteq\, \leqslant$ and so by the properties of relation lifting we obtain $Gr(Tf) \subseteq \overline{T}{\leqslant}$. In particular, we find that $\alpha \mathrel{\overline{T}{\leqslant}} (Tf)\alpha$; thus by our assumption we may conclude that $\nabla\alpha \leqslant \nabla(Tf)\alpha$. Also, observe that $a \leqslant b$ implies $fa \leq b$, for all $a \in \mathit{Base}(\alpha)$ and $b \in \mathit{Base}(\beta')$. Hence by Lemma 5.2.8 we may conclude from $\alpha \mathrel{\overline{T}{\leqslant}} \beta'$ that $(Tf)\alpha \mathrel{\overline{T}{\leq}} \beta'$, which gives $\nabla(Tf)\alpha \leq \nabla\beta'$. Combining our observations thus far, by Fact 5.4.2 it follows from

$\nabla\alpha \leqslant \nabla(Tf)\alpha$ and $\nabla(Tf)\alpha \leq \nabla\beta'$ that $\nabla\alpha \leqslant \nabla\beta'$ indeed. Thus our assumption (5.41) is justified indeed.

Turning to the proof itself, consider the map $h : PBase(\alpha) \to L$ given by

$$h(A) := \bigwedge \left(\{\neg a \mid a \in A\} \cup \{fa \mid a \notin A\}\right).$$

Our first observation is that, since by assumption $\neg a \vee fa = 1_{\mathbb{L}}$ for each $a \in Base(\alpha)$, we may infer that

$$1_{\mathbb{L}} = \bigwedge\{\neg a \vee fa \mid a \in Base(\alpha)\},$$

a straightforward application of the (finitary) distributive law yields that

$$1_{\mathbb{L}} = \bigvee\{h(A) \mid A \in PBase(\alpha)\}. \tag{5.42}$$

Define $X \subseteq L$ to be the range of h, so that we may think of h as a surjection $h : PBase(\alpha) \to X$, and read (5.42) as saying that $1 = \bigvee X$. Using Lemma 5.3.10(5), from the latter observation we may infer that

$$1_{V_T\mathbb{L}} = \bigvee\{\nabla\xi \mid \xi \in TX\}. \tag{5.43}$$

However, from $h : PBase(\alpha) \to X$ being surjective we may infer that $Th : TPBase(\alpha) \to TX$ is also surjective, so that we may read (5.43) as

$$1_{V_T\mathbb{L}} = \bigvee\{\nabla Th(\Phi) \mid \Phi \in TPBase(\alpha)\}. \tag{5.44}$$

This leads us to the key observation in our proof: We may partition the set $\{Th(\Phi) \mid \Phi \in TPBase(\alpha)\}$ into elements γ such that $\nabla\gamma \leq \nabla\beta$, and elements γ satisfying $\nabla\alpha \wedge \nabla\gamma = 0_{V_T\mathbb{L}}$.

1. Claim. Let $\Phi \in TPBase(\alpha)$.
(a) If $(\alpha, \Phi) \in \overline{T}\not\in$, then $Th(\Phi)\,\overline{T}\!\leq \beta$;
(b) if $(\alpha, \Phi) \notin \overline{T}\not\in$, then $\nabla\alpha \wedge \nabla Th(\Phi) = 0_{V_T\mathbb{L}}$.

Proof of Claim For part (a), it is not hard to see that

$$a \notin A \Rightarrow h(A) \leq f(a), \text{ for all } a \in Base(\alpha), A \in PBase(\alpha).$$

From this it follows by Lemma 5.2.8 that

$$\alpha\,\overline{T}\not\in \Phi \Rightarrow Th(\Phi)\,\overline{T}\!\leq (Tf)(\alpha) = \beta.$$

For part (b), assume that $\nabla\alpha \wedge \nabla Th(\Phi) > 0_{V_T\mathbb{L}}$. It suffices to derive from this that $\alpha\,\overline{T}\not\in \Phi$.

Let $\leq'$ be the restriction of $\leq$ to the non-zero part of $\mathbb{L}$, that is, $\leq' := \leq\restriction_{L'\times L'}$, where $L' = L \setminus \{0_{\mathbb{L}}\}$. We claim that for all $\gamma, \delta \in TL$:

$$\nabla\gamma \wedge \nabla\delta > 0_{V_T\mathbb{L}} \;\Rightarrow\; (\gamma,\delta) \in \overline{T}{\geq'} \,;\overline{T}{\leq'}. \tag{5.45}$$

To see this, assume that $\nabla\gamma \wedge \nabla\delta > 0_{V_T\mathbb{L}}$, and observe that Lemma 5.3.5 yields the existence of a $\theta \in TL$ such that $\nabla\theta > 0_{V_T\mathbb{L}}$ and $\theta \,\overline{T}{\leq}\, \gamma, \delta$. It follows from Lemma 5.3.10(1) that γ, δ and θ all belong to TL', and so θ is witnesses to the fact that $(\gamma,\delta) \in \overline{T}{\geq'} \,;\overline{T}{\leq'}$.

By (5.45) and the assumption on α and Φ it follows that $(\alpha,\Phi) \in \overline{T}{\geq'} \,;\overline{T}{\leq'} \,; (GrTh)\breve{}$, and so by Fact 5.2.7 we obtain

$$(\alpha,\Phi) \in \overline{T}({\geq'} \,; {\leq'} \,; (Grh)\breve{}) \tag{5.46}$$

The crucial observation now is that

$${\geq'} \,; {\leq'} \,; (Grh)\breve{} \;\subseteq\; \not\in. \tag{5.47}$$

For a proof, take a pair $(a, A) \in L \times PL$ in the LHS of (5.47), and suppose for contradiction that $a \in A$. Then by definition of h we obtain $h(A) \leq \neg a$, so that $a \wedge h(A) = 0_{\mathbb{L}}$. But if $a \geq' ; \leq' ; (Grh)\breve{}\, A$, then there must be some b such that $b \leq' a, h(A)$, and by definition of $\leq'$ this can only be the case if $b > 0_{\mathbb{L}}$. This gives the desired contradiction.

Finally, by monotonicity of relation lifting, it is an immediate consequence of (5.46) and (5.47) that $\alpha \,\overline{T}{\not\in}\, \Phi$. This finishes the proof of the Claim.

On the basis of the Claim it is straightforward to finish the proof. Define

$$c := \bigvee \left\{ Th(\Phi) \mid \Phi \in TPBase(\alpha) \text{ such that } (\alpha,\Phi) \not\in \overline{T}{\not\in} \right\},$$

then we may calculate that

$$\begin{array}{ll}
c \vee \nabla\beta & \\
\geq c \vee \bigvee \left\{ Th(\Phi) \mid \Phi \in TPBase(\alpha) \text{ such that } (\alpha,\Phi) \in \overline{T}{\not\in} \right\} & \text{(Claim 1(a))} \\
= \bigvee \left\{ Th(\Phi) \mid \Phi \in TPBase(\alpha) \right\} & \text{(definition of } c\text{)} \\
= 1_{V_T\mathbb{L}} & \text{(equation (5.44))}
\end{array}$$

and

$$\begin{array}{ll}
\nabla\alpha \wedge c & \\
= \bigvee \left\{ \nabla\alpha \wedge Th(\Phi) \mid \Phi \in TPBase(\alpha) \text{ such that } (\alpha,\Phi) \not\in \overline{T}{\not\in} \right\} & \text{(distributivity)} \\
= \bigvee \left\{ 0_{V_T\mathbb{L}} \mid \Phi \in TPBase(\alpha) \text{ such that } (\alpha,\Phi) \not\in \overline{T}{\not\in} \right\} & \text{(Claim 1(b))} \\
= 0_{V_T\mathbb{L}} &
\end{array}$$

In other words, c witnesses that $\nabla\alpha \preccurlyeq_{V_T\mathbb{L}} \nabla\beta$. ∎

We now arrive at the main result of this subsection, namely, that the T-powerlocale construction preserves regularity and zero-dimensionality.

5.4.7. THEOREM. *Let $\mathbb{L}$ be a frame and let T be a standard, finitary, weak pullback-preserving functor.*

1. *If $\mathbb{L}$ is regular then so is $V_T\mathbb{L}$.*
2. *If $\mathbb{L}$ is zero-dimensional then so is $V_T\mathbb{L}$.*

Proof (1). By Corollary 5.3.27, it suffices to show that for all $\alpha \in TL$,

$$\nabla\alpha = \bigvee\{\nabla\beta \in V_T\mathbb{L} \mid \nabla\beta \lessdot \nabla\alpha\}. \tag{5.48}$$

Take $\alpha \in TL$; we see that

$$\begin{aligned}
\nabla\alpha &= \bigvee\{\nabla\beta \mid \beta \,\overline{T}\!\lessdot\, \alpha\} && \text{by Lemma 5.4.5,}\\
&\leq \bigvee\{\nabla\beta \mid \nabla\beta \lessdot_{V_T\mathbb{L}} \nabla\alpha\} && \text{by Lemma 5.4.6,}\\
&\leq \nabla\alpha && \text{since } \lessdot \subseteq \leq.
\end{aligned}$$

It follows that (5.48) holds, concluding the proof of part (1).

(2). Again by Corollary 5.3.27, it suffices to show that for all $\alpha \in TL$,

$$\nabla\alpha = \bigvee\{\nabla\beta \mid \nabla\beta \in C_{V_T\mathbb{L}},\, \nabla\beta \leq \nabla\alpha\}. \tag{5.49}$$

The main observation here is that

$$\forall\beta \in TC_{\mathbb{L}},\, \nabla\beta \in C_{V_T\mathbb{L}}. \tag{5.50}$$

To see why, recall that $C_{\mathbb{L}} := \{b \in L \mid b \lessdot b\}$, so that for all $b \in C_{\mathbb{L}}$, $b = b$ implies $b \lessdot b$. Consequently, by relation lifting,

$$\forall\beta \in TC_{\mathbb{L}},\, \beta \,\overline{T}\!\lessdot\, \beta.$$

It follows by Lemma 5.4.6 that (5.50) holds. Now

$$\begin{aligned}
\nabla\alpha &= \bigvee\{\nabla\beta \mid \beta \in TC_{\mathbb{L}},\, \beta \,\overline{T}\!\leq\, \alpha\} && \text{by Lemma 5.4.5(1),}\\
&\leq \bigvee\{\nabla\beta \in C_{V_T\mathbb{L}} \mid \beta \,\overline{T}\!\leq\, \alpha\} && \text{by (5.50),}\\
&\leq \bigvee\{\nabla\beta \in C_{V_T\mathbb{L}} \mid \nabla\beta \leq \nabla\alpha\} && \text{by } (\nabla 1),\\
&= \nabla\alpha && \text{by order theory.}
\end{aligned}$$

It now follows that (5.49) holds; consequently we see that (2) holds. ∎

5.4.2 Compactness

In this subsection, we will show that if $\mathbb{L}$ is compact and zero-dimensional, then so is $V_T\mathbb{L}$. Our proof strategy is as follows. Given a compact zero-dimensional frame $\mathbb{L}$, we will define a new construction $V_T^C\mathbb{L}$ which is guaranteed to be compact, and then we show that $V_T\mathbb{L} \simeq V_T^C\mathbb{L}$.

We define a flat site presentation $\langle TC_{\mathbb{L}}, \overline{T}{\leq}, \lhd_0^C\rangle$, where

$$\lhd_0^C := \{(\alpha, \lambda^T(\Phi)) \in TC_{\mathbb{L}} \times PTL \mid \alpha \,\overline{T}{\leq}\, T\bigvee(\Phi),\ \Phi \in TP_\omega C_{\mathbb{L}}\}.$$

Observe that we view $TC_{\mathbb{L}}$ as a substructure of TL, which is justified by the fact that $C_{\mathbb{L}}$ is a sublattice of $\mathbb{L}$ (Fact 5.4.4): this fact tells us that $\bigvee\colon PL \to L$ restricts to a function from $P_\omega C_{\mathbb{L}}$ to $C_{\mathbb{L}}$; consequently, by standardness of T $T\bigvee$ maps $TP_\omega C_{\mathbb{L}}$ to $TC_{\mathbb{L}}$. Below, we will need the following property of relation lifting with respect to ordered sets.

5.4.8. Lemma. *Let $\mathbb{P}$ be a poset with a top element* 1.
$\forall\beta \in TP$, $\exists\alpha \in T\{1\}$, $\beta \,\overline{T}{\leq}\, \alpha$;

Proof Consider the following function at the ground level: $f\colon P \to \{1\}$, where f is the constant function $f\colon b \mapsto 1$. Then for all $b \in P$, we have $b \leq f(b)$ and $f(b) \in \{1\}$. By relation lifting, we see that for all $\beta \in TP$, $\beta \,\overline{T}{\leq}\, Tf(\beta)$ and $Tf(\beta) \in T\{1\}$. The statement follows. ∎

5.4.9. Lemma. *Let $\mathbb{L}$ be a frame. Then $\langle TC_{\mathbb{L}}, \overline{T}{\leq}, \lhd_0^C\rangle$ is a flat site. Moreover, if T maps finite sets to finite sets then* $\mathrm{Fr}\langle TC_{\mathbb{L}}, \overline{T}{\leq}, \lhd_0^C\rangle$ *is a compact frame.*

Proof Because $C_{\mathbb{L}}$ is a meet-subsemilattice of $\mathbb{L}$, we can apply Lemma 5.3.23 to $TC_{\mathbb{L}}$. Now the proof that $\langle TC_{\mathbb{L}}, \overline{T}{\leq}, \lhd_0^C\rangle$ is a flat site is analogous to that of Lemma 5.3.25.

Now suppose that T maps finite sets to finite sets. Then for all $\Phi \in TP_\omega C_{\mathbb{L}}$, it follows by Fact 5.2.12(3) that $\lambda^T(\Phi)$ is finite. Consequently,

$$\forall\alpha \lhd_0^C \lambda^T(\Phi),\ \lambda^T(\Phi) \text{ is finite.}$$

Moreover, by Lemma 5.4.8,

$$TC_{\mathbb{L}} = {\downarrow_{TC_{\mathbb{L}}}} T\{1_{\mathbb{L}}\},$$

since $1_{\mathbb{L}} \in C_{\mathbb{L}}$ as $C_{\mathbb{L}}$ is a sublattice of $\mathbb{L}$. Now since we assumed that T maps finite sets to finite sets, $\{1_{\mathbb{L}}\}$ must be finite. It is now follows from a straight-forward generalization of [93, Proposition 11] that $\mathrm{Fr}\langle TC_{\mathbb{L}}, \overline{T}{\leq}, \lhd_0^C\rangle$ is a compact frame. (The only change we need to make to [93, Proposition 11] is to generalize from from using single finite trees to using disjoint unions of $|T\{1_{\mathbb{L}}\}|$-many trees, so that one can cover each element of $T\{1_{\mathbb{L}}\}$.) ∎

We define $V_T^C\mathbb{L} := \mathrm{Fr}\langle TC_{\mathbb{L}}, \overline{T}{\leq}, \lhd_0^C\rangle$, and for the time being we denote the insertion of generaters by $\heartsuit\colon TC_{\mathbb{L}} \to V_T^C\mathbb{L}$. Our goal is now to show that $V_T\mathbb{L} \simeq V_T^C\mathbb{L}$. We will use a shortcut, exploiting the fact that both $V_T\mathbb{L}$ and $V_T^C\mathbb{L}$ have flatsite presentations: we will define *suplattice* homomorphisms $f'\colon V_T\mathbb{L} \to V_T^C\mathbb{L}$ and $g'\colon V_T^C\mathbb{L} \to V_T\mathbb{L}$. We then show that $g' \circ f' = \mathrm{id}$ and $f' \circ g' = \mathrm{id}$, so that $V_T\mathbb{L}$ and $V_T^C\mathbb{L}$ are isomorphic as suplattices. It then follows from order theory that they are also isomorphic as frames. We start by defining a function $g\colon TC_{\mathbb{L}} \to V_T\mathbb{L}$, defined as

$$g\colon \alpha \mapsto \nabla\alpha.$$

5.4.10. LEMMA. *Let $\mathbb{L}$ be a frame. Then the fuction g defined above extends to a* suplattice *homomorphism $g'\colon V_T^C\mathbb{L} \to V_T\mathbb{L}$ such that $g' \circ \heartsuit = g$.*

$$\begin{tikzcd}
V_T^C\mathbb{L} \arrow[r, dotted, "g'"] & V_T\mathbb{L} \\
TC_{\mathbb{L}} \arrow[u, "\heartsuit"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Proof We need to show that $g\colon TC_{\mathbb{L}} \to V_T\mathbb{L}$ preserves the order on $TC_{\mathbb{L}}$ and preserves covers in to joins: if $\alpha \lhd_0^C \lambda^T(\Phi)$, where $\alpha \in TC_{\mathbb{L}}$, $\Phi \in TPC_{\mathbb{L}}$ and $\alpha\ \overline{T}{\leq}\ \bigvee(\Phi)$, then $g(\alpha) \leq \bigvee\{g(\beta) \mid \beta \in \lambda^T(\Phi)\}$. Both of these properties follow straightforwardly from the fact that $\langle TC_{\mathbb{L}}, \overline{T}{\leq}, \lhd_0^C\rangle$ is a substructure of $\langle TL, \overline{T}{\leq}, \lhd_0^{\mathbb{L}}\rangle$. ∎

The next step is to define the suplattice homomorphism $f'\colon V_T\mathbb{L} \to V_T^C\mathbb{L}$. This requires a little more work then the definition of $g'\colon V_T^C\mathbb{L} \to V_T\mathbb{L}$, beginning with the following lemma.

