Multiple-conclusion Rules, Hypersequents Syntax and Step Frames

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Abstract. We investigate proof theoretic properties of logical systems via algebraic methods. We introduce a calculus for deriving multiple-conclusion rules and show that it is a Hilbert style counterpart of hypersequent calculi. Using step-algebras we develop a criterion establishing the bounded proof property and finite model property for these systems. Finally, we show how this criterion can be applied to universal classes axiomatized by certain canonical rules, thus recovering and extending known results from both semantically and proof-theoretically inspired modal literature.

1 Introduction

In this paper we continue proof theoretic investigations of modal logic via algebraic methods which started in [3, 4]. In [3, 4] the *bounded proof property* (the *bpp*), which is a kind of analytic subformula property, was introduced as a measurement of robustness of proof systems. An algebraic criterion was developed in [3, 4] establishing whether a modal system axiomatized by standard rules possesses the bpp. Here we extend this research in two directions. First, we investigate more expressive proof systems axiomatized by multiple-conclusion rules for which we develop equivalent systems via hypersequent calculi and prove for them an algebraic criterion for the bpp. Second, for a large class of logics (stable logics) we systematically design proof systems that have the bpp. Thus, we are at a position to not only check whether a system is robust, but also to *design robust proof systems*.

Multiple-conclusion rules recently gained attention in the modal logic literature (see e.g., [2, 13, 15]), because they constitute an essential tool for investigating classes of algebras beyond varieties and because canonical formulae axiomatizations can be nicely developed within this framework. On the other hand and from a completely different research perspective, the proof-theoretic oriented community realized that standard sequent formalisms are insufficient

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to handle complex logics and moved to more expressive hypersequent calculi (compare for instance the simplicity of communication rules used for the logics of linear frames developed in [1] with the more complex systems needed for cut elimination in the traditional context [9, 11]).

In this paper we connect multiple-conclusion rules and hypersequent calculi. To our best knowledge, no explicit calculus for multiple-conclusion rules has been proposed so far. Note that for semantic investigations such as [2, 13], it is in fact sufficient to specify *abstractly* the properties that a rule system (seen just as a set of rules) should satisfy. On the other hand, a specific calculus for multiple-conclusion rules is needed if we want to make a close comparison with the hypersequent approach. This calculus will play the role of a Hilbert calculus for hyper-formulae, i.e. for the syntactic components of a hyper-sequent. We will introduce such a calculus and investigate it using the techniques developed in [3, 4]. These techniques, based on semantic analysis of 'step' structures, have been shown to be rather effective in establishing the bpp. Our long-term proposal is to apply these techniques to obtain bpp and the *finite model property* (the fmp), thus also decidability, for logics axiomatized by canonical formulae. In this paper, we report a first success in this direction, already covering bpp and fmp for a continuum of logics, including those recently analyzed in [16] via the hypersequent approach.

Proofs of the results from Section 2 will be deferred to the first appendix; proofs of the results from Sections 3,4 (requiring routine adjustments from the corresponding proofs in [3, 4]) will be included into the second appendix.

2 A calculus for derived multiple-conclusion rules

Modal formulae are built from propositional variables $x, y, \dots$ by using the Booleans $(\neg, \wedge, \vee, \rightarrow, 0, 1)$ and modal operators $(\Diamond, \Box)$. For simplicity, we take $\neg$, $\wedge$, $\Diamond$ as primitive connectives, the remaining ones being defined in the customary way (in particular, $\Box$ is defined as $\neg\Diamond\neg$). We shall also use parameters $a, b, \dots$ instead of variables whenever we want to stress that uniform substitution does not apply to them. Underlined letters stand for tuples of unspecified length and formed by distinct elements, thus for instance, we may use $\underline{x}$ for a tuple such as $x_1, \dots, x_n$. When we write $\phi(\underline{x})$ we want to stress that ϕ contains at most the variables $\underline{x}$ (and no parameters) and similarly when we write $\phi(\underline{a})$ we want to stress that ϕ contains at most the parameters $\underline{a}$ (and no variables). The same convention applies to sets of formulae: if Γ is a set of formulae and we write $\Gamma(\underline{a})$, we mean that all formulae in Γ are of the kind $\phi(\underline{a})$, etc. We may occasionally replace variables with parameters in a formula: for this, we use the following self-explanatory notation. For a formula $\phi(\underline{x})$ we write $\phi(\underline{a})$ to mean that $\phi(\underline{a})$ is obtained from $\phi(\underline{x})$ by replacing $\underline{x} = x_1, \dots, x_n$ (simultaneously and respectively) by $\underline{a} = a_1, \dots, a_n$. The modal complexity (or the modal degree) of a formula ϕ counts the maximum number of nested modal operators in ϕ (the precise definition is by an obvious induction).

We recall some background on modal algebras, see e.g., [6, Sec. 5.2] or [7, Sec. 7.6] for more details. A *modal algebra* $\mathfrak{A} = (A, \Diamond)$ is a Boolean algebra A endowed with a unary operator $\Diamond$ satisfying $\Diamond(x \vee y) = \Diamond x \vee \Diamond y, \Diamond 0 = 0$. Notice that, here and elsewhere, we use the same name for a connective and the corresponding operator in modal algebras (thus, for instance, 0 is zero, $\vee$ is join, etc.). In this way, propositional formulae can be identified with *terms* in the first order language of modal algebras.

From the semantic side, we have the notion of a frame; a *frame* $\mathfrak{F} = (W, R)$ is a set W endowed with a binary relation R. The *dual* of a frame $\mathfrak{F} = (W, R)$ is the modal algebra $\mathfrak{F}^* = (\wp(W), \Diamond_R)$, where $\wp(W)$ is the powerset Boolean algebra and $\Diamond_R$ is the semilattice morphism associated with R. The latter is defined as follows: for $S \subseteq W$, we have $\Diamond_R(S) = \{w \in W \mid R(w) \cap S \neq \emptyset\}$ (here $R(w)$ denotes $\{v \in W \mid (w, v) \in R\}$). It should be noticed that there is a real duality (in the categorical sense) between modal algebras and frames only if we restrict to finite modal algebras and finite frames. If we want a full duality working for arbitrary modal algebras, we must introduce some topological structures on our frames (see, e.g., [6, Sec. 5.5], [7, Sec. 7.5], [14, Ch. 4] or [17]). For the purposes of this paper, however, the duality between finite frames and finite modal algebras will suffice.

2.1 Multiple-conclusion rules

Normal modal logics are an adequate formalism to describe equational classes of modal algebras. However, in this paper we are interested in more general classes. A class of modal algebras is said to be:

(i) *a variety* iff it is the class of models of a set of equations, i.e., of sentences of the kind $\forall \underline{x}\ \phi(\underline{x}) = 1$, where ϕ is a modal formula (aka a term in the first order language of modal algebras);
(ii) *a quasi-variety* iff it is the the class of models of a set of implications of equations, i.e. of sentences of the kind $\forall \underline{x}\ (\bigwedge_{i=1}^{n} \phi_i(\underline{x}) = 1 \to \psi(\underline{x}) = 1)$, where $\phi_1, \ldots, \phi_n, \psi$ are modal formulae;
(iii) *a universal class* iff it is the class of models of a set of clauses, i.e., of sentences of the kind $\forall \underline{x}\ (\bigwedge_{i=1}^{n} \phi_i(\underline{x}) = 1 \to \bigvee_{j=1}^{m} \psi_j(\underline{x}) = 1)$, where $\phi_1, \ldots, \phi_n$, $\psi_1, \ldots, \psi_m$ are modal formulae.

In order to describe universal classes within a propositional modal language, we shall use multiple-conclusion rules; a *multiple-conclusion rule* (or just a *rule*) is a pair of finite sets of formulae $\langle \Gamma, S \rangle$. If $\Gamma = \{\phi_1, \ldots, \phi_n\}, S = \{\psi_1, \ldots, \psi_m\}$, we write the rule $\langle \Gamma, S \rangle$ as Γ / S or as

$$\frac{\gamma_1, \ \ \ldots, \ \ \gamma_n}{\delta_1 \mid \cdots \mid \delta_m}\ (R)$$

The formulae $\Gamma = \{\gamma_1, \ldots . \gamma_n\}$ are said to be the *premises* of the rule (R) and the formulae $S = \{\delta_1, \ldots, \delta_m\}$ are said to be the *conclusions* of the rule (R). The multiple-conclusion rule (R) is said to be an *inference rule* or a *single-conclusion rule* iff $m = 1$, i.e., iff it has a single conclusion. A modal algebra

$\mathfrak{A} = (A, \Diamond)$ validates the multiple-conclusion rule (R) iff it is a model of the clause $\forall \underline{x}\ (\bigwedge_{i=1}^{n} \phi_i(\underline{x}) = 1 \to \bigvee_{j=1}^{m} \psi_j(\underline{x}) = 1)$.

We recall the notion of a rule system from [13]:

Definition 1. *A set of multiple-conclusion rules $\mathcal{K}$ is said to be a* rule system *iff it satisfies the following conditions for every formula ϕ and for every finite sets of formulae Γ, Γ', S, S':*

(i) $\phi/\phi \in \mathcal{K}$;
(ii) *if $\Gamma/S, \phi \in \mathcal{K}$ and $\Gamma, \phi/S \in \mathcal{K}$, then $\Gamma/S \in \mathcal{K}$;*
(iii) *if $\Gamma/S \in \mathcal{K}$ then $\Gamma, \Gamma'/S, S' \in \mathcal{K}$;*
(iv) *if $\Gamma/S \in \mathcal{K}$ then for every substitution σ, we have that $\Gamma\sigma/S\sigma \in \mathcal{K}$.*

Above we used obvious conventions about set-theoretic union of finite sets of formulae (e.g., Γ, ϕ stands for $\Gamma \cup \{\phi\}$, moreover Γ, Γ' stands for $\Gamma \cup \Gamma'$, etc.); in addition, we used $\Gamma\sigma$ to denote the set resulting from the application of σ to all members of Γ.