5.4.11. LEMMA. *Let $\mathbb{L}$ be a compact frame. If $\alpha \in TC_{\mathbb{L}}$ and $\Phi \in TPC_{\mathbb{L}}$ such that $\alpha\ \overline{T}{\leq}\ T\bigvee(\Phi)$, then there exists $\Phi_\alpha \in TP_\omega C_{\mathbb{L}}$ such that $\Phi_\alpha\ \overline{T}{\subseteq}_L\ \Phi$ and $\alpha\ \overline{T}{\leq}\ T\bigvee(\Phi_\alpha)$.*

Proof Since $\mathbb{L}$ is compact, we can show that

$$\text{for all } a \in C_{\mathbb{L}},\ a \text{ is compact.} \tag{5.51}$$

After all, if $a \in C_{\mathbb{L}}$ and $A \in PL$ such that $a \leq \bigvee A$, then also $1 \leq a \vee \neg a \leq \bigvee A \cup \{\neg a\}$, so by compactness of $\mathbb{L}$, there exists a finite $A' \subseteq A$ such that $a \vee \neg a \leq \bigvee A' \cup \{\neg a\}$. Consequently, $a \leq \bigvee A'$. Since A was arbitrary, it follows that a is compact.

We define

$$S := (\leq ; Gr(\bigvee)^{\smile}) \restriction_{C_{\mathbb{L}} \times PC_{\mathbb{L}}};$$

so that $(a, A) \in S$ iff $a \in C_{\mathbb{L}}$, $A \in PC_{\mathbb{L}}$ and $a \leq \bigvee A$. By (5.51), we can define a function $h\colon S \to S$ where $h\colon (a, A) \mapsto (a', A')$ such that $a = a'$, $A' \subseteq A$, $a' \leq \bigvee A'$

(otherwise h would not be well-defined) and such that A' is finite, i.e. $A' \in P_\omega C_{\mathbb{L}}$. In other words, $h\colon S \to S$ is a function which assigns a finite subcover A' to a set of zero-dimensional opens A covering a zero-dimensional open element a . If we denote the projection functions of S as

$$C_{\mathbb{L}} \xleftarrow{p_1} S \xrightarrow{p_2} PC_{\mathbb{L}}$$

then we can encode the above-mentioned properties of h as follows:

$$\begin{aligned}
&\forall x \in S, p_1 \circ h(x) = p_1(x);\\
&\forall x \in S, p_2 \circ h(x) \subseteq p_2(x);\\
&\forall x \in S, p_2 \circ h(x) \in P_\omega C_{\mathbb{L}}.
\end{aligned}$$

By relation lifting, it follows that

$$\forall x \in TS, Tp_1 \circ Th(x) = Tp_1(x); \tag{5.52}$$

$$\forall x \in TS, Tp_2 \circ Th(x) \mathrel{\overline{T}{\subseteq}} Tp_2(x); \tag{5.53}$$

$$\forall x \in TS, Tp_2 \circ Th(x) \in TP_\omega C_{\mathbb{L}}. \tag{5.54}$$

Finally, observe that it follows by relation lifing that

$$\forall \alpha \in TC_{\mathbb{L}}, \forall \Phi \in TPC_{\mathbb{L}},\ \alpha \mathrel{\overline{T}{\leq}} \bigvee(\Phi) \text{ iff } \alpha \mathrel{\overline{T}S} \Phi.$$

Now take $\alpha \in TC_{\mathbb{L}}$ and $\Phi \in TPC_{\mathbb{L}}$ such that $\alpha \mathrel{\overline{T}{\leq}} \bigvee(\Phi)$. Then by the above, we have $\alpha \mathrel{\overline{T}S} \Phi$, so by definition of $\overline{T}$ there must exist some $x \in TS$ such that $Tp_1(x) = \alpha$ and $Tp_2(x) = \Phi$. We define $\Phi_\alpha := Tp_2 \circ Th(x)$; observe that $Tp_1 \circ Th(x) = Tp_1(x) = \alpha$ by (5.52). Since Th is a function from TS to TS, we see that $\alpha \mathrel{\overline{T}S} \Phi_\alpha$, so that $\alpha \mathrel{\overline{T}{\leq}} T\bigvee(\Phi_\alpha)$. Moreover by (5.53) $\Phi_\alpha \mathrel{\overline{T}{\subseteq}} \Phi$ and by (5.54), $\Phi_\alpha \in TP_\omega C_{\mathbb{L}}$. This concludes the proof. ∎

We now define a map $f\colon TL \to V_T^C\mathbb{L}$ by sending

$$f\colon \alpha \mapsto \bigvee\{\heartsuit\beta \mid \beta \in TC_{\mathbb{L}},\ \beta \mathrel{\overline{T}{\leq}} \alpha\}.$$

This will give us our suplattice homomorphism $f'\colon V_T\mathbb{L} \to V_T^C\mathbb{L}$.

5.4.12. Lemma. *If $\mathbb{L}$ is a compact zero-dimensional frame then $f\colon TL \to V_T^C\mathbb{L}$ defined above extends to a* suplattice *homomorphism $f'\colon V_T\mathbb{L} \to V_T^C\mathbb{L}$, where $f' \circ \nabla = f$.*

$$\begin{tikzcd}
V_T\mathbb{L} \arrow[r, dotted, "f'"] & V_T^C\mathbb{L} \\
TL \arrow[u, "\nabla"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Proof In order to show that $f\colon TL \to V_T^C\mathbb{L}$ extends to a suplattice homomorphism, we need to show that f preserves the order on TL and f transforms covers into joins, i.e. that for all $(\alpha, \lambda^T(\Phi)) \in \lhd_0$, where $\alpha \mathrel{\overline{T}{\leq}} T\bigvee(\Phi)$, we have $f(\alpha) \leq \bigvee\{f(\gamma) \mid \gamma \in \lambda^T(\Phi)\}$. To see why f is order-preserving, suppose that $\alpha_0, \alpha_1 \in TL$ and that $\alpha_0 \mathrel{\overline{T}{\leq}} \alpha_1$. Then

$$\begin{aligned} f(\alpha_0) &= \bigvee\{\heartsuit\beta \mid \beta \in TC_{\mathbb{L}},\, \beta \mathrel{\overline{T}{\leq}} \alpha_0\} && \text{by definition of } f, \\ &\leq \bigvee\{\heartsuit\beta \mid \beta \in TC_{\mathbb{L}},\, \beta \mathrel{\overline{T}{\leq}} \alpha_1\} && \text{since } \beta \mathrel{\overline{T}{\leq}} \alpha_0 \mathrel{\overline{T}{\leq}} \alpha_1 \Rightarrow \beta \mathrel{\overline{T}{\leq}} \alpha_1, \\ &= f(\alpha_1) && \text{by definition of } f. \end{aligned}$$

Before we go ahead and show that f tranforms covers $\alpha \lhd_0 \lambda^T(\Phi)$ into joins, we show that the expression $\bigvee\{f(\gamma) \mid \gamma \in \lambda^T(\Phi)\}$ can be simplified:

$$\forall \Phi \in TPL,\ \bigvee\{f(\gamma) \mid \gamma \in \lambda^T(\Phi)\} = \bigvee\{\heartsuit\beta \mid \beta \in \lambda^T(T\Downarrow(\Phi))\}. \tag{5.55}$$

To see why, observe that

$$\begin{aligned} &\bigvee\{f(\gamma) \mid \gamma \in \lambda^T(\Phi)\} \\ &= \bigvee\left\{\bigvee\{\heartsuit\beta \mid \beta \in TC_{\mathbb{L}},\, \beta \leq \gamma\} \mid \gamma \in \lambda^T(\Phi)\right\} && \text{by definition of } f, \\ &= \bigvee\left\{\bigvee\{\heartsuit\beta \mid \beta \in TC_{\mathbb{L}},\, \beta \leq \gamma\} \mid \gamma \mathrel{\overline{T}{\in}} \Phi\right\} && \text{by definition of } \lambda^T, \\ &= \bigvee\{\heartsuit\beta \mid \beta \in TC_{\mathbb{L}},\, \exists\gamma \mathrel{\overline{T}{\in}} \Phi,\, \beta \leq \gamma\} && \text{by associativity of } \textstyle\bigvee, \\ &= \bigvee\{\heartsuit\beta \mid \beta \in TC_{\mathbb{L}},\, \beta \mathrel{\overline{T}{\leq}} ; \overline{T}{\in}\ \Phi\} && \text{by def. of relation composition,} \\ &= \bigvee\{\heartsuit\beta \mid \beta \in \lambda^T(T\Downarrow(\Phi))\} && \text{by Lemma 5.4.5(3).} \end{aligned}$$

Let $\alpha \in TL$ and $\Phi \in TPL$ such that $\alpha \mathrel{\overline{T}{\leq}} T\bigvee(\Phi)$; we need to show that $f(\alpha) \leq \bigvee\{f(\gamma) \mid \gamma \in \lambda^T(\Phi)\}$. By (5.55) it suffices to show that

$$f(\alpha) \leq \bigvee\{\heartsuit\gamma \mid \gamma \in \lambda^T(T\Downarrow(\Phi))\}. \tag{5.56}$$

Recall that $f(\alpha) = \bigvee\{\heartsuit\beta \mid \beta \in TC_{\mathbb{L}},\, \beta \leq \alpha\}$. We will show that

$$\forall \beta \in TC_{\mathbb{L}},\ \beta \mathrel{\overline{T}{\leq}} \alpha \Rightarrow \heartsuit\beta \leq \bigvee\{\heartsuit\gamma \mid \gamma \in \lambda^T(T\Downarrow(\Phi))\}. \tag{5.57}$$

Suppose that $\beta \in TC_{\mathbb{L}}$ and that $\beta \mathrel{\overline{T}{\leq}} \alpha$. Then since we assumed that $\alpha \mathrel{\overline{T}{\leq}} T\bigvee(\Phi)$, it follows that $\beta \mathrel{\overline{T}{\leq}} T\bigvee(\Phi)$. By Lemma 5.4.5(2), we know that $T\bigvee(\Phi) = T\bigvee \circ T\Downarrow(\Phi)$, so we see that

$$\beta \mathrel{\overline{T}{\leq}} T\bigvee \circ T\Downarrow(\Phi).$$

Now since $T\Downarrow(\Phi) \in TPC_{\mathbb{L}}$, we can now apply Lemma 5.4.11 to conclude that there must be some $\Phi' \in TP_\omega C_{\mathbb{L}}$ such that $\Phi' \mathrel{\overline{T}{\subseteq}} T\Downarrow(\Phi)$ and $\beta \mathrel{\overline{T}{\leq}} \bigvee\Phi'$. Now it follows by definition of $\lhd_0^C$ that $\beta \lhd_0^C \lambda^T(\Phi')$. Now

$$\begin{aligned} \heartsuit\beta &\leq \bigvee\{\heartsuit\gamma \mid \gamma \in \lambda^T(\Phi')\} && \text{since } \beta \lhd_0^C \lambda^T(\Phi'), \\ &\leq \bigvee\{\heartsuit\gamma \mid \gamma \in \lambda^T(T\Downarrow(\Phi))\} && \text{by L. 5.3.24 since } \Phi' \mathrel{\overline{T}{\subseteq}} T\Downarrow(\Phi). \end{aligned}$$

Since $\beta \in TC_{\mathbb{L}}$ was arbitrary it follows that (5.57) holds; consequently, (5.56) holds so that we may indeed conclude that f transforms covers into joins. We conclude that $f\colon TL \to V_T^C\mathbb{L}$ extends to a suplattice homomorphism $f'\colon V_T\mathbb{L} \to V_T^C\mathbb{L}$. ∎

Now that we have established the existence of suplattice homomorphisms $f'\colon V_T\mathbb{L} \to V_T^C\mathbb{L}$ and $g'\colon V_T^C\mathbb{L} \to V_T\mathbb{L}$, we are ready to prove the theorem of this subsection.

5.4.13. THEOREM. *Let $T\colon \mathbf{Set} \to \mathbf{Set}$ be a standard, finitary, weak pullback-preserving set funtor which maps finite sets to finite sets and let $\mathbb{L}$ be a frame. If $\mathbb{L}$ is compact and zero-dimensional then so is $V_T\mathbb{L}$.*

Proof It follows by Theorem 5.4.7 that $V_T\mathbb{L}$ is zero-dimensional. To show that $V_T\mathbb{L}$ is compact, it suffices to show that $V_T\mathbb{L} \simeq V_T^C\mathbb{L}$ by Lemma 5.4.9. We will establish that $V_T\mathbb{L} \simeq V_T^C\mathbb{L}$ by showing that $g'\colon V_T^C\mathbb{L} \to V_T\mathbb{L}$ and $f'\colon V_T\mathbb{L} \to V_T^C\mathbb{L}$ are *suplattice* isomorphisms, because $g' \circ f' = \mathrm{id}_{V_T\mathbb{L}}$ and $f' \circ g' = \mathrm{id}_{V_T^C\mathbb{L}}$. This is sufficient since by order theory, any suplattice isomorphism is also a frame isomorphism. We begin by making the following claim:

$$\forall \alpha \in TL,\ g' \circ f(\alpha) = \nabla\alpha. \tag{5.58}$$

After all, if $\alpha \in TL$ then

$$\begin{aligned}
g' \circ f(\alpha) &= g'\left(\bigvee\{\heartsuit\beta \mid \beta \in TC_{\mathbb{L}},\ \beta\,\overline{T}{\leq}\,\alpha\}\right) && \text{by definition of } f,\\
&= \bigvee\{g'(\heartsuit\beta) \mid \beta \in TC_{\mathbb{L}},\ \beta\,\overline{T}{\leq}\,\alpha\} && \text{since } g' \text{ preserves } \textstyle\bigvee,\\
&= \bigvee\{g(\beta) \mid \beta \in TC_{\mathbb{L}},\ \beta\,\overline{T}{\leq}\,\alpha\} && \text{by Lemma 5.4.10},\\
&= \bigvee\{\nabla\beta \mid \beta \in TC_{\mathbb{L}},\ \beta\,\overline{T}{\leq}\,\alpha\} && \text{by definition of } g,\\
&= \nabla\alpha && \text{by Lemma 5.4.5(1).}
\end{aligned}$$

It follows that (5.58) holds. Conversely, we claim that

$$\forall \alpha \in TC_{\mathbb{L}},\ f' \circ g(\alpha) = \heartsuit\alpha. \tag{5.59}$$

This is also not hard to see. Take $\alpha \in TC_{\mathbb{L}}$, then

$$\begin{aligned}
f' \circ g(\alpha) &= f'(\nabla\alpha) && \text{by definition of } g,\\
&= f(\alpha) && \text{by Lemma 5.4.12},\\
&= \bigvee\{\heartsuit\beta \mid \beta \in TC_{\mathbb{L}},\ \beta\,\overline{T}{\leq}\,\alpha\} && \text{by definition of } f,\\
&= \heartsuit\alpha && \text{since } \heartsuit \text{ is order-preserving.}
\end{aligned}$$

It follows that (5.59) holds. Now we see that for all $\alpha \in TL$,

$$\begin{aligned}
g' \circ f'(\nabla\alpha) &= g' \circ f(\alpha) && \text{since } f' \circ \nabla = f,\\
&= \nabla\alpha && \text{by (5.58)},\\
&= \mathrm{id}_{V_T\mathbb{L}}(\nabla\alpha).
\end{aligned}$$

In other words, we see that $g' \circ f'$ and $\mathrm{id}_{V_T\mathbb{L}}$ agree on the generators of $V_T\mathbb{L}$; it follows that $g' \circ f' = \mathrm{id}_{V_T\mathbb{L}}$. An analogous argument shows that $f' \circ g' = \mathrm{id}_{V_T^C\mathbb{L}}$. We conclude that $V_T\mathbb{L}$ and $V_T^C\mathbb{L}$ are isomorphic as suplattices and consequently also as frames; it follows that $V_T\mathbb{L}$ is compact. ∎

5.5 Further work

In this final section of this chapter, we give a list of possible questions for further work. The first question, or rather, task, is to investigate concrete instances of the T-powerlocale, beyond the cases $T = P_\omega$ and $T = \mathrm{Id}$. Some other, more technical questions could be the following:

- A straightforward question is if we can prove, given reasonable assumptions about T, that V_T preserves compactness.

- For a follow-up question, consider the following. There is a dual equivalence between **KRegFrm**, the category of compact regular frames and frame homomorphisms, and **KHaus**, the category of compact Hausdorff spaces and continuous maps, as witnessed by the functors $\mathrm{pt}\colon \mathbf{KRegFr} \to \mathbf{KHaus}$ and $\Omega\colon \mathbf{KHaus} \to \mathbf{KRegFr}$. If we denote the classical Vietoris hyperspace functor by $K\colon \mathbf{KHaus} \to \mathbf{KHaus}$, then it is the case that $K \circ \mathrm{pt} \simeq \mathrm{pt} \circ V$. (All of the above can be found in [54, Ch. III].) Our question is now if it is possible, given a coalgebra functor $T\colon \mathbf{Set} \to \mathbf{Set}$ such that V_T preserves compactness (and regularity, by Theorem 5.4.7), to define a functor $K_T\colon \mathbf{KHaus} \to \mathbf{KHaus}$, such that $K_T \circ \mathrm{pt} \simeq \mathrm{pt} \circ V_T$. Palmigiano & Venema propose an approach to this problem via Chu spaces in [73].