Definition 2. *A (normal)* modal rule system *is a rule system containing tautologies and the distribution schema $\Box(\alpha_1 \to \alpha_2) \to (\Box\alpha_1 \to \Box\alpha_2)$ (as single-conclusion 0-premises rules) as well as necessitation ($\alpha/\Box\alpha$) and modus ponens ($\alpha, \alpha \to \beta/\beta$) rules.*

We say that a set of rules K *entails* or *derives* a rule Γ/S (written $K \vdash \Gamma/S$) iff Γ/S belongs to the smallest modal rule system $[K]$ containing K. The following algebraic completeness theorem is proved in [13] (but follows also from our results below):

Theorem 1. *Let K be a set of multiple-conclusion rules. Then $K \vdash \Gamma/S$ iff every modal algebra validating all rules in K validates also Γ/S.*

2.2 Hyper-formulae and hyper-proofs

We now design a calculus for recognizing syntactically the relation $K \vdash \Gamma/S$. We shall actually give two equivalent versions of such a calculus, the latter to be seen just as a Hilbert-style analogue of the well-known hypersequent calculi [1].

An *hyper-formula* is a finite set of propositional formulae written in the form

$$\alpha_1 \mid \cdots \mid \alpha_n. \tag{1}$$

We use letters $S, S_1, S', \ldots$ for hyper-formulae; the notation $S \mid S'$ means set union and $S \mid \alpha$ and $\alpha \mid S$ stand for $S \mid \{\alpha\}$ and $\{\alpha\} \mid S$, respectively.

Definition 3. *Let Γ be a set of propositional modal formulae and let K be a set of multiple-conclusion rules. A K-*hyper-proof *(or a K-*derivation *or just a* derivation*) under assumptions Γ is a finite list of hyper-formulae $S_1, \ldots, S_n$ such that each S_i in it matches one of the following requirements:*

(i) *S_i is of the kind $\alpha \mid S$, where $\alpha \in \Gamma$ or α is a tautology or α is an instance of the distribution schema;*
(ii) *S_i is obtained from hyper-formulae preceding it by applying a rule from K or the necessitation rule or the modus ponens rule.*

We write $\Gamma \vdash_K S$ to mean that there is a K-derivation ending with S.

An important remark is in order for (ii): when we say that S_i is obtained by applying an inference rule, *we include uniform substitution and weakening in the application of the rule.* Thus, if the rule is (R), when we say that S_i is obtained from (R), we mean that there is a substitution σ such that S_i is of the kind $S \mid \delta_1\sigma \mid \cdots \mid \delta_m\sigma$ and that there are $j_1, \ldots, j_n < i$ such that S_{j_1} is of the kind $S \mid \gamma_1\sigma$, and $\ldots$ and S_{j_n} is of the kind $S \mid \gamma_n\sigma$ (of course, this applies also to the case $n = 0$, i.e., to zero-premisses rules).

Theorem 2. *Let K be a set of multiple-conclusion rules. Then $\Gamma \vdash_K S$ iff the multiple-conclusion rule Γ/S is valid in every modal algebra validating K.*

Corollary 1. *Let K be a set of multiple-conclusion rules. For each multiple-conclusion rule Γ/S, we have $K \vdash \Gamma/S$ iff $\Gamma \vdash_K S$.*

Notice that Theorem 1 follows from Corollary 1 and Theorem 2.

2.3 Hypersequent syntax

A *sequent* is a pair of finite sets of formulae written as $\Gamma \Rightarrow \Delta$ and a hypersequent is a finite set of sequents written as

$$\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n. \tag{2}$$

In this paper, we are investigating proof theoretic facts that only depends on the modal degree of formulae and on proofs, thus *we view a sequent $\Gamma \Rightarrow \Delta$ as the formula $\bigwedge \Gamma \to \bigvee \Delta$ and a hypersequent* (2) *as the hyperformula*

$$\bigwedge \Gamma_1 \to \bigvee \Delta_1 \mid \cdots \mid \bigwedge \Gamma_n \to \bigvee \Delta_n. \tag{3}$$

Still, there is an important difference between hyperproofs according to Definition 3 and hypersequent calculi e.g., in [1]: once translated into our formalism, the difference is in the possibility of using rules having hyper-formulae (and not just formulae) as premises. We show here that this difference is immaterial because we can translate these more general rules and proofs into our formalism. The translation is effective, does not increase the modal degree of formulae involved in the proofs, but might be harmful for complexity.

We first introduce the definitions needed to make the comparison. A *hyper-rule* is a $n * 1$-tuple of hyper-formulae, written as $S_1, \ldots, S_k/S$, if H is a set of hyper-rules, Γ is a set of hyper-formulae and S is a hyper-formula. We say that S is *provable from Γ in H*, written $\Gamma \Vdash_H S$ iff there exists a finite list of hyper-formulae $S_1, \ldots, S_n$ (called a *derivation*) such that each S_i in it matches one of the following requirements:

(i) S_i is a hyper-formula containing a member of Γ, or a tautology, or a formula of the form $\Box(\alpha_1 \to \alpha_2) \to (\Box\alpha_1 \to \Box\alpha_2)$;
(ii) S_i is obtained from hyper-formulae preceding it by applying modus ponens rule $\alpha,\ \alpha \to \beta/\beta$, necessitation rule $\alpha/\Box\alpha$, or a hyper-rule from H.

Again, 'to apply a rule $S_1, \ldots, S_k/S$ to get S_i' means that there is a substitution σ such that S_i is of the kind $\tilde{S} \mid S\sigma$ and that there are $j_1, \ldots, j_n < i$ such that S_{j_1} is of the kind $\tilde{S} \mid S_1\sigma$ and ... and S_{j_k} is of the kind $\tilde{S} \mid S_k\sigma$.[3]

Proposition 1. *Let H be a finite set of hyper-rules. Then it is possible to produce a set of rules K such that for all $\Gamma, \tilde{S}$ we have $\Gamma \Vdash_H \tilde{S}$ iff $\Gamma \vdash_K \tilde{S}$.*

Proof. (Sketch, see the appendix for full details) Consider a hyper-rule $S_1, \ldots, S_k/S$ from H: to obtain K, we simply replace it with the set of rules $\gamma(S_1), \ldots, \gamma(S_n)/S$, varying γ among the functions that pick one formula from each S_i, for each $i = 1, \ldots, n$. ⊣

Next we give a few examples. In order to make a more direct link with the current literature, we will use the hypersequent syntax (Gentzen standard sequent rules for classical logic, as well as external structural rules will be always implicitly assumed below).

Example 1. An adequate calculus for **S4** comprises the following two rules (taken from [12])

$$\frac{\Box\Gamma \Rightarrow A_1 \mid \cdots \mid \Box\Gamma \Rightarrow A_n}{\Gamma', \Box\Gamma \Rightarrow \Delta, \Box A_1, \cdots, \Box A_n}\ (\Rightarrow\Box)$$

$$\frac{\Box A, A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}\ (T)$$

where, if $\Gamma = \{\phi_1, \ldots, \phi_n\}$, then $\Box\Gamma$ stands for $\{\Box\phi_1, \ldots, \Box\phi_n\}$.

Example 2. Let us now consider the universal class of *prime* **S4.3** algebras: these are the modal algebras validating the above rules and satisfying in addition the clause

$$\forall x\ \forall y\ (\Box x \leq \Box y \text{ or } \Box y \leq \Box x).$$

To axiomatize this class, we can add to the above rules the further rule

$$\frac{\tilde{\Gamma}, \Box\Gamma, \Box\Gamma' \Rightarrow \Delta \qquad \tilde{\Gamma}', \Box\Gamma', \Box\Gamma \Rightarrow \Delta'}{\tilde{\Gamma}, \Box\Gamma \Rightarrow \Delta \mid \tilde{\Gamma}', \Box\Gamma' \Rightarrow \Delta'}\ (Dich)$$

taken from [12]. Rule $(Dich)$ is nothing but a variant of the communication rule introduced in [1].

Example 3. For prime **S5** algebras, we can add to **S4**-rules the following rule taken from [1]

$$\frac{\Box\Gamma, \Gamma' \Rightarrow \Box\Delta, \Delta'}{\Box\Gamma \Rightarrow \Box\Delta \mid \Gamma' \Rightarrow \Delta'}\ (S5)$$

[3] This notion of a derivation avoids the introduction of side components (in the sense of [1]) when specifying rules: in fact, the side component S is introduced directly when applying the rule.

3 Bounded proofs and step frames

From now on, we shall make exclusive reference to the calculus explained in Definition 3. We call a *modal calculus* (or simply a *calculus*) a set of multiple-conclusion rules *where only formulae of modal degree at most one occur.*[4]

When we write $\Gamma \vdash^n_K S$ we mean that there is a K-hyperproof under assumptions Γ (see Definition 3) in which only formulae of modal complexity at most n occur. We are mostly interested in the semantic characterization of the following property:

Definition 4. *We say that a calculus K has the* bounded proof property *(bpp, for short) iff for every hyper-formula S of modal complexity at most n and for every Γ containing only formulae of modal complexity at most n, we have*

$$\Gamma \vdash_K S \quad \Rightarrow \quad \Gamma \vdash^n_K S.$$

It should be clear that the bpp for K implies the decidability of the relation $\Gamma \vdash^n_K S$ (and hence, according to Corollary 1, of derivability of rules in K). This is because we have a bounded search space for hyper-formulae occurring in a possible derivations and for possible substitutions instantiating rules from K: in fact, there are only finitely many non-provably equivalent formulae containing at most a given finite set of propositional variables and with modal complexity bounded by a given n (notice that in a proof witnessing $\Gamma \vdash^n_K S$ we can freely suppose that only the variables occurring in Γ, S occur, because extra variables can be uniformly replaced by, say, 0).

The following proposition shows that we can limit our consideration to formulae of complexity 1 when checking the bpp.

Proposition 2. *A calculus K has the bounded proof property iff for every hyper-formula S of modal complexity at most 1 and for every Γ containing only formulae of modal complexity at most 1, we have $\Gamma \vdash_K S \Rightarrow \Gamma \vdash^1_K S$.*

In the following, we shall adopt the equivalent formulation of the bpp suggested by the above proposition. We shall call finite sets Γ of formulae of modal complexity at most 1, *finite presentations*. It is useful to use parameters (see Section 2) to name the variables occurring in a finite presentation Γ: this is because in a K-hyperproof under assumptions Γ, the formulae in Γ are introduced in the derivation as they are (no substitution applies to them), whereas substitutions are applied to rules in K. Thus, we write $\Gamma(\underline{a})$ to emphasize that at most the parameters $\underline{a}$ occur in Γ and $\Gamma(\underline{a}) \vdash_K S(\underline{a})$ to emphasize that the tuple $\underline{a}$ includes all parameters occurring in both Γ, S.

[4] This property can be assumed without loss of generality, by applying the transformation suggested in [3] (that transformation does not increase the modal degree of proofs). In [3] another property is assumed on rules (namely that variables occurring in them have occurrences inside a modal operator). This property was assumed there to simplify the definition of evaluation in step algebras, but in the present more general context it can have unclear side effects, so we prefer not to assume it anymore.

3.1 Conservative one-step algebras and one-step frames

We first recall the definition of one-step modal algebras and one-step frames from [10] and [5], and define conservative one-step modal algebras and one-step frames.

Definition 5. *A* one-step modal algebra *is a quadruple* $\mathcal{A} = (A_0, A_1, i_0, \Diamond_0)$*, where* A_0, A_1 *are Boolean algebras,* $i_0 : A_0 \to A_1$ *is a Boolean morphism, and* $\Diamond_0 : A_0 \to A_1$ *is a semilattice morphism (i.e., it preserves only* $0, \vee$*). The algebras* A_0, A_1 *are called the* source *and the* target *Boolean algebras of the one-step modal algebra* $\mathcal{A}$*. We say that* $\mathcal{A}$ *is* conservative *iff* i_0 *is injective and the union of the images* $i_0(A_0) \cup \Diamond(A_0)$ *generates* A_1 *as a Boolean algebra.*

From the dual semantic point of view we have the following:

Definition 6. *A* one-step frame *is a quadruple* $\mathcal{S} = (W_1, W_0, f, R)$*, where* W_0, W_1 *are sets,* $f : W_1 \to W_0$ *is a map and* $R \subseteq W_1 \times W_0$ *is a relation between* W_1 *and* W_0*. We say that* $\mathcal{S}$ *is* conservative *iff* f *is surjective and the following condition is satisfied for all* $w_1, w_2 \in W_1$*:*

$$f(w_1) = f(w_2) \ \&\ R(w_1) = R(w_2) \quad \Rightarrow \quad w_1 = w_2. \tag{4}$$

Similarly to the case of Kripke frames, above we used the notation $R(w_1)$ to mean the set $\{v \in W_0 \mid (w_1, v) \in R\}$ (and similarly for $R(w_2)$). The dual of a finite one-step frame $\mathcal{S} = (W_1, W_0, f, R)$ is the one-step modal algebra $\mathcal{S}^* = (\wp(W_0), \wp(W_1), f^*, \Diamond_R)$, where f^* is the inverse image operation and $\Diamond_R$ is the semilattice morphism associated with R. The latter is defined as follows: for $S \subseteq W_0$, we have $\Diamond_R(S) = \{w \in W_1 \mid R(w) \cap S \neq \emptyset\}$. Conservativity also carries over from one-step frames to one-step modal algebras (see [3] for a proof of the following proposition):

Proposition 3. *A finite one-step frame* $\mathcal{S}$ *is conservative iff its dual one-step modal algebra* $\mathcal{S}^*$ *is conservative.*

To complete our list of definitions, let us observe that a one-step modal algebra $\mathcal{A} = (A_0, A_1, i_0, \Diamond_0)$ in which we have $A_0 = A_1$ and $i_0 = id$ is nothing but a modal algebra. Similarly, a one-step frame $\mathcal{S} = (W_1, W_0, f, R)$ where we have $W_0 = W_1$ and $f = id$ is just a frame. For clarity, we shall sometimes call modal algebras and frames *standard* or *plain* modal algebras and frames, respectively.

3.2 Inference validation in step algebras

We spell out what it means for a one-step modal algebra and a one-step frame to validate a modal calculus K and a finite presentation Γ (the definition requires little modifications with respect to [3, 4] because we do not restrict to reduced rules).

Let us fix two finite sets of variables $\underline{x} = x_1, \ldots, x_n$, $\underline{y} = y_1, \ldots, y_m$ and a finite set of parameters $\underline{a} = a_1, \ldots, a_m$ (either $\underline{x}, \underline{y}$ or $\underline{a}$ can be empty). An $\underline{a}$-*augmented* one-step modal algebra $\mathcal{A} = (A_0, A_1, i_0, \Diamond_0, \underline{\mathtt{a}})$ is a one-step modal algebra together with displayed elements $\underline{\mathtt{a}} = \mathtt{a_1}, \ldots, \mathtt{a_m} \in A_0$ (these elements will interpret parameters).

Given an $\underline{a}$-augmented one-step modal algebra as above, an $\mathcal{A}$-*valuation* is a map associating with each variable $x_i \in \underline{x}$ an element $\mathtt{v}(x_i) \in A_0$ and with each variable $y_j \in \underline{y}$ an element $\mathtt{v}(y_j) \in A_1$. For every formula $\phi(\underline{x})$ of complexity 0, we define $\phi^{\mathtt{v}0} \in A_0$ as follows:

$$x_i^{\mathtt{v}0} = \mathtt{v}(x_i) \ \ \text{(for every variable } x_i \in \underline{x}); \qquad a_i^{\mathtt{v}0} = \mathtt{a_i} \ \ (a_i \in \underline{a});$$
$$(\phi_1 * \phi_2)^{\mathtt{v}0} = \phi_1^{\mathtt{v}0} * \phi_2^{\mathtt{v}0} \ \ (* = \wedge, \vee); \qquad (\neg\phi)^{\mathtt{v}0} = \neg(\phi^{\mathtt{v}0}).$$

For every formula $\phi(\underline{x})$ of complexity 0, we define $\phi^{\mathtt{v}1} \in A_1$ as $i_0(\phi^{\mathtt{v}0})$. For every $\psi(\underline{x}, \underline{y})$ of complexity at most 1 in which the $\underline{y}$ do not have occurrences within the scope of a modal operator, $\psi^{\mathtt{v}1} \in A_1$ is defined as follows:

$$y_j^{\mathtt{v}1} = \mathtt{v}(y_j) \ \ \text{(for every variable } y_j \in \underline{y}); \qquad (\Diamond\phi(\underline{x}))^{\mathtt{v}1} = \Diamond(\phi^{\mathtt{v}0});$$
$$(\psi_1 * \psi_2)^{\mathtt{v}1} = \psi_1^{\mathtt{v}1} * \psi_2^{\mathtt{v}1} \ \ (* = \wedge, \vee); \qquad (\neg\psi)^{\mathtt{v}1} = \neg(\psi^{\mathtt{v}1}).$$

Definition 7. *Suppose that the formulae $\delta_1(\underline{x}, \underline{y}), \ldots, \delta_k(\underline{x}, \underline{y})$, $\gamma_1(\underline{x}, \underline{y}), \ldots, \gamma_l(\underline{x}, \underline{y})$ have modal degree at most one and that the variables $\underline{y}$ are the variables not occurring in them inside the scope of a modal operator. We say that a one-step modal algebra $\mathcal{A}$* validates the multiple-conclusion rule

$$\frac{\gamma_1, \ \ \ldots, \ \ \gamma_n}{\delta_1 \mid \cdots \mid \delta_m} \ (R)$$

iff for every $\mathcal{A}$-valuation $\mathtt{v}$, we have that if ($\phi_1^{\mathtt{v}1} = 1$ and $\cdots$ and $\phi_m^{\mathtt{v}1} = 1$), then ($\gamma_1^{\mathtt{v}1} = 1$ or $\cdots$ or $\gamma_n^{\mathtt{v}1} = 1$). We say that $\mathcal{A}$ validates a modal calculus K *(written $\mathcal{A} \models K$) iff $\mathcal{A}$ validates all inferences from K.*

Notice that it might well be that K_1 and K_2 are equivalent (in the sense that rules from K_1 are derivable in K_2 and vice versa), but that only one of them is validated by a given $\mathcal{A}$. This phenomenon, however, cannot happen in case $\mathcal{A}$ is standard (i.e., it is a modal algebra).

For formulae $\phi(\underline{a})$ where the variables $\underline{x}$ do not occur, the valuation $\mathtt{v}$ is not relevant. Thus, in such cases, we may write $\phi^{\underline{\mathtt{a}}0}, \phi^{\underline{\mathtt{a}}1}$ instead of $\phi^{\mathtt{v}0}, \phi^{\mathtt{v}1}$, respectively, to stress the fact that the augmentation $\underline{\mathtt{a}}$ is the essential part of the definition. We write $\mathcal{A} \models \phi(\underline{a})$ for $\phi^{\underline{\mathtt{a}}1} = 1$ and $\mathcal{A} \models S(\underline{a})$ iff there is a $\phi \in S$ such that $\mathcal{A} \models \phi$. We say that $\mathcal{A}$ *validates the presentation Γ* (in symbols, $\mathcal{A} \models \Gamma(\underline{a})$) iff we have that $\mathcal{A} \models \phi(\underline{a})$ for all $\phi(\underline{a}) \in \Gamma$.

The notion of an $\mathcal{S}$-*valuation* for a one-step frame $\mathcal{S}$ is the expected one, namely $\mathtt{v}$ is an $\mathcal{S}$-*valuation* iff it is an $\mathcal{S}^*$-*valuation.* In the same way the other notions introduced above (augmentation, $\phi^{\mathtt{v}0}, \phi^{\mathtt{v}1}$, validation of a presentation, of an inference, of an axiomatic system) can be extended by duality to one-step frames.