- A related question concerns the work in [66, 67]. In these papers, the authors define a functor $M_T\colon \mathbf{BA} \to \mathbf{BA}$ on Boolean algebras, similar to the way we defined $V_T\colon \mathbf{Fr} \to \mathbf{Fr}$ on frames. It is known [54, §§II-3 and II-4] that $\mathbb{L}$ is a compact and zero-dimensional frame iff $\mathbb{L} \simeq \mathcal{I}\,\mathbb{B}$ for some Boolean algebra $\mathbb{B}$, where $\mathcal{I}$ is the ideal completion functor. This raises the question if our V_T is equivalent to M_T of [67] modulo ideal completion, i.e. if $\mathcal{I} \circ M_T \simeq V_T \circ \mathcal{I}$.

- A rather technical question is the one raised in Remark 5.3.22: in what sense, if at all, is $\widehat{\rho}\colon V_T \to V_{T'}$ a right adjoint?

A different line of questions concerns constructive mathematics. In their current form, it is not entirely obvious where in the results we have presented there might be constructive issues. We mention two possible improvements:

- We have described the construction $V_T\mathbb{L}$ by taking a frame $\mathbb{L}$ and then giving a presentation of $V_T\mathbb{L}$. One could also define V_T on presentations of frames, rather than on the full frames being presented. This way, one can study frames entirely via their presentations, which is conceptually comparable to the approach of formal topology [25].

- Our results crucially use relation lifting, which seems to be tied to the category of sets rather much. If one would take a more axiomatic approach to the properties of $\lambda^T \colon TP \to PT$, it might be possible to escape **Set** and describe yet another generalization of V_T which can be constructed using other categories.

Appendix A

Preliminaries

In this appendix, we will briefly discuss some of the mathematical background knowledge that we rely on elsewhere in this dissertation. The presentation of this appendix is not linear: when explaining one subject, we will sometimes refer to another one which may lie further ahead in the text.

A.1 Set theory

Throughout this dissertation, we assume that the reader is familiar with elementary set theoretical notions such as membership $x \in X$, intersection $X \cap Y$, union $X \cup Y$ and set theoretic difference $X \setminus Y$. An exception is Chapter 5, where we use some extra notation which does not occur in the rest of the dissertation. The additional preliminaries for Chapter 5 are discussed in §5.2.

The only thing we would like to briefly mention at this point is the powerset construction. If $f\colon X \to Y$ is a function between sets X and Y, then by $f^{-1}\colon \mathcal{P}(Y) \to \mathcal{P}(X)$ we denote the *inverse image function*, which maps any set $U \subseteq Y$ to

$$f^{-1}(U) := \{x \in X \mid f(x) \in U\}.$$

At the same time, if $U \subseteq X$, then we define

$$f[U] := \{f(x) \mid x \in U\};$$

this yields a function $f[\cdot]\colon \mathcal{P}(X) \to \mathcal{P}(Y)$. Using these two mappings on subsets based on a function $f\colon X \to Y$, we can define *two* functors on **Set**, the category of sets, with the same action on objects, viz. sending X to $\mathcal{P}(X)$. The *covariant powerset functor* $\mathcal{P}\colon \mathbf{Set} \to \mathbf{Set}$ maps $f\colon X \to Y$ to $\mathcal{P}f\colon \mathcal{P}(X) \to \mathcal{P}(Y)$, where $\mathcal{P}f\colon U \mapsto f[U]$. The *contravariant powerset functor* $\breve{P}\colon \mathbf{Set} \to \mathbf{Set}^{op}$ maps $f\colon X \to Y$ to $\breve{P}f\colon \mathcal{P}(Y) \to \mathcal{P}(X)$, where $\breve{P}f\colon U \mapsto f^{-1}(U)$.

A.2 Category theory

Our main reference for category theory is Mac Lane [69]; alternatively, one can consult Adamek, Herrlich & Strecker [3].

A.2.1 Categories and functors

A *category* is a structure $\mathbf{C}$ consisting of a class of objects $X, Y, Z, \ldots$ and a binary function $\mathrm{Hom}_{\mathbf{C}}$ which assigns to any two objects X, Y a class of *morphisms*, or *arrows*, denoted $\mathrm{Hom}_{\mathbf{C}}(X, Y)$.[1] If $f \in \mathrm{Hom}_{\mathbf{C}}(X, Y)$ then we write $f\colon X \to Y$ or

$$X \xrightarrow{f} Y.$$

We require that if $(X, Y) \neq (X', Y')$, then $\mathrm{Hom}_{\mathbf{C}}(X, Y)$ and $\mathrm{Hom}_{\mathbf{C}}(X', Y')$ are disjoint. In other words, given an arrow f in $\mathbf{C}$, f has a unique *domain* X and *codomain* Y such that $f \in \mathrm{Hom}_{\mathbf{C}}(X, Y)$. Every category comes equipped with an associative *composition operation* $\circ$ for morphisms, and identity morphisms id_X, one for each object X. If $f\colon X \to Y$ and $g\colon Y \to Z$ then $g \circ f\colon X \to Z$ is a morphism from X to Z. Saying that composition is associative means that for all $f\colon X \to Y$, $g\colon Y \to Z$ and $h\colon Z \to W$, we have

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

The identity arrows id_X are characterized by the following property: for all arrows $f\colon X \to Y$ and $g\colon Y \to X$,

$$f \circ \mathrm{id}_X = f \text{ and } \mathrm{id}_X \circ g = g.$$

We also use the arrows to define the notion of *isomorphism*. An arrow $f\colon X \to Y$ is an isomorphism if there exists an arrow $g\colon Y \to X$ such that

$$gf = \mathrm{id}_X \text{ and } fg = \mathrm{id}_Y .$$

A.2.1. Example. As a prototypical example of a category, consider **Set**, which has as its objects the class of all sets, and as its arrows all functions between sets. Composition of arrows is then simply composition of functions; the identity arrows are the identity functions and isomorphisms are precisely the bijective functions. Note that this is not *the* prototypical example of a category, see [69, §I.2] for a list of further basic examples.

Let $\mathbf{C}$ and $\mathbf{C}'$ be categories. We say $\mathbf{C}'$ is a *subcategory* of $\mathbf{C}$ if every object of $\mathbf{C}'$ is also an object of $\mathbf{C}$, and if for all objects X, Y of $\mathbf{C}'$, $\mathrm{Hom}_{\mathbf{C}'}(X, Y) \subseteq \mathrm{Hom}_{\mathbf{C}}(X, Y)$. We say $\mathbf{C}'$ is a *full subcategory* of $\mathbf{C}$ if for all objects X, Y of $\mathbf{C}'$, $\mathrm{Hom}_{\mathbf{C}'}(X, Y) = \mathrm{Hom}_{\mathbf{C}}(X, Y)$.

[1] In all categories we consider in this dissertation, $\mathrm{Hom}_{\mathbf{C}}(X, Y)$ is a set rather than a class.

A.2.2. EXAMPLE. The category $\mathbf{Set}_f$, which has as its objects the class of all finite sets, and as its morphisms all functions between finite sets, forms a full subcategory of $\mathbf{Set}$.

Given a category $\mathbf{C}$, we can construct its *dual category* $\mathbf{C}^{op}$. The objects of $\mathbf{C}^{op}$ are simply those of $\mathbf{C}$. The arrows of $\mathbf{C}^{op}$ are in 1-1 correspondence $f \mapsto f^{op}$ with those of $\mathbf{C}$. The only difference between $\mathbf{C}$ and $\mathbf{C}^{op}$ is that if $f\colon X \to Y$ is an arrow of $\mathbf{C}$, then $f^{op}\colon Y \to X$ in $\mathbf{C}^{op}$ goes in the other direction. We can now define the composite of two arrows $f^{op}\colon Y \to X$ and $g^{op}\colon Z \to Y$ in $\mathbf{C}^{op}$ to be

$$f^{op} \circ g^{op} := (gf)^{op}.$$

It is easy to see that $(\mathrm{id}_X)^{op}$ is the identity arrow of X in $\mathbf{C}^{op}$, and that $\mathbf{C}^{op}$ indeed forms a category.

Because of the way $\mathbf{C}^{op}$ is defined in terms of $\mathbf{C}$, we can unravel statements about $\mathbf{C}^{op}$ into statements about $\mathbf{C}$. Using this fact, we can automatically define the dual version of any categorical concept. Usually, we will name such a dual concept by prefixing 'co-' to the name of the original concept. This process can be made much more precise, see [69, §II.1, II.2].

A.2.3. EXAMPLE. Let $\mathbf{C}$ be a category and let $F\colon \mathbf{C} \to \mathbf{C}$ be an endofunctor, i.e. a functor with the same domain and codomain. An F-algebra consists of an object X and a morphism $h\colon F(X) \to X$. The dual notion, that of an F-coalgebra, consists simply of an object X and a morphism $r\colon X \to F(X)$.

A *functor* is a structure-preserving map between categories. Concretely, if $\mathbf{C}, \mathbf{D}$ are categories then a functor $F\colon \mathbf{C} \to \mathbf{D}$ consists of an assignment of an object $F(X)$ to every object X of $\mathbf{C}$, and of an assignment of an arrow $F(f) \in \mathrm{Hom}_{\mathbf{D}}(FX, FY)$ to every arrow $f \in \mathrm{Hom}_{\mathbf{C}}(X, Y)$ such that $F(g \circ f) = F(g) \circ F(f)$ for all $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathbf{C}$ and $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all X in $\mathbf{C}$. If it is not visually confusing, we will sometimes omit parentheses, writing e.g. Ff instead of $F(f)$.

A.2.4. EXAMPLE. As a trivial example of a functor, observe that given any category $\mathbf{C}$ we can define the identity functor $\mathrm{Id}_{\mathbf{C}}\colon \mathbf{C} \to \mathbf{C}$ which leaves all objects and arrows unchanged.

A functor $F\colon \mathbf{C} \to \mathbf{D}^{op}$ is called a *contravariant functor from* $\mathbf{C}$ *to* $\mathbf{D}$; one can alternatively view F as an 'arrow-reversing' functor from $\mathbf{C}$ to $\mathbf{D}$. The notion of functor we introduced before is sometimes also called a *covariant functor*.

A *natural transformation* ν between functors $F, G\colon \mathbf{C} \to \mathbf{D}$ is a family of $\mathbf{D}$-morphisms $\nu_X\colon F(X) \to G(X)$, one for each object X of $\mathbf{C}$, such that for all $f\colon X \to Y$ in $\mathbf{C}$, the following diagram commutes:

$$\begin{tikzcd} X \arrow[d, "f"'] & F(X) \arrow[r, "\nu_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ Y & F(Y) \arrow[r, "\nu_Y"'] & G(Y) \end{tikzcd}$$

If ν is a natural transformation such that for each X, ν_X is an isomorphism, then we call ν a *natural isomorphism.* If there exists a natural isomorphism between two functors $F, G\colon \mathbf{C} \to \mathbf{D}$, we say F and G are *naturally isomorphic.*

A.2.2 Adjunctions of categories

One of the fundamental notions of category theory is that of *adjunctions* between categories. A pair of functors $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{D} \to \mathbf{C}$ is called an *adjunction*, abbreviated $F \dashv G$, if there exist natural transformations $\eta\colon \mathrm{Id}_{\mathbf{C}} \to GF$ and $\epsilon\colon FG \to \mathrm{Id}_{\mathbf{D}}$ such that for all $f\colon X \to GY$, there exists a unique $f'\colon F(X) \to Y$ such that $f = Gf' \circ \eta_X$, and for all $g\colon FX \to Y$, there exists a unique $g'\colon X \to G(Y)$ such that $g = \epsilon_Y \circ Fg'$:

$$
\begin{tikzcd}
 & GF(X) \arrow[d, dotted, "G(f')"] \\
X \arrow[ur, "\eta_X"] \arrow[r, "f"'] & G(Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X) \arrow[d, dotted, "f'"] \\
Y
\end{tikzcd}
\qquad
\begin{tikzcd}
FG(Y) \arrow[dr, "\epsilon_Y"] & \\
F(X) \arrow[u, dotted, "F(g')"] \arrow[r, "g"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
G(Y) \\
X \arrow[u, dotted, "g'"']
\end{tikzcd}
$$

A *dual adjunction* between categories $\mathbf{C}$ and $\mathbf{D}$ is simply an adjunction between $\mathbf{C}$ and $\mathbf{D}^{op}$.

We say to categories $\mathbf{C}, \mathbf{D}$ are *equivalent* if there exist functors $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{D} \to \mathbf{C}$ such that GF is naturally isomorphic to $\mathrm{Id}_{\mathbf{C}}$ and FG is naturally isomorphic to $\mathrm{Id}_{\mathbf{D}}$. If both F and G are contravariant, we say $\mathbf{C}$ and $\mathbf{D}$ are *dually equivalent.*

If $\mathbf{D}$ is a subcategory of $\mathbf{C}$, then the inclusion of $\mathbf{D}$ into $\mathbf{C}$ forms a functor $J\colon \mathbf{D} \to \mathbf{C}$. If this functor has a left adjoint $F\colon \mathbf{C} \to \mathbf{D}$, we say that $\mathbf{D}$ is a *reflective subcategory* of $\mathbf{C}$, and we call $F\colon \mathbf{C} \to \mathbf{D}$ a *reflector.* Dually, if $J\colon \mathbf{D} \to \mathbf{C}$ has a right adjoint $G\colon \mathbf{C} \to \mathbf{D}$, we say that $\mathbf{D}$ is a *co-reflective subcategory* of $\mathbf{C}$, and we call $G\colon \mathbf{C} \to \mathbf{D}$ a *co-reflector.*

A.3 Order theory and domain theory

In this section we will discuss some of the order theory and domain theory that we employ in this dissertation. Our main reference for order theory is Davey & Priestley [29]; our main reference for domain theory is Abramsky & Jung [1].

A.3.1 Pre-orders and partial orders

A *pre-order* is a structure $\mathbb{P} = \langle P, \leq \rangle$ where $\leq\, \subseteq P \times P$ is a binary relation which is

- reflexive: $\forall x \in \mathbb{P}$, $x \leq x$;

- transitive: $\forall x, y, z \in \mathbb{P}$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

(Note that we will sometimes write '$x \in \mathbb{P}$' rather than '$x \in P$'.) Observe that a preorder $\langle P, \leq \rangle$ can be seen as a small category with a set of objects P, such that between any two objects x, y there is at most one arrow. From this perspective, we get that $\mathrm{Hom}_{\mathbb{P}}(x, y) \neq \emptyset$ iff $x \leq y$. The notion of dual category now specializes to the notion of the *order dual* of a pre-order; we denote the order dual of $\mathbb{P}$ by $\mathbb{P}^{op}$. In other words, $\mathbb{P}^{op} := \langle P, \geq \rangle$, where $x \geq y :\Leftrightarrow y \leq x$. It is not hard to see that $\mathbb{P}^{op}$ is again a pre-order. If $x \in \mathbb{P}$, we define $\downarrow_{\mathbb{P}} x = \{y \in \mathbb{P} \mid y \leq x\}$; $\uparrow_{\mathbb{P}} x$ is defined dually. If it is clear what order we are dealing with we will omit the subscripts on $\downarrow$ and $\uparrow$. For $U \subseteq P$, we define $\downarrow U = \bigcup_{x \in U} \downarrow x$. If $U = \downarrow U$ we say that U is a *lower set*; dually, if $U = \uparrow U$ then U is an *upper set*. An equivalent characterization of lower sets is to say that U is a lower set iff for all $x \in U$ and all $y \leq x$, we have $y \in U$.

A *partial order* (or *partially ordered set*, or *poset*) is a pre-order $\langle P, \leq \rangle$ which satisfies the additional property that it is

- anti-symmetric: $\forall x, y \in \mathbb{P}$, if $x \leq y$ and $y \leq x$ then $x = y$.

Although most of the order-theoretic notions we will discuss below also can be defined for pre-orders (in fact, even for categories), we will restrict our attention below to partially ordered sets.

Given two posets $\mathbb{P}$ and $\mathbb{Q}$, a function $f \colon P \to Q$ is called *order-preserving* if for all $x, y \in P$ such that $x \leq y$, we have $f(x) \leq f(y)$. (This is the order-theoretic specialization of the categorical notion of being a functor.) We say $f \colon \mathbb{P} \to \mathbb{Q}$ is an *order embedding* if for all $x, y \in \mathbb{P}$, $x \leq y$ iff $f(x) \leq f(y)$. One can show that order embeddings between posets are necessarily injective maps. An *order isomorphism* is a surjective order-embedding. By **Pos** we denote the category of partially ordered sets and order-preserving maps.

A.3.1. FACT. *Let $\mathbb{P}, \mathbb{Q}$ be posets and let $f \colon P \to Q$ be a function. The following are equivalent:*

1. *f is order-preserving;*
2. *for all $U \subseteq P$, $f[\downarrow U] \subseteq \downarrow f[U]$;*
3. *for all $V \subseteq Q$, if V is a lower set then so is $f^{-1}(V)$.*

Products of posets are defined point-wise: if $\{\mathbb{P}_i \mid i \in I\}$ is a collection of posets, we define

$$\prod_I \mathbb{P}_i = \langle \textstyle\prod_I P_i, \leq \rangle,$$

where for $x, y \in \prod_I P_i$, we have $x \leq y$ iff for all $i \in I$, $x(i) \leq_i y(i)$.

A.3.2 Adjunctions of partially ordered sets

Since every poset is a category, we can specialize the categorical notion of adjunction we introduced in §A.2.2 to posets.