Example 4. For the systems **S4**, **S4.3**, **S5**, it can be shown (by applying the 'step' variant of modal correspondence theory [3,4]) that a conservative one-step frame $\mathcal{S} = (W_1, W_0, f, R)$

- validates the rules of Example 1 iff it is step-transitive and step-reflexive, where the latter means $f \subseteq R$ and the former means $R \subseteq f \circ \geq_R$ (here $\circ$ is relation composition and $w_1 \geq_R w_2$ is defined to be $R(w_1) \supseteq R(w_2)$);
- validates the rules of Example 2 iff it is step-transitive, step-reflexive and step-linear, where the latter means $\forall w_1, w_2 \in W_1\ (R(w_1) \subseteq R(w_2)$ or $R(w_2) \subseteq R(w_1))$;
- validates the rules of Example 3 iff $R(w_1) = R(w_2)$ for alll $w_1, w_2 \in W_1$.

We can specialize our notions to standard modal algebras and frames. An $\underline{a}$-augmentation in a modal algebra $\mathfrak{A} = (A, \Diamond)$ is a tuple $\underline{\mathsf{a}}$ of elements from the support of A, matching the length of $\underline{a}$. For frames $\mathfrak{F} = (W, R)$, we dually take a tuple from $\wp(W)$, i.e., a tuple of subsets. Given such $\underline{a}$-augmentation, we can define $\mathfrak{A} \models \Gamma(\underline{a})$ and $\mathfrak{F} \models \Gamma(\underline{a})$ for a presentation $\Gamma(\underline{a})$, just specializing the above definitions (standard modal algebras and frames are special one-step modal algebras and frames). Notice that $\mathfrak{F} \models \Gamma(\underline{a})$ is *global* validity in terms of the Kripke forcing from the modal logic literature, see e.g., [14, Sec. 3.1].

Proposition 4. *Let $\mathcal{A} = (A, B, i, \Diamond, \underline{\mathsf{a}})$ be an augmented conservative one-step modal algebra that validates the modal calculus K and the presentation $\Gamma(\underline{a})$. Then, for every hyper-formula $S(\underline{a})$, we have that $\Gamma \vdash^1_K S$ implies $\mathcal{A} \models S$.*

4 Semantic characterizations of the bpp and the fmp

In this section we first introduce the morphisms of one-step modal algebras and one-step frames.

Definition 8. *An* embedding *between one-step modal algebras $\mathcal{A} = (A_0, A_1, i_0, \Diamond_0)$ and $\mathcal{A}' = (A'_0, A'_1, i'_0, \Diamond'_0)$ is a pair of injective Boolean morphisms $h : A_0 \to A'_0$, $k : A_1 \to A'_1$ such that*

$$k \circ i_0 = i'_0 \circ h \qquad \text{and} \qquad k \circ \Diamond_0 = \Diamond'_0 \circ h\ . \tag{5}$$

Notice that, when $\mathcal{A}'$ is standard (i.e. $A'_1 = A'_0 =$ and $i'_0 = id$), h must be $k \circ i_0$ and (5) reduces to

$$k \circ \Diamond_0 = \Diamond'_0 \circ k \circ i_0. \tag{6}$$

For frames we have the dual definition. In the definition below, we use $\circ$ to denote relational composition: for $R_1 \subseteq X \times Y$ and $R_2 \subseteq Y \times Z$, we have $R_2 \circ R_1 := \{(x, z) \in X \times Z \mid \exists y \in Y\ ((x, y) \in R_1\ \&\ (y, z) \in R_2)\}$. Notice that the relational composition applies also when one or both of R_1, R_2 is a function.

Definition 9. *A* p-morphism *between step frames $\mathcal{F}' = (W'_1, W'_0, f', R')$ and $\mathcal{F} = (W_1, W_0, f, R)$ is a pair of surjective maps $\mu : W'_1 \to W_1$, $\nu : W'_0 \to W_0$ such that*

$$f \circ \mu = \nu \circ f' \qquad \text{and} \qquad R \circ \mu = \nu \circ R'. \tag{7}$$

Notice that, when $\mathcal{F}'$ is standard (i.e., $W_1' = W_0'$ and $f' = id$), ν must be $f \circ \mu$ and (7) reduces to

$$R \circ \mu = f \circ \mu \circ R'. \tag{8}$$

The following definitions introduce the semantic notions needed for our characterization of bpp.

Definition 10. *Let $\mathcal{A}_0 = (A_0, A_1, i_0, \Diamond_0)$ be a one-step modal algebra. A* one-step extension *of $\mathcal{A}_0$ is a one-step modal algebra $\mathcal{A}_1 = (A_1, A_2, i_1, \Diamond_1)$ satisfying $i_1 \circ \Diamond_0 = \Diamond_1 \circ i_0$. Dually, a* one-step extension *of the one-step frame $\mathcal{S}_0 = (W_1, W_0, f_0, R_0)$ is a one-step frame $\mathcal{S}_1 = (W_2, W_1, f_1, R_1)$ satisfying $R_0 \circ f_1 = f_0 \circ R_1$.*

Definition 11. *A class of one-step modal algebras has the* extension property *iff every conservative one-step modal algebra $\mathcal{A}_0 = (A_0, A_1, i_0, \Diamond_0)$ in the class has an extension $\mathcal{A}_1 = (A_1, A_2, i_1, \Diamond_1)$ such that i_1 is injective and $\mathcal{A}_1$ is also in the class. A class of one-step modal frames has the* extension property *iff every conservative one-step frame $\mathcal{S}_0 = (W_1, W_0, f_0, R_0)$ in the class has an extension $\mathcal{S}_1 = (W_2, W_1, f_1, R_1)$ such that f_1 is surjective and $\mathcal{S}_1$ is also in the class.*

Theorem 3. *A modal calculus K has the bpp iff the class of finite one-step modal algebras (equivalently, the class of finite one-step frames) validating K has the extension property.*

The characterization of the bpp from Theorem 3 may not be easy to handle, because in practical cases one would like to avoid managing one-step extensions and would prefer to work with standard frames instead. This is possible, if we combine the bpp with the finite model property.

Definition 12. *A modal calculus K has the (global)* finite model property*, the* fmp *for short, if for every tuple $\underline{a}$ of parameters, for every finite set of formulae $\Gamma(\underline{a})$ and for every hyper-formula $S(\underline{a})$ we have $\Gamma \not\vdash_K S$ iff there exists a finite $\underline{a}$-augmented modal algebra $\mathfrak{A}$ such that $\mathfrak{A} \models K$, $\mathfrak{A} \models \Gamma(\underline{a})$ and $\mathfrak{A} \not\models S(\underline{a})$ (equivalently, iff there exists a finite $\underline{a}$-augmented Kripke frame $\mathfrak{F}$ such that $\mathfrak{F} \models K$, $\mathfrak{F} \models \Gamma(\underline{a})$ and $\mathfrak{F} \not\models S(\underline{a})$).*

We are ready for a characterization result:

Theorem 4. *A modal calculus K has both the bpp and the fmp iff every finite conservative one-step frame validating K is a p-morphic image of a finite frame validating K (equivalently, iff every finite conservative one-step modal algebra validating K has an embedding into a finite modal algebra validating K).*

Example 5. Theorem 4 applies to all Examples 1 - 3. The construction is the same in all cases and it is rather straightforward: given a finite conservative step frame $\mathcal{S} = (W_1, W_0, f, R)$ validating the rules of the calculus, we can define $\mathfrak{F}' = (W', R')$ and μ so that condition (8) is satisfied as follows:

$$W' := W, \qquad \mu := id, \qquad w_1 R' w_2 :\Leftrightarrow R(w_1) \supseteq R(w_2).$$

5 Modal stable rules

Canonical formulae for transitive modal logics and intuitionistic logic were introduced by Zakharyaschev (see [7] for an overview) who proved that all transitive modal logics and all intermediate logics are axiomatizable by canonical formulae. Jérabek [13] defined *canonical rules*, which are multiple-conclusion rules generalizing canonical formulas. Jérabek used these rules for an alternative proof of decidability of admissible rules for intuitionistic logic and transitive modal logics **K4**, **S4**, **S4.3**, etc. However, there are non-transitive modal logics not axiomatizable by canonical rules. [2] defines *stable canonical rules*, which differ from Zakharyaschev's canonical formulae and Jérabek's canonical rules and proves that every modal logic (including non-transitive ones) is axiomatizable by these rules. In this section we will concentrate on logics axiomatizable by a special subclass of stable canonical rules.

Subframe logics are the logics whose frames are closed under taking subframes. Transitive subframe logics are axiomatizable by a special subclass of canonical formulae called *subframe formulae*, see, e.g., [7]. A similar restriction to stable canonical rules gives a class of *stable logics*. But stable logics are not necessarily transitive. Logics in this class are exactly the logics that are closed under order-preserving onto maps. Transitive subframe logics and stable logics enjoy the fmp. Transitive subframe logics enjoy the fmp because they admit selective filtration, and stable logics enjoy the fmp because they admit the standard filtration [2].

In this section we show that all stable logics admit an axiomatization that has the bounded proof property. As we will see below, stable canonical rules will not produce an axiomatization that has the bpp. However, we will modify these rules so that the obtained rules do possess the bpp. This provides a systematic method of producing infinitely many proof calculi that are good (enjoying the bpp) from the proof-theoretic point of view. We remark that Lahav [16] also considers a class of modal logics whose Kripke frames satisfy special first-order conditions. He introduces hypersequent calculi for these logics and proves that these calculi admit cut elimination. It is easy to see that the non-transitive logics studied in [16] are stable logics – their frame classes are closed under order-preserving onto maps. Thus, the class of logics we investigate in this section extends the class of logics studied in [16] in the non-transitive case. (The transitive logic **K4** is not stable. Proof theoretic aspects of stable logics over **K4** will be investigated in the forthcoming paper.) Note, however, that [16] studies cut elimination, whereas we work with the bpp only. Now, if cut elimination gives the subformula property as a by-product, the bpp follows trivially. The converse is not true: we might have the bpp without the subformula property. However, it should be noticed that the bpp is a strong evidence about the proof-theoretic robusteness of a system and supplies a loose notion of analiticity which is sufficient for decidability and which can hold for a wide class of calculi, including cases where the design of cut-eliminating systems looks very problematic.

We start by recalling the definition of modal stable rules. Let $\mathfrak{F} = (F, R_F)$ be a finite frame. For every $a \in F$ we introduce a new propositional variable x_a.

The *modal stable rule* of $\mathfrak{F}$ is

$$\frac{\bigvee_{i=1}^{n} x_{a_i}, \quad \bigwedge_{i\neq j} \neg(x_{a_i} \wedge x_{a_j}), \quad \bigwedge_{i=1}^{n}(x_{a_i} \to \Box \bigvee_{b\in R_F(a_i)} x_b)}{\neg x_{a_1} \mid \cdots \mid \neg x_{a_n}} \; (r_{\mathfrak{F}})$$

where we suppose that $F = \{a_1, \ldots, a_n\}$.

A *stable embedding* of a modal algebra $\mathfrak{A} = (A, \Diamond)$ into a modal algebra $\mathfrak{B} = (B, \Diamond)$ is an injective Boolean morphism $\mu : A \to B$ such that we have $\Diamond\mu(x) \leq \mu(\Diamond x)$ for all $x \in A$. For a frame $\mathfrak{F}$ we denote by $\mathfrak{F}^*$ its dual modal algebra and for an algebra $\mathfrak{A}$ we denote by $\mathfrak{A}_*$ its dual frame. Recall that a map $f : W \to W'$ between Kripke frames (W, R) and (W', R') is called *order-preserving* if for each $x, y \in W$ we have xRy implies $f(x)R'f(y)$.

The following proposition is proved in [2].

Proposition 5. *Let $\mathfrak{A} = (A, \Diamond)$ be a modal algebra. Then*

1. *$\mathfrak{A}$ does not validate $(r_{\mathfrak{F}})$ iff there is a stable embedding of $\mathfrak{F}^*$ into $\mathfrak{A}$.*
2. *$\mathfrak{A}$ does not validate $(r_{\mathfrak{F}})$ iff there is a surjective order-preserving map from $\mathfrak{A}_*$ onto $\mathfrak{F}$ (here $\mathfrak{A}_*$ is the descriptive frame dual to $\mathfrak{A}$).*

Our aim is to show that all modal calculi axiomatized by rules of the kind $(r_{\mathfrak{F}})$ have the bounded proof property. Rules $(r_{\mathfrak{F}})$, however, are not good for the bpp, see the counterexample below. We replace rules $(r_{\mathfrak{F}})$ by modified versions.

For each $a \in F$ we just add an extra propositional variable r_a and define the new rule $(r^+_{\mathfrak{F}})$ by

$$\frac{\bigvee_{i=1}^{n} x_{a_i}, \quad \bigwedge_{i\neq j} \neg(x_{a_i} \wedge x_{a_j}), \quad \bigwedge_{i=1}^{n}(x_{a_i} \to \Box r_{a_i}), \quad \bigwedge_{i=1}^{n}(r_{a_i} \to \bigvee_{b\in R_F(a_i)} x_b)}{\neg x_{a_1} \mid \cdots \mid \neg x_{a_n}}$$

(since we want the r_a to have at least one occurrence inside a modal operator, we might also add premisses like $\Box(r_a \vee \neg r_a)$ if needed).

Lemma 1. *Rules $(r^+_{\mathfrak{F}})$ and $(r_{\mathfrak{F}})$ are inter-derivable.*

Proof. On one side, $(r_{\mathfrak{F}})$ can be obtained from $(r^+_{\mathfrak{F}})$ by applying the substitution $r_{a_i} \mapsto \bigvee_{b\in R_F(a_i)} x_b$. On the other side, we apply necessitation and distribution to the premise $\bigwedge_{i=1}^{n}(r_{a_i} \to \bigvee_{b\in R(a_i)} x_b)$ and then transitivity of implication to obtain $\bigwedge_{i=1}^{n}(x_{a_i} \to \Box \bigvee_{b\in R_F(a_i)} x_b)$.

Notice that the above fragment of a derivation, when plugged into a hyperproof, may increase the modal degree (if the substitution used to apply the rule $(r_{\mathfrak{F}})$ replaces the x_b with formulae of, say, modal degree 1, we get formulae of modal degree 2 when we use $(r_{\mathfrak{F}})$ to simulate $(r^+_{\mathfrak{F}})$). This is why $(r^+_{\mathfrak{F}})$ is preferable to $(r_{\mathfrak{F}})$ from the point of view of the modal complexity analysis of proofs.

Theorem 5. *A modal calculus comprising only rules of the kind $(r^+_{\mathfrak{F}})$ enjoys the bpp and fmp.*

Proof. Let $\mathcal{S} = (W_1, W_0, f, R)$ be a finite conservative step frame. Consider the Kripke frame $(W_1, \tilde{R})$ where $\tilde{R}$ is defined by

$$w\tilde{R}w' \;\text{ iff }\; wRf(w') \tag{9}$$

(i.e. we have $\tilde{R} = f^o \circ R$, where f^o is the converse of f, seen as a relation). This is a finite Kripke frame having $\mathcal{S}$ as p-morphic image. In fact, (8) is satisfied by taking $\mu := id$ because $f \circ \tilde{R} = f \circ f^o \circ R = R$, (we used that $f \circ f^o = id$, which holds by the surjectivity of f).

We now show that $(W_1, \tilde{R})$ validates $(r_{\mathfrak{F}})$ (recall that $(r^+_{\mathfrak{F}})$ is equivalent to it in standard frames because the two rules are inter-derivable): to this aim, we prove that if there is a surjective R-preserving map μ from $(W_1, \tilde{R})$ onto $\mathfrak{F} = (F, R_F)$, then $\mathcal{S}$ does not validate $(r^+_{\mathfrak{F}})$. Suppose there is such a μ. Define now a valuation $\mathtt{v}$ by taking $\mathtt{v}(x_a) = \{w \mid \mu(w) = a\} \subseteq W_1$ and

$$\mathtt{v}(r_a) = \{v \in W_0 \mid \forall w\ (f(w) = v \Rightarrow aR_F\mu(w))\}.$$

The definition is well defined because the variables having at least an occurrence inside a modal operator are precisely the r_a's, so these variables are evaluated as subsets of W_0 and the other ones as subsets of W_1. Thus $\mathtt{v}$ evaluates to 1 the formulae $\bigvee_{i=1}^n x_{a_i}$ and $\bigwedge_{i \neq j} \neg(x_{a_i} \wedge x_{a_j})$, whereas $\neg x_{a_1}, \ldots, \neg x_{a_n}$ are not evaluated to 1 (because μ is surjective). It remains to check that for every $a \in F$, we have (i) $x_a^{\mathtt{v1}} \subseteq \Box r_a^{\mathtt{v1}}$ and (ii) $r_a^{\mathtt{v1}} \subseteq (\bigvee_{b \in R_F(a)} x_b)^{\mathtt{v1}}$. Now (i) holds by (9) and because μ is order-preserving: if $w \in x_a^{\mathtt{v1}}$ and wRv then $v \in r_a^{\mathtt{v1}}$ because if $f(w') = v$ then $w\tilde{R}w'$ and consequently $a = \mu(w)R_F\mu(w')$. To prove (ii), pick $w \in f^*(r_a)$; we have in particular $aR_F\mu(w)$, thus $w \in (\bigvee_{b \in R_F(a)} x_b)^{\mathtt{v1}}$. $\dashv$

From Lemma 1, we immediately obtain the following result from [2]:

Corollary 2. *A modal calculus comprising only rules of the kind* $(r_{\mathfrak{F}})$ *enjoys the finite model property.*

The following counterexample shows that we really need to replace $(r_{\mathfrak{F}})$ by $(r^+_{\mathfrak{F}})$ to get the bpp.

Example 6. Consider the two element reflexive chain

$$\mathfrak{F} \;:=\; \boxed{b \circlearrowleft \longrightarrow \circlearrowleft a}$$

The rule $(r_{\mathfrak{F}})$ simplifies to

$$\frac{x_a \to \Box x_a}{x_a \mid \neg x_a}$$

This rule is validated in a step frame $\mathcal{S} = (W_1, W_0, f, R)$ iff for every proper subset $a \subseteq W_0$ (i.e., for every subset different from $\emptyset, W_0$) there is $w \in W_1$ such that $f(w) \in a$ and $R(w) \not\subseteq a$. In a standard frame (W, S) this means that every pair of elements of W are connected via an S-path (to see this, consider as a the set of points which are reachable in $n \geq 0$ steps by any given point and show that such an a must be total). It is not difficult to check that putting

$W_1 := \{w_1, w_2\}, W_0 := \{v\}, f(w_1) := f(w_2) := v, R(w_1) := \{v\}, R(w_2) := \emptyset$, we obtain a finite conservative one-step frame that validates $(r_{\mathfrak{F}})$ but cannot be a p-morphic image of a standard Kripke frame validating it (because in the latter there cannot be terminal points and any pre-image of w_2 along a p-morphism must be such by (8)). Since the fmp holds for the modal calculus axiomatized by the rule $(r_{\mathfrak{F}})$ according to Corollary 2, it is clear that it is the bpp that fails for it (failure of the bpp can also be directly checked by using Theorem 3 instead of Theorem 4 and Corollary 2).