A.3.2. DEFINITION. Let $f\colon \mathbb{P} \to \mathbb{Q}$ and $g\colon \mathbb{Q} \to \mathbb{P}$ be order-preserving maps. We say f and g are an adjoint pair, abbreviated $f \dashv g$, if one of the following equivalent conditions holds:

- $\forall x \in P$, $\forall y \in Q$, $f(x) \leq y$ iff $x \leq g(y)$;
- $\mathrm{id}_{\mathbb{P}} \leq g \circ f$ and $f \circ g \leq \mathrm{id}_{\mathbb{Q}}$.

Adjoint pairs have many attractive mathematical properties; we list a few below. As a reference, consider [1, Prop. 3.1.12].

A.3.3. FACT. *Suppose that $f\colon \mathbb{P} \to \mathbb{Q}$ and $g\colon \mathbb{Q} \to \mathbb{P}$ form an adjoint pair, i.e. $f \dashv g$. Then*

1. *f preserves all existing suprema and g preserves all existing infima;*
2. *f is an order embedding iff g is surjective iff $g \circ f = \mathrm{id}_{\mathbb{P}}$;*
3. *f is surjective iff g is order embedding iff $f \circ g = \mathrm{id}_{\mathbb{Q}}$.*

A.3.3 Dcpo's

A non-empty subset $U \subseteq P$ of a partially ordered set $\mathbb{P}$ is called *directed* if for all $x, y \in U$, there exists a $z \in U$ such that $x \leq z$ and $y \leq z$. A *dcpo* $\mathbb{D} = \langle D, \leq, \rangle$ is a partial order such that for every directed non-empty $U \subseteq D$, U has a supremum or *join* in $\mathbb{D}$, denoted $\bigvee U$. By **Dcpo** we denote the category of dcpo's and *Scott-continuous* functions, i.e. functions which preserve directed joins. As a general reference for facts about dcpo's, we suggest Abramsky & Jung [1]. We will see later that Scott-continuity is indeed a continuity property. For now we record the following fact about finite products of dcpo's. As a reference for this fact, consider [1, Prop. 3.2.2 and Lemma 3.2.6]

A.3.4. FACT. *Let $\mathbb{D}_1, \ldots, \mathbb{D}_n, \mathbb{E}$ be dcpo's. Then the poset-product $\mathbb{D}_1 \times \cdots \times \mathbb{D}_n$ is also a dcpo. If $f\colon \mathbb{D}_1 \times \cdots \times \mathbb{D}_n \to \mathbb{E}$ is a function, then f preserves all directed joins in $\mathbb{D}_1 \times \cdots \times \mathbb{D}_n$ iff f preserves directed joins in each coordinate.*

A.3.4 Order and topology

Given a partial order $\mathbb{P}$, one can use the order to define several topologies on P, as we do in §A.7. As a general reference, we suggest the Compendium of Continuous Lattices[45, Ch. 2 and 3]. We will mention a few properties of one such topology, the Scott topology.

A.3.5. Definition. Let $\mathbb{P} = \langle P, \leq \rangle$ be a poset. A set $U \subseteq \mathbb{P}$ is Scott-open iff $P \setminus U$ is a lower set closed under all existing directed joins. By $\sigma^{\uparrow}(\mathbb{P})$ we denote the *Scott topology* of $\mathbb{P}$ [45, Ch. 2].

The following well-known fact states that Scott-continuity of maps between dcpo's is indeed a continuity property. As a reference, consider [1, Prop. 2.3.4].

A.3.6. Fact. *A function $f\colon \mathbb{D} \to \mathbb{E}$ between dcpo's preserves directed joins iff it is continuous with respect to the Scott topology.*

Many order-theoretic properties can be characterized topologically; this is a major topic in this dissertation. As another example consider the following.

A.3.7. Example. Given a poset $\mathbb{P} = \langle P, \leq \rangle$ define the *Alexandrov topology* to be the collection of all upper sets of $\mathbb{P}$ [54, §II-1.8]. A function $f\colon \mathbb{P} \to \mathbb{Q}$ between posets is order-preserving iff f is continuous with respect to the Alexandrov topology.

A.4 Lattice theory

A standard reference for lattice theory is Birkhoff [18]. Good introductory expositions can also be found in e.g. [23], [29] and [54, Ch. I].

A.4.1 Semilattices and suplattices

A $\vee$-*semilattice* is an algebra (see §A.6) $\mathbb{L} = \langle L, 0, \vee \rangle$ with a binary operation $\vee$ called 'join' and a constant 0 which satisfy the following equations:

(i) $x \vee x \approx x$ (idempotence) (ii) $x \vee y \approx y \vee x$ (commutativity)

(iii) $x \vee (y \vee z) \approx (x \vee y) \vee z$ (associativity)

(iv) $x \vee 0 \approx x$ (0 is the identity).

A $\vee$-semilattice is a poset under the ordering $a \leq b :\Leftrightarrow a \vee b = b$. Alternatively, one can characterize $\vee$-semilattices as posets in which any finite subset has a least upper bound; one can then define 0 to be the upper bound of the empty set, and $a \vee b$ to be the least upper bound of $\{a, b\}$. A $\vee$-*semilattice homomorphism* $f\colon \mathbb{L} \to \mathbb{M}$ is a map between semilattices $\mathbb{L}$, $\mathbb{M}$ which preserves $\vee$ and 0, i.e. such that $f(a \vee b) = f(a) \vee f(b)$ for all $a, b \in \mathbb{L}$ and such that $f(0) = 0$.

A complete $\vee$-semilattice or *suplattice* is a partially ordered set in which *each* subset has a least upper bound; alternatively, a suplattice is an algebra $\mathbb{L} = \langle L, \bigvee, 0\rangle$, where $\bigvee$ is an infinitary operation.

A.4.1. FACT. *Let $\mathbb{L}$ and $\mathbb{M}$ be suplattices and let $f\colon \mathbb{L} \to \mathbb{M}$ be a function. The following are equivalent:*

1. *f preserves all non-empty joins;*
2. *f preserves binary joins and directed joins.*

The order dual notion of a $\vee$-semilattice is that of a $\wedge$-semilattice. In a $\wedge$-semilattice $\mathbb{L} = \langle L, \wedge, 1\rangle$, the operation $\wedge$ (*meet*) determines a partial order $a \leq b :\Leftrightarrow a \wedge b = a$. Under this order $a \wedge b$ is the *greatest lower bound* of $\{a, b\}$.

A.4.2 Lattices and complete lattices

A *(bounded) lattice* is an algebra $\mathbb{L} = \langle L, \wedge, \vee, 0, 1\rangle$ which is simultaneously a $\wedge$-semilattice and a $\vee$-semilattice. Alternatively, a lattice is a partially ordered set in which each finite subset has both a least upper bound and a greatest lower bound.

A.4.2. REMARK. Throughout this dissertation we assume that lattices are bounded, i.e. that a lattice $\mathbb{L}$ always has a least element $0_{\mathbb{L}}$ and a greatest element $1_{\mathbb{L}}$, and that lattice homomorphisms preserve $0_{\mathbb{L}}$ and $1_{\mathbb{L}}$.

By **Lat** we denote the category of lattices and lattice homomorphisms; by **CLat** we denote the category of complete lattices and complete homomorphisms. The following fact is the order-theoretic version of a well-known result from category theory, known as the Adjoint Functor Theorem. As a reference for this order-theoretic version, consider [1, Prop. 3.1.13].

A.4.3. FACT. *If $f\colon \mathbb{L} \to \mathbb{M}$ is a map between complete lattices which preserves all joins, then there exists a unique $g\colon \mathbb{M} \to \mathbb{L}$ such that f and g form an adjoint pair $f \dashv g$.*

A.4.3 Distributive lattices, Heyting algebras and Boolean algebras

Introductory discussions of distributive lattices, Heyting algebras and Boolean algebras can be found in Johnstone [54, Ch. I]. For additional technical detail, see e.g. Balbes & Dwinger [7] for distributive lattices and Heyting algebras and Koppelberg [63] for Boolean algebras.

A lattice $\mathbb{L}$ is *distributive* if it satisfies the equation $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \vee z)$, or equivalently, if it satisfies the equation $x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z)$. We denote the category of distributive lattices and lattice homomorphisms by **DL**. If $\mathbb{L}$ is complete, we say $\mathbb{L}$ satisfies the $(\wedge, \bigvee)$-*distributive law* (also known as the infinite distributive law) if for all $a \in \mathbb{L}$ and for all $S \subseteq \mathbb{L}$, $a \wedge \bigvee S = \bigvee_{b \in S}(a \wedge b)$. Complete lattices which satisfy the $(\wedge, \bigvee)$-distributive law are known as *frames*, also see §5.2.4.

A *Heyting algebra* is an algebra $\mathbb{A} = \langle A, \wedge, \vee, \rightarrow, 0, 1\rangle$ such that $\langle A, \wedge, \vee, 0, 1\rangle$ is a distributive lattice and $\rightarrow$ is a binary operation, known as Heyting implication, such that

$$z \wedge x \leq y \text{ iff } z \leq x \rightarrow y.$$

Alternatively, one can say that $\rightarrow$ has to satisfy the following axioms:

(i)	$x \rightarrow x \approx 1$	(ii)	$x \wedge (x \rightarrow y) \approx x \wedge y$
(iii)	$y \wedge (x \rightarrow y) \approx y$	(iv)	$x \rightarrow (y \wedge z) \approx (x \rightarrow y) \wedge (x \rightarrow z)$

It is a fact of order-theory that adjoints of maps are unique; consequently, since $\rightarrow$ is defined via an adjointness property with respect to $\wedge$, a given distributive lattice $\mathbb{L}$ admits at most one Heyting implication such that $\langle L, \wedge, \vee, \rightarrow, 0, 1\rangle$ is a Heyting algebra. We denote the category of Heyting algebras and Heyting algebra homomorphisms by **HA**.

A *Boolean algebra* is an algebra $\mathbb{A} = \langle A, \wedge, \vee, \neg, 0, 1\rangle$ such that $\langle A, \wedge, \vee, 0, 1\rangle$ is a distributive lattice, and where $\neg$ is a unary operation, known as negation or complementation, satisfying the following equations:

$$x \vee \neg x \approx 1 \text{ and } x \wedge \neg x \approx 0.$$

As was the case with Heyting algebras, a distributive lattice $\mathbb{L}$ admits at most one negation operation making it into a Boolean algebra. In fact, every Boolean algebra is also a Heyting algebra, if we define

$$a \rightarrow b := \neg a \vee b.$$

We denote the category of Boolean algebras and Boolean algebra homomorphisms by **BA**.

A.5 Completions

In §A.3 and §A.4 we have seen several kinds of complete ordered structures. Completions of ordered structures, that is embeddings of ordered structures into complete structures play an important role in this dissertation. In this section we will discuss some properties of the filter and ideal completions and the MacNeille completion.

A.5.1 The ideal (and filter) completion of a pre-order

The ideal completion is a construction that allows one to embed any partial order into a dcpo. As references we suggest Plotkin [75, §6] and Abramsky & Jung [1, §2.2.6].

Ideals and filters

A.5.1. DEFINITION. Let $\mathbb{P} = \langle P, \leq \rangle$ be a partial order. An *ideal* of $\mathbb{P}$ is a non-empty, directed lower set $I \subseteq \mathbb{P}$. We define $\mathcal{I}\,\mathbb{P} = \langle \mathrm{Idl}(\mathbb{P}), \subseteq \rangle$ (the *ideal completion of* $\mathbb{P}$) to be the collection of ideals of $\mathbb{P}$ ordered by subset inclusion. Dually, a *filter* is a non-empty upper set $F \subseteq \mathbb{P}$ which is co-directed. We define $\mathcal{F}\,\mathbb{P} := \langle \mathrm{Filt}(\mathbb{P}), \supseteq \rangle$ (the *filter completion of* $\mathbb{P}$) to be the collection of filters of $\mathbb{P}$.

One can embed $\mathbb{P}$ in $\mathcal{I}\,\mathbb{P}$ by mapping $x \mapsto {\downarrow} x$. Moreover, given a order-preserving function $f \colon \mathbb{P} \to \mathbb{P}'$ and $I \in \mathcal{I}\,\mathbb{P}$, we define

$$\mathcal{I}\,f : I \mapsto {\downarrow} f[I] = \{x' \in \mathbb{P}' \mid \exists x \in I,\, x' \leq f(x)\},$$

which is a Scott-continuous function from $\mathcal{I}\,\mathbb{P}$ to $\mathcal{I}\,\mathbb{P}'$. It is easy to show that if $\mathbb{L}$ is a $\vee$-semilattice, then a non-empty lower set $I \subseteq \mathbb{L}$ is an ideal iff I is closed under $\vee$. Similarly, filters in $\wedge$-semilattices are non-empty upper sets closed under $\wedge$.

Basic properties of $\mathcal{I}$ and $\mathcal{F}$

The following fact is a consequence of the fact that the definitions of ideals and filters are order-symmetric. Given a poset $\mathbb{P}$, I is an ideal of $\mathbb{P}^{op}$ iff I is a filter of $\mathbb{P}$. This gives us an order anti-isomorphism between $\mathcal{I}(\mathbb{P}^{op})$ and $\mathcal{F}(\mathbb{P})$.

A.5.2. FACT. $\mathcal{I}(\mathbb{P}^{op}) \simeq (\mathcal{F}\,\mathbb{P})^{op}$.

We will now list several useful properties of the ideal completion. Naturally, all of these properties dualize to the filter completion.

A.5.3. FACT. *Let $\mathbb{P}$ be a partial order.*

1. *$\mathcal{I}$ is a functor* **Pos** $\to$ **Dcpo** *and ${\downarrow}\colon \mathrm{Id} \to \mathcal{I}$ is a natural transformation.*
2. *If $f \colon \mathbb{P} \to \mathbb{D}$ is an order-preserving map to a dcpo $\mathbb{D}$, then there exists a unique Scott-continuous $f' \colon \mathcal{I}\,\mathbb{P} \to \mathbb{D}$ such that $f' \circ {\downarrow}\{\cdot\} = f$; i.e. $\mathcal{I}\,\mathbb{P}$ is the free dcpo over $\mathbb{P}$.*

$$\begin{tikzcd} \mathcal{I}\,\mathbb{P} \arrow[r, dotted, "f'"] & \mathbb{D} \\ \mathbb{P} \arrow[u, "{\downarrow_{\mathbb{P}}}"] \arrow[ur, "f"'] & \end{tikzcd}$$

3. *If $\mathbb{P}$ is a join semilattice then $\mathcal{I}\,\mathbb{P}$ is a complete lattice.*
4. *If $\mathbb{P}$ and $\mathbb{P}'$ are join semilattices and if $f \colon \mathbb{P} \to \mathbb{P}'$ preserves binary joins, then $\mathcal{I}\,f \colon \mathcal{I}\,\mathbb{P} \to \mathcal{I}\,\mathbb{P}'$ preserves all non-empty joins.*
5. *If $f \colon \mathbb{P} \to \mathbb{Q}$ is an order embedding then so is $\mathcal{I}\,f$.*

The ideal completion, and dually, the filter completion, preserves finite products; see [1, Prop. 2.2.23].

A.5.4. FACT. *Let $\mathbb{P}, \mathbb{Q}$ be partial orders. There exists an order isomorphism $h\colon \mathcal{I}(\mathbb{P} \times \mathbb{Q}) \to \mathcal{I}\,\mathbb{P} \times \mathcal{I}\,\mathbb{Q}$ such that for all $x \in P$, $y \in Q$,*

$$h \circ {\downarrow_{\mathbb{P}\times\mathbb{Q}}}(x, y) = {\downarrow_{\mathbb{P}}}(x) \times {\downarrow_{\mathbb{Q}}}(y).$$

*(The map h simply sends (I_1, I_2) to $I_1 \times I_2$.) In other words, the functor $\mathcal{I}\colon$ **Pos** $\to$ **Dcpo** preserves finite products.*

$$\begin{tikzcd}
\mathcal{I}(\mathbb{P} \times \mathbb{Q}) \arrow[r, "h"] & \mathcal{I}\,\mathbb{P} \times \mathcal{I}\,\mathbb{Q} \\
\mathbb{P} \times \mathbb{Q} \arrow[u, "\downarrow_{\mathbb{P}\times\mathbb{Q}}"] \arrow[ur, "\downarrow_{\mathbb{P}} \times \downarrow_{\mathbb{Q}}"'] &
\end{tikzcd}$$

Moreover, $\sigma^{\uparrow}(\mathcal{I}\,\mathbb{P}) \times \sigma^{\uparrow}(\mathcal{I}\,\mathbb{Q}) = \sigma^{\uparrow}(\mathcal{I}(\mathbb{P} \times \mathbb{Q}))$.

Algebraic dcpo's

Let $\mathbb{D}$ be a dcpo. An element $p \in \mathbb{D}$ is *compact* if for all directed $S \subseteq \mathbb{D}$, $p \leq \bigvee S$ implies that there is some $x \in S$ such that $p \leq x$. We denote the set of compact elements of $\mathbb{D}$ by $\mathrm{K}\,\mathbb{D}$. We call $\mathbb{D}$ an *algebraic dcpo* if for each $x \in \mathbb{D}$, the set ${\downarrow}\, x \cap \mathrm{K}\,\mathbb{D}$ is directed and $x = \bigvee({\downarrow}\, x \cap \mathrm{K}\,\mathbb{D})$.