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A Proofs from Section 2

For the proof of the algebraic completeness Theorem 2, we need a couple of lemmas:

Lemma 2. *Weakening is admissible: we have* $\Gamma \vdash_K S \Rightarrow \Gamma \vdash_K S \mid S'$*, for every* S'*.*

Proof. Trivial by induction on the length of derivation. ⊣

Lemma 3. *Let* Γ *be a set of formulae,* α *a formula,* S *a hyper-formula and* K *a set of multiple-conclusion rules. If* $\Gamma \cup \{\alpha\} \vdash_K S$ *and* $\Gamma \vdash_K \alpha \mid S$*, then* $\Gamma \vdash_K S$*.*

Proof. Assume $\Gamma \vdash_K \alpha \mid S$. Using weakening, by induction on proof length, it is easy to see that $\Gamma \cup \{\alpha\} \vdash_K \tilde{S}$ implies $\Gamma \vdash_K S \mid \tilde{S}$ for every $\tilde{S}$. The claim now follows because $S \mid S$ is equal to S (hyper-formulae are defined as sets of formulae). ⊣

Theorem 2 *Let* K *be a set of multiple-conclusion rules. Then* $\Gamma \vdash_K S$ *iff the multiple-conclusion rule* Γ/S *is valid in every modal algebra validating* K*.*

Proof. One direction is trivial. For the other direction, let us suppose that $\Gamma \vdash_K S$ does not hold. By Zorn's lemma, pick $\tilde{\Gamma}$ to be a maximal set of formulae containing Γ such that $\tilde{\Gamma} \not\vdash_K S$. We claim that for every hyperformula $\alpha_1 \mid \cdots \mid \alpha_n$

$$\tilde{\Gamma} \vdash_K \alpha_1 \mid \cdots \mid \alpha_n \mid S \qquad \Rightarrow \qquad \exists i \;\; \tilde{\Gamma} \vdash_K \alpha_i. \tag{10}$$

In fact, if this does not hold, by the maximality of $\tilde{\Gamma}$, we have both that $\tilde{\Gamma} \vdash_K \alpha_1 \mid \cdots \mid \alpha_n \mid S$ and that $\tilde{\Gamma} \cup \{\alpha_1\} \vdash_K S$. By the above lemma, this implies $\tilde{\Gamma} \vdash_K \alpha_2 \mid \cdots \mid \alpha_n \mid S$. Repeating the argument n-times, we obtain $\tilde{\Gamma} \vdash_K S$, contradiction.

Now notice that Lemma 3 and the maximality of $\tilde{\Gamma}$ imply that if $\tilde{\Gamma} \vdash_K \alpha$, then $\alpha \in \tilde{\Gamma}$ and $\Box\alpha \in \tilde{\Gamma}$ (the latter is because necessitation rule is mentioned in condition (ii) of Definition (3)). In addition, $\tilde{\Gamma}$ contains Γ and is disjoint from S, by condition (i) of Definition (3). Thus, if we put

$$\alpha_1 \approx \alpha_2 \quad \Leftrightarrow \quad \alpha_1 \leftrightarrow \alpha_2 \in \tilde{\Gamma}$$

we can introduce on the set of equivalence classes a modal algebra structure $\mathfrak{A} = (A, \Diamond)$. Since Γ is included in $\tilde{\Gamma}$ and is disjoint from S, $\mathfrak{A}$ does not validate Γ/S. By the claim (10) and condition (ii) of Definition (3), it is evident that $\mathfrak{A}$ validates all rules from K. ⊣

Corollary 1 *Let* K *be a set of multiple-conclusion rules. For each multiple-conclusion rule* Γ/Δ*, we have* $K \vdash \Gamma/\Delta$ *iff* $\Gamma \vdash_K \Delta$*.*

Proof. If is sufficient to observe that (I) if $\Gamma \vdash_K \Delta$, then Γ/Δ belongs to every modal rule system $\mathcal{K}$ containing K and that (II) $\{\Gamma/\Delta \mid \Gamma \vdash_K \Delta\}$ is a modal rule system extending K.

Claim (II) is immediate from Lemmas 2, 3.

Claim (I) is by induction on the length of the K-hyper-proof witnessing $\Gamma \vdash_K \Delta$: for instance, if the K-hyper-proof ends with an application of the necessitation rule according to Definition 3(ii), then from $\Gamma/\alpha, S \in \mathcal{K}$ (this holds by induction hypothesis) and from the fact that the necessitation rule belongs to every modal rule system, from conditions (iii) and (ii) of Definition 1, we obtain $\Gamma/\Box\alpha, S \in \mathcal{K}$. [5] ⊣

We now fill the missing details for the proof of the

Proposition 1 *Let H be a finite set of hyper-rules. Then it is possible to produce a set of rules K such that for all $\Gamma, \tilde{S}$ we have $\Gamma \Vdash_H \tilde{S}$ iff $\Gamma \vdash_K \tilde{S}$.*

Proof. Consider a hyper-rule $S_1, \ldots, S_k/S$ from H: to obtain K, we simply replace it with the set of rules $\gamma(S_1), \ldots, \gamma(S_n)/S$, varying γ among the functions that pick one formula from each S_i, for each $i = 1, \ldots, n$.

The right-to-left claim of the proposition is immediate by weakening. To show the left-to-right direction, we use the argument below. Suppose H' is obtained from H by replacing the hyper-rule $S_1, \ldots, S_n/S$ with the pair of rules

$$S_1', S_2, \ldots, S_n/S, \qquad S_1'', S_2, \ldots, S_n/S \tag{11}$$

where we suppose that S_1', S_1'' are both not empty and such that $S_1 = S_1' \cup S_1''$. We claim that we have $\Gamma \Vdash_H S$ iff $\Gamma \Vdash_{H'} S$ (clearly, the statement of the proposition follows from an iterated application of this claim). Again that $\Gamma \Vdash_H \tilde{S} \Leftarrow \Gamma \Vdash_{H'} \tilde{S}$ holds is trivial by weakening. Now suppose that we have $\Gamma \Vdash_H \tilde{S}$. In the derivation witnessing this, there will possibly be lines labelled by $S_1\sigma \mid T, \ldots, S_n\sigma \mid T$ justifying a line labelled $S\sigma \mid T$ via the use of the hyper-rule $S_1, \ldots, S_n/S$. The derivation can be corrected so to use the rules (11) instead (iterated corrections will eliminate any use of the rule $S_1, \ldots, S_n/S$).[6] We first produce (by weakening) derivations of $S_2\sigma \mid S_1''\sigma \mid T$ and $\cdots$ and $S_n\sigma \mid S_1''\sigma \mid T$. These hyperformulae, combined with $S_1'\sigma \mid S_1''\sigma \mid T$ yield a derivation of $S_1''\sigma \mid S\sigma \mid T$ by applying the first hyper-rule from (11). By weakening again, we produce now derivations of $S_2\sigma \mid S\sigma \mid T$ and $\cdots$ and $S_n\sigma \mid S\sigma \mid T$. These hyperformulae, combined with $S_1''\sigma \mid S\sigma \mid T$ yields a derivation of $S\sigma \mid S\sigma \mid T$ by applying the second hyper-rule from (11) and we are done because $S\sigma \mid S\sigma \mid T$ is equal to $S\sigma \mid T$ (hyperformulae are sets, not multisets). ⊣

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[5] Notice that we added S to α because, according to the remark following Definition 3, when we apply the necessitation rule $\alpha/\Box\alpha$, then we deduce $\Box\alpha \mid S$ from a proof line containing the hyperformula $\alpha \mid S$.

[6] Applying multiset induction it is possible to show that the order of corrections does not matter. Alternatively, one can start correcting subderivations requiring a single use of the rule $S_1, \ldots, S_n/S$.

B Missed Proofs from Sections 3 and 4

In this appendix we collect proofs requiring minimal modifications to the proofs of the corresponding statements from [3, 4].

Proposition 2 *A calculus K has the bounded proof property iff for every hyper-formula S of modal complexity at most 1 and for every Γ containing only formulae of modal complexity at most 1, we have $\Gamma \vdash_K S \Rightarrow \Gamma \vdash^1_K S$.*

Proof. We define the modal complexity of a substitution σ to be the maximum of the complexities of the formulae $\sigma(x_i)$, varying x among propositional variables. Notice as a general fact that

(*) if σ has modal complexity at most k and ϕ has modal complexity at most l, then $\phi\sigma$ has modal complexity at most $k + l$.

Given our Γ, S of modal complexities at most n, we define Γ_i, S_i, σ_i $(0 \leq i \leq n-1)$ such that: (i) σ_i has modal complexity at most 1 and Γ_i, S_i have modal complexities at most $n-i$; (ii) $S_{i+1}\sigma_{i+1} = S_i$; (iii) $\Gamma_{i+1}\sigma_{i+1}$ is equal to the union of Γ_i with some tautologies of modal complexity at most 1; (iv) $\Gamma_{i+1} \vdash_K S_{i+1}$ iff $\Gamma_i \vdash_K S_i$.

We let Γ_0 be Γ, S_0 be S and σ_0 be the identity replacement. To define the $i+1$-th data, consider all subformulae of the kind $\Diamond\psi$ occurring in Γ_i, S_i, where ψ has complexity 0. For each such subformula, pick a fresh variable x_ψ, replace everywhere $\Diamond\psi$ by x_ψ in Γ_i, S_i. Then add $x_\psi \leftrightarrow \Diamond\psi$ to Γ_i and let σ_{i+1} be given by $\{x_\psi \mapsto \Diamond\psi\}_\psi$. Hence Properties (i)-(iv) hold.

Now suppose that $\Gamma \vdash_K S$. Then we have $\Gamma_{n-1} \vdash_K S_{n-1}$ by (iv) and also $\Gamma_{n-1} \vdash^1_S S_{n-1}$ by (i) and the hypothesis of the proposition. Next, if we apply σ_{n-1} to the proof certifying $\Gamma_{n-1} \vdash^1_{\mathrm{Ax}} S_{n-1}$, by (ii)-(iii) and (*), we obtain $\Gamma_{n-2} \vdash^2_{\mathrm{Ax}} S_{n-2}$. Repeating this for $\sigma_{n-1}, \ldots, \sigma_1$, we finally obtain $\Gamma \vdash^n_{\mathrm{Ax}} S$. ⊣

Proposition 4 *Let $\mathcal{A} = (A, B, i, \Diamond, \underline{\mathrm{a}})$ be an augmented conservative one-step modal algebra that validates the modal calculus K and the presentation $\Gamma(\underline{a})$. Then, for every hyper-formula $S(\underline{a})$, we have that $\Gamma \vdash^1_K S$ implies $\mathcal{A} \models S$.*

Proof. Let $S_1, \ldots, S_n$ be the derivation of S witnessing $\Gamma \vdash^1_K S$. Notice that all formulae in such a proof must have modal complexity at most one. In addition, we can freely suppose that variables do not occur in the proof (only parameters are there). If there are variables, they can be replaced by a tautology, still obtaining a proof witnessing $\Gamma \vdash^1_K S$, because variables do not occur in Γ, S. Recall from Definition 3 that each S_i belongs to Γ (modulo weakening) or is obtained from the previous S_j's by applying the rules of K, modus ponens, or necessitation.