A.5.5. FACT. *Let $\mathbb{D}$ be an algebraic dcpo. Then*

1. $\mathbb{D} \simeq \mathcal{I}(\mathrm{K}\,\mathbb{D})$*;*
2. *the set $\{{\uparrow}\, p \mid p \in \mathrm{K}\,\mathbb{D}\}$ forms a base for the Scott topology on $\mathbb{D}$.*

If $\mathbb{L}$ is a complete lattice, then $\mathrm{K}\,\mathbb{L}$ is closed under finite $\vee$. Consequently, ${\downarrow}\, x \cap \mathrm{K}\,\mathbb{L}$ is always directed in a lattice; a complete lattice is therefore algebraic iff for all $x \in \mathbb{L}$, $x = \bigvee({\downarrow}\, x \cap \mathrm{K}\,\mathbb{D})$.

A.5.2 The MacNeille completion of a pre-order

Let $\mathbb{P} = \langle P, \leq \rangle$ be a pre-order and let $U \subseteq P$. We define

$$\begin{aligned}
\mathrm{ub}(U) &:= \{x \in \mathbb{P} \mid U \subseteq {\downarrow}\, x\} && \text{(the upper bounds of } U\text{)},\\
\mathrm{lb}(U) &:= \{x \in \mathbb{P} \mid U \subseteq {\uparrow}\, x\} && \text{(the lower bounds of } U\text{)}.
\end{aligned}$$

A *cut* of $\mathbb{P}$ is a pair of sets (U, V), where $U, V \subseteq \mathbb{P}$, such that $U = \mathrm{lb}(V)$ and $V = \mathrm{ub}(U)$. We can order the set of all cuts of $\mathbb{P}$ by setting

$$(U_1, V_1) \leq (U_2, V_2) :\Leftrightarrow U_1 \subseteq U_2 \text{ (or equivalently, } V_1 \supseteq V_2\text{)}.$$

Under this order, the set of all cuts of $\mathbb{P}$ is a complete lattice which we denote by $\overline{\mathbb{P}}$. We call $\overline{\mathbb{P}}$ the *MacNeille completion of* $\mathbb{P}$ [70]. There exists a natural map $i_{\mathbb{P}} \colon \mathbb{P} \to \overline{\mathbb{P}}$ defined by

$$i_{\mathbb{P}} \colon x \mapsto (\downarrow x, \uparrow x).$$

A.5.6. Remark. Similarly to the canonical extension (Fact 2.1.9), the MacNeille completion can be characterized up to isomorphism of completions. Banaschewski & Bruns [9, §4] showed that if $e \colon \mathbb{P} \to \mathbb{C}$ is an order embedding of a poset $\mathbb{P}$ into a complete lattice $\mathbb{C}$ such that $e[\mathbb{P}]$ is both join-dense and meet-dense, then there exists a unique order-isomorphism $h \colon \mathbb{C} \to \overline{\mathbb{P}}$ such that $h \circ e = i_{\mathbb{P}}$.

A.5.7. Example. Perhaps the best-known example of a MacNeille completion is the embedding of the rational numbers $\mathbb{Q}$ into the reals $\mathbb{R}$. This is known as the completion of the rationals by Dedekind cuts; for this reason the MacNeille completion is also known as the Dedekind-MacNeille completion.

A.6 Universal algebra

Our main reference for universal algebra is Burris & Sankappanavar [23].

A.6.1 Ω-algebras

An *algebraic signature* consists of a set of function symbols Ω and an arity function $\mathrm{ar} \colon \Omega \to \mathbb{N}$, assigning a finite arity to each operation symbol. An Ω-*algebra* is a structure $\mathbb{A} = \langle A; (\omega_{\mathbb{A}})_{\omega \in \Omega} \rangle$ where for each $\omega \in \Omega$, $\omega_{\mathbb{A}}$ is a function $\omega_{\mathbb{A}} \colon A^{\mathrm{ar}(\omega)} \to A$. Given two Ω-algebras $\mathbb{A}$, $\mathbb{B}$, and a function $f \colon A \to B$, we say that f is an Ω-*algebra homomorphism* if for all $\omega \in \Omega$, the following diagram commutes:

$$\begin{tikzcd} A^n \arrow[r, "\omega_{\mathbb{A}}"] \arrow[d, "f^{[n]}"'] & A \arrow[d, "f"] \\ B^n \arrow[r, "\omega_{\mathbb{B}}"] & B \end{tikzcd}$$

where $n = \mathrm{ar}(\omega)$ and $f^{[n]} \colon A^n \to B^n$ denotes the function

$$(x_1, \ldots, x_n) \mapsto (f(x_1), \ldots, f(x_n))\,.$$

A.6.2 Homomorphic images, subalgebras and products

Fix an algebraic signature Ω.

Subalgebras

Let $\mathbb{A} = \langle A, (\omega_{\mathbb{A}})_{\omega\in\Omega}\rangle$ be an algebra and let $B \subseteq A$. We say that B is a subuniverse of $\mathbb{A}$ if for all $\omega \in \Omega$, $\omega_{\mathbb{A}}[B^{\mathrm{ar}(\omega)}] \subseteq B$; in other words, if B is closed under all operations $\omega_{\mathbb{A}}$. In that case, we can define operations $\omega_{\mathbb{B}} := (\omega_{\mathbb{A}}) \restriction B^{\mathrm{ar}(\omega)}$ for all $\omega \in \Omega$, and we call $\mathbb{B} = \langle B, (\omega_{\mathbb{B}})_{\omega\in\Omega}\rangle$ a *subalgebra of* $\mathbb{A}$. If $\mathcal{K}$ is a class of algebras, we denote the class of all subalgebras of algebras in $\mathcal{K}$ by $\mathrm{S}(\mathcal{K})$. If $\mathcal{K}$ consists of a single algebra $\mathbb{A}$, we denote the *set* of all its subalgebras by $\mathrm{S}(\mathbb{A})$.

Homomorphic images, quotients and congruences

Let $\mathbb{A}$ and $\mathbb{B}$ be algebras and let $f\colon \mathbb{A} \to \mathbb{B}$ be a homomorphism; if f is surjective then we call $\mathbb{B}$ a *homomorphic image of* $\mathbb{A}$. If $\mathcal{K}$ is a class of algebras, we denote the class of all homomorphic images of algebras in $\mathcal{K}$ by $\mathrm{H}(\mathcal{K})$. Given an algebra $\mathbb{A}$, each algebra $\mathbb{B}$ in the class of all its homomorphic images $\mathrm{H}(\mathbb{A})$ can be represented as a *quotient*, using a congruence on $\mathbb{A}$.

A *congruence* is an equivalence relation $\theta \subseteq A \times A$ on the underlying set of $\mathbb{A}$ such that for all $\omega \in \Omega$ and all $a_1, \dots, a_n, b_1, \dots, b_n \in A$, where $n = \mathrm{ar}(\omega)$,

$$\text{if } a_i \;\theta\; b_i \text{ for all } i \le n, \text{ then also } \omega_{\mathbb{A}}(a_1, \dots, a_n) \;\theta\; \omega_{\mathbb{A}}(b_1, \dots, b_n).$$

It is a consequence of the definition of congruences that if θ is a congruence of $\mathbb{A} = \langle A, (\omega_{\mathbb{A}})_{\omega\in\Omega}\rangle$, then we can define an algebra structure on A/θ, the set of θ-equivalence classes of A. We denote this algebra by $\mathbb{A}/\theta$. If we define $\mu_\theta\colon \mathbb{A} \to \mathbb{A}/\theta$ as $\mu_\theta\colon a \mapsto a/\theta$, then $\mu_\theta\colon \mathbb{A} \to \mathbb{A}/\theta$ is a surjective algebra homomorphism. We denote the poset of all congruences of $\mathbb{A}$, ordered by subset inclusion, by $\mathrm{Con}\,\mathbb{A}$. In fact, $\mathrm{Con}\,\mathbb{A}$ is always a complete lattice, so we may speak of the *congruence lattice of* $\mathbb{A}$.

A natural example of a congruence is provided by the *kernel* of a given homomorphism $f\colon \mathbb{A} \to \mathbb{B}$:

$$\ker f := \{(a, b) \in A \times A \mid f(a) = f(b)\}.$$

It is a consequence of the Isomorphism Theorems of universal algebra that if $f\colon \mathbb{A} \to \mathbb{B}$ is a surjective homomorphism, then there exists a unique isomorphism $h\colon \mathbb{B} \to \mathbb{A}/\ker f$ such that $h \circ f = \mu_{\ker f}$. In other words, every homomorphic image of $\mathbb{A}$ is naturally isomorphic to a quotient of $\mathbb{A}$.

Products and subdirect products

If $\{\mathbb{A}_i \mid i \in I\}$ is a set of algebras, then we can define an algebra structure on the product $\prod_I A_i$. If ω is an operation of arity n, then we define $\omega_{\prod_I \mathbb{A}_i}$ on $(\prod_I \mathbb{A}_i)^n$ coordinate-wise as follows:

$$\left(\omega_{\prod_I \mathbb{A}_i}(a_1, \dots, a_n)\right)(i) = \left(\omega_{\mathbb{A}_i}\left(a_1(i), \dots, a_n(i)\right)\right).$$

We denote the product of $\{\mathbb{A}_i \mid i \in I\}$ by $\prod_I \mathbb{A}_i$. We denote the corresponding projection homomorphisms by $\pi_j \colon \prod_I \mathbb{A}_i \to \mathbb{A}_j$. If $\mathcal{K}$ is a class of algebras, we denote the class of all products of algebras in $\mathcal{K}$ by $\mathrm{P}(\mathcal{K})$.

An algebra $\mathbb{A}$ is a *subdirect product* of a set of algebras $\{\mathbb{B}_i \mid i \in I\}$ if there exists a set of surjective algebra homomorphisms $p_i \colon \mathbb{A} \to \mathbb{B}_i$ such that for all $a, b \in \mathbb{A}$, if $p_i(a) = p_i(b)$ for all $i \in I$, then $a = b$. In this case we call $(p_i \colon \mathbb{A} \to \mathbb{B}_i)_I$ a subdirect decomposition of $\mathbb{A}$. If $\mathcal{K}$ is a class of algebras, we denote the class of all subdirect products of algebras in $\mathcal{K}$ by $\mathrm{P_S}(\mathcal{K})$.

A.6.3 Varieties

A *variety* is a class of algebras $\mathcal{K}$ which is closed under H, S and P, i.e. such that $\mathrm{H}(\mathcal{K}) \subseteq \mathcal{K}$, $\mathrm{S}(\mathcal{K}) \subseteq \mathcal{K}$ and $\mathrm{P}(\mathcal{K}) \subseteq \mathcal{K}$. It is a fact of universal algebra that if $\mathcal{K}$ is a class of algebras, then $\mathrm{HSP}(\mathcal{K})$ is the smallest variety containing $\mathcal{K}$. We call $\mathrm{HSP}(\mathcal{K})$ the *variety generated by* $\mathcal{K}$.

Congruence-distributive varieties

Recall that given an algebra $\mathbb{A}$, we defined $\mathrm{Con}\,\mathbb{A}$ to be the lattice of congruences of $\mathbb{A}$. If $\mathrm{Con}\,\mathbb{A}$ is a distributive lattice, we say that $\mathbb{A}$ is *congruence-distributive.* If $\mathcal{V}$ is a variety such that every $\mathbb{A}$ in $\mathcal{V}$ is congruence-distributive, we call $\mathcal{V}$ a *congruence-distributive variety.* The following fact is well-known, cf. [23, Theorem 12.3].

A.6.1. FACT. *If* $\mathbb{A}$ *has a lattice reduct, then* $\mathrm{Con}\,\mathbb{A}$ *is distributive.*

We say that a variety $\mathcal{V}$ is *finitely generated* if there exists a finite algebra $\mathbb{A}$ such that $\mathcal{V} = \mathrm{HSP}(\mathbb{A})$. An equivalent condition is to demand that there exists a finite set of finite algebras K such that $\mathcal{V} = \mathrm{HSP}(K)$. The following fact, for which we do not have an explicit reference, is a straightforward consequence of Jónsson's Lemma [23, Corollary 6.10].

A.6.2. FACT. *If* $\mathcal{V}$ *is a congruence-distributive finitely generated variety and if* $\mathcal{V}'$ *is a subvariety of* $\mathcal{V}$*, then* $\mathcal{V}'$ *is also finitely generated.*

Proof sketch Suppose that $\mathcal{V} = \mathrm{HSP}(\mathbb{A})$ for some finite algebra $\mathbb{A}$. Since $\mathcal{V}$ is congruence-distributive, it follows by Jónsson's Lemma [23, Cor. 6.10] that $\mathcal{V}_{SI}$, the subdirect irreducibles of $\mathcal{V}$, are contained in $\mathrm{HS}(\mathbb{A})$. Consequently, every subdirect irreducible of $\mathcal{V}$ is finite. Moreover, since $\mathrm{S}(\mathbb{A})$ is finite, $\mathrm{HS}(\mathbb{A})$ is essentially finite, meaning that there exists a finite set $K \subseteq \mathcal{V}_{SI}$ such that $\mathcal{V}_{SI} = \mathrm{I}(K)$, where $\mathrm{I}(K)$ is the class of all algebras isomorphic to an algebra in K. Now let $\mathcal{V}'$ be a subvariety of $\mathcal{V}$; then $\mathcal{V}'_{SI} \subseteq \mathcal{V}_{SI}$, so there must exist a necessarily finite $K' \subseteq K$ such that $\mathcal{V}'_{SI} = \mathrm{I}(K')$. It follows that $\mathcal{V}' = \mathrm{HSP}(K')$, so that $\mathcal{V}'$ is indeed finitely generated. ∎

Boolean products

We recall the definition of *Boolean products* of algebras from [23, §IV.8]. Let $\mathbb{A}$ be an algebra. A Boolean product decomposition of $\mathbb{A}$ is a subdirect decomposition of $\mathbb{A}$, i.e. a collection of surjective maps $(p_x\colon \mathbb{A} \to \mathbb{B}_x)_{x\in X}$ such that for all $a, b \in \mathbb{A}$, if $p_x(a) = p_x(b)$ for all $x \in X$ then $a = b$, satisfying the additional properties that:

- X is a Boolean space;
- for all $a, b \in \mathbb{A}$, the set $\{x \in X \mid p_x(a) = p_x(b)\}$ is clopen;
- for all $a, b \in \mathbb{A}$ and all $U \subseteq X$ clopen, there is a (unique) $c \in \mathbb{A}$ such that $p_x(c) = p_x(a)$ if $x \in U$ and $p_x(c) = p_x(b)$ otherwise.

A.6.3. Fact. *If* $\mathbb{A}$ *is a finite algebra, then* $\mathrm{HSP}(\mathbb{A}) = \mathrm{HSP}_{\mathrm{B}}(\mathbb{A})$.

Since this fact seems to be a folklore result, we will sketch the proof below.

Proof sketch First we show that $\mathrm{HSP} = \mathrm{HSP}_{\mathrm{B}}\mathrm{P}_{\mathrm{U}}$, where P_{U} stands for taking ultraproducts. It is easy to see that $\mathrm{HSP}_{\mathrm{B}}\mathrm{P}_{\mathrm{U}} \subseteq \mathrm{HSP}$. For the converse, we will show that $\mathrm{P} \subseteq \mathrm{P}_{\mathrm{B}}\mathrm{P}_{\mathrm{U}}$. Let $\prod_I \mathbb{A}_i$ be a product of algebras; we will show that $\prod_I \mathbb{A}_i$ can be seen as the Boolean product of *all* ultraproducts over $\{\mathbb{A}_i \mid i \in I\}$. Define X to be the set of ultrafilters over I; then X is a Boolean space in its Stone topology. For each $U \in X$, let $p_U\colon \prod_I \mathbb{A}_i \to \prod_I \mathbb{A}_i/U$ be the projection onto the ultraproduct $\prod_I \mathbb{A}_i/U$. We claim that

$$\big(p_U\colon \textstyle\prod_I \mathbb{A}_i \to \prod_I \mathbb{A}_i/U\big)_{U\in X}$$

is a Boolean product decomposition. It is not hard to show the following facts:

1. $(p_U\colon \prod_I \mathbb{A}_i \to \prod_I \mathbb{A}_i/U)_X$ is a subdirect product decomposition: if $a, b \in \prod_I \mathbb{A}_i$ and $a \neq b$, then there exists $i \in I$ such that $a(i) \neq b(i)$. If we take the principal ultrafilter $U := \{J \subseteq I \mid i \in J\}$ then we see that $p_U(a) \neq p_U(b)$.
2. For all $a, b \in \prod_I \mathbb{A}_i$, $\{U \in X \mid p_U(a) = p_U(b)\}$ is clopen: it follows from the definition of p_U that $p_U(a) = p_U(b)$ iff $\{i \in I \mid a(i) = b(i)\} \in U$, and $\big\{U \in X \mid \{i \in I \mid a(i) = b(i)\} \in U\big\}$ is clopen by Stone duality.
3. For all $a, b \in \prod_I \mathbb{A}_i$ and all clopen $O \subseteq X$, there exists a $c \in \prod_I \mathbb{A}_i$ such that $p_U(c) = p_U(a)$ if $U \in O$ and $p_U(c) = p_U(b)$ if $U \in X \setminus O$: since $O \subseteq X$ is clopen, by Stone duality there exists an $J \subseteq I$ such that $U \in O$ iff $J \in U$. Now define $c(i) := a(i)$ if $i \in J$ and $c(i) := b(i)$ if $i \in I \setminus J$.