The case of modus ponens is easy. Now assume that S_j is obtained from S_i by applying the rule of necessitation: thus S_i is $\phi \mid \tilde{S}$ and S_j is $\Box\phi \mid \tilde{S}$. Then ϕ is of complexity 0 (otherwise the complexity of S_j will be greater than one). The induction hypothesis yields that either $\mathcal{A} \models \tilde{S}(\underline{a})$ or $\mathcal{A} \models \phi_i$. The former case is trivial and in the latter we have $\phi_i^{\mathtt{a}1} = 1$, and $i(\phi_i^{\mathtt{a}0}) = 1$. Since $\mathcal{A}$ is conservative, i is injective, which yields $\phi_i^{\mathtt{a}0} = 1$, thus $(\Box\phi_i)^{\mathtt{a}1} = \Box\phi_i^{\mathtt{a}0} = 1$ and finally $\mathcal{A} \models S_j$.

For the case of inference rules, we argue as follows. Suppose that S_i is obtained from $S_{i_1}, \ldots, S_{i_k}$ by applying the reduced rule $\phi_1, \ldots, \phi_k/\psi_1 \mid \cdots \mid \psi_l$ from K. Thus, for some $\tilde{S}$ and for some substitution σ, we have that S_{i_1} is $\tilde{S} \mid \phi_1\sigma$, ..., S_{i_k} is $\tilde{S} \mid \phi_k\sigma$ and S_i is $\tilde{S} \mid \psi_1\sigma \mid \cdots \mid \psi_l\sigma$. Suppose that $\underline{x} = x_1, \ldots, x_n$ are the variables having in the rule at least an occurrence inside a modal operator and that $\underline{y} = y_1, \ldots, y_m$ are the remaining variables. Since every propositional variable $x_i \in \underline{x}$ occurring in the rule must have an occurrence inside a $\Diamond$ and the formulae occurring in the proof have modal complexity at most 1, the substitution σ must map such variables to formulae of complexity 0 (on the other hand, the $\underline{y}$ can be mapped into formulae of complexity 0 or 1). In other words, we have that $\sigma(x_i) = \theta_i(\underline{a})$ where the θ_i have modal complexity 0 ($1 \leq i \leq n$) and $\sigma(y_j) = \zeta_j(\underline{a})$, where the ζ_j have modal complexity 0 or 1 ($1 \leq j \leq m$). If we take a valuation $\mathtt{w}$ such that $\mathtt{w}(x_i) = \theta_i^{\mathtt{a}0}$ ($1 \leq i \leq n$) and $\mathtt{w}(y_j) = \zeta_j^{\mathtt{a}1}$ ($1 \leq i \leq m$), we can easily check by induction that, for every subformula δ of any formula occurring in the rule, we have that $\delta^{\mathtt{w}1}$ is equal to $(\delta\sigma)^{\underline{\mathtt{a}}1}$. Thus, from the induction hypothesis, in case $\mathcal{A}$ does not validate $\tilde{S}$, we have $(\phi_1\sigma)^{\mathtt{a}1} = 1, \ldots, (\phi_m\sigma)^{\mathtt{a}1} = 1$, that is $\phi_1^{\mathtt{w}1} = 1, \ldots, \phi_m^{\mathtt{w}1} = 1$. Since $\mathcal{A}$ validates the rules, we must have that there is $k = 1, \ldots, n$ such that $1 = \psi_j^{\mathtt{w}1} = (\psi_j\sigma)^{\mathtt{a}1}$, namely $\mathcal{A} \models S_i$. ⊣

The following lemma is immediate.

Lemma 4. *Let (h, k) be an embedding between one-step modal algebras $\mathcal{A} = (A_0, A_1, i_0, \Diamond_0)$ and $\mathcal{A}' = (A'_0, A'_1, i'_0, \Diamond'_0)$. Suppose they are both augmented and that for the respective interpretations $\underline{\mathtt{a}}, \underline{\mathtt{a}}'$ of the parameters $\underline{a} = a_1, \ldots, a_n$, we have $h(\underline{\mathtt{a}}) = \underline{\mathtt{a}}'$, that is, $h(\mathtt{a}_i) = \mathtt{a}'_i$ for all $i = 1, \ldots, n$. Then for every hyperformula $S(\underline{a})$, we have $\mathcal{A} \models S(\underline{a})$ iff $\mathcal{A}' \models S(\underline{a})$.*

Corollary 3. *Suppose that there is an embedding between the one-step modal algebras $\mathcal{A}$ and $\mathcal{A}'$. Then, if $\mathcal{A}'$ validates a modal calculus K, so does $\mathcal{A}$.*

We now introduce an important ingredient of our proofs, namely *diagrams*. These are adaptations to our step contexts of classical methods in mathematical logic, due to A. Robinson in the model-theoretic environment [8, Ch. 2.1] and due to Jankov and Fine in the modal logic environment (see, e.g., [14, Sec. 7.3], [7, Sec. 9.4]). These rules are also similar to stable canonical rules of [2].

Let $\mathcal{A} = (A, B, i, \Diamond)$ be a finite conservative one-step algebra. For each $\mathtt{a} \in A$ we introduce a parameter $p_{\mathtt{a}}$ (below we call $\underline{a}$ the tuple of such parameters). Let

$$\begin{aligned} \Gamma^0_{\mathcal{A}}(\underline{a}) := {} & \{p_{\mathtt{a}\vee\mathtt{b}} \leftrightarrow p_{\mathtt{a}} \vee p_{\mathtt{b}} : \mathtt{a}, \mathtt{b} \in A\} \cup \\ & \{p_{\mathtt{a}\wedge\mathtt{b}} \leftrightarrow p_{\mathtt{a}} \wedge p_{\mathtt{b}} : \mathtt{a}, \mathtt{b} \in A\} \cup \\ & \{p_{\neg\mathtt{a}} \leftrightarrow \neg p_{\mathtt{a}} : \mathtt{a} \in A\}. \end{aligned}$$

We augment $\mathcal{A}$ by interpreting every parameter $p_{\mathtt{a}}$ as $\mathtt{a}$. By the conservativity of $\mathcal{A}$, for every $\mathtt{b} \in B$, there is $\theta_{\mathtt{b}}$ such that $\mathtt{b}$ is equal to $\theta_{\mathtt{b}}^{\underline{\mathtt{a}}1}$. Notice that from our definitions, it follows in particular that for $\mathtt{a} \in A$, we have $\theta_{i(\mathtt{a})} = p_{\mathtt{a}}$.

Now let

$$\begin{aligned}\Gamma^1_{\mathcal{A}}(\underline{a}) := &\{\theta_{\mathtt{a}\vee\mathtt{b}} \leftrightarrow \theta_{\mathtt{a}} \vee \theta_{\mathtt{b}} : \mathtt{a},\mathtt{b} \in B\} \cup \\ &\{\theta_{\mathtt{a}\wedge\mathtt{b}} \leftrightarrow \theta_{\mathtt{a}} \wedge \theta_{\mathtt{b}} : \mathtt{a},\mathtt{b} \in B\} \cup \\ &\{\theta_{\neg\mathtt{a}} \leftrightarrow \neg\theta_{\mathtt{a}} : \mathtt{a} \in B\} \cup \\ &\{\theta_{\Diamond\mathtt{a}} \leftrightarrow \Diamond p_{\mathtt{a}} : \mathtt{a} \in A\}.\end{aligned}$$

The *positive diagram* of $\mathcal{A}$ is the set of formulae $\Gamma_{\mathcal{A}}(\underline{a}) := \Gamma^0_{\mathcal{A}}(\underline{a}) \cup \Gamma^1_{\mathcal{A}}(\underline{a})$ and the *negative diagram* of $\mathcal{A}$ is the set of formulae

$$\Delta_{\mathcal{A}}(\underline{a}) := \{\theta_{\mathtt{a}} \leftrightarrow \theta_{\mathtt{b}} : \mathtt{a} \neq \mathtt{b}, \text{ for } \mathtt{a},\mathtt{b} \in B\}.$$

In this paper, *we view the negative diagram as a hyper-formula.*

We say that an augmented modal algebra $\mathcal{C}$ refutes $\Gamma_{\mathcal{A}} \vdash \Delta_{\mathcal{A}}$ iff we have $\mathcal{C} \models \phi(\underline{a})$ forall $\phi(\underline{a}) \in \Gamma_{\mathcal{A}}$ and $\mathcal{C} \not\models \Delta_{\mathcal{A}}$. The following Lemma is proved in [3]:

Lemma 5. *Let $\mathcal{A}$ be a conservative finite one-step algebra with the natural augmentation $\underline{\mathtt{a}}$ (interpreting every parameter $p_{\mathtt{a}}$ to $\mathtt{a}$). Then*

1. *$\mathcal{A}$ refutes $\Gamma_{\mathcal{A}} \vdash \Delta_{\mathcal{A}}$.*
2. *For each conservative one-step algebra $(C_0, C_1, j, \Diamond)$, there is an augmentation $\underline{\mathtt{c}}$ such that $\mathcal{C} = (C_0, C_1, j, \Diamond, \underline{\mathtt{c}})$ refutes $\Gamma_{\mathcal{A}} \vdash \Delta_{\mathcal{A}}$ iff $\mathcal{A}$ is embeddable into $\mathcal{C}$.*

The following lemma (also proved in [3]) relates embeddings into standard algebras with extensions.