Since $\{\mathbb{A}_i \mid i \in I\}$ was arbitrary, it follows that $\mathrm{P} \subseteq \mathrm{P}_{\mathrm{B}}\mathrm{P}_{\mathrm{U}}$, so that indeed $\mathrm{HSP} = \mathrm{HSP}_{\mathrm{B}}\mathrm{P}_{\mathrm{U}}$. Returning to the statement of the Fact, if $\mathbb{A}$ is finite, then $\mathrm{P}_{\mathrm{U}}(\mathbb{A}) = \mathrm{I}(\mathbb{A})$; it follows that $\mathrm{HSP}(\mathbb{A}) = \mathrm{HSP}_{\mathrm{B}}(\mathbb{A})$. ∎

A.6.4 Terms and equations

Let X be a set of variables. The set $\mathrm{Ter}_\Omega(X)$ of Ω-*terms over* X is defined as the smallest set such that

- $X \subseteq \mathrm{Ter}_\Omega(X)$;
- for all $\omega \in \Omega$, for all $t_1, \ldots, t_{\mathrm{ar}(\omega)} \in \mathrm{Ter}_\Omega(X)$, we have $\omega(t_1, \ldots, t_{\mathrm{ar}(\omega)}) \in \mathrm{Ter}_\Omega(X)$.

If t is a term, we write $t(x_1, \ldots, x_n)$ to indicate that the variables occuring in t are among $\{x_1, \ldots, x_n\}$. If $t(x_1, \ldots, x_n)$ is an Ω-term and $\mathbb{A}$ is an Ω-algebra, then we define the *term function* $t_\mathbb{A} \colon \mathbb{A}^n \to \mathbb{A}$ inductively as

$$\begin{aligned}(x_i)_\mathbb{A} &:= \pi_i \colon \mathbb{A}^n \to \mathbb{A},\\ \left(\omega(t_1(\bar{x}), \ldots, t_k(\bar{x}))\right)_\mathbb{A} &:= \omega_\mathbb{A} \circ \left((t_1)_\mathbb{A} \times \cdots \times (t_k)_\mathbb{A}\right),\end{aligned}$$

where $\bar{x}$ denotes the tuple $x_1, \ldots, x_n$ and $k = \mathrm{ar}(\omega)$. If $s(x_1, \ldots, x_n)$ and $t(x_1, \ldots, x_n)$ are terms we say that the *equation* $s \approx t$ is *valid* on an algebra $\mathbb{A}$ (alternatively, $\mathbb{A}$ satisfies $s \approx t$ or $\mathbb{A} \models s \approx t$) if $s_\mathbb{A} = t_\mathbb{A}$, i.e. if the corresponding term functions coincide. If $\mathbb{A}$ is an ordered algebra, then $\mathbb{A}$ satisfies the *inequation* $s \preccurlyeq t$ if $s_\mathbb{A} \leq t_\mathbb{A}$.

A.7 General topology

Our main reference for general topology is Engelking [33]. Another standard text is Bourbaki [22].

A.7.1 Topological spaces

Let X be a set. A *topology on* X is a collection of subsets $\tau \subseteq \mathcal{P}(X)$, which we call τ-*open sets*, such that

- $\{\emptyset, X\} \subseteq \tau$;
- for all $\mathcal{S} \subseteq \tau$, $\bigcup \mathcal{S} \in \tau$;
- for all $U, V \in \tau$, $U \cap V \in \tau$.

We call $\langle X, \tau \rangle$ a *topological space*. If it is clear what the topology on X is we will not mention τ explicitly. We define $U \subseteq X$ to be *closed* if $X \setminus U$ is open, i.e. if $X \setminus U \in \tau$. If both U and $X \setminus U$ are open, then we say U is *clopen* (closed-and-open).

A.7.1. EXAMPLE. Given any set X, we can always define two somewhat trivial topologies on it. The space $\langle X, \{\emptyset, X\}\rangle$ is literally called the *trivial* topology; this topology has the smallest collection of open sets possible. On the other extreme we have the *discrete* topology $\langle X, \mathcal{P}(X)\rangle$, which is the topology on X with the largest possible collection of open sets.

Subspaces

If $\langle X, \tau\rangle$ is a topological space and $Y \subseteq X$ is a set, we can define a topology on Y, called the *subspace topology*, by defining

$$\tau_Y := \{U \cap Y \mid U \in \tau_X\}.$$

We then call $\langle Y, \tau_Y\rangle$ a *subspace* of $\langle X, \tau_X\rangle$.

Bases and subbases

Let X be a set. A *subbase* for a topology on X is a collection $\mathcal{B} \subseteq \mathcal{P}(X)$ such that

- $\bigcup \mathcal{B} = X$.

We call $\mathcal{B}$ a *base* if addionally,

- $\forall U, V \in \mathcal{B}$, $U \cap V = \bigcup\{W \in \mathcal{B} \mid W \subseteq U, V\}$.

If $\mathcal{B}$ is a subbase on X, then $\{\bigcap \mathcal{S} \mid \mathcal{S} \subseteq \mathcal{B} \text{ finite}\}$ is a base on X.

A.7.2. EXAMPLE. As a trivial example, observe that if $\langle X, \tau\rangle$ is a topological space, then τ is a base on X.

Let $\mathcal{B}$ be a base on X and let $\mathcal{B}'$ be a subbase on X. We define

$$\begin{aligned} \langle \mathcal{B}\rangle &= \{\textstyle\bigcup \mathcal{B}_0 \mid \mathcal{B}_0 \subseteq \mathcal{B}\}, \text{ and} \\ \langle \mathcal{B}'\rangle &= \big\{\textstyle\bigcup \mathcal{B}_0 \mid \{\mathcal{B}_0 \subseteq \{\bigcap \mathcal{S} \mid \mathcal{S} \subseteq \mathcal{B} \text{ finite}\}\big\}. \end{aligned}$$

It is a fact of general topology that $\langle \mathcal{B}\rangle$ and $\langle \mathcal{B}'\rangle$ as defined above are topologies on X. Given a topological space $\langle X, \tau\rangle$, we say $\mathcal{B}$ *is a (sub-) base for* $\langle X, \tau\rangle$ if $\tau = \langle \mathcal{B}\rangle$. If $\mathcal{B}$ is a base for $\langle X, \tau\rangle$, then it follows from the definition of $\langle \mathcal{B}\rangle$ that for all open sets $U \in \tau$ and for all $x \in U$, there exists a $V \in \mathcal{B}$ such that $x \in V \subseteq U$.

Let X be a set. One can show that the partial order

$$\big\langle \{\tau \subseteq \mathcal{P}(X) \mid \langle X, \tau\rangle \text{ is a topological space}\}, \subseteq \big\rangle$$

is a complete lattice. If τ_0 and τ_1 are topologies on X with bases $\mathcal{B}_0$ and $\mathcal{B}_1$, respectively, then

$$\{U \cap V \mid U \in \mathcal{B}_0, V \in \mathcal{B}_1\}$$

is a base for $\tau_0 \vee \tau_1$.

A.7.2 Continuity

Let $\langle X, \tau_X \rangle$ and $\langle Y, \tau_Y \rangle$ be topological spaces. A function $f\colon X \to Y$ is called (τ_X, τ_Y)-*continuous* if for all $U \in \tau_Y$, $f^{-1}(U) \in \tau_X$. If it is clear from the context what the topologies are, we simply say that f is continuous. We say that $f\colon X \to Y$ is (τ_X, τ_Y)-*open* if for all $U \in \tau_X$, $f[U] \in \tau_Y$. We say that $\langle X, \tau_X \rangle$ and $\langle Y, \tau_Y \rangle$ are *homeomorphic* if there exists an open and continuous bijective map $f\colon X \to Y$; this is the natural notion of isomorphism for topological spaces.

Let $f\colon Y \to X$ be a continuous function between spaces $\langle X, \tau_X \rangle$ and $\langle Y, \tau_Y \rangle$. If f is injective and if for all $U \in \tau_Y$, there exists a $U' \in \tau_X$ such that $U = f^{-1}(U')$, then we call $f\colon X \to Y$ a *homeomorphic embedding*. Under these circumstances, $\langle Y, \tau_Y \rangle$ is homeomorphic to $f[Y]$, where the latter set is equipped with the subspace topology inherited from $\langle X, \tau_X \rangle$.

One of the reasons it is practical to work with bases and subbases for topologies is that they make it easier to check whether a function $f\colon X \to Y$ between topological spaces $\langle X, \tau_X \rangle$ and $\langle Y, \tau_Y \rangle$ is continuous: If $\mathcal{B}$ is a subbase for τ_Y, then one can show that $f\colon X \to Y$ is continuous iff for all $U \in \mathcal{B}$, $f^{-1}(U) \in \tau_X$.

A.7.3. Lemma. *Let X, Y be sets, let σ_0, σ_1 be topologies on X, let τ_0, τ_1 be topologies on Y and let $f\colon X \to Y$ be a function.*

1. *If $f\colon X \to Y$ is (σ_0, τ_0)-continuous and if $\sigma_0 \subseteq \sigma_1$ and $\tau_0 \supseteq \tau_1$, then f is also (σ_1, τ_1)-continuous.*

2. *If $f\colon X \to Y$ is both (σ_0, τ_0)-continuous and (σ_1, τ_1)-continuous, then f is also $(\sigma_0 \vee \sigma_1, \tau_0 \vee \tau_1)$-continuous.*

Proof This is an easy exercise. ∎

A.7.3 Separation and compactness

In this subsection we list several properties of topological spaces. A topological space $\langle X, \tau \rangle$ satisfies the T_0 *separation axiom* if for all $x, y \in X$ such that $x \neq y$, there exists an open set $U \subseteq X$ such that either $x \in U \not\ni y$ or $x \notin U \ni y$. A topological space satisfies the T_1 *separation axiom* if for all $x, y \in X$ such that $x \neq y$, there exist open sets $U, V \subseteq X$ such that $x \in U \not\ni y$ and $x \notin V \ni y$. Finally, a topological space satisfies the T_2 *separation axiom*, also known as the *Hausdorff separation axiom*, if for all $x, y \in X$ such that $x \neq y$, there exist *disjoint* open sets $U, V \subseteq X$ such that $x \in U$ and $y \in V$. We will usually simply say that a space *is* T_0, Hausdorff, etc.

Let $\langle X, \tau \rangle$ be a topological space and let $Y \subseteq X$. We say Y is *compact* if for all $\mathcal{S} \subseteq \tau$ such that $Y \subseteq \bigcup \mathcal{S}$, there exists a finite $\mathcal{S}_0 \subseteq \mathcal{S}$ such that $Y \subseteq \bigcup \mathcal{S}_0$. If X itself is a compact set we call $\langle X, \tau \rangle$ a *compact space*.

Boolean spaces

A topological space $\langle X, \tau\rangle$ is *zero-dimensional* if $\langle X, \tau\rangle$ has a base of clopen sets. Equivalently, $\langle X, \tau\rangle$ is zero-dimensional if for all open sets $U \subseteq X$ and all $x \in U$, there exists a clopen $V \subseteq X$ such that $x \in V \subseteq U$. We call spaces that are both compact, Hausdorff and zero-dimensional *Boolean spaces*; these spaces are also called Stone spaces [54].

A.8 Duality for ordered Kripke frames

In §4.1.4, we discuss the discrete duality between $\mathbf{DLO}^+$, the category of semi-topological distributive lattices with operators and complete homomorphisms, and **OKFr**, the category of ordered Kripke frames and bounded morphisms. However, in §4.1.4 we only describe the functor $(\cdot)^+ \colon \mathbf{OKFr}^{op} \to \mathbf{DLO}^+$. Below we will describe the functor $(\cdot)_+ \colon \mathbf{DLO}^+ \to \mathbf{OKFr}^{op}$. As a general reference about DLO's we mention Goldblatt [47]; the details of the discrete duality we use are discussed by Gehrke et al. in [40, §2.3].

Before we can describe the functor $(\cdot)_+ \colon \mathbf{DLO}^+ \to \mathbf{OKFr}^{op}$, we need a few definitions. Let $\mathbb{A}$ be a semi-topological DLO. By $\mathrm{J}^\infty(\mathbb{A})$ we denote the *completely join-irreducible* elements of $\mathbb{A}$, i.e. those $p \in \mathbb{A}$ such that for all $S \subseteq \mathbb{A}$, if $p = \bigvee S$ then there exists an $x \in S$ such that $p = x$. Using order duality we can define the set of completely meet irreducible elements of $\mathbb{A}$, denoted $\mathrm{M}^\infty(\mathbb{A})$. In a semi-topological DLO, we can define an order-isomorphism $\kappa\colon \mathrm{J}^\infty(\mathbb{A}) \to \mathrm{M}^\infty(\mathbb{A})$, by sending $p \mapsto \bigvee(\mathbb{A} \setminus {\uparrow} p)$.

Recall from §4.1.1 that a semi-topological DLO is a complete bi-algebraic distributive lattice-based algebra $\mathbb{A} = \langle A, \wedge_{\mathbb{A}}, \vee_{\mathbb{A}}, 0_{\mathbb{A}}, 1_{\mathbb{A}}, \Diamond_{\mathbb{A}}, \Box_{\mathbb{A}}\rangle$, where $\Diamond_{\mathbb{A}} \colon \mathbb{A}^n \to \mathbb{A}$ is a complete normal n-ary operator and $\Box_{\mathbb{A}} \colon \mathbb{A}^m \to \mathbb{A}$ is a complete normal m-ary dual operator. We now define

$$\mathbb{A}_+ := \langle \mathrm{J}^\infty(\mathbb{A}), \leq, R_{\mathbb{A}_+}, Q_{\mathbb{A}_+}\rangle,$$

where

- $\langle \mathrm{J}^\infty(\mathbb{A}), \leq\rangle$ is the set of completely join-irreducibles of $\mathbb{A}$ with the order inherited from $\mathbb{A}$;
- $R_{\mathbb{A}_+}$ is an $n+1$-ary relation on $\mathrm{J}^\infty(\mathbb{A})$ such that
$$p\ R_{\mathbb{A}_+}\ (q_1, \ldots, q_n) \text{ iff } p \leq \Diamond_{\mathbb{A}}(q_1, \ldots, q_n);$$
- $Q_{\mathbb{A}_+}$ is an $m+1$-ary relation on $\mathrm{J}^\infty(\mathbb{A})$ such that
$$p\ Q_{\mathbb{A}_+}\ (q_1, \ldots, q_m) \text{ iff } \Box_{\mathbb{A}}(\kappa(q_1), \ldots, \kappa(q_m)) \leq \kappa(p).$$

If $f\colon \mathbb{A} \to \mathbb{B}$ is a morphism of $\mathbf{DLO}^+$, i.e. a complete DLO homomorphism, then we define $f_+\colon \mathbb{B}_+ \to \mathbb{A}_+$ as

$$f_+\colon p \mapsto \bigwedge\{a \in \mathbb{A} \mid p \leq f(a)\},$$

which is indeed a bounded morphism. One can show that f_+ is in fact the left adjoint of $f\colon \mathbb{A} \to \mathbb{B}$, restricted to $\mathrm{J}^\infty(\mathbb{B})$.

A.8.1. FACT. *The functors* $(\cdot)_+\colon \mathbf{DLO}^+ \to \mathbf{OKFr}^{op}$ *and* $(\cdot)^+\colon \mathbf{OKFr}^{op} \to \mathbf{DLO}^+$ *form a dual equivalence of categories. Each of the functors* $(\cdot)_+$ *and* $(\cdot)^+$ *map finite objects to finite objects. Moreover given a complete DLO homomorphism* $f\colon \mathbb{A} \to \mathbb{B}$ *between semi-topological DLO's,*

1. $f\colon \mathbb{A} \to \mathbb{B}$ *is surjective iff* $f_+\colon \mathbb{B}_+ \to \mathbb{A}_+$ *is an embedding of ordered Kripke frames;*
2. $f\colon \mathbb{A} \to \mathbb{B}$ *is injective iff* $f_+\colon \mathbb{B}_+ \to \mathbb{A}_+$ *is surjective.*

The isomorphisms witnessing the above Fact are obtained as follows: for a semi-topological DLO $\mathbb{A}$, we map $\mathbb{A}$ to $(\mathbb{A}_+)^+$ by sending

$$a \mapsto \{p \in \mathrm{J}^\infty(\mathbb{A}) \mid p \leq a\},$$

which is a lower set of $\mathbb{A}_+$. Conversely, given an ordered Kripke frame $\mathfrak{F}$, we map $\mathfrak{F}$ to $(\mathfrak{F}^+)_+$ by sending

$$x \mapsto {\uparrow} x,$$

which is a completely join-irreducible element of $\mathfrak{F}^+$.

Bibliography

[1] S. Abramsky and A. Jung. Domain theory. In *Handbook of logic in computer science*, volume 3, pages 1–168. Clarendon Press, Oxford, 1994.

[2] P. Aczel. *Non-well-founded sets*, volume 14 of *CSLI Lecture Notes*. Stanford University Center for the Study of Language and Information, Stanford, CA, 1988.

[3] J. Adámek, H. Herrlich, and G. E. Strecker. *Abstract and concrete categories*. Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York, 1990.

[4] J. Adámek and V. Trnková. *Automata and Algebras in Categories*. Kluwer Academic Publishers, 1990.

[5] L. W. Anderson. One dimensional topological lattices. *Proc. Amer. Math. Soc.*, 10:715–720, 1959.

[6] C. Areces and R. I. Goldblatt, editors. *Advances in Modal Logic 7, Nancy 2008*. College Publications, 2008.

[7] R. Balbes and P. Dwinger. *Distributive lattices*. University of Missouri Press, Columbia, Mo., 1974.

[8] B. Banaschewski. On profinite universal algebras. In *General topology and its relations to modern analysis and algebra, III (Proc. Third Prague Topological Sympos., 1971)*, pages 51–62. Academia, Prague, 1972.

[9] B. Banaschewski and G. Bruns. Categorical characterization of the MacNeille completion. *Arch. Math. (Basel)*, 18:369–377, 1967.

[10] J. Barwise and L. Moss. *Vicious circles*, volume 60 of *CSLI Lecture Notes*. CSLI Publications, Stanford, CA, 1996.

[11] F. Bellissima. On the lattice of extensions of the modal logics KAlt_n. *Arch. Math. Logic*, 27(2):107–114, 1988.

[12] G. W. Bergman. Problem list from Algebras, lattices and varieties: a conference in honor of Walter Taylor, University of Colorado, 15–18 August, 2004. *Algebra Universalis*, 55:509–526, 2006.

[13] G. Bezhanishvili. Varieties of monadic Heyting algebras. II. Duality theory. *Studia Logica*, 62(1):21–48, 1999.

[14] G. Bezhanishvili and N. Bezhanishvili. Profinite Heyting algebras. *Order*, 25(3):211–227, 2008.

[15] G. Bezhanishvili, M. Gehrke, R. Mines, and P. J. Morandi. Profinite completions and canonical extensions of Heyting algebras. *Order*, 23(2-3):143–161, 2006.

[16] G. Bezhanishvili and J. Vosmaer. Comparison of MacNeille, canonical, and profinite completions. *Order*, 25(4):299–320, 2008.

[17] M. Bílková, A. Palmigiano, and Y. Venema. Proof systems for the coalgebraic cover modality. In Areces and Goldblatt [6], pages 1–21.

[18] G. Birkhoff. *Lattice theory.* Third edition. American Mathematical Society Colloquium Publications, Vol. XXV. American Mathematical Society, Providence, R.I., 1967.

[19] P. Blackburn, M. de Rijke, and Y. Venema. *Modal logic*, volume 53 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, Cambridge, 2001.

[20] W. J. Blok and D. Pigozzi. Algebraizable logics. *Mem. Amer. Math. Soc.*, 77(396):vi+78, 1989.

[21] G. Boole. *The mathematical analysis of logic, being an essay towards a calculus of deductive reasoning.* Macmillan Barclay & Macmillan, 1847.

[22] N. Bourbaki. *Elements of mathematics. General topology. Part 1.* Hermann, Paris, 1966.

[23] S. N. Burris and H. P. Sankappanavar. A Course in Universal Algebra: the Millenium Edition. Available online at `http://www.math.uwaterloo.ca/~snburris/`.

[24] D. M. Clark, B. A. Davey, R. S. Freese, and M. Jackson. Standard topological algebras: syntactic and principal congruences and profiniteness. *Algebra Universalis*, 52(2-3):343–376, 2004.

[25] T. Coquand, G. Sambin, J. Smith, and S. Valentini. Inductively generated formal topologies. *Ann. Pure Appl. Logic*, 124(1-3):71–106, 2003.

[26] B. A. Davey, M .J. Gouveia, M. Haviar, and H. A. Priestley. Natural extensions and profinite completions of algebras. Algebra Universalis (in press).

[27] B. A. Davey, M. Haviar, and H. A. Priestley. Boolean topological distributive lattices and canonical extensions. *Appl. Categ. Structures*, 15(3):225–241, 2007.

[28] B. A. Davey, M. Jackson, M. Maróti, and R. N. McKenzie. Principal and syntactic congruences in congruence-distributive and congruence-permutable varieties. *J. Aust. Math. Soc.*, 85(1):59–74, 2008.

[29] B. A. Davey and H. A. Priestley. *Introduction to lattices and order*. Cambridge University Press, New York, second edition, 2002.

[30] D. H. J. de Jongh and A. S. Troelstra. On the connection of partially ordered sets with some pseudo-Boolean algebras. *Nederl. Akad. Wetensch. Proc. Ser. A 69=Indag. Math.*, 28:317–329, 1966.

[31] M. de Rijke and Y. Venema. Sahlqvist's theorem for Boolean algebras with operators with an application to cylindric algebras. *Studia Logica*, 54(1):61–78, 1995.

[32] J. M. Dunn, M. Gehrke, and A. Palmigiano. Canonical extensions and relational completeness of some substructural logics. *J. Symbolic Logic*, 70:713–740, 2005.

[33] R. Engelking. *General topology*, volume 6 of *Sigma Series in Pure Mathematics*. Heldermann Verlag, Berlin, second edition, 1989. Translated from the Polish by the author.

[34] M. Gehkre and J. Harding. Bounded lattice expansions. *J. Algebra*, 238:345–371, 2001.

[35] M. Gehrke, S. Grigorieff, and J.-É. Pin. Recognition and Stone duality, 2010. In preparation.

[36] M. Gehrke, J. Harding, and Y. Venema. MacNeille completions and canonical extensions. *Trans. Amer. Math. Soc.*, 358(2):573–590 (electronic), 2006.

[37] M. Gehrke, R. Jansana, and A. Palmigiano. Canonical extensions for congruential logics with the deduction theorem, 2010. To appear in Ann. Pure Appl. Logic.

[38] M. Gehrke and B. Jónsson. Bounded distributive lattices with operators. *Mathematica Japonica*, 2:207–215, 1994.

[39] M. Gehrke and B. Jónsson. Bounded distributive lattice expansions. *Mathematica Scandinavica*, 94:13–45, 2004.

[40] M. Gehrke, H. Nagahashi, and Y. Venema. A Sahlqvist theorem for distributive modal logic. *Ann. Pure Appl. Logic*, 131(1-3):65–102, 2005.

[41] M. Gehrke and H. A. Priestley. Canonical extensions and completions of posets and lattices. *Reports on Mathematical Logic*, 43:133–152, 2008.

[42] M. Gehrke and J. Vosmaer. Canonical extension and canonicity via dcpo presentations, 2009. Submitted to Theoretical Computer Science.

[43] M. Gehrke and J. Vosmaer. A view of canonical extension, 2010. Proceedings of the Eighth International Tbilisi Symposium on Language, Logic and Computation.

[44] S. Ghilardi and G. Meloni. Constructive canonicity in non-classical logics. *Ann. Pure Appl. Logic*, 86:1–32, 1997.

[45] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, and D.S. Scott. *A Compendium of Continuous Lattices*. Springer-Verlag, 1980.

[46] R. I. Goldblatt. Metamathematics of modal logic. *Rep. Math. Logic*, 6:41–77, 1976.

[47] R. I. Goldblatt. Varieties of complex algebras. *Ann. Pure Appl. Logic*, 44(3):173–242, 1989.

[48] R. I. Goldblatt. Mathematical modal logic: a view of its evolution. *J. Appl. Log.*, 1(5-6):309–392, 2003.

[49] M. J. Gouveia. A note on profinite completions and canonical extensions. Algebra Universalis (in press).

[50] J. Harding. On profinite completions and canonical extensions. *Algebra Universalis*, 55:293–296, 2006.

[51] J. Harding and G. Bezhanishvili. MacNeille completions of Heyting algebras. *Houston J. Math.*, 30(4):937–952 (electronic), 2004.

[52] C. Hermida and B. Jacobs. Structural induction and coinduction in a fibrational setting. *Information and Computation*, 145(2):107–152, 1998.

[53] D. Janin and I. Walukiewicz. Automata for the modal μ-calculus and related results. In *Mathematical foundations of computer science 1995 (Prague)*, volume 969 of *Lecture Notes in Comput. Sci.*, pages 552–562. Springer, Berlin, 1995.

[54] P. T. Johnstone. *Stone spaces*, volume 3 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1982.

[55] P.T. Johnstone. Vietoris locales and localic semi-lattices. In R.-E. Hoffmann, editor, *Continuous Lattices and their Applications*, number 101 in Pure and Applied Mathematics, pages 155–18. Marcel Dekker, 1985.

[56] P.T. Johnstone and S. J. Vickers. Preframe presentations present. In A. Carboni, M.C. Pedicchio, and G. Rosolini, editors, *Category Theory – Proceedings, Como 1990*, number 1488 in Lecture Notes in Mathematics, pages 193–212. Springer-Verlag, 1991.

[57] B. Jónsson. On the canonicity of Sahlqvist identities. *Studia Logica*, 53(4):473–491, 1994.

[58] B. Jónsson and A. Tarski. Boolean algebras with operators I. *Amer. J. Math.*, 73:891–939, 1951.

[59] B. Jónsson and A. Tarski. Boolean algebras with operators II. *Amer. J. Math.*, 74:127–162, 1952.

[60] A. Jung, M. A. Moshier, and S. J. Vickers. Presenting dcpos and dcpo algebras. In *Proceedings of the 24th Conference on the Mathematical Foundations of Programming Semantics (MFPS XXIV)*, volume 218 of *Electronic Notes in Theoretical Computer Science*, pages 209–229, 2008.

[61] K. A. Kearnes and R. Willard. Residually finite, congruence meet-semidistributive varieties of finite type have a finite residual bound. *Proc. Amer. Math. Soc.*, 127(10):2841–2850, 1999.

[62] C. Kissig and Y. Venema. Complementation of coalgebra automata. In A. Kurz, M. Lenisa, and A. Tarlecki, editors, *Algebra and Coalgebra in Computer Science (CALCO 2009)*, volume 5728 of *LNCS*, pages 81–96. Springer, 2009.

[63] S. Koppelberg. *Handbook of Boolean algebras. Vol. 1*. North-Holland Publishing Co., Amsterdam, 1989. Edited by J. Donald Monk and Robert Bonnet.

[64] M. Kracht. *Tools and techniques in modal logic*, volume 142 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1999.

[65] C. Kupke. *Finitary Coalgebraic Logics*. PhD thesis, Institute for Logic, Language and Computation, Universiteit van Amsterdam, 2006.

[66] C. Kupke, A. Kurz, and Y. Venema. Completeness of the finitary Moss logic. In Areces and Goldblatt [6], pages 193–217.

[67] C. Kupke, K. Kurz, and Y. Venema. Completeness for ∇, 2010.

[68] J. D. Lawson. Intrinsic topologies in topological lattices and semilattices. *Pacific J. Math.*, 44:593–602, 1973.

[69] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[70] H. M. MacNeille. Partially ordered sets. *Trans. Amer. Math. Soc.*, 42(3):416–460, 1937.

[71] L. Moss. Coalgebraic logic. *Annals of Pure and Applied Logic*, 96:277–317, 1999. (Erratum published *Ann.P.Appl.Log.* 99:241–259, 1999).

[72] K. Numakura. Theorems on compact totally disconnected semigroups and lattices. *Proc. Amer. Math. Soc.*, 8:623–626, 1957.

[73] A. Palmigiano and Y. Venema. Nabla algebras and Chu spaces. In T. Mossakowski, U. Montanari, and M. Haveraaen, editors, *Algebra and Coalgebra in Computer Science. Second International Conference, CALCO 2007, Bergen, Norway, August 20-24, 2007*, volume 4624 of *LNCS*, pages 394–408. Springer Verlag, 2007.

[74] D. Papert Strauss. Topological lattices. *Proc. London Math. Soc. (3)*, 18:217–230, 1968.

[75] G. Plotkin. Pisa notes (on domain theory). Available online at `http://homepages.inf.ed.ac.uk/gdp/`, 1983.

[76] H. A. Priestley. Catalytic distributive lattices and compact zero-dimensional topological lattices. *Algebra Universalis*, 19(3):322–329, 1984.

[77] L. Ribes and P. Zalesskii. *Profinite groups*, volume 40 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2000.

[78] J. J. M. M. Rutten. Universal coalgebra: a theory of systems. *Theoret. Comput. Sci.*, 249(1):3–80, 2000. Modern algebra and its applications (Nashville, TN, 1996).

[79] H. Sahlqvist. Completeness and correspondence in the first and second order semantics for modal logic. In *Proceedings of the Third Scandinavian Logic Symposium (Univ. Uppsala, Uppsala, 1973)*, pages 110–143. Stud. Logic Found. Math., Vol. 82, Amsterdam, 1975. North-Holland.

[80] D.S. Scott. Outline of a mathematical theory of computation. In *4th Annual Princeton Conference on Information Sciences and Systems*, pages 169–176, 1970.

[81] M. H. Stone. The theory of representations for Boolean algebras. *Trans. Amer. Math. Soc.*, 40(1):37–111, 1936.

[82] R. Street. The formal theory of monads. *Journal of Pure and Applied Algebra*, 2(2):149–168, 1972.

[83] A. Tarski. Zur Grundlegung der Boole'schen Algebra I. *Fund. Math.*, 24:177–198, 1935.

[84] W. Taylor. Some constructions of compact algebras. *Ann. Math. Logic*, 3:395–435, 1971.

[85] W. Taylor. Residually small varieties. *Algebra Universalis*, 2:33–53, 1972.

[86] M. Theunissen and Y. Venema. MacNeille completions of lattice expansions. *Algebra Universalis*, 57:143–193, 2007.

[87] S. K. Thomason. Semantic analysis of tense logics. *J. Symbolic Logic*, 37:150–158, 1972.

[88] V. Trnková. Relational automata in a category and theory of languages. In M. Karpinski, editor, *Fundamentals of computation theory: proceedings of the 1977 international FCT-conference*, volume 56 of *Lecture Notes in Computer Science*, pages 340–355. Springer, 1977.

[89] Y. Venema. Algebras and coalgebras. In *Handbook of Modal Logic*, pages 331–426. Elsevier, 2006.

[90] Y. Venema. Automata and fixed point logic: a coalgebraic perspective. *Information and Computation*, 204:637–678, 2006.

[91] S. J. Vickers. *Topology via logic*, volume 5 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, Cambridge, 1989.

[92] S. J. Vickers. Constructive points of powerlocales. *Mathematical Proceedings of the Cambridge Philosophical Society*, 122:207–222, 1997.

[93] S. J. Vickers. Compactness in locales and in formal topology. *Ann. Pure Appl. Logic*, 137(1-3):413–438, 2006.

[94] S. J. Vickers and C. F. Townsend. A universal characterization of the double powerlocale. *Theoret. Comput. Sci.*, 316(1-3):297–321, 2004.

[95] L. Vietoris. Bereiche zweiter Ordnung. *Monatsh. Math. Phys.*, 32(1):258–280, 1922.

[96] J. Vosmaer. Connecting the profinite completion and the canonical extension using duality. Master's thesis, University of Amsterdam, November 2006.

[97] J. Vosmaer. MacNeille completion and profinite completion can coincide on finitely generated modal algebras. *Algebra Universalis*, 61(3-4):449–453, 2009.

Index

adjunction
  of categories, 202
  of posets, 204
algebra, 210
  Boolean product, 213
  congruence, 211
  homomorphic image, 211
  product, 211
  quotient, 211
  subalgebra, 211
  subdirect product, 211, 212
alt_n, 136

base for a topology, 215
Base of a functor, 147, 150
binary tree functor, 147
Boolean algebra, 129, 207
Boolean algebra with operators (BAO), 133
  compact Hausdorff, 135
  profinite, 136
  semi-topological, 133
Boolean topological
  algebra, 60
  BAO, 135
  Boolean algebra, 129
  DLO, 115
  lattice, 65
  lattice-based algebra, 105
  space, *see* topology, Boolean
bounded morphism, 116

canonical extension type, 86
canonical extension of a lattice
  definition, 13
  fixed points, 14
  functor, 43
  non-continuous, 97
  products, 24
  topological characterization, 21
  via dcpo presentations, 48
  when profinite, 77
canonical extension of a lattice-based algebra
  definition, 86
  finite, 87
  functor, 87
  universal property, 94, 105
  when profinite, 99
canonical extension of a map
  algebra homomorphism, 87
  composition, 42, 78
  continuity, 31, 40, 74
  definition
    arbitrary map, 73
    order-preserving map, 29
  maximality, 31, 77
  order embedding, 32
  via dcpo presentations, 50
  via $\liminf$, 69

canonicity, 89
category, 200
  dual, 201
cocone, 117
  map of cocones, 117
colimit, 117
compact Hausdorff
  algebra, 55
  BAO, 135
  Boolean algebra, 129
  lattice, 63
compactification, 55, 136
completely join-irreducible, 26, 217
completion, 13, 207
complex algebra, 123
composition of relations vs functions, 144
cone, 57
  map of cones, 57
congruence, 211
congruence lattice, 211

dcpo, 204
  algebra, 90
  algebraic, 209
  presentation, 47
diagram, 57
  associated, 120
  directed, 117
directed set, 204
distributive lattice, 41, 128, 206
distributive lattice with operators (DLO), 113
  Boolean topological, 115
  profinite, 115, 121
  semi-topological, 114
duality
  BAO's, 133, 135
  Boolean algebras, 129
  distributive lattices, 128
  DLO's, 124, 217
  Heyting algebras, 130
  profinite DLO's, 120
  topological BAO's, 132

equation, 214
equivalence of categories, 202

filter, 208
  completion, 208
filter elements, 14
finitely generated variety, 77, 87, 89, 99, 212
flat site, 153
frame (lattice), 152
  compact, 155
  presentation, 152
functor, 201
  finitary, 146
  standard, 146
  weak pullback preserving, 145
    examples, 146

generated subframe, *see* Kripke frame

Heyting algebra, 130, 207
homeomorphic embedding, 216
homomorphism, 210

ideal, 208
  completion, 208
ideal elements, 14
insertion of generators, 153

Kripke frame
  definition, 116, 133
  duality, 217
  generated subframe, 119
  hereditarily finite, 121
  image-finite, 133
  intuitionistic, 130

lattice, 206
lattice reduct, 87
lattice-based algebra, 84
lifted members of a set, 151
limit, 58
limiting cone, 58

lower set, 203

MacNeille completion, 128, 209

natural transformation, 201
normal operator, 37

operator, 37
order signature, 85
order type, 85
order-preserving map, 203
ordered Kripke frame, *see* Kripke frame

partially ordered set, 203
poset, *see* partially ordered set
power set monad, 145
power set functor, 199
pre-order, 202
prime filter frame, 123
profinite
  BAO, 136
  DLO, 115, 121
profinite algebra, 58
profinite completion, 62, 94, 126
  when equal to canonical extension, 99
pullback, 145

relation lifting, 147
  examples, 148

Scott-continuity, 204
semilattice, 205
slim redistribution, 158
smooth
  lattice-based algebra, 86
  map, 39, 78
span, 148
Stone space, *see* topology, Boolean
subbase, 215
subcategory, 200
  co-reflective, 125, 202
  reflective, 202
substructure, 119
suplattice, 153, 206
  presentation, 153

T-powerlocale
  compactness, 191
  definition, 159
  functor, 171
  natural transformations, 173
  regularity, 185
  via flat sites, 179
  zero-dimensionality, 185
topological algebra, 54
topology, 214
  and order, 205
  Boolean, 217
  compactness, 216
  continuous function, 216
  δ-, 16, 21
  interval, 15
  Lawson, 65, 67
  Scott, 15, 205
  separation axioms, 216
  subspace, 215
  zero-dimensional, 217

ultrafilter extension
  embedding into, 138
ultrafilter frame, *see* prime filter frame, 137
universal algebra, 210
universal property, 94, 105
upper set, 203

variety, 212
  congruence-distributive, 212
  finitely generated, *see* finitely generated variety
Vietoris
  hyperspace, 141
  powerlocale, 154, 165

weak pullback, 145

Samenvatting

In dit proefschrift bespreken we drie onderwerpen: canonieke extensies van tralie-algebra's, Stone-dualiteit voor distributive tralies met operatoren, en een generalisatie van de puntvrije Vietoris powerlocale-constructie.