Lemma 6. *Let $\mathcal{A} = (A_0, A_1, i_0, \Diamond_0)$ be a one-step modal algebra and let $(k \circ i_0, k)$ be an embedding of it into a standard modal algebra $\mathfrak{A} = (A, \Diamond)$. Then the one-step modal algebra $\mathcal{A}' = (A_1, A, k, \Diamond \circ k)$ is an extension of $\mathcal{A}$. Moreover, if $\mathfrak{A}$ validates a modal calculus K, then so does $\mathcal{A}'$.*

Theorem 3 *A modal calculus K has the bpp iff the class of finite one-step modal algebras (equivalenly, the class of finite one-step frames) validating K has the extension property.*

Proof. Suppose that the class of one-step modal algebras validating K has the extension property and let $\Gamma(\underline{a})$ be a finite presentation (i.e., a finite set of formulae of modal complexity at most 1) such that $\Gamma \not\vdash^1_K S$ for a hyper-formula $S(\underline{a})$ of modal complexity at most 1. We adapt the Lindembaum-like construction of the proof of Theorem 2 to build a one-step modal algebra $\mathcal{A}_0 = (A_0, A_1, i_0, \Diamond_0)$ as follows.

By Zorn lemma, pick $\tilde{\Gamma}$ to be a maximal set of formulae containing Γ such that $\tilde{\Gamma} \not\vdash^1_K S$. By the same argument used in the proof of Theorem 2, we have that for every hyperformula $\alpha_1 \mid \cdots \mid \alpha_n$ of modal degree at most 1

$$\tilde{\Gamma} \vdash_K \alpha_1 \mid \cdots \mid \alpha_n \mid S \qquad \Rightarrow \qquad \exists i \;\; \tilde{\Gamma} \vdash_K \alpha_i. \tag{12}$$

Now if we put

$$\alpha_1 \approx \alpha_2 \quad \Leftrightarrow \quad \alpha_1 \leftrightarrow \alpha_2 \in \tilde{\Gamma}$$

we can introduce on the set of equivalence classes a step algebra structure $\mathcal{A}_0 = (A_0, A_1, i_0 \Diamond_0)$, by considering the equivalence classes of the formulae of complexity 0 and of complexity (0 or 1), respectively, with the obvious morphisms. Moreover, using the claim (12) and considerations similar to those employed in the proof of Theorem 2, we can prove the following: (i) i_0 is injective and $\mathcal{A}_0$ is conservative (because the formulae of modal complexity 1 can all be obtained as Boolean combinations of formulae of modal complexity 0 and of formulae of the kind $\Diamond\psi$, where ψ has complexity 0); (ii) $\mathcal{A}_0$ validates K by construction; (iii) if we define an augmentation by taking $\mathtt{a}$ to be the tuple of the equivalence classes in A_0 of the parameters $\underline{a}$, we have $(\bigwedge \Gamma)^{\mathtt{a}1} = 1$ and $\phi^{\mathtt{a}1} \neq 1$, for all $\phi \in S$.

By the extension property, there is an extension $\mathcal{A}_1 = (A_1, A_2, i_1, \Diamond_1)$, with injective i_1, also validating K. We can freely assume that $\mathcal{A}_1$ is conservative; otherwise, we replace A_2 by the subalgebra generated by the images of i_1 and $\Diamond_1$, which as a subalgebra also trivially validates K. If we continue in this way, we generate a chain

$$A_0 \xrightarrow{i_0} A_1 \to \cdots \to A_k \xrightarrow{i_k} A_{k+1} \to \cdots \tag{13}$$

of Boolean algebras equipped with semilattice morphisms

$$A_0 \xrightarrow{\Diamond_0} A_1 \to \cdots \to A_k \xrightarrow{\Diamond_k} A_{k+1} \to \cdots \tag{14}$$

satisfying the conditions $\Diamond_{k+1} \circ i_k = i_{k+1} \circ \Diamond_k$ and such that for every $k \geq 0$, the one-step modal algebra $(A_k, A_{k+1}, i_k, \Diamond_k)$ validates Ax. Thus, the Boolean algebra A obtained by taking the colimit of (13) can be endowed with a semilattice morphism $\Diamond : A \longrightarrow A$ in such a way that $\mathfrak{A} := (A, \Diamond)$ is a standard modal algebra validating Ax by construction. In this algebra, under the obvious augmentation obtained by composing the previous augmentation $\mathtt{a}$ with the inclusion of A_0 into the colimit, since the i_k are all injective, we have $\mathfrak{A} \models \Gamma(\underline{a})$ and $\mathfrak{A} \not\models S(\underline{a})$. Thus, we found an augmented (standard) modal algebra validating K, Γ but not S: this implies that $\Gamma \not\vdash_S S$ by Theorem 2.

Conversely, suppose that the bpp holds and take a conservative finite one-step modal algebra $\mathcal{A} = (A_0, A_1, i_0, \Diamond)$. Let $\underline{a}$ be a list of parameters naming the elements of A_0. Since $\mathcal{A}$ (with the natural augmentation) refutes $\Gamma_{\mathcal{A}} \vdash \Delta_{\mathcal{A}}$, by Lemma 4, we have that $\Gamma_{\mathcal{A}} \not\vdash^1_K \Delta_{\mathcal{A}}$. By the bpp, we have

$$\Gamma_{\mathcal{A}} \not\vdash_K \Delta_{\mathcal{A}}. \tag{15}$$

By Theorem 2, there is a modal algebra $\mathfrak{A}$ such that $\mathfrak{A} \not\models \Gamma_{\mathcal{A}}/\Delta_{\mathcal{A}}$. Then by Lemma 5, $\mathcal{A}$ is embeddable into $\mathfrak{A}$ (seen as a one-step algebra), via an embedding, say, $(k \circ i_0, k)$. We now apply Lemma 6 and conclude that $\mathcal{A}' = (A_1, A, k, \Diamond \circ k)$ is an extension of $\mathcal{A}$ validating K. Since $\mathcal{A}'$ needs not be finite, we can consider the one-step subalgebra $\mathcal{B} = (A_1, A_2, \tilde{k}, \tilde{\Diamond})$, where A_2 is the Boolean subalgebra generated by the images of A_1 under k and $\Diamond \circ k$ and where $\tilde{k}, \tilde{\Diamond}$ are $k, \Diamond \circ k$, respectively, restricted in their codomains. Since $\mathcal{A}$ has an extension $\mathcal{B}$ validating K, the result follows. ⊣

Theorem 4 *A modal calculus K has both the bpp and the fmp iff every finite conservative one-step frame validating K is a p-morphic image of some finite frame validating K (equivalently, iff every finite conservative one-step modal algebra validating K has an embedding into some finite modal algebra validating K).*

Proof. First assume that every finite conservative one-step modal algebra validating K has an embedding into some standard finite modal algebra validating K. Since this implies that the class of finite one-step modal algebras validating K has the extension property (by Lemma 6), K has the bpp by Theorem 3. Now, to show that fmp holds, suppose that $\Gamma \not\vdash_K S$, for a finite set $\Gamma(\underline{a})$ and for a hyper-formula $S(\underline{a})$. By applying the same method as in the proof of Proposition 2, we can freely assume that Γ, ϕ have complexity 1. Like in the proof of Theorem 2, pick $\tilde{\Gamma}(\underline{a})$ to be a maximal set of formulae such that $\Gamma \subseteq \tilde{\Gamma}$ and $\tilde{\Gamma} \not\vdash_K S$ and build the 'Lindembaum algebra' for $\tilde{\Gamma}, K$. This is the algebra defined in the following way: for formulae $\psi_1(\underline{a}), \psi_2(\underline{a})$, let us put $\psi_1 \approx \psi_2$ iff $\tilde{\Gamma} \vdash_K \psi_1 \leftrightarrow \psi_2$. This is clearly an equivalence relation and we can build an augmented modal algebra out of it by defining all operations on equivalence classes. In particular, the selected tuple $\underline{\mathsf{a}}$ will be the tuple of the equivalence classes of the $\underline{a}$. We obtain Boolean subalgebras $A_0, A_1, \dots$ by considering the equivalence classes of the formulae of modal complexity at most 0, 1, ... Then S is refuted in the augmented one-step modal algebra $\mathcal{A} = (A_0, A_1, i_0, \Diamond_0, \underline{\mathsf{a}})$ which is such that $\mathcal{A} \models \Gamma(\underline{a})$ and $\mathcal{A} \models K$. Here i_0 is inclusion and $\Diamond_0$ is the restriction of $\Diamond$ to A_0 in the domain and A_1 in the codomain. By our assumption, $\mathcal{A}$ embeds (via some k satisfying (6)) into a finite modal algebra $\mathfrak{A} = (A, \Diamond)$ validating K. We can augment $\mathfrak{A}$ by taking the $k(i_0(\underline{\mathsf{a}}))$ as the selected tuple interpreting the parameters $\underline{a}$. As embeddings preserve the interpretation of formulas (see Lemma 4), we have that $(A, \Diamond, k(i_0(\underline{\mathsf{a}})))$ also refutes $S(\underline{a})$, validates $\Gamma(\underline{a})$. Hence, $(A, \Diamond, k(i_0(\underline{\mathsf{a}})))$ is a countermodel to $\Gamma \vdash_K S$, and thus K has the finite model property.

Now suppose K has both the bpp and the fmp and let $\mathcal{A}$ be a finite conservative one-step modal algebra that validates K. Since $\mathcal{A}$ (with the natural augmentation) refutes $\Gamma_{\mathcal{A}} \vdash \Delta_{\mathcal{A}}$, by Lemma 4, we have that $\Gamma_{\mathcal{A}} \not\vdash^1_K \Delta_{\mathcal{A}}$. By the bpp, we have $\Gamma_{\mathcal{A}} \not\vdash_K \Delta_{\mathcal{A}}$ and by the fmp there exists an augmented finite modal algebra $\mathfrak{C}$ witnessing this; thus $\mathfrak{C}$ refutes $\Gamma_{\mathcal{A}} \vdash \Delta_{\mathcal{A}}$ and validates K. By Lemma 5, this implies that $\mathcal{A}$ is embedded into $\mathfrak{C}$. ⊣