In Hoofdstukken 2 en 3 onderzoeken verbanden tussen canonieke extensies van tralie-algebra's en *topologische algebra*, *pro-eindige completeringen* en *gericht-volledige partiële ordeningen (dcpo's)*. We geven een topologische karakterisering van de canonieke extensie van een tralie in §2.1.3, en een verbeterde karakterisering van canonieke extensies van orde-bewarende afbeeldingen als maximale continue extensies, alsmede andere continuïteitsresultaten, in §2.2. De verbetering in de resultaten in §2.2 schuilt in het feit dat we niet uitgaan van distributieve tralies, maar van willekeurige tralies. In §2.3 ronden we Hoofdstuk 2 af met een andere karakterisering van canonieke extensies, namelijk door middel van dcpo-presentaties. In Hoofdstuk 3 bespreken we canonieke extensies van willekeurige afbeeldingen en canonieke extensies van algebras, beide in relatie tot topologische algebra. In §3.3.2 laten we zien dat canonieke extensies van surjectieve algebrahomomorfismes tussen tralie-algebra's wederom homomorfismes zijn. We gebruiken dit gegeven in §3.4.1 om te laten zien dat de pro-eindige completering van een willekeurige tralie-algebra $\mathbb{A}$ gekarakteriseerd kan worden als een volledig quotiënt van de canonieke extensie van $\mathbb{A}$. Vervolgens onderzoeken we in §3.4.2 wanneer de pro-eindige completering van een tralie-algebra $\mathbb{A}$ samenvalt met de canonieke extensie van $\mathbb{A}$. We sluiten Hoofdstuk 3 af met enige resultaten betreffende een universele eigenschap van canonieke extensies met betrekking tot Boolese topologische algebra's in §3.4.3.

In Hoofdstuk 4 verdiepen wij ons in de discrete Stone-dualiteit voor semi-topologische distributieve tralies met operatoren (DLO's) en geordende Kripke-frames. In §4.1 behandelen we de dualiteit tussen *pro-eindige DLO's* en de daarmee corresponderende *erfelijk eindige geordende Kripke-frames*. We bespreken enkele speciale gevallen van deze dualiteit in §4.2, te weten discrete dualiteit voor distributieve tralies, Boolese algebra's, Heytingalgebra's en modale algebra's. Wij sluiten dit hoofdstuk af met een nadere blik op de voornoemde dualiteit in het geval van Boolese algebra's met operatoren (BAO's) in §4.3. In dit geval kunnen we niet alleen de pro-eindige BAO's karakteriseren middels Stone-dualiteit, maar ook de *compacte Hausdorff-* en *Boolese topologische BAO's*. We laten zien dat compacte Hausdorff-BAO's (en Boolese topologische BAO's) de dualen zijn van *eindig vertakkende Kripke-frames*. We passen deze kennis toe in een beschouwing van de inbedding van Kripke-frames in hun ultrafilterextensies in §4.3.2.

In Hoofdstuk 5 gebruiken we een *geometrische* variant van de *Carioca-axioma's voor coalgebraïsche modale logica* om een nieuwe beschrijving te geven van de *puntvrije Vietorisconstructie*. In §5.3.1 introduceren we de T-powerlocale-constructie, gegeven een endofunctor $T\colon \mathbf{Set} \to \mathbf{Set}$ op de categorie van verzamelingen. We laten vervolgens in §5.3.3 zien dat het P-powerlocale, waar P de covariante machtsverzamelingfunctor is, de oorspronkelijke Vietoris powerlocale-constructie geeft. In §5.3.4 laten we zien dat de T-powerlocale-constructie onderdeel is van een functor V_T op de categorie van frames. Daarnaast laten we zien hoe een natuurlijke transformatie van een functor T' naar T een natuurlijke transformatie van V_T naar $V_{T'}$ oplevert. In §5.3.5 laten we zien dat het T-powerlocale gepresenteerd kan worden door middel van flat sites; dit leidt tot een algebraïsch bewijs van het feit dat formules in onze geometrische coalgebraïsche modale logica disjunctieve normaalvormen hebben. Ten slotte laten we in §5.4 zien dat de T-powerlocale-constructie frame-eigenschappen als regulariteit, nul-dimensionaliteit en de combinatie van nul-dimensionaliteit en compactheid bewaart.

Abstract

In this dissertation we discuss three subjects: canonical extensions of lattice-based algebras, Stone duality for distrbutive lattices with operators, and a generalization of the point-free Vietoris powerlocale construction.

In Chapters 2 and 3, we study canonical extensions of lattice-based algebras in relation to *topological algebra*, *profinite completions* and *directed complete partial orders (dcpo's)*. We provide a topological characterization theorem for the canonical extension of a lattice in §2.1.3, and we give an improved characterization of canonical extensions of order-preserving maps as maximal continuous extensions, along with further continuitresults, in §2.2. The improvement in the results of §2.2 lies in the fact that they hold for arbitrary rather than distributive lattices. In §2.3, we show how canonical extensions of lattices can be characterized using dcpo presentations, concluding Chapter 2. In Chapter 3 we discuss canonical extensions of arbitrary maps and canonical extensions of lattice-based algebras, both in relation to topological algebra. In §3.3.2, we show that the canonical extenion of a surjective lattice-based algebra homomorphism is again an algebra homomorphism. We use this fact to show in §3.4.1 that the profinite completion of any lattice-based algebra $\mathbb{A}$ can be characterized as a complete quotient of the canonical extension of $\mathbb{A}$. Subsequently, in §3.4.2, we investigate necessary and sufficient circumstances for the profinite completion of $\mathbb{A}$ to be equal to the canonical extension of $\mathbb{A}$. We conclude Chapter 3 with a discussion of a universal property of canonical extensions with respect to Boolean topological algebras in §3.4.3.

In Chapter 4, we study discrete Stone duality for semi-topological distributive lattices with operators (DLO's) and ordered Kripke frames. In §4.1, we study the duality between *profinite DLO's* and the corresponding *hereditarily finite ordered Kripke frames*. We consider special cases of this duality in §4.2, namely distributive lattices, Boolean algebras, Heyting algebras and modal algebras. Finally, in §4.3, we show that if we restrict our attention to Boolean algebras with operators (BAO's) rather than DLO's, we can characterize not only profinite BAO's via

Stone duality, but also *compact Hausdorff* and *Boolean topological BAO's*. We show that compact Hausdorff BAO's (and Boolean topological BAO's) are dual to *image-finite Kripke frames*. We use this knowledge to study the embedding of Kripke frames into their ultrafilter extensions in §4.3.2.

In Chapter 5, we use a *geometric* version of the *Carioca axioms for coalgebraic modal logic* with the cover modality to give a new description of the *point-free Vietoris construction*. In §5.3.1 we introduce the T-powerlocale construction, where $T\colon \mathbf{Set} \to \mathbf{Set}$ is a weak-pullback preserving, standard, finitary endofunctor on the category of sets. We then go on to show that the P-powerlocale, where P is the covariant powerset functor, is the usual Vietoris powerlocale in §5.3.3. In §5.3.4 we show that the T-powerlocale construction yields a functor V_T on the category of frames, and we show how to lift natural transformations between set functors T' and T to natural transformations between T-powerlocale functors V_T and $V_{T'}$. In §5.3.5 we show that the T-powerlocale can be presented using a flat site presentation rather than an frame presentation; this gives us an algebraic proof for the fact that formulas in our geometric coalgebraic modal logic have a disjunctive normal form. Finally in §5.4, we show that the T-powerlocale construction preserves regularity, zero-dimensionality and the combination of zero-dimensionality and compactness.

Titles in the ILLC Dissertation Series:

ILLC DS-2001-01: **Maria Aloni**
Quantification under Conceptual Covers

ILLC DS-2001-02: **Alexander van den Bosch**
Rationality in Discovery - a study of Logic, Cognition, Computation and Neuropharmacology

ILLC DS-2001-03: **Erik de Haas**
Logics For OO Information Systems: a Semantic Study of Object Orientation from a Categorial Substructural Perspective

ILLC DS-2001-04: **Rosalie Iemhoff**
Provability Logic and Admissible Rules

ILLC DS-2001-05: **Eva Hoogland**
Definability and Interpolation: Model-theoretic investigations

ILLC DS-2001-06: **Ronald de Wolf**
Quantum Computing and Communication Complexity

ILLC DS-2001-07: **Katsumi Sasaki**
Logics and Provability

ILLC DS-2001-08: **Allard Tamminga**
Belief Dynamics. (Epistemo)logical Investigations

ILLC DS-2001-09: **Gwen Kerdiles**
Saying It with Pictures: a Logical Landscape of Conceptual Graphs

ILLC DS-2001-10: **Marc Pauly**
Logic for Social Software

ILLC DS-2002-01: **Nikos Massios**
Decision-Theoretic Robotic Surveillance

ILLC DS-2002-02: **Marco Aiello**
Spatial Reasoning: Theory and Practice

ILLC DS-2002-03: **Yuri Engelhardt**
The Language of Graphics

ILLC DS-2002-04: **Willem Klaas van Dam**
On Quantum Computation Theory

ILLC DS-2002-05: **Rosella Gennari**
Mapping Inferences: Constraint Propagation and Diamond Satisfaction

ILLC DS-2002-06: **Ivar Vermeulen**
A Logical Approach to Competition in Industries

ILLC DS-2003-01: **Barteld Kooi**
Knowledge, chance, and change

ILLC DS-2003-02: **Elisabeth Catherine Brouwer**
Imagining Metaphors: Cognitive Representation in Interpretation and Understanding

ILLC DS-2003-03: **Juan Heguiabehere**
Building Logic Toolboxes

ILLC DS-2003-04: **Christof Monz**
From Document Retrieval to Question Answering

ILLC DS-2004-01: **Hein Philipp Röhrig**
Quantum Query Complexity and Distributed Computing

ILLC DS-2004-02: **Sebastian Brand**
Rule-based Constraint Propagation: Theory and Applications

ILLC DS-2004-03: **Boudewijn de Bruin**
Explaining Games. On the Logic of Game Theoretic Explanations

ILLC DS-2005-01: **Balder David ten Cate**
Model theory for extended modal languages

ILLC DS-2005-02: **Willem-Jan van Hoeve**
Operations Research Techniques in Constraint Programming

ILLC DS-2005-03: **Rosja Mastop**
What can you do? Imperative mood in Semantic Theory

ILLC DS-2005-04: **Anna Pilatova**
A User's Guide to Proper names: Their Pragmatics and Semanics

ILLC DS-2005-05: **Sieuwert van Otterloo**
A Strategic Analysis of Multi-agent Protocols

ILLC DS-2006-01: **Troy Lee**
Kolmogorov complexity and formula size lower bounds

ILLC DS-2006-02: **Nick Bezhanishvili**
Lattices of intermediate and cylindric modal logics

ILLC DS-2006-03: **Clemens Kupke**
Finitary coalgebraic logics

ILLC DS-2006-04: **Robert Špalek**
Quantum Algorithms, Lower Bounds, and Time-Space Tradeoffs

ILLC DS-2006-05: **Aline Honingh**
The Origin and Well-Formedness of Tonal Pitch Structures

ILLC DS-2006-06: **Merlijn Sevenster**
Branches of imperfect information: logic, games, and computation

ILLC DS-2006-07: **Marie Nilsenova**
Rises and Falls. Studies in the Semantics and Pragmatics of Intonation

ILLC DS-2006-08: **Darko Sarenac**
Products of Topological Modal Logics

ILLC DS-2007-01: **Rudi Cilibrasi**
Statistical Inference Through Data Compression

ILLC DS-2007-02: **Neta Spiro**
What contributes to the perception of musical phrases in western classical music?

ILLC DS-2007-03: **Darrin Hindsill**
It's a Process and an Event: Perspectives in Event Semantics

ILLC DS-2007-04: **Katrin Schulz**
Minimal Models in Semantics and Pragmatics: Free Choice, Exhaustivity, and Conditionals

ILLC DS-2007-05: **Yoav Seginer**
Learning Syntactic Structure

ILLC DS-2008-01: **Stephanie Wehner**
Cryptography in a Quantum World

ILLC DS-2008-02: **Fenrong Liu**
Changing for the Better: Preference Dynamics and Agent Diversity

ILLC DS-2008-03: **Olivier Roy**
Thinking before Acting: Intentions, Logic, Rational Choice

ILLC DS-2008-04: **Patrick Girard**
Modal Logic for Belief and Preference Change

ILLC DS-2008-05: **Erik Rietveld**
Unreflective Action: A Philosophical Contribution to Integrative Neuroscience

ILLC DS-2008-06: **Falk Unger**
Noise in Quantum and Classical Computation and Non-locality

ILLC DS-2008-07: **Steven de Rooij**
Minimum Description Length Model Selection: Problems and Extensions

ILLC DS-2008-08: **Fabrice Nauze**
Modality in Typological Perspective

ILLC DS-2008-09: **Floris Roelofsen**
Anaphora Resolved

ILLC DS-2008-10: **Marian Counihan**
Looking for logic in all the wrong places: an investigation of language, literacy and logic in reasoning

ILLC DS-2009-01: **Jakub Szymanik**
Quantifiers in TIME and SPACE. Computational Complexity of Generalized Quantifiers in Natural Language

ILLC DS-2009-02: **Hartmut Fitz**
Neural Syntax

ILLC DS-2009-03: **Brian Thomas Semmes**
A Game for the Borel Functions

ILLC DS-2009-04: **Sara L. Uckelman**
Modalities in Medieval Logic

ILLC DS-2009-05: **Andreas Witzel**
Knowledge and Games: Theory and Implementation

ILLC DS-2009-06: **Chantal Bax**
Subjectivity after Wittgenstein. Wittgenstein's embodied and embedded subject and the debate about the death of man.

ILLC DS-2009-07: **Kata Balogh**
Theme with Variations. A Context-based Analysis of Focus

ILLC DS-2009-08: **Tomohiro Hoshi**
Epistemic Dynamics and Protocol Information

ILLC DS-2009-09: **Olivia Ladinig**
Temporal expectations and their violations

ILLC DS-2009-10: **Tikitu de Jager**
"Now that you mention it, I wonder…": Awareness, Attention, Assumption

ILLC DS-2009-11: **Michael Franke**
Signal to Act: Game Theory in Pragmatics

ILLC DS-2009-12: **Joel Uckelman**
More Than the Sum of Its Parts: Compact Preference Representation Over Combinatorial Domains

ILLC DS-2009-13: **Stefan Bold**
Cardinals as Ultrapowers. A Canonical Measure Analysis under the Axiom of Determinacy.

ILLC DS-2010-01: **Reut Tsarfaty**
Relational-Realizational Parsing

ILLC DS-2010-02: **Jonathan Zvesper**
Playing with Information

ILLC DS-2010-03: **Cédric Dégremont**
The Temporal Mind. Observations on the logic of belief change in interactive systems

ILLC DS-2010-04: **Daisuke Ikegami**
Games in Set Theory and Logic

ILLC DS-2010-05: **Jarmo Kontinen**
Coherence and Complexity in Fragments of Dependence Logic

ILLC DS-2010-06: **Yanjing Wang**
Epistemic Modelling and Protocol Dynamics

ILLC DS-2010-07: **Marc Staudacher**
Use theories of meaning between conventions and social norms

ILLC DS-2010-08: **Amélie Gheerbrant**
Fixed-Point Logics on Trees

ILLC DS-2010-09: **Gaëlle Fontaine**
Modal Fixpoint Logic: Some Model Theoretic Questions

ILLC DS-2010-10: **Jacob Vosmaer**
Logic, Algebra and Topology. Investigations into canonical extensions, duality theory and point-free topology.